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Resumen

En este trabajo se diseña e implementa un esquema de control tolerante a fallas para una clase de sistemas no lineales de múltiples entradas y salidas con dinámicas de orden entero. Este esquema se basa en la construcción de una forma canónica, mediante la cual se realiza el seguimiento de las salidas del sistema por medio de observadores. Entonces se determina un controlador dinámico que linealiza la dinámica del error de seguimiento, con lo que se rechazan los efectos de las fallas en el sistema. Este controlador utiliza estimaciones de las fallas, por lo que un diagnóstico de las mismas es indispensable. Este diagnóstico se realiza también mediante observadores, utilizando para su construcción el enfoque algebraico diferencial. Una vez desarrollada la metodología, se demuestra que el sistema en lazo cerrado es asintóticamente estable. El control tolerante a fallas propuesto es evaluado mediante simulaciones en dos modelos. Posteriormente, la metodología presentada se extiende a sistemas con dinámicas de orden fraccionario, donde el diagnóstico de fallas, el diseño del controlador dinámico y la prueba de estabilidad del sistema en lazo cerrado son debidamente adaptados por medio de las herramientas del cálculo fraccionario. Finalmente, se realiza un comparativo entre los resultados obtenidos en los modelos con dinámicas de orden fraccionario y entero.

Abstract

In this work a fault-tolerant control scheme is designed and implemented, for a class of multi-input multi-output nonlinear systems with integer-order dynamics. This scheme is based on the construction of a canonical form, whereby tracking of the system outputs is performed by means of observers. Then a dynamical controller is determined, which linearizes the tracking error dynamics, and thus the effects of the faults in the system are rejected. This controller uses estimations of the faults, hence a fault diagnosis is essential. This diagnosis is performed also by way of observers, using the differential algebraic approach to construct them. Once the methodology is developed, it is proved that the closed-loop system is asymptotically stable. The proposed fault-tolerant control is assessed by means of simulations in two models. Later, the presented methodology is extended to systems with fractional-order dynamics, where the fault diagnosis, the design the dynamical controller and the stability proof of the closed-loop system are properly adapted with the aid of the tools of fractional calculus. Finally, it is performed a comparison between the results obtained in the models with fractional and integer-order dynamics.

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Capítulo 1

Introducción

1.1. Estado del arte

1.1.1. Diagnóstico y control tolerante a fallas

Las fallas son señales que aparecen en diferentes tipos de sistemas físicos de interés principal en aplicaciones industriales o de ingeniería. Las fallas son consideradas como desviaciones no permitidas de una o más características o parámetros de un sistema con respecto a su condición nominal. Es por esto que las fallas tienen efectos perjudiciales sobre el desempeño correcto y las buenas condiciones del equipo físico, por lo que es imperativo diseñar e implementar métodos para detectar la aparición de estas señales y eliminar o disminuir sus consecuencias.

La acción de determinar de forma binaria la existencia de fallas, así como el instante en el que ocurren, se conoce como detección de fallas. Si también se especifica el tipo de fallas y se localizan, se conoce como aislamiento de fallas. Si por otro lado, se determina la magnitud y comportamiento respecto al tiempo de las fallas, dicha acción es conocida como identificación de fallas. Finalmente, cuando se llevan a cabo detección, aislamiento e identificación, se dice que se realiza un diagnóstico de fallas (DF).

El DF ha sido un área de interés para investigación desde hace tiempo. Existen muchos trabajos que estudian este problema desde el enfoque de sistemas de control, como [1, 66], que involucran generación de residuos, desacoplamiento de entradas no deseadas, y enfoques adaptables. Para sistemas lineales, también se ha aplicado el enfoque geométrico [37], donde se utiliza el concepto de subespacios de inobservabilidad en conjunto con la generación de residuos. Existen también muchos trabajos que estudian el DF en sistemas no lineales [19, 20, 67]. El enfoque geométrico para detección y aislamiento de fallas vía generación de residuos también ha sido extendido a sistemas no lineales, donde las distribuciones son consideradas como los equivalentes no lineales de los subespacios de inobservabilidad [10].

Como se mencionó, una vez diagnosticada la falla es importante implementar una estrategia de control para suprimir o reducir en la mayor medida posible sus efectos. Es por esto que surge la necesidad de un control tolerante a fallas (CTF) diseñado acorde al sistema en cuestión, que elimine el efecto causado en los sistemas por las fallas una vez que éstas han sido diagnosticadas. El CTF forma una parte esencial en muchas aplicaciones existentes en automatización e ingeniería [6, 25, 29, 60]. Existen también diversos enfoques para obtener dicho tipo de control; el estudio [46] provee de una revisión de literatura básica que cubre la mayor parte de las áreas que comprende el CTF. El libro [5] presenta un enfoque basado en modelos para el CTF. Particularmente en el campo de sistemas no lineales, se obtuvieron recientemente resultados favorables aplicando técnicas algebraicas [15, 22, 23, 25, 29].

1.1.2. Sistemas de orden fraccionario

En su definición básica, el cálculo fraccionario es la generalización matemática del cálculo clásico, la cual involucra integrales y derivadas cuyo orden no es entero. Esta teoría surgió históricamente el 30 de septiembre de 1695, fecha relativamente cercana al inicio de la teoría del cálculo clásico. En un carta dirigida a G. W. Leibniz, G. de L'Hopital le cuestiona, en referencia a su recientemente propuesta notación d^n/dx^n, qué sucedería si $n = 1/2$, a lo que Leibniz responde que conduciría a una paradoja de la cual algún día importantes consecuencias se obtendrían. A esto se debe la etimología de la teoría, aunque en un sentido estricto el cálculo fraccionario involucra integrales y derivadas de orden real, ya sea racional o irracional, o incluso complejo; actualmente el orden de estos operadores puede ser constante, variante en el tiempo, aleatorio o difuso.

El comentario de Leibniz llevó a que a finales del siglo XIX aparecieran integrales y derivadas de orden arbitrario en una forma casi definitiva, debido principalmente a J. Liouville, A. K. Grünwald, A. V. Letnikov y G. F. B. Riemann. Sin embargo, por casi tres siglos la teoría se desarrolló principalmente como un campo teórico puro, útil solamente para los matemáticos. No fue sino hasta la década de los 70 del siglo XX que el cálculo fraccionario comenzó a ser objeto de diversas conferencias y tratados especializados. En cuanto a la primera conferencia, el mérito se debe a B. Ross, quien luego de su tesis doctoral en el área, organizó la Primera Conferencia de Cálculo Fraccionario y sus Aplicaciones en 1974 [56]. Concerniente a la primera monografía, el mérito se atribuye a K. B. Oldham e I. Spanier, quienes después de una colaboración iniciada en 1968, publicaron un libro dedicado al cálculo fraccionario en 1974 [45]. Al día de hoy existe una buena cantidad de libros [11, 43, 44, 47, 49], así como artículos de revistas y congresos especializados en el área, ya sea enfocados en la parte teórica o en las aplicaciones al mundo real.

Los sistemas dinámicos de orden fraccionario, es decir, sistemas cuyo modelo matemático es representado mediante derivadas e integrales de orden no entero, han sido fuertemente estudiados en décadas recientes. Esto se debe a la gran cantidad de aplicaciones y problemas físicos multidisciplinarios que presentan dinámicas con derivadas e integrales fraccionarias, tales como ciencia de materiales [9], sistemas térmicos [16], sistemas mecánicos amortiguados [17], circuitos eléctricos [24], bioingeniería [28], comportamiento polimérico [42], problemas de difusión [45], electromagnetismo [55], finanzas [59], viscoelasticidad [61], electromecánica [68], etc. Además, en ciertos casos las ecuaciones fraccionarias dan mejores aproximaciones del comportamiento de los sistemas que las de orden entero.

También se han desarrollado aplicaciones del cálculo fraccionario en la teoría de control. A. Oustaloup estudió algoritmos de orden fraccionario para el control de sistemas dinámicos y demostró la superioridad del CRONE (control robusto de orden no entero) [58] sobre el controlador PID. Por otro lado, I. Podlubny propuso una generalización del controlador PID, llamado controlador $\mathrm{PI}^\lambda\mathrm{D}^\mu$ [50], con un integrador de orden λ y un derivador de orden μ. A partir de aquí otras técnicas del control han sido extendidas a órdenes fraccionarios, como el control por modos deslizantes, el control adaptable por modelo de referencia y el control por reset [44]. De igual forma, se han desarrollado diversos resultados de estabilidad basados en diferentes enfoques [27, 38, 57].

En particular, con relación al DF en sistemas fraccionarios, se han utilizado enfoques como la generación de residuos [3], el espacio generalizado de paridad dinámica [4] y los modos deslizantes [48]. En cuanto al CTF, se han utilizado técnicas como el control aditivo [7], el control robusto contra fallas en los actuadores [62] y el control $\mathrm{PI}^\lambda\mathrm{D}^\mu$ [63].

1.2. Contribución

En este trabajo se pretende implementar un esquema de CTF para una clase de sistemas no lineales de múltiples entradas y salidas (MIMO) de orden entero, para posteriormente extenderlo a sistemas de orden fraccionario.

Primeramente, debe realizarse un DF. Para ello, se utiliza el enfoque algebraico diferencial [30, 31]. En este enfoque, las fallas deben satisfacer la propiedad de observabilidad algebraica para poder ser diagnosticadas. Este enfoque utiliza observadores de orden reducido para obtener estimaciones efectivas de las fallas, permitiendo a su vez estimar múltiples fallas simultáneamente [33, 64].

Una vez diagnosticadas las fallas, se propone un CTF dinámico basado en el seguimiento de la salida. Para esto, el sistema en cuestión es llevado a una forma canónica, y definiendo un error de seguimiento de la salida, se obtiene una forma canónica de este error. A partir de esta forma canónica se construye un observador de alta ganancia, el cual es utilizado para controlar la dinámica del error de seguimiento; dado que es un sistema MIMO, se obtendrá un observador por cada una de las salidas del sistema. Por último, a partir de la dinámica de estos observadores, se obtienen las ecuaciones que constituyen el controlador dinámico que rechazará el efecto de las fallas al realizar el seguimiento de la salida.

Cabe recalcar que el controlador dinámico propuesto es capaz de linealizar la dinámica del error de seguimiento. Además, utiliza estimaciones de las fallas, por lo que se toman las obtenidas mediante el DF. Una vez desarrollada la metodología, se verifica que el sistema en lazo cerrado es asintóticamente estable.

Para probar la metodología propuesta, se llevan a cabo simulaciones en dos modelos. El primero es un ejemplo numérico, mientras que el segundo se trata del sistema de tres tanques Amira DTS200 [2]. En este sistema se pueden introducir fallas múltiples en sensores y actuadores, lo que lo convierte en una plataforma versátil. Es por este motivo que este sistema ha sido ampliamente usado para estudios concernientes al DF y CTF [15, 22, 23, 33].

Posteriormente, la metodología para el DF y el CTF en sistemas de orden entero es extendida a sistemas fraccionarios de orden conmensurado, es decir, donde todas las dinámicas fraccionarias tienen el mismo orden. En este caso se redefine la condición de observabilidad algebraica, y se propone un observador de orden reducido para fallas con dinámicas fraccionarias. De igual forma, los resultados de estabilidad que se utilizaron para el caso entero son adaptados para poder desarrollar la metodología y obtener un CTF dinámico fraccionario. En este caso, se verifica que el sistema en lazo cerrado es Mittag-Leffler estable. Posteriormente, la metodología propuesta para este caso se prueba en dos modelos con dinámicas fraccionarias: el oscilador de Van der Pol y un motor de corriente directa (CD). Finalmente, se realiza un comparativo de los resultados obtenidos con el modelo del motor de CD en el caso fraccionario con resultados del caso entero para el mismo sistema, utilizando diferentes valores del orden fraccionario.

1.3. Objetivos

1.3.1. Objetivo general

Diseñar e implementar técnicas para el control de sistemas de orden entero y fraccionario, con base en observadores de estado.

1.3.2. Objetivos particulares

- Diseñar e implementar un esquema de control dinámico tolerante a fallas basado en observación para sistemas de orden entero.
- Diseñar e implementar un esquema de control dinámico tolerante a fallas basado en observación para sistemas de orden fraccionario.

1.4. Estructura de la tesis

En el Capítulo 2 se introducen brevemente algunas definiciones del álgebra diferencial, la cual es la herramienta matemática que se utiliza para el diagnóstico de fallas, así como para el diseño del esquema de control tolerante a fallas. Se presentan conceptos como las extensiones de campos diferenciales, el elemento primitivo diferencial, las dinámicas no lineales y las formas canónicas.

En el Capítulo 3 se desarrolla la metodología propuesta para el DF y el CTF en sistemas de orden entero. Se presenta la clase de sistemas con la que se trabajará, la noción de observabilidad algebraica y el observador de orden reducido para llevar a cabo el DF. Posteriormente se define una forma canónica a partir del sistema nominal, y por medio de la dinámica del error de seguimiento de la salida, se construye un observador de alta ganancia, de cuya dinámica se obtiene el CTF dinámico. Se demuestra que el sistema en lazo cerrado es estable. Finalmente, se aplica la metodología propuesta en un ejemplo académico y en el modelo del sistema de tres tanques Amira DTS200.

En el Capítulo 4 se definen algunos conceptos básicos del cálculo fraccionario, así como de la teoría de los sistemas dinámicos fraccionarios. Se presentan las funciones gamma y Mittag-Leffler, y se definen las integrales y derivadas de orden fraccionario. A continuación se definen los sistemas fraccionarios de tipo conmensurado. Posteriormente se presentan algunos controladores desarrollados para sistemas dinámicos fraccionarios, así como algunos resultados de estabilidad existentes para sistemas lineales y no lineales.

En el Capítulo 5 se extiende la metodología propuesta en el Capítulo 3 para DF y CTF a sistemas fraccionarios de orden conmensurado. Se define la condición de observabilidad algebraica fraccionaria para sistemas conmensurados, y se propone un observador de orden reducido fraccionario. Los resultados de estabilidad presentados en el Capítulo 4 son utilizados para la dinámica lineal del error y la prueba de estabilidad del sistema en lazo cerrado, el cual en este caso se verifica que es Mittag-Leffler estable. Para probar la metodología desarrollada para sistemas fraccionarios, se realizan simulaciones en los modelos del oscilador de Van der Pol y de un motor de CD con dinámicas fraccionarias. Por último, se lleva a cabo un comparativo de los resultados obtenidos para el motor de CD con los resultados de simulación en el mismo sistema con dinámicas de orden entero, para distintos órdenes fraccionarios.

En la parte de Conclusiones se realizan comentarios finales y se menciona el trabajo futuro que puede derivar de lo que se ha desarrollado.

Por último, se agrega la Bibliografía consultada y un Anexo con los artículos publicados a partir de este trabajo, así como la carátula de los artículos que han sido sometidos y se encuentran en revisión.

Finalmente, cabe recalcar que en este trabajo las variables $x(t)$, $u(t)$, $y(t)$, $f(t)$, $\eta(t)$, $e(t)$, $\varepsilon(t)$, $\gamma(t)$ y $\xi(t)$ son funciones del tiempo, por lo que se prescindirá de esta notación explícita con el fin de simplificar la lectura.

Capítulo 2

Fundamentos de álgebra diferencial

En este capítulo se presentan brevemente las bases del álgebra diferencial, herramienta matemática utilizada en el diseño de controladores para sistemas lineales y no lineales. Algunos conceptos de esta teoría serán utilizados en el capítulo siguiente para la detección de fallas y para construir el esquema de control tolerante a fallas en el caso entero.

2.1. Álgebra diferencial

El álgebra diferencial fue introducida por J. F. Ritt en 1932 [54], con el fin de introducir la completud de la teoría de sistemas de ecuaciones algebraicas en la teoría de sistemas de ecuaciones diferenciales que son algebraicas en las incógnitas y sus derivadas. En ese momento, el álgebra abstracta maduraba de la mano de R. Dedekind, L. Kronecker, D. Hilbert, E. Noether y B. L. van der Waerden. Además de los trabajos de Ritt, las bases del álgebra diferencial pueden encontrarse también en la obra de E. R. Kolchin [26].

2.1.1. Campos diferenciales

La teoría de campos fue creada en la segunda mitad del siglo XIX con el fin de evitar grandes manipulaciones de ecuaciones algebraicas. De forma similar la geometría diferencial, de gran uso actual en control no lineal [21], fue en parte desarrollada para evitar la abundancia de índices.

Definición 2.1.1. *Un campo diferencial K es un campo conmutativo equipado con una derivación simple $d/dt =$ " $\cdot$ ". Esta derivación obedece las reglas usuales:*

$$\frac{d}{dt}(a+b) = \dot{a} + \dot{b}$$
$$\frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$$

$\forall\ a, b \in K$.

Definición 2.1.2. *Una constante de K es un elemento $c \in K$ tal que $\dot{c} = 0$. El conjunto de constantes de K es un subcampo de K, llamado el campo de constantes.*

Definición 2.1.3. *Una extensión de campo diferencial L/K se da por dos campos diferenciales K, L tales que:*

i) K es un subcampo de L,

ii) la derivación de K es la restricción a K de la derivación de L.

Sea E/F una extensión de campo (no diferencial), es decir, dos campos (conmutativos no diferenciales) E, F tales que $F \subseteq E$. Solamente dos situaciones son posibles.

Definición 2.1.4. *Se dice que un elemento de E es algebraico sobre F si y solo si satisface una ecuación algebraica con coeficientes en F. La extensión E/F es algebraica si y solo si cualquier elemento de E es algebraico sobre F.*

Ejemplo 2.1.5. *$\sqrt{2}$ es una raíz de $x^2 - 2 = 0$ y es algebraica sobre $\mathbb{Q}$.*

Definición 2.1.6. *Se dice que un elemento $a \in E$ es trascendental sobre F si y solo si no es algebraico sobre F. Esto implica que no existe un polinomio de una sola variable $p(x)$ sobre F tal que $p(a) = 0$. Se dice que la extensión E/F es trascendental si y solo si existe al menos un elemento de E que es trascendental sobre F.*

Ejemplo 2.1.7. *La extensión $\mathbb{R}/\mathbb{Q}$ es trascendental puesto que e y π son trascendentales sobre $\mathbb{Q}$.*

Para extensiones trascendentales, existen análogos no lineales de la dimensión y la base para espacios vectoriales.

Definición 2.1.8. *Se dice que un conjunto $\{\xi_i \mid i \in I\}$ de elementos en E es F-algebraicamente dependiente si y solo si existe al menos un polinomio $P(x_1, \ldots, x_v)$ sobre F, tal que $P(\xi_{i_1}, \ldots, \xi_{i_v}) = 0$. Un conjunto que no es F-algebraicamente dependiente se dice que es F-algebraicamente independiente.*

Definición 2.1.9. *Un conjunto F-algebraicamente independiente que es maximal respecto a la inclusión se conoce como base de trascendencia de E/F. La cardinalidad de este conjunto se conoce como grado de trascendencia de E/F, denotado como tr $d^\circ E/F$.*

Por tanto, una extensión E/F es algebraica si tr $d^\circ E/F = 0$.

De forma análoga, para una extensión diferencial L/K son posibles dos situaciones.

Definición 2.1.10. *Se dice que un elemento $\xi \in L$ es diferencialmente algebraico sobre K si y solo si satisface una ecuación diferencial $P(\xi, \dot{\xi}, \ldots, \xi^{\alpha}) = 0$, donde P es un polinomio sobre K en ξ y sus derivadas. La ecuación $P = 0$ se conoce como ecuación algebraica diferencial. Se dice que la extensión L/K es diferencialmente algebraica si y solo si cualquier elemento de L es diferencialmente algebraico sobre K.*

Definición 2.1.11. *Se dice que un elemento $a \in L$ es diferencialmente trascendental sobre K si y solo si no es algebraico sobre K. Esto significa que no existe una ecuación algebraica diferencial sobre K satisfecha por a. Se dice que la extensión L/K es diferencialmente trascendental si y solo si existe al menos un elemento de L que es diferencialmente trascendental sobre K.*

Definición 2.1.12. *Se dice que un conjunto $\{\xi_i \mid i \in I\}$ de elementos en L es diferencialmente K-algebraicamente dependiente si y solo si el conjunto de derivadas de cualquier orden $\{\xi_i^{v_i} \mid i \in I,\ v_i = 0, 1, 2, \ldots\}$ es K-algebraicamente dependiente. En otras palabras, los elementos $\{\xi_i\}$ satisfacen alguna ecuación algebraica diferencial. Un conjunto que no es diferencialmente K-algebraicamente dependiente se dice que es diferencialmente K-algebraicamente independiente.*

Definición 2.1.13. *Un conjunto de elementos diferencialmente K-algebraicamente independientes que es maximal respecto a la inclusión se conoce como base de trascendencia diferencial de L/K. La cardinalidad de este conjunto se conoce como grado de trascendencia diferencial de L/K, denotado como diff tr $d^\circ L/K$.*

Por tanto, la extensión L/K es diferencialmente algebraica si diff tr $d^\circ L/K = 0$.

Por otro lado, sea $u = \{u_i \mid i \in I\}$ una base de trascendencia diferencial de L/K. Nótese como $K\langle u \rangle$ el campo diferencial generado por K y los elementos de u. Los campos diferenciales L y $K\langle u \rangle$ no coinciden en general, pero la extensión $L/K\langle u \rangle$ es diferencialmente algebraica.

Teorema 2.1.14. *Una extensión diferencial finitamente generada es diferencialmente algebraica si y solo si su grado de trascendencia (no diferencial) es finito.*

En otras palabras, el grado de trascendencia es el número de condiciones iniciales que se requieren para calcular las soluciones de las ecuaciones algebraico diferenciales.

2.1.2. Elemento primitivo

Sea E/F una extensión algebraica (no diferencial) finitamente generada. Se tiene el llamado teorema del elemento primitivo:

Teorema 2.1.15. *Existe un solo elemento $\gamma \in E$, el cual es un elemento primitivo, tal que $E = F(\gamma)$; es decir, E es generado por F y γ.*

Sea L/K una extensión diferencialmente algebraica finitamente generada, y asúmase que K no es un campo de constantes. Se tiene el llamado teorema del elemento primitivo diferencial [26]:

Teorema 2.1.16. *Existe un solo elemento $\delta \in L$, el cual es un elemento primitivo diferencial, tal que $L = K(\delta)$; es decir, L es generado por K y δ.*

2.2. Enfoque algebraico de la dinámica no lineal

Definición 2.2.1. *Sea k un campo diferencial base dado. Sea $k\langle u\rangle$ el campo diferencial generado por k y los elementos de un conjunto finito $u = (u_1, \dots, u_m)$ de cantidades diferenciales. Una dinámica es una extensión diferencialmente algebraica finitamente generada $K/k\langle u\rangle$. Se dice que la entrada u es independiente si y solo si u es una base de trascendencia diferencial de K/k.*

Sea n el grado de trascendencia (finito, no diferencial) de $K/k\langle u\rangle$. Sea $\xi = (\xi_1, \dots, \xi_v)$, $v \geq n$, un conjunto finito de elementos en K, que contiene una base de trascendencia de $K/k\langle u\rangle$.

Cada una de las derivadas $\dot{\xi}_1, \dots, \dot{\xi}_v$ son $k\langle u\rangle$-algebraicamente dependientes sobre ξ:

$$\begin{gathered} A_1(\dot{\xi}_1, \xi, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \\ \vdots \\ A_v(\dot{\xi}_v, \xi, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \end{gathered}$$

donde $A_1, \dots, A_v$ son polinomios sobre k. Puede verse que las ecuaciones anteriores son implícitas. Asúmase que las variables toman valores reales (o complejos). Cuando la matriz jacobiana

$$\begin{pmatrix} \partial A_1/\partial \dot{\xi}_1 & & (0) \\ & \ddots & \\ (0) & & \partial A_v/\partial \dot{\xi}_v \end{pmatrix}$$

tiene rango pleno v, se obtienen del teorema de la función implícita las siguientes ecuaciones diferenciales:

$$\begin{gathered} \dot{\xi}_1 = a_1(\xi, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \\ \vdots \\ \dot{\xi}_v = a_v(\xi, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \end{gathered}$$

Esta forma explícita solo es válida localmente, en los dominios donde la matriz jacobiana tiene rango pleno.

La variable ξ es conocida como estado generalizado o simplemente estado. El entero v se conoce como su dimensión. Un estado mínimo (generalizado), es decir, un estado de dimensión mínima, es una base de trascendencia de $K/k\langle u\rangle$; su dimensión es n. Un estado de este tipo se caracteriza por la independencia $k\langle u\rangle$-algebraica de sus componentes.

Sean $x = (x_1, \dots, x_n)$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ dos estados mínimos. Cualquier componente de $\bar{x}$ es $k\langle u\rangle$-algebraicamente dependiente sobre los componentes de x. Deben existir polinomios $P_1, \dots, P_n$ sobre k tales que:

$$\begin{gathered} P_1(\bar{x}_1, x, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \\ \vdots \\ P_n(\bar{x}_n, x, u, \dot{u}, \dots, u^{(\alpha)}) = 0 \end{gathered}$$

También, por minimalidad, debido a que los componentes de x son $k\langle u\rangle$-algebraicamente dependientes sobre los de x, se puede mostrar que la matriz jacobiana

$$\begin{pmatrix} \partial P_1/\partial \bar{x}_1 & & (0) \\ & \ddots & \\ (0) & & \partial P_n/\partial \bar{x}_n \end{pmatrix}$$

debe tener rango completo.

La discusión anterior puede resumirse de la siguiente manera.

Definición 2.2.2. *Sea $K/k\langle u\rangle$ una dinámica, es decir, una extensión diferencialmente algebraica finitamente generada. Una representación de estado es, en general, implícita y puede obtenerse de forma explícita solamente de forma local. Un estado mínimo es una base de trascendencia (no diferencial) de $K/k\langle u\rangle$. Dos estados mínimos están relacionados por ecuaciones que involucran a las variables de control y un número finito de sus derivadas.*

2.3. Forma canónica controlable generalizada no lineal

Con el fin de aplicar el teorema del elemento primitivo diferencial a la dinámica no lineal $K/k\langle u\rangle$, se debe asumir que $k\langle u\rangle$ no es un campo de constantes. Este caso se da, por ejemplo, cuando el conjunto u de variables de control es no vacío e independiente.

Sea δ un elemento primitivo diferencial de $K/k\langle u\rangle$. Considérese la secuencia de derivadas $\delta, \dot{\delta}, \ldots, \delta^{(v)}, \ldots$.

Lema 2.3.1. *[14] El conjunto $(\delta, \dot{\delta}, \ldots, \delta^{(v)})$ es $k\langle u\rangle$-algebraicamente independiente (dependiente, respectivamente) si y solo si $v \leq n-1$ ($v \geq n$, respectivamente), donde $n =$ tr $d^\circ K/k\langle u\rangle$.*

Corolario 2.3.2. *[14] $(\delta, \dot{\delta}, \ldots, \delta^{(n-1)})$ es una base de trascendencia de $K/k\langle u\rangle$.*

Considérese:

$$C(\delta, \dot{\delta}, \ldots, \delta^{(n)}, u, \dot{u}, \ldots, u^{(\alpha)}) = 0$$

donde C es un polinomio sobre k. La selección $x_1 = \delta, x_2 = \dot{\delta}, \ldots, x_n = \delta^{(n-1)}$ es un estado mínimo de la dinámica $K/k\langle u\rangle$, lo que lleva a las ecuaciones

$$\begin{gathered} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ C(x_1, x_2, \ldots, x_n, \dot{x}_n, u, \dot{u}, \ldots, u^{(\alpha)}) = 0 \end{gathered}$$

Esto es lo que se conoce como forma canónica controlable generalizada global. También puede obtenerse una forma canónica controlable generalizada local:

$$\begin{gathered} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = c\ (x_1, x_2, \ldots, x_n, u, \dot{u}, \ldots, u^{(\alpha)}) \end{gathered}$$

En el siguiente capítulo se presentará una forma canónica desarrollada de forma similar, la cual será parte importante en el esquema de control con tolerancia a fallas.

Capítulo 3

Control tolerante a fallas en sistemas de orden entero

En este capítulo se desarrolla la metodología principal utilizada para el control tolerante a fallas en sistemas de orden entero, realizando la extensión a sistemas de orden fraccionario en un capítulo posterior. Primero se presenta el diagnóstico de fallas utilizando herramientas de álgebra diferencial. Posteriormente se presenta el método con el cual se obtendrá el controlador dinámico que se utilizará para rechazar el efecto de las fallas en el sistema, el cual se basa en formas canónicas y seguimiento de la salida. A continuación se realiza una prueba de estabilidad en el sistema en lazo cerrado. Finalmente, se aplica la metodología al modelo de un sistema de tres tanques.

3.1. Diagnóstico de fallas

En esta sección se presenta el enfoque utilizado para el diagnóstico de fallas, el cual se basa en herramientas del álgebra diferencial, así como el diseño del observador que se utilizará para el mismo fin.

Primero, considérese un sistema no linear con fallas, descrito por:

$$\begin{aligned} \dot{x} &= g(x,u,f) \\ y &= h(x,u) \end{aligned} \tag{3.1}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $u \in \mathbb{R}^m$ el vector de entradas (control), $f \in \mathbb{R}^q$ el vector de entradas desconocidas (fallas), $y \in \mathbb{R}^p$ el vector de salidas, y g, h son funciones analíticas.

El primer paso a considerar para rechazar los efectos de las fallas que aparecen en el sistema es el diagnóstico de fallas (DF). Con el fin de diseñar un método para reconstruir las fallas que aparecen en el sistema, debemos saber si las mismas son diagnosticables, es decir, si es que pueden reconstruirse. Para esto, se presenta la siguiente definición obtenida del álgebra diferencial.

Definición 3.1.1. *Sea $\{u, y\}$ un subconjunto de un campo diferencial G en una dinámica $G/k\langle u\rangle$. Se dice que un elemento en G es algebraicamente observable con respecto* *a* $\{u,y\}$ *si es algebraico sobre* $k\langle u,y\rangle$. *Por lo tanto, se dice que un estado* x *es algebraicamente observable si y sólo si es algebraicamente observable con respecto a* $\{u,y\}$. *Se dice que una dinámica* $G/k\langle u\rangle$ *con salida* y *en* G *es algebraicamente observable si y sólo si cualquier estado tiene esta propiedad.*

La observabilidad algebraica significa que la extensión de campo diferencial $G/k\langle u\rangle$ es algebraica, es decir, toda la información diferencial está contenida en $k\langle u\rangle$.

Con base en lo anterior, se presentan las siguientes definiciones.

Definición 3.1.2. *Un elemento* x *en el campo diferencial* G *se dice algebraicamente observable (con respecto a* $\{u,y\}$*) si satisface una ecuación algebraica diferencial con coeficientes sobre* $k\langle u,y\rangle$.

Comentario 3.1.3. *Si una variable cumple con la definición anterior, se dice que satisface la condición de observabilidad algebraica (OA). Tanto los estados como las fallas pueden satisfacer esta propiedad. En particular, se dice que cada falla que satisfaga la condición de OA es diagnosticable.*

Definición 3.1.4. *Se dice que un sistema en la forma (3.1) es diagnosticable si es posible estimar el vector de fallas* f *de las ecuaciones del sistema y las historias temporales de* u *y* y. *Es decir, el sistema es diagnosticable si todos los elementos de* f *son algebraicamente observables con respecto a* $\{u,y\}$.

La noción de observabilidad algebraica consiste en expresar las fallas como la solución de una ecuación polinomial que dependen de las entradas y las salidas del sistema y una cantidad finita de sus derivadas temporales, con coeficientes en el campo k:

$$f_i = P_i\left(u, \dot{u}, ..., y, \dot{y}, ...\right).$$

Con el fin de ilustrar la Definición 3.1.4, se presentan los siguientes ejemplos.

Ejemplo 3.1.5. *Considérese el siguiente sistema:*

$$\begin{aligned} \dot{x}_1 &= x_1x_2 + f_1 + u \\ \dot{x}_2 &= x_1 \\ y_1 &= x_1 + f_2 \\ y_2 &= x_2 \end{aligned} \tag{3.2}$$

Dado que f_1 *y* f_2 *satisfacen la propiedad mencionada en la Definición 3.1.2:*

$$\begin{aligned} f_1 &= \ddot{y}_2 - y_2\dot{y}_2 - u \\ f_2 &= y_1 - \dot{y}_2 \end{aligned}$$

el sistema (3.2) *es diagnosticable y las fallas pueden reconstruirse con el conocimiento de* u, y *y sus derivadas temporales.*

Ejemplo 3.1.6. *El sistema*

$$\begin{aligned}\dot{x}_1 &= (x_1 + x_2)(u + f) \\ \dot{x}_2 &= u \\ y &= x_1 + x_2\end{aligned} \tag{3.3}$$

es diagnosticable dado que

$$f = \frac{\dot{y} - u}{y} - u$$

Nótese que con este enfoque se pueden diagnosticar fallas aditivas y multiplicativas.

Por otro lado, el vector de fallas $f = (f_1, ..., f_q)$ puede verse como un estado con una dinámica desconocida $\Omega(x, u, f) : \mathbb{R}^{n+m+q} \to \mathbb{R}^q$. Entonces, para estimarlo, se extiende el vector de estado (inmersion [31]).

Como puede notarse, un observador Luenberger clásico, el cual necesita el conocimiento completo de la dinámica del sistema, no puede construirse porque el término $\Omega(x, u, f)$ es desconocido. Sin embargo, este problema puede resolverse usando un observador de orden reducido (OOR), debido a que éste puede ser implementado por medio de la propiedad de diagnosticabilidad de las fallas, y es asintóticamente estable. El siguiente lema describe la construcción de un OOR proporcional para el sistema (3.1).

Lema 3.1.7. *[31] Si se satisfacen las siguientes hipótesis:*
H1 $|\Omega_i(x, u, f)| \leq N_i \in \mathbb{R}^+$, $\forall i = 1, ..., q$.
H2 f_i *es algebraicamente observable sobre* $\mathbb{R}\langle u, y\rangle$, $\forall i = 1, ..., q$.
Entonces el sistema

$$\dot{\hat{f}}_i = k_i(f_i - \hat{f}_i), \quad 1 \leq i \leq q \tag{3.4}$$

es un observador de orden reducido para el sistema (3.1), donde $\hat{f}_i$ denota la estimación de la falla f_i y $k_i \in \mathbb{R}^+$ $\forall i = 1, ..., q$ son coeficientes positivos que determinan la tasa de convergencia deseada del observador.

Algunas veces las derivadas de las salidas aparecen en la ecuación algebraica de la falla, entonces es necesario utilizar una variable auxiliar para aproximarlas como se describe en el lema siguiente.

Lema 3.1.8. *[31] Si una falla f_i, $1 \leq i \leq q$ del sistema (3.1) es algebraicamente observable y puede escribirse de la siguiente forma:*

$$f_i = a_i \dot{y} + b_i(u, y) \tag{3.5}$$

donde $a_i \in \mathbb{R}^m$ es un vector constante y $b_i(u, y)$ es una función acotada, entonces existe una función $\gamma_i \in C^1$, tal que el observador de orden reducido (3.4) puede escribirse como el siguiente sistema asintóticamente estable:

$$\begin{aligned}\dot{\gamma}_i &= -k_i\gamma_i + k_i b_i(x, u) - k_i^2 a_i y, \quad \gamma_i(0) = \gamma_{i0} \in \mathbb{R} \\ \hat{f}_i &= \gamma_i + k_i a_i y.\end{aligned} \tag{3.6}$$

Comentario 3.1.9. *Esta metodología es recursiva; se pueden introducir tantas variables virtuales como se necesiten.*

Comentario 3.1.10. *El OOR también sirve como un estimador de derivadas. Si existen derivadas de la entrada de orden 2 o superior, considérese la derivada temporal a estimarse como $\eta = \dot{y}$. De acuerdo a (3.4), se propone la siguiente estructura del observador*

$$\dot{\hat{\eta}} = k_\eta(\eta - \hat{\eta}) \tag{3.7}$$

introduciendo el cambio de variable $\gamma = \hat{\eta} - k_\eta y$, y de (3.7), se tiene

$$\begin{aligned} \dot{\gamma} &= -k_\eta \hat{\eta} \\ &= -k_\eta \gamma - k_\eta^2 y \end{aligned} \tag{3.8}$$

el cual junto con $\hat{\eta}$ constituye un estimador asintótico para $\eta = \dot{y}$.

Ahora, considerando la dinámica del OOR, se definen las siguientes variables:

$$\hat{f}_{i\bar{l}} = \hat{f}_{\bar{l}}^{(i-1)} \qquad i = 1, ..., \mu_{\bar{l}} \tag{3.9}$$

por lo tanto se pueden escribir subsistemas de estimación de las fallas como sigue:

$$\dot{\hat{\mathbf{f}}}_{\bar{l}} = E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}\ (u, y, f), \qquad 1 \leq \bar{l} \leq q \tag{3.10}$$

donde los elementos de E se dan por:

$$E_{ks} = \begin{cases} 1 & \text{si } k = s - 1 \\ 0 & \text{en caso contrario} \end{cases} \tag{3.11}$$

y

$$\hat{\mathbf{f}}_{\bar{l}} = \left(\hat{f}_{1\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}\bar{l}}\right); \qquad \omega_{\bar{l}}\ (u, y, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_{\bar{l}}\ (f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}\bar{l}}) \end{pmatrix}.$$

3.2. Control dinámico tolerante a fallas

La clase de sistemas no lineales (3.1) puede transformarse en la siguiente Forma Canónica de Observabilidad Generalizada Multi-entrada Multi-salida ($\mathcal{FCOGMM}$) por medio del elemento primitivo diferencial:

$$\begin{aligned} \dot{\eta}_{ij} &= \eta_{i+1,j}, \qquad 1 \leq i \leq n-1 \\ \dot{\eta}_{nj} &= -L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, f, ..., f^{(\mu)}) \\ y_j &= \eta_{1j} \end{aligned} \tag{3.12}$$

donde L_j es una función C^1 de valores reales, $\eta_j = (\eta_{1j}, ..., \eta_{nj}) \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $f \in \mathbb{R}^q$, y enteros γ, $\mu \geq 0$.

Nótese que se han elegido como elementos primitivos diferenciales a las salidas y_j, $1 \leq j \leq p$, del sistema. Por tanto, esta $\mathcal{FCOGMM}$ se conforma de p subsistemas, uno por cada salida.

Sea $y_R \in \mathbb{R}^p$ un vector de referencias con funciones C^n como elementos. El problema de seguimiento de salida con tolerancia a fallas consiste en encontrar un controlador dinámico que depende del vector de referencias y sus derivadas, las variables de estado canónicas y, como se mencionó antes, una estimación $\hat{f}$ del vector de fallas y sus derivadas temporales, tal que el controlador fuerce a la salida y a converger hacia y_R.

Defínase el error de seguimiento como:

$$e_{1j} = y_j - y_{Rj}, \qquad 1 \leq j \leq p \tag{3.13}$$

Dado que η_{ij} es igual a la $(i-1)$-ésima derivada temporal de y_j, es decir $\eta_{ij} = y_j^{(i-1)}$, las variables de error (3.13) se reescriben como:

$$e_{1j} = \eta_{1j} - y_{Rj}, \qquad 1 \leq j \leq p \tag{3.14}$$

Estas variables de error definen la siguiente $\mathcal{FCOGMM}$:

$$\begin{aligned} e_j^{(i)} &= \eta_{i+1,j} - y_{Rj}^{(i)}, \qquad 1 \leq i \leq n-1 \\ e_j^{(n)} &= \dot{\eta}_{nj} - y_{Rj}^{(n)} = -L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} \end{aligned} \tag{3.15}$$

con $e_{ij} = e_j^{(i-1)}$, $1 \leq i \leq n$. A continuación se impone una dinámica lineal invariante en el tiempo al error de seguimiento:

$$e_j^{(n)} + \sum_{i=0}^{n-1} a_{i+1,j} e_j^{(i)} = 0 \tag{3.16}$$

y del sistema (3.15), (3.16) se reescribe como:

$$\dot{\eta}_{nj} - y_{Rj}^{(n)} + \sum_{i=1}^{n} a_{ij} \left[\eta_{ij} - y_{Rj}^{(i-1)} \right] = 0 \tag{3.17}$$

esto es:

$$-L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_{ij} \left[\eta_{ij} - y_{Rj}^{(i-1)} \right]. \tag{3.18}$$

Ahora, se puede definir una cadena de integradores del error de seguimiento como sigue:

$$\begin{aligned} \dot{e}_{ij} &= e_{i+1,j}, \qquad 1 \leq i \leq n-1 \\ \dot{e}_{nj} &= -\sum_{i=1}^{n} a_{ij} e_{ij} \end{aligned} \tag{3.19}$$

o de forma compacta:

$$\dot{\mathbf{e}}_j = F_j \mathbf{e}_j \tag{3.20}$$

con

$$-L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_{ij} e_{ij} \tag{3.21}$$

donde $\mathbf{e}_j = (e_{11}, ..., e_{nj})$, $\mathbf{y}_{\mathbf{R}j} = (y_{Rj}, \dot{y}_{Rj}, ..., y_{Rj}^{(n-1)})$, y

$$F_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \vdots & \cdots & 1 \\ -a_{1j} & -a_{2j} & \cdots & -a_{nj} \end{pmatrix}$$

El origen $\mathbf{e}_j = 0$ es un punto de equilibrio para el sistema (3.20) si F_j es Hurwitz.

Como se verá, el controlador dinámico dependerá de las variables del error de seguimiento. Dado que las derivadas temporales del error no se encuentran disponibles para medición, se utilizará un observador para estimar el vector de error. Primero, el sistema (3.20) se reescribe como:

$$\dot{\mathbf{e}}_j = E\mathbf{e}_j + \varphi_j\left(\mathbf{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \tag{3.22}$$

donde $\mathbf{e} = (\mathbf{e}_1, ..., \mathbf{e}_p)$, $\mathbf{y}_R = (\mathbf{y}_{R1}, ..., \mathbf{y}_{Rp})$, $\mathbf{u} = (u, \dot{u}, ..., u^{(\gamma)})$, $\hat{\mathbf{f}} = (\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu)})$, los elementos de E se dan por (3.11) y

$$\varphi_j\left(\mathbf{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} \end{pmatrix}$$

Entonces, la estimación $\hat{\mathbf{e}}_j$ se obtiene con el siguiente observador de alta ganancia (OAG) [18]:

$$\dot{\hat{\mathbf{e}}}_j = E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) - S_\infty^{-1} C^T C(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{3.23}$$

donde S_∞ es solución de la siguiente ecuación de Lyapunov:

$$S_\infty\left(E + \frac{\theta}{2} I\right) + \left(E^T + \frac{\theta}{2} I\right) S_\infty = C^T C \tag{3.24}$$

con $\theta > 0$ y $C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$. Los coeficientes de S_∞ se dan por:

$$(S_\infty)_{ks} = \frac{\alpha_{ks}}{\theta^{k+s-1}}$$

donde (α_{ks}) es una matriz simétrica definida positiva independiente de θ.

Por otro lado, sea $\hat{u}_l$ la solución de:

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{u}_l^{(\gamma_l)}, \hat{\mathbf{f}}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_{ij} \hat{e}_{ij} \tag{3.25}$$

donde $\hat{u}_l^{(\gamma_l)}$ es la derivada de más alto orden de la entrada en cuestión que se encuentra en la ecuación dada. Por lo tanto, la ecuación dinámica del controlador $\hat{u}_l$ es:

$$\hat{u}_l^{(\gamma_l)} = K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right). \tag{3.26}$$

A partir de los p subsistemas dinámicos se obtendrán las ecuaciones dinámicas para p controladores. Estos controladores permitirán realizar el seguimiento de la referencia en el sistema original, con tolerancia a fallas (es decir, eliminando el efecto de las fallas). Por lo tanto, la ecuación (3.23) se reescribe como:

$$\dot{\hat{\mathbf{e}}}_j = E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{3.27}$$

Si se definen las siguientes variables:

$$\hat{u}_{il} = \hat{u}_l^{(i-1)} \qquad i = 1, ..., \gamma_l \tag{3.28}$$

las ecuaciones de los controladores dinámicos pueden escribirse en la siguiente forma canónica:

$$\dot{\hat{\mathbf{u}}}_l = E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right), \qquad 1 \le l \le m \tag{3.29}$$

donde $\hat{\mathbf{u}}_l = (\hat{u}_{1l}, ..., \hat{u}_{\gamma_l l})$ y

$$\kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \end{pmatrix}$$

Por lo tanto, la dinámica del sistema en lazo cerrado se da por:

$$\begin{aligned} \dot{\hat{\mathbf{e}}}_j &= E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) \\ \dot{\hat{\mathbf{u}}}_l &= E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ \dot{\hat{\mathbf{f}}}_{\bar{l}} &= E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}\,(u, y, f) \end{aligned} \tag{3.30}$$

para $1 \le j \le p$, $1 \le l \le m$ y $1 \le \bar{l} \le q$. Este sistema se expresa en forma de cadenas de integradores como sigue:

$$\begin{aligned} \dot{\hat{e}}_{ij} &= \hat{e}_{i+1,j} - \psi_i\left(\theta_j\right)\left(\hat{e}_j - e_j\right) && 1 \le i \le n-1 \\ \dot{\hat{e}}_{nj} &= -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, ..., \hat{u}^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} - \theta_j^n \\ & && 1 \le j \le p \\ \dot{\hat{u}}_{il} &= \hat{u}_{i+1,l} && 1 \le i \le \gamma_l - 1 \\ \dot{\hat{u}}_{\gamma_l l} &= K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) && 1 \le l \le m \\ \dot{\hat{f}}_{i\bar{l}} &= \hat{f}_{i+1,\bar{l}} && 1 \le i \le \mu_{\bar{l}} - 1 \\ \dot{\hat{f}}_{\mu_{\bar{l}}\bar{l}} &= k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}\bar{l}}) && 1 \le \bar{l} \le q \end{aligned}$$

donde $\psi_i(\theta_j)$ es una función obtenida de S_∞^{-1}.

En estas cadenas de integradores se pueden apreciar las dinámicas de los controladores y las estimaciones de las fallas. Como puede observarse, las variables obtenidas a partir de estas dinámicas forman parte explícitamente en la dinámica de los errores de seguimiento, lo cual conlleva a la solución del problema de seguimiento multisalida.

Por último, defínase el error de observación como $\varepsilon_j = \hat{\mathbf{e}}_j - \mathbf{e}_j$, obteniéndose su dinámica a partir de las ecuaciones (3.22) y (3.27):

$$\dot{\varepsilon}_j = \left(E - S_\infty^{-1} C^T C\right) \varepsilon_j + \Phi_j(\varepsilon, \hat{\mathbf{e}}) \tag{3.31}$$

donde

$$\Phi_j(\varepsilon, \hat{\mathbf{e}}) = \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - \varphi_j\left(\hat{\mathbf{e}} - \varepsilon, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right)$$

A continuación se establece el siguiente resultado.

Teorema 3.2.1. *Sea el sistema (3.1) descrito en la $\mathcal{FCOGMM}$ (3.12) compuesta de p subsistemas. La dinámica de observación correspondiente al subsistema j se compone de $\hat{\mathbf{e}}_j$ y ε_j. Sea $f_{\bar{l}}$ diagnosticable para $1 \leq \bar{l} \leq q$ y estimada por medio de la dinámica de $\hat{f}_{\bar{l}}$. Sea $\hat{u}_l$ la solución de*

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{u}_l^{(\gamma_l)}, \hat{\mathbf{f}}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_{ij} \hat{e}_{ij}$$

Entonces, el sistema en lazo cerrado (3.30) con control $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ es asintóticamente estable.

Demostración. Considérese la siguiente función candidata de Lyapunov:

$$V(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}) = V_1(\hat{\mathbf{e}}_j) + V_2(\varepsilon_j) + V_3(\tilde{f}_{\bar{l}}) \tag{3.32}$$

con

$$V_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j; \quad V_2(\varepsilon_j) = \varepsilon_j^T S_\infty \varepsilon_j; \quad V_3(\tilde{f}_{\bar{l}}) = \tilde{f}_{\bar{l}}^2$$

donde $\tilde{f}_{\bar{l}} = f_{\bar{l}} - \hat{f}_{\bar{l}}$ es el error de estimación de la falla.

Sea P la solución a la ecuación de Lyapunov $F^T P + PF = -I$, entonces P es definida positiva, y se define $\|x\|_P = \sqrt{x^T P x}$.

Sea S_∞ la solución a la ecuación de Lyapunov (3.24). Entonces S_∞ es definida positiva, y se define $\|x\|_{S_\infty} = \sqrt{x^T S_\infty x}$.

Primero, se obtiene la derivada temporal del primer término de (3.32):

$$\dot{V}_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \dot{\hat{\mathbf{e}}}_j + \dot{\hat{\mathbf{e}}}_j^T P \hat{\mathbf{e}}_j \leq -\alpha \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C \varepsilon_j \tag{3.33}$$

donde $\alpha = 1/\lambda_{\text{máx}}(P)$. Notando que

$$\left\|\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C S_\infty^{-1} S_\infty \varepsilon_j\right\| \leq \rho(\theta) \|\hat{\mathbf{e}}_j\|_P \|\varepsilon_j\|_{S_\infty}$$

con $\rho(\theta) = \left\|S_\infty^{-1} C^T C S_\infty^{-1}\right\|$, se obtiene la siguiente desigualdad:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \leq -\left(\alpha \|\hat{\mathbf{e}}_j\|_P + 2\rho(\theta) \|\varepsilon_j\|_{S_\infty}\right) \|\hat{\mathbf{e}}_j\|_P \tag{3.34}$$

Sean d_1, d_2 números positivos tales que $\|\hat{\mathbf{e}}_j\|_P \geq d_1 \|\hat{\mathbf{e}}_j\|$ y $\|\varepsilon_j\|_{S_\infty} \geq d_2 \|\varepsilon_j\|$. Por tanto se tiene:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \leq -(\alpha d_1 \|\hat{\mathbf{e}}_j\| + 2\rho(\theta) d_2 \|\varepsilon_j\|) \|\hat{\mathbf{e}}_j\|_P \tag{3.35}$$

Ahora se obtiene la derivada temporal del segundo término de (3.32):

$$\dot{V}_2(\varepsilon_j) = \varepsilon_j^T S_\infty \dot{\varepsilon}_j + \dot{\varepsilon}_j^T S_\infty \varepsilon_j \leq -\theta \|\varepsilon_j\|_{S_\infty}^2 + 2\|\varepsilon_j\|_{S_\infty} \|\Phi_j(\varepsilon, \hat{\mathbf{e}})\|_{S_\infty} \tag{3.36}$$

donde se tomó en cuenta la ecuación de Lyapunov (3.24), la descomposición de Cholesky y el hecho de que $\varepsilon_j^T C^T C \varepsilon_j \geq 0$. Además, notando que $\Phi_j(\varepsilon, \hat{\mathbf{e}})$ es diferenciable, por la propiedad de Lipschitz se tiene que existe $\lambda > 0$ tal que:

$$\|\Phi_j(\varepsilon, \hat{\mathbf{e}})\|_{S_\infty} \leq \lambda \|\varepsilon_j\|_{S_\infty} \qquad 1 \leq j \leq p$$

por lo tanto:

$$\dot{V}_2(\varepsilon_j) \leq -(\theta - 2\lambda) \|\varepsilon_j\|_{S_\infty}^2 \tag{3.37}$$

De (3.37) se obtiene la siguiente desigualdad:

$$\frac{d\left(\|\varepsilon_j\|_{S_\infty}^2\right)}{dt} \leq -(\theta - 2\lambda) \|\varepsilon_j\|_{S_\infty}^2 \tag{3.38}$$

de donde se encuentra que:

$$\|\varepsilon_j\|_{S_\infty} \leq -e^{-\gamma t} \|\varepsilon_j(0)\|_{S_\infty} \tag{3.39}$$

con $\gamma = \theta/2 - \lambda$. Por lo tanto, $\dot{V}_2(\varepsilon_j) \leq 0$ si se satisface la condición $\lambda < \theta/2$.

De forma similar, de (3.34) se obtiene la siguiente desigualdad:

$$\frac{d\left(\|\hat{\mathbf{e}}_j\|_P^2\right)}{dt} \leq -\left(\alpha \|\hat{\mathbf{e}}_j\|_P + 2\rho(\theta) \|\varepsilon_j\|_{S_\infty}\right) \|\hat{\mathbf{e}}_j\|_P \tag{3.40}$$

de donde se encuentra que:

$$\|\hat{\mathbf{e}}_j\|_P \leq A e^{-\frac{\alpha}{2} t} + B e^{-\gamma t} \tag{3.41}$$

con

$$\begin{aligned} A &= \|\hat{e}_j(0)\|_P - B \\ B &= -\frac{\rho(\theta) \|\varepsilon_j(0)\|_{S_\infty}}{\alpha/2 - \gamma} \end{aligned}$$

Por consiguiente, se satisface que $\dot{V}_1(\hat{\mathbf{e}}_j) \leq 0$.

Por último, se obtiene la derivada temporal del tercer término de (3.32):

$$\begin{aligned} \dot{V}_3(\tilde{f}_{\bar{l}}) &= 2\tilde{f}_{\bar{l}} \dot{\tilde{f}}_{\bar{l}} \\ &= 2\tilde{f}_{\bar{l}} \left(\Omega_{\bar{l}} - k_{\bar{l}} \tilde{f}_{\bar{l}}\right) \\ &\leq 2\tilde{f}_{\bar{l}} \left(\frac{N_{\bar{l}}}{k_{\bar{l}}} - \tilde{f}_{\bar{l}}\right) \end{aligned} \tag{3.42}$$

y se impone la condición $N_{\bar{l}}/k_{\bar{l}} \to 0$ con $t \to \infty$, entonces

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq -2\tilde{f}_{\bar{l}}^2 \leq 0. \tag{3.43}$$

Por lo tanto, el sistema (3.1) con el control tolerante a fallas $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ es asintóticamente estable, para $1 \leq j \leq p$, $1 \leq l \leq m$ y $1 \leq \bar{l} \leq q$. □

Comentario 3.2.2. *Puede verse de (3.41) que*

$$\|\hat{\mathbf{e}}_j\|_P \leq (A + B)\, e^{-\text{mín}\left\{\frac{\alpha}{2},\gamma\right\}t} \tag{3.44}$$

Si se elige la condición $\theta/2 - \lambda = \gamma > \alpha/2$, puede elegirse θ de forma que para algún valor fijo:

$$\|\hat{\mathbf{e}}_j\|_P \leq (A + B)\, e^{-\frac{\alpha}{2}t} \tag{3.45}$$

Esto implica que el error de observación ε_j converge más rápido que el error de seguimiento estimado $\hat{\mathbf{e}}_j$, por lo que la dinámica de $\hat{\mathbf{e}}_j$ es una aproximación útil de la dinámica de $\mathbf{e}_j$.

3.3. Aplicación

3.3.1. Ejemplo numérico

Considérese el siguiente sistema no lineal

$$\begin{aligned} \dot{x}_1 &= x_1x_2 + f + u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_3f + u \\ y &= x_2. \end{aligned} \tag{3.46}$$

3.3.1.1. Diagnóstico de fallas

Puede observarse que el sistema (3.46) es diagnosticable, porque puede obtenerse el siguiente polinomio:

$$f - \ddot{y} + \dot{y}y + u = 0 \tag{3.47}$$

por lo tanto, puede construirse un observador para estimar f basado en la ecuación (3.6). Para esto, se propone el siguiente OOR:

$$\begin{aligned} \dot{\gamma}_2 &= -k_2(\hat{x}_1y + \hat{u} + \gamma_2 + k_2\hat{x}_1) \\ \hat{f} &= \gamma_2 + k_2\hat{x}_1 \end{aligned} \tag{3.48}$$

donde $\hat{x}_1$ es la estimación de $\dot{y}$, la cual se obtiene con

$$\begin{aligned} \dot{\gamma}_1 &= -k_1\left(\gamma_1 + k_1y\right) \\ \hat{x}_1 &= \gamma_1 + k_1y. \end{aligned} \tag{3.49}$$

3.3.1.2. Control tolerante a fallas

Por otro lado, eligiendo $\eta_1 = y$, el sistema es transformado en la $\mathcal{FCOGMM}$ (3.12):

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ \dot{\eta}_3 &= (\eta_2)^2 + \eta_2(\eta_1)^2 + \eta_1 u + \eta_1 f + \dot{u} + \dot{f}.\end{aligned} \tag{3.50}$$

A continuación se construye el OAG (3.23), donde

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

y la matriz S_∞^{-1} se escoge como:

$$S_\infty^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{pmatrix}.$$

Definiendo $e_1 = \eta_1 - y_R$ y $e_1^{(i-1)} = e_i$, la ecuación del OAG es:

$$\begin{aligned}\dot{\hat{e}}_1 &= \hat{e}_2 - 3\theta(\hat{e}_1 - e_1) \\ \dot{\hat{e}}_2 &= \hat{e}_3 - 3\theta^2(\hat{e}_1 - e_1) \\ \dot{\hat{e}}_3 &= (\hat{e}_2 + \dot{y}_R)^2 + (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 + (\hat{e}_1 + y_R)\hat{u} + (\hat{e}_1 + y_R)\hat{f} + \dot{\hat{u}} + \dot{\hat{f}} \\ &\quad - \dddot{y}_R - \theta^3(\hat{e}_1 - e_1) \quad = \quad -\sum_{i=1}^{3} a_i \hat{e}_i.\end{aligned} \tag{3.51}$$

Del sistema (3.51), la ecuación del controlador dinámico (3.26) se obtiene como:

$$\begin{aligned}\dot{\hat{u}} &= -\sum a_i \hat{e}_i - (\hat{e}_2 + \dot{y}_R)^2 - (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 - (\hat{e}_1 + y_R)\hat{u} \\ &\quad - (\hat{e}_1 + y_R)\hat{f} - \dot{\hat{f}} + \dddot{y}_R.\end{aligned} \tag{3.52}$$

Por lo tanto, definiendo $\hat{u} = \hat{u}_1$ y $\hat{f} = \hat{f}_1$, la cadena de integradores del sistema en lazo cerrado es:

$$\begin{aligned}\dot{\hat{e}}_1 &= \hat{e}_2 - 3\theta(\hat{e}_1 - e_1) \\ \dot{\hat{e}}_2 &= \hat{e}_3 - 3\theta^2(\hat{e}_1 - e_1) \\ \dot{\hat{e}}_3 &= (\hat{e}_2 + \dot{y}_R)^2 + (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 + (\hat{e}_1 + y_R)\hat{u}_1 + (\hat{e}_1 + y_R)\hat{f}_1 + \dot{\hat{u}}_1 \\ &\quad + \dot{\hat{f}}_1 - \dddot{y}_R - \theta^3(\hat{e}_1 - e_1) \\ \dot{\hat{u}}_1 &= -a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 - (\hat{e}_2 + \dot{y}_R)^2 - (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 \\ &\quad - (\hat{e}_1 + y_R)\hat{u}_1 - (\hat{e}_1 + y_R)\hat{f}_1 - \dot{\hat{f}}_1 + \dddot{y}_R \\ \dot{\hat{f}}_1 &= -k_2(\hat{x}_1 y + \hat{u}_1 + \gamma_2 + k_2\hat{x}_1) + k_2(-k_1(\gamma_1 + k_1 y) + k_1\dot{y}).\end{aligned}$$

3.3.1.3. Resultados de simulación

Se realizaron simulaciones para este sistema en MATLAB Simulink durante 15 segundos, utilizando la referencia variable $y_R = 0.1sen(t)$ y la falla $f = 50\,[1 + sen(0.2t)\, e^{-0.5t}]\,\mathcal{U}(t-5)$, donde $\mathcal{U}(t)$ es la función escalón. Los parámetros de diseño se eligieron como $\theta = 20$, $a_1 = 8000$, $a_2 = 1200$, $a_3 = 60$, $k_1 = k_2 = 1$.

La falla f y su estimación $\hat{f}$ se muestran en la Fig. 3.1. Puede verse que la parte transitoria de la señal del OOR termina rápidamente, y 5 segundos después de que la falla aparece, la estimación sigue a la señal real. La Fig. 3.2 muestra el índice de desempeño del OOR propuesto, el cual fue evaluado utilizando la siguiente funcional de costo:

$$J_t = \frac{1}{t+\varepsilon}\int_0^t \left\|\tilde{f}_{\bar{l}}\right\|^2 dt \tag{3.53}$$

donde $\varepsilon = 0.0001$. Este parámetro se utiliza para evitar singularidades cuando $t = 0$, y se elige lo suficientemente pequeño de forma que no altere significativamente el valor del índice.

La Fig. 3.3 muestra la señal del control tolerante a fallas utilizado. Finalmente, la Fig. 3.4 muestra cómo la salida y sigue la referencia y_R; puede verse que aproximadamente después de un segundo la salida sigue a la referencia sinusoidal. Los efectos de la falla f, los cuales aparecen a los 5 segundos, pueden apreciarse antes de ser eliminados.

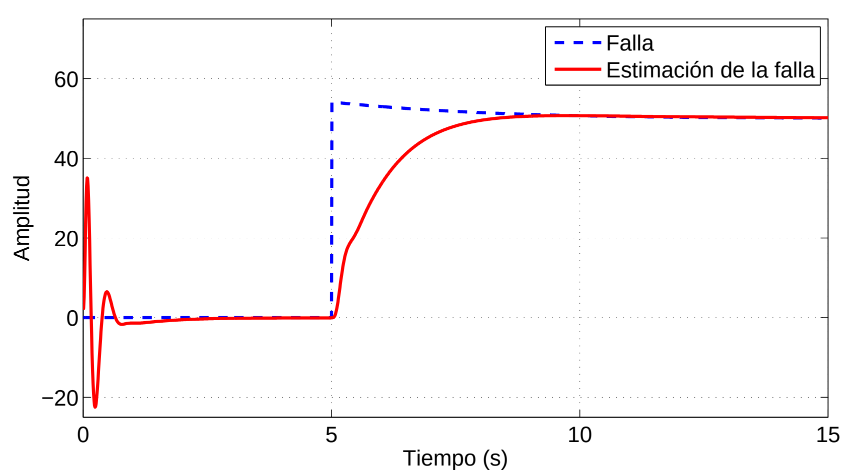

Figura 3.1: Estimación de la falla del ejemplo numérico.

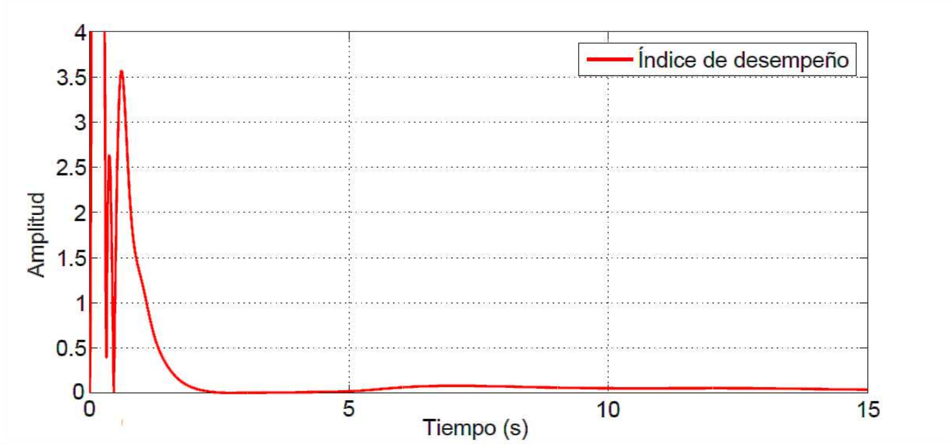

Figura 3.2: Evaluación del desempeño para la estimación de la falla del ejemplo numérico.

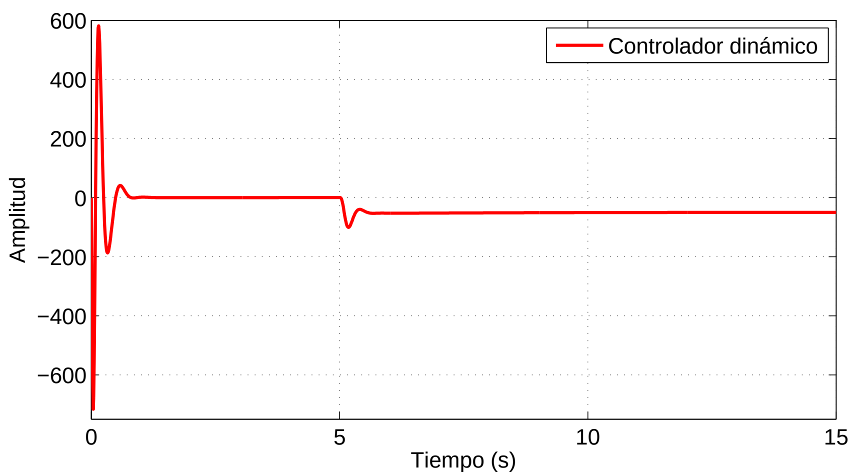

Figura 3.3: Controlador dinámico del ejemplo numérico.

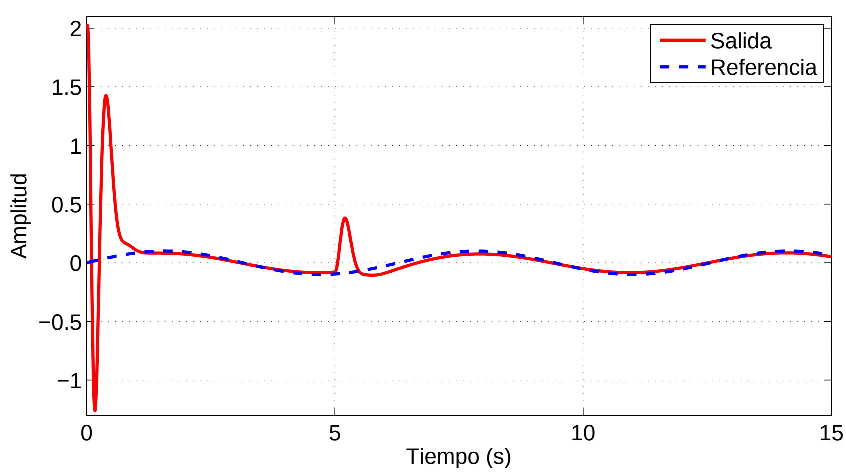

Figura 3.4: Seguimiento de la salida del ejemplo numérico.

3.3.2. Sistema de tres tanques

Considérese el modelo no lineal del sistema de tres tanques Amira DTS200 [2]:

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{A}\left(-q_{13} + u_1 + f_1\right) \qquad (3.54)\\
\dot{x}_2 &= \frac{1}{A}\left(q_{32} - q_{20} + u_2 + f_2\right)\\
\dot{x}_3 &= \frac{1}{A}\left(q_{13} - q_{32}\right)\\
y_1 &= x_2\\
y_2 &= x_3
\end{aligned}$$

donde

$$\begin{aligned}
q_{13} &= a_1 S\sqrt{2g(x_1 - x_3)}\\
q_{32} &= a_3 S\sqrt{2g(x_3 - x_2)}\\
q_{20} &= a_2 S\sqrt{2gx_2}
\end{aligned}$$

En este sistema, $u_i = q_i$, $i = 1, 2$ son las entradas manipulables de flujo, $x_i = h_i$, $i = 1, 2, 3$ son los niveles de los tanques, A es la sección transversal de los tanques, y los términos q_{ij} representan el flujo de agua del tanque i al tanque j. S es el área transversal de la tubería que interconecta cada tanque y los parámetros desconocidos a_i, $i = 1, 2, 3$ son los coeficientes de flujo de salida.

Es importante mencionar que el sistema posee cuatro regiones de operación, y la región considerada en este caso es $h_1 > h_3 > h_2 > 0$. Las características y variables del sistema y su modo de operación en la región mencionada se muestran en la Figura 3.5.

Del modelo (3.54), puede verse que x_1 no está disponible para su medición, sin embargo es algebraicamente observable de acuerdo a la Definición 3.1.2, lo que permite escribir el siguiente polinomio algebraico:

$$x_1 - y_2 - \frac{1}{2ga_1^2 S^2}\left(A\dot{y}_2 + a_3 S\sqrt{2g(y_2 - y_1)}\right)^2 = 0 \qquad (3.55)$$

Como se mencionó antes, los coeficientes de flujo a_1, a_2 y a_3 son desconocidos. Sin embargo puede verificarse que son algebraicamente observables. Por lo tanto, de (3.54) se obtienen las siguientes expresiones, las cuales están definidas en la región de interés ($h_1 > h_3 > h_2 > 0$):

$$a_1 = \frac{q_1 - A\dot{x}_1}{S\sqrt{2g(x_l - x_3)}} \qquad (3.56)$$

$$a_2 = \frac{q_1 + q_2 - A(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)}{S\sqrt{2g(x_2)}} \qquad (3.57)$$

$$a_3 = \frac{q_1 - A(\dot{x}_1 + \dot{x}_3)}{S\sqrt{2g(x_3 - x_2)}} \qquad (3.58)$$

Por lo tanto, si las mediciones de las salidas q_1 y q_2 y las variables de estado x_1, x_2 y x_3 del modelo nominal (sin la presencia de fallas) están disponibles, es posible estimar sus derivadas temporales y los estimados de los parámetros desconocidos a_1, a_2 y a_3 pueden obtenerse de (3.56) – (3.58).

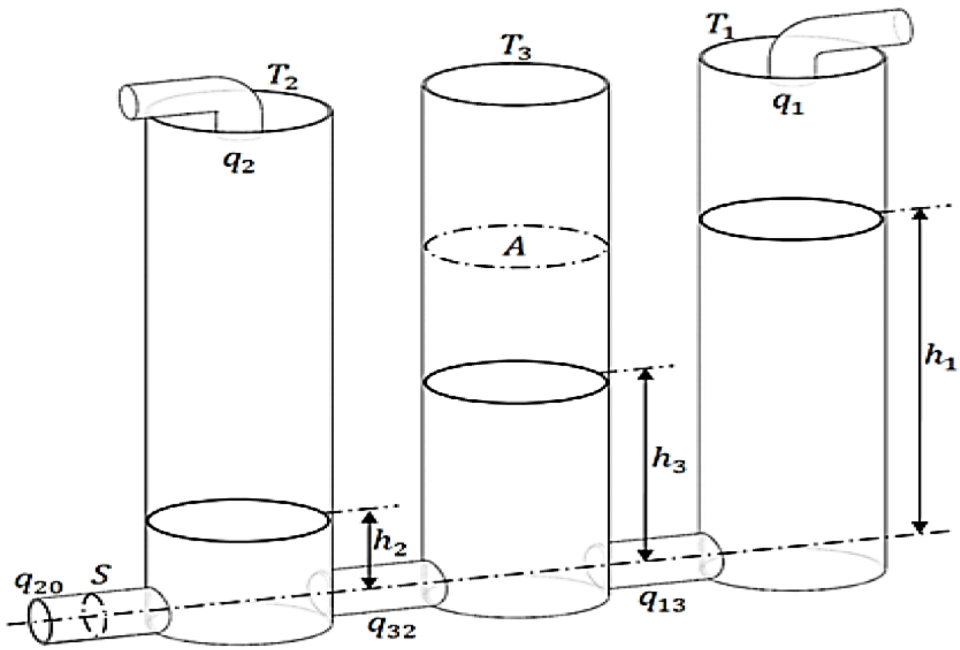

Figura 3.5: Diagrama esquemático del sistema Amira DTS200 operando en la región $h_1 > h_3 > h_2 > 0$.

3.3.2.1. Diagnóstico de fallas

A continuación se aplica la metodología propuesta. Primero, puede verse de (3.54) que f_1 y f_2 son diagnosticables, puesto que pueden obtenerse los siguientes polinomios:

$$f_1 - A\dot{x}_1 - a_1 S\sqrt{2g(x_1 - y_2)} + u_1 = 0 \tag{3.59}$$

$$f_2 - A\dot{y}_1 + a_3 S\sqrt{2g(y_2 - y_1)} - a_2 S\sqrt{2gy_1} + u_2 = 0 \tag{3.60}$$

por lo tanto, pueden construirse observadores para f_1 y f_2 basados en la ecuación (3.6). Para estimar la falla f_1, se propone el siguiente OOR:

$$\begin{aligned} \dot{\gamma}_1 &= -k_1(-q_{13} + u_1 + \gamma_1 + k_1 A\hat{x}_1) \\ \hat{f}_1 &= \gamma_1 + k_1 A\hat{x}_1. \end{aligned} \tag{3.61}$$

Como puede verse en la ecuación (3.55), x_1 es algebraicamente observable, por lo que se obtiene una estimación con el siguiente observador:

$$\begin{aligned} \dot{\gamma}_2 &= -k_2(\gamma_2 + k_2 y_2) \\ \hat{\zeta} &= \gamma_2 + k_2 y_2 \\ \hat{x}_1 &= y_2 + \frac{1}{2ga_1^2 S^2}\left(A\hat{\zeta} + q_{32}\right)^2 \end{aligned} \tag{3.62}$$

donde $\hat{\zeta}$ representa la estimación de $\dot{y}_2$.

Finalmente, para estimar la falla f_2, se utiliza el siguiente OOR:

$$\begin{aligned} \dot{\gamma}_3 &= -k_3(q_{32} - q_{20} + u_2 + \gamma_3 + k_3 A y_1) \\ \hat{f}_2 &= \gamma_3 + k_3 A y_1. \end{aligned} \tag{3.63}$$

3.3.2.2. Control tolerante a fallas

Por otro lado, se elige el cambio de variable $\eta_{1j} = y_j$, de forma que el sistema (3.54) sea transformado en la $\mathcal{FCOGMM}$ (3.12). Para $y_1 = \eta_{11}$, se tiene el siguiente subsistema:

$$
\begin{aligned}
\dot{\eta}_{11} &= \eta_{21} \\
\dot{\eta}_{21} &= \eta_{31} \\
\dot{\eta}_{31} &= \frac{1}{A}\left(a_3 S g^2 \frac{2(\eta_{12}-\eta_{11})(\eta_{32}-\eta_{31})-(\eta_{22}-\eta_{21})^2}{\sqrt{(2g(\eta_{12}-\eta_{11}))^3}}\right. \\
&\quad \left. -a_2 S g^2 \frac{2\eta_{11}\eta_{31}-\eta_{21}^2}{\sqrt{(2g\eta_{11})^3}} + \ddot{u}_2 + \ddot{f}_2\right)
\end{aligned}
\tag{3.64}
$$

y para $y_2 = \eta_{12}$ se tiene:

$$
\begin{aligned}
\dot{\eta}_{12} &= \eta_{22} \\
\dot{\eta}_{22} &= \eta_{32} \\
\dot{\eta}_{32} &= \frac{1}{A^2}\left(a_1^2 S^2 g^3 \frac{2(\dot{x}_1-\eta_{22})(x_1-\eta_{12})}{(2g(x_1-\eta_{12}))^2}\right. \\
&\quad \left. +a_1 S g^2 \frac{2(x_1-\eta_{12})}{\sqrt{(2g(x_1-\eta_{12}))^3}}\left(\dot{u}_1+\dot{f}_1\right)\right) \\
&\quad -\frac{1}{A}\left(a_1 S g^2 \frac{2\,(\eta_{32})\,(x_1-\eta_{12})+(\dot{x}_1-\eta_{22})^2}{\sqrt{(2g(x_1-\eta_{12}))^3}}\right. \\
&\quad \left. +a_3 S g^2 \frac{2(\eta_{12}-\eta_{11})(\eta_{32}-\eta_{31})-(\eta_{22}-\eta_{21})^2}{\sqrt{(2g(\eta_{12}-\eta_{11}))^3}}\right).
\end{aligned}
\tag{3.65}
$$

Estos subsistemas se utilizan para construir los OAG (3.27), donde las matrices E y C se definen como

$$
E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}
$$

y S_∞^{-1} se elige como sigue:

$$
S_\infty^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{pmatrix}.
$$

Definiendo $e_{11} = \eta_{11} - y_{R1}$, $e_{12} = \eta_{12} - y_{R2}$, $e_1^{(i-1)} = e_{i1}$ y $e_2^{(i-1)} = e_{i2}$, el OAG para $y_1 = \eta_{11}$ es:

$$
\begin{aligned}
\dot{\hat{e}}_{11} &= \hat{e}_{21} - 3\theta_1(\hat{e}_{11} - e_{11}) \qquad (3.66)\\
\dot{\hat{e}}_{21} &= \hat{e}_{31} - 3\theta_1^2(\hat{e}_{11} - e_{11})\\
\dot{\hat{e}}_{31} &= \frac{a_3 S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_{12}+y_{R2}) - (\hat{e}_{11}+y_{R1}))\right)^3}}\right)\times\\
&\quad \left(2\left((\hat{e}_{12}+y_{R2}) - (\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31}) - (\hat{e}_{22}-\hat{e}_{21})^2\right)\\
&\quad -\frac{a_2 S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_{11}+y_{R1}\right)\hat{e}_{31} - \hat{e}_{21}^2}{\sqrt{\left(2\left(\hat{e}_{11}+y_{R1}\right)\right)^3}}\right)\\
&\quad +\frac{1}{A}\left(\ddot{\hat{u}}_2 + \ddot{\hat{f}}_2\right) - \theta_1^3(\hat{e}_{11} - e_{11}) \quad = \quad -\sum_{i=1}^{3} s_i \hat{e}_{i1}
\end{aligned}
$$

y para $y_2 = \eta_{12}$:

$$
\begin{aligned}
\dot{\hat{e}}_{12} &= \hat{e}_{22} - 3\theta_2(\hat{e}_{12} - e_{12}) \qquad (3.67)\\
\dot{\hat{e}}_{22} &= \hat{e}_{32} - 3\theta_2^2(\hat{e}_{12} - e_{12})\\
\dot{\hat{e}}_{32} &= -\frac{a_3 S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_{12}+y_{R2}) - (\hat{e}_{11}+y_{R1}))\right)^3}}\right)\times\\
&\quad \left(2\left((\hat{e}_{12}+y_{R2}) - (\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31}) - (\hat{e}_{22}-\hat{e}_{21})^2\right)\\
&\quad -\frac{a_1 S\sqrt{g}}{A}\left(\frac{2\hat{e}_{32}(\hat{x}_1 - (\hat{e}_{12}+y_{R2})) + \left(\dot{\hat{x}}_1 - \hat{e}_{22}\right)^2}{\sqrt{\left(2(\hat{x}_1 - (\hat{e}_{12}+y_{R2}))\right)^3}}\right)\\
&\quad +\frac{a_1 S\sqrt{g}}{A^2}\left(\frac{1}{\sqrt{2(\hat{x}_1 - (\hat{e}_{12}+y_{R2}))}}\right)\left(\dot{\hat{u}}_1 + \dot{\hat{f}}_1\right)\\
&\quad +\frac{a_1^2 S^2 g}{2A^2}\left(\frac{\dot{\hat{x}}_1 - \hat{e}_{22}}{\hat{x}_1 - (\hat{e}_{12}+y_{R2})}\right) - \theta_2^3(\hat{e}_{12} - e_{12}) \quad = \quad -\sum_{i=1}^{3} t_i \hat{e}_{i2}.
\end{aligned}
$$

Dado que las referencias para este sistema son constantes, sus derivadas han sido despreciadas en estas ecuaciones, con el fin de simplificarlas. Por tanto, del subsistema (3.67) la dinámica de $\hat{u}_1$ se obtiene como:

$$
\begin{aligned}
\dot{\hat{u}}_1 &= \frac{A^2\sqrt{2(\hat{x}_1 - (\hat{e}_{12}+y_{R2}))}}{a_1 S\sqrt{g}}\left(-\sum_{i=1}^{3} t_i \hat{e}_{i2}\right) - \frac{a_1 S\sqrt{g}\left(\dot{\hat{x}}_1 - \hat{e}_{22}\right)}{\sqrt{2(\hat{x}_1 - (\hat{e}_{12}+y_{R2}))}}\\
&\quad +\frac{2A\hat{e}_{32}(\hat{x}_1 - (\hat{e}_{12}+y_{R2}) + A\left(\dot{\hat{x}}_1 - \hat{e}_{22}\right)^2}{2(\hat{x}_1 - (\hat{e}_{12}+y_{R2}))} \qquad (3.68)
\end{aligned}
$$

$$+\frac{a_3 A\sqrt{(\hat{x}_1-(\hat{e}_{12}+y_{R2}))}}{2a_1}\left(\frac{1}{\sqrt{((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1}))^3}}\right)\times$$
$$\left(2\left((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31})-(\hat{e}_{22}-\hat{e}_{21})^2\right)-\dot{\hat{f}}_1$$

y del subsistema (3.66), se obtiene la siguiente dinámica para $\hat{u}_2$:

$$\begin{aligned}\ddot{\hat{u}}_2 &= A\left(-\sum_{i=1}^{3}s_i\hat{e}_{i1}\right)+a_2S\sqrt{g}\left(\frac{2\left(\hat{e}_{11}+y_{R1}\right)\hat{e}_{31}-\hat{e}_{21}^2}{\sqrt{\left(2\left(\hat{e}_{11}+y_{R1}\right)\right)^3}}\right)\\ &\quad -a_3S\sqrt{g}\left(\frac{1}{\sqrt{\left(2((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1}))\right)^3}}\right)\times \\ &\quad \left(2\left((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31})-(\hat{e}_{22}-\hat{e}_{21})^2\right)-\ddot{\hat{f}}_2.\end{aligned} \tag{3.69}$$

Por lo tanto, definiendo $\hat{u}_1=\hat{u}_{11}$, $\hat{u}_2=\hat{u}_{12}$, $\dot{\hat{u}}_2=\hat{u}_{22}$, $\hat{f}_1=\hat{f}_{11}$, $\hat{f}_2=\hat{f}_{12}$ y $\dot{\hat{f}}_2=\hat{f}_{22}$, la cadena de integradores del sistema en lazo cerrado es:

$$\begin{aligned}\dot{\hat{e}}_{11} &= \hat{e}_{21}-3\theta_1(\hat{e}_{11}-e_{11})\\ \dot{\hat{e}}_{21} &= \hat{e}_{31}-3\theta_1^2(\hat{e}_{11}-e_{11})\\ \dot{\hat{e}}_{31} &= \frac{a_3S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1}))\right)^3}}\right)\times\\ &\quad \left(2\left((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31})-(\hat{e}_{22}-\hat{e}_{21})^2\right)\\ &\quad -\frac{a_2S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_{11}+y_{R1}\right)\hat{e}_{31}-\hat{e}_{21}^2}{\sqrt{\left(2\left(\hat{e}_{11}+y_{R1}\right)\right)^3}}\right)\\ &\quad +\frac{1}{A}\left(\dot{\hat{u}}_{22}+\dot{\hat{f}}_{22}\right)-\theta_1^3(\hat{e}_{11}-e_{11})\\ \dot{\hat{e}}_{12} &= \hat{e}_{22}-3\theta_2(\hat{e}_{12}-e_{12})\\ \dot{\hat{e}}_{22} &= \hat{e}_{32}-3\theta_2^2(\hat{e}_{12}-e_{12})\\ \dot{\hat{e}}_{32} &= -\frac{a_3S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1}))\right)^3}}\right)\times\\ &\quad \left(2\left((\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1})\right)(\hat{e}_{32}-\hat{e}_{31})-(\hat{e}_{22}-\hat{e}_{21})^2\right)\\ &\quad -\frac{a_1S\sqrt{g}}{A}\left(\frac{2\hat{e}_{32}(\hat{x}_1-(\hat{e}_{12}+y_{R2}))+\left(\dot{\hat{x}}_1-\hat{e}_{22}\right)^2}{\sqrt{\left(2(\hat{x}_1-(\hat{e}_{12}+y_{R2}))\right)^3}}\right)\\ &\quad +\frac{a_1S\sqrt{g}}{A^2}\left(\frac{1}{\sqrt{2(\hat{x}_1-(\hat{e}_{12}+y_{R2}))}}\right)\left(\dot{\hat{u}}_{11}+\dot{\hat{f}}_{11}\right)\end{aligned}$$

$$
\begin{aligned}
& +\frac{a_1^2 S^2 g}{2A^2}\left(\frac{\dot{\hat{x}}_1-\hat{e}_{22}}{\hat{x}_1-(\hat{e}_{12}+y_{R2})}\right)-\theta_2^3(\hat{e}_{12}-e_{12}) \\
\dot{\hat{u}}_{11} &= \frac{A^2\sqrt{2T_3}}{a_1 S\sqrt{g}}\left(-t_1\hat{e}_{12}-t_2\hat{e}_{22}-t_3\hat{e}_{32}\right)-\frac{a_1 S\sqrt{g}\dot{T}_3}{\sqrt{2T_3}} \\
& +\frac{a_3 A\sqrt{T_3}}{2a_1}\left(\frac{2T_1\ddot{T}_1-\left(\dot{T}_1\right)^2}{\sqrt{\left(T_1\right)^3}}\right) \\
& +\frac{2A\left(\hat{e}_{32}+\ddot{y}_{R2}\right)T_3+A\left(\dot{T}_3\right)^2}{2T_3}-\dot{\hat{f}}_{11} \\
\dot{\hat{u}}_{12} &= \hat{u}_{22} \\
\dot{\hat{u}}_{22} &= A\left(-s_1\hat{e}_{11}-s_2\hat{e}_{21}-s_3\hat{e}_{31}\right)-a_3 S\sqrt{g}\left(\frac{2T_1\ddot{T}_1-\left(\dot{T}_1\right)^2}{\sqrt{\left(2T_1\right)^3}}\right) \\
& +a_2 S\sqrt{g}\left(\frac{2T_2\ddot{T}_2-\left(\dot{T}_2\right)^2}{\sqrt{\left(2T_2\right)^3}}\right)-\dot{\hat{f}}_{22} \\
\dot{\hat{f}}_{11} &= -k_1(-q_{13}+\hat{u}_{11}+\gamma_1+k_1 A\hat{x}_1)+k_1 A\dot{\hat{x}}_1 \\
\dot{\hat{f}}_{12} &= \hat{f}_{22} \\
\dot{\hat{f}}_{22} &= -k_3(\dot{q}_{32}-\dot{q}_{20}+\hat{u}_{22}+\dot{\gamma}_3+k_3 A\dot{y}_1)+k_3 A\ddot{y}_1.
\end{aligned}
$$

donde

$$
\begin{aligned}
T_1 &= (\hat{e}_{12}+y_{R2})-(\hat{e}_{11}+y_{R1}) \\
T_2 &= \hat{e}_{11}+y_{R1} \\
T_3 &= \hat{x}_1-(\hat{e}_{12}+y_{R2}) \\
\dot{\hat{x}}_1 &= \hat{\zeta}+\frac{A\hat{\zeta}+q_{32}}{g a_1^2 S^2}\left(A\left(-k_2(\gamma_2+k_2 y_2)+k_2\dot{y}_2\right)+\dot{q}_{32}\right).
\end{aligned}
$$

3.3.2.3. Resultados de simulación

Se realizaron simulaciones para este sistema en MATLAB Simulink durante 1000 segundos, utilizando como referencias las señales $y_{R1}=0.06$ y $y_{R2}=0.11$ e introduciendo las fallas $f_1=0.01[1+sen(0.2te^{-0.01t})]\mathcal{U}(t-220)$ y $f_2=0.01[1+sen(0.05te^{-0.001t})]\mathcal{U}(t-300)$, donde $\mathcal{U}(t)$ es la función escalón. Los parámetros de diseño se eligieron como $\theta_1=\theta_2=1$, $s_1=t_1=1$, $s_2=t_2=3$, $s_3=t_3=3$, $k_1=1.85$, $k_2=0.3$, $k_3=22$.

Los parámetros del sistema son $A=0.0149\ m^2$ y $S=5\times10^{-5}\ m^2$, $a_1=0.4385$, $a_2=0.7774$ y $a_3=0.4435$.

Las Figs. 3.6 y 3.8 muestran los resultados de estimación de f_1 y f_2 respectivamente. Estas figuras muestan información de las fallas que puede utilizarse para establecer magnitudes críticas de las mismas, lo cual puede poner el proceso en riesgo y lo fuerza a tomar acciones correctivas a un nivel físico, como el reemplazo de sensores y actuadores. En las Figs. 3.7 y 3.9 se muestran los índices de desempeño de los OOR propuestos para estimar la fallas f_1 y f_2. Los índices fueron evaluados utilizando la misma funcional de costo del ejemplo numérico. La Fig. 3.10 muestra las señales del control tolerante a fallas utilizado. Finalmente, las Figs. 3.11 y 3.12 muestran el seguimiento de las salidas 1 y 2, respectivamente.

Comentario 3.3.1. *A partir del trabajo de este capítulo se publicó el artículo de revista [35]; de igual forma se presentaron en congresos internacionales las ponencias [41] y [65]. Estos trabajos se incluyen en el Anexo.*

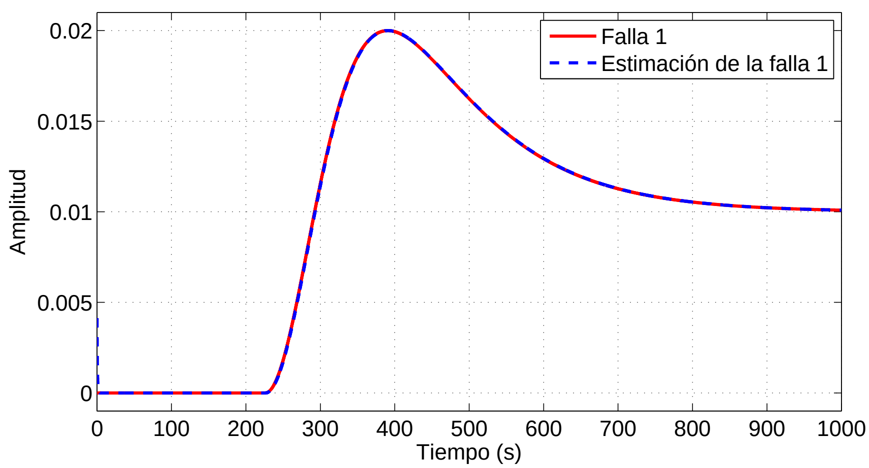

Figura 3.6: Estimación de la falla 1 del sistema de tres tanques.

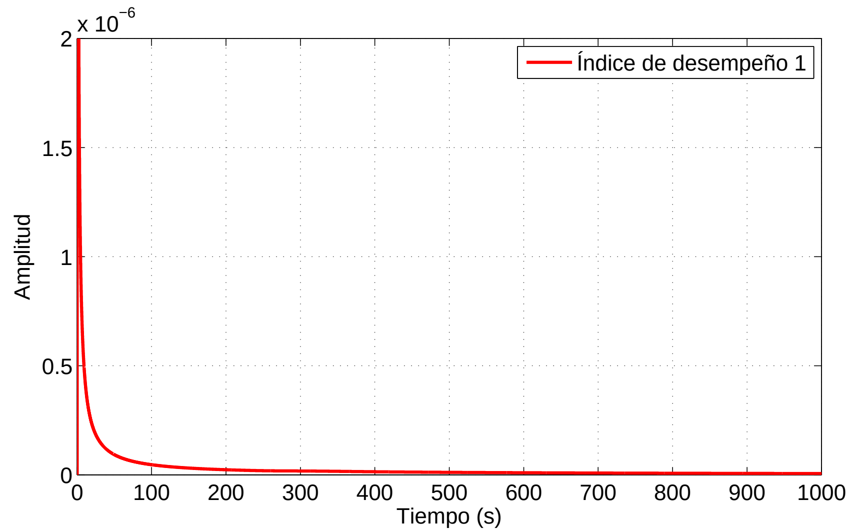

Figura 3.7: Evaluación de desempeño para la estimación de la falla 1 del sistema de tres tanques.

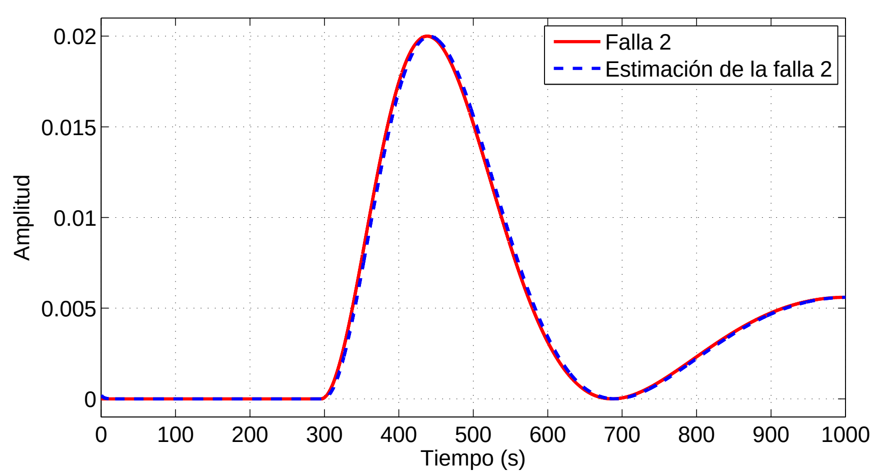

Figura 3.8: Estimación de la falla 2 del sistema de tres tanques.

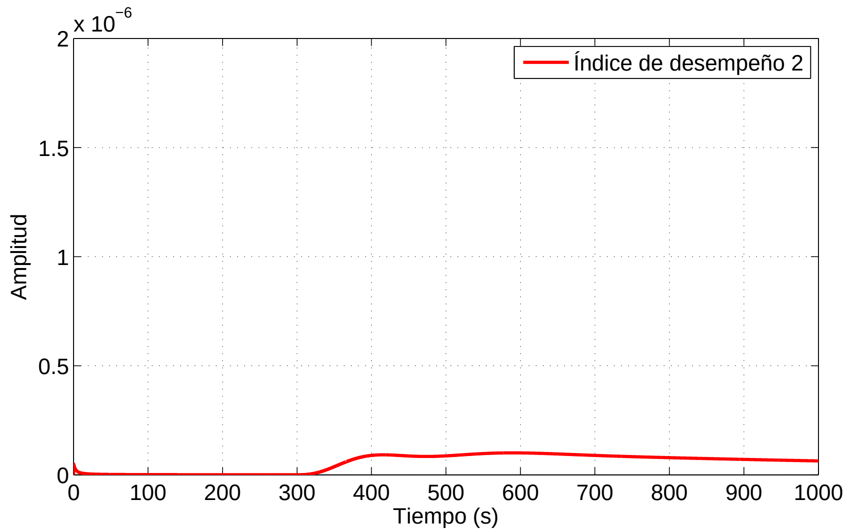

Figura 3.9: Evaluación de desempeño para la estimación de la falla 2 del sistema de tres tanques.

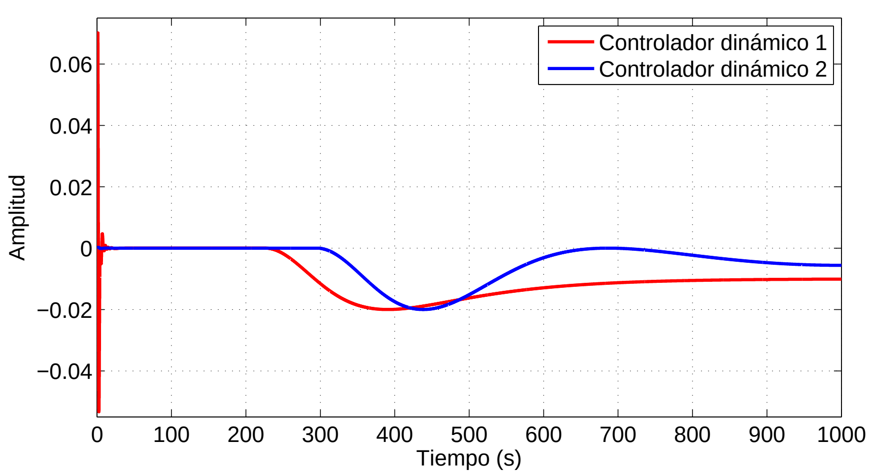

Figura 3.10: Controladores dinámicos del sistema de tres tanques.

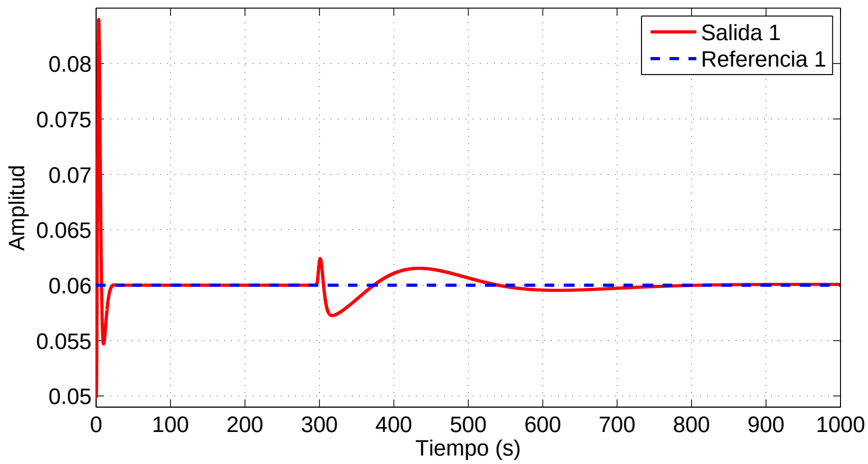

Figura 3.11: Seguimiento de la salida 1 del sistema de tres tanques.

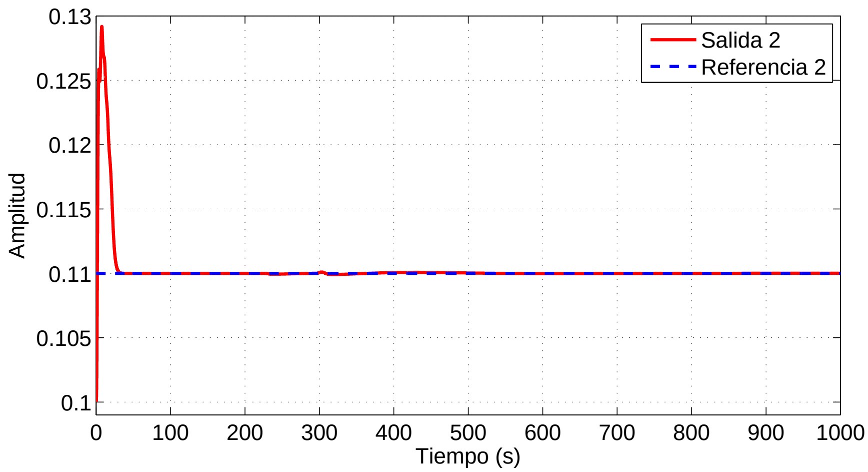

Figura 3.12: Seguimiento de la salida 2 del sistema de tres tanques.

Capítulo 4

Fundamentos de cálculo y sistemas dinámicos fraccionarios

En este capítulo se presentan las herramientas matemáticas básicas relacionadas con el cálculo fraccionario. Además, se da una introducción a la teoría de sistemas dinámicos con ecuaciones que presentan derivadas e integrales de orden fraccionario, así como algunos controladores de orden fraccionario que se han propuesto, y algunos resultados de estabilidad desarrollados para este tipo de sistemas.

4.1. Funciones utilizadas en cálculo fraccionario

4.1.1. Función gamma

La función gamma $\Gamma(\cdot)$ se define por la integral

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt,$$

la cual converge para argumentos reales positivos, o negativos no enteros, así como para argumentos complejos con parte real positiva, o negativa no entera. En particular, obsérvese que $\Gamma(1) = 1$.

Una de las propiedades básicas de la función gamma es que satisface la siguiente ecuación:

$$\Gamma(n+1) = n\Gamma(n)$$

De forma recursiva, se puede obtener:

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &\dots \\ &= n(n-1)(n-2)...\Gamma(1) \\ &= n(n-1)(n-2)...1 \\ &= n! \end{aligned}$$

Por lo tanto, la función gamma es una generalización de la función factorial, que permite incluir en el argumento valores no enteros e incluso complejos.

4.1.2. Función de Mittag-Leffler

Definición 4.1.1. *Sean $n_1, n_2 > 0$. La función E_{n_1,n_2} definida por*

$$E_{n_1,n_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kn_1 + n_2)}$$

siempre que la serie converja, es llamada la función de Mittag-Leffler de dos parámetros, con parámetros n_1 y n_2.

Nótese que

$$\begin{aligned} E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ &= e^z \end{aligned}$$

Por lo tanto, la función de Mittag-Leffler es una generalización de la función exponencial, y con la elección adecuada de los parámetros n_1 y n_2, puede ser usada para representar otro tipo de funciones, como son las funciones trigonométricas hiperbólicas y la función error.

4.2. Integrales y derivadas de orden fraccionario

4.2.1. Integral fraccionaria de Riemann-Liouville

Considérese la antiderivada o primitiva de orden entero de una función $f(t)$:

$$D^{-1}f(t) = \int_0^t f(x)dx.$$

Aplicando de nuevo el operador, se tiene:

$$D^{-2}f(t) = \int_0^t \int_0^x f(y)dydx.$$

Realizando un cambio de límites de integración, se obtiene:

$$\begin{aligned} D^{-2}f(t) &= \int_0^t \int_y^t f(y)dxdy \\ &= \int_0^t (t-y)f(y)dy. \end{aligned}$$

Similarmente, se obtiene:

$$
\begin{aligned}
D^{-3}f(t) &= \frac{1}{2}\int_0^t (t-y)^2 f(y)dy \\
&\vdots \\
D^{-n}f(t) &= \frac{1}{(n-1)!}\int_0^t f(y)(t-y)^{n-1}dy.
\end{aligned}
$$

Esta última ecuación es conocida como la fórmula de Cauchy para la integral iterada o repetida. Si se generaliza para el caso $n \in \mathbb{R}^+$, se tiene la siguiente definición.

Definición 4.2.1. *La integral fraccionaria de Riemann-Liouville de orden $\alpha > 0$ de una función $f(t)$ se define como*

$$
{}_0I_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau
$$

Para simplificar la notación, el operador integral fraccionario de Riemann-Liouville de orden α será representado como I^α.

4.2.2. Derivada fraccionaria de Riemann-Liouville

Considérese ahora la definición de la derivada de orden entero de una función $f(t)$:

$$
D^1 f(t) = \frac{df(t)}{dt} = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h}
$$

esto es, como el límite de una diferencia hacia atrás. De forma similar:

$$
\begin{aligned}
D^2 f(t) &= \frac{d^2f(t)}{dt^2} = \lim_{h\to 0}\frac{1}{h^2}[f(t)-2f(t-h)+f(t-2h)] \\
&\vdots \\
D^n f(t) &= \frac{d^nf(t)}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^n\binom{n}{k}f(t-kh)
\end{aligned}
$$

donde el coeficiente binomial se define como

$$
\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k!}
$$

Si se toma $n \in \mathbb{R}^+$ en esta última ecuación, se obtiene la definición de Grünwald-Letnikov de la derivada fraccionaria de orden n. Por otro lado, si se considera en esta definición una clase de funciones $f(t)$ con $m+1$ derivadas continuas en $t \geq 0$, se obtiene la siguiente definición.

Definición 4.2.2. *La derivada fraccionaria de Riemann-Liouville de orden α se define como:*

$$ {}_{0}^{RL}D_t^{\alpha} f(t) := D^m I^{m-\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau\right] $$

donde $m-1<\alpha<m, m\in\mathbb{N}$.

Esta definición juega un papel importante en el desarrollo de la teoría del cálculo fraccionario y posee varias aplicaciones en matemáticas puras, como soluciones de ecuaciones diferenciales, definiciones de clases de funciones, sumas de series, entre otras.

4.2.3. Derivada fraccionaria de Caputo

Algunos problemas de aplicación, como los existentes en la teoría de viscoelasticidad y la mecánica de sólidos, requieren la formulación de condiciones iniciales para sus modelos. Sin embargo, el enfoque de Riemann-Liouville conlleva a condiciones iniciales que contienen valores límite de derivadas fraccionarias; aunque estos problemas pueden ser resueltos matemáticamente, las soluciones son prácticamente inútiles dado que no hay interpretación física de este tipo de condiciones iniciales. La solución propuesta por M. Caputo para este problema es la siguiente.

Definición 4.2.3. *La derivada fraccionaria de Caputo se define como:*

$$ {}_{0}^{C}D_t^{\alpha} f(t) := I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau $$

donde $m-1<\alpha<m, m\in\mathbb{N}$.

La ventaja principal del enfoque de Caputo es que las condiciones iniciales de ecuaciones diferenciales que contienen derivadas de este tipo toman la misma forma que en el caso entero, las cuales poseen interpretaciones físicas conocidas. Sin embargo, hay que notar que la definición de Caputo requiere la integrabilidad de la m-ésima derivada de orden entero de la función $f(t)$.

Debido a que en este trabajo se manejarán aplicaciones a modelos de sistemas físicos, solamente se utilizará la definición de derivada fraccionaria de Caputo, cuyo operador de orden α será denotado como D^{α} para simplificar la notación.

4.3. Sistemas dinámicos fraccionarios

Como se ha mencionado, los sistemas dinámicos de orden fraccionario, en contraste con los sistemas de orden entero, han sido estudiados con mayor fuerza en las últimas décadas. Esto se debe a la gran cantidad de aplicaciones y fenómenos físicos que presentan dinámicas con derivadas e integrales fraccionarias, como problemas de difusión, viscoelasticidad, comportamiento polimérico, sistemas financieros, sistemas biológicos, sistemas mecánicos amortiguados, circuitos eléctricos, electroquímica, reología, fractales y propagación de calor.

En particular, en la teoría del control la mayor contribución ha sido el desarrollo de controladores PID generalizados, así como de otros controladores que involucran dinámicas fraccionarias, como el CRONE y controlador fraccionario por modos deslizantes.

4.3.1. Sistemas fraccionarios de orden conmensurado

Existen diferentes definiciones para este tipo de sistemas, e.g. la siguiente, que se encuentra en [11].

Definición 4.3.1. *La ecuación diferencial fraccionaria*

$$g(x, y(x), D_{\star 0}^{n_1} y(x), D_{\star 0}^{n_2} y(x), ..., D_{\star 0}^{n_k} y(x)) = 0$$

con $0 < n_1 < n_2 < ... < n_k$ *y cierta función* g *es llamada conmensurada si los números* $n_1, n_2, ..., n_k$ *son conmensurados, es decir, si los cocientes* n_μ/n_ν *son números racionales para todos* $\mu, \nu \in \{1, 2, ..., k\}$.

En este caso, el autor utiliza esta definición debido a que está relacionada con el uso tradicional del concepto que es común en teoría de números. Sin embargo, en este trabajo se utilizará la siguiente, definida en [47].

Definición 4.3.2. *Considérese el siguiente modelo en espacio de estados:*

$$\begin{aligned} D^{\boldsymbol{\alpha}} x &= Ax + Bu \\ y &= Cx \end{aligned}$$

donde $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ *y* $y \in \mathbb{R}^p$, $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_n]^T$ *es el vector de órdenes fraccionarios. Si* $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha \in \mathbb{R}$, *el sistema se llama de orden conmensurado, de otra forma es un sistema de orden inconmensurado.*

Para propósitos de este trabajo, considérese la siguiente clase de sistemas no lineales fraccionarios conmensurados con entradas desconocidas (fallas):

$$\begin{aligned} D^{\boldsymbol{\alpha}} x &= g(x, u, f) \\ y &= h(x, u) \end{aligned} \tag{4.1}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $u \in \mathbb{R}^m$ el vector de entradas (control), $f \in \mathbb{R}^q$ el vector de entradas desconocidas (fallas), $y \in \mathbb{R}^p$ el vector de salidas, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, g y h son funciones analíticas. En particular, en este trabajo se manejará $0 < \alpha < 1$.

4.4. Controladores de orden fraccionario

Los sistemas dinámicos fraccionarios habían sido estudiados en el contexto de los sistemas de control de forma mesurada, debido a la falta de métodos matemáticos apropiados para su análisis. En general, el dominio del tiempo había sido evitado, por lo que los primeros controladores de orden fraccionario se desarrollaron en el dominio de la frecuencia. Entre estos destacan el controlador PID de orden fraccionario y el CRONE, los cuales se describirán brevemente en las siguientes secciones.

Por otro lado, cabe mencionar que el cálculo fraccionario también se ha extendido a otro tipo de estrategias de control, como son los controles $\mathcal{H}_2$ y $\mathcal{H}_\infty$, compensadores de adelanto-atraso, control por modos deslizantes, control adaptable por modelo de referencia, control por reset y observadores de estado.

4.4.1. Controlador PID de orden fraccionario

I. Podlubny propuso una generalización del controlador PID, conocida como controlador PI$^\lambda$D$^\mu$ que, como su nombre lo indica, involucra un término integral de orden λ y un término derivativo de orden μ, donde $\lambda, \mu \in \mathbb{R}$.

Como está documentado en la teoría del controlador PID clásico (de orden entero), el término integral hace más lenta la respuesta del sistema y reduce su estabilidad relativa, pero elimina el error en estado estable. En el dominio del tiempo, disminuye el tiempo de crecimiento de la respuesta transitoria, aumentando a su vez el tiempo de establecimiento y el sobretiro. En el plano complejo, la acción integral desplaza el lugar geométrico de las raíces hacia el semiplano derecho. Finalmente, en el dominio de la frecuencia, genera un incremento de -20 dB/dec en las pendientes de las curvas de magnitud y un decremento de $\pi/2$ rad en la gráfica de fase.

Por otro lado, el término derivativo incrementa la estabilidad del sistema, pero magnifica los efectos del ruido a altas frecuencias. En el dominio del tiempo, puede observarse un decremento en el sobretiro y en el tiempo de asentamiento. En el plano complejo, la acción derivativa desplaza el lugar geométrico de las raíces hacia el semiplano izquierdo. Finalmente, en el dominio de la frecuencia, genera un incremento de 20 dB/dec en las pendientes de las curvas de magnitud y un adelanto de fase constante de $\pi/2$ rad.

Como se observa, en el caso entero los efectos globales de las acciones integral y derivativa dependen de las ganancias correspondientes elegidas, aunque generan el mismo tipo de resultados en general. En el dominio de Laplace, estas acciones pueden verse como variables con exponente inverso, pero unitario, siendo el orden 0 el caso proporcional. Es por esto que al elegir los órdenes de estas acciones como valores no enteros, se espera obtener efectos intermedios a los que se obtienen en el caso entero.

Por lo tanto, el objetivo del controlador PI$^\lambda$D$^\mu$ es obtener una respuesta deseada de la planta no solamente mediante la sintonización de las ganancias del mismo, sino además variando adecuadamente los órdenes de las acciones integral y derivativa. Se ha comprobado que mediante esta técnica se generan mejores resultados que con el PID clásico.

4.4.2. CRONE

CRONE es un acrónimo en francés que significa Control Robusto de Orden No Entero. Este control fue propuesto por A. Oustaloup y representa el primer marco teórico para la aplicación de sistemas fraccionarios en control automático. Algunas de las características principales de esta técnica son:

- Metodología basada en el dominio de la frecuencia.
- Control continuo o discreto de sistemas de múltiples entradas y salidas con perturbaciones.
- Uso de la realimentación unitaria.
- Robustez respecto a incertidumbres paramétricas.
- Control de sistemas de fase mínima y no mínima, plantas inestables o con modos de flexión mecánica, plantas variantes en el tiempo y plantas no lineales.

Existen tres generaciones de controladores CRONE. La primera se utiliza cuando la planta a controlar tiene una fase constante alrededor de una frecuencia de interés, y vuelve al lazo robusto ante cambios en la ganancia de la planta; sin embargo, no asegura un comportamiento asintótico. Su función de transferencia se da por:

$$C(s) = C_0 s^{\alpha}$$

con $\alpha, C_0 \in \mathbb{R}$.

Si la planta no tiene una fase constante, se utiliza el controlador CRONE de segunda generación, con la siguiente función de transferencia:

$$C(s) = \frac{F(s)}{G(s)} \qquad\qquad F(s) = \left(\frac{\omega_{cg}}{s}\right)^{\alpha}$$

donde ω_{cg} es la frecuencia de cruce en lazo abierto y $\alpha \in [1, 2]$.

Finalmente, el controlador CRONE de tercera generación considera otro tipo de incertidumbres en el modelo, como la colocación incorrecta de polos y ceros. Su objetivo principal es asegurar que la ganancia (o incluso el factor de amortiguamiento) en lazo cerrado nunca exceda cierto valor, aun cuando algunos parámetros de la planta varíen en cierto rango. También se ha demostrado que mediante esta técnica se generan mejores resultados que con el PID clásico.

4.5. Resultados de estabilidad para sistemas fraccionarios

En esta sección se presentan brevemente algunos resultados de estabilidad existentes desarrollados para sistemas dinámicos fraccionarios de orden conmensurado.

4.5.1. Sistemas lineales

Considérese la siguiente clase de sistemas lineales:

$$\begin{aligned} D^{\alpha}x &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{4.2}$$

Teorema 4.5.1. *[38] El sistema autónomo (4.2) es:*

- *asintóticamente estable si y sólo si* $|\arg(\lambda(A))| > \alpha\pi/2$. *En este caso, las componentes del estado decaen hacia 0 como* $t^{-\alpha}$.

- *estable si y sólo si es asintóticamente estable, o aquellos valores propios críticos que satisfacen* $|\arg(\lambda(A))| = \alpha\pi/2$ *tienen multiplicidad geométrica 1.*

4.5.2. Sistemas no lineales

Considérese la siguiente clase de sistemas no lineales:

$$D^{\alpha}x = f(t,x), \qquad \alpha \in (0,1] \tag{4.3}$$

A continuación se presentan algunos resultados referentes a pruebas de estabilidad para el sistema (4.3)

Definición 4.5.2. *Se dice que la solución del sistema (4.3) es Mittag-Leffler estable si*

$$\|x\| \leq \{m[x(0)]E_{\alpha,1}(-\lambda t^{\alpha})\}^{b}$$

$\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, *y* $m(x)$ *es localmente Lipschitz (con constante de Lipschitz* m_0*) en* $x \in \mathbb{B}$, *el cual es un subconjunto abierto de* $\mathbb{R}^n$.

Teorema 4.5.3. *[27] Sea* $x = 0$ *un punto de equilibrio para el sistema (4.3) y sea* $\mathbb{D} \subset \mathbb{R}^n$ *un dominio que contiene al origen. Sea* $V(t,x) : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ *una función continuamente diferenciable y localmente Lipschitz con respecto a* x *tal que*

$$\begin{aligned} \alpha_1\|x\|^{a} \leq V(t,x) &\leq \alpha_2\|x\|^{ab} \\ D^{\beta}V(t,x) &\leq -\alpha_3\|x\|^{ab} \end{aligned}$$

donde $t \geq 0, x \in \mathbb{D}, \beta \in (0,1), \alpha_1, \alpha_2, \alpha_3, a$ *y* b *son constantes positivas arbitrarias. Entonces* $x = 0$ *es Mittag-Leffler estable. Si las suposiciones se cumplen globalmente en* $\mathbb{R}^n$, *entonces* $x = 0$ *es globalmente Mittag-Leffler estable.*

Lema 4.5.4. *[12] Sea* $x \in \mathbb{R}^n$ *un vector de funciones diferenciables. Entonces, para cualquier instante de tiempo* $t \geq t_0$, *la siguiente relación se satisface*

$$\frac{1}{2}D^{\alpha}(x^T P x) \leq x^T P D^{\alpha}x, \qquad \forall\alpha \in (0,1], \forall t \geq t_0$$

donde $P \in \mathbb{R}^{n\times n}$ *es una matriz constante, simétrica y definida positiva.*

Capítulo 5

Control tolerante a fallas en sistemas de orden fraccionario

En este capítulo se aplica a sistemas de orden fraccionario la metodología propuesta anteriormente para construir controladores dinámicos tolerantes a fallas en sistemas de orden entero. Elementos básicos de la metodología conocida como el diagnóstico de fallas, la construcción de formas canónicas y el análisis de estabilidad son extendidos y analizados con base en las herramientas del cálculo fraccionario. La metodología se desarrolla para sistemas fraccionarios de orden conmensurado. El esquema se valida en dos modelos con dinámicas fraccionarias. Posteriormente se realiza un comparativo de resultados con sistemas de orden entero.

5.1. Diagnóstico de fallas

De igual forma que en el caso de orden entero, el primer paso a considerar para rechazar los efectos de las fallas que aparecen en el sistema es el DF, y con el fin de diseñar un método para reconstruir las fallas, debemos saber si son diagnosticables, pero tomando en cuenta que el sistema es fraccionario de orden conmensurado. Por tanto, se introduce la siguiente definición, en correspondencia con el caso entero.

Definición 5.1.1. *Una variable de estado $\eta_i \in \mathbb{R}$ satisface la condición de observabilidad algebraica fraccionaria (OAF) si es una función de las primeras $r_1, r_2 \in \mathbb{N}$ derivadas fracionarias secuenciales de u y y, es decir,*

$$\eta_i = \phi_i(u, D^{\alpha}u, D^{2\alpha}u, ..., D^{r_1\alpha}u, y, D^{\alpha}y, D^{2\alpha}y, ..., D^{r_2\alpha}y) \tag{5.1}$$

donde $\phi_i : \mathbb{R}^{(r_1+1)m} \times \mathbb{R}^{(r_2+1)p} \to \mathbb{R}$.

Comentario 5.1.2. *Tanto los estados como las fallas pueden satisfacer la condición de OAF. En particular, se dice que cada falla que satisfaga esta propiedad es diagnosticable.*

Para reconstruir las fallas, se utilizará un OOR fraccionario. El siguiente es la extensión fraccionaria de un OOR proporcional (OORP):

$$D^{\alpha}\hat{f}_i = k_i(f_i - \hat{f}_i), \qquad 1 \leq i \leq q.$$

Por otro lado, el siguiente es un OOR fraccionario que, además de la parte proporcional, agrega un término integral fraccionario. Este observador es conocido como OORPI$^{r\alpha}$.

$$D^{\alpha}\hat{f}_i = k_{i0}(f_i - \hat{f}_i) + \sum_{j=1}^{r'} k_{ij} I^{j\alpha}(f_i - \hat{f}_i)$$

Este observador, además de considerar un término correctivo proporcional al error de estimación de la falla, agrega términos integrales fraccionarios del error para mejorar la convergencia. El valor de r' se elige de forma que satisfaga condiciones de estabilidad deseadas para el observador.

Cabe recordar que este observador ha sido elegido porque no requiere conocer la dinámica de la falla, y además utiliza la propiedad de OAF definida para estimarla. Los términos $k_{ij} \in \mathbb{R}^+$ determinan la tasa de convergencia del observador. En particular, en este trabajo $r' = 1$, por lo que el OORPI$^{\alpha}$ que será utilizado es:

$$D^{\alpha}\hat{f}_i = k_{i0}(f_i - \hat{f}_i) + k_{i1} I^{\alpha}(f_i - \hat{f}_i) \tag{5.2}$$

Las Figs. 5.1 y 5.2 muestran un comparativo entre la estimación de una falla de tipo escalón hecha por un OORP y la estimación de la misma falla hecha por un OORPI$^{\alpha}$. Se puede observar que aunque genera un sobretiro mayor, el OORPI$^{\alpha}$ mejora la convergencia de la estimación.

Comentario 5.1.3. *De manera similar al caso entero, la dinámica desconocida de la falla f_i, esto es Ω_i, se supone que está acotada:*

$$D^{\alpha} f_i = \Omega_i \leq |\Omega_i| \leq N_i$$

donde $N_i \in \mathbb{R}^+$ es una cota superior para la dinámica de la falla.

Por último, considerando la dinámica del OORPI$^{\alpha}$, se definen las siguientes variables:

$$\hat{f}_{i\bar{l}} = D^{(i-1)\alpha}\hat{f}_{\bar{l}} \qquad i = 1, ..., \mu_{\bar{l}} \tag{5.3}$$

por lo tanto se pueden escribir subsistemas de estimación de las fallas como sigue:

$$D^{\alpha}\hat{\mathbf{f}}_{\bar{l}} = E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}\ (u, y, f), \qquad\qquad 1 \leq \bar{l} \leq q \tag{5.4}$$

donde $\hat{\mathbf{f}}_{\bar{l}} = (\hat{f}_{1\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}\bar{l}})$ y

$$\omega_{\bar{l}}\ (u, y, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_{i0} D^{(\mu_{\bar{l}}-1)\alpha}(f_{\bar{l}} - \hat{f}_{\bar{l}}) + k_{i1} D^{(\mu_{\bar{l}}-2)\alpha}(f_{\bar{l}} - \hat{f}_{\bar{l}}) \end{pmatrix}.$$

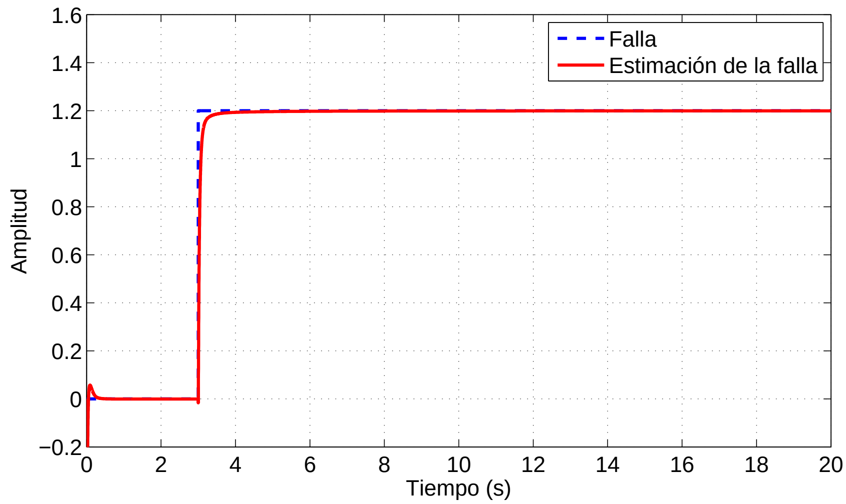

Figura 5.1: Estimación de una falla mediante un OORP.

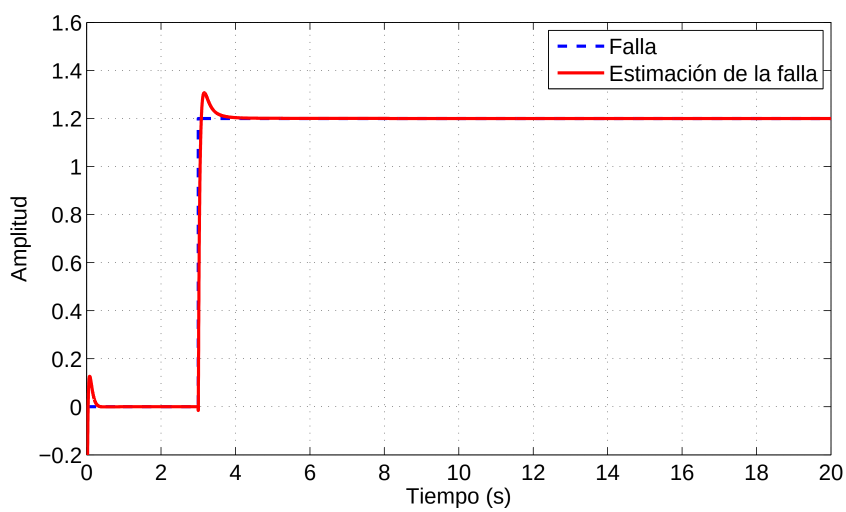

Figura 5.2: Estimación de una falla mediante un OORPI$^\alpha$.

5.2. Control dinámico tolerante a fallas

De igual forma que se hizo con la contraparte de orden entero, el sistema (4.1) se transforma en la siguiente Forma Canónica de Observabilidad Generalizada Multi-entrada Multi-salida Fraccionaria ($\mathcal{FCOGMMF}$):

$$\begin{aligned} D^{\alpha}\eta_{ij} &= \eta_{i+1,j}, \qquad 1 \leq i \leq n-1 \\ D^{\alpha}\eta_{nj} &= -L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha}u, f, ..., D^{\mu\alpha}f) \\ y_j &= \eta_{1j} \end{aligned} \tag{5.5}$$

donde L_j es una función C^1 de valores reales, $\eta_j = (\eta_{1j}, ..., \eta_{nj}) \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $f \in \mathbb{R}^q$, y enteros γ, $\mu \geq 0$. Esta $\mathcal{FCOGMMF}$ se conforma por p subsistemas, uno por cada salida.

Recordando que el error de seguimiento se define como:

$$\begin{aligned} e_{1j} &= y_j - y_{Rj}, \qquad 1 \leq j \leq p \\ &= \eta_{1j} - y_{Rj} \end{aligned}$$

se define la siguiente forma canónica fraccionaria del error:

$$\begin{aligned} D^{\alpha}e_{1j} &= \eta_{i+1,j} - D^{i\alpha}y_{Rj}, \qquad 1 \leq i \leq n-1 \\ D^{\alpha}e_{nj} &= D^{\alpha}\eta_{nj} - D^{n\alpha}y_{Rj} \\ &= -L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} \end{aligned} \tag{5.6}$$

imponiéndose de igual forma una dinámica lineal del error:

$$-L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} = -\sum_{i=1}^{n} a_{ij}\left[\eta_{ij} - D^{(i-1)\alpha}y_{Rj}\right].$$

Ahora, se puede definir una cadena de integradores del error de seguimiento como sigue

$$\begin{aligned} D^{\alpha}e_{ij} &= e_{i+1,j}, \qquad 1 \leq i \leq n-1 \\ D^{\alpha}e_{nj} &= -\sum_{i=1}^{n} a_{ij}e_{ij} \end{aligned}$$

o de forma compacta:

$$D^{\alpha}\mathbf{e}_j = F_j\mathbf{e}_j \tag{5.7}$$

donde

$$F_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \vdots & \cdots & 1 \\ -a_{1j} & -a_{2j} & \cdots & -a_{nj} \end{pmatrix}.$$

En este caso, con base en el Teorema 4.2.1, se tiene que el sistema (5.7) es estable si se satisface la siguiente propiedad:

$$|\arg(\lambda(F_j))| > \alpha\pi/2$$

Entonces, la estimación $\hat{\mathbf{e}}_j$ se obtiene con el siguiente OAG fraccionario:

$$D^{\alpha}\hat{\mathbf{e}}_j = E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \mathbf{u}, \hat{\mathbf{f}}\right) - S_{\infty}^{-1}C^T C(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{5.8}$$

donde S_∞ es solución de la siguiente ecuación de Lyapunov:

$$S_{\infty}\left(E + \frac{\theta}{2}I\right) + \left(E^T + \frac{\theta}{2}I\right)S_{\infty} = C^T C \tag{5.9}$$

con $\theta > 0$ y $C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$. Los coeficientes de S_∞ se dan por:

$$(S_{\infty})_{ks} = \frac{\alpha_{ks}}{\theta^{k+s-1}}$$

donde (α_{ks}) es una matriz simétrica definida positiva independiente de θ.

Por otro lado, sea $\hat{u}_l$ la solución de:

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, D^{\gamma_l\alpha}\hat{u}_l, \hat{\mathbf{f}}) - D^{n\alpha}y_{Rj} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij} \tag{5.10}$$

donde $D^{\gamma_l\alpha}\hat{u}_l$ es la derivada de más alto orden de la entrada en cuestión que se encuentra en la ecuación dada. Por lo tanto, la ecuación dinámica del controlador $\hat{u}_l$ es:

$$D^{\gamma_l\alpha}\hat{u}_l = K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \tag{5.11}$$

A partir de los p subsistemas dinámicos se obtendrán las ecuaciones dinámicas para p controladores. Estos controladores permitirán realizar el seguimiento de la referencia en el sistema original, con tolerancia a fallas (es decir, eliminando el efecto de las fallas). Por lo tanto, la ecuación (5.8) se reescribe como:

$$D^{\alpha}\hat{\mathbf{e}}_j = E\hat{\mathbf{e}}_j + \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_{\infty}^{-1}C^T C(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{5.12}$$

con

$$\hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, ..., D^{\gamma\alpha}\hat{u}, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} \end{pmatrix}.$$

Si se definen las siguientes variables:

$$\hat{u}_{il} = D^{(i-1)\alpha}\hat{u}_l \qquad i = 1, ..., \gamma_l \tag{5.13}$$

las ecuaciones de los controladores dinámicos pueden escribirse en la siguiente forma canónica:

$$D^{\alpha}\hat{\mathbf{u}}_l = E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right), \qquad 1 \le l \le m \tag{5.14}$$

donde $\hat{\mathbf{u}}_l = (\hat{u}_{1l}, ..., \hat{u}_{\gamma_l l})$ y

$$\kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_l(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}) \end{pmatrix}.$$

Por lo tanto, la dinámica del sistema en lazo cerrado se da por:

$$\begin{aligned} D^{\alpha}\hat{\mathbf{e}}_j &= E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_{\infty}^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) \\ D^{\alpha}\hat{\mathbf{u}}_l &= E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ D^{\alpha}\hat{\mathbf{f}}_{\bar{l}} &= E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}\,(u, y, f) \end{aligned} \tag{5.15}$$

para $1 \le j \le p$, $1 \le l \le m$ y $1 \le \bar{l} \le q$. Este sistema se expresa en forma de cadenas de integradores como sigue:

$$\begin{aligned} D^{\alpha}\hat{e}_{ij} &= \hat{e}_{i+1,j} - \psi_i(\theta_j)(\hat{e}_j - e_j) && 1 \le i \le n-1 \\ D^{\alpha}\hat{e}_{nj} &= -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} - \theta_j^n \\ & && 1 \le j \le p \\ D^{\alpha}\hat{u}_{il} &= \hat{u}_{i+1,l} && 1 \le i \le \gamma_l - 1 \\ D^{\alpha}\hat{u}_{\gamma_l l} &= K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) && 1 \le l \le m \\ D^{\alpha}\hat{f}_{i\bar{l}} &= \hat{f}_{i+1,\bar{l}} && 1 \le i \le \mu_{\bar{l}} - 1 \\ D^{\alpha}\hat{f}_{\mu_{\bar{l}}\bar{l}} &= k_{i0}D^{(\mu_{\bar{l}}-1)\alpha}(f_{\bar{l}} - \hat{f}_{\bar{l}}) + k_{i1}D^{(\mu_{\bar{l}}-2)\alpha}(f_{\bar{l}} - \hat{f}_{\bar{l}}) && 1 \le \bar{l} \le q \end{aligned}$$

donde $\psi_i(\theta_j)$ es una función obtenida de S_{∞}^{-1}.

Por último, defínase el error de observación como $\varepsilon_j = \hat{\mathbf{e}}_j - \mathbf{e}_j$, obteniéndose su dinámica a partir de las ecuaciones (5.6) y (5.12):

$$D^{\alpha}\varepsilon_j = \left(E - S_{\infty}^{-1}C^TC\right)\varepsilon_j + \Phi_j(\varepsilon, \hat{\mathbf{e}}) \tag{5.16}$$

donde

$$\Phi_j(\varepsilon, \hat{\mathbf{e}}) = \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - \varphi_j\left(\hat{\mathbf{e}} - \varepsilon, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right).$$

A continuación se establece el siguiente resultado.

Teorema 5.2.1. *Considérese el sistema (4.1) descrito en la $\mathcal{FCOGMMF}$ (5.5) compuesta de p subsistemas. La dinámica de observación correspondiente al subsistema j*

se compone de $\hat{\mathbf{e}}_j$ *y* ε_j*, para* $1 \leq j \leq p$*. Sea* $f_{\bar{l}}$ *diagnosticable y estimada por medio de la dinámica de* $\hat{f}_{\bar{l}}$*, para* $1 \leq \bar{l} \leq q$*. Sea* $\hat{u}_l$ *la solución de*

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, D^{\gamma_l \alpha}\hat{u}_l, \hat{\mathbf{f}}) - D^{n\alpha} y_{Rj} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij}$$

para $1 \leq l \leq m$*. Entonces, el origen del sistema en lazo cerrado (5.15) es Mittag-Leffler estable.*

Demostración. Considérese la siguiente función candidata de Lyapunov:

$$V(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j + \varepsilon_j^T S_\infty \varepsilon_j + \tilde{f}_{\bar{l}}^2 \tag{5.17}$$

donde $\tilde{f}_{\bar{l}} = f_{\bar{l}} - \hat{f}_{\bar{l}}$ es el error de estimación de la falla. Defínase $V_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j$, $V_2(\varepsilon_j) = \varepsilon_j^T S_\infty \varepsilon_j$ y $V_3(\tilde{f}_{\bar{l}}) = \tilde{f}_{\bar{l}}^2$, donde P y S_∞ son matrices constantes, simétricas, definidas positivas y soluciones de $F^T P + PF = -I$ y (5.9) respectivamente. Sean $\|x\|_P = \sqrt{x^T P x}$ y $\|x\|_{S_\infty} = \sqrt{x^T S_\infty x}$. Nótese que $V(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}})$ satisface la primera desigualdad del Teorema 4.5.3 dado que:

$$\begin{array}{rcccl} \alpha_{11}\|\hat{\mathbf{e}}_j\| & \leq & V_1(\hat{\mathbf{e}}_j) & \leq & \alpha_{21}\|\hat{\mathbf{e}}_j\| \\ \alpha_{12}\|\varepsilon_j\| & \leq & V_2(\varepsilon_j) & \leq & \alpha_{22}\|\varepsilon_j\| \\ \alpha_{13}\|\tilde{f}_{\bar{l}}\| & \leq & V_3(\tilde{f}_{\bar{l}}) & \leq & \alpha_{23}\|\tilde{f}_{\bar{l}}\| \end{array} \tag{5.18}$$

con $\alpha_{11} = \lambda_{\text{mín}}(P)$, $\alpha_{12} = \lambda_{\text{mín}}(S_\infty)$, $\alpha_{13} = 1$, $\alpha_{21} = \frac{1}{2}(\lambda_{\text{mín}}(P) + \lambda_{\text{máx}}(P))$, $\alpha_{22} = \frac{1}{2}(\lambda_{\text{mín}}(S_\infty) + \lambda_{\text{máx}}(S_\infty))$, $\alpha_{23} = \sup(\|\tilde{f}_{\bar{l}}\|)$, y $a = b = 1$.

Por la propiedad de linealidad de la derivada de Caputo y el Lema 4.5.4, se sigue que

$$D^\alpha V = D^\alpha V_1 + D^\alpha V_2 + D^\alpha V_3 \leq 2\hat{\mathbf{e}}_j^T P D^\alpha \hat{\mathbf{e}}_j + 2\varepsilon_j^T S_\infty D^\alpha \varepsilon_j + 2\tilde{f}_{\bar{l}}^T D^\alpha \tilde{f}_{\bar{l}}. \tag{5.19}$$

Por otro lado, nótese que:

$$\begin{array}{rcl} D^\alpha V_1(\hat{\mathbf{e}}_j) & \leq & 2\hat{\mathbf{e}}_j^T P D^\alpha \hat{\mathbf{e}}_j = 2\hat{\mathbf{e}}_j^T P \left[F\hat{\mathbf{e}}_j - S_\infty^{-1} C^T C \varepsilon_j\right] \\ & = & \hat{\mathbf{e}}_j^T \left(F^T P + PF\right)\hat{\mathbf{e}}_j - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C \varepsilon_j \\ & = & -\|\hat{\mathbf{e}}_j\|^2 - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C S_\infty^{-1} S_\infty \varepsilon_j \\ & \leq & -\|\hat{\mathbf{e}}_j\|^2 + \bar{K}\|\hat{\mathbf{e}}_j\|_{P^*}\|\varepsilon_j\|_{S_\infty^*}\rho(\theta) \\ & \leq & -\left(1 - \bar{K} d_1 d_2 \|\varepsilon_j\|\rho(\theta)\right)\|\hat{\mathbf{e}}_j\| \end{array}$$

donde $\bar{K} > 0$, $P^* = PP^T$, $S_\infty^* = S_\infty S_\infty^T$, $d_1 = \sqrt{\lambda_{\max}(P^*)}$, $d_2 = \sqrt{\lambda_{\max}(S_\infty^*)}$ y $\rho(\theta) = \|S_\infty^{-1} C^T C S_\infty^{-1}\|$.

Por lo tanto, se obtiene

$$D^\alpha V_1(\hat{\mathbf{e}}_j) \leq -\delta_{31}\|\hat{\mathbf{e}}_j\| \tag{5.20}$$

con $\delta_{31} = 1 - \bar{K} d_1 d_2 \|\varepsilon_j\|\rho(\theta)$.

De forma similar:

$$\begin{aligned}
D^{\alpha}V_2\left(\varepsilon_j\right) &\leq 2\varepsilon_j^T S_\infty D^\alpha \varepsilon_j = 2\varepsilon_j^T S_\infty \left[E_\theta \varepsilon_j + \Phi_j\left(\varepsilon, \hat{\mathbf{e}}\right)\right] \\
&= \varepsilon_j^T \left[E^T S_\infty + S_\infty E - C^T C\right]\varepsilon_j - \varepsilon_j^T C^T C \varepsilon_j + 2\varepsilon_j^T S_\infty \Phi_j\left(\varepsilon, \hat{\mathbf{e}}\right) \\
&\leq -\theta \|\varepsilon_j\|^2_{S_\infty} + 2\varepsilon_j^T S_\infty \Phi_j\left(\varepsilon, \hat{\mathbf{e}}\right) \\
&\leq -\theta\lambda_{\min}\left(S_\infty\right)\|\varepsilon_j\|^2 + 2\gamma\|\varepsilon_j\|^2_{S_\infty} \\
&\leq -\left(\theta\lambda_{\min}\left(S_\infty\right) - 2\gamma\lambda_{\max}\left(S_\infty\right)\right)\|\varepsilon_j\|
\end{aligned}$$

donde $\gamma^2 = \bar{\lambda}$, la cual es una constante tal que $\|\Phi_j\left(\varepsilon, \hat{\mathbf{e}}\right)\|^2_{S_\infty} \leq \bar{\lambda}\|\varepsilon_j\|^2_{S_\infty}$, dado que $\Phi_j(\varepsilon, \hat{\mathbf{e}})$ es Lipschitz y $\Phi_j(0, \hat{\mathbf{e}}) = 0$.

Entonces, se obtiene

$$D^{\alpha}V_2\left(\varepsilon_j\right) \leq -\delta_{32}\|\varepsilon_j\| \tag{5.21}$$

con $\delta_{32} = \theta\lambda_{\min}\left(S_\infty\right) - 2\gamma\lambda_{\max}\left(S_\infty\right)$.

Finalmente:

$$\begin{aligned}
D^{\alpha}V_3(\tilde{f}_{\bar{l}}) &\leq 2\tilde{f}_{\bar{l}} D^\alpha \tilde{f}_{\bar{l}} = 2\tilde{f}_{\bar{l}}\left(\Omega_{\bar{l}} - k_{\bar{l}0}\tilde{f}_{\bar{l}} - k_{\bar{l}1} I^\alpha \tilde{f}_{\bar{l}}\right) \\
&\leq 2\tilde{f}_{\bar{l}}\Omega_{\bar{l}} - 2k_{\bar{l}1}\tilde{f}_{\bar{l}} I^\alpha \tilde{f}_{\bar{l}} \\
&\leq 2N_{\bar{l}}\|\tilde{f}_{\bar{l}}\| - 2k_{\bar{l}1}\|\tilde{f}_{\bar{l}}\||I^\alpha \tilde{f}_{\bar{l}}| \\
&\leq -\left(2k_{\bar{l}1}|I^\alpha \tilde{f}_{\bar{l}}| - 2N_{\bar{l}}\right)\|\tilde{f}_{\bar{l}}\|.
\end{aligned}$$

Entonces, se obtiene

$$D^{\alpha}V_3(\tilde{f}_{\bar{l}}) \leq -\delta_{33}\|\tilde{f}_{\bar{l}}\| \tag{5.22}$$

con $\delta_{33} = 2k_{\bar{l}1}|I^\alpha \tilde{f}_{\bar{l}}| - 2N_{\bar{l}}$.

Por lo tanto, si $\delta_{31},\ \delta_{32},\ \delta_{33} > 0$, del Teorema 4.5.3 y las desigualdades (5.18-5.22), se concluye que el origen del sistema (5.15) es Mittag-Leffler estable. □

5.3. Aplicación

5.3.1. Oscilador de Van der Pol

Considérese la versión modificada del oscilador fraccionario de orden conmensurado de Van der Pol [47]:

$$\begin{aligned}
D^\alpha x_1 &= x_2 \\
D^\alpha x_2 &= -x_1 - \varepsilon(x_1^2 - 1)x_2
\end{aligned}$$

Añadiendo una entrada de control y una entrada desconocida y, seleccionando el primer estado como una salida medible a controlar, el sistema con el que se trabajará es:

$$\begin{aligned}
D^\alpha x_1 &= x_2 + u \\
D^\alpha x_2 &= -x_1 - \varepsilon(x_1^2 - 1)x_2 + f \\
y &= x_1
\end{aligned} \tag{5.23}$$

En este caso, el objetivo de control es el seguimiento por parte de la salida (el primer estado) de una trayectoria deseada, lo que hace que el oscilador presente un comportamiento caótico deseado en el plano fase, incluso en presencia de la falla.

5.3.1.1. Diagnóstico de la falla

En esta sección se diseñará el OORPI$^{\alpha}$ para reconstruir la falla en el modelo del oscilador de Van der Pol. Primero, se debe determinar si la falla f es diagnosticable, es decir, si satisface la condición de OAF.

De (5.23) se obtiene el siguiente polinomio:

$$f = D^{\alpha}x_2 + y + \varepsilon(y^2 - 1)x_2 \tag{5.24}$$

Puede observarse que el estado x_2 también satisface la condición de OAF. De (5.23):

$$x_2 = D^{\alpha}y - u \tag{5.25}$$

por lo tanto, el estado x_2 es fraccionariamente algebraicamente observable y puede ser reconstruido. Entonces, la falla es diagnosticable. Primero, se diseña un OORPI$^{\alpha}$ para estimar x_2:

$$D^{\alpha}\hat{x}_2 = k_{10}(D^{\alpha}y - u - \hat{x}_2) + k_{11}I^{\alpha}(D^{\alpha}y - u - \hat{x}_2)$$

Definiendo una variable auxiliar γ_1 como

$$\gamma_1 = \hat{x}_2 - k_{10}y$$

el OORPI$^{\alpha}$ para obtener la estimación del estado es

$$\begin{aligned} D^{\alpha}\gamma_1 &= -k_{10}\left(u + \gamma_1 + k_{10}y\right) + k_{11}y - k_{11}I^{\alpha}\left(u + \gamma_1 + k_{10}y\right) && (5.26) \\ \hat{x}_2 &= \gamma_1 + k_{10}y && (5.27) \end{aligned}$$

Ahora, se diseña el OORPI$^{\alpha}$ para estimar la falla:

$$D^{\alpha}\hat{f} = k_{20}(D^{\alpha}\hat{x}_2 + y + \varepsilon(y^2 - 1)\hat{x}_2 - \hat{f}) + k_{21}I^{\alpha}(D^{\alpha}\hat{x}_2 + y + \varepsilon(y^2 - 1)\hat{x}_2 - \hat{f})$$

Definiendo una variable auxiliar γ_2 como

$$\gamma_2 = \hat{f} - k_{20}\hat{x}_2$$

el OORPI$^{\alpha}$ para obtener la estimación de la falla es

$$\begin{aligned} D^{\alpha}\gamma_2 &= -k_{20}(-y - \varepsilon(y^2 - 1)\hat{x}_2 + \gamma_2 + k_{20}\hat{x}_2) + k_{21}\hat{x}_2 && (5.28) \\ &\quad -k_{21}I^{\alpha}(-y - \varepsilon(y^2 - 1)\hat{x}_2 + \gamma_2 + k_{20}\hat{x}_2) \\ \hat{f} &= \gamma_2 + k_{20}\hat{x}_2 && (5.29) \end{aligned}$$

5.3.1.2. Control tolerante a fallas

Con el fin de aplicar el controlador tolerante a fallas, debemos encontrar la $\mathcal{FCOGM}$ $\mathcal{MF}$, para lo cual primero tenemos que obtener las n derivadas fraccionarias de la salida:

$$\begin{aligned} D^{\alpha}y &= x_2 + u \\ D^{2\alpha}y &= D^{\alpha}x_2 + D^{\alpha}u \\ &= -y - \varepsilon(y^2-1)x_2 + D^{\alpha}u + f \end{aligned} \tag{5.30}$$

Ahora el error de seguimiento se describe en forma canónica:

$$\begin{aligned} e_1 &= y - y_R \\ D^{\alpha}e_1 &= D^{\alpha}y - D^{\alpha}y_R = e_2 \\ D^{\alpha}e_2 &= D^{2\alpha}e_1 = D^{2\alpha}y - D^{2\alpha}y_R \end{aligned} \tag{5.31}$$

y se obtiene un OAGF:

$$\begin{aligned} D^{\alpha}\hat{e}_1 &= \hat{e}_2 - 2\theta(\hat{e}_1 - e_1) \\ D^{\alpha}\hat{e}_2 &= D^{2\alpha}y - D^{2\alpha}y_R - \theta^2(\hat{e}_1 - e_1) \\ &= -y - \varepsilon(y^2-1)x_2 + D^{\alpha}u + \hat{f} - D^{2\alpha}y_R - \theta^2(\hat{e}_1 - e_1) = -\sum_{i=1}^{2} a_i\hat{e}_i \end{aligned} \tag{5.32}$$

Finalmente, de (5.32) se obtiene la dinámica del controlador fraccionario tolerante a fallas:

$$D^{\alpha}\hat{u} = -a_1\hat{e}_1 - a_2\hat{e}_2 + y + \varepsilon(y^2-1)\hat{x}_2 - \hat{f} + D^{2\alpha}y_R \tag{5.33}$$

5.3.1.3. Resultados de simulación

Se realizaron simulaciones durante 60 segundos en el modelo del sistema, para lo cual se utilizó el bloque de derivada fraccionaria para Simulink del Ninteger Toolbox de MATLAB [13]. Se seleccionó $\alpha = 0.9$. La referencia se eligió como $y_R(t) = 2sin(t)$. La falla se eligió como $f(t) = cos(t)$ apareciendo a partir de 20 s. El valor del parámetro del sistema es $\varepsilon = 0.1$. Los parámetros de diseño (ganancias) se escogieron como $\theta = 20$, $a_1 = 400$, $a_2 = 40$, $k_{10} = k_{20} = 30$ y $k_{11} = k_{21} = 1$.

La Fig. 5.3 muestra los resultados del DF con el OORPI$^{\alpha}$. Puede verse que la falla estimada sigue a la señal de la falla real en un tiempo muy corto. El índice de desempeño del OORPI$^{\alpha}$ se evaluó con la siguiente funcional de costo:

$$J_t = \frac{1}{t+\epsilon}\int_0^t \left\|\tilde{f}\right\|^2 dt$$

donde $\epsilon = 0.0001$. El índice de desempeño se muestra en la Fig. 5.4; puede verse que el error de diagnóstico tiene una magnitud decreciente incluso en presencia de la falla. Por otro lado, la Fig. 5.5 muestra la señal del controlador dinámico fraccionario que permite el seguimiento de la salida. La Fig. 5.6 muestra la señal de la salida y con CTF utilizando el controlador dinámico fraccionario diseñado. Puede verse que la salida sigue la referencia en aproximadamente 1 segundo y, cuando la falla aparece, sus efectos son eliminados inmediatamente. Finalmente, la Fig. 5.7 muestra el retrato fase del oscilador de Van der Pol con $y = x_1$ y $\hat{x}_2$.

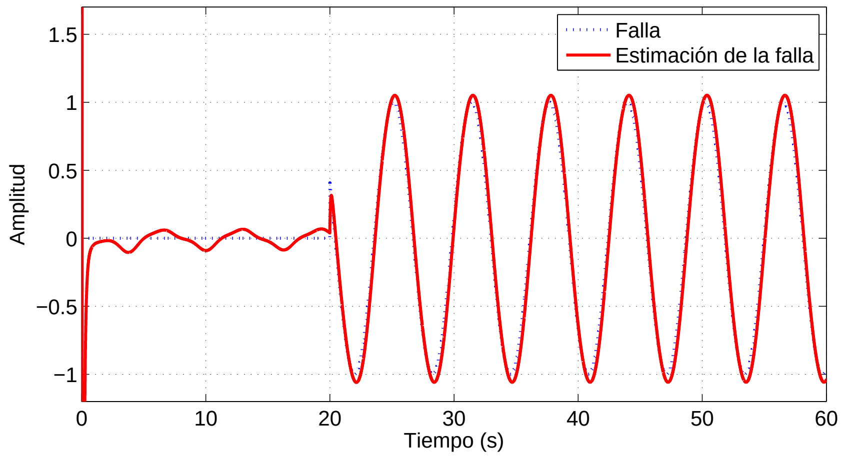

Figura 5.3: Diagnóstico de la falla del oscilador de Van der Pol.

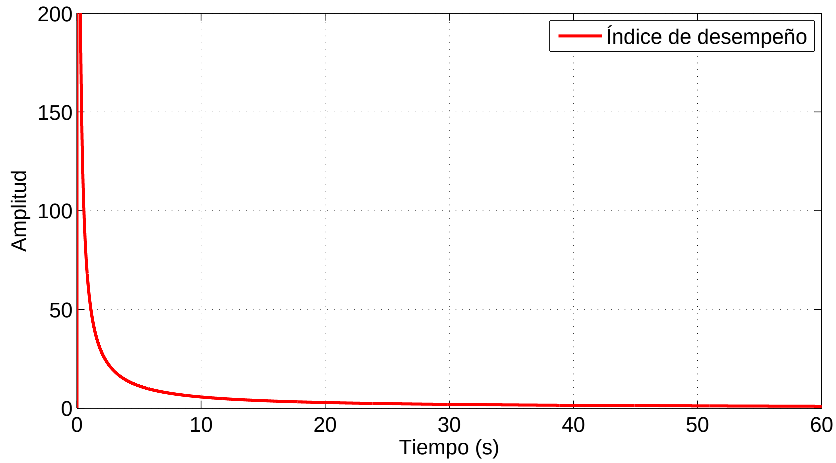

Figura 5.4: Índice de desempeño del diagnóstico de la falla del oscilador de Van der Pol.

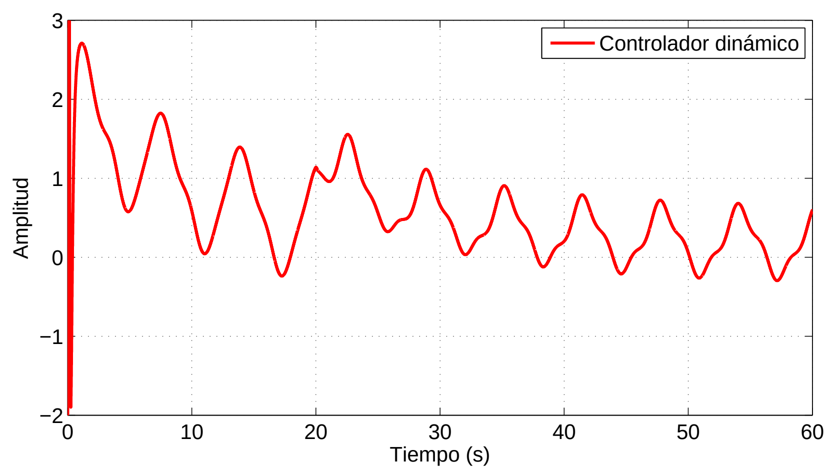

Figura 5.5: Controlador dinámico tolerante a fallas del oscilador de Van der Pol.

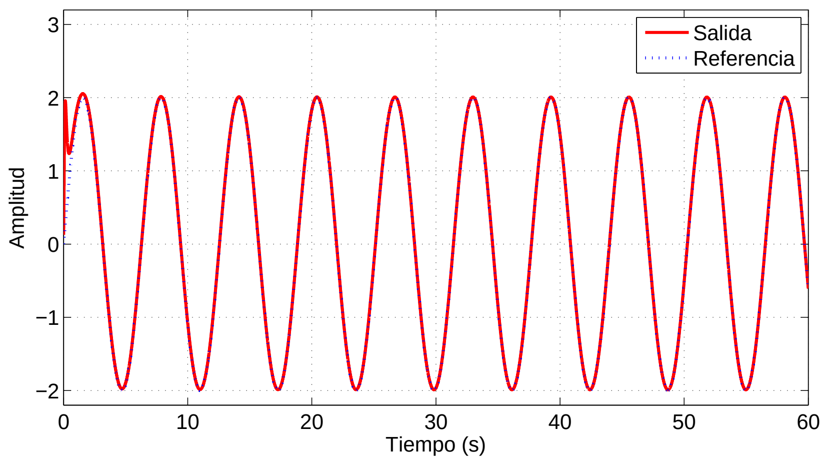

Figura 5.6: Seguimiento de la salida del oscilador de Van der Pol.

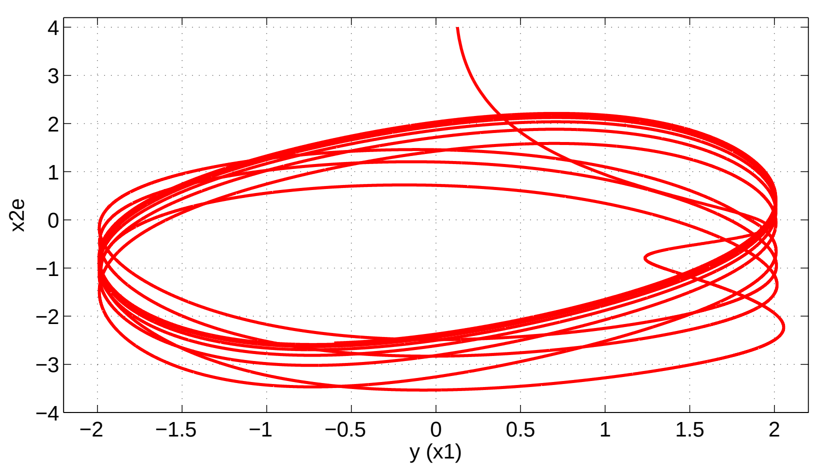

Figura 5.7: Retratro fase del oscilador de Van der Pol.

5.3.2. Motor de CD

En este apartado se presenta el modelo fraccionario conmensurado de un motor de CD:

$$\begin{aligned} D^{\alpha}x &= \omega \\ D^{\alpha}\omega &= \frac{1}{J}\left[c\phi i_a - T_L\right] \\ D^{\alpha}i_a &= \frac{1}{L_a}\left[V_a - R_a i_a - c\phi\omega\right]. \end{aligned} \tag{5.34}$$

La Fig. 5.8 muestra un diagrama del circuito eléctrico del motor en cuestión.

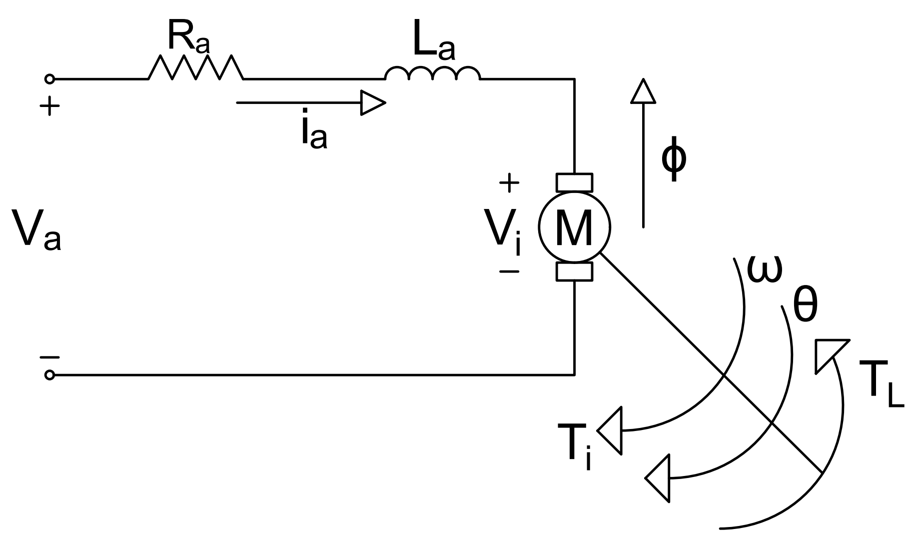

Figura 5.8: Circuito eléctrico del motor de CD

Las variables del sistema son:

Símbolo	Variable	Unidades
V_a	Voltaje de armadura	V
i_a	Corriente de armadura	A
ω	Velocidad angular	rpm

Los parámetros del sistema son:

Símbolo	Parámetro	Unidades
R_a	Resistencia de la armadura	Ω
L_a	Inductancia de la armadura	H
ϕ	Flujo magnético	Vs
J	Momento de inercia total	kgm^2
T_L	Par de carga	Nm

Además

$$c\phi\omega = V_i \qquad\qquad c\phi i_a = T_i$$

donde $c\phi$ es una constante del motor, V_i es el voltaje inducido y T_i es el par electromagnético. Por otro lado, considérese un estado adicional x tal que $I^\alpha\omega = x$ (sistema adaptado de [8]).

Comentario 5.3.1. *Si $\alpha = 1$ en esta integral, x representa la posición angular, la integral de orden entero de ω. Dado que $\alpha \neq 1$, la variable x no representa la posición angular, sino simplemente la integral de orden fraccionario de ω.*

Las variables de estado se eligen como $x_1 = x$, $x_2 = \omega$, $x_3 = i_a$, $u = V_a$ y $y = \omega$. Además, considérese una falla aditiva f acoplada a la entrada. Por tanto, el modelo que se utilizará es:

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ D^\alpha x_3 &= \frac{1}{L_a}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ y &= x_2. \end{aligned} \tag{5.35}$$

Puede verse que el objetivo de control es mantener la velocidad del motor en un valor nominal, incluso en presencia de fallas, las cuales en este caso representan variaciones en el voltaje de entrada.

5.3.2.1. Diagnóstico de la falla

En esta sección se diseñará el OORPI$^\alpha$ para reconstruir la falla en el modelo del motor de CD. Primero, se debe determinar si la falla f es diagnosticable, es decir, si satisface la condición de OAF.

De (5.35) se obtiene el siguiente polinomio:

$$f = c\phi y + R_a x_3 + L_a D^\alpha x_3 - u \tag{5.36}$$

Se puede observar que el estado x_3 también debe estimarse de alguna forma, por lo que en este caso también debe satisfacer la condición de OAF. De (5.35) se obtiene:

$$x_3 = \frac{1}{c\phi}\left[J D^\alpha y + T_L\right] \tag{5.37}$$

por lo tanto, el estado x_3 es fraccionariamente algebraicamente observable y también puede reconstruirse. Entonces, la falla es diagnosticable. Ahora se procede a diseñar el OORPI$^\alpha$ para estimarla:

$$\begin{aligned} D^\alpha \hat{f} &= k_{10}(f - \hat{f}) + k_{11} I^\alpha (f - \hat{f}) \\ &= k_{10}(c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \end{aligned} \tag{5.38}$$

Con el fin de eliminar la derivada fraccionaria de $\hat{x}_3$ en el término proporcional, se propone la variable auxiliar γ_1 y se define la estimación de la falla como:

$$\hat{f} = \gamma_1 + k_{10} L_a \hat{x}_3 \tag{5.39}$$

y se tiene que:

$$\begin{aligned} D^\alpha \gamma_1 &= D^\alpha \hat{f} - k_{10} L_a D^\alpha \hat{x}_3 \\ &= k_{10}(c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) - k_{10} L_a D^\alpha \hat{x}_3. \end{aligned} \tag{5.40}$$

Por lo tanto, el OORPI$^\alpha$ que se usará para estimar la falla es:

$$\begin{aligned} D^\alpha \gamma_1 &= k_{10}(c\phi y + R_a \hat{x}_3 - u - \gamma_1 - k_{10} L_a \hat{x}_3) + k_{11} L_a \hat{x}_3 \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 - u - \gamma_1 - k_{10} L_a \hat{x}_3) \\ \hat{f} &= \gamma_1 + k_{10} L_a \hat{x}_3 \end{aligned} \tag{5.41}$$

donde $\hat{x}_3$ se obtiene con:

$$\hat{x}_3 = \frac{1}{c\phi} \left[J D^\alpha y + T_L \right]. \tag{5.42}$$

Con el fin de estimar la derivada de orden α de y en (5.42), se define $\xi = D^\alpha y$ y se propone el siguiente OORPI$^\alpha$:

$$\begin{aligned} D^\alpha \hat{\xi} &= k_{\xi 0}(\xi - \hat{\xi}) + k_{\xi 1} I^\alpha (\xi - \hat{\xi}) \\ &= k_{\xi 0}(D^\alpha y - \hat{\xi}) + k_{\xi 1} I^\alpha (D^\alpha y - \hat{\xi}) \end{aligned} \tag{5.43}$$

Ahora, se introduce la variable auxiliar γ_ξ y se define la variable estimada $\hat{\xi}$ como:

$$\hat{\xi} = \gamma_\xi + k_{\xi 0} y$$

Entonces se tiene:

$$\begin{aligned} D^\alpha \gamma_\xi &= D^\alpha \hat{\xi} - k_{\xi 0} D^\alpha y \\ &= -k_{\xi 0} \hat{\xi} + k_{\xi 1} I^\alpha (D^\alpha y - \hat{\xi}) \\ &= k_{\xi 0} \left(-\gamma_\xi - k_{\xi 0} y \right) + k_{\xi 1} I^\alpha (D^\alpha y - \gamma_\xi - k_{\xi 0} y). \end{aligned}$$

Por lo tanto, la estimación de $\hat{x}_3$ se obtiene con:

$$\hat{x}_3 = \frac{1}{c\phi} \left[J \hat{\xi} + T_L \right] \tag{5.44}$$

donde $\hat{\xi}$ se obtiene con el siguiente OORPI$^\alpha$:

$$\begin{aligned} D^\alpha \gamma_\xi &= -k_{\xi 0} \left(\gamma_\xi + k_{\xi 0} y \right) + k_{\xi 1} y - k_{\xi 1} I^\alpha (\gamma_\xi + k_{\xi 0} y) \\ \hat{\xi} &= \gamma_\xi + k_{\xi 0} y. \end{aligned} \tag{5.45}$$

5.3.2.2. Control tolerante a fallas

Con el fin de aplicar el controlador tolerante a fallas, debemos encontrar la $\mathcal{FCOGM}$ $\mathcal{MF}$, para lo cual primero tenemos que obtener las n derivadas fraccionarias de la salida:

$$\begin{aligned} D^{\alpha}y &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ D^{2\alpha}y &= \frac{c\phi}{J}D^{\alpha}x_3 = \frac{c\phi}{L_aJ}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ D^{3\alpha}y &= \frac{c\phi}{L_aJ}[-c\phi D^{\alpha}x_2 - R_a D^{\alpha}x_3 + D^{\alpha}u + D^{\alpha}f] \\ &= \frac{c\phi}{L_aJ}[R_a c\phi y - c\phi D^{\alpha}y + R_a^2 x_3 - R_a u - R_a f + D^{\alpha}u + D^{\alpha}f]. \end{aligned} \tag{5.46}$$

Ahora se crea una forma canónica del error de seguimiento:

$$\begin{aligned} e_1 &= y - y_R \\ D^{\alpha}e_1 &= D^{\alpha}y - D^{\alpha}y_R = e_2 \\ D^{\alpha}e_2 &= D^{2\alpha}e_1 = D^{2\alpha}y - D^{2\alpha}y_R = e_3 \\ D^{\alpha}e_3 &= D^{3\alpha}e_1 = D^{3\alpha}y - D^{3\alpha}y_R \end{aligned} \tag{5.47}$$

de forma que se puede construir un OAGF para él:

$$\begin{aligned} D^{\alpha}\hat{e}_1 &= \hat{e}_2 - 3\theta\left(\hat{e}_1 - e_1\right) \\ D^{\alpha}\hat{e}_2 &= \hat{e}_3 - 3\theta^2\left(\hat{e}_1 - e_1\right) \\ D^{\alpha}\hat{e}_3 &= \frac{c\phi}{L_aJ}[R_a c\phi y - c\phi D^{\alpha}y + R_a^2\hat{x}_3 - R_a\hat{u} - R_a\hat{f} + D^{\alpha}\hat{u} + D^{\alpha}\hat{f}] - D^{3\alpha}y_R \\ &- \theta^3\left(\hat{e}_1 - e_1\right) = -\sum_{i=1}^{3} a_i\hat{e}_i. \end{aligned} \tag{5.48}$$

De (5.48) se obtiene la siguiente dinámica del controlador tolerante a fallas:

$$\begin{aligned} D^{\alpha}\hat{u} &= \frac{L_aJ}{c\phi}\left(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + D^{3\alpha}y_R\right) - R_a c\phi y + c\phi D^{\alpha}y \\ &- R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - D^{\alpha}\hat{f}. \end{aligned} \tag{5.49}$$

Éste es el controlador que eliminará los efectos de la falla en el sistema; sin embargo, puede verse que también se necesita una estimación de la derivada de orden α de y, por lo que se usa la obtenida con (5.45). Por tanto, la dinámica del controlador tolerante a fallas que se usará es:

$$\begin{aligned} D^{\alpha}\hat{u} &= \frac{L_aJ}{c\phi}(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + D^{3\alpha}y_R) - R_a c\phi y + c\phi\hat{\xi} \\ &- R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - D^{\alpha}\hat{f}. \end{aligned} \tag{5.50}$$

Comentario 5.3.2. *Nótese que la dinámica de $D^{\alpha}\hat{f}$ se obtiene de la dinámica del OORPI$^{\alpha}$ utilizado para el DF.*

Por lo tanto, definiendo $\hat{u} = \hat{u}_1$ y $\hat{f} = \hat{f}_1$, la cadena de integradores del sistema en lazo cerrado es:

$$
\begin{aligned}
D^{\alpha}\hat{e}_1 &= \hat{e}_2 - 3\theta\left(\hat{e}_1 - e_1\right) \\
D^{\alpha}\hat{e}_2 &= \hat{e}_3 - 3\theta^2\left(\hat{e}_1 - e_1\right) \\
D^{\alpha}\hat{e}_3 &= \frac{c\phi}{L_a J}[R_a c\phi y - c\phi\hat{\xi} + R_a^2\hat{x}_3 - R_a\hat{u} - R_a\hat{f} + D^{\alpha}\hat{u} + D^{\alpha}\hat{f}] - D^{3\alpha}y_R \\
&\quad - \theta^3\left(\hat{e}_1 - e_1\right) \\
D^{\alpha}\hat{u}_1 &= \frac{L_a J}{c\phi}(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + D^{3\alpha}y_R) - R_a c\phi y + c\phi\hat{\xi} - R_a^2\hat{x}_3 + R_a\hat{u}_1 + R_a\hat{f}_1 \\
&\quad - D^{\alpha}\hat{f}_1 \\
D^{\alpha}\hat{f}_1 &= k_{10}(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) + k_{11}L_a\hat{x}_3 \\
&\quad + k_{11}I^{\alpha}(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) \\
&\quad + \frac{k_{10}L_a}{c\phi}\left[J(-k_{\xi 0}\left(\gamma_\xi + k_{\xi 0}y\right) + k_{\xi 1}y - k_{\xi 1}I^{\alpha}(\gamma_\xi + k_{\xi 0}y) + k_{\xi 0}\hat{\xi}) + T_L\right].
\end{aligned}
$$

5.3.2.3. Resultados de simulación

Se realizaron simulaciones durante 20 segundos en el modelo del sistema, para lo cual se utilizó el bloque de derivada fraccionaria para Simulink del Ninteger Toolbox de MATLAB [13]. Se utilizaron los siguientes valores de los parámetros:

R_a	2.13 Ω
L_a	0.00484 H
$c\phi$	0.0683 Vs
J	0.0001148 kgm^2
T_L	0.0608 Nm
V_a	12 V

Se seleccionó $\alpha = 0.9$. La referencia se ajustó en $y_R = 177$. La falla se eligió como $f = 0.1V_a$, apareciendo a partir de 10 s. Los parámetros de diseño adimensionales (ganancias) se escogieron como $\theta = 2000$, $a_1 = 8000$, $a_2 = 1200$, $a_3 = 60$, $k_{10} = k_{\xi 0} = 20$ y $k_{11} = k_{\xi 1} = 100$.

La Fig. 5.9 muestra los resultados del diagnóstico con el OORPI$^{\alpha}$. Puede verse que la falla estimada sigue la señal de la falla real en un tiempo corto. El índice de desempeño del OORPI$^{\alpha}$ se evaluó de igual forma con la siguiente funcional de costo:

$$J_t = \frac{1}{t+\epsilon}\int_0^t \left\|\tilde{f}\right\|^2 dt$$

donde $\epsilon = 0.0001$. El índice de desempeño se muestra en la Fig. 5.10; puede verse que el error de diagnóstico tiene una magnitud aceptable incluso en presencia de la falla. Por otro lado, la Fig. 5.11 muestra la señal del controlador dinámico fraccionario que lleva a cabo el seguimiento de la salida. Finalmente, la Fig. 5.12 muestra la salida y con CTF utilizando el controlador dinámico fraccionario diseñado. Puede verse que el sistema sigue a la referencia a los 3 s, y cuando la falla aparece, sus efectos son eliminados en aproximadamente 2 s.

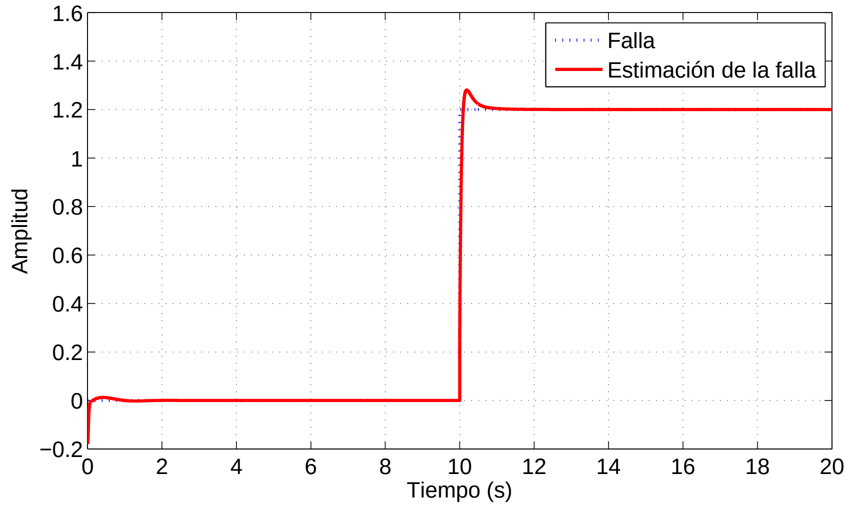

Figura 5.9: Diagnóstico de la falla del motor de CD.

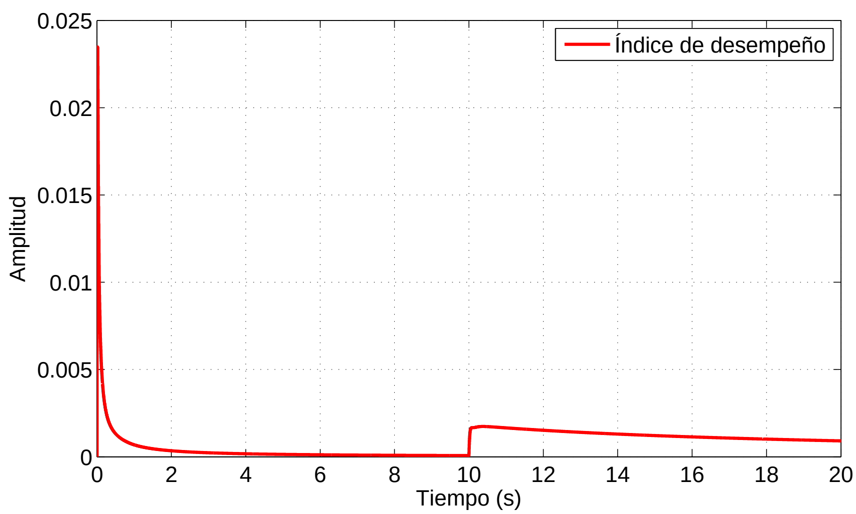

Figura 5.10: Índice de desempeño del diagnóstico de la falla del motor de CD.

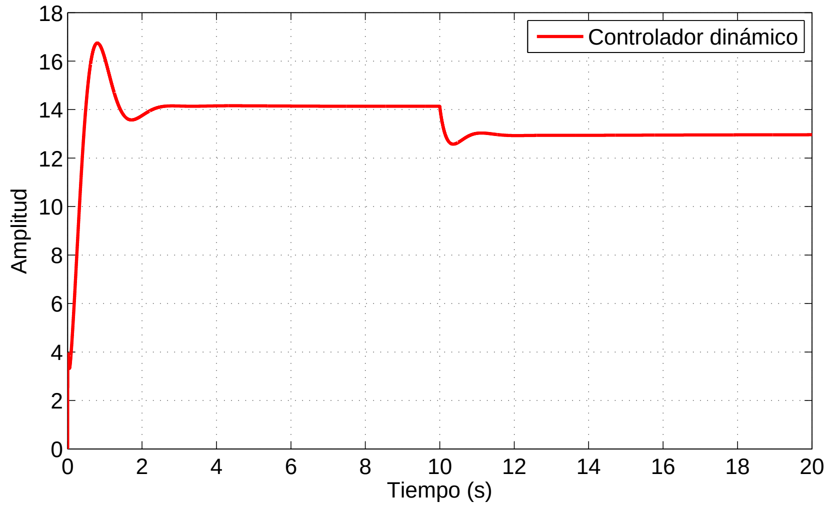

Figura 5.11: Controlador dinámico tolerante a fallas del motor de CD.

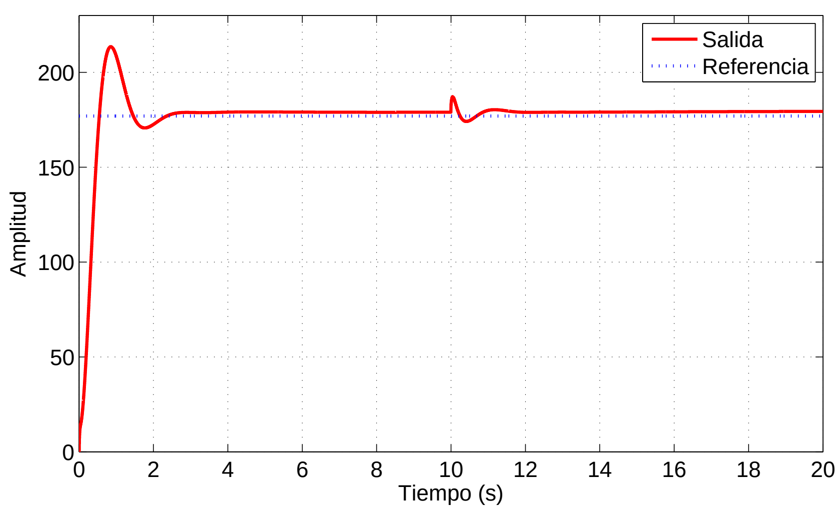

Figura 5.12: Seguimiento de la salida del motor de CD.

5.3.3. Comparación de resultados con el caso entero

En este apartado se realiza un comparativo de los resultados obtenidos en la sección anterior con los obtenidos al aplicar la metodología al motor de CD pero con dinámicas de orden entero, para lo que se utilizará el esquema propuesto en el Capítulo 3.

Considérese el modelo de orden entero del motor de CD:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{J}\left[c\phi i_a - T_L\right] \\ \dot{i}_a &= \frac{1}{L_a}\left[V_a - R_a i_a - c\phi\omega\right]. \end{aligned} \tag{5.51}$$

Las variables del sistema son:

Símbolo	Variable	Unidades
V_a	Voltaje de armadura	V
i_a	Corriente de armadura	A
θ	Posición angular	rad
ω	Velocidad angular	rpm

Los parámetros del sistema son:

Símbolo	Parámetro	Unidades
R_a	Resistencia de la armadura	Ω
L_a	Inductancia de la armadura	H
ϕ	Flujo magnético	Vs
J	Momento de inercia total	kgm^2
T_L	Par de carga	Nm

Las variables de estado se eligen como $x_1 = \theta$, $x_2 = \omega$, $x_3 = i_a$, $u = V_a$ y $y = \omega$. Además, considérese una falla aditiva f acoplada a la entrada. Por tanto, el modelo que se utilizará es:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ \dot{x}_3 &= \frac{1}{L_a}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ y &= x_2. \end{aligned} \tag{5.52}$$

5.3.3.1. Diagnóstico de la falla

De (5.52) se obtiene el siguiente polinomio:

$$f = c\phi y + R_a x_3 + L_a \dot{x}_3 - u \tag{5.53}$$

Se puede observar que el estado x_3 también debe estimarse de alguna forma, por lo que en este caso también debe ser algebraicamente observable. De (5.52) se obtiene:

$$x_3 = \frac{1}{c\phi}\left[J\dot{y} + T_L\right] \tag{5.54}$$

por lo tanto, el estado x_3 es algebraicamente observable y también puede reconstruirse. Entonces, la falla es diagnosticable.

Con el fin de relizar un mejor comparativo con el caso fraccionario, para el diagnóstico de fallas en esta sección se utilizará el siguiente OORPI de orden entero:

$$\dot{\hat{f}}_i = k_{i0}(f_i - \hat{f}_i) + k_{i1}I(f_i - \hat{f}_i) \tag{5.55}$$

donde I denota una integración de orden entero.

Por tanto, se procede a diseñar el OORPI para estimar la falla:

$$\begin{aligned} \dot{\hat{f}} &= k_{10}(f - \hat{f}) + k_{11}I(f - \hat{f}) \\ &= k_{10}(c\phi y + R_a\hat{x}_3 + L_a\dot{\hat{x}}_3 - u - \hat{f}) + k_{11}I(c\phi y + R_a\hat{x}_3 + L_a\dot{\hat{x}}_3 - u - \hat{f}) \end{aligned} \tag{5.56}$$

Con el fin de eliminar la derivada de $\hat{x}_3$ en el término proporcional, se propone la variable auxiliar γ_1 y se define la estimación de la falla como:

$$\hat{f} = \gamma_1 + k_{10}L_a\hat{x}_3 \tag{5.57}$$

y se tiene que:

$$\begin{aligned} \dot{\gamma}_1 &= \dot{\hat{f}} - k_{10}L_a\dot{\hat{x}}_3 \\ &= k_{10}(c\phi y + R_a\hat{x}_3 + L_a\dot{\hat{x}}_3 - u - \hat{f} + k_{11}I(c\phi y + R_a\hat{x}_3 + L_a\dot{\hat{x}}_3 - u - \hat{f}) - k_{10}L_a\dot{\hat{x}}_3. \end{aligned} \tag{5.58}$$

Por lo tanto, el OORPI que se usará para estimar la falla es:

$$\begin{aligned} \dot{\gamma}_1 &= k_{10}(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) + k_{11}L_a\hat{x}_3 \\ &\quad + k_{11}I(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) \\ \hat{f} &= \gamma_1 + k_{10}L_a\hat{x}_3 \end{aligned} \tag{5.59}$$

donde $\hat{x}_3$ se obtiene con:

$$\hat{x}_3 = \frac{1}{c\phi}\left[J\dot{y} + T_L\right]. \tag{5.60}$$

Con el fin de estimar la derivada de y en (5.42), se define $\xi = \dot{y}$ y se propone el siguiente OORPI:

$$\begin{aligned} \dot{\hat{\xi}} &= k_{\xi 0}(\xi - \hat{\xi}) + k_{\xi 1}I(\xi - \hat{\xi}) \\ &= k_{\xi 0}(\dot{y} - \hat{\xi}) + k_{\xi 1}I(\dot{y} - \hat{\xi}) \end{aligned} \tag{5.61}$$

Ahora, se introduce la variable auxiliar γ_ξ y se define la variable estimada $\hat{\xi}$ como:

$$\hat{\xi} = \gamma_\xi + k_{\xi 0}y$$

Entonces se tiene:

$$\begin{aligned}\dot{\gamma}_\xi &= \dot{\hat{\xi}} - k_{\xi 0}\dot{y} \\ &= -k_{\xi 0}\hat{\xi} + k_{\xi 1}I(\dot{y} - \hat{\xi}) \\ &= k_{\xi 0}\left(-\gamma_\xi - k_{\xi 0}y\right) + k_{\xi 1}I(\dot{y} - \gamma_\xi - k_{\xi 0}y).\end{aligned}$$

Por lo tanto, la estimación de $\hat{x}_3$ se obtiene con:

$$\hat{x}_3 = \frac{1}{c\phi}\left[J\hat{\xi} + T_L\right] \tag{5.62}$$

donde $\hat{\xi}$ se obtiene con el siguiente OORPI:

$$\begin{aligned}\dot{\gamma}_\xi &= -k_{\xi 0}\left(\gamma_\xi + k_{\xi 0}y\right) + k_{\xi 1}y - k_{\xi 1}I(\gamma_\xi + k_{\xi 0}y) \\ \hat{\xi} &= \gamma_\xi + k_{\xi 0}y.\end{aligned} \tag{5.63}$$

5.3.3.2. Control tolerante a fallas

Con el fin de aplicar el controlador tolerante a fallas, debemos encontrar la $\mathcal{FCOGM}$ $\mathcal{M}$, para lo cual primero tenemos que obtener las n derivadas de la salida:

$$\begin{aligned}\dot{y} &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ \ddot{y} &= \frac{c\phi}{J}\dot{x}_3 = \frac{c\phi}{L_aJ}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ \dddot{y} &= \frac{c\phi}{L_aJ}[-c\phi\dot{x}_2 - R_a\dot{x}_3 + \dot{u} + \dot{f}] \\ &= \frac{c\phi}{L_aJ}[R_a c\phi y - c\phi\dot{y} + R_a^2 x_3 - R_a u - R_a f + \dot{u} + \dot{f}].\end{aligned} \tag{5.64}$$

Ahora se crea una forma canónica del error de seguimiento:

$$\begin{aligned}e_1 &= y - y_R \\ \dot{e}_1 &= \dot{y} - \dot{y}_R = e_2 \\ \dot{e}_2 &= \ddot{e}_1 = \ddot{y} - \dot{y}_R = e_3 \\ \dot{e}_3 &= \dddot{e}_1 = \dddot{y} - \dddot{y}_R\end{aligned} \tag{5.65}$$

de forma que se puede construir un OAG para él:

$$\begin{aligned}\dot{\hat{e}}_1 &= \hat{e}_2 - 3\theta\left(\hat{e}_1 - e_1\right) \\ \dot{\hat{e}}_2 &= \hat{e}_3 - 3\theta^2\left(\hat{e}_1 - e_1\right) \\ \dot{\hat{e}}_3 &= \frac{c\phi}{L_aJ}[R_a c\phi y - c\phi\dot{y} + R_a^2\hat{x}_3 - R_a\hat{u} - R_a\hat{f} + \dot{\hat{u}} + \dot{\hat{f}}] - \dddot{y}_R - \theta^3\left(\hat{e}_1 - e_1\right) \\ &= -\sum_{i=1}^{3} a_i\hat{e}_i.\end{aligned} \tag{5.66}$$

De (5.66) se obtiene la siguiente dinámica del controlador tolerante a fallas:

$$\begin{aligned}\dot{\hat{u}} &= \frac{L_a J}{c\phi}\left(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + \dddot{y}_R\right) - R_a c\phi y + c\phi\dot{y} \\ &- R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - \dot{\hat{f}}.\end{aligned} \tag{5.67}$$

Éste es el controlador que eliminará los efectos de la falla en el sistema; sin embargo, puede verse que también se necesita una estimación de la derivada de y, por lo que se usa la obtenida con (5.63). Por tanto, la dinámica del controlador tolerante a fallas que se usará es:

$$\begin{aligned}\dot{\hat{u}} &= \frac{L_a J}{c\phi}(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + \dddot{y}_R) - R_a c\phi y + c\phi\hat{\xi} \\ &-R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - \dot{\hat{f}}.\end{aligned} \tag{5.68}$$

Por lo tanto, definiendo $\hat{u} = \hat{u}_1$ y $\hat{f} = \hat{f}_1$, la cadena de integradores del sistema en lazo cerrado es:

$$\begin{aligned}\dot{\hat{e}}_1 &= \hat{e}_2 - 3\theta\left(\hat{e}_1 - e_1\right) \\ \dot{\hat{e}}_2 &= \hat{e}_3 - 3\theta^2\left(\hat{e}_1 - e_1\right) \\ \dot{\hat{e}}_3 &= \frac{c\phi}{L_a J}[R_a c\phi y - c\phi\hat{\xi} + R_a^2\hat{x}_3 - R_a\hat{u} - R_a\hat{f} + \dot{\hat{u}} + \dot{\hat{f}}] - \dddot{y}_R - \theta^3\left(\hat{e}_1 - e_1\right) \\ \dot{\hat{u}}_1 &= \frac{L_a J}{c\phi}(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + \dddot{y}_R) - R_a c\phi y + c\phi\hat{\xi} - R_a^2\hat{x}_3 + R_a\hat{u}_1 + R_a\hat{f}_1 - \dot{\hat{f}}_1 \\ \dot{\hat{f}}_1 &= k_{10}(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) + k_{11}L_a\hat{x}_3 \\ &+k_{11}I(c\phi y + R_a\hat{x}_3 - u - \gamma_1 - k_{10}L_a\hat{x}_3) \\ &+\frac{k_{10}L_a}{c\phi}\left[J(-k_{\xi 0}\left(\gamma_\xi + k_{\xi 0}y\right) + k_{\xi 1}y - k_{\xi 1}I(\gamma_\xi + k_{\xi 0}y) + k_{\xi 0}\hat{\xi}) + T_L\right].\end{aligned}$$

5.3.3.3. Resultados de simulación

Se realizaron simulaciones para este sistema en MATLAB Simulink durante 20 segundos. Se utilizaron los siguientes valores de los parámetros:

R_a	2.13 Ω
L_a	0.00484 H
$c\phi$	0.0683 Vs
J	0.0001148 kgm^2
T_L	0.0608 Nm
V_a	12 V

La referencia se ajustó en $y_R = 177$. La falla se eligió como $f = 0.1V_a$, apareciendo a partir de 10 s. Los parámetros de diseño adimensionales (ganancias) se escogieron como $\theta = 2000$, $a_1 = 8000$, $a_2 = 1200$, $a_3 = 60$, $k_{10} = k_{\xi 0} = 20$ y $k_{11} = k_{\xi 1} = 100$.

Al igual que en el caso fraccionario, tanto el diagnóstico de la falla como el seguimiento de la salida fueron satisfactorios, por lo que a continuación se realizará la comparación de los resultados obtenidos. Para propósitos de comparación, se realizaron simulaciones del modelo fraccionario con los valores de α de 0.9, 0.93 y 0.97, siendo el caso entero equivalente al valor de $\alpha = 1$.

La Fig. 5.13 compara el estado x_1 en los cuatro casos, correspondiendo a la posición angular en el caso entero y a la integral fraccionaria de orden α de la velocidad angular en el caso fraccionario. Se puede observar que en el caso entero aparece al inicio una oscilación mayor que deriva en una recta con mayor inclinación, mientras que en el caso fraccionario aparece una curva más uniforme y de menor pendiente, la cual va incrementándose cada vez más, acercándose hacia la recta del caso entero conforme incrementa el valor de α.

En la Fig. 5.14 se muestra la comparación de la salida, es decir, el estado x_2 en los cuatro casos, correspondiendo a la velocidad angular en todos ellos. Se puede observar que, aunque todas las señales se estabilizan en un valor cercano incluso en presencia de la falla a los 10 segundos, el caso entero presenta en la parte transitoria una sobreoscilación de amplitud mucho más grande y que se estabiliza en un tiempo mayor. De igual forma, se observa que en el caso fraccionario la sobreoscilación aumenta de magnitud conforme el valor de α se acerca a la unidad.

La Fig. 5.15 compara el estado x_3 en los cuatro casos, correspondiendo a la corriente de armadura en todos ellos. Similar a lo que sucede con la salida, el caso entero presenta un transitorio con un sobretiro y un tiempo de asentamiento mayores, coincidiendo las señales de ambos casos en la parte estable; también, en el caso fraccionario el sobretiro se incrementa al aumentar el valor de α.

En la Fig. 5.16 se muestra la comparación de la estimación de la falla $\hat{f}$ en los cuatro casos. De forma parecida a lo que se ha observado, en el caso entero aparece una oscilación transitoria mayor al inicio de la señal (aún sin falla); cuando aparece la falla, en todos los casos aparece un sobretiro que se estabiliza en poco tiempo, pero que es mayor cuando el valor de α es mayor.

La Fig. 5.17 compara el índice de desempeño del diagnóstico de la falla en los cuatro casos. El índice se evaluó con la funcional de costo:

$$J_t = \frac{1}{t+\epsilon}\int_0^t \left\|\tilde{f}\right\|^2 dt$$

donde $\epsilon = 0.0001$. Debido al comportamiento oscilante del caso entero, se aprecia que, aunque en todos los casos se tiende a cero, el índice es mucho mayor en el caso entero, y va disminuyendo su amplitud conforme decrece el valor de α.

Finalmente, en la Fig. 5.18 se muestra la comparación entre la señal del control dinámico tolerante a fallas $\hat{u}$ en los cuatro casos. Como se observa, los comportamientos presentados en el caso entero hacen que esta señal presente mayores oscilaciones y tiempos de asentamiento que en los casos fraccionarios, en los cuales se sigue observando un mayor sobretiro cuando el valor de α es mayor.

Se puede concluir de estos resultados que, al utilizar las dinámicas fraccionarias, el modelo del motor de DC simula poseer una especie de amortiguamiento, el cual va decreciendo conforme aumenta el valor de α. Es por esto que en este caso sus señales, tanto la velocidad angular como la corriente de armadura, presentan pequeñas oscilaciones que se estabilizan en poco tiempo, a diferencia de los resultados obtenidos con la dinámica entera. Esto podría parecer una ventaja, ya que podría concluirse que el sistema se comporta mejor con la dinámica fraccionaria; sin embargo, hay que tomar en cuenta cómo realizar la implementación de un modelo de este tipo, además del significado físico de las variables y dinámicas involucradas. En vez de considerarlo como una mejor opción, el modelo fraccionario puede tomarse en conjunto con un sistema de control como una alternativa a los controladores clásicos, con el fin de obtener diversas respuestas con un comportamiento deseado diferente a los existentes.

Por otro lado, se observa el efecto del término integral del OORPI en el modelo con dinámicas de orden entero, que produce que aparezcan sobreoscilaciones de gran magnitud. Ha de notarse cómo es que en los casos fraccionarios la presencia de un término integral no incrementa en una proporción tan grande el sobretiro (ver Figs. 5.1 y 5.2) y, aunque pudiera representar una mejor alternativa en algunos sistemas, habría de tomarse en cuenta también el método numérico empleado para implementar los operadores fraccionarios, puesto que se podrían obtener resultados diversos.

Comentario 5.3.3. *A partir del trabajo de este capítulo se presentó en un congreso internacional la ponencia [39], además de que se tiene un artículo de revista aceptado [40]. Tanto la ponencia presentada como el artículo aceptado se muestran en el Anexo.*

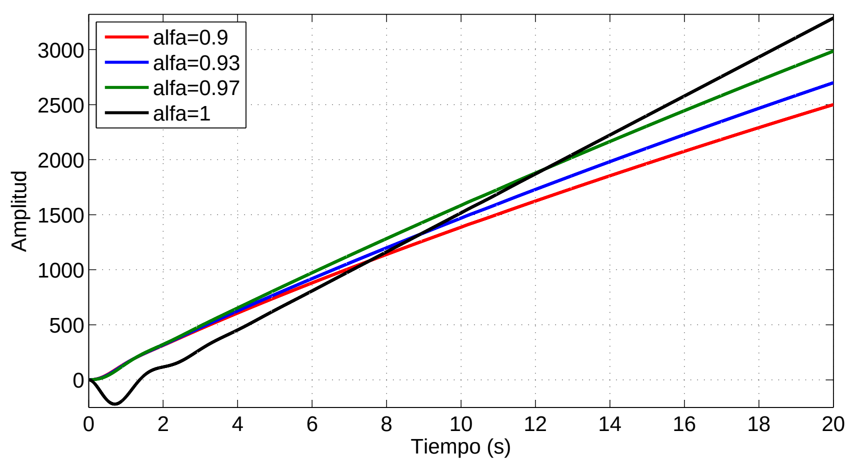

Figura 5.13: Comparación del estado x_1 del motor de CD entre el caso entero y el caso fraccionario

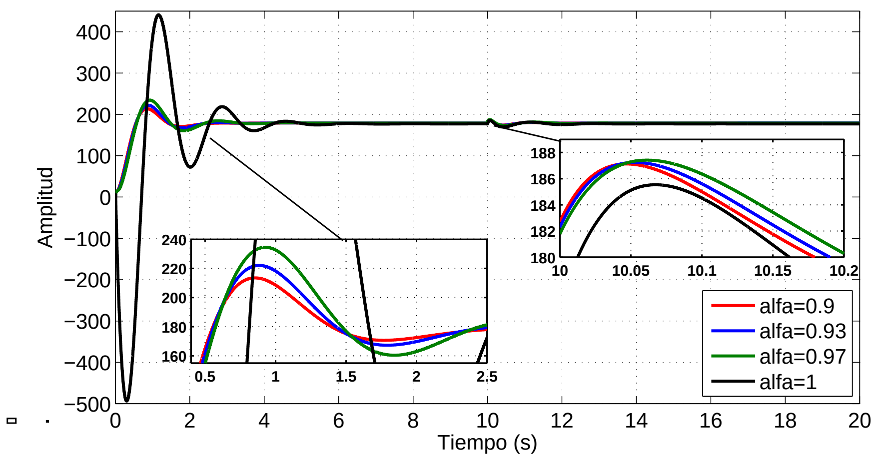

Figura 5.14: Comparación de la salida (estado x_2) del motor de CD entre el caso entero y el caso fraccionario

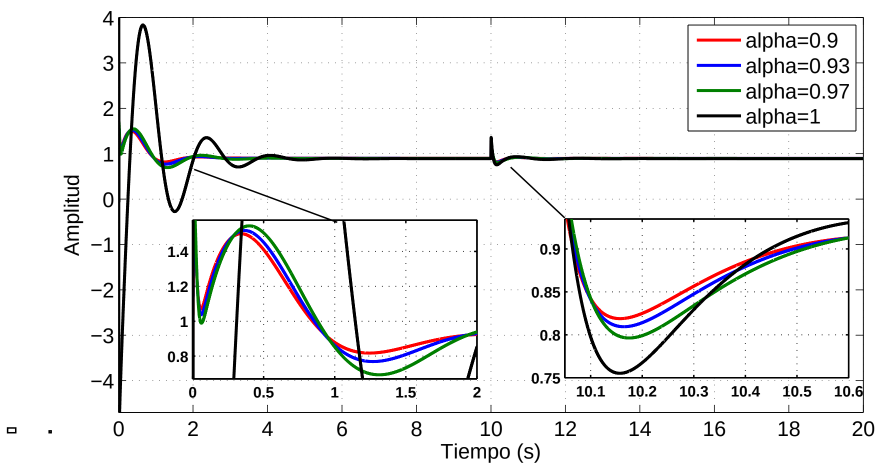

Figura 5.15: Comparación del estado x_3 del motor de CD entre el caso entero y el caso fraccionario

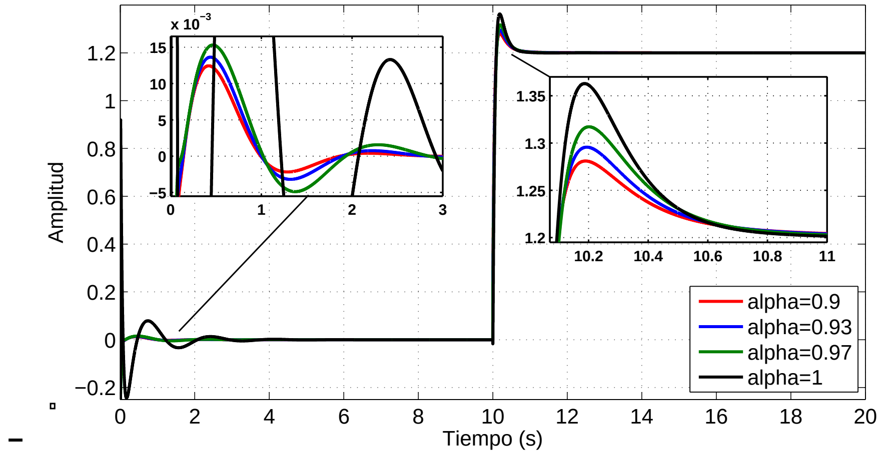

Figura 5.16: Comparación de la estimación de la falla $\hat{f}$ del motor de CD entre el caso entero y el caso fraccionario

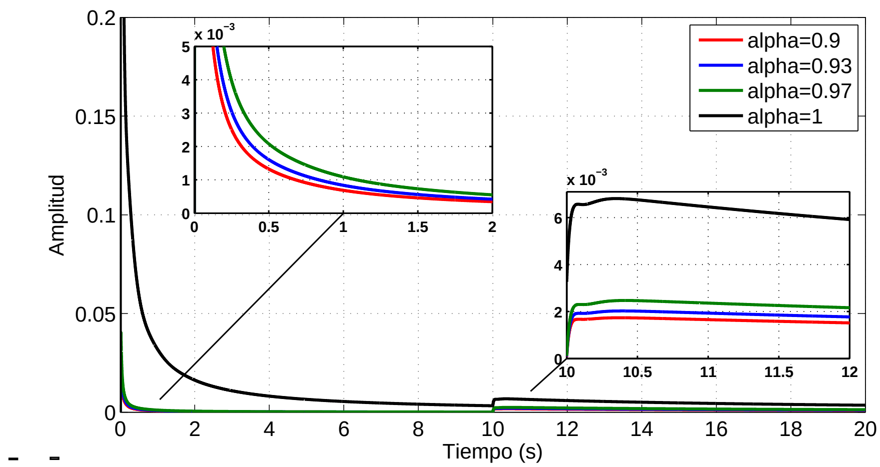

Figura 5.17: Comparación del índice de desempeño del diagnóstico de la falla $\hat{f}$ del motor de CD entre el caso entero y el caso fraccionario

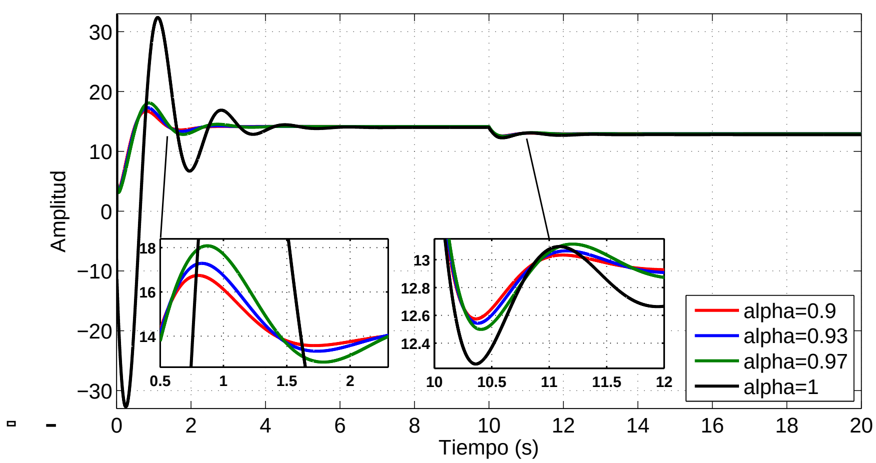

Figura 5.18: Comparación del controlador dinámico tolerante a fallas $\hat{u}$ del motor de CD entre el caso entero y el caso fraccionario

Conclusiones y trabajo futuro

Conclusiones

En este trabajo se propuso un esquema de control tolerante a fallas para sistemas de orden entero y fraccionario. Este esquema está basado en la obtención de la Forma Canónica de Observabilidad Generalizada Multi-entrada Multi-salida a partir del sistema nominal, para con ella construir un observador de alta ganancia para controlar la dinámica del error de seguimiento de la salida. A partir de la dinámica de este observador, se obtuvieron las ecuaciones que constituyen el controlador dinámico tolerante a fallas, que para rechazar los efectos de éstas utiliza estimaciones obtenidas a partir del diagnóstico de las mismas. El diagnóstico se llevo a cabo mediante observadores de orden reducido, siempre y cuando las fallas satisficieran la propiedad de observabilidad algebraica. El controlador dinámico logra el seguimiento de la salida al linealizar la dinámica del error, la cual además es estabilizada. Por medio del análisis de Lyapunov, se concluyó que el sistema en lazo cerrado es asintóticamente estable. La metodología se aplicó mediante simulaciones a un ejemplo académico y al sistema de tres tanques Amira DTS200. En ambos casos se comprobó que, con la selección adecuada de ganancias, las fallas eran diagnosticadas con éxito, utilizando una funcional de costo para evaluar el desempeño del observador de orden reducido. Esto conllevó a que el seguimiento de las salidas se realizara con resultados satisfactorios y que además, los efectos de las fallas en las salidas fueran eliminados en un tiempo corto.

Por otro lado, se extendió la metodología propuesta para el control tolerante a fallas en sistemas fraccionarios de orden conmensurado. Para el diagnóstico de fallas se definió la propiedad de observabilidad algebraica fraccionaria para sistemas de orden conmensurado, y se utilizó un observador de orden reducido fraccionario con un término integral para la estimación de fallas con dinámicas fraccionarias. Se utilizó la Forma Canónica de Observabilidad Generalizada Multi-entrada Multi-salida Fraccionaria para la obtención del controlador dinámico, y se verificó que el sistema en lazo cerrado es Mittag-Leffler estable. La metodología extendida se aplicó mediante simulaciones en los modelos fraccionarios conmensurados del oscilador de Van der Pol y de un motor de CD, verificándose de igual forma que tanto el diagnóstico de la falla como el seguimiento de la salida en ambos casos fueron llevados a cabo satisfactoriamente. Finalmente, se realizó un comparativo de resultados para el motor de CD entre el caso fraccionario con tres valores diferentes del orden y el caso entero. Se observaron cuáles son las ventajas y los aspectos a tomar en cuenta en la utilización de dinámicas de orden fraccionario en el modelo del motor, considerándose éstas como una alternativa de control a los resultados existentes en el caso entero.

Trabajo futuro

Algunos de los temas en los que se puede continuar la investigación a partir de lo desarrollado en este trabajo son:

a) Control tolerante a fallas en sistemas fraccionarios de orden inconmensurado. Es de gran interés extender la metodología propuesta a este tipo de sistemas, donde deben tomarse ciertas consideraciones de diseño y análisis de estabilidad al ser diferentes los órdenes de las derivadas de las ecuaciones de los estados.

b) Control tolerante a fallas en sistemas con fallas no diagnosticables (con el enfoque algebraico diferencial). Dado que se trabajó con fallas que satisficieran las condiciones de observabilidad algebraica, sería interesante abordar sistemas con fallas que no cumplan estas condiciones, como sucede en los sistemas no diferencialmente planos y los sistemas liouvilianos, entre otros.

c) Control tolerante a fallas no acopladas a la entrada. En esta tesis se trabajaron principalmente fallas aditivas acopladas a la entrada, que representan errores en el comportamiento de los actuadores. Sería de interés trabajar también con fallas multiplicativas, así como otro tipo de perturbaciones como ruido de medición (fallas en las salidas) e incertidumbre paramétrica.

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Anexo: Publicaciones

A continuación se presentan las publicaciones generadas a partir de este trabajo.

Publicaciones en revistas aceptadas

Martínez-Guerra, R., Trejo-Zúñiga, I., & Meléndez-Vázquez, F. (2017). A dynamical controller with Fault-tolerance: real-time experiments. Journal of the Franklin Institute 354, 3378–3404.

Meléndez-Vázquez, F., Martínez-Fuentes, O., & Martínez-Guerra, R. Fractional fault-tolerant dynamical controller for a class of Commensurate-order Fractional systems. International Journal of Systems Science, aceptado.

Ponencias en congresos internacionales

Meléndez-Vázquez, F., Trejo-Zúñiga, I., & Martínez-Guerra, R. (2015). Fault-tolerant asymptotic output tracking: An application to the three-tank system. Proceedings of the 12th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE), Mexico City, Mexico, October 28-30, 2015, 182–187.

Trejo-Zúñiga, I., Meléndez-Vázquez, F., & Martínez-Guerra, R. (2016). Fault-tolerant dynamical controller with some experimental results. Proceedings of the 2016 American Control Conference (ACC), Boston, MA, USA, July 6-8, 2016, 7549–7554.

Meléndez-Vázquez, F., & Martínez-Guerra, R. (2016). Fractional fault-tolerant dynamical controller for the fractional model of a DC motor. Third Mexican Workshop in Fractional Calculus, Zacatecas, Zacatecas, México, September 11-16, 2016, poster.

Publicación en revista sometida

Meléndez-Vázquez, F., & Martínez-Guerra, R. A reduced-order fractional integral observer for synchronization and anti-synchronization of fractional-order chaotic systems.

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A dynamical controller with fault-tolerance: Real-time experiments

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Abstract

We propose a dynamical controller that is based on the high-gain and the reduced-order observers, is fault-tolerant and is obtained by means of a multi-input multi-output generalized observability canonical form generated from a differential primitive element. The dynamical controller is able to linearize the tracking errors, achieving ultimate uniform boundedness with measurement noise. To accomplish this, a fault diagnosis is required, involving additive and multiplicative faults, which have to be reconstructed simultaneously and online. Some real-time results are presented to illustrate the effectiveness of the proposed methodology.

1. Introduction

Controls with fault tolerance (FT) have a key role in many applications in automation and engineering [1–4]. There are many different approaches to achieve such control, for instance the survey [5] provides a basic literature review covering most areas of fault-tolerant control (FTC). The book [6] presents a model-based approach for FTC. Particularly in the field of nonlinear systems, encouraging results were recently obtained applying algebraic techniques [2,3,7–9]. This paper proposes to solve this problem the construction of a fault-tolerant

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dynamical controller capable of linearizing the tracking errors dynamics; the controller is obtained transforming the system into a multi-input multi-output generalized observability canonical form ($\mathcal{MMGOCF}$) represented as a chain of integrators.

For MIMO systems with faults, the dynamical controller is dependent on each of them, forcing the implementation of a fault diagnosis (FD) method. Then, if the system has multiple faults (additive and multiplicative), the FD method has to be able to reconstruct each fault simultaneously and online.

Fault diagnosis has been a research area for a long time. There are numerous works that study this problem, as [10,11], involving residual generation, disturbance decoupling, and adaptive approaches. For linear systems, the geometric approach has also been applied [12], where the concept of unobservability subspaces is used together with residual generators. There are also several papers that study fault detection and diagnosis in nonlinear systems [13–15]. For example, in [14] a Gaussian particle filter is used for estimation purpose; as another example, an adaptive estimation algorithm for recursive estimation of parameters related to faults is the way that the paper [15] deals with the problem. Also, the geometric approach for failure detection and isolation via residual generation has been extended to nonlinear systems [16], where the distribution tools are considered as the main ingredients of the unobservability subspaces. Furthermore, the fault diagnosis under the data-driven framework has received great attention recently [17,18].

The solution proposed to the problem of diagnosis of additive and multiplicative faults is performed using the differential algebraic approach [19,20]. In the book [20] the fault detection and diagnosis problem in nonlinear systems is presented using differential and algebraic tools. This framework uses the reduced-order observer (model-free type) to achieve an effective estimation of the faults, while giving the possibility to estimate online and simultaneously several faults [21,22].

On the other hand, in order to implement tracking on the MIMO system, it is necessary to use also an observer. A high-gain observer (HGO) [23] is employed for estimating the tracking error dynamics and assure its stability. Then, it is proven that the overall system is ultimate uniformly bounded in the presence of measurement noise [24].

Furthermore, some real-time results are presented using the Amira DTS200 system [25]. The Amira system provides an opportunity to introduce multiple faults in sensors and actuators, making it a very versatile system (benchmark); this is why it has been widely used for experimental studies on FD and FTC [7–9,21]. In particular, in [7] the fault diagnosis is realized via algebraic estimation of derivatives, that yields estimates of the residuals (fault indicators), whereas in this paper the faults are not only detected but also diagnosed as they are reconstructed via an observer. Another difference lies in the control laws. While in [7] is used a nonlinear extension of a classic proportional-integral (PI) controller, that is independent of the faults, in the current paper is proposed a dynamical controller. This dynamical controller seeks to stabilize the tracking error system and depends on simultaneous fault diagnosis (fault-tolerant dynamical controller), which leads to the elimination of the effects of the faults in the system. As far as we know, this technique has not been employed in literature.

This paper is organized as follows. In Section 2, the definition of diagnosable system and the FD problem are presented. In Section 3 the canonical forms for the MIMO system are introduced, as well as the HGO used for tracking and the method to obtain the dynamical controller. In Section 4 the closed-loop system is shown, along with the proof of stability. In Section 5 the method is applied on a numerical example and on the Amira system, the results are presented and discussed. Finally, in Section 6 the paper is closed with some concluding remarks.

2. Fault diagnosis

The dynamical controller proposed for fault-tolerant tracking is dependent on the multiple faults that appear in the system. Thus, the need for a FD is a consequence of the proposed method, and a diagnosis capable to reconstruct each fault is required.

Firstly, consider a nonlinear systems with faults, described by:

$$\begin{aligned} \dot{x}(t) &= g(x, u, f) \\ y(t) &= h(x, u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ the control input vector, $f \in \mathbb{R}^r$ is an unknown input vector (fault vector), $y \in \mathbb{R}^p$ is the output vector, and g and h are assumed to be analytic functions.

As mentioned above the estimation of the faults $(\hat{f})$ that appear in the system is needed. To achieve this in the differential algebraic framework, the system has to satisfy the diagnosability property.

Definition 1. An element f in the differential field $k\langle u, y\rangle$ is said to be algebraically observable with respect to u and y if it satisfies a differential equation with coefficients over $k\langle u, y\rangle$ (k is a constant field, u and y are differential quantities).

Definition 2. A system of the form (1) is said to be diagnosable if it is possible to estimate the fault f from the system equations and the time histories of the data u and y. This is, the system is diagnosable if f is algebraically observable with respect to u and y.

The notion of algebraic observability is to express the faults as polynomial equations dependent on the inputs and outputs of the system and finitely many time derivatives, with coefficients in k:

$$f_{\bar{l}} = P_{\bar{l}}(u, \dot{u}, \ldots, y, \dot{y}, \ldots)$$

In order to illustrate Definition 2, two examples are presented.

Example 1. Let us consider the following nonlinear system.

$$\begin{aligned} \dot{x}_1 &= x_1x_2 + f_1 + u \\ \dot{x}_1 &= x_1 \\ y_1 &= x_1 + f_2 \\ y_2 &= x_2 \end{aligned} \tag{2}$$

Since f_1 and f_2 satisfy Definition 2

$$\begin{aligned} f_1 &= \ddot{f}_2 - y_2\dot{y}_2 - u \\ f_2 &= y_1 - \dot{y}_2 \end{aligned}$$

the system (2) is diagnosable and the faults can be reconstructed from the knowledge of u, y and their time derivatives.

Example 2. The system

$$\begin{aligned}\dot{x}_1 &= (x_1 + x_2)(u + f)\\ \dot{x}_2 &= u\\ y &= x_1 + x_2\end{aligned} \tag{3}$$

is a diagnosable system since

$$f = \frac{\dot{y} - u}{y} - u$$

It is clear that systems with additive and multiplicative faults can be addressed using the proposed algebraic approach.

Moreover, the unknown fault vector $f = (f_1, \ldots, f_r)$ can be seen as a state with an uncertain dynamics $\Omega(x, u, f) : \mathbb{R}^{n+m+r} \to \mathbb{R}^r$. Then, in order to estimate it, the state vector is extended (immersion [20]).

As it can be seen a classic Luenberger observer, which needs full knowledge of the system dynamics, cannot be constructed because the term $\Omega(x, u, f)$ is unknown. However this problem can be solved using a reduced-order observer (ROO), because it can be implemented from the algebraic observability property of the faults, and is asymptotically stable. Next lemma describes the construction of a proportional ROO for (1).

Lemma 1. *[20] If the following hypotheses are satisfied:*
Assumption 1 $|\Omega_{\bar{l}}(x, u, f)| \leq N_{\bar{l}} \in \mathbb{R}^+ \ \forall \bar{l} = 1, \ldots, r.$
Assumption 2 *$f(t)$ is algebraically observable over $\mathbb{R}\langle u, y\rangle$.*
Then the system

$$\dot{\hat{f}}_{\bar{l}} = k_{\bar{l}}(f_{\bar{l}} - \hat{f}_{\bar{l}}), \quad 1 \leq \bar{l} \leq r \tag{4}$$

is an asymptotic reduced-order observer for system in Eq. (1), where $\hat{f}_{\bar{l}}$ denotes the estimate of fault $f_{\bar{l}}$ and $k_{\bar{l}} \in \mathbb{R}^+ \ \forall \bar{l} = 1, \ldots, r$ are positive coefficients that determine the desired convergence rate of the observer.

Sometimes the output derivatives appear in the algebraic equation of the fault, then it is necessary to use an auxiliary variable to approximate them as described in the next lemma.

Lemma 2. *[20] If a fault signal $f_{\bar{l}}$, $1 \leq \bar{l} \leq r$ of system (1) is algebraically observable and can be written in the following form:*

$$f_{\bar{l}} = a_{\bar{l}}\dot{y} + b_{\bar{l}}(u, y) \tag{5}$$

where $a_{\bar{l}} = [a_1, \ldots, a_m] \in \mathbb{R}^m$ is a constant vector and $b_{\bar{l}}(u, y)$ is a bounded function, then there exists a function $\gamma_{\bar{l}} \in C^1$, such that the reduced-order observer in Eq. (4) can be written as the following asymptotically stable system:

$$\begin{aligned}\dot{\gamma}_{\bar{l}} &= -k_{\bar{l}}\gamma_{\bar{l}} + k_{\bar{l}}b_{\bar{l}}(x, u) - k_{\bar{l}}^2 a_{\bar{l}}y, \quad \gamma_{\bar{l}}(0) = \gamma_{\bar{l}0} \in \mathbb{R}\\ \hat{f}_{\bar{l}} &= \gamma_{\bar{l}} + k_{\bar{l}}a_{\bar{l}}y\end{aligned} \tag{6}$$

Remark 1. This methodology is recursive; we can introduce as many virtual variables as needed.

Remark 2. The ROO also serves as an estimator of derivatives. If there are output time derivatives of order 2 or higher, consider the time derivative to be estimated $\eta = \dot{y}$. According to Eq. (4), we propose the observer structure

$$\dot{\hat{\eta}} = k_\eta(\eta - \hat{\eta}) \tag{7}$$

introducing the change of variable $\gamma = \hat{\eta} - k_\eta y$, and from Eq. (7), we have

$$\begin{aligned}\dot{\gamma} &= -k_\eta \hat{\eta} \\ &= -k_\eta \gamma - k_\eta^2 y\end{aligned} \tag{8}$$

which constitutes with $\hat{\eta}$ an asymptotic estimator for $\eta = \dot{y}$.

Now, considering the ROO dynamics, the following variables are defined:

$$\hat{f}_i^{\bar{l}} = \hat{f}_{\bar{l}}^{(i-1)} \qquad i = 1, ..., \mu_{\bar{l}} \tag{9}$$

then the fault estimation subsystems are written as:

$$\dot{\hat{\mathbf{f}}}_{\bar{l}}(t) = E\hat{\mathbf{f}}_{\bar{l}}(t) + \omega_{\bar{l}}(u, y, f), \quad 1 \le \bar{l} \le r \tag{10}$$

where

$$\hat{\mathbf{f}}_{\bar{l}} = \left(\hat{f}_1^{\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}\right); \qquad \omega_{\bar{l}}(u, y, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) \end{pmatrix}$$

Remark 3. The ROO is able to reconstruct the states, the faults and their time derivatives, provided that they satisfy the conditions of Lemma 1. Given that this procedure can be done online, this is useful to perform real-time applications.

3. Fault-tolerant multi-output tracking problem

A nonlinear system described by Eq. (1) can be represented by the following $\mathcal{MMGOCF}$ due to the differential primitive element:

$$\begin{aligned}\dot{\eta}_i^j &= \eta_{i+1}^j, \quad 1 \le i \le n-1 \\ \dot{\eta}_n^j &= -L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, f, ..., f^{(\mu)}) \\ y_j &= \eta_1^j\end{aligned} \tag{11}$$

where L_j is a C^1 real-valued function, $\eta_j = (\eta_1^j, ..., \eta_n^j) \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $f \in \mathbb{R}^r$, and some integers γ, $\mu \ge 0$. This $\mathcal{MMGOCF}$ is constituted of p subsystems, one for each output y_j, $1 \le j \le p$.

Let $y_R(t) \in \mathbb{R}^p$ be a reference output vector with C^n function elements. The output tracking problem with FT consists in finding a dynamical controller that depends on the reference output vector $y_R(t)$ and its time derivatives $y_R^{(i)}(t)$, the state variables η_i^j of the canonical system, and, as previously mentioned, an estimation $\hat{f}$ of the fault vector and its time derivatives, such that the controller locally forces $y(t)$ to converge towards $y_R(t)$.

Define the output tracking error as:

$$e_1^j = y_j - y_{Rj}, \qquad 1 \le j \le p \tag{12}$$

Given that η_i^j is equal to the $(i-1)$ th time derivative of y_j, that is $\eta_i^j = y_j^{(i-1)}$, for $1 \le i \le n$ and $1 \le j \le p$, the error variables are rewritten as:

$$e_1^j = \eta_1^j - y_{Rj}, \quad 1 \le j \le p \tag{13}$$

The p output errors define the following $\mathcal{MMGOCF}$:

$$\begin{aligned} e_j^{(i)}(t) &= \eta_{i+1}^j - y_{Rj}^{(i)}, \quad 1 \le i \le n-1 \\ e_j^{(n)}(t) &= \dot{\eta}_n^j - y_{Rj}^{(n)}(t) = -L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} \end{aligned} \tag{14}$$

with $e_i^j(t) = e_j^{(i-1)}(t),\ 1 \le i \le n$. Now, a linear time-invariant (LTI) dynamics is imposed for the tracking error:

$$e_j^{(n)}(t) + \sum_{i=0}^{n-1} a_{i+1}^j e_j^{(i)}(t) = 0 \tag{15}$$

and from system in Eq. (14), Eq. (15) is rewritten as:

$$\dot{\eta}_n^j - y_{Rj}^{(n)}(t) + \sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}(t)\right] = 0 \tag{16}$$

that is:

$$-L_j(\eta_1, ..., \eta_p, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}\right] \tag{17}$$

We can obtain a chain of integrators of the error as follows:

$$\begin{aligned} \dot{e}_i^j &= e_{i+1}^j, \qquad 1 \le i \le n-1 \\ \dot{e}_n^j &= -\sum_{i=1}^{n} a_i^j e_i^j \end{aligned} \tag{18}$$

or in a compact form:

$$\dot{\mathbf{e}}_j = F_j \mathbf{e}_j \tag{19}$$

and

$$-L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j e_i^j \tag{20}$$

where $\mathbf{e}_j = (e_1^1, ..., e_n^j),\ y_{Rj} = (y_{Rj}, \dot{y}_{Rj}, ..., y_{Rj}^{(n-1)}),$ and

$$F_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \vdots & \cdots & 1 \\ -a_1^j & -a_2^j & \cdots & -a_n^j \end{pmatrix}$$

The origin $\mathbf{e}_j = 0$ is an equilibrium point for system in Eq. (19) if F_j is Hurwitz.

Furthermore, the controller depends also on the tracking errors. To estimate them, an observer is used. Firstly, system in Eq. (19) is rewritten as:

$$\dot{\mathbf{e}}_j(t) = E\mathbf{e}_j(t) + \varphi_j\left(\mathbf{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \tag{21}$$

where $\mathbf{e} = (\mathbf{e}_1, ..., \mathbf{e}_p)$, $\mathbf{y}_R = (\mathbf{y}_{R1}, ..., \mathbf{y}_{Rp})$, $\mathbf{u} = (u, \dot{u}, ..., u^{(\gamma)})$, $\hat{\mathbf{f}} = (\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu)})$, the elements of E are given by:

$$E_{ks} = \begin{cases} 1 & if \ k = s-1 \\ 0 & otherwise \end{cases}$$

and

$$\varphi_j\left(\mathbf{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., u^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) \\ -y_{Rj}^{(n)} \end{pmatrix}$$

Then, the estimation $\hat{\mathbf{e}}_j(t)$ is obtained by the following HGO [23]:

$$\dot{\hat{\mathbf{e}}}_j(t) = E\hat{\mathbf{e}}_j(t) + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \tag{22}$$

where S_∞ is the solution to the equation:

$$S_\infty\left(E + \frac{\theta}{2}I\right) + \left(E^T + \frac{\theta}{2}I\right)S_\infty = C^T C \tag{23}$$

with $\theta > 0$ and $C = (1 \quad 0 \quad \ldots \quad 0)$. The coefficients of S_∞ are given by:

$$(S_\infty)_{ks} = \frac{\alpha_{ks}}{\theta^{k+s-1}}$$

where α_{ks} is a symmetric positive definite matrix independent of θ.

Furthermore, let $\hat{u}_l$ be the solution to

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{u}_l^{(\gamma_l)}, \hat{\mathbf{f}}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j \hat{e}_i^j \tag{24}$$

where $\hat{u}_l^{(\gamma_l)}$ is the highest order derivative of the given input found in the equation. Thus, the dynamical equation for controller $\hat{u}_l$ is:

$$\hat{u}_l^{(\gamma_l)} = K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \tag{25}$$

These controllers yield tracking in the original MIMO system, with fault tolerance (eliminates the effects of the faults). So, Eq. (22) is rewritten as:

$$\dot{\hat{\mathbf{e}}}_j(t) = E\hat{\mathbf{e}}_j(t) + \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \tag{26}$$

with

$$\hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, \ldots, \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, \ldots, \hat{u}^{(\gamma)}, \hat{f}, \ldots, \hat{f}^{(\mu)}) \\ -y_{Rj}^{(n)} \end{pmatrix}$$

Moreover, define the observation error as $\varepsilon_j(t) = \hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)$, and the following dynamics is obtained from Eqs. (21) and (26):

$$\dot{\varepsilon}_j(t) = \left(E - S_\infty^{-1}C^T C\right)\varepsilon_j(t) + \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \tag{27}$$

where

$$\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) = \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - \varphi_j\left(\underbrace{\hat{\mathbf{e}} - \varepsilon}_{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right)$$

Finally, if the following variables are defined:

$$\hat{u}_i^l = \hat{u}_l^{(i-1)} \quad i = 1, \ldots, \gamma_l \tag{28}$$

then the dynamical controller subsystems are written as follows:

$$\dot{\hat{\mathbf{u}}}_l(t) = E\hat{\mathbf{u}}_l(t) + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right), \qquad 1 \le l \le m \tag{29}$$

where $\hat{\mathbf{u}}_l = \left(\hat{u}_1^l, \ldots, \hat{u}_{\gamma_l}^l\right)$ and

$$\kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \end{pmatrix}$$

4. Stability analysis of the closed-loop system

The closed-loop dynamics is given by:

$$\begin{aligned} \dot{\hat{\mathbf{e}}}_j(t) &= E\hat{\mathbf{e}}_j(t) + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \\ \dot{\varepsilon}_j(t) &= \left(E - S_\infty^{-1}C^T C\right)\varepsilon_j(t) + \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \\ \dot{\hat{\mathbf{u}}}_l(t) &= E\hat{\mathbf{u}}_l(t) + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ \dot{\hat{\mathbf{f}}}_{\bar{l}}(t) &= E\hat{\mathbf{f}}_{\bar{l}}(t) + \omega_{\bar{l}}(u, y, f) \end{aligned} \tag{30}$$

for $1 \leq j \leq p$, $1 \leq l \leq m$ and $1 \leq \bar{l} \leq r$. Developing the equations for $\hat{\mathbf{e}}_j$, $\hat{\mathbf{u}}_l$ and $\hat{\mathbf{f}}_{\bar{l}}$, the following chain of integrators is obtained:

$$
\begin{aligned}
\dot{\hat{e}}_i^j &= \hat{e}_{i+1}^j - \psi_i(\theta_j)(\hat{e}_j - e_j) \qquad 1 \leq i \leq n-1 \\
\dot{\hat{e}}_n^j &= -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, ..., \hat{u}^{(\gamma)}, \hat{f}, ..., \hat{f}^{(\mu)}) - y_{Rj}^{(n)} - \theta_j^n, 1 \leq j \leq p \\
\dot{\hat{u}}_i^l &= \hat{u}_{i+1}^l \qquad 1 \leq i \leq \gamma_l - 1 \\
\dot{\hat{u}}_{\gamma_l}^l &= K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \qquad 1 \leq l \leq m \\
\dot{\hat{f}}_i^{\bar{l}} &= \hat{f}_{i+1}^{\bar{l}} \qquad 1 \leq i \leq \mu_{\bar{l}} - 1 \\
\dot{\hat{f}}_{\mu_{\bar{l}}}^{\bar{l}} &= k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) \qquad 1 \leq \bar{l} \leq r
\end{aligned}
$$

where $\psi_i(\theta_j)$ is a function obtained from S_∞^{-1}.

In this chain of integrators, the dynamics of the controllers and the fault estimations can be appreciated. As it can be seen, the variables obtained from these dynamics take part explicitly in the tracking error dynamics, leading to the solution of the multi-output tracking problem.

Now the main result of this paper is stated.

Theorem 1. *Let system in Eq. (1) be described in the $\mathcal{MMGOCF}$ Eq. (11) is composed of p subsystems. The observation dynamics corresponding to subsystem j are $\hat{\mathbf{e}}_j(t)$ and $\boldsymbol{\varepsilon}_j(t)$. Let $f_{\bar{l}}(t)$ be diagnosable for $1 \leq \bar{l} \leq r$ and estimated by means of the dynamics of $\dot{\hat{\mathbf{f}}}_{\bar{l}}(t)$. Let $\hat{u}_l(t)$ be the solution to*

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{u}_l^{(\gamma_l)}, \hat{\mathbf{f}}) - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j \hat{e}_i^j$$

Then, closed-loop system in Eq. (30) with control $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ is asymptotically stable.

Proof. Consider the following Lyapunov function:

$$V(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}) = V_1(\hat{\mathbf{e}}_j) + V_2(\varepsilon_j) + V_3(\tilde{f}_{\bar{l}}) \tag{31}$$

with

$$V_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j; \quad V_2(\varepsilon_j) = \varepsilon_j^T S_\infty \varepsilon_j; \quad V_3(\tilde{f}_{\bar{l}}) = \tilde{f}_{\bar{l}}^T I \tilde{f}_{\bar{l}}$$

where $\tilde{f}_{\bar{l}} = f_{\bar{l}} - \hat{f}_{\bar{l}}$. Besides, P is the solution to $F^T P + PF = -I$. Then P is positive-definite and define $\|x\|_P = \sqrt{x^T P x}$. Let S_∞ be the solution to Eq. (23), then S_∞ is positive-definite and denote $\|x\|_{S_\infty} = \sqrt{x^T S_\infty x}$.

Now, taking the derivative with respect to time of the first term of Eq. (31):

$$\dot{V}_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \dot{\hat{\mathbf{e}}}_j + \dot{\hat{\mathbf{e}}}_j^T P \hat{\mathbf{e}}_j \leq -\alpha \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C \varepsilon_j \tag{32}$$

where $\alpha = 1/\lambda_{\max}(P)$. Noting that

$$\left\|\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C S_\infty^{-1} S_\infty \varepsilon_j\right\| \leq \rho(\theta) \|\hat{\mathbf{e}}_j\|_P \|\varepsilon_j\|_{S_\infty}$$

with $\rho(\theta) = \|S_\infty^{-1} C^T C S_\infty^{-1}\|$, then the next inequality is obtained:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \le -(\alpha \|\hat{\mathbf{e}}_j\|_P + 2\rho(\theta)\|\varepsilon_j\|_{S_\infty})\|\hat{\mathbf{e}}_j\|_P \tag{33}$$

Let d_1, d_2 be positive numbers such that $\|\hat{\mathbf{e}}_j\|_P \ge d_1\|\hat{\mathbf{e}}_j\|$ and $\|\varepsilon_j\|_{S_\infty} \ge d_2\|\varepsilon_j\|$. Thus the inequality is rewritten as:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \le -(\alpha d_1 \|\hat{\mathbf{e}}_j\| + 2\rho(\theta) d_2 \|\varepsilon_j\|)\|\hat{\mathbf{e}}_j\|_P \tag{34}$$

and this yields to

$$\dot{V}_1(\hat{\mathbf{e}}_j) \le 0$$

Taking the derivative with respect to time of the second term of Eq. (31):

$$\dot{V}_2(\varepsilon_j) = \varepsilon_j^T S_\infty \dot{\varepsilon}_j + \dot{\varepsilon}_j^T S_\infty \varepsilon_j \le -\theta\|\varepsilon_j\|_{S_\infty}^2 + 2\|\varepsilon_j\|_{S_\infty}\|\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))\|_{S_\infty} \tag{35}$$

where it was taken into account in Eq. (23), the Cholesky decomposition and the fact that $\varepsilon_j^T C^T C \varepsilon_j \ge 0$. Besides, noting that $\Phi_j(\boldsymbol{\varepsilon}(t), \hat{\mathbf{e}}(t))$ is differentiable, by the Lipschitz property:

$$\|\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))\|_{S_\infty} \le \lambda\|\varepsilon_j(t)\|_{S_\infty} \qquad 1 \le j \le p$$

thus:

$$\dot{V}_2(\varepsilon_j) \le -(\theta - 2\lambda)\|\varepsilon_j\|_{S_\infty}^2 \tag{36}$$

with $\lambda < \theta/2$.

From Eq. (36) the following inequality is obtained:

$$\frac{d(\|\varepsilon_j\|_{S_\infty}^2)}{dt} \le -(\theta - 2\lambda)\|\varepsilon_j\|_{S_\infty}^2 \tag{37}$$

that yields:

$$\|\varepsilon_j\|_{S_\infty} \le -e^{-\gamma t}\|\varepsilon_j(0)\|_{S_\infty} \tag{38}$$

with $\gamma = \theta/2 - \lambda$.

Similarly, from Eq. (33) the next equation is achieved:

$$\frac{d\left(\|\hat{\mathbf{e}}_j\|_P^2\right)}{dt} \le -(\alpha\|\hat{\mathbf{e}}_j\|_P + 2\rho(\theta)\|\varepsilon_j\|_{S_\infty})\|\hat{\mathbf{e}}_j\|_P \tag{39}$$

this yields:

$$\|\hat{\mathbf{e}}_j\|_P \le Ae^{-\frac{\alpha}{2}t} + Be^{-\gamma t} \tag{40}$$

with

$$A = \|\hat{e}_j(0)\|_P - B$$
$$B = -\frac{\rho(\theta)\|\varepsilon_j(0)\|_{S_\infty}}{\alpha/2 - \gamma}$$

Finally, taking the derivative with respect to time of the third term of Eq. (31):

$$V_3(\tilde{f}_{\bar{l}}) = \tilde{f}_{\bar{l}}^T I \tilde{f}_{\bar{l}} = \tilde{f}_{\bar{l}}^2 \tag{41}$$

$$\dot{V}_3(\tilde{f}_{\bar{l}}) = 2\tilde{f}_{\bar{l}}\dot{\tilde{f}}_{\bar{l}} \leq 2\tilde{f}_{\bar{l}}^T\left(\frac{N_{\bar{l}}}{k_{\bar{l}}} - \tilde{f}_{\bar{l}}\right) \tag{42}$$

and the condition $N_{\bar{l}}/k_{\bar{l}} \to 0$ with $t \to \infty$ is imposed, then

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq -2\tilde{f}_{\bar{l}}^2 \tag{43}$$

Thus, the system (1) with control $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ is asymptotically stable, for $1 \leq j \leq p$, $1 \leq l \leq m$ and $1 \leq \bar{l} \leq r$. $\square$

The following remark shows the relation of the convergence velocity between the tracking error and the observation error.

Remark 4. It can be seen from Eq. (40) that

$$\|\hat{\mathbf{e}}_j\|_P \leq (A+B)e^{-\min\{\frac{\alpha}{2},\gamma\}t} \tag{44}$$

Selecting the condition $\theta/2 - \lambda = \gamma > \alpha/2$, θ can be chosen such that for a fixed value:

$$\|\hat{\mathbf{e}}_j\|_P \leq (A+B)e^{-\frac{\alpha}{2}t} \tag{45}$$

This implies that the observation error $\boldsymbol{\varepsilon}_j(t)$ converges faster than the estimated tracking error $\hat{\mathbf{e}}_j(t)$.

Note that the following remark establishes the stability region obtained considering a measurement noise in the output, using the Uniform Ultimate Boundedness theorem [24].

Remark 5. Consider the above problem but with a bounded deterministic noise δ such that $\|\delta\| \leq \delta^+$, $\delta^+ > 0$, with corrupted measurement, i.e. $y + \delta$. In this case the observation error is defined as $\varepsilon_j = \hat{\mathbf{e}}_j - \mathbf{e}_j - \boldsymbol{\delta}_j$, where $\boldsymbol{\delta}_j = col(\delta_j \quad 0 \quad \dots \quad 0)$ and $\delta_j \in \mathbb{R}$. In this case, the dynamics of the HGO in Eq. (26) is given by:

$$\begin{aligned}\dot{\hat{\mathbf{e}}}_j &= E\hat{\mathbf{e}}_j + \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j - \boldsymbol{\delta}_j) \\ &= E\hat{\mathbf{e}}_j + \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) + S_\infty^{-1}C^TC\boldsymbol{\delta}_j\end{aligned} \tag{46}$$

and thus the observation dynamics is written as follows:

$$\begin{aligned}\dot{\varepsilon}_j &= E\varepsilon - S_\infty^{-1}C^TC\varepsilon + \hat{\varphi}_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - \varphi_j\left(\mathbf{e}, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ &\quad + S_\infty^{-1}C^TC\boldsymbol{\delta}_j = E_\theta\varepsilon_j + \Phi_j(\varepsilon, \hat{\mathbf{e}}) + W_j\end{aligned} \tag{47}$$

where $W_j = S_\infty^{-1}C^TC\boldsymbol{\delta}_j$.

Hence, the derivative of the second term of the Lyapunov function in Eq. (31), i.e. $V_2(\varepsilon_j) = \varepsilon_j^TS_\infty\varepsilon_j$, is:

$$\begin{aligned}\dot{V}_2(\varepsilon_j) &= \dot{\varepsilon}_j^TS_\infty\varepsilon_j + \varepsilon_j^TS_\infty\dot{\varepsilon}_j \\ &\leq -\theta\varepsilon_j^TS_\infty\varepsilon_j + 2\varepsilon_j^TS_\infty\Phi_j + 2\varepsilon_j^TS_\infty W_j \\ &\leq -\theta\|\varepsilon_j\|_{S_\infty}^2 + 2\|\varepsilon_j\|_{S_\infty}\|\Phi_j\|_{S_\infty} + 2\|\varepsilon_j\|_{S_\infty}\|W_j\|_{S_\infty} \\ &\leq -(\theta - 2\lambda)\|\varepsilon_j\|_{S_\infty}^2 + 2\|\varepsilon_j\|_{S_\infty}\|W_j\|_{S_\infty}\end{aligned} \tag{48}$$

Since the term W_j is given by [20]

$$W_j = S_\infty^{-1} C^T C \boldsymbol{\delta}_j = \begin{pmatrix} -n\theta & 0 & \dots & 0 \\ -\frac{n(n-1)}{2!}\theta^2 & 0 & \dots & 0 \\ -\frac{n(n-1)(n-2)}{2!}\theta^3 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{n(n-1)}{2!}\theta^{n-2} & 0 & \dots & 0 \\ -n\theta^{n-1} & 0 & \dots & 0 \\ -\theta^n & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \delta_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$W_j = \begin{pmatrix} -n\theta\delta_j \\ -\frac{n(n-1)}{2!}\theta^2\delta_j \\ -\frac{n(n-1)(n-2)}{2!}\theta^3\delta_j \\ \vdots \\ -\frac{n(n-1)}{2!}\theta^{n-2}\delta_j \\ -n\theta^{n-1}\delta_j \\ -\theta^n\delta_j \end{pmatrix} \tag{49}$$

given that the noise is deterministic and bounded, i.e. $\|\delta_j\| \le \delta^+$, $n \in \mathbb{Z}^+$ is finite and $\theta > 0$, then $\exists\, \Gamma > 0$ finite, such that $\|W_j\|_{S_\infty} \le \Gamma$. Therefore:

$$\dot{V}_2(\varepsilon_j) \le -(\theta - 2\lambda)\|\varepsilon_j\|^2_{S_\infty} + 2\Gamma\|\varepsilon_j\|_{S_\infty} \tag{50}$$

Now, applying the Rayleigh–Ritz inequality:

$$\dot{V}_2(\varepsilon_j) \le -(\theta - 2\gamma)\lambda_{\min}(S_\infty)\|\varepsilon_j\|^2 + 2\Gamma\sqrt{\lambda_{\max}(S_\infty)}\|\varepsilon_j\| \tag{51}$$

Finally, applying the Uniform Ultimate Boundedness Theorem [24] it is concluded that $\varepsilon_j(t)$ is bounded uniformly by any initial state $\varepsilon_j(0)$ and remains in a compact set $B_b = \{\varepsilon_j : \|\varepsilon_j\| \le b,\ b > 0\}$, where the ultimate bound is defined as

$$b = \sqrt{\frac{\lambda_{\max}(S_\infty)}{\lambda_{\min}(S_\infty)}}\left(\frac{2\Gamma\sqrt{\lambda_{\max}(S_\infty)}}{(\theta - 2\gamma)\lambda_{\min}(S_\infty)}\right) \tag{52}$$

Furthermore, from a similar analysis, the derivative of the first term of the Lyapunov function in Eq. (31), i.e. $V_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j$ is:

$$\begin{aligned} \dot{V}_1(\hat{\mathbf{e}}_j) &= \hat{\mathbf{e}}_j^T P \dot{\hat{\mathbf{e}}}_j + \dot{\hat{\mathbf{e}}}_j^T P \hat{\mathbf{e}}_j \\ &\le -(\alpha d_1\|\hat{\mathbf{e}}_j\| + 2\rho(\theta)d_2\|\varepsilon_j\| - 2\Gamma)\|\hat{\mathbf{e}}_j\|_P \end{aligned} \tag{53}$$

and we obtain the following ultimate bound for $\hat{\mathbf{e}}_j(t)$:

$$b = \sqrt{\frac{(\lambda_{\max}(P))^3}{\lambda_{\min}(P)d_1^2}}\left(-\frac{4\rho(\theta)d_2\Gamma\sqrt{\lambda_{\max}(S_\infty)}}{(\theta - 2\gamma)\lambda_{\min}(S_\infty)} + 2\Gamma\right) \tag{54}$$

So, in the presence of measurement noise, $\varepsilon_j(t)$ and $\hat{\mathbf{e}}_j(t)$ are uniform ultimate bounded; the effect of this can be seen in the experimental results (see Fig. 12).

5. A numerical example and a real-time application

5.1. Numerical example

Consider the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_1x_2 + f + u\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= x_3 f + u\\ y &= x_2\end{aligned} \tag{55}$$

Choosing $\eta_1 = y$, the system can be transformed into the $\mathcal{MMGOCF}$ in Eq.(11):

$$\begin{aligned}\dot{\eta}_1 &= \eta_2\\ \dot{\eta}_2 &= \eta_3\\ \dot{\eta}_3 &= (\eta_2)^2 + \eta_2(\eta_1)^2 + \eta_1 u + \eta_1 f + \dot{u} + \dot{f}\end{aligned} \tag{56}$$

Now, the tracking error HGO in Eq. (22) is built, where

$$E = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}; \quad C = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$$

and matrix S_∞^{-1} is chosen as:

$$S_\infty^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3\\ 3\theta^2 & 5\theta^3 & 2\theta^4\\ \theta^3 & 2\theta^4 & \theta^5\end{pmatrix}$$

Defining $e_1 = \eta_1 - y_R$ and $e_1^{(i-1)} = e_i$, the tracking error HGO is written as:

$$\begin{aligned}\dot{\hat{e}}_1 &= \hat{e}_2 - 3\theta(\hat{e}_1 - e_1)\\ \dot{\hat{e}}_2 &= \hat{e}_3 - 3\theta^2(\hat{e}_1 - e_1)\\ \dot{\hat{e}}_3 &= (\hat{e}_2 + \dot{y}_R)^2 + (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 + (\hat{e}_1 + y_R)\hat{u} + (\hat{e}_1 + y_R)\hat{f} + \dot{\hat{u}} + \dot{\hat{f}}\\ &\quad - \dddot{y}_R - \theta^3(\hat{e}_1 - e_1) = -\sum_{i=1}^{3} a_i\hat{e}_i\end{aligned} \tag{57}$$

From system in Eq. (57), the dynamical equation of the controller in Eq. (25) is obtained as:

$$\dot{\hat{u}} = -\sum a_i\hat{e}_i - (\hat{e}_2 + \dot{y}_R)^2 - (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 - (\hat{e}_1 + y_R)\hat{u} - (\hat{e}_1 + y_R)\hat{f} - \dot{\hat{f}} + \dddot{y}_R \tag{58}$$

Defining $\hat{u} = \hat{u}_1$, the dynamics of the controller is:

$$\dot{\hat{u}}_1 = -(\hat{e}_2 + \dot{y}_R)^2 - (\hat{e}_2 + \dot{y}_R)(\hat{e}_1 + y_R)^2 - (\hat{e}_1 + y_R)\hat{u} - (\hat{e}_1 + y_R)\hat{f} - \dot{\hat{f}} + \dddot{y}_R \tag{59}$$

Furthermore, it can be seen that system in Eq. (55) is diagnosable, because the following polynomial can be obtained:

$$f - \ddot{y} + \dot{y}y + u = 0 \tag{60}$$

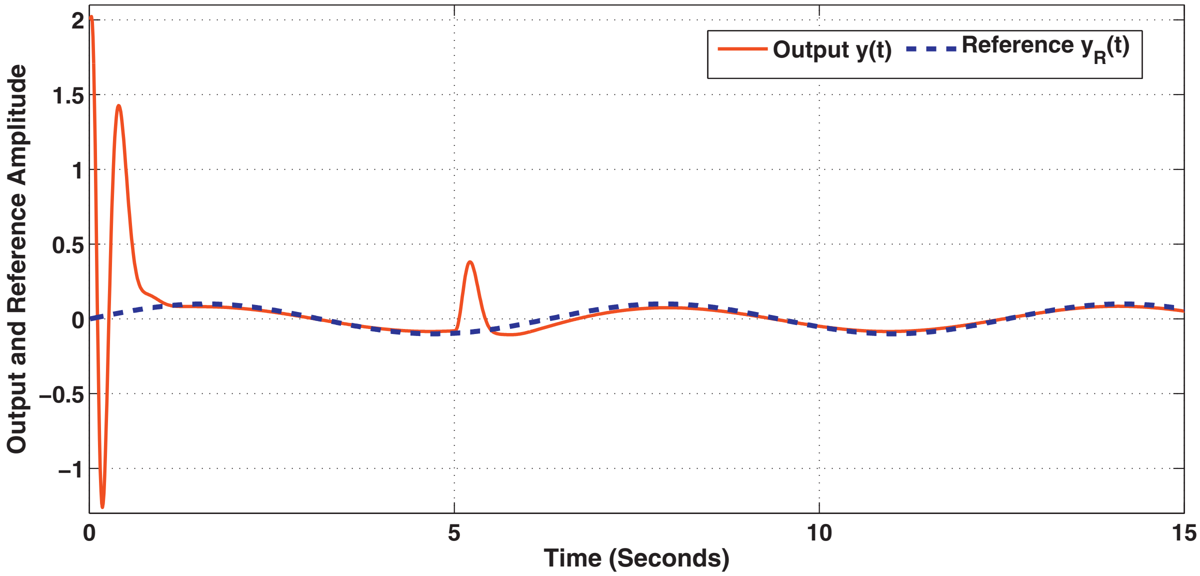

Fig. 1. Output tracking of the numerical example.

thus an observer for estimating f can be built based on Eq. (6). For this, the following ROO is proposed:

$$\begin{aligned}\dot{\gamma}_2 &= -k_2(\hat{x}_1 y + \hat{u} + \gamma_2 + k_2\hat{x}_1)\\ \hat{f} &= \gamma_2 + k_2\hat{x}_1\end{aligned} \tag{61}$$

where $\hat{x}_1$ is the estimation of $\dot{y}$, which is obtained with

$$\begin{aligned}\dot{\gamma}_1 &= -k_1(\gamma_1 + k_1 y)\\ \hat{x}_1 &= \gamma_1 + k_1 y\end{aligned} \tag{62}$$

Defining $\hat{f} = \hat{f}_1$, the dynamics of the fault is:

$$\dot{\hat{f}}_1 = -k_2(\hat{x}_1 y + \hat{u} + \gamma_2 + k_2\hat{x}_1) + k_2(-k_1(\gamma_1 + k_1 y) + k_1\hat{x}_1) \tag{63}$$

Simulations were made for this system over 15 s, using the time-variant reference $y_R(t) = 0.1\sin(t)$ and the fault $f(t) = 50[1 + \sin(0.2te^{-0.5t})]\ \mathcal{U}(t-5)$, where $\mathcal{U}(t)$ is the step function. The design parameters were chosen as $\theta = 20$, $a_1 = 8000$, $a_2 = 1200$, $a_3 = 60$, $k_1 = k_2 = 1$.

Fig. 1 shows how the output $y(t)$ follows the reference $y_R(t)$; it can be seen that approximately after one second the output follows the sinusoidal reference. However, the effect of $f(t)$, which begins at 5 s, can be appreciated before it is eliminated.

Fig. 2 shows the fault-tolerant control signal used. It can be appreciated that the controller uses less energy when the fault appears; this is due to the nature of the fault, whose amplitude has a positive value. The controller seeks to compensate it, and the elimination of the effects of the fault can be seen in the output tracking graph.

The fault $f(t)$ and its estimation $\hat{f}(t)$ are shown in Fig. 3. It can be seen that the ROO transient part ends quickly, and 5 s after the fault appears, the estimation follows the real signal.

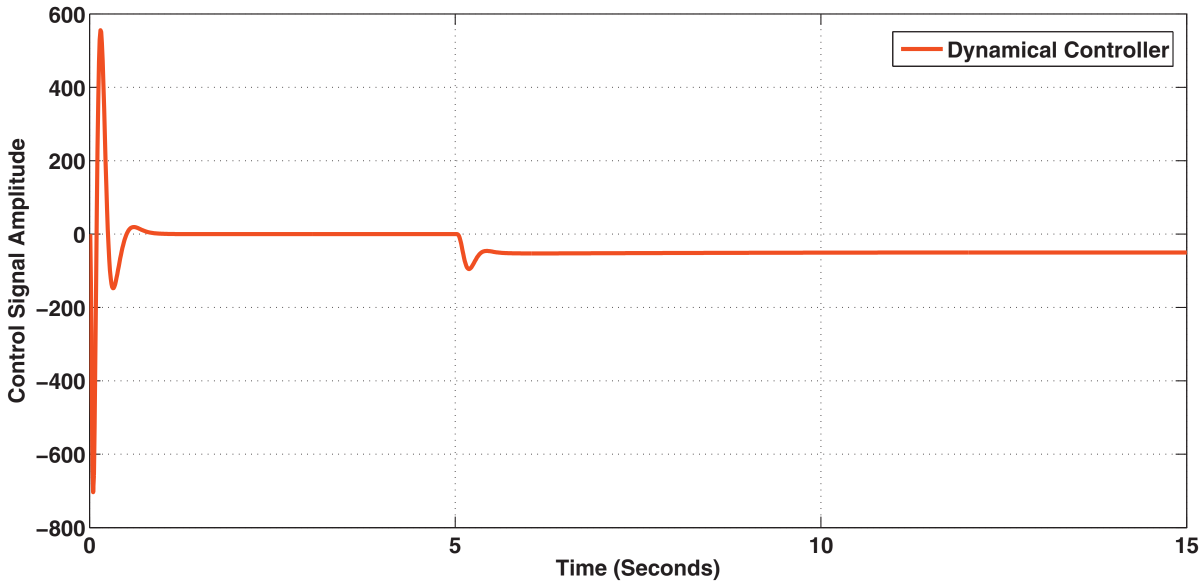

Fig. 2. Dynamical controller of the numerical example.

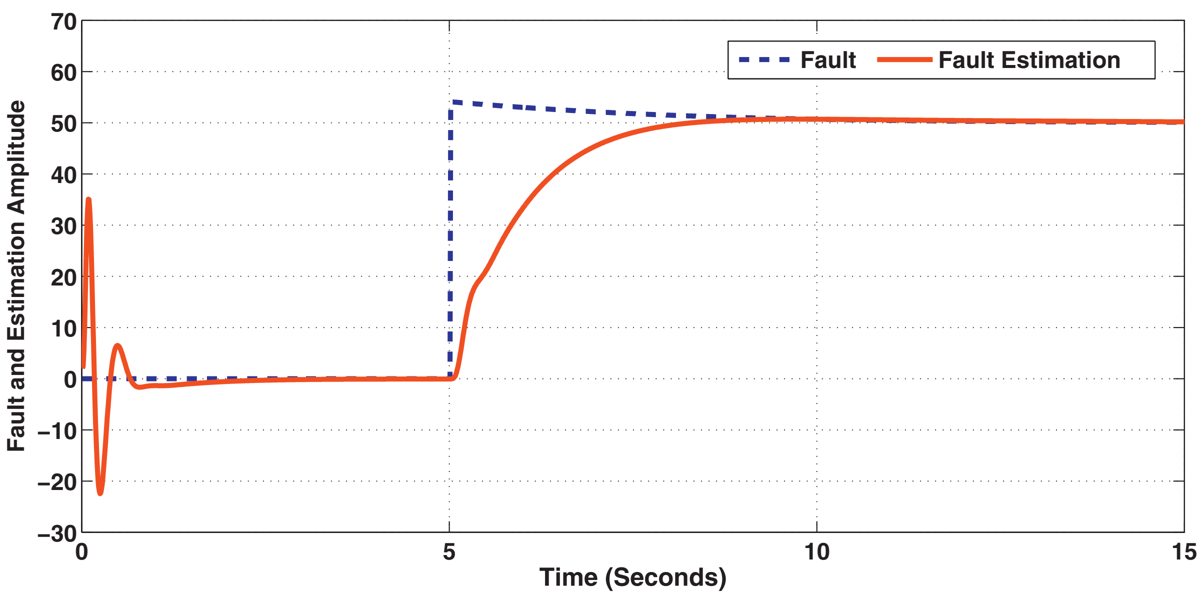

Fig. 3. Fault estimation of the numerical example.

Finally, Fig. 4 shows the performance index of the ROO proposed to estimate the fault, which was evaluated using the following cost functional:

$$J_t = \frac{1}{t+\varepsilon}\int_0^t \|\tilde{f}_{\bar{l}}(t)\|^2 dt \tag{64}$$

where $\varepsilon = 0.0001$. This parameter is used to avoid singularities when $t = 0$, and is chosen sufficiently small so that it does not alter significantly the value of the index.

Remark 6. A comparative analysis of fault diagnosis with different estimators was performed and is presented in Figs. 5–7. There were considered a sliding mode differentiator [26] and an algebraic observer [27]); these algorithms have shown the ability to reconstruct the fault, but

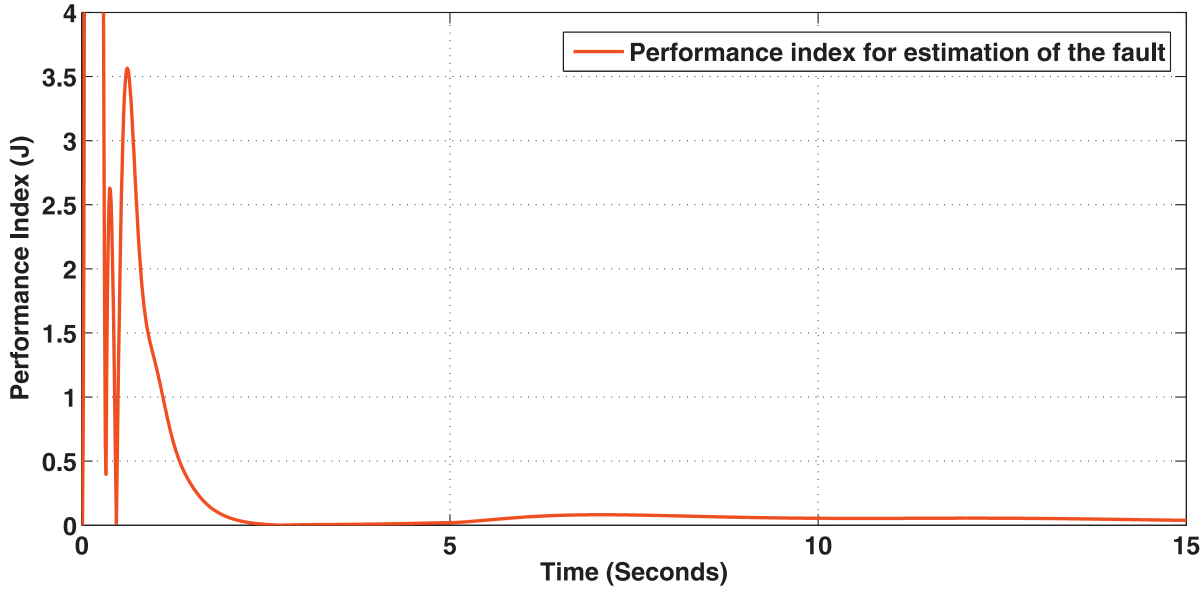

Fig. 4. Performance evaluation for the fault estimation of the numerical example.

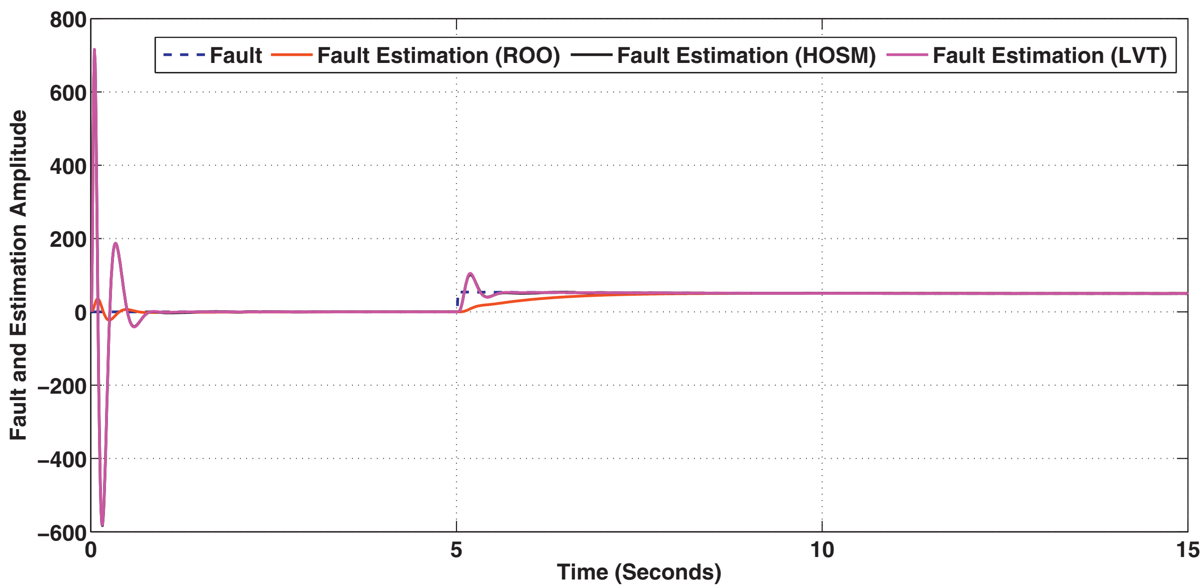

Fig. 5. Fault estimation: comparative analysis.

with notable differences. It can be seen that the counterparts of the ROO show an overshoot ten times greater than the amplitude of the fault. Finally, it is worth highlighting that the ROO also shows a better performance index (Fig. 7). For the mentioned reasons, the ROO has been chosen to realize the fault diagnosis in simulations and in the implementation in real-time.

5.2. Real-time application

Consider the nonlinear Amira system model [25]:

$$\dot{x}_1 = \frac{1}{A}(-q_{13} + u_1 + f_1) \tag{65}$$

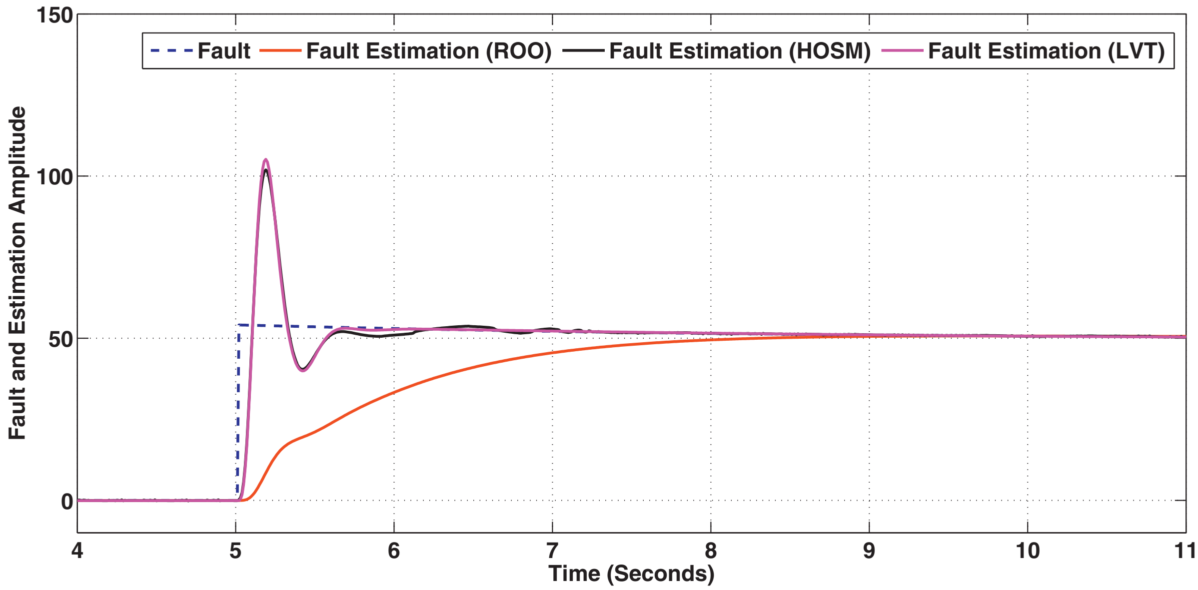

Fig. 6. Fault estimation: comparative analysis (Zoom).

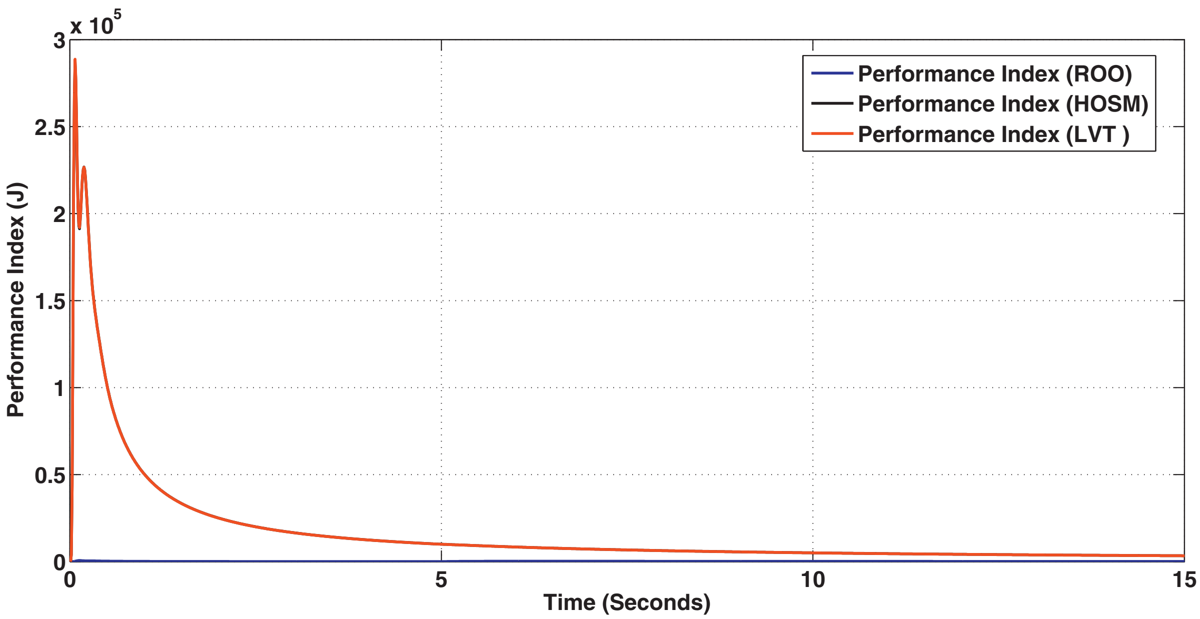

Fig. 7. Performance evaluation: comparative analysis.

$$\dot{x}_2 = \frac{1}{A}(q_{32} - q_{20} + u_2 + f_2)$$
$$\dot{x}_3 = \frac{1}{A}(q_{13} - q_{32})$$
$$y_1 = x_2$$
$$y_2 = x_3$$

with

$$q_{13} = a_1 S\sqrt{2g(x_1 - x_3)}$$

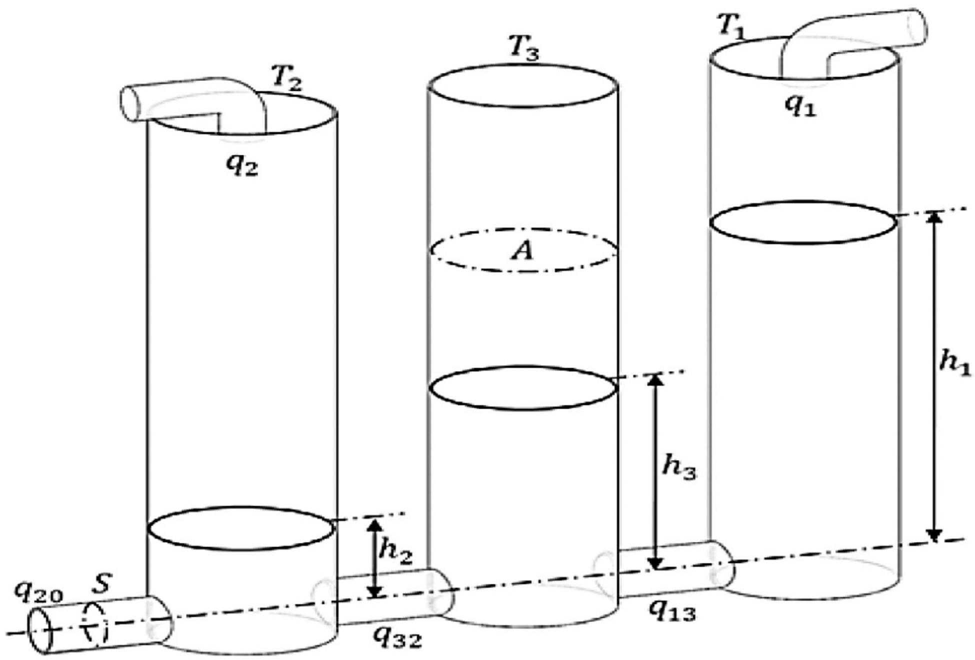

Fig. 8. Schematic diagram of the Amira DTS200 working in the region $h_1 > h_3 > h_2 > 0$.

$$q_{32} = a_3 S\sqrt{2g(x_3 - x_2)}$$
$$q_{20} = a_2 S\sqrt{2gx_2}$$

In this system, $u_i = q_i,\ i = 1, 2$ are the manipulable input flows, $x_i = h_i,\ i = 1, 2, 3$ are the levels of each tank, A is the cross section of the tanks, and the terms q_{ij} represent the water flow from tank i to tank j. S is the cross-sectional area of the pipe that interconnects each tank and the unknown parameters $a_i,\ i = 1, 2, 3$ are the output flow coefficients.

It is important to mention that the system has four operation regions, and the region considered for this work is $h_1 > h_3 > h_2 > 0$. To strengthen the information, the characteristics and variables of the system and how the system operates on the desired region are shown on Fig. 8. A picture of the real system used is shown on Fig. 9.

From the model (65), it can be seen that x_1 is not available for measurement, however it is algebraically observable according to Definition 2, which allows to write the following algebraic polynomial:

$$x_1 - y_2 - \frac{1}{2ga_1^2S^2}(A\dot{y}_2 + a_3S\sqrt{2g(y_2 - y_1)})^2 = 0 \tag{66}$$

5.2.1. Parameter estimation

As it is mentioned above, the flow transfer coefficients a_1, a_2 and a_3 are not known, however it can be verified that they are algebraically observable, that is, they satisfy differential algebraic equations with coefficients in $k\langle u,\ y\rangle$ (see definition of algebraically observable parameter [20]). Thus, from Eq. (65) the following expressions, which are defined in the region of interest ($h_1 > h_3 > h_2 > 0$) are obtained:

$$a_1 = \frac{q_1 - A\dot{x}_1}{S\sqrt{2g(x_l - x_3)}} \tag{67}$$

Fig. 9. Amira DTS200 three-tank system benchmark.

$$a_2 = \frac{q_1 + q_2 - A(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)}{S\sqrt{2g(x_2)}} \tag{68}$$

$$a_3 = \frac{q_1 - A(\dot{x}_1 + \dot{x}_3)}{S\sqrt{2g(x_3 - x_2)}} \tag{69}$$

Thus, if the measurements of the inputs q_1 and q_2 and the state variables x_1, x_2 and x_3 from the nominal model (without the presence of faults) are available, it is possible to estimate their time derivatives and from Eqs. (67)–(69) the estimates of the uncertain parameters a_1, a_2 and a_3 can be obtained.

5.2.2. Experimental results

Now, the proposed method is applied. Firstly, the change of variables $\eta_1^j = y_j$ is selected, so the system in Eq. (65) can be transformed into the $\mathcal{MMGOCF}$ in Eq. (11). For $y_1 = \eta_1^1$, the subsystem is:

$$\begin{aligned}
\dot{\eta}_1^1 &= \eta_2^1 \\
\dot{\eta}_2^1 &= \eta_3^1 \\
\dot{\eta}_3^1 &= \frac{1}{A}\left(a_3 S g^2 \frac{2(\eta_1^2 - \eta_1^1)(\eta_3^2 - \eta_3^1) - (\eta_2^2 - \eta_2^1)^2}{\sqrt{\left(2g(\eta_1^2 - \eta_1^1)\right)^3}} - a_2 S g^2 \frac{2\eta_1^1\eta_3^1 - \left(\eta_2^1\right)^2}{\sqrt{\left(2g\eta_1^1\right)^3}} + \ddot{u}_2 + \ddot{f}_2 \right)
\end{aligned} \tag{70}$$

and for $y_2 = \eta_1^2$:

$$
\begin{aligned}
\dot{\eta}_1^2 &= \eta_2^2 \\
\dot{\eta}_2^2 &= \eta_3^2 \\
\dot{\eta}_3^2 &= \frac{1}{A^2}\left(a_1^2 S^2 g^3 \frac{2(\dot{x}_1-\eta_2^2)(x_1-\eta_1^2)}{\left(2g(x_1-\eta_1^2)\right)^2} + a_1 S g^2 \frac{2(x_1-\eta_1^2)}{\sqrt{\left(2g(x_1-\eta_1^2)\right)^3}}(\dot{u}_1+\dot{f}_1)\right) \\
&\quad - \frac{1}{A}\left(a_1 S g^2 \frac{2\left(\eta_3^2\right)(x_1-\eta_1^2)+(\dot{x}_1-\eta_2^2)^2}{\sqrt{\left(2g(x_1-\eta_1^2)\right)^3}} + a_3 S g^2 \frac{2(\eta_1^2-\eta_1^1)(\eta_3^2-\eta_3^1)-(\eta_2^2-\eta_2^1)^2}{\sqrt{\left(2g(\eta_1^2-\eta_1^1)\right)^3}}\right)
\end{aligned} \tag{71}
$$

These subsystems allow the construction of the tracking error HGO in Eq. (26), where the matrices E and C are defined as

$$
E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}
$$

and S_∞^{-1} is chosen as follows:

$$
S_\infty^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{pmatrix}
$$

Defining $e_1^1 = \eta_1^1 - y_{R1}$, $e_1^2 = \eta_1^2 - y_{R2}$, $e_1^{(i-1)} = e_i^1$ and $e_2^{(i-1)} = e_i^2$, the HGO for $y_1 = \eta_1^1$ is:

$$
\begin{aligned}
\dot{\hat{e}}_1^1 &= \hat{e}_2^1 - 3\theta_1(\hat{e}_1^1 - e_1^1) \\
\dot{\hat{e}}_2^1 &= \hat{e}_3^1 - 3\theta_1^2(\hat{e}_1^1 - e_1^1) \\
\dot{\hat{e}}_3^1 &= \frac{a_3 S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1}))\right)^3}}\right) \\
&\quad \times \left(2\left((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1})\right)\left(\hat{e}_3^2-\hat{e}_3^1\right)-\left(\hat{e}_2^2-\hat{e}_2^1\right)^2\right) \\
&\quad - \frac{a_2 S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_1^1+y_{R1}\right)\left(\hat{e}_3^1\right)-\left(\hat{e}_2^1\right)^2}{\sqrt{\left(2\left(\hat{e}_1^1+y_{R1}\right)\right)^3}}\right) + \frac{1}{A}\left(\ddot{\hat{u}}_2+\ddot{\hat{f}}_2\right) - \theta_1^3(\hat{e}_1^1-e_1^1) = -\sum_{i=1}^{3} s_i \hat{e}_i^1
\end{aligned} \tag{72}
$$

and for $y_2 = \eta_1^2$:

$$
\begin{aligned}
\dot{\hat{e}}_1^2 &= \hat{e}_2^2 - 3\theta_2(\hat{e}_1^2 - e_1^2) \\
\dot{\hat{e}}_2^2 &= \hat{e}_3^2 - 3\theta_2^2(\hat{e}_1^2 - e_1^2) \\
\dot{\hat{e}}_3^2 &= -\frac{a_3 S\sqrt{g}}{A}\left(\frac{1}{\sqrt{\left(2((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1}))\right)^3}}\right) \\
&\quad \times \left(2\left((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1})\right)\left(\hat{e}_3^2-\hat{e}_3^1\right)-\left(\hat{e}_2^2-\hat{e}_2^1\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{a_1S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_3^2\right)(\hat{x}_1-(\hat{e}_1^2+y_{R2}))+\left(\dot{\hat{x}}_1-\hat{e}_2^2\right)^2}{\sqrt{\left(2(\hat{x}_1-(\hat{e}_1^2+y_{R2}))\right)^3}}\right)\\
&+\frac{a_1S\sqrt{g}}{A^2}\left(\frac{1}{\sqrt{2(\hat{x}_1-(\hat{e}_1^2+y_{R2}))}}\right)(\dot{\hat{u}}_1+\dot{\hat{f}}_1)\\
&+\frac{a_1^2S^2g}{2A^2}\left(\frac{\dot{\hat{x}}_1-\hat{e}_2^2}{\hat{x}_1-\left(\hat{e}_1^2+y_{R2}\right)}\right)-\theta_2^3(\hat{e}_1^2-e_1^2)=-\sum_{i=1}^{3}t_i\hat{e}_i^2
\end{aligned}
\tag{73}
$$

Given that the references for this system are constant, their derivatives have been neglected in these equations. Consequently, from subsystem in Eq. (73) the dynamics of $\hat{u}_1$ is:

$$
\begin{aligned}
\dot{\hat{u}}_1=&\frac{A^2\sqrt{2(\hat{x}_1-\left(\hat{e}_1^2+y_{R2}\right)}}{a_1S\sqrt{g}}\left(-\sum_{i=1}^{3}t_i\hat{e}_i^2\right)-\frac{a_1S\sqrt{g}\left(\dot{\hat{x}}_1-\hat{e}_2^2\right)}{\sqrt{2(\hat{x}_1-(\hat{e}_1^2+y_{R2}))}}\\
&+\frac{2A\left(\hat{e}_3^2\right)(\hat{x}_1-\left(\hat{e}_1^2+y_{R2}\right)+A(\dot{\hat{x}}_1-\hat{e}_2^2)^2}{2(\hat{x}_1-(\hat{e}_1^2+y_{R2}))}\\
&+\frac{a_3A\sqrt{(\hat{x}_1-(\hat{e}_1^2+y_{R2}))}}{2a_1}\left(\frac{1}{\sqrt{((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1}))^3}}\right)\\
&\times\left(2\left(\left(\hat{e}_1^2+y_{R2}\right)-(\hat{e}_1^1+y_{R1})\right)\left(\hat{e}_3^2-\hat{e}_3^1\right)-\left(\hat{e}_2^2-\hat{e}_2^1\right)^2\right)-\dot{\hat{f}}_1
\end{aligned}
\tag{74}
$$

and from subsystem in Eq. (72), the dynamics of $\hat{u}_2$ is given by:

$$
\begin{aligned}
\ddot{\hat{u}}_2=&A\left(-\sum_{i=1}^{3}s_i\hat{e}_i^1\right)+a_2S\sqrt{g}\left(\frac{2\left(\hat{e}_1^1+y_{R1}\right)\left(\hat{e}_3^1\right)-\left(\hat{e}_2^1\right)^2}{\sqrt{\left(2\left(\hat{e}_1^1+y_{R1}\right)\right)^3}}\right)\\
&-a_3S\sqrt{g}\left(\frac{1}{\sqrt{\left(2((\hat{e}_1^2+y_{R2})-(\hat{e}_1^1+y_{R1}))\right)^3}}\right)\\
&\times\left(2\left(\left(\hat{e}_1^2+y_{R2}\right)-\left(\hat{e}_1^1+y_{R1}\right)\right)(\hat{e}_3^2-\hat{e}_3^1)-\left(\hat{e}_2^2-\hat{e}_2^1\right)^2\right)-\ddot{\hat{f}}_2
\end{aligned}
\tag{75}
$$

Defining $\hat{u}_1=\hat{u}_1^1$, $\hat{u}_2=\hat{u}_1^2$ and $\dot{\hat{u}}_2=\hat{u}_2^2$, the chains of integrators of the controllers are:

$$
\begin{aligned}
\dot{\hat{u}}_1^1=&\frac{A^2\sqrt{2T_3}}{a_1S\sqrt{g}}\left(\dddot{y}_{R2}-\sum_{i=1}^{3}t_i\hat{e}_i^2\right)-\frac{a_1S\sqrt{g}\dot{T}_3}{\sqrt{2T_3}}+\frac{a_3A\sqrt{T_3}}{2a_1}\left(\frac{2T_1\ddot{T}_1-(\dot{T}_1)^2}{\sqrt{(T_1)^3}}\right)\\
&+\frac{2A(\hat{e}_3^2+\ddot{y}_{R2})T_3+A\left(\dot{T}_3\right)^2}{2T_3}-\dot{\hat{f}}_1\\
\dot{\hat{u}}_1^2=&\hat{u}_2^2
\end{aligned}
$$

$$\dot{\hat{u}}_2^2 = A\left(\dddot{y}_{R1} - \sum_{i=1}^{3} s_i \hat{e}_i^1\right) - a_3 S\sqrt{g}\left(\frac{2T_1\ddot{T}_1 - (\dot{T}_1)^2}{\sqrt{(2T_1)^3}}\right) + a_2 S\sqrt{g}\left(\frac{2T_2\ddot{T}_2 - (\dot{T}_2)^2}{\sqrt{(2T_2)^3}}\right) - \ddot{\hat{f}}_2 \tag{76}$$

Furthermore, it can be seen from Eq. (65) that f_1 and f_2 are diagnosable, because the following polynomials can be obtained:

$$f_1 - A\dot{x}_1 - a_1 S\sqrt{2g(x_1 - y_2)} + u_1 = 0 \tag{77}$$

$$f_2 - A\dot{y}_1 + a_3 S\sqrt{2g(y_2 - y_1)} - a_2 S\sqrt{2gy_1} + u_2 = 0 \tag{78}$$

thus, the observers for f_1 and f_2 can be built based on Eq. (6). For estimating fault f_1, the following ROO is proposed:

$$\begin{aligned}\dot{\gamma}_1 &= -k_1(-q_{13} + u_1 + \gamma_1 + k_1 A\hat{x}_1)\\ \hat{f}_1 &= \gamma_1 + k_1 A\hat{x}_1\end{aligned} \tag{79}$$

As can be seen on Eq. (66), x_1 is algebraically observable, so an estimate is obtained with:

$$\begin{aligned}\dot{\gamma}_2 &= -k_2(\gamma_2 + k_2 y_2)\\ \hat{\zeta} &= \gamma_2 + k_2 y_2\\ \hat{x}_1 &= y_2 + \frac{1}{2ga_1^2 S^2}(A\hat{\zeta} + q_{32})^2\end{aligned} \tag{80}$$

where $\hat{\zeta}$ represents the estimation of $\dot{y}_2$.

Finally, for estimating fault f_2, the following ROO is used:

$$\begin{aligned}\dot{\gamma}_3 &= -k_3(q_{32} - q_{20} + u_2 + \gamma_3 + k_3 A y_1)\\ \hat{f}_2 &= \gamma_3 + k_3 A y_1\end{aligned} \tag{81}$$

Defining $\hat{f}_1 = \hat{f}_1^1$, $\hat{f}_2 = \hat{f}_1^2$ and $\dot{\hat{f}}_2 = \hat{f}_2^2$, the chains of integrators of the faults are:

$$\begin{aligned}\dot{\hat{f}}_1^1 &= -k_1(-q_{13} + u_1 + \gamma_1 + k_1 A\hat{x}_1) + k_1 A\dot{\hat{x}}_1\\ \dot{\hat{f}}_1^2 &= \hat{f}_2^2\\ \dot{\hat{f}}_2^2 &= -k_3(\dot{q}_{32} - \dot{q}_{20} + \dot{u}_2 + \dot{\gamma}_3 + k_3 A\dot{y}_1) + k_3 A\ddot{y}_1\end{aligned} \tag{82}$$

with

$$\dot{\hat{x}}_1 = \hat{\zeta} + \frac{A\hat{\zeta} + q_{32}}{ga_1^2 S^2}(A(-k_2(\gamma_2 + k_2 y_2) + k_2\dot{y}_2) + \dot{q}_{32}) \tag{83}$$

Real-time experiments were made in the Amira DTS200 system over 3000 s, using the signals $y_{R1}(t) = 0.06$ and $y_{R2}(t) = 0.11$ as the references and introducing the additive faults $f_1(t) = 1 \times 10^{-6}(1 + sin(0.2te^{-0.01t}))\mathcal{U}(t - 220)$ and $f_2(t) = 1 \times 10^{-6}(1 + sin(0.05te^{-0.001t}))\mathcal{U}(t - 300)$, where $\mathcal{U}(t)$ is the step function. The design parameters were chosen as $\theta_1 = \theta_2 = 1$, $s_1 = t_1 = 1$, $s_2 = t_2 = 3$, $s_3 = t_3 = 3$, $k_1 = 1.85$, $k_2 = 0.3$, $k_3 = 22$.

The parameters of the system are $A = 0.0149$ m^2 and $S = 5 \times 10^{-5}$ m^2. The unknown parameters a_1, a_2 and a_3 were estimated without the presence of faults, using the following

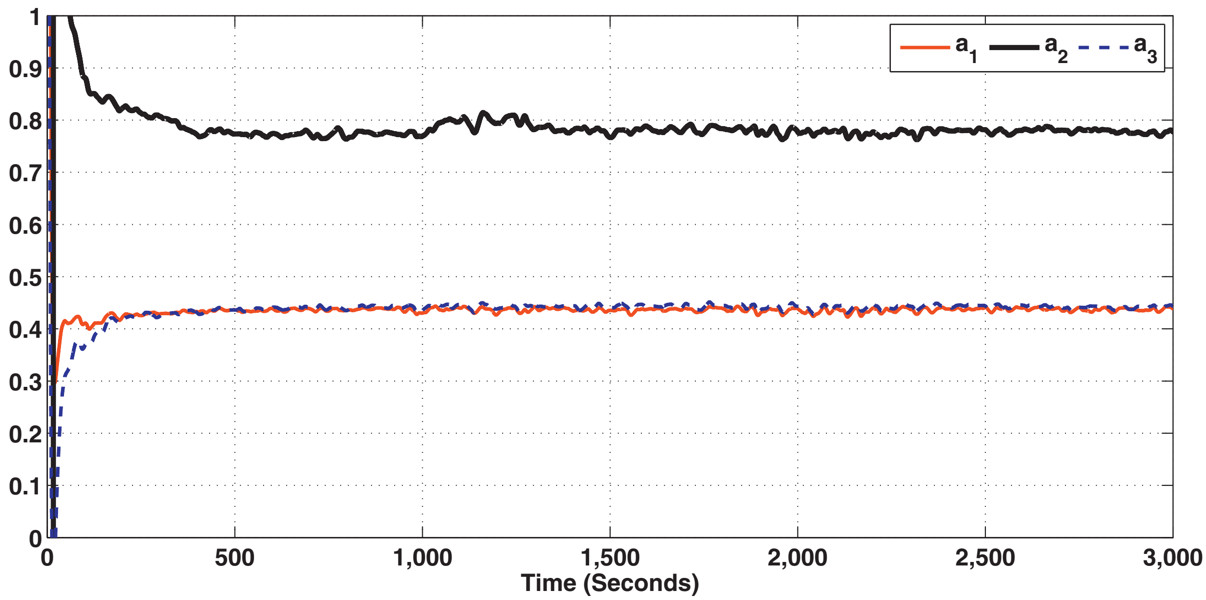

Fig. 10. Parameter identification of the Amira DTS200.

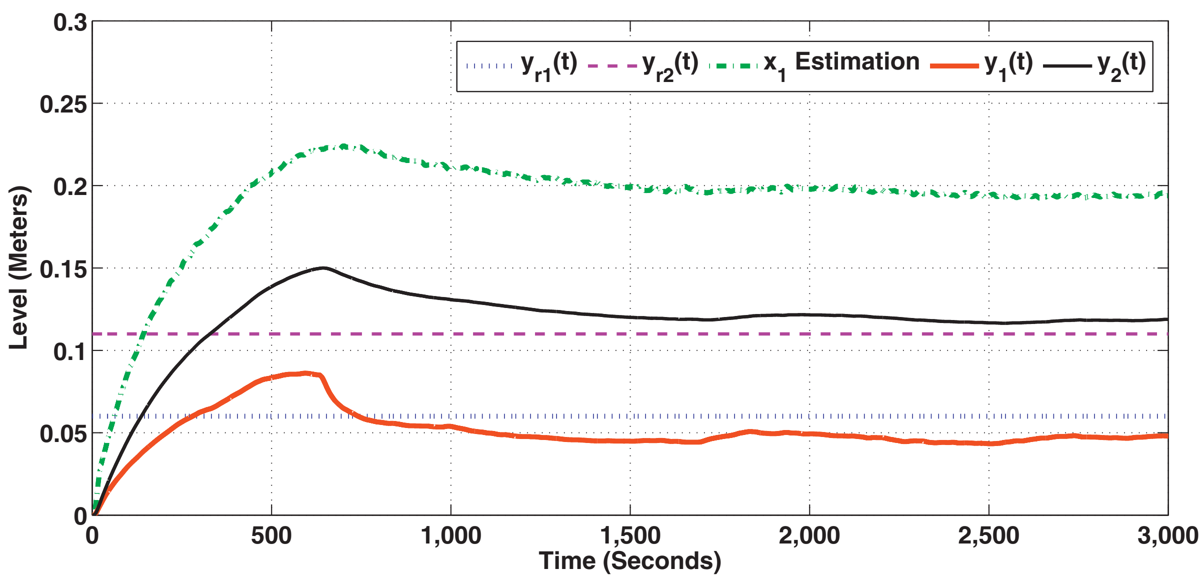

Fig. 11. Tank levels and multi-output tracking of the Amira DTS200 without fault-tolerant control.

values for the input flows: $q_1 = 0.00002$ m$^3/s$ and $q_2 = 0.000015$ m$^3/s$. The identification process, shown in Fig. 10, was performed along 3000 s, and the flow parameters were obtained as $a_1 = 0.4385$, $a_2 = 0.7774$ and $a_3 = 0.4435$.

Figs. 11 and 12 compare the behavior of the tracking with and without fault-tolerance. Fig. 11 illustrates how the effects of the faults affect tracking when a non-fault-tolerant control is applied. On the other hand, Fig. 12 shows how using the fault-tolerant control improves the tracking of the references, while suppressing the effects of both faults. Besides, as stated, the system operates in the desired region. The estimation $\hat{x}_1(t)$ is also displayed.

Figs. 13 and 15 show the estimation results of $f_1(t)$ and $f_2(t)$ respectively. These figures throw graphical valuable information, such as the magnitude of the faults and the time when

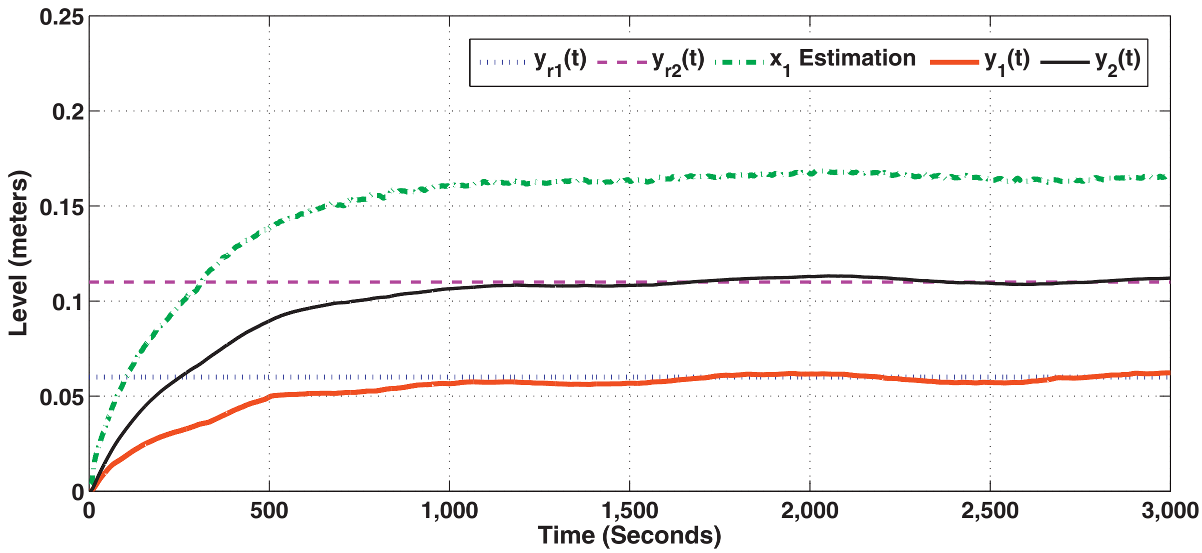

Fig. 12. Tank levels and multi-output tracking of the Amira DTS200 with fault-tolerant control.

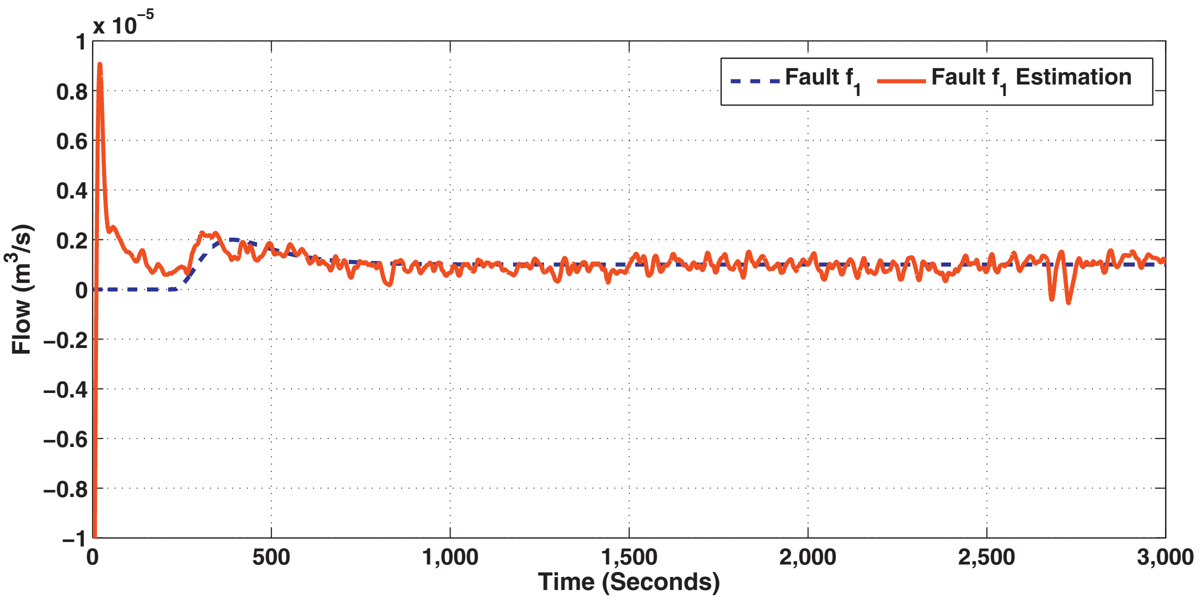

Fig. 13. Fault 1 estimation of the Amira DTS200.

they present. This information is extremely important because it can establish a critical magnitude of the fault, which may put the process on risk and forces to take corrective actions at a physical level, like sensors or actuators replacing.

Finally, in Figs. 14 and 16 are shown the performance indices of the ROO proposed to estimate the faults f_1 and f_2. The indices were evaluated using the same cost functional used in the numerical example.

Remark 7. In order to reduce measurement noise, a second-order low-pass Butterworth filter was used, which has the following transfer function:

$$G_f(s) = \frac{1}{32s^2 + 8s + 1} \tag{84}$$

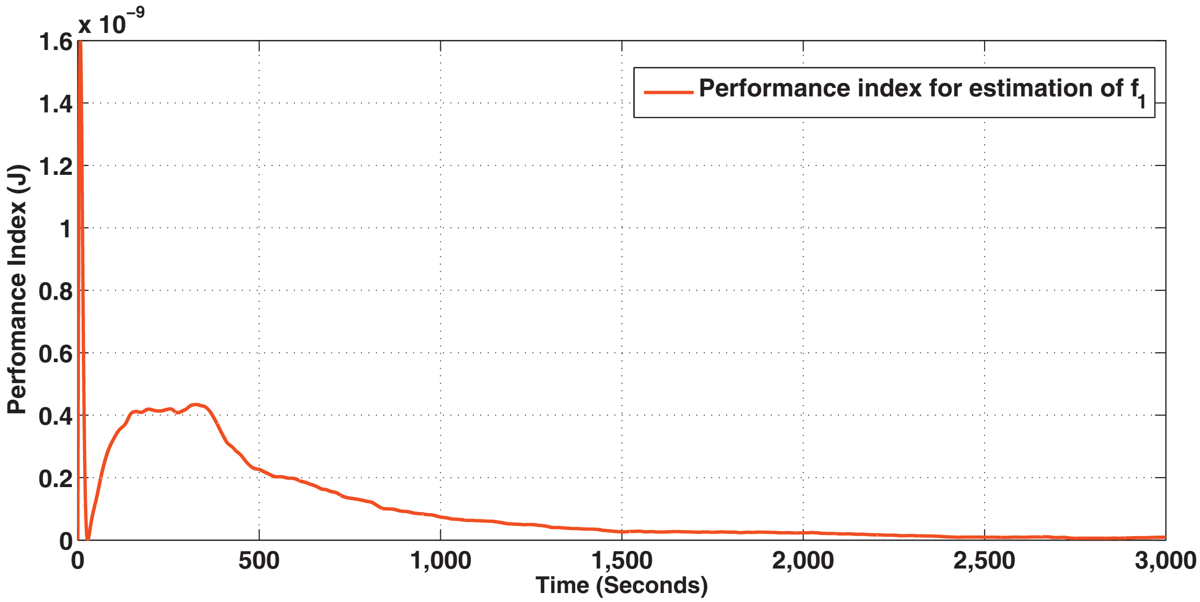

Fig. 14. Performance evaluation for the fault 1 estimation of the Amira DTS200.

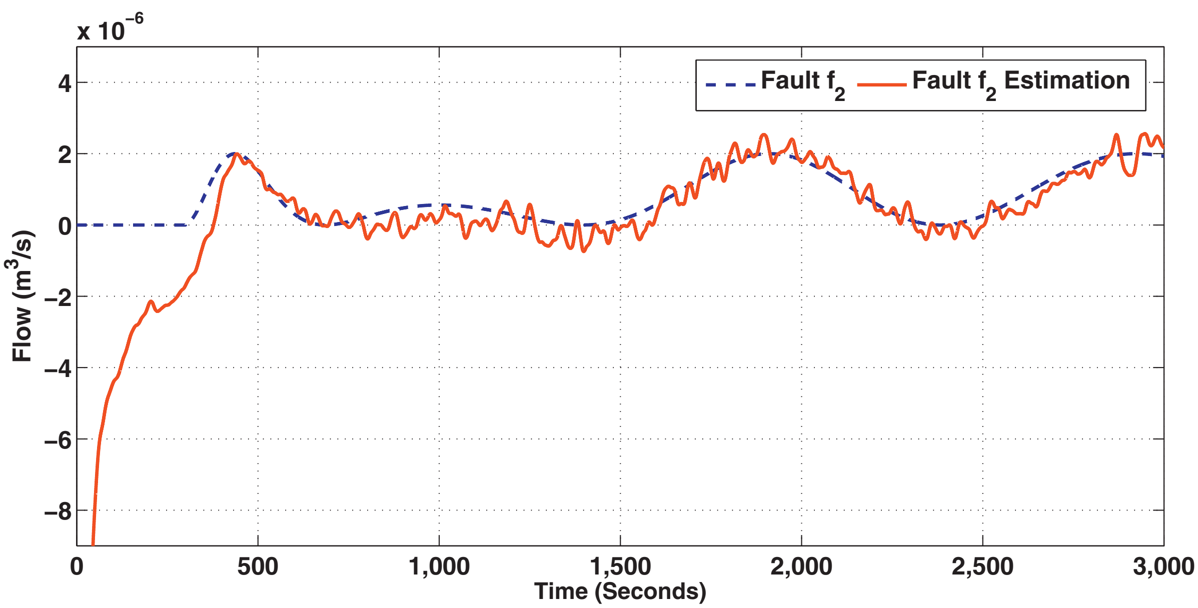

Fig. 15. Fault 2 estimation of the Amira DTS200.

Filter gains were designed considering a cutoff frequency of 0.03 Hz based on open-loop experiments on the system, where it can be seen a dominant (slow) time constant of 300 s. The smoothness of the real signals is a result of the implementation of this filter.

6. Concluding remarks

This work presented a novel dynamical controller with fault tolerance to achieve tracking for nonlinear MIMO systems. The system is transformed to a $\mathcal{MMGOCF}$ as a chain of integrators, which will be used to compute the dynamical controllers and the estimated

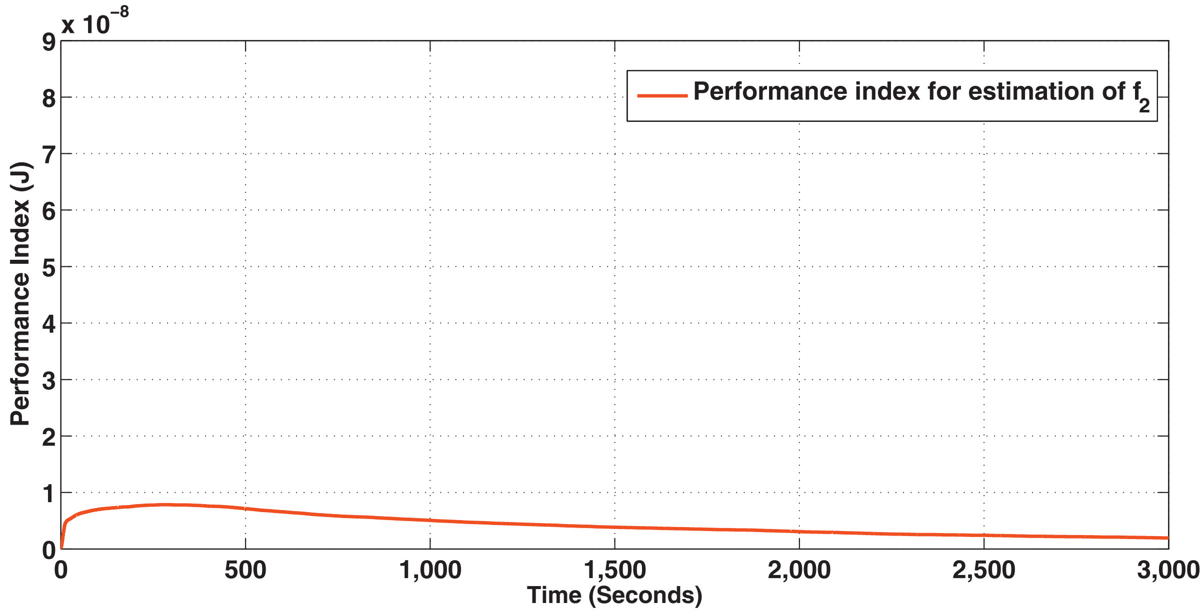

Fig. 16. Performance evaluation for the fault 2 estimation of the Amira DTS200.

faults. The controller is capable of suppressing the negative effects of the faults in the system, simultaneously in the presence of multiple faults. It was verified using the Lyapunov approach that the closed-loop system accomplishes asymptotic stability (without noise), and ultimate uniform boundedness considering measurement noise.

Not only fault detection and isolation were performed, but a complete fault diagnosis was implemented in real-time, that was capable of estimating multiple faults, satisfying the requirements of the controller proposed. In the application on the Amira system, the effects of the faults were compensated due to the structure of the controller, while the multi-output tracking was performed. Also, a parameter identification was carried out in order to determine the unknown parameters of the system.

Fault-tolerant control leads to a more efficient controller due to the use of an estimation of the fault, in the sense that less energy is spent to overcome the effects of the fault. The energy levels used by the control depend in a great extent on the dynamical behavior and physical nature of the fault.

Finally, it is worth to mention the ease of implementation in real-time of the proposed idea. As can be seen on the experiments performed on the Amira system, the diagnosis led to a successful fault-tolerant multi-output tracking. Fulfilling the needs of the fault-tolerant dynamical controller showed how the effects of the faults were compensated online.

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Fractional fault-tolerant dynamical controller for a class of Commensurate-order Fractional systems

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ABSTRACT
A fault-tolerant control scheme is proposed for a class of commensurate-order fractional nonlinear systems that consists of two fractional-order observers (hybrid scheme). The diagnosis of the faults is performed by means of a model-free fractional proportional integral reduced-order observer that uses the fractional algebraic observability property. It is designed a fractional dynamical controller obtained in a natural way from the dynamics of a fractional high-gain observer, which is constructed from a Fractional Generalized Observability Canonical Form; the controller performs output tracking, thus eliminating the effects of the faults. A stability analysis on the overall system demonstrates that the origin is Mittag-Leffler stable. The proposed methodology is assessed by means of simulations on the fractional models of the Van der Pol oscillator and a DC motor.

KEYWORDS
Fractional fault-tolerant dynamical controller, fractional commensurate-order systems, fractional proportional integral reduced-order observer, fractional high-gain observer, Fractional Generalized Observability Canonical Form.

1. Introduction

Fractional calculus is the mathematical generalization of classical calculus, that comprises integrals and derivatives of real order, let it be rational or irrational, or even complex. This theory has been developed slowly over two centuries, but the first recent monograph devoted to fractional calculus is given by K. B. Oldham and J. Spanier (Oldham & Spanier, 1974), and nowadays a vast amount of literature dedicated to it exists.

On the other hand fractional-order dynamical systems, in contrast to the classical integer-order systems, have been strongly studied in the last decades. This is due to the great amount of applications and physical problems that present dynamics with fractional derivatives and integrals, such as materials science (De Espindola et al., 2005), thermal systems (Gabano & Poinot, 2011), diffusion problems (Oldham & Spanier, 1974), viscoelasticity (Shaw et al., 1997), polymer behavior (Metzler et al., 1995), finance (Scalas et al., 2000), bioengineering (Magin, 2006), damped mechanical systems (Gaul et al., 1991), electrical circuits (Kaczorek & Rogowski, 2015), electromagnetism (Rosales et al., 2011), electromechanics (Yu et al., 2013), etc. In particular, in control theory the generalization of the PID controller and other controllers that involve frac-

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tional dynamics have been developed, such as the CRONE (Sabatier, Oustaloup et al., 2002) and the fractional sliding mode controller (Monje et al., 2010; Podlubny, 1999). Some recent specific studies regarding fractional order systems can be found, for example one that deals with the existence of solutions of a class of fractional stochastic differential equations with delays (Bao & Cao, 2017); other paper proposes an identification algorithm for a time-delay fractional-order system with measurement noise (Gao, 2016); other work deals with the analysis of bifurcations in a delayed fractional complex-valued neural network (Huang et al., 2017); in another article the implementation of some classes of commensurate fractional-order transfer functions is performed with fractional-order capacitors (Tavakoli-Kakhki, 2017); other work investigates the problem of on line parameter estimation for fractional-order linear systems by making an extension of the gradient algorithm (Wei et al., 2015); in another paper an analysis to some approximations of the fractional order Van der Pol Oscillator is performed (Xiao et al., 2017).

Furthermore, systems with fault tolerance, that consists of fault diagnosis (FD) and fault-tolerant control (FTC), are essential in practical and industrial applications, due to the great impact that the effects of the faults have on the correct performance and good conditions of physical equipment. Regarding integer-order systems, there are some surveys involving FD (Alcorta-García & Frank, 2013; Martínez-Guerra & Mata-Machuca, 2014a; Willsky, 1976), while there exist some monographs concerning FTC (Blanke et al., 2003; Patton, 1997). Some recent specific examples of FD and FTC for integer-order systems can be found, for instance one where a FTC approach is designed that is able to simultaneously compensate for actuator faults, model mismatch and parameter variations in aircraft systems (Fekih, 2014); other paper addresses the problem of fault reconstruction and FTC in linear systems subject to actuator faults via learning observers, designing a reconfigurable fault-tolerant controller (Jia et al., 2016); in another work it is proposed a FTC scheme for linear systems that uses the linear fractional transformation and LMI techniques to handle mismatched uncertainties, and adaptive techniques to compensate actuator faults (Liu et al., 2017); in other article reduced-order functional observers are used for state estimation of linear systems with input delays, which has potential applications to fault detection (Mohajerpoor et al., 2015).

Thus, given the increasing number of models with fractional dynamics, and also the importance of developing FD and FTC methods for monitoring the performance of systems, the interest in studying fractional-order systems with faults has emerged. Currently, regarding FD in fractional-order systems various approaches have been proposed; for example, a Generalized Fractional Observer Scheme that allows residual generation is used for fault detection and isolation (Aribi et al., 2013); in another work the generalized dynamic parity space method and the Luenberger diagnosis observer are used for fault detection in fractional-order thermal systems (Aribi et al., 2014); other article proposes an estimation scheme for disturbances and faults using fractional sliding mode observation (Pisano & Usai, 2011); another paper deals with estimation via fractional extended Kalman filter with Lévy noises (Sun et al., 2017). Concerning fractional FTC, some other techniques have been utilized, such as an additive control used for fault-tolerance with the aid of a fractional Luenberger observer (Chouki et al., 2015); in other paper a robust FTC against uncertainties and actuator faults is proposed, where the robustness of the scheme is given via solution of LMIs (Shen et al., 2013); in another work an Auto tuned fractional-order PID controller is used to design a FTC scheme for an Autonomous Underwater Vehicle (Talange & Joshi, 2016).

The aim of this work is to develop an alternative FTC methodology for a class of commensurate-order fractional nonlinear systems; the proposed scheme is hybrid, because it involves two fractional-order observers. First, FD is performed via a fractional proportional reduced-order observer plus an integral term, which has as a main advantage with respect to the approaches given in the literature that it is model-free (it does not require to know the structure of the system), and uses a property called fractional algebraic observability; also, the integral term improves the convergence of the estimation of the faults. Furthermore, for an estimation of the output tracking error a fractional high-gain observer is proposed, inspired by the one from Gauthier et al. for integer order systems (Gauthier et al., 1992); the dynamics of this observer is obtained from the nominal system transformed into a Fractional Generalized Observability Canonical Form (FGOCF). Then a fractional-order dynamical controller is obtained in a natural way, whose purpose is to track desired outputs, hence eliminating the effects of the faults that enter the system. It is worth to note that this controller uses estimations of the faults obtained with the FD. Also, it is proven that the origin of the system in closed-loop with the fractional dynamical controller is Mittag-Leffler stable. Lastly, the methodology proposed is assessed with its application to two commensurate-order fractional models: the Van der Pol oscillator and a DC motor. As far as we know the papers discussed above, as well as the current literature, do not deal with the FD and FTC problems in fractional-order systems as they have been addressed in this work.

The reminder of the paper is organized as follows. Section 2 presents some fractional calculus tools that will be used to develop the proposed method. Section 3 introduces how the fractional FD is performed by means of the fractional proportional integral reduced-order observer. Section 4 shows the main methodology to build the fractional FTC, with the fractional high-gain observer and the fractional dynamical controller. In Section 5 the stability of the origin of the closed-loop system is analyzed. Section 6 describes the applications to the commensurate fractional models of the Van der Pol oscillator and the DC motor. Finally, some concluding remarks are given in Section 7.

2. Fractional calculus tools

In this section are defined briefly some tools from fractional calculus that are used in this work.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f(t)$ is defined as

$$ {}_0I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau $$

where $\Gamma(\cdot)$ is the Gamma function. The operator will be denoted as I^{α} in order to simplify the notation.

On the other hand, there are several definitions of fractional derivatives; the one that is going to be used in this work is the following.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $f(t)$ is defined as

$$ {}_0^C D_t^\alpha f(t) := I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau $$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$, and $f^{(m)}(t)$ is the classical $m-th$ derivative of $f(t)$.

The Caputo derivative is selected since with it, the integer-order derivatives of the initial conditions can be defined, which are adequate for physical systems; also, the Caputo derivative of a constant is zero, as in the integer-order case. So, as it is the only fractional derivative used, the Caputo derivative operator will be denoted henceforth as D^α in order to simplify the notation.

Furthermore, consider the following class of fractional nonlinear systems:

$$ \begin{aligned} D^{\boldsymbol{\alpha}} x(t) &= g(x,u,f) \\ y(t) &= h(x,u) \end{aligned} \tag{1} $$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the available output vector, $u \in \mathbb{R}^m$ is the known input or control vector, $f \in \mathbb{R}^q$ is the unknown input or fault vector, $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_n]^T$ is the vector of fractional orders, and g and h are analytic functions.

Specifically, in this work commensurate-order fractional systems are considered. There are different definitions for these kind of systems; here is going to be used the definition given by I. Petráš (Petráš, 2011).

Definition 2.3. Consider the class of fractional nonlinear systems (1). If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha \in \mathbb{R}$, system (1) is called a commensurate-order system, otherwise it is an incommensurate-order system.

In this work is considered $\alpha \in (0,1)$.

Definition 2.4. (Diethelm, 2010) Let $n_1, n_2 > 0$. The function E_{n_1,n_2} defined by

$$ E_{n_1,n_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kn_1 + n_2)} \tag{2} $$

whenever the series converges, is called the *two-parameter Mittag-Leffler function* with parameters n_1 and n_2.

The Mittag Leffler function is a generalization of the exponential function; in particular, note that $E_{1,1}(z) = e^z$. For this reason, the solution of the fractional order systems and their stability conditions involve this function, as shown in the following definition.

Definition 2.5. (Li et al., 2010) The solution of system $D^\alpha x(t) = f(t,x)$ is said to be Mittag-Leffler stable if

$$ \|x(t)\| \leq \{m[x(0)]E_{\alpha,1}(-\lambda t^\alpha)\}^b $$

$\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz (with Lipschitz constant m_0) on $x \in \mathbb{B}$, an open subset of $\mathbb{R}^n$.

Theorem 2.6. *(Li et al., 2010) Let $x = 0$ be an equilibrium point for the system $D^{\alpha}x(t) = f(t, x)$ and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\begin{aligned} \alpha_1 \|x\|^a \leq V(t, x(t)) &\leq \alpha_2 \|x\|^{ab} \\ D^{\beta} V(t, x(t)) &\leq -\alpha_3 \|x\|^{ab} \end{aligned}$$

where $t \geq 0,\ x \in \mathbb{D},\ \beta \in (0, 1),\ \alpha_1,\ \alpha_2,\ \alpha_3,\ a$ and b arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.

Lemma 2.7. *(Duarte-Mermoud et al., 2015) Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any time instant $t \geq t_0$, the following relationship holds*

$$\frac{1}{2} D^{\alpha}(x^T(t) P x(t)) \leq x^T(t) P D^{\alpha} x(t) \qquad \forall \alpha \in (0, 1)$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, symmetric and positive definite matrix.

In what follows, the fractional FD concepts are introduced.

3. Fractional fault diagnosis

In this section is presented the condition that the faults have to satisfy for being diagnosed, and the main tool to obtain estimates of them.

In order to design a method to reconstruct the faults that appear in the system, it has to be determined if they are diagnosable, but taking into account that the system in question is of commensurate fractional order. Thus, the fractional algebraic observability property is introduced.

Definition 3.1. A variable $\eta_i \in \mathbb{R}$ is said to be fractionally algebraically observable if it is a function of the first $r_1, r_2 \in \mathbb{N}$ sequential fractional derivatives, respectively, of the known input u and the available output y, i.e.,

$$\eta_i = \phi_i(u, D^{\alpha}u, D^{2\alpha}u, ..., D^{r_1\alpha}u, y, D^{\alpha}y, D^{2\alpha}y, ..., D^{r_2\alpha}y) \tag{3}$$

where $\phi_i : \mathbb{R}^{(r_1+1)m} \times \mathbb{R}^{(r_2+1)p} \to \mathbb{R}$.

Remark 3.2. If a variable fits into the above definition, it is said that it satisfies the fractional algebraic observability (FAO) condition. Both states and faults may satisfy this property. In particular, each fault that satisfies the FAO condition is said to be diagnosable.

Once being diagnosed, the faults are reconstructed by means of a fractional proportional integral reduced-order observer (PI$^{r\alpha}$ROO) (Cruz et al., 2015):

$$D^{\alpha} \hat{f}_i = k_{i0}(f_i - \hat{f}_i) + \sum_{j=1}^{r'} k_{ij} I^{j\alpha}(f_i - \hat{f}_i)$$

where $\hat{f}_i$ is an estimation of the fault f_i and the terms $k_{ij} \in \mathbb{R}^+$ determine the convergence rate of the observer.

This observer considers a proportional corrective term of the fault estimation error, and fractional integral terms of the error to improve its convergence. The value r' is chosen in order to fulfill the stability conditions for the observer.

This observer has been chosen because it is model-free (it does not require to know the dynamics of the fault) and it uses the FAO condition defined to estimate the fault. In particular, in this work $r' = 1$ is used, that is to say

$$D^\alpha \hat{f}_i = k_{i0}(f_i - \hat{f}_i) + k_{i1} I^\alpha (f_i - \hat{f}_i) \tag{4}$$

Remark 3.3. The unknown fractional dynamics of fault f_i, i.e. Ω_i, is assumed to be bounded:

$$D^\alpha f_i = \Omega_i \leq \|\Omega_i\| \leq N_i \tag{5}$$

where $N_i \in \mathbb{R}^+$ is an upper bound for the fault dynamics.

Remark 3.4. The real fault f_i cannot be used directly since is unavailable for measurement, so its estimation $\hat{f}_i$ is used in all the existing dynamics. Due to the speed of the PI$^\alpha$ROO, the convergence time can be neglected and the estimation of the faults are used in place of the real faults.

4. Fault-tolerant control

In this section, the main methodology for constructing the FTC is presented. The nominal system is transformed into a fractional canonical form, in order to build an output error tracking fractional observer. Then, the dynamics of the controller is obtained.

The multivariable case is considered. The available outputs of the system are chosen as fractional differential primitive elements. Defining $\eta_{ij} = D^{(i-1)\alpha} y_j$, the class of nonlinear systems (1) can be represented by the following Fractional Generalized Observability Canonical Form (FGOCF) (Martínez-Guerra & Mata-Machuca, 2014b):

$$\begin{aligned} D^\alpha \eta_{ij} &= \eta_{i+1,j}, \qquad 1 \leq i \leq n-1 \\ D^\alpha \eta_{nj} &= -L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha} u, \hat{f}, ..., D^{\mu\alpha} \hat{f}) \\ y_j &= \eta_{1j}, \qquad 1 \leq j \leq p \end{aligned} \tag{6}$$

where L_j is a C^1 real-valued function, $\eta_j = (\eta_{1j}, ..., \eta_{nj})$, $y = (y_1, ..., y_p)$, $u \in \mathbb{R}^m$, $\hat{f} \in \mathbb{R}^q$, and some constants $\gamma, \mu \in \mathbb{R}^+$. Actually, this FGOCF consists in p subsystems of the form (6), one for each output.

The FTC proposed will consist in the tracking of the output error with respect to a given reference $y_R = (y_{R1}, ..., y_{Rp})$, by means of an observer-based dynamical controller built from the fractional canonical form of that error. As it can be seen, the dynamics of the canonical form (6) depends on the faults, so the estimations $\hat{f}_i$ obtained with the FD are used instead.

Let the output tracking error be $e_{1j} = y_j - y_{Rj}$. Given that $\eta_{ij} = D^{(i-1)\alpha} y_j$, the error variables are rewritten as:

$$e_{1j} = \eta_{1j} - y_{Rj}, \qquad 1 \le j \le p$$

Noting that

$$\begin{aligned} D^{\alpha} e_{1j} &= D^{\alpha}\eta_{1j} - D^{\alpha} y_{Rj} \\ &= \eta_{2j} - D^{\alpha} y_{Rj} \end{aligned}$$

this change of variable defines the following fractional canonical form:

$$\begin{aligned} D^{\alpha} e_{ij} &= \eta_{i+1,j} - D^{i\alpha} y_{Rj}, \qquad 1 \le i \le n-1 \\ D^{\alpha} e_{nj} &= D^{\alpha}\eta_{nj} - D^{n\alpha} y_{Rj} = -L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha} y_{Rj} \end{aligned} \tag{7}$$

Now, a linear time-invariant dynamics for the tracking error is imposed:

$$D^{\alpha} e_{nj} + \sum_{i=0}^{n-1} a_{ij} e_{ij} = 0, \qquad a_{ij} > 0 \tag{8}$$

From system (7), (8) is rewritten as:

$$D^{\alpha}\eta_{nj} - D^{n\alpha} y_{Rj} + \sum_{i=1}^{n} a_{ij}\left[\eta_{ij} - D^{(i-1)\alpha} y_{Rj}\right] = 0 \tag{9}$$

that is equivalent to

$$-L_j(\eta_1, ..., \eta_p, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha} y_{Rj} = -\sum_{i=1}^{n} a_{ij}\left[\eta_{ij} - D^{(i-1)\alpha} y_{Rj}\right] \tag{10}$$

A chain of integrators of the error can be obtained as follows:

$$\begin{aligned} D^{\alpha} e_{ij} &= e_{i+1,j}, \qquad 1 \le i \le n-1 \\ D^{\alpha} e_{nj} &= -\sum_{i=1}^{n} a_{ij} e_{ij} \end{aligned} \tag{11}$$

or in a vector form

$$D^{\alpha}\mathbf{e}_j = F_j \mathbf{e}_j \tag{12}$$

and

$$-L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha} y_{rj} = -\sum_{i=1}^{n} a_{ij} e_{ij} \tag{13}$$

where $\mathbf{e}_j = (e_{1j}, ..., e_{nj})$, $\mathbf{y}_{Rj} = (y_{Rj}, D^{\alpha} y_{Rj}, ..., D^{(n-1)\alpha} y_{Rj})$, and

$$F_j = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & \vdots & \dots & 1 \\ -a_{1j} & -a_{2j} & \dots & -a_{nj} \end{pmatrix}$$

System (12) is stable if the following property is satisfied (Petráš, 2011):

$$|\arg(\lambda(F_j))| > \alpha\frac{\pi}{2} \tag{14}$$

Given that dynamics (13) depends on the tracking errors, and only the first one is available for measurement, an observer is used to estimate the rest. Firstly, system (12) is rewritten as:

$$D^{\alpha}\mathbf{e}_j = E\mathbf{e}_j + \varphi_j\left(\mathbf{e}, \mathbf{y}_R, \mathbf{u}, \hat{\mathbf{f}}\right) \tag{15}$$

where $\mathbf{e} = (\mathbf{e}_1, ..., \mathbf{e}_p)$, $\mathbf{y}_R = (\mathbf{y}_{R1}, ..., \mathbf{y}_{Rp})$, $\mathbf{u} = (u, ..., D^{\gamma\alpha}u)$, $\hat{\mathbf{f}} = \left(\hat{f}, ..., D^{\mu\alpha}\hat{f}\right)$, the elements of E are given by:

$$E_{ks} = \begin{cases} 1 & \text{if } k = s-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi_j(\mathbf{e}, \mathbf{y}_R, \mathbf{u}, \mathbf{f}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\mathbf{e}_1 + \mathbf{y}_{R1}, ..., \mathbf{e}_p + \mathbf{y}_{Rp}, u, ..., D^{\gamma\alpha}u, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} \end{pmatrix}$$

The estimation $\hat{\mathbf{e}}_j$ of the tracking error vector is obtained by the following fractional high-gain observer (FHGO):

$$D^{\alpha}\hat{\mathbf{e}}_j = E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, \mathbf{u}, \hat{\mathbf{f}}\right) - S_{\infty}^{-1}C^T C(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{16}$$

where S_{∞} is the solution to the equation:

$$S_{\infty}\left(E + \frac{\theta}{2}I\right) + \left(E^T + \frac{\theta}{2}I\right)S_{\infty} = C^T C \tag{17}$$

with $\theta > 0$ and $C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$. The coefficients of S_{∞} are given by:

$$(S_{\infty})_{ks} = \frac{a_{ks}}{\theta^{k+s-1}}$$

where (a_{ks}) is a symmetric positive definite matrix independent of θ.

Now, the dynamics of the fault-tolerant controllers is obtained from the equation of the tracking error observer (16). Consider the following equation:

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, D^{\gamma_l \alpha}\hat{u}_l, \hat{\mathbf{f}}) - D^{n\alpha} y_{Rj} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij} \tag{18}$$

where $D^{\gamma_l \alpha}\hat{u}_l$ is the highest order fractional derivative of the input found in the equation.

From the Implicit Function Theorem for fractional differential equations (Benchohra & Souid, 2015; Nieto et al., 2015; Tidke & Mahajan, 2017), we obtain:

$$D^{\gamma_l \alpha}\hat{u}_l = K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha} y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \tag{19}$$

with solution $\hat{u}_l$, obtained numerically from a chain of integrators (see (21) and (22)); this variable represents the fault tolerant controller, with $1 \leq l \leq m$.

These controllers yield tracking in the original system, with fault tolerance (elimination of the effects of the faults). So, equation (16) is rewritten as:

$$D^{\alpha}\hat{\mathbf{e}}_j = E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_{\infty}^{-1}C^T C(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{20}$$

with

$$\varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, ..., D^{\gamma\alpha}\hat{u}, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha} y_{Rj} \end{pmatrix}$$

5. Stability analysis of the closed-loop system

In this section, a stability analysis of the origin of the overall system is performed, taking into account the results from fractional calculus.

If the following variables are defined:

$$\hat{u}_{il} = D^{(i-1)\alpha}\hat{u}_l \qquad i = 1, ..., \gamma_l \tag{21}$$

then, considering the controller dynamics (19), the dynamical controller subsystems are written as follows:

$$D^{\alpha}\hat{\mathbf{u}}_l = E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha} y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right), \qquad 1 \leq l \leq m \tag{22}$$

where $\hat{\mathbf{u}}_l = (\hat{u}_{1l}, ..., \hat{u}_{\gamma_l l})$ and

$$\kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha} y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha} y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \end{pmatrix}$$

Now the following variables are defined:

$$\hat{f}_{i\bar{l}} = D^{(i-1)\alpha}\hat{f}_{\bar{l}} \qquad i = 1, ..., \mu_{\bar{l}} \tag{23}$$

then, considering the PI$^\alpha$ROO dynamics (4), the fault estimation subsystems are written as:

$$D^\alpha \hat{\mathbf{f}}_{\bar{l}} = E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}(u, y, f), \qquad 1 \le \bar{l} \le q \tag{24}$$

where $\hat{\mathbf{f}}_{\bar{l}} = \left(\hat{f}_{1\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}\bar{l}}\right)$ and

$$\omega_{\bar{l}}(u, y, f) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ D^{(\mu_{\bar{l}}-1)\alpha}k_{\bar{l}0}(f_{\bar{l}} - \hat{f}_{\bar{l}}) + D^{(\mu_{\bar{l}}-1)\alpha}k_{\bar{l}1}(I^\alpha f_{\bar{l}} - I^\alpha \hat{f}_{\bar{l}}) \end{pmatrix}$$

Hence, the closed-loop dynamics is given by:

$$\begin{aligned} D^\alpha \hat{\mathbf{e}}_j &= E\hat{\mathbf{e}}_j + \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) \\ D^\alpha \hat{\mathbf{u}}_l &= E\hat{\mathbf{u}}_l + \kappa_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ D^\alpha \hat{\mathbf{f}}_{\bar{l}} &= E\hat{\mathbf{f}}_{\bar{l}} + \omega_{\bar{l}}(u, y, f) \end{aligned} \tag{25}$$

for $1 \le j \le p$, $1 \le l \le m$ and $1 \le \bar{l} \le q$. Writing explicitly these equations, the following chain of integrators is obtained:

$$\begin{aligned} D^\alpha \hat{e}_{ij} &= \hat{e}_{i+1,j} - \psi_i(\theta_j)(\hat{e}_j - e_j) \qquad 1 \le i \le n-1 \\ D^\alpha \hat{e}_{nj} &= -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_{R1}, ..., \hat{\mathbf{e}}_p + \mathbf{y}_{Rp}, \hat{u}, ..., D^{\gamma\alpha}\hat{u}, \hat{f}, ..., D^{\mu\alpha}\hat{f}) - D^{n\alpha}y_{Rj} - \theta_j^n \\ &\quad 1 \le j \le p \\ D^\alpha \hat{u}_{il} &= \hat{u}_{i+1,l} \qquad 1 \le i \le \gamma_l - 1 \\ D^\alpha \hat{u}_{\gamma_l,l} &= K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, D^{n\alpha}y_{Rj}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \qquad 1 \le l \le m \\ D^\alpha \hat{f}_{i,\bar{l}} &= \hat{f}_{i+1,\bar{l}} \qquad 1 \le i \le \mu_{\bar{l}} - 1 \\ D^\alpha \hat{f}_{\mu_{\bar{l}},\bar{l}} &= D^{(\mu_{\bar{l}}-1)\alpha}k_{\bar{l}0}(f_{\bar{l}} - \hat{f}_{\bar{l}}) + D^{(\mu_{\bar{l}}-1)\alpha}k_{\bar{l}1}(I^\alpha f_{\bar{l}} - I^\alpha \hat{f}_{\bar{l}}) \qquad 1 \le \bar{l} \le q \end{aligned}$$

where $\psi_i(\theta_j)$ is a function obtained from S_∞^{-1}.

In this chain of integrators, the dynamics of the controllers and the fault estimations can be appreciated. As it can be seen, the variables obtained from these dynamics take part explicitly in the tracking error dynamics, leading to the solution of the tracking problem.

Moreover, define the observation error as $\varepsilon_j = \hat{\mathbf{e}}_j - \mathbf{e}_j$, and the following dynamics is obtained from equations (16) and (20):

$$D^\alpha \varepsilon_j = \left(E - S_\infty^{-1}C^TC\right)\varepsilon_j + \Phi_j(\varepsilon, \hat{\mathbf{e}}) \tag{26}$$

where

$$\Phi_j(\varepsilon, \hat{\mathbf{e}}) = \varphi_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) - \varphi_j\left(\hat{\mathbf{e}} - \varepsilon, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right)$$

Now the main result of this paper is stated.

Theorem 5.1. *Let system (1) be described in the FGOCF (6) composed of p subsystems. Let the observation dynamics corresponding to subsystem j be composed of $\hat{\mathbf{e}}_j$ and ε_j, for $1 \leq j \leq p$. Let $f_{\bar{l}}$ be diagnosable and estimated by means of the dynamics of $\hat{f}_{\bar{l}}$, for $1 \leq \bar{l} \leq q$. Let $\hat{u}_l$ be the solution to*

$$-L_j(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, D^{\gamma_l \alpha}\hat{u}_l, \hat{\mathbf{f}}) - y_{Rj}^{(n\alpha)} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij}$$

for $1 \leq l \leq m$. Then, the origin of the closed-loop system (25) is Mittag-Leffler stable.

Proof. Consider the following Lyapunov function candidate:

$$V\left(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}\right) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j + \varepsilon_j^T S_\infty \varepsilon_j + \tilde{f}_{\bar{l}}^2 \tag{27}$$

where $\tilde{f}_{\bar{l}} = f_{\bar{l}} - \hat{f}_{\bar{l}}$ is the fault estimation error. Define $V_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j$, $V_2(\varepsilon_j) = \varepsilon_j^T S_\infty \varepsilon_j$ and $V_3(\tilde{f}_{\bar{l}}) = \tilde{f}_{\bar{l}}^2$. P, S_∞ are constant, square, symmetric, positive definite matrices and solutions of $F^T P + PF = -I$ and (17) respectively. Let $\|x\|_P = \sqrt{x^T P x}$ and $\|x\|_{S_\infty} = \sqrt{x^T S_\infty x}$. Note that $V\left(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}\right)$ satisfies the first inequality of Theorem 2.6 since:

$$\begin{array}{rcccl} \alpha_{11}\|\hat{\mathbf{e}}_j\| & \leq & V_1(\hat{\mathbf{e}}_j) & \leq & \alpha_{21}\|\hat{\mathbf{e}}_j\| \\ \alpha_{12}\|\varepsilon_j\| & \leq & V_2(\varepsilon_j) & \leq & \alpha_{22}\|\varepsilon_j\| \\ \alpha_{13}\|\tilde{f}_{\bar{l}}\| & \leq & V_3(\tilde{f}_{\bar{l}}) & \leq & \alpha_{23}\|\tilde{f}_{\bar{l}}\| \end{array} \tag{28}$$

with $\alpha_{11} = \lambda_{\min}(P)$, $\alpha_{12} = \lambda_{\min}(S_\infty)$, $\alpha_{13} = 1$, $\alpha_{21} = \frac{1}{2}(\lambda_{\min}(P) + \lambda_{\max}(P))$, $\alpha_{22} = \frac{1}{2}(\lambda_{\min}(S_\infty) + \lambda_{\max}(S_\infty))$, $\alpha_{23} = \sup(\|\tilde{f}_{\bar{l}}\|)$, and $a = b = 1$.

By the linearity property of the Caputo Derivative and using Lemma 2.7, it follows that

$$D^\alpha V = D^\alpha V_1 + D^\alpha V_2 + D^\alpha V_3 \leq 2\hat{\mathbf{e}}_j^T P D^\alpha \hat{\mathbf{e}}_j + 2\varepsilon_j^T S_\infty D^\alpha \varepsilon_j + 2\tilde{f}_{\bar{l}} D^\alpha \tilde{f}_{\bar{l}} \tag{29}$$

On the other hand note that:

$$\begin{array}{rcl} D^\alpha V_1(\hat{\mathbf{e}}_j) & \leq & 2\hat{\mathbf{e}}_j^T P D^\alpha \hat{\mathbf{e}}_j = 2\hat{\mathbf{e}}_j^T P\left[F\hat{\mathbf{e}}_j - S_\infty^{-1} C^T C \varepsilon_j\right] \\ & = & \hat{\mathbf{e}}_j^T\left(F^T P + PF\right)\hat{\mathbf{e}}_j - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C \varepsilon_j \\ & = & -\|\hat{\mathbf{e}}_j\|^2 - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C S_\infty^{-1} S_\infty \varepsilon_j \\ & \leq & -\|\hat{\mathbf{e}}_j\|^2 + \bar{K}\|\hat{\mathbf{e}}_j\|_{P^*}\|\varepsilon_j\|_{S_\infty^*}\rho(\theta) \\ & \leq & -\left(1 - \bar{K} d_1 d_2 \|\varepsilon_j\| \rho(\theta)\right)\|\hat{\mathbf{e}}_j\| \end{array}$$

where $\bar{K} > 0$, $P^* = PP^T$, $S_\infty^* = S_\infty S_\infty^T$, $d_1 = \sqrt{\lambda_{\max}(P^2)}$, $d_2 = \sqrt{\lambda_{\max}(S_\infty^2)}$ and $\rho(\theta) = \|S_\infty^{-1} C^T C S_\infty^{-1}\|$.

Then, we obtain

$$D^\alpha V_1(\hat{\mathbf{e}}_j) \leq -\delta_{31}\|\hat{\mathbf{e}}_j\| \tag{30}$$

with $\delta_{31} = 1 - \bar{K} d_1 d_2 \|\varepsilon_j\| \rho(\theta)$.

In a similar way:

$$\begin{aligned}
D^\alpha V_2(\varepsilon_j) &\leq 2\varepsilon_j^T S_\infty D^\alpha \varepsilon_j = 2\varepsilon_j^T S_\infty \left[E_\theta \varepsilon_j + \Phi_j(\varepsilon, \hat{\mathbf{e}})\right] \\
&= \varepsilon_j^T \left[E^T S_\infty + S_\infty E - C^T C\right] \varepsilon_j - \varepsilon_j^T C^T C \varepsilon_j + 2\varepsilon_j^T S_\infty \Phi_j(\varepsilon, \hat{\mathbf{e}}) \\
&\leq -\theta\|\varepsilon_j\|_{S_\infty}^2 + 2\varepsilon_j^T S_\infty \Phi_j(\varepsilon, \hat{\mathbf{e}}) \ \ (\text{since } \|\Phi_j(\varepsilon, \hat{\mathbf{e}})\|_{S_\infty}^2 \leq \bar{\lambda}\|\varepsilon_j\|_{S_\infty}^2,\ \bar{\lambda} = \gamma^2) \\
&\leq -\theta\lambda_{\min}(S_\infty)\|\varepsilon_j\|^2 + 2\gamma\|\varepsilon_j\|_{S_\infty}^2 \\
&\leq -\left(\theta\lambda_{\min}(S_\infty) - 2\gamma\lambda_{\max}(S_\infty)\right)\|\varepsilon_j\|
\end{aligned}$$

Then, we obtain

$$D^\alpha V_2(\varepsilon_j) \leq -\delta_{32}\|\varepsilon_j\| \tag{31}$$

with $\delta_{32} = \theta\lambda_{\min}(S_\infty) - 2\gamma\lambda_{\max}(S_\infty)$.

Finally:

$$\begin{aligned}
D^\alpha V_3(\tilde{f}_{\bar{l}}) &\leq 2\tilde{f}_{\bar{l}} D^\alpha \tilde{f}_{\bar{l}} = 2\tilde{f}_{\bar{l}}\left(\Omega_{\bar{l}} - k_{\bar{l}0}\tilde{f}_{\bar{l}} - k_{\bar{l}1} I^\alpha \tilde{f}_{\bar{l}}\right) \\
&\leq 2\tilde{f}_{\bar{l}}\Omega_{\bar{l}} - 2k_{\bar{l}1}\tilde{f}_{\bar{l}} I^\alpha \tilde{f}_{\bar{l}} \\
&\leq 2N_{\bar{l}}\|\tilde{f}_{\bar{l}}\| - 2k_{\bar{l}1}\|\tilde{f}_{\bar{l}}\||I^\alpha \tilde{f}_{\bar{l}}| \\
&\leq -\left(2k_{\bar{l}1}|I^\alpha \tilde{f}_{\bar{l}}| - 2N_{\bar{l}}\right)\|\tilde{f}_{\bar{l}}\|
\end{aligned}$$

Then, we obtain

$$D^\alpha V_3(\tilde{f}_{\bar{l}}) \leq -\delta_{33}\|\tilde{f}_{\bar{l}}\| \tag{32}$$

with $\delta_{33} = 2k_{\bar{l}1}|I^\alpha \tilde{f}_{\bar{l}}| - 2N_{\bar{l}}$.

Therefore, if $\delta_{31},\ \delta_{32},\ \delta_{33} > 0$, from Theorem 2.6 and equations (28-32), it is concluded that the origin of system (25) is Mittag-Leffler stable. □

6. Application to commensurate-order fractional systems

In this section the proposed methodology is assessed by means of numerical simulations in two commensurate-order fractional systems.

6.1. *Van der Pol oscillator*

Consider the modified version of the fractional-order Van der Pol oscillator (Petráš, 2011):

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= -x_1 - \varepsilon(x_1^2 - 1)x_2 \end{aligned}$$

Adding a control input and a fault and selecting the first state as a measurable output to be controlled, the system to work with is:

$$\begin{aligned} D^\alpha x_1 &= x_2 + u \\ D^\alpha x_2 &= -x_1 - \varepsilon(x_1^2 - 1)x_2 + f \\ y &= x_1 \end{aligned} \tag{33}$$

In this case, the control aim is tracking of a desired trajectory by the output (in this case the first state), thus making the oscillator to present a desired chaotic behavior in the phase plane, even in the presence of faults.

6.1.1. Fault diagnosis

First, the PI$^\alpha$ROO for reconstructing the fault is designed, for which it must be determined if f is diagnosable, i.e. if it satisfies the FAO condition. From (33) the following polynomial is obtained:

$$f = D^\alpha x_2 + y + \varepsilon(y^2 - 1)x_2 \tag{34}$$

It can be observed that state x_2 satisfies also the FAO condition. From (33):

$$x_2 = D^\alpha y - u \tag{35}$$

thus, state x_2 is fractionally algebraically observable and can be reconstructed. Hence, the fault is fractionally diagnosable. First, a PI$^\alpha$ROO for estimating x_2 is designed:

$$D^\alpha \hat{x}_2 = k_{10}(D^\alpha y - u - \hat{x}_2) + k_{11} I^\alpha (D^\alpha y - u - \hat{x}_2)$$

Defining an auxiliary variable γ_1 as:

$$\gamma_1 = \hat{x}_2 - k_{10} y$$

the PI$^\alpha$ROO to obtain the estimation of the state is:

$$D^\alpha \gamma_1 = -k_{10}\left(u + \gamma_1 + k_{10} y\right) + k_{11} y - k_{11} I^\alpha \left(u + \gamma_1 + k_{10} y\right) \tag{36}$$

$$\hat{x}_2 = \gamma_1 + k_{10} y \tag{37}$$

Now, the PI$^\alpha$ROO for estimating the fault is designed:

$$D^\alpha \hat{f} = k_{20}(D^\alpha \hat{x}_2 + y + \varepsilon(y^2 - 1)\hat{x}_2 - \hat{f}) + k_{21} I^\alpha (D^\alpha \hat{x}_2 + y + \varepsilon(y^2 - 1)\hat{x}_2 - \hat{f})$$

Defining an auxiliary variable γ_2 as

$$\gamma_2 = \hat{f} - k_{20}\hat{x}_2$$

the PI$^\alpha$ROO to obtain the estimation of the fault is:

$$\begin{aligned} D^\alpha\gamma_2 &= -k_{20}(-y - \varepsilon(y^2-1)\hat{x}_2 + \gamma_2 + k_{20}\hat{x}_2) + k_{21}\hat{x}_2 \\ &\quad -k_{21}I^\alpha(-y - \varepsilon(y^2-1)\hat{x}_2 + \gamma_2 + k_{20}\hat{x}_2) \end{aligned} \tag{38}$$

$$\hat{f} = \gamma_2 + k_{20}\hat{x}_2 \tag{39}$$

6.1.2. Fault-tolerant control

Now, a fractional fault-tolerant controller is designed from the FGOCF, obtaining the following:

$$\begin{aligned} D^\alpha y &= x_2 + u \\ D^{2\alpha}y &= D^\alpha x_2 + D^\alpha u \\ &= -y - \varepsilon(y^2-1)x_2 + D^\alpha u + f \end{aligned} \tag{40}$$

So the tracking error is described in a canonical form:

$$\begin{aligned} e_1 &= y - y_R \\ D^\alpha e_1 &= D^\alpha y - D^\alpha y_R = e_2 \\ D^\alpha e_2 &= D^{2\alpha}e_1 = D^{2\alpha}y - D^{2\alpha}y_R \end{aligned} \tag{41}$$

and a FHGO can be obtained:

$$\begin{aligned} D^\alpha\hat{e}_1 &= \hat{e}_2 - 2\theta(\hat{e}_1 - e_1) \\ D^\alpha\hat{e}_2 &= D^{2\alpha}y - D^{2\alpha}y_R - \theta^2(\hat{e}_1 - e_1) \\ &= -y - \varepsilon(y^2-1)x_2 + D^\alpha u + \hat{f} - D^{2\alpha}y_R - \theta^2(\hat{e}_1 - e_1) = -\sum_{i=1}^{2} a_i\hat{e}_i \end{aligned} \tag{42}$$

Finally, from equation (42), the dynamics of the fractional fault-tolerant controller is obtained:

$$D^\alpha\hat{u} = -a_1\hat{e}_1 - a_2\hat{e}_2 + y + \varepsilon(y^2-1)\hat{x}_2 - \hat{f} + D^{2\alpha}y_R \tag{43}$$

6.1.3. Simulation results

Simulations were performed over 60 seconds in Matlab-Simulink® in the model of the system. The value $\alpha = 0.9$ is selected. The reference is set as $y_R(t) = 2sin(t)$. The fault is set to be $f(t) = cos(t)$ beginning at 20 seconds. The value of the parameter of the system is $\varepsilon = 0.1$. The design parameters (gains) are chosen as $\theta = 20$, $a_1 = 400$, $a_2 = 40$, $k_{10} = k_{20} = 30$ and $k_{11} = k_{21} = 1$.

Figure 1 shows the FD results with the PI$^\alpha$ROO. It can be seen that the estimated fault follows the signal of the real fault in a very short time. The performance index

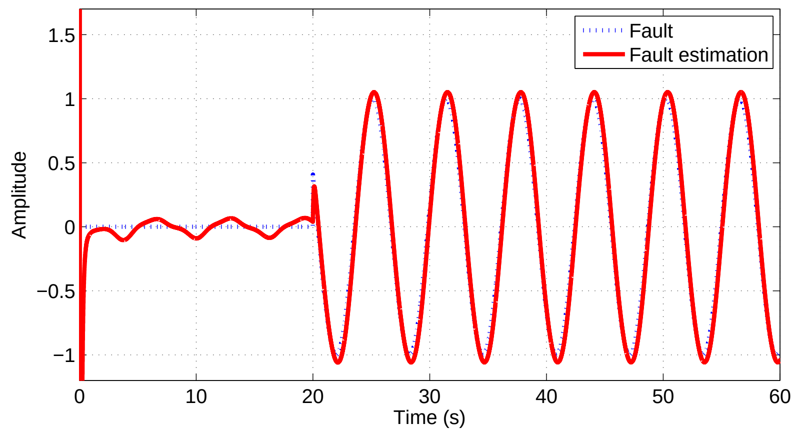

Figure 1. Fault diagnosis for the Van der Pol system.

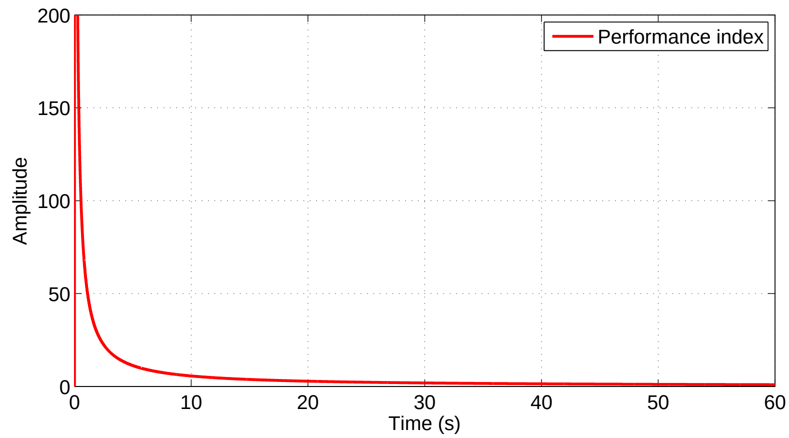

Figure 2. Performance index of the fault diagnosis for the Van der Pol system.

of the PI$^{\alpha}$ROO was evaluated using the following cost functional:

$$J_t = \frac{1}{t+\epsilon}\int_0^t \left\|\tilde{f}\right\|^2 dt$$

where $\epsilon = 0.0001$. The performance index is shown in Fig. 2; it can be seen that the diagnosis error has a diminishing magnitude even in the presence of the fault. Furthermore, Fig. 3 shows the signal of the fractional dynamical controller that yields the output tracking. Fig. 4 shows the signal of the output y with FTC using the fractional dynamical controller designed. It can be seen that the output follows the reference approximately in 1 second, and when the fault appears, its effects are eliminated immediately. Fig. 5 shows the phase portrait of the Van der Pol oscillator with $y = x_1$ and $\hat{x}_2$. Finally, Fig. 6 shows a scheme of the Van der Pol system in closed-loop with the fractional fault diagnosis observer and the fractional fault-tolerant dynamical controller (Matlab-Simulink$^{\circledR}$).

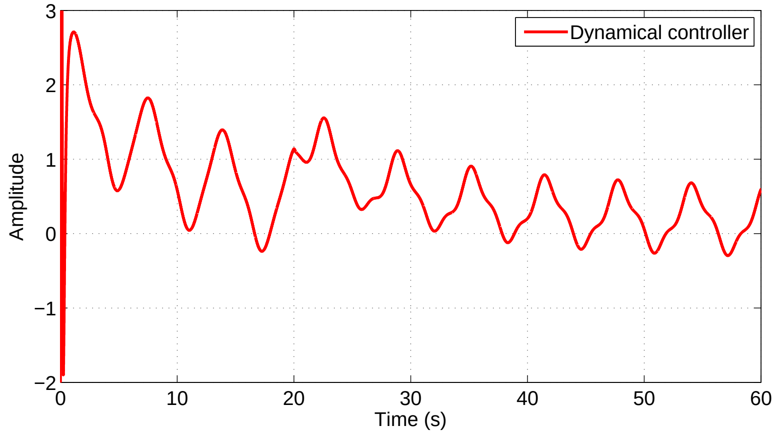

Figure 3. Fault-tolerant dynamical controller for the Van der Pol system.

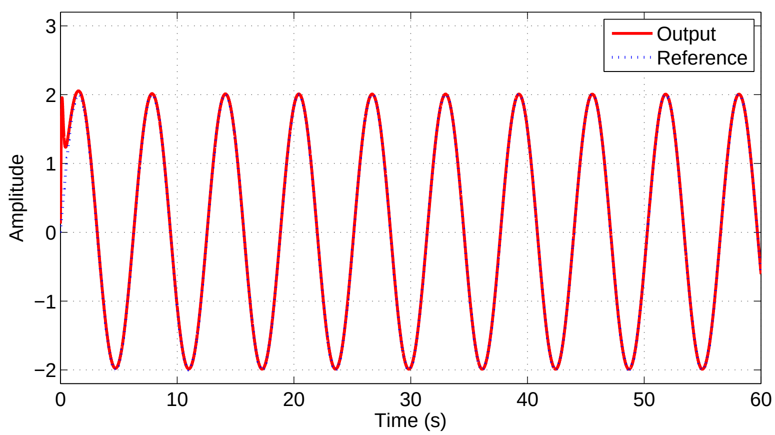

Figure 4. Output tracking for the Van der Pol system.

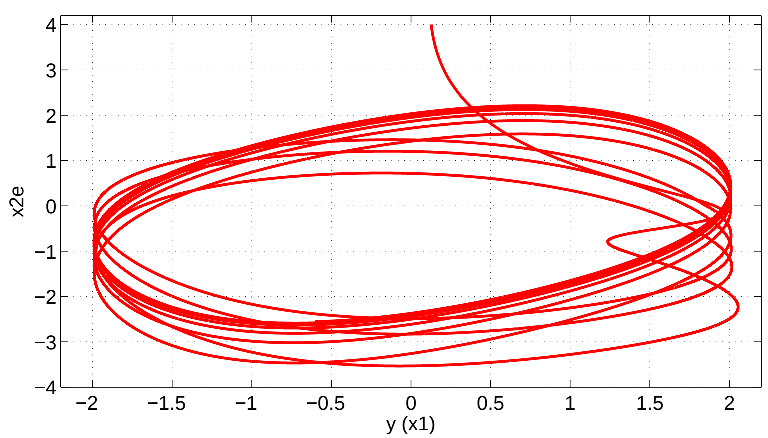

Figure 5. Phase portrait of the Van der Pol system.

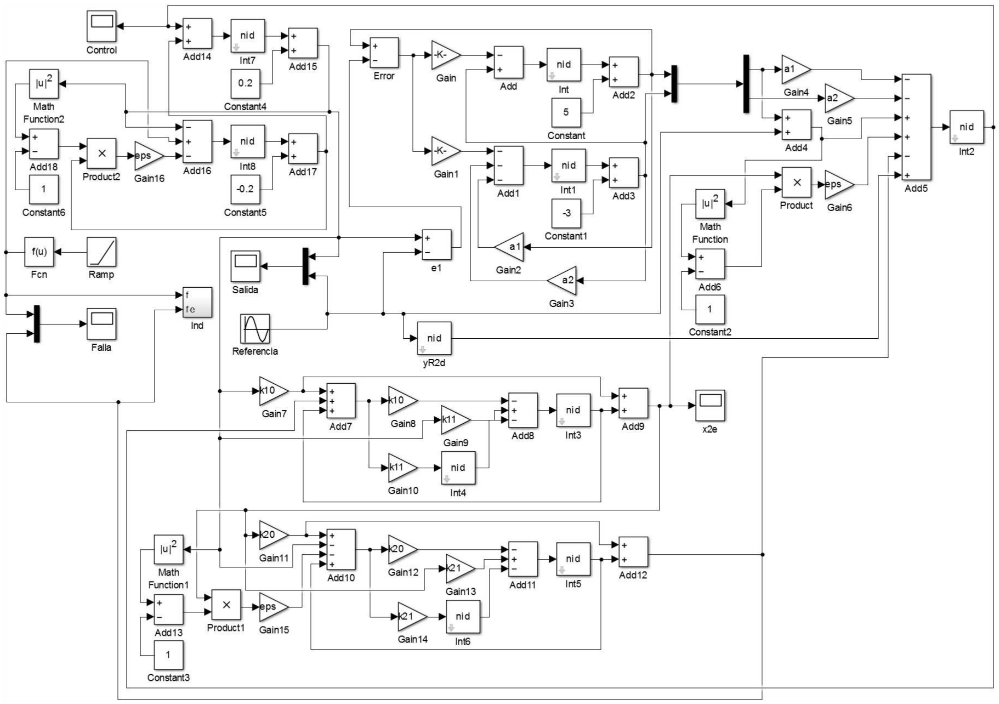

Figure 6. Closed-loop system of the Van der Pol oscillator (Matlab-Simulink®).

6.2. *Model of a DC motor*

Here is presented a fractional model of a DC motor:

$$
\begin{aligned}
D^{\alpha}x(t) &= \omega(t) \\
D^{\alpha}\omega(t) &= \frac{1}{J}\left[c\phi i_a(t) - T_L\right] \\
D^{\alpha}i_a(t) &= \frac{1}{L_a}\left[V_a - R_a i_a(t) - c\phi\omega(t)\right]
\end{aligned} \tag{44}
$$

The variables of the system are given by

Symbol	Variable	Units
V_a	Armature voltage	V
i_a	Armature current	A
ω	Angular velocity	rpm

The parameters of the system are as follows:

Symbol	Parameter	Units
R_a	Armature resistance	Ω
L_a	Armature inductance	H
ϕ	Magnetic flux	Vs
J	Total moment of inertia	kgm^2
T_L	Load torque	Nm

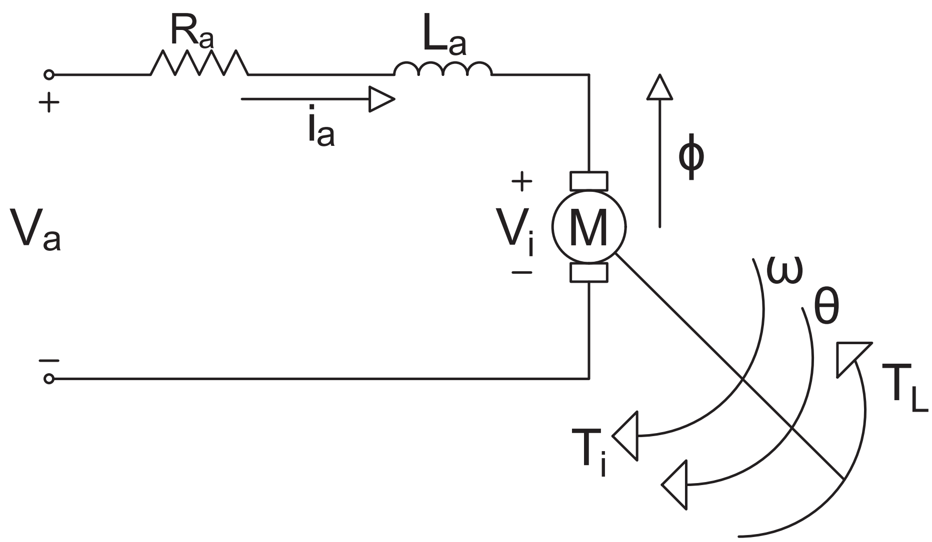

Figure 7. DC Motor.

Besides

$$c\phi\omega(t) = V_i \qquad c\phi i_a(t) = T_i$$

where $c\phi$ is a motor constant, V_i is the induced voltage and T_i is the electromagnetic torque. Futhermore, consider an state $x(t)$ such that $I^\alpha \omega = x$ (augmented system of (Cipin et al., 2013)).

Remark 6.1. If α is set as 1 in this integral, x would be the angular position, the integral of ω. Since $\alpha \neq 1$, the variable x does not represent the angular position, but only the fractional-order integral of ω.

Then, the state-space variables are chosen as $x_1 = x$, $x_2 = \omega$, $x_3 = i_a$, $u = V_a$ and $y = \omega$. Also, consider an additive fault f coupled to the input. So, the model to be used is:

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ D^\alpha x_3 &= \frac{1}{L_a}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ y &= x_2 \end{aligned} \tag{45}$$

It can be seen that the control aim is to maintain the motor speed in the nominal value, even in the presence of faults, which in this case are variations in the input voltage.

6.2.1. *Fault diagnosis*

Now, the PI$^\alpha$ROO for reconstructing the fault is designed, but first it must be determined if f is diagnosable, i.e. if it satisfies the FAO condition. From equation (45) the following polynomial is obtained:

$$f = c\phi y + R_a x_3 + L_a D^\alpha x_3 - u \tag{46}$$

It can be observed that state x_3 satisfies also the FAO condition. From (45):

$$x_3 = \frac{1}{c\phi}\left[J D^\alpha y + T_L\right] \tag{47}$$

thus, state x_3 is fractionally algebraically observable and can be reconstructed. Hence, the fault is fractionally diagnosable. Now, the PI$^\alpha$ROO is designed:

$$\begin{aligned} D^\alpha \hat{f} &= k_{10}(f - \hat{f}) + k_{11} I^\alpha (f - \hat{f}) \\ &= k_{10}(c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \end{aligned} \tag{48}$$

In order to deal with the fractional derivative of $\hat{x}_3$ in the proportional term, the auxiliary variable γ_1 is proposed and the estimated fault is defined as:

$$\hat{f} = \gamma_1 + k_{10} L_a \hat{x}_3 \tag{49}$$

and

$$\begin{aligned} D^\alpha \gamma_1 &= D^\alpha \hat{f} - k_{10} L_a D^\alpha \hat{x}_3 \\ &= k_{10}(c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \hat{f}) - k_{10} L_a D^\alpha \hat{x}_3 \end{aligned} \tag{50}$$

So, the PI$^\alpha$ROO to estimate the fault is:

$$\begin{aligned} D^\alpha \gamma_1 &= k_{10}(c\phi y + R_a \hat{x}_3 - u - \gamma_1 - k_{10} L_a \hat{x}_3) \\ &\quad + k_{11} I^\alpha (c\phi y + R_a \hat{x}_3 + L_a D^\alpha \hat{x}_3 - u - \gamma_1 - k_{10} L_a \hat{x}_3) \\ \hat{f} &= \gamma_1 + k_{10} L_a \hat{x}_3 \end{aligned} \tag{51}$$

where $\hat{x}_3$ is given by:

$$\hat{x}_3 = \frac{1}{c\phi}\left[J D^\alpha y + T_L\right] \tag{52}$$

In order to estimate the α-order derivative of y, the variable $\xi = D^\alpha y$ is defined and the following PI$^\alpha$ROO is proposed:

$$\begin{aligned} D^\alpha \hat{\xi} &= k_{\xi 0}(\xi - \hat{\xi}) + k_{\xi 1} I^\alpha (\xi - \hat{\xi}) \\ &= k_{\xi 0}(D^\alpha y - \hat{\xi}) + k_{\xi 1} I^\alpha (D^\alpha y - \hat{\xi}) \end{aligned} \tag{53}$$

Now the auxiliary variable γ_ξ is introduced and the estimated variable $\hat{\xi}$ is defined as:

$$\hat{\xi} = \gamma_\xi + k_{\xi 0} y \tag{54}$$

Then

$$\begin{aligned} D^\alpha \gamma_\xi &= D^\alpha \hat{\xi} - k_{\xi 0} D^\alpha y \\ &= -k_{\xi 0}\hat{\xi} + k_{\xi 1} I^\alpha (D^\alpha y - \hat{\xi}) \\ &= k_{\xi 0}\left(-\gamma_\xi - k_{\xi 0} y\right) + k_{\xi 1} I^\alpha (D^\alpha y - \gamma_\xi - k_{\xi 0} y) \end{aligned} \tag{55}$$

Hence, the estimation of $\hat{x}_3$ is obtained with:

$$\hat{x}_3 = \frac{1}{c\phi}\left[J\hat{\xi} + T_L\right] \tag{56}$$

6.2.2. Fault-tolerant control

Now, a fractional fault-tolerant controller is designed from the FGOCF, obtaining the following:

$$\begin{aligned} D^\alpha y &= \frac{1}{J}\left[c\phi x_3 - T_L\right] \\ D^{2\alpha} y &= \frac{c\phi}{J} D^\alpha x_3 = \frac{c\phi}{L_a J}\left[-c\phi x_2 - R_a x_3 + u + f\right] \\ D^{3\alpha} y &= \frac{c\phi}{L_a J}\left[-c\phi D^\alpha x_2 - R_a D^\alpha x_3 + D^\alpha u + D^\alpha f\right] \\ &= \frac{c\phi}{L_a J}\left[R_a c\phi y - c\phi D^\alpha y + R_a^2 x_3 - R_a u - R_a f + D^\alpha u + D^\alpha f\right] \end{aligned} \tag{57}$$

So the tracking error is described in a canonical form:

$$\begin{aligned} e_1 &= y - y_R \\ D^\alpha e_1 &= D^\alpha y - D^\alpha y_R = e_2 \\ D^\alpha e_2 &= D^{2\alpha} e_1 = D^{2\alpha} y - D^{2\alpha} y_R \\ D^\alpha e_3 &= D^{3\alpha} e_1 = D^{3\alpha} y - D^{3\alpha} y_R \end{aligned} \tag{58}$$

and a FHGO can be obtained:

$$\begin{aligned} D^\alpha \hat{e}_1 &= \hat{e}_2 - 3\theta\left(\hat{e}_1 - e_1\right) \\ D^\alpha \hat{e}_2 &= \hat{e}_3 - 3\theta^2\left(\hat{e}_1 - e_1\right) \\ D^\alpha \hat{e}_3 &= \frac{c\phi}{L_a J}\left[R_a c\phi y - c\phi D^\alpha y + R_a^2 \hat{x}_3 - R_a \hat{u} - R_a \hat{f} + D^\alpha \hat{u} + D^\alpha \hat{f}\right] - D^{3\alpha} y_R \\ & -\theta^3\left(\hat{e}_1 - e_1\right) = -\sum_{i=1}^{3} a_i \hat{e}_i \end{aligned} \tag{59}$$

From equation (59), the dynamics of the fractional fault-tolerant controller is obtained:

$$D^{\alpha}\hat{u} = \frac{L_a J}{c\phi}\left(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + D^{3\alpha}y_R\right) - R_a c\phi y + c\phi y^{(\alpha)} - R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - D^{\alpha}\hat{f} \tag{60}$$

This controller eliminates the effects of the fault in the system; however, it can be seen that an estimation of the α-order derivative of y is needed. Thus, the dynamics of the fault-tolerant controller is the following:

$$D^{\alpha}\hat{u} = \frac{L_a J}{c\phi}\left(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + D^{3\alpha}y_R\right) - R_a c\phi y + c\phi\hat{\xi} - R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - D^{\alpha}\hat{f} \tag{61}$$

Remark 6.2. Note that the dynamics of $D^{\alpha}\hat{f}$ is obtained from the dynamics of the PI$^{\alpha}$ROO used for FD.

6.2.3. Simulation results

Simulations were performed over 20 seconds in the model of the system, with the following values of the parameters:

R_a	2.13 Ω
L_a	0.00484 H
$c\phi$	0.0683 Vs
J	0.0001148 kgm^2
T_L	0.0608 Nm
V_a	12 V

The value $\alpha = 0.9$ is selected. The reference is set as $y_R = 177$ rpm. The fault is set to be $f = 0.1V_a$ beginning at 10 seconds. The dimensionless design parameters (gains) are chosen as $\theta = 2000$, $a_1 = 8000$, $a_2 = 1200$, $a_3 = 60$, $k_{10} = k_{\xi 0} = 20$ and $k_{11} = k_{\xi 1} = 100$.

Figure 8 shows the FD results with the PI$^{\alpha}$ROO. It can be seen that the estimated fault follows the signal of the real fault in a short time. The performance index of the PI$^{\alpha}$ROO was evaluated using the same cost functional:

$$J_t = \frac{1}{t+\epsilon}\int_0^t \left\|\tilde{f}\right\|^2 dt$$

with $\epsilon = 0.0001$. The performance index is shown in Fig. 9; it can be seen that the diagnosis error has a small magnitude even in the presence of the fault. Furthermore, Fig. 10 shows the signal of the fractional dynamical controller that yields the output tracking. Fig. 11 shows the signal of the output y with FTC using the fractional dynamical controller. It can be seen that the system follows the reference in 3 seconds, and when the fault appears, its effects are eliminated in approximately 2 seconds. Finally, Fig. 12 shows a scheme of the motor system in closed-loop with the fractional fault diagnosis observer and the fractional fault-tolerant dynamical controller (Matlab-Simulink®).

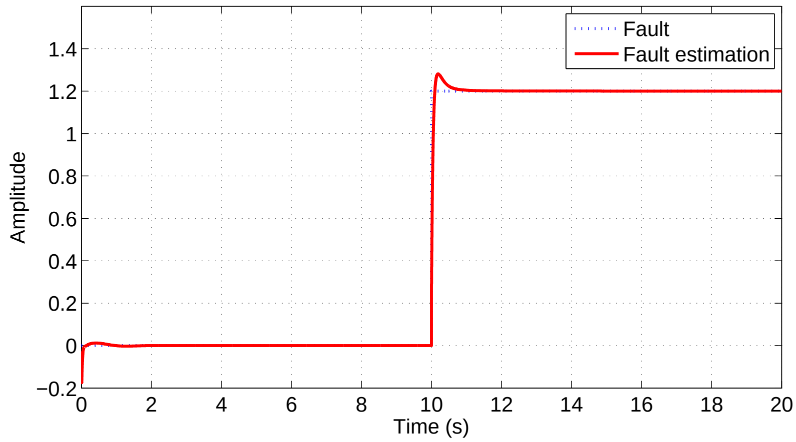

Figure 8. Fault diagnosis for the DC motor.

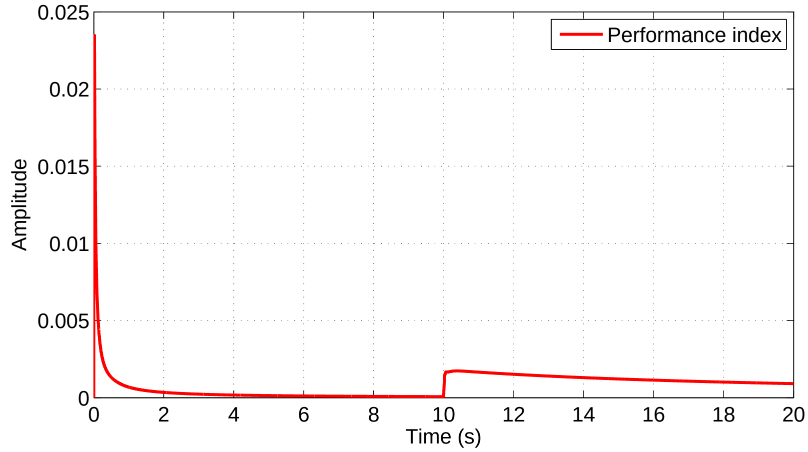

Figure 9. Performance index of the fault diagnosis for the DC motor.

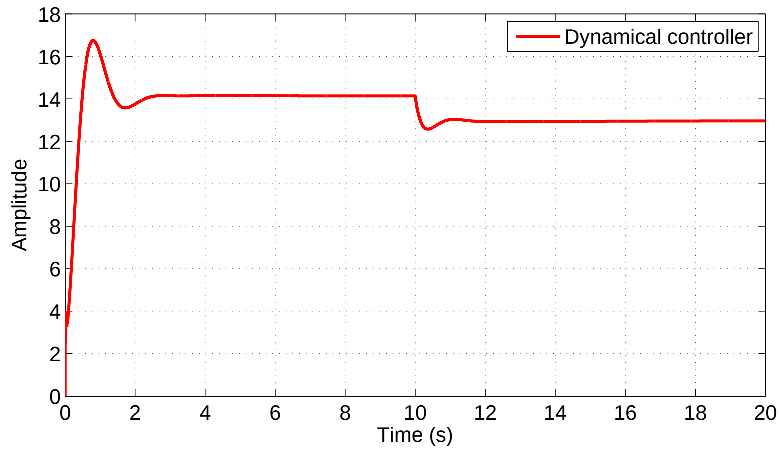

Figure 10. Fault-tolerant dynamical controller for the DC motor.

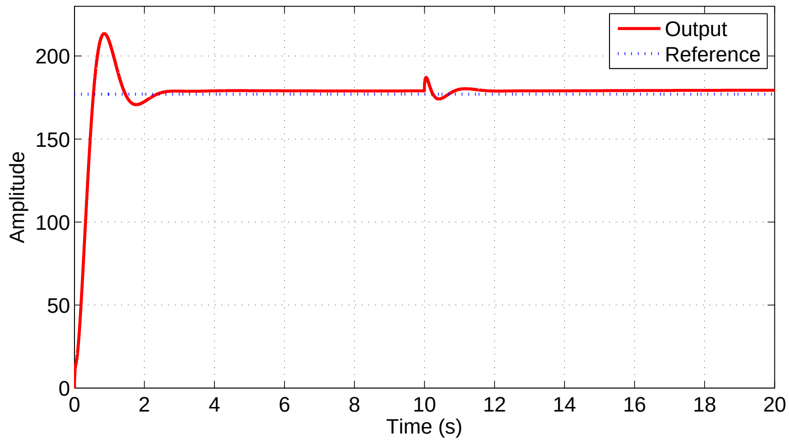

Figure 11. Output tracking for the DC motor.

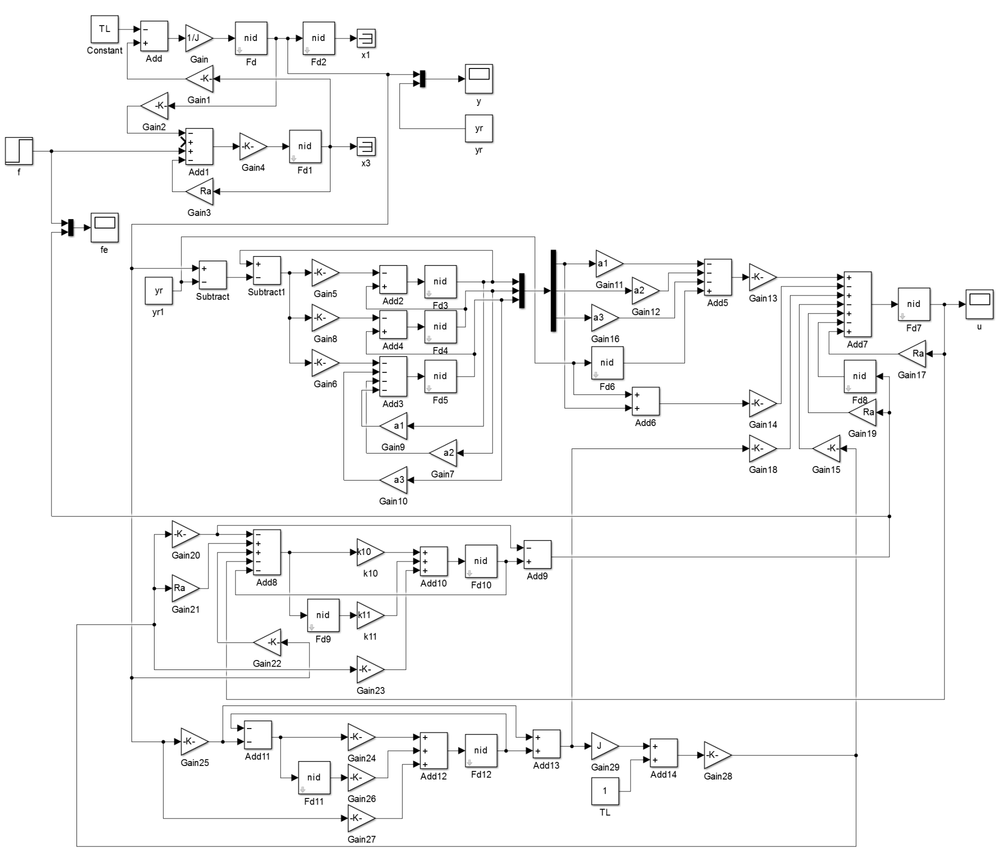

Figure 12. Closed-loop system of the DC motor (Matlab-Simulink®).

7. Concluding remarks

In this work, a fault-tolerant control scheme was proposed for a class of commensurate-order fractional nonlinear systems. This FTC scheme is hybrid, in the sense that it consists of a fractional proportional integral reduced-order observer and a fractional high-gain observer. A fractional fault-tolerant dynamical controller is designed in order to achieve output tracking in the presence of faults; the controller is obtained in a natural way from a Fractional Generalized Observability Canonical Form of the output tracking error. It is worth to mention that this controller uses estimations of the faults to eliminate simultaneously their effects. It is verified that the origin of the system in closed-loop with the fractional dynamical controller is Mittag-Leffler stable. The proposed methodology is assessed in the commensurate fractional models of the Van der Pol oscillator and a DC motor, where it can be seen by means of the performance indices that the faults are estimated successfully, and thus this leads to output tracking in each system.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Fault-Tolerant Asymptotic Output Tracking: An Application to the Three-Tank System

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***Abstract*—An observer-based fault tolerant control for output tracking of a class of nonlinear systems is proposed. A Multi-output Generalized Observability Canonical Form ($\mathcal{MGOCF}$) is presented, this form allows to obtain a dynamical controller that depends on estimations of the faults. The dynamical controller is capable of linearizing the tracking errors, which achieves asymptotic stability. Finally, with the purpose of illustrating the effectiveness of the suggested approach, an application on the three-tank system is shown.**

I. Introduction

For more than three decades the fault diagnosis has been studied, several papers dealing with this problem can be found, for instance the survey [1], involving residual generation, disturbance decoupling, and adaptive approaches. In the area of nonlinear systems there exist a wide number of works related with fault detection and diagnosis where different techniques are applied. The current paper tackles the problem of diagnosis via the differential algebraic approach [8]. This technique allows to use the reduced-order observer to yield a very effective estimation of the faults; at the same time, it also gives the possibility to estimate simultaneously several faults [9] [11], which is necessary to attack successfully the output tracking problem addressed here.

Safety and performance of processes have become a major issue in control engineering practice. Fault-tolerant control plays an important role in this context. For instance, the book [3] presents a model-based approach for fault diagnosis and fault-tolerant control.

The proposed solution for attacking this problem is to obtain a dynamical controller transforming the system into a $\mathcal{MGOCF}$, represented as a chain of integrators, and that is easily implemented. It is worth of mention that this controller depends on the faults, making the fault diagnosis extremely important. To illustrate the proposed solution, the Amira DTS200 three-tank system (TTS) is considered [2]. This system has been widely used for experimental studies on fault diagnosis [4] [6] [7].

This work is organized as follows: in Section 2, the development of the canonical forms for the systems are presented, as well as the high-gain observers for tracking and the method to obtain the dynamical controller. The diagnosis problem is tackled in Section 3 using the reduced-order observer. In Section 4 the interconnected system is presented, along with the stability analysis of the system in closed-loop. In Section 5 the proposed method is applied on the TTS. Finally, in Section 6 the paper is closed with some concluding remarks.

II. Fault-Tolerant Output Tracking Problem

Consider the class of nonlinear systems with faults described by the following equations:

$$\begin{aligned}\dot{x}(t) &= g(x,u,f) \\ y(t) &= h(x,u)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the known input vector, called "control vector", $f \in \mathbb{R}^r$ is an unknown input vector called "fault vector", $y \in \mathbb{R}^p$ is the output vector, and g and h are assumed to be analytic functions.

In general, a class of systems represented by (1) can be transformed into the following $\mathcal{MGOCF}$ as a consequence of the differential primitive element for nonlinear systems ([8]):

$$\begin{aligned}\dot{\eta}_i^j &= \eta_{i+1}^j, \qquad 1 \le i \le n-1 \\ \dot{\eta}_n^j &= -L_j(\eta_1,...,\eta_p,\mathbf{u}_1,...,\mathbf{u}_m,\mathbf{f_1},...,\mathbf{f_r}) \\ y_j &= \eta_1^j\end{aligned} \tag{2}$$

where L_j is a C^1 real-valued function, $\eta_j = (\eta_1^j,...,\eta_n^j) \in \mathbb{R}^n$, $y = (y_1,...,y_p) \in \mathbb{R}^p$, $\mathbf{u}_l = (u_l, \dot{u}_l..., u_l^{(\gamma_l)})$, $1 \le l \le m$, $\mathbf{f}_{\bar{l}} = (f_{\bar{l}}, \dot{f}_{\bar{l}}..., f_{\bar{l}}^{(\mu_{\bar{l}})})$, $1 \le \bar{l} \le r$, and some integers γ_l, $\mu_{\bar{l}} \ge 0$. This $\mathcal{MGOCF}$ is composed of p subsystems, one for each output y_j, $1 \le j \le p$.

Let $y_R(t) = (y_{R1},...,y_{Rp}) \in \mathbb{R}^p$ be a reference output vector which elements are differentiable at least n times ($y_R(t)$ is of class C^n). The fault-tolerant output tracking problem consists of finding a dynamical controller described by a time-varying scalar ordinary differential equation, which has as inputs: a) the output reference vector $y_R(t)$, together with its time derivatives $y_R^{(i)}(t)$, b) the state coordinates η_i^j of the canonical system, and c) an estimation $\hat{f}$ of the fault vector and its time derivatives obtained by means of an observer, such that the controller locally forces $y(t)$ to converge asymptotically towards $y_R(t)$.

Let the output tracking error function $e_j(t)$ be defined as the difference between $y_j(t)$ and $y_{Rj}(t)$:

$$e_j(t) = y_j(t) - y_{Rj}(t) \tag{3}$$

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Since η_i^j is equal to the $(i-1)$th time derivative of $y_j(t)$, that is $\eta_i^j = y_j^{(i-1)}$, for $1 \leq i \leq n$ and $1 \leq j \leq p$, the canonical form of the error is

$$\begin{aligned} e_j^{(i)}(t) &= \eta_{i+1}^j - y_{Rj}^{(i)}, \qquad 1 \leq i \leq n-1 \qquad (4)\\ e_j^{(n)}(t) &= \dot{\eta}_n^j - y_{Rj}^{(n)}(t) = -L_j(\eta, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) \\ &\quad + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \end{aligned}$$

where

$$\begin{aligned} \eta &= (\eta_1, ..., \eta_p), \qquad \mathbf{u}_l = (u_l, \dot{u}_l, ..., u_l^{(\gamma_l)}), \\ \bar{\mathbf{u}}_l &= (u_l, \dot{u}_l, ..., u_l^{(\gamma_l - 1)}), \qquad \bar{\hat{\mathbf{f}}} = \left(\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu-1)}\right), \\ \hat{f} &= \left(\hat{f}_1, ..., \hat{f}_r\right), \qquad \hat{f}^{(\mu)} = \left(\hat{f}_1^{(\mu_1)}, ..., \hat{f}_r^{(\mu_r)}\right) \end{aligned}$$

The variable $u_l^{(\gamma_l)}$ represents the input with the highest derivative in the corresponding j block.

By requiring a linear time-invariant autonomous dynamics for the tracking error function,

$$e_j^{(n)}(t) + \sum_{i=0}^{n-1} a_{i+1}^j e_j^{(i)}(t) = 0 \qquad (5)$$

it follows from (4) that (5) may be rewritten as

$$\dot{\eta}_n^j - y_{Rj}^{(n)}(t) + \sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}(t)\right] = 0 \qquad (6)$$

that is

$$\begin{aligned} &-L_j(\eta, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} \qquad (7)\\ &\qquad = y_{Rj}^{(n)} - \sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}\right] \end{aligned}$$

Let $e_i^j(t) = e_j^{(i-1)}(t)$, $1 \leq i \leq n$ be the components of an error vector $e_j(t) = (e_1^j(t), e_2^j(t), ..., e_n^j(t))^T$. Thus, it is obtained for $1 \leq j \leq p$

$$\begin{aligned} \dot{e}_i^j &= e_{i+1}^j, \qquad 1 \leq i \leq n-1 \\ \dot{e}_n^j &= -\sum_{i=1}^{n} a_i^j e_i^j \end{aligned}$$

or in a compact form

$$\dot{\mathbf{e}}_j = F_j \mathbf{e}_j \qquad (8)$$

and

$$\begin{aligned} &-L_j(\mathbf{e}_1 + \mathbf{y}_R, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \\ &\qquad = -\sum_{i=1}^{n} a_i^j e_i^j \qquad (9) \end{aligned}$$

where $\mathbf{e}_1 = (e_1^1, ..., e_1^p)$, $\mathbf{y}_R = (y_{R1}, ..., y_{Rp})$, and

$$F_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ : & : & \cdots & : \\ 0 & : & \cdots & 1 \\ -a_1^j & -a_2^j & \cdots & -a_n^j \end{pmatrix} \qquad (10)$$

Assuming that F is Hurwitz, the origin $\mathbf{e}_j = 0$ is an asymptotically stable equilibrium point for system (8). Moreover, the dynamical controller depends on the tracking errors, so they have to be estimated by means of an observer. Thus, system (8) is rewritten as:

$$\dot{\mathbf{e}}_j(t) = E\mathbf{e}_j(t) + \varphi_j\left(\mathbf{e}_1, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \qquad (11)$$

where $\mathbf{u} = (\mathbf{u}_1, ..., \mathbf{u}_m)$, $\hat{\mathbf{f}} = (\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu)})$, the elements of E are given by

$$E_{ks} = \begin{cases} 1 & \text{if } k = s-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi_j(\cdot) = col\begin{pmatrix} 0 & \ldots & 0 & -L_j(\cdot) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \end{pmatrix}$$

Hence, the estimation $\hat{\mathbf{e}}_j(t)$ of the tracking error $\mathbf{e}_j(t) = \mathbf{y}_j(t) - \mathbf{y}_{Rj}(t)$ can be given by the following exponential linear observer [5]:

$$\dot{\hat{\mathbf{e}}}_j(t) = E\hat{\mathbf{e}}_j(t) + \varphi_j(\cdot) - S_\infty^{-1}C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \qquad (12)$$

where S_∞ is the solution to the equation

$$S_\infty\left(E + \frac{\theta}{2}I\right) + \left(E^T + \frac{\theta}{2}I\right)S_\infty = C^T C$$

with $\theta > 0$ and coefficients $(S_\infty)_{ks} = \alpha_{ks}/\theta^{k+s-1}$, where α_{ks} is a symmetric positive definite matrix which is independent of θ, and $C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$.

Furthermore, let $\hat{u}_l$ be the solution to

$$-L_j(\cdot) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j \hat{e}_i^j$$

Thus, an explicit expression for the controller $\hat{u}_l$ can be written as follows:

$$\hat{u}_l^{(\gamma_l)} = K_l\left(\hat{\mathbf{e}}_j, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \hat{\mathbf{f}}\right) \qquad (13)$$

Applying these controllers to the original system yields the multi-output tracking, while eliminating the effects of the faults. So, the equation of the tracking error observer (12) is now

$$\dot{\hat{\mathbf{e}}}_j(t) = E\hat{\mathbf{e}}_j(t) + \varphi_j(\cdot) - S_\infty^{-1}C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \qquad (14)$$

with

$$\varphi_j(\cdot) = col\begin{pmatrix} 0 & \ldots & 0 & -L_j(\cdot) + \hat{u}_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \end{pmatrix}$$

On the other hand, defining the observation error as $\varepsilon_j(t) = \hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)$, the following dynamics is obtained from equations (14) and (11):

$$\dot{\varepsilon}_j(t) = \left(E - S_\infty^{-1}C^T C\right)\varepsilon_j(t) + \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \qquad (15)$$

where

$$\begin{aligned} \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) &= \varphi_j\left(\hat{\mathbf{e}}_1, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\ &\quad - \varphi_j\left(\hat{\mathbf{e}}_1 - \varepsilon_1, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \end{aligned}$$

Now, the dynamics of the controller proposed (13) is considered. If the following variables are defined

$$\hat{u}_i^l = \hat{u}_l^{(i-1)} \qquad 1 \le i \le \gamma_l$$

the dynamical controllers subsystems are written as follows

$$\dot{\hat{\mathbf{u}}}_l(t) = E\hat{\mathbf{u}}_l(t) + \kappa_l\left(\hat{\mathbf{e}}_j, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right), \qquad 1 \le l \le m \tag{16}$$

where $\hat{u}_l = \left(\hat{u}_1^l, ..., \hat{u}_{\gamma_l}^l\right)$, and

$$\kappa_l\left(\hat{\mathbf{e}}_j, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) = col\left(\begin{array}{cccc} 0 & \dots & 0 & K_l(\cdot) \end{array}\right)$$

III. Fault diagnosis

In order to apply the controller equation (13), the estimation of the faults $(\hat{f}\,)$ that appear in the system is needed; to obtain this, the system has to be diagnosable.

Definition 1. *An element f in $k\langle u, y\rangle$ is said to be algebraically observable if f satisfies a differential equation with coefficients over $k\langle u, y\rangle$. The system is diagnosable if f is observable with respect to u and y.*

The algebraic observability notion requires that each fault component be able to be written as a solution of a polynomial equation in $f_{\bar{l}}$ and finitely many time derivatives of u and y, with coefficients in k:

$$H_{\bar{l}}\left(f_{\bar{l}}, u, \dot{u}, ..., y, \dot{y}, ...\right) = 0$$

Consider system (1). The fault vector $f = (f_1, ..., f_r)$ is unknown and it can be seen as a state with an uncertain dynamics $\Omega(x, u, f) : \mathbb{R}^{n+m+r} \to \mathbb{R}^r$. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector (immersion [8]).

This problem can be solved using the next lemma [8], that describes the construction of a proportional reduced-order observer for (1).

Lemma 1. *The system*

$$\dot{\hat{f}}_{\bar{l}} = k_{\bar{l}}(f_{\bar{l}} - \hat{f}_{\bar{l}}), \quad 1 \le \bar{l} \le r \tag{17}$$

is an asymptotic reduced-order observer for system (1), where $\hat{f}_{\bar{l}}$ denotes the estimate of fault $f_{\bar{l}}$ and $k_{\bar{l}} \in \mathbb{R}^+$ are positive coefficients that determine the desired convergence rate of the observer.

Now, considering the dynamics generated by the reduced-order observers (17) for the estimated faults, the following variables are defined

$$\hat{f}_i^{\bar{l}} = \hat{f}_{\bar{l}}^{(i-1)} \qquad 1 \le i \le \mu_{\bar{l}}$$

then the fault estimation subsystems are written as

$$\dot{\hat{\mathbf{f}}}_{\bar{l}}(t) = E\hat{\mathbf{f}}_{\bar{l}}(t) + \omega_{\bar{l}}(u, y, f), \qquad 1 \le \bar{l} \le r \tag{18}$$

where $\hat{\mathbf{f}}_{\bar{l}} = \left(\hat{f}_1^{\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}\right)$, and

$$\omega_{\bar{l}}(u, y, f) = col\left(\begin{array}{cccc} 0 & \dots & 0 & k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) \end{array}\right)$$

IV. Asymptotic stability of the closed-loop system

The dynamics of the closed-loop system, composed of the estimated tracking errors $\hat{\mathbf{e}}_j$, the observation errors ε_j, the controllers $\hat{\mathbf{u}}_l$ and the estimated faults $\hat{\mathbf{f}}_{\bar{l}}$ is given by

$$\begin{aligned}
\dot{\hat{\mathbf{e}}}_j(t) &= E\hat{\mathbf{e}}_j(t) + \varphi_j\left(\hat{\mathbf{e}}_1, \mathbf{y}_R, y_{Rj}^{(n)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \\
&\quad - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \\
\dot{\varepsilon}_j(t) &= \left(E - S_\infty^{-1}C^TC\right)\varepsilon_j(t) + \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \\
\dot{\hat{\mathbf{u}}}_l(t) &= E\hat{\mathbf{u}}_l(t) + \kappa_l\left(\hat{\mathbf{e}}_j, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \\
\dot{\hat{\mathbf{f}}}_{\bar{l}}(t) &= E\hat{\mathbf{f}}_{\bar{l}}(t) + \omega_{\bar{l}}(u, y, f)
\end{aligned} \tag{19}$$

Developing these equations for $\hat{\mathbf{e}}_j$, $\hat{\mathbf{u}}_l$ and $\hat{\mathbf{f}}_{\bar{l}}$, the following chain of integrators is obtained:

$$\begin{aligned}
\dot{\hat{e}}_i^j &= \hat{e}_{i+1}^j - f_i(\theta_j)(\hat{e}_j - e_j) & & 1 \le i \le n-1 \\
\dot{\hat{e}}_n^j &= -L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_R, \hat{\mathbf{u}}_1, ..., \bar{\hat{\mathbf{u}}}_l, ..., \hat{\mathbf{u}}_m, \bar{\hat{\mathbf{f}}}) \\
&\quad + \hat{u}_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} - \theta_j^n & & 1 \le j \le p \\
\dot{\hat{u}}_i^l &= \hat{u}_{i+1}^l \qquad 1 \le i \le \gamma_l - 1 \\
\dot{\hat{u}}_{\gamma_l}^l &= K_l\left(\hat{\mathbf{e}}_j, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \hat{\mathbf{f}}\right); & & 1 \le l \le m \\
\dot{\hat{f}}_i^{\bar{l}} &= \hat{f}_{i+1}^{\bar{l}} \qquad 1 \le i \le \mu_{\bar{l}} - 1 \\
\dot{\hat{f}}_{\mu_{\bar{l}}}^{\bar{l}} &= k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) & & 1 \le \bar{l} \le r
\end{aligned}$$

Now, the following theorem can be established.

Theorem 1. *Suppose system (1) described in the $\mathcal{MGOCF}$, composed of p subsystems as defined by (2). Let $\hat{\mathbf{e}}_j(t)$ and $\varepsilon_j(t)$ be the observation dynamics corresponding to subsystem j. Let $f_{\bar{l}}(t)$ be diagnosticable for $1 \le \bar{l} \le r$ and estimated by means of the dynamics of $\dot{\hat{\mathbf{f}}}_{\bar{l}}(t)$. Let $\hat{u}_l(t)$ be the solution to*

$$-L_j(\hat{\mathbf{e}}_1 + \mathbf{y}_R, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} = -\sum_{i=1}^{n} a_i^j \hat{e}_i^j$$

Then, the closed loop system (19) with control $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ is asymptotically stable.

Sketch of the proof. Consider the next Lyapunov function:

$$V(\hat{\mathbf{e}}_j, \varepsilon_j, \tilde{f}_{\bar{l}}) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j + \varepsilon_j^T S_\infty \varepsilon_j + \tilde{f}_{\bar{l}}^T I \tilde{f}_{\bar{l}} \tag{20}$$

where $\tilde{f}_{\bar{l}} = f_{\bar{l}} - \hat{f}_{\bar{l}}$. First, taking the derivative with respect to time of the first term of (20):

$$\dot{V}_1(\hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_j^T P \dot{\hat{\mathbf{e}}}_j + \dot{\hat{\mathbf{e}}}_j^T P \hat{\mathbf{e}}_j \le -\alpha \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j - 2\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C \varepsilon_j$$

where $P, Q > 0$, $\alpha = 1/\lambda_{\max}(P)$ and define $\|x\|_P = \sqrt{x^T P x}$ and $\|x\|_{S_\infty} = \sqrt{x^T S_\infty x}$. Noting that

$$\left\|\hat{\mathbf{e}}_j^T P S_\infty^{-1} C^T C S_\infty^{-1} S_\infty \varepsilon_j\right\| \le \rho(\theta) \|\hat{\mathbf{e}}_j\|_P \|\varepsilon_j\|_{S_\infty}$$

with $\rho(\theta) = \left\| S_\infty^{-1} C^T C S_\infty^{-1} \right\|$, then the next inequality is obtained:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \leq -\left(\alpha \left\| \hat{\mathbf{e}}_j \right\|_P - 2\rho(\theta) \left\| \varepsilon_j \right\|_{S_\infty} \right) \left\| \hat{\mathbf{e}}_j \right\|_P \tag{21}$$

Let d_1, d_2 be positive numbers such that $\left\| \hat{\mathbf{e}}_j \right\|_P \geq d_1 \left\| \hat{\mathbf{e}}_j \right\|$ and $d_2 \left\| \varepsilon_j \right\| \geq \left\| \varepsilon_j \right\|_{S_\infty}$. Thus the inequality is rewritten as:

$$\dot{V}_1(\hat{\mathbf{e}}_j) \leq -\left(\alpha d_1 \left\| \hat{\mathbf{e}}_j \right\| - 2\rho(\theta) d_2 \left\| \varepsilon_j \right\| \right) \left\| \hat{\mathbf{e}}_j \right\|_P$$

and this yields to

$$\dot{V}_1(\hat{\mathbf{e}}_j) \leq 0$$

The derivative of the term $V_2(\varepsilon_j)$ gives

$$\begin{aligned} \dot{V}_2(\varepsilon_j) &= \varepsilon_j^T S_\infty \dot{\varepsilon}_j + \dot{\varepsilon}_j^T S_\infty \varepsilon_j \leq -\theta \left\| \varepsilon_j \right\|_{S_\infty}^2 \\ &\quad +2 \left\| \varepsilon_j \right\|_{S_\infty} \left\| \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \right\|_{S_\infty} \end{aligned}$$

Noting that $\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))$ is differentiable, by the Lipschitz property

$$\left\| \Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t)) \right\|_{S_\infty} \leq \lambda \left\| \varepsilon_j(t) \right\|_{S_\infty}$$

for $1 \leq j \leq p$; thus

$$\dot{V}_2(\varepsilon_j) \leq -(\theta - 2\lambda) \left\| \varepsilon_j \right\|_{S_\infty}^2 \tag{22}$$

From (22) the following inequality is obtained

$$\frac{d\left(\left\| \varepsilon_j \right\|_{S_\infty}^2 \right)}{dt} \leq -(\theta - 2\lambda) \left\| \varepsilon_j \right\|_{S_\infty}^2, \qquad \theta - 2\lambda > 0$$

that yields $\left\| \varepsilon_j \right\|_{S_\infty} \leq -e^{-\gamma t} \left\| \varepsilon_j(0) \right\|_{S_\infty}$, $\gamma = \theta/2 - \lambda$.

Similarly, from (21) the next equation is achieved

$$\frac{d\left(\left\| \hat{\mathbf{e}}_j \right\|_P^2 \right)}{dt} \leq -\left(\alpha \left\| \hat{\mathbf{e}}_j \right\|_P - 2\rho(\theta) \left\| \varepsilon_j \right\|_{S_\infty} \right) \left\| \hat{\mathbf{e}}_j \right\|_P$$

Then

$$\left\| \hat{\mathbf{e}}_j \right\|_P \leq A e^{-\frac{\alpha}{2} t} + B e^{-\gamma t} \tag{23}$$

where

$$A = \left\| \hat{e}_j(0) \right\|_P - B \qquad B = \frac{\rho(\theta) \left\| \varepsilon_j(0) \right\|_{S_\infty}}{\alpha/2 - \gamma}$$

Finally, the derivative of the term $V_3(\tilde{f}_{\bar{l}})$ gives

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq 2 \tilde{f}_{\bar{l}}^T \left(\frac{N_{\bar{l}}}{k_{\bar{l}}} - \tilde{f}_{\bar{l}} \right)$$

and the condition $N_{\bar{l}}/k_{\bar{l}} \to 0$ with $t \to \infty$ is imposed, then

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq -2 \tilde{f}_{\bar{l}}^2$$

Therefore, the system (1) with control $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_p)$ is asymptotically stable, for $1 \leq j \leq p$, $1 \leq l \leq m$ and $1 \leq \bar{l} \leq r$. □

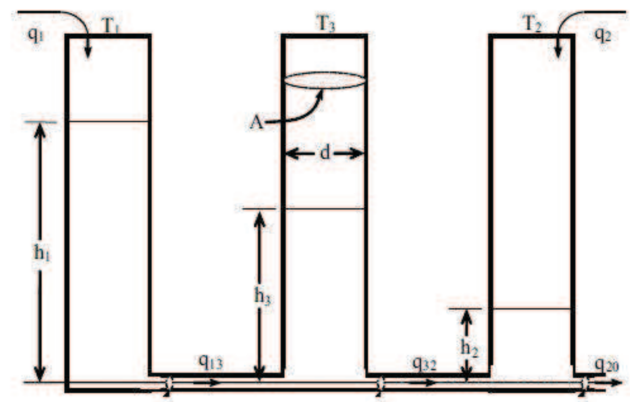

Fig. 1. Schematic diagram of the Three-Tank System.

V. Application to the three-tank system

Consider the nonlinear three-tank system (TTS), shown in Figure 1

$$\begin{aligned} \dot{x}_1 &= \frac{1}{A}(-q_{13} + u_1 + f_1) \\ \dot{x}_2 &= \frac{1}{A}(q_{32} - q_{20} + u_2 + f_2) \\ \dot{x}_3 &= \frac{1}{A}(q_{13} - q_{32}) \\ y_1 &= x_2 \\ y_2 &= x_3 \end{aligned} \tag{24}$$

where $u_i = q_i$, $i = 1, 2$ are the manipulable input flows, $x_i = h_i$, $i = 1, 2, 3$ are the levels of each tank, A is the cross section of the tanks, and $q_{ij} = a_i S \sqrt{2g(x_i - x_j)}$ represent the water flow from tank i to tank j, with $x_0 = 0$. S is the cross-sectional area of the pipe that interconnects each tank and a_i, $i = 1, 2, 3$ are the output flow coefficients.

It can be seen that x_1 is not available for measurement, but it is algebraically observable according to Definition 1

$$x_1 - y_2 - \frac{1}{2g a_1^2 S^2} \left(A\dot{y}_2 + a_3 S \sqrt{2g(y_2 - y_1)} \right)^2 = 0$$

Selecting $\eta_1^j = y_j$, the system can be transformed into a $\mathcal{MGOCF}$. For $y_1 = \eta_1^1$ the subsystem is

$$\begin{aligned} \dot{\eta}_i^1 &= \eta_{i+1}^1, \qquad i = 1, 2 \\ \dot{\eta}_3^1 &= \frac{1}{A}\left(a_3 S g^2 \left(\frac{2(\eta_1^2 - \eta_1^1)(\eta_3^2 - \eta_3^1) - (\eta_2^2 - \eta_2^1)^2}{\sqrt{\left(2g(\eta_1^2 - \eta_1^1)\right)^3}} \right) \right. \\ &\qquad \left. - a_2 S g^2 \left(\frac{2\eta_1^1 \eta_3^1 - \left(\eta_2^1\right)^2}{\sqrt{\left(2g\eta_1^1\right)^3}} \right) + \ddot{u}_2 + \ddot{f}_2 \right) \end{aligned}$$

and for $y_2 = \eta_1^2$

$$\dot{\eta}_i^2 = \eta_{i+1}^2, \qquad i = 1, 2$$

$$
\begin{aligned}
\dot{\eta}_3^2 &= \frac{1}{A^2}\left(a_1^2 S^2 g^3 \left(\frac{2(\dot{x}_1-\eta_2^2)(x_1-\eta_1^2)}{\left(2g(x_1-\eta_1^2)\right)^2}\right)\right. \\
&\quad + a_1 S g^2 \left(\frac{2(x_1-\eta_1^2)}{\sqrt{\left(2g(x_1-\eta_1^2)\right)^3}}\right)\left(\dot{u}_1+\dot{f}_1\right)\Bigg) \\
&\quad - \frac{1}{A}\left(a_1 S g^2 \left(\frac{2(\eta_3^2)(x_1-\eta_1^2)+(\dot{x}_1-\eta_2^2)^2}{\sqrt{\left(2g(x_1-\eta_1^2)\right)^3}}\right)\right. \\
&\quad \left. + a_3 S g^2 \left(\frac{2(\eta_1^2-\eta_1^1)(\eta_3^2-\eta_3^1)-(\eta_2^2-\eta_2^1)^2}{\sqrt{\left(2g(\eta_1^2-\eta_1^1)\right)^3}}\right)\right)
\end{aligned}
$$

Now, the tracking error observers (12) for each input are built, choosing

$$
S_\infty^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{pmatrix}
$$

Defining $e_1^1 = \eta_1^1 - y_{R1}$, $e_1^2 = \eta_1^2 - y_{R2}$, $e_1^{(i-1)} = e_i^1$ and $e_2^{(i-1)} = e_i^2$, the observer for $y_1 = \eta_1^1$ is

$$
\begin{aligned}
\dot{\hat{e}}_i^1 &= \hat{e}_{i+1}^1 - 3\theta_1^i(\hat{e}_1^1 - e_1^1), \qquad i=1,2 \\
\dot{\hat{e}}_3^1 &= \frac{1}{A}\left(\ddot{\hat{u}}_2 + \ddot{\hat{f}}_2\right) + \frac{a_3 S\sqrt{g}}{A}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{(2T_1)^3}}\right) - \\
&\quad \frac{a_2 S\sqrt{g}}{A}\left(\frac{2T_2\ddot{T}_2 - \left(\dot{T}_2\right)^2}{\sqrt{(2T_2)^3}}\right) - \dddot{y}_{R1} - \theta_1^3(\hat{e}_1^1 - e_1^1) \\
&= -\sum_{i=1}^{3} s_i \hat{e}_i^1
\end{aligned}
\tag{25}
$$

and for $y_2 = \eta_1^2$

$$
\begin{aligned}
\dot{\hat{e}}_i^2 &= \hat{e}_{i+1}^2 - 3\theta_2^i(\hat{e}_1^2 - e_1^2), \qquad i=1,2 \\
\dot{\hat{e}}_3^2 &= -\frac{a_1 S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_3^2 + \ddot{y}_{R2}\right)T_3 + \left(\dot{T}_3\right)^2}{\sqrt{(2T_3)^3}}\right) \\
&\quad -\frac{a_3 S\sqrt{g}}{A}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{(2T_1)^3}}\right) + \frac{a_1^2 S^2 g \dot{T}_3}{2A^2 T_3} \\
&\quad + \frac{a_1 S\sqrt{g}}{A^2\sqrt{2T_3}}\left(\dot{\hat{u}}_1 + \dot{\hat{f}}_1\right) - \dddot{y}_{R2} - \theta_2^3(\hat{e}_1^2 - e_1^2) \\
&= -\sum_{i=1}^{3} t_i \hat{e}_i^2
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
T_1 &= \left(\hat{e}_1^2 + y_{R2}\right) - \left(\hat{e}_1^1 + y_{R1}\right) \\
T_2 &= \left(\hat{e}_1^1 + y_{R1}\right) \\
T_3 &= \hat{x}_1 - \left(\hat{e}_1^2 + y_{R2}\right)
\end{aligned}
$$

Consequently, from subsystem (26), the dynamics of $\hat{u}_1$ is

$$
\begin{aligned}
\dot{\hat{u}}_1 &= \frac{A^2\sqrt{2T_3}}{a_1 S\sqrt{g}}\left(\dddot{y}_{R2} - \sum_{i=1}^{3} t_i \hat{e}_i^2\right) - \frac{a_1 S\sqrt{g}\dot{T}_3}{\sqrt{2T_3}} \\
&\quad + \frac{a_3 A\sqrt{T_3}}{2a_1}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{(T_1)^3}}\right) \\
&\quad + \frac{2A\left(\hat{e}_3^2 + \ddot{y}_{R2}\right)T_3 + A\left(\dot{T}_3\right)^2}{2T_3} - \dot{\hat{f}}_1
\end{aligned}
$$

and from subsystem (25), the dynamics of $\hat{u}_2$ is given by

$$
\begin{aligned}
\ddot{\hat{u}}_2 = A\left(\dddot{y}_{R1} - \sum_{i=1}^{3} s_i \hat{e}_i^1\right) - a_3 S\sqrt{g}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{(2T_1)^3}}\right) \\
+ a_2 S\sqrt{g}\left(\frac{2T_2\ddot{T}_2 - \left(\dot{T}_2\right)^2}{\sqrt{(2T_2)^3}}\right) - \ddot{\hat{f}}_2
\end{aligned}
$$

Furthermore, it can be seen that system (24) is diagnosable

$$
\begin{aligned}
f_1 - A\dot{x}_1 - a_1 S\sqrt{2g(x_1 - y_2)} + u_1 &= 0 \\
f_2 - A\dot{y}_1 + a_3 S\sqrt{2g(y_2 - y_1)} - a_2 S\sqrt{2g y_1} + u_2 &= 0
\end{aligned}
$$

thus, the observers for f_1 and f_2 can be built based on equation (17). In order to estimate fault f_1, the following reduced-order observer is proposed:

$$
\begin{aligned}
\dot{\gamma}_1 &= -k_1(-q_{13} + u_1 + \gamma_1 + k_1 A\hat{x}_1) \\
\hat{f}_1 &= \gamma_1 + k_1 A\hat{x}_1
\end{aligned}
$$

where the estimation of x_1 is obtained with

$$
\begin{aligned}
\dot{\gamma}_2 &= -k_2(\gamma_2 + k_2 y_2) \\
\hat{\zeta} &= \gamma_2 + k_2 y_2 \\
\hat{x}_1 &= y_2 + \frac{1}{2g a_1^2 S^2}\left(A\hat{\zeta} + q_{32}\right)^2
\end{aligned}
$$

where $\hat{\zeta}$ is an estimation of $\dot{y}_2$.

And finally to estimate the fault f_2, the following reduced-order observer is used:

$$
\begin{aligned}
\dot{\gamma}_3 &= -k_3(q_{32} - q_{20} + u_2 + \gamma_3 + k_3 A y_1) \\
\hat{f}_2 &= \gamma_3 + k_3 A y_1
\end{aligned}
$$

Simulations were made for this system over 1000 seconds, using the signals $y_{R1}(t) = 0.15$, $y_{R2}(t) = 0.2$, $f_1(t) = 0.01(1 + sin(0.2te^{-0.01t}))\ \forall t > 225$, $f_2(t) = 0.01(1 + sin(0.05te^{-0.001t}))\ \forall t > 296$. The parameters of the system are $A = 0.0149\ m^2$, $S = 5 \times 10^{-5}\ m^2$, $a_1 = 0.393874$, $a_2 = 0.685827$, $a_3 = 0.414357$. The design parameters were chosen as $\theta_1 = \theta_2 = 1$, $s_1 = t_1 = 1$, $s_2 = t_2 = 3$, $s_3 = t_3 = 3$, $k_1 = k_2 = k_3 = 0.1$.

Figures 2 and 3 show how the outputs $y_1(t)$ and $y_2(t)$ follow the references $y_{R1}(t)$ and $y_{R2}(t)$ respectively; the transient response and the effects of both faults can be appreciated as well. It can be seen that the effects of the faults are more

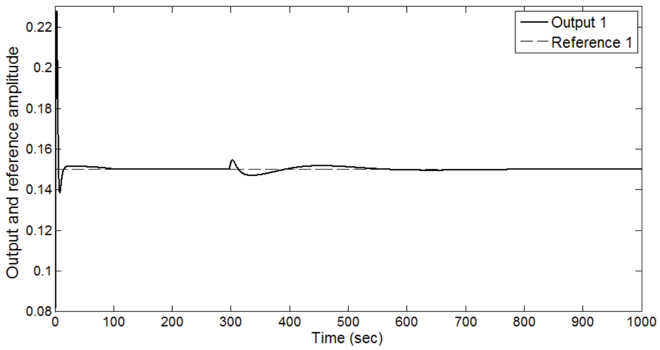

Fig. 2. Output 1 tracking of the TTS.

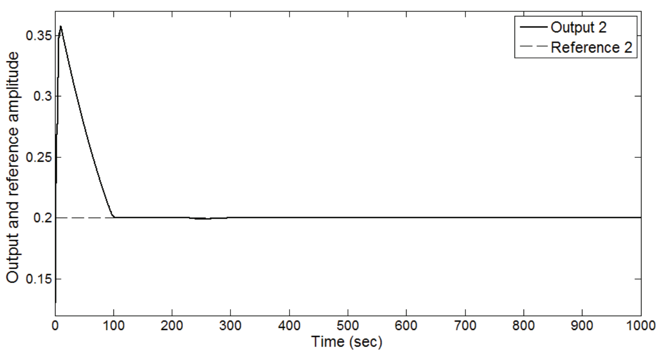

Fig. 3. Output 2 tracking of the TTS.

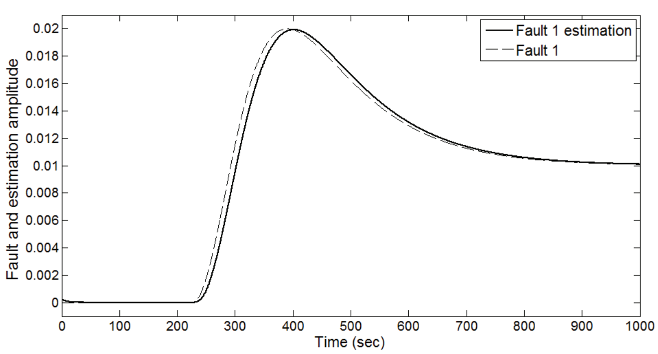

Fig. 4. Fault estimation 1 of the TTS.

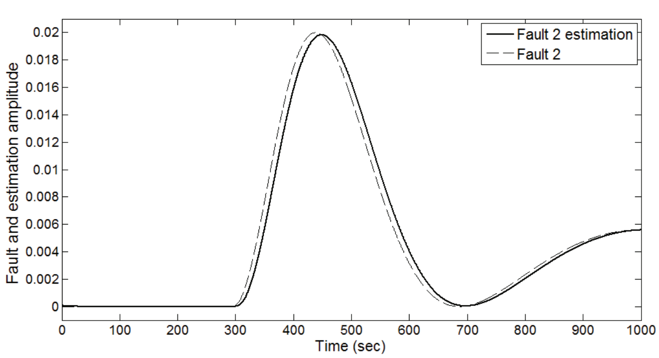

Fig. 5. Fault estimation 2 of the TTS.

notorious in output 1, but output 2 has a bigger transient response.

Figures 4 and 5 show the faults $f_1(t)$ and $f_2(t)$ with their estimations $\hat{f}_1(t)$ and $\hat{f}_2(t)$. It can be seen that both faults are estimated with a small error, thus leading to the tracking of the two references.

VI. Concluding remarks

An asymptotically stable fault-tolerant dynamical controller for nonlinear systems that can be transformed into the $\mathcal{MGOCF}$ is presented. The system is transformed into a chain of integrators, from where the dynamical controllers and the estimated faults are obtained. As the proposed controller depends on the fault estimations, the negative effects generated by the faults are suppressed simultaneously, while achieving tracking for the system outputs. In this work not only fault detection and isolation, but a complete fault diagnosis was performed; this fulfilled the needs of the proposed fault-tolerant dynamical controller. Finally, it is worth to mention the ease of implementation of the proposed idea. The method allows a safe performance of the process, as well as the fault diagnosis gives a chance to act in response of the malfunction of the components of the system and to avoid economic losses.

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Fault-Tolerant Dynamical Controller with Some Experimental Results

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Abstract— **A practical application of a fault-tolerant dynamical controller with the purpose of tracking or regulation is shown. This observer-based fault-tolerant control is obtained by means of a generalized observability canonical form multi-input multi-output. The dynamical controller is capable of linearizing the tracking errors, achieving asymptotic stability. To accomplish this, a fault diagnosis is required for systems with additive or multiplicative faults. These faults are reconstructed online and simultaneously. To illustrate the effectiveness of the suggested approach a real-time application is presented.**

I. Introduction

Fault-tolerant control (FTC) plays an important role in control engineering practice. There exist many different approaches to obtain this kind of control, the works [1]-[3] provide a basic literature review covering most areas of FTC. In the field of nonlinear systems, encouraging results were obtained applying algebraic techniques [4]-[6]. In this paper, the proposed solution for attacking this problem is to obtain a dynamical controller transforming a multi-input multi-output (MIMO) system in a generalized observability canonical form multi-input multi-output (GOCFMM), represented as a chain of integrators.

For systems with multiple faults, this dynamical controller is dependent of each of them, forcing the implementation of a fault diagnosis (FD) able to reconstruct each fault simultaneously and online. The FD has been studied for a long time, numerous papers dealing with this problem can be found. Particularly, in the area of nonlinear systems there exist a wide number of works related with fault detection and diagnosis where different techniques are applied. In this work the problem of diagnosis is solved using the differential algebraic approach [7]-[9], which permits to estimate simultaneously several faults using a reduced-order observer (ROO), necessary to attack successfully the output tracking problem.

Furthermore, to illustrate the proposed solution, the Amira DTS200 three-tank system (TTS) is considered [10], the TTS provides an opportunity to introduce multiple faults in sensors and actuators, making it a very versatile system; this is why the TTS has been widely used for experimental studies on FD and FTC [4]-[8].

This paper is organized as follows: in Section 2, the definition of diagnosable system and the FD problem are presented. In Section 3 the GOCFMM for the MIMO system is introduced, as well as the high-gain observers (HGO) [11] used for tracking and the method to obtain the dynamical controller. In Section 4 the closed-loop system is shown, along with the proof of asymptotic stability. In Section 5 the method is applied on the TTS and the results are presented and discussed. Finally, in Section 6 the paper is closed with some conclusions and observations.

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II. Fault diagnosis

The FD is an important part of the proposed method, the dynamical controller will be dependent of multiple faults, therefore, a diagnosis capable to reconstruct each fault is required.

Firstly, consider a nonlinear systems with faults, described by the following equations:

$$\begin{aligned} \dot{x}(t) &= g(x,u,f) \\ y(t) &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ the input vector, called "control vector", $f \in \mathbb{R}^r$ is an unknown input vector called "fault vector", $y \in \mathbb{R}^p$ is the output vector, and g and h are assumed to be analytic functions.

As mentioned above the estimation of the faults ($\hat{f}$) that appear in the system is needed; to obtain this, the system has to satisfy the diagnosability property.

Definition 1: An element f in the differential field $k\langle u,y\rangle$ is said to be algebraically observable if it satisfies a differential equation with coefficients over $k\langle u,y\rangle$. The system is diagnosable if f is observable with respect to u and y.

The algebraic observability notion requires that each fault component be able to be written as a solution of a polynomial equation in f, and finitely many time derivatives of u and y, with coefficients in k:

$$H\left(f,u,\dot{u},...,y,\dot{y},...\right)=0$$

Consider system (1). The fault vector $f=(f_1,...,f_r)$ is unknown and it can be seen as a state with an uncertain dynamics $\Omega\left(x,u,f\right):\mathbb{R}^{n+m+r}\to\mathbb{R}^r$. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector (immersion [7]).

This problem can be solved using the next lemma [7], that describes the construction of a ROO for (1).

Lemma 1: The system

$$\dot{\hat{f}}_{\bar{l}} = k_{\bar{l}}(f_{\bar{l}}-\hat{f}_{\bar{l}}),\quad 1\leq \bar{l}\leq r \tag{2}$$

is an asymptotic ROO for system (1), where $\hat{f}_{\bar{l}}$ denotes the estimate of fault $f_{\bar{l}}$ and $k_{\bar{l}}\in\mathbb{R}^+$ are positive coefficients.

Now, considering the dynamics generated by the ROO (2) for the estimated faults, the following variables are defined

$$\hat{f}_i^{\bar{l}} = \hat{f}_{\bar{l}}^{(i-1)} \qquad 1\leq i\leq \mu_{\bar{l}}$$

then the fault estimation subsystems are written as

$$\dot{\hat{\mathbf{f}}}_{\bar{l}}(t) = E\hat{\mathbf{f}}_{\bar{l}}(t) + \omega_{\bar{l}}(u, y, f), \qquad 1 \leq \bar{l} \leq r \qquad (3)$$

where $\hat{\mathbf{f}}_{\bar{l}} = \left(\hat{f}_1^{\bar{l}}, ..., \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}\right)$, E is given by

$$E_{ks} = \begin{cases} 1 & \text{if } k = s-1 \\ 0 & \text{otherwise} \end{cases}$$

and $\omega_{\bar{l}}(u, y, f) = col\begin{pmatrix} 0 & \dots & 0 & k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)} - \hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) \end{pmatrix}$

III. Fault tolerant control and the output tracking problem

In general, a class of nonlinear systems represented by (1) can be transformed into the following GOCFMM for MIMO systems as a consequence of the differential primitive element [7]:

$$\begin{aligned} \dot{\eta}_i^j &= \eta_{i+1}^j, \qquad 1 \leq i \leq n-1 \\ \dot{\eta}_n^j &= -L_j(\eta_1, ..., \eta_p, \mathbf{u}_1, ..., \mathbf{u}_m \mathbf{f_1}, ..., \mathbf{f_r}) \qquad (4) \\ y_j &= \eta_1^j \end{aligned}$$

where L_j is a C^1 real-valued function, $\eta_j = (\eta_1^j, ..., \eta_n^j) \in \mathbb{R}^n$, $y = (y_1, ..., y_p) \in \mathbb{R}^p$, $\mathbf{u}_l = (u_l, \dot{u}_l ..., u_l^{(\gamma_l)})$, $1 \leq l \leq m$, $\mathbf{f}_{\bar{l}} = (f_{\bar{l}}, \dot{f}_{\bar{l}} ..., f_{\bar{l}}^{(\mu_{\bar{l}})})$, $1 \leq \bar{l} \leq r$, and some integers γ_l, $\mu_{\bar{l}} \geq 0$. This GOCFMM is composed of p subsystems, one for each output y_j, $1 \leq j \leq p$.

Let $y_R(t) = (y_{R1}, ..., y_{Rp}) \in \mathbb{R}^p$ be a reference output vector which elements are differentiable at least n times ($y_R(t)$ is of class C^n). The fault-tolerant output tracking problem consists of finding a dynamical controller described by a time-varying scalar ordinary differential equation, which has as inputs: a) the output reference vector $y_R(t)$, together with its time derivatives $y_R^{(i)}(t)$, b) the state coordinates η_i^j of the canonical system, and c) an estimation $\hat{f}$ of the fault vector and its time derivatives obtained by means of an observer, such that the controller locally forces $y(t)$ to converge asymptotically towards $y_R(t)$.

Let the output tracking error function $e_j(t)$ be defined as the difference between $y_j(t)$ and $y_{Rj}(t)$:

$$e_j(t) = y_j(t) - y_{Rj}(t) \qquad (5)$$

Since η_i^j is equal to the $(i-1)$th time derivative of $y_j(t)$, that is $\eta_i^j = y_j^{(i-1)}$, for $1 \leq i \leq n$ and $1 \leq j \leq p$, the GOCFMM of the error is

$$\begin{aligned} e_j^{(i)}(t) &= \eta_{i+1}^j - y_{Rj}^{(i)}, \qquad 1 \leq i \leq n-1 \qquad (6) \\ e_j^{(n)}(t) &= \dot{\eta}_n^j - y_{Rj}^{(n)}(t) = -L_j(\eta, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) \\ &\quad + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \end{aligned}$$

where

$$\eta = (\eta_1, ..., \eta_p), \qquad \mathbf{u}_l = (u_l, \dot{u}_l, ..., u_l^{(\gamma_l)}),$$

$$\bar{\mathbf{u}}_l = (u_l, \dot{u}_l, ..., u_l^{(\gamma_l - 1)}), \qquad \bar{\hat{\mathbf{f}}} = \left(\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu-1)}\right),$$

$$\hat{f} = \left(\hat{f}_1, ..., \hat{f}_r\right), \qquad \hat{f}^{(\mu)} = \left(\hat{f}_1^{(\mu_1)}, ..., \hat{f}_r^{(\mu_r)}\right)$$

The variable $u_l^{(\gamma_l)}$ represents the input with the highest derivative in the corresponding j block.

By requiring a linear time-invariant autonomous dynamics for the tracking error function,

$$e_j^{(n)}(t) + \sum_{i=0}^{n-1} a_{i+1}^j e_j^{(i)}(t) = 0 \qquad (7)$$

it follows from (6) that (7) may be rewritten as

$$\dot{\eta}_n^j - y_{Rj}^{(n)}(t) + \sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}(t)\right] = 0 \qquad (8)$$

that is

$$\begin{aligned} -L_j(\eta, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} \qquad (9) \\ = y_{Rj}^{(n)} - \sum_{i=1}^{n} a_i^j \left[\eta_i^j - y_{Rj}^{(i-1)}\right] \end{aligned}$$

Let $e_i^j(t) = e_j^{(i-1)}(t)$, $1 \leq i \leq n$ be the components of an error vector $e_j(t) = (e_1^j(t), e_2^j(t), ..., e_n^j(t))^T$. Thus, it is obtained for $1 \leq j \leq p$

$$\begin{aligned} \dot{e}_i^j &= e_{i+1}^j, \qquad 1 \leq i \leq n-1 \\ \dot{e}_n^j &= -\sum_{i=1}^{n} a_i^j e_i^j \end{aligned}$$

or in a compact form

$$\dot{\mathbf{e}}_j = F_j \mathbf{e}_j \qquad (10)$$

$$\begin{aligned} -L_j(\mathbf{e}_1 + \mathbf{y}_R, \mathbf{u}_1, ..., \bar{\mathbf{u}}_l, ..., \mathbf{u}_m, \bar{\hat{\mathbf{f}}}) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \\ = -\sum_{i=1}^{n} a_i^j e_i^j \qquad (11) \end{aligned}$$

where $\mathbf{e}_1 = (e_1^1, ..., e_1^p)$, $\mathbf{y}_R = (y_{R1}, ..., y_{Rp})$, and

$$F_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ : & : & \cdots & : \\ 0 & : & \cdots & 1 \\ -a_1^j & -a_2^j & \cdots & -a_n^j \end{pmatrix} \qquad (12)$$

Assuming that F is Hurwitz, the origin $\mathbf{e}_j = 0$ is an asymptotically stable equilibrium point for system (10). Moreover, the dynamical controller depends on the tracking errors, so they have to be estimated by means of an observer. Thus, system (10) is rewritten as:

$$\dot{\mathbf{e}}_j(t) = E\mathbf{e}_j(t) + \varphi_j\left(\mathbf{e}_1, \mathbf{y}_R, y_{Rj}^{(n)}, \mathbf{u}, \hat{\mathbf{f}}\right) \qquad (13)$$

where $\mathbf{u} = (\mathbf{u}_1, ..., \mathbf{u}_m)$, $\hat{\mathbf{f}} = (\hat{f}, \dot{\hat{f}}, ..., \hat{f}^{(\mu)})$, and

$$\varphi_j(\cdot) = col\begin{pmatrix} 0 & \dots & 0 & -L_j(\cdot) + u_l^{(\gamma_l)} + \hat{f}^{(\mu)} - y_{Rj}^{(n)} \end{pmatrix}$$

Hence, the estimation $\hat{\mathbf{e}}_j(t)$ of the tracking error $\mathbf{e}_j(t) = \mathbf{y}_j(t) - \mathbf{y}_{Rj}(t)$ can be given by the following exponential linear observer [11]:

$$\dot{\hat{\mathbf{e}}}_j(t) = E\hat{\mathbf{e}}_j(t) + \varphi_j(\cdot) - S_\infty^{-1} C^T C(\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)) \qquad (14)$$

where S_∞ is the solution to the equation

$$S_\infty\left(E+\frac{\theta}{2}I\right)+\left(E^T+\frac{\theta}{2}I\right)S_\infty=C^TC$$

with $\theta>0$ and coefficients $(S_\infty)_{ks}=\alpha_{ks}/\theta^{k+s-1}$, where α_{ks} is a symmetric positive definite matrix independent of θ, and $C=\begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}$.

Furthermore, let $\hat{u}_l$ be the solution to

$$-L_j(\cdot)+u_l^{(\gamma_l)}+\hat{f}^{(\mu)}-y_{Rj}^{(n)}=-\sum_{i=1}^{n}a_i^j\hat{e}_i^j$$

Thus, an explicit expression for the controller $\hat{u}_l$ can be written as follows:

$$\hat{u}_l^{(\gamma_l)}=K_l\left(\hat{\mathbf{e}}_j,\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u}_1,...,\bar{\mathbf{u}}_l,...,\mathbf{u}_m,\hat{\mathbf{f}}\right)\qquad(15)$$

Applying these controllers to the original system yields the multi-output tracking, while eliminating the effects of the faults. So, the equation of the tracking error observer (14) is now

$$\dot{\hat{\mathbf{e}}}_j(t)=E\hat{\mathbf{e}}_j(t)+\hat{\varphi}_j(\cdot)-S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j(t)-\mathbf{e}_j(t))\quad(16)$$

with

$$\hat{\varphi}_j(\cdot)=col\begin{pmatrix}0 & \dots & 0 & -L_j(\cdot)+\hat{u}_l^{(\gamma_l)}+\hat{f}^{(\mu)}-y_{Rj}^{(n)}\end{pmatrix}$$

On the other hand, defining the observation error as $\varepsilon_j(t)=\hat{\mathbf{e}}_j(t)-\mathbf{e}_j(t)$, the following dynamics is obtained from equations (16) and (13):

$$\dot{\varepsilon}_j(t)=\left(E-S_\infty^{-1}C^TC\right)\varepsilon_j(t)+\Phi_j(\varepsilon(t),\hat{\mathbf{e}}(t))\qquad(17)$$

where

$$\begin{aligned}\Phi_j(\varepsilon(t),\hat{\mathbf{e}}(t)) &= \hat{\varphi}_j(\hat{\mathbf{e}}_1,\mathbf{y}_R,y_{Rj}^{(n)},\hat{\mathbf{u}},\hat{\mathbf{f}})\\ &\quad -\varphi_j(\underbrace{\hat{\mathbf{e}}_1-\varepsilon_1}_{e_1},\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u},\hat{\mathbf{f}})\end{aligned}$$

Now, the dynamics of the controller proposed (15) is considered. If the following variables are defined

$$\hat{u}_i^l=\hat{u}_l^{(i-1)}\qquad 1\le i\le\gamma_l$$

the dynamical controllers subsystems are written as follows

$$\dot{\hat{\mathbf{u}}}_l(t)=E\hat{\mathbf{u}}_l(t)+\kappa_l\left(\hat{\mathbf{e}}_j,\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u},\hat{\mathbf{f}}\right),\qquad 1\le l\le m\qquad(18)$$

where $\hat{u}_l=\left(\hat{u}_1^l,...,\hat{u}_{\gamma_l}^l\right)$, and

$$\kappa_l\left(\hat{\mathbf{e}}_j,\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u},\hat{\mathbf{f}}\right)=col\begin{pmatrix}0 & \dots & 0 & K_l(\cdot)\end{pmatrix}$$

IV. Asymptotic stability of the closed-loop system

The dynamics of the closed-loop system, composed of the estimated tracking errors $\hat{\mathbf{e}}_j$, the observation errors ε_j, the controllers $\hat{\mathbf{u}}_l$ and the estimated faults $\hat{\mathbf{f}}_{\bar{l}}$ is given by

$$\begin{aligned}\dot{\hat{\mathbf{e}}}_j(t) &= E\hat{\mathbf{e}}_j(t)+\varphi_j\left(\hat{\mathbf{e}}_1,\mathbf{y}_R,y_{Rj}^{(n)},\hat{\mathbf{u}},\hat{\mathbf{f}}\right)\\ &\quad -S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j(t)-\mathbf{e}_j(t))\\ \dot{\varepsilon}_j(t) &= \left(E-S_\infty^{-1}C^TC\right)\varepsilon_j(t)+\Phi_j(\varepsilon(t),\hat{\mathbf{e}}(t))\\ \dot{\hat{\mathbf{u}}}_l(t) &= E\hat{\mathbf{u}}_l(t)+\kappa_l\left(\hat{\mathbf{e}}_j,\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u},\hat{\mathbf{f}}\right)\\ \dot{\hat{\mathbf{f}}}_{\bar{l}}(t) &= E\hat{\mathbf{f}}_{\bar{l}}(t)+\omega_{\bar{l}}(u,y,f)\end{aligned}\qquad(19)$$

Developing these equations for $\hat{\mathbf{e}}_j$, $\hat{\mathbf{u}}_l$ and $\hat{\mathbf{f}}_{\bar{l}}$, the following chain of integrators is obtained:

$$\begin{aligned}\dot{\hat{e}}_i^j &= \hat{e}_{i+1}^j-\psi_i(\theta_j)(\hat{e}_j-e_j) && 1\le i\le n-1\\ \dot{\hat{e}}_n^j &= -L_j(\hat{\mathbf{e}}_1+\mathbf{y}_R,\hat{\mathbf{u}}_1,...,\bar{\hat{\mathbf{u}}}_l,...,\hat{\mathbf{u}}_m,\bar{\hat{\mathbf{f}}})\\ &\quad +\hat{u}_l^{(\gamma_l)}+\hat{f}^{(\mu)}-y_{Rj}^{(n)}-\theta_j^n && 1\le j\le p\\ \dot{\hat{u}}_i^l &= \hat{u}_{i+1}^l && 1\le i\le\gamma_l-1\\ \dot{\hat{u}}_{\gamma_l}^l &= K_l\left(\hat{\mathbf{e}}_j,\mathbf{y}_R,y_{Rj}^{(n)},\mathbf{u}_1,...,\bar{\mathbf{u}}_l,...,\mathbf{u}_m,\hat{\mathbf{f}}\right); && 1\le l\le m\\ \dot{\hat{f}}_i^{\bar{l}} &= \hat{f}_{i+1}^{\bar{l}} && 1\le i\le\mu_{\bar{l}}-1\\ \dot{\hat{f}}_{\mu_{\bar{l}}}^{\bar{l}} &= k_{\bar{l}}(f_{\bar{l}}^{(\mu_{\bar{l}}-1)}-\hat{f}_{\mu_{\bar{l}}}^{\bar{l}}) && 1\le\bar{l}\le r\end{aligned}$$

where $\psi_i(\theta_j)$ is a function obtained from S_∞^{-1} [7].

Now, the following theorem can be established.

Theorem 1: Suppose the system (1) described in the GOCFMM, composed of p subsystems as defined by (4). Let $\hat{\mathbf{e}}_j(t)$ and $\varepsilon_j(t)$ be the observation dynamics corresponding to subsystem j. Let $f_{\bar{l}}(t)$ be diagnosticable for $1\le\bar{l}\le r$ and estimated by means of the dynamics of $\dot{\hat{\mathbf{f}}}_{\bar{l}}(t)$. Let $\hat{u}_l(t)$ be the solution to

$$-L_j(\hat{\mathbf{e}}_1+\mathbf{y}_R,\mathbf{u}_1,...,\bar{\mathbf{u}}_l,...,\mathbf{u}_m,\bar{\hat{\mathbf{f}}})+u_l^{(\gamma_l)}+\hat{f}^{(\mu)}-y_{Rj}^{(n)}=-\sum_{i=1}^{n}a_i^j\hat{e}_i^j$$

Then, the closed loop system (19) with control $\hat{u}=(\hat{u}_1,...,\hat{u}_p)$ is asymptotically stable.

Proof: [Sketch] Consider the next Lyapunov function:

$$V(\hat{\mathbf{e}}_j,\varepsilon_j,\tilde{f}_{\bar{l}})=\hat{\mathbf{e}}_j^TP\hat{\mathbf{e}}_j+\varepsilon_j^TS_\infty\varepsilon_j+\tilde{f}_{\bar{l}}^TI\tilde{f}_{\bar{l}}\qquad(20)$$

where $\tilde{f}_{\bar{l}}=f_{\bar{l}}-\hat{f}_{\bar{l}}$. First, taking the derivative with respect to time of the first term of (20):

$$\dot{V}_1(\hat{\mathbf{e}}_j)=\hat{\mathbf{e}}_j^TP\dot{\hat{\mathbf{e}}}_j+\dot{\hat{\mathbf{e}}}_j^TP\hat{\mathbf{e}}_j\le-\alpha\hat{\mathbf{e}}_j^TP\hat{\mathbf{e}}_j-2\hat{\mathbf{e}}_j^TPS_\infty^{-1}C^TC\varepsilon_j$$

where $P,S_\infty^{-1}>0$, $\alpha=1/\lambda_{\max}(P)$ and define $\|x\|_P=\sqrt{x^TPx}$ and $\|x\|_{S_\infty}=\sqrt{x^TS_\infty x}$. Noting that

$$\left\|\hat{\mathbf{e}}_j^TPS_\infty^{-1}C^TCS_\infty^{-1}S_\infty\varepsilon_j\right\|\le\rho(\theta)\|\hat{\mathbf{e}}_j\|_P\|\varepsilon_j\|_{S_\infty}$$

with $\rho(\theta)=\left\|S_\infty^{-1}C^TCS_\infty^{-1}\right\|$, then:

$$\dot{V}_1(\hat{\mathbf{e}}_j)\le-\left(\alpha\|\hat{\mathbf{e}}_j\|_P-2\rho(\theta)\|\varepsilon_j\|_{S_\infty}\right)\|\hat{\mathbf{e}}_j\|_P\qquad(21)$$

Let d_1, d_2 be positive numbers such that $\|\hat{\mathbf{e}}_j\|_P\ge d_1\|\hat{\mathbf{e}}_j\|$ and $d_2\|\varepsilon_j\|\ge\|\varepsilon_j\|_{S_\infty}$. Thus the inequality is rewritten as:

$$\dot{V}_1(\hat{\mathbf{e}}_j)\le-\left(\alpha d_1\|\hat{\mathbf{e}}_j\|-2\rho(\theta)d_2\|\varepsilon_j\|\right)\|\hat{\mathbf{e}}_j\|_P$$

and this yields to

$$\dot{V}_1(\hat{\mathbf{e}}_j)\le 0$$

The derivative of the term $V_2(\varepsilon_j)$ gives

$$\begin{aligned}\dot{V}_2(\varepsilon_j) &= \varepsilon_j^T S_\infty \dot{\varepsilon}_j + \dot{\varepsilon}_j^T S_\infty \varepsilon_j \leq -\theta \left\|\varepsilon_j\right\|_{S_\infty}^2 \\ &+2\left\|\varepsilon_j\right\|_{S_\infty} \left\|\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))\right\|_{S_\infty}\end{aligned}$$

Noting that $\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))$ is differentiable, by the Lipschitz property

$$\left\|\Phi_j(\varepsilon(t), \hat{\mathbf{e}}(t))\right\|_{S_\infty} \leq \lambda \left\|\varepsilon_j(t)\right\|_{S_\infty}$$

for $1 \leq j \leq p$; thus

$$\dot{V}_2(\varepsilon_j) \leq -(\theta - 2\lambda)\left\|\varepsilon_j\right\|_{S_\infty}^2 \tag{22}$$

From (22) the following inequality is obtained

$$\frac{d\left(\left\|\varepsilon_j\right\|_{S_\infty}^2\right)}{dt} \leq -(\theta - 2\lambda)\left\|\varepsilon_j\right\|_{S_\infty}^2, \qquad \theta - 2\lambda > 0$$

that yields $\left\|\varepsilon_j\right\|_{S_\infty} \leq -e^{-\gamma t}\left\|\varepsilon_j(0)\right\|_{S_\infty}$, $\gamma = \theta/2 - \lambda$.

Similarly, from (21) the next equation is achieved

$$\frac{d\left(\left\|\hat{\mathbf{e}}_j\right\|_P^2\right)}{dt} \leq -\left(\alpha\left\|\hat{\mathbf{e}}_j\right\|_P - 2\rho(\theta)\left\|\varepsilon_j\right\|_{S_\infty}\right)\left\|\hat{\mathbf{e}}_j\right\|_P$$

Then

$$\left\|\hat{\mathbf{e}}_j\right\|_P \leq Ae^{-\frac{\alpha}{2}t} + Be^{-\gamma t} \tag{23}$$

where

$$A = \left\|\hat{e}_j(0)\right\|_P - B \qquad\qquad B = \frac{\rho(\theta)\left\|\varepsilon_j(0)\right\|_{S_\infty}}{\alpha/2 - \gamma}$$

Finally, the derivative of the term $V_3(\tilde{f}_{\bar{l}})$ gives

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq 2\tilde{f}_{\bar{l}}^T\left(\frac{N_{\bar{l}}}{k_{\bar{l}}} - \tilde{f}_{\bar{l}}\right)$$

and the condition $N_{\bar{l}}/k_{\bar{l}} \to 0$ with $t \to \infty$ is imposed, then

$$\dot{V}_3(\tilde{f}_{\bar{l}}) \leq -2\tilde{f}_{\bar{l}}^2$$

Therefore, system (1) with the control $\hat{u} = (\hat{u}_1, ..., \hat{u}_p)$ is asymptotically stable, for $1 \leq j \leq p$, $1 \leq l \leq m$ and $1 \leq \bar{l} \leq r$. ■

V. Real-time application to the TTS

1) System model: Consider the nonlinear three-tank system (TTS), shown in Figure 1

$$\begin{aligned}\dot{x}_1 &= \frac{1}{A}(-q_{13} + u_1 + f_1) \qquad (24)\\ \dot{x}_2 &= \frac{1}{A}(q_{32} - q_{20} + u_2 + f_2) \\ \dot{x}_3 &= \frac{1}{A}(q_{13} - q_{32}) \\ y_1 &= x_2 \\ y_2 &= x_3\end{aligned}$$

where $u_i = q_i$, $i = 1, 2$ are the input flows, $x_i = h_i$, $i = 1, 2, 3$ are the levels of each tank, A is the cross section of the tanks, and $q_{ij} = a_i S\sqrt{2g(x_i - x_j)}$ represent the water flow from tank i to tank j, with $x_0 = 0$. S is the cross-sectional area of the pipe that interconnects each tank and a_i, $i = 1, 2, 3$ are the output flow coefficients. It can be seen that

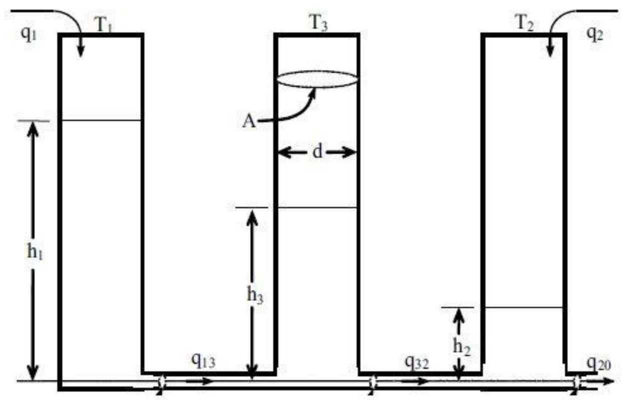

Fig. 1. Schematic diagram of the Three-Tank System.

x_1 is not available for measurement, but it is algebraically observable according to Definition 1

$$x_1 - y_2 - \frac{1}{2ga_1^2S^2}\left(A\ddot{y}_2 + a_3S\sqrt{2g(y_2 - y_1)}\right)^2 = 0$$

2) Experimental results: Now, the proposed method is applied. Firstly, the change of variables $\eta_1^j = y_j$ is selected, so the system (24) can be transformed into the GOCFMM (4). For $y_1 = \eta_1^1$, the subsystem is:

$$\begin{aligned}\dot{\eta}_i^1 &= \eta_{i+1}^1, \qquad i = 1, 2 \\ \dot{\eta}_3^1 &= \frac{1}{A}\left(a_3Sg^2\left(\frac{2(\eta_1^2 - \eta_1^1)(\eta_3^2 - \eta_3^1) - (\eta_2^2 - \eta_2^1)^2}{\sqrt{(2g(\eta_1^2 - \eta_1^1))^3}}\right)\right. \\ &\left. -a_2Sg^2\left(\frac{2\eta_1^1\eta_3^1 - (\eta_2^1)^2}{\sqrt{(2g\eta_1^1)^3}}\right) + \ddot{u}_2 + \ddot{f}_2\right)\end{aligned}$$

and for $y_2 = \eta_1^2$

$$\begin{aligned}\dot{\eta}_i^2 &= \eta_{i+1}^2, \qquad i = 1, 2 \\ \dot{\eta}_3^2 &= \frac{1}{A^2}\left(a_1^2S^2g^3\left(\frac{2(\dot{x}_1 - \eta_2^2)(x_1 - \eta_1^2)}{(2g(x_1 - \eta_1^2))^2}\right)\right. \\ &\left. +a_1Sg^2\left(\frac{2(x_1 - \eta_1^2)}{\sqrt{(2g(x_1 - \eta_1^2))^3}}\right)\left(\dot{u}_1 + \dot{f}_1\right)\right) \\ &-\frac{1}{A}\left(a_1Sg^2\left(\frac{2(\eta_3^2)(x_1 - \eta_1^2) + (\dot{x}_1 - \eta_2^2)^2}{\sqrt{(2g(x_1 - \eta_1^2))^3}}\right)\right. \\ &\left. +a_3Sg^2\left(\frac{2(\eta_1^2 - \eta_1^1)(\eta_3^2 - \eta_3^1) - (\eta_2^2 - \eta_2^1)^2}{\sqrt{(2g(\eta_1^2 - \eta_1^1))^3}}\right)\right)\end{aligned}$$

Now the tracking error observers (14) for each input are built. Defining $e_1^1 = \eta_1^1 - y_{R1}$, $e_1^2 = \eta_1^2 - y_{R2}$, $e_1^{(i-1)} = e_i^1$ and $e_2^{(i-1)} = e_i^2$, the observer for $y_1 = \eta_1^1$ is

$$\dot{\hat{e}}_i^1 = \hat{e}_{i+1}^1 - 3\theta_1^i(\hat{e}_1^1 - e_1^1), \quad i = 1, 2 \tag{25}$$

$$\dot{\hat{e}}_3^1 = \frac{1}{A}\left(\ddot{\hat{u}}_2 + \ddot{\hat{f}}_2\right) + \frac{a_3 S\sqrt{g}}{A}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{\left(2T_1\right)^3}}\right) - \dddot{y}_{R1}$$
$$-\frac{a_2 S\sqrt{g}}{A}\left(\frac{2T_2\ddot{T}_2 - \left(\dot{T}_2\right)^2}{\sqrt{\left(2T_2\right)^3}}\right) - \theta_1^3(\hat{e}_1^1 - e_1^1) = -\sum_{i=1}^{3} s_i \hat{e}_i^1$$

and for $y_2 = \eta_1^2$

$$\dot{\hat{e}}_i^2 = \hat{e}_{i+1}^2 - 3\theta_2^i(\hat{e}_1^2 - e_1^2), \qquad i = 1, 2$$
$$\dot{\hat{e}}_3^2 = -\frac{a_1 S\sqrt{g}}{A}\left(\frac{2\left(\hat{e}_3^2 + \ddot{y}_{R2}\right)T_3 + \left(\dot{T}_3\right)^2}{\sqrt{\left(2T_3\right)^3}}\right)$$
$$-\frac{a_3 S\sqrt{g}}{A}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{\left(2T_1\right)^3}}\right) + \frac{a_1^2 S^2 g\dot{T}_3}{2A^2 T_3} - \dddot{y}_{R2}$$
$$+\frac{a_1 S\sqrt{g}}{A^2\sqrt{2T_3}}\left(\dot{\hat{u}}_1 + \dot{\hat{f}}_1\right) - \theta_2^3(\hat{e}_1^2 - e_1^2) = -\sum_{i=1}^{3} t_i \hat{e}_i^2 \qquad (26)$$

where

$$\begin{aligned} T_1 &= \left(\hat{e}_1^2 + y_{R2}\right) - \left(\hat{e}_1^1 + y_{R1}\right) \\ T_2 &= \left(\hat{e}_1^1 + y_{R1}\right) \\ T_3 &= \hat{x}_1 - \left(\hat{e}_1^2 + y_{R2}\right) \end{aligned}$$

Hence, from subsystem (26), the dynamics of $\hat{u}_1$ is

$$\begin{aligned} \dot{\hat{u}}_1 &= \frac{A^2\sqrt{2T_3}}{a_1 S\sqrt{g}}\left(\dddot{y}_{R2} - \sum_{i=1}^{3} t_i \hat{e}_i^2\right) - \frac{a_1 S\sqrt{g}\dot{T}_3}{\sqrt{2T_3}} \\ &+ \frac{a_3 A\sqrt{T_3}}{2a_1}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{\left(T_1\right)^3}}\right) \\ &+ \frac{2A\left(\hat{e}_3^2 + \ddot{y}_{R2}\right)T_3 + A\left(\dot{T}_3\right)^2}{2T_3} - \dot{\hat{f}}_1 \end{aligned}$$

and from subsystem (25), the dynamics of $\hat{u}_2$ is given by

$$\ddot{\hat{u}}_2 = A\left(\dddot{y}_{R1} - \sum_{i=1}^{3} s_i \hat{e}_i^1\right) - a_3 S\sqrt{g}\left(\frac{2T_1\ddot{T}_1 - \left(\dot{T}_1\right)^2}{\sqrt{\left(2T_1\right)^3}}\right)$$
$$+a_2 S\sqrt{g}\left(\frac{2T_2\ddot{T}_2 - \left(\dot{T}_2\right)^2}{\sqrt{\left(2T_2\right)^3}}\right) - \ddot{\hat{f}}_2$$

Furthermore, it can be seen that system (24) is diagnosable

$$\begin{aligned} f_1 - A\dot{x}_1 - a_1 S\sqrt{2g(x_1 - y_2)} + u_1 &= 0 \\ f_2 - A\dot{y}_1 + a_3 S\sqrt{2g(y_2 - y_1)} - a_2 S\sqrt{2gy_1} + u_2 &= 0 \end{aligned}$$

thus, the observers for f_1 and f_2 can be built based on equation (2). In order to estimate fault f_1, the following ROO is proposed:

$$\begin{aligned} \dot{\gamma}_1 &= -k_1(-q_{13} + u_1 + \gamma_1 + k_1 A\hat{x}_1) \\ \hat{f}_1 &= \gamma_1 + k_1 A\hat{x}_1 \end{aligned}$$

where the estimation of x_1 is obtained with

$$\begin{aligned} \dot{\gamma}_2 &= -k_2(\gamma_2 + k_2 y_2) \\ \hat{\zeta} &= \gamma_2 + k_2 y_2 \\ \hat{x}_1 &= y_2 + \frac{1}{2ga_1^2 S^2}\left(A\hat{\zeta} + q_{32}\right)^2 \end{aligned}$$

where $\hat{\zeta}$ is an estimation of $\dot{y}_2$.

And finally to estimate f_2 the following ROO is used:

$$\begin{aligned} \dot{\gamma}_3 &= -k_3(q_{32} - q_{20} + u_2 + \gamma_3 + k_3 A y_1) \\ \hat{f}_2 &= \gamma_3 + k_3 A y_1 \end{aligned}$$

Real-time experiments were made in the Amira DTS200 TTS system over 3000 seconds, using the signals $y_{R1}(t) = 0.06$ and $y_{R2}(t) = 0.11$ as the references and introducing the additive faults $f_1(t) = 1 \times 10^{-6}(1 + sin(0.2te^{-0.01t}))\mathcal{U}(t - 220)$ and $f_2(t) = 1 \times 10^{-6}(1 + sin(0.05te^{-0.001t}))\mathcal{U}(t - 300)$, where $\mathcal{U}(t)$ is the step function. The design parameters were chosen as $\theta_1 = \theta_2 = 1$, $s_1 = t_1 = 1$, $s_2 = t_2 = 3$, $s_3 = t_3 = 3$, $k_1 = 1.85$, $k_2 = 0.3$, $k_3 = 22$.

The parameters of the system are $A = 0.0149\ m^2$ and $S = 5 \times 10^{-5}\ m^2$. The unknown parameters were chosen as $a_1 = 0.4385$, $a_2 = 0.7774$ and $a_3 = 0.4435$ [9].

Figures 2 and 3 compare the behavior of the tracking with and without fault-tolerance. Figure 2 illustrates how the effects of the faults affect tracking when a non-fault-tolerant control is applied. On the other hand, Figure 3 shows how using the fault-tolerant control improves the tracking of the references, while suppressing the effects of both faults. Besides, as stated, the system operates in the desired region. The estimation $\hat{x}_1(t)$ is also displayed.

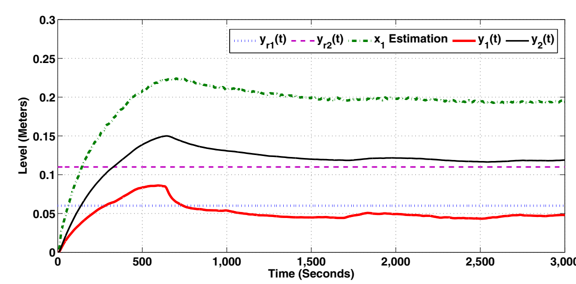

Fig. 2. Tank levels and multi-output tracking of the TTS without fault-tolerant control.

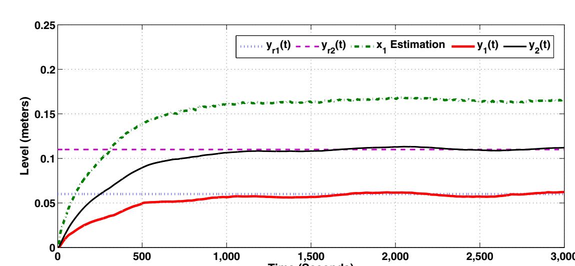

Fig. 3. Tank levels and multi-output tracking of the TTS with fault-tolerant control.

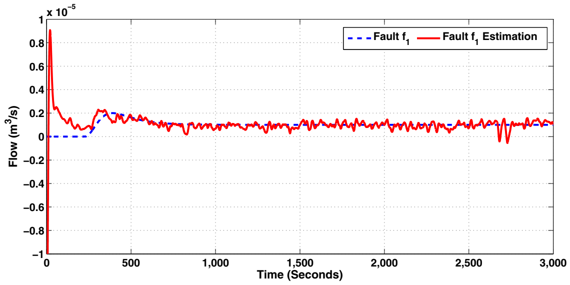

Fig. 4. Fault 1 estimation of the TTS.

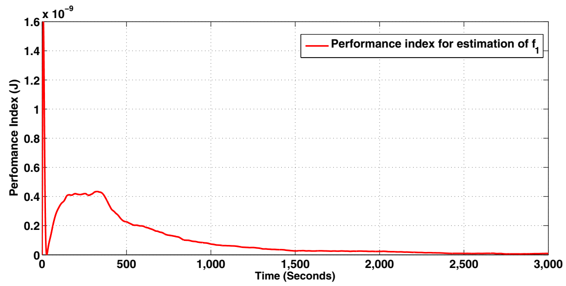

Fig. 5. Performance evaluation for the fault 1 estimation of the TTS.

Figures 4 and 6 show the estimation results of $f_1(t)$ and $f_2(t)$ respectively. These figures throw graphical valuable information, such as the magnitude of the faults and the time when they appear.

Finally, Figures 5 and 7 show the performance indices of the ROO proposed to estimate the faults. The indices were evaluated using the following cost functional:

$$J_t = \frac{1}{t+\varepsilon}\int_0^t \left\| \tilde{f}_{\bar{l}}(t) \right\|^2 dt$$

where $\varepsilon = 0.0001$.

VI. Conclusions

A new alternative fault-tolerant control is presented for nonlinear MIMO systems that can be transformed into a GOCFMM. The system in question is rewritten as a chain of integrators, from where the dynamical controllers and the estimated faults are obtained. As the proposed controller depends on the fault estimations, the negative effects generated by the faults are suppressed, while achieving tracking for all the system outputs. It was verified that the closed-loop system is asymptotically stable.

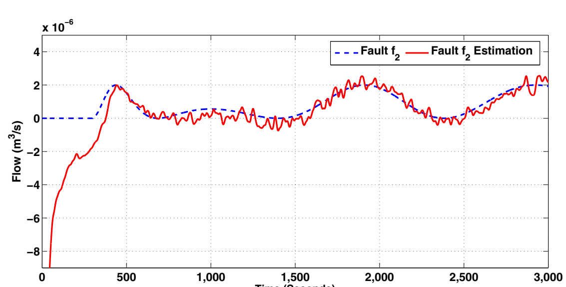

Fig. 6. Fault 2 estimation of the TTS.

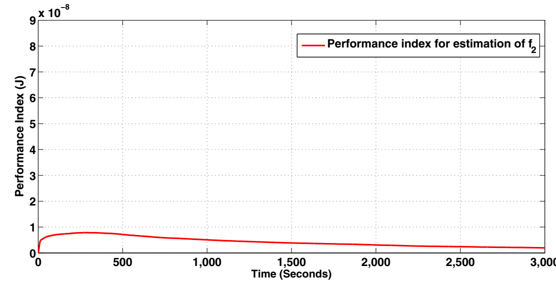

Fig. 7. Performance evaluation for the fault 2 estimation of the TTS.

In this work a complete fault diagnosis was implemented in real-time, that was capable of estimating the faults simultaneously. This fulfilled the needs of the fault-tolerant dynamical controller. In the application on the TTS, the effects of the faults were compensated online, while the multi-output tracking was performed.

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Fractional fault-tolerant dynamical controller for the fractional model of a DC motor

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Abstract: A fractional fault-tolerant dynamical controller is proposed for the commensurate fractional model of a DC motor. After performing an estimation of the faults, the rejection of their effects is done considering a fractional canonical form, which enables to perform output tracking via a fractional high-gain observer; this allows to define the fractional fault-tolerant controller. A stability analysis is performed on the overall system. Simulation results are shown to assess the proposed methodology.

Keywords: Fractional order systems, fault tolerance, fractional canonical form, dynamical controller.

Introduction

Controls with fault tolerance have a key role in many applications in automation and engineering; given the increasing application of fractional calculus in these systems, it is important to consider fault-tolerant systems with fractional dynamics. There are many different approaches to achieve such control; here it is proposed a dynamical controller for commensurate fractional systems that uses output tracking and estimations of the faults in order to eliminate their effects.

Objective

To design a fault-tolerant dynamical controller for the fractional model of a DC motor, by means of a fractional canonical form, that guarantees that the closed-loop is stable.

Methodology

Consider the following class of commensurate fractional nonlinear systems:

$$\begin{aligned} x^{(\alpha)}(t) &= g(x,u,f) \\ y(t) &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f \in \mathbb{R}^q$. Choosing the outputs as fractional differential primitive elements, this system can be represented by the following fractional generalized observability canonical form (FGOCF):

$$\begin{aligned} \eta_{ij}^{(\alpha)} &= \eta_{i+1,j} \qquad\qquad 1 \le i \le n-1 \\ \eta_{nj}^{(\alpha)} &= -L_j(\eta_1,\dots,\eta_p,u,\dots,u^{(\gamma)},f,\dots,f^{(\mu)}) \\ y_j &= \eta_{1j} \end{aligned} \tag{2}$$

The fault-tolerant control (FTC) will consist in the tracking of each output y_j with respect to a given reference y_{Rj}, by means of a dynamical controller obtained from an observer in the canonical form of the error. As it can be seen, the dynamics depends on the faults, so we will obtain estimations of them with the following fractional proportional integral reduced-order observer (FPIROO) [1]:

$$\hat{f}_i^{(\alpha)} = k_{i0}\left(f_i - \hat{f}_i\right) + \sum_{j=1}^{r'_i} k_{ij}\left(f_i^{(-j\alpha)} - \hat{f}_i^{(-j\alpha)}\right) \tag{3}$$

The faults can be estimated if they are diagnosable, this is, if they satisfy the following property [1]:

Definition. A variable $f_i \in \mathbb{R}$ satisfies the fractional algebraic observability (FAO) condition if it is a function of the first $r \in \mathbb{N}$ sequential fractional derivatives of the available output, i.e $f_i = \phi_i(y, y^{(\alpha)}, y^{(2\alpha)}, \dots, y^{(r\alpha)})$.

Let the output tracking error be $e_{1j} = y_j - y_{Rj} = \eta_{1j} - y_{Rj}$, this defines the following FGOCF for the tracking error:

$$\begin{aligned} e_{ij}^{(\alpha)} &= \eta_{i+1,j} - y_{Rj}^{(i\alpha)} \qquad\qquad 1 \le i \le n-1 \\ e_{nj}^{(\alpha)} &= -L_j(\eta_1,\dots,\eta_p,u,\dots,u^{(\gamma)},f,\dots,f^{(\mu)}) - y_{Rj}^{(n\alpha)} \end{aligned} \tag{4}$$

Imposing a linear time-invariant dynamics for the error, we obtain in a compact form:

$$\mathbf{e}_j^{(\alpha)} = E\mathbf{e}_j + \varphi_j(\mathbf{e}, \mathbf{y_R}, \mathbf{u}, \mathbf{f}) = F\mathbf{e}_j \tag{5}$$

with

$$-L_j\left(\eta_1,\dots,\eta_p,u,\dots,u^{(\gamma)},f,\dots,f^{(\mu)}\right) - y_{Rj}^{(n\alpha)} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij}$$

$$E = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \varphi_j(\cdot) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -L_j(\cdot) - y_{Rj}^{(n\alpha)} \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{1j} & -a_{2j} & \cdots & -a_{nj} \end{pmatrix}$$

The estimation $\hat{\mathbf{e}}_j$ of the tracking error is obtained by the following fractional high-gain observer (FHGO):

$$\hat{\mathbf{e}}_j^{(\alpha)} = E\hat{\mathbf{e}}_j + \varphi_j(\hat{\mathbf{e}}, \mathbf{y}_R, \mathbf{u}, \hat{\mathbf{f}}) - S_\infty^{-1}C^TC(\hat{\mathbf{e}}_j - \mathbf{e}_j) \tag{6}$$

The dynamics of the fault-tolerant controller is obtained from the FHGO (6):

$$\hat{u}_l^{(\gamma_l)} = K_l\left(\hat{\mathbf{e}}, \mathbf{y}_R, y_{Rj}^{(n\alpha)}, \hat{\mathbf{u}}, \hat{\mathbf{f}}\right) \tag{7}$$

Define the observation error as $\varepsilon_j = \hat{\mathbf{e}}_j - \mathbf{e}_j$, and the following dynamics is obtained from (5) and (6):

$$\varepsilon_j^{(\alpha)} = (E - S_\infty^{-1}C^TC)\varepsilon_j + \Phi(\varepsilon, \hat{\mathbf{e}}) \tag{8}$$

Theorem. Let system (1) be described in the FGOCF (2) composed of p subsystems. The observation dynamics corresponding to subsystem j are $\hat{\mathbf{e}}_j(t)$ and $\varepsilon_j(t)$. Let $f_i(t)$ be diagnosable for $1 \le i \le$ q and estimated by means of the dynamics of $\hat{f}_i(t)$. Let $\hat{u}_l(t)$ be the solution to

$$-L_j\left(\hat{\mathbf{e}}, \mathbf{y}_R, \hat{\mathbf{u}}, \hat{u}_l^{(\gamma_l)}, \hat{\mathbf{f}}\right) - y_{Rj}^{(n\alpha)} = -\sum_{i=1}^{n} a_{ij}\hat{e}_{ij}$$

Then, the closed-loop system with control $\hat{u} = (\hat{u}_1, \dots, \hat{u}_p)$ is stable.

Proof [sketch]. We propose the following Lyapunov function:

$$V(t) = V_1(\hat{\mathbf{e}}_j) + V_2(\varepsilon_j) + V_3(\tilde{f}_i) = \hat{\mathbf{e}}_j^T P \hat{\mathbf{e}}_j + \varepsilon_j^T S_\infty \varepsilon_j + \tilde{f}_i^T I \tilde{f}_i$$

And we obtain the following inequalities [2]:

$$V_1^{(\alpha)} \le -\left(ad_1\|\hat{\mathbf{e}}_j\| + 2\rho(\theta)d_2\|\varepsilon_j\|\right)\|\hat{\mathbf{e}}_j\|_P \le 0$$

$$V_2^{(\alpha)} \le -\theta\|\varepsilon_j\|_{S_\infty}^2 + 2\|\varepsilon_j\|_{S_\infty}\|\Phi_j\|_{S_\infty} \le -(\theta - 2\lambda)\|\varepsilon_j\|_{S_\infty}^2 \le 0$$

$$V_3^{(\alpha)} \le 2\tilde{f}_i^T\left(N_i/k_i - \tilde{f}_i\right) \le 2\tilde{f}_i^T \le 0 \qquad \blacksquare$$

Results

The methodology has been applied to the following fractional model of a DC motor [3]:

$$\begin{aligned} i_a^{(\alpha)}(t) &= 1/L_a[V_a - R_a i_a(t) - c\phi\omega(t)] \\ \omega^{(\alpha)}(t) &= 1/J[c\phi i_a(t) - T_L] \end{aligned}$$

The state-space model which is used is the following:

$$\begin{aligned} x_1^{(\alpha)} &= x_2 \\ x_2^{(\alpha)} &= 1/J[c\phi x_3 - T_L] \\ x_3^{(\alpha)} &= 1/L_a[-c\phi x_2 - R_a x_3 + u + f] \end{aligned}$$

The fault can be estimated by means of the FPIROO (3) because it is diagnosable:

$$f = c\phi y + R_a x_3 + L_a x_3^{(\alpha)} - u \qquad\qquad x_3 = 1/c\phi\left[J y^{(\alpha)} + T_L\right]$$

The FHGO (6) for tracking for this system is:

$$\begin{aligned} \hat{e}_1^{(\alpha)} &= \hat{e}_2 - 3\theta(\hat{e}_1 - e_1) \\ \hat{e}_2^{(\alpha)} &= \hat{e}_3 - 3\theta^2(\hat{e}_1 - e_1) \\ \hat{e}_3^{(\alpha)} &= c\phi/L_a J\left[R_a c\phi y - c\phi y^{(\alpha)} + R_a^2\hat{x}_3 - R_a\hat{u} - R_a\hat{f} + \hat{u}^{(\alpha)} + \hat{f}^{(\alpha)}\right] - y_R^{(3\alpha)} \\ &\quad -\theta^3(\hat{e}_1 - e_1) = -a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 \end{aligned}$$

from where the following fractional fault-tolerant controller (7) is obtained:

$$\hat{u}^{(\alpha)} = L_a J/c\phi\left(-a_1\hat{e}_1 - a_2\hat{e}_2 - a_3\hat{e}_3 + y_R^{(3\alpha)}\right) - R_a c\phi y + c\phi\hat{\xi} - R_a^2\hat{x}_3 + R_a\hat{u} + R_a\hat{f} - \hat{f}^{(\alpha)}$$

Simulations were performed with the parameters found in [3] and $\alpha = 0.9, \theta = 2000, a_1 = 8000, a_2 = 1200, a_3 = 60, k_1 = 10, k_2 = 25, y_R = 177$.

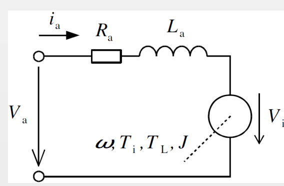

Figure 1. Circuit of the DC Motor

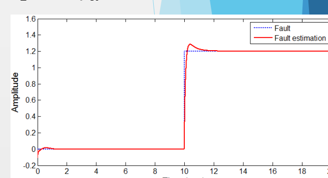

Figure 2. Fault diagnosis

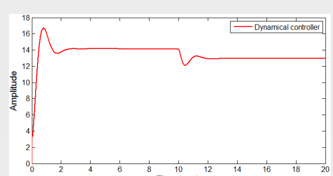

Figure 3. Dynamical controller

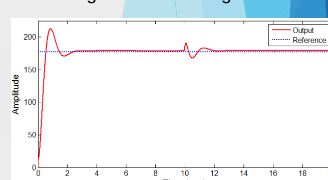

Figure 4. Output tracking

Concluding remarks

A fractional faul-tolerant dynamical controller was designed for commensurate fractional systems. It was obtained from a fractional canonical form of the output tracking error. This controller uses estimations of the faults to eliminate their effects. The overall system is stable.

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A reduced-order fractional integral observer for synchronization and anti-synchronization of fractional-order chaotic systems

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Abstract

In this paper, the problems of synchronization and anti-synchronization are solved for commensurate and incommensurate fractional chaotic systems. A reduced-order fractional integral observer is proposed for fractional systems satisfying a fractional algebraic observability condition. This observer is used as a slave system, whose states are synchronized with the ones from the chaotic system, which acts as a master. The observer uses a reduced set of measurable signals from the master system, solving also the anti-synchronization problem as a straightforward extension of the synchronization one. It is proven that the proposed observer is Mittag-Leffler stable. Numerical simulations on the fractional Lorenz and Rössler systems assess the performance of the proposed methodology.

Keywords: Reduced-order fractional integral observer; fractional synchronization; fractional anti-synchronization; fractional algebraic observability; fractional chaotic systems.

1. Introduction

Synchronization occurs when oscillatory (or repetitive) systems via some kind of interaction adjust their behaviors relative to one another so as to

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