CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS

AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE CONTROL AUTOMÁTICO

“Stability Conditions for Neutral Type Time-Delay Systems via the Delay Lyapunov Matrix”

T H E S I S

Presented by

MARCO ANTONIO GOMEZ ALVAREZ

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Thesis advisors:

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Prof. Aleksei Valerievich Egorov

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“Condiciones de Estabilidad para Sistemas con Retardos de Tipo Neutral mediante la Matriz de Lyapunov”

T E S I S

Que presenta

MARCO ANTONIO GOMEZ ALVAREZ

Para obtener el grado de

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Directores de tesis:

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Contents

Notation

$\mathbb{R}$	Real numbers set
$\mathbb{R}^n$	Space of n-dimension vectors with entries in $\mathbb{R}$.
$\mathbb{R}^{n\times n}$	Space of $n \times n$ matrices with entries in $\mathbb{R}$.
$\mathbf{Re}(s)$	Real part of an imaginary number s.
i	denotes the imaginary unit.
$C^{(1)}\left([a,b],\mathbb{R}^n\right)$	Space of $\mathbb{R}^n$-valued continuously differentiable functions on the interval $[a,b]$.
$PC\left([a,b],\mathbb{R}^n\right)$	Space of $\mathbb{R}^n$-valued piecewise continuous functions on the interval $[a,b]$.
$PC^{(1)}\left([a,b],\mathbb{R}^n\right)$	Space of $\mathbb{R}^n$-valued piecewise continuously differentiable functions on the interval $[a,b]$.
I_n	Identity matrix of dimension $n \times n$.
$0_{n\times n}$	Zero matrix of dimension $n \times n$.
A^T	Transpose of matrix A.
$A > 0$	Symmetric positive definite matrix A.
$A \geq 0$	Symmetric positive semidefinite matrix A.
$A \not> 0$	Symmetric matrix A no positive definite.
$A \not\geq 0$	Symmetric matrix A no positive semidefinite.
$\lambda_{min}(A)$	Minimum eigenvalue of the matrix A.
$\lambda_{max}(A)$	Maximum eigenvalue of the matrix A.
$[A_{ij}]_{i,j}^r$	Square block matrix with i-th row and j-th column element A_{ij}
$k = \overline{a,b}$	Means that k takes all the consecutive entire values from a to b.
$\lceil r \rceil$	Ceil function, which maps r to the least integer greater or equal to r.
$\Vert\cdot\Vert$	Euclidian norm for vectors and matrices.
$\Vert\varphi\Vert_h$	Supremum norm of the function φ, defined as $\Vert\varphi\Vert_h = \sup\limits_{\theta\in[-h,0]} \Vert\varphi(\theta)\Vert$.
$\mathcal{O}(\tau)$	Set defined as $\mathcal{O}(\tau) := [0,\infty)\backslash\{\tau + ih\}_{i=0}^{\infty}$, where $\tau \in \mathbb{R}$.
$x_t(\varphi)$	Restriction of the solution $x(t,\varphi)$ to the interval $[t-h,t]$, i.e. $x_t(\varphi) : \theta \to x(t+\theta,\varphi),\;\; \theta \in [-h,0]$.
$g(t+0)$	Evaluation of the function g at point t on the right-hand side.
$g(t-0)$	Evaluation of the function g at point t on the left-hand side.

Resumen

Las dinámica de los sistemas con retardo de tipo neutral no solo depende de los estados presentes, sino también de la razón de cambio de estados pasados. La relevancia del análisis de estabilidad de esta clase de sistemas se debe principalmente al hecho de que son adecuados para modelar una amplia variedad de fenómenos en diferentes campos de la ciencia y la tecnología. Entre los diferentes enfoques existentes para estudiar la estabilidad de tales sistemas, las funcionales de Lyapunov-Krasovskii han mostrado ser una herramienta poderosa en las últimas decadas.

En este trabajo de tesis, condiciones necesarias y suficientes de estabilidad para sistemas con retardo de tipo neutral son obtenidas en el marco de trabajo de funcionales de Lyapunov-Krasovskii de tipo completo. Las condiciones de estabilidad presentadas se distinguen por estar expresdas en términos de la matriz de Lyapunov y por requerir solamente un número finito de operaciones matemáticas para ser empleadas. Este resultado extiende el criterio de estabilidad de Lyapunov ampliamente conocido para sistemas libres de retardos a sistemas con retardos.

Abstract

The dynamic of neutral type time-delay systems not only depends on present states, but on past state rates of change as well. The relevance of the stability analysis of this class of systems is mainly due to the fact that they are suitable for modeling a wide variety of phenomena in different science and technology fields. Among the diverse approaches for studying the stability of such systems, the Lyapunov-Krasovskii functionals have shown to be a powerful tool in the last decades.

In the present thesis work, necessary and sufficient stability conditions for time-delay systems of neutral type are obtained in the Lyapunov-Krasovskii functionals of complete type framework. The distinctive of the presented stability conditions are that they are given in terms of the so-called delay Lyapunov matrix and only require a finite number of mathematical operations in order to be tested. This result extends the well-known Lyapunov stability criterion for delay free systems to the time-delay case.

Introduction

In general, a physical phenomena whose dynamic depends only on the present state can be modeled by differential equations. However, there exist dynamical systems in which its behaviour is affected by the past. These phenomena are described by differential-difference equations, also known in the literature as time-delay systems. There are different classes of linear differential-difference equations depending on their structure. In order to clarify this classification, consider the equation

$$\frac{d}{dt}(x(t) + dx(t-h)) = a_0 x(t) + a_1 x(t-h) + a_2 \int_{-h}^{0} f(\theta) x(t+\theta) d\theta,$$

where d, a_0, a_1 and a_2 are real numbers, $f(\theta)$, $\theta \in [-h, 0]$, is a continuous function and $h > 0$ is the delay. If $d = 0$ and $a_2 = 0$ then the differential-difference equation is of retarded type and if $d \neq 0$ and $a_2 = 0$, then it is called of neutral type. For the case in which $a_2 \neq 0$, it is said that differential-difference equation is of distributed type.

In this thesis work, we focus on time-delay systems of neutral type. As one can see, neutral type time-delay systems are a generalization of the retarded type ones and are characterized by the fact that the state rate of change depends not only on present states, but also on past state rates of change, i.e., there exist delays in the derivatives of the dynamical equation. This special feature makes this class of system suitable for modelling a variety of phenomena but at the same time makes its study more complex.

Some physical phenomena that are described by differential-difference equations of neutral type are presented next. They show their usefulness in different fields of the science and engineering.

- *Passivity-based PI control.* The σ-stability analysis of the proportional-integral control of a passive linear system with delayed communication channel is addressed in Castaños et al. (2017). The problem is stated as follows. Consider the linear system

  $$\begin{aligned} \dot{x}(t) =& -ax(t) + bu_1(t) \\ y_1(t) =& x(t), \end{aligned}$$

  where $u_1(t)$ denotes the input and $y_1(t)$ the output. The next PI controller is proposed:

  $$\begin{aligned} \dot{z}(t) =& u_0(t), \\ y_0(t) =& k_p u_0(t) + k_i z(t), \end{aligned}$$

  where $u_0(t)$ and $y_0(t)$ are the controller input and output, respectively.

  The introduction of delays in the communication channel and the application of the so-called scattering transformation leads to the closed-loop transfer function (see Figure 1)

  $$\frac{y_1(s)}{\overline{y}_1(s)} = \frac{2d\,(k_p s + k_i)\, b e^{-h_1 s}}{p_2(s)s^2 + p_1(s)s + p_0(s)}, \tag{1}$$

where d is an arbitrary parameter and

$$\begin{aligned} p_2(s) =& (1+e^{-sh})d + (1-e^{-sh})k_p \\ p_1(s) =& (1+e^{-sh})(bk_p+a)d + (1-e^{-sh})(bd^2+ak_p+k_i) \\ p_0(s) =& (1+e^{-sh})bk_i d + (1-e^{-sh})ak_i. \end{aligned}$$

We observe that the term p_2 includes exponential terms, which represent delayed terms in the time domain. It means that the stability properties of the closed-loop are determined by a quasipolynomial of neutral type.

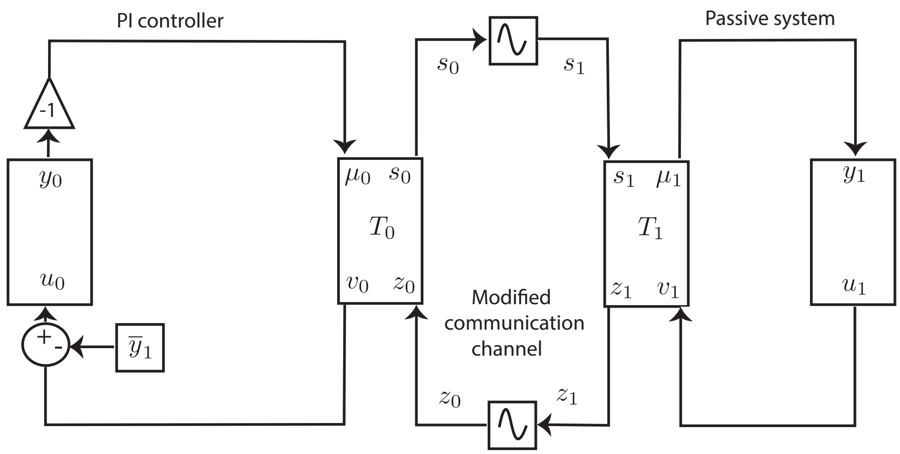

Figure 1: Proportional-integral control of a passive linear system. The blocks denoted by T_0 and T_1 correspond to the Scattering transformation.

- *The Rijke tube*. A general scheme of a Rijke tube, which is a tube with an embedded heat source, is illustrated in Figure 2. The pressure is increased by the air passing through the wire mesh and propagated along the tube, returning and affecting itself at the heating area. In other words, the pressure at a time T is affected by itself from earlier time instants. The interest in the study of the Rijke tube comes from the thermoacoustic instability phenomenon, arisen from the dynamic exchange between the heat release and the pressure variations, which indeed appears in combustors of gas turbines and aero engines.

  The obtention of a model of the thermoacoustic instability phenomenon in the Rijke tube is presented in detail in Zalluhgolu et al. (2016). The stability properties of a simplified model are described by a quasipolynomial of the form

$$p(s) = p_1(s)s + p_0(s)$$

  where

$$\begin{aligned} p_1(s) =& 2Abc^2\rho\left(R_uR_de^{-(h_u+h_d)s}-1\right), \\ p_0(s) =& -R_da(1-\gamma)e^{-h_ds} - R_ua(\gamma-1)e^{-h_us} \\ &+ R_uR_d\left(2Ac\rho + a\gamma - a\right)e^{(-h_u+h_d)s} + a - a\gamma - 2Ac_1^2\rho. \end{aligned}$$

  Here, A is the tube cross-sectional area, b is the heat release time constant, c and ρ are related to the wave speed and the air density, respectively, a represents the heat release gain, γ is the heat capacity ratio, and R_u and R_d are the acoustic reflection coefficients. The delays h_u and h_d arises from the round-trip travel times of the acoustic waves and are related with x_d and x_u (see

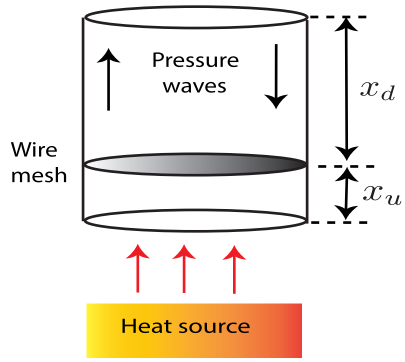

Figure 2: Rijke tube scheme

Figure 2). One observes that the quasypolinomial $p(s)$ is of neutral type, as the highest order term contain an exponential function.

- *Predator-prey model*. The well known logistic equation, given by

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right),$$

where r is the intrinsic growth rate of the specie $x(t)$ and K is the environment capacity for $x(t)$, has been useful for describing the oscillation of single-species population sizes in constant environments.

In order to obtain a better description of the physical phenomena by the model, modifications have been carried out. Some of them are the addition of delays and a dynamic term in the per capita growth rate. These modifications lead to the neutral time-delay logistic equation (Gopalsamy and Zhang (1988))

$$\dot{x}(t) = rx(t)\left(1 - \frac{(x(t-h) + \rho\dot{x}(t-h))}{K}\right).$$

Inspired by the preceding equation, the following non-linear neutral type with multiple delays predator-prey model is proposed in Kuang (1991):

$$\begin{aligned}\dot{y}(t) =& ry(t)\left(1 - \frac{y(t-\tau) + \rho\dot{y}(t-\tau)}{K}\right) - z(t)\rho(y(t))\\ \dot{z}(t) =& z(t)\left(-\alpha + \beta\rho(y(t-h))\right),\end{aligned}$$

where τ and h are positive delays, α and β are positive constants, $y(t)$ and $z(t)$ denote the prey and predator species, respectively and the function $p(y)$ is known as the predator response function. The first equation describes the interaction between the prey growth and the presence of the predator. The second one states the growth of the predator population, obviously affected by the prey population.

The stability problem whose relevance mainly relies on practical reasons is a shared interest in the examples mentioned above. The knowledge of the parameters for which a system is stable provides evident advantages for its operation, although there may be obviously other important factors, as the well enough approximated model. In addition to the practical reasons, we must say that the complexity in the stability study of neutral type time-delay systems makes them attractive to be studied from a purely theoretical perspective.

There exist diverse methods for studying the stability properties of neutral type delay systems. Most of them have been developed first for retarded differential-difference equations, and then the no always trivial generalization to the neutral case has been carried out. According to the used mathematical tools, the stability analysis methods can be classified into two approaches: frequency and temporal.

The frequency domain techniques rely on the continuity property of the roots location with respect to variations of system parameters. A consequence of this property are the stability/instability switches that occur when roots are located on the imaginary axis. For neutral systems, the continuity property of the roots depends on the stability of the difference equation. Some representative methods based on this principle are mentioned in what follows. The well known D-subdivision method introduced by Neimark (1949) for retarded systems can be applied to the neutral case, however, as it is shown by Boese (1998), the difference equation stability has to be taken into account. A more geometric perspective is used to study in detail the stability crossing curves for systems with two and three delays in Gu et al. (2005) and Gu and Naghnaeian (2011). The so called Cluster Treatment of Characteristic roots (CTCR) based on the Rekasius bilinear transformation is presented in Olgac and Sipahi (2004). The extension of these results to two and three delays is addressed in Sipahi and Olgac (2006), Olgac et al. (2008), Sipahi et al. (2010). Sweeping-test frequency by using frequency-dependent matrices is proposed in Chen (1995) (see Chapter 2 and 3 in Gu et al. (2003) for a better explanation of this method). A similar technique is introduced in Fu et al. (2006) with some numerical computation improvement. Numerical algorithms for the computation of the roots location have also been developed, see, for instance, Michiels and Vyhlídal (2005) and Chapter 2 of Michiels and Niculescu (2014).

The time domain techniques are based on the ideas introduced by Razumikhin (1956) and Krasovskii (1963), which extend the Lyapunov stability method for delay free differential equations. Through this thesis work, we use the Krasovskii approach, that is based on the proposal of functionals instead of functions. A comprehensive mathematical framework of this theory is exposed in the books Hale and Lunel (1993) and Kolmanovskii and Myshkis (1999). The main difficult that this approach faces is the adequate choice of the functionals. Indeed, the construction of Lyapunov-Krasovskii candidate functionals have led to the obtention of sufficient stability conditions of LMI type more or less conservative (see Chapter 5 in Niculescu (2001) and Chapter 3 in Fridman (2014) for the basic ideas). Reduction of the conservatism has been the subject of a vast number of contributions. A recent contribution in this direction is presented in Seuret and Gouaisbaut (2015), where an asymptotic non-conservative set of integral inequalities is introduced to propose more suitable functionals. However, up to the best knowledge of the author, the extension to the neutral case have not been done yet.

A systematic method for the computation of Lyapunov-Krasovskii functionals that consists in the prescription of a negative derivative was first proposed by Castelan and Infante (1979). Decades later, the so-called functionals of complete type were introduced. Functionals of complete type are characterized by capturing the complete state in its derivative and by being defined by the delay Lyapunov matrix. They were stated first by Rodriguez et al. (2004) for the one delay case and the integration by parts of the functional obtained there allowed presenting in Kharitonov (2005), under the differentiability assumption of the initial functions, a new expression, known as "New form for Lyapunov functionals". The multiple delays scalar case was addressed by Velázquez-Velázquez and Kharitonov (2009) and some elemental ideas for the multi-variable case were presented by Ochoa et al. (2009) and Ochoa et al. (2012). For an extensive study of these functionals the reader is referred to the book Kharitonov (2013).

The availability of the analogous of the Lyapunov matrix for neutral time-delay systems has allowed the extension of well-known results for delay free systems to the time-delay case. See, for instance, the estimation of exponential decay rate (Kharitonov (2005)), the computation of the critical parameters of the system (Ochoa et al. (2013)), the introduction of a predictor control scheme for systems with input delay (Kharitonov (2015)) and robust stability analysis (Alexandrova (2018)), just to mention a few.

An open problem is the extension of the stability criterion for linear systems, which is given in terms of the positivity of the Lyapunov matrix V, solution of the Lyapunov equation $A^T V + VA = -W$. The analogous of this result in the time-delay case has been object of study in numerous contributions in the last years. The first contribution in this direction was introduced by Mondié (2012), where a stability criterion depending on the delay Lyapunov function for the retarded type scalar equation was obtained. Multi-variable retarded and distributed type systems were addressed in Egorov and Mondié (2014) and Cuvas and Mondié (2016), where a family of necessary stability conditions that depend on the delay Lyapunov matrix was provided.

A first stability criterion within this approach for systems with multiple pointwise delays was presented in Egorov (2014) (see Egorov et al. (2017) for the distributed case). However, the sufficiency part is only theoretical, as one has to make an infinite number of mathematical operations in order to check the stability condition. In this case, adopting the terminology used in Egorov (2016), we say that the stability criterion is *infinite*. A first attempt to overcome this problem was recently reported in Egorov (2016), where a new stability criterion depending on the delay Lyapunov matrix and the fundamental matrix of the system is introduced. Its main distinctive feature is that it is *finite*, i.e., it is such that a finite number of mathematical operations is required to test it. Nonetheless, the introduction of the fundamental matrix may demand a greater computational effort in the test of the condition.

The relevance of the study of neutral type time-delay systems and the missing Lyapunov matrix based stability conditions for them in the literature lead us to the main objective of this thesis work: *Obtention of a* finite *stability criterion* uniquely *given in terms of the delay Lyapunov matrix for neutral type time-delay systems*. It is worthy of mention that even for the retarded type case the same problem has only partially solved, as the finite stability criterion introduced in Egorov (2016) also depends on the fundamental matrix of the system.

The manuscript is organized in six chapters. In Chapter 1, some particular aspects of neutral type time-delay systems, such as the smoothing and stability properties, that reveal the complexity in their study, are exposed.

Chapter 2 is devoted to the introduction of basic concepts of the system, the Lyapunov-Krasovskii functionals of complete type and the delay Lyapunov matrix. There, the introduction of a new Cauchy formula allows the relaxation of the space of the initial functions in the computation of the "New form for Lyapunov functionals", which is key in obtaining the subsequent results.

In Chapter 3, necessary stability conditions depending on the delay Lyapunov matrix for neutral type systems with one delay are presented. The key components for their attainment are the relaxation of the initial function space in the computation of the Lyapunov-Krasovskii functional of complete type carried out in Chapter 2 and the introduction of new delay Lyapunov matrix properties. Some examples illustrate the effectiveness of our approach and a comparison with other stability methods is discussed.

Chapter 4 is devoted to the non-trivial generalization of the one delay case results of Chapter 3 to the multiple commensurate case. The "New form of the Lyapunov functional" for this case is presented, up to the knowledge of the author, for the first time. We also introduce a stability equivalence between neutral type systems and difference equations in continuous time. Although the essential ideas for obtaining the results in this chapter are the same as those used in Chapter 3, the technical computations are more complex.

In Chapter 5, the main result of the thesis work is addressed: finite stability criteria given in terms of the delay Lyapunov matrix for neutral type time-delay systems with a single delay. There, two different stability criteria are presented: one depends on the delay Lyapunov and fundamental matrices, while the other only on the delay Lyapunov matrix. The seminal ideas of the result in this chapter are based on the ones introduced by Egorov (2016) and Alexandrova and Zhabko (2016). Finally, conclusions and future work are outlined in Chapter 6.

The specific contributions of this thesis work are summarized next:

1. Computation of the functional of complete type via a new Cauchy formula for neutral type systems with single and multiple commensurate delays (Section 2.3 and Section 4.2).

2. Theorem 3.2 and Theorem 4.4: Necessary stability conditions depending on the delay Lyapunov matrix for neutral type systems with single and multiple commensurate delays.

3. Theorem 5.3: *Finite* stability criterion in terms of the delay Lyapunov and fundamental matrices for neutral type systems with single delay.

4. Theorem 5.4 and Corollary 5.1: *Finite* stability criterion in terms of the delay Lyapunov matrix for neutral and retarded type systems with single delay.

We end the introductory part with a compilation of the papers derived from this work:

Journal papers:

1. A Lyapunov matrix based stability criterion for a class of time-delay systems,
   Marco A. Gomez, Alexey V. Egorov and Sabine Mondié
   Vestnik Sankt-Peterburgskogo Universiteta, Prikladnaya Matematika, Informatika, Protsessy Upravleniya, 13(4), (2017) pp. 407–416.

2. Necessary stability conditions for neutral-type systems with multiple commensurate delays,
   Marco A. Gomez, Alexey V. Egorov and Sabine Mondié
   International Journal of Control, https://doi.org/10.1080/00207179.2017.1384574 (2017).

3. Necessary stability conditions for neutral type systems with a single delay,
   Marco A. Gomez, Alexey V. Egorov and Sabine Mondié
   IEEE Transactions on Automatic Control, 62(9), (2016) pp. 4691–4697.

4. Necessary exponential stability conditions for linear periodic time-delay systems,
   Marco A. Gomez, Gilberto Ochoa and Sabine Mondié
   International Journal of Robust and Nonlinear Control, 26(18), (2016) pp. 3996–4007.

Conference papers:

1. Computation of the region of attraction for a class of nonlinear neutral type delay systems,
   Marco A. Gomez, Alexey V. Egorov and Sabine Mondié
   In 20th IFAC World Congress: Toulouse, France, July 2017.

2. Scanning the space of parameters for stability regions of neutral type delay systems: A Lyapunov matrix approach,
   Marco A. Gomez, Carlos Cuvas, Sabine Mondié and Alexey V. Egorov
   In IEEE 55th Conference on Decision and Control (CDC): Las Vegas, NV, USA, December 2016.

3. Obtention of the functional of complete type for neutral type delay systems via a new Cauchy formula,
   Marco A. Gomez, Alexey V. Egorov and Sabine Mondié
   In 13th IFAC Workshop on Time Delay Systems TDS 2016: Istanbul, Turkey, June 2016.

Chapter 1

Particularities of neutral type time-delay systems

Neutral type time-delay systems differ from the retarded type by some special features. This chapter is dedicated to present some of these particular aspects, which indeed reveal the complexity of this class of systems. The smoothing properties are exposed in Section 1.1. In Section 1.2, we briefly explain the role of the difference equation stability in the study of the stability of neutral systems. Finally, the non-equivalence between asymptotic and exponential stability is illustrated by one example in Section 1.3.

1.1 Smoothing property

Neutral type time-delay systems are known for the absence of the smoothing property. In order to illustrate this, we first consider the retarded type equation

$$\dot{x}(t) = x(t-1),$$

and the initial function given by

$$\varphi(t) = 1, \ -1 \leq t \leq 0.$$

Applying the step-by-step method (see Bellman and Cooke (1963)), one obtains, for $0 \leq t \leq 1$,

$$x(t) = t,$$

for $1 \leq t \leq 2$

$$x(t) = 1 + (t-1) + \frac{(t-2)^2}{2},$$

and by induction,

$$x(t) = \sum_{k=0}^{N} \frac{(t-k)^k}{k!}, \ N \leq t \leq N+1, \ N = 0, 1, \ldots$$

It follows from the previous equation that $\dot{x}(t)$ is a continuous function except uniquely at $t = 0$, (indeed, $\dot{x}(-0) = 0$ and $\dot{x}(+0) = 1$) the second derivative of $x(t)$ is discontinuous at $t = 2$, the third derivative is not continuous at $t = 3$, and so on. In this way, the solution $x(t)$ becomes smoother on every interval $[Nh, (N+1)h]$. The same can be said in general for every initial function $\varphi \in C^{(1)}([-1, 0], \mathbb{R})$. As shown next, this is one property that neutral type system does not have.

Consider the neutral type system

$$\dot{x}(t) = \dot{x}(t-1).$$

We take an arbitrary initial function $\varphi \in C^{(1)}([-1,0],\mathbb{R})$ and apply once again the step-by-step method. For $0 \leq t < 1$,

$$\int_0^t \dot{x}(s)ds = \int_0^t \dot{x}(s-1)ds$$
$$x(t) = \varphi(0) - \varphi(-1) + \varphi(t-1).$$

For the interval $1 \leq t < 2$, we get

$$x(t) = 2\left(\varphi(0) - \varphi(-1)\right) + \varphi(t-2).$$

By induction, one can obtain a general solution, which is given by

$$x(t) = j(\varphi(0) - \varphi(-1)) + \varphi(t-j),\ (j-1) \leq t < j,\ j = 1, 2, \ldots.$$

From the previous equation, one can note that if $\dot{\varphi}(-1) \neq \dot{\varphi}(0)$ is not satisfied, then the function $\dot{x}(t)$ contains discontinuity points not only at $t = 0$ as in the retarded type case, but also at $t = j$, $j = 1, 2, \ldots$.

1.2 Difference equation stability

As was indicated in the introduction, neutral type systems are characterized by involving a difference equation in the derivative. For instance, consider the linear system

$$\frac{d}{dt}(x(t) - Dx(t-h)) = A_0 x(t) + A_1 x(t-h), \tag{1.1}$$

where $A_0, A_1, D \in \mathbb{R}^{n\times n}$ and h is the delay. A necessary stability condition for this system is the stability of the difference equation

$$x(t) = Dx(t-h).$$

Indeed, let s_d be an eigenvalue of D. It generates a chain of eigenvalues of system (1.1) of the form (see Chapter 12 of Bellman and Cooke (1963) or Chapter 6 of Kharitonov (2013))

$$s_r = \widehat{s} + i\frac{2r\pi}{h} + \xi_r,\ r = \pm 1, \pm 2, \ldots,$$

where $\widehat{s}$ is a complex number such that $s_d = e^{-\widehat{s}h}$ and $\xi_r \to 0$ as $|r| \to \infty$. It implies that

$$\lim_{|r|\to\infty} \mathbf{Re}(s_r) \to \mathbf{Re}(\widehat{s}),$$

and in turn that, if $|s_d| > 1$, system (1.1) is unstable. This fact motivates the classical Schur stability assumption of the matrix D (i.e., that all its eigenvalues lie inside the unit circle) in the stability analysis methods of neutral type systems.

For neutral type-delay systems with m delays the stability study of a difference equation of the form

$$x(t) = \sum_{i=1}^{m} D_i x(t-h_i), \tag{1.2}$$

where $D_i \in \mathbb{R}^{n\times n}$, has to be considered. In this case, equation (1.2) is stable if and only if (Hale and Lunel (1993))

$$\sup \mathbf{Re}(s) : \left\{ \det\left(sI - \sum_{i=1}^{m} se^{-sh_i} D_i \right) = 0 \right\} < 0. \tag{1.3}$$

In the next section, a consequence arising from a non exponentially stable difference equation is illustrated by one example.

1.3 On the exponential stability

Unlike the retarded type systems, there may exist systems of neutral type with all their characteristic roots in the left half plane that are not exponentially stable. The following example illustrates this.

Example 1.1. *(Fridman (2014), Hale and Lunel (2002)) Let the neutral system be*

$$\frac{d}{dt}\left(x(t)+x(t-1)\right)=-x(t). \tag{1.4}$$

We show first that system (1.4) *has all its characteristic roots on the left half plane. The quasipolynomial is given by*

$$p(s)=s+1+se^{-s},$$

which can be written as

$$\bar{p}(s)=1+e^{-s}+\frac{1}{s}. \tag{1.5}$$

Observe that $s=0$ is not a characteristic root. Considering the root $s=\alpha+i\beta$, we get

$$e^{-\alpha}e^{-i\omega}=-1-\frac{1}{\alpha+i\omega},$$

and taking the module on both sides, we arrive at

$$e^{-\alpha}=\left|1+\frac{1}{\alpha+i\omega}\right|\geq\left|1+\frac{\alpha}{\alpha^2+\omega^2}\right|.$$

For $\alpha=0$, we get the equality $1=\left|1+\dfrac{1}{i\omega}\right|$ which is not satisfied for finite ω. For $\alpha>0$, clearly

$$e^{-\alpha}\ngeq\left|1+\frac{\alpha}{\alpha^2+\omega^2}\right|,$$

which means that the characteristic roots have no positive real part. The characteristic roots of the quasipolynomial $p(s)$ are shown in Figure 1.1.

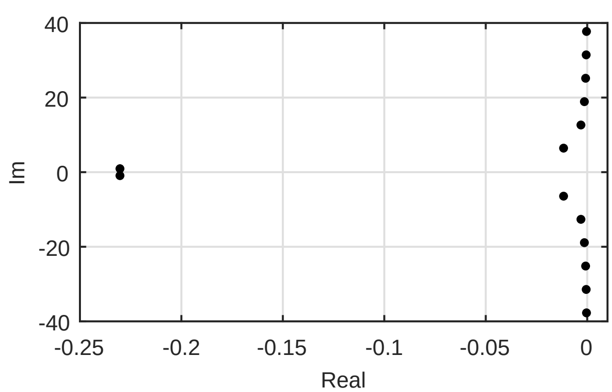

Figure 1.1: Characteristic roots of system (1.4)

Notice now that the roots of the difference equation quasipolynomial

$$p_d(s_d)=1+e^{-s_d},$$

are given by

$$s_{d_k}=(2k+1)\pi i,\ k=0,\pm1,\pm2\ \ldots, \tag{1.6}$$

i.e., $\mathbf{Re}(s_d) = 0$.

It is well known that for real constants c_1 *and* c_2, $c_1 < \mathbf{Re}(s) < c_2$, *which implies that on this band, the function* $|e^s|$ *is bounded and therefore* $e^s + 1 \to 0$ *as* $|s| \to \infty$. *By equation* (1.5)*, it means that there is a chain of eigenvalues* s_k *of system* (1.4) *and eigenvalues* s_{d_k} *of* $1 + e^{-s}$ *such that* $s_k - s_{d_k} \to 0$ *as* $k \to \infty$. *Hence, it follows from the equality in* (1.6)*, that system* (1.4) *has eigenvalues approaching to the imaginary axis, which shows that despite its characteristic roots are in the left half plane, the system is not exponentially stable.*

1.4 Conclusion

We present some subtleties of systems of neutral type that reveal their complexity. Discontinuities of the solution expose the careful analysis demanded for their study. It is shown that the stability analysis requires the stability study of difference equations in continuous time, and that for some cases exponential and asymptotic stability are not equivalent.

Chapter 2

Preliminary concepts: Lyapunov-Krasovskii functionals of complete type

We consider a linear neutral type time-delay system of the form

$$\frac{d}{dt}\left(x(t) - Dx(t-h)\right) = A_0 x(t) + A_1 x(t-h),\ t \geq 0. \tag{2.1}$$

The solution $x(\cdot, \varphi)$ of system (2.1) satisfies the following:

1. $x(\theta, \varphi) = \varphi(\theta)$, $\theta \in [-h, 0]$.
2. It is piecewise continuous and satisfies system (2.1) on $t \in [0, \infty)$ almost everywhere.
3. Sewing condition: the function $x(t, \varphi) - Dx(t-h, \varphi)$ is continuous with respect to t (right continuous at $t = 0$).

For initial functions φ from the space $PC\left([-h, 0], \mathbb{R}^n\right)$, we assume right-continuity. In this case, the solutions are right-continuous everywhere.

The main contributions of the thesis work rely on the Lyapunov-Krasovskii functionals of complete type framework developed in the last decade for systems of the form (2.1). In this chapter, we provide the elemental concepts of this framework, which is used in the successive chapters. For the sake of completeness, we include the proofs of some basic results.

The chapter is organized as follows. In the next section, we give some basic definitions concerning to the fundamental matrix. Section 2.2 is dedicated to present the so-called delay Lyapunov matrix and its basic properties. In Section 2.3, a new Cauchy formula that requires neither the differentiability nor even the continuity of the initial function is introduced and used for computing the functional of complete type. The relaxation of the space of initial functions is crucial for the obtention of the necessary stability conditions in Chapter 3 since an important element is the choice of a particular class of piecewise continuous initial function that depends on the fundamental matrix.

2.1 Basic definitions

A key element in the subsequent results is the fundamental matrix of system (2.1), denoted by $K(t)$. It is defined as follows:

Definition 2.1 (Bellman and Cooke (1963)). *The fundamental matrix $K(t)$ satisfies the equation*

$$\frac{d}{dt}\left(K(t) - K(t-h)D\right) = K(t)A_0 + K(t-h)A_1, \tag{2.2}$$

with the initial conditions $K(\theta) = 0$ for $\theta \in [-h, 0)$, $K(0) = I$, and the sewing condition, i.e.,

$$K(t) - K(t-h)D \;\textit{ is continuous for }\; t > 0,$$

and right-continuous at $t = 0$.

From the Laplace transform of equation (2.2), one can see that the fundamental matrix is also a solution of the equation

$$\frac{d}{dt}\left(K(t) - DK(t-h)\right) = A_0K(t) + A_1K(t-h),\; t \geq 0,\; a.e.$$

From similar arguments of those exposed in Section 1.1, the fundamental matrix includes discontinuity points at $t = jh$, $j = 0, 1, 2, \ldots$ since $K(-0) \neq K(+0)$. The jumps are described in the following lemma.

Lemma 2.1 (Bellman and Cooke (1963), Kharitonov (2013)). *The fundamental matrix $K(t)$ has jumps at points jh, $j = 0, 1, 2, \ldots$ and their size values are determined by*

$$\Delta K(jh) = D^j, \tag{2.3}$$

where $\Delta K(jh) = K(jh + 0) - K(jh - 0)$.

Moreover, the value of $K(t)$ at a discontinuity point coincides with its right-hand side, i.e.

$$K(jh) = K(jh + 0).$$

Proof. Let us prove by induction that (2.3) holds. The case $j = 0$ immediately follows from the initial condition $K(0) = I$. As the difference $K(t) - K(t-h)D$ is continuous, then $\Delta K(t) - \Delta K(t-h)D = 0$ for $t \geq 0$. It implies that

$$\Delta K(t) = \Delta K(t-h)D, \tag{2.4}$$

and that for any number $q > 0$

$$\Delta K(qh) = D^q.$$

Consider $t = (q+1)h$ in (2.4). From the previous equality we have that

$$\Delta K((q+1)h) = \Delta K(qh)D = D^{q+1}.$$

The above equality finishes the prove. □

The Cauchy formula for system (2.1) introduced in Bellman and Cooke (1963) is determined by the fundamental matrix. We recall it in the next lemma.

Lemma 2.2 (Bellman and Cooke (1963)). *Given an initial function $\varphi \in PC^{(1)}\left([-h, 0], \mathbb{R}^n\right)$, system* (2.1) *admits the solution*

$$x(t, \varphi) = \left(K(t) - K(t-h)D\right)\varphi(0) + \int_{-h}^{0} K(t-h-\theta)\left(D\varphi'(\theta) + A_1\varphi(\theta)\right) d\theta. \tag{2.5}$$

Proof. Set $\xi \in (0, t)$, where $t > 0$. Note that

$$\begin{aligned}
J(\xi) = \frac{\partial}{\partial \xi}\left(K(t-\xi) - K(t-\xi-h)D\right)x(\xi, \varphi) &= -\left(K(t-\xi)A_0 + K(t-\xi-h)A_1\right)x(\xi, \varphi) \\
&\quad + K(t-\xi)\left(Dx'(\xi-h, \varphi) + A_0x(\xi, \varphi) + A_1x(\xi-h, \varphi)\right) - K(t-\xi-h)Dx'(\xi, \varphi)
\end{aligned}$$
$$= -K(t-\xi-h)A_1x(\xi, \varphi) + K(t-\xi)Dx'(\xi-h, \varphi) + K(t-\xi)A_1x(\xi-h, \varphi) - K(t-\xi-h)Dx'(\xi, \varphi).$$

Integrating from 0 to t, we get

$$x(t,\varphi)-(K(t)-K(t-h)D)\varphi(0)=-\int_0^t K(t-\xi-h)\left(A_1x(\xi,\varphi)+Dx'(\xi,\varphi)\right)d\xi$$
$$+\int_0^t K(t-\xi)\left(A_1x(\xi-h,\varphi)+Dx'(\xi-h,\varphi)\right)d\xi$$
$$=\int_t^0 K(t-\xi-h)\left(A_1x(\xi,\varphi)+Dx'(\xi,\varphi)\right)d\xi+\int_{-h}^{t-h} K(t-\xi-h)\left(A_1x(\xi,\varphi)+Dx'(\xi,\varphi)\right)d\xi.$$

Equality (2.5) is directly deduced from the previous equation and from the fact that $x(t)=\varphi(t)$ for $t\in[-h,0]$ and $K(\theta)=0$ for $\theta<0$. $\square$

In what follows, we use the next definition of exponential stability.

Definition 2.2 (Bellman and Cooke (1963)). *System* (2.1) *is said to be exponentially stable, if every solution of the system satisfies the inequality*

$$\|x(t,\varphi)\|\leq\gamma e^{-\sigma t}\|\varphi\|_h,\ t\geq 0,$$

for $\sigma>0$ and $\gamma\geq 1$.

2.2 The delay Lyapunov matrix

Under exponential stability assumption of system (2.1), the matrix function

$$U(\tau)=\int_0^\infty K^T(t)WK(t+\tau)dt,\ \tau\in\mathbb{R}, \tag{2.6}$$

is defined as the delay Lyapunov matrix associated with a symmetric matrix W (see Rodriguez et al. (2004), Kharitonov (2005) and Kharitonov (2013)). The Lyapunov matrix is continuous for $\tau\in\mathbb{R}$ (Lemma 6.3 in Kharitonov (2013)), continuously differentiable on $\tau\in\mathbb{R}\backslash\Omega$, where $\Omega=\{jh\mid j=0,\pm1,\ldots\}$, and satisfies the following properties:

1. Dynamic
$$U'(\tau)-U'(\tau-h)D=U(\tau)A_0+U(\tau-h)A_1,\ \tau\geq 0,\ \tau\in\mathbb{R}\backslash\Omega. \tag{2.7}$$
2. Symmetry
$$U^T(\tau)=U(-\tau),\ \ \tau\geq 0. \tag{2.8}$$
3. Algebraic
$$A_0^TU(0)+U(0)A_0+A_1^TU(h)+U(-h)A_1$$
$$-\left(A_0^TU(-h)+A_1^TU(0)\right)D-D^T\left(U(h)A_0+U(0)A_1\right)=-W. \tag{2.9}$$

The algebraic property can also be written as

$$-W=\Delta U'(0)-D^T\Delta U'(0)D, \tag{2.10}$$

where $\Delta U'(0)=U'(+0)-U'(-0)$.

Because of the fundamental matrix jumps at $t=jh$, $j=0,1,2,\ldots$, the first derivative of the delay Lyapunov matrix is written as (Kharitonov (2005))

$$U'(\tau)=\int_{\mathcal{O}(-\tau)}K^T(t)W\frac{d}{d\tau}K(t+\tau)dt+\sum_{j=0}^{\infty}K^T(jh-\tau)W\Delta K(jh),\ \ \tau\in\mathbb{R}\backslash\Omega. \tag{2.11}$$

An expression for the second derivative is obtained as follows. Consider first the change of variable $\xi = t+\tau$ in the preceding equality:

$$U'(\tau) = \int_{\mathcal{O}(0)} K^T(\xi-\tau) W \frac{d}{d\xi} K(\xi) d\xi + \sum_{j=0}^{\infty} K^T(jh-\tau) W \Delta K(jh), \ \tau \in \mathbb{R}\backslash\Omega.$$

Differentiating with respect to τ, we get

$$U''(\tau) = \frac{d}{d\tau}\int_{\mathcal{O}(0)} K^T(\xi-\tau) W \frac{d}{d\xi} K(\xi) d\xi + \sum_{j=0}^{\infty} \frac{d}{d\tau} K^T(jh-\tau) W \Delta K(jh), \ \tau \in \mathbb{R}\backslash\Omega. \tag{2.12}$$

The above expressions will be useful in Subsection 2.3.2.

It follows from equation (2.6) and the discontinuities of the fundamental matrix that the first derivative of the delay Lyapunov matrix presents jumps, which are characterized in the next lemma.

Lemma 2.3 (Kharitonov (2005))**.** *Let system* (2.1) *be exponentially stable. The jump size values of the first derivative of the Lyapunov matrix $U(\tau)$ at points $j = 0, 1, 2, \ldots$, are given by*

$$\Delta U'(jh) = \Delta U'(0) D^j, \tag{2.13}$$

where $\Delta U'(jh) = U'(jh+0) - U'(jh-0)$.

Proof. From the algebraic property (2.10), we have that $\Delta U'(0)$ satisfies

$$\Delta U'(0) = -\sum_{j=0}^{\infty} \left(D^j\right)^T W D^j.$$

Indeed,

$$\Delta U'(0) - D^T \Delta U'(0) D = -\sum_{j=0}^{\infty} \left(D^j\right)^T W D^j + \sum_{j=0}^{\infty} \left(D^{j+1}\right)^T W D^{j+1} = -W.$$

Now, by equation (2.11), for $l = 0, 1, 2, \ldots$

$$\begin{aligned} U'(lh+0) - U'(lh-0) &= \sum_{j=0}^{\infty} \left(K(jh-lh-0) - K(jh-lh+0)\right)^T W \Delta K(jh) \\ &= -\sum_{j=0}^{\infty} \left(\Delta K((j-l)h)\right)^T W \Delta K(jh). \end{aligned}$$

Consider the change of variable $k = j - l$ in the sum. From the fact that $K(\theta) = 0$ for $\theta < 0$ and equation (2.3), we arrive at

$$\Delta U'(lh) = -\sum_{k=-l}^{\infty} \Delta K(kh)^T W \Delta K((k+l)h) = -\sum_{k=0}^{\infty} (D^k)^T W D^{k+l} = \Delta U'(0) D^l, \ l = 0, 1, \ldots.$$

□

Now, let us introduce a new definition of the delay Lyapunov matrix. This definition, based on the basic properties, allows one to avoid the assumption of the exponential stability of system (2.1).

Definition 2.3 (Kharitonov (2013))**.** *The delay Lyapunov matrix $U(\tau)$, $\tau \in \mathbb{R}$, of system* (2.1)*, associated with a given symmetric matrix W, is a continuous matrix satisfying the dynamic* (2.7)*, symmetry* (2.8) *and algebraic* (2.9) *properties.*

We enunciate the conditions under which the Lyapunov matrix exists and is unique. In order to do this, we first remind the following definition.

Definition 2.4 (Kharitonov (2013)). *System* (2.1) *satisfies the Lyapunov condition if there exists* $\varepsilon > 0$ *satisfying*

$$|s_1 + s_2| > \varepsilon,$$

i.e., if the following holds:

1. *System* (2.1) *does not have eigenvalues* s_0 *such that* $-s_0$ *is also an eigenvalue.*

2. *The matrix* D *does not have eigenvalues* s_d *such that* s_d^{-1} *is also an eigenvalue.*

Theorem 2.1. *(Kharitonov (2013)) System* (2.1) *admits a unique Lyapunov matrix associated with a given symmetric matrix* W *if and only if the Lyapunov condition holds.*

The case in which the Lyapunov condition is not satisfied is recalled in the next theorem.

Theorem 2.2. *(Kharitonov (2013)) If system* (2.1) *does not satisfy the Lyapunov condition, then there exists a symmetric matrix* W *for which equation* (2.7) *has no solution that satisfies properties* (2.8) *and* (2.9).

2.3 Lyapunov-Krasovskii functional of complete type

The functional of complete type for neutral type delay systems is first introduced in Rodriguez et al. (2004) and it is determined by

$$v(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) \left(W_1 + (h+\theta)W_2\right) \varphi(\theta) d\theta, \ \varphi \in PC^{(1)}\left([-h, 0], \mathbb{R}^n\right), \tag{2.14}$$

where

$$\begin{aligned} v_0(\varphi) = \varphi^T(0) &\left(U(0) - D^T U(h) - U(-h)D + D^T U(0) D\right) \varphi(0) \\ &+ 2\varphi^T(0) \int_{-h}^{0} \left(U(-h-\theta) - D^T U(-\theta)\right) \left(D\varphi'(\theta) + A_1\varphi(\theta)\right) d\theta \\ &+ \int_{-h}^{0} \left(D\varphi'(\theta_1) + A_1\varphi(\theta_1)\right)^T \int_{-h}^{0} U(\theta_1 - \theta_2) \left(D\varphi'(\theta_2) + A_1\varphi(\theta_2)\right) d\theta_2 d\theta_1. \end{aligned} \tag{2.15}$$

In general, the procedure for obtaining an explicit expression of the functional $v_0(\varphi)$, under the assumption that the system is exponentially stable, consists of two steps:

1. The prescription of a negative quadratic time derivative of the functional.

2. Its integration and the substitution of the Cauchy formula (2.5).

By assuming that the initial functions belong to the space of continuous functions, i.e., $\varphi \in C^{(1)}\left([-h, 0], \mathbb{R}^n\right)$, and integrating by parts, the functional $v_0(\varphi)$ is transformed to the following expression in Kharitonov (2005):

$$\begin{aligned}
v_0(\varphi) = &\left(\varphi(0) - D\varphi(-h)\right)^T U(0) \left(\varphi(0) - D\varphi(-h)\right) \\
&+ 2\left(\varphi(0) - D\varphi(-h)\right)^T \int_{-h}^{0} \left(U'(-h-\theta)D + U(-h-\theta)A_1\right)\varphi(\theta)d\theta \\
&+ \int_{-h}^{0}\varphi^T(\theta_1)\int_{-h}^{0} A_1^T U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2 d\theta_1 \\
&- \int_{-h}^{0}\varphi^T(\theta_1)\int_{-h}^{0}\Big(D^T U'(\theta_1-\theta_2)A_1 - A_1^T U'(\theta_1-\theta_2)D\Big)\varphi(\theta_2)d\theta_2 d\theta_1 \\
&- \int_{-h}^{0}\varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0} U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2 + \int_{\theta_1+0}^{0} U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2\right)d\theta_1 \\
&- \int_{-h}^{0}\varphi^T(\theta)D^T\Delta U'(0)D\varphi(\theta)d\theta. \qquad (2.16)
\end{aligned}$$

One observes that in (2.16) there are not derivatives on the initial functions, but there are on the delay Lyapunov matrix. This transformed functional is called "New form of the functional" and it is key in the obtention of the main result of Chapter 3. However, in order to use it, a relaxation of the space to which initial functions belong is needed. The next subsections are devoted to this task.

2.3.1 A new Cauchy formula

In this section, we present a new Cauchy formula of system (2.1) given in terms of the fundamental matrix. It is shown that this formula is a generalization of (2.5).

Theorem 2.3. *Given an initial function $\varphi \in PC\left([-h,0],\mathbb{R}^n\right)$, the solution $x(t,\varphi)$ of system* (2.1) *is determined by*

$$\begin{aligned}
x(t,\varphi) = &K(t)\left(\varphi(0) - D\varphi(-h)\right) + \int_{-h}^{0} K(t-h-\theta)A_1\varphi(\theta)d\theta \\
&+ \frac{d}{dt}\left(\int_{-h}^{0} K(t-\theta-h)D\varphi(\theta)d\theta\right),\ t \geq 0. \qquad (2.17)
\end{aligned}$$

Proof. We consider the term

$$J(\xi) = \left(K(t-\xi) - K(t-\xi-h)D\right)\int_0^{\xi} x(\theta)d\theta,\ \xi \in [0,t),\ t > 0.$$

Integrating system (2.1), we get the expression

$$x(t) = Dx(t-h) + \varphi(0) - D\varphi(-h) + A_0\int_0^t x(\theta)d\theta + A_1\int_{-h}^{t-h} x(\theta)d\theta,\ t \geq 0.$$

By using (2.2) and the above equality, we get

$$\begin{aligned}
\frac{d}{d\xi}J(\xi) = &-K(t-\xi-h)A_1\int_0^{\xi} x(\theta)d\theta \\
&+ K(t-\xi)\Big(\varphi(0) - D\varphi(-h) + Dx(\xi-h) + A_1\int_0^{\xi} x(\theta-h)d\theta\Big) - K(t-\xi-h)Dx(\xi).
\end{aligned}$$

Integrating from 0 to t, we arrive at

$$\begin{aligned}
\int_0^t x(\theta)d\theta = &-\int_0^t K(t-\xi-h)A_1\int_0^{\xi} x(\theta)d\theta d\xi + \int_0^t K(t-\xi)A_1\int_0^{\xi} x(\theta-h)d\theta d\xi \\
&- \int_0^t K(t-\xi-h)Dx(\xi)d\xi + \int_0^t K(t-\xi)Dx(\xi-h)d\xi + \int_0^t K(t-\xi)\left(\varphi(0) - D\varphi(-h)\right)d\xi,
\end{aligned}$$

and by the change of variables $\eta = t-\xi-h$ and $\eta = t-\xi$, in the first and second integrals, respectively, we get

$$-\int_0^t K(t-\xi-h)A_1\int_0^\xi x(\theta)d\theta d\xi + \int_0^t K(t-\xi)A_1\int_0^\xi x(\theta-h)d\theta d\xi =$$
$$= -\int_{-h}^{t-h}\int_0^{t-\eta-h} K(\eta)A_1 x(\theta)d\theta d\eta + \int_0^t\int_{-h}^{t-\eta-h} K(\eta)A_1 x(\theta)d\theta d\eta.$$

As $K(\eta)=0$ for $\eta<0$, the last equality yields

$$-\int_{-h}^{t-h}\int_0^{t-\eta-h} K(\eta)A_1 x(\theta)d\theta d\eta + \int_0^t\int_{-h}^{t-\eta-h} K(\eta)A_1 x(\theta)d\theta d\eta =$$
$$= \int_{t-h}^t\int_{-h}^{t-\eta-h} K(\eta)A_1\varphi(\theta)d\theta d\eta + \int_0^{t-h}\int_{-h}^0 K(\eta)A_1\varphi(\theta)d\theta d\eta,$$

hence,

$$\int_0^t x(\theta)d\theta = \int_{t-h}^t\int_{-h}^{t-\eta-h} K(\eta)A_1\varphi(\theta)d\theta d\eta + \int_0^{t-h}\int_{-h}^0 K(\eta)A_1\varphi(\theta)d\theta d\eta$$
$$+ \int_{-h}^0 K(t-\theta-h)D\varphi(\theta)d\theta + \int_0^t K(\theta)d\theta\left(\varphi(0)-D\varphi(-h)\right).$$

Differentiating with respect to time we obtain (2.17). □

We show now that in fact the Cauchy formula given in Lemma 2.2 is a special case of the one introduced in Theorem 2.3. In order to do this consider an initial function φ differentiable almost everywhere on $[-h,0]$ and differentiate the third term in (2.17):

$$\frac{d}{dt}\left(\int_{-h}^0 K(t-\theta-h)D\varphi(\theta)d\theta\right) = -\frac{d}{dt}\left(\int_{t+h}^t K(\xi-h)D\varphi(t-\xi)d\xi\right)$$
$$= -K(t-h)D\varphi(0) + K(t)D\varphi(-h) + \int_{-h}^0 K(t-\theta-h)D\varphi'(\theta)d\theta.$$

Substituting the preceding equality in (2.17) we arrive at the Cauchy formula (2.5).

2.3.2 Technical results

Some results that are useful in the computation of the complete type functional are in order. Let us introduce the terms:

$$J_1(\xi) = \int_0^\infty K^T(t-\xi-h)W\frac{d}{dt}\left(\int_{-h}^0 K(t-\theta-h)D\varphi(\theta)d\theta\right)dt, \tag{2.18}$$

$$J_2 = \int_0^\infty \frac{d}{dt}\left(\int_{-h}^0 \varphi^T(\theta_1)D^TK^T(t-\theta_1-h)d\theta_1\right)W\frac{d}{dt}\left(\int_{-h}^0 K(t-\theta_2-h)D\varphi(\theta_2)d\theta_2\right)dt. \tag{2.19}$$

Observe that these terms are determined by an improper integral and the fundamental matrix of the system. Following the definition (2.6), we prove that they can be expressed in terms of the delay Lyapunov matrix.

Proposition 2.1. *For $\xi\in\mathbb{R}$,*

$$J_1(\xi) = \int_{-h}^0 U'(\xi-\theta)D\varphi(\theta)d\theta.$$

Proof. We write the right hand side of (2.18) as follows:

$$\int_0^{\infty} K^T(t-\xi-h)W\frac{d}{dt}\left(\int_{-h}^{0} K(t-\theta-h)D\varphi(\theta)d\theta\right)dt =$$
$$= \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} K^T(t-\xi-h)W\left(\frac{d}{dt}\int_{-h}^{t-(j+1)h-0} K(t-\theta-h)D\varphi(\theta)d\theta\right.$$
$$\left. + \frac{d}{dt}\int_{t-(j+1)h+0}^{0} K(t-\theta-h)D\varphi(\theta)d\theta\right)dt$$
$$= \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} K^T(t-\xi-h)W\left(\int_{-h}^{t-(j+1)h-0} \frac{d}{dt}K(t-\theta-h)D\varphi(\theta)d\theta\right.$$
$$\left. + \int_{t-(j+1)h+0}^{0} \frac{d}{dt}K(t-\theta-h)D\varphi(\theta)d\theta + \Delta K(jh)D\varphi(t-(j+1)h)\right)dt.$$

Then, by changing the integration order, and as

$$\frac{d}{dt}K(t-\theta-h) = -\frac{d}{d\theta}K(t-\theta-h),$$

we arrive at

$$J_1(\xi) = \int_{-h}^{0}\left(-\int_{\mathcal{O}(\theta)} K^T(t-\xi-h)W\frac{d}{d\theta}K(t-\theta-h)dt + \sum_{j=0}^{\infty} K^T(\theta+jh-\xi)W\Delta K(jh)\right)D\varphi(\theta)d\theta.$$

By the change of variable $\eta = t-\xi-h$ in the first integral and by using (2.11), we get

$$J_1(\xi) = \int_{-h}^{0}\left(-\int_{\mathcal{O}(\theta-\xi)} K^T(\eta)W\frac{d}{d\theta}K(\eta+\xi-\theta)d\eta + \sum_{j=0}^{\infty} K^T(\theta+jh-\xi)W\Delta K(jh)\right)D\varphi(\theta)d\theta$$
$$= \int_{-h}^{0} U'(\xi-\theta)D\varphi(\theta)d\theta.$$

□

Proposition 2.2. *The following equality holds:*

$$J_2 = -\int_{-h}^{0}\varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0} U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2 + \int_{\theta_1+0}^{0} U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2\right)d\theta_1$$
$$- \int_{-h}^{0}\varphi^T(\theta)D^T\Delta U'(0)D\varphi(\theta)d\theta.$$

Proof. We rewrite the right hand side of the term J_2 in (2.19) as follows:

$$J_2 = \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h}\frac{d}{dt}\left(\int_{-h}^{0}\varphi^T(\theta_1)D^TK^T(t-\theta_1-h)d\theta_1\right)W\left(\int_{-h}^{t-(j+1)h-0}\frac{d}{dt}K(t-\theta_2-h)D\varphi(\theta_2)d\theta_2\right.$$
$$\left. + \int_{t-(j+1)h+0}^{0}\frac{d}{dt}K(t-\theta_2-h)D\varphi(\theta_2)d\theta_2 + \Delta K(jh)D\varphi(t-(j+1)h)\right)dt$$

Changing the integration order, we get

$$J_2 = \int_{-h}^{0}\int_{\mathcal{O}(\theta_2)}\frac{d}{dt}\left(\int_{-h}^{0}\varphi^T(\theta_1)D^TK^T(t-\theta_1-h)d\theta_1\right)W\frac{d}{dt}K(t-\theta_2-h)D\varphi(\theta_2)dtd\theta_2$$
$$+ \int_{-h}^{0}\sum_{j=0}^{\infty}\frac{d}{dt}\left(\int_{-h}^{0}\varphi^T(\theta_1)D^TK^T(t+jh-\theta_1)d\theta_1\right)W\Delta K(jh)D\varphi(t)dt.$$

Introducing the change of variable $\xi = t - \theta_2 - h$ in the first term, we have

$$
\begin{aligned}
J_2 = & \int_{-h}^{0}\int_{\mathcal{O}(0)} \frac{d}{d\xi}\left(\int_{-h}^{0}\varphi^T(\theta_1)D^T K^T(\xi+\theta_2-\theta_1)d\theta_1\right) W \frac{d}{d\xi}K(\xi)D\varphi(\theta_2)d\xi d\theta_2 \\
& + \int_{-h}^{0}\sum_{j=0}^{\infty}\frac{d}{d\theta_2}\left(\int_{-h}^{0}\varphi^T(\theta_1)D^T K^T(\theta_2+jh-\theta_1)d\theta_1\right) W\Delta K(jh)D\varphi(\theta_2)d\theta_2 \\
= & \int_{-h}^{0}\varphi^T(\theta_1)D^T\int_{-h}^{0}\frac{d}{d\theta_2}\left(\int_{\mathcal{O}(0)}K^T(\xi+\theta_2-\theta_1)W\frac{d}{d\xi}K(\xi)d\xi\right)D\varphi(\theta_2)d\theta_2 d\theta_1 \\
& + \int_{-h}^{0}\sum_{j=0}^{\infty}\int_{-h}^{\theta_2-0}\varphi^T(\theta_1)D^T\frac{d}{d\theta_2}K^T(\theta_2+jh-\theta_1)d\theta_1 W\Delta K(jh)D\varphi(\theta_2)d\theta_2 \\
& + \int_{-h}^{0}\sum_{j=0}^{\infty}\int_{\theta_2+0}^{0}\varphi^T(\theta_1)D^T\frac{d}{d\theta_2}K^T(\theta_2+jh-\theta_1)d\theta_1 W\Delta K(jh)D\varphi(\theta_2)d\theta_2 \\
& + \int_{-h}^{0}\varphi^T(\theta)D^T\left(\sum_{j=0}^{\infty}(\Delta K(jh))^T W\Delta K(jh)\right)D\varphi(\theta)d\theta.
\end{aligned}
$$

By changing the limits in the second term and applying the expressions for $U'(\tau)$ and $U''(\tau)$ given by (2.11) and (2.12), respectively, we get

$$
\begin{aligned}
J_2 = & -\int_{-h}^{0}\varphi^T(\theta_1)D^T\left(\int_{-h}^{\theta_1-0}U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2 + \int_{\theta_1+0}^{0}U''(\theta_1-\theta_2)D\varphi(\theta_2)d\theta_2\right)d\theta_1 \\
& - \int_{-h}^{0}\varphi^T(\theta)D^T\Delta U'(0)D\varphi(\theta)d\theta.
\end{aligned}
$$

□

2.3.3 Computation of the functional

In this section the transformed functional (2.16), obtained by assuming that $\varphi \in C^{(1)}([-h,0],\mathbb{R}^n)$ and integrating by parts in Kharitonov (2005), is computed directly with the Cauchy formula (2.17). The difference in using this formula is that now *the initial functions belong to the space of piecewise continuous functions.*

System (2.1) is assumed to be exponentially stable. We look for a functional $v_0(\varphi)$ such that

$$
\frac{d}{dt}v_0(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi),\;\; t\geq 0,\; \varphi\in PC([-h,0],\mathbb{R}^n), \tag{2.20}
$$

where the matrix W is symmetric positive definite. Integrating the above expression from 0 to $T>0$, we get

$$
v_0(x_T(\varphi)) - v_0(\varphi) = -\int_0^T x^T(t,\varphi)Wx(t,\varphi)dt.
$$

As system (2.1) is exponentially stable, $x_T(\varphi)\to 0$ when $T\to\infty$, and $v_0(0)=0$, we arrive at

$$
v_0(\varphi) = \int_0^{\infty} x^T(t,\varphi)Wx(t,\varphi)dt,\; \varphi\in PC([-h,0],\mathbb{R}^n).
$$

Substituting the Cauchy formula (2.17) into the previous equation we have

$$
\begin{aligned}
v_0(\varphi) &= (\varphi(0) - D\varphi(-h))^T \int_0^\infty K^T(t)WK(t)dt\,(\varphi(0) - D\varphi(-h)) \\
&\quad + 2\,(\varphi(0) - D\varphi(-h))^T \int_{-h}^0 \int_0^\infty K^T(t)WK(t-\theta-h)dtA_1\varphi(\theta)d\theta \\
&\quad + \int_{-h}^0 \int_{-h}^0 \varphi^T(\theta_1)A_1^T \int_0^\infty K^T(t-\theta_1-h)WK(t-\theta_2-h)dtA_1\varphi(\theta_2)d\theta_2 d\theta_1 \\
&\quad + 2\,(\varphi(0) - D\varphi(-h))^T J_1(-h) + 2\int_{-h}^0 \varphi^T(\theta_1)A_1^T J_1(\theta_1)d\theta_1 + J_2. \qquad (2.21)
\end{aligned}
$$

By using the expressions obtained in Proposition 2.1 and Proposition 2.2, and the definition of the delay Lyapunov matrix (2.6) in equation (2.21), we get the same expression as (2.16), which is rewritten in a simpler form as follows:

$$
\begin{aligned}
v_0(\varphi) &= (\varphi(0) - D\varphi(-h))^T U(0)\,(\varphi(0) - D\varphi(-h)) + 2\,(\varphi(0) - D\varphi(-h))^T \int_{-h}^0 F_1(-h-\theta)\varphi(\theta)d\theta \\
&+ \int_{-h}^0 \int_{-h}^0 \varphi^T(\theta_1)F_2(\theta_1-\theta_2)\varphi(\theta_2)d\theta_2 d\theta_1 - \int_{-h}^0 \varphi^T(\theta)D^T\Delta U'(0)D\varphi(\theta)d\theta,\ \varphi \in PC\left([-h,0],\mathbb{R}^n\right),
\end{aligned} \tag{2.22}
$$

where

$$
\begin{aligned}
F_1(\tau) &= \begin{cases} U(\tau)A_1 + U'(\tau)D, & \tau \in \mathbb{R}\backslash\Omega, \\ 0, & \tau \in \Omega, \end{cases} \\
F_2(\tau) &= \begin{cases} A_1^T F_1(\tau) - D^T F_1'(\tau), & \tau \in \mathbb{R}\backslash\Omega, \\ 0, & \tau \in \Omega, \end{cases}
\end{aligned} \tag{2.23}
$$

The functional of complete type is given by (Kharitonov (2013))

$$
v(\varphi) = v_0(\varphi) + \int_{-h}^0 \varphi^T(\theta)\left(W_1 + (h+\theta)W_2\right)\varphi(\theta)d\theta,\ \varphi \in PC\left([-h,0],\mathbb{R}^n\right), \tag{2.24}
$$

where $v_0(\varphi)$ is determined by (2.22). The computation of the functional is obtained under the assumption of exponential stability of system (2.1). In Kharitonov (2013), it is shown that this assumption can be dropped by using Definition 2.3 of the delay Lyapunov matrix.

We now remind the main characteristic of the functional $v(\varphi)$, which is that under exponential stability assumption, it satisfies a quadratic lower bound.

Theorem 2.4. *(Kharitonov (2013)) Consider the constants*

$$
u_0 = \|U(0)\|,\ u_1 = \sup_{\tau\in(0,h)} \|F_1(\tau)\|,\ u_2 = \sup_{\tau\in(0,h)} \|F_2(\tau)\|.
$$

Let system (2.1) *be exponentially stable, given symmetric positive definite matrices W_i, $i = 0,1,2$, and positive numbers α_1 and α_2, the functional of complete type* (2.24) *satisfies:*

$$
\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2,\ \varphi \in PC\left([-h,0],\mathbb{R}^n\right),
$$

where

$$
\alpha_2 = u_0(1+\|D\|)^2 + 2hu_1(1+\|D\|) + h^2u_2 + h\left(\|W_1\| + \frac{h}{2}\|W_2\| + \left\|D^T\Delta U'(0)D\right\|\right)
$$

and α_1 is such that the following pencil matrix is positive semidefinite

$$
L(\alpha_1) = \begin{pmatrix} W_0 & 0 \\ 0 & W_1 \end{pmatrix} + \alpha_1 \begin{pmatrix} A_0 + A_0^T & A_1 - A_0^T D \\ A_1^T - D^T A_0 & -D^T A_1 - A_1^T D \end{pmatrix}.
$$

Proof. Consider the functional

$$\widehat{v}(\varphi) = v(\varphi) - \alpha_1 \|\varphi(0) - D\varphi(-h)\|^2.$$

In order to prove that the functional v satisfies a quadratic lower bound it is enough to prove that $\widehat{v}(\varphi) \geq 0$. Differentiating with respect to time, we get

$$\frac{d}{dt}\widehat{v}(x_t) = -x^T(t)W_0x(t) - x^T(t-h)W_1x(t-h) - \int_{t-h}^{t} x^T(\theta)W_2x(\theta)d\theta \\ - 2\alpha(x(t) - Dx(t-h))^T(A_0x(t) + A_1x(t-h)) \geq -\widehat{x}^T(t)L(\alpha)\widehat{x}(t),$$

where $\widehat{x}^T(t) = \begin{pmatrix} x^T(t) & x^T(t-h) \end{pmatrix}$. As system is exponentially stable and $L(\alpha_1) \geq 0$, we have

$$\widehat{v}(\varphi) \geq \int_0^\infty \widehat{x}^T(t)L(\alpha_1)\widehat{x}(t)dt \geq 0,$$

hence

$$v(\varphi) \geq \alpha_1 \|\varphi(0) - D\varphi(-h)\|^2.$$

The upper bound is obtained by estimating the one of each term of the functional v as follows:

$$v(\varphi) \leq (1 + \|D\|)^2 u_0 \|\varphi\|_h^2 + 2(1 + \|D\|)u_1 \int_{-h}^{0} \|\varphi(\theta)\|d\theta \|\varphi\|_h + u_2 \int_{-h}^{0}\int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta \\ + \|D^T \Delta U'(0) D\| \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta + \|W_1\| \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta + \|W_2\| \int_{-h}^{0} (h+\theta)\|\varphi(\theta)\|^2 d\theta \leq \alpha_2 \|\varphi\|_h^2$$

□

2.4 Conclusion

In this chapter, we provide the basic framework for our work. Elementary concepts of the fundamental matrix and the delay Lyapunov matrix are recalled, and the functional of complete type for neutral type delay systems is constructed without neither continuity nor differentiability assumptions on the initial functions. The presented approach is key for the obtention of the main result in the next chapter and sets the path for the determination of the functional corresponding to the neutral multiple delay case in Chapter 4.

Chapter 3

Necessary stability conditions: Single delay case

In this chapter, we provide necessary stability conditions for neutral type delay systems of the form (2.1). The particularity of these conditions is that, as in the delay-free case, they depend only on the delay Lyapunov matrix.

The ideas for the attainment of these conditions are inspired by those used for the retarded type case in Egorov and Mondié (2014), which rely on:

1. An appropriate choice for the class of piecewise initial functions depending on the fundamental matrix.

2. The introduction of new properties of the delay Lyapunov matrix.

The result concerning the computation of the functional (2.22), which was originally proved for differentiable initial functions, had to be relaxed in Chapter 2 because of the required piecewise initial function in terms of the fundamental matrix. The complex nature of neutral type systems demands a careful treatment in the proofs of the new properties of the delay Lyapunov matrix.

The chapter is organized as follows. In the next section, an auxiliary functional which is not of complete type but it has a quadratic lower bound is introduced. The new properties of the delay Lyapunov matrix that relates it with the fundamental matrix are given in Section 3.2. In Section 3.4, the necessary stability conditions are stated and, in Section 3.5, they are illustrated by some examples. Finally, we end the chapter with a comparison of the proposed approach with other stability analysis methods and some conclusions.

3.1 An auxiliary functional

The following seminorm is considered:

$$\|\varphi\|_{\mathcal{H}} = \sqrt{\|\varphi(0) - D\varphi(-h)\|^2 + \int_{-h}^{0} \|\varphi(\theta)\|^2 d\theta}.$$

We introduce an auxiliary functional based on $v_0(\varphi)$. This functional is simpler than the one of complete type yet it has a quadratic lower bound. It is given by

$$v_1(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) W \varphi(\theta) d\theta, \; \varphi \in PC\left([-h, 0], \mathbb{R}^n\right), \tag{3.1}$$

where $v_0(\varphi)$ is determined by (2.22) and $W > 0$. In view of equation (2.20), the derivative of $v_1(\varphi)$ along the solutions of system (2.1) is

$$\frac{d}{dt}v_1(x_t(\varphi)) = -x^T(t-h,\varphi)Wx(t-h,\varphi),\ t \geq 0.$$

In the next theorem it is shown that the functional $v_1(\varphi)$ admits a quadratic lower bound.

Theorem 3.1. *Let system* (2.1) *be exponentially stable, there exists a number* $\beta > 0$ *such that*

$$v_1(\varphi) \geq \beta \|\varphi\|_{\mathcal{H}}^2,\ \varphi \in PC([-h,0],\mathbb{R}^n).$$

Proof. Since matrices W_j, $j = 0,1,2$, are positive definite and $(h+\theta) \in [0,h]$ for $\theta \in [-h,0]$, Theorem 2.4 implies that

$$v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta)\left(W_1 + hW_2\right)\varphi(\theta)d\theta \geq \alpha_1 \left\|\varphi(0) - D\varphi(-h)\right\|^2.$$

Adding the term $\int_{-h}^{0} \varphi^T(\theta)W_0\varphi(\theta)d\theta$ on both sides, and taking into account that $W = W_0 + W_1 + hW_2$, we get

$$v_1(\varphi) \geq \alpha_1 \left\|\varphi(0) - D\varphi(-h)\right\|^2 + \int_{-h}^{0} \varphi^T(\theta)W_0\varphi(\theta)d\theta.$$

The result follows by setting $\beta = \min\left\{\alpha_1, \lambda_{min}(W_0)\right\} > 0$. □

3.2 New properties of the delay Lyapunov matrix

The new properties of the Lyapunov matrix presented below are of primary importance in obtaining the main result in this chapter. The first one is the analogue of the dynamic property (2.7) for negative arguments.

Lemma 3.1. *For* $\tau < 0$, $\tau \in \mathbb{R} \setminus \Omega$

$$U'(\tau) - D^TU'(\tau+h) = -A_0^TU(\tau) - A_1^TU(\tau+h). \tag{3.2}$$

Proof. From the symmetry property (2.8), we have

$$\left(U(\tau) - D^TU(\tau+h)\right)^T = U(-\tau) - U(-\tau-h)D,$$

then by equation (2.7),

$$\begin{aligned}\frac{d}{d\tau}\left(U(\tau) - D^TU(\tau+h)\right)^T &= -\frac{d}{d\tau}\left(U(-\tau) - U(-\tau-h)D\right)\\ &= -U(-\tau)A_0 - U(-\tau-h)A_1,\ \tau < 0.\end{aligned}$$

Transposing both sides, we get the result in (3.2). □

A general dynamic property is proved next using (2.7) and (3.2).

Lemma 3.2 (Generalized dynamic property)**.** *For* $\tau \in \mathbb{R}\setminus\Omega$,

$$U'(\tau) - U'(\tau-h)D = U(\tau)A_0 + U(\tau-h)A_1 + K^T(-\tau)W. \tag{3.3}$$

Proof. Let us introduce the function

$$G(\widehat{\tau}) = \begin{cases} -U'(\widehat{\tau}) + D^T U'(\widehat{\tau}+h) - A_0^T U(\widehat{\tau}) - A_1^T U(\widehat{\tau}+h) - WK(\widehat{\tau}), & \widehat{\tau} \in \mathbb{R}\backslash\Omega, \\ \lim_{\theta\to\widehat{\tau}+0} G(\theta), & \widehat{\tau} \in \Omega. \end{cases}$$

It follows from the dynamic property for negative arguments (3.2) that $G(\widehat{\tau}) = 0$, $\widehat{\tau} < 0$. Observe that, for $\widehat{\tau} = +0$, by (3.2) and continuity of $U(\tau)$, $\tau \in \mathbb{R}$, $G(+0)$ can be written as

$$G(+0) = -U'(+0) + D^T U'(h+0) + U'(-0) - D^T U'(h-0) - W.$$

Applying equation (2.13) and algebraic property (2.10), we get $G(+0) = 0$.

To prove (3.3), we have to show that $G(\widehat{\tau}) = 0$ on $(0,\infty)$. For this, observe first that for $\widehat{\tau} \geq 0$, dynamic property (2.7) implies that

$$\begin{aligned} G(\widehat{\tau}) - G(\widehat{\tau}-h)D = -(U(\widehat{\tau})A_0 + U(\widehat{\tau}-h)A_1) + D^T(U(\widehat{\tau}+h)A_0 + U(\widehat{\tau})A_1) - A_0^T(U(\widehat{\tau}) - U(\widehat{\tau}-h)D) \\ - A_1^T(U(\widehat{\tau}+h) - U(\widehat{\tau})D) - W(K(\widehat{\tau}) - K(\widehat{\tau}-h)D). \end{aligned}$$

By definitions of $U(\tau)$ and $K(\tau)$, the previous difference is continuous on $\widehat{\tau} \geq 0$ and

$$\begin{aligned} G'(\widehat{\tau}) - G'(\widehat{\tau}-h)D &= -(U'(\widehat{\tau})A_0 + U'(\widehat{\tau}-h)A_1) \\ &+ D^T(U'(\widehat{\tau}+h)A_0 + U'(\widehat{\tau})A_1) - A_0^T(U(\widehat{\tau})A_0 + U(\widehat{\tau}-h)A_1) - A_1^T(U(\widehat{\tau}+h)A_0 + U(\widehat{\tau})A_1) \\ &- W(K(\widehat{\tau})A_0 + K(\widehat{\tau}-h)A_1),\ \widehat{\tau} \geq 0. \end{aligned} \tag{3.4}$$

Hence, expression in (3.4) can be rewritten as a neutral time delay system of the form

$$G'(\widehat{\tau}) - G'(\widehat{\tau}-h)D = G(\widehat{\tau})A_0 + G(\widehat{\tau}-h)A_1,\ \ \widehat{\tau} \geq 0,\ a.e.$$

As the initial condition is $G(\widehat{\tau}) = 0$, $\widehat{\tau} \in [-h, 0]$, the previous system has a unique trivial solution $G(\widehat{\tau}) = 0$ on $\widehat{\tau} \in (0,\infty)$. Thus, we have proved that $G(\widehat{\tau}) = 0$ for $\widehat{\tau} \in \mathbb{R}$. Finally, transposing $G(\widehat{\tau})$ and setting $\tau = -\widehat{\tau}$ allow to arrive at (3.3). □

Lemma 3.3. *For $\tau \geq 0$, $\xi \in \mathbb{R}$,*

$$\begin{aligned} U(\tau+\xi) =& U(\xi)\left(K(\tau) - DK(\tau-h)\right) + \int_{-h}^{0} F_1(\xi-\theta-h)K(\tau+\theta)d\theta \\ &+ \int_{-\tau}^{0} K^T(\theta-\xi)WK(\theta+\tau)d\theta. \end{aligned} \tag{3.5}$$

Proof. We introduce the continuous function

$$P(\tau) = U(\tau+\xi) - U(\xi)\left(K(\tau) - DK(\tau-h)\right) - \int_{\tau-h}^{\tau} F_1(\bar{\tau}-\theta-h)K(\theta)d\theta - \int_0^{\tau} K^T(\theta-\xi-\tau)WK(\theta)d\theta,$$

which can be written, by the definition of $F_1(\tau)$ in (**??**), as

$$\begin{aligned} P(\tau) &= U(\tau+\xi) - U(\xi)\left(K(\tau) - DK(\tau-h)\right) \\ &- \int_{\tau-h}^{\tau} \left(U(\xi+\tau-\theta-h)A_1 + U'(\xi+\tau-\theta-h)D\right)K(\theta)d\theta - \int_0^{\tau} K^T(\theta-\xi-\tau)WK(\theta)d\theta, \end{aligned}$$

In order to prove the lemma, it is enough to show that $P(\tau) = 0$. Introduce the change of variable $\xi = \bar{\tau} - \tau$:

$$\begin{aligned} P(\tau) &= U(\bar{\tau}) - U(\bar{\tau}-\tau)\left(K(\tau) - DK(\tau-h)\right) \\ &- \int_{\tau-h}^{\tau} \left(U(\bar{\tau}-\theta-h)A_1 + U'(\bar{\tau}-\theta-h)D\right)K(\theta)d\theta - \int_0^{\tau} K^T(\theta-\bar{\tau})WK(\theta)d\theta. \end{aligned}$$

By differentiating $P(\tau)$, we obtain

$$P'(\tau) = \Big(U(\bar{\tau}-\tau)A_1 + U'(\bar{\tau}-\tau)D - U'(\bar{\tau}-\tau)D - U(\bar{\tau}-\tau)A_1\Big)K(\tau-h)$$
$$+ \Big(U'(\bar{\tau}-\tau) - U(\bar{\tau}-\tau)A_0 - U(\bar{\tau}-\tau-h)A_1 - U'(\bar{\tau}-\tau-h)D - K^T(\tau-\bar{\tau})W\Big)K(\tau)$$

Clearly, the first summand is zero, while the second one is zero by equation (3.3). It implies that $P(\tau)$ is a constant function. The result follows by the continuity of the function P and the fact that $P(0) = 0$. □

In what follows, a number of special cases of equation (3.5) that are instrumental in the proof of Lemma 3.4 are presented.

Corollary 3.1. *For $\tau \geq 0$ and $\xi \geq 0$,*

$$U(\tau+\xi) = U(\xi)\Big(K(\tau) - DK(\tau-h)\Big) + \int_{-h}^{0} F_1(\xi-\theta-h)K(\tau+\theta)d\theta. \tag{3.6}$$

Proof. The result follows from the fact that $s - \xi \geq 0$ for $\xi \geq 0$ and $s \in [-\tau, 0]$ in equation (3.5). □

Corollary 3.2. *For $\tau \geq 0$, the Cauchy formula for the Lyapunov matrix is*

$$U(\tau) = U(0)\Big(K(\tau) - DK(\tau-h)\Big) + \int_{-h}^{0} F_1(-\theta-h)K(\tau+\theta)d\theta. \tag{3.7}$$

Corollary 3.3. *For $\xi \in (0,h)$ and $\tau \geq 0$,*

$$\begin{aligned} U'(\tau+\xi) &= U'(\xi)\Big(K(\tau) - DK(\tau-h)\Big) \\ &\quad + \int_{-h}^{0} U'(\xi-h-s)A_1K(\tau+s)ds + \int_{-h}^{\xi-h-0} U''(\xi-h-s)DK(\tau+s)ds \\ &\quad + \int_{\xi-h+0}^{0} U''(\xi-h-s)DK(\tau+s)ds + \Delta U'(0)DK(\tau+\xi-h), \ a.e. \end{aligned} \tag{3.8}$$

Proof. The result is obtained by differentiating equation (3.5) with respect to ξ, and using the fact that $K(s)$ is continuous on $s \in (0,h)$. □

3.3 Bilinear functional

The result proved in Lemma 3.4 below is obtained through the above corollaries and the introduction of the bilinear functional

$$\begin{aligned} z(\varphi_1,\varphi_2) &= (\varphi_1(0) - D\varphi_1(-h))^T U(0)(\varphi_2(0) - D\varphi_2(-h)) + (\varphi_1(0) - D\varphi_1(-h))^T \int_{-h}^{0} F_1(-h-\theta)\varphi_2(\theta)d\theta \\ &\quad + \int_{-h}^{0} \varphi_1^T(\theta)F_1^T(-h-\theta)d\theta(\varphi_2(0) - D\varphi_2(-h)) + \int_{-h}^{0}\int_{-h}^{0} \varphi_1^T(\theta_1)F_2(\theta_1-\theta_2)\varphi_2(\theta_2)d\theta_2 d\theta_1 \\ &\quad - \int_{-h}^{0} \varphi_1^T(\theta)\Delta U'(0)\varphi_2(\theta)d\theta, \ \varphi_1,\varphi_2 \in PC\left([-h,0],\mathbb{R}^n\right). \end{aligned} \tag{3.9}$$

It follows directly from (2.10) that $v_1(\varphi) = z(\varphi,\varphi)$.

Lemma 3.4. *For any $\tau_1, \tau_2 \in [0,h]$ and $\mu, \eta \in \mathbb{R}^n$,*

$$z(K(\tau_1 + \cdot)\mu, K(\tau_2 + \cdot)\eta) = \mu^T U(\tau_2 - \tau_1)\eta.$$

Proof. Observe first that for $\tau \geq 0$ and $\xi \in (0,h)$, by equations (3.6) and (3.8)

$$\begin{aligned} F_1^T(-\tau-\xi) = A_1^T U(\tau+\xi) - D^T U'(\tau+\xi) = F_1^T(-\xi)\left(K(\tau) - DK(\tau-h)\right) \\ + \int_{-h}^{0} F_2(\xi-\theta-h)K(\tau+\theta)d\theta - D^T \Delta U'(0) DK(\tau+\xi-h),\ a.e. \end{aligned} \quad (3.10)$$

Now, rewrite (3.9) as follows:

$$\begin{aligned} &z(K(\tau_1 + \cdot)\mu, K(\tau_2 + \cdot)\eta) = \\ &\quad \mu^T \Bigg((K(\tau_1) - DK(\tau_1 - h))^T \left(U(0)(K(\tau_1) - DK(\tau_1 - h)) + \int_{-h}^{0} F_1(-h-\theta)K(\tau_2+\theta)d\theta \right) \\ &\quad + \int_{-h}^{0} K^T(\tau_1+\theta_1)\left(F_1^T(-h-\theta_1)(K(\tau_2) - DK(\tau_2-h)) + \int_{-h}^{0} F_2(\theta_1-\theta_2)K(\tau_2+\theta_2)d\theta_2 \right) d\theta_1 \\ &\quad - \int_{-h}^{0} K^T(\tau_1+\theta)\Delta U'(0) K(\tau_2+\theta)d\theta \Bigg)\eta. \end{aligned}$$

By using (3.7) in the first term, and (3.10) in the second one, we get

$$\begin{aligned} z(K(\tau_1+\cdot)\mu, K(\tau_2+\cdot)\eta) = \mu^T \Bigg((K(\tau_1) - DK(\tau_1-h))^T U(\tau_1) + \int_{-h}^{0} K^T(\tau_1+\theta_1) F_1^T(-\tau_2-h-\theta_1)d\theta_1 \\ + \int_{-h}^{0} K^T(\tau_1+\theta)\left(D^T \Delta U'(0) D - \Delta U'(0) \right) K(\tau_2+\theta)d\theta \Bigg)\eta. \end{aligned}$$

By applying equation (2.10) in the third term,

$$\begin{aligned} z(K(\tau_1+\cdot)\mu, K(\tau_2+\cdot)\eta) = \mu^T \Bigg((K(\tau_1) - DK(\tau_1-h))^T U(\tau_1) + \int_{-h}^{0} K^T(\tau_1+\theta_1) F_1^T(-\tau_2-h-\theta_1)d\theta_1 \\ + \int_{-h}^{0} K^T(\tau_1+\theta) W K(\tau_2+\theta)d\theta \Bigg)\eta. \end{aligned}$$

Finally, by property (3.5), we arrive at the desired result. □

3.4 Necessary stability conditions

We are now ready to prove the main result of this chapter. Let us introduce the following function:

$$\psi_r(\theta) = \sum_{j=1}^{r} K(\tau_j + \theta)\gamma_j,\ \theta \in [-h, 0], \quad (3.11)$$

where $\tau_j \in [0,h]$ and $\gamma_j \in \mathbb{R}^n$, $j = \overline{1,r}$, are constant arbitrary vectors. In order to illustrate function (3.11), we consider the particular case $k(t) = e^{-0.5t}$, $t \in [0,h]$, with $r = 4$, $\gamma_1 = \gamma_3 = 0.5$ and $\gamma_2 = \gamma_4 = 1$. It is depicted in Figure 3.1.

The necessary stability conditions for system (2.1) are established with the help of Theorem 3.1 and Lemma 3.4.

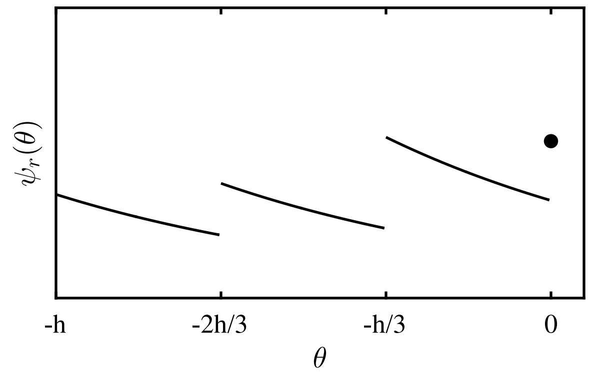

Figure 3.1: Particular case of function (3.11)

Theorem 3.2. *If system (2.1) is exponentially stable, then*

$$\mathcal{K}_r(\tau_1, \ldots, \tau_r) > 0, \tag{3.12}$$

for $\tau_k \in [0, h]$, $k = \overline{1, r}$, *and* $\tau_i \neq \tau_j$ *if* $i \neq j$. *Here*

$$\mathcal{K}_r(\tau_1, \ldots, \tau_r) = \begin{pmatrix} U(0) & U(\tau_2 - \tau_1) & \ldots & U(\tau_r - \tau_1) \\ U^T(\tau_2 - \tau_1) & U(0) & \ldots & U(\tau_r - \tau_2) \\ \vdots & & \ddots & \vdots \\ U^T(\tau_r - \tau_1) & U^T(\tau_r - \tau_2) & \ldots & U(0) \end{pmatrix}. \tag{3.13}$$

Proof. Observe that $v_1(\varphi) = z(\varphi, \varphi)$. Substituting into the bilinear functional (3.9) the particular initial function (3.11) gives

$$v_1(\psi_r) = z(\psi_r, \psi_r) = \sum_{k=1}^{r} \sum_{j=1}^{r} z(K(\tau_k + \cdot)\gamma_k, K(\tau_j + \cdot)\gamma_j).$$

In view of Lemma 3.4, this is

$$v_1(\psi_r) = \sum_{k=1}^{r} \sum_{j=1}^{r} \gamma_k^T U(\tau_j - \tau_k)\gamma_j.$$

The above can be rewritten in matrix form as

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r(\tau_1, \ldots, \tau_r)\gamma, \tag{3.14}$$

with $\gamma = \begin{pmatrix} \gamma_1^T & \ldots & \gamma_r^T \end{pmatrix}^T$. From Theorem 3.1, we get

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r(\tau_1, \ldots, \tau_r)\gamma \geq \beta \left\| \psi_r \right\|_{\mathcal{H}}^2, \ \beta > 0.$$

We prove now that if $\gamma \neq 0$, then $\|\psi_r\|_{\mathcal{H}} > 0$. Assume that $0 = \tau_0 \leq \tau_1 < \ldots < \tau_r$ and observe that if $\gamma \neq 0$, then there is at least one vector $\gamma_i \neq 0$. For the case $\gamma_q \neq 0$ with $\gamma_{q+1} = \ldots = \gamma_r = 0$, we have

$$\psi_r(\theta) = e^{A_0(\tau_q + \theta)}\gamma_q, \ \theta \in [-\tau_q, -\tau_q + \Delta_q),$$

where $\Delta_q = \min\{\tau_q - \tau_{q-1}, h\}$. If $q = 1$ and $\gamma_2 = \ldots = \gamma_r = 0$ with $\tau_0 = \tau_1 = 0$ we have $\psi_r(0) = \gamma_1 \neq 0$. In all the cases $\|\psi_r\|_{\mathcal{H}} > 0$, hence,

$$\mathcal{K}_r(\tau_1, \ldots, \tau_r) > 0.$$

□

As in the retarded type case in Egorov and Mondié (2015), the previous theorem enables us to show that the maximum of the norm of the delay Lyapunov matrix is achieved at zero.

Lemma 3.5. *Let the nominal system* (2.1) *be exponentially stable. Then*

$$\|U(\tau)\| < \|U(0)\|\,, \tau \in [-h, 0) \cup (0, h].$$

Proof. Theorem 3.2 implies that $U(0) > 0$, therefore $\|U(0)\| = \lambda_{max}\left(U(0)\right)$ and there exists a matrix $B > 0$ such that $U(0) = B^2$, $B = (U(0))^{1/2}$. From the stability condition of Theorem 3.2 we have

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} U(0) & U(\tau) \\ U^T(\tau) & U(0) \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & B^{-1}U(\tau)B^{-1} \\ B^{-1}U^T(\tau)B^{-1} & I \end{pmatrix} > 0,$$

which in turn implies that

$$\left\|B^{-1}U(\tau)B^{-1}\right\| < 1.$$

Then,

$$\|U(\tau)\| = \left\|B\left(B^{-1}U(\tau)B^{-1}\right)B\right\| < \|B\|^2 = \lambda_{max}(B^TB) = \|U(0)\|\,,\ \tau \in (0, h].$$

For $\tau \in [-h, 0)$ the result follows by the symmetry property (2.8). □

3.5 Examples

In this section, we illustrate the stability conditions of Theorem 3.2 by some examples. The general procedure for using this theorem is the following:

1. Compute the delay Lyapunov matrix $U(\tau)$, $\tau \in [0, h]$, associated with a positive definite matrix W. This can be done by using the semianalytic method introduced in Kharitonov (2005) and Kharitonov (2013) (see Appendix C).

2. Set an arbitrary number r and choose the parameters $\tau_i \in [0, h]$, $i = \overline{1, r}$.

3. Construct the corresponding matrix $\mathcal{K}_r(\tau_1, \dots, \tau_r)$ and check if it is positive definite.

In the examples below, the stability maps in the space of parameters of some systems are constructed. The isolated points on the figures correspond to the points where the conditions of Theorem 3.2, evaluated at 80×80 points in the space of parameters, hold, and the solid lines are the stability boundaries determined by the D-subdivision method (Neimark (1949)). Here, the delay Lyapunov matrix is computed for $W = I_n$ by using the semianalytic method.

Notice that as the stability condition (3.12) is only necessary, it may be satisfied for a given number r at points where the system is actually unstable (see Example 3.3). In order to discard those points, we use the computed stability boundaries: if regions delimited by boundaries contain loci for which the condition test both positively and negatively (i.e., $\mathcal{K}_r > 0$ and $\mathcal{K}_r \not> 0$), the whole region is unstable.

Example 3.1. *Consider the scalar neutral type equation (Kolmanovskii and Myshkis (1999)):*

$$\frac{d}{dt}\left(x(t) - dx(t-h)\right) = ax(t) + bx(t-h),\ h = 1. \tag{3.15}$$

We illustrate first step by step the procedure mentioned at the beginning of the section. In order to do this, we consider $a = -2$, $b = 0$ and $d = -0.9$.

1. *Compute the delay Lyapunov function $u(\tau)$, $\tau \in [0, 1]$.*

2. *Set $r = 2$ with $\tau_1 = 0$ and $\tau_2 = \tau \in (0, h]$.*

3. *With the Lyapunov function previously computed, construct the matrix*

$$\mathcal{K}_2(0,\tau) = \begin{pmatrix} u(0) & u(\tau) \\ u(\tau) & u(0) \end{pmatrix}.$$

For instance, for $\tau = 0.5$, this matrix takes the following numerical values:

$$\mathcal{K}_2(0,0.5) = \begin{pmatrix} 0.5663 & 0.0214 \\ 0.0214 & 0.5663 \end{pmatrix}.$$

One can see that it is positive definite. In order to use Theorem 3.2 one should construct the matrix $\mathcal{K}_2(0,\tau)$ and check if it is positive definite for each $\tau \in (0,1]$.

Based on the previously said, we now compute the stability map of system (3.15). *The isolated points on Figure 3.2 and Figure 3.3 correspond to the points where condition*

$$\mathcal{K}_2(0,\tau) > 0, \;\; \tau \in (0,1], \tag{3.16}$$

holds in the space of parameters (a,b) and (b,d), respectively.

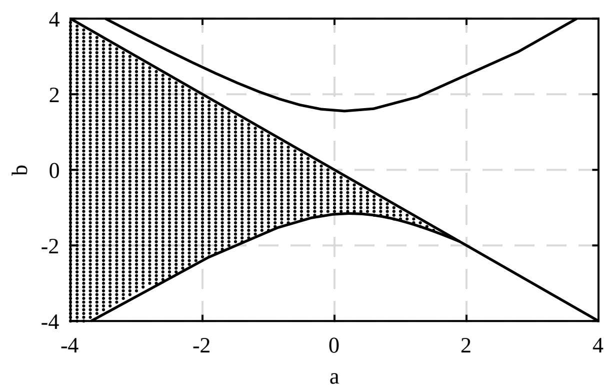

Figure 3.2: System (3.15) with $d = -0.9$, condition (3.16)

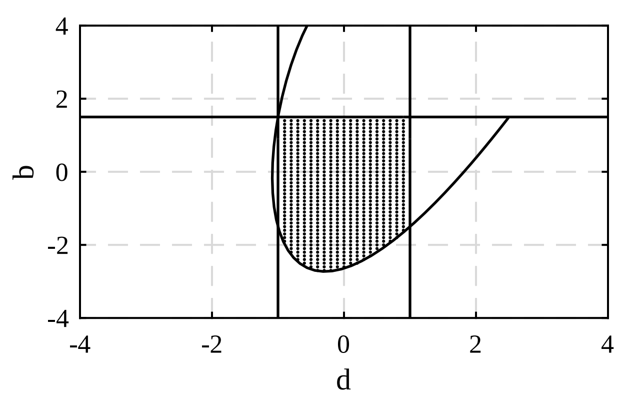

Figure 3.3: System (3.15) with $a = -1.5$, condition (3.16)

Example 3.2. *The PD control of the system described by the transfer function*

$$H(s) = \frac{15s^2 + 3s - 20}{125s^3 + 70s^2 + 10s + 8} e^{-2s},$$

introduced in Méndez (2011), is described by system (2.1) *with delay* $h = 2$ *and matrices*

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{15}{125}k_d \end{pmatrix}, \ A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{8}{125} & -\frac{10}{125} & -\frac{70}{125} \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{20}{125}k_p & -\frac{1}{125}(3k_p - 20k_d) & -\frac{8}{125}(15k_p + 3k_d) \end{pmatrix},$$

where parameters k_p *and* k_d *are the proportional and derivative gains, respectively. The points where the condition*

$$\mathcal{K}_2(0, \tau) > 0, \ \ \tau \in (0, 2], \tag{3.17}$$

holds in the space of parameters (k_p, k_d) *are depicted on Figure 3.4. As in Example 3.1, one observes that the exact stability region is achieved with the tested condition.*

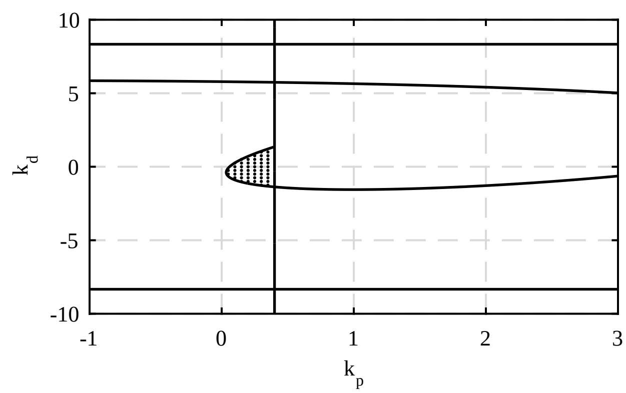

Figure 3.4: Example 3.2, condition (3.17)

Example 3.3. *The σ-stability analysis of the proportional-integral control of a passive linear system leads to studying a quasipolynomial of neutral type (Castaños et al. (2017)). Its time domain representation is of the form* (2.1)*, with matrices*

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{\alpha_2}{\alpha_1} \end{pmatrix},\ A_0 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & \alpha_1 \\ -\sigma^2\alpha_1 + \sigma\beta_1 - \gamma_1 & -\beta_1 + 2\sigma\alpha_1 \end{pmatrix},$$

and

$$A_1 = \frac{1}{\alpha_1}\begin{pmatrix} 0 & 0 \\ -\sigma^2\alpha_2 + \sigma\beta_2 - \gamma_2 & -\beta_2 + 2\sigma\alpha_2 \end{pmatrix},$$

where

$$\begin{aligned}
&\alpha_1 = d + k_p;\ \gamma_1 = bk_i d + ak_i;\\
&\alpha_2 = (d - k_p)\, e^{\sigma h};\ \gamma_2 = (bk_i d - ak_i)e^{\sigma h};\\
&\beta_1 = (bk_p + a)\, d + bd^2 + ak_p + k_i;\\
&\beta_2 = \big((bk_p + a)d - bd^2 - ak_p - k_i\big)\, e^{\sigma h}.
\end{aligned}$$

For the parameter numerical values

$$a = 0.4,\ b = 50,\ h = 0.2,\ d = 0.8,\ \sigma = 0.3,$$

the stability of the difference operator imposes in the D-subdivisions the additional condition $|k_p| < 26.67$. The points where the condition

$$\mathcal{K}_2(0,\tau) > 0,\ \ \tau \in (0, 0.2], \tag{3.18}$$

and condition

$$\mathcal{K}_3(0,\tau,h) > 0,\ \ \tau \in (0, 0.2), \tag{3.19}$$

hold in the space of gains (k_p, k_i) are depicted on Figure 3.5 and Figure 3.6, respectively. Observe that Figure 3.5 shows a region delimited by stability boundaries where the stability test of Theorem 3.2 gives a positive result for some points and negative for others, thus the region is unstable.

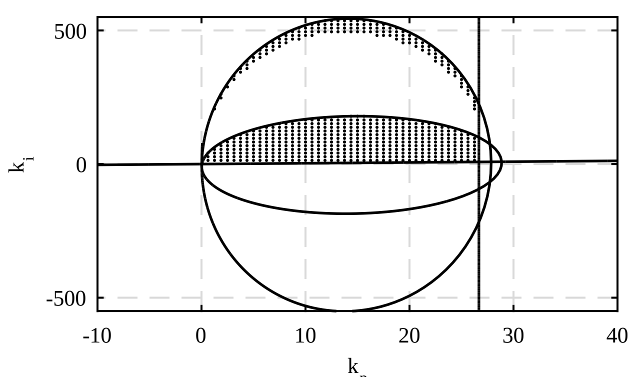

Figure 3.5: Example 3.3, condition (3.18)

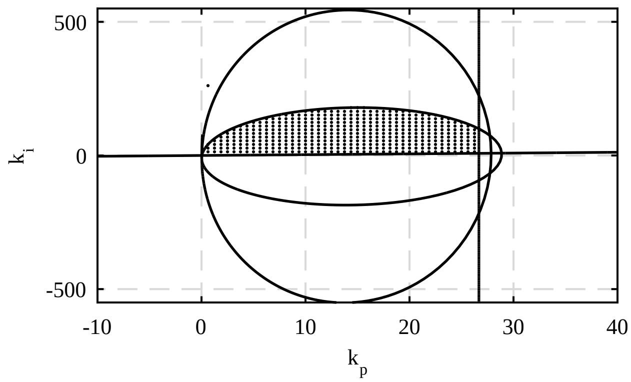

Figure 3.6: Example 3.3, condition (3.19)

3.6 Discussion

A comparison with other delay systems stability analysis methods is in order. The stability conditions in (3.12) and the frequency domain techniques (see, for example, Neimark (1949), Olgac and Sipahi (2004), Ochoa et al. (2013)) actually complement each other. As shown in the presented examples, it is possible to reduce the conservatism of the necessary stability conditions (3.12) by increasing the number r and know the exact stability zone by using the boundaries obtained via D-subdivision method. Notice also that the test we present can substitute the root direction analysis carried out on the stability boundaries, and the involved accounting process of the number of unstable roots of the characteristic quasi-polynomial in each region.

Clearly, the location of the roots can also be obtained by the direct computation of the roots based on existing numerical algorithms (see, Michiels and Vyhlídal (2005) and Chapter 2 in Michiels and Niculescu (2014) for a brief overview of such methods). However, these two approaches in the frequency domain do not provide a Lyapunov-Krasovskii functional at stable points of the space of parameters. LMI type sufficient stability conditions known to introduce conservatism (see Chapter 5 in Niculescu (2001) and Chapter 3 in Fridman (2014)), do give a functional, but only at points where an LMI solution is obtained.

On the contrary, at each point in the space of parameters where the stability test (3.12) of neutral type system (2.1) is conclusive, we are able to present the functional of complete type (2.24), which is defined by the delay Lyapunov matrix associated with the matrix W. This enables using this functional in a variety of applications such as estimation of the region of attraction of non-linear systems (Gomez et al. (2017)), robust stability analysis with respect to system parameters and uncertain/time-varying delay, exponential estimates of the system response (Kharitonov (2005)), analysis of the predictor-based control scheme for neutral type systems with input delay (Kharitonov (2015)), and so on.

3.7 Conclusion

Necessary stability conditions for neutral time-delay systems that depend exclusively on the delay Lyapunov matrix are provided. The result is obtained by the combination of new properties of the delay Lyapunov matrix with an appropriate choice of initial functions. The presented examples illustrate the efficiency of this new tool for the stability analysis of systems of neutral type with a single delay.

There are two striking facts worthy of mention. The first is that the stability conditions have the same form as those obtained for pointwise and distributed retarded systems, and the second one is the reduction of the conservatism of the necessary stability conditions by increasing the number r. Indeed, the presented examples show that there are cases in which there exists a number r for which the exact stability zone in the space of parameters can be achieved. This last observation is related to the results of sufficiency presented in Chapter 5.

Chapter 4

Necessary stability conditions: Multiple commensurate delay case

Consider a neutral type delay system with multiple delays of the form

$$\frac{d}{dt}\left(\sum_{j=0}^{m} D_j x(t-h_j)\right) = \sum_{j=0}^{m} A_j x(t-h_j),\ t \geq 0, \tag{4.1}$$

where $D_0 = I_n$, $D_1, \ldots, D_m, A_0, \ldots A_m \in \mathbb{R}^{n\times n}$, $0 = h_0 < h_1 < \ldots < h_m$ and $h_j = jh$ for $j = \overline{0,m}$, where $h > 0$ is the basic delay. The solution $x(\cdot,\varphi)$ of system (4.1) satisfies:

1. $x(\theta,\varphi) = \varphi(\theta)$, $\theta \in [-mh, 0]$.

2. It is piecewise continuous and satisfies system (4.1) on $t \in [0,\infty)$ almost everywhere.

3. Sewing condition: the function $\sum_{j=0}^{m} D_j x(t-jh,\varphi)$ is continuous with respect to t (right-continuous at zero).

The initial function is considered from the space $PC([-mh,0],\mathbb{R}^n)$ and it is assumed to be right-continuous. In this case, the solution is right-continuous everywhere. We equip the space of initial functions with the seminorm

$$\|\varphi\|_{\mathcal{H}} = \sqrt{\left\|\sum_{j=0}^{m} D_j\varphi(-jh)\right\|^2 + \int_{-mh}^{0} \|\varphi(\theta)\|^2 d\theta}.$$

In this chapter, we extend the results presented in Chapter 2 and Chapter 3 for one delay to the multiple delay case. In the next section we provide the new Cauchy formula for systems of the form (4.1). The Lyapunov-Krasovskii functionals of complete type as well as the definition of the delay Lyapunov matrix are introduced in Section 4.2. Section 4.3 is devoted to the obtention of the new properties of the delay Lyapunov matrix and their relation with a bilinear functional. The necessary stability conditions for system (4.1) are presented in Section 4.4 and an equivalence in the stability sense between neutral type systems and the difference equation is established in Section 4.5. Finally, some illustrative examples ends the chapter.

4.1 Cauchy formula

The fundamental matrix $K(t)$ of system (4.1) is the solution of the equation

$$\frac{d}{dt}\left(\sum_{j=0}^{m} D_j K(t-jh)\right) = \sum_{j=0}^{m} A_j K(t-jh), \tag{4.2}$$

with the following initial conditions:

1. For $\theta = 0$, $K(\theta) = I_n$.
2. For $\theta < 0$, $K(\theta) = 0$.

Moreover, $\sum_{j=0}^{m} D_j K(t-jh)$ is continuous for $t > 0$ and right-continuous at $t = 0$. The matrix $K(t)$ is also such that

$$\frac{d}{dt}\left(\sum_{j=0}^{m} K(t-jh) D_j\right) = \sum_{j=0}^{m} K(t-jh) A_j,\ a.e.$$

Similarly to Lemma 2.1, it follows from the sewing condition that the fundamental matrix has discontinuities at points $t = jh$, $j = 0, 1, \ldots$.

In the next lemma we state the new Cauchy formula for system (4.1) introduced for the single delay case in Theorem 2.3.

Lemma 4.1. *Given an initial function $\varphi \in PC\left([-mh, 0], \mathbb{R}^n\right)$, the solution $x(t, \varphi)$ of system* (4.1) *is*

$$\begin{aligned} x(t,\varphi) = K(t)\sum_{j=0}^{m} D_j\varphi(-jh) &+ \sum_{j=1}^{m}\int_{-jh}^{0} K(t-\theta-jh)A_j\varphi(\theta)d\theta \\ &- \sum_{j=1}^{m}\frac{d}{dt}\left(\int_{-jh}^{0} K(t-\theta-jh)D_j\varphi(\theta)d\theta\right),\ t \geq 0. \end{aligned} \tag{4.3}$$

Proof. Observe that the integral form of system (4.1) is given by

$$x(t) = \sum_{j=0}^{m} D_j\varphi(-jh) - \sum_{j=1}^{m} D_j x(t-jh) + \sum_{j=0}^{m} A_j \int_0^t x(\theta - jh)d\theta.$$

Fix $t > 0$ and consider the term

$$J(\xi) = \sum_{j=0}^{m} K(t-\xi-jh)D_j \int_0^{\xi} x(\theta)d\theta,\ \xi \in [0, t].$$

Differentiating with respect to ξ on $[0, t]$, we have

$$\begin{aligned} \frac{d}{d\xi}J(\xi) &= -\sum_{j=0}^{m} K(t-jh-\xi)A_j\int_0^{\xi} x(\theta)d\theta + \sum_{j=0}^{m} K(t-\xi-jh)D_j x(\xi) \\ &= -\sum_{j=1}^{m} K(t-jh-\xi)A_j\int_0^{\xi} x(\theta)d\theta + K(t-\xi)\left(\sum_{j=0}^{m} D_j\varphi(-jh) - \sum_{j=1}^{m} D_j x(\xi-jh) + \sum_{j=1}^{m} A_j\int_0^{\xi} x(\theta-jh)d\theta\right) \\ &\quad + \sum_{j=1}^{m} K(t-\xi-jh)D_j x(\xi). \end{aligned}$$

Integrating from 0 to t, we arrive at

$$\int_0^t \frac{d}{d\xi}J(\xi)d\xi = -\sum_{j=1}^{m}\int_0^t K(t-jh-\xi)A_j\int_0^{\xi}x(\theta)d\theta d\xi + \sum_{j=1}^{m}\int_0^t K(t-\xi)A_j\int_0^{\xi}x(\theta-jh)d\theta d\xi$$
$$+\int_0^t K(t-\xi)\left(\sum_{j=0}^{m}D_j\varphi(-jh)-\sum_{j=1}^{m}D_jx(\xi-jh)\right)d\xi+\sum_{j=1}^{m}\int_0^t K(t-\xi-jh)D_jx(\xi)d\xi.$$

By changing the variables $\eta = t - jh - \xi$ and $\eta = t - \xi$ in the first two terms, respectively, we get

$$\sum_{j=1}^{m}\int_{t-jh}^{-jh}\int_0^{t-jh-\eta}K(\eta)A_jx(\theta)d\theta d\eta+\sum_{j=1}^{m}\int_0^t\int_0^{t-\eta}K(\eta)A_jx(\theta-jh)d\theta d\eta$$
$$=\sum_{j=1}^{m}\int_0^{t-jh}\int_{t-jh-\eta}^{0}K(\eta)A_jx(\theta)d\theta d\eta+\int_0^t\sum_{j=1}^{m}\int_{-jh}^{t-\eta-jh}K(\eta)A_jx(\theta)d\theta d\eta$$
$$=\sum_{j=1}^{m}\int_0^{t-jh}\int_{t-jh-\eta}^{0}K(\eta)A_jx(\theta)d\theta d\eta+\sum_{j=1}^{m}\left(\int_0^{t-jh}\int_{-jh}^{t-\eta-jh}K(\eta)A_jx(\theta)d\theta d\eta+\int_{t-jh}^{t}\int_{-jh}^{t-\eta-jh}K(\eta)A_jx(\theta)d\theta d\eta\right)$$
$$=\sum_{j=1}^{m}\int_0^{t-jh}\int_{-jh}^{0}K(\eta)A_j\varphi(\theta)d\theta d\eta+\sum_{j=1}^{m}\int_{t-jh}^{t}\int_{-jh}^{t-\eta-jh}K(\eta)A_j\varphi(\theta)d\theta d\eta.$$

As $\int_0^t \frac{d}{d\xi}J(\xi)d\xi = \int_0^t x(\theta)d\theta$, then

$$\int_0^t x(\theta)d\theta = \sum_{j=1}^{m}\int_0^{t-jh}\int_{-jh}^{0}K(\eta)A_j\varphi(\theta)d\theta d\eta+\sum_{j=1}^{m}\int_{t-jh}^{t}\int_{-jh}^{t-\eta-jh}K(\eta)A_j\varphi(\theta)d\theta d\eta$$
$$+\int_0^t K(t-\xi)\left(\sum_{j=0}^{m}D_j\varphi(-jh)-\sum_{j=1}^{m}D_jx(\xi-jh)\right)d\xi+\sum_{j=1}^{m}\int_0^t K(t-\xi-jh)D_jx(\xi)d\xi.$$

We obtain (4.3) by differentiating the previous expression with respect to t. □

4.2 Lyapunov-Krasovskii functionals of complete type

Let us introduce the definition of the delay Lyapunov matrix.

Definition 4.1. *(Ochoa et al. (2009)) The delay Lyapunov matrix $U(\tau)$, $\tau \in \mathbb{R}$, associated with a given symmetric matrix W is a continuous matrix, continuously differentiable on $\mathbb{R}\backslash\Omega$, where $\Omega = \{jh\,|j = 0, \pm 1, \pm 2, \ldots\}$, and satisfies the equations*

$$\sum_{j=0}^{m}U'(\tau-jh)D_j=\sum_{j=0}^{m}U(\tau-jh)A_j,\ \tau\geq 0, \tag{4.4}$$

$$U(\tau)=U^T(-\tau), \tag{4.5}$$

$$-W=\sum_{j=0}^{m}\sum_{k=0}^{m}A_j^TU((j-k)h)D_k+\sum_{j=0}^{m}\sum_{k=0}^{m}D_j^TU((j-k)h)A_k, \tag{4.6}$$

called, dynamic, symmetry and algebraic properties, respectively.

We look for a functional such that

$$\frac{d}{dt}v_0(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi),\ t \geq 0,\ W > 0. \tag{4.7}$$

The functional $v_0(\varphi)$ is computed under the exponential stability assumption of system (4.1) and by using the Cauchy formula (4.3) (see Gomez et al. (2016c) and Kharitonov (2013) for the one delay case). The lengthy computations are developed in Appendix A. The computed functional is determined by

$$v_0(\varphi) = \sum_{j=0}^{m}\sum_{k=0}^{m}\varphi^T(-jh)D_j^T U(0) D_k \varphi(-kh) + 2\sum_{j=0}^{m}\sum_{k=1}^{m}\varphi^T(-jh)D_j^T\int_{-kh}^{0}F_k^{(1)}(-\theta-kh)\varphi(\theta)d\theta$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{m}\int_{-jh}^{0}\int_{-kh}^{0}\varphi^T(\theta_1)F_{jk}^{(2)}(\theta_1-\theta_2+jh-kh)\varphi(\theta_2)d\theta_2 d\theta_1 - \sum_{j=1}^{m}\sum_{k=1}^{m}\int_{0}^{h}\varphi^T(\theta-jh)F_{jk}^{(3)}\varphi(\theta-jh)d\theta,$$

with

$$F_k^{(1)}(\tau) = \begin{cases} U(\tau)A_k - U'(\tau)D_k, & \tau \in \mathbb{R}\backslash\Omega, \\ 0, & \tau\in\Omega \end{cases}$$
$$F_{jk}^{(2)}(\tau) = \begin{cases} A_j^T F_k^{(1)}(\tau) + D_j^T \frac{d}{d\tau}F_k^{(1)}(\tau), & \tau \in \mathbb{R}\backslash\Omega \\ 0, & \tau\in\Omega \end{cases}$$
$$F_{jk}^{(3)} = \sum_{p=j}^{m}\sum_{q=k}^{m} D_p^T \Delta U'((k-j-q+p)h)D_q,$$

where $\Delta U'(\xi) = U'(\xi+0) - U'(\xi-0)$ and

$$U(\tau) = \int_0^\infty K^T(t)WK(t+\tau)dt,\ \tau\in\mathbb{R},$$

which satisfies Definition 4.1.

In the next theorem, we drop the exponential stability assumption of system (4.1) by using Definition 4.1. The proof is given in Appendix B.

Theorem 4.1. *Let $U(\tau)$, $\tau\in\mathbb{R}$, be the delay Lyapunov matrix. The functional $v_0(\varphi)$ satisfies* (4.7).

The functional of complete type is given by

$$v(\varphi) = v_0(\varphi) + \sum_{j=1}^{m}\int_{-jh}^{0}\varphi^T(\theta)\left(W_j + (jh+\theta)W_{m+j}\right)\varphi(\theta)d\theta,\ \varphi\in PC\left([-mh,0],\mathbb{R}^n\right),$$

where the delay Lyapunov matrix is associated to the matrix $W = W_0 + \sum_{j=1}^{m}\left(W_j + jhW_{m+j}\right)$. The functional $v(\varphi)$ satisfies

$$\frac{d}{dt}v(x_t) = -\sum_{j=0}^{m}x^T(t-jh)W_j x(t-jh) - \sum_{j=1}^{m}\int_{-jh}^{0}x^T(t+\theta)W_{m+j}x(t+\theta)d\theta, \tag{4.8}$$

and under the exponential stability assumption, it admits the quadratic lower bound given in the next theorem.

Theorem 4.2. *Let system* (4.1) *be exponentially stable. For given positive definite matrices W_j, $j=\overline{0,2m}$, there exists a positive number α such that*

$$v(\varphi) \geq \alpha\left\|\sum_{j=0}^{m}D_j\varphi(-jh)\right\|^2,\ \varphi\in PC\left([-mh,0],\mathbb{R}^n\right).$$

Proof. The proof is similar to the one for one delay case presented in Kharitonov (2013). Let us introduce the functional

$$\widehat{v}(\varphi) = v(\varphi) - \widehat{\alpha}\left\|\sum_{j=0}^{m} D_j\varphi(-jh)\right\|^2, \ \varphi \in PC\left([-mh,0],\mathbb{R}^n\right).$$

By equality (4.8), we have

$$\frac{d}{dt}\widehat{v}(x_t) \leq -\chi^T(t)L(\widehat{\alpha})\chi(t),$$

where $\chi(t) = \begin{bmatrix} x^T(t) & x^T(t-h) & \dots & x^T(t-mh)\end{bmatrix}^T$ and

$$L(\widehat{\alpha}) = \begin{pmatrix} W_0 & 0 & \dots & 0 \\ 0 & W_1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & W_m \end{pmatrix} + \widehat{\alpha}\begin{pmatrix} A_0 + A_0^T & A_1 + A_0^T D_1 & \dots & A_m + A_0^T D_m \\ D_1^T A_0 + A_1^T & D_1^T A_1 + A_1^T D_1 & \dots & D_1^T A_m + A_1^T D_m \\ \vdots & & \ddots & \vdots \\ D_m^T A_0 + A_m^T & D_m^T A_1 + A_m^T D_1 & \dots & D_m^T A_m + A_m^T D_m^T \end{pmatrix}.$$

Observe that there exists $\widehat{\alpha} = \alpha > 0$ such that the $L(\alpha) \geq 0$, as the first summand is positive definite. As system (4.1) is exponentially stable,

$$\widehat{v}(\varphi) \geq \int_0^{\infty} \chi^T(t)L(\alpha)\chi(t)dt \geq 0.$$

The result straightforwardly follows from the above inequality. □

In order to obtain the necessary stability conditions, we use a functional similar to the one in (3.1) presented in Chapter 3 (see, Egorov and Mondié (2014) and Gomez et al. (2016b)):

$$v_1(\varphi) = v_0(\varphi) + \int_{-mh}^{0} \varphi^T(\theta)W\varphi(\theta)d\theta, \ \varphi \in PC\left([-mh,0],\mathbb{R}^n\right). \tag{4.9}$$

From the equality $\frac{d}{dt}v_0(x_t) = -x^T(t,\varphi)Wx(t,\varphi)$, we have

$$\frac{d}{dt}v_1(x_t) = -x^T(t-mh,\varphi)Wx(t-mh,\varphi), \ t \geq 0.$$

The functional v_1 admits the following quadratic lower bound.

Theorem 4.3. *Let system* (4.1) *be exponentially stable, then there exists a positive number β such that*

$$v_1(\varphi) \geq \beta \left\|\varphi\right\|_{\mathcal{H}}^2, \ \varphi \in PC\left([-mh,0],\mathbb{R}^n\right).$$

Proof. Set $\beta = \min\left\{\alpha, \lambda_{min}\left(W_0\right)\right\}$, where α is given by Theorem 4.2. By Theorem (4.2),

$$v_1(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta)(W_0 + W_1 + hW_2)\varphi(\theta)d\theta \geq \alpha\|\varphi(0) - D\varphi(-h)\|^2 + \int_{-h}^{0} \varphi^T(\theta)W_0\varphi(\theta)d\theta,$$

which implies that $v_1(\varphi) \geq \beta \left\|\varphi\right\|_{\mathcal{H}}^2$. □

4.3 Auxiliary results

In this section, we provide some auxiliary results, namely new properties of the delay Lyapunov matrix and their relation with a bilinear functional.

4.3.1 New properties of the delay Lyapunov function

We introduce first the dynamic property for negative arguments.

Lemma 4.2. *For* $\tau < 0$, $\tau \in \mathbb{R}\backslash\Omega$,

$$\sum_{j=0}^{m} D_j^T U'(\tau + jh) = -\sum_{j=0}^{m} A_j^T U(\tau + jh). \tag{4.10}$$

Proof. From the dynamic property (4.4) and symmetry property (4.5), we have

$$\left(\sum_{j=0}^{m} D_j^T U'(\tau + jh)\right)^T = -\sum_{j=0}^{m} U'(-\tau - jh) D_j,\ \tau < 0,\ \tau \in \mathbb{R}\backslash\Omega$$

therefore,

$$\sum_{j=0}^{m} D_j^T U'(\tau + jh) = -\sum_{j=0}^{m} A_j^T U(\tau + jh).$$

□

Next, an alternative form of writing the algebraic property (4.6) is introduced. It is worthy of mention that in Velázquez-Velázquez and Kharitonov (2009) this result was presented for the scalar case under the assumption of exponential stability.

Lemma 4.3. *The algebraic property* (4.6) *can be written as*

$$\sum_{j=0}^{m} \Delta U'(-jh) D_j = -W, \tag{4.11}$$

where $\Delta U'(\tau) = U'(\tau + 0) - U'(\tau - 0)$.

Proof. By continuity of the delay Lyapunov matrix, we have

$$\sum_{j=0}^{m} \Delta U'(\tau - jh) D_j = \sum_{j=0}^{m} U(\tau - jh + 0) A_j - \sum_{j=0}^{m} U(\tau - jh - 0) A_j = 0, \quad \tau > 0. \tag{4.12}$$

By dynamic properties (4.4) and (4.10), the left hand side of algebraic property (4.6) can be written as

$$\begin{aligned}
&\sum_{j=0}^{m}\left(\sum_{k=0}^{m} A_k^T U((k-j)h)\right) D_j + \sum_{k=0}^{m} D_k^T \left(\sum_{j=0}^{m} U((k-j)h) A_j\right) \\
&\quad = -\sum_{j=0}^{m}\left(\sum_{k=0}^{m} D_k^T U'((k-j)h - 0)\right) D_j + \sum_{k=0}^{m} D_k^T \left(\sum_{j=0}^{m} U'((k-j)h + 0) D_j\right) \\
&\quad = \sum_{k=0}^{m}\sum_{j=0}^{m} D_k^T \Delta U'((k-j)h) D_j = \sum_{j=0}^{m} D_0^T \Delta U'(-jh) D_j.
\end{aligned}$$

The last equality holds true by (4.12). Taking into account that $D_0 = I_n$, we finish the proof. □

A generalization of the dynamic properties (4.4) and (4.10) is introduced next.

Lemma 4.4. *For* $\tau \in \mathbb{R}\backslash\Omega$,

$$\sum_{j=0}^{m} U'(\tau - jh) D_j = \sum_{j=0}^{m} U(\tau - jh) A_j + K^T(-\tau) W.$$

Proof. Let us introduce the function

$$F(\widehat{\tau}) = \begin{cases} \sum_{j=0}^{m} D_j^T U'(\widehat{\tau} + jh) + \sum_{j=0}^{m} A_j^T U(\widehat{\tau} + jh) + WK(\widehat{\tau}), & \widehat{\tau} \in \mathbb{R}\backslash\Omega, \\ \lim_{\theta \to \widehat{\tau}+0} F(\theta), & \tau \in \Omega. \end{cases}$$

Note that $F(\widehat{\tau}) = 0$ for $\widehat{\tau} \in [-mh, 0)$ as $K(\theta) = 0$ for $\theta < 0$ and by the dynamic property (4.4). We now prove that it also holds for $\widehat{\tau} = 0$. By continuity of $U(\tau)$, $\tau \in \mathbb{R}$,

$$\sum_{j=0}^{m} A_j^T U(+0 + jh) = \sum_{j=0}^{m} A_j^T U(-0 + jh) = -\sum_{j=0}^{m} D_j^T U'(-0 + jh),$$

hence, by the preceding equation and the algebraic property (4.11),

$$\begin{aligned} F(0) &= \sum_{j=0}^{m} D_j^T U'(+0 + jh) - \sum_{j=0}^{m} D_j^T U'(-0 + jh) + W \\ &= \sum_{j=0}^{m} D_j^T \Delta U'(jh) + W = 0. \end{aligned}$$

Let us prove that $\sum_{k=0}^{m} F(\widehat{\tau} - kh) D_k$ is continuous at every point for $\widehat{\tau} > 0$. Observe that

$$\begin{aligned} \sum_{k=0}^{m} F(\widehat{\tau} - kh) D_k = \sum_{j=0}^{m} D_j^T & \left(\sum_{k=0}^{m} U(\widehat{\tau} + jh - kh) A_k \right) \\ & + \sum_{k=0}^{m} \sum_{j=0}^{m} A_j^T U(\widehat{\tau} + jh - kh) D_k + W \sum_{k=0}^{m} K(\widehat{\tau} - kh) D_k, \ \widehat{\tau} \geq 0. \end{aligned}$$

According to the definitions of the matrices $K(\tau)$ and $U(\tau)$, the above sum is continuous everywhere on $\widehat{\tau} \geq 0$. As every summand of the function $F(\widehat{\tau})$ satisfies an equation of the form (4.4), it satisfies the delay equation

$$\sum_{j=0}^{m} F'(\widehat{\tau} - jh) D_j = \sum_{j=0}^{m} F(\widehat{\tau} - jh) A_j, \ \widehat{\tau} \geq 0, \ a.e.$$

Since $F(\widehat{\tau}) = 0$ for $\widehat{\tau} \in [-mh, 0]$, then the unique solution of the previous system is the trivial one, i.e,

$$\sum_{i=0}^{m} D_i^T U'(\widehat{\tau} + ih) + \sum_{i=0}^{m} A_i^T U(\widehat{\tau} + ih) + WK(\widehat{\tau}) = 0.$$

Transposition and setting $\tau = -\widehat{\tau}$, give the result. □

Lemma 4.5. *For $\tau \geq 0$ and $\xi \in \mathbb{R}$,*

$$U(\tau + \xi) = U(\xi) \sum_{j=0}^{m} D_j K(\tau - jh) + \sum_{j=1}^{m} \int_{-jh}^{0} F_j^{(1)}(\xi - s - jh) K(\tau + s) ds + \int_{-\tau}^{0} K^T(s - \xi) W K(\tau + s) ds. \tag{4.13}$$

Proof. Fix $\bar{\tau} \in \mathbb{R}$, and introduce the continuous function

$$\begin{aligned} G(\tau) =& U(-\bar{\tau}) - U(-\bar{\tau} - \tau) \sum_{i=0}^{m} D_i K(\tau - ih) - \sum_{j=1}^{m} \int_{-jh}^{0} F_j^{(1)}(-\bar{\tau} - \tau - s - jh) K(\tau + s) ds \\ & - \int_{-\tau}^{0} K^T(s + \bar{\tau} + \tau) W K(\tau + s) ds. \end{aligned}$$

In order to prove the lemma, it is enough to show that $G(\tau)=0$. By the change of variable $\eta=s+\tau$ in the integral terms, we obtain

$$\begin{aligned}G(\tau)=U(-\bar{\tau})-U(-\bar{\tau}-\tau)\sum_{j=0}^{m}D_jK(\tau-jh)&\\ -\sum_{j=1}^{m}\int_{\tau-jh}^{\tau}F_j^{(1)}(-\bar{\tau}-\eta-jh)K(\eta)d\eta-\int_{0}^{\tau}K^T(\eta+\bar{\tau})WK(\eta)d\eta.&\end{aligned}$$

Differentiating this function and using equation (4.2), we arrive at

$$\begin{aligned}G'(\tau)=&U'(-\bar{\tau}-\tau)\sum_{j=0}^{m}D_jK(\tau-jh)-U(-\bar{\tau}-\tau)\sum_{j=0}^{m}A_jK(\tau-jh)-\sum_{j=1}^{m}F_j^{(1)}(-\bar{\tau}-\tau-jh)K(\tau)\\ &+\sum_{j=1}^{m}F_j^{(1)}(-\bar{\tau}-\tau)K(\tau-jh)-K^T(\tau+\bar{\tau})WK(\tau)\\ =&\left(-\sum_{j=0}^{m}F_j^{(1)}(-\bar{\tau}-\tau-jh)-K^T(\bar{\tau}+\tau)W\right)K(\tau)\\ &+\sum_{j=1}^{m}\Big(U'(-\bar{\tau}-\tau)D_j-U(-\bar{\tau}-\tau)A_j+F_j^{(1)}(-\bar{\tau}-\tau)\Big)K(\tau-jh),\ \tau\geq 0,\ \tau\in\mathbb{R}\backslash\Omega.\end{aligned}$$

By the definition of $F_i^{(1)}$ and Lemma 4.4, we have that $G'(\tau)=0$, $\tau\geq 0$, $\tau\in\mathbb{R}\backslash\Omega$, which implies that $G(\tau)$ is constant. Now, as $G(0)=0$ and G is continuous, then $G(\tau)=0$ for $\tau\geq 0$ and the lemma is proved. □

Corollary 4.1. *For $\tau\geq 0$ and $\xi\geq 0$,*

$$U(\tau+\xi)=U(\xi)\sum_{j=0}^{m}D_jK(\tau-jh)+\sum_{j=1}^{m}\int_{-jh}^{0}F_j^{(1)}(\xi-s-jh)K(\tau+s)ds. \tag{4.14}$$

Corollary 4.2. *For $\tau\geq 0$ and $\xi\in(lh,(l+1)h)$, $l=0,1,\ldots$,*

$$\begin{aligned}U'(\tau+\xi)=U'(\xi)\sum_{j=0}^{m}D_jK(\tau-jh)+\sum_{j=1}^{m}\sum_{p=1}^{j}\int_{-ph}^{(1-p)h}\frac{d}{d\xi}F_j^{(1)}(\xi-s-jh)K(\tau+s)ds&\\ -\sum_{j=1}^{m}\sum_{p=1}^{j}\Delta U'((l+p-j)h)D_jK(\tau+\xi-(l+p)h),\ a.e.&\end{aligned}\tag{4.15}$$

Proof. The next expression follows from the fact that the matrix $K(t)$ is continuous for $t\in(lh,(l+1)h)$ and from the definition of $F_j^{(1)}$:

$$\begin{aligned}&\frac{d}{d\xi}\int_{-jh}^{0}F_j^{(1)}(\xi-s-jh)K(\tau+s)ds\\ &=\sum_{p=1}^{j}\int_{-ph}^{(1-p)h}\frac{d}{d\xi}F_j^{(1)}(\xi-s-jh)K(\tau+s)ds-\sum_{p=1}^{j}\Delta U'((l+p-j)h)D_jK(\tau+\xi-(l+p)h),\ a.e.\end{aligned}$$

Differentiating (4.14) with respect to ξ and using the previous equation, we get (4.15). □

Lemma 4.6. *For $\tau \geq 0$ and $\xi \in (lh, (l+1)h)$, $l = 0, 1, \ldots$,*

$$\left(F_j^{(1)}(-\tau-\xi)\right)^T = \left(F_j^{(1)}(-\xi)\right)^T \sum_{k=0}^{m} D_k K(\tau - kh) + \sum_{k=1}^{m} \int_{-kh}^{0} F_{jk}^{(2)}(\xi - s - kh) K(\tau + s) ds$$
$$- \sum_{k=1}^{m} \sum_{q=1}^{k} D_j^T \Delta U'((l+q-k)h) D_k K(\tau + \xi - (l+q)h), \ a.e. \quad (4.16)$$

Proof. The result is obtained directly by applying equations (4.14) and (4.15) as follows

$$\begin{aligned}(F_j^{(1)}(-\tau-\xi))^T &= A_j^T U(\tau+\xi) + D_j^T U'(\tau+\xi)\\ &= (F_j^{(1)}(-\xi))^T \sum_{k=0}^{m} D_k K(\tau - kh) + \sum_{k=1}^{m} \int_{-kh}^{0} F_{jk}^{(2)}(\xi - s - kh) K(\tau + s) ds\\ &\quad - \sum_{k=1}^{m} \sum_{q=1}^{k} D_j^T \Delta U'((l+q-k)h) D_k K(\tau + \xi - (l+q)h), \ a.e.\end{aligned}$$

□

4.3.2 Bilinear functional

Let us introduce the bilinear functional

$$\begin{aligned} z(\varphi_1, \varphi_2) &= \sum_{j=0}^{m} \sum_{k=0}^{m} \varphi_1^T(-jh) D_j^T U(0) D_k \varphi_2(-kh) + \sum_{j=0}^{m} \sum_{k=1}^{m} \varphi_1^T(-jh) D_j^T \int_{-kh}^{0} F_k^{(1)}(-\theta - kh) \varphi_2(\theta) d\theta\\ &\quad + \sum_{k=0}^{m} \sum_{j=1}^{m} \int_{-jh}^{0} \varphi_1^T(\theta) (F_j^{(1)}(-\theta - jh))^T d\theta D_k \varphi_2(-kh)\\ &\quad + \sum_{j=1}^{m} \sum_{k=1}^{m} \int_{-jh}^{0} \int_{-kh}^{0} \varphi_1^T(\theta_1) F_{jk}^{(2)}(\theta_1 - \theta_2 + jh - kh) \varphi_2(\theta_2) d\theta_2 d\theta_1\\ &\quad - \sum_{j=1}^{m} \sum_{k=1}^{m} \int_{0}^{h} \varphi_1^T(\theta - jh) F_{jk}^{(3)} \varphi_2(\theta - kh) d\theta + \int_{-mh}^{0} \varphi_1^T(\theta) W \varphi_2(\theta) d\theta, \ \varphi_1, \varphi_2 \in PC\left([-mh, 0], \mathbb{R}^n\right). \end{aligned}$$

We show that for functions given in terms of the fundamental matrix the bilinear functional reduces to a simple expression depending uniquely on the delay Lyapunov matrix.

Lemma 4.7. *For $\tau_1, \tau_2 \in [0, mh]$,*

$$z(K(\tau_1 + \cdot)\gamma, K(\tau_2 + \cdot)\eta) = \gamma^T U(\tau_2 - \tau_1)\eta. \quad (4.17)$$

Proof. Observe first that for any function φ,

$$\begin{aligned}&\sum_{j=1}^{m} \sum_{k=1}^{m} \int_{0}^{h} \varphi(\theta - jh) F_{jk}^{(3)} \varphi(\theta - kh) d\theta\\ &\qquad = \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{q=1}^{k} \sum_{p=1}^{j} \int_{-qh}^{(1-q)h} \varphi(\theta + (q-p)h) D_j^T \Delta U'((q-p-k+j)h) D_k \varphi(\theta) d\theta. \quad (4.18)\end{aligned}$$

Consider now the functions $\bar{\varphi}_1(\theta) = K(\tau_1 + \theta)\gamma$ and $\bar{\varphi}_2(\theta) = K(\tau_2 + \theta)\eta$. Substituting $\bar{\varphi}_1$ and $\bar{\varphi}_2$ and rearranging some terms in $z(\bar{\varphi}_1, \bar{\varphi}_2)$, we have

$$\begin{aligned}
&z(K(\tau_1+\cdot)\gamma, K(\tau_2+\cdot)\eta)\\
&\quad=\gamma^T\Bigg(\sum_{j=0}^{m}K^T(\tau_1-jh)D_j^T\left(\sum_{k=0}^{m}U(0)D_kK(\tau_2-kh)+\sum_{k=1}^{m}\int_{-kh}^{0}F_k^{(1)}(-\theta-kh)K(\tau_2+\theta)d\theta\right)\\
&\qquad+\sum_{j=1}^{m}\sum_{p=1}^{j}\int_{-ph}^{(1-p)h}K^T(\tau_1+\theta_1)\Bigg((F_j^{(1)}(-\theta_1-jh))^T\sum_{k=0}^{m}D_kK(\tau_2-kh)\\
&\qquad+\sum_{k=1}^{m}\int_{-kh}^{0}F_{jk}^{(2)}(\theta_1-\theta_2+jh-kh)K(\tau_2+\theta_2)d\theta_2\Bigg)d\theta_1\\
&\qquad-\sum_{j=1}^{m}\sum_{k=1}^{m}\int_{0}^{h}K^T(\tau_1+\theta-jh)F_{jk}^{(3)}K(\tau_2+\theta-kh)d\theta+\int_{-mh}^{0}K^T(\tau_1+\theta)WK(\tau_2+\theta)d\theta\Bigg)\eta.
\end{aligned}$$

Applying (4.14) in the first summand, (4.16) with $\xi=\theta_1+jh$, $l=j-p$ and (4.18), in the second one, we get

$$\begin{aligned}
&z(K(\tau_1+\cdot)\gamma, K(\tau_2+\cdot)\eta)=\gamma^T\Bigg(\sum_{j=0}^{m}K^T(\tau_1-jh)D_j^TU(\tau_2)\\
&+\sum_{j=1}^{m}\sum_{p=1}^{j}\int_{-ph}^{(1-p)h}K^T(\tau_1+\theta_1)(F_j^{(1)}(-\tau_2-\theta_1-jh))^Td\theta_1+\int_{-mh}^{0}K^T(\tau_1+\theta)WK(\tau_2+\theta)d\theta\Bigg)\eta,
\end{aligned}$$

and the result is obtained by using equation (4.13). □

4.4 Necessary stability conditions for neutral type systems

We are now ready to present the necessary stability conditions for system (4.1). They are obtained with the help of the results introduced in the previous sections.

Theorem 4.4. *Let system* (4.1) *be exponentially stable and* $\tau_k\in[0,mh]$, $k=\overline{1,r}$, *such that* $\tau_i\neq\tau_j$ *if* $i\neq j$, *then*

$$\mathcal{K}_r(\tau_1,\ldots,\tau_r)>0, \tag{4.19}$$

where

$$\mathcal{K}_r(\tau_1,\ldots,\tau_r)=\begin{pmatrix}U(0) & U(\tau_2-\tau_1) & \ldots & U(\tau_r-\tau_1)\\ U^T(\tau_2-\tau_1) & U(0) & \ldots & U(\tau_r-\tau_2)\\ \vdots & & \ddots & \vdots\\ U^T(\tau_r-\tau_1) & U^T(\tau_r-\tau_2) & \ldots & U(0)\end{pmatrix}.$$

Proof. We use the same initial function as the one given by (3.11):

$$\psi_r(\theta)=\sum_{j=1}^{r}K(\tau_j+\theta)\gamma_j,\ \theta\in[-mh,0],$$

where $\gamma_j\in\mathbb{R}^n$, $j=\overline{1,m}$, are arbitrary constant vectors. Since $v_1(\varphi)=z(\varphi,\varphi)$ for every φ, then

$$v_1(\psi_r)=z(\psi_r,\psi_r)=\sum_{k=1}^{r}\sum_{j=1}^{r}z(K(\tau_k+\cdot)\gamma_i,K(\tau_j+\cdot)\gamma_j).$$

From equality (4.17), we have

$$v_1(\psi_r) = \sum_{k=1}^{r}\sum_{j=1}^{r} \gamma_i^T U(\tau_j - \tau_k)\gamma_j = \gamma^T \mathcal{K}_r(\tau_1, \dots, \tau_r)\gamma,$$

where $\gamma = \begin{pmatrix}\gamma_1^T & \dots & \gamma_r^T\end{pmatrix}^T$ and ny Theorem 4.3, we get

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r(\tau_1, \dots, \tau_r)\gamma \geq \beta \left\|\psi_r\right\|_{\mathcal{H}}^2, \ \beta > 0.$$

Finally, we show that $\|\psi_r\|_{\mathcal{H}} \neq 0$ for $\gamma \neq 0$. Assume that $\|\psi_r\|_{\mathcal{H}} = 0$, by right-continuity of the initial function ψ_r and the definition of $\|\cdot\|_{\mathcal{H}}$, $\psi_r(\theta) = 0$, on $\theta \in [-h, 0]$, which contradicts to $\gamma \neq 0$. Hence,

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r(\tau_1, \dots, \tau_r)\gamma > 0,$$

which implies

$$\mathcal{K}_r(\tau_1, \dots, \tau_r) > 0.$$

□

4.5 Necessary stability conditions for difference equations

The relevance of the study of the difference equation in continuous time comes from the usefulness in modelling diverse physical phenomenons. The stability analysis of this class of equations has been subject of many contributions.

In this section, we provide a result that allows to use the stability conditions of Theorem 4.4 to study the stability of difference equations. Consider the equation

$$y(t) = \sum_{j=1}^{m} B_j y(t - h_j), \ t \geq 0, \tag{4.20}$$

where $B_j \in \mathbb{R}^{n\times n}$.

We say that system (4.20) is exponentially stable if there exist $\mu > 0$ and $\alpha > 0$ such that

$$\|y(t, \phi)\| \leq \mu e^{-\alpha t} \sup_{\theta\in[-mh,0]} \|\phi(\theta)\|,$$

with $\phi \in PC\left([-mh, 0], \mathbb{R}^n\right)$.

Let us introduce the neutral type delay system

$$\frac{d}{dt}\left(x(t) - \sum_{j=1}^{m} B_j x(t - h_j)\right) = Mx(t) - M\sum_{j=1}^{m} B_j x(t - h_j), \tag{4.21}$$

where $M \in \mathbb{R}^{n\times n}$. In the next lemma, we state the relation between systems (4.20) and (4.21) in the stability sense.

Lemma 4.8. *Let matrix M be Hurwitz stable. System* (4.20) *is exponentially stable if and only if* (4.21) *does.*

Proof. The characteristic equation of system (4.21) is given by

$$\begin{aligned} p(s) = \det\left(sI_n - \sum_{j=1}^{m} B_j s e^{-sh_j} - M + M\sum_{j=1}^{m} B_j e^{-sh_j}\right) &= \det\left((sI_n - M)\left(I_n - \sum_{j=1}^{m} B_j e^{-sh_j}\right)\right) \\ &= \det(sI_n - M)\det\left(I_n - \sum_{j=1}^{m} B_j e^{-sh_j}\right). \end{aligned}$$

This equality implies that system (4.21) is exponentially stable if and only if (4.20) is. □

Theorem 4.5. *Let matrix M be Hurwitz stable. If system* (4.20) *is exponentially stable, then conditions of Theorem 4.4 holds for system* (4.21).

Proof. By observing that system (4.21) is a special case of (4.1), the result follows directly from Lemma 4.8. □

4.6 Examples

In this section, we illustrate the necessary stability conditions stated in Theorem 4.4 by some examples. We follow the same procedure as the presented in Section 3.5 of Chapter 3.

The delay Lyapunov matrix is computed by the semianalytic method (see, for instance, Ochoa et al. (2009) and Appendix C) and the conditions are tested at 80×80 points of the space of parameters. Some techniques introduced in Gomez et al. (2016a) that are recalled in Appendix C are used to reduce the computational effort. In each example, the continuous lines correspond to the stability boundaries obtained by the D-subdivision method (Neimark (1949)).

We first consider the scalar systems with three delays presented in Olgac et al. (2008). The particularity of this example is that the third delay is a sum of the other two.

Example 4.1. *Let a system of the form* (4.1) *with scalar coefficients be*

$$\frac{d}{dt}\left(x(t) + 0.8x(t-\eta_1) + 0.15x(t-\eta_2)\right) = x(t) - 1.5x(t-\eta_1) + 2x(t-\eta_2) - 5x(t-(\eta_1+\eta_2)). \quad (4.22)$$

Consider condition (4.19) *with $r = 4$:*

$$\mathcal{K}_4(\tau_1, \tau_2, \tau_3, \tau_4) > 0, \quad (4.23)$$

where $\tau_1 = 0$, $\tau_2 = \tau$, $\tau_3 = \dfrac{\eta_{max}}{2}$ and $\tau_4 = \dfrac{\eta_{max}}{2} + \tau$, with $\tau \in \left(0, \dfrac{\eta_{max}}{2}\right)$ and $\eta_{max} = \eta_1 + \eta_2$.

We compute the stability map in the space of parameters (η_1, η_2). In order to do this, we establish as basic delay $h = 0.01$ and choose values of η_1 and η_2 such that $\eta_1 = jh$ and $\eta_2 = kh$, where $j, k = \overline{0, N}$. Here $N = 80$, i.e. the condition is tested at 80×80 points of the space of parameters. For every $j, k = \overline{0, N}$, system (4.22) *can be expressed as a system of the form* (4.1) *with delays h_j.*

Figure 4.1 shows the points in the space of parameters (η_1, η_2), where (4.23) *holds. We observe that*

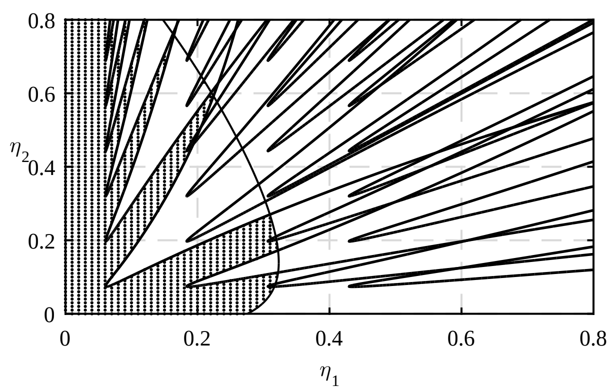

Figure 4.1: Example 4.1, condition (4.23)

in this case, the exact stability region is achieved.

The second example concerns a linearized neutral delay predator-prey model introduced in Kuang (1991).

Example 4.2. *The first approximation of the model proposed in Kuang (1991) is given by a neutral system of the form*

$$\frac{d}{dt}\left(x(t)+\begin{pmatrix}\epsilon & 0\\ 0 & 0\end{pmatrix}x(t-\eta_2)\right)=\begin{pmatrix}-a & -c\\ 0 & 0\end{pmatrix}x(t)+\begin{pmatrix}0 & 0\\ d & 0\end{pmatrix}x(t-\eta_1)+\begin{pmatrix}-b & 0\\ 0 & 0\end{pmatrix}x(t-\eta_2), \quad (4.24)$$

where the parameter numerical values are considered to be

$$a=2,\ b=0.4,\ c=1,\ d=3.5,\ \eta_2=0.5.$$

Let us compute the stability map in the space of parameters (η_1,ϵ)*. We take* $h=0.025$ *as the basic delay and consider values of* η_1 *such that* $\eta_1=jh$, $j=\overline{1,N}$, *where* $N=80$. *In this case, the delay* $\eta_2=20h$. *For every* $j=\overline{1,N}$, *system* (4.24) *can be written as a system of the form* (4.1).

Consider $\eta_{max}=\max\{\eta_1,\eta_2\}$. *The points in the space of parameters* (η_1,ϵ)*, where*

$$\mathcal{K}_2(0,\tau)>0,\ \tau\in(0,\eta_{max}], \quad (4.25)$$

and

$$\mathcal{K}_4(\tau_1,\tau_2,\tau_3,\tau_4)>0, \quad (4.26)$$

with $\tau_1=0$, $\tau_2=\tau$, $\tau_3=\dfrac{\eta_{max}}{2}$, $\tau_4=\dfrac{\eta_{max}}{2}+\tau$, *and* $\tau\in\left(0,\dfrac{\eta_{max}}{2}\right)$, *hold are depicted in Figures 4.2 and 4.3, respectively. This example allows us to observe that the exact stability zone in the space of parameters is achieved by increasing the parameter* r *from* $r=2$ *to* $r=4$.

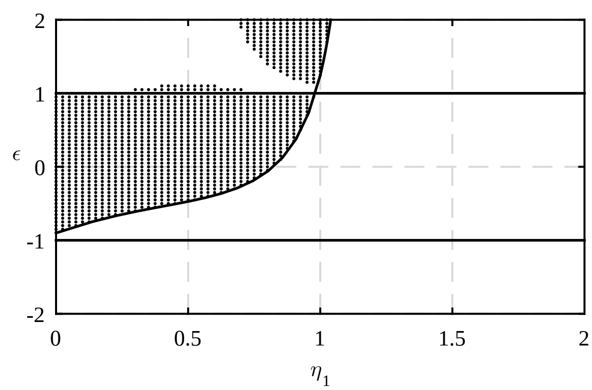

Figure 4.2: Example 4.2, condition (4.25)

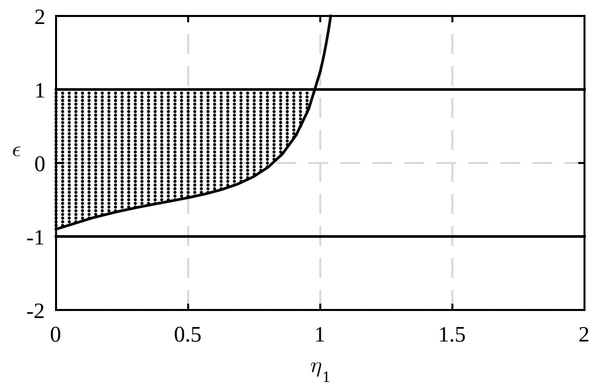

Figure 4.3: Example 4.2, condition (4.26)

The next example is taken from Rocha et al. (2016). We use Theorem 4.5 to determine the stability map in the space of the system parameters.

Example 4.3. *Let us consider the difference equation in continuous time* (4.20) *with* $h_1 = 0.5$*,* $h_2 = 1$ *and* $h_3 = h_{max} = 1.5$ *and matrices*

$$B_1 = \begin{pmatrix} 0.3 & 1 \\ -1 & 0.3 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0.5ab^2 \\ 0.2b & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0.2a & 0 \\ 0 & 0.5a^2b \end{pmatrix}. \tag{4.27}$$

In order to use Theorem 4.5, we consider a neutral type time-delay system of the form (4.21) *with* $M = -I$ *and matrices* B_i*,* $i = 1, 2, 3$*, given by* (4.27)*. The points where condition*

$$\mathcal{K}_4(\tau_1, \tau_2, \tau_3, \tau_4) > 0, \tag{4.28}$$

with $\tau_1 = 0$*,* $\tau_2 = \tau$*,* $\tau_3 = \dfrac{h_{max}}{2}$*,* $\tau_4 = \dfrac{h_{max}}{2} + \tau$*, and* $\tau \in \left(0, \dfrac{h_{max}}{2}\right)$*, holds in the space of parameters* (a, b) *are depicted on Figure 4.4. Figure 4.5 shows the points where the next condition holds:*

$$\mathcal{K}_6(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) > 0, \tag{4.29}$$

where $\tau_1 = 0$*,* $\tau_2 = \tau$*,* $\tau_3 = \dfrac{h_{max}}{3}$*,* $\tau_4 = \dfrac{h_{max}}{3} + \tau$*,* $\tau_5 = \dfrac{2h_{max}}{3}$ *and* $\tau_6 = \dfrac{2h_{max}}{3} + \tau$*, where* $\tau \in \left(0, \dfrac{h_{max}}{3}\right)$*. The exact stability zone is achieved by this condition.*

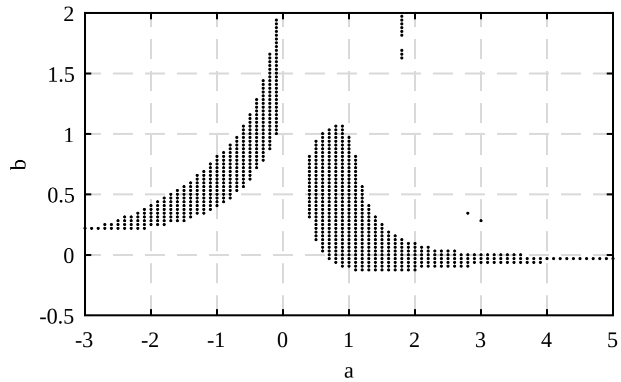

Figure 4.4: Example 4.3, condition (4.28).

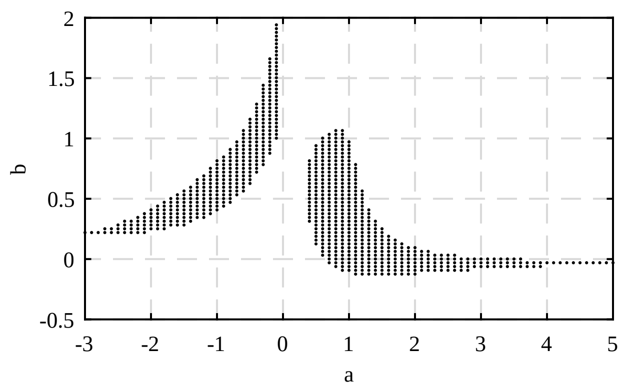

Figure 4.5: Example 4.3, condition (4.29).

4.7 Conclusion

Inspired by the ideas of Chapter 3, necessary stability conditions for neutral type systems with multiple commensurate delays are presented. They preserve the same form as those obtained for a single delay. The result is based on two preliminary results which are relevant in their own right: the computation of the functional of complete type by a new Cauchy formula and the proof of new properties of the delay Lyapunov matrix. Some examples show the efficiency of this new approach in assessing the stability of neutral type systems with multiple delays.

Additionally, the stability equivalence between a particular neutral type system and difference equation in continuous time is obtained. This result suggests the application of stability tools known for neutral type systems, particularly the necessary conditions presented in this contribution, to the stability analysis of difference equations.

Chapter 5

Necessary and sufficient stability conditions: Single delay case

In this chapter, we present two different *finite* stability criteria for neutral type systems with single delay, which are, analogously to the delay free case, given in terms of the positivity of a matrix that depends on the delay Lyapunov matrix. Their main characteristic is that a *finite* number of mathematical operations are needed to be tested.

The result is obtained by following similar arguments to the used in the retarded type case by Egorov (2016): Inspired by the recent work presented by Alexandrova and Zhabko (2016), we use initial functions from a compact set and approximate them by functions of the form (3.11), which were used for obtaining the necessary stability conditions in Chapter 3. Then, an estimate of the approximation error is computed and employed in order to get a finite number of mathematical operations in the criteria.

The chapter is organized as follows. In Section 5.1 some basic facts on the system are reminded for the convenience of the reader. Quadratic upper and lower bounds for the functional v_1 introduced in Chapter 3 are provided in Section 5.2. In Section 5.3 some auxiliary results related to the compact set previously mentioned are presented. The main results of the chapter are introduced in Section 5.4 and Section 5.5: finite stability criteria given in terms of the delay Lyapunov matrix. They are illustrated by some academic examples in Section 5.6 and their differences are discussed in Section 5.7. Finally, the chapter ends with some final remarks.

5.1 Basic facts on the system

For the convenience of the reader, throughout the chapter, we recall some elements previously presented. We consider the system introduced in Chapter 3:

$$\frac{d}{dt}\left(x(t) - Dx(t-h)\right) = A_0x(t) + A_1x(t-h), \tag{5.1}$$

where $A_0,\ A_1$ and D are matrices in $\mathbb{R}^{n\times n}$, and $h > 0$ is the delay.

5.1.1 Fundamental matrix

The fundamental matrix $K(t)$ of system (5.1) satisfies the equation

$$\frac{d}{dt}\left(K(t) - DK(t-h)\right) = A_0K(t) + A_1K(t-h), \tag{5.2}$$

with the initial condition $K(t) = I_n$ for $t = 0$ and $K(t) = 0$ for $t < 0$. Lemma 2.1 states that the fundamental matrix has jumps at $t = jh$, $j = 0, 1, 2, \ldots$, i.e.,

$$\Delta K(jh) = D^j,\ j = 0, 1, 2 \ldots,$$

and that the value of the fundamental matrix at points $t = jh$, coincides with the right-hand side value, i.e. $K(jh) = K(jh + 0)$.

From the previously recalled, it follows that, for $t \in [0, h]$, the fundamental matrix is given by

$$K(t) = \begin{cases} e^{A_0 t}, & t \in [0, h), \\ e^{A_0 h} + D & t = h. \end{cases} \tag{5.3}$$

5.1.2 Schur stability

Next, we introduce a formal definition of stability of matrix D.

Definition 5.1. *(Niculescu (2001)) We say that matrix D is Schur stable if all its eigenvalues are inside the unit circle.*

As was shown in Chapter 1, the stability of matrix D is a necessary stability condition for the stability of neutral type systems. Indeed, a well-known assumption in the stability study of system (5.1) is the Schur stability of matrix D (see, for instance, Hale and Lunel (1993) and Fridman (2014)). An upper estimate of the norm of this matrix is required in the subsequent results and one way for computing it is given in the next lemma.

Lemma 5.1 (Kharitonov et al. (2006))**.** *A Schur stable matrix D admits the following upper bound:*

$$\|D^k\| \leq d\rho^k,$$

with $\rho \in (0, 1)$ and $d = \sqrt{\dfrac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}}$, where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix solution of

$$D^T Q D - \rho^2 Q < 0.$$

5.1.3 Spectral abscissa

Let $s \in \mathbb{C}$ be an eigenvalue of system (5.1). The spectral abscissa of system (5.1) is defined as (Michiels and Niculescu (2014)):

$$c = \sup\left\{\mathbf{Re}(s) | \det(sI_n - De^{-sh}s - A_0 - A_1 e^{-sh}) = 0\right\}.$$

The following lemma provides an estimate of c.

Lemma 5.2. *Assume that matrix D is Schur stable. Every eigenvalue $s \in \mathbb{C}$ of system* (5.1) *satisfies*

$$\mathbf{Re}(s) \leq \frac{d}{1-\rho}\left(\|A_0\| + \|A_1\|\right),$$

where d and ρ are determined by Lemma 5.1.

Proof. If $\mathbf{Re}(s) \leq 0$, the upper bound obviously holds. Let s be an eigenvalue of system (5.1) with $\mathbf{Re}(s) > 0$ and consider its corresponding eigenvector $c \in \mathbb{C}^n$. The following equality holds:

$$s\left(I_n - e^{-sh}D\right)c = \left(A_0 + A_1 e^{-sh}\right)c. \tag{5.4}$$

As matrix D is Schur stable, the inverse of the matrix $I_n - e^{-sh}D$ exists. Indeed, since every eigenvalue of the matrix D is inside the unit circle, then

$$\det\left(I_n - e^{-sh}D\right) = \det\left(e^{-sh}(e^{sh}I_n - D)\right) = e^{-nsh}\det\left(e^{sh}I_n - D\right) \neq 0.$$

Notice that

$$\|c\| \leq \|(I_n - e^{-sh}D)^{-1}\|\|(I_n - e^{-sh}D)c\|,$$

therefore, from equation (5.4),

$$\frac{|s|\|c\|}{\|(I_n - e^{-sh}D)^{-1}\|} \leq |s|\|(I_n - e^{-sh}D)c\| \leq (\|A_0\| + \|A_1\|)\|c\|,$$

which implies that

$$|s| \leq (\|A_0\| + \|A_1\|)\|(I_n - e^{-sh}D)^{-1}\|.$$

We now estimate the multiplier of the previous expression. Consider the following equality:

$$(I_n - e^{-sh}D)^{-1} = I_n + \sum_{k=1}^{\infty} e^{-ksh}D^k.$$

As $|e^{-ksh}| < 1$ and by Lemma 5.1, $\|D^k\| \leq d\rho^k$, we get,

$$\|(I_n - e^{-sh}D)^{-1}\| = \left\|\sum_{k=0}^{\infty} e^{-ksh}D^k\right\| \leq \sum_{k=0}^{\infty} d\rho^k = \frac{d}{1-\rho},$$

hence

$$\mathbf{Re}(s) \leq |s| \leq \frac{d}{1-\rho}\left(\|A_0\| + \|A_1\|\right).$$

□

5.2 Functional v_1

The following key functional was introduced in Chapter 3:

$$v_1(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta)W\varphi(\theta)d\theta,\ W > 0,\ \varphi \in PC\left([-h,0],\mathbb{R}^n\right), \tag{5.5}$$

where $v_0(\varphi)$, given by equation (2.22), is determined by the delay Lyapunov matrix associated with the matrix W. The derivative of the functional $v_1(\varphi)$ along the solutions of the system is

$$\frac{d}{dt}v_1(x_t(\varphi)) = -x^T(t-h,\varphi)Wx(t-h,\varphi),\ \varphi \in PC\left([-h,0],\mathbb{R}^n\right). \tag{5.6}$$

We also recall the bilinear functional introduced in Chapter 3:

$$\begin{aligned} z(\varphi_1,\varphi_2) &= \left(\varphi_1(0) - D\varphi_1(-h)\right)^T U(0)\left(\varphi_2(0) - D\varphi_2(-h)\right) + \left(\varphi_1(0) - D\varphi_1(-h)\right)^T \int_{-h}^{0} F_1(-h-\theta)\varphi_2(\theta)d\theta \\ &+ \int_{-h}^{0} \varphi_1^T(\theta)F_1^T(-h-\theta)d\theta\left(\varphi_2(0) - D\varphi_2(-h)\right) + \int_{-h}^{0}\int_{-h}^{0} \varphi_1^T(\theta_1)F_2(\theta_1-\theta_2)\varphi_2(\theta_2)d\theta_2 d\theta_1 \\ &- \int_{-h}^{0} \varphi_1^T(\theta)\Delta U'(0)\varphi_2(\theta)d\theta,\ \varphi_1,\varphi_2 \in PC\left([-h,0],\mathbb{R}^n\right), \end{aligned} \tag{5.7}$$

We provide next an upper bound for the previously mentioned functionals.

Lemma 5.3. *For any $\varphi_1, \varphi_2 \in PC([-h,0],\mathbb{R}^n)$,*

$$|v_1(\varphi)| \leq \beta_2 \|\varphi\|_h^2,$$
$$|z(\varphi_1,\varphi_2)| \leq \beta_2 \|\varphi_1\|_h \|\varphi_2\|_h,$$

where

$$\beta_2 = (1 + \|D\|)^2 \|U(0)\| + 2h(1+\|D\|) f_1 + h^2 f_2 + h\|\Delta U'(0)\|,$$

with

$$f_1 = \sup_{\tau \in (0,h)} \|F_1(\tau)\|, \ f_2 = \sup_{\tau \in (0,h)} \|F_2(\tau)\|.$$

Proof. We estimate an upper bound of each term of the bilinear functional (5.7) as follows. For the first term, we have:

$$\left| (\varphi_1(0) - D\varphi_1(-h))^T U(0) (\varphi_2(0) - D\varphi_2(-h)) \right| \leq (1 + \|D\|)^2 \|U(0)\| \|\varphi_1\|_h \|\varphi_2\|_h.$$

The next two summands are bounded by the same estimate:

$$\left| (\varphi_1(0) - D\varphi_1(-h))^T \int_{-h}^{0} F_1(-h-\theta)\varphi_2(\theta) d\theta \right| \leq f_1(1+\|D\|)h\|\varphi_1\|_h\|\varphi_2\|_h$$

$$\left| \int_{-h}^{0} \varphi_1^T(\theta) F_1^T(-h-\theta) d\theta \, (\varphi_2(0) - D\varphi_2(-h)) \right| \leq f_1(1+\|D\|)h\|\varphi_1\|_h\|\varphi_2\|_h.$$

For the double integral term, we get,

$$\left| \int_{-h}^{0} \int_{-h}^{0} \varphi_1^T(\theta_1) F_2(\theta_1 - \theta_2) \varphi_2(\theta_2) d\theta_2 d\theta_1 \right| \leq h^2 f_2 \|\varphi_1\|_h \|\varphi_2\|_h.$$

Finally, the last term is bounded as

$$\left| \int_{-h}^{0} \varphi_1^T(\theta) \Delta U'(0) \varphi_2(\theta) d\theta \right| \leq h\|\Delta U'(0)\| \|\varphi_1\|_h \|\varphi_2\|_h.$$

The upper bound for the bilinear functional $z(\cdot,\cdot)$ and $v_1(\cdot)$ directly follows from the previous estimates and from the equality $v_1(\varphi) = z(\varphi,\varphi)$, respectively. □

In the next theorem, it is shown that the functional $v_1(\varphi)$ satisfies a quadratic lower bound of the form $v_1(\varphi) \geq \beta_1 \|\varphi(0) - D\varphi(-h)\|^2$ and that one can estimate the number $\beta_1 > 0$ when system is stable.

Theorem 5.1. *Assume that matrix D is Schur stable. System* (5.1) *is exponentially stable if and only if the Lyapunov condition holds and for a number $\beta_1 > 0$,*

$$v_1(\varphi) \geq \beta_1 \|\varphi(0) - D\varphi(-h)\|^2, \ \varphi \in PC([-h,0],\mathbb{R}^n). \tag{5.8}$$

Furthermore, if system (5.1) *is exponentially stable,*

$$v_1(\varphi) \geq \beta_A^\star \|\varphi(0) - D\varphi(-h)\|^2, \ \varphi \in PC([-h,0],\mathbb{R}^n),$$

where $\beta_A^\star = \dfrac{\beta}{2}$ and $\beta > 0$ is such that

$$P(\beta) = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} + \beta \begin{pmatrix} A_0^T + A_0 & -A_0^T D + A_1 \\ -D^T A_0 + A_1^T & -A_1^T D - D^T A_1 \end{pmatrix} \geq 0. \tag{5.9}$$

Proof. Necessity: Consider the functional

$$\widetilde{v}_1(\varphi) = v_1(\varphi) - \frac{1}{2}\int_{-h}^{0} \varphi^T(\theta) W \varphi(\theta) d\theta - \frac{\beta}{2}\|\varphi(0) - D\varphi(-h)\|^2.$$

Differentiating this functional with respect to the time, we obtain

$$\begin{aligned}\frac{d}{dt}\widetilde{v}_1(x_t) =& - x^T(t-h)Wx(t-h) - \frac{1}{2}x^T(t)Wx(t) + \frac{1}{2}x^T(t-h)Wx(t-h) \\ & - \beta\left(A_0 x(t) + A_1 x(t-h)\right)^T\left(x(t) - Dx(t-h)\right) \\ &= -\frac{1}{2}\hat{x}^T(t)P(\beta)\hat{x}(t),\end{aligned}$$

where $\hat{x}(t) = \begin{pmatrix} x(t)^T & x^T(t-h)\end{pmatrix}^T$. As system (5.1) is stable, $x(t) \to 0$ as $t \to \infty$, and

$$\lim_{t\to\infty}\int_0^t \frac{d}{ds}\widetilde{v}_1(x_s)ds = -\widetilde{v}_1(\varphi),$$

hence,

$$\widetilde{v}_1(\varphi) = \lim_{t\to\infty}\int_0^t \frac{1}{2}\hat{x}^T(s)P(\beta)\hat{x}(s)ds \geq 0,$$

which implies that

$$v_1(\varphi) \geq \frac{1}{2}\int_{-h}^{0}\varphi^T(\theta)W\varphi(\theta)d\theta + \frac{\beta}{2}\|\varphi(0) - D\varphi(-h)\|^2 \geq \frac{\beta}{2}\|\varphi(0) - D\varphi(-h)\|^2.$$

Sufficiency: Because of equation (5.6),

$$v_1(\varphi) \geq v_1(x_t) \geq \beta_1\left\|x(t,\varphi) - Dx(t-h,\varphi)\right\|^2. \tag{5.10}$$

By Lemma 5.3, there is a number $\beta_2 > 0$ such that $v_1(\varphi) \leq \beta_2\left\|\varphi\right\|_h^2$, then

$$\beta_1\left\|x(t,\varphi) - Dx(t-h,\varphi)\right\|^2 \leq v_1(\varphi) \leq \beta_2\left\|\varphi\right\|_h^2,$$

and therefore

$$\left\|x(t,\varphi) - Dx(t-h,\varphi)\right\| \leq \sqrt{\frac{\beta_2}{\beta_1}}\left\|\varphi\right\|_h.$$

It implies that

$$x(t,\varphi) = Dx(t-h,\varphi) + \xi(t),$$

where ξ is such that

$$\left\|\xi(t)\right\| \leq \sqrt{\frac{\beta_2}{\beta_1}}\left\|\varphi\right\|_h,\ t \geq 0.$$

By iterating $k-1$ times the previous equation, for $t \in [(k-1)h, kh)$, we get

$$x(t,\varphi) = D^k x(t-kh,\varphi) + \sum_{j=0}^{k-1} D^j\xi(t-jh).$$

As matrix D is Schur stable, by Lemma 5.1, there exist $d \geq 1$ and $\rho \in (0,1)$ such that $\left\|D^k\right\| \leq d\rho^k$. Then,

$$\left\|x(t,\varphi)\right\| \leq \left(d\rho^k + \sum_{j=0}^{k-1} d\rho^j\sqrt{\frac{\beta_2}{\beta_1}}\right)\left\|\varphi\right\|_h,\ t \geq 0.$$

Therefore,

$$\|x(t,\varphi)\| \leq \frac{d}{1-\rho}\sqrt{\frac{\beta_2}{\beta_1}}\|\varphi\|_h,\ t \geq 0,$$

which means that system (5.1) is stable in the Lyapunov sense. In addition, the stability is exponential because the Lyapunov condition holds. □

Remark 5.1. *Since the greater the number $\beta_A^\star$ is, the less computational effort (see Theorem 5.3), we take β as the first value for which the determinant of the matrix $P(\beta)$ is zero.*

5.3 Auxiliary results

We define the set of initial functions

$$\mathcal{S} := \left\{\varphi \in C^{(1)}\left([-h,0],\mathbb{R}^n\right) \middle| \|\varphi\|_h = \|\varphi(0)\| = 1,\ \|\varphi'\| \leq \mu M\right\},$$

with $M = \|A_0\| + \|A_1\|$ and $\mu = \dfrac{d}{1-\rho}$, where the numbers d and ρ are given by Lemma 5.1.

We present some auxiliary results related to the set $\mathcal{S}$ that are key in the attainment of the stability conditions in Section 5.4 and Section 5.5. We introduce first a stability condition in terms of the functional (5.5) for initial functions from the set $\mathcal{S}$ and then, we show that any arbitrary function that belongs to the set $\mathcal{S}$ can be approximated by a function of the form (3.11).

5.3.1 Sufficient stability condition in terms of the functional v_1

The next lemma is useful for proving the main theorem of this section.

Lemma 5.4. *Let E_1 and E_2 be real matrices in $\mathbb{R}^{n\times n}$. If $\det(E_1 + iE_2) = 0$, then there exist two vectors c_1 and c_2 such that*

1. $(E_1 + iE_2)(c_1 + ic_2) = 0$.
2. $\|c_1\| = 1$.
3. $\|c_2\| \leq 1$.
4. $c_1^T c_2 = 0$.

Proof. Since $\det(E_1 + iE_2) = 0$, there exists a complex vector $\xi_1 + i\xi_2 \neq 0$ such that

$$(E_1 + iE_2)(\xi_1 + i\xi_2) = 0.$$

Introduce now the following vectors

$$\begin{aligned}\widehat{c}_1 =& \xi_1 + b\xi_2\\ \widehat{c}_2 =& -b\xi_1 + \xi_2,\end{aligned}$$

where b is a real number, and observe that

$$(E_1 + iE_2)(\widehat{c}_1 + i\widehat{c}_2) = 0.$$

Consider the product

$$\widehat{c}_1^T\widehat{c}_2 = (1-b^2)\xi_1^T\xi_2 - b\left(\|\xi_1\|^2 - \|\xi_2\|^2\right).$$

If $\xi_1^T\xi_2 = 0$, we set $b = 0$, otherwise, we take any real solution of the following quadratic equation, which always has two:

$$b^2 + b\frac{\|\xi_1\|^2 - \|\xi_2\|^2}{\xi_1^T\xi_2} - 1 = 0.$$

In both cases, we get $\widehat{c}_1^T\widehat{c}_2 = 0$. Now, at least one of the vectors $\widehat{c}_1$ and $\widehat{c}_2$ is nonzero and the desired vectors c_1 and c_2 can be constructed as follows: if $\|\widehat{c}_1\| \geq \|\widehat{c}_2\|$, we have

$$c_1 = \frac{\widehat{c}_1}{\|\widehat{c}_1\|},\ c_2 = \frac{\widehat{c}_2}{\|\widehat{c}_1\|},$$

and if $\|\widehat{c}_1\| < \|\widehat{c}_2\|$,

$$c_1 = \frac{\widehat{c}_2}{\|\widehat{c}_2\|},\ c_2 = -\frac{\widehat{c}_1}{\|\widehat{c}_2\|}.$$

$\square$

The basic idea of the following result is borrowed from Alexandrova and Zhabko (2016).

Theorem 5.2. *Assume that matrix D is Schur stable. System* (5.1) *is exponentially stable if the Lyapunov condition holds and there exists $\beta_1 > 0$ such that for any initial function $\varphi \in \mathcal{S}$*

$$v_1(\varphi) \geq \beta_1. \tag{5.11}$$

Proof. Assume by contradiction that system (5.1) is not exponentially stable but the Lyapunov condition and inequality (5.11) hold. It means that there exists an eigenvalue $s = \alpha + i\beta$ with $\alpha > 0$, and two vectors $c_1,\ c_2 \in \mathbb{R}^n$ that satisfy conditions of Lemma 5.4 such that

$$x(t,\varphi) = e^{\alpha t}\phi(t),\ \phi(t) = \cos{(\beta t)}\, c_1 - \sin{(\beta t)}\, c_2,\ t \in (-\infty, \infty), \tag{5.12}$$

is a solution of system (5.1). The initial function corresponding to solution (5.12) is given by

$$\varphi(\theta) = x(\theta, \varphi),\ \theta \in [-h, 0].$$

Let us prove first that $\varphi \in \mathcal{S}$. By Lemma 5.4, notice that $\|\varphi(0)\| = 1$ and $\|\phi(t)\|^2 = \cos^2(\beta t)\|c_1\|^2 + \sin^2(\beta t)\|c_2\|^2 \leq 1$. The last inequality implies that $\max\limits_{t\in\mathbb{R}} \|\phi(t)\| = 1$, hence

$$\|x(t)\| = e^{\alpha t}\, \|\phi(t)\| \leq \|\varphi(0)\| = 1,\ t \leq 0,$$

and particularly $\|x(\theta)\| = \|\varphi(\theta)\| \leq 1$ for $\theta \in [-h, 0]$. Now, since $x(t,\varphi)$ satisfies (5.1) for $t \in (-\infty, \infty)$, we have

$$\|\dot{x}(t) - D\dot{x}(t-h)\| \leq \|A_0\|\, \|x(t)\| + \|A_1\|\, \|x(t-h)\| \leq M,\ t \leq 0.$$

The previous expression means that there is a function ξ that satisfies $\|\xi(t)\| \leq M$ for $t \leq 0$ and

$$y(t) = Dy(t-h) + \xi(t), \tag{5.13}$$

where $y(t) = \dot{x}(t)$. Notice that

$$y(t) = \sum_{j=0}^{\infty} D^j \xi(t - jh)$$

satisfies (5.13). Indeed, by substituting it into (5.13) we get

$$y(t) - Dy(t-h) = \sum_{j=0}^{\infty} D^j\xi(t-jh) - \sum_{j=0}^{\infty} D^{(j+1)}\xi(t-(j+1)h) = \sum_{j=0}^{\infty} D^j\xi(t-jh) - \sum_{j=1}^{\infty} D^j\xi(t-jh) = \xi(t).$$

As matrix D is Schur stable, the sum converges and, by Lemma 5.1, there are constants $\rho \in (0,1)$ and $d \geq 1$ such that $\left\|D^j\right\| \leq d\rho^j$. Then, we obtain

$$\|y(t)\| \leq \sum_{j=0}^{\infty} d\rho^j M = \mu M,\ t \leq 0.$$

From the previous inequality we arrive at

$$\|\varphi'(\theta)\| = \|y(\theta)\| \leq \mu M,\ \theta \in [-h, 0].$$

Now, from equation equation (5.6),

$$v_1(\varphi) = v_1(x_T) + \int_{-h}^{T-h} x^T(t,\varphi) W x(t,\varphi) dt,\ W > 0, \tag{5.14}$$

where $T = 2\pi/\beta$ if $\beta \neq 0$, and $T = 1$ if $\beta = 0$. Since T is the period of the function $\phi(t)$, we have $x(T+\theta) = e^{\alpha T}\varphi(\theta)$ and

$$v_1(x_T(\varphi)) = e^{2\alpha T} v_1(\varphi),$$

which implies that,

$$v_1(\varphi) = -\frac{1}{e^{2\alpha T}-1}\int_{-h}^{T-h} x^T(t,\varphi_s) W x(t,\varphi_s) dt \leq -\frac{\lambda_{\min}(W)}{e^{2\alpha T}-1}\int_{-h}^{T-h} \|x(t)\|^2 dt$$

By Lemma 5.4,

$$\begin{aligned}\int_{-h}^{T-h} \|x(t)\|^2 dt = \int_{-h}^{T-h} e^{2\alpha t}\left(\cos^2(\beta t)\|c_1\|^2 + \sin^2(\beta t)\|c_2\|^2\right) dt \geq \int_{-h}^{T-h} e^{2\alpha t}\cos^2(\beta t) dt \\ = \frac{e^{-2\alpha h}\left(e^{2\alpha T}-1\right)}{4\alpha}\left(\cos^2(\beta h) + \frac{\left(\alpha\cos(\beta h) - \beta\sin(\beta h)\right)^2}{\alpha^2+\beta^2}\right)\end{aligned}$$

Therefore,

$$v_1(\varphi) \leq -\frac{\lambda_{\min}(W) e^{-2\alpha h}}{4\alpha}\left(\cos^2(\beta h) + \frac{\left(\alpha\cos(\beta h) - \beta\sin(\beta h)\right)^2}{\alpha^2+\beta^2}\right).$$

The previous inequality contradicts the assumption and ends the proof. □

5.3.2 Approximation of the set $\mathcal{S}$

The function

$$\psi_r(\theta) = \sum_{j=1}^{r} K(\tau_j + \theta)\gamma_j, \tag{5.15}$$

previously introduced in (3.11), has played a key role in the obtention of the necessary stability conditions in Chapter 3 and Chapter 4. In this section, we show that it is possible to approximate any function from the set $\mathcal{S}$ by a function of the form (5.15).

In order to do this, we construct the function ψ_r in (5.15) as suggested by Egorov (2016) for the retarded type case:

1. Set $\tau_j = (j-1)\delta_r$, where $\delta_r = \dfrac{h}{r-1}$ and $r \geq 2$.
2. Choose vectors γ_j, $j = \overline{1,r}$, such that

$$\psi_r(-\tau_j) = \varphi(-\tau_j). \tag{5.16}$$

An estimate of the error $R_r = \varphi - \psi_r$ considering the function ψ_r constructed by the above steps is given in the next lemma.

Lemma 5.5. *For every $\varphi \in \mathcal{S}$*

$$\|R_r\|_h = \|\varphi - \psi_r\|_h \leq \varepsilon_r, \tag{5.17}$$

where

$$\varepsilon_r = \frac{(\mu M + L)e^{Lh}}{1/\delta_r + L},$$

and L is the Lipschitz constant of $K(t)$ on $t \in (0, h)$, i.e., it is such that $\|K'(t)\| \leq L$.

Proof. By equation (5.16), in particular $R_r(-\tau_j) = \varphi(-\tau_j) - \psi_r(-\tau_j) = 0$, hence,

$$\|R_r\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta) - \psi_r(\theta)\| = \max_{j\in\{2,\dots,r\}} \sup_{\theta\in(-\tau_j,-\tau_{j-1})} \|\varphi(\theta) - \psi_r(\theta)\|.$$

As $\|\varphi'(\theta)\| \leq \mu M$, then

$$\|\varphi(\theta) - \varphi(-\tau_j)\| \leq \mu M(\theta + \tau_j),\ \theta \in (-\tau_j, -\tau_{j-1}). \tag{5.18}$$

Observe that $\|K'(t)\| \leq L$, for $t \in (0, h)$, implies that

$$\|K(t_1) - K(t_2)\| \leq L|t_1 - t_2|,\ t_1, t_2 \in (0, h). \tag{5.19}$$

Now, take a number $j \in \{2, \dots, r\}$. From expressions (5.16), (5.18) and (5.19), we obtain the next sequence of inequalities:

$$\begin{aligned}
\|\varphi(\theta) - \psi_r(\theta)\| =& \|\varphi(\theta) - \varphi(-\tau_j) + \psi_r(-\tau_j) - \psi_r(\theta)\| \\
\leq& \|\varphi(\theta) - \varphi(-\tau_j)\| + \|\psi_r(-\tau_j) - \psi_r(\theta)\| \\
\leq& \mu M(\theta + \tau_j) + \sum_{k=j}^{r} \|K(\tau_k - \tau_j) - K(\tau_k + \theta)\|\|\gamma_k\| \\
\leq& (\tau_j + \theta)\left(\mu M + L\sum_{k=j}^{r}\|\gamma_k\|\right),\ \theta \in (-\tau_j, -\tau_{j-1}).
\end{aligned}$$

We look for an upper bound estimate of $\|\gamma_k\|$. By equation (5.16), we have

$$\|\gamma_r\| = \|\varphi(-\tau_r)\| \leq \|\varphi(0)\| = 1,$$

and

$$\sum_{k=j}^{r} K(\tau_k - \tau_j)\gamma_k = \gamma_j + \sum_{k=j+1}^{r} K(\tau_k - \tau_j)\gamma_k = \varphi(-\tau_j),\ j = \overline{2, r-1}.$$

In view of the previous equality, we obtain

$$\begin{aligned}
\|\gamma_j\| =& \left\|\varphi(-\tau_j) - \sum_{k=j+1}^{r} K(\tau_k - \tau_j)\gamma_k\right\| \\
=& \left\|\varphi(-\tau_j) - \varphi(-\tau_{j+1}) + \psi_r(-\tau_{j+1}) - \sum_{k=j+1}^{r} K(\tau_k - \tau_j)\gamma_k\right\| \\
\leq& \|\varphi(-\tau_j) - \varphi(-\tau_{j+1})\| + \sum_{k=j+1}^{r} \|K(\tau_k - \tau_{j+1}) - K(\tau_k - \tau_j)\|\,\|\gamma_k\| \\
\leq& \mu M\delta_r + L\delta_r \sum_{k=j+1}^{r} \|\gamma_k\|.
\end{aligned}$$

Using the preceding inequality, one can prove by induction that

$$\|\gamma_j\| \leq \delta_r \left(\mu M + L\right)\left(1+\delta_r L\right)^{r-j-1}, \; j = \overline{2, r-1},$$

therefore,

$$\mu M + L\sum_{k=2}^{r} \|\gamma_k\| \leq \mu M + L\delta_r \left(\mu M + L\right)\sum_{k=2}^{r-1} \left(1+\delta_r\right)^{r-k-1} + L.$$

Expanding the sum and rearranging terms in the right hand side, we arrive at

$$\mu M + L\sum_{k=2}^{r} \|\gamma_k\| \leq \left(\mu M + L\right)\left(1+L\delta_r\right)^{r-2}.$$

Finally, since $(\tau_i + \theta) \leq \delta_r$ for $\theta \in (-\tau_i, -\tau_{i-1})$, we have

$$\begin{aligned}
\|R_r\|_h &\leq \max_{j\in\{2,\ldots,r\}} \sup_{\theta\in(-\tau_j,-\tau_{j-1})} \left(\mu M + L\sum_{k=j}^{r} \|\gamma_k\|\right)(\tau_j+\theta) \\
&\leq \left(\mu M + L\sum_{k=2}^{r} \|\gamma_k\|\right)\delta_r \leq \left(\mu M + L\right)\left(1+L\delta_r\right)^{r-2}\delta_r \\
&= \frac{\delta_r(\mu M + L)}{1+L\delta_r}\left(1+\frac{Lh}{r-1}\right)^{r-1} \leq \frac{(\mu M + L)e^{Lh}}{1/\delta_r + L}.
\end{aligned}$$

□

Remark 5.2. *The estimate of the supremum norm of the error R_r is of the same form as the one obtained in Egorov (2016) for the retarded type case, except for the term μ, which indeed is related with the matrix D.*

5.4 Fundamental and Lyapunov matrices based stability criterion

As in Subsection 5.3.2, we consider

$$\tau_j = (j-1)\delta_r, \; j = \overline{1, r}, \; r \geq 2, \tag{5.20}$$

and the matrices with constant coefficients

$$\mathcal{K}_r = \left[U(\tau_j - \tau_k)\right]_{k,j=1}^{r} = \left[U\left(\frac{j-k}{r-1}h\right)\right]_{k,j=1}^{r},$$

and

$$\mathcal{A}_r = \begin{pmatrix} K^T(\tau_1)K(\tau_1) & K^T(\tau_1)K(\tau_2) & \ldots & K^T(\tau_1)K(\tau_r-0) \\ K^T(\tau_2)K(\tau_1) & K^T(\tau_2)K(\tau_2) & \ldots & K^T(\tau_2)K(\tau_r-0) \\ \vdots & & \ddots & \vdots \\ K^T(\tau_r-0)K(\tau_1) & K^T(\tau_r-0)K(\tau_2) & \ldots & K^T(\tau_r-0)K(\tau_r-0) \end{pmatrix}.$$

Henceforth, we assume that $\mathcal{K}_1 = U(0)$. We recall that, from equality (3.14),

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r \gamma,$$

with $\gamma = \begin{pmatrix}\gamma_1^T & \ldots & \gamma_r^T\end{pmatrix}^T$.

In the next theorem, we provide a stability criterion for the particular case in which $\|D\| < 1$, which is based on the delay Lyapunov matrix and the fundamental matrix of the system.

Theorem 5.3. *Assume that matrix D satisfies $\|D\| < 1$. System* (5.1) *is exponentially stable if and only if the Lyapunov condition and the following hold:*

$$\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A} > 0, \tag{5.21}$$

where

$$r_A = 1 + \left\lceil e^{Lh} h \left(\mu M + L\right) \left(\alpha_A + \sqrt{\alpha_A(\alpha_A + 1)}\right) - Lh \right\rceil, \tag{5.22}$$

with $\alpha_A = \dfrac{\beta_2}{\beta_1(1 - \|D\|)^2}$. *Here,* $\beta_1 \in (0, \beta_A^\star)$ *and* β_2 *is given by Lemma 5.3.*

Proof. Necessity: Consider the function (5.15). By the initial conditions of the fundamental matrix and from the fact that $\tau_{r_A} = \tau_r = h$, the following chain of equalities holds for every $\gamma_j \in \mathbb{R}^n$, $j = \overline{1, r_A}$:

$$\begin{aligned}
\psi_{r_A}(0) - D\psi_{r_A}(-h) &= \sum_{j=1}^{r_A} K(\tau_j)\gamma_j - D\sum_{j=1}^{r_A} K(\tau_j - h)\gamma_j = \sum_{j=1}^{r_A} K(\tau_j)\gamma_j - DK(\tau_{r_A} - h)\gamma_{r_A} \\
&= \sum_{j=1}^{r_A - 1} K(\tau_j)\gamma_j + K(h)\gamma_{r_A} - D\gamma_{r_A} = \sum_{j=1}^{r_A - 1} K(\tau_j)\gamma_j + K(h-0)\gamma_{r_A},
\end{aligned}$$

which implies that

$$\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 = \gamma^T \mathcal{A}_{r_A}\gamma,$$

where $\gamma = \begin{pmatrix}\gamma_1^T & \dots & \gamma_r^T\end{pmatrix}^T$. By Theorem 5.1 and the previous identity, we have, for every $\gamma \in \mathbb{R}^{nr_A}$ such that $\gamma^T \mathcal{A}_{r_A}\gamma > 0$,

$$\gamma^T \mathcal{K}_{r_A}\gamma - \beta_1 \gamma^T \mathcal{A}_{r_A}\gamma = v_1(\psi_{r_A}) - \beta_1\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 > v_1(\psi_{r_A}) - \beta_A^\star\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 \geq 0.$$

For the case in which $\gamma^T \mathcal{A}_{r_A}\gamma = 0$, $\gamma \neq 0$, the inequality $\gamma^T \mathcal{K}_{r_A}\gamma - \beta_1\gamma^T \mathcal{A}_{r_A}\gamma > 0$ remains true, since from Theorem 3.2, for every number r_A, $\mathcal{K}_{r_A} > 0$.

Sufficiency: Consider a function $\varphi \in \mathcal{S}$ and $R_{r_A} = \varphi - \psi_{r_A}$. Observe that

$$\begin{aligned}
v_1(\varphi) &= z(\psi_{r_A} + R_{r_A}, \psi_{r_A} + R_{r_A}) \\
&= z(\psi_{r_A}, \psi_{r_A}) + 2z(\psi_{r_A}, R_{r_A}) + z(R_{r_A}, R_{r_A}) \\
&= v_1(\psi_{r_A}) + 2z(\varphi, R_{r_A}) - v_1(R_{r_A}).
\end{aligned}$$

By construction, $\psi_{r_A}(0) = \varphi(0)$, $\psi_{r_A}(-h) = \varphi(-h)$ and $\|\varphi\|_h = 1$, hence, from Lemma 5.3 and the fact that $\|D\| < 1$, we get

$$\begin{aligned}
v_1(\varphi) =& v_1(\psi_{r_A}) - \beta_1\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 + \beta_1\|\varphi(0) - D\varphi(-h)\|^2 + 2z(\varphi, R_{r_A}) - v_1(R_{r_A}) \\
\geq& v_1(\psi_{r_A}) - \beta_1\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 + \beta_1(1 - \|D\|)^2\|\varphi\|_h^2 - 2\beta_2\|R_{r_A}\|_h - \beta_2\|R_{r_A}\|_h^2 \\
=& v_1(\psi_{r_A}) - \beta_1\|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 + \beta_1(1 - \|D\|)^2 - 2\beta_2\|R_{r_A}\|_h - \beta_2\|R_{r_A}\|_h^2.
\end{aligned}$$

For a number r_A given by (5.22), it follows from Lemma 5.5 that

$$\|R_{r_A}\|_h \leq \frac{(\mu M + L)e^{Lh} h}{r_A - 1 + Lh} \leq \frac{\bar{\beta}_1}{\beta_2 + \sqrt{\beta_2(\beta_2 + \bar{\beta}_1)}},$$

where $\bar{\beta}_1 = (1 - \|D\|)^2\beta_1$. From the above inequality, we have

$$\bar{\beta}_1 - 2\beta_2\|R_{r_A}\|_h - \beta_2\|R_{r_A}\|_h^2 \geq 0.$$

Indeed,

$$\begin{aligned}
&\bar{\beta}_1 - 2\beta_2\|R_{r_A}\|_h - \beta_2\|R_{r_A}\|_h^2 \geq \\
&\geq \frac{1}{(\beta_2 + \sqrt{\beta_2(\beta_2 + \bar{\beta}_1)})^2}\left(\bar{\beta}_1\left(\beta_2 + \sqrt{\beta_2(\beta_2 + \bar{\beta}_1)}\right)^2 - 2\beta_2\bar{\beta}_1\left(\beta_2 + \sqrt{\beta_2(\beta_2 + \bar{\beta}_1)}\right) - \beta_2\bar{\beta}_1^2\right) = 0
\end{aligned}$$

Therefore,

$$v_1(\varphi) \geq v_1(\psi_{r_A}) - \beta_1 \|\psi_{r_A}(0) - D\psi_{r_A}(-h)\|^2 = \gamma^T \mathcal{K}_{r_A} \gamma - \beta_1 \gamma^T \mathcal{A}_{r_A} \gamma \geq \lambda_{\min}\left(\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}\right) \|\gamma\|^2.$$

Notice that, as $1 = \|\psi_{r_A}(0)\|^2 = \gamma^T \left[K^T(\tau_i)K(\tau_j)\right]_{i,j=1}^{r_A} \gamma$,

$$1 \leq \lambda_{\max}\left(\left[K^T(\tau_i)K(\tau_j)\right]_{i,j=1}^{r_A}\right) \|\gamma\|^2, \tag{5.23}$$

which implies that there exists a number $\tilde{\gamma} > 0$ such that $\|\gamma\| \geq \tilde{\gamma}$, and in turn that

$$v_1(\varphi) \geq \widetilde{\beta},$$

with $\widetilde{\beta} = \lambda_{\min}\left(\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}\right) \widetilde{\gamma}^2 > 0$. By Theorem 5.2 and the previous inequality, we conclude that system (5.1) is exponentially stable. □

Remark 5.3. *For the sake of illustration, some particular cases of the stability criterion* (5.21) *are considered next . If $r_A = 2$,*

$$\mathcal{K}_2 - \beta_1 \mathcal{A}_2 = \begin{pmatrix} U(0) & U(h) \\ U^T(h) & U(0) \end{pmatrix} - \beta_1 \begin{pmatrix} I_n & K(h-0) \\ K^T(h-0) & K^T(h-0)K(h-0) \end{pmatrix} > 0,$$

if $r_A = 3$,

$$\mathcal{K}_3 - \beta_1 \mathcal{A}_3 =$$

$$= \begin{pmatrix} U(0) & U\left(\frac{h}{2}\right) & U(h) \\ U^T\left(\frac{h}{2}\right) & U(0) & U\left(\frac{h}{2}\right) \\ U^T(h) & U^T\left(\frac{h}{2}\right) & U(0) \end{pmatrix} - \beta_1 \begin{pmatrix} I_n & K\left(\frac{h}{2}\right) & K(h-0) \\ K^T\left(\frac{h}{2}\right) & K^T\left(\frac{h}{2}\right)K\left(\frac{h}{2}\right) & K^T\left(\frac{h}{2}\right)K(h-0) \\ K^T(h-0) & K^T(h-0)K\left(\frac{h}{2}\right) & K^T(h-0)K(h-0) \end{pmatrix} > 0,$$

and so on.

5.5 Lyapunov matrix based stability criterion

In this section, we provide a new stability criterion that, unlike the one presented in Section 5.4, is given uniquely in terms of the delay Lyapunov matrix and does not require the assumption $\|D\| < 1$. The cornerstone of the new stability criterion is the following instability result.

Lemma 5.6. *Assume that matrix D is Schur stable. If system* (5.1) *is unstable, there exists $\varphi \in \mathcal{S}$ such that*

$$v_1(\varphi) \leq -\beta^\star,$$

with

$$\beta^\star = \frac{\lambda_{\min}(W)}{4\mu M} e^{-2\mu M h} \cos^2(bh),$$

where b exists and is unique on $\left(0, \dfrac{\pi}{2h}\right)$, and satisfies

$$1 - \frac{\mu M}{\sqrt{(\mu M)^2 + b^2}} - \cos^2(bh) = 0.$$

Proof. As system (5.1) is unstable, there exists an eigenvalue $s = \alpha + i\beta$ with $\alpha > 0$ and $\beta \geq 0$, and two vectors $c_1,\, c_2 \in \mathbb{R}^n$ that fulfill the conditions of Lemma 5.4 such that the following expression is a solution of system (5.1):

$$x(t,\varphi) = e^{\alpha t}\phi(t),\ \phi(t) = \cos\beta t c_1 - \sin\beta t c_2,\ t \in (-\infty, \infty), \tag{5.24}$$

which corresponds to the initial function

$$\varphi(\theta) = x(\theta, \varphi),\ \theta \in [-h, 0].$$

Notice that the same solution is used in the proof of Theorem 5.2. There, it is shown that $\varphi \in \mathcal{S}$ and that

$$v_1(\varphi) \leq -\frac{\lambda_{\min}(W)e^{-2\alpha h}}{4\alpha} f(\beta),$$

where

$$\begin{aligned} f(\beta) =& \cos^2(\beta h) + \frac{(\alpha\cos(\beta h) - \beta\sin(\beta h))^2}{\alpha^2+\beta^2} \\ =& 1 + \frac{\alpha}{\alpha^2+\beta^2}\left(\alpha\cos(2\beta h) - \beta\sin(2\beta h)\right). \end{aligned}$$

We focus now on providing a lower bound different from zero for the function $f(\beta)$. Notice first that

$$f(\beta) \geq \cos^2(\beta h),\ \beta \geq 0,$$

and that, since $\alpha \leq \dfrac{d}{1-\rho}\left(\|A_0\| + \|A_1\|\right) = \mu M$ by Lemma 5.2,

$$\begin{aligned} f(\beta) =& 1 + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos(2\beta h) - \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin(2\beta h)\right) \\ \geq& 1 - \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \geq 1 - \frac{\mu M}{\sqrt{(\mu M)^2+\beta^2}},\ \beta \geq 0. \end{aligned}$$

In particular,

$$f(\beta) \geq \cos^2(bh),\ 0 \leq \beta \leq b < \frac{\pi}{2h},$$

and

$$f(\beta) \geq 1 - \frac{\mu M}{\sqrt{(\mu M)^2+b^2}},\ \beta \geq b > 0$$

As

$$1 - \frac{\mu M}{\sqrt{(\mu M)^2+b^2}} - \cos^2(bh) = 0,$$

we have

$$\cos^2(bh) = 1 - \frac{\mu M}{\sqrt{(\mu M)^2+b^2}},$$

hence

$$f(\beta) \geq \cos^2(bh),\ \beta \geq 0.$$

Since

$$g(\beta) = 1 - \frac{\mu M}{\sqrt{(\mu M)^2+\beta^2}} - \cos^2(\beta h),\ \beta \in \left(0, \frac{\pi}{2h}\right),$$

is an increasing function, and

$$g(0) = -1 \ \text{ and } \ g\left(\frac{\pi}{2h}\right) = 1 - \left(1 + \left(\frac{\pi}{2h\mu M}\right)^2\right)^{-1/2} > 0,$$

the number b on $(0, \pi/2h)$ such that $g(b) = 0$ exists and is unique. Therefore, from the previously obtained and the fact that $0 < \alpha \leq \mu M$, we get the desired result:

$$v_1(\varphi) \leq -\frac{\lambda_{\min}(W)e^{-2\alpha h}}{4\alpha} f(\beta) \leq -\frac{\lambda_{\min}(W)e^{-2\mu M h}}{4\mu M}\cos^2(bh) = -\beta^\star.$$

□

We are now ready to present the main result of this section. We consider the same block matrix $\mathcal{K}_r$ introduced in Section 5.4.

Theorem 5.4. *Assume that matrix D is Schur stable. System* (5.1) *is exponentially stable if and only if the Lyapunov condition and the following hold:*

$$\mathcal{K}_r > 0, \tag{5.25}$$

where

$$r = 1 + \left\lceil e^{Lh} h \left(\mu M + L\right) \left(\alpha^\star + \sqrt{\alpha^\star(\alpha^\star + 1)}\right) - Lh \right\rceil, \tag{5.26}$$

with $\alpha^\star = \dfrac{\beta_2}{\beta^\star}$. Here, $\beta^\star$ is determined by Lemma 5.6 and β_2 is given by Lemma 5.3.

Proof. The necessity directly follows from Theorem 3.2. In order to prove the sufficiency, we assume by contradiction that system (5.1) is unstable but the Lyapunov condition and the conditions of the theorem hold. Similarly to the proof of sufficiency of Theorem 5.3, consider $\varphi \in \mathcal{S}$ and $R_r = \varphi - \psi_r$, then

$$v_1(\varphi) = v_1(R_r + \psi_r) = v_1(\psi_r) + 2z(\varphi, R_r) - v_1(R_r).$$

By Lemma 5.3 and Lemma 5.6,

$$\begin{aligned} v_1(\psi_r) &= v_1(\varphi) - 2z(\varphi, R_r) + v_1(R_r) \\ &\leq -\beta^* + 2\beta_2 \|R_r\|_h + \beta_2 \|R_r\|_h^2. \end{aligned}$$

By using Lemma 5.5 and considering the number r given by (5.26), we have that

$$\|R_r\|_h \leq \frac{(\mu M + L)e^{Lh}}{1/\delta_r + L} \leq \frac{\beta^\star}{\sqrt{\beta_2(\beta_2 + \beta^\star)} + \beta_2},$$

which implies that

$$-\beta^\star + 2\beta_2 \|R_r\|_h + \beta_2 \|R_r\|_h^2 \leq 0.$$

In fact,

$$\begin{aligned} &-\beta^\star + 2\beta_2 \|R_r\|_h + \beta_2 \|R_r\|_h^2 \leq \\ &\leq \frac{1}{(\beta_2 + \sqrt{\beta_2(\beta_2 + \beta^\star)})^2} \left(-\beta^\star \left(\beta_2 + \sqrt{\beta_2(\beta_2 + \beta^\star)}\right)^2 + 2\beta_2\beta^\star \left(\beta_2 + \sqrt{\beta_2(\beta_2 + \beta^\star)}\right) + \beta_2 (\beta^\star)^2 \right) = 0 \end{aligned}$$

Finally, from the previous inequality, we get

$$v_1(\psi_r) = \gamma^T \mathcal{K}_r \gamma \leq 0.$$

The obtained contradiction finishes the proof. □

Remark 5.4. *If $r = 2$,*

$$\mathcal{K}_2 = \begin{pmatrix} U(0) & U(h) \\ U^T(h) & U(0) \end{pmatrix} > 0,$$

if $r = 3$,

$$\mathcal{K}_3 = \begin{pmatrix} U(0) & U\left(\frac{h}{2}\right) & U(h) \\ U^T\left(\frac{h}{2}\right) & U(0) & U\left(\frac{h}{2}\right) \\ U^T(h) & U^T\left(\frac{h}{2}\right) & U(0) \end{pmatrix} > 0,$$

if $r = 4$,

$$\mathcal{K}_4 = \begin{pmatrix} U(0) & U\left(\frac{h}{3}\right) & U\left(\frac{2h}{3}\right) & U(h) \\ U^T\left(\frac{h}{3}\right) & U(0) & U\left(\frac{h}{3}\right) & U\left(\frac{2h}{3}\right) \\ U^T\left(\frac{2h}{3}\right) & U^T\left(\frac{h}{3}\right) & U(0) & U\left(\frac{h}{3}\right) \\ U^T(h) & U^T\left(\frac{2h}{3}\right) & U^T\left(\frac{h}{3}\right) & U\left(0\right) \end{pmatrix} > 0.$$

The stability criterion given in Theorem 5.4 remains true for the case in which $D = 0_{n\times n}$, i.e., for retarded type systems. It is worthy of mention that even for the retarded type case no *finite* stability criterion depending *uniquely* on the delay Lyapunov matrix has been presented in the literature until now. The result is stated in the next corollary.

Corollary 5.1. *The system*

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h)$$

is exponentially stable if and only if the Lyapunov condition and the condition (5.25) *hold, with* $\mu = 1$.

Remark 5.5. *Notice that the numbers r_A and r, which depend on the parameters of the system, determine the size of the matrices $\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}$ and $\mathcal{K}_r$, respectively. In both cases, the numbers r_A and r decrease (or increase) as the delay does. For $h = 0$, $r_A = 1$ and $r = 1$, and the Lyapunov matrix stability criterion for delay free systems is recovered, i.e., $\mathcal{K}_1 = U(0) > 0$.*

5.6 Examples

In this section, a couple of academic examples illustrate the stability criteria previously introduced. We compute the delay Lyapunov matrix $U(\tau)$ associated with the matrix $W = I_n$ via the semianalytic method. The stability tests are corroborated by the spectral abscissa computed via the QPmR (Quasi-Polynomial Mapping Based Rootfinder) algorithm introduced in Vyhlídal and Zítek (2009).

The general procedure in order to use Theorem 5.3 for the stability study of systems of the form (5.1), is basically the same as the one required to use Theorem 5.4. Indeed, notice that, although the the numbers r_A and r are different, their formulae have the same form. The procedure is summarized next:

1. Compute the delay Lyapunov matrix $U(\tau)$, $\tau \in [0, h]$, associated with a positive definite matrix W.

2. Compute the number r_A (or number r) according to formula (5.22) (formula (5.26)). Number $\beta_A^\star$ in (5.22) and $\beta^\star$ in (5.26) are computed from Theorem 5.1 and Lemma 5.6, respectively. The rest of the parameters in both cases are obtained in the same way: number β_2 is computed from Lemma 5.3, number $\mu = d/(1-\rho)$ is calculated by the LMI of Lemma 5.1 and the Lipschitz constant L of the fundamental matrix can be estimated from equation (5.3) by $L = \|A_0\| e^{\|A_0\| h}$.

3. Construct the block matrices $\mathcal{K}_{r_A}$ (or $\mathcal{K}_r$) and $\mathcal{A}_{r_A}$ with the previously computed delay Lyapunov matrix $U(\tau)$, $\tau \in [0, h]$, and with the expression for the fundamental matrix given by equation (5.3), respectively.

4. Check positivity of the matrix $\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}$ (or $\mathcal{K}_r$).

Example 5.1. *We analyze system* (5.1) *with delay $h = 1$ and matrices*

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}, \; A_0 = \begin{pmatrix} -0.1 & 0 \\ 0 & p \end{pmatrix}, \; A_1 = \begin{pmatrix} -0.2 & 0.1 \\ 0.1 & 0 \end{pmatrix}, \tag{5.27}$$

for two different values of the parameter $p \in \mathbb{R}$. The parameters to be calculated in order to use Theorems 5.3 and 5.4 are shown next. For $p = -0.2$,

$$\beta_A^\star = 1.14, \; \beta^\star = 0.0881 \, \beta_2 = 6.6450 \,, \mu = 1.1292, \; M = 0.4414, \; L = 0.2443,$$

and for $p = 0.1$,

$$\beta_A^\star = 1.435, \; \beta^\star = 0.1621, \; \beta_2 = 7.4615 \,, \mu = 1.1292, \; M = 0.3414, \; L = 0.3414.$$

Table 5.1: Stability test of system (5.27)

Parameter p	Number r_A	Number r	$\lambda_{min}\left(\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}\right)$	$\lambda_{\min}(\mathcal{K}_r)$	Spectral abscissa
$p = -0.2$	$r_A = 15$	$r = 145$	0.0181	0.0017	-0.0518
$p = 0.1$	$r_A = 9$	$r = 53$	-54.7716	-239.2499	0.1968

Table 5.1 shows the computed number r_A and r for system (5.27)*, and the minimum eigenvalue of the symmetric matrix $\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}$ and $\mathcal{K}_r$. According to the obtained results, it follows from Theorems 5.3 and 5.4 that system* (5.27) *is exponentially stable for $p = -0.2$ and unstable for $p = 0.1$.*

Example 5.2. *We consider now a system of the form* (5.1) *of dimension $n = 3$ with matrices*

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & -0.05 \end{pmatrix}, \; A_0 = \begin{pmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ -0.05 & -0.1 & p \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0. & 0 \\ 0 & 0 & 0 \\ -0.01 & -0.05 & -0.07 \end{pmatrix}, \tag{5.28}$$

where p is an arbitrary real number. We study the stability of the system for different values of (p, h). The numerical values to be calculated in order to use Theorems 5.3 and 5.4 for different system parameter values are shown below. For $(p, h) = (-0.3, 0.2)$,

$$\beta_A^\star = 0.545, \; \beta^\star = 0.1752, \; \beta_2 = 167.4706 \; , \; M = 0.6788, \; L = 0.6667,$$

for $(p, h) = (0.1, 0.2)$,

$$\beta_A^\star = 0.7850, \; \beta^\star = 0.2077, \; \beta_2 = 14.6523, \; M = 0.6064, \; L = 0.5767,$$

and for $(p, h) = (0.1, 0.5)$,

$$\beta_A^\star = 0.7850, \; \beta^\star = 0.1047, \; \beta_2 = 14.8688, \; M = 0.6064, \; L = 0.6741.$$

As there is no change in matrix D, the number $\mu = 1.1292$ remains equal in all cases. Notice that the increase of the delay impacts on the value of $\beta^\star$, but not on $\beta_A^\star$. This shows that the number r is doubly affected by the delay in comparison with r_A.

In Table 5.2, the computed numbers r_A and r for different parameters of system (5.28) *are shown. There, one can observe that the numbers r and r_A increase as the delay h does. By Theorems 5.3 and 5.4 one can conclude that system* (5.28) *is exponentially stable for $(p, h) = (-0.3, 0.2)$ and unstable for $(p, h) = (0.1, 0.2)$ and $(p, h) = (0.1, 0.5)$.*

Table 5.2: Stability test of system (5.28)

Parameters (p, h)	Number r_A	Number r	$\lambda_{min}\left(\mathcal{K}_{r_A} - \beta_1 \mathcal{A}_{r_A}\right)$	$\lambda_{\min}(\mathcal{K}_r)$	Spectral abscissa
$(-0.3, 0.2)$	$r_A = 1268$	$r = 3133$	3.9×10^{-5}	1.5×10^{-5}	-0.0380
$(0.1, 0.2)$	$r_A = 68$	$r = 202$	-839.17	-2.3×10^3	0.0926
$(0.1, 0.5)$	$r_A = 92$	$r = 543$	-1.08×10^3	-5.9×10^3	0.0978

5.7 Discussion

There are basically two differences between the stability criterion of Theorem 5.3 and Theorem 5.4. The first and most obvious one is that the stability condition of Theorem 5.3 is given in terms not only of the delay Lyapunov matrix, but of the fundamental matrix as well. The second difference consists in the computation of the numbers r_A (formula (5.22)) and r (formula (5.26)). Both formulae have the same form, however, the numbers α_A and $\alpha^\star$ are computed by using $\beta_A^\star$ in equation (5.22) and $\beta^\star$ in equation (5.26), respectively. This detail impact in the number of operations required in each criterion, indeed, the examples presented in the above section show that $r_A < r$, which means that the dimension of the required matrix to be tested in Theorem 5.3 is less than the one in Theorem 5.4.

Up to our knowledge, Theorem 5.3 and Theorem 5.4 introduced here are the only stability criteria for neutral type systems existing within the time-domain approach literature. The LMI approach only provides sufficient stability conditions, and although the introduced conservatism can be reduced by increasing the numerical complexity (see, for instance, Seuret and Gouaisbaut (2015)), up to our knowledge there are no theoretical results about the necessity. The finite stability criteria of Theorems 5.3 and 5.4 and the LMI type sufficient stability conditions contrast in the fact that the first requires an algorithm to compute the delay Lyapunov matrix and the second demands to use an optimization software. Another difference is that in the stability criteria introduced here, the numerical complexity not only depends on the dimension of the system, as is in the LMI type stability conditions, but also on the numerical values of the system parameters (see Remark 5.5).

5.8 Conclusions

Finite stability criteria, which generalize the well-known Lyapunov based stability conditions for delay free systems, are provided. They are given in terms of the positivity of a block matrix determined by the delay Lyapunov matrix. Although the dimension of the block matrix tends to be very large, a fact of major theoretical significance is that the new stability conditions only require a finite number of mathematical operations.

Chapter 6

Conclusions and future work

A new tool for the stability analysis of neutral type time-delay systems is presented. The main contributions of this work are listed next:

1. A new approach for the computation of the Lyapunov-Krasovskii functional of complete type is proposed. It is based on a new Cauchy formula and its main characteristic is that it does not require the differentiability of the initial functions assumption.

2. A family of necessary stability conditions for neutral type systems with one delay are obtained. The particularity of these conditions is that they depend exclusively on the delay Lyapunov matrix. The obtention of the result is based on the relaxation of the differentiability of the initial functions assumption in the computation of the Lyapunov-Krasovskii functional of complete type and the introduction of new properties of the delay Lyapunov matrix. A number of examples illustrate how they can be used with frequency domain tools in order to study the stability of neutral type systems. A notable fact is that these stability conditions preserve the same form as those for pointwise and distributed retarded systems.

3. The results obtained for neutral type systems with one delay are extended to the multiple delay case. This extension involves the computation of the Lyapunov-Krasovskii functional of complete type for the multivariable case, which had not been presented in the literature until now. A number of illustrative examples that are known to be challenging by their multiple delay nature shows the efficiency of these conditions.

4. *Finite* stability criteria for neutral type systems with a single delay are provided. The main hallmarks is that they are given in terms of the positivity of a block matrix constructed with the delay Lyapunov matrix and that only require a finite number of mathematical operations to be tested.

The following is considered as future work:

1. Reduction of the conservatism in the estimates involved in the computation of the numbers r_A in Theorem 5.3 and r in Theorem 5.4.

2. Extension of the stability criteria of Theorem 5.3 and Theorem 5.4 to the multiple delay case.

3. Dropping the assumption $\|D\| < 1$ in the stability criteria of Theorem 5.3.

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Appendix A

Computation of the functional v_0

In order to compute the functional v_0 we assume that system (4.1) is exponentially stable. Then, the delay Lyapunov matrix can be written as

$$U(\tau) = \int_0^\infty K^T(t) W K(t+\tau) dt.$$

First and second derivative of the delay Lyapunov matrix are given by

$$U'(\tau) = \int_{\mathcal{O}(-\tau)} K^T(t) W \frac{d}{d\tau} K(t+\tau) dt + \sum_{l=0}^{\infty} K^T(lh-\tau) W \Delta K(lh), \ \tau \in \mathbb{R} \backslash \Omega, \tag{A.1}$$

$$U''(\tau) = \frac{d}{d\tau}\left(\int_{\mathcal{O}(0)} K^T(\xi-\tau) W \frac{d}{d\xi} K(\xi) d\xi + \sum_{l=0}^{\infty} K^T(lh-\tau) W \Delta K(lh) \right), \ \tau \in \mathbb{R} \backslash \Omega. \tag{A.2}$$

The following equality is useful in what follows. For $t \in [lh, (l+1)h]$, $l = 0, 1, \ldots$,

$$\begin{aligned} &\frac{d}{dt} \int_{-ih}^{0} K(t-\theta-ih) D_i \varphi(\theta) d\theta \\ &\qquad = \sum_{p=1}^{i} \int_{\substack{-ph \\ \theta \neq t-(l+p)h}}^{(1-p)h} \frac{d}{dt} K(t-\theta-ih) D_i \varphi(\theta) d\theta + \sum_{p=1}^{i} \Delta K((l+p-i)h) D_i \varphi(t-(l+p)h). \end{aligned} \tag{A.3}$$

Before computing the functional v_0, we first present two useful equalities in the next propositions. Let us introduce the terms

$$J_{ij}^{(1)}(\xi) = \int_0^\infty K^T(t-\xi-ih) W \frac{d}{dt}\left(\int_{-jh}^{0} K(t-\theta-jh) D_j \varphi(\theta) d\theta \right) dt, \ i = \overline{0,m}, \ j = \overline{1,m}$$

$$J_{ij}^{(2)} = \int_0^\infty \frac{d}{dt}\left(\int_{-ih}^{0} K(t-\theta_1-ih) D_i \varphi(\theta_1) d\theta_1 \right)^T W \left(\int_{-jh}^{0} K(t-\theta_2-jh) D_j \varphi(\theta_2) d\theta_2 \right) dt, \ i,j = \overline{1,m}$$

Proposition A.1. *For $\xi \in [-mh, 0]$,*

$$J_{ij}^{(1)}(\xi) = \sum_{q=1}^{j} \int_{-qh}^{(1-q)h} U'(\xi-\theta+ih-jh) D_i \varphi(\theta) d\theta, \ i = \overline{0,m}, \ j = \overline{1,m}. \tag{A.4}$$

Proof. Note first that $\int_0^\infty f(s)ds = \sum_{l=0}^{\infty}\int_{lh}^{(l+1)h} f(s)ds$. By (A.3), we have

$$J_{ij}^{(1)}(\xi) = \sum_{l=0}^{\infty}\int_{lh}^{(l+1)h} K^T(t-\xi-ih)W \times \sum_{q=1}^{j}\left(\int_{\substack{-qh\\ \theta\neq t-(l+q)h}}^{(1-q)h} \frac{d}{dt}K(t-\theta-jh)D_j\varphi(\theta)d\theta + \Delta K((l+q-j)h)D_j\varphi(t-(l+q)h)\right) dt$$

Change of the integration order yields

$$J_{ij}^{(1)}(\xi) = \sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\mathcal{O}(\theta)} K^T(t-\xi-ih)W\frac{d}{dt}K(t-\theta-jh)dtD_j\varphi(\theta)d\theta + \sum_{q=1}^{j}\sum_{l=0}^{\infty}\int_{lh}^{(l+1)h} K^T(t-\xi-ih)W\Delta K((l+q-j)h)D_j\varphi(t-(l+p)h)dt.$$

By applying the change of variable $\eta = t-\xi-ih$ in the first term, and $\theta = t-(l+q)h$ in the second one, we obtain

$$J_{ij}^{(1)}(\xi) = \sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\mathcal{O}(\theta-\xi)} K^T(\eta)W\frac{d}{d\eta}K(\eta+\xi-\theta+ih-jh)d\eta D_j\varphi(\theta)d\theta + \sum_{q=1}^{j}\sum_{l=0}^{\infty}\int_{-qh}^{(1-q)h} K^T(\theta-\xi+(l+q)h-ih)W\Delta K((l+q-j)h)D_j\varphi(\theta)d\theta.$$

Now, we change the variable of the sum over the index l as follows. Set $k = l+q-j$, and as $q-j\leq 0$, we get

$$J_{ij}^{(1)}(\xi) = \sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\mathcal{O}(\theta-\xi)} K^T(\eta)W\frac{d}{d\eta}K(\eta+\xi-\theta+ih-jh)d\eta D_j\varphi(\theta)d\theta + \sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\sum_{k=0}^{\infty} K^T(kh+\theta-\xi+jh-ih)W\Delta K(kh)D_j\varphi(\theta)d\theta.$$

Applying (A.1) we observe that the previous expression is equal to

$$J_{ij}^{(1)}(\xi) = \sum_{q=1}^{j}\int_{-qh}^{(1-q)h} U'(\xi-\theta+ih-jh)D_j\varphi(\theta)d\theta,$$

which ends the proof. □

Proposition A.2. *For* $i,j=\overline{1,m}$,

$$J_{ij}^{(2)} = \sum_{p=1}^{i}\sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\substack{-ph\\ \theta_1\neq\theta_2+(q-p)h}}^{(1-p)h} \varphi^T(\theta_1)D_i^TU''(\theta_1-\theta_2+ih-jh)D_j\varphi(\theta_2)d\theta_2d\theta_1 - \sum_{q=1}^{j}\sum_{p=1}^{i}\int_{-qh}^{(1-q)h} \varphi^T(\theta+(q-p)h)D_i^T\Delta U'((q-p-j+i)h)D_j\varphi(\theta)d\theta \quad (A.5)$$

Proof. Applying equation (A.3), the right hand side of (A.5) can be written as

$$J_{ij}^{(2)} = \sum_{l=0}^{\infty} \int_{lh}^{(l+1)h} \frac{d}{dt} \left(\int_{-ih}^{0} K(t-\theta_1-ih) D_i \varphi(\theta_1) d\theta_1 \right)^T W \Bigg(\sum_{q=1}^{j} \int_{\substack{-qh \\ \theta_2 \neq t-(l+q)h}}^{(1-q)h} \frac{d}{dt} K(t-\theta_2-jh) D_j \varphi(\theta_2) d\theta_2$$
$$+ \sum_{q=1}^{j} \Delta K((l+q-j)h) D_j \varphi(t-(l+q)h) \Bigg) dt$$

By changing the integration order we have

$$J_{ij}^{(2)} = \sum_{q=1}^{j} \int_{-qh}^{(1-q)h} \int_{\mathcal{O}(\theta_2)} \frac{d}{dt} \left(\int_{-ih}^{0} K(t-\theta_1-ih) D_i \varphi(\theta_1) d\theta_1 \right)^T W \frac{d}{dt} K(t-\theta_2-jh) dt D_j \varphi(\theta_2) d\theta_2$$
$$+ \sum_{l=0}^{\infty} \sum_{q=1}^{j} \int_{lh}^{(l+1)h} \frac{d}{dt} \left(\int_{-ih}^{0} K(t-\theta_1-ih) D_i \varphi(\theta_1) d\theta_1 \right)^T W \Delta K((l+q-j)h) D_j \varphi(t-(l+q)h) dt$$

Consider the change of variable $\eta = t - \theta_2 - jh$ in the first summand:

$$J_{ij}^{(21)} = \sum_{q=1}^{j} \int_{-qh}^{(1-q)h} \int_{\mathcal{O}(0)} \frac{d}{d\eta} \left(\int_{-ih}^{0} \varphi^T(\theta_1) D_i^T K^T(\eta+\theta_2-\theta_1+jh-ih) d\theta_1 \right) W \frac{d}{d\eta} K(\eta) d\eta D_j \varphi(\theta_2) d\theta_2$$
$$= \sum_{q=1}^{j} \int_{-qh}^{(1-q)h} \int_{\mathcal{O}(0)} \frac{d}{d\theta_2} \left(\int_{-ih}^{0} \varphi^T(\theta_1) D_i^T K^T(\eta+\theta_2-\theta_1+jh-ih) d\theta_1 \right) W \frac{d}{d\eta} K(\eta) d\eta D_j \varphi(\theta_2) d\theta_2$$
$$= \int_{-ih}^{0} \sum_{q=1}^{j} \int_{-qh}^{(1-q)h} \varphi^T(\theta_1) D_i^T \frac{d}{d\theta_2} \left(\int_{\mathcal{O}(0)} K^T(\eta+\theta_2-\theta_1+jh-ih) W \frac{d}{d\eta} K(\eta) d\eta \right) D_j \varphi(\theta_2) d\theta_2 d\theta_1 \tag{A.6}$$

Applying equation (A.3) to the second summand of $J_{ij}^{(2)}$, we get

$$J_{ij}^{(22)} = \sum_{p=1}^{i} \sum_{q=1}^{j} \sum_{l=0}^{\infty} \int_{lh}^{(l+1)h} \int_{\substack{-ph \\ \theta_1 \neq t-(l+p)h}}^{(1-p)h} \varphi^T(\theta_1) D_i^T \frac{d}{dt} K^T(t-\theta_1-ih) d\theta_1 W \Delta K((l+q-j)h) D_j \varphi(t-(l+q)h) dt$$
$$+ \sum_{q=1}^{j} \sum_{p=1}^{i} \sum_{l=0}^{\infty} \int_{lh}^{(l+1)h} \varphi^T(t-(l+p)h) D_i^T \Delta K^T((l+p-i)h) W \Delta K((l+q-j)h) D_j \varphi(t-(l+q)h) dt$$

By the change of variable $\theta_2 = t - (l+q)h$ in the first term and $\theta = t - (l+q)h$ in the second one of $J_{ij}^{(22)}$, we arrive at

$$J_{ij}^{(22)} = \sum_{p=1}^{i} \sum_{q=1}^{j} \sum_{l=0}^{\infty} \int_{-qh}^{(1-q)h} \int_{\substack{-ph \\ \theta_1 \neq \theta_2+(q-p)h}}^{(1-p)h} \varphi^T(\theta_1) D_i^T \frac{d}{d\theta_2} K^T(\theta_2-\theta_1+lh+qh-ih) W \Delta K((l+q-j)h) D_j \varphi(\theta_2) d\theta_1 d\theta_2$$
$$+ \sum_{p=1}^{i} \sum_{q=1}^{j} \sum_{l=0}^{\infty} \int_{-qh}^{(1-q)h} \varphi^T(\theta+(q-p)h) D_i^T \Delta K^T((l+p-i)h) W \Delta K((l+q-j)h) D_j \varphi(\theta) d\theta$$

Finally, change of variable $k = l + q - j$ over the index l allows to arrive at

$$J_{ij}^{(22)} = \sum_{p=1}^{i}\sum_{q=1}^{j}\sum_{k=0}^{\infty}\int_{-qh}^{(1-q)h}\int_{\substack{-ph \\ \theta_1\neq\theta_2+(q-p)h}}^{(1-p)h} \varphi^T(\theta_1)D_i^T\frac{d}{d\theta_2}K^T(\theta_2-\theta_1+kh+jh-ih)W\Delta K(kh)D_j\varphi(\theta_2)d\theta_1 d\theta_2$$

$$+\sum_{q=1}^{j}\sum_{p=1}^{i}\sum_{k=0}^{\infty}\int_{-qh}^{(1-q)h}\varphi^T(\theta+(q-p)h)D_i^T\Delta K^T((k-q+p+j-i)h)W\Delta K(kh)D_j\varphi(\theta)d\theta$$

By using (A.2), and the identity $\Delta U'(jh) = -\sum_{k=0}^{\infty}\Delta K^T((k-j)h)W\Delta K(kh)$ we arrive at the result:

$$J_{ij}^{(21)} + J_{ij}^{(22)} = \sum_{p=1}^{i}\sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\substack{-ph \\ \theta_1\neq\theta_2+(q-p)h}}^{(1-p)h}\varphi^T(\theta_1)D_i^T U''(\theta_1-\theta_2+ih-jh)D_j\varphi(\theta_2)d\theta_2 d\theta_1$$

$$-\sum_{q=1}^{j}\sum_{p=1}^{i}\int_{-qh}^{(1-q)h}\varphi^T(\theta+(q-p)h)D_i^T\Delta U'((q-p-j+i)h)D_j\varphi(\theta)d\theta$$

□

We look for a functional such that

$$\frac{d}{dt}v_0(x_t(\varphi)) = -x^T(t,\varphi)Wx(t,\varphi),\ t\geq 0,\ W>0. \tag{A.7}$$

By integrating the preceding equation, it follows by the exponential stability assumption on system (4.1) that

$$v_0(\varphi) = \int_0^{\infty} x^T(t,\varphi)Wx(t,\varphi)dt,\ t\geq 0.$$

Then, substitution of Cauchy formula (2.17) yields

$$v_0(\varphi) = \sum_{i=0}^{m}\sum_{j=0}^{m}\varphi^T(-ih)D_i^T\int_0^{\infty}K^T(t)WK(t)dtD_j\varphi(-jh)$$

$$+2\sum_{i=0}^{m}\sum_{j=1}^{m}\varphi^T(-ih)D_i^T\left(\int_0^{\infty}K^T(t)W\int_{-jh}^{0}K(t-\theta-jh)A_i\varphi(\theta)d\theta dt - J_{ij}(-ih)\right)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{\infty}\int_{-ih}^{0}\varphi^T(\theta_1)A_i^T\left(\int_{-jh}^{0}K^T(t-\theta_1-ih)WK(t-\theta_2-jh)A_j\varphi(\theta_2)d\theta_2 - J_{ij}(\theta_1)\right)d\theta_1 dt$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-jh}^{0}J_{ij}^T(\theta)A_j\varphi(\theta)d\theta + \sum_{i=1}^{m}\sum_{j=1}^{m}J_{ij}^{(2)}.$$

By using (A.4) and (A.5) we arrive at

$$v_0(\varphi) = \sum_{i=0}^{m}\sum_{j=0}^{m}\varphi^T(-ih)D_i^T U(0)D_j\varphi(-jh)$$

$$+2\sum_{i=0}^{m}\sum_{j=1}^{m}\sum_{q=1}^{j}\varphi^T(-ih)D_i^T\int_{-qh}^{(1-q)h}\left(U(-\theta-jh)A_j - U'(-\theta-jh)D_j\right)\varphi(\theta)d\theta$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=1}^{j}\int_{-ih}^{0}\varphi^T(\theta_1)A_i^T\int_{-qh}^{(1-q)h}\left(U(\theta_1-\theta_2+ih-jh)A_j - U'(\theta_1-\theta_2-jh+ih)D_j\right)\varphi(\theta_2)d\theta_2 d\theta_1$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{-ih}^{0}\varphi^T(\theta_1)D_i^TU'(\theta_1-\theta_2-jh+ih)A_j\varphi(\theta_2)d\theta_2d\theta_1$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{p=1}^{i}\sum_{q=1}^{j}\int_{-qh}^{(1-q)h}\int_{\substack{-ph\\ \theta_1\neq\theta_2+(q-p)h}}^{(1-p)h}\varphi^T(\theta_1)D_i^TU''(\theta_1-\theta_2+ih-jh)D_j\varphi(\theta_2)d\theta_2d\theta_1$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=1}^{j}\sum_{p=1}^{i}\int_{-qh}^{(1-q)h}\varphi^T(\theta+(q-p)h)D_i^T\Delta U'((q-p-j+i)h)D_j\varphi(\theta)d\theta.$$

Applying definitions of the functions $F_j^{(1)}$ and $F_{ij}^{(2)}$ we get

$$v_0(\varphi)=\sum_{i=0}^{m}\sum_{j=0}^{m}\varphi^T(-ih)D_i^TU(0)D_j\varphi(-jh)+2\sum_{i=0}^{m}\sum_{j=1}^{m}\varphi^T(-ih)D_i^T\int_{-jh}^{0}F_j^{(1)}(-\theta-jh)\varphi(\theta)d\theta$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-ih}^{0}\int_{-jh}^{0}\varphi^T(\theta_1)F_{ij}^{(2)}(\theta_1-\theta_2+ih-jh)\varphi(\theta_2)d\theta_2d\theta_1$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=1}^{j}\sum_{p=1}^{i}\int_{-qh}^{(1-q)h}\varphi^T(\theta+(q-p)h)D_i^T\Delta U'((q-p-j+i)h)D_j\varphi(\theta)d\theta$$

Appendix B

Proof of Theorem 4.1

In order to prove Theorem 4.1, we introduce first the following useful properties:

(i) For $\tau \geq 0$

$$\sum_{j=0}^{m} F_j^{(1)}(\tau - jh) = 0. \tag{B.1}$$

(ii) For $\tau < 0$

$$\sum_{i=0}^{m} F_{ij}^{(2)}(\tau + ih) = 0,\ j = \overline{1, m}. \tag{B.2}$$

Equation (B.1) follows directly from the dynamic property (4.4). We prove (B.2). From the dynamic property (4.4) and symmetry property (4.5),

$$- A_j^T \sum_{i=0}^{m} U'(-\tau - ih) D_i - D_j^T \sum_{i=0}^{m} U''(-\tau - ih) D_i =$$
$$= -A_j^T \sum_{i=0}^{m} U(-\tau - ih) A_i - D_j^T \sum_{i=0}^{m} U'(-\tau - ih) A_i,\ \tau < 0.$$

Transposing both sides yields

$$\sum_{i=0}^{m} D_i^T \frac{d}{d\tau} F_j^{(1)}(\tau + ih) = -\sum_{i=0}^{m} A_i^T F_j^{(1)}(\tau + ih),$$

which implies that

$$\sum_{i=0}^{m} \left(D_i^T \frac{d}{d\tau} F_j^{(1)}(\tau + ih) + A_i^T F_j^{(1)}(\tau + ih) \right) = 0,\ \tau < 0.$$

Consider now the functional v_0:

$$v_0(x_t) = v_0^{(1)}(t) + v_0^{(2)}(t) + v_0^{(3)}(t) + v_0^{(4)}(t),$$

where

$$
\begin{aligned}
v_0^{(1)}(t) =& \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T U(0) D_j x(t-jh),\\
v_0^{(2)}(t) =& 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T \int_{-jh}^{0} F_j^{(1)}(-\theta-jh)x(t+\theta)d\theta,\\
v_0^{(3)}(t) =& \sum_{i=1}^{m}\sum_{j=1}^{m} \int_{-ih}^{0}\int_{-jh}^{0} x^T(t+\theta_1)F_{ij}^{(2)}(\theta_1-\theta_2+ih-jh)x(t+\theta_2)d\theta_2 d\theta_1,\\
v_0^{(4)}(t) =& -\sum_{i=1}^{m}\sum_{j=1}^{m} \int_{0}^{h} x^T(t+\theta-ih)F_{ij}^{(3)} x(t+\theta-jh)d\theta.
\end{aligned}
$$

We differentiate each term:

$$
\frac{d}{dt}v_0^{(1)}(t) = 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T U(0) A_j x(t-jh).
$$

Before differentiating the second one, we compute the next derivative:

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{t-jh}^{t} F_j^{(1)}(t-\theta-jh)x(\theta)d\theta\right) = \sum_{q=1}^{j}\frac{d}{dt}\left(\int_{t-qh+0}^{t-(q-1)h-0} F_j^{(1)}(t-\theta-jh)x(\theta)d\theta\right)\\
&= \sum_{q=1}^{j} F_j^{(1)}((q-j-1)h+0)x(t-(q-1)h) - \sum_{q=1}^{j} F_j^{(1)}((q-j)h-0)x(t-qh) + \int_{t-jh}^{t}\frac{d}{dt}F_j^{(1)}(t-\theta-jh)x(\theta)d\theta.
\end{aligned}
$$

The non-integral terms are grouped as follows:

$$
\begin{aligned}
&\sum_{q=0}^{j-1} F_j^{(1)}((q-j)h+0)x(t-qh) - \sum_{q=1}^{j} F_j^{(1)}((q-j)h-0)x(t-qh)\\
&\qquad = F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh) + \sum_{q=1}^{j}\Delta F_j^{(1)}((q-j)h)x(t-qh)\\
&\qquad\qquad = F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh) - \sum_{q=1}^{j}\Delta U'((q-j)h)D_j x(t-qh).
\end{aligned}
$$

Returning to the derivative of $v_0^{(2)}$:

$$\frac{d}{dt}v_0^{(2)}(t) = 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)A_i^T \int_{t-jh}^{t} F_j^{(1)}(-\theta-jh)x(\theta)d\theta$$
$$+2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T \frac{d}{dt}\left(\int_{t-jh}^{t} F_j^{(1)}(t-\theta-jh)x(\theta)d\theta\right)$$
$$= 2\sum_{i=0}^{m} x^T(t-ih)D_i^T\left(\sum_{j=1}^{m} F_j^{(1)}(-jh+0)x(t) - \sum_{j=1}^{m} F_j^{(1)}(+0)x(t-jh) - \sum_{j=1}^{m}\sum_{q=1}^{j}\Delta U'((q-j)h)D_j x(t-qh)\right)$$
$$+2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)A_i^T\int_{t-jh}^{t} F_j^{(1)}(t-\theta-jh)x(\theta)d\theta + 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T \int_{\substack{t-jh\\ t-\theta\in\mathbb{R}\setminus\Omega}}^{t} \frac{d}{dt}F_j^{(1)}(t-\theta-jh)x(\theta)d\theta$$
$$= 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T\left(F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh) - \sum_{q=j}^{m}\Delta U'((j-q)h)D_q x(t-jh)\right)$$
$$+2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta-jh)x(\theta)d\theta.$$

Now, we differentiate the term $v_0^{(3)}$:

$$\frac{d}{dt}v_0^{(3)} = \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{d}{dt}\left(\int_{t-ih}^{t}\int_{t-jh}^{t} x^T(\theta_1)F_{ij}^{(2)}(\theta_1-\theta_2+ih-jh)x(\theta_2)d\theta_2 d\theta_1\right)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-jh}^{t} x^T(t)F_{ij}^{(2)}(t-\theta_2+ih-jh)x(\theta_2)d\theta_2 - \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-jh}^{t} x^T(t-ih)F_{ij}^{(2)}(t-\theta_2-jh)x(\theta_2)d\theta_2$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-ih}^{t} x^T(\theta_1)F_{ij}^{(2)}(\theta_1-t+ih-jh)x(t)d\theta_1 - \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-ih}^{t} x^T(\theta_1)F_{ij}^{(2)}(\theta_1-t+ih)x(t-jh)d\theta_1$$
$$= 2\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-jh}^{t} x^T(t)F_{ij}^{(2)}(t-\theta+ih-jh)x(\theta)d\theta - 2\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-jh}^{t} x^T(t-ih)F_{ij}^{(2)}(t-\theta-jh)x(\theta)d\theta.$$

Finally,

$$\frac{d}{dt}v_0^{(4)}(t) = -\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{d}{dt}\left(\int_{t}^{t+h} x^T(\theta-ih)F_{ij}^{(3)}x(\theta-jh)d\theta\right)$$
$$= -\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t-(i-1)h)F_{ij}^{(3)}x(t-(j-1)h)) + \sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t-ih)F_{ij}^{(3)}x(t-jh)$$
$$= -\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} x^T(t-ih)F_{i+1,j+1}^{(3)}x(t-jh) + \sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t-ih)F_{ij}^{(3)}x(t-jh)$$
$$= \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)\Big(F_{ij}^{(3)} - F_{i+1,j+1}^{(3)}\Big)x(t-jh) - \sum_{i=0}^{m} x^T(t-ih)F_{i,0}^{(3)}x(t) - \sum_{j=1}^{m} x^T(t)F_{0j}^{(3)}x(t-jh)$$

under the assumption that $F_{ij}^{(3)} = 0$, if i or j is equal to $m+1$. By definition of $F_{ij}^{(3)}$ and properties (4.11) and (4.12),

$$F_{0j}^{(3)} = -WD_j, \quad j = \overline{0,m}.$$

Hence,

$$\frac{d}{dt}v_0^{(4)}(t) = \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)\Big(F_{ij}^{(3)} - F_{i+1,j+1}^{(3)}\Big)x(t-jh) - x^T(t)Wx(t) + 2x^T(t)W\sum_{j=0}^{m} D_j x(t-jh)$$

$$= 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T\left(\sum_{q=j}^{m}\Delta U'((j-q)h)D_q\right)x(t-jh)$$

$$-\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T\Delta U'(0)D_j x(t-jh) - x^T(t)Wx(t) + 2x^T(t)W\sum_{j=0}^{m} D_j x(t-jh).$$

The sum of the preceding derivatives can be grouped in integral and non-integral terms, denoted by IT and NIT, respectively:

$$\frac{d}{dt}v_0(x_t) = IT + NIT.$$

Let us prove first that $IT = 0$. We have that IT is given by

$$IT = 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta-jh)x(\theta)d\theta$$

$$+ 2\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta+ih-jh)x(\theta)d\theta - 2\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t-ih)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta-jh)x(\theta)d\theta$$

$$= 2\sum_{j=1}^{m} x^T(t)\int_{t-jh}^{t} F_{0j}^{(2)}(t-\theta-jh)x(\theta)d\theta + 2\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta+ih-jh)x(\theta)d\theta$$

$$= 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t)\int_{t-jh}^{t} F_{ij}^{(2)}(t-\theta+ih-jh)x(\theta)d\theta.$$

By equality (B.2), we get $IT = 0$.

We now address the term NIT. It is determined by

$$NIT = 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T U(0)A_j x(t-jh)$$

$$+ 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T\left(F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh) - \sum_{q=j}^{m}\Delta U'((j-q)h)D_q x(t-jh)\right)$$

$$+ 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T\left(\sum_{q=j}^{m}\Delta U'((j-q)h)D_q\right)x(t-jh)$$

$$-\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T\Delta U'(0)D_j x(t-jh) - x^T(t)Wx(t) + 2x^T(t)W\sum_{j=0}^{m} D_j x(t-jh)$$

$$= -x^T(t)Wx(t) + 2x^T(t)W\sum_{j=0}^{m} D_j x(t-jh) + 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T U(0)A_j x(t-jh)$$

$$+ 2\sum_{i=0}^{m}\sum_{j=1}^{m} x^T(t-ih)D_i^T\left(F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh)\right)$$

$$+ 2\sum_{i=0}^{m} x^T(t-ih)D_i^T\left(\sum_{q=0}^{m}\Delta U'(-qh)D_q\right)x(t) - \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T\Delta U'(0)D_j x(t-jh).$$

By property (4.11),

$$NIT = -x^T(t)Wx(t) + 2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T U(0)A_j x(t-jh)$$

$$+2\sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T \left(F_j^{(1)}(-jh+0)x(t) - F_j^{(1)}(+0)x(t-jh)\right) - \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T \Delta U'(0)D_j x(t-jh).$$

From equality (B.1), we finally get

$$NIT = -x^T(t)Wx(t) + \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)D_i^T \left(2U(0)A_j - 2F_j^{(1)}(+0) - \Delta U'(0)D_j\right) x(t-jh)$$

$$= -x^T(t)Wx(t) + \sum_{i=0}^{m}\sum_{j=0}^{m} x^T(t-ih)\left(D_i^T U'(+0)D_j + D_i^T U'(-0)D_j\right) x(t-jh) = -x^T(t)Wx(t).$$

Appendix C

Computation of the delay Lyapunov matrix

We remind the semianalytic procedure for the computation of the delay Lyapunov matrix for systems of neutral type with commensurate delays and provide some techniques for reducing the computation effort in testing the stability conditions presented in Chapter 4. Let us consider a system of the form (4.1).

We introduce some notation. Let B and C be real matrices of arbitrary dimension. The Kronecker product $B \otimes C$ is defined by

$$B \otimes C = \begin{pmatrix} b_{11}C & \dots & b_{1n}C \\ \vdots & \ddots & \vdots \\ b_{m1}C & \dots & b_{mn}C \end{pmatrix},$$

and the product denoted by $B \circ C$ as

$$B \circ C = \begin{pmatrix} b_1^T c_{11} & \dots & b_1^T c_{1n} \\ \vdots & \ddots & \vdots \\ b_1^T c_{m1} & \dots & b_1^T c_{mn} \\ \vdots & \ddots & \vdots \\ b_m^T c_{11} & \dots & b_m^T c_{1n} \\ \vdots & \ddots & \vdots \\ b_m^T c_{m1} & \dots & b_m^T c_{mn} \end{pmatrix}.$$

The next properties hold:

$$\mathrm{vec}(BXC) = \left(C^T \otimes B\right) \mathrm{vec}(X), \tag{C.1}$$

and

$$\mathrm{vec}(BXC) = (B \circ C)\, \mathrm{vec}(X), \tag{C.2}$$

where X is a matrix of compatible dimensions with B and C, and $\mathrm{vec}(\cdot)$ represents the vectorization of a matrix.

C.1 Semianalityc procedure

Let us introduce the following auxiliary matrices:

$$X_j(\xi) = U(\xi + jh),\ \xi \in [0, h],\ j = -m, \dots, 0, m-1. \tag{C.3}$$

Proposition C.1. *The auxiliary matrices* (C.3) *satisfy the next system*

$$\begin{aligned}\sum_{i=0}^{m} X'_{j-i}(\xi)D_i &= \sum_{i=0}^{m} X_{j-i}(\xi)A_i,\ j=\overline{0,m-1},\ \xi\in(0,h),\\ \sum_{i=0}^{m} D_i^T X'_{i+j}(\xi) &= -\sum_{i=0}^{m} A_i^T X_{i+j}(\xi),\ j=\overline{-m,-1},\ \xi\in(0,h),\end{aligned} \tag{C.4}$$

with the boundary conditions

$$\begin{aligned}&X_{j+1}(0)=X_j(h),\ j=\overline{-m,m-2},\\ &-W=\sum_{i=0}^{m}\left(D_i^T X_0(0)A_i + A_i^T X_0(0)D_i\right)+\sum_{i=0}^{m-1}\sum_{j=i+1}^{m}\Big(D_j^T X_{j-i-1}(h)A_i + D_i^T X_{j-i-1}^T(h)A_j\\ &\qquad + A_i^T X_{j-i-1}^T(h)D_j + A_j^T X_{j-i-1}(h)D_i\Big).\end{aligned} \tag{C.5}$$

Proof. From the dynamic property (4.4), we have that

$$\sum_{i=0}^{m} X'_{j-i}(\xi)D_i = \sum_{i=0}^{m} U'(\xi+(j-i)h)D_i = \sum_{i=0}^{m} U(\xi+(j-i)h)A_i,\ j=\overline{0,m-1},\ \xi\in(0,h).$$

By the dynamic property (4.10),

$$\sum_{i=0}^{m} D_i^T X'_{i+j}(\xi) = \sum_{i=0}^{m} D_i^T U'(\xi+(i+j)h) = -\sum_{i=0}^{m} A_i^T U(\xi+(i+j)h),\ j=\overline{-m,-1},\ \xi\in(0,h).$$

The first boundary condition in (C.5) directly follows from the symmetric property (4.5) and the second one from the algebraic property (4.6). $\square$

In order to solve system (C.4) with boundary conditions (C.5), we vectorize the system. Let us consider the vector

$$X^T(\xi)=\left[\mathrm{vec}^T(X_{m-1}(\xi)) \quad \dots \quad \mathrm{vec}^T(X_{-m}(\xi))\right].$$

Using (C.1) and (C.2), the dynamic system (C.4) can be written as follows:

$$L_1X'(\xi)=L_2X(\xi),\ \xi\in(0,h), \tag{C.6}$$

where

$$L_1=\begin{pmatrix} \bar D_0 & \bar D_1 & \dots & \bar D_m & 0 & \dots & 0 & 0\\ 0 & \bar D_0 & \dots & \bar D_{m-1} & \bar D_m & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots 0 & 0 & \dots & \bar D_{m-1} & \bar D_m & \\ \tilde D_0 & \tilde D_1 & \dots & \tilde D_m & 0 & \dots & 0 & 0\\ 0 & \tilde D_0 & \dots & \tilde D_{m-1} & \tilde D_m & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & \tilde D_{m-1} & \tilde D_m \end{pmatrix}$$

and

$$L_2=\begin{pmatrix} \bar A_0 & \bar A_1 & \dots & \bar A_m & 0 & \dots & 0 & 0\\ 0 & \bar A_0 & \dots & \bar A_{m-1} & \bar D_m & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots 0 & 0 & \dots & \bar A_{m-1} & \bar A_m & \\ \tilde A_0 & \tilde A_1 & \dots & \tilde A_m & 0 & \dots & 0 & 0\\ 0 & \tilde A_0 & \dots & \tilde A_{m-1} & \tilde A_m & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & \tilde A_{m-1} & \tilde A_m \end{pmatrix}$$

with $\bar{D}_i = D_i^T \otimes I$, $\bar{A}_i = A_i^T \otimes I$, $\tilde{D}_i = -I \otimes D_i^T$ and $\tilde{A}_i = -I \otimes A_i^T$. The dimension of the system is of $2mn^2 \times 2mn^2$. The boundary conditions (C.5) can be expressed, after vectorization, as

$$MX(0) + NX(h) = -W_v, \tag{C.7}$$

where W_v is the vectorization of W, and matrices M and N are defined as follows:

$$M = \left(\begin{array}{c|c} M_{11} & 0_{(2m-1)n^2 \times n^2} \\ \hline M_{21} & 0_{n^2 \times n^2} \end{array}\right), \; N = \left(\begin{array}{c|c} 0_{(2m-1)n^2 \times n^2} & N_{12} \\ \hline H_{0,m} & N_{22} \end{array}\right),$$

with $M_{11} = -N_{12} = I_{(2m-1)n^2}$,

$$M_{21} = \begin{pmatrix} 0_{n^2 \times (m-1)n^2} & G & 0_{n^2 \times (m-1)n^2} \end{pmatrix},$$
$$N_{22} = \left(\sum_{i=0}^{1} H_{i,i+(m-1)} \quad \sum_{i=0}^{2} H_{i,i+(m-2)} \cdots \sum_{i=0}^{m-1} H_{i,i+1} \quad 0_{n^2 \times mn^2} \right).$$

Here,

$$G = \sum_{i=0}^{m} \left(\left(A_i^T \otimes D_i^T\right) + \left(D_i^T \otimes A_i^T\right)\right), \text{ and}$$
$$H_{i,j} = \left(A_i^T \otimes D_j^T\right) + \left(A_i^T \circ A_j\right) + \left(A_i^T \circ D_j\right) + \left(D_i^T \otimes A_j^T\right)$$

From (C.6), we have that

$$X(\xi) = e^{L\xi} X(0), \quad \xi \in [0, h],$$

where $L = L_1^{-1} L_2$. Invertibility justification of matrix L_1 can be found in Gohberg and Lerer (1976). The initial condition $X(0)$ is determined by the previous equation and (C.7) as follows:

$$X(0) = (M + Ne^{Lh})^{-1} W_v.$$

Having a solution for X, by the definition of the matrices X_j, we have a solution for the matrix $U(\tau)$, $\tau \in [-mh, mh]$.

C.2 Reduction of the computational effort: two delays case

In this section, we present some techniques that reduce the computational effort in testing the stability conditions presented in Chapter 4, when the chart is obtained for more than one delay as parameters, which is a more challenging case. We present here the ideas for two delays systems of the form:

$$\frac{d}{dt}\left(\sum_{j=0}^{2} D_j x(t - h_j)\right) = \sum_{j=0}^{2} A_j x(t - h_j), \quad t \geq 0.$$

Organize the scanning for a delay free system (C.4)-(C.5) of smaller dimension:

1. Take $N + 1$ equidistant points with the same step Δ on each axis.
2. Set a counter for every axis, this is $h_1^j = j\Delta$, and $h_2^i = i\Delta$, $i, j = \overline{0, N}$. Set the basic delay as $h = \gcd(i, j)\Delta$.
3. Compute the values $m_1 = h_1^j / h$ and $m_2 = h_2^i / h$. The dimension of the delay free system is $2mn^2$, where $m = \max\{m_1, m_2\}$.

4. Obtain the Lyapunov matrix through the auxiliary matrices X_j for the particular case of two delays:

$$X_j(\xi) = U(\xi + jh),\ \xi \in [0,h],\ j = \overline{-m, m-1}.$$

with derivatives, for $j \geq 0$,

$$\frac{d}{d\xi}\left(X_j(\xi) + D_1 X_{j-m_1}(\xi) + D_2 X_{j-m_2}(\xi)\right) = X_j(\xi)A_0 + X_{j-m_1}(\xi)A_1 + X_{j-m_2}(\xi)A_2,$$

and for $j < 0$,

$$\frac{d}{d\xi}\left(X_j'(\xi) + D_1^T X_{j+m_1}(\xi) + D_2^T X_{j+m_2}(\xi)\right) = -A_0^T X_j(\xi) - A_1^T X_{j+m_1}(\xi) - A_2^T X_{j+m_2}(\xi).$$

The boundary conditions are

$$X_{j+1}(0) = X_j(h),\ j = \overline{-m, m-2},$$

$$-W = \sum_{i=0}^{2}\left(D_i^T X_0(0)A_i + A_i^T X_0(0)D_i\right)$$

$$+\sum_{i=0}^{1}\sum_{j=i+1}^{2}\left(D_j^T X_{m_j-m_i-1}(h)A_i + D_i^T X_{m_j-m_i-1}^T(h)A_j + A_i^T X_{m_j-m_i-1}^T(h)D_j + A_j^T X_{m_j-m_i-1}(h)D_i\right).$$

This procedure minimizes the dimension of the delay free system. Indeed, the natural choice $h = \Delta$ gives a delay free system of dimension $2n^2N$ for every pair of values h_1 and h_2, which implies a greater computational effort. For example, when $h_1 = 8\Delta$ and $h_2 = 12\Delta$, it is clearly better to have a delay free of dimension $2 \times 3 \times n^2$ than of $2 \times 12 \times n^2$.

Additionally, notice that for two particular values of h_1 and h_2, the form of the delay free system (C.4)-(C.5) is the same for every pair of values ih_1, ih_2, $i = 1, 2, ...$ in the space of parameters. The computation of matrix exponentials are reduced by using the fact that

$$e^{Lih} = \left(e^{Lh}\right)^i, i = 1, 2, ...$$