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T E S I S

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Resumen

En este trabajo estudiamos problemas de controlabilidad para ecuaciones parabólicas lineales y semilineales. La tesis está dividida en dos partes.

En la primera parte, estudiamos algunos problemas de control jerárquico para sistemas de ecuaciones parábolicas. En el Capítulo 2, presentamos una estrategia de Stackelberg para control robusto de una ecuación de calor semilineal. Por un lado, tenemos un control, llamado *líder*, encargado de la controlabilidad a cero del sistema. Por el otro, tenemos un control, llamado *seguidor*, que resuelve un problema de control robusto. Para este último, buscamos un punto silla de un funcional de costo, con lo que éste es insensible a una clase de perturbaciones. En el Capítulo 3, presentamos una estrategia de Stackelberg-Nash para el control de un sistema acoplado de ecuaciones parabólicas. Aquí, el líder resuelve un problema de controlabilidad a cero, mientras que los seguidores, resuelven un equilibrio de Nash correspondiente a un problema de optimización multiobjetivo.

En la segunda parte, estudiamos el problema de controles insensibilizantes para ecuaciones parabólicas semidiscretas. En particular, probamos la $\phi(h)-$controlabilidad a cero del sistema en cascada que surge de la reformulación del problema de control insensibilizante. Aquí, $\phi(h)$ es una función adecuada del parámetro de discretización h, tal que $\lim_{h\to 0}\phi(h) = 0$. Además, presentamos diversos experimentos numéricos para la aproximación de controles insensibilizantes usando el método de unicidad de Hilbert (HUM, por sus siglas en inglés).

Abstract

This work is devoted to the study of some controllability problems concerning linear and semilinear parabolic systems. The dissertation is divided into two parts.

In the first part, we study some hierarchic control problems for linear and semilinear parabolic equations. In Chapter 2, we present a robust Stackelberg strategy for a semilinear heat quation. More precisely, we have one control, called the *leader*, that is responsible for a null controllability property. Additionally, we have a control, named the *follower* that solves a robust control objective. That means that we seek for a saddle point of a cost functional. In this way, the objective for the follower control is insensitive to a broad class of external disturbances. In Chapter 3, we present a Stackelberg-Nash strategy to control a system of coupled parabolic equations. In this case, we consider one leader control and two follower controls. As before, the leader deals with a null controllability objective. On the other hand, we look for a Nash equilibrium for the followers, corresponding to a multi-objective optimization strategy.

In the second part, we study the insensitizing control problem for semi-discrete parabolic equations. In particular, we prove the $\phi(h)$-null controllability of the cascade system arising from the reformulation of the insensitizing problem. Here, $\phi(h)$ is a suitable function of the discretization parameter h such that $\lim_{h\to 0} \phi(h) = 0$. We perform several numerical experiments approximating the insensitizing controls by using the Hilbert Uniqueness Method (HUM).

Contents

Chapter 1

Introduction

There are two main branches of control theory for partial differential equations (PDE), which seem to proceed in very different directions, but they are in fact complementary. The first one, the optimal control theory, deals with the problem of finding a control function for a given system such that an optimality criterion is achieved. The second one studies the capability of driving the system from a given initial state to a desired final target. We refer to this property as controllability.

When working in either context, we can formulate a control system of a PDE as

$$\begin{cases} y'(t) = f(t, y(t), v(t)), \quad t > 0, \\ y(0) = y^0, \end{cases} \tag{1.1}$$

where $y^0 \in \mathcal{X}$ is the initial datum, $t \mapsto y(t) \in \mathcal{X}$ is the state of the system, and $t \mapsto v(t) \in \mathcal{V}$ is the control function exerted on the system. The Banach spaces $\mathcal{X}$ and $\mathcal{V}$ are called the space state and the space of admissible controls, respectively. We denote by $y(t; y^0, v)$ the solution to (1.1) at time t for given $(y^0, v) \in \mathcal{X} \times \mathcal{V}$.

Assume that system (1.1) is well-posed, this is, there exists a unique solution y depending continuously on the data. For a real-valued and positive function $G = G(y, v)$, we define the cost functional $J(v) = G(y(v), v)$ and consider the optimization problem

$$\min_{v \in \mathcal{V}} J(v). \tag{1.2}$$

The minimum, if it exists, is called the optimal control. Hence, in optimal control theory, we study (among other things) the conditions to determine if the problem (1.2) has a solution, if it is unique or not, global or local, etcetera.

On the other hand, the fundamental question of controllability is: given two states $y^0 \in \mathcal{X}$ and $y^1 \in \mathcal{X}$ of system (1.1), does it exists a function $v \in \mathcal{V}$ such that it can *steer* the system from y^0 to y^1 on a fixed time $T > 0$? As we shall see, the word *steer* may be interpreted in different ways.

Definition 1. *Let* $T > 0$.

- ***Exact controllability.*** *System (1.1) is exactly controllable at time T if for all $(y^0, y^1) \in \mathcal{X} \times \mathcal{X}$, there exists $v \in \mathcal{V}$ such that*

$$y(T; y^0, v) = y^1.$$

- ***Approximate controllability.*** *System (1.1) is approximately controllable at time T if for all $(y^0, y^1) \in \mathcal{X} \times \mathcal{X}$, and every $\varepsilon > 0$, there exists $v \in \mathcal{V}$ such that*

$$\|y(T; y^0, v) - y^1\|_{\mathcal{X}} < \varepsilon.$$

- ***Controllability to trajectories.*** *System (1.1) is controllable to trajectories at time T if for every $(y^0, \widehat{y}^0) \in \mathcal{X} \times \mathcal{X}$ and $\widehat{v} \in \mathcal{V}$, there exists $v \in \mathcal{V}$ such that*

$$y(T; y^0, v) = y(T, \widehat{y}^0, \widehat{v}).$$

- ***Null controllability.*** *System (1.1) is null controllable at time T if for all $y^0 \in \mathcal{X}$, there exists $v \in \mathcal{V}$ such that*

$$y(T; y^0, v) = 0.$$

The nature of the system (1.1) will strongly influence to determine any property of controllability. For instance, in the finite dimension case, all of the controllability properties are by now well understood for linear and nonlinear systems. Indeed, for a linear time-invariant system, a necessary and sufficient condition is the well-known Kalman criterion (see, for instance, [23]), which allows proving that all controllability notions are equivalent.

However, in infinite dimension, the controllability will depend on the particular properties of the equation under study. For instance, it is well known that the wave equation can be approximately controllable at time T and not null controllable for any positive time, while the transport equation may be null controllable at a time T and not approximately controllable at this time. Also, the heat equation and, more generally, the parabolic systems are not exactly controllable at time T. This is due to the regularizing effect of the heat equation.

1.1 Some controllability results for the heat equation

In this dissertation, we are interested in studying some controllability problems for linear and semilinear parabolic systems. Here, we recall some well-known results on the controllability of the linear heat equation.

Here and all along the report we will consider $\Omega \subset \mathbb{R}^N$ a bounded domain with $\partial\Omega \in C^2$. We write $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ with $T > 0$ fixed. Let us consider the controlled heat equation

$$\begin{cases} y_t - \Delta y = v\chi_\omega, & \text{in Q}, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

In (1.3), $y = y(x,t)$ is the state, y^0 is the initial datum, and $v = v(x,t)$ is a control function applied on the open set $\omega \subset \Omega$. We denote by χ_ω the characteristic function of the set ω. We assume that $y^0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, thus (1.3) admits a unique solution (see, for instance, [27])

$$y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

From Definition 1, we write the controllability properties for the heat equation as follows:

Definition 2. *System (1.3) is* ***null controllable at time*** T *if for all* $y^0 \in L^2(\Omega)$*, there exists* $v \in L^2(\omega \times (0,T))$ *such that*

$$y(T) = 0.$$

Definition 3. *System (1.3) is* ***approximately controllable at time*** T *if for all* $y^0, y^1 \in L^2(\Omega)$ *and any* $\varepsilon > 0$*, there exists* $v \in L^2(\omega \times (0,T))$ *such that*

$$\|y(T) - y^1\|_{L^2(\Omega)} < \varepsilon.$$

Note that by linearity, the solution y to (1.3) can be decomposed into the sum of the uncontrolled solution and the solution starting from initial datum zero. More precisely, let us consider

$$y = \hat{y} + \bar{y},$$

where $\hat{y}$ and $\bar{y}$ are solution to the linear systems

$$\begin{cases} \hat{y}_t - \Delta \hat{y} = v\chi_\omega, & \text{in Q}, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0) = 0, & \text{in } \Omega, \end{cases} \qquad \begin{cases} \bar{y}_t - \Delta \bar{y} = 0, & \text{in Q}, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = y^0, & \text{in } \Omega. \end{cases}$$

In particular, since $y \in C([0,T; L^2(\Omega)])$ we have that for $t = T$

$$y(T) = L_T v + S_T y^0,$$

where

$$L_T \in \mathcal{L}(L^2(\omega \times (0,T)); L^2(\Omega)) \quad \text{defined as} \quad L_T v = \hat{y}(T)$$

and

$$S_T \in \mathcal{L}(L^2(\Omega)) \quad \text{defined as} \quad S_T y^0 = \bar{y}(T).$$

With these notations, we have the following result:

Proposition 4. *System (1.3) is*

- *null controllable at time T if and only if* $\mathrm{Im} S_T \subset \mathrm{Im} L_T$*,*
- *approximately controllable at time T if and only if* $\mathrm{Im} S_T$ *is dense in* $L^2(\Omega)$*.*

In order to write this result in a more suitable form, we need the following theorems

Theorem 5. *Let E and F be two Hilbert spaces and let $L : D(L) \subset E \to F$ be a linear operator densely defined and closed. Then $N(L^*)^\perp = \overline{\mathrm{Im}(L)}$. In particular,* $\mathrm{Im}(L)$ is dense in F if and only if L^* is injective.

Theorem 6. *Let E, F and G be three Hilbert spaces and $K : G \to F$, $L : E \to F$ be bounded linear operators. Then*

$$\mathrm{Im}(K) \subset \mathrm{Im}(L)$$

if and only if

$$\text{for some } c > 0 \text{ and all } f \in F^* \quad \|K^* f\|_G \leq c\|L^* f\|_E.$$

We refer the reader to [11] for the proof of Theorem 5 (see Corollary II.17 and Theorem II.19) and to [64, Th. 2.2, p. 208] for the proof of Theorem 6.

Applying Theorems 5 and 6 with $E = L^2(\omega \times (0,T))$, $F = G = L^2(\Omega)$, $L = L_T$ and $K = S_T$, we can express the controllability properties of system (1.3) as follows.

Proposition 7. *The system (1.3) is*

- ***null controllable at time** T if and only if there exists a constant C such that*

$$\forall \varphi^T \in L^2(\Omega), \quad \|S_T^* \varphi^T\|_{L^2(\Omega)} \leq C\|L_T^* \varphi^T\|_{L^2(\omega\times(0,T))}.$$

- ***approximately controllable at time** T if and only if*

$$\forall \varphi^T \in L^2(\Omega), \quad L_T^* \varphi^T = 0 \Rightarrow \varphi^T = 0,$$

In the practice, the properties of L_T^* and S_T^* can be characterized in terms of the adjoint system to (1.3). Let us consider the backward heat equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & \text{in Q}, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T, & \text{in } \Omega. \end{cases} \tag{1.4}$$

For all $\varphi^T \in L^2(\Omega)$, system (1.4) admits a unique solution $\varphi \in C([0,T]; L^2(\Omega))$. Multiplying (1.4) by y solution to (1.3) in $L^2(Q)$ and integrating by parts, we obtain

$$\int_\Omega \varphi^T y(T) - \int_\Omega \varphi(0) y^0 = \iint_Q \varphi v \chi_\omega.$$

With this relation, it is not difficult to see that the adjoint operators of S_T and L_T are

$$\begin{aligned} S_T^* : L^2(\Omega) &\to L^2(\Omega) \\ \varphi^T &\mapsto \varphi(0), \end{aligned} \qquad \begin{aligned} L_T^* : L^2(\Omega) &\to L^2(\omega \times (0,T)) \\ \varphi^T &\mapsto \varphi\chi_\omega. \end{aligned}$$

In this way, we can formulate the controllability results for system (1.3) in terms of observability properties for the adjoint system (1.4). We summarize them in the following Theorem

Theorem 8. *System* (1.3) *is*

- ***null controllable at time T*** *if and only if there exists a constant C such that for every $\varphi^T \in L^2(\Omega)$, the solution φ to* (1.4) *satisfies*

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\varphi|^2, \tag{1.5}$$

- ***approximately controllable at time T*** *if and only if, for all $\varphi^T \in L^2(\Omega)$, the solution φ to* (1.4) *verifies the unique continuation principle: if $\varphi = 0$ in $\omega \times (0,T)$, then $\varphi^T = 0$.*

Remark 9. Inequality (61) is called *observability inequality.*

1.2 Null controllability for the heat equation: Carleman estimates

For the heat equation, the null controllability problem has been solved by Lebeau & Robbiano [46] and Fursikov & Imanuvilov [33] by employing different methods. In fact, the following result was proved

Theorem 10. *The heat equation* (1.3) *is null controllable for all $T > 0$ and all non-empty open set $\omega \subset \Omega$.*

Here, we briefly recall the techniques used in [33] to prove the null controllability of the heat equation. According to Theorem 8, if we are able to obtain an observability inequality for the adjoint system (1.4), then the heat equation (1.3) is null controllable

Originally introduced in [20] to prove a unique continuation property, the Carleman estimates are weighted inequalities which are very effective tools to prove controllability properties for a wide variety of problems. Following [30], we will prove inequality (61) by using Carleman estimates.

Let $\tilde{\omega}$ be a non-empty subset satisfying $\tilde{\omega} \subset\subset \omega$. Set for any $m > 1$ the weight functions

$$\alpha(x,t) = \frac{e^{2\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}$$

for $(x,t) \in Q$, where $\eta^0 \in C^2(\bar{\Omega})$ is such that

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad |\nabla\eta^0| > 0 \quad \overline{\Omega\backslash\tilde{\omega}}.$$

For the existence of such function see [33, Lemma 1.1].

Theorem 11 ([33]). *There exist constants $C > 0$, $\lambda_0 \geq 1$ and $s_0 > 0$ such that, for any solution z to*

$$\begin{cases} -z_t - \Delta z = F, & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z^T, & \text{in } \Omega. \end{cases}$$

with $F \in L^2(Q)$ and $z^T \in L^2(\Omega)$, we have

$$s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} \left(|z_t|^2 + |\Delta z|^2\right) + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla z|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |z|^2 \leq C \left(s^3 \lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |z|^2 + \iint_Q e^{-2s\alpha} |F|^2 \right), \tag{1.6}$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2)$.

We apply the Carleman inequality (1.6) to the adjoint system (1.4). Fixing $\lambda = \lambda_0$, we obtain

$$\iint_Q e^{-2s\alpha} t^{-3} (T-t)^{-3} |\varphi|^2 \leq C \iint_{\omega \times (0,T)} e^{-2s\alpha} t^{-3} (T-t)^{-3} |\varphi|^2 \tag{1.7}$$

for all $s \geq s_0(T + T^2)$. Then, it is not difficult to see that

$$e^{-2s_0(T+T^2)\alpha} t^{-3} (T-t)^{-3} \geq e^{-2C(1+1/T)} \frac{1}{T^6} \quad \text{in} \quad \Omega \times (T/4, 3T/4),$$
$$e^{-2s_0(T+T^2)\alpha} t^{-3} (T-t)^{-3} \leq e^{-C(1+1/T)} \frac{1}{T^6} \quad \text{in} \quad \Omega \times (0, T).$$

Using the above inequalities in (1.7) we obtain

$$\iint_{\Omega \times (T/4, 3T/4)} |\varphi|^2 \leq C \iint_{\omega \times (0,T)} |\varphi|^2, \tag{1.8}$$

where C depends only on Ω, ω and T. From the equation verified by φ and classical energy estimates, we get

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq \frac{2}{T} \iint_{\Omega \times (T/4, 3T/4)} |\varphi|^2. \tag{1.9}$$

Putting together (1.8) and (1.9) gives the desired observability inequality.

The ideas and tools presented here will serve as a basis for proving the controllability results of this thesis. In fact, we will prove observability inequalities for complex coupled systems, but the way of proceeding will be the same.

1.3 The heat equation and the penalized HUM method

Although apparently are different, the problems of optimal control and controllability are closely related. In the framework of controllability, if one control exists it is certainly not unique. The Hilbert uniqueness method (HUM), originally introduced in [34], aims to formulate the control problem as an optimization problem, which consists in characterize and build the minimal L^2-norm control. We follow the spirit of [15] to present some results of the HUM approach.

Similar to Section 1.1, we denote for any $(v, y^0) \in L^2(\omega \times (0,T)) \times L^2(\Omega)$, the solution y to (1.3) at time T as

$$L_T(v|y^0) = y(T),$$

where $L_T \in \mathcal{L}(L^2(\omega \times (0,T) \times L^2(\Omega); L^2(\Omega))$.

For any $\delta \geq 0$, we define the (possible empty) closed convex set

$$\mathrm{Adm}(y_0, \delta) = \left\{ v \in L^2(\omega \times (0,T)) : \|L_T(v|y^0)\|_{L^2(\Omega)} \leq \delta \right\}.$$

With this notation, we can rewrite the null controllability property as follows:

Definition 12. *System (1.3) is null controllable at time T if for all $y^0 \in L^2(\Omega)$,*

$$\mathrm{Adm}(y^0, 0) \neq \emptyset.$$

In this situation $v \in \mathrm{Adm}(y^0, 0)$ is called a null-control associated with the initial datum y^0.

As mentioned before, the HUM approach consists in finding the control v with minimal norm. More precisely, for any $\delta \geq 0$ such that $\mathrm{Adm}(y^0, \delta)$ is not empty we define $v^\delta \in \mathrm{Adm}(y^0, \delta)$ as the unique control satisfying

$$F(v^\delta) = \inf_{v \in \mathrm{Adm}(y^0, \delta)} F(v), \tag{1.10}$$

where

$$F(v) = \frac{1}{2} \iint_{\omega \times (0,T)} |v|^2, \quad \forall v \in L^2(\omega \times (0,T)).$$

In the minimization problem (1.10), in the case where $\mathrm{Adm}(y^0, 0) \neq \emptyset$, the control v^0 is called the null-control associated with the initial data y^0.

The minimization problem (1.10) can be handled by duality theory, but the dual functional associated with (1.10) is not coercive in the usual dual state space $L^2(\Omega)$. In fact, a much large space is required. This issue leads to difficulties when using this approach for numerical purposes.

To avoid such problems, it is convenient to introduce a penalized version to the problem. For any $\varepsilon > 0$, we define the quadratic cost functional

$$F_\varepsilon(v) = \frac{1}{2} \iint_{\omega \times (0,T)} |v|^2 + \frac{1}{2\varepsilon} \int_\Omega |L_T(v|y^0)|^2, \quad \forall v \in L^2(\omega \times (0,T)), \tag{1.11}$$

that we wish to minimize onto the whole space $L^2(\omega \times (0,T))$.

The study of this optimization problem as a function of ε makes possible to recover the null controllability of the heat equation. Moreover, the penalization technique will play a key role in the numerical results presented in Chapter 4.

For any $\varepsilon > 0$, the functional (1.11) has a unique minimizer on $L^2(\omega \times (0,T))$, denoted as v_ε. This is due to the fact that F_ε is strictly convex, continuous and coercive. The minimizer is characterized by the following Euler-Lagrange equation

$$\iint_{\omega\times(0,T)} v_\varepsilon \tilde{v} + \frac{1}{\varepsilon}\left(L_T(v_\varepsilon|y^0), L_T(\tilde{v}|0)\right)_{L^2(\Omega)} = 0, \quad \forall \tilde{v} \in L^2(\omega \times (0,T)).$$

Using results of the Fenchel-Rockafellar theory (see, for instance, [26]), we can obtain an associated dual problem. For any $\varepsilon > 0$, we define the cost functional

$$J_\varepsilon(\varphi^T) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2 + \frac{\varepsilon}{2}\|\varphi^T\|^2_{L^2(\Omega)} + \int_\Omega y^0\varphi(0), \quad \forall \varphi^T \in L^2(\Omega) \tag{1.12}$$

where φ is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & \text{in Q}, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T, & \text{in } \Omega. \end{cases} \tag{1.13}$$

Again, from standard arguments of convex optimization, we have that J_ε has a unique minimizer $\varphi^T_\varepsilon \in L^2(\Omega)$ for any $\varepsilon > 0$. In this case, the minimizer is characterized by the equation

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon \tilde{\varphi} + \varepsilon\left(\varphi^T_\varepsilon, \tilde{\varphi}\right)_{L^2(\Omega)} + (y_0, \tilde{\varphi}(0))_{L^2(\Omega)} = 0, \quad \forall \tilde{\varphi}^T \in L^2(\Omega),$$

where we have denoted φ_ε as the solution to (1.4) with initial datum φ^T_ε. We have the following important result:

Proposition 13. *For any $\varepsilon > 0$, the minimizers v_ε and φ^T_ε of the functionals F_ε and J_ε, respectively, are related through the formulas*

$$v_\varepsilon = \varphi_\varepsilon \chi_\omega,$$

and

$$L_T(v_\varepsilon|y_0) = y(T) = -\varepsilon\varphi^T_\varepsilon.$$

As a consequence, we have

$$\inf_{v\in L^2(\omega\times(0,T))} F_\varepsilon = F_\varepsilon(v_\varepsilon) = -J_\varepsilon(\varphi^T_\varepsilon) = -\inf_{\varphi^T\in L^2(\Omega)} J_\varepsilon.$$

We can express the null controllability of (1.3), for a given initial datum y^0, in terms of the behavior of the penalized problem (1.11):

Theorem 14. *System (1.3) is null controllable from the initial datum y^0 if and only if we have*

$$M_{y^0}^2 := 2 \sup_{\varepsilon>0} \left(\inf_{v \in L^2(\omega\times(0,T))} F_\varepsilon \right) < +\infty \tag{1.14}$$

In this case, we have

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq M_{y^0},$$
$$\|L_T(v_\varepsilon|y^0)\|_{L^2(\Omega)} \leq M_{y^0}\sqrt{\varepsilon}.$$

Moreover, we have $\|v^0\|_{L^2(\omega\times(0,T))} = M_{y^0}^2$ and

$$v_\varepsilon \to v^0 \quad \textit{as} \quad \varepsilon \to 0.$$

Remark 15. For $\varepsilon_1 > \varepsilon_2 > 0$ and $v \in L^2(\omega \times (0,T))$ we have that $F_{\varepsilon_2}(v) \geq F_{\varepsilon_1}(v)$. Then, it follows that the supremum with respect to ε in (1.14) its in fact the limit when $\varepsilon \to 0$ of $\inf_v F_\varepsilon$.

In the same way, by analyzing the penalized problems, we can prove a more standard statement of the equivalence between observability and controllability.

Proposition 16. *System (1.3) is null controllable for any $y^0 \in L^2(\Omega)$ if and only if there exists $C_{obs} > 0$ such that*

$$\|\varphi(0)\|_{L^2(\Omega)}^2 \leq C_{obs}^2 \iint_{\omega\times(0,T)} |\varphi|^2, \quad \forall \varphi^T \in L^2(\Omega),$$

where φ is the solution to (1.4).

Remark 17. The proof of Proposition 16 is based on the analysis of the penalized HUM functionals instead of the more general functional analysis results presented in Section 1.1.

1.4 Hierarchic control: a multi-objective optimization problem

Optimization problems arise in many applications of engineering and mathematics. Traditionally, such problems deal with a single objective: minimize cost, maximize benefit. When studying more realistic and complex situations, it is desirable to include several different (and even contradictory) control objectives. Therefore, the introduction of multi-objective optimization is essential. Different notions of multi-objective problems were introduced in economics and game theory, see [56], [57], [60]

To fix ideas in the context of PDE control, consider system (1.3) and let us introduce the control point of view. We would like to choose v in order to achieve two different objectives:

1. null controllability, i.e., $y(T) = 0$, and
2. find the best control v such that y is "not too far from" a desired target y_d.

As above, we can associate a cost functional for the first objective and design the control v subject to the null control controllability constraint. On the other hand, consider the quadratic cost functional

$$J_2(v) = \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |y - y_d|^2 + \frac{\beta}{2}\iint_{\omega\times(0,T)} |v|^2, \tag{1.15}$$

where $\beta > 0$ is a constant, $\mathcal{O}_d \subset \Omega$ is a non empty open subset and $y_d \in L^2(\mathcal{O}_d \times (0,T))$ is given. Then, the second control objective is to achieve the optimization problem

$$\min_{v\in L^2(\omega\times(0,T))} J_2(v). \tag{1.16}$$

This classical optimal control problem has been thoroughly studied, see, for instance, [48, 63], and the references within. Since the functional J_2 is continuous, strictly convex and coercive, there exists a unique element $\bar{v} \in L^2(\omega \times (0,T))$ such that (1.16) holds.

It is clear that the control problems 1-2 are well posed and have a solution. Nevertheless, it is not clear how to design the control v satisfying both control objectives. In fact, it is impossible in general to choose the same v fulfilling the control problems 1-2.

We propose here to employ the concept of **hierarchic control** to address the multi-objective problem. Originally introduced in [50, 51] by J.-L. Lions, this methodology uses the notion of Stackelberg optimal control. In the framework of game theory, the Stackelberg competition (see [60]) is a non-cooperative decision problem where one of the participants enforce its strategy on the other participants. The enforcing player is called the **leader** and the other players are called the **followers**.

According to the original work of Lions, to apply this idea we divide first the control set ω into two parts

$$\omega = \omega_1 \cup \omega_2, \quad \text{up to a set of measure } \omega,$$

where ω_i are open sets of Ω and $\omega_1 \cap \omega_2 = \emptyset$. If χ_{ω_i} denotes the characteristic function of ω_i and v^i denotes the restriction of v to $\omega_i \times (0,T)$, then (1.3) becomes

$$\begin{cases} y_t - \Delta y = v^1\chi_{\omega_1} + v^2\chi_{\omega_2}, & \text{in Q},\\ y = 0 \quad \text{on } \Sigma, \\ y(0) = y^0, & \text{in } \Omega. \end{cases} \tag{1.17}$$

For given $v^i \in L^2(\omega_i \times (0,T))$, $i = 1,2$, system (1.17) admits a unique solution that we denote as $y(v^1, v^2) = y(x,t;v^1,v^2)$.

Following [51], we will design first the follower control v^2. To this end, consider the problem

$$\begin{cases} \text{given } v^1 \in L^2(\omega_1 \times (0,T)), \text{ find} \\ \min\limits_{v^2} J_2(v^1, v^2) \end{cases} \tag{1.18}$$

where

$$J_2(v^1, v^2) = \frac{1}{2} \iint_{\mathcal{O}_d \times (0,T)} |y - y_d|^2 + \frac{\beta}{2} \iint_{\omega_2 \times (0,T)} |v^2|^2, \tag{1.19}$$

and y is the solution to (1.17). Problem (1.18) admits a unique solution for each $v^1 \in L^2(\omega_1 \times (0,T))$. Indeed, note that v^1 participates in the minimization problem by means of the variable y, but it can be seen as a constant during the process. Therefore, we write the solution as

$$v^2 = \mathcal{F}(v^1). \tag{1.20}$$

In the second step, we replace v^2 as given in (1.20) and obtain

$$y(v^1, \mathcal{F}(v^1)). \tag{1.21}$$

Then, in a very natural fashion, we address the problem

$$\begin{cases} \text{find } v^1 \in L^2(\omega_1 \times (0,T)), \text{ which minimizes} \\ J_1(v^1) = \dfrac{1}{2} \displaystyle\iint_{\omega_1 \times (0,T)} |v^1|^2 \quad \text{subject to} \quad y(\cdot, T; v^1, \mathcal{F}(v^1)) = 0. \end{cases} \tag{1.22}$$

In this way, the Stackelberg problem we consider here is:

Step 1. Given v^1 (the leader), we choose v^2 (the follower) using (1.18)–(1.20).

Step 2. We choose the leader v^1 using (1.22).

Remark 18. The Stackelberg problem described above will be the basis for the problems studied in Chapters 2 and 3. There, we will study hierarchic control problems for other more general equations. Additionally, we are interested in changing the optimization problem in step 1. Indeed, we are going to consider two generalizations for the classical optimal control problem (1.15)–(1.16).

Remark 19. Step (1.20) is known as characterization and commonly involves identifying the optimal point with a suitable adjoint system. Note that this characterization is then substituted in (1.21) and needs to be considered in the optimization problem for the leader control. This process involves additional difficulties to a classic null controllability problem.

1.5 Main contributions

In this section, we present a summary of the main results obtained during the realization of this research. It is divided into two parts. In the first, we present new results on hierarchic control for some linear and semilinear parabolic systems. In the second, we study the insensitizing control problem from a numerical point of view.

1.5.1 Robust Stackelberg controllability for a semilinear heat equation

In Chapter 2, we are interested in controlling the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = h\chi_\omega + v\chi_{\mathcal{O}} + \psi & \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \quad y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.23}$$

where $f \in C^2$ is a globally Lipschitz function such that $f(0) = 0$ and $\psi \in L^2(Q)$ is an undesired perturbation disturbing the control objectives. Here, we have denoted h as the leader and v as the follower.

Using the idea of hierarchic control described above, we want to get $y(T) = 0$ using the minimal L^2-norm control h and to "stay near" a desired state y_d with the control v but, unlike the case explained in Section 1.4, there is now a perturbation affecting the performance of the system.

In the case where $\psi = 0$, this problem has been solved in [6]. In chapter 2, we will combine the concept of hierarchic control with the concept of robust control appearing in optimal control problems (see, for instance, [8, 9, 10]). As far as we know, the idea of combining robustness with a Stackelberg strategy is new in literature.

To include the effect of the perturbation ψ in the hierarchic problem, we will change the functional (1.19) to

$$J_r(h,\psi,v) = \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 dxdt + \frac{1}{2}\left[\ell^2 \iint_{\mathcal{O}\times(0,T)} |v^2| dxdt - \gamma^2 \iint_Q |\psi|^2 dxdt\right].$$

where $\ell, \gamma > 0$ are constants and $y_d \in L^2(\mathcal{O}_d \times (0,T))$ is given.

Instead of only looking for the control v of minimal norm, we look also for the worst possible disturbance $\psi \in L^2(Q)$. Given a leader control h, we look for an optimal pair $(\bar{v}, \bar{\psi})$ such that

$$J_r(h,\bar{v},\bar{\psi}) = \min_{v\in L^2(\mathcal{O}\times(0,T))} \sup_{\psi\in L^2(Q)} J_r(h,v,\psi) = \max_{\psi\in L^2(Q)} \inf_{v\in L^2(\mathcal{O}\times(0,T))} J_r(h,v,\psi). \tag{1.24}$$

In this way, we obtain the best control v which works in the presence of the worst disturbance ψ. Once the pair $(\bar{v}, \bar{\psi})$ has been identified, we replace it in (1.23) and write

$$y(h, \bar{v}(h), \bar{\psi}(h)).$$

Then, we solve the problem

$$\begin{cases} \text{find } h \in L^2(\omega\times(0,T)), \text{ which minimizes} \\ J(h) = \dfrac{1}{2}\displaystyle\iint_{\omega\times(0,T)} |h|^2 \quad \text{subject to} \quad y(\cdot,T;h,\bar{v}(h),\bar{\psi}(h)) = 0. \end{cases} \tag{1.25}$$

We have the following result:

Theorem 20. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$ and $N \leq 6$. Let $f \in C^2$ be a globally Lipschitz function verifying $f(0) = 0$ and $f'' \in L^\infty(\mathbb{R})$. Then, there exist γ_0, ℓ_0 and a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $\gamma > \gamma_0$, $\ell > \ell_0$, $y_0 \in L^2(\Omega)$, and any $y_d \in L^2(Q)$ verifying*

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y_d|^2 < +\infty, \tag{1.26}$$

there exists a Stackelberg strategy $(h, \bar{\psi}, \bar{v})$ for the optimization problems (1.24)–(1.25)

Observe that in the problem (1.24), the optimization is carried out in the unbounded set $L^2(\mathcal{O} \times (0,T)) \times L^2(Q)$. From a practical point of view, it could be more interesting to optimize over some convex, closed and bounded sets $\mathcal{V}_{ad}$, Ψ_{ad} defining the sets of admissible controls and admissible perturbations, respectively.

In this spirit, we get the following result:

Theorem 21. *Let us assume that $f(y) = ay$ for some $a = a(x,t) \in L^\infty(Q)$ and that $\omega \cap \mathcal{O}_d \neq \emptyset$. Then, there exist γ_0, ℓ_0 and a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $\gamma > \gamma_0$, $\ell > \ell_0$, $y_0 \in L^2(\Omega)$, and $y_d \in L^2(\mathcal{O}_d \times (0,T))$ verifying (1.26), there exist a leader control h and a unique associated point $(\bar{v}, \bar{\psi}) \in \mathcal{V}_{ad} \times \Psi_{ad}$ verifying the optimization problems (1.24)–(1.25).*

1.5.2 Stackelberg-Nash controllability for some parabolic systems

There are several papers considering multi-objective optimization in the context of control of PDE (see, for instance, [6, 7, 41, 47, 50, 51]). Most of the previous works have one thing in common: they deal with hierarchical control of a single equation. In the second chapter, we are interested in developing a Stackelberg strategy where the system dynamics is given by a non-scalar system of parabolic equations. These type of systems are particularly studied in mathematical biology (c.f. [22, 24, 55]).

Let us consider

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11} y_1 + a_{12} y_2 = h_1 \chi_\omega + v^1 \chi_{\omega_1} + v^2 \chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21} y_1 + a_{22} y_2 = h_2 \chi_\omega & \text{in } Q, \\ y_j = 0 \text{ on } \Sigma, \ j = 1,2, \\ y_j(x,0) = y_j^0(x) \text{ in } \Omega, \ j = 1,2, \end{cases} \tag{1.27}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q)$ and $y_j^0 \in L^2(\Omega)$ are given.

In system (3.1), $y = (y_1, y_2)^t$ is the state, $v^j = v^j(x,t)$ and $h_j = h_j(x,t)$ are the follower and leader control functions, respectively. We write $h = (h_1, h_2)^t$ to abridge the notation.

Note that we have increased the number of follower and leader controls. In this part, we want that each follower control satisfies its own optimization criterium. That is, given $h \in [L^2(\omega \times (0,T))]^2$, we look for a solution to the problem

$$\min_{v^i \in L^2(\omega_i \times (0,T))} J_i(h, v^1, v^2), \quad i = 1,2, \tag{1.28}$$

where

$$\begin{aligned} J_i(h,v^1,v^2) = \frac{\alpha_i}{2}\iint_{\mathcal{O}_{d\times(0,T)}} \left(|y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2\right) dxdt \\ + \frac{\mu_i}{2}\iint_{\omega_i\times(0,T)} |v^i|^2 dxdt,\ i=1,2, \end{aligned} \tag{1.29}$$

with $\alpha_i, \mu_i > 0$ and $y^i_{j,d}$ are given functions.

To solve simultaneously the optimization problems (1.28) it is necessary to define a concept of solution.

Definition 22. *Let $h \in [L^2(\omega \times (0,T))]^2$ be given. The pair $(\bar{v}^1, \bar{v}^2)$ is called a Nash equilibrium for (1.29) if*

$$\begin{aligned} J_1(h,\bar{v}^1,\bar{v}^2) \le J_1(h,v^1,\bar{v}^2), \quad \forall v^1 \in L^2(\omega_1\times(0,T)), \\ J_2(h,\bar{v}^1,\bar{v}^2) \le J_2(h,\bar{v}^1,v^2), \quad \forall v^2 \in L^2(\omega_2\times(0,T)). \end{aligned} \tag{1.30}$$

Therefore, we look for conditions to ensure the existence of a unique Nash equilibrium $(\bar{v}^1, \bar{v}^2)$. If we can do that, we replace it in (1.27) to obtain

$$y(h, v^1(h), v^2(h)).$$

Following the hierarchic methodology, we finally solve for the leader problem

$$\begin{cases} \text{find } h_1, h_2 \in L^2(\omega\times(0,T)), \text{ which minimizes} \\ J(h) = \dfrac{1}{2}\displaystyle\iint_{\omega\times(0,T)} \left(|h_1|^2 + |h_2|^2\right) \\ \text{subject to} \quad y(\cdot, T; h, \bar{v}^1(h), \bar{v}^2(h)) = 0. \end{cases} \tag{1.31}$$

We have the following result:

Theorem 23. *Suppose that $\omega \cap \mathcal{O}_d \neq \emptyset$ and that μ_i, $i=1,2$, are large enough. Then, there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $y^i_{j,d} \in L^2(\mathcal{O}_d \times (0,T))$ satisfying*

$$\iint_{\mathcal{O}_d\times(0,T)} \rho^2 |y^i_{j,d}|^2 dxdt < +\infty, \quad i,j = 1,2, \tag{1.32}$$

and any $y^0 \in L^2(\Omega)^2$, there exists a Stackelberg-Nash strategy $(h_1, h_2, \bar{v}^1, \bar{v}^2)$ for the optimization problems (1.28) and (1.31).

One of the most challenging problems when dealing with non-scalar systems is to control many equations with few controls. Observe that we obtained the null controllability of (1.27) by means of two leader controls.

In the pure controllability framework, i.e., when $v^1 = v^2 = 0$, M. González-Burgos and L. de Teresa (see [37]) proved that (1.27) is null controllable when $h_2 \equiv 0$ if the coefficient a_{21} satisfies a sign condition (see eq. (1.34)). We expected to use this condition to prove that Theorem 23 is also valid if $h_2 = 0$. However, it was not possible for the follower functionals defined as (1.29).

In Chapter 3, we will prove that penalizing the action of the followers, we can eliminate the action of the leader control h_2. Consider the modified follower functionals

$$\begin{aligned} J_i(h, v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_d \times (0,T)} \left(|y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2\right) dxdt \\ + \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} \rho_*^2 |v^i|^2 dxdt, \ i = 1, 2, \end{aligned} \tag{1.33}$$

where $\rho_* = \rho_*(t)$ is a suitable weight function to be clarified. In this case, we have the following:

Theorem 24. *Suppose that $\mathcal{O}_d \cap \omega \neq \emptyset$, μ_i, $i = 1, 2$, are large enough and assume $h_2 \equiv 0$. If*

$$a_{21} \geq a_0 > 0 \quad \text{or} \quad -a_{21} \geq a_0 > 0 \quad \text{in } (\mathcal{O}_d \cap \omega) \times (0, T), \tag{1.34}$$

then there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that if (1.32) holds, then for any $y^0 \in L^2(\Omega)^2$ there exists a Stackelberg-Nash strategy $(h_1, \bar{v}^1, \bar{v}^2)$ for the functionals given by (1.33).

1.5.3 Insensitizing controls for the semi-discrete heat equation

In the second part of this thesis, we study the insensitizing control problem. This problem, originally introduced by Lions [49] reads a follows.

Let us consider

$$\begin{cases} y_t - \Delta y = \xi + v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x, 0) = y_0(x) + \tau \bar{y} \end{cases} \tag{1.35}$$

where $\xi \in L^2(Q)$ is a given source term and $v = v(x, t)$ is the control. We suppose that the initial condition is partially known, that is, y_0 is a known approximate value to $y(\cdot, 0)$ and $\tau \bar{y}$ is an unknown error affecting this approximation. Here $\bar{y}$ is a function in $L^2(\Omega)$ with norm equal to 1 and $\tau \in \mathbb{R}$ is small.

The question arising in the insensitizing control problem is that if there exists a control v such that we can obtain measurements that are independent of the initial condition. More precisely, let $\mathcal{O} \subset \Omega$ be a non-empty set and consider the functional

$$\Psi(y(x, t; \tau, v)) = \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} |y(x, t; \tau, v)|^2 \tag{1.36}$$

where $y(\cdot,\cdot;\tau,v)$ is the solution to (1.35) associated to τ and v. The problem of **insensitizing** the functional Ψ is to find a control v such that the energy of the system in the observation set $\mathcal{O}$ is locally insensitive to the small perturbations $\tau\bar{y}$. In other words,

Definition 25. *We say that the control v insensitizes the functional Ψ given by (1.36) if*

$$\left.\frac{\partial\Psi(y(\cdot,\cdot;\tau,v))}{\partial\tau}\right|_{\tau=0}=0 \qquad \forall\bar{y}\in L^2(\Omega),\ \|\bar{y}\|_{L^2(\Omega)}=1.$$

In [12] it was proved that the control v insensitizes Ψ if and only if v solves the non-standard null controllability problem

$$\begin{cases} y_t-\Delta y=\xi+v\chi_\omega & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(x,0)=y_0(x) & \text{in } \Omega,\end{cases}$$

$$\begin{cases} -q_t-\Delta q=y\chi_{\mathcal{O}} & \text{in } Q,\\ q=0 & \text{on } \Sigma,\\ q(x,T)=0 & \text{in } \Omega,\end{cases}$$

$$q(x,0)=0 \quad \text{in } \Omega.$$

The insensitizing control problem has been studied for a wide variety of systems and from different perspectives, see, for instance, [12, 13, 14, 21, 40, 43, 62].

In this thesis, we are interested in studying this problem from a numerical point of view. We consider the 1-D semi-discrete system

$$\begin{cases} \partial_t y_h+\mathcal{A}^{\mathfrak{M}}y_h=\mathbf{1}_\omega v_h+\xi_h & \text{in } Q=(0,L)\times(0,T),\\ y_h=0 & \text{on } \Sigma=\{0,L\}\times(0,T),\\ y_h(0)=y_{h,0}+\tau w_{h,0} & \text{in } (0,L).\end{cases} \tag{1.37}$$

Here $\mathcal{A}^{\mathfrak{M}}$ is the discrete approximation of $\mathcal{A}:=-\partial_x^2$ for a mesh $\mathfrak{M}$ with step size h. As in the continuous case, the insensitizing problem is equivalent to steer $q_h(0)$ to 0 where (y_h,q_h) is the solution to

$$\begin{cases} \partial_t y_h+\mathcal{A}^{\mathfrak{M}}y_h=\mathbf{1}_\omega v_h+\xi_h & \text{in } Q,\\ -\partial_t q_h+\mathcal{A}^{\mathfrak{M}}q_h=\mathbf{1}_{\mathcal{O}}y_h & \text{in } Q,\\ y_h=q_h=0 & \text{on } \Sigma,\\ y_h(0)=y_h^0,\quad q_h(T)=0 & \text{in } (0,L).\end{cases} \tag{1.38}$$

By means of discrete Carleman inequalities, we are going to study controllability properties for (1.38). In fact, we will see that we can build a semi-discrete control v_h such that

$$|q_h(0)|_{L^2(\Omega)}\le Ce^{-C/h}\|\xi_h\|_{L^2(e_{\mathcal{M}})} \tag{1.39}$$

where $L^2(e_{\mathcal{M}})$ is a weighted space to be clarified. This means that we will reach a target with size depending on the discretization step h.

On the other hand, following the ideas of [15], we will use the penalized HUM methodology to connect the penalization term of a suitable cost functional to the discretization step h. In this way, we will be able to compute numerical approximations of insensitizing controls for (1.38) verifying (1.39). Several experiments regarding well-known insensitizing results in the continuous case are discussed in Chapter 4.

Part I

On hierarchic control problems for parabolic equations

Chapter 2

Robust Stackelberg controllability for a semilinear heat equation

2.1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded open set with boundary $\partial\Omega \in C^2$. For $T > 0$, we denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Let ω and $\mathcal{O}$ be nonempty subsets of Ω with $\omega \cap \mathcal{O} = \emptyset$. We consider the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = h\chi_\omega + v\chi_{\mathcal{O}} + \psi & \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \quad y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

where f is a globally Lipschitz-continuous function, $y_0 \in L^2(\Omega)$ is a given initial datum and $\psi \in L^2(Q)$ is an unknown perturbation.

In (2.1), $y = y(x,t)$ is the state and $h = h(x,t)$, $v = v(x,t)$ are two different control functions acting on the system through ω and $\mathcal{O}$, respectively.

We want to choose the controls v and h in order to achieve two different optimal objectives:

1. solve for the "best" control v such that y is "not too far" from a desired target y_d which is effective even in the presence of the "worst" disturbance ψ, and
2. find the minimal L^2-norm control h such that $y(\cdot,T) = 0$.

The first problem, introduced in [9] for the linearized Navier-Stokes system, looks for a control such that a cost functional achieves its minimum for the worst disturbance. Solving for such control is a way of achieving system robustness: a control which works even in the presence of the worst disturbance ψ will also be robust to a class of other possible perturbations. This approach is useful in physical systems in which unpredictable disturbances are common.

The second problem is a classical null controllability problem. It has been thoroughly studied in the recent years for a wide variety of systems described by partial differential equations, see for instance [30].

When dealing with multi-objective optimization problems, a concept of a solution needs to be clarified. There are different equilibrium concepts (see [56, 57, 60]) which determine a strategy leading to choice good controls. In the framework of control of PDEs, there are several works applying successfully these strategies, see [7, 36, 41, 47, 50, 51].

Here, we use the so-called hierarchic control introduced by Lions in [51] to achieve the desired goals. This technique uses the notion of Stackelberg optimization. Below, we will explain both the robust control and the null controllability problem and then how we will apply the hierarchic control methodology to solve the multi-objective optimization problem.

2.1.1 The control problem

Let $\mathcal{O}_d \subset \Omega$ be an open set representing an observation domain. Let us introduce the cost functional

$$J_r(\psi, v; h) = \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |y - y_d|^2 dxdt + \frac{1}{2}\left[\ell^2 \iint_{\mathcal{O}\times(0,T)} |v|^2 dxdt - \gamma^2 \iint_Q |\psi|^2 dxdt\right]. \tag{2.2}$$

where $\ell, \gamma > 0$ are constants and $y_d \in L^2(\mathcal{O}_d \times (0,T))$ is given. This functional describes the robust control problem. We seek to simultaneously maximize J_r with respect to ψ and minimize it with respect to v, while maintaining the state y "close enough" to a desired target y_d in $\mathcal{O}_d \times (0,T)$. Note that the functional (2.2) generalizes some classical optimization problems (see, for instance, [48, 63]).

As explained in [10], one can intuitively consider the problem as a game between a designer looking for the best control v and a malevolent disturbance ψ spoiling the control objective. The parameter ℓ^2 may be interpreted as the price of the control to the designer: the limit $\ell \to \infty$ corresponds to a prohibitively expensive control and results in $v \to 0$ in the minimization with respect to v. On the other hand, the parameter γ^2 may be interpreted as the magnitude of the perturbation that the problem can afford. The $\gamma \to \infty$ limit results in $\psi \to 0$ in the maximization with respect to ψ.

The robust control problem is considered to be solved when a saddle point $(\overline{v}, \overline{\psi})$ is reached. As we will see further, for $\gamma > \gamma_0$ and $\ell > \ell_0$, where γ_0, ℓ_0 are some critical values, we obtain the existence and uniqueness of the saddle point.

The second problem we aim to solve is to find the minimal norm control satisfying a null controllability constraint. More precisely, we look for a control $h \in L^2(\omega \times (0,T))$ minimizing

$$J(h) = \frac{1}{2}\iint_{\omega\times(0,T)} |h|^2 dxdt \quad \text{subject to} \quad y(\cdot, T) = 0. \tag{2.3}$$

It is well-known that for nonlinear terms f satisfying a global Lipschitz condition, the semilinear heat equation is null controllable (see, for instance, [30, 32, 33]). The proof combines an observability inequality for a suitable adjoint linear system and a fixed point technique. We will use a similar argument to deduce the null controllability within the hierarchic control framework.

Now, we are in position to describe the hierarchic control strategy to solve the optimization problems associated to the cost functionals (2.2) and (2.3). According to the formulation originally introduced by H. von Stackelberg [60], we denote h as the *leader* control and v as the *follower* control.

First we assume that the state is well defined in function of the controls, the perturbation and the initial condition, that is, there exists $y = y(h, v, \psi)$ uniquely determined by h, v, ψ, and y_0. Then, the hierarchic control method follows two steps:

1. The follower v assumes that the leader h has made a choice, that is, given $h \in L^2(\omega \times (0,T))$ we look for an optimal pair (v, ψ) such that is a saddle point to (2.2). Formally defined:

   Definition 26. *Let $h \in L^2(\omega \times (0,T))$ be fixed. The control $\bar{v} \in \mathcal{V}_{ad}$, the disturbance $\bar{\psi} \in \Psi_{ad}$ and the associated state $\bar{y} = \bar{y}(h, \bar{v}, \bar{\psi})$ solution to (2.1) are said to solve the robust control problem when a saddle point $(\bar{\psi}, \bar{v})$ of the cost functional (2.2) is achieved, that is*

   $$J_r(\bar{v}, \psi; h) \leq J_r(\bar{v}, \bar{\psi}; h) \leq J_r(v, \bar{\psi}; h), \quad \forall (v, \psi) \in \mathcal{V}_{ad} \times \Psi_{ad}. \tag{2.4}$$

   Here, $\mathcal{V}_{ad}$ and Ψ_{ad} are non-empty, closed, convex, and bounded or unbounded sets defining the set of admissible controls and perturbations, respectively.

   Under certain conditions, we will see that there exists a unique pair $(\bar{v}, \bar{\psi})$ and $\bar{y} = \bar{y}(h, \bar{v}, \bar{\psi})$ satisfying (2.4).

2. Once the saddle point has been identified for each leader control h, we look for an optimal control $\hat{h}$ such that

   $$J(\hat{h}) = \min_h J(h) \tag{2.5}$$

   subject to

   $$\bar{y}(\cdot, T; h, \bar{v}(h), \bar{\psi}(h)) = 0. \tag{2.6}$$

Remark 27. • As in [51], we use the hierarchic control strategy to reduce the original multi-objective optimization problem to solving the mono-objective problems (2.4) and (2.5)–(2.6). However, in the second minimization problem the optimal strategy of the follower is fixed and its characterization needs to be considered. Indeed, the follower anticipates the leader's strategy and reacts optimally to its action, then if the leader wants to optimize its objective it has to take into account the optimal response of the follower.

- Here, we use the fact that $\omega \cap \mathcal{O} = \emptyset$. Observe that, in practice, the leader control cannot decide explicitly what to do at the points in the domain of the followers. Indeed, if this assumption is not true, once the leader has been chosen, the follower is modifying the leader at those points.

2.1.2 Main results

The first result concerning the robust hierarchic control is the following one:

Theorem 28. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$ and $N \leq 6$. Let $f \in C^2(\mathbb{R})$ be a globally Lipschitz function verifying $f(0) = 0$ and $f'' \in L^\infty(\mathbb{R})$. Then, there exist γ_0, ℓ_0 and a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $\gamma > \gamma_0$, $\ell > \ell_0$, $y_0 \in L^2(\Omega)$, and any $y_d \in L^2(\mathcal{O}_d \times (0,T))$ verifying*

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y_d|^2 < +\infty, \tag{2.7}$$

there exist a leader control h and a unique associated saddle point $(\bar{v}, \bar{\psi})$ such that the corresponding solution to (2.1) satisfies (2.6).

As usual in the robust control problems, the assumption on γ means that the possible disturbances spoiling the control objectives must have moderate L^2-norms. Indeed, if this condition is not met we cannot prove the existence of the saddle point (34). On the other hand, the assumption on the target y_d means that it approaches 0 as $t \to T$. This is a common feature in some null controllability problems (see, for instance, [61, 6]).

Within this framework, we are also interested in proving a hierarchic result when the follower control v and the perturbation ψ belong to some bounded sets. To this end, let E_1 and E_2 be two non-empty, closed intervals such that $0 \in E_i$. We define the set of admissible controls by

$$\mathcal{V}_{ad} = \left\{ v \in L^2(\mathcal{O} \times (0,T)) : v(x,t) \in E_1 \text{ for a.e. } (x,t) \in \mathcal{O} \times (0,T) \right\}, \tag{2.8}$$

and the set of admissible perturbations by

$$\Psi_{ad} = \left\{ \psi \in L^2(Q) : \psi(x,t) \in E_2 \text{ for a.e. } (x,t) \in Q \right\}. \tag{2.9}$$

Defined in this way, the sets $\mathcal{V}_{ad}$ and Ψ_{ad} are non-empty, closed, convex, bounded sets of $L^2(\mathcal{O} \times (0,T))$ and $L^2(Q)$, respectively.

We will carry out the optimization problem in the set $\mathcal{V}_{ad} \times \Psi_{ad}$ and restrict ourselves to the linear case. The controllability result is the following:

Theorem 29. *Let us assume that $f(y) = ay$ for some $a = a(x,t) \in L^\infty(Q)$ and that $\omega \cap \mathcal{O}_d \neq \emptyset$. Then, there exist γ_0, ℓ_0 and a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $\gamma > \gamma_0$, $\ell > \ell_0$, $y_0 \in L^2(\Omega)$, and any $y_d \in L^2(\mathcal{O}_d \times (0,T))$ verifying (2.7), there exist a leader control h and a unique associated saddle point $(\bar{v}, \bar{\psi}) \in \mathcal{V}_{ad} \times \Psi_{ad}$ such that the corresponding solution to (2.1) satisfies (2.6).*

The above theorem allows us to consider more practical situations. In real-life applications, we may desire to constrain the controls due to the maximum and minimum limits of the actuators. On the other hand, we would like to take into consideration the perturbations affecting the system from a family of functions a priori known, without the necessity to look for the optimal performance over a large set of disturbances.

The rest of the chapter is organized as follows. In section 2.2, we study the corresponding part to the robust control problem. In fact, we will see that provided a sufficiently large value of γ, there exists an optimal pair $(\bar{v}, \bar{\psi})$ that can be chosen for any leader control. Then, in section 3, once the follower strategy has been fixed, we proceed to obtain the leader control h verifying the null controllability problem. We devote section 2.5 to prove Theorem 29.

2.2 The robust control problem

2.2.1 Existence of the saddle point

We devote this section to solve the minimization problem concerning the robust control problem. To do this, we will follow the spirit of [10]. Here, we present results needed to prove the existence and uniqueness of the saddle point, as well as its characterization. In this stage, we assume that the leader has made a choice h, so we will keep it fixed all along this section.

It is well-known (see, for instance, [45]) that for a globally Lipschitz function f and any $y_0 \in L^2(\Omega)$, any $(h, v) \in L^2(\omega \times (0,T)) \times L^2(\mathcal{O} \times (0,T))$ and any $\psi \in L^2(Q)$, system (2.1) admits a unique weak solution $y \in W(0,T)$, where

$$W(0,T) := \left\{ y \in L^2(0,T; H_0^1(\Omega)),\, y_t \in L^2(0,T; H^{-1}(\Omega)) \right\}.$$

Moreover, y satisfies an estimate of the form

$$\|y\|_{W(0,T)} \leq C \left(\|y_0\|_{L^2(\Omega)} + \|h\|_{L^2(\omega\times(0,T))} + \|v\|_{L^2(\mathcal{O}\times(0,T))} + \|\psi\|_{L^2(Q)} \right), \tag{2.10}$$

where $C > 0$ does not depend on ψ, h, v nor y_0. If, in addition, $y_0 \in H_0^1(\Omega)$, then (2.1) admits a unique solution $y \in W_2^{2,1}(Q)$, where

$$W_2^{2,1}(Q) := \left\{ y \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)),\, y_t \in L^2(0,T; L^2(\Omega)) \right\}.$$

Remark 30. In order to obtain a solution to (2.1) it is sufficient to consider a locally Lipschitz function f. Moreover, if f satisfies a particular growth at infinity, system (2.1) is null controllable in the classical sense, see [32]. At this level, the assumptions on f in Theorem 28 are too strong. However, they will be essential for the differentiability results for the functional (2.2).

The main goal of this section is to proof the existence of a solution $(\bar{v}, \bar{\psi})$ to the robust control problem of Definition 26. The result is based on the following:

Proposition 31. *Let J be a functional defined on $X \times Y$, where X and Y are convex, closed, non-empty, unbounded sets. If*

1. *$\forall v \in X$, $\psi \mapsto J(v,\psi)$ is concave and upper semicontinuous,*
2. *$\forall \psi \in Y$, $v \mapsto J(v,\psi)$ is convex and lower semicontinuous,*
3. *$\exists v_0 \in X$ such that $\lim_{\|\psi\|_Y \to \infty} J(v_0,\psi) = -\infty$,*
4. *$\exists \psi_0 \in Y$ such that $\lim_{\|v\|_X \to \infty} J(v,\psi_0) = +\infty$,*

then $\mathcal{J}$ possesses at least one saddle point $(\bar{v},\bar{\psi})$ and

$$J(\bar{v},\bar{\psi}) = \min_{v\in X} \sup_{\psi \in Y} J(v,\psi) = \max_{\psi\in Y} \inf_{v\in X} J(v,\psi).$$

The proof can be found on [26, Prop. 2.2, p. 173]. We intend to apply Proposition 31 to the functional (2.2) with $X = L^2(\mathcal{O}\times(0,T))$ and $Y = L^2(Q)$. In order to establish conditions 1–4 for our problem, we need to study first the differentiability of the solution to (2.1) with respect to the data. We have the following results:

Lemma 32. *Let f be as in Theorem 28 and $h \in L^2(\omega\times(0,T))$ be given. Then, the operator $G : (v,\psi) \to y$ solution to (2.1) is continuously Fréchet differentiable from $L^2(\mathcal{O}\times(0,T)) \times L^2(Q) \mapsto W_2^{2,1}(Q)$. The directional derivate in every direction (v',ψ') is given by*

$$G'(v,\psi)(v',\psi') = w \tag{2.11}$$

where w is the solution to the linear system

$$\begin{cases} w_t - \Delta w + f'(y)w = v'\chi_{\mathcal{O}} + \psi', & \text{in } Q,\\ w = 0 \quad \text{on } \Sigma, \quad w(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.12}$$

with $y = G(v,\psi)$ solution to (2.1).

Proof. We derive in a straightforward manner the first derivative of the operator G and its characterization. The arguments used here are by now classic, see for instance [29], [58].

Given $(v',\psi') \in L^2(\mathcal{O}\times(0,T)) \times L^2(Q)$ and $\tau \in (0,1)$, we will prove the Fréchet-differentiability of G by showing the convergence of $w^\tau \to w$ as $\tau \to 0$ where $w^\tau := (y^\tau - y)/\tau$ for $\tau \neq 0$, with $y = y(h,v,\psi)$, $y^\tau = y(h, v+\tau v', \psi + \tau\psi')$ and w is the solution to the linear problem (2.12). From a simple computation we obtain that w^τ satisfies

$$\begin{cases} w_t^\tau - \Delta w^\tau + \frac{1}{\tau}\left(f(y^\tau) - f(y)\right) = v'\chi_{\mathcal{O}} + \psi', & \text{in } Q,\\ w^\tau = 0 \quad \text{on } \Sigma, \quad w^\tau(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.13}$$

Since f is continuously differentiable, we can use the mean value theorem to deduce that

$$g^\tau(x,t) = \frac{1}{\tau}\left(f(y^\tau) - f(y)\right) = f'(\tilde{y}^\tau)w^\tau \tag{2.14}$$

where $\tilde{y}^\tau = y - \theta_\tau(y^\tau - y)$ with $\theta_\tau \in (0,1)$. Replacing (2.14) in (2.13) and then multiplying by w^τ in $L^2(\Omega)$, it is not difficult to see

$$\|w^\tau(t)\|_{L^2(\Omega)} + \|\nabla w^\tau\|_{L^2(0,T;H_0^1(\Omega))} \leq C\left(\|v'\|_{L^2(0,T;L^2(\mathcal{O}))} + \|\psi'\|_{L^2(Q)}\right), \quad \forall t \in [0,T], \quad \forall \tau \in (0,1).$$

Hence, the sequence $\{w^\tau\}$ is bounded in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Taking into account the above estimate, together with (2.13), (2.14) and since f is globally Lipschitz, we conclude that there exists a positive constant C independent of τ, v' and ψ' such that

$$\|w^\tau\|_{W(0,T)} \leq C\left(\|v'\|_{L^2(0,T;L^2(\mathcal{O}))} + \|\psi'\|_{L^2(Q)}\right), \quad \forall \tau \in (0,1).$$

On the other hand, w^τ can be viewed as the solution of an initial boundary-value problem with right-hand side term $\tilde{g} := v'\chi_{\mathcal{O}} + \psi' - g^\tau$ and zero initial datum. Thus, from classical energy estimates,

$$\|w^\tau\|_{W_2^{2,1}(Q)} \leq C\|\tilde{g}\|_{L^2(Q)}, \quad \forall \tau \in (0,1),$$

for some constant C independent of τ, v' and ψ'. In view of the expression of g^τ and from the the global Lipschitz property of f, we deduce the existence of a positive constant still denoted by C, independent of τ, v' and ψ', such that

$$\|w^\tau\|_{W_2^{2,1}(Q)} \leq C\left(\|v'\|_{L^2(0,T;L^2(\mathcal{O}))} + \|\psi'\|_{L^2(Q)}\right), \quad \forall \tau \in (0,1).$$

Since the space $W_2^{2,1}(Q)$ is reflexive, we have that (extracting a subsequence)

$$w^\tau \rightharpoonup \hat{w} \quad \text{weakly in } W_2^{2,1}(Q), \tag{2.15}$$

as $\tau \to 0$, for some element $\hat{w}$. Now, from the continuity of y with respect to the data (see Eq. (2.10)), and combining (2.15) with the fact that $W_2^{2,1}(Q) \subset L^2(Q)$ with compact imbedding, we get

$$g^\tau \to f'(y)\hat{w} \quad \text{in } L^2(Q). \tag{2.16}$$

Taking the weak limit in (2.13) and using (2.16) is not difficult to see that $\hat{w}$ is solution to (2.12). On the other hand, since $W_2^{2,1}(Q) \subset L^2(0,T;H_0^1(\Omega))$ with compact imbedding, we have that the convergence is strong in this space. □

Lemma 33. *Under assumptions of Lemma 32. The operator $G : (v,\psi) \to y$ solution to (2.1) is twice continuously Fréchet differentiable from $L^2(\mathcal{O} \times (0,T)) \times L^2(Q) \mapsto W(0,T)$. Moreover, the second derivative of G at (v,ψ) is given by the expression*

$$G''(v,\psi)[(v_1,\psi_1),(v_2,\psi_2)] = z \tag{2.17}$$

where z is the unique weak solution to the problem

$$\begin{cases} z_t - \Delta z + f'(y)z = -f''(y)w_1w_2 & \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.18}$$

with $y = G(v,\psi)$ solution to (2.1), and where w_i is the solution to (2.12) in the direction (v_i,ψ_i).

Proof. We follow will the arguments of [63] and use the implicit function theorem to deduce that G is twice continuously Fréchet differentiable. We rewrite system (2.1) as follows:

$$y = G_Q(v, \psi - f(y)) + G_0(h, y_0) \tag{2.19}$$

where $G_Q \in \mathcal{L}(L^2(\mathcal{O}\times(0,T))\times L^2(Q); W(0,T))$ and $G_0 \in \mathcal{L}(L^2(\omega\times(0,T))\times L^2(\Omega); W(0,T))$. More specifically, we rewrite y as

$$y = y_1 + y_2 \tag{2.20}$$

where y_1 and y_2 are solution to

$$\begin{cases} y_{1,t} - \Delta y_1 = v\chi_{\mathcal{O}} + \psi - f(y) & \text{in Q} \\ y_1 = 0 \quad \text{on } \Sigma, \quad y_1(x,0) = 0 & \text{in } \Omega \end{cases}, \quad \begin{cases} y_{2,t} - \Delta y_2 = h\chi_\omega & \text{in Q} \\ y_2 = 0 \quad \text{on } \Sigma, \quad y_2(x,0) = y_0 & \text{in } \Omega. \end{cases}$$

Equivalently, we express equation (2.19) in the form

$$0 = y - G_Q(v, \psi - f(y)) + G_0 y_0 =: F(y, v, \psi).$$

In this way, F is twice continuously Fréchet differentiable from $L^2(\mathcal{O} \times (0,T)) \times L^2(Q) \times L^2(\Omega)$ into $W(0,T)$. Indeed, G_Q and G_0 are continuous linear mappings and the operator $y \mapsto f(y)$ is twice continuously Fréchet differentiable.

On the other hand, the derivative $\partial_y F(y, v, \psi)$ is surjective. In fact,

$$\partial_y F(y, v, \psi) = \overline{w}$$

is equivalent to

$$\overline{w} = \overline{y} + G_Q(0, -f'(y)\overline{y}).$$

Setting $\zeta = \overline{y} - \overline{w}$ and using the definition of the mapping G_Q, we have that the above equation is equivalent to the problem

$$\begin{cases} \zeta_t - \Delta\zeta = f'(y)\zeta + f'(y)\overline{w} & \text{in } Q, \\ \zeta = 0 \quad \text{on } \Sigma, \quad \zeta(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.21}$$

Thanks to the assumptions on f, for every $\overline{w} \in L^2(Q)$, system (2.21) has a unique solution $\zeta \in W(0,T)$. Hence, by the implicit function theorem, the equation $F(y, v, \psi) = 0$ has a unique solution $y = y(v, \psi)$ in some open neighborhood of any arbitrarily chosen point $(\widetilde{y}, \widetilde{v}, \widetilde{\psi})$. Moreover, the implicit function theorem yields that G inherits the smoothness properties of F, therefore G is twice continuously Fréchet differentiable.

To obtain the characterization of the second derivate, we note from (2.19) that

$$y = G(v, \psi) = G_Q(v, \psi - f(G(v, \psi))) + G_0 y_0.$$

Then, differentiating on both sides of the above equation with respect to (v, ψ) in the direction (v_1, ψ_1), we get

$$G'(v, \psi)(v_1, \psi_1) = -G_Q\big(f'(G(v, \psi))[G'(v, \psi)(v_1, \psi_1)]\big) + G_Q(v_1, \psi_1).$$

Repeating the process in the direction (v_2, ψ_2) yields

$$\begin{aligned} G''(v,\psi)[(v_1,\psi_1),(v_2,\psi_2)] = &- G_Q \left\{ f''(G(v,\psi)) \left(G'(v,\psi)(v_1,\psi_1)\right) \left(G'(v,\psi)(v_2,\psi_2)\right) \right. \\ &\left. + f'(G(v,\psi)) G''(v,\psi)[(v_1,\psi_1),(v_2,\psi_2)] \right\}. \end{aligned}$$

Setting $y = G(v,\psi)$, $w_i = G'(v,\psi)(v_i,\psi_i)$ and $z = G''(v,\psi)[(v_1,\psi_1),(v_2,\psi_2)]$ in the previous equation, we obtain that

$$z = -G_Q \left\{ f''(y) w_1 w_2 + f'(y) z \right\}.$$

Therefore, from the definition of G_Q, we conclude that z is solution to (2.18). □

With Lemmas 32 and 33, we are ready to proof one of the main result of this section:

Proposition 34. *Under assumptions of Lemma 32. Let $y_0 \in L^2(\Omega)$ and $h \in L^2(\omega \times (0,T))$ be given. Then, for γ and ℓ sufficiently large, there exists a saddle point $(\bar{v}, \bar{\psi}) \in L^2(\mathcal{O} \times (0,T)) \times L^2(Q)$ and $\bar{y} = \bar{y}(h, \bar{v}, \bar{\psi})$ such that*

$$J_r(\bar{v}, \psi; h) \leq J_r(\bar{v}, \bar{\psi}; h) \leq J_r(v, \bar{\psi}; h), \quad \forall (v,\psi) \in L^2(\mathcal{O} \times (0,T)) \times L^2(Q).$$

Proof. In order to prove the existence of the saddle point $(\bar{v}, \bar{\psi})$ we will verify conditions 1–4 from Proposition 31.

Condition 1. By Lemma 32, and since the norm is lower semicontinuous, the map $\psi \mapsto \mathcal{J}(v,\psi)$ is upper semicontinuous. To check the concavity, we will show that

$$\mathcal{G}(\tau) = J_r\left(v, \psi + \tau\psi'\right) \tag{2.22}$$

is concave with respect to τ near $\tau = 0$, that is, $\mathcal{G}''(0) < 0$. Using the notation previously introduced, we set $y := G(v, \psi + \tau\psi')$. In view of the results of Lemmas 32 and 33, we have that $\mathcal{G}(\tau)$ is a composition of twice differentiable maps. Then it can be readily verified that

$$\begin{aligned} \mathcal{G}'(\tau) = &\iint_{\mathcal{O}_d \times (0,T)} \left(G(v,\psi+\tau\psi') - y_d\right) G'(v,\psi+\tau\psi')(0,\psi') dxdt \\ &- \gamma^2 \iint_Q \left(\psi + \tau\psi'\right) \psi' dxdt. \end{aligned}$$

A further differentiation with respect to τ yields

$$\begin{aligned} \mathcal{G}''(\tau) = &\iint_{\mathcal{O}_d \times (0,T)} \left(G(v,\psi+\tau\psi') - y_d\right) G''(v,\psi+\tau\psi') \left[(0,\psi'),(0,\psi')\right] dxdt \\ &+ \iint_{\mathcal{O}_d \times (0,T)} \left|G'(v,\psi+\tau\psi')(0,\psi')\right|^2 dxdt - \gamma^2 \iint_Q |\psi'|^2 dxdt. \end{aligned} \tag{2.23}$$

We define $y' := G(v, \psi + \tau\psi')(0, \psi')$ and $y'' := G''(v, \psi + \tau\psi')[(0, \psi'), (0, \psi')]$ which, according to Lemmas 32 and 33, are solution to

$$\begin{cases} y'_t - \Delta y' + f'(y) y' = \psi' & \text{in } Q, \\ y' = 0 \quad \text{on } \Sigma, \quad y'(x, 0) = 0 & \text{in } \Omega, \end{cases} \tag{2.24}$$

$$\begin{cases} y''_t - \Delta y'' + f'(y) y'' = -f''(y)|y'|^2 & \text{in } Q, \\ y'' = 0 \quad \text{on } \Sigma, \quad y''(x, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.25}$$

Then, we rewrite (2.23) as

$$\mathcal{G}''(\tau) = \iint_{\mathcal{O}_d \times (0,T)} (y - y_d) y'' dx dt + \iint_{\mathcal{O}_d \times (0,T)} |y'|^2 dx dt - \gamma^2 \iint_Q |\psi'|^2 dx dt. \tag{2.26}$$

Now, we will see that for sufficiently large γ the last term in the above equation dominates and thus $\mathcal{G}''(0) < 0$ for $(v, \psi) \in L^2(\mathcal{O} \times (0,T)) \times L^2(Q)$.

We begin by estimating the second term. Thanks to the assumptions on f, there exists $L > 0$ be such that $|f'(s)| + |f''(s)| \leq L$, $\forall s \in \mathbb{R}$. Since the linear system (2.24) has a unique solution $y' \in W(0,T)$ for any $\psi' \in L^2(Q)$, we can obtain

$$\iint_{\mathcal{O}_d \times (0,T)} |y'|^2 dx dt \leq C_1 \iint_Q |\psi'|^2 dx dt. \tag{2.27}$$

for some $C_1 > 0$ only depending on Ω, $\mathcal{O}_d$, L and T.

To compute the first term, we need an estimate for y''. We multiply (2.25) by y'' in $L^2(\Omega)$ and integrate by parts, whence

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |y''|^2 dx + \int_\Omega |\nabla y''|^2 dx &= -\int_\Omega f'(y)|y''|^2 dx - \int_\Omega f''(y)|y'|^2 y'' \\ &\leq L \int_\Omega |y''|^2 dx + L \int_\Omega |y'|^2 |y''| \end{aligned}$$

and using Gronwall's and Poincare's inequality, we obtain

$$\iint_Q |y''|^2 dx dt \leq C \iint_Q |y'|^2 |y''| dx dt. \tag{2.28}$$

Applying Hölder inequality in the above expression yields

$$\iint_Q |y''|^2 dx dt \leq C \|y'\|^2_{L^{2p'}(0,T;L^{2q'}(\Omega))} \|y''\|_{L^p(0,T;L^q(\Omega))}, \tag{2.29}$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. To bound the right-hand side of the previous inequality, the idea is to find p and q such that

$$y'' \in L^p(0,T;L^q(\Omega)), \quad y' \in L^{2p'}(0,T;L^{2q'}(\Omega)).$$

First, recall that y' is more regular than $W(0,T)$. In fact, from classical results (see, for instance, [27, 52]) we have that $y' \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ with $y'_t \in L^2(0,T;L^2(\Omega))$. Moreover, we have the estimate

$$\|y'\|_{L^\infty(0,T;H_0^1(\Omega))} + \|y'\|_{L^2(0,T;H^2(\Omega))} + \|y'_t\|_{L^2(0,T;L^2(\Omega))} \leq C\|\psi'\|_{L^2(Q)}. \tag{2.30}$$

In view of (2.30), it is reasonable to look for conditions such that the following embedding holds

$$L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^{2p'}(0,T;L^{2q'}(\Omega)). \tag{2.31}$$

Let X and Y be Banach spaces. From well-known interpolation results (see, for instance, [59]), we have

$$L^{p_0}(0,T;X)) \cap L^{p_1}(0,T;Y) \hookrightarrow L^{p_\theta}(0,T;B), \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{2.32}$$

with $0 < \theta < 1$ and where B is the intermediate space of class θ (with respect to X and Y), that is, B is the space verifying

$$\|g\|_B \leq C\|g\|_X^{1-\theta}\|g\|_Y^\theta, \quad \forall g \in X \cap Y, \quad 0 < \theta < 1,$$

for some C.

From (2.31) and (2.32), we deduce that

$$\frac{1}{2p'} = \frac{\theta}{2}. \tag{2.33}$$

On the other hand, from classical Sobolev embedding results, we have

$$H^2(\Omega) \hookrightarrow L^{\frac{2N}{N-4}}(\Omega), \tag{2.34}$$
$$H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega). \tag{2.35}$$

for some maximal N to be determined. Then, the space $L^{2q'}(\Omega)$ is an intermediate space with respect to (2.34) and (2.35) if

$$\frac{1}{2q'} = \frac{(N-4)\theta}{2N} + \frac{(N-2)(1-\theta)}{2N}, \quad 0 < \theta < 1. \tag{2.36}$$

Setting p' to a fixed value such that $\theta \in (0,1)$ and replacing (2.33) into (2.36), we obtain

$$q' = \frac{p'N}{p'(N-2)-2}, \tag{2.37}$$

and from (2.33) and (2.37), we deduce that

$$p = p'/(p'-1) \quad \text{and} \quad q = p'N/(2p'+2). \tag{2.38}$$

Thus, from (2.31), (2.29) and estimate (2.30), we get

$$\iint_Q |y''|^2 dxdt \le C\|\psi'\|^2_{L^2(Q)}\|y''\|_{L^p(0,T;L^q(\Omega))}.$$

It remains to verify that $L^2(0,T;L^2(\Omega)) \hookrightarrow L^p(0,T;L^q(\Omega))$. From (2.38), it is not difficult to see that this is true if $p \le 2, q \le 2$, that is, if $2 \le p'$ and $N \le 4(p'+1)/p'$. Setting $p' = 2$, we get $N \le 6$ and thus the estimate

$$\|y''\|_{L^2(0,T;L^2(\Omega))} \le C_2\|\psi'\|^2_{L^2(Q)}. \tag{2.39}$$

Putting together (2.26), (2.27) and (2.39) yields

$$\mathcal{G}''(0) \le (C_2 + C_1 - \gamma^2)\|\psi'\|^2_{L^2(Q)}, \quad \forall \psi' \in L^2(Q),\ \psi' \ne 0.$$

Therefore, for γ large enough we have $\mathcal{G}''(0) < 0$ and therefore $\psi \mapsto J(v,\psi)$ is strictly concave.

Condition 2. By Lemma 32, and since the norm is lower semicontinuous, the map $v \mapsto J(v,\psi)$ is lower semicontinuous. In order to show convexity, it is sufficient to prove that

$$\mathcal{G}(\tau) = J(v + \tau v', \psi)$$

is convex with respect to τ near $\tau = 0$, that is, $\mathcal{G}''(0) > 0$. Arguing as above, we obtain

$$\mathcal{G}''(\tau) = \iint_{\mathcal{O}_d\times(0,T)} (y - y_d)y''dxdt + \iint_{\mathcal{O}_d\times(0,T)} |y'|^2 dxdt + \ell^2 \iint_{\omega\times(0,T)} |v'|^2 dxdt. \tag{2.40}$$

where we have denoted $y' = G(v + \tau v', \psi)(v', 0)$ and $y'' = G''(v + \tau v', \psi)[(v', 0), (v', 0)]$. Note that estimates for y' and y'' can be obtained in the same way as in the proof of Condition 1 by putting v' instead of ψ' in (2.24)–(2.25). Then, it is not difficult to see that

$$\mathcal{G}''(0) \ge (l^2 - C_2 - C_1)\|v\|^2_{L^2(\mathcal{O}\times(0,T))}, \quad \forall v \in L^2(\mathcal{O}\times(0,T)),\ v \ne 0.$$

Thus, under the assumption that ℓ is large enough, $v \mapsto J(v,\psi)$ is strictly convex.

Condition 3. Taking $v = 0$ and using formulas (2.19)–(2.20) for $y = y(0,\psi)$ we obtain

$$\begin{aligned} J_r(0,\psi;h) &= \iint_{\mathcal{O}_d\times(0,T)} |y_1 + y_2 - y_d|^2 dxdt - \frac{\gamma^2}{2}\iint_Q |\psi|^2 dxdt \\ &\le -\frac{\gamma^2}{2}\|\psi\|^2_{L^2(Q)} + C\|\psi\|^2_{L^2(Q)} + C_3, \end{aligned}$$

where C_3 is a positive constant only depending on y_0, h and y_d. Hence, for a sufficiently large value of γ, condition 3 holds.

Condition 4. Taking $\psi = 0$ in (2.2) we get

$$J_r(v,0;h) \ge \frac{\ell^2}{2}\iint_{\mathcal{O}\times(0,T)} |v|^2 dxdt,$$

and condition 4 follows immediately. This ends the proof of Proposition 34.

□

2.2.2 Characterization of the saddle point

The existence of a saddle point $(\bar{v}, \bar{\psi})$ for the functional J_r implies that

$$\frac{\partial J_r}{\partial v}(\bar{v}, \bar{\psi}) = 0 \quad \text{and} \quad \frac{\partial J_r}{\partial \psi}(\bar{v}, \bar{\psi}) = 0, \tag{2.41}$$

so our task is to find such expressions. Indeed, is not difficult to see that

$$\left(\frac{\partial J_r}{\partial v}(v, \psi), (v_1, 0) \right) = \iint_{\mathcal{O}_d \times (0,T)} (y - y_d) w_v dxdt + \ell^2 \iint_{\mathcal{O} \times (0,T)} v v_1 dxdt \tag{2.42}$$

$$\left(\frac{\partial J_r}{\partial \psi}(v, \psi), (0, \psi_1) \right) = \iint_{\mathcal{O}_d \times (0,T)} (y - y_d) w_\psi dxdt - \gamma^2 \iint_Q \psi \psi_1 dxdt \tag{2.43}$$

where w_v and w_ψ are the directional derivatives of y solution to (2.1) in the directions $(v_1, 0)$ and $(0, \psi_1)$, respectively. To determine the solution of the robust control, we define the adjoint state to system (2.12)

$$\begin{cases} -q_t - \Delta q + f'(y) q = (y - y_d) \chi_{\mathcal{O}_d} & \text{in } Q, \\ q = 0 \quad \text{on } \Sigma, \quad q(x, T) = 0 & \text{in } \Omega. \end{cases} \tag{2.44}$$

We have the following result:

Lemma 35. *Let $y = y(h, v, \psi) \in W(0,T)$ be the solution to (2.1). Let w be the solution to (2.12) with $(v_1, \psi_1) \in L^2(\mathcal{O} \times (0,T)) \times L^2(Q)$ and q be the solution to (2.44). Then*

$$\iint_{\mathcal{O}_d \times (0,T)} (y - y_d) w \, dxdt = \iint_{\mathcal{O} \times (0,T)} q v_1 dxdt + \iint_Q q \psi_1 dxdt. \tag{2.45}$$

Proof. We multiply (2.44) by w in $L^2(Q)$ and integrate by parts, more precisely

$$\begin{aligned} \iint_Q (y - y_d) \chi_{\mathcal{O}_d} q dxdt &= \iint_Q \left(-q_t - \Delta q + f'(y) q \right) w \, dxdt \\ &= - \int_\Omega q w \, dx \Big|_0^T + \iint_Q q \left(w_t - \Delta w + f'(y) w \right) dxdt. \end{aligned}$$

Upon substituting the initial data for q and w and the right-hand side of (2.12) in the above equation, we obtain (2.45). □

Replacing (2.45) in (2.42), with $\psi_1 = 0$ and taking an arbitrary $v_1 \in L^2(\mathcal{O} \times (0,T))$ we get

$$\left(\frac{\partial J_r}{\partial v}(v, \psi), (v_1, 0) \right) = \iint_{\mathcal{O} \times (0,T)} q v_1 dxdt + \ell^2 \iint_{\mathcal{O} \times (0,T)} v v_1 dxdt, \quad \forall v_1 \in L^2(\mathcal{O} \times (0,T)).$$

In particular, we deduce

$$\frac{\partial J_r}{\partial v}(v,\psi) = (q+\ell^2 v)|_{\mathcal{O}}. \tag{2.46}$$

Analogously, from (2.45) and (2.43) with $v_1 = 0$ and $\psi_1 \in L^2(Q)$ as arbitrary we have

$$\left(\frac{\partial J_r}{\partial \psi}(v,\psi),(0,\psi_1)\right) = \iint_Q q\psi_1 dxdt - \gamma^2 \iint_Q \psi\psi_1 dxdt, \quad \forall \psi_1 \in L^2(Q).$$

whence

$$\frac{\partial J_r}{\partial \psi}(v,\psi) = q - \gamma^2\psi. \tag{2.47}$$

Proposition 36. *Let $h \in L^2(\omega \times (0,T))$ and $y_0 \in L^2(\Omega)$ be given. Let $(\bar{v},\bar{\psi})$ be a solution to the robust control problem in Definition 26. Then*

$$\bar{v} = -\frac{1}{\ell^2}q\,|_{\mathcal{O}} \quad \text{and} \quad \bar{\psi} = \frac{1}{\gamma^2}q \tag{2.48}$$

where q is found from the solution (y,q) to the coupled system

$$\begin{cases} y_t - \Delta y + f(y) = h\chi_\omega - \frac{1}{\ell^2}q\chi_{\mathcal{O}} + \frac{1}{\gamma^2}q & \text{in } Q,\\ -q_t - \Delta q + f'(y)q = (y-y_d)\chi_{\mathcal{O}_d} & \text{in } Q,\\ y = q = 0 & \text{on } \Sigma,\\ y(x,0) = y_0(x), \quad q(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{2.49}$$

which admits a unique solution for sufficiently large γ and ℓ.

Proof. The existence of the solution to the robust control problem is ensured by Proposition 34 provided the parameters γ and ℓ are large enough. A necessary condition for $(\bar{v},\bar{\psi})$ to be a saddle point of J_r is given in (2.41), therefore from (2.46) and (2.47) we conclude that (2.48)–(2.49) holds.

To check uniqueness assume that $(\bar{v},\bar{\psi})$ and $(\tilde{v},\tilde{\psi})$ are two different saddle points in $L^2(\omega \times (0,T)) \times L^2(Q)$. Then, from the strict convexity and strict concavity proved in Proposition 34, we have

$$\mathcal{J}(\tilde{v},\tilde{\psi}) < \mathcal{J}(\bar{v},\tilde{\psi}) < \mathcal{J}(\bar{v},\bar{\psi}).$$

On the other hand,

$$\mathcal{J}(\bar{v},\bar{\psi}) < \mathcal{J}(\tilde{v},\bar{\psi}) < \mathcal{J}(\tilde{v},\tilde{\psi}).$$

These lead to a contradiction, and therefore the saddle point $(\bar{v},\bar{\psi})$ is unique. □

Summarizing, what we found in this section is that given a leader control h, there exists a unique solution to the robust control problem stated in Definition 26. Moreover, it is characterized by the coupled system (2.49). However, this characterization added a second equation coupled to the original system, so we need to take into account system (2.49) to obtain a solution to the leader's minimization problem (see Remark 27).

2.3 The null controllability problem: the observability inequality

Once the optimal strategy for the follower control has been chosen (see Section 2.2.2), the next step in the hierarchic methodology is to obtain an optimal control $\hat{h}$ such that

$$J(\hat{h}) = \min_h J(h) \quad \text{subject to} \quad y(\cdot,T) = 0. \tag{2.50}$$

where y can be found from the solution (y,q) to (2.49). We start by proving an observability inequality for the adjoint system to the linearized version of (2.49)

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = \theta\chi_{\mathcal{O}_d} & \text{in } Q, \\ \theta_t - \Delta\theta + c\theta = -\frac{1}{\ell^2}\varphi\chi_{\mathcal{O}} + \frac{1}{\gamma^2}\varphi & \text{in } Q, \\ \varphi = \theta = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^T(x), \quad \theta(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{2.51}$$

where $a, c \in L^\infty(Q)$ and $\varphi^T \in L^2(\Omega)$. Such inequality will be the main tool to conclude the proof of Theorem 28.

The main result of this section is the following one:

Proposition 37. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$ and that γ and ℓ are large enough. There exist a positive constant C only depending on Ω, ω, $\mathcal{O}$, $\mathcal{O}_d$, $\|a\|_\infty$, $\|c\|_\infty$, and T, and a weight function $\rho = \rho(t)$ blowing up at $t = T$ only depending on Ω, ω, $\mathcal{O}_d$, $\|a\|_\infty$, $\|c\|_\infty$ and T such that, for any $\varphi^T \in L^2(\Omega)$, the solution (φ,θ) to (2.51) satisfies*

$$\int_\Omega |\varphi(0)|^2 dx + \iint_Q \rho^{-2}|\theta|^2 dxdt \le C \iint_{\omega\times(0,T)} |\varphi|^2 dxdt. \tag{2.52}$$

We postpone the proof of this result until the end of this section. The main tool to prove Proposition 37 is a well-known Carleman inequality for linear parabolic systems.

First, let us introduce several weight functions that will be useful in the reminder of this section. We introduce a special function whose existence is guaranteed by the following result [33, Lemma 1.1].

Lemma 38. *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open subset. Then, there exists $\eta^0 \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \eta^0(x) > 0 \quad \textit{all } x \in \Omega, \qquad \eta^0|_{\partial\Omega} = 0, \\ |\nabla\eta^0| > 0 \quad \textit{for all } x \in \overline{\Omega\backslash\mathcal{B}}. \end{cases}$$

For $\lambda > 0$ a parameter, we introduce the weight functions

$$\alpha(x,t) = \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda\left(2\|\eta^0\|_\infty + \eta^0(x)\right)}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}. \tag{2.53}$$

For $m \in \mathbb{R}$ and a parameter $s > 0$, we will use the following notation to abridge estimates:

$$\begin{aligned} I_m(s,\lambda;z) &:= \iint_Q e^{-2s\alpha}(s\xi)^{m-2}\lambda^{m-1}|\nabla z|^2 + \iint_Q e^{-2s\alpha}(s\xi)^m\lambda^{m+1}|z|^2, \\ I_{m,\mathcal{B}}(s,\lambda;z) &:= \iint_{\mathcal{B}\times(0,T)} e^{-2s\alpha}(s\xi)^m\lambda^{m+1}|z|^2. \end{aligned} \tag{2.54}$$

First, we state a Carleman estimate, due to [42], for solutions to the heat equation:

Lemma 39. *Let $\mathcal{B} \subset\subset \Omega$ be a nonempty open subset. For any $m \in \mathbb{R}$, there exist positive constants s_m, λ_m, and C_m such that, for any $s \geq s_m$, $\lambda \geq \lambda_m$, $F \in L^2(Q)$ and every $z^0 \in L^2(\Omega)$, the solution z to*

$$\begin{cases} z_t - \Delta z = F & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = z^0(x) & \text{in } \Omega, \end{cases}$$

satisfies

$$I_m(s,\lambda;z) \leq C_m\left(I_{m,\mathcal{B}}(s,\lambda;z) + \iint_Q e^{-2s\alpha}(s\lambda\xi)^{m-3}|F|^2 dxdt\right). \tag{2.55}$$

Furthermore, C_m only depends on ω, $\mathcal{B}$ and m and s_m can be taken of the form $s_m = \sigma_m(T+T^2)$ where σ_m only depends on ω, $\mathcal{B}$ and m.

Remark 40. Note that by changing t for $T-t$, Lemma 39 remains valid for linear backward in time systems. Therefore, we can apply it interchangeably in what follows.

The observability inequality (2.52) is consequence of a global Carleman inequality and some energy estimates. We present below a Carleman inequality for the solutions to system (2.51):

Proposition 41. *Under assumptions of Proposition 37. There exist positive constants constant C and σ_2 such that the solution (φ,θ) to (2.51) satisfies*

$$I_3(s,\lambda;\varphi) + I_3(s,\lambda;\theta) \leq C\iint_{\omega\times(0,T)} e^{-2s\alpha}s^7\lambda^8\xi^7|\varphi|^2. \tag{2.56}$$

for any $s \geq s_2 = \sigma_2(T+T^2+T^2(\|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|a-c\|_\infty^{1/2}))$, any $\lambda \geq C$ and every $\varphi^T \in L^2(\Omega)$.

Proof. Hereinafter C will denote a generic positive constant that may change from line to line. We start by applying Carleman inequality (2.55) to each equation in system (2.51) with $m = 3$, $\mathcal{B} = \omega' \subset\subset \omega_0 := \omega \cap \mathcal{O}_d$ and add them up, hence

$$\begin{aligned} &I_3(s,\lambda;\varphi) + I_3(s,\lambda;\theta) \\ &\quad\leq C\Bigg(I_{3,\omega'}(s,\lambda;\varphi) + I_{3,\omega'}(s,\lambda;\theta) + \iint_Q e^{-2s\alpha}|\theta\chi_{\mathcal{O}_d}|^2 dxdt \\ &\qquad + \iint_Q e^{-2s\alpha}|-\tfrac{1}{\ell^2}\varphi\chi_{\mathcal{O}} + \tfrac{1}{\gamma^2}\varphi|^2 + \iint_Q e^{-2s\alpha}\|a\|_\infty^2|\varphi|^2 + \iint_Q e^{-2s\alpha}\|c\|_\infty^2|\theta|^2\Bigg). \end{aligned}$$

Taking the parameter s large enough we can absorb some of the lower order terms in the right-hand side of the above expression. More precisely, there exists a constant $\sigma_1 > 0$, such that

$$I_3(s,\lambda;\varphi) + I_3(s,\lambda;\theta) \leq C\left(I_{3,\omega'}(s,\lambda;\varphi) + I_{3,\omega'}(s,\lambda;\theta) + \iint_Q e^{-2s\alpha}|\theta\chi_{\mathcal{O}_d}|^2 + \iint_Q e^{-2s\alpha}\left| -\tfrac{1}{\ell^2}\varphi\chi_{\mathcal{O}} + \tfrac{1}{\gamma^2}\varphi\right|^2\right)$$

is valid for every

$$s \geq s_1 = \sigma_1(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|c\|_\infty^{2/3})). \tag{2.57}$$

Then, taking the parameter λ large enough we get

$$I_3(s,\lambda;\varphi) + I_3(s,\lambda;\theta) \leq C\left(I_{3,\omega'}(s,\lambda;\varphi) + I_{3,\omega'}(s,\lambda;\theta)\right). \tag{2.58}$$

for every $s \geq s_1$ and $\lambda \geq C$.

The next step is to eliminate the local term on the right hand side corresponding to θ. We will reason out as in [37] and [61]. We consider a function $\zeta \in C_0^\infty(\mathbb{R}^N)$ verifying:

$$0 \leq \zeta \leq 1 \text{ in } \Omega, \quad \zeta \equiv 1 \quad \text{in } \omega', \quad \operatorname{supp}\zeta \subset \omega_0, \tag{2.59}$$

$$\frac{\Delta\zeta}{\zeta^{1/2}} \in L^\infty(\Omega), \quad \frac{\nabla\zeta}{\zeta^{1/2}} \in L^\infty(\Omega)^N. \tag{2.60}$$

Such function exists. It is sufficient to take $\zeta = \tilde{\zeta}^4$ with $\tilde{\zeta} \in C_0^\infty(\Omega)$ veryfing (2.59).

Let $s \geq s_1$ with s_1 given in (2.57). We define $u := e^{-2s\alpha}s^3\lambda^4\xi^3$. Multiplying the equation satisfied by φ in (2.51) by $u\zeta\theta$, integrating by parts over Q and taking into account that $u(x,0)$ vanishes in Ω we obtain

$$\begin{aligned}\iint_Q u\zeta|\theta|^2\chi_{\mathcal{O}_d} =& \iint_Q (a-c)\varphi\theta u\zeta + \iint_Q \varphi\theta\zeta\partial_t u - \iint_Q \varphi\theta\Delta(u\zeta) \\ &- 2\iint_Q \nabla(u\zeta)\cdot\nabla\theta\,\varphi + \frac{1}{\gamma^2}\iint_Q |\varphi|^2 u\zeta \\ :=& I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{2.61}$$

Let us estimate each I_i, $1 \leq i \leq 4$, we keep the last term as it is. From Hölder and Young inequalities, we readily obtain

$$I_1 = \iint_Q (a-c)\varphi\theta u\zeta \leq \delta_1\iint_Q u\zeta|\theta|^2 + \frac{1}{4\delta_1}\|a-c\|_\infty^2\iint_Q u\zeta|\varphi|^2. \tag{2.62}$$

for any $\delta_1 > 0$. Observe that

$$\begin{aligned}|\partial_t u| &\leq 3s^3\lambda^4\xi^2\xi_t e^{-2s\alpha} + 2s^3\lambda^4\xi^3 e^{-2s\alpha}s\alpha_t, \\ &\leq CTs^3\lambda^4\xi^4 e^{-2s\alpha} + CTs^4\lambda^4\xi^5 e^{-2s\alpha}, \\ &\leq CTs^4\lambda^4\xi^5 e^{-2s\alpha},\end{aligned}$$

where we have used that $\alpha_t \leq CT\xi^2$. Then, we can estimate

$$\begin{aligned}|I_2| &\leq \iint_Q |\varphi||\theta||\partial_t u|\zeta \leq CT \iint_Q s^4\lambda^4\xi^5 e^{-2s\alpha}|\varphi||\theta|\zeta \\ &\leq \delta_2 \iint_Q u\zeta|\theta|^2 + \frac{CT^2}{\delta_2}\iint_Q s^5\lambda^4\xi^7 e^{-2s\alpha}|\varphi|^2\zeta \\ &\leq \delta_2 \iint_Q u\zeta|\theta|^2 + \frac{C}{\delta_2}\iint_Q s^7\lambda^4\xi^7 e^{-2s\alpha}|\varphi|^2\zeta \end{aligned} \tag{2.63}$$

for any $\delta_2 > 0$, where we have used in the last line that $s \geq \sigma_1 T$.

In order to estimate I_3, we compute first

$$\Delta\left(e^{-2s\alpha}s^3\lambda^4\xi^3\zeta\right) = \Delta\left(e^{-2s\alpha}s^3\lambda^4\xi^3\right)\zeta + \Delta\zeta e^{-2s\alpha}s^3\lambda^4\xi^3 + 2\nabla(e^{-2s\alpha}s^3\lambda^4\xi^3)\cdot\nabla\zeta \tag{2.64}$$

and

$$|\Delta\left(e^{-2s\alpha}s^3\lambda^4\xi^3\right)| \leq Ce^{-2s\alpha}s^5\lambda^6\xi^5, \tag{2.65}$$

$$|\nabla\left(e^{-2s\alpha}s^3\lambda^4\xi^3\right)| \leq Ce^{-2s\alpha}s^4\lambda^5\xi^4, \tag{2.66}$$

where the above inequalities follow from the fact that

$$\partial_i\alpha = -\partial_i\xi = -C\lambda\partial_i\eta^0\xi \leq C\lambda\xi.$$

Then, from (2.64)–(2.66) and using (2.60), we obtain

$$\begin{aligned}|I_3| \leq &C\iint_Q |\varphi||\theta|e^{-2s\alpha}s^5\lambda^6\xi^5\zeta + C\iint_Q |\varphi||\theta|e^{-2s\alpha}s^3\lambda^4\xi^3\zeta^{1/2} \\ &+ C\iint_Q |\varphi||\theta|e^{-2s\alpha}s^4\lambda^5\xi^4\zeta^{1/2}.\end{aligned}$$

Using Hölder and Young inequalities and (2.59) yield

$$\begin{aligned}|I_3| \leq &\delta_3\iint_Q u\zeta|\theta|^2 + \frac{C}{\delta_3}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^7\lambda^8\xi^7|\varphi|^2 \\ &+ \frac{C}{\delta_3}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^3\lambda^4\xi^3|\varphi|^2 + \frac{C}{\delta_3}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^5\lambda^6\xi^5|\varphi|^2\end{aligned}$$

for some $\delta_3 > 0$. Note that $\xi^{-1} \leq CT^2/4$, then, for any $\nu, \mu \in \mathbb{N}$ with $\nu \geq \mu$ we have

$$(s\xi)^\mu = s^\mu\xi^\nu\xi^{\mu-\nu} \leq Cs^\mu\xi^\nu(T^2/4)^{-(\mu-\nu)} \leq Cs^\nu\xi^\nu, \tag{2.67}$$

since $s \geq CT^2$. Hence,

$$|I_3| \leq \delta_3\iint_Q u\zeta|\theta|^2 + \frac{C}{\delta_3}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^7\lambda^8\xi^7|\varphi|^2. \tag{2.68}$$

Using (2.60), (2.66) and (2.67), we estimate I_4 as

$$
\begin{aligned}
|I_4| &\leq C \iint_Q e^{-2s\alpha} \left(s^3\lambda^4\xi^3|\nabla\theta||\varphi|\zeta^{1/2} + s^4\lambda^5\xi^4|\nabla\theta||\varphi|\zeta \right) \\
&\leq \varepsilon \iint_Q e^{-2s\alpha} s\lambda^2\xi|\nabla\theta|^2 + \frac{C}{\varepsilon} \iint_{\omega_0\times(0,T)} e^{-2s\alpha} s^7\lambda^8\xi^7|\varphi|^2
\end{aligned}
\tag{2.69}
$$

for $\varepsilon > 0$.

Setting $\delta_i = 1/6$, $1 \leq i \leq 3$, and $\varepsilon = \frac{1}{4C}$ with C the constant in (2.58), and upon substituting estimates (2.62)-(2.63) and (2.68)–(2.69) in (2.61), we obtain

$$
\begin{aligned}
\iint_Q e^{-2s\alpha} s^3\lambda^4\xi^3|\theta|^2\chi_{\mathcal{O}_d} \leq & C \iint_{\omega_0\times(0,T)} e^{-2s\alpha} \left[\|a-c\|_\infty^2 s^3\lambda^4\xi^3|\varphi|^2 + s^7\lambda^8\xi^7|\varphi|^2 \right] \\
& + \frac{1}{2C} \iint_Q e^{-2s\alpha} s\lambda^2\xi|\nabla\theta|^2 + \frac{1}{\gamma^2} \iint_Q e^{-2s\alpha} s^3\lambda^4\xi^3|\varphi|^2.
\end{aligned}
\tag{2.70}
$$

Thus, in view of (2.58)–(2.59) and (2.70), we obtain

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha} \left(s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2 \right) + \iint_Q e^{-2s\alpha} \left(s\lambda^2\xi|\nabla\theta|^2 + s^3\lambda^4\xi^3|\theta|^2 \right) \\
&\quad \leq C\|a-c\|_\infty^2 \iint_{\omega_0\times(0,T)} e^{-2s\alpha} s^3\lambda^4\xi^3|\varphi|^2 + C \iint_{\omega_0\times(0,T)} e^{-2s\alpha} s^7\lambda^8\xi^7|\varphi|^2 \\
&\quad + \frac{C}{\gamma^2} \iint_Q e^{-2s\alpha} s^3\lambda^4\xi^3|\varphi|^2.
\end{aligned}
$$

Taking $s \geq CT^2\|a-c\|_\infty^{1/2}$, the above inequality now reads

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha} \left(s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2 \right) + \iint_Q e^{-2s\alpha} \left(s\lambda^2\xi|\nabla\theta|^2 + s^3\lambda^4\xi^3|\theta|^2 \right) \\
&\quad \leq C \iint_{\omega_0\times(0,T)} e^{-2s\alpha} s^7\lambda^8\xi^7|\varphi|^2 + \frac{C}{\gamma^2} \iint_Q e^{-2s\alpha} s^3\lambda^4\xi^3|\varphi|^2.
\end{aligned}
\tag{2.71}
$$

for every $s \geq s_2$ with

$$
s_2 = \sigma_2(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|c\|_\infty^{2/3} + \|a-c\|_\infty^{1/2})).
\tag{2.72}
$$

for some σ_2 only depending on Ω, ω and $\mathcal{O}_d$.

Observe that the last term in (2.71) has the same power of s, λ and ξ as in the corresponding term on the left-hand side. Thus, provided γ is large enough, we can absorb it into the right-hand side. Finally, since $\omega_0 \subset \omega$, we obtain the desired inequality (2.56). Therefore the proof is complete. □

Now, we are going to improve inequality (2.56) in the sense that the weight functions do not vanish at $t = 0$. First, let us consider the function

$$l(t) = \begin{cases} T^2/4 & \text{for} \quad 0 \leq t \leq T/2, \\ t(T-t) & \text{for} \quad T/2 \leq t \leq T, \end{cases}$$

and the functions

$$\beta(x,t) = \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{l(t)}, \quad \phi(x,t) = \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{l(t)},$$
$$\beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t), \quad \phi^*(t) = \min_{x\in\overline{\Omega}} \phi(x,t).$$

With these definitions, we have the following:

Proposition 42. *Let s and λ as in Proposition 53 and ℓ, γ be large enough. Then there exists a positive constant C depending on Ω, ω, ω_d, s, λ, $\|a\|_\infty$, $\|c\|_\infty$ and T such that*

$$\begin{aligned} \|\varphi(0)\|^2_{L^2(\Omega)} + \iint_Q e^{-2s\beta^*}(\phi^*)^3|\varphi|^2 dxdt + \iint_Q e^{-2s\beta^*}(\phi^*)^3|\theta|^2 dxdt \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\beta}\phi^7|\varphi|^2 dxdt, \end{aligned} \tag{2.73}$$

for any $\varphi^T \in L^2(\Omega)$, where (φ, θ) is the associated solution to (2.51).

Proof. The proof is standard and relies on several well-known arguments [31]. First, by construction $\alpha = \beta$ and $\xi = \phi$ in $\Omega \times (T/2, T)$, hence

$$\begin{aligned} &\int_{T/2}^T \int_\Omega e^{-2s\alpha}\xi^3|\varphi|^2 + \int_{T/2}^T \int_\Omega e^{-2s\alpha}\xi^3|\theta|^2 \\ &= \int_{T/2}^T \int_\Omega e^{-2s\beta}\phi^3|\varphi|^2 + \int_{T/2}^T \int_\Omega e^{-2s\beta}\phi^3|\theta|^2. \end{aligned}$$

Therefore, from (2.56) and the definition of β and γ we obtain

$$\int_{T/2}^T \int_\Omega e^{-2s\beta}\phi^3|\varphi|^2 + \int_{T/2}^T \int_\Omega e^{-2s\beta}\phi^3|\theta|^2 \leq C \iint_{\omega\times(0,T)} e^{-2s\beta}\phi^7|\varphi|^2. \tag{2.74}$$

On the other hand, for the domain $\Omega \times (0, T/2)$, we will use energy estimates for system (2.51). In fact, let us introduce a function $\eta \in C^1([0,T])$ such that

$$\eta = 1 \text{ in } [0, T/2], \quad \eta = 0 \text{ in } [3T/4, T], \quad |\eta'(t)| \leq C/T.$$

Using classical energy estimates for $\eta\varphi$ solution to the first equation of system (2.51) we obtain

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(0,T/2;H^1_0(\Omega))} \leq C \left(\frac{1}{T^2}\|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \|\eta\theta\|^2_{L^2(0,3T/4;L^2(\Omega))} \right).$$

From the definition of η, Poincaré inequality and adding $\|\theta\|^2_{L^2(0,T/2;L^2(\Omega))}$ on both sides of the previous inequality we have

$$\begin{aligned}&\|\varphi(0)\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\theta\|^2_{L^2(0,T/2;L^2(\Omega))}\\ &\leq C\left(\|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \|\theta\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \|\theta\|^2_{L^2(0,T/2;L^2(\Omega))}\right).\end{aligned} \tag{2.75}$$

In order to eliminate the term $\|\theta\|^2_{L^2(0,T/2;L^2(\Omega))}$ in the right hand side, we use standard energy estimates for the second equation in (2.51), thus

$$\begin{aligned}\iint_{\Omega\times(0,T/2)} |\theta|^2 &\leq C\left(\frac{1}{\gamma^4}\iint_Q |\varphi|^2 + \frac{1}{\ell^4}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\right)\\ &\leq \frac{C}{\min\{\gamma^4,\ell^4\}}\iint_Q |\varphi|^2.\end{aligned} \tag{2.76}$$

Replacing (2.76) in (2.75) and since γ and ℓ are large enough we obtain

$$\begin{aligned}\|\varphi(0)\|^2_{L^2(\Omega)}+&\|\varphi\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\theta\|^2_{L^2(0,T/2;L^2(\Omega))}\\ &\leq C\left(\|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega))} + \|\theta\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right).\end{aligned} \tag{2.77}$$

Using (2.74) to estimate the terms in the right hand side of (2.77) and taking into account that the weight functions are bounded in $[0, 3T/4]$ we have the estimate

$$\begin{aligned}\|\varphi(0)\|^2_{L^2(\Omega)} + \int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\phi^3|\varphi|^2 + \int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\phi^3|\theta|^2\\ \leq C\left(\iint_{\omega\times(0,T)} e^{-2s\beta}\phi^7|\varphi|^2\right).\end{aligned}$$

This estimate, together with (2.74), and the definitions of ϕ^* and β^* yield the desired inequality (2.73). □

Proof of Proposition 37. The observability inequality (2.52) follows immediately from Proposition 42. Indeed, let us set $s = s_2$ as in (2.72) and define $\rho(t) = e^{s\beta^*}$. Thus $\rho(t)$ is a non-decreasing strictly positive function blowing up at $t = T$ that depends on Ω, ω, $\mathcal{O}_d$, $\|a\|_\infty$, $\|c\|_\infty$ and T, but can be chosen independently of $\mathcal{O}$, ℓ and γ.

We obtain energy estimates with this new function for θ solution to the second equation of (2.51). More precisely

$$\begin{aligned}\iint_Q \rho^{-2}|\theta|^2 dxdt &\leq C\left(\frac{1}{\gamma^4}\iint_Q \rho^{-2}|\varphi|^2 dxdt + \frac{1}{\ell^4}\iint_{\mathcal{O}\times(0,T)} \rho^{-2}|\varphi|^2 dxdt\right)\\ &\leq C\iint_Q \rho^{-2}|\varphi|^2 dxt\end{aligned}$$

Since $e^{-2s\beta}\phi^7 \leq C$ for all $(x,t) \in Q$ and noting that the right hand side of the previous inequality is comparable to the left hand side of inequality (2.73) up to a multiplicative constant, we obtain (2.52). This concludes the proof of Proposition 37. □

2.4 Proof of Theorem 28

In this section, we will end the proof of Theorem 28. We have already determined an optimal strategy for the follower control (see Proposition 36). It remains to obtain an strategy for the leader control h such that (y, q) solution to (2.49) verifies $y(T) = 0$.

The proof is inspired by well-known results on the controllability of nonlinear systems (see, for instance, [65, 61, 14, 28]) where controllability properties for linear problems and suitable fixed point arguments are the main ingredients.

Proof of Theorem 28. We start by proving the existence of a leader control h for a linearized version of (2.49). In fact, for given $a, c \in L^\infty(Q)$, $y_0 \in L^2(Q)$ and $y_d \in L^2(\mathcal{O}_d \times (0,T))$, we consider the linear system

$$\begin{cases} y_t - \Delta y + ay = h\chi_\omega - \frac{1}{\ell^2} q\chi_{\mathcal{O}} + \frac{1}{\gamma^2} q & \text{in } Q, \\ -q_t - \Delta q + cq = (y - y_d)\chi_{\mathcal{O}_d} & \text{in } Q, \\ y = q = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \quad q(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{2.78}$$

and the corresponding adjoint system (2.51). Then, the following result holds

Proposition 43. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$. Let C and ρ as in Proposition 37. For any $\varepsilon > 0$, any $y_0 \in L^2(\Omega)$, and any $y_d \in L^2(\mathcal{O}_d \times (0,T))$ such that*

$$\iint_Q \rho^2 |y_d|^2 dxdt < +\infty$$

there exists a leader control $h_\varepsilon \in L^2(\omega \times (0,T))$ such that the associated solution $(y_\varepsilon, q_\varepsilon)$ to (2.78) satisfies

$$\|y_\varepsilon(T)\|_{L^2(\Omega)} \leq \varepsilon \tag{2.79}$$

Moreover, the controls $\{h_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L^2(\omega \times (0,T))$, namely

$$\|h_\varepsilon\|_{L^2(\omega\times(0,T))} \leq \sqrt{C}\left(\|y_0\|_{L^2(\Omega)} + \|\rho y_d\|_{L^2(Q)}\right), \quad \forall \varepsilon > 0. \tag{2.80}$$

Proof. The proof is by now well-known. For the sake of completeness, we sketch some of the steps. For any fixed $\varepsilon > 0$, consider

$$\mathcal{F}_\varepsilon(\varphi^T) = \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2 dxdt + \varepsilon\|\varphi^T\|_{L^2(\Omega)} + \int_\Omega y_0\varphi(0)dx - \iint_{\mathcal{O}_d\times(0,T)} \theta y_d \, dxdt, \tag{2.81}$$

where (φ, θ) is the solution to (2.51) with initial datum $\varphi^T \in L^2(\Omega)$. It can be verified that (2.81) is continuous and strictly convex. From Hölder and Young inequalities and using the observability inequality (2.52) is not difficult to see that

$$\mathcal{F}_\varepsilon(\varphi^T) \geq \frac{1}{4}\iint_{\omega\times(0,T)} |\varphi|^2 dxdt + \varepsilon\|\varphi^T\|_{L^2(\Omega)} - C\left(\|y_0\|^2_{L^2(\Omega)} + \|\rho y_d\|^2_{L^2(Q)}\right),$$

hence (2.81) is also coercive. Consequently, $\mathcal{F}_\varepsilon$ reaches its minimum at a unique point $\varphi_\varepsilon^T \in L^2(\Omega)$. When $\varphi_\varepsilon^T \neq 0$, the optimality condition can be computed, that is

$$\begin{aligned}&\iint_{\omega\times(0,T)} \varphi_\varepsilon \varphi \, dxdt + \left(\frac{\varphi_\varepsilon^T}{\|\varphi_\varepsilon^T\|}, \varphi^T\right)_{L^2(\Omega)} \\ &+ \int_\Omega y_0 \varphi(0) dx - \iint_{\mathcal{O}_d\times(0,T)} y_d \theta \, dxdt = 0, \quad \forall \varphi^T \in L^2(\Omega),\end{aligned} \tag{2.82}$$

where $(\varphi_\varepsilon, \theta_\varepsilon)$ is the solution to (2.51) with initial condition φ_ε^T. Set $h_\varepsilon = \varphi_\varepsilon \chi_\omega$, then $(y_\varepsilon, q_\varepsilon)$ solution to (2.78) associated to this control verifies (2.79). To conclude, observe that setting $\varphi^T = \varphi_\varepsilon^T$ in (2.82) and using the observability inequality (2.52) yields estimate (2.80). □

Now, we will apply a fixed point argument to prove an approximate controllability result for the nonlinear system (2.49). For a given globally Lipschitz function $f \in C^2(\mathbb{R})$ verifying $f(0) = 0$, we can write

$$f(s) = g(s)s, \quad \forall s \in \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function defined by

$$g(s) = \int_0^1 f'(\sigma s) \, d\sigma.$$

The continuity of f and f' and the density of $C_c^\infty(Q)$ in $L^2(Q)$ allow to see that $g(z)$ and $f'(z)$ belong to $L^\infty(Q)$ for every $z \in L^2(Q)$.

For each $z \in L^2(Q)$, let us consider the linear system (2.78) with $a = a_z = g(z)$ and $c = c_z = f'(z)$. Thanks to the hypothesis on f, there exists M such that

$$\|a_z\|_\infty, \|c_z\|_\infty \leq M, \quad \forall z \in L^2(Q). \tag{2.83}$$

In view of Proposition (43), for any given $\varepsilon > 0$ there exists a leader control $h_z \in L^2(\omega \times (0,T))$ such that the solution (y_z, q_z) to (2.78) corresponding to a_z, c_z satisfies

$$\|y_z(T)\|_{L^2(\Omega)} < \varepsilon.$$

Moreover, we have the estimate (uniform with respect to ε and z)

$$\|h_z\|_{L^2(\omega\times(0,T))} \leq \sqrt{C}(\|y_0\|_{L^2(\Omega} + \|\rho y_d\|_{L^2(Q)}), \quad \forall z \in L^2(Q), \tag{2.84}$$

where C only depends on Ω, $\mathcal{O}_d$, $\mathcal{O}$, M and T and ρ only depends on Ω, $\mathcal{O}_d$, M and T.

We consider the mapping $\Lambda : L^2(Q) \to L^2(Q)$ defined by $\Lambda z = y_z$ with (y_z, q_z) the solution to (2.78) associated to the potentials a_z, c_z, and the control h_z provided by Proposition 43. By means of the Schauder fixed point theorem, we will deduce that Λ possesses at least one fixed point. It can be proved that if ℓ and γ are large enough then (2.78) has a unique solution $y_z \in W(0,T)$ veryfing

$$\|y_z\|_{W(0,T)} \leq C\left(1 + \|h\|_{L^2(\omega\times(0,T))}\right), \tag{2.85}$$

where C only depends on Ω, $\mathcal{O}$, $\mathcal{O}_d$, γ, ℓ, K, y_0, y_d and T. In view of (2.83)–(2.85), we deduce that Λ maps $L^2(Q)$ into a bounded set of $W(0,T)$. This space is compacty embbeded in $L^2(Q)$, therefore it exists a fixed compact set K such that

$$\Lambda(L^2(Q)) \subset K.$$

It can be readily verified that Λ is also a continuous map from $L^2(Q)$ into $L^2(Q)$. Therefore, we can use Schauder fixed point theorem to ensure that Λ has at least one fixed point $y = y_\varepsilon$, where $(y_\varepsilon, q_\varepsilon)$ together with the control $h_\varepsilon = h_{y_\varepsilon}$ solve

$$\begin{cases} y_{\varepsilon,t} - \Delta y_\varepsilon + g(y_\varepsilon)y_\varepsilon = h_\varepsilon\chi_\omega - \frac{1}{\ell^2}q_\varepsilon\chi_{\mathcal{O}} + \frac{1}{\gamma^2}q_\varepsilon & \text{in } Q,\\ -q_{\varepsilon,t} + \Delta q_\varepsilon + f'(y_\varepsilon)q_\varepsilon = (y_\varepsilon - y_d)\chi_{\mathcal{O}_d} & \text{in } Q,\\ y_\varepsilon = q_\varepsilon = 0 & \text{on } \Sigma,\\ y_\varepsilon(x,0) = y_0(x), \quad q_\varepsilon(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{2.86}$$

verifying (2.79).

To conclude the proof of Theorem 28, we will pass to the limit in (2.86) and (2.79). Thanks to (2.84), the control h_ε is uniformly bounded in $L^2(\omega\times(0,T))$. Since (2.83) holds, the solution $(y_\varepsilon, q_\varepsilon)$ lies in a bounded set of $W(0,T)\times W(0,T)$ and therefore in a compact set of $L^2(Q)\times L^2(Q)$. Then, up to a subsequence, we have

$$\begin{aligned} &h_\varepsilon \rightharpoonup h \quad \text{weakly in} \quad L^2(\omega\times(0,T)),\\ &(y_\varepsilon, q_\varepsilon) \to (y,q) \quad \text{in} \quad L^2(Q)\times L^2(Q),\\ &y_\varepsilon(T) \to y(T) \quad \text{in} \quad L^2(\Omega), \end{aligned}$$

for some $h \in L^2(\omega\times(0,T))$ and some $(y,q) \in W(0,T)\times W(0,T)$. Due to the continuity of g, we can pass to the limit in (2.86), thus (y,q) solves (2.49) with leader control h and initial datum y_0. Moreover, passing to the limit in (2.79) we conclude that $y(\cdot,T) = 0$. Therefore the proof is complete. □

2.5 Proof of Theorem 29

In the previous sections, we proved the existence of a robust Stackelberg control for a nonlinear system when $(v,\psi) \in L^2(\mathcal{O}\times(0,T))\times L^2(Q)$. Here, we will follow the arguments to show that a similar result can be obtained when the follower control v and the perturbation ψ belong to the bounded sets (2.8)–(2.9), respectively.

As stated in the theorem, we consider the linear system

$$\begin{cases} y_t - \Delta y + ay = h\chi_\omega + v\chi_{\mathcal{O}} + \psi & \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \quad y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{2.87}$$

where $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$ is given.

It is clear that for given $y_0 \in L^2(\Omega)$, any $h \in L^2(\omega \times (0,T))$ and each $(v, \psi) \in \mathcal{V}_{ad} \times \Psi_{ad}$, system (2.87) admits a unique solution $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$.

As before, we begin by proving the existence of a saddle point $(\bar{v}, \bar{\psi})$ for the cost functional (2.2). The following result will give us conditions to determine its existence:

Proposition 44 (Prop. 2.1, p. 171, [26])**.** *Let J be a functional defined on $X \times Y$, where X and Y are convex, closed, non-empty, bounded sets. If*

1. *$\forall v \in X$, $\psi \mapsto J(v,\psi)$ is concave and upper semicontinous,*
2. *$\forall \psi \in Y$, $v \mapsto J(v,\psi)$ is convex and lower semicontinous,*

then J possesses at least one saddle point $(\bar{v}, \bar{\psi})$ and

$$\mathcal{J}(\bar{v}, \bar{\psi}) = \min_{v \in X} \sup_{\psi \in Y} J(v,\psi) = \max_{\psi \in Y} \inf_{v \in X} J(v,\psi).$$

We will apply Proposition 31 to (2.2) with $X = \mathcal{V}_{ad}$ and $Y = \Psi_{ad}$. In fact, verifying the conditions 1–2 will be easier than in the nonlinear case. Recall that in the first part of the hierarchic control the leader control h is fixed. First, we have the following:

Lemma 45. *Let $h \in L^2(\omega \times (0,T))$ and $y_0 \in L^2(\Omega)$ be given. The mapping $(v,\psi) \mapsto y(v,\psi)$ from $\mathcal{V}_{ad} \times \Psi_{ad}$ into $L^2(0,T; H_0^1(\Omega))$ is affine, continuous, and has Gâteau derivative $y'(v',\psi')$ in every direction $(v',\psi') \in L^2(\mathcal{O} \times (0,T)) \times L^2(Q)$. Moreover, the derivative $y'(v',\psi')$ solves the linear system*

$$\begin{cases} y'_t - \Delta y' + ay' = v'\chi_{\mathcal{O}} + \psi' & \text{in } Q, \\ y' = 0 \quad \text{on } \Sigma, \quad y'(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.88}$$

Proof. The fact that $(v,\psi) \mapsto y(v,\psi)$ is affine and continuous follows from the linearity of (2.87) and well-known energy estimates for the heat equation. In the same way, thanks to linearity, the existence of the Gâteau derivative and its characterization can be obtained by letting λ tends to 0 in the expression $y^\lambda := (y(v + \lambda v', \psi + \lambda \psi') - y(v,\psi))/\lambda$. ☐

With this lemma, we are in position to check conditions 1–2 of Proposition 44. This will give the existence of at most one saddle point of functional (2.2).

Proposition 46. *Let $y_0 \in L^2(\Omega)$ and $h \in L^2(\omega \times (0,T))$ be given. Then, for γ sufficiently large, we have that*

1. *$\forall\psi \in \Psi_{ad}$, $v \mapsto J_r(v,\psi)$ is strictly convex lower semicontinuous,*

2. *$\forall v \in \mathcal{V}_{ad}$, $\psi \mapsto J_r(v,\psi)$ is strictly concave upper semicontinuous.*

Proof. Condition 1. Thanks to Lemma 45, the map $v \mapsto J_r(v,\psi)$ is lower semicontinuous. Since $v \mapsto y(v,\psi)$ is linear, the strict convexity of J_r can be readily verified.

Condition 2. Also, by Lemma 45, the map $\psi \mapsto J_r(v,\psi)$ is upper semicontinuous. To prove the concavity, we will argue as in the nonlinear case. To this end, consider

$$\mathcal{G}(\tau) = J_r(v, \psi + \tau\psi').$$

Then, it is sufficient to prove that $\mathcal{G}(\tau)$ is concave with respect to τ. We compute

$$\mathcal{G}'(\tau) = \iint_{\mathcal{O}_d\times(0,T)} (y + \tau y' - y_d)y' - \gamma^2 \iint_Q (\psi + \tau\psi')\psi',$$

where y' is solution to (2.88) with $v' = 0$. It is clear that y' is independent of τ, hence

$$\mathcal{G}''(\tau) = \iint_{\mathcal{O}_d\times(0,T)} |y'|^2 - \gamma^2 \iint_Q |\psi'|^2.$$

From classical energy estimates for the heat equation, we obtain

$$\mathcal{G}''(\tau) \le -(\gamma^2 - C)\|\psi'\|^2_{L^2(Q)}, \quad \forall \psi' \in L^2(Q),$$

where C is a positive constant only depending Ω, $\mathcal{O}_d$, $\|a\|_\infty$ and T. Then, for a sufficiently large value γ, we have $\mathcal{G}''(\tau) < 0$, $\forall \tau \in \mathbb{R}$. Thus, the function $\mathcal{G}$ is strictly concave, and the strict concavity of $\psi \mapsto J_r(v,\psi)$ follows immediately. This concludes the proof. □

Combining the statements of Propositions 44 and 46, we are able to deduce the existence of at most one saddle point $(\bar{v},\bar{\psi}) \in \mathcal{V}_{ad} \times \Psi_{ad}$. Unlike the nonlinear case, the solution $(\bar{v},\bar{\psi})$ to the robust control problem may not necessarily satisfy (2.41), unless it is located in the interior of the domain $\mathcal{V}_{ad} \times \Psi_{ad}$.

To characterize in this case the solution to the control problem, we use the fact that if $(\bar{v},\bar{\psi})$ is a saddle point of J, then

$$J_r(\bar{v},\bar{\psi}) \le J_r((1-\lambda)\bar{v} + \lambda v, \bar{\psi}), \quad \forall v \in \mathcal{V}_{ad},$$

or equivalently

$$0 \le J_r(\bar{v} + \lambda(v - \bar{v})) - J_r(\bar{v},\bar{\psi}), \quad \forall v \in \mathcal{V}_{ad}.$$

Dividing by λ and taking the limit as $\lambda \to 0$, we obtain from the above expression

$$0 \le \iint_{\mathcal{O}_d\times(0,T)} (y - y_d)\hat{y} + \ell^2 \iint_{\mathcal{O}\times(0,T)} \bar{v}(v - \bar{v}), \tag{2.89}$$

where y is the solution to (2.87) evaluated in $(\bar{v},\bar{\psi})$ and $\hat{y}$ stands for the directional derivative (2.88) in the direction $(v-\bar{v},0)$. We introduce the adjoint state q solution to the linear system

$$\begin{cases} -q_t-\Delta q+aq=(y-y_d)\chi_{\mathcal{O}_d} & \text{in } Q,\\ q=0 \quad \text{on } \Sigma, \quad q(x,T)=0 & \text{in } \Omega. \end{cases} \tag{2.90}$$

Multiplying (2.90) by $\hat{y}$ and integrating by parts in $L^2(Q)$, it is not difficult to see that we can rewrite (2.89) as

$$0\leq \iint_{\mathcal{O}\times(0,T)}(q+\ell^2\bar{v})(v-\bar{v}), \quad \forall v\in \mathcal{V}_{ad}.$$

Also, from the properties of the saddle point $(\bar{v},\bar{\psi})$, we have

$$J_r(\bar{v},(1-\lambda)\bar{\psi}+\lambda\psi)\leq \mathcal{J}(\bar{v},\bar{\psi}), \quad \forall \psi\in\Psi_{ad}.$$

Arguing as above, we deduce that

$$\iint_{\mathcal{O}_d\times(0,T)}(y-y_d)\tilde{y}-\gamma^2\iint_{\mathcal{O}\times(0,T)}\bar{\psi}(\psi-\bar{\psi})\leq 0, \tag{2.91}$$

where y is the solution to (2.87) evaluated in $(\bar{v},\bar{\psi})$ and $\tilde{y}$ denotes the directional derivative (2.88) in the direction $(0,\psi-\bar{\psi})$. If we multiply (2.90) by $\tilde{y}$ and integrate by parts in $L^2(Q)$, we can rewrite (2.91) as

$$\iint_Q (q-\gamma^2\bar{\psi})(\psi-\bar{\psi})\leq 0, \quad \forall \psi\in\Psi_{ad}.$$

In this way, we have that $(\bar{v},\bar{\psi})$ satisfies the robust control problem (2.4) if $(y,p,\bar{v},\bar{\psi})$ satisfies the following optimality system:

$$\begin{cases} y_t-\Delta y+ay=h\chi_\omega+\bar{v}\chi_{\mathcal{O}}+\bar{\psi} & \text{in } Q,\\ -q_t-\Delta q+aq=(y-y_d)\chi_{\mathcal{O}_d} & \text{in } Q,\\ y=q=0 \quad \text{on } \Sigma, \quad y(x,0)=y_0(x),\ q(x,T)=0 & \text{in } \Omega, \end{cases} \tag{2.92}$$

$$\bar{v}\in\mathcal{V}_{ad}, \quad \bar{\psi}\in\Psi_{ad}, \tag{2.93}$$

$$\iint_{\mathcal{O}\times(0,T)}(q+\ell^2\bar{v})(v-\bar{v})\geq 0, \quad \forall v\in\mathcal{V}_{ad}, \tag{2.94}$$

$$\iint_Q (q-\gamma^2\bar{\psi})(\psi-\bar{\psi})\leq 0, \quad \forall\psi\in\Psi_{ad}. \tag{2.95}$$

From the hierarchic control methodology, the next step is obtain a leader control h such that y solution to the coupled system (2.92) satisfies $y(T)=0$. The idea is to apply the results from Section 2.4. We follow the spirit of [6].

First, note that $\bar{\psi}$ satisfying the variational inequality (2.95) can be written as the projection onto the convex set Ψ_{ad}, that is,

$$\bar{\psi} = \Pi_{\Psi_{ad}}\left(\tfrac{1}{\gamma^2}q\right).$$

The same is true for (2.94). In this case, we have

$$\bar{v} = \Pi_{\mathcal{V}_{ad}}\left(-\tfrac{1}{\ell^2}q|_{\mathcal{O}}\right).$$

In view of this, the optimality system (2.92)–(2.95) now reads

$$\begin{cases} y_t - \Delta y + ay = h\chi_\omega + \Pi_{\mathcal{V}_{ad}}(-\frac{1}{\ell^2}q|_{\mathcal{O}})\chi_{\mathcal{O}} + \Pi_{\Psi_{ad}}(\frac{1}{\gamma^2}q) & \text{in } Q, \\ -q_t - \Delta q + aq = (y - y_d)\chi_{\mathcal{O}_d} \quad \text{in } Q, \\ y = q = 0 \quad \text{on } \Sigma, \\ y(x,0) = y_0(x),\ q(x,T) = 0 \quad \text{in } \Omega, \end{cases} \tag{2.96}$$

As in the semilinear case, we will analyze the null controllability of (2.96) by means of a fixed point method. To do this, note that for every $z \in L^2(Q)$, $\Pi_{\Psi_{ad}}$ can be expressed in the form $\Pi_{\Psi_{ad}}(z) = \rho(z)z$ where the function $\rho(z)$ is defined as

$$\rho(z) = \begin{cases} 1, & \text{if } z(x,t) \in E_2 \\ \Pi_{E_2}(z)/z, & \text{otherwise.} \end{cases}$$

for a.e. $(x,t) \in Q$. Here, Π_{E_2} denotes the projection of $\mathbb{R}$ onto the interval E_2.

Defined in this way, $z \mapsto \rho(z)$ is continuous on $L^2(Q)$ and $\|\rho(z)\|_\infty \leq 1$, $\forall z \in L^2(Q)$. Analogously, we can define a function σ such that $\Pi_{\mathcal{V}_{ad}}$ can be expressed in the form $\Pi_{\mathcal{V}_{ad}} = \sigma(z)z$ for every $z \in L^2(\mathcal{O} \times (0,T))$.

Therefore, the controllability problem is now to find $h \in L^2(\omega \times (0,T))$ such that the solution to

$$\begin{cases} y_t - \Delta y + ay = h\chi_\omega - \tilde{\sigma}(q)\frac{1}{\ell^2}q\chi_{\mathcal{O}} + \tilde{\rho}(q)\frac{1}{\gamma^2}q & \text{in } Q, \\ -q_t - \Delta q + aq = (y - y_d)\chi_{\mathcal{O}_d} \quad \text{in } Q, \\ y = q = 0 \quad \text{on } \Sigma, \\ y(x,0) = y_0(x),\ q(x,T) = 0 \quad \text{in } \Omega, \end{cases} \tag{2.97}$$

verifies $y(T)$=0. In system (2.97), $\tilde{\sigma}(q)$ stands for $\tilde{\sigma}(q) = \sigma(\frac{1}{\gamma^2}q|_{\mathcal{O}})$ while $\tilde{\rho}(q)$ denotes $\tilde{\rho}(q) = \rho(\frac{1}{\gamma^2}q)$. We will establish the null controllability for (2.97) arguing as in section 2.4.

For each $\tilde{q} \in L^2(Q)$, let us consider the linear system

$$\begin{cases} y_t - \Delta y + ay = h\chi_\omega - \tilde{\sigma}(\tilde{q})\frac{1}{\ell^2}q\chi_{\mathcal{O}} + \tilde{\rho}(\tilde{q})\frac{1}{\gamma^2}q & \text{in } Q, \\ -q_t - \Delta q + aq = (y - y_d)\chi_{\mathcal{O}_d} \quad \text{in } Q, \\ y = q = 0 \quad \text{on } \Sigma, \\ y(x,0) = y_0(x),\ q(x,T) = 0 \quad \text{in } \Omega. \end{cases} \tag{2.98}$$

In this case, adapting the arguments in Section 2.3, is not difficult to obtain an observability inequality (see Eq. 2.52) for the solutions to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = \theta\chi_{\mathcal{O}_d} & \text{in } Q, \\ \theta_t - \Delta\theta + a\theta = -\frac{1}{\ell^2}\tilde{\sigma}(\tilde{q})\varphi\chi_{\mathcal{O}} + \frac{1}{\gamma^2}\tilde{\rho}(\tilde{q})\varphi & \text{in } Q, \\ \varphi = \theta = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^T(x), \quad \theta(x,0) = 0 & \text{in } \Omega. \end{cases}$$

With this new observability estimate and following Section 2.4, we can build a control $\tilde{h}$ associated to each $\tilde{q} \in L^2(Q)$ such that

$$\|\tilde{y}(T)\|_{L^2(\Omega)} < \varepsilon, \tag{2.99}$$

where we have denoted by $\tilde{y}$ the first component of $(\tilde{y}, \tilde{q})$ solution to (2.98) with this control. Moreover, the control $\tilde{h}$ satisfies

$$\|\tilde{h}\|_{L^2(\omega\times(0,T))} \leq C, \tag{2.100}$$

for some $C > 0$ that can be chosen independently of γ and ℓ.

Thanks to (2.100), the controlled solution $(\tilde{y}, \tilde{q})$ is uniformly bounded in $W(0,T) \times W(0,T)$. Therefore, we can deduce that the mapping $\tilde{q} \mapsto q$ has at least one fixed point. The rest of the proof follows as in Section 2.4.

Chapter 3

Stackelberg-Nash controllability for linear parabolic coupled systems

3.1 Introduction

Let Ω be an open and bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2 and ω be an open and nonempty subset of Ω. Given $T > 0$, we consider the following system of coupled parabolic PDEs with leader control localized in ω and follower controls localized in $\omega_1, \omega_2 \subset \Omega$ with $\omega_i \cap \omega = \emptyset$. More precisely

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h_1\chi_\omega + v^1\chi_{\omega_1} + v^2\chi_{\omega_2} \quad \text{in } Q = \Omega \times (0,T), \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = h_2\chi_\omega \quad \text{in } Q, \\ y_j = 0 \text{ on } \Sigma = \partial\Omega \times (0,T),\ j = 1,2, \\ y_j(x,0) = y_j^0(x) \text{ in } \Omega,\ j = 1,2, \end{cases} \tag{3.1}$$

where $a_{ij} = a_{ij}(x,t) \in L^\infty(Q)$ and $y_j^0 \in L^2(\Omega)$ are given.

In system (3.1), $y = (y_1, y_2)^t$ is the state, $v^j = v^j(x,t)$ and $h_j = h_i(x,t)$ are the follower and leader control functions, respectively, while χ_ω and χ_{ω_i} denote the characteristic functions of ω and ω_j.

Observe that for each $h_j \in L^2(\omega \times (0,T))$, $v_j \in L^2(\omega_j \times (0,T))$, and $y_{j,0} \in L^2(\Omega)$, $j = 1,2$, system (3.1) admits a unique weak solution $y \in [C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)]^2$, hereinafter denoted as

$$y = y(x,t; h_1, h_2, v^1, v^2).$$

In the case where only a (leader) control is exerted on ω, there exist several papers devoted to the controllability of non-scalar parabolic systems, see for instance [1], [2], [39], or [4] for a recent survey on the controllability of coupled parabolic problems. In particular, in [37] the authors proved that system (3.1) is null controllable whenever a single control is applied in the first equation of the coupled system, as long as a_{21} has a fixed sign on an open subset of ω. Indeed, when dealing with the null controllability of system (3.1), it is not necessary to apply a second control h_2 to obtain $y_1(T) = y_2(T) = 0$.

Now, we introduce the control point of view where we assume that we have a hierarchy in our wishes and we will describe the Stackelberg-Nash strategy for system (3.1). Let $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset \Omega$ be open subsets, representing the observation domains of the followers, which are localized arbitrarily in Ω. Define the functionals

$$\begin{aligned} J_i(h_1, h_2, v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d \times (0,T)}} |y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2 dxdt \\ + \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} |v^i|^2 dxdt, \ i = 1, 2, \end{aligned} \tag{3.2}$$

and the main functional

$$J(h_1, h_2) = \frac{1}{2} \iint_{\omega \times (0,T)} |h_1|^2 dxdt + \frac{1}{2} \iint_{\omega \times (0,T)} |h_2|^2 dxdt, \tag{3.3}$$

where $\alpha_i, \mu_i > 0$ are constants and $y^i_d = (y^i_{1,d}, y^i_{2,d})^t$ is a given function in $L^2(\mathcal{O}_{1,d} \times (0,T)) \times L^2(\mathcal{O}_{2,d} \times (0,T))$.

The main objective is to choose $h = (h_1, h_2)^t$ minimizing J subject to the null controllability constraint

$$y(\cdot, T; h_1, h_2, v^1, v^2) = 0 \quad \text{in } \Omega. \tag{3.4}$$

The second objective is the following. Given the functions h and y^i_d, we want to choose the control v^i minimizing J_i. Intuitively, this is that throughout the interval $t \in (0,T)$

$$\begin{gathered} y(x, t; h_1, h_2, v) \text{ “do not deviate much” from } y^i_d(x,t), \\ \text{in the observability domain } \mathcal{O}_{i,d}. \end{gathered} \tag{3.5}$$

To achieve simultaneously (3.4) and (3.5), the control process can be described as follows:

- For a fixed leader control $h = (h_1, h_2)^t$, find controls $(\overline{v}^1, \overline{v}^2)$ (depending on h) and the corresponding state solution $y = y(h_1, h_2, \overline{v}^1, \overline{v}^2)$ to (3.1) satisfying the Nash equilibrium related to the functionals (J_1, J_2). That is, given h, find $(\overline{v}^1, \overline{v}^2)$ such that

$$\begin{aligned} J_1(h, \overline{v}^1, \overline{v}^2) &\le J_1(h, v^1, \overline{v}^2), \quad \forall v^1 \in L^2(\omega_1 \times (0,T)), \\ J_2(h, \overline{v}^1, \overline{v}^2) &\le J_2(h, \overline{v}^1, v^2), \quad \forall v^2 \in L^2(\omega_2 \times (0,T)), \end{aligned}$$

  or equivalently

$$J_1\left(h, \overline{v}^1, \overline{v}^2\right) = \min_{v^1} J_1\left(h, v^1, \overline{v}^2\right), \tag{3.6}$$

$$J_2\left(h, \overline{v}^1, \overline{v}^2\right) = \min_{v^2} J_2\left(h, \overline{v}^1, v^2\right). \tag{3.7}$$

Any pair $(\overline{v}^1, \overline{v}^2)$ satisfying (3.6)–(3.7) is called a Nash equilibrium for (J_1, J_2). Thanks to the linearity of system (3.1), J_1 and J_2 are strictly convex functionals. Then $(\overline{v}^1, \overline{v}^2)$ is a Nash equilibrium with respect to (J_1, J_2) if and only if

$$\left(\frac{\partial J_1}{\partial v^1}(h, \overline{v}^1, \overline{v}^2), v^1\right) = 0 \quad \forall v^1 \in L^2(\omega_1 \times (0,T)), \tag{3.8}$$

$$\left(\frac{\partial J_2}{\partial v^2}(h, \overline{v}^1, \overline{v}^2), v^2\right) = 0 \quad \forall v^2 \in L^2(\omega_2 \times (0,T)). \tag{3.9}$$

- After identifying the Nash equilibrium and the associated state $y = y(h, \overline{v}^1(h), \overline{v}^2(h))$ for each $h = (h_1, h_2)^t$, we look for an optimal control $\widehat{h}$ such that

$$J(\widehat{h}_1, \widehat{h}_2) = \min_{h_1, h_2} J\left(h_1, h_2, \overline{v}^1(h), \overline{v}^2(h)\right) \tag{3.10}$$

  subject to the restriction

$$y(\cdot, T; h_1, h_2, \overline{v}^1(h), \overline{v}^2(h)) = 0 \quad \text{in } \Omega. \tag{3.11}$$

3.1.1 Main results

Within this spirit, the main contributions of this chapter can be stated as follows. Assume that

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d}, \tag{3.12}$$

denoted in the following sections as $\mathcal{O}_d$. Our first result is the following:

Theorem 47. *Suppose that (3.12) holds and that μ_i, $i = 1, 2$, are large enough. Then, there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that for any $y_d^i \in [L^2(\mathcal{O}_d \times (0,T))]^2$ satisfying*

$$\iint_{\mathcal{O}_d \times (0,T)} \rho^2 |y_{j,d}^i|^2 dxdt < +\infty, \quad i, j = 1, 2, \tag{3.13}$$

and any $y^0 \in L^2(\Omega)^2$, there exists a control $h = (h_1, h_2) \in [L^2(\omega \times (0,T))]^2$ and the corresponding Nash equilibrium $(\bar{v}^1, \bar{v}^2)$ such that the solution of (3.1) satisfies (3.11).

When dealing with the controllability of non-scalar parabolic systems, one of the main questions is if it is possible to control many equations with few controls. There are various positive answers in the context of controllability problems (see [4] for a survey on this topic). Therefore, in the case of hierarchic control, it is natural to ask if we can remove the action of one of the leader controls.

To this purpose, consider the modified follower functionals

$$\begin{aligned} J_i(h, v^1, v^2) = {} & \frac{\alpha_i}{2} \iint_{\mathcal{O}_d \times (0,T)} |y_1 - y_{1,d}^i|^2 + |y_2 - y_{2,d}^i|^2 dxdt \\ & + \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} \rho_*^2 |v^i|^2 dxdt, \ i = 1, 2, \end{aligned} \tag{3.14}$$

where $\rho_* = \rho_*(t)$ is a suitable weight function to be clarified below. We will prove that by introducing this new function penalizing the action of the followers, we can eliminate the action of the leader control h_2. More precisely, we have the following:

Theorem 48. *Suppose that (3.12) holds, μ_i, $i = 1, 2$, are large enough and assume $h_2 \equiv 0$. If*

$$a_{21} \geq a_0 > 0 \quad \text{or} \quad -a_{21} \geq a_0 > 0 \quad \text{in } (\mathcal{O}_d \cap \omega) \times (0, T), \tag{3.15}$$

there exists a positive function $\rho = \rho(t)$ blowing up at $t = T$ such that if (3.13) holds, then for any $y^0 \in L^2(\Omega)^2$ there exists a Stackelberg-Nash strategy $(h_1, \bar{v}^1, \bar{v}^2)$ for the functionals given by (3.3) and (3.14), with h_1 subject to $y_1(T) = y_2(T) = 0$.

Remark 49. Some remarks are in order.

- The hierarchical control is largely motivated by applications where more than one objective is desirable in the behavior of the system under study. For instance, if $y = y(x, t)$ represents the concentration of a chemical product, the methodology is to reach the state 0 by means of a control h acting on ω, but at the same time try to keep the concentration near a reasonable quantity in $\mathcal{O}_d$ along the time interval $(0, T)$ by means of control v.

- Just as in [6], the condition $\rho y^i_{j,d} \in L^2(Q)$ seems natural and it means that the follower objectives $y^i_{j,d}$ approach 0 as $t \to T$. This is because the leader control h should not find any obstruction to control the system. It remains an open problem to verify if this condition is necessary, even in the scalar case.

- Equation (3.15) is exactly the sign condition employed on [37] to prove the null controllability of (3.39) when a single control is applied. Moreover, such condition can be applied repeatedly to study the null controllability for non-scalar parabolic problems in cascade form, see [37].

- It remains as an open problem, in [6] and here, to eliminate the condition $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$. Intuitively it should be more difficult to drive the solution close to two different objectives in the same subset than close to two different ones in different subsets.

- Unlike other papers as [41] (in the scalar case) or [7] (in the coupled case), we are supposing that the follower controls are being applied in some sets ω_i disjoint of the leader set ω. This leads to a more realistic situation, because otherwise once the followers choose a policy, the leader modifies its behavior at the same points.

The rest of the chapter is organized as follows. We devote sections 3.2 and 3.3 to prove Theorem 47. In the first one, we give sufficient conditions for the existence and uniqueness of Nash equilibrium, as well as its characterization, while in the second, we prove that the leader controls solve the problem of null controllability. In section 3.4, we prove Theorem 48. Lastly, we present some concluding remarks in section 3.5.

3.2 Nash equilibrium

3.2.1 Existence and uniqueness

In this section, we recall an existence and uniqueness result concerning the Nash equilibrium in the sense of (3.8)–(3.9) (see, for instance, [25]). We follow the same spirit as in [41] to present the result. Here, no hypotheses are required regarding the control sets ω_i and ω or the observation sets $\mathcal{O}_{i,d}$, so we keep the notation from the problem formulation.

Consider the functionals given by (3.2) and define the functional spaces

$$\mathcal{H}_i = L^2(\omega_i \times (0,T)),\ \ i=1,2,$$
$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2.$$

as well as the operator

$$\Lambda_i \in \mathcal{L}(\mathcal{H}_i, L^2(Q)^2) \quad \text{defined as} \quad \Lambda_i v^i = y^i,$$

where $y^i = (y_1^i, y_2^i)^t$ is solution of

$$\left\{\begin{array}{l} y_{1,t}^i - \Delta y_1^i + a_{11} y_1^i + a_{12} y_2^i = v^i \chi_{\omega_i} \quad \text{in } Q, \\ y_{2,t}^i - \Delta y_2^i + a_{21} y_1^i + a_{22} y_2^i = 0 \quad \text{in } Q, \\ y_j^i(0) = 0 \text{ in } \Omega, \quad y_j^i = 0 \text{ on } \Sigma,\ j=1,2. \end{array}\right.$$

With this notation, for any $h \in [L^2(\omega \times (0,T))]^2$, we write the solution of (3.1) as follows

$$y = \Lambda_1 v^1 + \Lambda_2 v^2 + q(h),$$

where $q(h) = (q_1(h), q_2(h))$ solves the system

$$\left\{\begin{array}{l} q_{1,t} - \Delta q_1 + a_{11} q_1 + a_{12} q_2 = h_1 \chi_{\omega} \quad \text{in } Q, \\ q_{2,t} - \Delta q_2 + a_{21} q_1 + a_{22} q_2 = h_2 \chi_{\omega} \quad \text{in } Q, \\ q_j(0) = y_j^0 \text{ in } \Omega, \quad q_j = 0 \text{ on } \Sigma,\ j=1,2. \end{array}\right.$$

Then, the functionals (3.2) can be rewritten as

$$J_i(h, v^1, v^2) = \frac{\mu_i}{2} \iint_{\omega_i \times (0,T)} |v^i|^2 dxdt + \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} \|\Lambda_1 v^1 + \Lambda_2 v^2 - \tilde{y}_d^i\|^2 dxdt, \quad \text{for } i=1,2,$$

where $\widetilde{y}_d^i = y_d^i - q(h)|_{\mathcal{O}_{i,d}}$, $(i,j=1,2)$ and $\|\cdot\|$ stands for the usual Euclidian norm. We have that $(\overline{v}^1, \overline{v}^2)$ is a Nash equilibrium if and only if satisfies (3.8)–(3.9), this is

$$\mu_i \iint_{\omega_i \times (0,T)} \overline{v}^i v^i dxdt + \alpha_i \iint_{\mathcal{O}_{i,d} \times (0,T)} \left(\Lambda_1 \overline{v}^1 + \Lambda_2 \overline{v}^2 - \tilde{y}_d^i\right) \cdot \Lambda_i v^i dxdt = 0, \tag{3.16}$$

for $i = 1,2$ and for any $(v^1, v^2) \in \mathcal{H}$. It follows that

$$\mu_i \left(\overline{v}^i, v^i\right)_{\omega_i \times (0,T)} + \alpha_i \left(\Lambda_i^* \left[(\Lambda_1 \overline{v}^1 + \Lambda_2 \overline{v}^2)\big|_{\mathcal{O}_{i,d}} - \tilde{y}_d^i \right], v^i \right)_{\omega_i \times (0,T)} = 0,$$

where $(\cdot,\cdot)_{\mathcal{A}}$ denotes the internal product in $L^2(\mathcal{A})$ and $\Lambda_i^* \in \mathcal{L}([L^2(Q)]^2, \mathcal{H}_i)$ is the adjoint operator of Λ_i. Hence,

$$\mu_i \overline{v}^i + \alpha_i \Lambda_i^* \left[(\Lambda_1 \overline{v}^1 + \Lambda_2 \overline{v}^2)\big|_{\mathcal{O}_{i,d}} \right] = \alpha_i \Lambda_i^* \tilde{y}_d^i, \quad \text{in } \mathcal{H}_i, \quad i = 1,2.$$

For all $v = (v^1, v^2)$, we define the operator $R = (R_1, R_2) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ as

$$R_i v := \mu_i v^i + \alpha_i \Lambda_i^* \left[(\Lambda_1 v^1 + \Lambda_2 v^2) \chi_{\mathcal{O}_{i,d}} \right],$$

for each $i = 1,2$. Therefore, $\overline{v} = (\overline{v}^1, \overline{v}^2)$ is a Nash equilibrium if and only if

$$R\overline{v} = \left(\alpha_1 \Lambda_1^* \tilde{y}_d^1, \alpha_2 \Lambda_2^* \tilde{y}_d^2 \right)^t, \quad i = 1,2, \tag{3.17}$$

where the right hand side is a given fixed element of $\mathcal{H}$. Let us calculate

$$\begin{aligned}(Rv, v)_{\mathcal{H}} = \sum_{i=1}^{2} \mu_i \|v^i\|^2_{L^2(\omega_i \times (0,T))} &+ \alpha_1 \left(\Lambda_1 v^1 + \Lambda_2 v^2, \Lambda_1 v^1 \right)_{\mathcal{O}_{1,d} \times (0,T)} \\ &+ \alpha_2 \left(\Lambda_1 v^1 + \Lambda_2 v^2, \Lambda_2 v^2 \right)_{\mathcal{O}_{2,d} \times (0,T)}. \end{aligned} \tag{3.18}$$

We have the following result:

Proposition 50. *Assume that μ_1 and μ_2 are sufficiently large (see Eq. (3.19) below). Then, for each $h = (h_1, h_2) \in [L^2(\omega \times (0,T))]^2$, there exists a unique Nash equilibrium $(\overline{v}^1(h), \overline{v}^2(h))$ in the sense of (3.8)–(3.9).*

Proof. By developing the product of cross terms in (3.18) and applying Young's inequality to them, we obtain

$$\begin{aligned}(Rv, v)_{\mathcal{H}} \geq \mu_1 \|v^1\|^2_{\mathcal{H}_1} + \mu_2 \|v^2\|^2_{\mathcal{H}_2} &- \frac{\alpha_1}{4} \|\Lambda_2 \chi_{\mathcal{O}_{1,d}}\|^2_{\mathcal{H}_{1,d}} \|v^2\|^2_{\mathcal{H}_2} \\ &- \frac{\alpha_2}{4} \|\Lambda_1 \chi_{\mathcal{O}_{2,d}}\|^2_{\mathcal{H}_{2,d}} \|v^1\|^2_{\mathcal{H}_1},\end{aligned}$$

where $\|\cdot\|_{\mathcal{H}_{i,d}}$ denotes the norm in the space $\mathcal{L}(\mathcal{H}_{3-i}, L^2(\mathcal{O}_{i,d} \times (0,T)))$ for $i = 1,2$. Then, for parameters μ_1 and μ_2 large enough such that

$$\begin{aligned} 4\mu_1 &> \alpha_2 \|\Lambda_1 \chi_{\mathcal{O}_{2,d}}\|^2_{\mathcal{H}_{2,d}}, \\ 4\mu_2 &> \alpha_1 \|\Lambda_2 \chi_{\mathcal{O}_{1,d}}\|^2_{\mathcal{H}_{1,d}}, \end{aligned} \tag{3.19}$$

we get

$$(Rv, v)_{\mathcal{H}} \geq \gamma \|v\|^2_{\mathcal{H}}, \quad \gamma = \min_{i=1,2} \left\{ \mu_i - \frac{\alpha_{3-i}}{4} \|\Lambda_i \chi_{\omega_{3-i,d}}\|^2_{\mathcal{H}_{3-i,d}} \right\} > 0. \tag{3.20}$$

We define the functional $a(v,u) : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ as

$$a(v,u) = (Rv,u)_{\mathcal{H}}.$$

Then, from the definition of R and the estimation (3.20), we have that a is a continuous and coercive bilinear form. Applying the Lax-Milgram theorem (see e.g. [11]), we conclude that for all $\phi \in \mathcal{H}$, there exists a unique element $v \in \mathcal{H}$ such that

$$a(v,u) = (\phi,u) \quad \forall u \in \mathcal{H},$$

Indeed, setting $\phi = \left(\alpha_1 \Lambda_1^* \widetilde{y}_d^1, \alpha_2 \Lambda_2^* \widetilde{y}_d^2\right)^t$ (see Eq. (3.17)) we obtain the desired result. Thus, we have proved the existence and uniqueness of the Nash equilibrium related to (J_1, J_2). $\square$

3.2.2 Characterization of the Nash equilibrium

We have shown that for μ_1 and μ_2 large enough, there exist a unique Nash equilibrium for (J_1, J_2). We want to express it in terms of a new adjoint variable. In view of (3.16), we have that $(\overline{v}^1, \overline{v}^2)$ is a Nash equilibirum (in the sense of (3.8)–(3.9)) if and only if

$$\begin{aligned} &\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} \left(y_1 - y_{1,d}^i\right) \widehat{y}_1^i + \left(y_2 - y_{2,d}^i\right) \widehat{y}_2^i dxdt \\ +&\mu_i \iint_{\omega_i\times(0,T)} \overline{v}^i \widehat{v}^i dxdt = 0, \quad \forall \widehat{v}^i \in L^2(\omega_i \times (0,T)),\ i = 1,2, \end{aligned} \tag{3.21}$$

where $\widehat{y}^i = \left(\widehat{y}_1^i, \widehat{y}_2^i\right)^t$ is the solution of system

$$\begin{cases} \widehat{y}_{1,t}^i - \Delta \widehat{y}_1^i + a_{11}\widehat{y}_1^i + a_{12}\widehat{y}_2^i = \widehat{v}^i \chi_{\omega_i} & \text{in } Q, \\ \widehat{y}_{2,t}^i - \Delta \widehat{y}_2^i + a_{21}\widehat{y}_1^i + a_{22}\widehat{y}_2^i = 0 & \text{in } Q, \\ \widehat{y}_j^i(0) = 0 \text{ in } \Omega, \quad \widehat{y}_j^i = 0 \text{ on } \Sigma,\ j = 1,2. \end{cases} \tag{3.22}$$

Let us introduce the adjoint state to (3.22), that is, $p^i = \left(p_1^i, p_2^i\right)^t$ solution of

$$\begin{cases} -p_{1,t}^i - \Delta p_1^i + a_{11}p_1^i + a_{21}p_2^i = \alpha_i \left(y_1 - y_{1,d}^i\right) \chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ -p_{2,t}^i - \Delta p_2^i + a_{12}p_1^i + a_{22}p_2^i = \alpha_i \left(y_2 - y_{2,d}^i\right) \chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ p_j^i(T) = 0 \text{ in } \Omega, \quad p_j^i = 0 \text{ on } \Sigma,\ j = 1,2. \end{cases} \tag{3.23}$$

If we multiply (3.23) by $\widehat{y}^i$ in $L^2(Q)^2$ and integrate by parts, we obtain

$$\iint_Q \alpha_i \left(y_1 - y_{1,d}^i\right) \chi_{\mathcal{O}_{i,d}} \widehat{y}_1^i - a_{21} p_2^i \widehat{y}_1^i dxdt = \iint_Q p_1^i \left(\widehat{v}^i \chi_{\omega_i} - a_{12}\widehat{y}_2^i\right) dxdt,$$

$$\iint_Q \alpha_i \left(y_2 - y_{2,d}^i\right) \chi_{\mathcal{O}_{i,d}} \widehat{y}_2^i dxdt = \iint_Q (-a_{21} p_2^i \widehat{y}_1^i + a_{12} p_1^i \widehat{y}_2^i) dxdt.$$

Adding up the above expressions and replacing on (3.21) we have

$$\iint_{\omega_i\times(0,T)} p_1^i\widehat{v}^i dxdt + \mu_i \iint_{\omega_i\times(0,T)} \overline{v}^i\widehat{v}^i dxdt = 0,$$

which implies that

$$(p_1^i + \mu_i\overline{v}^i)|_{\omega_i} = 0.$$

Therefore, given $h \in [L^2(\omega \times (0,T))]^2$, the pair $(\overline{v}^1, \overline{v}^2)$ is a Nash equilibrium for problem (3.6)–(3.7) if and only if

$$\overline{v}^i = -\frac{1}{\mu_i} p_1^i|_{\omega_i}, \quad i = 1, 2,$$

where p_1 can be found from (y, p^i) solution to the coupled system

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h_1\chi_\omega - \frac{1}{\mu_1}p_1^1\chi_{\omega_1} - \frac{1}{\mu_2}p_1^2\chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = h_2\chi_\omega & \text{in } Q, \\ -p_{1,t}^i - \Delta p_1^i + a_{11}p_1^i + a_{21}p_2^i = \alpha_i\left(y_1 - y_{1,d}^i\right)\chi_{\mathcal{O}_{i,d}} & \text{in } Q, \\ -p_{2,t}^i - \Delta p_2^i + a_{12}p_1^i + a_{22}p_2^i = \alpha_i\left(y_2 - y_{2,d}^i\right)\chi_{\mathcal{O}_{1,d}} & \text{in } Q, \\ y_j(0) = y_j^0,\ p_j^i(T) = 0, \quad y_j = p_j^i = 0 \text{ on } \Sigma,\ i,j = 1,2. \end{cases} \tag{3.24}$$

3.3 Proof of Theorem 47

Recall that the main goal in the hierarchic methodology is to prove the null controllability of (y_1, y_2) at time T. However, the computation of the follower controls satisfying (3.6)-(3.7) added four additional equations coupled to the original system under study. Hence, we now look for a pair $(h_1, h_2) \in [L^2(\omega \times (0,T))]^2$ such that the solution of (3.44) satisfies (3.10)-(3.11).

It is classical by now that null controllability is related to the observability of a proper adjoint system (see, for instance, [64], [30]). For our particular case, let us consider the adjoint system

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \alpha_1\theta_1^1\chi_{\mathcal{O}_{1,d}} + \alpha_2\theta_1^2\chi_{\mathcal{O}_{2,d}} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{12}\varphi_1 + a_{22}\varphi_2 = \alpha_1\theta_2^1\chi_{\mathcal{O}_{1,d}} + \alpha_2\theta_2^2\chi_{\mathcal{O}_{1,d}} & \text{in } Q, \\ \theta_{1,t}^i - \Delta\theta_1^i + a_{11}\theta_1^i + a_{12}\theta_2^i = -\frac{1}{\mu_i}\varphi_1\chi_{\omega_i} & \text{in } Q, \\ \theta_{2,t}^i - \Delta\theta_2^i + a_{21}\theta_1^i + a_{22}\theta_2^i = 0 & \text{in } Q, \\ \varphi_j(T) = f_j,\ \theta_j^i(0) = 0 \text{ in } \Omega, \quad \varphi_j = \theta_j^i = 0 \text{ on } \Sigma,\ i,j = 1,2. \end{cases} \tag{3.25}$$

The main task is to prove an observability inequality for system (3.25). Taking into consideration the assumption (3.12) we can simplify the previous system as follows

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = (\alpha_1\theta_1^1 + \alpha_2\theta_1^2)\chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{12}\varphi_1 + a_{22}\varphi_2 = (\alpha_1\theta_2^1 + \alpha_2\theta_2^2)\chi_{\mathcal{O}_d} & \text{in } Q, \\ \theta_{1,t}^i - \Delta\theta_1^i + a_{11}\theta_1^i + a_{12}\theta_2^i = -\frac{1}{\mu_i}\varphi_1\chi_{\omega_i} & \text{in } Q, \\ \theta_{2,t}^i - \Delta\theta_2^i + a_{21}\theta_1^i + a_{22}\theta_2^i = 0 & \text{in } Q, \\ \varphi_j(T) = f_j,\ \theta_j^i(0) = 0 \text{ in } \Omega, \quad \varphi_j = \theta_j^i = 0 \text{ on } \Sigma,\ j = 1, 2. \end{cases} \tag{3.26}$$

We have the following result:

Proposition 51. *Under assumptions of Theorem 47, there exist a positive constant C and a positive weight function $\rho = \rho(t)$ blowing up at $t = T$ such that*

$$\begin{aligned} \|\varphi_1(0)\|^2_{L^2(\Omega)} + \|\varphi_2(0)\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \iint_Q \rho^{-2} \left(|\theta_1^i|^2 + |\theta_2^i|^2\right) dxdt \\ \leq C \left(\iint_{\omega\times(0,T)} \left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt \right), \end{aligned} \tag{3.27}$$

for any $(f_1, f_2) \in [L^2(\Omega)]^2$, where (φ, θ^i) is the associated solution to (3.28).

Remark 52. It remains an open problem if the required observability inequality holds true when $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$. See [5] for some results in this direction for the scalar case.

The proof of Proposition 51 relies on various well-known arguments. For the moment, suppose that the proposition holds and let us end the proof of Theorem 47. There are several ways to prove that inequality (3.27) implies the existence of a pair (h_1, h_2) of minimum norm. We sketch one of them. First, we prove that

$$\|(f_1, f_2)\|^2_W = \iint_{\omega\times(0,T)} \left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt,$$

where (φ_1, φ_2) are the first two components of the solution to (3.28) defines a norm in $[L^2(\Omega)]^2$. This can be readily verified by means of Proposition 55 below or directly from (3.27), providing a unique continuation property. Then, we define W as the completion of $[L^2(\Omega)]^2$ with this norm and set

$$\begin{aligned} \mathcal{J}(f_1, f_2) =& \frac{1}{2}\|(f_1, f_2)\|^2_W + \int_\Omega y_1^0\varphi_1(0)dx + \int_\Omega y_2^0\varphi_2(0)dx \\ &- \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_d\times(0,T)} \left(\theta_1^i y_{1,d}^i + \theta_2^i y_{2,d}^i\right) dxdt, \end{aligned}$$

where (φ, θ^i) is the solution to (3.28). It is clear that $\mathcal{F}$ is continuous and strictly convex. Moreover, the observability inequality (3.27) allows to prove that

$$\mathcal{J}(f_1, f_2) \geq \frac{1}{4}\|(f_1, f_2)\|_W^2 - C\left(\int_\Omega |y_1^0|^2 dx + \int_\Omega |y_2^0|^2 dx + \sum_{i=1}^{2} \alpha_i^2 \iint_Q \rho^2 \left(|y_{1,d}^i|^2 + |y_{2,d}^i|^2\right) dxdt\right),$$

where C and ρ are provided by Proposition 51. Therefore, $\mathcal{J}$ is coercive in W. Note that here, we have used the growth assumption (3.13). Consequently, from classical results (see, for instance, [30]), the existence of a minimizer $(\widehat{f}_1, \widehat{f}_2)$ solution to

$$\mathcal{J}(\widehat{f}_1, \widehat{f}_2) = \min_{(f_1, f_2) \in W} \mathcal{J}(f_1, f_2)$$

is guaranteed. Thus, the pair $(h_1, h_2) = (\widehat{\varphi}_1 \chi_\omega, \widehat{\varphi}_2 \chi_\omega)$, where $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ is the solution to (3.28) corresponding to this minimizer solves the leader problem (3.10)–(3.11). This concludes the proof of Theorem 47.

3.3.1 Proof of the observability inequality

This section is devoted to the proof of Proposition 51. The observability inequality (3.27) is consequence of a global Carleman inequality and some energy estimates. Here, we follow the spirit of Section 2.3 to present the result.

In view of assumption (3.12), we may simplify (3.26) as

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \psi_1 \chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{12}\varphi_1 + a_{22}\varphi_2 = \psi_2 \chi_{\mathcal{O}_d} & \text{in } Q, \\ \psi_{1,t} - \Delta\psi_1 + a_{11}\psi_1 + a_{12}\psi_2 = -\left(\frac{\alpha_1}{\mu_1}\chi_{\omega_1} + \frac{\alpha_2}{\mu_2}\chi_{\omega_2}\right)\varphi_1 & \text{in } Q, \\ \psi_{2,t} - \Delta\psi_2 + a_{21}\psi_1 + a_{22}\psi_2 = 0 & \text{in } Q, \\ \varphi_j(T) = f_j,\ \psi_j(0) = 0 \text{ in } \Omega, \quad \varphi_j = \psi_j = 0 \text{ on } \Sigma,\ j = 1, 2, \end{cases} \tag{3.28}$$

where $\psi_j = \alpha_1 \theta_j^1 + \alpha_2 \theta_j^2$ for $j = 1, 2$. Using the notation introduced in (2.53)–(2.54), we present below a Carleman inequality for the solutions to system (3.28). This will be essential to prove the observability inequality (3.27).

Proposition 53. *Under assumptions of Theorem 47. There exist positive constants C and σ_1 such that (φ, ψ) solution to (3.28) satisfies*

$$\begin{aligned} &I_3(s, \lambda; \varphi_1) + I_3(s, \lambda; \varphi_2) + I_3(s, \lambda; \psi_1) + I_3(s, \lambda; \psi_2) \\ &\qquad \leq C\left(\iint_{\omega \times (0,T)} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt\right). \end{aligned} \tag{3.29}$$

for any $s \geq s_1 = \sigma_1(T + T^2 + T^2[\max_{1\leq i,j\leq 2} \|a_{ij}\|_\infty^{2/3}])$, *any* $\lambda \geq C$ *and every* $(f_1, f_2) \in [L^2(\Omega)]^2$.

Proof. Let us define $\omega_0 = \omega \cap \mathcal{O}_d$. Since $\omega_0 \neq \emptyset$, there exists some subset $\omega' \subset\subset \omega_0$. We start by applying Carleman inequality (2.55) to each equation in system (3.28) with $m = 3$ and $\mathcal{B} = \omega'$. By adding them up, we obtain

$$\begin{aligned}
&I_3(s,\lambda;\varphi_1) + I_3(s,\lambda;\varphi_2) + I_3(s,\lambda;\psi_1) + I_3(s,\lambda;\psi_2)\\
&\quad\leq C\left(\sum_{j=1}^{2}\left(I_{3,\omega'}(s,\lambda;\varphi_j) + I_{3,\omega'}(s,\lambda;\psi_j)\right) + \sum_{j=1}^{2}\iint_Q e^{-2s\alpha}|\psi_j\chi_{\mathcal{O}_d}|^2 dxdt\right.\\
&\qquad + \iint_Q e^{-2s\alpha}\left|-\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} - \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2}\right|^2 dxdt\\
&\qquad \left.+ \sum_{i=1}^{2}\sum_{j=1}^{2}\iint_Q e^{-2s\alpha}\left(\|a_{ji}\|_\infty^2|\varphi_j|^2 + \|a_{ij}\|_\infty^2|\psi_j|^2\right)dxdt\right).
\end{aligned}$$

Taking the parameters s and λ large enough we can absorb the lower order terms into the left-hand side in the previous inequality. More precisely, we have

$$\begin{aligned}
&I_3(s,\lambda;\varphi_1) + I_3(s,\lambda;\varphi_2) + I_3(s,\lambda;\psi_1) + I_3(s,\lambda;\psi_2)\\
&\qquad\qquad \leq C\left(\sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\varphi_j) + \sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\psi_j)\right),
\end{aligned} \tag{3.30}$$

valid for every $\lambda \geq C$ and every

$$s \geq s_1 = \sigma_1(T + T^2 + T^2[\max_{1\leq i,j\leq 2}\|a_{ij}\|_\infty^{2/3}]).$$

The next step is to eliminate the local terms corresponding to ψ_1 and ψ_2. We will reason out as in Chapter 2. We consider a function $\zeta \in C^\infty(\mathbb{R}^N)$ verifying:

$$\begin{gathered}
0 \leq \zeta \leq 1 \text{ in } \Omega, \quad \zeta \equiv 1 \quad \text{in } \omega', \quad \operatorname{supp}\zeta \subset \omega_0,\\
\frac{\Delta\zeta}{\zeta^{1/2}} \in L^\infty(\Omega), \quad \frac{\nabla\zeta}{\zeta^{1/2}} \in L^\infty(\Omega)^N.
\end{gathered} \tag{3.31}$$

Define $u := e^{-2s\alpha}s^3\lambda^4\xi^3$. Then, we multiply the equations satisfied by φ_1 and φ_2 in system (3.28) by $u\zeta\psi_1$ and $u\zeta\psi_2$, respectively, and integrate over Q. We add those expressions to obtain

$$\begin{aligned}
\iint_Q u\zeta\left(|\psi_1|^2 + |\psi_2|^2\right)\chi_{\mathcal{O}_d} &= \iint_Q u\zeta\psi_1(-\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2)\\
&\quad + \iint_Q u\zeta\psi_2(-\varphi_{2,t} - \Delta\varphi_1 + a_{12}\varphi_1 + a_{22}\varphi_2).
\end{aligned} \tag{3.32}$$

Following the arguments in Section 2.3, we can integrate several times with respect to the time and space variables in the right hand side of the above expression to obtain the following

$$\begin{aligned} I_{3,\omega'}(s,\lambda;\psi_1)+I_{3,\omega'}(s,\lambda;\psi_2) &\leq \varepsilon C_A\left(I_3(s,\lambda;\psi_1)+I_3(s,\lambda;\psi_2)\right) \\ &+C_{\varepsilon,A}\left(\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^7\lambda^8\xi^7|\varphi_1|^2+\iint_{\omega_0\times(0,T)} e^{-2s\alpha}s^7\lambda^8\xi^7|\varphi_2|^2\right), \end{aligned} \tag{3.33}$$

where $\varepsilon>0$ and C_A, $C_{\varepsilon,A}$ are new constants only depending on Ω, ω', ω and $\|a_{ij}\|_\infty$. Replacing (3.33) in (3.50) with ε small enough and noting that $\omega_0\subset\omega$, we obtain the desired inequality. This concludes the proof of Proposition 53. □

As in Section 2.3, we are going to improve inequality (3.29) in the sense that the weight functions do not vanish at $t=0$. We consider the function

$$l(t)=\begin{cases} T^2/4 & \text{for} \quad 0\leq t\leq T/2, \\ t(T-t) & \text{for} \quad T/2\leq t\leq T, \end{cases}$$

and the functions

$$\begin{aligned} &\beta(x,t)=\frac{e^{4\lambda\|\eta^0\|_\infty}-e^{\lambda(2\|\eta^0\|_\infty+\eta^0(x))}}{l(t)}, \quad \gamma(x,t)=\frac{e^{\lambda(2\|\eta^0\|_\infty+\eta^0(x))}}{l(t)}, \\ &\beta^*(t)=\max_{x\in\overline{\Omega}}\beta(x,t), \quad \gamma^*(t)=\min_{x\in\overline{\Omega}}\gamma(x,t). \end{aligned}$$

With these definitions, we have the following

Proposition 54. *Let s and λ as in Proposition 53 and μ_i be large enough. Then there exists a positive constant C depending on Ω, ω, ω_d, s, λ and T such that*

$$\begin{aligned} &\|\varphi_1(0)\|^2_{L^2(\Omega)}+\|\varphi_2(0)\|^2_{L^2(\Omega)}+\iint_Q e^{-2s\beta^*}(\gamma^*)^3\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt \\ &\quad+\iint_Q e^{-2s\beta^*}(\gamma^*)^3\left(|\psi_1|^2+|\psi_2|^2\right)dxdt\leq C\left(\iint_{\omega\times(0,T)} e^{-2s\beta}\gamma^7\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt\right), \end{aligned} \tag{3.34}$$

for any $(f_1,f_2)\in[L^2(\Omega)]^2$, where (φ,ψ) is the associated solution to (3.28).

Proof. We follow the arguments of the proof of Proposition 42. First, by construction $\alpha=\beta$ and $\xi=\gamma$ in $\Omega\times(T/2,T)$, hence

$$\begin{aligned} &\int_{T/2}^T\int_\Omega e^{-2s\alpha}\xi^3\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt+\int_{T/2}^T\int_\Omega e^{-2s\alpha}\xi^3\left(|\psi_1|^2+|\psi_2|^2\right)dxdt \\ &=\int_{T/2}^T\int_\Omega e^{-2s\beta}\gamma^3\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt+\int_{T/2}^T\int_\Omega e^{-2s\beta}\gamma^3\left(|\psi_1|^2+|\psi_2|^2\right)dxdt \end{aligned}$$

Therefore, from (3.29) and the definition of β and γ we obtain

$$\int_{T/2}^{T}\int_{\Omega} e^{-2s\beta}\gamma^3\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt+\int_{T/2}^{T}\int_{\Omega} e^{-2s\beta}\gamma^3\left(|\psi_1|^2+|\psi_2|^2\right)dxdt \leq C\left(\iint_{\omega_0\times(0,T)} e^{-2s\beta}\gamma^7\left(|\varphi_1|^2+|\varphi_2|^2\right)dxdt\right) \tag{3.35}$$

On the other hand, for the domain $\Omega\times(0,T/2)$, we will use energy estimates for system (3.28). In fact, let us introduce a function $\eta\in C^1([0,T])$ such that

$$\eta=1 \text{ in } [0,T/2],\quad \eta=0 \text{ in } [3T/4,T],\quad |\eta'(t)|\leq C/T.$$

Using classical energy estimates for $\eta\varphi_1$ and $\eta\varphi_2$ solution to the first and second equation of system (3.28) we obtain

$$\begin{aligned}&\|\varphi_1(0)\|^2_{L^2(\Omega)}+\|\varphi_2(0)\|^2_{L^2(\Omega)}+\|\varphi_1\|^2_{L^2(0,T/2;H^1_0(\Omega))}+\|\varphi_2\|^2_{L^2(0,T/2;H^1_0(\Omega))}\\&\leq C\left(\frac{1}{T^2}\|\varphi_1\|^2_{L^2(T/2,3T/4;L^2(\Omega))}+\frac{1}{T^2}\|\varphi_2\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right.\\&\qquad\left.+\|\eta\psi_1\|^2_{L^2(0,3T/4;L^2(\Omega))}+\|\eta\psi_2\|^2_{L^2(0,3T/4;L^2(\Omega))}\right).\end{aligned}$$

From the definition of η and adding $\|\psi_j\|^2_{L^2(0,T/2;L^2(\Omega))}$ on both sides of the previous inequality we have

$$\begin{aligned}&\|\varphi_1(0)\|^2_{L^2(\Omega)}+\|\varphi_2(0)\|^2_{L^2(\Omega)}+\sum_{i=1}^{2}\|\varphi_j\|^2_{L^2(0,T/2;L^2(\Omega))}+\sum_{j=1}^{2}\|\psi_j\|^2_{L^2(0,T/2;L^2(\Omega))}\\&\leq C\left(\sum_{j=1}^{2}\|\varphi_j\|^2_{L^2(T/2,3T/4;L^2(\Omega))}+\sum_{j=1}^{2}\|\psi_j\|^2_{L^2(T/2,3T/4;L^2(\Omega))}+\sum_{j=1}^{2}\|\psi_j\|^2_{L^2(0,T/2;L^2(\Omega))}\right).\end{aligned} \tag{3.36}$$

In order to eliminate the terms $\|\psi_j\|^2_{L^2(0,T/2;L^2(\Omega))}$ in the right hand side, we use standard energy estimates for the third and fourth equation in (3.28), thus

$$\iint_{\Omega\times(0,T/2)}(|\psi_1|^2+|\psi_2|^2)dxdt\leq C\left(\frac{\alpha_1^2}{\mu_1^2}+\frac{\alpha_2^2}{\mu_2^2}\right)\iint_{\Omega\times(0,T/2)}|\varphi_1|^2dxdt. \tag{3.37}$$

Replacing (3.37) in (3.38) and since μ_i, $i=1,2$, are large enough we obtain

$$\begin{aligned}&\|\varphi_1(0)\|^2_{L^2(\Omega)}+\|\varphi_2(0)\|^2_{L^2(\Omega)}+\sum_{i=1}^{2}\|\varphi_j\|^2_{L^2(0,T/2;L^2(\Omega))}+\sum_{j=1}^{2}\|\psi_j\|^2_{L^2(0,T/2;L^2(\Omega))}\\&\leq C\left(\sum_{j=1}^{2}\|\varphi_j\|^2_{L^2(T/2,3T/4;L^2(\Omega))}+\sum_{j=1}^{2}\|\psi_j\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right).\end{aligned} \tag{3.38}$$

Using (3.35) to estimate the first four terms in the right hand side of (3.38) and taking into account that the weight functions are bounded in $[0, 3T/4]$ we have the estimate

$$\|\varphi_1(0)\|^2_{L^2(\Omega)} + \|\varphi_2(0)\|^2_{L^2(\Omega)} + \int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\gamma^3\left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt$$
$$+ \int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\gamma^3\left(|\psi_1|^2 + |\psi_2|^2\right) dxdt \le C\left(\iint_{\omega\times(0,T)} e^{-2s\beta}\gamma^7\left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt\right).$$

This estimate, together with (3.35), and the definitions of γ^* and β^* yield the desired inequality (3.34). □

Now we conclude the proof of Proposition 51. To this end, define $\rho(t) = e^{s\beta^*}$. Thus, $\rho(t)$ is a non-decreasing strictly positive function blowing up at $t = T$. We obtain energy estimates with this new weight function for (θ_1^i, θ_2^i) solution to the third and fourth equation of system (3.26). More precisely,

$$\iint_Q \rho^{-2}(|\theta_1^i|^2 + |\theta_2^i|^2) dxdt \le C \iint_{\omega_i\times(0,T)} \rho^{-2}|\varphi_1|^2 dxdt, \quad i = 1, 2.$$

Since $e^{-2s\beta}\gamma^7 \le C$ for all $(x,t) \in Q$ and noting that the right hand side of the previous inequality is comparable to the left hand side of inequality (3.34) up to a multiplicative constant, we obtain (3.27). This concludes the proof of Proposition 51.

3.4 Proof of Theorem 48

In this section, we present a Stackelberg-Nash strategy where only one leader control is applied. As mentioned before, one important subject in the controllability of non-scalar systems is the possibility to control many equations with few controls.

Let us consider system (3.1) with $h_2 \equiv 0$, namely,

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h\chi_\omega + v^1\chi_{\omega_1} + v^2\chi_{\omega_2} \quad \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = 0, \quad \text{in } Q, \\ y_j(x,0) = y_j^0(x) \text{ in } \Omega, \quad y_j = 0 \text{ on } \Sigma,\ j = 1,2, \end{cases} \tag{3.39}$$

where $a_{ij} \in L^\infty(Q)$ and $y_j^0 \in L^2(\Omega)$ are given. Observe that in (3.39) the leader and follower controls act only on the right-hand side of the first equation.

In order to achieve the hierarchic control described in Section 3.1, we need to introduce some changes in the functionals to be minimized. In fact, the modification to the main functional is straightforward, we consider in this case

$$J(h) = \frac{1}{2}\iint_{\omega\times(0,T)} |h|^2 dxdt. \tag{3.40}$$

On the other hand, we will modify the follower functionals by adding a weighted norm in (3.2). To this end, consider $\alpha(x,t)$ as in (2.53). We write

$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \qquad \widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t) \tag{3.41}$$

Then, for a given function $\rho_* = \rho_*(t)$ verifying

$$\rho_*(t) \geq e^{s\alpha*/2}, \tag{3.42}$$

we take

$$\begin{aligned} J_i(h,v^1,v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_d\times(0,T)} |y_1 - y^i_{1,d}|^2 + |y_2 - y^i_{2,d}|^2 dxdt \\ + \frac{\mu_i}{2} \iint_{\omega_i\times(0,T)} \rho_*^2 |v^i|^2 dxdt,\ i=1,2, \end{aligned} \tag{3.43}$$

where $\alpha_i, \mu_i > 0$ are constants and $y^i_d = (y^i_{1,d}, y^i_{2,d})^t$ are given functions in $L^2(\omega_{i,d}\times(0,T))$, $i=1,2$.

Note that a function $\rho_*(t)$ satisfying (3.42) must blow up as $t\to 0$ and $t\to T$. Therefore, minimizing the functionals (3.43) will lead to follower controls v^i vanishing at $t=0$ and $t=T$. This subtle change allow us to eliminate the local term corresponding to φ_2 in (3.34) and then we can obtain an observability inequality with only φ_1 in the right-hand side.

By adapting the methods discussed in Section 3.2, it can be readily verified that the pair $(\bar{v}_1, \bar{v}_2)$ is a Nash equilibrium for (3.43) if and only if

$$\bar{v}_i = -\frac{1}{\mu_i}\rho_*^{-2} p^i_1,\ i=1,2,$$

where p^i_j, y_j, $i,j=1,2$, are solution to the system:

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h\chi_\omega - \frac{1}{\mu_1}\rho_*^{-2}p^1_1\chi_{\omega_1} - \frac{1}{\mu_2}\rho_*^{-2}p^2_1\chi_{\omega_2} & \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = 0 & \text{in } Q, \\ -p^i_{1,t} - \Delta p^i_1 + a_{11}p^i_1 + a_{21}p^i_2 = \alpha_i\left(y_1 - y^i_{1,d}\right)\chi_{\mathcal{O}_d} & \text{in } Q, \\ -p^i_{2,t} - \Delta p^i_2 + a_{12}p^i_1 + a_{22}p^i_2 = \alpha_i\left(y_2 - y^i_{2,d}\right)\chi_{\mathcal{O}_d} & \text{in } Q, \\ y_j(0) = y^0_j,\ p^i_j(T) = 0, \quad y_j = p^i_j = 0 \text{ on } \Sigma,\ i,j=1,2. \end{cases} \tag{3.44}$$

Indeed, since $\rho_* \geq C$ for some $C>0$ independent of μ_i, we can obtain bounds similar to (3.19) and apply the results of Proposition 50 to obtain the existence and uniqueness of the Nash equilibrium. This proves the first part of Theorem 48.

To finish the proof, we need to establish an appropriate observability estimate. This can be obtained by following exactly the proof in Section 3.3.1 but introducing the weight function ρ_*^{-2}. We prove here a Carleman inequality for the "simplified" adjoint system to (3.44) (see the change of variable introduced in Eq. (3.28)). To do this, consider the system:

$$\begin{cases} -\varphi_{1,t} - \Delta\varphi_1 + a_{11}\varphi_1 + a_{21}\varphi_2 = \psi_1\chi_{\mathcal{O}_d} & \text{in } Q, \\ -\varphi_{2,t} - \Delta\varphi_2 + a_{12}\varphi_1 + a_{22}\varphi_2 = \psi_2\chi_{\mathcal{O}_d} & \text{in } Q, \\ \psi_{1,t} - \Delta\psi_1 + a_{11}\psi_1 + a_{12}\psi_2 = -\rho_*^{-2}\left(\frac{\alpha_1}{\mu_1}\chi_{\omega_1} + \frac{\alpha_2}{\mu_2}\chi_{\omega_2}\right)\varphi_1 & \text{in } Q, \\ \psi_{2,t} - \Delta\psi_2 + a_{21}\psi_1 + a_{22}\psi_2 = 0 & \text{in } Q, \\ \varphi_j(T) = f_j,\ \psi_j(0) = 0 \text{ in } \Omega, \quad \varphi_j = \psi_j = 0 \text{ on } \Sigma,\ j = 1,2. \end{cases} \tag{3.45}$$

We have the following result:

Proposition 55. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and that μ_i, $i = 1,2$, are sufficiently large. There exists a positive constant C such that the solution (φ, ψ) to (3.45) satisfies*

$$\begin{aligned} \iint_Q e^{-2s\alpha}(s\xi)^3(|\varphi_1|^2 + |\varphi_2|^2)dxdt + \iint_Q e^{-2s\alpha}(s\xi)^3(|\psi_1|^2 + |\psi_2|^2)dxdt \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2 dxdt. \end{aligned} \tag{3.46}$$

for every $(f_1, f_2) \in [L^2(\Omega)]^2$.

Proof. The first part of the proof is similar to the proof of Proposition 53. We define $\omega_0 := \omega \cap \mathcal{O}_d$ and consider subsets ω', $\tilde{\omega}$ such that $\omega' \subset\subset \tilde{\omega} \subset\subset \omega_0$.

Note that systems (3.45) and (3.28) are the same except for the third equation. We apply Carleman inequality (2.55) to each equation in (3.45) with $m = 3$ and $\mathcal{B} = \omega'$. Adding them up and arguing as in the proof of Proposition 53 we can use the parameters s and λ to absorb the lower order terms. More precisely, we obtain

$$\begin{aligned} &I_3(s,\lambda;\varphi_1) + I_3(s,\lambda;\varphi_2) + I_3(s,\lambda;\psi_1) + I_3(s,\lambda;\psi_2) \\ &\leq C\left(\sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\varphi_j) + \sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\psi_j)\right. \\ &\qquad \left. + \iint_Q e^{-2s\alpha}|-\rho_*^{-2}(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2 dxdt\right), \end{aligned} \tag{3.47}$$

for all λ and s large enough.

We will estimate the last term in the above expression. From (3.41) and (3.42), we have that

$$\iint_Q e^{-2s\alpha}|-\rho_*^{-2}(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2 dxdt \leq \iint_Q e^{-4s\alpha}|(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2. \tag{3.48}$$

On the other hand, it is not difficult to see that for all $\varepsilon > 0$ and any $M \geq 0$, there exists $C_{\varepsilon,M} > 0$ such that

$$e^{s\alpha^*} \leq C e^{s(1+\varepsilon)\widehat{\alpha}} s^M \lambda^M \xi^M,$$

thus, taking $\varepsilon = 1$ and $M = 3/2$ in the above expression, we deduce from (3.48) that

$$\begin{aligned}\iint_Q e^{-4s\alpha} |(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2 &\leq \iint_Q e^{-4s\widehat{\alpha}} |(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2 \\ &\leq C \iint_Q e^{-2s\alpha^*} s^3\lambda^3\xi^3 |(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2 \\ &\leq C \iint_Q e^{-2s\alpha} s^3\lambda^3\xi^3 |(\tfrac{\alpha_1}{\mu_1}\varphi_1\chi_{\omega_1} + \tfrac{\alpha_2}{\mu_2}\varphi_1\chi_{\omega_2})|^2. \quad (3.49)\end{aligned}$$

Note that the power of λ on the above estimate is lower than in the term $I_3(s,\lambda;\varphi_1)$. We substitute (3.49) in (3.47) and since μ_i are large enough, we can take $\lambda \geq C$ (with $C > 0$ not depending on μ_i) to absorb the remaining terms and obtain

$$\begin{aligned}&I_3(s,\lambda;\varphi_1) + I_3(s,\lambda;\varphi_2) + I_3(s,\lambda;\psi_1) + I_3(s,\lambda;\psi_2) \\ &\quad \leq C\left(\sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\varphi_j) + \sum_{j=1}^{2} I_{3,\omega'}(s,\lambda;\psi_j)\right).\end{aligned} \tag{3.50}$$

Proceeding exactly as before (see Eqs. (3.31)–(3.33)), we can eliminate from the right-hand side the local terms in ψ_1 and ψ_2, thus obtaining

$$\begin{aligned}&I_3(s,\lambda;\varphi_1) + I_3(s,\lambda;\varphi_2) + I_3(s,\lambda;\psi_1) + I_3(s,\lambda;\psi_2) \\ &\quad \leq C\left(\iint_{\tilde{\omega}\times(0,T)} e^{-2s\alpha} s^7\lambda^8\xi^7 \left(|\varphi_1|^2 + |\varphi_2|^2\right) dxdt\right).\end{aligned} \tag{3.51}$$

Up to here, we have the same result as in Proposition 53. Now, we want to eliminate the local terms corresponding to φ_2 in the right-hand side of (3.51). Condition (3.15) and the weight function ρ_* will be useful in this step.

We fix s and λ to a sufficiently large values. Given the subset $\tilde{\omega}$, we consider a function $\tilde{\eta} \in C^\infty(\mathbb{R}^N)$ verifying:

$$0 \leq \tilde{\eta} \leq 1 \text{ in } \Omega, \quad \tilde{\eta} \equiv 1 \text{ in } \tilde{\omega}, \quad \operatorname{supp}\tilde{\eta} \subset \omega_0$$

$$\frac{\Delta\tilde{\eta}}{\tilde{\eta}^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\nabla\tilde{\eta}}{\tilde{\eta}^{1/2}} \in L^\infty(\Omega)^N.$$

We denote $\tilde{u} = e^{-2s\alpha}(s\xi)^7$ to abridge the notation. Recall that the coefficient a_{21} satisfies (3.15) and, for simplicity, assume that $a_{21} \geq a_0$ in $\omega \cap \mathcal{O}_d \times (0,T)$. We multiply the equation satisfied by φ_1 in system (3.45) by $u\tilde{\eta}\varphi_2$ and integrate in Q. We obtain

$$\begin{aligned} a_0 \iint_{\tilde{\omega}\times(0,T)} e^{-2s\alpha}(s\xi)^7|\varphi_2| \leq& \iint_Q \tilde{u}\tilde{\eta}a_{21}|\varphi_2|^2 \\ =& \iint_Q (\varphi_{1,t} + \Delta\varphi_1 - a_{11}\varphi_1)\tilde{u}\tilde{\eta}\varphi_2 + \iint_Q \psi_1\chi_{\mathcal{O}_d}\tilde{u}\tilde{\eta}\varphi_2 \\ :=& \sum_{n=1}^{4} K_n. \end{aligned} \tag{3.52}$$

We proceed to estimate each of the terms K_i. We have

$$\begin{aligned} |K_1| =& \left|\iint_Q e^{-2s\alpha}(s\xi)^7\tilde{\eta}\varphi_2\varphi_{1,t}dxdt\right| \\ =& \left|\iint_Q (e^{-2s\alpha}(s\xi)^7\tilde{\eta}\varphi_2)_t\varphi_1 dxdt\right| \\ \leq& \varepsilon_1 \iint_Q e^{-2s\alpha}(s\xi)^{-1}|\varphi_{2,t}|^2dxdt + \varepsilon_2 \iint_Q e^{-2s\alpha}(s\xi)^3|\varphi_2|^2dxdt \\ &+ (C_{\varepsilon_1} + C_{\varepsilon_2}) \iint_{\omega_0\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2dxdt, \end{aligned}$$

for some $\varepsilon_1, \varepsilon_2 > 0$. Then, it is not difficult to see that

$$\begin{aligned} |K_2| =& \left|\iint_Q e^{-2s\alpha}(s\xi)^7\tilde{\eta}\varphi_2\Delta\varphi_1 dxdt\right| \\ =& \left|\iint_Q \Delta(e^{-2s\alpha}(s\xi)^7\tilde{\eta}\varphi_2)\varphi_1 dxdt\right| \\ \leq& \varepsilon_3 \iint_Q e^{-2s\alpha}(s\xi)^{-1}|\Delta\varphi_2|^2dxdt + C_{\varepsilon_3}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2dxdt \\ &+ \varepsilon_4 \iint_Q e^{-2s\alpha}s\xi|\nabla\varphi_2|^2dxdt + C_{\varepsilon_4}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2dxdt \\ &+ \varepsilon_5 \iint_Q e^{-2s\alpha}(s\xi)^3|\varphi_2|^2dxdt + C_{\varepsilon_5}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2dxdt. \end{aligned}$$

The estimate of K_3 is straightforward. For K_4 we get

$$\begin{aligned} |K_4| =& \left|\iint_Q e^{-2s\alpha}(s\xi)^7\tilde{\eta}\varphi_2\psi_1 dxdt\right| \\ \leq& \frac{1}{2}\iint_Q e^{-4s\alpha}(s\xi)^{14}\tilde{\eta}|\varphi_2|^2dxdt + \frac{1}{2}\iint_Q |\psi_1|^2dxdt. \end{aligned}$$

Observe that given $\varepsilon_6 > 0$, we have $e^{-2s\alpha}(s\xi)^{11} < \varepsilon_6$ for s large enough, whence

$$|K_4| \leq \frac{\varepsilon_6}{2}\iint_Q e^{-2s\alpha}(s\xi)^3\tilde{\eta}|\varphi_2|^2dxdt + \frac{1}{2}\iint_Q |\psi_1|^2dxdt.$$

Putting all the estimates together, and choosing appropriate constants ε_i, $i = 1, \dots, 6$, we obtain from (3.51) and (3.52)

$$\begin{aligned}\iint_Q e^{-2s\alpha}(s\xi)^3(|\varphi_1|^2 + |\varphi_2|^2)dxdt + \iint_Q e^{-2s\alpha}(s\xi)^3(|\psi_1|^2 + |\psi_2|^2)dxdt \\ \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}(s\xi)^{15}|\varphi_1|^2dxdt + C\iint_Q |\psi_1|^2 dxdt.\end{aligned} \tag{3.53}$$

To eliminate the last term in the right hand side of the previous equation, we obtain energy estimates for the third and fourth equation in system (3.45), more precisely

$$\begin{aligned}\iint_Q (|\psi_1|^2 + |\psi_2|^2)dxdt &\leq C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2}\right)\iint_Q |\varphi_1\rho_*^{-2}|^2dxdt, \\ &\leq C\left(\frac{\alpha_1^2}{\mu_1^2} + \frac{\alpha_2^2}{\mu_2^2}\right)\iint_Q e^{-2s\alpha^*}|\varphi_1|^2dxdt.\end{aligned}$$

Since $e^{-2s\alpha^*} \leq e^{-2s\alpha}$ and provided that μ_i are large enough, we can put the above estimate in (3.53) and absorb the remaining term into the left hand side. Therefore the proof is complete. □

With the new Carleman estimate (3.46), we can obtain an observability inequality following the procedure of Section 3.3.1 . Such inequality will only have φ_1 as an observation term in the right-hand side and will imply the null controllability of (3.44). This concludes the proof of Theorem 48.

3.5 Concluding remarks

The first main result of this chapter can be easily extended to the control problem

$$\begin{cases} y_{1,t} - \Delta y_1 + a_{11}y_1 + a_{12}y_2 = h_1\chi_{\omega_1} + v\chi_{\mathcal{O}} \quad \text{in } Q, \\ y_{2,t} - \Delta y_2 + a_{21}y_1 + a_{22}y_2 = h_2\chi_{\omega_2} \quad \text{in } Q, \\ y_j(x,0) = y_{j,0} \text{ in } \Omega, \quad y_j = 0 \text{ on } \Sigma, \ j = 1,2. \end{cases} \tag{3.54}$$

as long as $\omega_1 \cap \omega_2 \neq \emptyset$. Indeed, it is enough to consider a set $\omega_0 \subset\subset \omega_1 \cap \omega_2$ and then apply the results of this paper to this new set to obtain a hierarchic control result. However, the same is not true when $\omega_1 \cap \omega_2 = \emptyset$. The techniques shown in this chapter fail to obtain an observability inequality as (3.27) since we cannot use Carleman estimates with different weights (related to ω_1 and ω_2) and eliminate all the local terms that appear on the right hand side. Indeed, to eliminate some of them we will need an upper estimation on the first Carleman weight by the second, and to eliminate the others we will need the contrary. This is due that we have a system of four equations fully coupled.

On the other hand, one would expect to use results of simultaneous control (see, for instance, [53]) when the same control is applied in both equations, that is, $h = h_1 = h_2$. By means of the transformation $\tilde{y}_1 = y_1 + y_2$, $\tilde{y}_2 = y_1 - y_2$ one can transform (3.54) into

$$\begin{cases} \tilde{y}_{1,t} - \Delta\tilde{y}_1 + \tilde{a}_{11}\tilde{y}_1 + \tilde{a}_{12}\tilde{y}_2 = h\chi_\omega + v\chi_\mathcal{O} \quad \text{in } Q, \\ \tilde{y}_{2,t} - \Delta\tilde{y}_2 + \tilde{a}_{21}\tilde{y}_1 + a_{22}\tilde{y}_2 = v\chi_\mathcal{O} \quad \text{in } Q, \\ \tilde{y}_j(x,0) = \tilde{y}_{j,0} \text{ in } \Omega, \quad y_j = 0 \text{ on } \Sigma, \ j = 1,2, \end{cases}$$

for some new coefficients $\tilde{a}_{ij} \in L^\infty(Q)$. However, the result of hierarchic control when only a single control is applied uses the modified functional (3.14) which is different from the one we have employed in the result with two controls (see Theorem 47). Moreover, we can design the follower control v but when returning to the original variable, the follower objective is no longer fulfilled.

The hierarchic control is an interesting and challenging problem because there are many available configurations (where the leader and follower controls may be placed) and several controllability constraints that may be imposed. As discussed in [6], some problems have been solved for the scalar problem, but other difficulties arise when dealing with coupled systems. Thus, the results are far from being complete.

Part II

On numerical results for the insensitizing control problem

Chapter 4

Insensitizing controls for the heat equation: a numerical approach

4.1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded and open set with boundary $\partial\Omega \in \mathcal{C}^2$. Let $T > 0$ and ω be an open and non empty subset of Ω. We consider the following parabolic system

$$\begin{cases} \partial_t y - \Delta y + f(y) = \mathbf{1}_\omega v + \xi \quad \text{in } Q = \Omega \times (0,T), \\ y = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y_0 + \tau w_0 \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where f is a globally Lipschitz-continuous function, ξ and y_0 are given in $L^2(Q)$ and $L^2(\Omega)$, respectively. In (4.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is a control function supported in ω.

The data of equation (4.1) are incomplete in the following sense:

- $w_0 \in L^2(\Omega)$ is unknown and $|w_0|_{L^2(\Omega)} = 1$,
- $\tau \in \mathbb{R}$ is unknown and small enough.

Let Ψ be a differentiable functional defined on the set of solutions to (4.1). We say that the control h insensitizes $\Psi(y)$ if

$$\left| \frac{\partial \Psi(y(x,t;v,\tau))}{\partial \tau} \Big|_{\tau=0} \right| = 0, \quad \forall w_0 \in L^2(\Omega) \text{ with } |w_0|_{L^2(\Omega)} = 1. \tag{4.2}$$

When (4.2) holds the functional Ψ is locally insensitive to the perturbation τw_0. There are several possible choices of Ψ. One possible choice of Ψ is the square of the L^2-norm of the state in some observation subset $\mathcal{O} \subset \Omega$, namely,

$$\Psi(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} y^2 \, dx dt. \tag{4.3}$$

It is by now well known that the insensitivity condition (4.2) is equivalent to a null control problem. This equivalence is given in the following result.

Proposition 56. *Let us consider the following cascade system of heat equations:*

$$\begin{cases} \partial_t y - \Delta y + f(y) = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0 & \text{in } \Omega, \end{cases} \tag{4.4}$$

$$\begin{cases} -\partial_t q - \Delta q + f'(y) q = \mathbf{1}_{\mathcal{O}} y & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

Then, the insensitivity condition (4.2) is equivalent to find v such that

$$q(0) = 0. \tag{4.6}$$

Observe that (4.6) is precisely a null controllability property for the cascade system (4.4)-(4.5). However, this situation is more complex than a standard control problem. In fact, two main difficulties arise. On one hand, the control v acts indirectly on the equation satisfied by q by means of the variable y. On the other, note that (4.4) is forward in time while (4.5) is backward in time. The irreversibility of the heat equation imposes additional difficulties that do not appear in classical cascade systems in which both equations are in the same direction of time (see [37]).

This problem, originally addressed by Lions [49], has been thoroughly studied in different contexts. In [12], the authors relaxed condition (4.2) as follows: given $\varepsilon > 0$, the control v is said to ε-insensitize Ψ if

$$\left| \frac{\partial \Psi(y(x,t;v,\tau))}{\partial \tau} \Big|_{\tau=0} \right| \leq \varepsilon.$$

More precisely, ε-insensitivity is equivalent to $|q(0)|_{L^2(\Omega)} \leq \varepsilon$, which corresponds to an approximate controllability problem, instead of a null control problem. In this context, the authors proved the existence of such controls in the presence of both unknown initial and boundary data. In [61], two main results are given. On one hand, the author proved that we cannot expect the existence of insensitizing controls for every $y_0 \in L^2(\Omega)$ when $\Omega \backslash \overline{\omega} \neq \emptyset$, even if $f = 0$. On the other hand, for $y_0 = 0$ and a suitable hypothesis on the source term ξ, the author proved the existence of insensitizing controls such that (4.2) holds. This result was generalized in [13] and [14] to nonlinearities with certain superlinear growth and nonlinear terms depending on the state y and its gradient. Regarding the class of initial data y_0 that can be insensitized, the work of de Teresa and Zuazua [62] gives different results of positive and negative nature. More recently, there are many works within the context of insensitizing controls for other functionals rather than (4.3) and equations of different nature. For instance, in [39], the author considers a functional involving the gradient of the state for a linear heat system and in [38] treats the case of the curl of the solution for a Stokes system. In [40] and [21], the authors studied the insensitizing controls of the Navier-Stokes equation and the Boussinesq system.

4.1.1 Statement of the problem

In this article, we are interested in studying the insensitizing control problem from another perspective. The main goal of this paper is to present methods and results concerning the numerical computation of insensitizing controls for one dimensional parabolic problems.

Basically, the strategy is as follows: first we build a semi-discrete approximation of the PDE under study and by means of semi-discrete Carleman estimates we deduce a “relaxed” observability inequality. This allows us to establish the existence of insensitizing controls within this framework. Then, we will use the penalized HUM approach discussed in [15] to actually compute the controls.

We begin by considering the following 1-D semi-discrete system

$$\begin{cases} \partial_t y_h + \mathcal{A}^{\mathfrak{M}} y_h + f(y_h) = \mathbf{1}_\omega v_h + \xi_h & \text{in } Q = (0,L)\times(0,T), \\ y_h = 0 \quad \text{on } \Sigma = \{0,L\}\times(0,T), \\ y_h(0) = y_{h,0} + \tau w_{h,0} & \text{in } (0,L). \end{cases} \tag{4.7}$$

where f is a C^1 globally Lipschitz-continuous function, with $f(0) = 0$. Here $\mathcal{A}^{\mathfrak{M}}$ is the discrete approximation of $\mathcal{A} := -\partial_x^2$ for a mesh $\mathfrak{M}$ with step size h. These notions will be precisely introduced below. As in the continuous case, we are interested in proving the existence of controls that insensitize the functional

$$\Psi(y_h) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |y_h|^2 dxdt, \tag{4.8}$$

where y_h is the solution to (4.7). Following the ideas of the continuous case, it can be proved that the insensitizing control problem for (4.7) is equivalent to steer $q_h(0)$ to 0 where (y_h, q_h) is the solution to

$$\begin{cases} \partial_t y_h + \mathcal{A}^{\mathfrak{M}} y_h + f(y_h) = \mathbf{1}_\omega v_h + \xi_h & \text{in } Q, \\ -\partial_t q_h + \mathcal{A}^{\mathfrak{M}} q_h + f'(y_h) q_h = \mathbf{1}_{\mathcal{O}} y_h & \text{in } Q, \\ y_h = q_h = 0 \quad \text{on } \Sigma, \\ y_h(0) = y_h^0, \quad q_h(T) = 0 & \text{in } (0,L). \end{cases} \tag{4.9}$$

To accomplish this, we follow the strategy outlined in [61], but taking into account the particularities associated with the semi-discrete nature of the problem. In fact, in a first step, we will study controllability properties of the linearized version of (4.9). Then, a fixed point argument allow us to obtain the controllability result for the nonlinear system.

Using a series of tools developed in [16, 17, 19], we are able to prove an observability inequality of the form

$$\iint_Q e^{-\frac{\mathcal{M}}{t}} |z_h|^2 \le C_{obs}\left(\|z_h\|^2_{L^2(\omega\times(0,T))} + e^{-\frac{C}{h}} |p_h^0|^2_{L^2(\Omega)} \right),$$

valid for every solution of the adjoint linear system

$$\begin{cases} -\partial_t z_h + \mathcal{A}^{\mathfrak{M}} z_h + a_h z_h = \mathbf{1}_{\mathcal{O}} p_h & \text{in } Q, \\ \partial_t p_h + \mathcal{A}^{\mathfrak{M}} p_h + b_h p_h = 0 & \text{in } Q, \\ z_h = p_h = 0 & \text{in } \Sigma, \\ z(T) = 0, \quad p(0) = p_h^0 & \text{in } (0, L). \end{cases}$$

Note that there is an additional term in the right hand side of the inequality (as compared with the one obtained in the continuous case, see Eq. (8) in [61]). In fact, because of the presence of this term we refer to it as a *relaxed* observability inequality. Indeed, as discussed in [16], [19], in some cases this term cannot be avoided. This is connected to an obstruction of the null controllability of the semi-discrete heat equation, as pointed out by a counter-example due to O. Kavian, see for instance [66]. The study of relaxed observability estimates for discretized parabolic equations was initiated by [44]. We refer to [15] for a review.

Actually, with the previous inequality we are able to prove that there exists v_h with $\|v_h\|_{L^2(\omega\times(0,T))} \leq C$, for some positive constant C not depending on h, such that

$$|q_h(0)|_{L^2(\Omega)} \leq C\sqrt{\phi(h)}\|\xi_h\|_{L^2(e_{\mathcal{M}})},$$

where $L^2(e_{\mathcal{M}})$ is a weighted space to be clarified and $\phi(h)$ is a function of the discretization parameter such that

$$\liminf_{h\to 0} \frac{\phi(h)}{e^{-C/h^2}} > 0. \tag{4.10}$$

This means we do not exactly achieve null controllability at the discrete level, nevertheless we reach a small target, whose size goes exponentially to zero as the mesh size $h \to 0$.

Thus we speak of $\Phi(h)$-insensitizing controls, which should not be confused with the notion of ε-insensitivity (as discussed in [12], [43]): here, the size of the neighborhood reached by the solution at time T is not fixed, but is a function of the discretization step.

4.1.2 Discrete settings and notation

Following [16] and [19], we establish the framework of the discrete setting to clarify the exposition of the results. In particular, the notation introduced on those articles, allows to carry out most of the computations in a very intuitive manner. In particular, it enables to emulate as close as possible the continuous insensitizing problem as addressed for instance in [61], [14].

As mentioned above, we restrict our analysis to semi-discrete systems in one dimension space.

Let us set $\Omega = (0, L)$ and consider the elliptic operator $\mathcal{A} = -\partial_x^2$ with homogeneous Dirichlet boundary conditions. We introduce finite differences approximations of the operator $\mathcal{A}$. Let $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = L$. We refer to this discretization as to the primal mesh $\mathfrak{M} := \{x_i : i = 1, \ldots, N\}$. We define $|\mathfrak{M}| := N$. We set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$ and $x_{i+\frac{1}{2}} = (x_{i+1} - x_i)/2$, $i = 1, \ldots, N$. We call $\overline{\mathfrak{M}} := \{x_{i+\frac{1}{2}} : i = 0, \ldots, N\}$ the dual mesh and we set $h_i = (h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}})/2 = x_{i+\frac{i}{2}} - x_{i-\frac{1}{2}}$, $i = 1, \ldots, N$.

We denote by $\mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}}$ the sets of discrete functions defined on $\mathfrak{M}$ and $\overline{\mathfrak{M}}$, respectively. If $u \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}}$), we denote by u_i (resp. $u_{i+\frac{1}{2}}$) its value corresponding to x_i (resp. $x_{i+\frac{1}{2}}$). For $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{i=1}^{N} \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} u_i \in L^\infty(\Omega).$$

Since no confusion is possible, by abuse of notation, we shall often write u instead of $u^{\mathfrak{M}}$. Additionally, for $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$\int_\Omega u := \int_\Omega u^{\mathfrak{M}}(x)dx = \sum_{i=1}^{N} h_i u_i.$$

For some $u \in \mathbb{R}^{\mathfrak{M}}$, we shall need to associate boundary conditions $u^{\partial\mathfrak{M}} = \{u_0, u_{N+1}\}$. The set of such extended discrete functions is denoted by $\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_0 = u_{N+1} = 0$, in short $u^{\partial\mathfrak{M}} = 0$ or even $u|_{\partial\Omega} = 0$.

For $u \in \mathbb{R}^{\overline{\mathfrak{M}}}$ we define

$$u^{\overline{\mathfrak{M}}} = \sum_{i=0}^{N} \mathbf{1}_{[x_i, x_{i+1}]} u_{i+\frac{1}{2}} \in L^\infty(\Omega).$$

As above, for $u \in \mathbb{R}^{\overline{\mathfrak{M}}}$, we set

$$\int_\Omega u := \int_\Omega u^{\mathfrak{M}}(x)dx = \sum_{i=0}^{N} h_{i+\frac{1}{2}} u_{i+\frac{1}{2}}.$$

In the same manner, we define the following L^2-inner product on $\mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}}$)

$$(u, v)_{L^2(\Omega)} = \int_\Omega u\, v = \int_\Omega u^{\mathfrak{M}}(x) v^{\mathfrak{M}}(x)dx.$$
$$\left(\text{resp.} \quad (u, v)_{L^2(\Omega)} = \int_\Omega u\, v = \int_\Omega u^{\overline{\mathfrak{M}}}(x) v^{\overline{\mathfrak{M}}}(x)dx.\right)$$

The associated norms will be denoted by $|u|_{L^2(\Omega)}$. We use similar definitions and notations for functions restricted to the domains $\mathcal{O}$ and ω.

For semi-discrete functions $u(t)$ in $\mathbb{R}^{\mathfrak{M}}$ (or $\mathbb{R}^{\overline{\mathfrak{M}}}$) for all $t \in (0,T)$, we define the following L^2-norm

$$\|u(t)\|_{L^2(Q)} = \left(\int_0^T\!\!\int_\Omega |u(t)|^2 dt\right)^{1/2}.$$

Endowing the space of semi-discrete functions $L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ (resp. $L^2(0,T;\mathbb{R}^{\overline{\mathfrak{M}}})$) with this norm yields a Hilbert space.

Analogously, we shall define the space $L^\infty(0,T;\mathbb{R}^{\mathfrak{M}})$ (resp. $L^\infty(0,T;\mathbb{R}^{\overline{\mathfrak{M}}})$) by means of the norm

$$\|u(t)\|_{L^\infty(Q)} = \operatorname*{ess\,sup}_{t\in(0,T)} \left(\sup_{i\in\{1,\dots,N\}} |u_i(t)| \right).$$

Similarly, we shall use such norms for spaces of semi-discrete functions defined on (or restricted to) the domains $\omega \times (0,T)$ or $\mathcal{O} \times (0,T)$.

In order to manipulate the discrete functions, we define the following translation operators for indices:

$$(\tau^+ u)_{i+\frac{1}{2}} := u_{i+1}, \quad (\tau^- u)_{i+\frac{1}{2}} := u_i, \quad i = 0,\dots,N.$$

A first-order difference operator D_i and an averaging operator A_i are then given by

$$\begin{aligned} (Du)_{i+\frac{1}{2}} &:= \frac{1}{h_{i+\frac{1}{2}}}(\tau^+ u - \tau^- u)_{i+\frac{1}{2}}, \\ (Au)_{i+\frac{1}{2}} &= \tilde{u}_{i+\frac{1}{2}} := \frac{1}{2}(\tau^+ u + \tau^- u)_{i+\frac{1}{2}}. \end{aligned} \tag{4.11}$$

Both map $\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ into $\mathbb{R}^{\overline{\mathfrak{M}}}$.

Likewise, we define on the dual mesh translation operators $\tau^\pm$ as follows

$$(\tau^+ u)_i := u_{i+\frac{1}{2}}, \quad (\tau^- u)_i := u_{i-\frac{1}{2}}, \quad i = 1,\dots,N.$$

Then, a difference operator $\overline{D}$ and an averaging operator $\overline{A}$ (both mapping $\mathbb{R}^{\overline{\mathfrak{M}}}$ into $\mathbb{R}^{\mathfrak{M}}$) are given by

$$\begin{aligned} (\overline{D}u)_i &:= \frac{1}{h_i}(\tau^+ u - \tau^- u)_i \\ (\overline{A}u)_i &= \overline{u}_i := \frac{1}{2}(\tau^+ u + \tau^- u)_i \end{aligned} \tag{4.12}$$

Note that there is no need for boundary conditions here.

A continuous function f defined on $\overline{\Omega}$ can be sampled on the primal mesh, that is, $f^{\mathfrak{M}} = \{f(x_i) : i = 1,\dots,N\}$, which we identify to

$$f^{\mathfrak{M}} = \sum_{i=1}^N \mathbf{1}_{[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]} f_i, \quad f_i = f_i(x_i),\ i = 1,\dots,N.$$

We also set

$$f^{\partial\mathfrak{M}} = \{f(x_0), f(x_{N+1})\} = \{f(0), f(L)\},$$
$$f^{\mathfrak{M}\cup\partial\mathfrak{M}} = \{f(x_i) : i = 0, \dots, N+1\}.$$

The function f can also be sampled on the dual mesh, i.e., $f^{\overline{\mathfrak{M}}} = \{f(x_{i+\frac{1}{2}}) : i = 0, \dots, N\}$, which we identify to

$$f^{\overline{\mathfrak{M}}} = \sum_{i=0}^{N} \mathbf{1}_{[x_i, x_{i+1}]} f_{i+\frac{1}{2}}, \quad f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}),\ i = 0, \dots, N.$$

In the sequel, we will use the symbol f for both the continuous function and its sampling on the primal or dual mesh. Indeed, from the context, one will be able to deduce the appropriate sampling. For example, with u defined on the primal mesh $\mathfrak{M}$, in an expression like $\overline{D}(\rho D u)$ where $\rho : \bar{\Omega} \to \mathbb{R}$ is a given function, it is clear that the function ρ is sampled on the dual mesh $\overline{\mathfrak{M}}$ as Du is defined on this mesh and the operator $\overline{D}$ acts on functions defined on this mesh as well.

Remark 57. In the sequel, we use meshes with constant discretization steps to simplify the notation. In this case, $h_i = h$ and $h_{i+\frac{1}{2}} = h$, $\forall i$. Thus, we can write $x_i = ih$ and $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$. The introduction for more general meshes is possible, see [17] for a detailed discussion.

4.1.3 Statement of the main results

Hereinafter we omit the subscript h, in the case of discrete functions, for the sake of concision. Also we use $\iint_Q u = \int_0^T \int_\Omega u(t)dt$. With the notation we have introduced, a suitable finite-difference approximation of the elliptic operator $\mathcal{A}y = -\partial_x^2 y$ with homogeneous Dirichlet boundary conditions is $\mathcal{A}^{\mathfrak{M}} y = -\overline{D}(Dy)$ for $y \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ satisfying $y^{\partial\mathfrak{M}} = 0$, so that

$$(\mathcal{A}^{\mathfrak{M}} y)_i = -\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right), \quad i = 1, \dots, N.$$

We introduce the weight $e_{\mathcal{M}}(t) = \exp(\mathcal{M} t^{-1})$ and define the Hilbert space

$$L^2(e_{\mathcal{M}}) = \left\{ f \in L^2(Q) : \iint_Q e_{\mathcal{M}} |f|^2 < \infty \right\},$$

endowed with the natural norm.

Remark 58. Any $f \in L^2(Q)$ compactly supported in $\bar{\Omega} \times (0, T]$, belongs to $L^2(e_{\mathcal{M}})$.

We now state our main insensitivity result:

Theorem 59. *Let $f \in C^1(\mathbb{R})$ be globally Lipschitz with $f(0)=0$. Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Then, there exists a positive constant $\mathcal{M}$ depending on Ω, ω, $\mathcal{O}$ and T such that for any $\xi \in L^2(e_{\mathcal{M}})$, any h chosen sufficiently small and any $\phi(h)$ verifying (4.10), one can find a semi-discrete control function $v \in L^2(Q)$ uniformly bounded as*

$$\|v\|_{L^2(\omega\times(0,T))} \leq C_{obs}\|\xi\|_{L^2(e_{\mathcal{M}})},$$

with C_{obs} given in (4.15), such that the functional given by (4.8) is $\phi(h)$-insensitized.

Remark 60. Some remarks are in order:

- Roughly speaking, the condition $y_0 = 0$ is due to the fact that the first equation in (4.9) is forward in time and the second one is backward in time. Most of the results regarding insensitizing controls assume this condition. We refer the reader to [62] for a compendium on the possible initial conditions that can be insensitized. As suggested on that work, the answer is not obvious.

- Observe that the condition $f(0) = 0$ is in concordance with the assumption on ξ.

- The assumption $\omega \cap \mathcal{O} \neq \emptyset$ is essential to prove an observability inequality (see Eq. (4.14) below), which is the main ingredient in the proof of Theorem 59. In the continuous and linear case, there are some results on the controllability of non-scalar parabolic systems when $\omega \cap \mathcal{O} = \emptyset$. In [3], the authors proved several null controllability results for a 1-D coupled parabolic system in which both equations are forward in time. In that work, some new interesting phenomena appear, such the minimal time for controllability or the geometrical dependence of the sets ω and $\mathcal{O}$.

- Also, in [43] the authors prove that in the continuous insensitizing problem, the assumption on $\omega \cap \mathcal{O}$ may be omited. Nevertheless, one can only achieve an ε-insensitizing result. The insensitivity problem when $\omega \cap \mathcal{O} = \emptyset$ remains as an open problem, both in the continuous and semi-discrete case.

- Additionally, we may ask for simultaneous $\phi(h)$-null and $\phi(h)$-insensitizing controls, that is, to control (y, q) solution to (4.9) such that

  $$|y(T)|_{L^2(\Omega)} + |q(0)|_{L^2(\Omega)} \leq C\sqrt{\phi(h)}\left(\iint_Q e^{\frac{\mathcal{M}'}{t(T-t)}}|\xi|^2\right)^{1/2}.$$

  for a (possibly) different constant $\mathcal{M}'$. Observe that we only need to impose an extra condition on ξ at time $t = T$.

The rest of the chapter is organized as follows: in section 4.2 we prove an observability inequality that is indeed one of the main results on this chapter. In section 4.3, we prove our main theorem. Finally, we devote section 4.4 to make an extensive discussion on numerical methods for the computation of insensitizing controls.

4.2 The observability inequality

In this section we prove an observability inequality that is the semi-discrete counterpart of the presented in [61] or [14]. This result will be the main tool in the proof of Theorem 59. As mentioned in the introduction, the $\phi(h)$-insensitivity problem is equivalent to find a uniformly bounded control v such that

$$|q(0)|_{L^2(\Omega)} \leq C\sqrt{\phi(h)}\|\xi\|_{L^2(e_{\mathcal{M}})},$$

where (y,q) is the solution to (4.9). It is well-known that controllability properties for system (4.9) are related to the observability of the linear adjoint system, in this case, given by

$$\begin{cases} -\partial_t z + \mathcal{A}^{\mathfrak{M}} z + az = \mathbf{1}_{\mathcal{O}} p \quad \text{in } Q, \\ \partial_t p + \mathcal{A}^{\mathfrak{M}} p + bp = 0 \quad \text{in } Q, \\ z = p = 0 \quad \text{on } \Sigma, \\ z(T) = 0, \quad p(0) = p^0 \quad \text{in } \Omega. \end{cases} \tag{4.13}$$

Thus, the main result in this section is the following:

Proposition 61. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists positive constants C_0, C_1, C_{obs} and $\mathcal{M}$ such that for all $T > 0$ and all potential functions a and b, under the condition $h \leq \min(h_0, h_1)$ with*

$$h_1 = C_0\left(1 + \tfrac{1}{T} + (\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/6})\right)^{-1},$$

for every $p^0 \in \mathbb{R}^{\mathfrak{M}}$, the corresponding solution (z,p) to (4.13) satisfies

$$\iint_Q \exp(-\tfrac{\mathcal{M}}{t})|z|^2 dxdt \leq C_{obs}^2 \left(\|z\|^2_{L^2(\omega\times(0,T))} + e^{\frac{-C_1}{h}} \left|p^0\right|^2_{L^2(\Omega)}\right), \tag{4.14}$$

where

$$C_{obs} = \exp\left[C\left(1 + \tfrac{1}{T} + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/6} + T(1 + \|a\|_\infty + \|b\|_\infty)\right)\right], \tag{4.15}$$

and

$$\mathcal{M} = C\left[1 + T + T(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/6})\right].$$

The main tool to prove this Proposition is a uniform Carleman estimate for semi-discrete parabolic operators. This strategy was originally developed in [19]. The goal is to mimic at the discrete level various techniques from the analysis of PDE control problems.

To this end, it s necessary to introduce an auxiliary function ψ fulfilling the following assumption. The construction of such function is classical. Interested readers can see [33, 19] for additional remarks on this function.

Assumption 62. *Let $\mathcal{B}_0 \subset\subset \mathcal{B}$ be a nonempty open set. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\overline{\Omega}$ in $\mathbb{R}^n$. The function $x \mapsto \psi(x)$ is in $\mathcal{C}^p(\overline{\tilde{\Omega}}, \mathbb{R})$, p sufficiently large, and satisfies form some $c > 0$*

$$\psi > 0 \quad \text{in } \tilde{\Omega}, \qquad |\nabla \psi| \geq c \quad \text{in } \tilde{\Omega} \backslash \mathcal{B}_0,$$
$$\text{and} \quad \partial_{n_x} \psi(x) \leq -c < 0, \quad \text{for } x \in V_{\partial\Omega},$$

where $V_{\partial\Omega}$ is a sufficiently small neighborhood of $\partial\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n_x is extended from $\partial\Omega$.

Now, let $K > \|\psi\|_\infty$ and set

$$\begin{aligned} &\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \\ &r(t,x) = e^{s(t)\varphi(x)}, \quad \rho(t,x) = (r(t,x))^{-1} \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} s(t) &= \tau\theta(t), \quad \tau > 0, \\ \theta(t) &= \frac{1}{(t+\delta T)(T+\delta T - t)} \end{aligned}$$

for $0 < \delta < 1/2$. The parameter δ is introduced to avoid singularities at time $t = 0$ and $t = T$. Further comments are provided in [19].

We recall below the Carleman estimate for semi-discrete parabolic operators of the form $P_{\pm}^{\mathfrak{M}} = \partial_t \pm \mathcal{A}^{\mathfrak{M}}$. We use the following notation to abridge the estimates:

$$\begin{aligned} I_\tau(u) :=& \tau^{-1} \|\theta^{-1/2} e^{\tau\theta\varphi} \overline{D}(Du)\|_{L^2(Q)}^2 + \tau^{-1} \|\theta^{-1/2} e^{\tau\theta\varphi} \partial_t u\|_{L^2(Q)}^2 \\ &+ \tau \left(\|\theta^{1/2} e^{\tau\theta\varphi} Du\|_{L^2(Q)}^2 + \|\theta^{1/2} e^{\tau\theta\varphi} \overline{Du}\|_{L^2(Q)}^2 \right) + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} u\|_{L^2(Q)}^2 \end{aligned}$$

Theorem 63. *Let $\mathcal{B}$ be an open subset of Ω. Let a function ψ satisfying Assumption 62. We define φ according to (4.16). For the parameter $\lambda \geq 1$ sufficiently large, there exist C_0, $\tau_0 \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$, depending on $\mathcal{B}$, $\mathcal{B}_0$ and λ such that*

$$\begin{aligned} I_\tau(u) \leq C &\left(\|e^{\tau\theta\varphi} P_{\pm}^{\mathfrak{M}} u\|_{L^2(Q)}^2 + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} u\|_{L^2(\mathcal{B}\times(0,T))}^2 \right) \\ &+ Ch^{-2} \left(\left| e^{\tau\theta\varphi} u|_{t=0} \right|_{L^2(\Omega)}^2 + |e^{\tau\theta\varphi} u|_{t=T}|_{L^2(\Omega)}^2 \right), \end{aligned}$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$, $0 < \delta \leq 1/2$, $\tau h (\delta T^2)^{-1} \leq \varepsilon_0$, and $u \in \mathscr{C}^1([0,T]; \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}})$ satisfying $u|_{\partial\Omega\times(0,T)} = 0$.

Remark 64. Unlike [19], note that we have added the term $\tau^{-1}\|\theta^{-1/2} e^{\tau\theta\varphi} \overline{D}(Du)\|_{L^2(Q)}^2$ in the left-hand side of the Carleman inequality. It follows from the fact that $\overline{D}(Du) = P_{\pm}^{\mathfrak{M}} u \pm \partial_t u$ and

$$\tau^{-1} \|\theta^{-1/2} e^{\tau\theta\varphi} \overline{D}(Du)\|_{L^2(Q)}^2 \leq 2\tau^{-1} \|\theta^{-1/2} e^{\tau\theta\varphi} P_{\pm}^{\mathfrak{M}} u\|_{L^2(Q)}^2 + 2\tau^{-1} \|\theta^{-1/2} e^{\tau\theta\varphi} \partial_t u\|_{L^2(Q)}^2.$$

Now we are in position to prove the observability inequality. To manipulate the operators such as D, $\overline{D}$ and also to provide estimates for the successive application of such operators on the weight functions, we have summarized the main discrete calculus rules in Appendix A. We state only the most useful results to accomplish the proof of Proposition 61. For a rigorous discussion on these features we refer the reader to [16], [19].

Proof of Proposition 61. The structure of the proof is similar to [61] and [14]. We have divided the proof in four steps. We keep track of the dependences of the constants.

Step 1. Let us consider two open sets B_1 and B_2 such that $B_1 \subset\subset B_2 \subset \omega \cap \mathcal{O}$. We begin by applying Theorem 63 to the solution p of (4.13) with $P_+^{\mathfrak{M}} = -bp$ and $\mathcal{B} = B_1$, namely

$$
\begin{aligned}
I_\tau(p) \leq C &\left(\|e^{\tau\theta\varphi}\, bp\|^2_{L^2(Q)} + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi}\, p\|^2_{L^2(B_1\times(0,T))} \right) \\
&+ Ch^{-2} \left(\left| e^{\tau\theta\varphi} p_{|t=0} \right|^2_{L^2(\Omega)} + \left| e^{\tau\theta\varphi} p_{|t=T} \right|^2_{L^2(\Omega)} \right),
\end{aligned}
$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$ and $\tau h (\delta T^2)^{-1} \leq \varepsilon_0$. As $1 \leq C\theta T^2$, it can be readily obtained

$$
I_\tau(p) \leq C\tau^3 \|\theta^{3/2} e^{\tau\theta\varphi}\, p\|^2_{L^2(B_1\times(0,T))} + Ch^{-2} \left(\left| e^{\tau\theta\varphi} p_{|t=0} \right|^2_{L^2(\Omega)} + \left| e^{\tau\theta\varphi} p_{|t=T} \right|^2_{L^2(\Omega)} \right) \tag{4.17}
$$

for $\tau_1 \geq \tau_0$ sufficiently large and $\tau \geq \tau_1(T + T^2 + T^2\|b\|_\infty^{2/3})$.

Next, we apply Theorem 63 to the solution z to (4.13) with $\mathcal{B} = B_1$ and $P_-^{\mathfrak{M}} = az - \mathbf{1}_{\mathcal{O}} p$, hence

$$
\begin{aligned}
I_\tau(z) \leq C &\left(\|e^{\tau\theta\varphi}\, az\|^2_{L^2(Q)} + \|e^{\tau\theta\varphi} p\|_{L^2(\mathcal{O}\times(0,T))} + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi}\, z\|^2_{L^2(B_1\times(0,T))} \right) \\
&+ Ch^{-2} \left| e^{\tau\theta\varphi} z_{|t=0} \right|^2_{L^2(\Omega)},
\end{aligned}
$$

where we have used the fact that $z(T) = 0$. Reasoning as before, it is not difficult to see that

$$
\begin{aligned}
I_\tau(z) \leq C &\left(\|e^{\tau\theta\varphi} p\|_{L^2(\mathcal{O}\times(0,T))} + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi}\, z\|^2_{L^2(B_1\times(0,T))} \right) \\
&+ Ch^{-2} \left| e^{\tau\theta\varphi} z_{|t=0} \right|^2_{L^2(\Omega)},
\end{aligned} \tag{4.18}
$$

for $\tau \geq \tau_2(T + T^2 + T^2\|a\|_\infty^{2/3})$. Then, combining (4.17) and (4.18), we readily obtain

$$
\begin{aligned}
I_\tau(z) + I_\tau(p) \leq C &\left(\tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} z\|^2_{L^2(Q)} + \tau^3 \|\theta^{3/2} e^{\tau\theta\varphi} p\|^2_{L^2(Q)} \right) \\
&+ Ch^{-2} \left(\left| e^{\tau\theta\varphi} p_{|t=0} \right|^2_{L^2(\Omega)} + \left| e^{\tau\theta\varphi} p_{|t=T} \right|^2_{L^2(\Omega)} \right) \\
&+ Ch^{-2} \left| e^{\tau\theta\varphi} z_{|t=0} \right|^2_{L^2(\Omega)},
\end{aligned} \tag{4.19}
$$

for all τ_3 sufficiently large and

$$\tau \geq \tau_3(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3})). \tag{4.20}$$

Step 2. We proceed to obtain an inequality which bounds p with respect to z. For this, we consider a function $\eta \in C^\infty(\Omega)$ such that

$$0 \leq \eta \leq 1 \text{ in } \Omega, \quad \eta = 1 \text{ in } B_1, \quad \operatorname{supp} \eta \subset B_2 \subset \omega \cap \mathcal{O}, \tag{4.21}$$

$$\tfrac{\overline{D}(D\eta)}{\eta^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \tfrac{\overline{D\eta}}{\eta^{1/2}} \in L^\infty(\Omega). \tag{4.22}$$

Let τ be as in (4.20). We multiply the equation satisfied by z in (4.13) by $\eta s^3 r^2 p$. Then, we have

$$\begin{aligned}\iint_{B_1\times(0,T)} s^3 r^2 |p|^2 &\leq \iint_{\mathcal{O}\times(0,T)} \eta s^3 r^2 |p|^2 \\ &= \iint_Q (a-b) z \eta s^3 r^2 p + \iint_Q (-\partial_t z + \mathcal{A}^{\mathfrak{M}} z + bz)\eta s^3 r^2 p \\ &= \sum_{i=1}^4 I_n,\end{aligned} \tag{4.23}$$

where we recall that $s = \tau\theta$ and $r = e^{s\varphi}$.

Let us estimate each I_n, $1 \leq n \leq 3$. We keep the term I_4 as it will be useful later. Hereinafter, C will denote a generic positive constant which may change from line to line. First, using Hölder and Young inequalities we have

$$\begin{aligned} I_1 &= \iint_Q (a-b) z\eta s^3 r^2 p \\ &\leq \gamma_0 \iint_Q \eta s^3 r^2 |p|^2 + \frac{1}{4\gamma_0}\|a-b\|_\infty^2 \iint_Q \eta s^3 r^2 |z|^2, \end{aligned} \tag{4.24}$$

for any $\gamma_0 > 0$. On the other hand, integrating with respect to t we obtain that

$$\begin{aligned} I_2 &= -\iint_Q \partial_t z \eta s^3 r^2 p \\ &= -\int_\Omega z\eta s^3 r^2 p\Big|_0^T + \iint_Q z\eta\partial_t(s^3 r^2 p) \\ &= \int_\Omega z(0)\eta s^3(0) r^2(0) p(0) + \iint_Q z\eta \partial_t(s^3 r^2) p + \iint_Q z\eta s^3 r^2 \partial_t p \\ &:= I_{21} + I_{22} + I_{23}, \end{aligned} \tag{4.25}$$

where we have used the fact that $z(T) = 0$.

Remark 65. Unlike the continous case, note that $r(0) \neq 0$, so we have the additional term I_{21}.

First, we estimate I_{21} as follows

$$\begin{aligned} I_{21} &= \int_\Omega z(0)\tau^3 \left(\frac{1}{(\delta T)(T+\delta T)}\right)^3 e^{\frac{2\tau\varphi(x)}{(\delta T)(T+\delta T)}} p(0) \\ &\le \int_\Omega |z(0)| \frac{\tau^3}{\delta^3 T^6} e^{-\frac{C\tau}{\delta T^2}} |p(0)|. \end{aligned}$$

Therefore

$$|I_{21}| \le \frac{1}{2}\frac{\tau^2}{\delta^2 T^4}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + \frac{1}{2}\frac{\tau^4}{\delta^4 T^8}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}},$$

where we have applied Young and Hölder inequalities. From the conditions of Theorem 63, we have $\frac{\tau h}{\delta T^2} \le \varepsilon_0$, then

$$|I_{21}| \le Ch^{-2}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \tag{4.26}$$

Now, we estimate I_{22}. Note that

$$\begin{aligned} \partial_t(\theta^3 r^2) &= \partial_t\theta^3 r^2 + \theta^3\partial_t r^2 \\ &= 3\theta^2\theta' r^2 + 2\theta^3 r r' \\ &= 3\theta^2(2t-T)\theta^2 r^2 + 2\theta^3 r(r\tau\theta'\varphi(x)) \\ &= 3\theta^4(2t-T)r^2 + 2\theta^3 r^2\tau(2t-T)\theta^2\varphi(x) \\ &\le 3\theta^4 T r^2 + 2\theta^5 r^2\tau T|\varphi(x)|. \end{aligned}$$

Since $\tau \ge CT$

$$\left|\partial_t(\theta^3 r^2)\right| \le C\theta^4\tau r^2 + C\theta^5\tau^2 r^2.$$

With the estimate above, we have

$$\begin{aligned} |I_{22}| &\le \tau^3 \iint_Q \eta|z||\partial_t(\theta^3 r^2)||p| \\ &\le C\tau^3 \iint \eta|z|(\theta^4\tau r^2 + \theta^5\tau^2 r^2)|p| \\ &= C\iint_Q \eta|z|(s^4 r^2 + s^5 r^2)|p|. \end{aligned}$$

Applying Hölder and Young inequalities, we get

$$|I_{22}| \le 2\gamma_0 \iint_Q s^3 r^2\eta|p|^2 + \frac{C}{\gamma_0}\iint_Q s^7 r^2\eta|z|^2. \tag{4.27}$$

We keep the term I_{23} as it will be useful later.

In order to estimate I_3, we integrate by parts using the discrete integration formula

$$
\begin{aligned}
I_3 &= \iint_Q \mathcal{A}^{\mathfrak{M}} z\eta s^3 r^2 p = -\iint_Q \overline{D}(Dz)\eta s^3 r^2 p \\
&= -\iint_Q s^3 z \overline{D}(D(\eta r^2 p)).
\end{aligned}
$$

We compute

$$
\begin{aligned}
\overline{D}\left(D(\eta r^2 p)\right) &= \overline{D}\left(\widetilde{\eta r^2} Dp + D\left(\eta r^2\right)\tilde{p}\right) \\
&= \overline{\widetilde{\eta r^2}}\,\overline{D}(Dp) + \overline{D}(\widetilde{\eta r^2})\overline{Dp} + \overline{D}(D(\eta r^2))\overline{\widetilde{p}} + \overline{D\left(\eta r^2\right)}\,\overline{D}(\widetilde{p}) \\
&= \overline{\widetilde{\eta r^2}}\,\overline{D}(Dp) + \overline{D}(D(\eta r^2))\overline{\widetilde{p}} + 2\overline{D(\eta r^2)}\,\overline{Dp}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_3 = &-\iint_Q s^3 z\left(\overline{\widetilde{\eta r^2}}\,\overline{D}(Dp) + \overline{D}(D(\eta r^2))\overline{\widetilde{p}}\right) - 2\iint_Q s^3 z\overline{D(\eta r^2)}\,\overline{Dp} \\
&=: I_{31} + I_{32}.
\end{aligned}
$$

We proceed to estimate I_{31}. The double averaged functions can be computed with formula (A.1), that is

$$
\overline{\widetilde{\eta r^2}}\,\overline{D}(Dp) + \overline{D}(D(\eta r^2))\overline{\widetilde{p}} = \eta r^2\overline{D}(Dp) + \left(p + \frac{h^2}{2}\overline{D}(Dp)\right)\overline{D}\left(D(\eta r^2)\right). \tag{4.28}
$$

We develop

$$
\begin{aligned}
\overline{D}\left(D(\eta r^2)\right) &= \overline{D}\left(\widetilde{\eta} Dr^2 + D\eta\,\widetilde{r^2}\right) \\
&= \overline{\widetilde{\eta}}\,\overline{D}(Dr^2) + \overline{D}\widetilde{\eta}\overline{Dr^2} + \overline{D}(D\eta)\overline{\widetilde{r^2}} + \overline{D\eta}\,\overline{D}\widetilde{r^2} \\
&= \overline{\widetilde{\eta}}\,\overline{D}(Dr^2) + \overline{\widetilde{r^2}}\overline{D}(D\eta) + 2\overline{D\eta}\,\overline{Dr^2}.
\end{aligned}
$$

We use again (A.1) and after a straightforward computation

$$
\begin{aligned}
\overline{D}\left(D(\eta r^2)\right) &= \left(\eta + \frac{h^2}{2}\overline{D}(D\eta)\right)\overline{D}(Dr^2) + r^2\overline{D}(D\eta) + 2\overline{D\eta}\,\overline{Dr^2} \\
&= \left(\eta + \frac{h^2}{2}\overline{D}(D\eta)\right)\left(2r\overline{D}(Dr) + 2\overline{Dr}^2 + \frac{h^2}{2}\overline{D}(Dr)^2\right) \\
&\quad + r^2\overline{D}(D\eta) + 2\overline{D\eta}\left(2r\overline{Dr} + h^2\overline{D}(Dr)\overline{Dr}\right) \\
&= r^2\overline{D}(D\eta) + 2\eta r\overline{D}(Dr) + 2\eta\overline{Dr}^2 + \frac{h^2}{2}\eta\overline{D}(Dr)^2 \\
&\quad + h^2 r\overline{D}(D\eta)\overline{D}(Dr) + h^2\overline{D}(D\eta)\overline{Dr}^2 + \frac{h^4}{4}\overline{D}(D\eta)\overline{D}(Dr)^2 \\
&\quad + 4r\overline{D\eta}\,\overline{Dr} + 2h^2\overline{D\eta}\,\overline{D}(Dr)\overline{Dr} \\
&=: \widetilde{I}_1(r).
\end{aligned} \tag{4.29}
$$

Thus, we can group together the all the terms of I_{31} (see eq. (4.28) and (4.29)), to obtain

$$\begin{aligned} I_{31} &= -\iint_Q s^3 z\eta r^2 \overline{D}(Dp) - \iint_Q s^3 zp\widetilde{I}_1(r) - \frac{h^2}{2}\iint_Q s^3 z\overline{D}(Dp)\widetilde{I}_1(r) \\ &=: I_{31}^{(1)} + I_{31}^{(2)} + I_{31}^{(3)}. \end{aligned} \tag{4.30}$$

We will keep the first term of the above expression. In order to estimate the second one, we take into account the result of Proposition 78 and properties (4.22). Reasoning as before, we can estimate the second term by

$$|I_{31}^{(2)}| \leq 9\gamma_0 \iint_Q s^3 r^2 \eta |p|^2 + \frac{C}{\gamma_0}\iint_{B_2\times(0,T)} s^{11} r^2 |z|^2. \tag{4.31}$$

The same procedure may be applied to estimate the term $I_{31}^{(3)}$. In that case, we obtain for any $\gamma_1 > 0$

$$|I_{31}^{(3)}| \leq 9\gamma_1 \iint_Q s^{-1} r^2 \eta |\overline{D}(Dp)|^2 + \frac{C}{\gamma_1}\iint_{B_2\times(0,T)} s^{15} r^2 |z|^2. \tag{4.32}$$

On the other hand, we compute

$$\begin{aligned} 2\overline{D(\eta r^2)} &= 2\overline{D}(\widetilde{\eta r^2}) \\ &= 2\overline{D}\left(\widetilde{\eta}\widetilde{r^2} + \frac{h^2}{4} D\eta D r^2\right) \\ &= 2\left(\overline{\widetilde{\eta}}\,\overline{D}(\widetilde{r^2}) + \overline{\widetilde{r^2}}\,\overline{D}(\widetilde{\eta}) + \frac{h^2}{4}\overline{D\eta}\,\overline{D}(Dr^2) + \frac{h^2}{4}\overline{D}(D\eta)\overline{Dr^2}\right). \end{aligned}$$

Arguing as in the previous steps

$$\begin{aligned} 2\overline{D(\eta r^2)} &= 2\left(\eta\overline{Dr^2} + r^2\overline{D\eta} + \frac{h^2}{2}\overline{D}(D\eta)\overline{Dr^2} + \frac{h^2}{2}\overline{D\eta}\,\overline{D}(Dr^2)\right) \\ &= 2\Big(2\eta r\overline{Dr} + h^2\eta\overline{D}(Dr)\overline{Dr} + r^2\overline{D\eta} + h^2\overline{D}(D\eta) r\overline{Dr} \\ &\quad + \frac{h^4}{2}\overline{D}(D\eta)\overline{D}(Dr)\overline{Dr} + h^2\overline{D\eta}\, r\overline{D}(Dr) + \frac{h^4}{4}\overline{D\eta}\,\overline{D}(Dr)^2 \\ &\quad + h^2\overline{D\eta}\,\overline{Dr}^2\Big) \\ &=: \widetilde{I}_2(r). \end{aligned}$$

Replacing the above expression in I_{32} we obtain

$$I_{32} = -\iint_Q s^3 z\overline{Dp}\,\widetilde{I}_2(r).$$

Similarly to the previous development, we estimate I_{32} as

$$|I_{32}| \leq 8\gamma_2 \iint_Q sr^2\eta|\overline{D}p|^2 + \frac{C}{\gamma_2}\iint_{B_2\times(0,T)} s^{13}r^2|z|^2, \tag{4.33}$$

for any $\gamma_2 > 0$. Notice that the terms I_4, I_{23} and $I_{31}^{(1)}$ (see eq. (4.23), (4.25) and (4.30)) satisfy the equation solved by p, that is

$$\iint_Q z\eta s^3 r^2 \left(\partial_t p - \overline{D}(Dp) + bp\right) = 0.$$

By means of equations (4.26) and (4.27) we get

$$\begin{aligned}|I_2| \leq\ & 2\gamma_0 \iint_Q s^3r^2\eta|p|^2 + Ch^{-2}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ &+ \frac{C}{\gamma_0}\iint_Q s^7r^2\eta|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}.\end{aligned} \tag{4.34}$$

We put together (4.31), (4.32) and (4.33), obtaining

$$\begin{aligned}|I_3| \leq\ & 9\gamma_0 \iint_Q s^3r^2\eta|p|^2 + 9\gamma_1\iint_Q s^{-1}r^2\eta|\overline{D}(Dp)|^2 + 8\gamma_2\iint_Q sr^2\eta|\overline{D}p|^2 \\ &+ C\left(\frac{1}{\gamma_0}+\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\iint_{B_2\times(0,T)} s^{15}r^2|z|^2.\end{aligned} \tag{4.35}$$

Taking estimates (4.24), (4.34) and (4.35) in equation (4.23) and using (4.21), we obtain

$$\begin{aligned}\iint_{\mathcal{B}_1\times(0,T)} s^3r^2|p|^2 \leq\ & 12\gamma_0\iint_Q s^3r^2|p|^2 + 9\gamma_1\iint_Q s^{-1}r^2|\overline{D}(Dp)|^2 + 8\gamma_2\iint_Q sr^2|\overline{D}p|^2 \\ &+ C\left(\frac{1}{\gamma_0}+\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\iint_{B_2\times(0,T)} r^2\left[s^3\|a-b\|_\infty^2|z|^2 + s^{15}|z|^2\right] \\ &+ Ch^{-2}\int_\Omega |z(0)|^2e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}.\end{aligned}$$

Thus, replacing the above expression in (4.19) and taking γ_i small enough, we select τ as in (4.20) to obtain

$$\begin{aligned}I_\tau(z) + I_\tau(p) \leq\ & C\|a-b\|_\infty^2 \iint_{B_2\times(0,T)} s^3r^2|z|^2 + C\iint_{B_2\times(0,T)} s^{15}r^2|z|^2 \\ &+ Ch^{-2}\left(\left|e^{\tau\theta\varphi}p_{|t=0}\right|^2_{L^2(\Omega)} + \left|e^{\tau\theta\varphi}p_{|t=T}\right|^2_{L^2(\Omega)} + \left|e^{\tau\theta\varphi}z_{|t=0}\right|^2_{L^2(\Omega)}\right) \\ &+ Ch^{-2}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-4}\int_\Omega |p(0)|^2e^{-\frac{C\tau}{\delta T^2}}.\end{aligned}$$

Taking $\tau \geq CT^2\|a-b\|^{1/6}$ and returning to the original notation, we rewrite the above inequality as

$$\begin{aligned} I_\tau(z) + I_\tau(p) \leq C \iint_{B_2\times(0,T)} & e^{2\theta\tau\varphi}\tau^{15}\theta^{15}|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ & + Ch^{-2}\left(\left| e^{\tau\theta\varphi} p_{|t=0}\right|^2_{L^2(\Omega)} + \left| e^{\tau\theta\varphi} p_{|t=T}\right|^2_{L^2(\Omega)} + \left| e^{\tau\theta\varphi} z_{|t=0}\right|^2_{L^2(\Omega)}\right) \\ & + Ch^{-2}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}}, \end{aligned} \tag{4.36}$$

valid for every

$$\tau \geq \tau_4\left(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/6})\right)$$

with τ_4 large enough.

Step 3. Here, we use standard energy estimates for the heat equation to bound the last four terms in inequality (4.36).

As $\theta(T) = \theta(0) = (T^2(1+\delta)\delta)^{-1}$, we have $e^{\tau\theta\varphi|_{t=0}} = e^{\tau\theta\varphi|_{t=T}} \leq e^{C\frac{\tau}{\delta T^2}\sup_{x\in\bar\Omega}\varphi}$ and we compute

$$\begin{aligned} I_\tau(z) + I_\tau(p) \leq C \iint_{B_2\times(0,T)} & e^{2\theta\tau\varphi}\tau^{15}\theta^{15}|z|^2 + Ch^{-2}\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \\ & + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} + Ch^{-2}\int_\Omega |p(T)|^2 e^{-\frac{C\tau}{\delta T^2}}, \end{aligned} \tag{4.37}$$

as $\sup_{x\in\bar\Omega}\varphi < 0$. From energy estimates for p solution to the second equation in system (4.13), for $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$|p(t_2)|^2_{L^2(\Omega)} \leq e^{2\|b\|_\infty(t_2-t_1)}|p(t_1)|^2_{L^2(\Omega)}. \tag{4.38}$$

In particular, we obtain

$$\int_\Omega |p(T)|^2 e^{-\frac{C\tau}{\delta T^2}} \leq C\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \tag{4.39}$$

On the other hand, from energy estimates for z solution to the first equation in (4.13), we get for $t \in [0,T]$

$$|z(t)|^2_{L^2(\Omega)} \leq \int_t^T e^{2(1+\|a\|_\infty)(s-t)}|p(s)|^2_{L^2(\mathcal{O})}ds, \tag{4.40}$$

whence

$$\int_\Omega |z(0)|^2 \leq C\iint_{\mathcal{O}\times(0,T)} |p|^2.$$

Using (4.38) it is not difficult to see that

$$\int_\Omega |z(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \leq C\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \tag{4.41}$$

Replacing accordingly (4.39) and (4.41) in inequality (4.37) we obtain

$$I_\tau(z) + I_\tau(p) \leq C \iint_{B_2\times(0,T)} e^{2\theta\tau\varphi}\tau^{15}\theta^{15}|z|^2 \quad + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \tag{4.42}$$

Step 4. In the last part of the proof, we use energy estimates and inequality (4.42) to obtain a modified Carleman inequality with weight functions not decaying at $t = T$. To this end, first we fix

$$\tau = \tau_4\left(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b|\|_\infty^{2/3} + \|a-b\|^{1/6})\right).$$

Let us consider

$$l(t) = \begin{cases} (t+\delta T)(T+\delta T - t) & \text{for } 0 \leq t \leq T/2, \\ (T/2+\delta T)^2 & \text{for } T/2 \leq t \leq T, \end{cases}$$

and the following associated function

$$\sigma(t) = \frac{1}{l(t)}.$$

By construction, $\theta = \sigma$ for $t \in [0, T/2]$, so that

$$\begin{aligned} &\int_0^{T/2}\!\!\int_\Omega e^{2\tau\theta\varphi}\theta^3|z|^2 + \int_0^{T/2}\!\!\int_\Omega e^{2\tau\theta\varphi}\theta^3|p|^2 \\ &= \int_0^{T/2}\!\!\int_\Omega e^{2\tau\sigma\varphi}\sigma^3|z|^2 + \int_0^{T/2}\!\!\int_\Omega e^{2\tau\sigma\varphi}\sigma^3|p|^2. \end{aligned}$$

Then, it follows from (4.42) that

$$\begin{aligned} &\int_0^{T/2}\!\!\int_\Omega e^{2\tau\sigma\varphi}\sigma^3|z|^2 + \int_0^{T/2}\!\!\int_\Omega e^{2\tau\sigma\varphi}\sigma^3|p|^2 \\ &\qquad \leq C\iint_{B_2\times(0,T)} e^{2\tau\theta\varphi}\tau^{15}\sigma^{15}|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}. \end{aligned} \tag{4.43}$$

Now, consider a function $\nu \in C^1([0,T])$ such that

$$\nu = 0 \text{ in } [0, T/4], \quad \nu = 1 \text{ in } [T/2, T], \quad |\nu'| \leq C/T.$$

Applying classical energy estimates for the heat equation to system (4.13) with function ν, that is for both νz and νp, we obtain

$$\|\nu z\|^2_{L^2(\Omega\times(T/4,T))} \leq Ce^{2(1+\|a\|_\infty)T}\left(\frac{1}{T}\|z\|^2_{L^2(\Omega\times(T/4,T/2))} + \|\nu p\|^2_{L^2(\mathcal{O}\times(T/4,T))}\right),$$

and

$$\|\nu p\|^2_{L^2(\Omega\times(T/4,T))} \leq Ce^{2\|b\|_\infty T}\left(\frac{1}{T}\|p\|^2_{L^2(\Omega\times(T/4,T/2))}\right).$$

Combining both inequalities and bearing in mind the definition of ν we get

$$\|z\|^2_{L^2(\Omega\times(T/2,T))} \le Ce^{2(1+\|a\|_\infty+\|b\|_\infty)T}\left(\|z\|^2_{L^2(\Omega\times(T/4,T/2))}+\|p\|^2_{L^2(\Omega\times(T/4,T/2))}\right).$$

Since $\sigma(t)$ is constant on $(T/2,T)$ we can introduce the weight function on the l.h.s. of the above inequality, thus

$$\int_{T/2}^T\int_\Omega e^{2\tau\sigma\hat{\varphi}}\sigma^3|z|^2 \le Ce^{2(1+\|a\|_\infty+\|b\|_\infty)T}\left(\|z\|^2_{L^2(\Omega\times(T/4,T/2))}+\|p\|^2_{L^2(\Omega\times(T/4,T/2))}\right).$$

Taking into account that the weight function σ is bounded in $[T/4,T/2]$ we can estimate the right-hand side terms by means of (4.43) and obtain that

$$\int_{T/2}^T\int_\Omega e^{2\tau\sigma\hat{\varphi}}\sigma^3|z|^2 \\ \le Ce^{2(1+\|a\|_\infty+\|b\|_\infty)T}\left(\iint_{B_2\times(0,T)} e^{2\tau\theta\varphi}\tau^{15}\sigma^{15}|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}\right).$$

This inequality together with (4.43) gives

$$\int_0^T\int_\Omega e^{2\tau\sigma\hat{\varphi}}\sigma^3|z|^2 \\ \le Ce^{2(1+\|a\|_\infty+\|b\|_\infty)T}\left(\iint_{B_2\times(0,T)} e^{2\tau\theta\varphi}\tau^{15}\sigma^{15}|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}\right).$$

It can be readily verified by means of the definition of σ that

$$Ce^{-\frac{8c_0\tau}{Tt}} \le Ce^{-2c_0\tau\gamma} \le e^{2\tau\sigma\varphi}\sigma^3,$$

where

$$\gamma = \begin{cases} 1/(t(T-t)) & 0\le t\le T/2, \\ 4/T^2 & T/2\le t\le T. \end{cases}$$

This, together with the fact that $\sigma \ge (T+\delta T^2)^{-1}$ yields

$$\int_0^T\int_\Omega e^{-\frac{8c_0\tau}{Tt}}|z|^2 \\ \le CT^6 e^{2(1+\|a\|_\infty+\|b\|_\infty)T}\left(\iint_{B_2\times(0,T)} e^{2\tau\theta\varphi}\tau^{15}\sigma^{15}|z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}}\right).$$

We define $c_0 := \inf_{x\in\bar{\Omega}}\varphi(x)$. Then

$$e^{2\tau\theta\varphi}\tau^{15}\theta^{15} \le e^{-2c_0\tau\theta}\tau^{15}\theta^{15} := f(t) = \frac{1}{g(t)}$$

for $t \in [0,T]$. We compute the derivative of $g(t)$, that is

$$g'(t) = \tau^{-15} e^{2c_0\tau\theta}(T-2t)\theta^{-13}(-2c_0\tau + 15\theta^{-1}).$$

Thus, for

$$\tau \geq \frac{15T^2(1/4+\delta(1+\delta))}{2c_0} \geq \frac{15T^2}{8c_0}$$

the function $g(t)$ is strictly decreasing in $(0,T/2)$ and strictly increasing in $(T/2,T)$. In particular, this implies that for $t \in [0,T]$, the function f is bounded as

$$f(t) \leq f(T/2) = e^{-\frac{8c_0\tau}{T^2}} \tau^{15} 2^{30} T^{-15}.$$

Taking

$$\tau \geq \tau_5 \left(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|^{1/6}) \right)$$

where $\tau_5 = \max\{\tau_4, 15/(8c_0)\}$ we obtain

$$\begin{aligned}\iint_Q & e^{-\frac{8c_0\tau}{Tt}} |z|^2 \\ & \leq C e^{C\left((1+\|a\|_\infty+\|b\|_\infty)T + \frac{\tau}{T^2}\right)} \left(\iint_{B_2\times(0,T)} |z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\tau}{\delta T^2}} \right).\end{aligned} \tag{4.44}$$

To conclude the proof, we recall the conditions from Theorem 63:

$$\frac{\tau h}{\delta T^2} \leq \varepsilon_0 \quad \text{and} \quad h \leq h_0.$$

They need to be fullfilled along with $\delta \leq \delta_1$. We take

$$\tau = \tau_5 \left(T + T^2 + T^2(\|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|^{1/6}) \right) \tag{4.45}$$

and define h_1 as

$$h_1 = \frac{\varepsilon_0}{\tau_5}\delta_1 \left(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/6} \right)^{-1}.$$

We choose $h \leq \min\{h_0, h_1\}$ and $\delta = h\delta_1/h_1 \leq \delta_1$. With these, we can find $\frac{\tau h}{\delta T^2} = \varepsilon_0$ and moreover, from (4.44) we have

$$\begin{aligned}\iint_Q & e^{-\frac{8c_0\tau}{Tt}} |z|^2 \\ & \leq C e^{C\left((1+\|a\|_\infty+\|b\|_\infty)T + \frac{\tau}{T^2}\right)} \left(\iint_{B_2\times(0,T)} |z|^2 + Ch^{-4}\int_\Omega |p(0)|^2 e^{-\frac{C\varepsilon_0}{h}} \right),\end{aligned}$$

which gives

$$\iint_Q e^{-\frac{8c_0\tau}{Tt}}|z|^2 \leq e^{C'\left((1+\|a\|_\infty+\|b\|_\infty)T+\frac{\tau}{T^2}\right)}\left(\iint_{B_2\times(0,T)}|z|^2+\int_\Omega |p(0)|^2 e^{-\frac{C''\varepsilon_0}{h}}\right).$$

Finally, setting τ as in (4.45) and recalling that $B_2 \subset \omega$, we obtain the desired result. □

4.3 Proof of Theorem 59

We devote this section to prove the existence of controls insensitizing the L^2-norm of the observation of the solution of (4.9). The proof follows the same spirit as other well-known results for controllability of nonlinear systems (see [28], [32], [61], ...). We start with the existence of h-insensitizing controls for a linearized version of (4.9), that is, for given $a \in L^\infty(Q)$, $b \in L^\infty(Q)$ and $\xi \in L^2(Q)$, we consider the linear system

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ -\partial_t q + \mathcal{A}^{\mathfrak{M}} y + bq_h = \mathbf{1}_{\mathcal{O}} y_h & \text{in } Q, \\ y = q = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad q_h(T) = 0 & \text{in } (0, L). \end{cases} \tag{4.46}$$

and the corresponding adjoint system (4.13). The following result holds:

Proposition 66. *For $T > 0$, there exists a map $L_{(T;a,b)} : L^2(0,T;\mathbb{R}^{\mathfrak{M}}) \to L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ such that if $h \leq \min\{h_0, h_1\}$ with h_1 as given in Proposition 61, for all source term $\xi \in L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ satisfying*

$$\|\xi\|_{L^2(e_{\mathcal{M}})} < \infty \tag{4.47}$$

there exits a semi-discrete control function v given by $v = L_{(T;a,b)}(\xi)$ such that the solution to (4.46) satisfies

$$|q(0)|_{L^2(\Omega)} \leq C_{obs} e^{-\frac{C}{h}} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

and

$$\|v\|_{L^2(Q)} \leq C_{obs} \|\xi\|_{L^2(e_{\mathcal{M}})},$$

with C_{obs} as given in Proposition 61.

Proof. Consider the adjoint system (4.13). The relaxed observability inequality of Proposition 61 gives

$$\iint_Q \exp(-\tfrac{\mathcal{M}}{t})|z|^2 dxdt \leq C_{obs}^2 \left(\|z\|^2_{L^2(\omega\times(0,T))} + \varepsilon\, |p_0|^2_{L^2(\Omega)}\right), \tag{4.48}$$

with $\varepsilon = \phi(h) = e^{-C_1/h}$. We introduce the functional

$$J(p^0) = \frac{1}{2}\iint_{\omega\times(0,T)} |z|^2 + \frac{\varepsilon}{2}|p_0|^2_{L^2(\Omega)} + \iint_Q z\xi. \tag{4.49}$$

The functional J is continuous, strictly convex and coercive on a finite dimiensional space, thus it admits a unique minimizer that we denote as p_0^{opt}. We denote by (z^{opt}, p^{opt}) the associated solution of the adjoint problem (4.13) with this initial data.

We compute the Euler-Lagrange equation for this minimization problem, namely

$$\iint_{\omega\times(0,T)} z^{opt}z + \varepsilon\langle p_0^{opt}, p_0\rangle_{L^2(\Omega)} + \iint_Q \xi z = 0, \quad \forall p_0 \in \mathbb{R}^{\mathfrak{M}}. \tag{4.50}$$

where (z,p) is the associated solution to the data p_0. We set the control $v = L_{T;a,b}(p_0) = \mathbf{1}_\omega z^{opt}$ and consider the solution (y,q) to the controlled problem

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega z^{opt} + \xi, \\ -\partial_t q + \mathcal{A}^{\mathfrak{M}} y + bq = \mathbf{1}_{\mathcal{O}} y, \\ y = q = 0, \\ y(0) = 0, \quad q(T) = 0. \end{cases}$$

Multiplying the above equation by (z,p) and integrating by parts we obtain

$$(q(0), p_0)_{L^2(\Omega)} = \iint_{\omega\times(0,T)} z^{opt}z + \iint_Q \xi z$$

for any $p_0 \in \mathbb{R}^{\mathfrak{M}}$. Substituting this expression in (4.50) we deduce that

$$q(0) = -\varepsilon p_0^{opt}. \tag{4.51}$$

On the other hand, we take $p_0 = p_0^{opt}$ in (4.50), then

$$\|z^{opt}\|^2_{L^2(\omega\times(0,T))} + \varepsilon|p_0^{opt}|^2_{L^2(\Omega)} = -\iint_Q \xi z.$$

Since ξ satisfies (4.47), we introduce the weight function in the right hand side of the above inequality, thus

$$\|z^{opt}\|^2_{L^2(\omega\times(0,T))} + \varepsilon|p_0^{opt}|^2_{L^2(\Omega)} \leq \left(\iint_Q e^{\frac{\mathcal{M}}{t}}|\xi|^2\right)^{1/2}\left(\iint_Q e^{\frac{-\mathcal{M}}{t}}|z|^2\right)^{1/2}.$$

With the observability inequality (4.48) we have

$$\|z^{opt}\|^2_{L^2(\omega\times(0,T))} + \varepsilon|p_0^{opt}|^2_{L^2(\Omega)} \leq C_{obs}^2 \iint_Q e^{\frac{\mathcal{M}}{t}}|\xi|^2.$$

This yields

$$\|v\|_{L^2(\omega\times(0,T))} = \|z^{opt}\|_{L^2(\omega\times(0,T))} \leq C_{obs}\left(\iint_Q e^{\frac{\mathcal{M}}{t}}|\xi|^2\right)^{1/2}$$

and

$$\varepsilon^{1/2}|p_0^{opt}|_{L^2(\Omega)} \leq C_{obs}\left(\iint_Q e^{\frac{\mathcal{M}}{t}}|\xi|^2\right)^{1/2}. \tag{4.52}$$

Hence, the linear map

$$\begin{aligned} L_{(T;a,b)} : L^2(\Omega) &\to L^2(\omega\times(0,T)),\\ p_0 &\mapsto y, \end{aligned}$$

is well defined and continuous. Finally, with (4.51) and (4.52)

$$|q(0)|_{L^2(\Omega)} \leq C_{obs}e^{-C/h}\left(\iint_Q e^{\frac{\mathcal{M}}{t}}|\xi|^2\right)^{1/2},$$

which concludes the proof. □

Proof of Theorem 59. Let us define

$$g(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s\neq 0,\\ f'(0) & \text{if } s = 0.\end{cases}$$

The assumption on f guarantee that g and f' are both continuous and bounded functions. We set $\mathcal{Z} = L^2(0,T;\mathbb{R}^{\mathfrak{M}})$. For $\zeta\in\mathcal{Z}$ we consider the linear control problem

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}}y + g(\zeta)y = \mathbf{1}_\omega v + \xi & \text{in } Q,\\ -\partial_t q + \mathcal{A}^{\mathfrak{M}}q + f'(\zeta)q = \mathbf{1}_{\mathcal{O}}y & \text{in } Q,\\ y = q = 0 & \text{on } \Sigma,\\ y(0) = 0, \quad q(T) = 0 & \text{in } (0,L).\end{cases} \tag{4.53}$$

We set $a_\zeta = g(\zeta)$ and $b_\zeta = f'(\zeta)$. Indeed, we have

$$\|a_\zeta\|_\infty, \|b_\zeta\|_\infty \leq K, \quad \forall\zeta\in\mathcal{Z}. \tag{4.54}$$

Then, we apply Proposition 66, with h chosen sufficiently small, i.e. $h\leq\min(h_0,h_1)$ with

$$h_1 = C\left(1+\tfrac{1}{T}+(K^{2/3}+K^{1/6})\right)^{-1},$$

and denote by $v_\zeta = L_{(T;a_\zeta,b_\zeta)}(\xi)$ and (y_ζ,q_ζ) the associated control function and controlled solution. We have

$$|q_\zeta(0)|_{L^2(\Omega)} \leq Ce^{-C_1/h}\|\xi\|_{L^2(e_{\mathcal{M}})}, \quad v_\zeta\leq C\|\xi\|_{L^2(e_{\mathcal{M}})}. \tag{4.55}$$

where $C_1 > 0$ and $C = \exp\left[C\left(1 + \frac{1}{T} + K^{2/3} + K^{1/6} + T(1+K)\right)\right]$, uniform with respect to ζ and the discretization parameter h. Now, we consider a a map

$$\begin{aligned} \Lambda : \mathcal{Z} &\to \mathcal{Z}, \\ \zeta &\mapsto y_\zeta, \end{aligned}$$

where y_ζ is the solution to (4.53) associated to $a_\zeta = g(\zeta)$ and $b_\zeta = f'(\zeta)$, with v_ζ as in (4.55). By classical regularity results for the heat equation we obtain

$$\|y_\zeta\|_{L^2(Q)} \leq e^{C(1+T+T(\|a_\zeta\|_\infty + \|b_\zeta\|_\infty))} \left(\|\xi\|_{L^2(Q)} + \|\mathbf{1}_\omega v_\zeta\|_{L^2(Q)}\right),$$

and taking into account (4.54) and (4.55), we deduce that the image of Λ is bounded. Following the methods of [12] and [28], it can be verified that Λ is continuous and compact from $\mathcal{Z}$ into itself. Therefore, applying Schauder's fixed point theorem, there exists $y \in \mathcal{Z}$ such that $\Lambda(y) = y$. Setting $v = L_{(T;a_y,b_y)}(\xi)$ we obtain

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}} y + f(y) = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ -\partial_t q + \mathcal{A}^{\mathfrak{M}} q + f'(y) q = \mathbf{1}_{\mathcal{O}} y & \text{in } Q, \\ y = q = 0 \quad \text{on } \Sigma, \\ y(0) = 0, \quad q(T) = 0 \quad \text{in } (0, L), \end{cases}$$

which concludes the proof as we have found a control v that drives the solution of the semilinear semi-discrete parabolic system to a final state $q(0)$ satisfying the estimates (4.55). □

4.4 The penalized Hilbert Uniqueness Method and its application to the insensitizing control problem

The main goal of this section is to present methods and results concerning the numerical computation of insensitizing controls for the heat equation. The strategy that we follow here was initially introduced in the seminal work of Lions and Glowinski, see [34, 35], and it is based on a formulation of the control problem as an adequate optimization problem. This method, referred as the Hilbert Uniqueness Method (hereinafter HUM), has been successfully applied in its penalized version to the numerical computation of null controls for parabolic problems in [18], [15]. This version allows to circumvent some issues related with the computation of the controls, such as the lack of coercivity of the dual functional, that leads to severe problems when numerical methods are applied (see e.g. [54] for a detailed discussion). But most important, as noted in [15], the penalized HUM is almost problem independent (at least in the linear case). In the practice two numerical schemes are used to solve forward and backward equations and can be chosen at the user's convenience.

As noted in Proposition 56, the insensitizing problem is equivalent to a null control problem for a cascade system of equations. Thus the proposal here is to employ the penalized HUM to characterize and build the minimal L^2-norm control satisfying a convenient minimization problem.

To this end, we consider a standard fully-discrete scheme for the heat equation with unknown data. For $M > 0$, we set $\delta t = T/M$ and we consider an implicit Euler scheme with respect to the time variable, namely

$$\begin{cases} y^0 = y_0 + \tau w_0, \\ \frac{y^{n+1}-y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^n = v^{n+1}\chi_\omega + \xi^{n+1}, \quad \forall n \in [\![0, M-1]\!], \end{cases} \tag{4.56}$$

where $v_{\delta t} = (v^n)_{1\leq n\leq M}$ is a fully-discrete control function whose cost, that is the discrete $L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})$-norm, is defined by

$$\|v_{\delta t}\|_{L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})} := \left(\sum_{n=1}^{M} \delta t |v^n|_{L^2(\Omega)}\right)^{1/2}.$$

Consider the functional

$$\Psi(y^n) = \frac{1}{2}\sum_{n=1}^{M} \delta t \int_{\mathcal{O}} |y^n|^2 \tag{4.57}$$

defined on the set of solutions to (4.56). Then, our desire is to insensitize the functional (4.57), that is to find $v_{\delta t}$ such that

$$\frac{\partial \Psi(y^n(\cdot,\cdot;\tau, v_{\delta_t}))}{\partial \tau}\Big|_{\tau=0} = 0, \quad \forall w_0 \in L^2(\Omega), \quad |w_0|_{L^2(\Omega)} = 1. \tag{4.58}$$

As in the semi-discrete and continuous cases, we have the following result

Proposition 67. *Let us consider the following cascade system of heat equations:*

$$\begin{cases} y^0 = y_0, \\ \frac{y^{n+1}-y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, \quad \forall n \in [\![0, M-1]\!], \end{cases} \tag{4.59}$$

$$\begin{cases} q^{M+1} = 0, \\ \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} q^n = \mathbf{1}_{\mathcal{O}} y^n, \quad \forall n \in [\![1, M]\!]. \end{cases} \tag{4.60}$$

Then, the insensitizing condition (4.58) is equivalent to

$$q^1 = 0.$$

Proof. Recall that

$$\Psi(y^n) = \frac{1}{2}\sum_{n=1}^{M} \delta t \int_{\mathcal{O}} |y^n|^2.$$

Taking the derivative with respect to τ and evaluating at $\tau = 0$ we get that (4.58) is precisely

$$\sum_{n=1}^{M} \delta t \int_{\mathcal{O}} y^n w^n = 0 \tag{4.61}$$

for every $w_0 \in \mathbb{R}^{\mathfrak{M}}$, $|w_0|_{L^2(\Omega)} = 1$, where y^n is the solution corresponding to $\tau = 0$ and w^n is the derivative of y^n solution to (4.56) at $\tau = 0$. More precisely, y^n solves

$$\begin{cases} y^0 = y_0, \\ \frac{y^{n+1}-y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} = \mathbf{1}_\omega v^{n+1} + \xi^{n+1}, \quad n \in [\![0, M-1]\!], \end{cases}$$

and w^n solves

$$\begin{cases} w^0 = w_0, \\ \frac{w^{n+1}-w^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} w^{n+1} = 0, \quad n \in [\![0, M-1]\!]. \end{cases} \tag{4.62}$$

We multiply (4.62) by a sequence $(q^{n+1})_{0 \le n \le M-1}$ in $L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{M}})$, that is,

$$\sum_{n=0}^{M-1} \delta t \left(\frac{w^{n+1}-w^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} w^{n+1}, q^{n+1} \right)_{L^2(\Omega)} = 0.$$

After rearranging some terms and from the fact that $\mathcal{A}^{\mathfrak{M}}$ is a symmetric operator we obtain

$$\begin{aligned} &- \left(w^0, q^1\right)_{L^2(\Omega)} + \sum_{n=1}^{M-1} \delta t \left(w^n, \frac{q^n - q^{n+1}}{\delta t} \right)_{L^2(\Omega)} \\ &+ \left(w^M, q^M\right)_{L^2(\Omega)} + \sum_{n=1}^{M} \delta t \left(w^n, \mathcal{A}^{\mathfrak{M}} q^n \right)_{L^2(\Omega)} = 0. \end{aligned}$$

Adding and subtracting the term $\left(w^M, q^M - q^{M-1}\right)_{L^2(\Omega)}$ in the above expression we get

$$- \left(w^0, q^1\right)_{L^2(\Omega)} + \sum_{n=1}^{M} \delta t \left(w^n, \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} q^n \right)_{L^2(\Omega)} + \left(w^M, q^{M+1}\right)_{L^2(\Omega)} = 0. \tag{4.63}$$

We set $q^{M+1} = 0$ and take $(q^n)_{1 \le n \le M}$ as the solution of the adjoint system to (4.62) corresponding to a second member $\mathbf{1}_{\mathcal{O}} y^n$, in other words, the sequence q^n solves

$$\begin{cases} q^{M+1} = 0, \\ \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} q^n = \mathbf{1}_{\mathcal{O}} y^n, \quad n \in [\![1, M]\!]. \end{cases}$$

Consequently, we substitute in (4.63) and thus

$$\sum_{n=1}^{M} \delta t \left(w^n, y^n\right)_{L^2(\mathcal{O})} = \left(w^0, q^1\right)_{L^2(\Omega)}.$$

Hence (4.61) is equivalent to ask

$$(w_0, q^1) = 0 \quad \forall w_0 \in \mathbb{R}^{\mathfrak{M}}, \quad |w_0|_{L^2(\Omega)} = 1,$$

that is

$$q^1 = 0.$$

□

With the above notation and following the methodology of the penalized HUM (see for instance [15]), we introduce the primal fully-discrete functional

$$F_{\varepsilon,h,\delta t}(v_{\delta t}) = \frac{1}{2}\sum_{n=1}^{M}\delta t\int_{\omega}|v^n|^2 + \frac{1}{2\varepsilon}|q^1|^2_{L^2(\Omega)} \tag{4.64}$$

that we wish to minimize onto the whole space $L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})$.

The first step in the analysis is to identify the correct dual functional.

Proposition 68. *For any $\varepsilon > 0$, we define the functional*

$$J_{\varepsilon,h,\delta t}(p_0) = \frac{1}{2}\sum_{n=1}^{M}\delta t\int_{\omega}|z^n|^2 + \frac{\varepsilon}{2}|p_0|^2_{L^2(\Omega)} + \sum_{n=1}^{M}\delta t\,(\xi^n, z^n)_{L^2(\Omega)} + \left(y_0, z^1\right)_{L^2(\Omega)}, \tag{4.65}$$

where the sequence $(z^n,p^n)_n$ is the solution to the following adjoint problem

$$\begin{cases} z^{M+1} = 0, \\ \frac{z^n - z^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} z^n = \mathbf{1}_{\mathcal{O}} p^n, \quad n \in [\![1, M]\!], \end{cases} \tag{4.66}$$

$$\begin{cases} p^0 = p_0, \\ \frac{p^{n+1} - p^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} p^{n+1} = 0, \quad n \in [\![0, M-1]\!]. \end{cases} \tag{4.67}$$

The functionals $F_{\varepsilon,h,\delta t}$ and $J_{\varepsilon,h,\delta t}$ are in duality, in the sense that their respective minimizers $v_{\varepsilon,\delta t} \in L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})$ and $p_{0,\varepsilon} \in \mathbb{R}^{\mathfrak{M}}$ satisfy

$$\inf_{L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})} F_{\varepsilon,h,\delta t} = F_{\varepsilon,h,\delta t}(v_{\varepsilon,\delta t}) = -J_{\varepsilon,h,\delta t}(p_{0,\varepsilon}) = -\inf_{\mathbb{R}^{\mathfrak{M}}} J_{\varepsilon,h,\delta t}.$$

As a consequence

$$v_{\varepsilon,\delta t} = (z^n_\varepsilon)_{1\le n\le M},$$

where z^n_ε is the solution to (4.66)-(4.67) with initial data $p_0 = p_{0,\varepsilon}$.

Remark 69. We have taken into account an initial condition $y_0 \neq 0$ in (4.59) to compute the dual functional. This will be useful at a numerical level to illustrate different results already known about the class of initial data that can be insensitized. Note that when $y_0 = 0$, the functional (4.65) is in fact the fully-discrete version of (4.49)

Proof. Let us introduce the following operator

$$L \in \mathcal{L}(L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}});\mathbb{R}^{\mathfrak{M}}) \quad \text{defined as} \quad L v_{\delta t} = Q^1, \tag{4.68}$$

where (Y^n, Q^n) is the solution to

$$\begin{cases} Y^0 = 0, \\ \frac{Y^{n+1} - Y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} Y^{n+1} = \mathbf{1}_\omega v^{n+1}, \quad \forall n \in [\![0, M-1]\!], \end{cases}$$

$$\begin{cases} Q^{M+1} = 0, \\ \frac{Q^n - Q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} Q^n = \mathbf{1}_{\mathcal{O}} Y^n, \quad \forall n \in [\![1, M]\!]. \end{cases}$$

With this notation, we rewrite (4.64) as

$$\begin{aligned} F_{\varepsilon,\delta t} &= \frac{1}{2}\sum_{n=1}^{M}\delta t\int_{\omega}|v^n|^2 + \frac{1}{2\varepsilon}|Lv_{\delta t} + \mathring{q}^1|^2_{L^2(\Omega)} \\ &:= \bar{F}(v_{\delta t}) + G(Lv_{\delta t}), \end{aligned}$$

where the sequence $(\mathring{y}^n, \mathring{q}^n)_n$ stands for the *free* solution to (4.59)-(4.60), that is, the solution with given data ξ^n and y_0 but $(v^n)_n \equiv 0$. More precisely,

$$\begin{cases} \mathring{y}^0 = y_0, \\ \frac{\mathring{y}^{n+1}-\mathring{y}^n}{\delta t} + \mathcal{A}^{\mathfrak{M}}\mathring{y}^{n+1} = \xi^{n+1}, \quad \forall n \in [\![0, M-1]\!], \end{cases} \tag{4.69}$$

$$\begin{cases} \mathring{q}^{M+1} = 0, \\ \frac{\mathring{q}^{n}-\mathring{q}^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}}\mathring{q}^{n} = \mathbf{1}_{\mathcal{O}}\mathring{y}^n, \quad \forall n \in [\![1, M]\!]. \end{cases} \tag{4.70}$$

Using the duality theory of Fenchel and Rockafellar (see [26]) we have the equality

$$\inf_{L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})}(\bar{F}(v_{\delta t}) + G(Lv_{\delta t})) = -\inf_{\mathbb{R}^{\mathfrak{M}}}(\bar{F}^*(L^*p_0) + G^*(-p_0)),$$

where L^* denotes the adjoint operator of L and F^* is the conjugate function of F, i.e.,

$$F^*(\sigma) = \sup_{\widehat{\sigma}}\{(\sigma, \widehat{\sigma}) - F(\widehat{\sigma})\}. \tag{4.71}$$

We multiply (4.66) by $(Y^n)_n$ in $L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})$

$$\sum_{n=1}^{M}\delta t\,(p^n, Y^n)_{L^2(\mathcal{O})} = \sum_{n=1}^{M}\delta t\left(\frac{z^n - z^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}}z^n, Y^n\right)_{L^2(\Omega)}.$$

and after rearranging some terms we obtain

$$\begin{aligned} \sum_{n=0}^{M-1}\delta t\left(p^{n+1}, Y^{n+1}\right)_{L^2(\mathcal{O})} &= \left(z^1, Y^0\right)_{L^2(\Omega)} - \left(z^{M+1}, Y^n\right)_{L^2(\Omega)} \\ &\quad + \sum_{n=0}^{M-1}\delta t\left(z^{n+1}, \frac{Y^{n+1} - Y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}}Y^{n+1}\right)_{L^2(\Omega)}. \end{aligned}$$

Substituting initial conditions and the equation satisfied by $(Y^n)_n$ in the above expression yields

$$\sum_{n=0}^{M-1}\delta t\left(p^{n+1}, Y^{n+1}\right)_{L^2(\mathcal{O})} = \sum_{n=0}^{M-1}\delta t\left(z^{n+1}, v^{n+1}\right)_{L^2(\omega)}. \tag{4.72}$$

Now, we multiply (4.67) by $(Q^{n+1})_n$ in $L^2_{\delta t}(0,T;\mathbb{R}^{\mathfrak{M}})$

$$\sum_{n=1}^{M} \delta t \left(\frac{p^{n+1}-p^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} p^{n+1}, Q^{n+1} \right)_{L^2(\Omega)} = 0,$$

and proceeding as before we readily obtain

$$\sum_{n=0}^{M-1} \delta t \left(p^{n+1}, Y^{n+1} \right)_{L^2(\mathcal{O})} = \left(p_0, Q^1 \right)_{L^2(\Omega)}. \tag{4.73}$$

Combining (4.72) and (4.73) we get

$$\begin{aligned}\sum_{n=0}^{M-1} \delta t \left(z^{n+1}, \mathbf{1}_\omega v^{n+1} \right)_{L^2(\Omega)} &= (p_0, Q^1)_{L^2(\Omega)} \\ &= (p_0, L v_{\delta t})_{L^2(\Omega)}.\end{aligned}$$

Then, from the definition of the operator L we conclude that

$$\sum_{n=0}^{M-1} \delta t \left(z^{n+1}, v^{n+1} \right)_{L^2(\omega)} = \sum_{n=0}^{M-1} \delta t (L^* p_0, v^{n+1})_{L^2(\Omega)}$$

whence

$$L^* p_0 = (\mathbf{1}_\omega z^n)_{1 \le n \le M}. \tag{4.74}$$

Using definition (4.71) is not difficult to see that

$$\bar{F}^* = \bar{F},$$

so

$$\bar{F}^*(L^* p_0) = \frac{1}{2} \sum_{n=1}^{M} \delta t \int_\omega |z^n|^2. \tag{4.75}$$

Now, we compute the term $G^*(p_0)$ as

$$\begin{aligned} G^*(p_0) &= \sup_{\hat{p}_0} \left\{ (p_0, \hat{p}_0)_{L^2(\Omega)} - G(\hat{p}_0) \right\} \\ &= \sup_{\hat{p}_0} \left\{ (p_0, \hat{p}_0)_{L^2(\Omega)} - \tfrac{1}{2\varepsilon} |\hat{p}^0 + \mathring{q}^1|^2_{L^2(\Omega)} \right\} \\ &= \sup_{\hat{p}_0} \left\{ (p_0, \hat{p}_0 + \mathring{q}^1)_{L^2(\Omega)} - (p_0, \mathring{q}^1)_{L^2(\Omega)} - \tfrac{1}{2\varepsilon} \left| \hat{p}^0 + \mathring{q}^1 \right|^2_{L^2(\Omega)} \right\}. \end{aligned}$$

Setting $\tilde{p}_0 = \hat{p}_0 + \mathring{q}_1$

$$G^*(p_0) = -\left(p_0, \mathring{q}^1 \right)_{L^2(Q)} + \sup_{\tilde{p}_0} \left\{ (p_0, \tilde{p}_0)_{L^2(\Omega)} - \tfrac{1}{2\varepsilon} \left| \tilde{p}_0 \right|^2_{L^2(\Omega)} \right\}.$$

The supremum is attained when $\tilde{p}_0 = \varepsilon p_0$ therefore

$$G^*(-p_0) = \left(p_0, \mathring{q}^1\right)_{L^2(\Omega)} + \frac{\varepsilon}{2}|p_0|^2_{L^2(\Omega)}. \tag{4.76}$$

To finish the proof, we rewrite $(p_0, \mathring{q}^1)$ in terms of the data of the problem. We multiply the equation satisfied by $(\mathring{q}^n)_n$ by $(p^n)_n$

$$\sum_{n=1}^{M} \delta t \left(\frac{\mathring{q}^n - \mathring{q}^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} \mathring{q}^n, p^n \right)_{L^2(\Omega)} = \sum_{n=1}^{M} \delta t \left(\mathring{y}^n, p^n\right)_{L^2(\mathcal{O})}.$$

Reasoning as before we obtain

$$\left(\mathring{q}^1, p_0\right)_{L^2(\Omega)} = \sum_{n=1}^{M} \delta t \left(\mathring{y}^n, p^n\right)_{L^2(\mathcal{O})}. \tag{4.77}$$

On the other hand, we have

$$\sum_{n=0}^{M-1} \delta t \left(\frac{\mathring{y}^{n+1} - \mathring{y}^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} \mathring{y}^{n+1}, z^{n+1} \right)_{L^2(\Omega)} = \sum_{n=0}^{M-1} \delta t \left(\xi^{n+1}, z^{n+1}\right)_{L^2(\Omega)},$$

and proceeding as before we get

$$\sum_{n=1}^{M} \delta t \left(\mathring{y}^n, p^n\right)_{\mathcal{O}} = \sum_{n=1}^{M} \delta t \left(\xi^n, z^n\right)_{L^2(\Omega)} + \left(y_0, z^1\right)_{L^2(\Omega)}. \tag{4.78}$$

We combine (4.77) and (4.78) and replace in (4.76). With that equation and (4.75) we obtain the desired result. □

Following the ideas in [15], we consider the penalized HUM where the parameter $\varepsilon = \phi(h)$ is connected to the discretization parameter h. When properly selected, the method yields to a satisfactory approximation of an insensitizing control for the original problem.

4.5 Numerical results

4.5.1 Computational method

We devote this section to address the actual computation of the fully-discrete insensitizing controls. As noted in Proposition 68, such controls are the minimizers of $F_{\varepsilon,\delta t}$ but may be also be computed by minimizing the dual functionals $J_{\varepsilon,\delta t}$. Since the dual functionals are defined on the finite dimensional space $\mathbb{R}^{\mathfrak{M}}$, instead of the larger space $L^2_{\delta t}(0, T; \mathbb{R}^{\mathfrak{M}})$, it is convenient to apply optimization algorithms to the dual functionals.

Since these functionals are quadratic and coercive, the conjugate gradient algorithm is a natural choice to address the minimization problem. To apply this method it is necessary to compute, at each iteration, the gradient of $J_{\varepsilon,\delta t}$.

Proposition 70. *For any $h > 0$, $\delta t > 0$, $\varepsilon > 0$ and any $p_0 \in \mathbb{R}^{\mathfrak{M}}$, we have*

$$\nabla J_{\varepsilon,h,\delta t}(p_0) = L\left(\mathbf{1}_\omega z\right) + \varepsilon p_0 + L_1(\xi, y_0), \tag{4.79}$$

where L stands for the operator (4.68) and

$$L_1 \in \mathcal{L}(L^2(0,T;\mathbb{R}^{\mathfrak{M}}) \times \mathbb{R}^{\mathfrak{M}}, \mathbb{R}^{\mathfrak{M}}) \quad \textit{defined as} \quad L_1(\xi, y_0) = \mathring{q}^1.$$

Proof. We begin by computing the directional derivative of (4.65), i.e.

$$\sum_{n=1}^{M} \delta t \int_\omega z^n \bar{z}^n + \varepsilon \left(p_0, \bar{p}_0\right)_{L^2(\Omega)} + \sum_{n=1}^{M} \delta t \left(\xi^n, \bar{z}^n\right)_{L^2(\Omega)} + \left(y_0, \bar{z}(0)\right)_{L^2(\Omega)}, \ \forall \bar{p}_0 \in \mathbb{R}^{\mathfrak{M}}, \tag{4.80}$$

where $(\bar{z}^n, \bar{p}^n)_n$ is the solution to (4.66)-(4.67) with initial condition $\bar{p}_0$. We rewrite this derivative in a suitable manner. Using (4.74), we have for the first term that

$$\sum_{n=1}^{M} \delta t \int_\omega z^n L^* \bar{p}_0 = \left(L(\mathbf{1}_\omega z), \bar{p}_0\right)_{L^2(\Omega)}. \tag{4.81}$$

Then, we multiply $(\mathring{y}^n, \mathring{q}^n)_n$ solution to the uncontrolled system (4.69)–(4.70) by $(\bar{z}^n, \bar{p}^n)_n$ in $L^2(0,T;\mathbb{R}^{\mathfrak{M}}) \times L^2(0,T;\mathbb{R}^{\mathfrak{M}})$. Using integration by parts we get

$$\sum_{n=1}^{M} \delta t (\xi^n, \bar{z}^n)_{L^2(\Omega)} + (y_0, \bar{z}(0))_{L^2(\Omega)} = (\mathring{q}^1, \bar{p}_0)_{L^2(\Omega)}. \tag{4.82}$$

Putting together (4.80)–(4.82) yield the desired result. □

The actual computation of the gradient must be regarded as follows. The last term in (4.79), which does not depend on p_0, is actually the solution at final time of the uncontrolled problem (4.69)-(4.70) with given data y_0 and ξ. This computation can be carried once at the beginning of the program and stored in memory. The second term is easy to compute.

The first term of the gradient require several steps to be computed:

1. In the first step, we solve the adjoint problem with the initial datum p_0. This is achieved in two steps. We begin by solving the homogeneous forward system

$$\begin{cases} p^0 = p_0, \\ \frac{p^{n+1} - p^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} p^{n+1} = 0, \quad n \in [\![0, M-1]\!]. \end{cases}$$

   Then, we solve the backwards system with second member $\mathbf{1}_{\mathcal{O}} p^n$

$$\begin{cases} z^{M+1} = 0, \\ \frac{z^n - z^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} z^n = \mathbf{1}_{\mathcal{O}} p^n, \quad n \in [\![1, M]\!], \end{cases}$$

2. Then, we restrict the solution of $(z^n)_n$ to the domain ω and set $v^n = \mathbf{1}_\omega z^n$. This gives a control in $L^2_{\delta t}$ in $L^2(0,T;\mathbb{R}^{\mathfrak{M}})$.

3. Afterwards, we proceed to compute the solution $(y^n)_n$ with this control and zero initial data. More precisely, we solve

$$\begin{cases} y^0 = 0, \\ \frac{y^{n+1}-y^n}{\delta t} + \mathcal{A}^{\mathfrak{M}} y^{n+1} = \mathbf{1}_\omega z^{n+1}, \quad \forall n \in [\![0, M-1]\!]. \end{cases}$$

   Finally, we solve for the backward problem with second member $\mathbf{1}_{\mathcal{O}} y^n$

$$\begin{cases} q^{M+1} = 0, \\ \frac{q^n - q^{n+1}}{\delta t} + \mathcal{A}^{\mathfrak{M}} q^n = \mathbf{1}_{\mathcal{O}} y^n, \quad \forall n \in [\![1, M]\!]. \end{cases}$$

Remark 71. Note that the procedure to compute the control for a given problem requires basically two solvers to deal with the numerical computation of forward and backward parabolic equations. This modularity allows to test different cases without major changes in the program.

4.5.2 Some experiments

We present here some results obtained from the application of the penalized HUM to the problem of insensitizing controls. In accordance with the discussion in section 4.4, we use the standard finite-difference scheme on a uniform mesh of the domain $\Omega = (0,1)$. We denote by N the number of points in the mesh. We use the implicit Euler time discretization and denote by M the number of time intervals. It has been shown in [15] that the results in those kind of problems does not depend too much on the time step, as soon as it is chosen to ensure at least the same accuracy as the space discretization. Here we will always take $M = 2000$.

We consider the following problem with a control time $T = 1$

$$\begin{cases} \partial_t y - 0.1\partial_x^2 y = \mathbf{1}_\omega v + \xi, \\ -\partial_t q - 0.1\partial_x^2 q = \mathbf{1}_{\mathcal{O}} y, \\ y(0,t) = q(0,t) = 0, \quad y(1,t) = q(1,t) = 0, \\ y(x,0) = y_0(x), \quad q(x,T) = 0. \end{cases} \tag{4.83}$$

This problem will serve to study a broad class of insensitizing problems. We apply below the HUM methodology in different contexts that have been studied in the continuous case over the years.

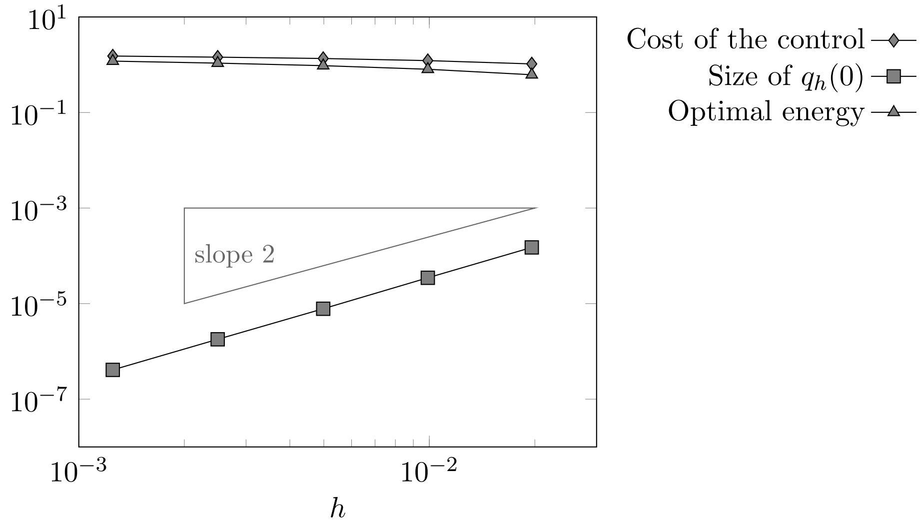


Figure 4.1: Convergence properties of the method for insensitizing problem.

The insensitizing problem

The first positive result on the existence of insensitizing controls for (4.83) was developed in [61]. Here, we extended that result to the case of $\phi(h)$-insensitizing controls for the semi-discrete heat equation.

We begin by testing the case of the localized domain $\omega = (0, 0.5)$. To verify the hypotheses of the main results in [61] and here, we set $\mathcal{O} = (0.3, 0.8)$ and $y_0(x) \equiv 0$. The source term ξ is selected as $\xi(x,t) = \mathbf{1}_{\Omega\times(0.2,1)}(x,t)$. This ensures that $\xi \in L^2(e_{\mathcal{M}})$. As discussed above, we choose the penalization term ε as a function of h. In particular, we choose $\varepsilon = \phi(h) = h^4$. In fact, we use this penalization term for all the simulations performed in this section. We refer the reader to [15] for a detailed discussion on the selection of the function $\phi(h)$.

In figure 4.1, we observe the numerical results for the controllability of system (4.83). As expected and according to the results presented in this chapter, the size of the final state $q(0)$ behaves like $\sqrt{\phi(h)} = h^2$ and, moreover, we see that the cost of the control and the optimal energy remain bounded as $h \to 0$. Recall that in the insensitizing problem, we look for a control such that the functional

$$\Psi(y) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} y^2 dxdt, \tag{4.84}$$

defined on the solutions of

$$\begin{cases} \partial_t y - 0.1\partial_x^2 y = \mathbf{1}_\omega v + \xi \\ y(0,t) = y(1,t) = 0, \quad y(x,0) = y_0 + \tau w_0. \end{cases}$$

is locally insensitive to the perturbation τw_0. To illustrate this fact, we use the computed control v in the above system and test for different values of τ and some initial data w_0. In figure 4.2 we observe the value of the insensitizing functional (4.84) for small values of τ ranging from -0.5 to 0.5. As expected, we observe that the value of $\Phi(y)$ for the controlled solution achieves its minimum when $\tau = 0$.

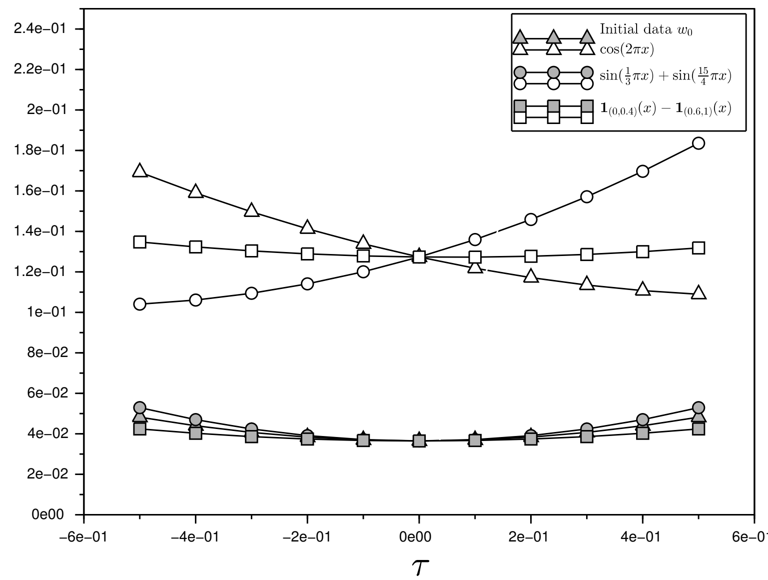


Figure 4.2: Value of $\Phi(y)$ for different parameters τ and initial data w_0. Gray markers: controlled solution. White markers: uncontrolled solution.

The dependence on ξ

It has been widely discussed if the hypothesis on the source term, namely $\xi \in L^2(e_{\mathcal{M}})$, is indeed necessary to obtain the insensitizing result. Modifying the program accordingly, we prove for different source terms of the form

$$\xi(x,t) = \exp\left(-\frac{\mathcal{M}}{t}\right) \mathbf{1}_{\Omega}(x), \tag{4.85}$$

In figure 4.3 we illustrate the effect of ξ during simulations. For some values of $\mathcal{M}$ we mantain the controllability result, but as these values decrease to 0 the convergence rate of the target $q(0)$ is approximately h and not h^2. Similarly, the optimal energy and the control seems to behave like $h^{-0.5}$. As discussed in [15], such behaviors might correspond to the case where the continuous problem is approximately but not null controllable (and it may even depend on the time T). In the insensitizing framework, this means that we are in the context of ε-insensitizing (see for instance [12]). In these cases, further investigation is desirable.

Simultaneous insensitizing and null control

In the continuous case, we can ask for simultaneous null and insensitizing controls, that is, we look for a uniformly bounded control $v \in L^2(\omega \times (0,T))$ such that

$$y(T) = 0 \quad \text{and} \quad q(0) = 0. \tag{4.86}$$

For the semi-discrete case, we have an analogous concept. Consider the linear semi-discrete system

$$\begin{cases} \partial_t y + \mathcal{A}^{\mathfrak{M}} y = \mathbf{1}_\omega v + \xi & \text{in } Q, \\ -\partial_t q + \mathcal{A}^{\mathfrak{M}} q = \mathbf{1}_{\mathcal{O}} y & \text{in } Q, \\ y = q = 0 & \text{on } \Sigma, \\ y(0) = 0, \quad q(T) = 0 & \text{in } (0,L). \end{cases}$$

Following the proof of Proposition 61, we can obtain the observability inequality

$$\iint_Q e^{-\frac{\mathcal{M}}{t(T-t)}} |z|^2 \le C \left(\iint_{\omega \times (0,T)} |z|^2 + e^{-C/h} \left(|z_F|^2_{L^2(\Omega)} + |p_0|^2_{L^2(\Omega)} \right) \right), \tag{4.87}$$

for the solutions to the adjoint system

$$\begin{cases} -\partial_t z + \mathcal{A}^{\mathfrak{A}} z = \mathbf{1}_{\mathcal{O}} p & \text{in } Q, \\ \partial_t p + \mathcal{A}^{\mathfrak{M}} p = 0 & \text{in } Q, \\ z = p = 0 & \text{on } \Sigma \\ z(T) = z_F, \quad p(0) = p_0. \end{cases}$$

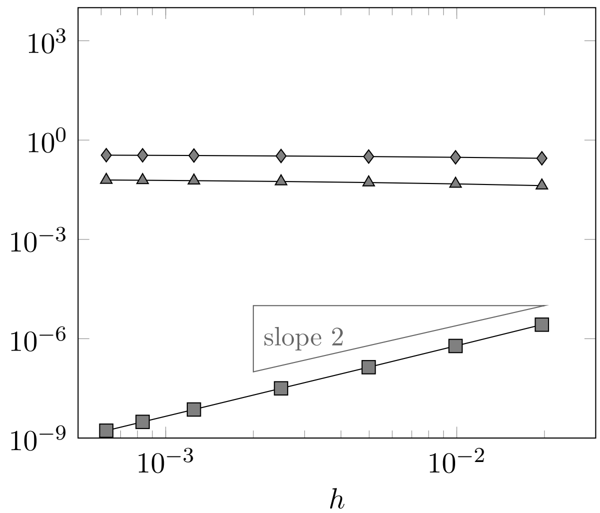
Remark 72. Note that weight function in the left-hand side of (4.87) vanishes at $t = 0$ and $t = T$.

Adapting the results of Section 4.3, we can prove the simultaneous insensitizing and null control by minimizing the dual functional

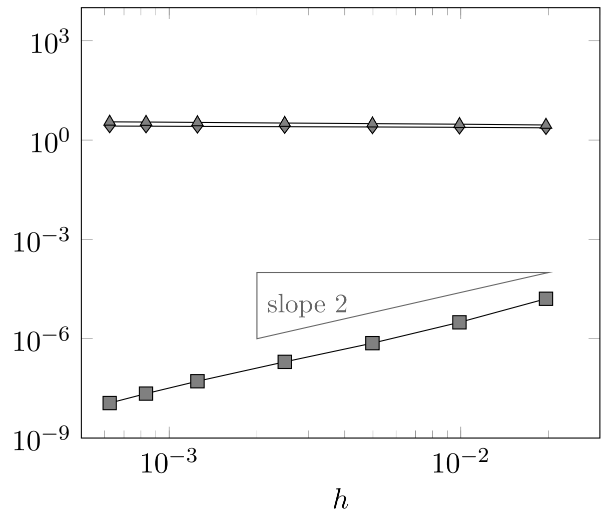
$$J_\varepsilon(z_F, p_0) = \frac{1}{2} \iint_{\omega \times (0,T)} |z|^2 + \frac{\varepsilon}{2} \left(|z_F|^2_{L^2(\Omega)} + |p_0|^2_{L^2(\Omega)} \right) + \iint_Q \xi z. \tag{4.88}$$

More precisely, for any $\xi \in L^2(Q)$ such that

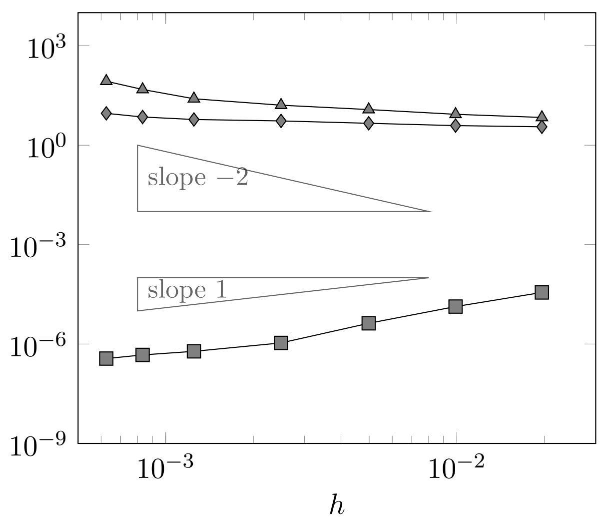
$$\iint_Q e^{\frac{\mathcal{M}}{t(T-t)}} |\xi|^2 < +\infty, \tag{4.89}$$



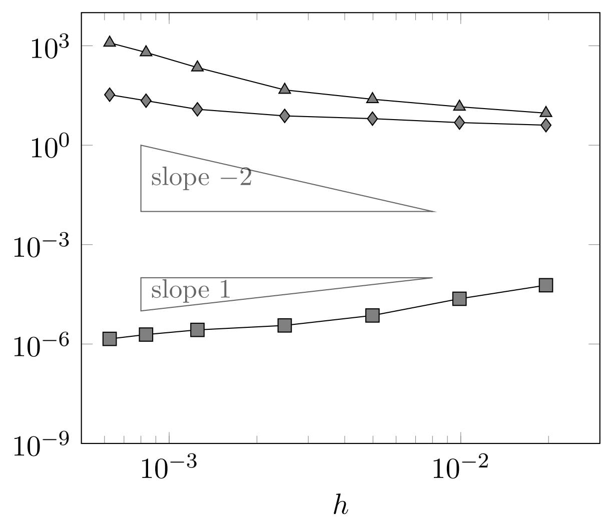
(a) $\mathcal{M} = 1$



(b) $\mathcal{M} = 0.1$



(c) $\mathcal{M} = 0.01$



(d) $\mathcal{M} = 0.0$

Figure 4.3: Different values of $\mathcal{M}$ in the source term.

then the inequality (4.87) together with the minimization of (4.88) yield

$$|y(T)|_{L^2(\Omega)} + |q(0)|_{L^2(\Omega)} \le Ce^{-C/h}\left(\iint_Q e^{\frac{\mathcal{M}}{t(T-t)}}|\xi|^2\right),$$
$$\|v\|_{L^2(Q)} \le C\left(\iint_Q e^{\frac{\mathcal{M}}{t(T-t)}}|\xi|^2\right).$$

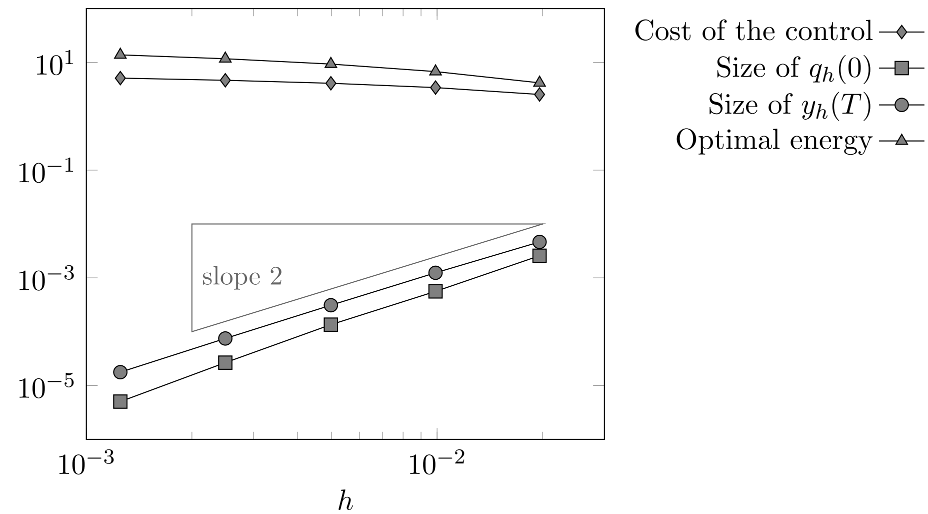


Figure 4.4: Simultaneous insensitizing and null-control

In this case, we can make numerical simulations to illustrate the simultaneous null and insensitizing controls. As before, we take $\omega = (0, 0.5)$, $\mathcal{O} = (0.3, 0.8)$ and $y_0(x) = 0$. For this test, we choose the source term as

$$\xi(x,t) = \mathbf{1}_{\Omega\times(0.2,0.8)}(x,t)$$

which verifies the integrability condition (4.89). In figure 4.4, we observe that the size of the computed targets $y(T)$ and $q(0)$ behaves as expected, i.e., $\sqrt{\phi(h)} = h^2$. Moreover, the norm of the computed control remains bounded as $h \to 0$.

The class of initial data that can be insensitized

The insensitizing results in Theorem 59 and Theorem 1 in [61] use the fact that $y_0(x) = 0$. There are very few results identifying the class of initial data that can be insensitized. In [62], the authors studied some geometric configurations in which the subdomain $\mathcal{O}$ to be insensitized and the control set ω play a key role.

When $\mathcal{O} \subset \omega$, one may obtain the following inequality

$$\int_\Omega |\partial_x z(x,0)|^2 \le C\iint_{\omega\times(0,T)} (|\partial_t z|^2 + |\partial_x^2 z|^2),$$

for solutions to the adjoint system

$$\begin{cases} -\partial_t z - \partial_x^2 z = \mathbf{1}_{\mathcal{O}} p, \\ \partial_t p - \partial_x^2 p = 0, \\ z(0,t) = p(0,t) = 0, \quad p(1,t) = z(1,t) = 0, \\ z(x,T) = 0, \quad p(x,0) = p_0. \end{cases}$$

In this case, we recover a Sobolev norm on $z(\cdot, 0)$ and hence the insensitization can be achieved for initial data in a Sobolev space. We illustrate this fact in figure 4.5. For this experiment we have used that $\omega = (0.3, 0.8)$, $\mathcal{O} = (0.4, 0.6)$, $\xi = 0$ and $y_0(x) = \mathbf{1}_{(0.2,0.7)}(x)$.

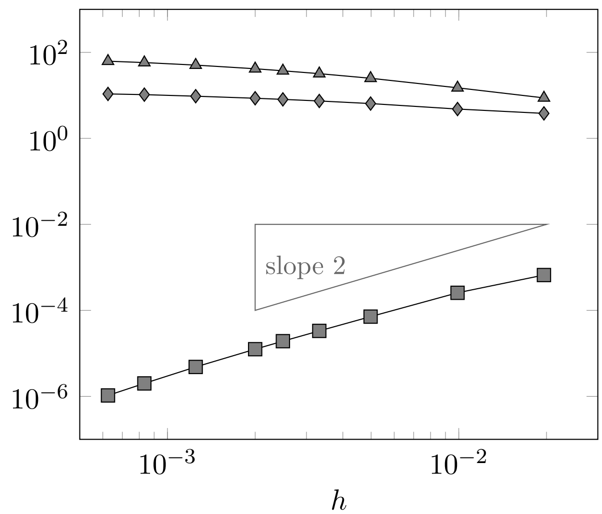
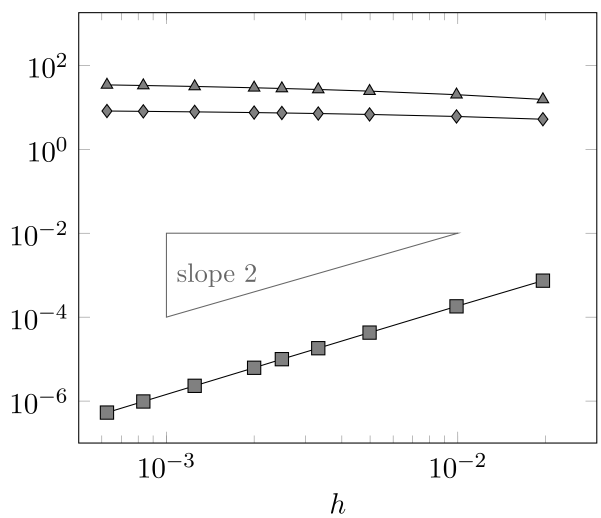


Figure 4.5: The case where $\mathcal{O} \subset \omega$

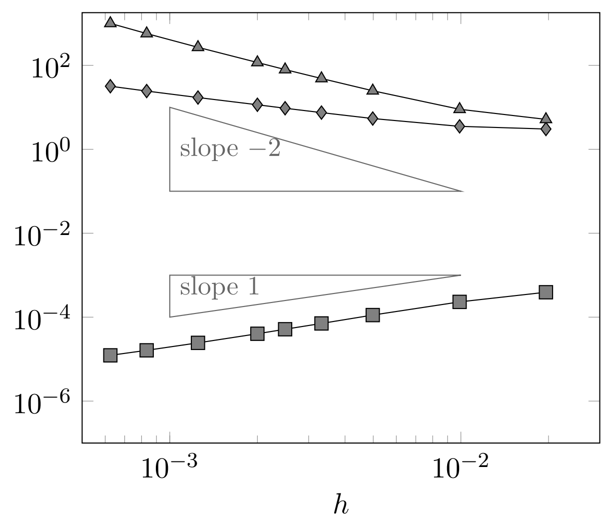
On the other hand, Theorem 2.2 in [62] states that when $\mathcal{O} = \Omega$ it is possible to insensitize initial data of the form $y_0 = \sum_{j=1}^{\infty} b_j \varphi_j$ with

$$\sum_{j=1}^{\infty} e^{B\sqrt{\lambda_j}} b_j^2 < \infty, \quad B > 0,$$

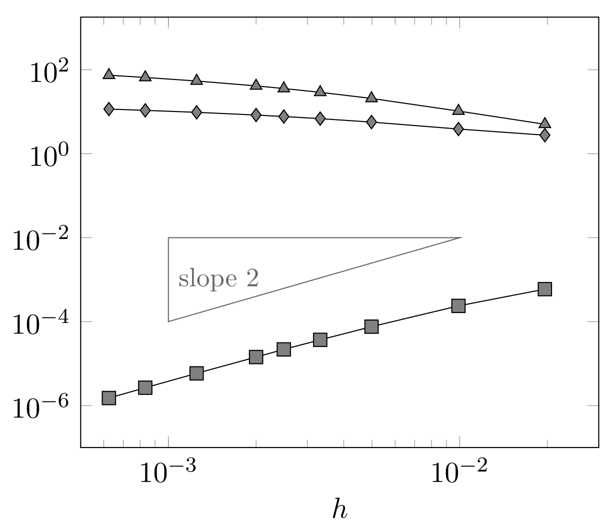
where λ_j and φ_j are the eigenvalues and eigenfunctions of the Dirichlet Laplacian, respectively. In figure 4.6, we present some experiments with different initial data. In the first case and third, we select initial data satisfying the above condition and, as expected, we observe that the convergence ratio of $q(0)$ is $\sqrt{\phi(h)} = h^2$. In the other case, observe that the size of the target actually goes to 0 but at a lower rate, while the optimal energy blows up as h^{-2}. This indicates that the system is approximately, but not null controllable.



(a) $y_0(x) = \sin(\pi x)$



(b) $y_0(x) = \sin^2(\pi x)$



(c) $y_0(x) = \sin^3(\pi x)$

Figure 4.6: The case where $\mathcal{O} = \Omega$

The case $\omega \cap \mathcal{O} = \emptyset$

As in other insensitizing results (see e.g. [61], [13], [40], ...), we use the fact that $\omega \cap \mathcal{O} \neq \emptyset$ in order to locally estimate p in terms of z. Without this hypothesis, we would not be able to obtain the observability inequality (4.14).

In [43], the authors proved that system (4.83) can be actually ε-insensitized when $\omega \cap \mathcal{O} = \emptyset$, for any $y_0 \in L^2(\Omega)$ and $\xi \in L^2(Q)$. Adapting the program, we are able to test different geometric configurations of ω and $\mathcal{O}$.

In the following experiment we set $\omega = (0, 0.5)$, $\mathcal{O} = (0.8, 1)$, $y_0(x) = \sin^2(\pi x)$, and $\xi(x,t) = 0$. In figure 4.7, we observe the size of the computed target $q(0)$ with a penalization term $\phi(h) = h^4$. The computed target decreases to 0 as $h^{0.6}$ instead of the expected rate h^2. Since only a result of ε-insensitizing is known for the continuous case, the problem may not be null-controllable or the numerical approximation may require a stronger condition on the penalization function ϕ (see [15]). Moreover, new phenomena (as minimal-time controllability) associated to the fact that $\omega \cap \mathcal{O} = \emptyset$ may arise, see [3]. In any case, further investigation is desirable.

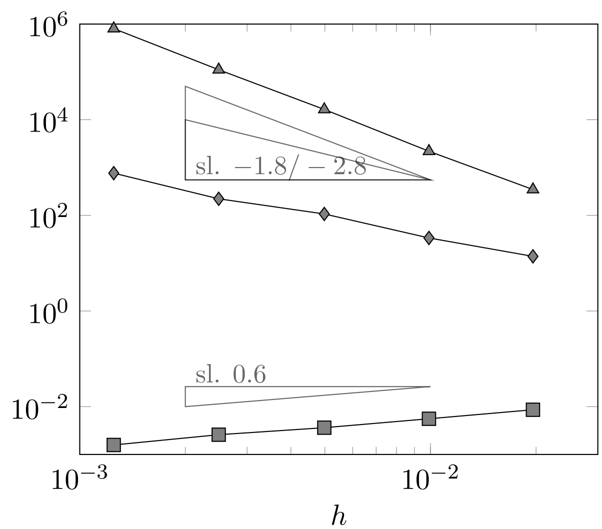


Figure 4.7: The case where $\mathcal{O} \cap \omega = \emptyset$

Appendix A

Some discrete calculus results

The objective of this appendix is to provide a summary of calculus rules for discrete operators such as D, $\overline{D}$ and also to provide estimates for successive applications of such operators on the weight functions. We state here the results without proof. For a detailed reading we refer to [16].

To avoid cumbersome notation we introduce the following continuous difference and averaging operators. For a function f defined on $\mathbb{R}$ we set:

$$\tau^+ f(x) := f(x + \tfrac{h}{2}), \qquad \tau^- f(x) := f(x - \tfrac{h}{2}),$$
$$\mathsf{D} f := \frac{1}{h}(\tau^+ - \tau^-) f, \qquad \mathsf{A} f = \hat{f} := \frac{1}{2}(\tau^+ + \tau^-) f.$$

Discrete versions of the results we give below will be natural, indeed, with the notation given in the introduction, for a function f continuously defined on $\mathbb{R}$, the discrete function Df is in fact $\mathsf{D}f$ sampled on the dual mesh $\overline{\mathfrak{M}}$, and $\overline{D}f$ is $\mathsf{D}f$ sampled on the primal mesh $\mathfrak{M}$. We use similar meanings for averaging symbols $\tilde{f}$, $\bar{f}$ (see (4.12), (4.11)), and for more general combinations: for instance $\widetilde{\overline{D}Df}$ will be the function $\widehat{\mathsf{DD}f}$ sampled on $\overline{\mathfrak{M}}$.

A.1 Discrete calculus formulae

Lemma 73. *Let the functions f_1 and f_2 be continuously defined over $\mathbb{R}$. We have*

$$\mathsf{D}(f_1 f_2) = \mathsf{D}(f_1)\hat{f}_2 + \hat{f}_1 \mathsf{D} f_2.$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}}$ is

$$D(f_1 f_2) = D(f_1)\tilde{f}_2 + \tilde{f}_1 D(f_2), \quad \overline{D}(g_1 g_2) = \overline{D}(g_1)\bar{g}_2 + \bar{g}_1 \overline{D}(g_2)$$

Lemma 74. *Let the functions f_1 and f_2 be continuously defined over $\mathbb{R}$. We have*

$$\widehat{f_1 f_2} = \hat{f}_1 \hat{f}_2 + \frac{h^2}{4}\mathsf{D}(f_1)\mathsf{D}(f_2)$$

The translation of the result to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}}$ is

$$\widetilde{f_1 f_2} = \tilde{f}_1 \tilde{f}_2 + \frac{h^2}{4} D(f_1) D(f_2), \quad \overline{g_1 g_2} = \bar{g}_1 \bar{g}_2 + \frac{h^2}{4} \overline{D}(g_1) \overline{D}(g_2).$$

Lemma 75. *Let the funciton f be continuously defined over $\mathbb{R}$. We have*

$$A^2 f := \hat{\hat{f}} = f + \frac{h^2}{2} \mathsf{D}\mathsf{D} f \tag{A.1}$$

The following proposition covers discrete integration by parts:

Proposition 76. *Let $f \in \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ and $g \in \mathbb{R}^{\overline{\mathfrak{M}}}$. Then,*

$$\int_\Omega f(\overline{D} g) = -\int_\Omega (Df) g + f_{N+1} g_{N+\frac{1}{2}} - f_0 g_{\frac{1}{2}},$$
$$\int_\Omega f \overline{g} = \int_\Omega \tilde{f} g - \frac{h}{2} f_{N+1} g_{N+\frac{1}{2}} - \frac{h}{2} f_0 g_{\frac{1}{2}}.$$

A.2 Some results related to the weight functions

We present here two technical results related to discrete operations performed on the Carleman weight functions. These are of particular interest in the demonstration of Proposition 61. We refer the reader to [16], [19] for a complete review of the results and their proofs.

Lemma 77. *Let f be a smooth function defined on $\mathbb{R}$. We have*

$$\mathsf{D}^j f = \partial_x^j f + C_j' h^2 \int_{-1}^{1} (1 - |\sigma|)^{j+1} \partial_x^{j+2} f(. + l_j \sigma h) d\sigma,$$
$$\mathsf{A}^j f = f + C_j h^2 \int_{-1}^{1} (1 - |\sigma|) \partial_x^2 f(. + l_j \sigma h) d\sigma, \quad j = 1, 2, \ l_1 = \frac{1}{2}, \ l_2 = 1.$$

We set $r = e^{s\varphi}$ and $\rho = r^{-1}$. The positive parameters s and h will be large and small respectively. We highlight the dependence on s, h and λ in the following estimate. We assume $s \geq 1$ and $\lambda \geq 1$.

Proposition 78. *Provided $sh \leq \mathfrak{K}$, we have*

$$r \mathsf{A}^j \mathsf{D} \rho = r \partial_x \rho + s \mathcal{O}_{\lambda, \mathfrak{K}}\left((sh^2)\right) = s \mathcal{O}_{\lambda, \mathfrak{K}}(1), \quad j = 0, 1,$$
$$r \mathsf{D}^2 \rho = r \partial_x^2 \rho + s^2 \mathcal{O}_{\lambda, \mathfrak{K}}\left((sh^2)\right) = s^2 \mathcal{O}_{\lambda, \mathfrak{K}}(1).$$

The same estimates hold with ρ and r interchanged.

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