CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE CONTROL AUTOMÁTICO

Ecuaciones en diferencias en tiempo continuo:
un enfoque de funcionales de Lyapunov-Krasovskii

T E S I S

QUE PRESENTA

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Para obtener el grado de

Maestro en Ciencias

en la Especialidad de Control Automático

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Ciudad de México **Agosto, 2016**

Abstract

A study on the stability of continuous-time difference equations is presented. Based on the construction of the analogous of the delay Lyapunov matrix for this class of systems, a complete type Lyapunov-Krasovskii functional is proposed that allows to present necessary conditions of stability. Examples are given throughout the work to illustrate the construction of the new Lyapunov matrix and the obtained necessary conditions of stability.

Resumen

Se presenta un análisis sobre la estabilidad de ecuaciones en diferencias en tiempo continuo. Con base en la construcción del análogo de la matriz de Lyapunov para esta clase de sistemas, se propone una funcional de Lyapunov-Krasovskii de tipo completo que permite presentar condiciones necesarias de estabilidad. A lo largo del trabajo se muestran ejemplos que ilustran la construcción de la nueva matriz de Lyapunov y las condiciones necesarias de estabilidad obtenidas.

Acknowledgements

I would like to express my gratitude to:

- Dr. Sabine Mondié for her commitment to this work from the very beginning, her corrections and observations, and her always respectful direction.

- Dr. Michael Di Loreto for his complete disposition and his help during my stay at the Ampère Laboratory, UMR CNRS 5005, INSA de Lyon, France. For his observations, recommendations and guidance.

- The doctors at DCA in CINVESTAV, for all of their classes and help during the last two years.

- CONACYT for the scholarship, grants and proyect 18075, for allowing me and many others to continue learning and preparing for the work of being a world-class researcher.

- My parents, siblings, girlfriend and friends for their total support.

Contents

Notations and Symbols

$\mathbb{R}$	Field of real numbers
$\|x\|$	Euclidian norm of a vector $x \in \mathbb{R}^n$
$\mathbb{N}_0$	Set of natural numbers, including zero
$\mathbb{Z}$	Set of integers
$\mathbb{R}^n$	Space of $n-$vectors with entries in $\mathbb{R}$
i	Imaginary unit, $i^2 = -1$
$0_{n\times n}$	Zero $n \times n$ matrix
I_n	$n \times n$ Identity matrix
$\|A\|$	Induced norm of a matrix A, $\|A\| = \max_{\|x\|=1} \|Ax\|$
$\mathcal{C}([-h,0),\mathbb{R}^n)$	Space of $\mathbb{R}^n-$valued continuous functions on $[-h,0)$
$\mathcal{PC}([-h,0),\mathbb{R}^n)$	Space of $\mathbb{R}^n-$valued piecewise continuous functions on $[-h,0)$
0_h	$\mathbb{R}^n$-valued trivial function, $0_h(\theta) = 0 \in \mathbb{R}^n$, $\theta \in [-h,0]$
$f(t+0)$	Right-hand-side limit of $f(t)$ at a point t, $f(t+0) = \lim_{\epsilon\to 0} f(t+\|\epsilon\|)$
$f(t-0)$	Left-hand-side limit of $f(t)$ at a point t, $f(t-0) = \lim_{\epsilon\to 0} f(t-\|\epsilon\|)$
$\|\varphi\|_h$	Uniform norm, $\|\varphi\|_h = \sup_{-h\le\theta<0} \|\varphi(\theta)\|$
$x'(t)$	First derivative of $x(t)$
$x''(t)$	Second derivative of $x(t)$
x_t	Restriction of $x(t)$, $x_t : \theta \to x(t+\theta), \theta \in [-h,0)$
A^T	Transpose of matrix a A
$A > 0 (A \geq 0)$	Symmetric matrix A is positive definite (positive semidefinite)
$\lambda(A)$	Eigenvalue of a matrix A
$\lambda_{\max}(A)$, $\lambda_{\min}(A)$	Maximum, minimum eigenvalue of a symmetric matrix A
$\rho(A)$	Spectral radius of a square matrix A
$A \otimes B$	Kronecker product of matrices A and B
$\mathrm{vec}(A)$	Vector of stacked columns of a matrix A

Introduction

Automatic Control is a field of Engineering Sciences that focuses on the analysis and synthesis of the behavior of dynamic systems. The main tool to achieve these purposes is mathematics, which provides resources to manipulate symbols in order to develop strategies that modify a given dynamic system so that a desired objective is accomplished.

This objective is often associated to taking the response of the system to an *equilibrium point*, which, once it is reached, remains as it is in the absence of perturbations. Equilibrium points are distinguished by a preeminent property called *stability*. This property is commonly characterized in the sense of Lyapunov, a Russian engineer and mathematician who established the basis of the theory called after him. If the behavior of the response of a system starting from a point close to the equilibrium point remains nearby, the equilibrium point is stable; else, it is unstable.

In many areas of study, problems appear that can be modeled (mathematically represented) by the so-called *difference equations*, which relate a measurement obtained from a system in a given time with previous measurements in equally spaced time instants in the past. These mathematical models are oversimplifications of the actual phenomena, but they can be sufficient in order to achieve the aforementioned control purposes.

In this work, we deal with the analysis of the stability of a more general form of these difference equations known as *difference equations in continuous time*, in which the past measurement times, now called *delays*, need not be equally spaced or even commensurate (integer multiples of a fixed delay). As is customary in the literature of Automatic Control, the mathematical models will be referred to as systems, keeping in mind that they are not the actual systems but their symbolic representations.

Throughout this Thesis, the Lyapunov-Krasovskii strategy is pursued in an analogous manner to the one that V. Kharitonov [1], S. Mondié and A. Egorov [2] have followed for time-delay systems. This strategy will lead the reader to our main result, Necessary Conditions for stability of continuous-time difference equations.

Motivation and previous work

In the context of Engineering Sciences, motivations to study these equations come from sampled-data systems, neutral time-delay systems (see [3], [1], and [4]), difference equations with distributed delays (examples in [5]), conservation laws modeled by first order hyperbolic partial differential equations in which a transport phenomenon occurs ([3], [6]), and other classes of linear systems with distributed parameters, which have been shown in [7] to admit a representation in the form of difference equations in continuous time.

Physical examples of these systems include wave equations (a particular case of hyperbolic partial differential equations), whose applications range widely, from acoustics (see [8], [9], and [10]), electrical engineering (e.g. the telegraph equation in [11], [12]), mechanics (examples in [13]), communications, etc (see [14], [15], and [16]). In Chap. 1, an example of the modeling of a physical system in terms of difference equations in continuous time is given.

The stability analysis for these systems has been subject of study for many years, and spectral conditions for stability independent of the delays have been proposed in [17], [18], and [3]. In the general case, these spectral conditions are difficult to verify. The need to find stability conditions that are numerically easier to handle arises and the Lyapunov-Krasovskii approach, which yields conditions in the form of linear matrix inequalities (LMI), has proven useful. Among the first works that use this approach is [19], in which, using quadratic Lyapunov-Krasovskii functionals, conditions for L_2 asymptotic stability independent of the delays have been obtained. This approach has been extended in [20] and [21].

More recently, there has been increasing interest in the construction of Lyapunov-Krasovskii functionals that provide constructive exponential decay rate estimates and reduce conservatism, as can be seen in [22] and [23].

In the study of time-delay systems, the stability problem has also been approached by means of a Lyapunov-Krasovskii functional with a prescribed derivative that has led to the definition of the so-called Lyapunov delay matrix in [24]. Moreover, a functional, named of complete type, that satisfies a quadratic lower bound when the system is stable, was proposed. Necessary conditions of stability of time-delay systems in terms of the Lyapunov delay matrix have been presented in [25], [26], [2]. In this work we present the analogous case of these necessary conditions for continuous-time difference equations.

Objectives

The main objectives of this work are:

- To propose a Lyapunov-Krasovskii functional with prescribed derivative under the assumption of exponential stability of the systems under study (Chap. 2).
- To analyze the matrix-valued function that defines this functional. This is called the *Lyapunov matrix* given that it is a counterpart of the matrix solution of the classical Lyapunov matrix equation for delay-free systems. From this analysis, it is shown that the assumption of exponential stability is not necessary for the existence of this Lyapunov matrix (Chap. 2).
- To present a complete type functional that fulfills Sufficient Conditions for the stability of difference equations in continuous time (Chap. 3).
- To introduce a method for the computation of the above mentioned Lyapunov matrix (Chap. 4).
- To present Necessary Conditions for the stability of difference equations in continuous time (Chap. 5).

Chapter 1

Difference equations in continuous time: preliminaries

Introduction

In the context of time-invariant linear systems, we present a state-space representation of a particular type of system as follows

$$\begin{cases} x(t) &= \sum_{j=1}^{m} A_j x(t-h_j) + \sum_{j=0}^{m} B_j u(t-h_j), \quad t \geq 0 \\ y(t) &= \sum_{j=0}^{m} C_j x(t-h_j) + \sum_{j=0}^{m} D_j u(t-h_j), \quad t \geq 0, \end{cases} \tag{1.1}$$

which resembles that of time-delay systems treated in [1], but having difference equations instead of the differential type. We call this type of system a *difference equation in continuous time*, for the resemblance it has with the difference equations that are normally defined in discrete time in the classic literature of linear systems [27].

Here, $x(t) \in \mathbb{R}^n$ is called the instantaneous state at time $t \geq 0$, $h_1, \ldots, h_m$ are the delays, with $0 = h_0 < h_1 < \cdots < h_m = H$. Moreover, A_j are $n \times n$ real matrices for $j = 1, \ldots, m$, and $B_j \in \mathbb{R}^{n \times q}$, $C_j \in \mathbb{R}^{p \times n}$ and $D_j \in \mathbb{R}^{p \times q}$ for $j = 0, \ldots, m$.

Considering $u(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^p$ to be the input and output of system (1.1), respectively, we have as motivations to study this class of equations the approximation of linear systems with distributed parameters discussed in [7], including conservation laws, as well as neutral time-delay systems and sampled-data systems, among others.

For instance, let us consider the distortionless RLCG-electrical transmission line [28] in Fig. 1.1.

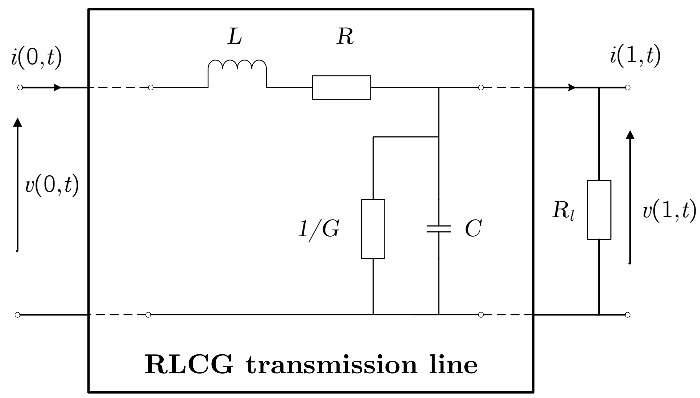


Figure 1.1: RLCG-electrical transmission line

The electrical dissipative components are assumed to satisfy $\alpha = \frac{R}{L} = \frac{C}{G}$, where R, C, L and G are, respectively, the resistance, capacitance, inductance and conductance of the finite length line (we consider a unit length). The following first-order linear partial differential equations describe the Kirchoff's laws of the line:

$$Li_t(z,t) = -Ri(z,t) - v_z(z,t), \tag{1.2}$$

$$Cv_t(z,t) = -Gv(z,t) - i_z(z,t), \tag{1.3}$$

for $0 \leq z \leq 1$ and $t \geq 0$, where $i(z,t)$ and $v(z,t)$ stand for the current and voltage in position z and time t, respectively. The boundary conditions are

$$v(1,t) = R_l i(1,t), \quad u(t) = v(0,t), \tag{1.4}$$

where R_l is the load resistance connected to the line, and $u(t)$ is the input voltage. The d'Alembert solutions for (1.2)-(1.3) are

$$i(z,t) = \mathrm{e}^{-\alpha t}\frac{\phi(z-\nu t) - \psi(z+\nu t)}{2\sigma} \tag{1.5}$$

$$v(z,t) = \mathrm{e}^{-\alpha t}\frac{\phi(z-\nu t)+\psi(z+\nu t)}{2}, \tag{1.6}$$

here, $\nu = \frac{1}{\sqrt{LC}}$ stands for the velocity of wave propagation, $\sigma = \sqrt{\frac{L}{C}}$ is known as the wave impedance of the line, ϕ and ψ are arbitrary (smooth) functions. The boundary conditions (1.4) yield

$$\psi(1+\nu t) = \kappa\phi(1-\nu t) \tag{1.7}$$

$$2u(t) = \mathrm{e}^{-\alpha t}\left(\phi(-\nu t)+\psi(\nu t)\right), \tag{1.8}$$

where $\kappa = \frac{R_l-\sigma}{R_l+\sigma}$. We define a new variable in the vector form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathrm{e}^{-\alpha(t-h)}\phi(t-\nu t) \\ \mathrm{e}^{-\alpha t}\phi(-\nu t) \end{bmatrix}, \tag{1.9}$$

with $h = \frac{1}{\nu}$. Using this variable, equations (1.7)-(1.8) are written as

$$x(t) = Ax(t-h) + Bu(t), \tag{1.10}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $a = \kappa\mathrm{e}^{-2\alpha h}$.

We retrieve from (1.9) the physical state variables:

$$v(1,t) = (1+\kappa)\mathrm{e}^{-\alpha h}x_1(t)$$

$$\sigma i(0,t) = 2x_2(t) - u(t).$$

Therefore, the response of the RLCG-electric line to input voltage is modeled by the difference equation in continuous time in (1.10). Notice that, since $|\kappa| \le 1$, $|a| < 1$ is related to the dissipative components of the line. This fact implies that the matrix A has eigenvalues with module strictly less than one, which is related to the asymptotic stability of the system.

As a second illustration, let us analyze the link between continuous-time difference equations and other time-delay systems, which have been more extensively studied.

Linear time-delay systems

The following is the representation of a linear time-delay system with commensurate delays

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-jh) + \sum_{j=0}^{m} B_j u(t-jh), \quad t \geq 0 \tag{1.11}$$

$$y(t) = \sum_{j=0}^{m} C_j x(t-jh) + \sum_{j=0}^{m} D_j u(t-jh), \quad t \geq 0 \tag{1.12}$$

where h is strictly positive, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$ is the input of the system and $y(t) \in \mathbb{R}^p$ is the output of the system. The matrices A_j, B_j, C_j and D_j for $j = 0, \ldots, m$ are real matrices of the appropriate dimensions and the delays $h_j = jh$ are the so-called point-wise delays. For any continuous initial condition $\varphi(\theta)$, $\theta \in [-mh, 0]$, there exists a unique solution for (1.11)-(1.12).

Neutral type systems

A neutral type system (see [4], [29], and the reference therein) admits the following state-space representation

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-jh) + \sum_{j=0}^{m} B_j u(t-jh) + \sum_{j=1}^{m} E_j \dot{x}(t-jh), \quad t \geq 0 \tag{1.13}$$

$$y(t) = \sum_{j=0}^{m} C_j x(t-jh) + \sum_{j=0}^{m} D_j u(t-jh) + \sum_{j=1}^{m} F_j y(t-j\theta), \quad t \geq 0, \tag{1.14}$$

with initial condition $\varphi \in \mathcal{C}([-mh, 0], \mathbb{R}^n)$ and E_j, F_j, real matrices for $j = 1, \ldots, m$. Notice that for $E_j = 0$ and $F_j = 0$, we obtain a time-delay system. The response of the system is unique for any initial condition $\varphi(\theta)$, $\theta \in [-mh, 0]$.

Now, we are interested in finding a representation of the neutral type systems as coupled differential-difference systems.

First, we define the delay operator $\varsigma : x(t) \to x(t-h)$. This operator induces $\varsigma^j(x(t)) = x(t-jh)$. The neutral type system (1.13)-(1.14) can be written, in terms of the operator ς as

$$\dot{x}(t) = \sum_{j=0}^{m} A_j \varsigma^j(x(t)) + \sum_{j=0}^{m} B_j \varsigma^j(u(t)) + \sum_{j=1}^{m} E_j \varsigma^j(\dot{x}(t)). \tag{1.15}$$

We define

$$x_1(t) = \left(I_n - \sum_{j=1}^{m} E_j \varsigma^j \right) x(t), \quad x_2(t) = x(t).$$

Using these variables, the neutral equation (1.15) now has the form

$$\begin{aligned} \dot{x}_1(t) &= \sum_{j=0}^{m} A_j \varsigma^j(x_2(t)) + \sum_{j=0}^{m} B_j \varsigma^j(u(t)), \\ x_2(t) &= x_1(t) + \sum_{j=1}^{m} E_j \varsigma^j(x_2(t)), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \dot{x}_1(t) &= \sum_{j=0}^{m} A_j x_2(t-jh) + \sum_{j=0}^{m} B_j u(t-jh), \\ x_2(t) &= x_1(t) + \sum_{j=1}^{m} E_j x_2(t-jh). \end{aligned}$$

This is a case of the more general coupled difference-differential equation

$$\begin{cases} \dot{x}_1(t) &= \sum_{j=0}^{m} A_j x_1(t-jh) + \sum_{j=0}^{m} E_j x_2(t-jh) + \sum_{j=0}^{m} B_j u(t-jh), \\ x_2(t) &= \sum_{j=0}^{m} \tilde{A}_j x_1(t-jh) + \sum_{j=0}^{m} \tilde{E}_j x_2(t-jh) + \sum_{j=0}^{m} \tilde{B}_j u(t-jh), \\ y(t) &= \sum_{j=0}^{m} C_j x_1(t-jh) + \sum_{j=0}^{m} \tilde{C}_j x_2(t-jh) + \sum_{j=0}^{m} D_j u(t-jh). \end{cases}$$

The two previous illustrations show the utility of the continuous-time difference equations in engineering sciences and their applications. First, in a reinterpretation of a classic problem involving partial differencial equations, and later in the analysis of a subject of interest in the literature of delay systems.

In this research work, we will study, in the framework of the Lyapunov Krasovskii Theory, difference equations in continuous time of the form

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j), \quad t \geq 0, \tag{1.16}$$

which is the particular autonomous case of (1.1).

Let $\varphi : [-H, 0) \to \mathbb{R}^n$ be an initial function. We assume that the function belongs to the space $\mathcal{PC}([-H, 0), \mathbb{R}^n)$, of piecewise continuous functions defined on the segment $[-H, 0)$. Let $x(t, \varphi)$ stand for the solution of system (1.16) under the initial condition

$$x(\theta, \varphi) = \varphi(\theta), \quad \theta \in [-H, 0),$$

and let $x_t(\varphi)$ denote the restriction of the solution to the segment $[t - H, t)$,

$$x_t(\varphi) : \theta \to x(t + \theta, \varphi), \quad \theta \in [-H, 0).$$

In the remainder of this work we will be using the Euclidean norm for vectors and the induced matrix norm for matrices. For elements of the space $\mathcal{PC}([-H, 0), \mathbb{R}^n)$ we use the uniform norm

$$\|\varphi(\theta)\|_h = \sup_{\theta \in [-H, 0)} \|\varphi(\theta)\|,$$

and the L_2 norm

$$\|\varphi\|_{L_2} = \int_{-H}^{0} \|\varphi(\theta)\| \mathrm{d}\theta.$$

Let us write next a few comments on the delays in difference equations.

Preliminary remarks on delays.

Let $h_1, h_2, \ldots, h_m$ be a set of ordered delays such that $0 < h_1 < h_2 < \cdots < h_m$.

The delays are said to be *commensurate* if there exist $h > 0$ and positive integers $p_1, \ldots, p_m$ such that

$$h_j = p_j h, \quad j = 1, \ldots, m.$$

The delays are said to be *rationally dependent* if there exist γ_j in $\mathbb{Z}$ not all of which are zero such that

$$\sum_{j=1}^{m} \gamma_j h_j = 0.$$

If such integers do not exist, the delays are said to be rationally independent, that is, the only $m-$tuple of integers $\gamma_j \in \mathbb{Z}$ such that $\sum_{j=1}^{m} \gamma_j h_j = 0$ is the trivial solution in which every integer γ_j is zero. Let $\varphi \in \mathcal{C}([-h_m, 0))$ be an initial condition. The times when jump discontinuities occur are characterized by the propagation of the first jump which arises at time $t_0 = 0$, with

$$\Delta(t_0) = x(t_0 + 0, \varphi) - x(t_0 - 0, \varphi) = \sum_{j=1}^{m} A_j \varphi(-h_j) - \varphi(-0).$$

Since the instantaneous state $x(t, \varphi)$ is a copy of its past values, the times of discontinuity are propagated by positive integer combinations of the delays. Namely, denoting $\{t_n\}_{n \in \mathbb{N}_0}$ the times of discontinuity, we have

$$t_n = p_{n1} h_1 + \cdots + m_{nm} h_m,$$

for some $p_{nk} \in \mathbb{N}_0$, $k = 1, \ldots, m$. The times of discontinuity are defined iteratively by

$$t_n = \min_{p_{nj} \in \mathbb{N}_0} \left\{ \sum_{j=1}^{m} p_{nj} h_j \quad : \quad t_n > t_{n-1} \right\}$$

with $t_0 = 0$.

We define

$$\delta_n = t_{n+1} - t_n, \quad n \in \mathbb{N}_0.$$

The set $\mathcal{G} = \{\delta_n, n \in \mathbb{N}_0\}$ is an additive subgroup of $(\mathbb{R}, +)$, and is therefore either in the form $h\mathbb{Z}$ for $h > 0$, or dense in $\mathbb{R}$. The first case arises when the delays are commensurate, while the second case arises when the delays are rationally independent.

Few comments are in order. First, the explicit dependency structure of the delays has to be taken into account in the analysis. For this, assume that the delays $(h_1, \ldots, h_m)$ are linear combinations of the form

$$h_i = \sum_{j=1}^{N} \gamma_{ij} \eta_j, \quad i = 1, \ldots, m,$$

where $\eta_j > 0$, $\gamma_{ij} \in \mathbb{N}_0$, and $N \leq m$.

The question is then to reformulate the initial equation with delays h_i by another equation in the delays η_j. For this, notice that the elements $(\eta_1, \ldots, \eta_N)$ can always be assumed to be rationally independent. Indeed, if this is not the case, there exist $\rho_j \in \mathbb{Z}$, for $j = 1, \ldots, N$ not all identically zero such that

$$\rho_1 \eta_1 + \cdots + \rho_N \eta_N = 0.$$

There exists at least i_0 in $\{1, \ldots, N\}$ such that $\rho_{i_0} > 0$. Therefore,

$$\eta_{i_0} = -\frac{\sum_{i \neq i_0} \rho_i \eta_i}{\rho_{i_0}}.$$

Define $\tilde{\eta}_i = \frac{\eta_i}{\rho_{i_0}}$, for $i = 1, \ldots, N$, $i \neq i_0$. It follows that

$$\eta_{i_0} = -\sum_{i \neq i_0} \rho_i \tilde{\eta}_i.$$

We conclude that $\eta_i = \rho_{i_0} \tilde{\eta}_i$ and that η_{i_0} is a linear combination of the delays $\tilde{\eta}_i$, for $i \neq i_0$. In other words, the delays h_i, for $i = 1, \ldots, m$, can be written with respect to $\tilde{\eta}_i$, that is, for any $i = 1 \ldots, m$,

$$\begin{aligned} h_i &= \sum_{j=1}^{N} \gamma_{ij} \eta_j \\ &= \sum_{j \neq i_0} \gamma_{ij} \rho_{i_0} \tilde{\eta}_j - \gamma_{ii_0} \sum_{j \neq i_0} \rho_j \tilde{\eta}_j \\ &= \sum_{j \neq i_0} \left(\gamma_{ij} \rho_{i_0} - \gamma_{ii_0} \rho_j \right) \tilde{\eta}_j. \end{aligned}$$

In other words, we have reduced the number of independent delays which generate the initial delays h_i. Hence, without loss of generality, the elements $(\eta_1, \ldots, \eta_N)$ can be taken as rationally independent. The second question now is to reformulate the initial difference equation in the delays h_i into another form with delays η_i. For this, consider the plant

$$x(t) = A_1 x(t - h_1) + A_2 x(t - h_2) + A_3 x(t - h_1 - 2h_2),$$

where delays h_1 and h_2 are rationally independent, $x(t) \in \mathbb{R}^n$, and $A_j \in \mathbb{R}^{n \times n}$, for $j = 1, 2, 3$.

Define

$$\chi(t) = \begin{bmatrix} x(t) \\ x(t-h_2) \\ x(t-2h_2) \end{bmatrix}.$$

It follows that

$$\chi(t) = \begin{bmatrix} A_1 & 0_{n\times n} & A_3 \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix} \chi(t-h_1) + \begin{bmatrix} A_2 & 0_{n\times n} & 0_{n\times n} \\ I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_n & 0_{n\times n} \end{bmatrix} \chi(t-h_2)$$

which is a difference equation with independent delays (that is $\eta_1 = h_1$ and $\eta_2 = h_2$ with the notations above), where the explicit dependency structure on delays was taken into account. With this as a background, we can introduce the concepts needed in order to construct a Lyapunov-Krasovskii functional that allows us to conclude on the stability of system (1.16).

The following sections will address the representation of an explicit expression for the solutions of system (1.16) in terms of their initial functions.

1.1 Fundamental matrix

In order to derive this equation, we will make use of the fundamental matrix of system (1.16). This matrix has been defined for linear systems of different nature [12], and has shown to be of great use in the study of their dynamics.

Definition 1.1. *For the system (1.16), we define the continuous-time difference matrix equation*

$$K(t) = \sum_{j=1}^{m} K(t-h_j)A_j + I_n, \quad t \geq 0, \tag{1.17}$$

with the initial condition

$$K(\theta) = 0_{n\times n} \qquad \text{for } \theta \in [-H, 0). \tag{1.18}$$

Such a matrix $K(t)$ is called the fundamental matrix of system (1.16). From its initial condition we have that it is well-defined for all $t \geq 0$ and it is straightforward to show that it is also unique.

Definition 1.2. *We also introduce the following matrix function*

$$\bar{K}(t) = K(t) + K_0, \quad t \geq -H, \tag{1.19}$$

where

$$K_0 = \left(\sum_{j=1}^{m} A_j - I_n \right)^{-1}, \tag{1.20}$$

assuming that λ is not a root of $\det\left(I_n - \sum_{j=1}^{m} A_j \mathrm{e}^{-\lambda h_j}\right) = 0$. *We will refer to this matrix in forthcoming sections.*

Lemma 1.1. *The matrix $\bar{K}(t)$ defined in (1.19) satisfies the following equation*

$$\bar{K}(t) = \sum_{j=1}^{m} \bar{K}(t - h_j) A_j, \quad t \geq 0. \tag{1.21}$$

Proof. We use the definition of $\bar{K}(t)$ in (1.19) and equation (1.17) to write

$$\bar{K}(t) = \sum_{j=1}^{m} K(t - h_j) A_j + I_n + K_0, \quad t \geq 0,$$

we replace $K(t - h_j)$ with $\bar{K}(t - h_j) - K_0$, and obtain

$$\bar{K}(t) = \sum_{j=1}^{m} \bar{K}(t - h_j) A_j + I_n + K_0 - \sum_{j=1}^{m} K_0 A_j, \quad t \geq 0,$$

grouping the terms of the preceding equation, it follows that

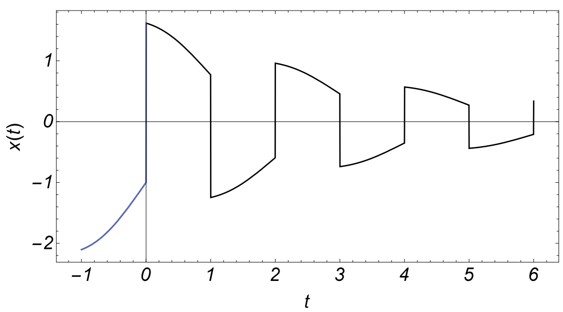
$$\bar{K}(t) = \sum_{j=1}^{m} \bar{K}(t - h_j) A_j + I_n - K_0 \left(\sum_{j=1}^{m} A_j - I_n \right), \quad t \geq 0,$$

equation (1.27) follows from the definition of K_0 in (1.20). □

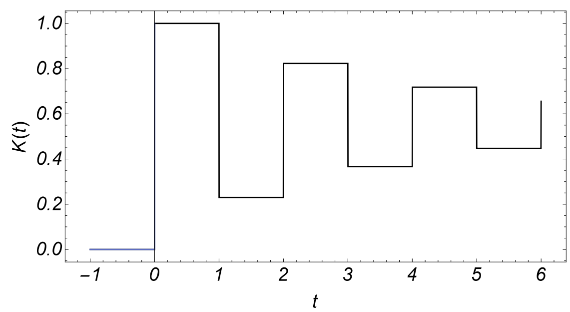
Example 1. One delay system. *In Fig. 1.2 (a) we show the time response of a one-dimensional system described by a particular case of equation (1.16), namely*

$$x(t) = ax(t-H), \quad t \geq 0$$
$$x(\theta) = \varphi(\theta), \qquad \theta < 0, \tag{1.22}$$

with $a = -0.77$*,* $H = 1$*, and* $\varphi(\theta) = 0.5\sin(2\theta) - \mathrm{e}^{-0.5\theta}$*. Fig 1.2 (b) shows its corresponding fundamental function* $K(t)$*.*



(a) Time response of system (1.22)



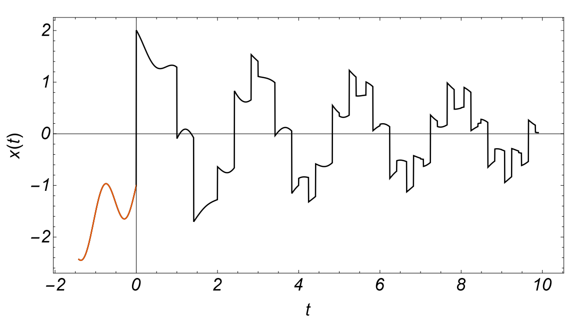
(b) Fundamental function $K(t)$ of system (1.22)

Figure 1.2: Graphic representations of the dynamics of system (1.22)

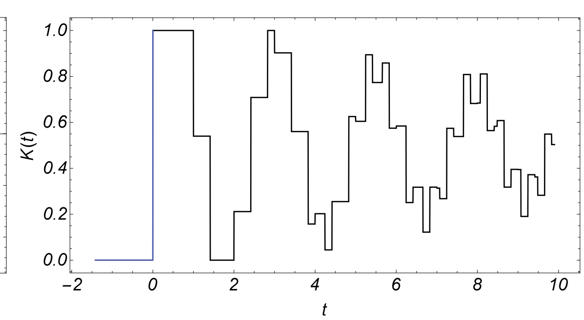
Example 2. System with two non commensurate delays *In Fig. 1.3 (a) we show the time response of the one-dimensional system described by*

$$x(t) = a_1 x(t-1) + a_2 x(t-\sqrt{2}), \quad t \geq 0$$
$$x(\theta) = \varphi(\theta), \qquad \theta < 0 \tag{1.23}$$

with $a_1 = -0.46$*,* $a_2 = -0.54$*, and* $\varphi(\theta) = 0.5\sin(6\theta) - \mathrm{e}^{-0.5\theta}$*. Fig 1.3 (b) shows its corresponding fundamental function* $K(t)$*.*



(a) Time response of system (1.23)



(b) Fundamental function $K(t)$ of system (1.23)

Figure 1.3: Graphic representations of the dynamics of system (1.23)

We continue to characterize the fundamental matrix with the following property

Lemma 1.2. *The fundamental matrix $K(t)$ defined in (1.17) satisfies the matrix equation*

$$K(t) = \sum_{j=1}^{m} A_j K(t-h_j) + I_n, \quad t \geq 0. \tag{1.24}$$

This does not mean that matrix $K(t)$ commutes individually with the coefficient matrices A_j, $j = \overline{1,m}$.

Proof. To verify this, we introduce the matrix $Q(t)$, which is the unique solution to

$$Q(t) = \sum_{j=1}^{m} A_j Q(t-h_j) + I_n, \quad t \geq 0 \tag{1.25}$$

$$Q(\theta) = 0_{n\times n}, \quad \theta < 0.$$

Consider the following identity:

$$\int_{-0}^{t} K(t-s)Q(s)\mathrm{d}s = \int_{0}^{t+0} K(t-\theta)Q(\theta)\mathrm{d}\theta. \tag{1.26}$$

Replacing $K(t-s)$ on the left-hand side (l.h.s.) of (1.26) with (1.17), and $Q(\theta)$ on the right-hand side (r.h.s.) of the same equation with (1.25), we have

$$\sum_{j=1}^{m}\int_{-0}^{t} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{-0}^{t} Q(s)\mathrm{d}s = \sum_{j=1}^{m}\int_{0}^{t+0} K(t-\theta)A_jQ(\theta-h_j)\mathrm{d}\theta + \int_{0}^{t+0} K(t-\theta)\mathrm{d}\theta,$$

the change of variable $s = \theta - h_j$ on the first integral of the r.h.s. of the preceding equation yields

$$\sum_{j=1}^{m}\int_{-0}^{t} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{-0}^{t} Q(s)\mathrm{d}s = \sum_{j=1}^{m}\int_{-h_j}^{t-h_j+0} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{0}^{t+0} K(t-\theta)\mathrm{d}\theta,$$

subtracting $\sum_{j=1}^{m}\int_{-0}^{t-h_j+0} K(t-s-h_j)A_jQ(s)\mathrm{d}s$ from both sides of this equation, we obtain

$$\sum_{j=1}^{m}\int_{t-h_j+0}^{t} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{-0}^{t} Q(s)\mathrm{d}s = \sum_{j=1}^{m}\int_{-h_j}^{-0} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{0}^{t+0} K(t-\theta)\mathrm{d}\theta,$$

the change of variable $s = t - \theta$ on the second integral of the r.h.s. gives

$$\sum_{j=1}^{m} \int_{t-h_j+0}^{t} K(t-s-h_j)A_jQ(s)\mathrm{d}s + \int_{-0}^{t} Q(s)\mathrm{d}s = \sum_{j=1}^{m} \int_{-h_j}^{-0} K(t-s-h_j)A_jQ(s)\mathrm{d}s - \int_{t+0}^{0} K(s)\mathrm{d}s.$$

Notice that for $s \in (t-h_j, t]$, $K(t-s-h_j) = 0_{n\times n}$, and for $s \in [-h_j, 0)$, $Q(s) = 0_{n\times n}$. Then, we arrive at

$$\int_{-0}^{t} Q(s)\mathrm{d}s = \int_{0}^{t+0} K(s)\mathrm{d}s.$$

Taking the time derivative on both sides, yields

$$Q(t) = K(t+0), \qquad t \geq 0,$$

and, given that $K(t)$ is right-continuous, we arrive at (1.24). □

Corollary 1.1. *The matrix $\bar{K}(t)$ defined in (1.19) satisfies the following equation*

$$\bar{K}(t) = \sum_{j=1}^{m} A_j\bar{K}(t-h_j), \quad t \geq 0. \tag{1.27}$$

Proof. We use the result in (1.24), the definition of $\bar{K}(t)$ in (1.19) and the same arguments used in Lemma 1.1 to prove this assertion. □

1.2 Cauchy formula

Next, we present the Cauchy formula for system (1.16). This formula, also known as variation of constants formula, see Bellman and Cooke [12], provides an expression of the solution of system (1.16) in terms of the fundamental matrix.

Theorem 1.1. *Given an initial function $\varphi \in \mathcal{PC}([-H,0),\mathbb{R}^n)$ the following equality holds:*

$$x(t,\varphi) = \sum_{j=1}^{m} \int_{-h_j}^{0} \frac{\mathrm{d}}{\mathrm{d}t}K(t-\theta-h_j)A_j\varphi(\theta)\mathrm{d}\theta, \quad t \geq 0, \tag{1.28}$$

where the integral is in the Lebesgue sense. This equality is known as the Cauchy formula for system (1.16).

Proof. Let us consider the following equation:

$$\int_0^t K(t-\theta)x(\theta)\mathrm{d}\theta = \int_0^{t+0} K(t-\theta)x(\theta)\mathrm{d}\theta. \tag{1.29}$$

Using the expression for $K(t)$ in (1.17) and the expression for $x(t)$ in (1.16) on the l.h.s. and the r.h.s. of (1.29), respectively, gives

$$\sum_{j=1}^{m}\int_0^t K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta + \int_0^t x(\theta)\mathrm{d}\theta = \sum_{j=1}^{m}\int_0^{t+0} K(t-\theta)A_jx(\theta-h_j)\mathrm{d}\theta. \tag{1.30}$$

Applying the change of variable $\xi = \theta - h_j$ on the r.h.s. of (1.30), the equation becomes

$$\sum_{j=1}^{m}\int_0^t K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta + \int_0^t x(\theta)\mathrm{d}\theta = \sum_{j=1}^{m}\int_{-h_j}^{t-h_j+0} K(t-\xi-h_j)A_jx(\xi)\mathrm{d}\xi. \tag{1.31}$$

Replacing ξ with θ and taking into consideration that $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$, the r.h.s. of (1.31) can be modified so that it becomes

$$\begin{aligned}\sum_{j=1}^{m}\int_0^t K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta + \int_0^t x(\theta)\mathrm{d}\theta = &\sum_{j=1}^{m}\int_0^{t-h_j+0} K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta \\ &+\sum_{j=1}^{m}\int_{-h_j}^{0} K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta.\end{aligned} \tag{1.32}$$

Now, subtracting the first term on the r.h.s. of (1.32) from both sides of the equation, yields

$$\sum_{j=1}^{m}\int_{t-h_j+0}^{t} K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta + \int_0^t x(\theta)\mathrm{d}\theta = \sum_{j=1}^{m}\int_{-h_j}^{0} K(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta.$$

Notice that, for $\theta \in (t-h_j, t]$, the expression $t-\theta-h_j \in [-h_j, 0)$. Therefore, $K(t-\theta-h_j) = 0_{n\times n}$. Moreover, $x(\theta) = \varphi(\theta)$ on $\theta \in [-h_j, 0)$. Thus, the preceding equation can be rewritten as

$$\int_0^t x(\theta)\mathrm{d}\theta = \sum_{j=1}^{m}\int_{-h_j}^{0} K(t-\theta-h_j)A_j\varphi(\theta)\mathrm{d}\theta. \tag{1.33}$$

Taking the first derivative with respect to t on both sides of (1.33) yields

$$x(t) = \sum_{j=1}^{m} \int_{-h_j}^{0} \frac{\mathrm{d}}{\mathrm{d}t} K(t - \theta - h_j) A_j \varphi(\theta) \mathrm{d}\theta.$$

□

1.3 Predictor

Next, we present a predictor formula for system (1.16).

Lemma 1.3. *With the knowledge of the state, x_t, of system (1.16) at a given time t, we can compute a value for $x(t+\tau)$, $\tau \geq 0$ using the following formula:*

$$x(t+\tau, x_t) = \sum_{j=1}^{m} \int_{-h_j}^{0} K'(\tau - \theta - h_j) A_j x(t+\theta)\, d\theta. \tag{1.34}$$

Proof. We can prove this Lemma with the same steps we used to prove Theorem 1.1, replacing the identity (1.29) with

$$\int_{0}^{\tau} K(\tau - \theta) x(t+\theta) \mathrm{d}\theta = \int_{0}^{\tau+0} K(\tau - \theta) x(t+\theta) \mathrm{d}\theta,$$

which yields

$$\int_{0}^{\tau} x(t+\theta) \mathrm{d}\theta = \sum_{j=1}^{m} \int_{-h_j}^{0} K(\tau - \theta - h_j) A_j x(t+\theta) \mathrm{d}\theta.$$

We take the derivative with respect to τ on both sides of the preceding equation to obtain

$$x(t+\tau) = \sum_{j=1}^{m} \int_{-h_j}^{0} K'(\tau - \theta - h_j) A_j x(t+\theta) \mathrm{d}\theta.$$

Thus, we arrive at (1.34). □

1.4 Stability of difference equations in continuous time

1.4.1 Stability definitions

The following definitions concerning the class of linear systems under study are found in [19].

Definition 1.3. *System (1.16) is said to be*

i) stable (resp. L_2-stable) if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\varphi\|_h < \delta$ implies $\|x(t,\varphi)\| < \varepsilon$ (resp. $\|x(t,\varphi)\|_{L_2} < \varepsilon$), for any $t \geq 0$.

ii) L_2-asymptotically stable if it is L_2-stable and, for any bounded initial function $\varphi \in \mathcal{PC}([-H,0),\mathbb{R}^n)$,

$$\lim_{t\to\infty} \|x_t(\varphi)\|_{L_2} = 0,$$

iii) asymptotically stable if it is stable and, for any bounded initial function $\varphi \in \mathcal{PC}([-H,0),\mathbb{R}^n)$,

$$\lim_{t\to\infty} \|x(t,\varphi)\| = 0,$$

iv) L_2-exponentially stable if it is L_2-asymptotically stable and if there exist $\alpha \geq 0$ and $\mu > 0$ such that,

$$\|x_t(\varphi)\|_{L_2} \leq \alpha \mathrm{e}^{-\mu t}\|\varphi\|_h, \quad \forall t \geq 0.$$

v) exponentially stable if it is asymptotically stable and if there exist $\alpha \geq 0$ and $\mu > 0$ such that,

$$\|x_t(\varphi)\| \leq \alpha \mathrm{e}^{-\mu t}\|\varphi\|_h, \quad \forall t \geq 0.$$

As it can be seen, exponential stability implies L_2-exponential stability, and this in turn implies asymptotic stability. However, the converse is false, in general. Furthermore, these definitions hold for a given set of delays $\{h_1,\ldots,h_m\}$. If these properties hold independently of the delays, we say that system (1.16) is stable in *the delays [19].*

1.4.2 Spectral analysis of asymptotic stability

From [3] we have the following necessary and sufficient condition for L_2-asymptotic stability in the delays of system (1.16):

$$\sup\left\{\rho\left(\sum_{j=1}^{m} \mathrm{e}^{i\theta_j} A_j\right), \quad \theta_j \in [0, 2\pi]\right\} < 1. \tag{1.35}$$

This condition, in the scalar case is equal to

$$\sum_{j=1}^{m} |a_j| < 1. \tag{1.36}$$

If the delays in system (1.16) are commensurate, i.e., there exists $h > 0$ such that $h_j = p_j h$, $p_j \in \mathbb{Z}$, $j = 1, \ldots, m$, then it admits the following representation with a single delay $h > 0$ (see [19])

$$X(t) = AX(t-h), \qquad t \geq 0, \tag{1.37}$$

where $X(t) \in \mathbb{R}^{p_m n}$, and A is the following $p_m n \times p_m n$ companion matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_1 & 0 & \cdots & 0 & A_i & 0 & \cdots & 0 & A_m \\ I & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & \ddots & & & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

For this case the stability criterion for discrete linear systems (see Appendix B) holds. A generalization of the stability condition in (1.35) that takes into account the algebraic multiplicity of the eigenvalues of A is given by

Theorem 1.2. *[30] System (1.37) is*

i) asymptotically stable *if and only if* $\rho(A) < 1$*,*

ii) stable *if and only if* $\rho(A) \leq 1$*, and for any eigenvalue* $|\lambda_k| = 1$*,* $\operatorname{rank}(A - \lambda_k I) = n - q_k$*, where* q_k *is the algebraic multiplicity of* λ_k*.*

1.4.3 Lyapunov-Krasovskii sufficient conditions of exponential stability

In this section we present sufficient conditions for stability of system (1.16) given by the Lyapunov Krasovskii approach, which uses a functional whose upper right-hand derivative is defined by

$$D^+v(x_t(\varphi)) = \limsup_{h\to 0^+} \frac{v(x_{t+h}(\varphi) - v(x_t(\varphi))}{h}.$$

This upper right-hand derivative of v is named the Dini derivative [31]. Let us remind that if the Dini derivative of a continuous function $f(\cdot)$ is not positive, then, this function does not increase. The following is a result in which the L_2-stability of system (1.16) is established.

Theorem 1.3. *Assume that there exists a continuous functional $v : \mathcal{PC}([-H,0),\mathbb{R}^n) \to \mathbb{R}$ such that $t \mapsto v(x_t(\varphi))$ is (upper right-hand) differentiable for all $t \geq 0$ and such that*

1. $\exists \quad \alpha_1 > 0$ *s.t.* $\forall t \geq 0,\ \alpha_1 \|x_t(\varphi)\|^2_{L_2} \leq v(x_t(\varphi))$,
2. $\exists \quad \alpha_2 \geq 0$ *s.t.* $v(\varphi) \leq \alpha_2 \|\varphi\|^2_h$,
3. $\exists \quad \sigma > 0$ *s.t.* $\forall t \geq 0,\ D^+v(x_t(\varphi)) \leq -2\sigma v(x_t)$.

Then (1.16) is L_2-exponentially stable, that is

$$\|x_t(\varphi)\|_{L_2} \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\varphi\|_h \mathrm{e}^{-\sigma t}, \quad t \geq 0.$$

Proof. From assumption (3) we obtain

$$D^+v(x_t(\varphi)) + 2\sigma v(x_t(\varphi)) \leq 0,$$

this leads to

$$v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

Assumptions (1) and (2) yield

$$\|x_t(\varphi)\|_{L_2}^2 \leq \frac{\alpha_2}{\alpha_1}\|\varphi\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0,$$

which ends the proof. $\square$

Chapter 2

Functional with prescribed derivative

With Theorem 1.3 as a motivation, we start searching for quadratic functionals that satisfy the theorem conditions. For this purpose, we follow a strategy based on the direct Lyapunov method. We begin by selecting a desired time derivative and then computing the functional whose time derivative along the solution of (1.16) corresponds to the selected one. The linearity and time invariance of the system allows us to begin with the case where the desired derivative is a quadratic form.

2.1 Form of the functional

In this section we assume that the system (1.16) is stable. Under this assumption, we are able to construct, with the help of the Cauchy formula (1.28), a functional with prescribed derivative.

We first define a quadratic functional $v_0(\varphi)$, $\varphi \in \mathcal{PC}([-H,0),\mathbb{R}^n)$, that satisfies the equality

$$D^+ v_0(x_t) = -x^T(t,\varphi)Wx(t,\varphi), \quad t \geq 0, \tag{2.1}$$

along the solutions of system (1.16). Here W is a given positive definite matrix. Integrating equation (2.1) from $t=0$ to $t=T>0$ we obtain

$$v_0(x_T(\varphi)) - v_0(\varphi) = -\int_0^T x(t,\varphi)Wx(t,\varphi)\mathrm{d}t.$$

Since system (1.16) is exponentially stable, $x_T(\varphi) \to 0_h$ as $T \to \infty$, and we arrive at the expression

$$v_0(\varphi) = \int_0^{\infty} x^T(t,\varphi) W x(t,\varphi) \mathrm{d}t,$$

the fact that $x(t)$ is right-continuous gives

$$v_0(\varphi) = \int_{-0}^{\infty} x^T(t,\varphi) W x(t,\varphi) \mathrm{d}t.$$

The exponential stability of system (1.16) implies that the improper integral on the right-hand side of the previous equality is well defined. Replacing $x(t,\varphi)$ with the r.h.s. of the Cauchy formula (1.28), we have

$$v_0(\varphi) = \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}\int_{-h_j}^{0}\varphi^T(\xi)A_i^T\int_{-0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}K^T(t-\xi-h_i)W\frac{\mathrm{d}}{\mathrm{d}t}K(t-\theta-h_j)\mathrm{d}tA_j\varphi(\theta)\mathrm{d}\theta\mathrm{d}\xi. \tag{2.2}$$

The equality

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t-\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}K(t-\tau),$$

allows us to express (2.2) as

$$v_0(\varphi) = \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}\int_{-h_j}^{0}\varphi^T(\xi)A_i^T\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-0}^{\infty}K^T(t-\xi-h_i)WK(t-\theta-h_j)\mathrm{d}tA_j\varphi(\theta)\mathrm{d}\theta\mathrm{d}\xi. \tag{2.3}$$

We define the following matrix

$$U(\tau) \triangleq \int_{-0}^{\infty} K^T(t) W \bar{K}(t+\tau)\mathrm{d}t, \qquad \tau \in \mathbb{R}, \tag{2.4}$$

where $\bar{K}(t)$ is introduced in (1.19). This matrix will play an important role in the following sections. In analogy with the delay-free case, and other cases of delay systems reported in the literature, this real valued matrix function is called the Lyapunov matrix of system (1.16).

Let us prove that the matrix $U(\tau)$ is well defined for $\tau \in \mathbb{R}$.

Lemma 2.1. *Given $\tau_0 \in \mathbb{R}$, the improper integral (2.4) converges absolutely and uniformly with respect to $\tau \in [\tau_0, \infty)$.*

Proof. From equation (1.27) we see that the columns of matrix $\bar{K}(t)$ are solutions of (1.16) with specific initial conditions. Therefore, under the assumption that system (1.16) is exponentially stable, we can say that this matrix admits an upper exponential estimate of the form

$$\|\bar{K}(t)\| \leq \gamma \mathrm{e}^{-\sigma t}, \qquad t \geq 0. \tag{2.5}$$

It follows directly from (2.5) that, for $t \geq 0$

$$\begin{aligned}\|K^T(t)W\bar{K}(t+\tau)\| &= \left\|\left(\bar{K}^T(t) - K_0^T\right)W\bar{K}(t+\tau)\right\| \\ &\leq \gamma^2\|W\|\mathrm{e}^{-\sigma(2t+\tau)} + \gamma\,\|WK_0\|\,\mathrm{e}^{-\sigma(t+\tau)}.\end{aligned}$$

Now, let $\tau \in [\tau_0, \infty)$; then, the inequality

$$\int_{-0}^{\infty} \|K^T(t)W\bar{K}(t+\tau)\|\mathrm{d}t \leq \frac{\gamma^2}{2\gamma}\|W\|\mathrm{e}^{-\sigma\tau_0}\left(1 + \frac{2}{\gamma}\,\|K_0\|\right),$$

proves the statement. □

Observe that, under the change of variable $s = t - \xi - h_i$, the expression

$$\int_{-0}^{\infty} K^T(t-\xi-h_i)WK(t-\theta-h_j)\mathrm{d}t,$$

appearing in (2.3), becomes

$$\begin{aligned}\int_{-\xi-h_i-0}^{\infty} K^T(s)WK(s+\xi+h_i-\theta-h_j)\mathrm{d}s &= \int_{-0}^{\infty} K^T(s)W\bar{K}(s+\xi+h_i-\theta-h_j)\mathrm{d}s \\ &\quad - \int_{-0}^{\infty} K^T(s)WK_0\mathrm{d}s \\ &\quad + \int_{-\xi-h_i-0}^{-0} K^T(s)WK(s+\xi+h_i-\theta-h_j)\mathrm{d}s.\end{aligned}$$

From the definition of $U(\tau)$ in (2.4), we get

$$\begin{aligned}\int_{-0}^{\infty} K^T(t-\xi-h_i)WK(t-\theta-h_j)\mathrm{d}t =& U(-\theta-h_j+\xi+h_i)\\ &+\int_{-\xi-h_i-0}^{-0} K^T(s)WK(s+\xi+h_i-\theta-h_j)\mathrm{d}s\\ &-\int_{-0}^{\infty} K^T(s)WK_0\mathrm{d}s.\end{aligned}$$

Returning to (2.3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-0}^{\infty} K^T(t-\xi-h_i)WK(t-\theta-h_j)\mathrm{d}t =& \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}U(-\theta-h_j+\xi+h_i)\\ &-\frac{\mathrm{d}}{\mathrm{d}\xi}\int_{-\xi-h_i-0}^{-0} K^T(s)WK'(s-\theta-h_j+\xi+h_i)\mathrm{d}s\\ &-\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-0}^{\infty} K^T(s)\mathrm{d}sWK_0.\end{aligned}$$

As $\xi \in [-h_i, 0]$, $s \in [-h_i, 0)$ and $K(s) = 0_{n\times n}$. We observe also that the terms inside the second integral on the r.h.s. of the preceding equation are constant with respect to θ and ξ. Then, we arrive at the following expression

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-0}^{\infty} K^T(t-\xi-h_i)WK(t-\theta-h_j)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}U(-\theta-h_j+\xi+h_i)\\ &= -U''(-\theta-h_j+\xi+h_i),\end{aligned}$$

which allows us to write the functional (2.2) as

$$v_0(\varphi) = -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}\int_{-h_j}^{0} \varphi^T(\xi)A_i^TU''(-\theta-h_j+\xi+h_i)A_j\varphi(\theta)\mathrm{d}\theta\mathrm{d}\xi. \tag{2.6}$$

2.2 Properties of the delay Lyapunov matrix

In this section we prove some properties of the matrix function $U(\tau)$, $\tau \in [-H, H]$, defined in (2.4), and of its derivative's jump discontinuities, denoted as $\Delta U'(\tau)$.

2.2.1 Properties of $U(\tau)$

Lemma 2.2. *Let system (1.28) be exponentially stable. We define the $n \times n$ antisymmetric matrix*

$$P \triangleq \int_{-0}^{\infty} \bar{K}^T(\tau) W K_0 \mathrm{d}\tau - \int_{-0}^{\infty} K_0^T W \bar{K}(\tau) \mathrm{d}\tau \tag{2.7}$$

with K_0 defined in (1.20). Then, the Lyapunov matrix (2.4) associated to the positive definite matrix W, satisfies the following properties:

1. *Symmetry property:*

$$U(-\tau) = U^T(\tau) + P - \tau K_0^T W K_0, \qquad \tau \in [-H, H] \tag{2.8}$$

2. *Dynamic property:*

$$U(\tau) = \sum_{j=1}^{m} U(\tau - h_j) A_j, \quad \tau \geq 0, \tag{2.9}$$

Proof. 1. **Symmetry property**

The matrix $U(\tau)$ satisfies

$$\begin{aligned} U(-\tau) &= \int_{-0}^{\infty} K^T(t) W \bar{K}(t-\tau) \mathrm{d}t = \int_{-\tau-0}^{\infty} K^T(\xi+\tau) W \bar{K}(\xi) \mathrm{d}\xi \\ &= \int_{-\tau-0}^{\infty} \left(\bar{K}(\xi+\tau) - K_0\right)^T W \left(K(\xi) + K_0\right) \mathrm{d}\xi \\ &= U^T(\tau) + \int_{-\tau-0}^{-0} \bar{K}^T(\xi+\tau) W K(\xi) \mathrm{d}\xi + \int_{-\tau-0}^{\infty} \left(\bar{K}^T(\xi+\tau) W K_0 - K_0^T W \bar{K}(\xi)\right) \mathrm{d}\xi. \end{aligned}$$

Consider the case $\tau \geq 0$. Since the matrix $K(\xi) = 0_{n\times n}$, $\xi \in [-\tau, 0)$, the first integral term on the right-hand side of the preceding equation equals zero, then we rewrite the equality as

$$U(-\tau) = U^T(\tau) + \int_{-0}^{\infty} \bar{K}^T(\xi) W K_0 \mathrm{d}\xi - \int_{-0}^{\infty} K_0^T W \bar{K}(\xi) \mathrm{d}\xi - \int_{-\tau-0}^{-0} K_0^T W \bar{K}(\xi) \mathrm{d}\xi.$$

The definition of P given in (2.7) and the fact that for $\xi \in [-H, 0)$, $\bar{K}(\xi) = K_0$ lead us to equation (2.8) for $\tau \in [0, H)$.

Consider now the case $\tau \in [-H, 0)$, the equality

$$U(\tau) = U^T(-\tau) + P + \int_{-0}^{\tau-0} K_0^T W \bar{K}(\xi)\mathrm{d}\xi,$$

is satisfied. We transpose the preceding equation and obtain that (2.8) also holds, thus ending the proof.

2. **Dynamic property**

From the definition of the fundamental matrix, (1.17), and the Lyapunov matrix (2.4), we have

$$\begin{aligned} U(\tau) &= \int_{-0}^{\infty} K^T(t) W \bar{K}(t+\tau)\mathrm{d}t \\ &= \int_{-0}^{\infty} K^T(t) W \left(\sum_{j=1}^{m} \bar{K}(t+\tau-h_j)A_j \right) \mathrm{d}t \\ &= \sum_{j=1}^{m} \int_{-0}^{\infty} K^T(t) W \bar{K}(t+\tau-h_j)\mathrm{d}t A_j. \end{aligned}$$

Since $U(\tau - h_j) = \int_{-0}^{\infty} K^T(t) W \bar{K}(t+\tau-h_j)$, we arrive at the result in (2.9).

□

2.2.2 Computation of matrix P

Lemma 2.3. *The matrix P defined in (2.7) satisfies the following equation*

$$P = K_0^T \left[\sum_{j=1}^{m} h_j \left(W K_0 A_j - A_j^T K_0^T W \right) \right] K_0, \tag{2.10}$$

Proof. We begin by taking the Laplace transform of equation (1.27), this is

$$\begin{aligned} \hat{\bar{K}}(s) &= \sum_{j=1}^{m} \int_{-0}^{\infty} \bar{K}(t-h_j)\mathrm{e}^{-st}\mathrm{d}t A_j \\ &= \sum_{j=1}^{m} \hat{\bar{K}}(s)\mathrm{e}^{-sh_j} A_j + \sum_{j=1}^{m} \int_{-h_j}^{-0} \bar{K}(t)\mathrm{e}^{-s(t+h_j)}\mathrm{d}t A_j \\ &= \sum_{j=1}^{m} \hat{\bar{K}}(s)\mathrm{e}^{-sh_j} A_j + \frac{1}{s} K_0 \sum_{j=1}^{m} \left(1 - \mathrm{e}^{-sh_j}\right) A_j, \end{aligned}$$

which is equivalent to

$$\hat{\bar{K}}(s) = \frac{1}{s} K_0 \sum_{j=1}^{m} \left(1 - \mathrm{e}^{-sh_j}\right) A_j \left(I_n - \sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} \right)^{-1}. \tag{2.11}$$

Given that

$$\begin{aligned} K_0 \sum_{j=1}^{m} A_j &= K_0 \left(\sum_{j=1}^{m} A_j - I_n + I_n \right) \\ &= K_0 \left(\sum_{j=1}^{m} A_j - I_n \right) + K_0 \\ &= I_n + K_0, \end{aligned}$$

we can rewrite equation (2.11) as

$$\hat{\bar{K}}(s) = \frac{1}{s} \left(I_n + K_0 \left(I_n - \sum_{j=1}^{m} \mathrm{e}^{-sh_j} A_j \right) \right) \left(I_n - \sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} \right)^{-1},$$

which yields

$$\hat{\bar{K}}(s) = \frac{1}{s} \left\{ K_0 - \left(\sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} - I_n \right)^{-1} \right\}.$$

Now, we define the matrix function

$$R(t) = - \int_{-0}^{t} K_0^T W \bar{K}(\tau) \mathrm{d}\tau,$$

which has the following Laplace Transform

$$\hat{R}(s) = \frac{1}{s^2} K_0^T W \left\{ \left(\sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} - I_n \right)^{-1} - K_0 \right\}.$$

From the definition of P in (2.7), we see that

$$P = \lim_{t \to \infty} \left\{ R(t) - R^T(t) \right\}.$$

Making use of the Final Value Theorem, we find that

$$P = \lim_{s \to 0} \left\{ s\hat{R}(s) - s\hat{R}^T(s) \right\}.$$

It can be readily verified that

$$P = \lim_{s\to 0}\frac{1}{s}\left\{K_0^T W\left(\sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} - I_n\right)^{-1} - \left(\sum_{j=1}^{m} A_j^T \mathrm{e}^{-sh_j} - I_n\right)^{-1} W K_0\right\}.$$

The series expansion of the term $\left(\sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} - I_n\right)^{-1}$ allows us to conclude that

$$\begin{aligned}
\lim_{s\to 0}\frac{1}{s}\left(\sum_{j=1}^{m} A_j \mathrm{e}^{-sh_j} - I_n\right)^{-1} &= \lim_{s\to 0}\frac{1}{s}\left(\sum_{j=1}^{m} A_j\left(1 - sh_j\right) - I_n\right)^{-1} \\
&= \lim_{s\to 0}\frac{1}{s}\left(K_0^{-1}\left(I_n - sK_0\sum_{j=1}^{m} h_j A_j\right)\right)^{-1} \\
&= \lim_{s\to 0}\frac{1}{s}\left(I_n - sK_0\sum_{j=1}^{m} h_j A_j\right)^{-1} K_0 \\
&= \lim_{s\to 0}\frac{1}{s}\left(I_n + sK_0\sum_{j=1}^{m} h_j A_j\right) K_0.
\end{aligned}$$

Therefore,

$$P = \lim_{s\to 0}\frac{1}{s}\left\{K_0^T W\left(I_n + sK_0\sum_{j=1}^{m} h_j A_j\right) K_0 - K_0^T\left(I_n + sK_0\sum_{j=1}^{m} h_j A_j\right)^T W K_0\right\},$$

and equation (2.10) follows. □

2.2.3 Properties of $\Delta U'(\tau)$

From the definition of the matrix function $U(\tau)$ in (2.4), we have limitations in order to use it for practical applications. First, we find that this definition is only applicable to exponentially stable systems. Second, the definition is of little help in order to find an analytic expression of the function. We will use the function $\Delta U'(\tau)$, defined as $U'(\tau+0) - U'(\tau-0)$ to assist us in finding a new definition of the Lyapunov matrix that is valid for unstable systems as well, and can be easily computed.

Now, we present properties of the matrix function $\Delta U'(\tau)$, $\tau \in [-H, H]$. These properties will be used to show that matrix $U(\tau)$ exists regardless of the stability of system (1.16).

Lemma 2.4. *Let system (1.16) be exponentially stable, the Lyapunov matrix (2.4) associated to a positive definite matrix W satisfies the following properties:*

1. *Symmetry property:*

$$\Delta U'(-\tau) = [\Delta U'(\tau)]^T, \tag{2.12}$$

2. *Dynamic property:*

$$\Delta U'(\tau) = \begin{cases} \sum_{j=1}^{m} \Delta U'(\tau - h_j)A_j, & \tau > 0, \\ \\ \sum_{j=1}^{m} A_j^T \Delta U'(\tau + h_j), & \tau < 0, \end{cases} \tag{2.13}$$

3. *Generalized algebraic property:*

$$\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta U'(\tau + h_i - h_j)A_j - \Delta U'(\tau) = \begin{cases} W\Delta K(\tau), & \tau \geq 0 \\ \\ \Delta K^T(-\tau)W, & \tau < 0. \end{cases} \tag{2.14}$$

Proof. 1. **Symmetry property**

From the definition of the Lyapunov matrix (2.4) we have,

$$U'(\tau) = \int_{-0}^{\infty} K^T(t)WK'(t+\tau)\mathrm{d}t, \qquad \tau \in \mathbb{R},$$

where the integral is in the Lebesgue sense, it is a right-continuous function.

The fundamental matrix $K(t)$ is a constant function except at discontinuity points depending on the delays h_j, $j = \overline{1,m}$. Defining the set of discontinuity instants of $K(t)$ as $\mathscr{I}_{\mathscr{K}} = \{t_\kappa\}_{\kappa \in \mathbb{N}_0}$, where

$$t_\kappa \triangleq \min_{p_\kappa^1,\dots,p_\kappa^m} \left\{ \sum_{j=1}^{m} p_\kappa^j h_j \mid t_\kappa > t_{\kappa-1}, \quad p_\kappa^j \in \mathbb{N}_0 \right\}, \tag{2.15}$$

we can write $U'(\tau)$ as

$$U'(\tau) = \sum_{\kappa \geq 0} \int_{t_\kappa - \tau - 0}^{t_\kappa - \tau + 0} K^T(t) W K'(t+\tau) \mathrm{d}t,$$

which yields

$$U'(\tau) = \sum_{\kappa \geq 0} K^T(t_\kappa - \tau) W \Delta K(t_\kappa). \tag{2.16}$$

We have for $\Delta U'(\tau) = U'(\tau + 0) - U'(\tau - 0)$, $\tau \in \mathbb{R}$

$$\Delta U'(\tau) = -\sum_{\kappa \geq 0} \Delta K^T(t_\kappa - \tau) W \Delta K(t_\kappa). \tag{2.17}$$

From the definition of $\Delta K(t)$, we have that at least one of the terms of the previous sum is not zero only when $t_\kappa - \tau \in \mathscr{I}_{\mathscr{K}}$ for some $\kappa \in \mathbb{N}_0$. Otherwise, $\Delta U'(\tau) = 0$. We define the set of values of $\bar{\tau} \in [a, b]$ such that for at least one $\kappa \in \mathbb{N}_0$, $t_\kappa - \bar{\tau} \in \mathscr{I}_{\mathscr{K}}$ as

$$\mathscr{I}_{\mathscr{U}\,[a,b]} = \{\bar{\tau}\}_{\bar{\tau} \in [a,b]}. \tag{2.18}$$

Defining the variable $t_q = t_\kappa - \bar{\tau}$, and

$$q(\bar{\tau}) = \left\{ q \in \mathbb{N}_0 \mid t_q = \begin{cases} \bar{\tau}, & \bar{\tau} \geq 0, \\ 0, & \bar{\tau} < 0 \end{cases} \right\},$$

we obtain

$$\Delta U'(\bar{\tau}) = -\sum_{q \geq q(-\bar{\tau})} \Delta K^T(t_q) W \Delta K(t_q + \bar{\tau}).$$

If $\bar{\tau} \geq 0$, then $q(-\bar{\tau}) = 0$ and

$$\Delta U'(\bar{\tau}) = -\sum_{q \geq 0} \Delta K^T(t_q) W \Delta K(t_q + \bar{\tau}), \tag{2.19}$$

otherwise, we can write

$$\Delta U'(\bar{\tau}) = -\sum_{q\geq 0} \Delta K^T(t_q) W \Delta K(t_q + \bar{\tau}) + \sum_{0\leq q<q(-\bar{\tau})} \Delta K^T(t_q) W \Delta K(t_q + \bar{\tau}).$$

As $t_q + \bar{\tau} < 0$, for $0 \leq q < q(-\bar{\tau})$; then, $\Delta K(t_q + \bar{\tau}) = 0_{n\times n}$, and (2.19) holds.

Given the two definitions of $\Delta U'(\tau)$ that we have found in (2.17) and (2.19), it is straightforward to verify that property (2.12) is satisfied in the case $t_q - \tau \in \mathscr{I}_{\mathscr{K}}$, for at least one $q \in \mathbb{N}_0$. Otherwise, $t_q - \tau \notin \mathscr{I}_{\mathscr{K}}$, $t_q + \tau \notin \mathscr{I}_{\mathscr{K}}$ for all $q \in \mathbb{N}_0$, and $\Delta U'(\tau) = 0_{n\times n} = (\Delta U'(-\tau))^T$.

2. **Dynamic property**

Let us remember the definition of the fundamental matrix, (1.17), we have

$$K(t) = \begin{cases} \sum_{j=1}^{m} K(t-h_j)A_j + I_n, & t \geq 0 \\ \\ 0_{n\times n}, & t < 0, \end{cases}$$

therefore, for $t > 0$ we have that

$$\begin{aligned} \Delta K(t) = K(t+0) - K(t-0) &= \sum_{j=1}^{m} K(t+0-h_j)A_j + I_n - \sum_{j=1}^{m} K(t-0-h_j)A_j - I_n, \\ &= \sum_{j=1}^{m} \left(K(t+0-h_j) - K(t-0-h_j)\right) A_j, \\ &= \sum_{j=1}^{m} \Delta K(t-h_j)A_j. \end{aligned}$$

For $t = 0$

$$\Delta K(0) = K(+0) - K(-0) = I_n.$$

Summarizing these results, we have

$$\Delta K(t) = \begin{cases} \sum_{j=1}^{m} \Delta K(t-h_j)A_j, & t > 0 \\ I_n, & t = 0, \\ 0_{n\times n}, & t < 0. \end{cases} \tag{2.20}$$

Considering the definition of $\Delta U'(\tau)$ in (2.19) we can write

$$\Delta U'(\tau) = -\sum_{\kappa\geq 0} \Delta K^T(t_\kappa) W \Delta K(t_\kappa + \tau).$$

In view of the dynamics of $\Delta K(t)$ given in (2.20), it follows that

$$\begin{aligned} \Delta U'(\tau) &= -\sum_{\kappa\geq 0} \Delta K^T(t_\kappa) W \sum_{j=1}^{m} \Delta K(t_\kappa + \tau - h_j)A_j, \\ &= -\sum_{j=1}^{m}\sum_{\kappa\geq 0} \Delta K^T(t_\kappa) W \Delta K(t_\kappa + \tau - h_j)A_j, \\ &= \sum_{j=1}^{m} \Delta U'(\tau - h_j)A_j, \qquad \tau > 0, \end{aligned} \tag{2.21}$$

and we arrive at the result for $\tau > 0$ in (2.13).

Consider $\tau < 0$, so that $-\tau > 0$, and $\Delta U'(-\tau)$ satisfies equation (2.21) as follows

$$\Delta U'(-\tau) = \sum_{j=1}^{m} \Delta U'(-\tau - h_j)A_j, \quad \tau < 0,$$

as the matrix $\Delta U'(-\tau)$ satisfies the symmetry property (2.12), we can apply it on both sides and obtain

$$\left(\Delta U'(\tau)\right)^T = \sum_{j=1}^{m} \left(\Delta U'(\tau + h_j)\right)^T A_j, \quad \tau < 0.$$

Transposition of this equation ends the proof.

3. **Generalized algebraic property** We will consider both terms on the left-hand side of equation (2.14) separately. For $\Delta U'(\tau)$, $\tau \geq 0$, defined by equation (2.19), we can write

$$\Delta U'(\tau) = -\sum_{\kappa>0} \Delta K^T(t_\kappa) W \Delta K(t_\kappa + \tau) - \Delta K^T(0) W \Delta K(\tau).$$

Then, using the dynamics of $\Delta K(t)$ described in (2.20), we have

$$\begin{aligned}\Delta U'(\tau) &= -\sum_{\kappa>0} \Delta K^T(t_\kappa) W \Delta K(t_\kappa + \tau) - \Delta K^T(0) W \Delta K(\tau) \\ &= -\sum_{\kappa>0} \sum_{i=1}^{m} A_i^T \Delta K^T(t_\kappa - h_i) W \sum_{j=1}^{m} \Delta K(t_\kappa + \tau - h_j) A_j - W \Delta K(\tau),\end{aligned}$$

which is equal to

$$\Delta U'(\tau) = -\sum_{\kappa>0} \sum_{i=1}^{m} \sum_{j=1}^{m} A_i^T \Delta K^T(t_\kappa - h_i) W \Delta K(t_\kappa + \tau - h_j) A_j - W \Delta K(\tau). \tag{2.22}$$

Now, for $\sum_{i=1}^{m} \sum_{j=1}^{m} A_i^T \Delta U'(\tau + h_i - h_j) A_j$, consider again equation (2.19)

$$\Delta U'(\tau + h_i - h_j) = -\sum_{\kappa \geq 0} \Delta K^T(t_\kappa) W \Delta K(t_\kappa + \tau + h_i - h_j),$$

the change of variable $t_q = t_\kappa + h_i$ allows us to write this equation as

$$\begin{aligned}\Delta U'(\tau + h_i - h_j) = & -\sum_{q \geq q(h_i)} \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j) \\ = & -\sum_{q \geq 0} \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j) \\ & + \sum_{0 \leq q < q(h_i)} \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j).\end{aligned}$$

The last term is canceled given that $t_q - h_i < 0$ and $\Delta K(t) = 0$, for $t < 0$. Finally, we arrive at

$$\Delta U'(\tau + h_i - h_j) = -\sum_{q \geq 0} \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j). \tag{2.23}$$

As a consequence,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta U'(\tau + h_i - h_j)A_j = & -\sum_{q>0}\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j)A_j \\ & -\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta K^T(-h_i) W \Delta K(\tau - h_j)A_j\end{aligned}$$

the last term is equal to zero as $\Delta K(-h_i) = 0$. Thus, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta U'(\tau + h_i - h_j)A_j = -\sum_{q>0}\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta K^T(t_q - h_i) W \Delta K(t_q + \tau - h_j)A_j, \tag{2.24}$$

Subtracting (2.22) from (2.24) we arrive at the case $\tau \geq 0$ in (2.14). We now consider the case $\tau < 0$. Equation

$$\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \Delta U'(-\tau + h_i - h_j)A_j - \Delta U'(-\tau) = W \Delta K(-\tau)$$

holds. Then, applying the symmetry property on both sides of this equation we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \left(\Delta U'(\tau - h_i + h_j)\right)^T A_j - \left(\Delta U'(\tau)\right)^T = W \Delta K(-\tau).$$

We can interchange i for j, and transpose both sides to end the proof.

$\square$

2.3 Proof by derivation without stability assumption

In the following result, we prove that the derivative of the functional v_0 defined in (2.6) is indeed (2.1). The importance of this result is that, in contrast with the strategy followed thus far for finding the functional, no stability assumption of system (1.16) is made. Therefore, we conclude that the functional is valid for stable as well as for unstable systems. The proof of this result relies on the use of the properties defined in Lemma 2.4.

Theorem 2.1. *Let the matrix $\tilde{U}(\tau), \tau \in [-H, H]$, satisfy properties (2.8), (2.9), and (2.12)-(2.14). If we use the following definition of a functional $\tilde{v}_0(x_t)$,*

$$\tilde{v}_0(x_t) = -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-h_i}^{t}\int_{t-h_j}^{t} x^T(\xi)A_i^T\tilde{U}''(-\theta-h_j+\xi+h_i)A_jx(\theta)\,d\theta\,d\xi. \tag{2.25}$$

Then, this functional is such that along the solutions of system (1.16) the following equality holds:

$$D^+\tilde{v}_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

Proof. Given a solution x_t of system (1.16), we consider the upper-right hand derivative of $v_0(x_t)$ as defined in (2.25), this is

$$\begin{aligned} D^+\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{i=1}^{m}\sum_{j=1}^{m}A_i^T\int_{t-h_j}^{t}\tilde{U}''(t+h_i-\theta-h_j)A_jx(\theta)\mathrm{d}\theta \\ & +\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(t-h_i)A_i^T\int_{t-h_j}^{t}\tilde{U}''(t-\theta-h_j)A_jx(\theta)\mathrm{d}\theta \\ & -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-h_i}^{t}x^T(\xi)A_i^T\tilde{U}''(-t+\xi+h_i-h_j)\mathrm{d}\xi A_jx(t) \\ & +\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-h_i}^{t}x^T(\xi)A_i^T\tilde{U}''(-t+\xi+h_i)\mathrm{d}\xi A_jx(t-h_j). \end{aligned}$$

The preceding equation is equal to

$$\begin{aligned} D^+\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{i=1}^{m}\sum_{j=1}^{m}A_i^T\int_{-h_j}^{0}\tilde{U}''(h_i-\theta-h_j)A_jx(t+\theta)\mathrm{d}\theta \\ & +\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(t-h_i)A_i^T\int_{-h_j}^{0}\tilde{U}''(-\theta-h_j)A_jx(t+\theta)\mathrm{d}\theta \\ & -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}x^T(t+\xi)A_i^T\tilde{U}''(\xi+h_i-h_j)\mathrm{d}\xi A_jx(t) \\ & +\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-h_i}^{0}x^T(t+\xi)A_i^T\tilde{U}''(\xi+h_i)\mathrm{d}\xi A_jx(t-h_j), \end{aligned}$$

equivalently, as $\tilde{U}(\tau)$ is a piecewise linear function,

$$\begin{aligned}
D^{+}\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{i=1}^{m}\sum_{j=1}^{m}A_i^T\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,(-h_j,0]}}\int_{-\bar{\tau}-h_j}^{-\bar{\tau}-h_j+0}\tilde{U}''(h_i-\theta-h_j)A_jx(t+\theta)\mathrm{d}\theta \\
& +\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(t-h_i)A_i^T\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,(-h_j,0]}}\int_{-\bar{\tau}-h_j}^{-\bar{\tau}-h_j+0}\tilde{U}''(-\theta-h_j)A_jx(t+\theta)\mathrm{d}\theta \\
& -\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,[0,h_i)}}\int_{\bar{\tau}-h_i-0}^{\bar{\tau}-h_i}x^T(t+\xi)A_i^T\tilde{U}''(\xi+h_i-h_j)\mathrm{d}\xi A_jx(t) \\
& +\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,[0,h_i)}}\int_{\bar{\tau}-h_i-0}^{\bar{\tau}-h_i}x^T(t+\xi)A_i^T\tilde{U}''(\xi+h_i)\mathrm{d}\xi A_jx(t-h_j),
\end{aligned}$$

where $\mathscr{I}_{\mathscr{U}\,[a,b]}$ is defined in (2.18).

Using the fact that the matrix $\tilde{U}'(t)$ is right-continuous, that is, $\tilde{U}'(t+0)=\tilde{U}'(t)$, we have that $\Delta\tilde{U}'(t)=\tilde{U}'(t)-\tilde{U}'(t-0)$, then we get

$$\begin{aligned}
D^{+}\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{i=1}^{m}\sum_{j=1}^{m}A_i^T\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,(-h_j,0]}}\Delta\tilde{U}'(\bar{\tau}+h_i)A_jx(t-\bar{\tau}-h_j) \\
& +\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(t-h_i)A_i^T\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,(-h_j,0]}}\Delta\tilde{U}'(\bar{\tau})A_jx(t-\bar{\tau}-h_j) \\
& -\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,[0,h_i)}}x^T(t+\bar{\tau}-h_i)A_i^T\Delta\tilde{U}'(\bar{\tau}-h_j)A_jx(t) \\
& +\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}\,[0,h_i)}}x^T(t+\bar{\tau}-h_i)A_i^T\Delta\tilde{U}'(\bar{\tau})A_jx(t-h_j).
\end{aligned}$$

Now, we use equation (1.16), and the fact that $\sum_{-a<j\leq b}G(j)=\sum_{-a<j<b}G(j)+G(b)$, to rewrite the preceding equation as

$$
\begin{aligned}
D^+\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T \sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(-h_j,0)} \Delta\tilde{U}'(\bar{\tau}+h_i)A_j x(t-\bar{\tau}-h_j) \\
& -x^T(t)\sum_{i=1}^{m} A_i^T\Delta\tilde{U}'(h_i)x(t) + \sum_{j=1}^{m} x^T(t)\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(-h_j,0)} \Delta\tilde{U}'(\bar{\tau})A_j x(t-\bar{\tau}-h_j) \\
& +x^T(t)\Delta\tilde{U}'(0)x(t) - \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(0,h_i)} x^T(t+\bar{\tau}-h_i)A_i^T\Delta\tilde{U}'(\bar{\tau}-h_j)A_j x(t) \\
& -\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(t)A_i^T\Delta\tilde{U}'(h_i-h_j)A_j x(t) \\
& +\sum_{i=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(0,h_i)} x^T(t+\bar{\tau}-h_i)A_i^T\Delta\tilde{U}'(\bar{\tau})x(t) + \sum_{i=1}^{m} x^T(t)A_i^T\Delta\tilde{U}'(h_i)x(t).
\end{aligned}
$$

Grouping terms, we obtain

$$
\begin{aligned}
D^+\tilde{v}_0(x_t) = \quad & -x^T(t)\sum_{j=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(-h_j,0)} \left[\sum_{i=1}^{m} A_i^T\Delta\tilde{U}'(\bar{\tau}+h_i) - \Delta\tilde{U}'(\bar{\tau})\right] A_j x(t-\bar{\tau}-h_j) \\
& -\sum_{i=1}^{m}\sum_{\bar{\tau}\in\mathscr{I}_{\mathscr{U}}(0,h_i)} x^T(t+\bar{\tau}-h_i)A_i^T\left[\sum_{j=1}^{m}\Delta\tilde{U}'(\bar{\tau}-h_j)A_j - \Delta\tilde{U}'(\bar{\tau})\right] x(t) \\
& -x^T(t)\left[\sum_{i=1}^{m} A_i^T\Delta\tilde{U}'(h_i) - \sum_{i=1}^{m} A_i^T\Delta\tilde{U}'(h_i)\right] x(t) \\
& -x^T(t)\left[\sum_{i=1}^{m}\sum_{j=1}^{m} A_i^T\Delta\tilde{U}'(h_i-h_j)A_j - \Delta\tilde{U}'(0)\right] x(t).
\end{aligned}
$$

The first and second terms are equal to zero because property (2.13) is satisfied. The expression in brackets of the fourth term equals W, given that property (2.14) is also satisfied, hence

$$D^+\tilde{v}_0(x_t) = -x^T(t)Wx(t),$$

and the result follows.

$\square$

Definition 2.1. *Let the $n\times n$ matrix $U(\tau)$ satisfy Eq. (2.13). We say that it is a Lyapunov matrix of system (1.16) associated with a symmetric matrix W if it also satisfies properties (2.8), (2.9), (2.12), and(2.14).*

Conclusion

In this chapter we have presented the form $v_0(x_t)$ of the functional with prescribed derivative $D^+v_0(x_t) = -x^T(t)Wx(t)$.

We have shown that this expression is valid for stable and unstable systems indistinctly.

It is noticeable that the functional is completely determined by the so-called delay Lyapunov matrix, which satisfies a number of important properties. In contrast with differential systems, as those studied in [1], the properties of the jumps in the derivative of $U(\tau)$ will play a significant role in the remainder of our work.

Chapter 3

Complete type functional

In this chapter we present a new functional, called of complete type. By adding new terms to $v_0(x_t)$, it is possible to prove that the new functional admits quadratic lower and upper bounds, which provide constructive L_2-exponential decay rate estimates of the commensurate case of (1.16).

3.1 Form of the complete type functional

First, we obtain a quadratic upper bound for the functional $v_0(x_t)$. We consider only the case of commensurate delays.

We define the *basic delay*, h, as the greatest common divisor of the delays h_i, $i = \overline{1,m}$, and p_i as the integer such that $p_i h = h_i$. Then, we can write the functional $v_0(\varphi)$ in (2.6) as

$$\begin{aligned} v_0(\varphi) &= -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{h_i}\int_0^{h_j}\varphi^T(\xi-h_i)A_i^T U''(\xi-\theta)A_j\varphi(\theta-h_j)\mathrm{d}\theta\mathrm{d}\xi \\ &= -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{h_j}\sum_{\bar\tau\in\mathscr{I}_{\mathscr{U}}(-\theta,h_i-\theta]}\int_{\bar\tau+\theta-0}^{\bar\tau+\theta}\varphi^T(\xi-h_i)A_i^T U''(\xi-\theta)\mathrm{d}\xi A_j\varphi(\theta-h_j)\mathrm{d}\theta \\ &= -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{h_j}\sum_{\bar\tau\in\mathscr{I}_{\mathscr{U}}(-\theta,h_i-\theta]}\varphi^T(\theta+\bar\tau-h_i)A_i^T\Delta U'(\bar\tau)A_j\varphi(\theta-h_j)\mathrm{d}\theta. \end{aligned}$$

The change of variable $\bar{\sigma} = \bar{\tau} + \theta$ in the last equation yields

$$v_0(\varphi) = -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{h_j} \sum_{\bar{\sigma}\in\mathscr{I}_{\mathscr{U}\,(0,h_i]}} \varphi^T(\bar{\sigma}-h_i)A_i^T\Delta U'(\bar{\sigma}-\theta)A_j\varphi(\theta-h_j)\mathrm{d}\theta.$$

For the commensurate delay case, $\mathscr{I}_{\mathscr{U}\,(0,H]} = \{h, 2h, \ldots, p_m h\}$, therefore,

$$v_0(\varphi) = -\sum_{i=1}^{m}\sum_{j=1}^{m}\int_0^{h_j}\sum_{q=1}^{p_i}\varphi^T(qh-h_i)A_i^T\Delta U'(qh-\theta)A_j\varphi(\theta-h_j)\mathrm{d}\theta. \tag{3.1}$$

Now, we introduce the notations

$$\mu = \sup_{\tau\in[0,H]}\|-\Delta U'(\tau)\|, \quad a_j = \|A_j\|, \quad j=\overline{1,m}.$$

Let us prove that the non-negative constant μ exists from the definition of $\Delta U'(\tau)$ in (2.19). We have that

$$-\Delta U'(\tau) = \sum_{q\geq 0}\Delta K^T(t_q)W\Delta K(t_q+\tau).$$

From equation (1.24) we find, for $\Delta K(t) = K(t+0) - K(t-0)$,

$$\begin{aligned}\Delta K(t) &= \sum_{j=1}^{m} A_j\Delta K(t-h_j), \quad t>0\\ \Delta K(0) &= I_n.\end{aligned}$$

Therefore, as the columns of $\Delta K(t)$ satisfy equation (1.16), if the system is exponentially stable $\Delta K(t)$ admits an upper exponential estimate of the form

$$\|\Delta K(t)\| \leq \gamma \mathrm{e}^{-\sigma t}, \quad t\geq 0.$$

Consequently,

$$\|-\Delta U'(\tau)\| \leq \gamma^2\|W\|\mathrm{e}^{-\sigma\tau}\sum_{q\geq 0}\mathrm{e}^{-2\sigma t_q}, \qquad \tau\geq 0. \tag{3.2}$$

Since we are studying the case of commensurate delays, each discontinuity time $t_q \in \mathscr{I}_{\mathscr{K}}$ defined in (2.15) is a multiple of the base delay h. Thus, the inequality (3.2) is equivalent to

$$\begin{aligned}\|-\Delta U'(\tau)\| &\leq \gamma^2 \|W\| \mathrm{e}^{-\sigma\tau} \sum_{q\geq 0} \mathrm{e}^{-2\sigma q h} \\ &\leq \gamma^2 \|W\| \mathrm{e}^{-\sigma\tau} \frac{1}{1-\mathrm{e}^{-2\sigma h}}, \qquad \tau \geq 0.\end{aligned}$$

Therefore, μ exists.

Now, we write an upper bound for $v_0(\varphi)$ as follows

$$v_0(\varphi) \leq \max_{i\in[1,m]} \frac{a_i^2 \mu}{2} \left\{ \sum_{j=1}^{m} \int_{-h_j}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta + \sum_{j=1}^{m} \sum_{q=0}^{p_j} \|\varphi(-qh)\|^2 \right\}. \tag{3.3}$$

With this in mind, we propose a complete type functional in the following theorem.

Theorem 3.1. *Given the symmetric matrices W_q, $q = \overline{0, 2p_m}$, let us define the functional*

$$\begin{aligned} w(\varphi) =& \sum_{j=1}^{m} \sum_{q=0}^{p_j} \varphi^T(-qh) W_q \varphi(-qh) \\ &+ \sum_{j=1}^{m} \sum_{q=1}^{p_j} \int_{-qh}^{0} \varphi^T(\theta) W_{p_m+q} \varphi(\theta) \mathrm{d}\theta. \end{aligned} \tag{3.4}$$

If there exists a Lyapunov matrix $U(\tau)$ associated with the matrix

$$W = W_0 + \sum_{j=1}^{m} \sum_{q=1}^{p_j} \left(W_q + qh W_{p_m+q} \right),$$

and $v_0(\varphi)$ is the functional (2.6) defined by this Lyapunov matrix, then the upper right-hand derivative of the functional

$$v(\varphi) = v_0(\varphi) + \sum_{j=1}^{m} \sum_{q=1}^{p_j} \int_{-qh}^{0} \varphi^T(\theta) \left(W_q + (qh+\theta) W_{p_m+q} \right) \varphi(\theta) \mathrm{d}\theta \tag{3.5}$$

along the solutions of system (1.16) is such that

$$D^+ v(x_t) = -w(x_t), \quad t \geq 0.$$

Proof. We know that

$$D^+v_0(x_t) = -x^T(t)Wx(t) = -x^T(t)\left\{W_0 + \sum_{j=1}^{m}\sum_{q=1}^{p_j}(W_q + qhW_{p_m+q})\right\}x(t).$$

Let us call $Q_j(t)$ the terms under the summation sign in equation (3.5). A simple change of the integration variable yields the equality

$$\begin{aligned} Q_j(t) = & \sum_{q=1}^{p_j}\int_{-qh}^{0} x^T(t+\theta)\left(W_q + (qh+\theta)W_{p_m+q}\right)x(t+\theta)\,d\theta \\ = & \sum_{q=1}^{p_j}\int_{t-qh}^{t} x^T(\xi)\left(W_q + (qh+\xi-t)W_{p_m+q}\right)x(\xi)\,d\theta, \end{aligned}$$

therefore,

$$\begin{aligned} \sum_{j=1}^{m} D^+Q_j(t) =& x^T(t)\left[\sum_{j=1}^{m}\sum_{q=1}^{p_j}(W_q + qhW_{p_m+q})\right]x(t) \\ & - \sum_{j=1}^{m}\sum_{q=1}^{p_j}\left[x^T(t-qh)W_qx(t-qh) + \int_{-qh}^{0} x^T(t+\xi)W_{p_m+q}x(t+\xi)\,d\theta\right]. \end{aligned}$$

The theorem statement follows directly from the preceding expressions for the time derivatives. □

Definition 3.1. *We say that functional (3.5) is of the complete type if the matrices W_q, $q = 0, 1, \dots, 2p_m$ are positive definite.*

3.2 Quadratic lower and upper bounds

Lemma 3.1. *Let system (1.16) be exponentially stable. Given the positive-definite matrices W_q, $q = 0, 1, \dots, 2p_m$, there exists a positive constant α_1 such that the complete type functional (3.5) satisfies the inequality*

$$\alpha_1\|\varphi\|_{L_2}^2 \le v(\varphi), \quad \varphi \in PC([-H, 0), \mathbb{R}^n).$$

Proof. We consider a modified functional of the form

$$\tilde{v}(\varphi) = v(\varphi) - \alpha \|\varphi\|_{L_2}^2 = v(\varphi) - \alpha \int_{-H}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where $\alpha \in \mathbb{R}$. Then

$$D^+ \tilde{v}(x_t) = -\tilde{w}(x_t), \quad t \geq 0.$$

Here,

$$\tilde{w}(x_t) = w(x_t) + \alpha \left[x^T(t)x(t) - x^T(t-H)x(t-H)\right]$$

$$\geq \begin{pmatrix} x^T(t) & x^T(t-H) \end{pmatrix} \mathcal{L}(\alpha) \begin{pmatrix} x(t) \\ x(t-H) \end{pmatrix}.$$

The matrix $\mathcal{L}(\alpha)$ is given by the following expression

$$\mathcal{L}(\alpha) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_{p_m} \end{pmatrix} + \alpha \begin{pmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & -I_n \end{pmatrix}.$$

Because the matrices W_q, $q = 0, 1, \ldots, 2p_m$, are positive definite, there exists $\alpha_1 > 0$ such that the matrix $\mathcal{L}(\alpha_1)$ is positive definite. Indeed, we find that $0 < \alpha_1 \leq \lambda_{\min}(W_{p_m})$, and we conclude that

$$\tilde{w}(x_t) \geq 0.$$

The exponential stability of system (1.16) makes it possible to present $\tilde{v}(\varphi)$ in the form

$$\tilde{v}(\varphi) = \int_{-0}^{\infty} \tilde{w}(x(t,\varphi))\mathrm{d}t \geq 0,$$

therefore, we conclude that $\alpha_1 \|\varphi\|_{L_2}^2 \leq v(\varphi)$.

□

Remark 1. *Observe that if we had only considered $v_0(x_t)$ ($W_q = 0$, $q = 0, 1, \ldots, 2p_m$) it would not have been possible to find an L_2 lower quadratic bound.*

Lemma 3.2. *Given the symmetric matrices W_q, $q = 0, 1, \dots, 2p_m$, assume that system (1.16) admits a Lyapunov matrix associated with the matrix*

$$W = W_0 + \sum_{j=1}^{m} \sum_{q=1}^{p_j} \left(W_q + qhW_{p_m+q}\right).$$

Then, there exists a positive constant α_2 such that the complete type functional (3.5) satisfies the inequality

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2.$$

Proof. The following estimates hold for the terms of functional (3.5):

For $v_0(\varphi)$,

$$v_0(\varphi) \leq \max_{i\in[1,m]} \frac{a_i^2 \mu}{2} \sum_{j=1}^{m} \left(h_j + (p_j+1)p_j/2\right) \|\varphi\|_h^2.$$

For $j = \overline{1, m}$,

$$\begin{aligned} Q_j(t) = & \sum_{q=1}^{p_j} \int_{-qh}^{0} \varphi^T(\theta) \left(W_q + (qh+\theta)W_{p_m+q}\right) \varphi(\theta)\, d\theta \\ \leq & \left(\sum_{q=1}^{p_j} \lambda_{\max} \left(W_q + qhW_{p_m+q}\right) qh \right) \|\varphi\|_h^2. \end{aligned}$$

As a result, we arrive at an upper estimation of the form

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2,$$

where

$$\alpha_2 = \max_{i\in[1,m]} \frac{a_i^2 \mu}{2} \sum_{j=1}^{m} \left(h_j + (p_j+1)p_j/2\right) + \sum_{j=1}^{m} \sum_{q=1}^{p_j} \lambda_{\max} \left(W_q + qhW_{p_m+q}\right) qh.$$

□

The terms added to $v_0(x_t)$ in order to get the complete type functional $v(x_t)$ defined in (3.5) were shown to be relevant, as they allowed to establish the existence of a quadratic lower bound for $v(x_t)$. This bound will turn out to be crucial for finding exponential estimates of the solution as shown in Theorem 1.3.

3.3 Exponential estimates of the response of difference equations in continuous time

3.3.1 L_2-exponential stability (case of commensurate delays)

The upper and lower bounds of the functional $v(x_t)$ that we found in the previous section, together with the statement in Theorem 1.3 allow us to write a quadratic upper bound for $\|x_t(\varphi)\|_{L_2}$ as follows

$$\|x_t(\varphi)\|_{L_2} \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|\mathrm{e}^{-\sigma t}, \quad t \geq 0$$

where $\alpha_2 \geq \max_{i\in[1,m]} \frac{a_i^2\mu}{2} \sum_{j=1}^{m} \left(h_j + (p_j+1)p_j/2\right) + \sum_{j=1}^{m}\sum_{q=1}^{p_j} \lambda_{\max}\left(W_q + qhW_{p_m+q}\right) qh$, $\alpha_1 \leq \lambda_{\min}(W_{p_m})$.

For the constant σ, we have that

$$D^+v(x_t) + 2\sigma v(x_t) \leq 0.$$

Therefore,

$$\left(\min_{q\in[1,p_m]} \lambda_{min}(W_{p_m+q}) - 2\sigma\left(\max_{i\in[1,m]}\frac{a_i^2\mu}{2} + \max_{q\in[1,p_m]}\lambda_{\max}\left(W_q + qhW_{p_m+q}\right)\right)\right)$$
$$\times \sum_{j=1}^{m}\sum_{q=1}^{p_j}\int_{-qh}^{0}\|x(t+\theta)\|^2\mathrm{d}\theta + \left(\min_{q\in[1,p_m]}\lambda_{\min}(W_q) - \sigma\max_{i\in[1,m]} a_i^2\mu\right)\sum_{j=1}^{m}\sum_{q=0}^{p_j}\|x(t-qh)\|^2 \quad \geq 0.$$

If σ is such that the terms of the preceding inequality are both greater than zero, then the conditions of Theorem 1.3 are fulfilled, and we have an expression for the L_2-exponential estimate for systems with commensurate delays.

3.3.2 Exponential stability (case of commensurate delays)

The following result (Lemma 7 in [23]) allows us to find exponential estimates of the response of system (1.16) in terms of the constants defined in the preceding section.

Lemma 3.3. *[23] Assume that there exists $\beta \leq 0$ and $\sigma > 0$ such that, for any $\varphi \in \mathcal{PC}([-H, 0), \mathbb{R}^n)$ and for any $t \geq 0$,*

$$\|x_t(\varphi)\|_{L_2} \leq \beta \|\varphi\|_h \mathrm{e}^{-\sigma t}.$$

Then,

$$\|x(t, \varphi)\| \leq \frac{\beta}{\sqrt{H}} \|\varphi\|_h \mathrm{e}^{-\sigma t}, \quad t \geq 0.$$

In this case, $\beta = \sqrt{\frac{\alpha_2}{\alpha_1}}$.

Conclusion

In this chapter, a Complete Type Lyapunov-Krasovskii functional was introduced in order to fulfill the sufficient conditions for the stability given by Theorem 1.3. The existence of an upper bound of the matrix function $\Delta U'(\tau)$ defined in (2.19) was proven for the commensurate case, and constructive exponential estimates of the L_2 and the Euclidean norms of the solutions of (1.16) were obtained. The non-commensurate delay case is substantially more complex to analyze and it is beyond the scope of this work.

Chapter 4

Construction of the Lyapunov matrix

Providing an effective numerical procedure for constructing the matrices $\Delta U'(\tau)$ and $U(\tau)$ is indeed a crucial step of our approach. Only in this case, we will be able to use the theoretical results we present. Now, we present a strategy that can be followed in order to construct matrix $U(\tau)$ that satisfies properties (2.8),(2.9), and (2.12)-(2.14).

4.1 Single-delay case

For the difference equation

$$\begin{aligned} x(t) &= Ax(t-H), \quad t \geq 0, \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-H, 0), \end{aligned} \tag{4.1}$$

with $x(t) \in \mathbb{R}^n$, the fundamental matrix is given by

$$K(t) = K(t-H)A + I_n, \quad t \geq 0, \tag{4.2}$$

with initial condition

$$K(\theta) = 0_{n\times n}, \quad \theta \in [-H, 0). \tag{4.3}$$

For any positive definite matrix W, the Lyapunov matrix (2.4) satisfies the properties (2.8)-(2.9).

We consider the following equalities, given by the dynamic property (2.9), for $\xi \in [0, H]$,

$$\begin{aligned} U(\xi) =& U(\xi - H)A, \\ U^T(H - \xi) =& A^T U^T(\xi). \end{aligned}$$

Applying the symmetry property (2.8) to the second equation we have

$$U(\xi) = U(\xi - H)A, \tag{4.4}$$

$$U(\xi - H) = A^T U(\xi) - \left(K_0^{-1}\right)^T P - \left(\xi I_n + H K_0^T\right) W K_0. \tag{4.5}$$

where P defined in (2.10), is given by

$$P = H K_0^T W K_0^2 - H (K_0^2)^T W K_0.$$

We define the following variables, for $\xi \in [0, H]$,

$$\begin{aligned} Y(\xi) &= U(\xi), \\ Z(\xi) &= U(\xi - H). \end{aligned} \tag{4.6}$$

Writing (4.4)-(4.5) in terms of the variables defined in (4.6), the following system of equations is obtained

$$\begin{aligned} Y(\xi) - Z(\xi)A &= 0 \\ Z(\xi) - A^T Y(\xi) &= -\left(K_0^{-1}\right)^T P - \left(\xi I_n + H K_0^T\right) W K_0. \end{aligned} \tag{4.7}$$

Next, we use some properties of the Kronecker product (see Appendix A) to solve for the matrices $Y(\xi)$ and $Z(\xi)$. Define the vectors $y(\xi) = \mathrm{vec}(Y(\xi))$ and $z(\xi) = \mathrm{vec}(Z(\xi))$. Then, the system of equations (4.7) can be rewritten in algebraic form as

$$\begin{bmatrix} I_n \otimes I_n & -A^T \otimes I_n \\ -I_n \otimes A^T & I_n \otimes I_n \end{bmatrix} \begin{bmatrix} y(\xi) \\ z(\xi) \end{bmatrix} = \begin{bmatrix} 0_{n^2} \\ -\mathrm{vec}(\left(K_0^{-1}\right)^T P + \left(\xi I_n + H K_0^T\right) W K_0) \end{bmatrix}. \tag{4.8}$$

Using the matrix inversion lemma [32], we have that the preceding system of equations has unique solution if the matrix

$$I_n \otimes I_n - A^T \otimes A^T$$

is invertible, that is, if none of the eigenvalues of A lies on the unit circle of the complex plane.

Recalling the definitions in (4.6), we obtain, for $\xi \in [0, H]$,

$$u(\xi) = \mathrm{vec}(U(\xi)) = -\left(I_n \otimes I_n - A^T \otimes A^T\right)^{-1} \mathrm{vec}\left(\left(K_0^{-1}\right)^T PA + \left(\xi I_n + HK_0^T\right) WK_0 A\right),$$

$$u(\xi - H) = -\left(I_n \otimes I_n - A^T \otimes A^T\right)^{-1} \mathrm{vec}\left(\left(K_0^{-1}\right)^T P + \left(\xi I_n + HK_0^T\right) WK_0\right),$$

with

$$P = HK_0^T W K_0^2 - H(K_0^2)^T W K_0.$$

Example 3. *In Fig. 4.1 we present the construction of $U(\tau)$, $\tau \in [-H, H]$ associated to a one-dimensional system, for a given positive definite matrix $W = I_2$, and a given parameter matrix* $A = \begin{bmatrix} -0.9375 & 1.11844 \\ 0.3732 & -1.3009 \end{bmatrix}$.

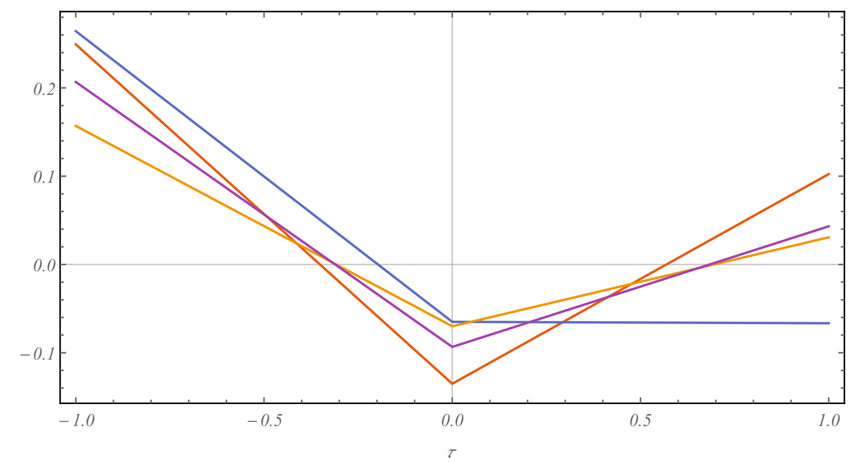


Figure 4.1: Graph of $U(\tau)$, for a two-dimensional system of the form (4.1). The elements of $U(\tau)$ appear as follows: $U_{11}(\tau)$ (——), $U_{12}(\tau)$(——), $U_{21}(\tau)$ (——), and $U_{22}(\tau)$ (——).

4.2 Multiple commensurate delays case

It is clear that the case of commensurate delays can be reduced to the one delay case, however, it seems important to address it as a multiple-delay system as a preparatory step to the case of multiple non commensurate delays.

For the case of a difference equation in continuous time with multiple commensurate delays of the form

$$\begin{aligned} x(t) &= \sum_{j=1}^{m} A_j x(t-jh), \quad t \geq 0, \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-mh, 0), \end{aligned} \tag{4.9}$$

where h is known as the basic delay, the fundamental matrix is given by

$$K(t) = \sum_{j=1}^{m} K(t-jh)A_j + I_n, \quad t \geq 0, \tag{4.10}$$

with initial condition

$$K(\theta) = 0_{n\times n}, \quad \theta \in [-mh, 0). \tag{4.11}$$

We consider the following equalities, given by the dynamic and the symmetry properties, (2.9) and (2.8), respectively. For $\xi \in [0, h]$,

$$U(kh+\xi) = \sum_{j=1}^{m} U\left((k-j)h+\xi\right) A_j, \quad k = 0, 1, \dots, m-1, \tag{4.12}$$

$$\begin{aligned} U(\xi - kh) = &\sum_{j=1}^{m} A_j^T U\left(\xi + (j-k)h\right) - \left(K_0^{-1}\right)^T P \\ &- \left((\xi - kh)I_n + \sum_{j=1}^{m} h_j A_j^T K_0^T\right) W K_0, \quad k = 1, 2, \dots, m. \end{aligned} \tag{4.13}$$

where P is defined in (2.10), namely

$$P = K_0^T \left[\sum_{j=1}^{m} jh\left(W K_0 A_j - A_j^T K_0^T W\right)\right] K_0. \tag{4.14}$$

Let us define the auxiliary matrices, for $\xi \in [0, h]$

$$Y_k(\xi) = U(kh+\xi), \quad k \in \{-m, -m+1, \dots, 0, \dots, m-1\}. \tag{4.15}$$

First, we observe that the equations in (4.12), for $k \in \{0, 1, \dots, m-1\}$, rewritten in the variables introduced in (4.15) are

$$Y_k(\xi) = \sum_{j=1}^{m} Y_{k-j}(\xi) A_j, \qquad k \in \{0, 1, \dots, m-1\}. \tag{4.16}$$

Now, assume that $k = \overline{1, m}$; then, the equations in (4.13) are rewritten as

$$Y_{-k}(\xi) = \sum_{j=1}^{m} A_j^T Y_{-k+j}(\xi) - \left(K_0^{-1}\right)^T P - \left((\xi - kh) I_n + \sum_{j=1}^{m} h_j A_j^T K_0^T\right) W K_0, \qquad k = \overline{1, m}. \tag{4.17}$$

Observe that (4.16)-(4.17) is a system of $2m$ algebraic equations with $2m$ unknowns defined in (4.15). To solve this system, we use Kronecker products along with vectorization techniques (see Appendix A). We define the vectors $y_k(\xi) = \mathrm{vec}(Y_k(\xi))$, $k \in \{-m, 1-m, \dots, -1, 0, 1, \dots, m-1\}$, $\xi \in [0, h]$. Then, the system of equations (4.16)-(4.17) can be rewritten as

$$y_k(\xi) = \begin{cases} \sum_{j=1}^{m} \left(A_j^T \otimes I_n\right) y_{k-j}(\xi), & k \geq 0 \\ \\ \sum_{j=1}^{m} \left(I_n \otimes A_j^T\right) y_{k+j}(\xi) & \\ -\mathrm{vec}\left\{ \left(K_0^{-1}\right)^T P + \left((\xi + kh) I_n + \sum_{j=1}^{m} h_j A_j^T K_0^T\right) W K_0 \right\}, & k < 0. \end{cases} \tag{4.18}$$

Corollary 4.1. *If the system of equations (4.16)-(4.17) admits a unique solution*

$$\{Y_{m-1}(\xi), Y_{m-2}(\xi), \dots, Y_0(\xi), \dots, Y_{-m}(\xi)\}, \quad \xi \in [0, h],$$

then, there exists a unique Lyapunov matrix $U(\tau)$ associated with the matrix W. This matrix is defined on $[0, H]$ by the equalities

$$U(kh + \xi) = Y_j(\xi), \quad \xi \in [0, h], \quad k = 0, 1, \dots, m-1.$$

Example 4. Two-delay system *Consider the following system*

$$x(t) = A_1 x(t-1) + A_2 x(t-3/2), \qquad t \geq 0$$
$$x(\theta) = \varphi(\theta), \qquad \theta \in [-3/2, 0), \tag{4.19}$$

where $x(t) \in \mathbb{R}^n$ and the basic delay, h, is equal to $1/2$, given that it is the maximum number that divides 1 and $3/2$. We solve for $U(\tau)$, $\tau \in [-3/2, 3/2]$, by finding the solution of the system of equations (4.16)-(4.17), using Kronecker products, as follows:

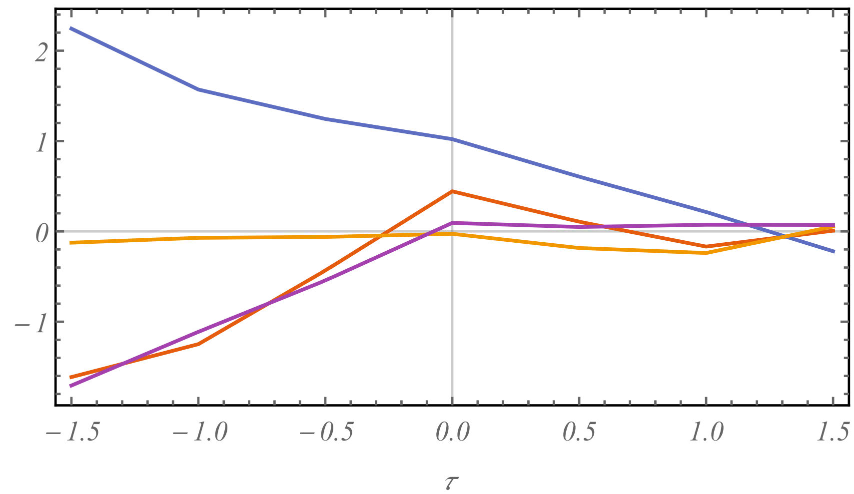
$$\begin{pmatrix} I_{n^2} & 0_{n^2\times n^2} & -A_1^T\otimes I_n & -A_2^T\otimes I_n & 0_{n^2\times n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & I_{n^2} & 0_{n^2\times n^2} & -A_1^T\otimes I_n & -A_2^T\otimes I_n & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & 0_{n^2\times n^2} & I_{n^2} & 0_{n^2\times n^2} & -A_1^T\otimes I_n & -A_2^T\otimes I_n \\ -I_n\otimes A_2^T & -I_n\otimes A_1^T & 0_{n^2\times n^2} & I_{n^2} & 0_{n^2\times n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & -I_n\otimes A_2^T & -I_n\otimes A_1^T & 0_{n^2\times n^2} & I_{n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & 0_{n^2\times n^2} & -I_n\otimes A_2^T & -I_n\otimes A_1^T & 0_{n^2\times n^2} & I_{n^2} \end{pmatrix} \begin{pmatrix} y_2(\xi) \\ y_1(\xi) \\ y_0(\xi) \\ y_{-1}(\xi) \\ y_{-2}(\xi) \\ y_{-3}(\xi) \end{pmatrix} = - \begin{pmatrix} 0_{n^2\times 1} \\ 0_{n^2\times 1} \\ 0_{n^2\times 1} \\ \mathrm{vec}\Big(\big(K_0^{-1}\big)^T P + \big(\xi I_n + 1/2 K_0^T\big) W K_0\Big) \\ \mathrm{vec}\Big(\big(K_0^{-1}\big)^T P + \big(\xi I_n + K_0^T\big) W K_0\Big) \\ \mathrm{vec}\Big(\big(K_0^{-1}\big)^T P + \big(\xi I_n + 3/2 K_0^T\big) W K_0\Big) \end{pmatrix},$$

where $y_k(\xi) = U(kh+\xi)$, $\xi \in [0, 1/2)$, $P = K_0^T \left[\sum_{j=1}^{2} hp_j \left(W K_0 A_j - A_j^T K_0^T W\right)\right] K_0$, $p_1 = 2$, and $p_2 = 3$. Plots of $U(\tau) \in \mathbb{R}^2$, $\tau \in [-3/2, 3/2]$, are shown in Fig. 4.2 for $W = I_2$, $K_0 = (A_1 + A_2 - I_2)^{-1}$, and different values of A_1 and A_2.

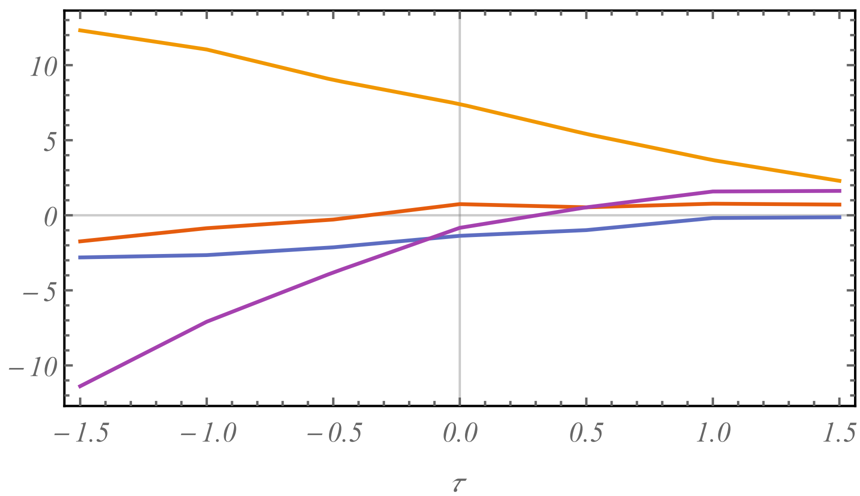
Fig 4.2(a) corresponds to $A_1 = \begin{bmatrix} -0.4 & -0.3 \\ 0.1 & 0.15 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 0.1 & 0.25 \\ -0.9 & -0.1 \end{bmatrix}$,

Fig 4.2(b) corresponds to $A_1 = \begin{bmatrix} 0.1 & -0.7 \\ 0.6 & 0.8 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 0.5 & -0.25 \\ -0.4 & -0.2 \end{bmatrix}$,

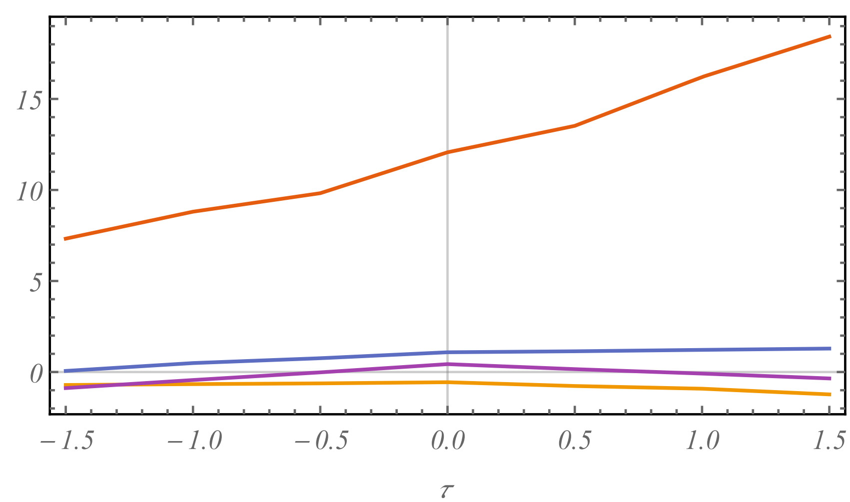
finally, Fig 4.2(c) corresponds to $A_1 = \begin{bmatrix} 1.1 & 0 \\ -0.4 & 0 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 0.25 & -0.125 \\ -0.4 & -0.5 \end{bmatrix}$.



(a)



(b)



(c)

Figure 4.2: $U(\tau)$, $\tau \in [-7/2, 7/2)$ related to the two-dimensional system of two commensurate delays (4.19). The elements of $U(\tau)$ appear as $U_{11}(\tau)$ (—), $U_{12}(\tau)$ (—), $U_{21}(\tau)$ (—), and $U_{22}(\tau)$ (—).

4.3 Non-commensurate delay case: proposal of an approximation

In this section, we analyze a strategy to approximate the matrix function $U_{nc}(\tau)$ related to a system of two non commensurate delays. We will do so by computing the matrices $U_s(\tau)$ related to systems of commensurate delays that tend to the non commensurate case. The intuition behind this approach is that a sufficiently good approximation obtained from the systems with commensurate delays is attainable, since it will become noticeable that the sequence of the matrix functions $U_s(\tau)$ tends to a continuous function. This will be better understood through an example.

For the sake of clarity we use the same system as in Example 2, this is, the system described by the following equation.

$$\begin{aligned} &x(t) = a_1 x(t-1) + a_2 x(t-\sqrt{2}), \quad t \geq 0, \\ &x(\theta) = \varphi(\theta), \quad \theta \in \left[-\sqrt{2}, 0\right). \end{aligned} \tag{4.20}$$

The fundamental function of this system is given by

$$\begin{aligned} &K(t) = K(t-1)a_1 + K(t-\sqrt{2})a_2 + 1, \quad && t \geq 0, \\ &K(\theta) = 0, && \theta < 0, \end{aligned} \tag{4.21}$$

Consider the continued fraction representation [33] of $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

which can also be written as

$$\sqrt{2} = [1; 2, 2, 2, \dots] \tag{4.22}$$

A rational approximation of the number $\sqrt{2}$ is given by a finite number of twos on the l.h.s. of equation (4.22), allowing us to find an approximation of $U_{nc}(\tau)$ in the same way we have done for the multiple delay case. Using this finite approximation of the irrational number, we have the following system of equations:

$$\begin{aligned} y_k(\xi) = U_s(kh+\xi) &= \sum_{j=1}^{2} U_s(kh - h_j + \xi)a_j, \\ &= \sum_{j=1}^{2} y_{k-\frac{h_j}{h}}(\xi)a_j, \qquad\qquad k \geq 0. \end{aligned} \tag{4.23}$$

$$y_k(\xi) = \sum_{j=1}^{2} a_j y_{k+\frac{h_j}{h}}(\xi) - k_0 p - \left(\xi + kh + \sum_{j=1}^{2} h_j a_j k_0\right) w k_0, \quad k < 0 \tag{4.24}$$

Where

$w > 0$,

$k_0 = 1/(a_1 + a_2 - 1)$,

$h_1 = 1$,

$h_2 = [1; 2, 2, \cdots, 2]$,

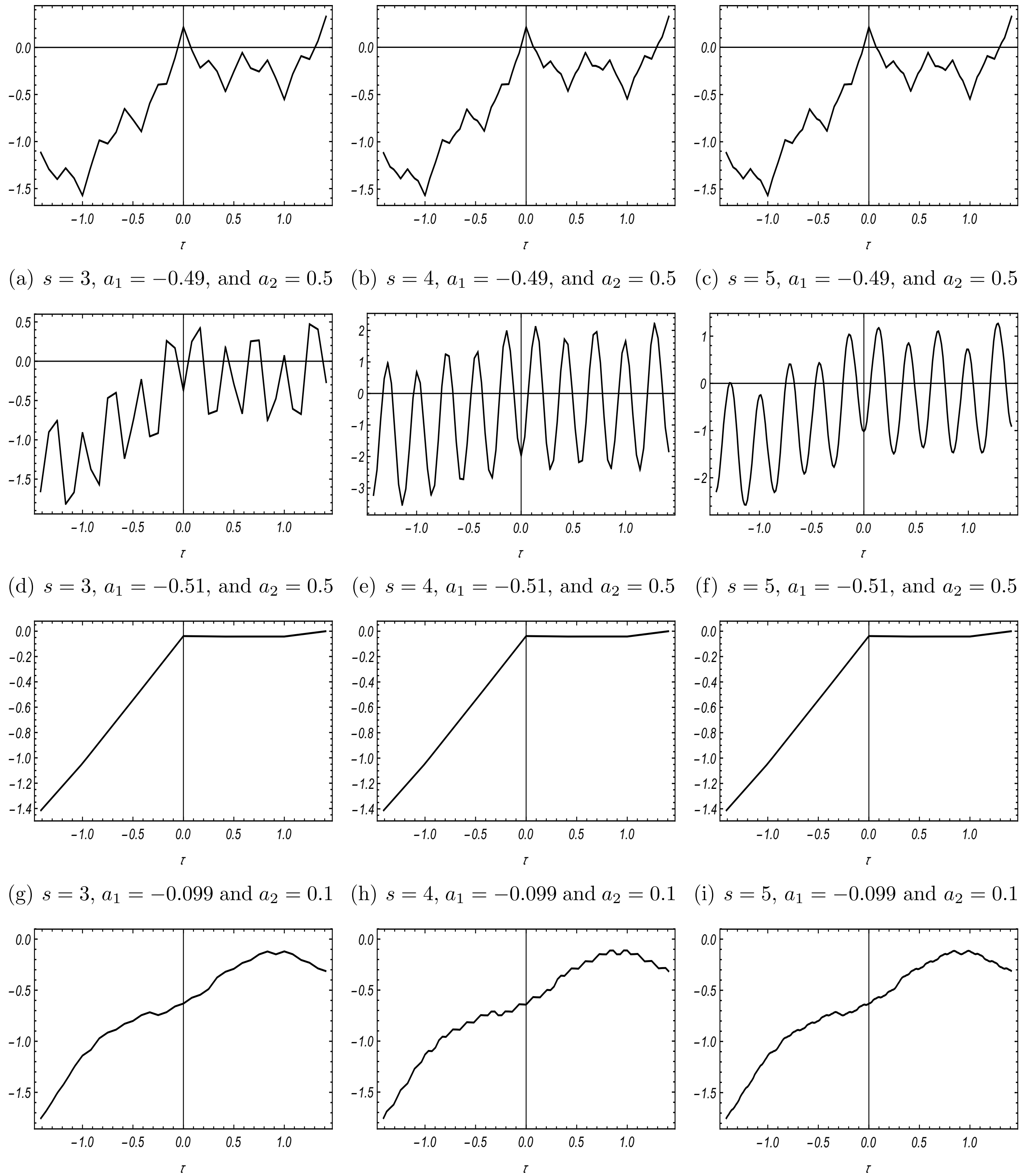
$h = \gcd(h_1, h_2)$, is the basic delay,

$p = -p = 0$,

$k \in \{-h_2/h, -h_2/h+1, \ldots, 0, \ldots, h_2/h-2, h_2/h-1, h_2/h\}$.

Then, the solution for the function $U_s(kh+\xi)$, $\xi \in [0,h]$, is obtained by solving the system of equations (4.23)-(4.24).

Graphs of the scalar function $U_s(\tau)$, $\tau \in [-H, H]$ are shown in Fig. 4.3 for different values of a_1, a_2, and s is the number of twos on the continued fraction notation of h_2. Notice that as s increases, the function $U_s(\tau)$ converges to a continuous function.



(a) $s = 3$, $a_1 = -0.49$, and $a_2 = 0.5$ (b) $s = 4$, $a_1 = -0.49$, and $a_2 = 0.5$ (c) $s = 5$, $a_1 = -0.49$, and $a_2 = 0.5$

(d) $s = 3$, $a_1 = -0.51$, and $a_2 = 0.5$ (e) $s = 4$, $a_1 = -0.51$, and $a_2 = 0.5$ (f) $s = 5$, $a_1 = -0.51$, and $a_2 = 0.5$

(g) $s = 3$, $a_1 = -0.099$ and $a_2 = 0.1$ (h) $s = 4$, $a_1 = -0.099$ and $a_2 = 0.1$ (i) $s = 5$, $a_1 = -0.099$ and $a_2 = 0.1$

(j) $s = 3$, $a_1 = -0.99$ and $a_2 = 1$ (k) $s = 4$, $a_1 = -0.99$ and $a_2 = 1$ (l) $s = 5$, $a_1 = -0.99$ and $a_2 = 1$

Figure 4.3: Results of the construction of $U(\tau)$ for the rational approximation of the system of two non-commensurate delays in (4.20).

Conclusion

In this chapter, we have presented the construction methodology (the numerical computation) of the matrix function $U(\tau)$ using the properties of this matrix introduced in Chap. 2, for the cases of a single delay and multiple commensurate delays.

For the case of non commensurate delays, we have resorted to an approximation given by a close enough commensurate case. This topic deserves further research efforts in the future, as an analysis of the error of approximation ought to be made.

Chapter 5

Necessary stability conditions

In the study of time-delay systems, necessary conditions of stability were introduced by S. Mondié and A. Egorov in [25], [26], [2]. Based on the delay Lyapunov Matrix, they proposed a complete type functional that satisfies a lower quadratic bound if the system is stable. Rewriting this condition in terms depending solely on the delay Lyapunov matrix, they arrived at necessary conditions of stability that become less conservative as a parameter r increases. The analogous of this result for continuous-time difference equations is presented here. A comparison between existing conditions of stability and the new necessary conditions is presented in the examples at the end of the chapter.

5.1 Preliminary results

5.1.1 A new functional

We introduce in this section a new functional derived from $v_0(x_t)$ defined in (2.6). This functional will allow us to find necessary conditions for the stability of system (1.16) depending on the Lyapunov matrix.

Consider the quadratic functional

$$v_1(\varphi) = v_0(\varphi) + \int_{-H}^{0} \varphi^T(\theta) W \varphi(\theta) \mathrm{d}\theta. \tag{5.1}$$

Its upper right-hand derivative along the solutions of system (1.16) is equal to

$$D^+v_1(x_t(\varphi)) = -x^T(t-H,\varphi)Wx(t-H,\varphi).$$

The following is a result of necessary conditions for the stability of system (1.16) depending on the recently defined functional $v_1(x_t)$.

Theorem 5.1. *If system (1.16) is exponentially stable, then there exists $\alpha > 0$ such that*

$$v_1(\varphi) \geq \alpha\|\varphi\|_{L_2}^2. \tag{5.2}$$

Proof. Using the same argument that in the proof of Lemma 3.1, we define a functional $\tilde{v}(\varphi)$ such that

$$\tilde{v}(\varphi) = v_1(\varphi) - \alpha\|\varphi\|_{L_2}^2,$$

where $\alpha \in \mathbb{R}$. Then,

$$D^+\tilde{v}(x_t) = -\tilde{w}(x_t), \quad t \geq 0.$$

Consequently,

$$\tilde{w}(x_t) = x^T(t-H)Wx(t-H) + \alpha\|x(t)\|^2 - \alpha\|x(t-H)\|^2.$$

Therefore, the asymptotic stability of system (1.16) implies $\alpha < \lambda_{\min}(W)$.

□

5.1.2 Cauchy formula for $K(t)$

Lemma 5.1. *Given the fundamental matrix $K(t)$ defined in (1.17) and (1.24), the following equality holds:*

$$K(t+\tau) = K(t) + \sum_{j=1}^{m}\int_{-h_j}^{0}\frac{\mathrm{d}}{\mathrm{d}t}K(t+\theta-h_j)A_jK(\theta+\tau)d\theta, \tag{5.3}$$

here, the integral is in the Lebesgue sense.

Proof. Let us consider the identity, for $t \geq 0$ and $\tau \geq 0$

$$\int_0^\tau K(t+\tau-\theta)K(\theta)\mathrm{d}\theta = \int_{-0}^\tau K(t+\tau-s)K(s)\mathrm{d}s.$$

We use the formula in (1.17) and the formula in (1.24) to arrive at the following equation

$$\begin{aligned}\sum_{j=1}^m \int_0^\tau K(t+\tau-\theta)A_jK(\theta-h_j)\mathrm{d}\theta + \int_0^\tau K(t+\tau-\theta)\mathrm{d}\theta = &\sum_{j=1}^m \int_{-0}^\tau K(t+\tau-s-h_j)A_jK(s)\mathrm{d}s\\ &+ \int_{-0}^\tau K(s)\mathrm{d}s,\end{aligned}$$

the change of variable $s=\theta-h_j$ on the first integral of the l.h.s. of the preceding equation yields

$$\begin{aligned}\sum_{j=1}^m \int_{-h_j}^{\tau-h_j} K(t+\tau-s-h_j)A_jK(s)\mathrm{d}s + \int_0^\tau K(t+\tau-\theta)\mathrm{d}\theta = &\sum_{j=1}^m \int_{-0}^\tau K(t+\tau-s-h_j)A_jK(s)\mathrm{d}s\\ &+ \int_{-0}^\tau K(s)\mathrm{d}s.\end{aligned}$$

This equation can be written as follows

$$\int_0^\tau K(t+\tau-\theta)\mathrm{d}\theta = \sum_{j=1}^m \int_{\tau-h_j}^\tau K(t+\tau-s-h_j)A_jK(s)\mathrm{d}s + \int_{-0}^\tau K(s)\mathrm{d}s.$$

As $s\in[-h_j,0)$, $K(s)=0_{n\times n}$. The change of variable $\theta=s-\tau$ on the first integral of the r.h.s., together with the change of variable $s=t+\tau-\theta$ on the integral of the l.h.s. of this equation yield

$$\int_t^{t+\tau} K(s)\mathrm{d}s = \sum_{j=1}^m \int_{-h_j}^0 K(t-\theta-h_j)A_jK(\theta+\tau)\mathrm{d}\theta + \int_0^\tau K(s)\mathrm{d}s.$$

Taking the derivative with respect to t on both sides of the previous equation we get

$$K(t+\tau)-K(t) = \sum_{j=1}^m \int_{-h_j}^0 \frac{\mathrm{d}}{\mathrm{d}t}K(t-\theta-h_j)A_jK(\theta+\tau)\mathrm{d}\theta.$$

Thus, we arrive at (5.3). □

Remark 2. *Notice that equation (5.3) holds for any $t\in\mathbb{R}$ and any $\tau\geq 0$.*

5.1.3 A crucial bilinear functional

Let us define the following functional depending on two initial conditions, which will play an important role in the proof connecting the necessary condition (5.2) for the stability of (1.16):

$$z(\varphi, \psi) = \int_{-H}^{\infty} x^T(t, \varphi) W x(t, \psi) \mathrm{d}t. \tag{5.4}$$

In the next lemma, we show that for special initial functions depending on the fundamental matrix $K(t)$, and constant vectors, this bilinear functional reveals a dependence on the Lyapunov delay matrix function.

Lemma 5.2. *For any τ_k, $\tau_l \in (0, H)$, and initial functions*

$$\varphi_0(\xi) = \sum_{k=1}^{r} K(\tau_k + \xi)\gamma_k, \quad \xi \in [-H, 0),$$

$$\psi_0(\theta) = \sum_{l=1}^{r} K(\tau_l + \theta)\gamma_l, \quad \theta \in [-H, 0)$$

the bilinear functional (5.4) can be written as follows

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r} \sum_{l=1}^{r} \gamma_k^T F(\tau_k, \tau_l)\gamma_l, \tag{5.5}$$

where $F(\tau_k, \tau_l) = U(0) - U(-\tau_k) - U(\tau_l) + U(\tau_l - \tau_k)$.

Proof. Replacing $x(t, \cdot)$ with the r.h.s. of the Cauchy formula (1.28), we get the following equation:

$$\begin{aligned} z(\varphi_0, \psi_0) = & \sum_{k=1}^{r} \sum_{l=1}^{r} \int_{-H}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-h_i}^{0} \int_{-h_j}^{0} \gamma_k^T K^T(\tau_k + \xi) A_i^T \frac{\mathrm{d}}{\mathrm{d}t} K^T(t - \xi - h_i) W \\ & \times \frac{\mathrm{d}}{\mathrm{d}t} K(t - \theta - h_j) A_i K(\tau_l + \theta)\gamma_l \mathrm{d}\theta \mathrm{d}\xi \mathrm{d}t \end{aligned} \tag{5.6}$$

Using equation (5.3), equation (5.6) becomes

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-H}^{\infty} \left(K^T(t+\tau_k) - K^T(t)\right) W \left(K(t+\tau_l) - K(t)\right) \mathrm{d}t\gamma_l.$$

The fact that $K(t+\tau_k) - K(t) = \int_{-\tau_k}^{0} K'(t-\xi)\mathrm{d}\xi$, allows us to write

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-H}^{\infty} \left(\int_{-\tau_k}^{0} K'(t-\xi)\mathrm{d}\xi\right)^T W \left(\int_{-\tau_l}^{0} K'(t-\theta)\mathrm{d}\theta\right) \mathrm{d}t\gamma_l,$$

we now substitute $K'(t-\theta)$ with $-\frac{\mathrm{d}}{\mathrm{d}\theta}K(t-\theta)$ and $K'(t-\xi)$ with $-\frac{\mathrm{d}}{\mathrm{d}\xi}K(t-\xi)$, to obtain

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-\tau_k}^{0}\int_{-\tau_l}^{0} \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-H}^{\infty} K^T(t-\xi)WK(t-\theta)\mathrm{d}t\mathrm{d}\theta\mathrm{d}\xi\gamma_l,$$

the change of variable $\theta = t - \xi$ on the innermost integral of the preceding equation yields

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-\tau_k}^{0}\int_{-\tau_l}^{0} \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-H-\xi}^{\infty} K^T(t)WK(t+\xi-\theta)\mathrm{d}t\mathrm{d}\theta\mathrm{d}\xi\gamma_l.$$

We use the definition of $U(\tau)$ in (2.4) to obtain

$$\begin{aligned} z(\varphi_0, \psi_0) = &\sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-\tau_k}^{0}\int_{-\tau_l}^{0} \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(U(\xi-\theta) + \int_{-H-\xi}^{-0} K^T(t)WK(t+\xi-\theta)\mathrm{d}t\right)\mathrm{d}\theta\mathrm{d}\xi\gamma_l \\ &-\sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-\tau_k}^{0}\int_{-\tau_l}^{0} \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{-0}^{\infty} K^T(t)\mathrm{d}tWK_0\mathrm{d}\theta\mathrm{d}\xi\gamma_l. \end{aligned}$$

For the first integral with respect to t in the preceding equation we have that on the interval $t \in [-H-\xi, 0)$, $K(t) = 0_{n\times n}$, and the second term of the r.h.s. is zero given that $K(t)$ is constant with respect to ξ and θ. Then, we can write

$$z(\varphi_0, \psi_0) = -\sum_{k=1}^{r}\sum_{l=1}^{r}\gamma_k^T \int_{-\tau_k}^{0}\int_{-\tau_l}^{0} U''(\xi-\theta)\mathrm{d}\theta\mathrm{d}\xi\gamma_l.$$

It can be readily verified that

$$z(\varphi_0, \psi_0) = \sum_{k=1}^{r} \sum_{l=1}^{r} \gamma_k^T \left(U(0) - U(-\tau_k) - U(\tau_l) + U(\tau_l - \tau_k) \right) \gamma_l.$$

Thus, arriving at (5.5). □

5.2 Main result

In this section, we present necessary stability conditions for system (1.16). The proof of the result is achieved by using a particular case of the generalized Cauchy formula and the bilinear functional introduced in the previous section, in conjunction with the lower bound on the functional $v_1(x_t)$ presented in section 5.1.1. We also discuss the special case of commensurate delays, in particular, the connection with stability results for discrete delays. Finally, some academic examples validate our theoretical results.

5.2.1 Necessary stability conditions for difference equations in continuous time

Let us introduce the matrix-valued function determined by the Lyapunov matrix

$$\mathscr{K}_r(\tau_1, \ldots, \tau_r) = \left\{ F(\tau_k, \tau_l) \right\}_{k,l=1}^{r}, \tag{5.7}$$

where the expression $\{A_{ij}\}_{i,j=1}^{r}$ corresponds to a block matrix, where the element lying in the i-th row and the j-th column is $A_{ij} \in \mathbb{R}^{n \times n}$ $(i, j = 1, \ldots, r)$. Now, we present a family of necessary stability conditions with increasing complexity depending on the parameter r in (5.7).

Theorem 5.2. *If system (1.16) is exponentially stable, then*

$$\mathscr{K}_r(\tau_1, \ldots, \tau_r) > 0 \tag{5.8}$$

where $\tau_i \in (0, H]$, $i = \overline{1, r}$, *and* $\tau_k \neq \tau_l$, *if* $k \neq l$.

Proof. We begin by proving that the matrix $\mathscr{K}_r$ is symmetric. Given the definition (5.7), it suffices to show that

$$F(\tau_k, \tau_l) = F^T(\tau_l, \tau_k). \tag{5.9}$$

Matrix $F(x, y)$ is defined as

$$F(x, y) = U(0) - U(y) - U(-x) + U(y - x), \tag{5.10}$$

then, for $F(\tau_l, \tau_k)$ we have

$$F(\tau_l, \tau_k) = U(0) - U(-\tau_l) - U(\tau_k) + U(\tau_k - \tau_l).$$

The symmetry property (2.8) allows us to write

$$F(\tau_l, \tau_k) = U^T(0) - U^T(\tau_l) - U^T(-\tau_k) + U^T(\tau_l - \tau_k),$$

thus, we arrive at (5.9).

From the definition of $z(\cdot, \cdot)$ in (5.4), it is straightforward to see that the functional $v_1(\varphi)$ defined in (5.1) equals $z(\varphi, \varphi)$. Then, it follows that, for

$$\tilde{\varphi}(\theta) = \sum_{k=1}^{r} \varphi_k(\theta), \quad \theta \in [-H, 0),$$

where $\varphi_k(\theta) = K(\tau_k + \theta)\gamma_k$, with $\gamma_k \in \mathbb{R}^n$, $k = \overline{1, r}$, the functional $v_1(\tilde{\varphi})$ is equal to

$$v_1(\tilde{\varphi}) = z(\tilde{\varphi}, \tilde{\varphi}) = \sum_{k=1}^{r} \sum_{l=1}^{r} z(\varphi_k, \varphi_l),$$

equivalently,

$$v_1(\tilde{\varphi}) = \sum_{k=1}^{r} \sum_{l=1}^{r} \gamma_k^T F(\tau_k, \tau_l)\gamma_l = \gamma^T \mathscr{K}_r(\tau_1, \ldots, \tau_r)\gamma,$$

where $\gamma = \left[\gamma_1^T, \gamma_2^T, \ldots, \gamma_r^T\right]^T$. As system (1.16) is assumed to be exponentially stable, $v_1(\tilde{\varphi}) \geq \beta\|\tilde{\varphi}\|_{L_2}^2$ by Theorem 5.1, it remains to show that $\|\tilde{\varphi}\|_{L_2} > 0$ if $\gamma \neq 0$.

It is easy to check that $K(t) = I_n$ for $t \in [0, h_1)$. Suppose, $0 < \tau_0 < \tau_1 < \cdots < \tau_r$. If $\gamma \neq 0$, there exists at least one $\gamma_q \neq 0$. If $\gamma_q \neq 0$, then $\tilde{\varphi}(\theta) = \varphi_q(\theta) = K(\tau_q + \theta)\gamma_q = \gamma_q \neq 0$ for $\theta \in [-\tau_q, -\tau_q + \Delta_q)$, where $\Delta_q = \min\{\tau_q - \tau_{q-1}, h_1\} > 0$. Since τ_q is arbitrary, we have that $\|\tilde{\varphi}\|_{L_2} > 0$, which proves the Theorem. □

5.2.2 Discussion of the case of commensurate delays

It is well known that in the case of commensurate delays, the difference equation (1.16) in continuous time can be viewed as a discrete system.

Single delay case

First, we consider the case of a single delay, which evidently is of commensurate type. In the result below, we prove that it is possible to recover from our conditions, the classical discrete Lyapunov equation.

$$x(t) = Ax(t - H), \quad t \geq 0, \tag{5.11}$$

with the initial condition $x(\theta) = \varphi(\theta)$, $\theta \in [-H, 0)$.

It has been shown in the literature [31] (the proof of this result is given in Appendix B) that a stability criterion for system (1.16) is the following

Theorem 5.3. *System (5.11) is asymptotically stable if and only if for a positive definite matrix Q, the solution P of the discrete Lyapunov equation*

$$A^T PA - P = -Q, \tag{5.12}$$

is such that P is positive definite.

Clearly, it is important to prove that our main result, Theorem 5.2, implies in the sense of necessity the statement of Theorem 5.3.

In other words, we must show that condition (5.8) of Theorem 5.2 implies the existence of a positive definite matrix P, satisfying the discrete Lyapunov equation (5.12).

Proposition 1. *If the system (5.11) is exponentially stable, then the condition (5.8) implies that the conditions of Theorem 5.3 are satisfied.*

Proof. Notice first that the matrix $U(\tau)$ that defines $\mathscr{K}_r$ in (5.8) of Theorem 5.2 satisfies the generalized algebraic property (2.14). In the special case $i, j = 1$ and $\tau = 0$ this is

$$A^T \Delta U'(0) A - \Delta U'(0) = W. \tag{5.13}$$

Consider now condition (5.8) of Theorem 5.2. In this case, this is

$$\mathscr{K}_1(H) = F(H, H) = 2U(0) - U(-H) - U(H) > 0.$$

This can be written as

$$\mathscr{K}_1(H) = U(0) - U(-H) - (U(H) - U(0)) > 0.$$

Given that $U(\tau)$ is a piecewise linear function, $U(0) - U(-H) = U'(-0)H$ and $U(H) - U(0) = U'(+0)H$, then we can write the preceding inequality as

$$\mathscr{K}_1(H) = H \left(U'(-0) - U'(+0)\right) > 0,$$

using the fact that $\Delta U'(0) = U'(+0) - U'(-0)$ we arrive at

$$\mathscr{K}_1(H) = -H \Delta U'(0) > 0.$$

Since H is positive, we have that, if the system is stable, $\Delta U'(0)$ is a negative definite matrix, or equivalently $-\Delta U'(0) > 0$.

Clearly, if we rewrite (5.13) as

$$A^T \left(-\Delta U'(0)\right) A - \left(-\Delta U'(0)\right) = -W,$$

then, by taking $P \equiv -\Delta U'(0)$, we have recovered the necessary conditions of Theorem 5.3 for the discrete case. □

Case with multiple commensurate delays

In the case of commensurate delays, we are able to provide a simpler form of the necessary stability conditions proved in this chapter.

Corollary 5.1. *Consider a system of the form (4.9) with multiple commensurate delays. A necessary stability condition for this system is:*

$$- \left\{ \Delta U'\left((i-j)h \right) \right\}_{i,j=1}^{m} > 0. \tag{5.14}$$

Proof. We start from the definition of $F(kh, lh)$ in (5.10)

$$\begin{aligned} F(kh, lh)) &= U(0) - U(lh) - U(-kh) + U\left((l-k)h \right) \\ &= -\left(U(lh) - U(0) \right) + \left(U\left((l-k)h \right) - U(-kh) \right). \end{aligned}$$

Because $U(\tau)$ is a piecewise continuous linear function, we have that

$$\begin{aligned} F(kh, lh)) - h &\left[\sum_{p=0}^{l-1} U'(ph+0) - \sum_{p=0}^{l-1} U'\left((p-k+1)h - 0 \right) \right] \\ &= -h \sum_{p=0}^{l-1} \left(U'(ph+0) - U'\left((p-k+1)h - 0 \right) \right); \end{aligned}$$

hence, we can write the expression $U'(ph+0) - U'\left((p-k+1)h - 0 \right)$ as

$$\sum_{q=p-k+1}^{p} \Delta U'(qh).$$

Then, we obtain that

$$F(kh, lh) = -h \sum_{p=0}^{l-1} \sum_{q=p-k+1}^{p} \Delta U'(qh).$$

With the changes of variable $i = p + 1$ and $j = i - q$ we get

$$F(kh, lh) = -h \sum_{i=1}^{l} \sum_{j=1}^{k} \Delta U' \left((i-j)h\right).$$

It is easy to show that the preceding equation implies

$$\mathscr{K}_m(h, 2h, \ldots, mh) = -hT^T \left\{\Delta U' \left((i-j)h\right)\right\}_{i,j=1}^{m} T,$$

where $T = \begin{bmatrix} I_n & I_n & \cdots & I_n \\ 0_{n\times n} & I_n & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \cdots & I_n \end{bmatrix}$ is a block matrix composed of $m \times m$ matrices of size $n \times n$.

The result follows from the fact that T is an orthogonal transformation. $\square$

5.3 Examples

We present the following examples illustrating the use of our main result in order to find the stability regions in the space of parameters for different types of systems.

Example 5. One-delay system.

Let us consider the following system, for $x(t) \in \mathbb{R}$.

$$x(t) = a_1 x(t-h) + a_2 x(t-2h). \tag{5.15}$$

It is equivalent to the following 2-dimensional system

$$\chi(t) = \underbrace{\begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}}_{A} \chi(t-h).$$

For the parameter space a_2 vs. a_1, we compare the region where the condition for asymptotic stability of equation (5.15) (i.e. $\rho(A) < 1$, equivalently, all solutions λ for $det(\lambda I - A) = 0$ are inside the unit circle) is satisfied (solid line), against the region marked by points in which the pair (a_1, a_2) satisfies $-\Delta U'(0) > 0$. The resulting plot is shown in Fig. 5.1. The construction of the delay Lyapunov matrix of this system is addressed in Section 4.1.

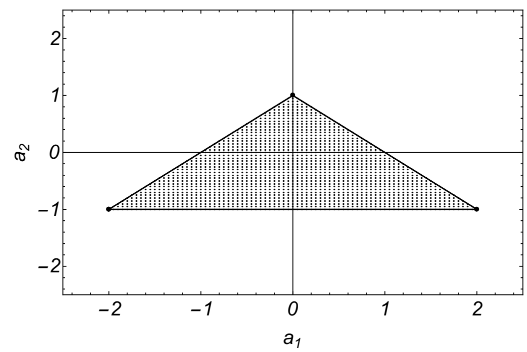


Figure 5.1: Space of parameters a_1 vs. a_2 satisfying $-\Delta U'(0) > 0$, for system (5.15).

Now, we will define a particular case of the matrix $\mathscr{K}_r$, this is

$$\mathscr{K}_r\left(\frac{H}{r}, 2\frac{H}{r}, \ldots, H\right) = \left\{F\left(k\frac{H}{r}, l\frac{H}{r}\right)\right\}_{k,l=1}^{r}, \tag{5.16}$$

from which the conditions in the forthcoming examples will follow.

Example 6. System with multiple commensurate delays.

In Fig. 5.2 (a), we present the regions in the space of parameters a_1 vs. a_2 where equation

$$x(t) = a_1 x(t-1) + a_2 x(t-3/2), \qquad t \geq 0$$

$$x(\theta) = \varphi(\theta), \qquad \theta \in [-3/2, 0),$$

is stable, that is, where

$$\max_{i\in[1,3]} |\lambda_i(A)| < 1,$$

with A given by

$$A = \begin{bmatrix} 0 & a_1 & a_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In Fig. 5.1 (b)-(d) we plot the pairs (a, b) *satisfying the following necessary conditions of stability*

- *Condition 1:*

$$\mathscr{K}_1\,(H) > 0. \tag{5.17}$$

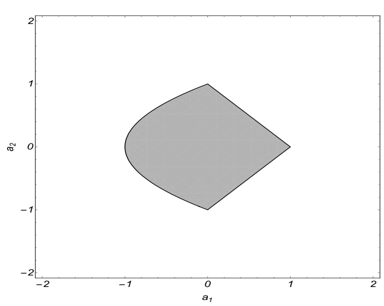
- *Condition 2:*

$$\mathscr{K}_2\left(\frac{H}{2}, H\right) > 0. \tag{5.18}$$

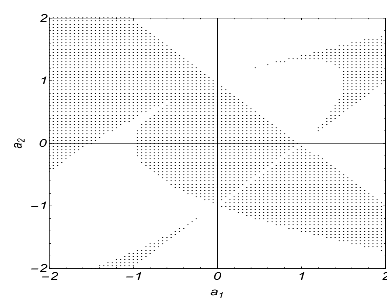
- *Condition 3:*

$$\mathscr{K}_3\left(\frac{H}{3}, 2\frac{H}{3}, H\right) > 0. \tag{5.19}$$

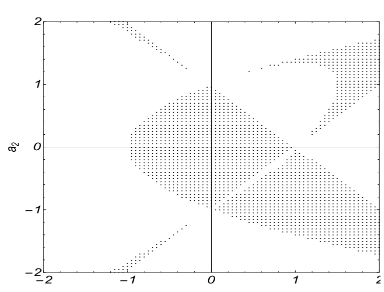
The construction of the delay Lyapunov matrix of the multidimensional case of this system is addressed in Example 4, Section 4.2.



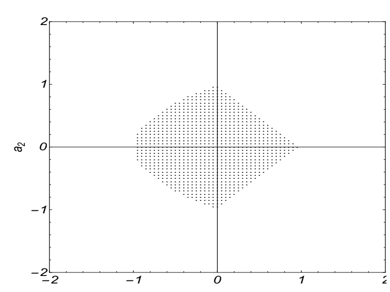
(a) Pairs (a_1, a_2) where necessary and sufficient conditions are satisfied



(b) Pairs (a_1, a_2) where (5.17) is satisfied



(c) Pairs (a_1, a_2) where (5.18) is satisfied



(d) Pairs (a_1, a_2) where (5.19) is satisfied

Figure 5.2: Stability region for equation (4.19).

This example illustrates the fact that increasing the value of r in $\mathscr{K}_r(\cdot)$ reduces the region in the space of parameters where the necessary conditions are satisfied. For any value of r bigger than 3, the resulting graph will not change with respect to Fig. 5.2 (d).

Example 7. Two-dimensional system with multiple commensurate delays.

In Fig. 5.3, we present the regions in the space of parameters a vs. b where the following equation:

$$x(t) = \underbrace{\begin{bmatrix} 0.3 & 1 \\ -1 & 0.3 \end{bmatrix}}_{A_1} x(t-1/2) + \underbrace{\begin{bmatrix} 0 & 0.5ab^2 \\ 0.2b & 0 \end{bmatrix}}_{A_2} x(t-1) + \underbrace{\begin{bmatrix} 0.2a & 0 \\ 0 & 0.5a^2b \end{bmatrix}}_{A_3} x(t-3/2) \quad t \geq 0, \tag{5.20}$$

with initial condition

$$x(\theta) = \varphi(\theta), \qquad \theta \in [-3/2, 0),$$

satisfies the following necessary conditions of stability:

- *Condition 1:*

$$-\Delta U'(0) > 0. \tag{5.21}$$

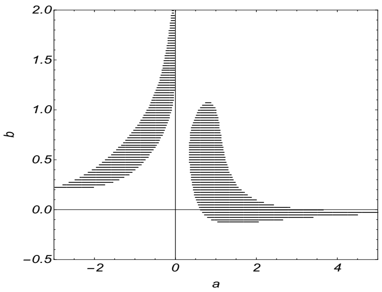
- *Condition 2:*

$$-\begin{bmatrix} \Delta U'(0) & \Delta U'(1/2) \\ \Delta U'(-1/2) & \Delta U'(0) \end{bmatrix} > 0. \tag{5.22}$$

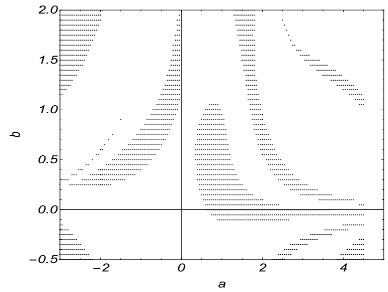
- *Condition 3:*

$$-\begin{bmatrix} \Delta U'(0) & \Delta U'(1/2) & \Delta U'(1) \\ \Delta U'(-1/2) & \Delta U'(0) & \Delta U'(1/2) \\ \Delta U'(-1) & \Delta U'(-1/2) & \Delta U'(0) \end{bmatrix} > 0. \tag{5.23}$$

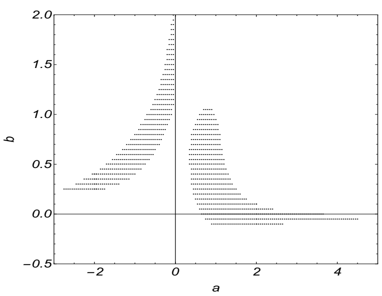
The construction of the delay Lyapunov matrix of this system is addressed in Example 4, Section 4.2.



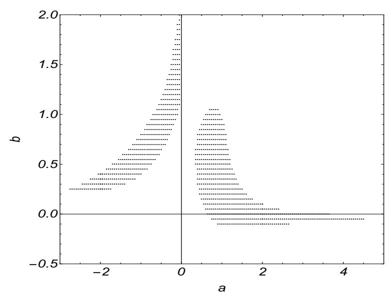
(a) Pairs (a, b) where necessary and sufficient conditions are satisfied



(b) Pairs (a, b) where (5.25) is satisfied



(c) Pairs (a, b) where (5.22) is satisfied



(d) Pairs (a, b) where (5.26) is satisfied

Figure 5.3: Stability region for equation (5.20).

For this example, the necessary and sufficient conditions are obtained via the Lyapunov equation for discrete systems (5.12), where matrix A is given by

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \\ I_2 & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & I_2 & 0_{2\times 2} \end{bmatrix}.$$

The necessary conditions were written in terms of the matrix $\Delta U'(k/2)$, $k = 0, \pm 1, \pm 2$. The reduction of the region in which the necessary conditions are satisfied as r increases is evident from the graphs.

Example 8. System with two non-commensurate delays. *We present in Fig. 5.4 the regions in the space of parameters a_1 vs. a_2 where equation*

$$
\begin{aligned}
&x(t) = a_1 x(t-1) + a_2 x(t-\sqrt{2}), \quad t \geq 0, \\
&x(\theta) = \varphi(\theta), \quad \theta \in \left[-\sqrt{2}, 0\right).
\end{aligned}
\tag{5.24}
$$

introduced in Section 4.3 is stable [3], that is $|a_1|+|a_2|<1$ *(Fig. 5.4 (a)), against the pairs* (a_1, a_2) *satisfying the following necessary conditions of stability dependent on* $\mathscr{K}_r(\cdot)$*, defined in (5.16)*

- *Condition 1:*

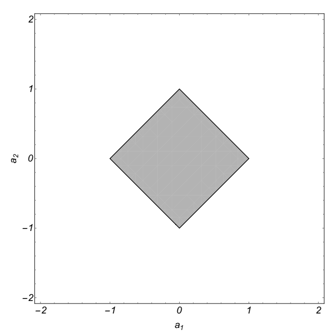
$$\mathscr{K}_1(h_2) > 0, \qquad s = 3. \tag{5.25}$$

- *Condition 2:*

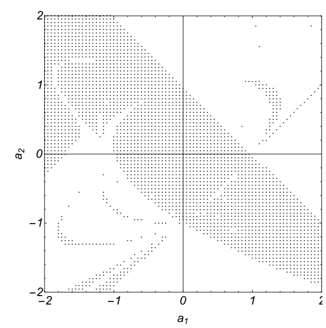
$$\mathscr{K}_s(h, 2h, \ldots, h_2) > 0, \qquad s = 3. \tag{5.26}$$

- *Condition 3:*

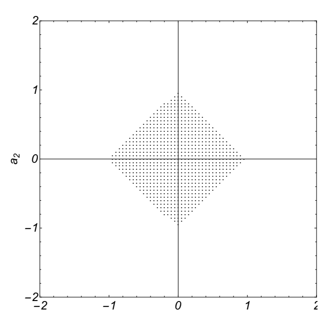
$$\mathscr{K}_s(h, 2h, \ldots, h_2) > 0, \qquad s = 6. \tag{5.27}$$



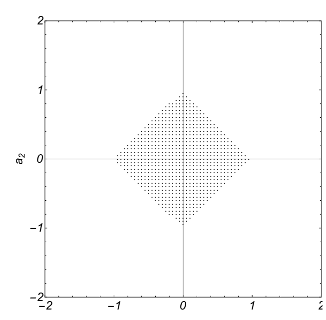
(a) Pairs (a_1, a_2) where necessary and sufficient conditions are satisfied



(b) Pairs (a_1, a_2) where (5.25) is satisfied



(c) Pairs (a_1, a_2) where (5.26) is satisfied



(d) Pairs (a_1, a_2) where (5.27) is satisfied

Figure 5.4: Stability region for equation (5.24).

We can see from this example that the region in the space of parameters for which the necessary conditions of stability are satisfied is practically the same as the necessary and sufficient conditions shown in Fig. 5.4 (a). This approximation improves as r increases and s also increases.

5.4 Comparison with sufficient conditions of stability

In this section, we compare the necessary conditions of stability for continuous-time difference equations presented in this work, against other conditions of stability in the literature. We will obtain the graphs of the regions in the space of parameters for which the conditions are satisfied and comment on the results.

The following sufficient condition for stability of system (1.16) appears in [19] [29]

$$\sum_{j=1}^{m} \|A_j\| < 1. \tag{5.28}$$

As can be expected, this condition is very conservative in the sense that in many cases where the system is stable, (5.28) does not hold, which makes it of little use in the analysis of stability.

Based on the Lyapunov-Krasovskii approach, the following result is a consequence of Theorem 2 and Lemma 7 in [23].

Consider that there exist $n \times n$ symmetric positive definite matrices P_j, $j = \overline{1,m}$ that define the symmetric matrix

$$-M_0 = A_c^T \mathrm{diag}(P_1, \dots, P_m) A_c - \mathrm{diag}(P_1, \dots, P_m), \tag{5.29}$$

where A_c is the $nm \times nm$ companion matrix,

$$A_c = \begin{bmatrix} A_1 & A_2 & \cdots & A_N \\ I_n & & & \\ & \ddots & & \\ & & I_n & 0_{n\times n} \end{bmatrix}, \tag{5.30}$$

and $\mathrm{diag}(P_1, \cdots, P_m)$ is defined by the following block diagonal matrix:

$$\mathrm{diag}(P_1, \cdots, P_m) = \begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_m \end{bmatrix}.$$

Lemma 5.3. *Assume that there exist $n \times n$ symmetric positive definite matrices P_j, $j = \overline{1, m}$, and $\mu > 0$, such that*

$$-M_\mu = -M_0 + \mathrm{diag}\left(\left(1 - \mathrm{e}^{-2\mu h_j}\right)(P_j - P_{j+1}), \ldots, \left(1 - \mathrm{e}^{-2\mu h_m}\right)(P_m - P_{m+1})\right), \tag{5.31}$$

where M_0 is defined in (5.29). Then, the system (1.16) is exponentially stable with exponential decay rate μ.

Proof. Obvious from Theorem 2 and Lemma 7 in [23]. □

Another, more complex, sufficient condition of stability was proposed in [23] Theorem 8, with the advantage that it provides a constructive exponential estimate of the response of system (1.16). A rephrasing of this theorem is written in the form of an LMI as follows:

Theorem 5.4. *[23] Assume that there exist a $n \times n$ symmetric positive definite matrix S_m, $n \times n$ symmetric positive semidefinite matrices S_i, for $i = \overline{1, m-1}$, $2n \times 2n$ symmetric positive semidefinite matrices P_j and Q_j, for $j = 2, \cdots, m$ defined as*

$$P_j = \begin{bmatrix} P_{11}^{(j)} & P_{12}^{(j)} \\ * & P_{22}^{(j)} \end{bmatrix}, \quad Q_j = \begin{bmatrix} Q_{11}^{(j)} & Q_{12}^{(j)} \\ * & Q_{22}^{(j)} \end{bmatrix},$$

(where $*$ represents symmetric terms) with respective blocks of size $n \times n$, and $\mu > 0$ such that

$$
-\Lambda_\mu = \begin{bmatrix} \Psi_\mu & \Upsilon_\mu \\ \Upsilon_\mu^T & \Phi_\mu \end{bmatrix} \tag{5.32}
$$

is a $2(m-1)n \times 2(m-1)n$ *negative semidefinite matrix, where*

$$
\begin{aligned}
\Psi_\mu &= A_c^T \mathfrak{W} A_c - \mathfrak{W}_P - \mathfrak{W}_Q \\
\Phi_\mu &= \operatorname{diag}\left(\mathrm{e}^{-2\mu\kappa_j}\left(Q_{22}^{(j)} - P_{11}^{(j)}\right)\right)
\end{aligned}
$$

$$
\Upsilon_\mu = A_c^T \begin{bmatrix} \mathrm{e}^{-2\mu\kappa_2} Q_{12}^{(2)} & \cdots & \mathrm{e}^{-2\mu\kappa_m} Q_{12}^{(m)} \\ 0_{(m-1)n\times n} & \cdots & 0_{(m-1)n\times n} \end{bmatrix} - \begin{bmatrix} 0_{n\times(m-1)n} \\ \operatorname{diag}\left(\mathrm{e}^{-2\mu\kappa_j} P_{12}^{(j)T}\right) \end{bmatrix}
$$

$$
\mathfrak{W} = \begin{bmatrix} \Xi & P_{12}^{(2)} & P_{12}^{(3)} & \cdots & P_{12}^{(m)} \\ * & P_{22}^{(2)} & & & \\ * & & P_{22}^{(3)} & & \\ \vdots & & & & P_{22}^{(m)} \end{bmatrix}, \quad \Xi = \sum_{j=1}^{m} S_j + \sum_{j=2}^{m} \left(P_{11}^{(j)} + \mathrm{e}^{-2\mu\kappa_j} Q_{11}^{(j)}\right)
$$

$$
\mathfrak{W}_P = \operatorname{diag}\left(\mathrm{e}^{-2\mu h_1} S_1, \mathrm{e}^{-2\mu\kappa_2} P_{22}^{(2)} + \mathrm{e}^{-2\mu h_2} S_2, \ldots, \mathrm{e}^{-2\mu\kappa_m} P_{22}^{(m)} + \mathrm{e}^{-2\mu h_m} S_m\right)
$$

$$
\mathfrak{W}_Q = \begin{bmatrix} \mathrm{e}^{-2\mu h_2} Q_{11}^{(2)} & \mathrm{e}^{-2\mu h_2} Q_{12}^{(2)} & & & \\ * & \mathrm{e}^{-2\mu h_2} Q_{22}^{(2)} + \mathrm{e}^{-2\mu h_3} Q_{11}^{(3)} & \mathrm{e}^{-2\mu h_3} Q_{12}^{(3)} & & \\ & \ddots & \ddots & \ddots & \\ & & & \mathrm{e}^{-2\mu h_{m-1}} Q_{22}^{(m-1)} + \mathrm{e}^{-2\mu h_m} Q_{11}^{(m)} & \mathrm{e}^{-2\mu h_m} Q_{22}^{(m)} \\ & & & * & \mathrm{e}^{-2\mu h_m} Q_{22}^{(m)} \end{bmatrix},
$$

and $\kappa_j = h_j - h_{j-1}$, *for* $j = \overline{2,m}$. *Then, the system (1.16) is exponentially stable with exponential decay rate* μ.

In the following example, we find the regions where the sufficient conditions of stability described above hold, and compare them with the regions found using the necessary conditions proposed in this work. It is important to notice that greater conservatism of the sufficient conditions implies that the conditions of stability will hold for smaller regions in the space of parameters, whereas for the necessary conditions smaller regions correspond to less conservative conditions.

Example 9. Closed-loop system

Consider the difference equation

$$x(t) = A_1 x(t-h_1) + A_2 x(t-h_2) + Bu(t),$$

with $h_1 = 1$, $h_2 = \pi$, *and*

$$A_1 = \begin{bmatrix} -0.4 & -0.3 \\ 0.1 & 0.15 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0.25 \\ -0.9 & -0.1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Making

$$u(t) = \begin{bmatrix} a_1 & 0 \end{bmatrix} x(t-h_1) + \begin{bmatrix} 0 & a_2 \end{bmatrix} x(t-h_2),$$

we have the following closed-loop system

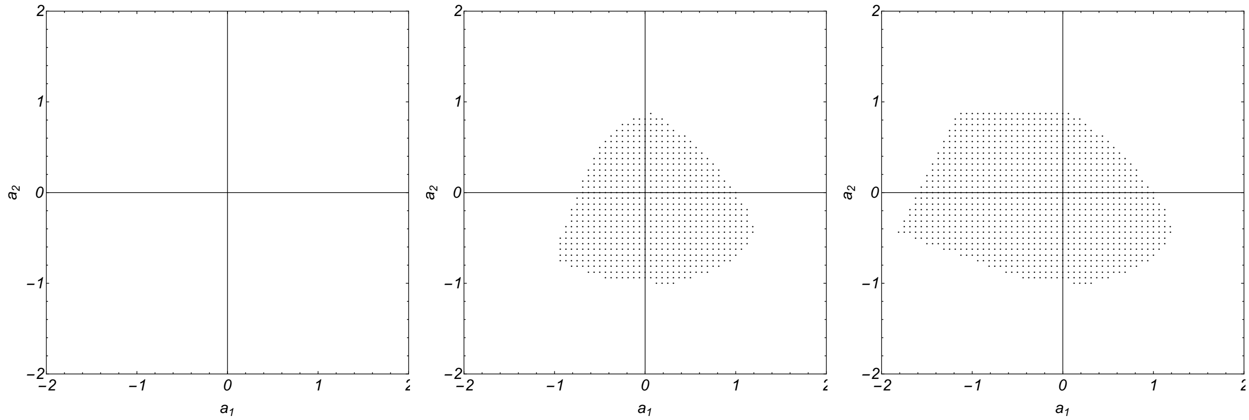
$$x(t) = \bar{A}_1 x(t-h_1) + \bar{A}_2 x(t-h_2), \tag{5.33}$$

with

$$\bar{A}_1 = \begin{bmatrix} -0.4 & -0.3 \\ 0.1 + a_1 & 0.15 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0.1 & 0.25 \\ -0.9 & -0.1 + a_2 \end{bmatrix}.$$

We wish to compare the regions in the space of parameters a_1 *vs.* a_2 *given by the sufficient conditions of stability in Lemma 5.3 and Lemma 5.4 against the region corresponding to the necessary conditions given in Theorem 5.2.*

We show in Fig. 5.5 (a) the plot of the pairs (a_1, a_2) *that satisfy equation (5.28); in Fig. 5.5 (b), the pairs corresponding to the stability condition in Lemma 5.3; and, in Fig. 5.5 (c), the region in the space of parameters for which the conditions in Lemma 5.4 hold. Finally, in Fig. 5.6 (a)-(c) we show the regions of stability that correspond to the necessary conditions given by the matrix* $\mathscr{K}_r(\cdot)$ *defined in (5.16), obtained by means of approximations of* $U(\tau)$ *parametrized by* s *and* r*. Here, the parameter* s *is the amount of numbers that define the finite continued fraction that approximates* π*, that is,* $\pi \approx [3; \underbrace{7, 15, 1, 292, 1, \ldots}_{s}]$ *[34].*

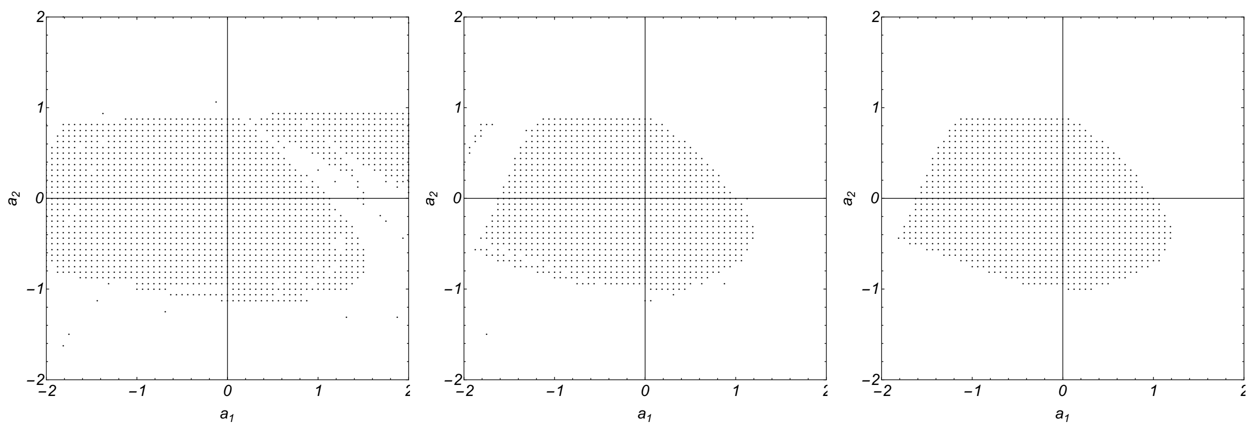


(a) Pairs (a_1, a_2) where the sufficient condition in (5.28) is satisfied

(b) Pairs (a_1, a_2) where the sufficient condition in Lemma 5.3 is satisfied

(c) Pairs (a_1, a_2) where the sufficient condition in Theorem 5.4 is satisfied

Figure 5.5: Stability regions for equation (4.19), found using sufficient conditions for exponential stability.



(a) Pairs (a_1, a_2) where the necessary condition (5.8) is satisfied, for $r = 1$, $s = 1$

(b) Pairs (a_1, a_2) where the necessary condition (5.8) is satisfied, for $r = 7$, $s = 1$.

(c) Pairs (a_1, a_2) where the necessary condition (5.8) is satisfied, for $r = 22$, $s = 1$

Figure 5.6: Stability regions for equation (4.19) found using necessary conditions for L_2-exponential stability.

The figures show that the more complex stability condition in Theorem 5.4 is less conservative than the stability conditions in 5.3 and equation (5.28), which is not useful for the analysis of the closed-loop system (5.33). In contrast, the regions of stability found using the necessary conditions in (5.8), are less conservative for higher values of r with a fixed value of s. We can have a very good idea of the region of exact stability from Fig. 5.5(c) and Fig. 5.6 (c).

5.5 σ-stability of the response of differential equations in continuous time.

Lemma 5.4. *The response of system (1.16) has an exponential decay σ if and only if the system*

$$y(t) = \sum_{j=1}^{m} \mathrm{e}^{\sigma h_j} A_j y(t-h_j), \quad t \geq 0,$$

is asymptotically stable.

Proof. We introduce the new variable

$$y(t) = \mathrm{e}^{\sigma t} x(t), \quad t \geq 0.$$

As $x(t) = \mathrm{e}^{-\sigma t} y(t)$, then

$$\begin{aligned} y(t) &= \mathrm{e}^{\sigma t} \sum_{j=1}^{m} A_j \mathrm{e}^{-\sigma (t-h_j)} y(t-h_j) \\ &= \sum_{j=1}^{m} A_j \mathrm{e}^{\sigma h_j} y(t-h_j). \end{aligned}$$

Clearly, if $y(t)$ is asymptotically stable, then $x(t)$ decreases with exponential decay σ. □

We use this result to extend Example 9 in order to find regions in the space of parameters that allow a given exponential estimate of the solution of the closed-loop system (5.33).

Example 10. *In this example we compare the regions of σ-stability of the solutions of the closed-loop system (5.33) given by the sufficient conditions in Lemma 5.3 and Theorem 5.4, against the regions obtained using the necessary conditions in Theorem 5.2.*

For a given $\sigma > 0$, we make $\mu = \sigma$ in Lemma 5.3 and find if the sufficient conditions of stability hold for system (5.33). We also set $\mu = \sigma$ in Theorem 5.4 in order to find whether or not system (5.33) parametrized by a_1 and a_2 satisfies the sufficient conditions of stability.

In order to make use of the necessary conditions in (5.8), we introduce the variable

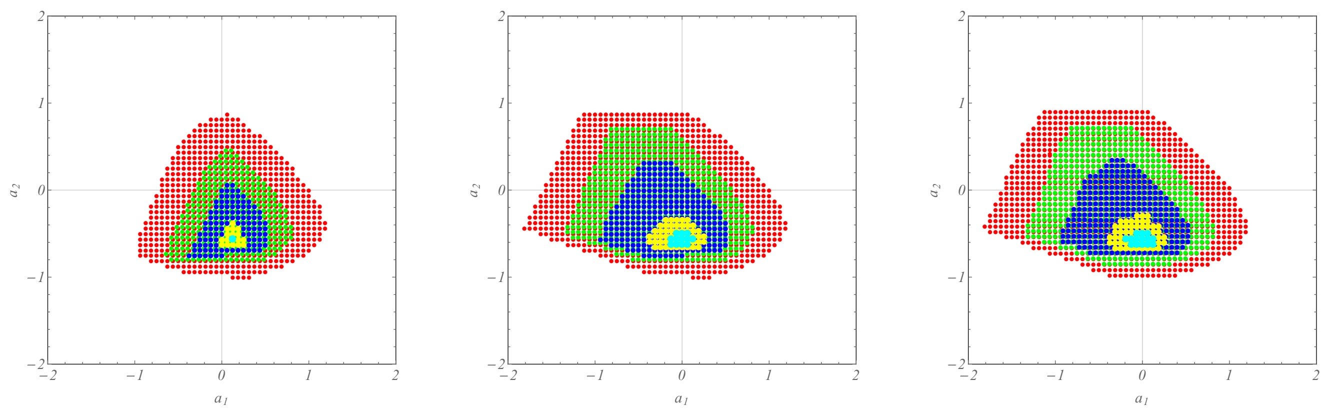
$$y(t) = e^{\sigma t} x(t), \quad t \geq 0,$$

so that the analysis is made with respect to the system

$$y(t) = \sum_{j=1}^{m} A_j e^{\sigma h_j} y(t - h_j).$$

Then, if $y(t)$ is stable, $x(t)$ is σ-stable, and the necessary conditions hold.

In Fig. 5.7(a) (resp. (b), (c)) we show the regions in the space of parameters (a_1, a_2) that satisfy the sufficient conditions for stability in Lemma 5.3 (Theorem 5.4, Theorem 5.2), for $\sigma = 0$ (●), $\sigma = 0.06$ (●), $\sigma = 0.108$ (●), $\sigma = 0.16$ (●), and $\sigma = 0.175$ (●).



(a) Pairs (a_1, a_2) where the sufficient condition in Lemma 5.3 is satisfied

(b) Pairs (a_1, a_2) where the sufficient condition in Theorem 5.4 is satisfied

(c) Pairs (a_1, a_2) where the necessary condition (5.8) is satisfied, for $r = 22$, $s = 1$

Figure 5.7: σ-stability regions for equation (5.33). For different values of σ, $\sigma = 0$ (●), $\sigma = 0.06$ (●), $\sigma = 0.108$ (●), $\sigma = 0.16$ (●), and $\sigma = 0.175$ (●).

With this example it is possible to verify that a state feedback of a system described with continuous-time different equations may be useful in order to change the decay rate of the system, so it approaches the equilibrium more rapidly. Once again, we practically have the exact conditions of stability for the example discussed, since the necessary conditions given by Theorem 5.2 and the sufficient conditions given by Theorem 5.4 yield almost equivalent regions. Notice that it is possible to stabilize an unstable system with this state-feedback approach.

Conclusion

We were able to find necessary conditions for the stability of difference equations in continuous following a strategy used for time-delay systems. These necessary conditions depend on the matrix function $U(\tau)$ introduced in Chap. 2. An alternative form to write these conditions was presented, with respect to $\Delta U'(\tau)$, for the commensurate case.

It was shown that the necessary condition for the one-delay case is also sufficient, and that it is the same as the well-known Lyapunov equation for discrete time systems.

Additionally, examples were discussed in which the necessary conditions help find the stability region in the space of parameters of given continuous-time difference equations, we emphasized the role of the parameter r in reducing the conservatism of the conditions.

Finally, a brief discussion on sufficient conditions of stability found in the literature was established so that we were able to compare our result with previous works. This comparison is made in Examples 9 and 10.

Conclusion

Necessary conditions for asymptotic L_2-stability of linear difference equations in continuous time were obtained via the use of a Lyapunov-Krasovskii functional with prescribed upper right-hand derivative. The matrix function that defines such a functional was analyzed and its properties were discussed. This properties allowed for the construction of the Lyapunov matrix $U(\tau)$, for stable as well as for unstable systems. An idea on the approximation of this matrix for the non commensurate case is presented, and the necessary conditions are written in terms independent of the commensurability of the delays.

Moreover, a complete type functional was proposed such that L_2-exponential estimates of the trajectories of the system can be obtained explicitly in the case of commensurate delays. Some examples are shown that illustrate the construction of the matrix function $U(\tau)$ and the application of the necessary conditions to find regions of stability in the space of parameters.

Future Work

The fundamentals of this work may be useful in order to address control problems concerning difference equations in continuous time. For instance, the predictor formula (1.34) can be used in problems in which there is a delay in the input of a system defined by this class of equations.

In order to find exponential estimates of the response of difference equations in continuous time with non-commensurate delays, the issue of convergence of the Lyapunov matrix $U(\tau)$, or a numerical approximation within a margin of error, needs to be elucidated. Once this problem is solved, results for parametric robustness can be obtained using a more appropriate complete type functional.

The stability conditions thus far presented in previous works and in this one, make it possible to implement control strategies for physical systems that can be modeled by continuous-time difference equations.

The discontinuous nature of the response of this type of systems makes it a problem that can be addressed in the distribution sense, an effort in order to find conditions of stability with the corresponding mathematical tools, should be made.

Appendix A

Kronecker Product

The Kronecker Product is defined for two matrices of arbitrary size over any ring. For our purposes we will only consider the field of the real numbers, and real matrices will be denoted by $\mathbb{R}^{m\times n}$, where m and n are the number of rows and columns of the matrix, respectively.

Definition A.1. *The Kronecker product of the matrix $A \in \mathbb{R}^{p\times q}$ with the matrix $B \in \mathbb{R}^{r\times s}$ is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}. \tag{A.1}$$

Other names for the Kronecker product include tensor product, direct product or left direct product. The applications of this operation range widely, in order to start exploring some of them, we introduce the notation of the vec$(\cdot)$ operator.

Definition A.2. *For any matrix $A = \{a_{ij}\} \in \mathbb{R}^{m\times n}$, the* vec$(\cdot)$ *operator is defined as*

$$\mathrm{vec}(A) = \Big[a_{11}, \cdots, a_{m1}, a_{12}, \cdots a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}\Big]^T, \tag{A.2}$$

i.e. the entries of A are stacked columnwise, forming a vector of length mn.

A.1 Properties of the Kronecker Product

The following are some of the most useful properties of the Kronecker product, they are stated and proven in the basic literature about matrix analysis (e.g. [35]).

A.1.1 Basic Properties

KR 1. *Multiplication with a scalar*

$$(\mu A) \otimes B = A \otimes (\mu B) = \mu(A \otimes B), \quad \forall \mu \in \mathbb{R}, A \in \mathbb{R}^{p \times q}, B \in \mathbb{R}^{r \times s}.$$

KR 2. *Transpose of Kronecker product*

$$(A \otimes B)^T = A^T \otimes B^T, \quad \forall A \in \mathbb{R}^{p \times q}, B \in \mathbb{R}^{r \times s}.$$

KR 3. *The Kronecker product is associative, i.e.*

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C = A \otimes B \otimes C, \quad \forall A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{r \times s}.$$

KR 4. *The Kronecker product is right-distributive, i.e.*

$$(A + B) \otimes C = A \otimes C + B \otimes C, \quad \forall A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{r \times s}.$$

KR 5. *The Kronecker product is left-distributive, i.e.*

$$A \otimes (B + C) = A \otimes B + A \otimes C, \quad \forall A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{r \times s}.$$

KR 6. *The product of two Kronecker products yields another Kronecker product:*

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \quad \forall A \in \mathbb{R}^{p \times q}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{q \times k}, D \in \mathbb{R}^{s \times l}.$$

KR 7. *The trace of the Kronecker product of two matrices is the product of the traces of the* *matrices, i.e.*

$$\begin{aligned}\operatorname{trace}(A \otimes B) &= \operatorname{trace}(B \otimes A)\\ &= \operatorname{trace}(A)\operatorname{trace}(B), \quad \forall A \in \mathbb{R}^{p\times q}, B \in \mathbb{R}^{r\times s}.\end{aligned}$$

KR 8. *The determinant of the Kronecker product satisfies:*

$$\begin{aligned}\det(A \otimes B) &= \det(B \otimes A)\\ &= (\det(A))^n(\det(B))^m, \quad \forall A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}.\end{aligned}$$

This implies that $A \otimes B$ is not singular if and only if both A and B are nonsingular.

KR 9. *If $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ are not singular then*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

This property follows directly from **KR 6.**

A.1.2 Matrix Equations and the Kronecker Product

The Kronecker product can be used to present linear equations in which the unknowns are matrices. An example of these equations is

$$AXB = C,$$

which is equivalent to the following:

$$(B^T \otimes A)\operatorname{vec}(X) = \operatorname{vec}(C)$$

Appendix B

Stability of Discrete-Time Systems

Consider the discrete-time linear system

$$x[k+1] = Ax[k], \tag{B.1}$$

where $x \in \mathbb{R}$ and $A \in \mathbb{R}^{n \times n}$. The asymptotic stability of this system depends on the eigenvalues of matrix A and it is achieved when all of them are located inside the unit circle in the complex plane.

Definition B.1. *The matrix A is called Schur stable if and only if*

$$|\lambda_i| < 1, \forall i = \overline{1, n},$$

where λ_i's are the eigenvalues of matrix A. In particular it is true that

$$\max_i |\lambda_i| < 1,$$

where $\max_i |\lambda_i|$ *is known as* $\rho(A)$*, the* spectral radius *of A.*

Theorem B.1. *[31] Considering the system (B.1), the following conditions are equivalent:*

1. *The matrix A is Schur stable*

2. *Given any matrix $Q = Q^T > 0$ there exists a positive definite matrix $P = P^T$ satisfying the discrete-time matrix Lyapunov equation*

$$A^T PA - P = -Q \tag{B.2}$$

Proof. Let us show that $1 \implies 2$. Let A be Schur stable, and let us take any positive definite matrix Q. Take the matrix $P = \sum_{j=0}^{\infty}(A^T)^j QA^j$, which is well defined by the asymptotic stability of A, and $P = P^T > 0$ by definition. Substitute P into (B.2)

$$A^T PA - P = A^T \left(\sum_{j=0}^{\infty}(A^T)^j QA^j \right) A - \sum_{j=0}^{\infty}(A^T)^j QA^j,$$

equivalently

$$A^T PA - P = \left(\sum_{j=1}^{\infty}(A^T)^j QA^j \right) - \sum_{j=0}^{\infty}(A^T)^j QA^j = -Q.$$

In order to show that $2 \implies 1$, consider the Lyapunov function $V(x) = x^T Px$, we have that

$$\begin{aligned} V(x[k+1]) - V(x[k]) &= x^T[k]A^T PAx[k] - x^T[k]Px[k] \\ &= x^T[k] \left(A^T PA - P \right) x[k] \\ &= -x^T[k]Qx[k] < 0. \end{aligned}$$

Therefore, the sequence $[V(x[k])]_{k\in\mathbb{N}}$ is strictly decreasing and bounded from below, from the stability Theorem of Lyapunov it follows that matrix A is Schur stable.

$\square$

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