Análisis de estabilidad de sistemas homogéneos en presencia de retardos

TESIS

Que presenta

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Contents

Resumen

Para el análisis de estabilidad de sistemas no lineales con retardos, se introduce el concepto de homogeneidad como una opción alternativa para aquelllos sistemas que admiten la solución trivial pero cuya estabilidad asintótica falla para un sistema aproximado de estos. Este documento se centra en el análisis de sistemas homogéneos con retardo a través del enfoque de Lyapunov-Krasovskii, basado en un funcional introducido en una reciente investigación. El análisis se divide en dos casos. El primero está dedicado a la dilatación ponderada, donde presentamos condiciones de estabilidad para esta clase de sistemas no lineales y su región de atracción, así como el análisis de robustez y la estimación de soluciones. El segundo caso, que es un caso particular del primero cuando las ponderaciones son iguales a uno, hace referencia a la dilatación estándar, donde encontramos cotas de robustez para el sistema perturbado y presentamos la estimación de las soluciones. En ambos casos, se presentan y discuten algunos ejemplos para validar los resultados encontrados.

Abstract

For stability analysis of non-linear systems, the homogeneity concept is introduced as an alternative option to systems that admits the trivial solution whose asymptotic stability fails for the linear approximation. This document focuses on analysing homogeneous time-delay systems via Lyapunov-Krasovskii framework, based on a recently introduced functional. The analysis is divided into two cases. The first one is devoted to weighted dilation, where we present stability conditions for this class of nonlinear systems and its attraction region as well as robustness analysis and the estimate of solutions. The second one, which is a particular case from the first one when weights are equal to one, is concerned to standard dilation and we find robustness bounds for the perturbed system and present the estimate of the solutions. In both cases, some examples are presented and discussed to validate the results found.

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Chapter 1

Introduction

Homogeneous systems have received sustained attention from the systems and control community. The homogeneous or weighted homogeneous concept [1] may indeed be useful in the study of the stability of the trivial solution of systems with zero linear approximation [2]. The homogeneous first approximation has the important property that the state of the system can be analysed on a sphere around the trivial solution by re-scaling it. This transformation which does not change the system behaviour, simplifies the analysis. Recently addressed topics include the stability analysis [3], [4], robustness analysis [5] as well as observers and controllers design [4].

A phenomenon that appears in many physical dynamic systems is the presence of time delays. Some well known examples are biological systems [6], communication networks, transports phenomenons, dynamic population growth [6] and mechanical systems [7], among others. The previous observations have lead to the extensions of the concept of homogeneity to time-delay systems. The main contributions in this direction include delay-independent stability conditions [8], [9], estimates of the solutions [10] and robustness analysis [11]. The homogeneity concept allowed establishing a remarkable result: if the delay-free system is asymptotically stable, then the homogeneous delay system is locally asymptotically stable for all delays. It is worth mentioning that the above mentioned result is proved for homogeneous systems with standard dilation in the Lyapunov-Razumikhin framework.

1.1   Problem Statement

The stability analysis of dynamic systems with homogeneous and delayed right hand side is usually studied in the Lyapunov-Razumikhin framework. To carry out this study, a Lyapunov function (which is often the one found for the delay free system) is proposed and differentiated along the system trajectories. Then, sufficient conditions are obtained from the Razumikhin condition. Motivated by the complete type functional for homogeneous time-delay systems with standard dilation presented in [12] in the Lyapunov-Krasovskii framework, our aim is to generalise this functional to the context of systems having weighted homogeneous right-hand side and to use it to solve some problems of interest.

1.2   Thesis Objectives

The main purpose of this work is to introduce the complete type functional for homogeneous time-delay systems with weighted dilation and to study its properties.

More precisely, our specific objectives are:

- To determine the robustness bounds of systems submitted to additive perturbations.

- To find estimates of the solutions.

- To present estimates of the domain of attraction of weighted homogeneous time-delay systems.

- To present the detailed constants involved in existing results in the Lyapunov-Razumikhin framework [10, 13].

- To compare the Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches.

1.3 Methodology

Our work combines the concepts and results from nonlinear systems analysis, in particular those for the special class of homogeneous systems with the well known tools for the time domain study of time delay systems, namely, the Lyapunov-Razumikhin approach which is based on Lyapunov functions plus the Razumikhin conditions, and the Lyapunov-Krasovskii framework, which relies on functionals.

1.4 Thesis structure

This document is organised as follows. The second chapter starts with a review of the theoretical preliminaries on homogeneous systems. It is followed by a review of recent results on the study of systems with homogeneous right-hand side with delay both in the Lyapunov-Razumikhin framework and in the Lyapunov-Krasovskii one. In Chapter 3, we introduce the Lyapunov-Krasovskii functional for time-delay systems with weighted dilation via Lyapunov-Krasovskii framework and we apply it to the problem of robustness with respect to disturbances and to the estimates of the solutions. In Chapter 4, we revisit the stability analysis in the Lyapunov-Krasovskii framework presented in [12]. We extend it to the robustness analysis and we also complete the contribution on estimates of the solutions, by fully characterising all constants involved in the estimates. Finally, Chapter 5 is devoted to concluding remarks and future research directions.

1.5 Publications

The main results of our work on the analysis of systems with homogeneous and delayed right hand side are currently submitted to international journals:

- Gerson Portilla, Irina Alexandrova, Sabine Mondié and Aleksei Zhabko. *Estimates for solutions of homogeneous time-delay systems: Comparison of Lyapunov-Krasovskii and Lyapunov-Razumikhin techniques.* System and Control Letters, Submitted.

- Gerson Portilla, Irina Alexandrova, and Sabine Mondié. *A Lyapunov-Krasovskii functional for weighted homogeneous time-delay systems.* Developing.

Chapter 2

Theoretical preliminaries

In this chapter, we present theoretical preliminaries concerning homogeneous systems and their properties. Besides, we recall two fundamentals theorems of time-delay systems analysis and some previous results for homogeneous time-delay systems via the Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches.

2.1 Homogeneous systems

To put into context the terminology that will be used by this document, we present a short review of homogeneous systems.

Definition 1. *[1]. The dilation linear operator is defined as*

$$\delta_\varepsilon^r(x) := (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n), \ \forall \varepsilon > 0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is a set of coordinates. The constants $r_i > 0$ are the weights of the coordinates.

Definition 2. *[1]. For any $r_i > 0$ for all $i \in \overline{1,n}$ and $x, y \in \mathbb{R}^n$, a function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called δ^r-homogeneous if*

$$V(\delta_\varepsilon^r(x)) = \varepsilon^k V(x) \tag{2.2}$$

holds for some $k \in \mathbb{R}$ and all $\varepsilon > 0$. The vector function $f(x, y) = (f_1(x, y), \ldots, f_n(x, y)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is called δ^r-homogeneous if every component f_i satisfies the relation

$$f_i(\delta_\varepsilon^r(x), \delta_\varepsilon^r(y)) = \varepsilon^{k+r_i} f_i(x, y) \tag{2.3}$$

for some $k \in \mathbb{R}$. In both cases, the constant k is called the degree of homogeneity.

Definition 3. *[1]. A δ^r-homogeneous norm is a map $x \mapsto \|x\|_{r,p}$ defined by*

$$\|x\|_{r,p} := \left(\sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}, \ \forall x \in \mathbb{R}^n, \ p \geq 1. \tag{2.4}$$

The set $S_{r,p} = \{x : \|x\|_{r,p} = 1\}$ is the corresponding δ^r-homogeneous unit sphere.

An important property of the homogeneous norm is that $\|\delta_\varepsilon^r(x)\|_{r,p} = \varepsilon \|x\|_{r,p}$, in other words, it is an homogeneous function of degree one. Furthermore, it is not a norm in the usual sense, since it does not satisfy the norm properties of scalability and the triangle inequality.

If all $r_i = 1$, we write δ^1. This case is called *standard dilation* [1]. Thus, equation (2.3) is reduced to

$$f_i(\delta_\varepsilon^1(x), \delta_\varepsilon^1(y)) = \varepsilon^\mu f_i(x, y), \tag{2.5}$$

where $\mu = k + 1 > 0$ and since this value is the same in every component f_i, then this is called the degree of homogeneity for the *standard dilation*. Additionally, it is straightforward to see that in this case, the homogeneous norm reduces to the euclidean norm with $p = 2$.

Lemma 1. *Every component of the vector function $f(x, y)$ in (2.3), $x, y \in \mathbb{R}^n$, admits a bound of the form*

$$|f_i(x, y)| \leq m_i \left(\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i} \right) \tag{2.6}$$

$$m_i = \max_{\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i} = 1} |f_i(x, y)| > 0.$$

Proof. Due to homogeneity, we can work in the unit sphere. With the dilation constant $\varepsilon = \frac{1}{\|x\|_{r,p}}$, we have

$$\|\delta_\varepsilon^r(x)\|_{r,p} = \varepsilon \|x\|_{r,p} = 1. \tag{2.7}$$

Now, consider the function $g(x)$, whose components admit a bound in the unit sphere of the form

$$|g_i(\delta_\varepsilon^r(x))| \leq C_i, \; C_i = \max_{\|x\|_{r,p} = 1} |g_i(x)|,$$

and this maximum value C_i exists due to continuity of $g(x)$. By homogeneity, we have

$$|\varepsilon^{k+r_i} g_i(x)| \leq C_i,$$

with homogeneity degree k. Taking ε as in (2.7), we get

$$|g_i(x)| \leq \|x\|_{r,p}^{k+r_i} C_i. \tag{2.8}$$

Consider now two vector arguments in the function $f_i(x, y)$, and take

$$\varepsilon = \frac{1}{(\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i})^{1/k+r_i}}. \tag{2.9}$$

It is readily seen that if $\|y\| = 0$ or $\|x\| = 0$, (2.9) satisfies the upper bound of a function with one vector argument as in (2.8). Then, $f_i(x, y)$ admits a bound of the form

$$|f_i(\delta_\epsilon^r(x), \delta_\epsilon^r(y))| \leq m_i,$$

where

$$m_i = \max_{\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i} = 1} |f_i(x, y)|.$$

By homogeneity, we have

$$\left| \frac{1}{\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i}} f_i(x, y) \right| \leq m_i,$$

equivalently,

$$|f_i(x, y)| \leq m_i(\|x\|_{r,p}^{k+r_i} + \|y\|_{r,p}^{k+r_i}).$$

$\square$

Since the derivative is also δ^r-homogeneous [3], of homogeneity degree $k + r_i - r_j$, there exists a constant $\eta_{ij} > 0$ such that

$$\left| \frac{\partial f_i(x, y)}{\partial x_j} \right| \le \eta_{ij} \left(\|x\|_{r,p}^{k+r_i-r_j} + \|y\|_{r,p}^{k+r_i-r_j} \right), \tag{2.10}$$

whose proof follows from Lemma 1. It follows from the *standard dilation* that (2.6) and (2.10) can be reduced to

$$\|f(x, y)\| \le m \left(\|x\|^\mu + \|y\|^\mu \right) \tag{2.11}$$

$$m = \max_{\|x\|^\mu + \|y\|^\mu = 1} \|f(x, y)\| > 0,$$

and

$$\left\| \frac{\partial f(x, y)}{\partial x} \right\| \le \eta \left(\|x\|^{\mu-1} + \|y\|^{\mu-1} \right), \ \eta > 0, \tag{2.12}$$

respectively.

Consider the delay-free system

$$\dot{x} = f(x(t), x(t)), \ x(t) \in \mathbb{R}^n, \tag{2.13}$$

where the vector function $f(x(t), x(t))$ is continuously differentiable and δ^r-homogeneous of homogeneous degree $k > 0$, for every $r_i > 0$. In the sequel, the following assumption is made.

Assumption 1. *The trivial solution of system* (2.13) *is asymptotically stable.*

Next, we remind an important theorem on systems of the form (2.13) satisfying Assumption 1.

Theorem 1. *[1]. Let system* (2.13) *be such that the origin is a locally asymptotically stable equilibrium. Assume that the vector function f is δ^r-homogeneous of degree k for some $r_i > 0$. Then, for any $z \in \mathbb{N}$ and any $\gamma > z \cdot \max_i\{r_i\}$, there exists a positive definite δ^r-homogeneous of degree γ and of class C^z Lyapunov function V, whose negative definite derivative along the solutions of* (2.13), $\dot{V} = \left(\frac{\partial V(x)}{\partial x} \right)^T f(x, x)$, *is δ^r-homogeneous of degree $k + \gamma$.*

Theorem 1 establishes that is possible to find a positive definite δ^r-homogeneous Lyapunov function $V(x)$ for (2.13). It is assumed that this function $V(x)$ has a homogeneity degree $\gamma \ge 2 \cdot \max_i\{r_i\}$, thus this function admits a lower and upper bound [1], i.e. there exists values $\alpha_0, \alpha_1 > 0$ for $i, j = \overline{1, n}$ such that

$$\alpha_0 \|x\|_{r,p}^\gamma \le V(x) \le \alpha_1 \|x\|_{r,p}^\gamma. \tag{2.14}$$

Since $V(x)$ is δ^r-homogeneous, $\frac{\partial V(x)}{\partial x_i}$ and $\frac{\partial^2 V(x)}{\partial x_i \partial x_j}$ are also δ^r-homogeneous of degree $\gamma - r_i \ge 0$ and $\gamma - r_i - r_j \ge 0$ [3], respectively. Hence, there exists values $\beta_i, \psi_{ij} > 0$ for $i, j = \overline{1, n}$ such that

$$\left| \frac{\partial V(x)}{\partial x_i} \right| \le \beta_i \|x\|_{r,p}^{\gamma - r_i}, \ \left| \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right| \le \psi_{ij} \|x\|_{r,p}^{\gamma - r_i - r_j}. \tag{2.15}$$

It also follows from Theorem 1 that the time derivative of $V(x)$ along the solutions of system (2.13) satisfies, for a constant $\text{w} > 0$ [3], the equation

$$\frac{dV(x(t))}{dt} = \left(\frac{\partial V(x)}{\partial x} \right)^T f(x, x) = -W(x) \le -\text{w} \|x\|_{r,p}^{\gamma + k}. \tag{2.16}$$

For *standard dilation*, let the right-hand side of (2.13) and $V(x)$ be δ^1-homogeneous. Assume that $\gamma \geq 2$, then the relation (2.14) holds for some $\alpha_0, \alpha_1 > 0$ and $\frac{\partial V(x)}{\partial x}$ and $\frac{\partial^2 V(x)}{\partial x^2}$ are δ^1-homogeneous of homogeneous degree $\gamma - 1 \geq 0$ and $\gamma - 2 \geq 0$, respectively. Hence there exists values $\beta, \psi > 0$ such that

$$\left\| \frac{\partial V(x)}{\partial x} \right\| \leq \beta \|x\|^{\gamma-1}, \quad \left\| \frac{\partial^2 V(x)}{\partial x^2} \right\| \leq \psi \|x\|^{\gamma-2}. \tag{2.17}$$

Moreover, the time derivative of $V(x)$ along the solutions of system (2.13) satisfies, for a constant $w > 0$, the equation

$$\frac{dV(x(t))}{dt} = \left(\frac{\partial V(x)}{\partial x} \right)^T f(x,x) = -W(x) \leq -w\|x\|^{\gamma+\mu-1}. \tag{2.18}$$

2.2 Time-delay systems

In the present document, we study a special class of nonlinear systems with delays, those that are called weighted homogeneous. More precisely, we analyse systems of the form

$$\dot{x}(t) = f(x(t), x(t-h)), \tag{2.19}$$

with initial function

$$x(\theta) = \varphi(\theta), \varphi \in PC([-h,0], \mathbb{R}^n).$$

Here, $x(t) \in \mathbb{R}^n$ and $h > 0$ is a constant delay, the vector function $f(x,y)$ is continuously differentiable and δ^r-homogeneous of homogeneity degree $k > 0$ for all $x, y \in \mathbb{R}^n$. The space of $\mathbb{R}^n$ valued piecewise continuous functions on $[-h, 0]$ is denoted $PC([-h,0], \mathbb{R}^n)$. This space is endowed with the norm

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|,$$

where $\|\varphi(\theta)\|$ stands for the euclidean norm. In computations it turns out to be more convenient to use the following homogeneous norm

$$\|\varphi\|_{\mathscr{H}} = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|_{r,p},$$

where $\|\varphi(\theta)\|_{r,p}$ stands for the typical homogeneous norm. The solution of system (2.19) and the restriction of the solution to the segment $[t-h, t]$, corresponding to the initial function φ are respectively denoted as $x(t)$ and x_t. If the initial condition is important, we write $x(t, \varphi)$ and $x_t(\varphi) := x(t+\theta, \varphi)$, respectively. Next, we remind two of the fundamental results of time-delay systems stability analysis.

Theorem 2. *(Lyapunov–Krasovskii Theorem)[14]. Suppose that the right-hand side of* (2.19) *is continuous and bounded. If there exists a continuously differentiable functional* $v(\varphi)$, *such that*

$$u_1(\|\varphi(0)\|) \leq v(\varphi) \leq u_2(\|\varphi\|_h), \tag{2.20}$$

$$\frac{dv(x_t)}{dt} \leq -w(\|x(t)\|), \tag{2.21}$$

in the neighbourhood $\|\varphi\|_h < H_1$ *and* $\|x_t\|_h < H_2$, *respectively. Here,* u_1, u_2 *and* w *are continuous non-decreasing functions, where additionally* $u_1(s)$ *and* $u_2(s)$ *for* $s > 0$ *and* $u_1(0) = u_2(0) = 0$, *furthermore, if* $w(s) > 0$ *for* $s > 0$, *then the trivial solution of* (2.19) *is asymptotically stable.*

Theorem 3. *(Lyapunov-Razumikhin Theorem)[14]. Suppose that the right-hand side of* (2.19) *is continuous and bounded. If there exists a continuous function $V(x)$, such that*

$$u_1(\|x\|) \leq V(x) \leq u_2(\|x\|), \tag{2.22}$$

in the neighbourhood $\|x\| < H$, and

$$\frac{dV(x(t))}{dt} \leq -w(\|x(t)\|) \ if \ V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))) \ \forall \theta \in [-h, 0], \tag{2.23}$$

where $u_1(s)$, $u_2(s)$ and $w(s)$ are positive for $s > 0$ and $p(s) > s$ for $s > 0$, then the trivial solution of (2.19) *is asymptotically stable.*

We will recall some important results of homogeneous time-delay systems from Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches. For the sake of completeness and for comparison purposes, we remind the detailed proofs in the appendices, including explicit formulas for the involved constants.

2.2.1 Previous results in the Lyapunov-Razumikhin framework

The vast majority of the current results dealing with homogeneous time-delay systems uses stability analysis tools based on the Lyapunov-Razumikhin approach. Thus, we recall next some of the main results.

Theorem 4. [11]. *Let $k > 0$. If system* (2.13) *is asymptotically stable, then the trivial solution of delay system* (2.19) *is also asymptotically stable for any delay $h \geq 0$.*

Theorem 5. [13]. *Let the right-hand side of* (2.13) *and* (2.19) *be δ^1-homogeneous and $\mu > 1$. If system* (2.13) *is asymptotically stable, then the trivial solution of delay system* (2.19) *is also asymptotically stable for any delay $h \geq 0$.*

The estimates of the domain of attraction and of the system response obtained in the Lyapunov-Razumikhin framework in [11, 10, 15] for *standard dilation*, where the Lyapunov function of the delay free system (2.13) is used.

Given $\alpha > 1$, introduce the Razumikhin condition

$$V(x(\xi)) < \alpha V(x(t)), \quad \xi \in [t - 2h, t], \quad t \geq h. \tag{2.24}$$

It is shown in [11] that there exists $\delta > 0$ and $k_5 = k_5(\delta) > 0$ such that the time derivative of $V(x)$ satisfies

$$\frac{dV(x(t))}{dt} \leq -k_5 \|x(t)\|^{\gamma+\mu-1} \tag{2.25}$$

along the solutions of system (2.19) which obey the Razumikhin condition (2.24) and $\|x_t\|_h \leq \delta$. Define the values

$$\kappa = \left(\frac{\alpha_0}{\alpha_1}\right)^{\frac{1}{\gamma}}, \quad K = \left(1 + (\mu-1)mh(\kappa\delta)^{\mu-1}\right)^{\frac{1}{\mu-1}}.$$

Theorem 6. [11]. *Let Δ be the root of the equation*

$$\Delta + mh\Delta^\mu = \frac{\kappa\delta}{K}. \tag{2.26}$$

If system (2.13) *is asymptotically stable, then the set of initial functions satisfying $\|\varphi\|_h < \Delta$ is contained in the region of attraction of the trivial solution of system* (2.19).

Remark 1. *It follows from the proof of Theorem 6 that if $\|\varphi\|_h < \Delta$, then $\|x(t, \varphi)\| < \delta$ for any $t \geq 0$.*

Theorem 7. [10, 15]. *If system (2.13) is asymptotically stable, then there exist $\tilde{c}_1, \tilde{c}_2 > 0$ such that the solutions of system (2.19) with $\|\varphi\|_h < \Delta$, where Δ is the root of equation (2.26), admit an estimate of the form*

$$\|x(t, \varphi)\| \leq \frac{\tilde{c}_1 \|\varphi\|_h}{(1 + \tilde{c}_2 \|\varphi\|_h^{\mu-1} t)^{\frac{1}{\mu-1}}}, \quad t \geq 0. \tag{2.27}$$

The constants k_5, δ, $\tilde{c}_1$, $\tilde{c}_2$ are specified in Appendix A.

2.2.2   Previous results in the Lyapunov-Krasovskii framework

In recent research, a functional for analysing the stability of homogeneous time-delay systems in the Lyapunov-Krasovskii framework for *standard dilation* was presented. This functional is of the form:

$$v(\varphi) = V(\varphi(0)) + \left(\frac{\partial V(x)}{\partial x}\right)^T \bigg|_{x=\varphi(0)} \times \int_{-h}^{0} f(\varphi(0), \varphi(\theta))d\theta$$

$$+ \int_{-h}^{0} (\mathrm{w}_1 + (h+\theta)\mathrm{w}_2)\|\varphi(\theta)\|^{\gamma+\mu-1}d\theta. \tag{2.28}$$

Here $\mathrm{w}_1, \mathrm{w}_2 > 0$ are such that $\mathrm{w}_0 = \mathrm{w} - \mathrm{w}_1 - h\mathrm{w}_2 > 0$. It was shown in [12] that the functional (2.28) admits a lower and an upper bound.

Lemma 2. [12]. *There exist $a_1, a_2 > 0$ such that functional (2.28) admits a lower bound of the form*

$$v(\varphi) \geq a_1 \|\varphi(0)\|^\gamma + a_2 \int_{-h}^{0} \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta \tag{2.29}$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_1)$,

$$H_1 = \left(\frac{\alpha_0}{h\beta m(1 + \chi^{-2\mu})}\right)^{\frac{1}{\mu-1}},$$
$$a_1 = \alpha_0 - h\beta m(1 + \chi^{-2\mu})\delta^{\mu-1},$$
$$a_2 = \mathrm{w}_1 - k_2 m\chi^{2(\gamma-1)}.$$

The constant $\chi > 0$ is chosen in such a way that $a_2 > 0$.

Now, we present an upper bound for $v(\varphi)$.

Lemma 3. *There exist $b_1, b_2 > 0$ such that functional (2.28) admits an upper bound of the form*

$$v(\varphi) \leq b_1 \|\varphi(0)\|^\gamma + b_2 \int_{-h}^{0} \|\varphi(\theta)\|^\gamma d\theta, \tag{2.30}$$

if $\|\varphi\|_h \leq \delta$, with

$$b_1 = \alpha_1 + 2hm\beta\delta^{\mu-1},$$
$$b_2 = (m\beta + \mathrm{w}_1 + h\mathrm{w}_2)\delta^{\mu-1}.$$

It also holds that

$$v(\varphi) \leq b \left(\|\varphi(0)\|^{\gamma} + \int_{-h}^{0} \|\varphi(\theta)\|^{\gamma} d\theta \right), \quad b = \max\{b_1, b_2\}. \tag{2.31}$$

It follows from Lemma 3 that $v(\varphi)$ also admits an upper bound of the form

$$v(\varphi) \leq \alpha_1 \|\varphi(0)\|^{\gamma} + b_3 \|\varphi\|_h^{\gamma+\mu-1}, \tag{2.32}$$

where $b_3 = (2\beta m + \mathrm{w}_1 + h\mathrm{w}_2)h$.

With the help of the previous Lemmas, we have established that the functional (2.28) satisfies bounds of the form (2.20) required in Theorem 2 in the neighbourhood H_1. In view of Theorem 2, the time derivative of functional $v(\varphi)$ along the system trajectories satisfies the derivative condition (2.21) [12].

Lemma 4. *[12]. There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional (2.28) along the solutions of system (2.19), admits a bound of the form*

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+\mu-1} - c_2 \|x(t-h)\|^{\gamma+\mu-1} - c_3 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta, \tag{2.33}$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_2)$,

$$H_2 = \left(\min \left\{ \frac{\mathrm{w}_0}{4hL}, \frac{\mathrm{w}_1}{2hL}, \frac{\mathrm{w}_2}{2L} \right\} \right)^{\frac{1}{\mu-1}},$$

where $L = m\eta\beta + m^2\psi$, $c_1 = \mathrm{w}_0 - 4hL\delta^{\mu-1}$, $c_2 = \mathrm{w}_1 - 2hL\delta^{\mu-1}$ and $c_3 = \mathrm{w}_2 - 2L\delta^{\mu-1}$.

It follows from (2.33) that

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL\delta^{\mu-1})\|x(t)\|^{\gamma+\mu-1} - (\mathrm{w}_2 - 2L\delta^{\mu-1}) \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta,$$

equivalently, taking $c_1 = \mathrm{w}_0 - 4hL\delta_2^{\mu-1}$ and $c_3 = \mathrm{w}_2 - 2L\delta_2^{\mu-1}$, in such a way that c_1 and c_3 be positive, we have

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+\mu-1} - c_3 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta.$$

Defining $c = \min\{c_1, c_3\}$ we obtain

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|^{\gamma+\mu-1} + \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta \right). \tag{2.34}$$

The following result on the estimate of the domain of attraction is established in [12].

Theorem 8. [12]. *Let Δ be a positive root of equation*

$$\alpha_1 \Delta^{\gamma} + b_3 \Delta^{\gamma+\mu-1} = a_1 \delta^{\gamma}. \tag{2.35}$$

If system (2.13) is asymptotically stable, then the set of initial functions $\|\varphi\|_h < \Delta$ is the estimate of the attraction region of the trivial solution of (2.19).

Remark 2. *It follows from the proof of Theorem 8 that if $\|\varphi\|_h < \Delta$, then $\|x(t, \varphi)\| < \delta$ for any $t \geq 0$.*

Scalar Case

Consider a scalar equation of the form

$$\dot{x}(t) = \eta_1 x^\mu(t) + \eta_2 x^\mu(t-h), \tag{2.36}$$

where the constants $\eta_1, \eta_2 \in \mathbb{R}$, $x \in \mathbb{R}$, $h \in \mathbb{R}^+$ and $\mu > 1$ is an odd entire number. Assume that $\eta_1 + \eta_2 < 0$ which implies the asymptotic stability of the trivial solution of (2.36). Take $w = -2(\eta_1 + \eta_2) > 0$, $V(x) = x^2$, and use the following modification of functional (2.28) presented in [12]:

$$v(\varphi) = \varphi^2(0) + 2\eta_2 \varphi(0) \int_{-h}^0 \varphi^\mu(\theta)d\theta \tag{2.37}$$

$$+\eta_2^2 \left(\int_{-h}^0 \varphi^\mu(\theta)d\theta \right)^2 + \int_{-h}^0 (w_1 + (h+\theta)w_2)\varphi^{\mu+1}(\theta)d\theta.$$

Here, $w_1, w_2 > 0$ are such that $w_0 = w - w_1 - hw_2 > 0$. For functional (2.37), the lower bound (2.29) takes the form

$$v(\varphi) \geq a_1 \varphi^2(0) + a_2 \int_{-h}^0 \varphi^{\mu+1}(\theta)d\theta, \tag{2.38}$$

where $|\varphi|_h \leq \delta$. Here, $\delta \in (0, H_1)$, and

$$H_1 = \left(\frac{w_1 \chi^2}{|\eta_2|} \right)^{\frac{1}{\mu-1}}, \quad 0 < \chi < \sqrt{\frac{1}{h|\eta_2|}},$$

$$a_1 = 1 - \chi^2 h|\eta_2| > 0, \quad a_2 = w_1 - |\eta_2|\chi^{-2}\delta^{\mu-1} > 0.$$

The upper bound (2.31) for functional (2.28) is of the form

$$v(\varphi) \leq b\left(\varphi^2(0) + \int_{-h}^0 \varphi^2(\theta)d\theta \right), \quad |\varphi|_h \leq \delta,$$

where $b = \max\{b_1, b_2\}$,

$$b_1 = 1 + |\eta_2|h,$$
$$b_2 = \left(|\eta_2|(1+|\eta_2|h)\delta^{\mu-1} + w_1 + hw_2 \right)\delta^{\mu-1}.$$

The functional admits also an upper bound

$$v(\varphi) \leq \varphi^2(0) + b_3|\varphi|_h^{\mu+1},$$

where $b_3 = (2|\eta_2| + \eta_2^2 h\delta^{\mu-1} + w_1 + hw_2)h$. Finally, the time derivative of functional (2.37) along the solutions of equation (2.36) satisfies

$$\frac{dv(x_t)}{dt} \leq -c\left(x^{\mu+1}(t) + \int_{-h}^0 x^{\mu+1}(t+\theta)d\theta \right)$$

in the neighbourhood $|x_t|_h \leq \delta$, where $\delta \in (0, H_2)$,

$$H_2 = \left(\min\left\{ \frac{w_0}{hL}, \frac{w_2}{L} \right\} \right)^{\frac{1}{\mu-1}}, \quad L = |\eta_2||\eta_1 + \eta_2|,$$
$$c_1 = w_0 - hL\delta^{\mu-1} > 0, \quad c_2 = w_2 - L\delta^{\mu-1} > 0,$$
$$c = \min\{c_1, c_2\}.$$

Similarly to Theorem 8, if Δ is a positive root of equation

$$\Delta^2 + b_3\Delta^{\mu+1} = a_1\delta^2, \tag{2.39}$$

then the set of initial functions $|\varphi|_h < \Delta$ is the estimate of the region of attraction of the trivial solution of (2.36).

Chapter 3

Analysis of time-delay systems with weighted dilation

As discussed in Chapter 2, there exists two approaches to analyse time-delay systems in the time domain. The first one is the Lyapunov-Razumikhin approach, which has allowed getting some important results concerning delay-independent stability conditions. This approach is based on a Lyapunov function satisfying the Razumikhin condition. Since only the delay-free system fulfils a Lyapunov function, it makes this condition strong enough. The second one is the Lyapunov-Krasovskii approach based on functionals, which allows taking into account the whole state of the time-delay system. Thus, taking advantage of this feature, this chapter is devoted to extending the analysis for weighted homogeneous time-delay systems using the Lyapunov-Krasovskii approach, especially leading on applications as perturbation analysis and estimates of the solutions.

The bounds of the functional and the region of attraction to ensure asymptotic stability for the homogeneous time-delay systems and a particular case are the topics of Section 3.1. In Section 3.2, we restrict our attention to the case when we consider the effect of an additive perturbed term. We give some conditions on the parameters for the perturbed term to preserve asymptotic stability. In Section 3.3, we present the estimate of the solutions for homogeneous time-delay systems with weighted dilation with the help of the bounds of the functional found in the previous sections. The last section deals with an illustrative example.

3.1 Weighted Homogeneous System

In this section, we extend the results on the Lyapunov-Krasovskii functional for the standard dilation introduced in [12] to the case of systems with weighted homogeneous right-hand side with delay. The expression of the functional is now:

$$v(\varphi) = V(\varphi(0)) + \left(\frac{\partial V(x)}{\partial x}\right)^{T}\bigg|_{x=\varphi(0)} \times \int_{-h}^{0} f(\varphi(0), \varphi(\theta))d\theta$$

$$+ \int_{-h}^{0} (\mathrm{w}_1 + (h+\theta)\mathrm{w}_2)\|\varphi(\theta)\|_{r,p}^{\gamma+k}d\theta. \quad (3.1)$$

Here $\mathrm{w}_1, \mathrm{w}_2 > 0$ are such that $\mathrm{w}_0 = \mathrm{w} - \mathrm{w}_1 - h\mathrm{w}_2 > 0$. As a preliminary step, we compute bounds on the functional $v(\varphi)$ and its time derivative. For the sake of clarity, the three summand of (3.1) are denoted as $I_1(\varphi)$, $I_2(\varphi)$ and $I_3(\varphi)$.

Lemma 5. *There exist $a_1, a_2 > 0$ such that functional* (3.1) *admits a lower bound of the form*

$$v(\varphi) \geq a_1 \|\varphi(0)\|_{r,p}^{\gamma} + a_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma+k} d\theta, \tag{3.2}$$

in the neighbourhood $\|\varphi\|_{\mathscr{H}} \leq \delta$. *Here,* $\delta \in (0, H_1)$,

$$H_1 = \left(\frac{\alpha_0}{\sum_{i=1}^{n} h \beta_i m_i \left(1 + \chi^{-2(k+r_i)}\right)} \right)^{\frac{1}{k}},$$

$$a_1 = \alpha_0 - \sum_{i=1}^{n} h \beta_i m_i \left(1 + \chi^{-2(k+r_i)}\right) \delta^k,$$

$$a_2 = \mathrm{w}_1 - \sum_{i=1}^{n} \beta_i m_i \chi^{2(\gamma - r_i)}.$$

The constant $\chi > 0$ *is chosen in such a way that* $a_2 > 0$.

Proof. In order to determine a lower bound, we analyse each summand of (3.1). The lower bound of $I_1(\varphi)$ is

$$I_1(\varphi) \geq \alpha_0 \|\varphi(0)\|_{r,p}^{\gamma}. \tag{3.3}$$

We now look for a lower bound of $I_2(\varphi)$. We first seek an upper bound of $|I_2(\varphi)|$

$$|I_2(\varphi)| \leq \sum_{i=1}^{n} \beta_i \|\varphi(0)\|_{r,p}^{\gamma - r_i} \times \int_{-h}^{0} m_i \left(\|\varphi(0)\|_{r,p}^{k+r_i} + \|\varphi(\theta)\|_{r,p}^{k+r_i} \right) d\theta$$

$$\leq \sum_{i=1}^{n} h \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma+k} + \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma - r_i} \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{k+r_i} d\theta.$$

Now, we introduce a parameter $\chi > 0$ in the second summand of the previous inequality

$$|I_2(\varphi)| \leq \sum_{i=1}^{n} h \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma+k} + \beta_i m_i \chi^{\gamma - k - 2r_i} \left(\frac{\|\varphi(0)\|_{r,p}}{\chi} \right)^{\gamma - r_i} \int_{-h}^{0} (\chi \|\varphi(\theta)\|_{r,p})^{k+r_i} d\theta.$$

Using inequality $|d|^z |e|^g \leq |d|^{z+g} + |e|^{z+g}$ for any $z, g > 1$ and $d, e \geq 0$, we get

$$|I_2(\varphi)| \leq \sum_{i=1}^{n} \left(h \beta_i m_i + \frac{h \beta_i m_i}{\chi^{2(k+r_i)}} \right) \|\varphi(0)\|_{r,p}^{\gamma+k} + \beta_i m_i \chi^{2(\gamma - r_i)} \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma+k} d\theta,$$

which implies that a lower bound for $I_2(\varphi)$ is

$$I_2(\varphi) \geq -|I_2(\varphi)| \geq -\left(\sum_{i=1}^{n} \left(h \beta_i m_i + \frac{h \beta_i m_i}{\chi^{2(k+r_i)}} \right) \|\varphi(0)\|_{r,p}^{\gamma+k} + \beta_i m_i \chi^{2(\gamma - r_i)} \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma+k} d\theta \right). \tag{3.4}$$

Finally, the lower bound of $I_3(\varphi)$ is given by

$$I_3(\varphi) \geq \mathrm{w}_1 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma+k} d\theta. \tag{3.5}$$

Then, in view of (3.3), (3.4) and (3.5), $v(\varphi)$ admits an estimate of the form

$$v(\varphi) \geq \left(\alpha_0 - \sum_{i=1}^{n} h \beta_i m_i \left(1 + \chi^{-2(k+r_i)}\right) \|\varphi(0)\|_{r,p}^{k} \right) \|\varphi(0)\|_{r,p}^{\gamma}$$

$$+ \left(\mathrm{w}_1 - \sum_{i=1}^{n} \beta_i m_i \chi^{2(\gamma - r_i)} \right) \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta.$$

We choose $\chi > 0$ such that $\left(\mathrm{w}_1 - \sum_{i=1}^{n} \beta_i m_i \chi^{2(\gamma - r_i)} \right) > 0$. Next, we need to define the neighbourhood, where $v(\varphi)$ has a positive definite lower bound. Clearly, this bound is positive in the neighbourhood

$$\|\varphi\|_{\mathscr{H}} \leq \delta < \left(\frac{\alpha_0}{\sum_{i=1}^{n} h \beta_i m_i \left(1 + \chi^{-2(k+r_i)} \right)} \right)^{\frac{1}{k}} = H_1.$$

It follows that

$$v(\varphi) \geq a_1 \|\varphi(0)\|_{r,p}^{\gamma} + a_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta,$$

with $a_1 = \alpha_0 - \sum_{i=1}^{n} h \beta_i m_i \left(1 + \chi^{-2(k+r_i)} \right) \delta^k$ and $a_2 = \mathrm{w}_1 - \sum_{i=1}^{n} \beta_i m_i \chi^{2(\gamma - r_i)}$. □

It is worthy of mention that $v(\varphi)$ also admits a bound of the form

$$v(\varphi) \geq a_1 \|\varphi(0)\|_{r,p}^{\gamma}. \tag{3.6}$$

Lemma 6. *There exist $b_1, b_2 > 0$ such that functional (3.1) admits an upper bound of the form*

$$v(\varphi) \leq b_1 \|\varphi(0)\|_{r,p}^{\gamma} + b_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma} d\theta, \tag{3.7}$$

if $\|\varphi\|_{\mathscr{H}} \leq \delta$, with

$$b_1 = \alpha_1 + \sum_{i=1}^{n} 2h \beta_i m_i \delta^k,$$

$$b_2 = \left(\mathrm{w}_1 + h\mathrm{w}_2 + \sum_{i=1}^{n} \beta_i m_i \right) \delta^k.$$

Proof. Now, we seek an upper bound for $v(\varphi)$. Observe first that

$$I_1(\varphi) \leq \alpha_1 \|\varphi(0)\|_{r,p}^{\gamma}. \tag{3.8}$$

It follows from (2.6) and (2.15) that

$$I_2(\varphi) \leq \sum_{i=1}^{n} \beta_i \|\varphi(0)\|_{r,p}^{\gamma - r_i} \times \int_{-h}^{0} m_i \left(\|\varphi(0)\|_{r,p}^{k+r_i} + \|\varphi(\theta)\|_{r,p}^{k+r_i} \right) d\theta$$

$$\leq \sum_{i=1}^{n} h \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma + k} + \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma - r_i} \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{k+r_i} d\theta.$$

Using inequality $|d|^z |e|^g \leq |d|^{z+g} + |e|^{z+g}$ for any $z, g > 1$ and $d, e \geq 0$, we have

$$I_2(\varphi) \leq \sum_{i=1}^{n} 2h \beta_i m_i \|\varphi(0)\|_{r,p}^{\gamma + k} + \beta_i m_i \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta. \tag{3.9}$$

Finally, we estimate the term $I_3(\varphi)$. Indeed,

$$I_3(\varphi) \leq (\mathrm{w}_1 + h\mathrm{w}_2) \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta. \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$v(\varphi) \leq \left(\alpha_1 + \sum_{i=1}^{n} 2h\beta_i m_i \|\varphi(0)\|_{r,p}^k\right) \|\varphi(0)\|_{r,p}^\gamma + \left(\mathrm{w}_1 + h\mathrm{w}_2 + \sum_{i=1}^{n} \beta_i m_i\right) \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma+k} d\theta.$$

If we take a neighbourhood $\|\varphi\|_{\mathscr{H}} \leq \delta$, we arrive at

$$v(\varphi) \leq \left(\alpha_1 + \sum_{i=1}^{n} 2h\beta_i m_i \delta^k\right) \|\varphi(0)\|_{r,p}^\gamma + \left(\mathrm{w}_1 + h\mathrm{w}_2 + \sum_{i=1}^{n} \beta_i m_i\right) \delta^k \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^\gamma d\theta,$$

or,

$$v(\varphi) \leq b_1 \|\varphi(0)\|_{r,p}^\gamma + b_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^\gamma d\theta,$$

where $b_1 = \alpha_1 + \sum_{i=1}^{n} 2h\beta_i m_i \delta^k$ and $b_2 = \left(\mathrm{w}_1 + h\mathrm{w}_2 + \sum_{i=1}^{n} \beta_i m_i\right) \delta^k$. $\qquad\square$

It also holds that

$$v(\varphi) \leq b\left(\|\varphi(0)\|_{r,p}^\gamma + \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^\gamma d\theta\right), \quad b = \max\{b_1, b_2\}. \tag{3.11}$$

It follows from the proof of Lemma 6 that $v(\varphi)$ also admits an upper bound of the form

$$v(\varphi) \leq \alpha_1 \|\varphi(0)\|^\gamma + b_3 \|\varphi\|_{\mathscr{H}}^{\gamma+k}, \tag{3.12}$$

where $b_3 = \left(\mathrm{w}_1 + h\mathrm{w}_2 + \sum_{i=1}^{n} 2h\beta_i m_i\right) h$.

We will provide an alternative proof to the stability result achieved in the Razumikhin framework (Theorem 1 in [13]). The new proof is possible thank to the above Lyapunov-Krasovskii functional. In addition to the functional bounds (3.2) found in the neighbourhood H_1, we must now find a neighbourhood where the time derivative of $v(\varphi)$ along the solutions of system (2.19) satisfies the derivative condition (2.21).

Lemma 7. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional* (3.1) *along the solutions of system* (2.19)*, admits a bound of the form*

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+k} - c_2 \|x(t-h)\|^{\gamma+k} - c_3 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+k} d\theta, \tag{3.13}$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_2)$,

$$H_2 = \left(\min\left\{\frac{\mathrm{w}_0}{4hL}, \frac{\mathrm{w}_1}{2hL}, \frac{\mathrm{w}_2}{2L}\right\}\right)^{\frac{1}{k}},$$

where $L = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_i m_j \eta_{ij} + m_i \psi_{ij} m_j)$, $c_1 = \mathrm{w}_0 - 4hL\delta^k$, $c_2 = \mathrm{w}_1 - 2hL\delta^k$ and $c_3 = \mathrm{w}_2 - 2L\delta^k$.

Proof. We differentiate each of three summands of (3.1). Observe that

$$\left.\frac{dI_1(x_t)}{dt}\right|_{(2.19)} = \frac{d}{dt} V(x(t)) = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T f(x(t), x(t-h)).$$

Next, we differentiate the term $I_2(\varphi)$

$$\left.\frac{dI_2(x_t)}{dt}\right|_{(2.19)} = \frac{d}{dt}\left(\left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{-h}^{0} f(x(t), x(t+\theta)) d\theta\right)$$

$$= \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T (f(x(t), x(t)) - f(x(t), x(t-h))) + \sum_{j=1}^{2} \Lambda_j,$$

where

$$\Lambda_1 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^{t} \frac{\partial f(x(t), x(s))}{\partial x(t)} ds \times f(x(t), x(t-h))$$

$$\Lambda_2 = (f(x(t), x(t-h)))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2}\right) \int_{-h}^{0} f(x(t), x(t+\theta)) d\theta.$$

Finally, we address the term $I_3(\varphi)$,

$$\left.\frac{dI_3(x_t)}{dt}\right|_{(2.19)} = \frac{d}{dt} \left(\int_{-h}^{0} (\mathrm{w}_1 + (h+\theta)\mathrm{w}_2) \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \right)$$

$$= (\mathrm{w}_1 + h\mathrm{w}_2) \|x(t)\|_{r,p}^{\gamma+k} - \mathrm{w}_1 \|x(t-h)\|_{r,p}^{\gamma+k} - \mathrm{w}_2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta.$$

Adding the three summands of the derivative, we obtain

$$\left.\frac{dv(x_t)}{dt}\right|_{(2.19)} = -\mathrm{w}_0 \|x(t)\|_{r,p}^{\gamma+k} - \mathrm{w}_1 \|x(t-h)\|_{r,p}^{\gamma+k} - \mathrm{w}_2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta + \sum_{j=1}^{2} \Lambda_j.$$

Using (2.6), (2.10), (2.14), (2.15) and (2.16), we get the following upper bounds

$$\Lambda_1 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i m_j \|x(t)\|_{r,p}^{\gamma-r_i} (\|x(t)\|_{r,p}^{k+r_j} + \|x(t-h)\|_{r,p}^{k+r_j})$$

$$\times \int_{-h}^{0} \eta_{ij} (\|x(t)\|_{r,p}^{k+r_i-r_j} + \|x(t+\theta)\|_{r,p}^{k+r_i-r_j}) d\theta$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i m_j \eta_{ij} (4h \|x(t)\|_{r,p}^{\gamma+2k} + 2h \|x(t-h)\|_{r,p}^{\gamma+2k} + 2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+2k}),$$

$$\Lambda_2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} m_i \psi_{ij} (\|x(t)\|_{r,p}^{k+r_i} + \|x(t-h)\|_{r,p}^{k+r_i})$$

$$\times \|x(t)\|_{r,p}^{\gamma-r_i-r_j} \int_{-h}^{0} m_j (\|x(t)\|_{r,p}^{k+r_j} + \|x(t+\theta)\|_{r,p}^{k+r_j}) d\theta$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} m_i \psi_{ij} m_j (4h \|x(t)\|_{r,p}^{\gamma+2k} + 2h \|x(t-h)\|_{r,p}^{\gamma+2k} + 2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+2k} d\theta).$$

In this way, we obtain the inequality

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL \|x(t)\|_{r,p}^{k}) \|x(t)\|_{r,p}^{\gamma+k} - (\mathrm{w}_1 - 2hL \|x(t-h)\|_{r,p}^{k}) \|x(t-h)\|_{r,p}^{\gamma+k}$$

$$- (\mathrm{w}_2 - 2L \|x_t\|_{\mathscr{H}}^{k}) \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta, \quad (3.14)$$

where $L = \sum_{i=1}^{n} \sum_{j=1}^{n} (\beta_i m_j \eta_{ij} + m_i \psi_{ij} m_j)$. This bound is positive in the neighbourhood

$$\|x_t\|_{\mathscr{H}} \leq \delta < \left(\min \left\{ \frac{\mathrm{w}_0}{4hL}, \frac{\mathrm{w}_1}{2hL}, \frac{\mathrm{w}_2}{2L} \right\} \right)^{\frac{1}{k}} = H_2.$$

□

It follows from (3.14) that

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL\delta^k)\|x(t)\|_{r,p}^{\gamma+k} - (\mathrm{w}_2 - 2L\delta^k) \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta,$$

equivalently, taking $c_1 = \mathrm{w}_0 - 4hL\delta^k$ and $c_2 = \mathrm{w}_2 - 2L\delta^k$, in such a way that c_1 and c_2 positive, we have

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|_{r,p}^{\gamma+k} - c_2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta.$$

Defining $c = \min\{c_1, c_2\}$ we obtain

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|_{r,p}^{\gamma+k} + \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \right). \tag{3.15}$$

Summarizing, we have proved that the functional $v(\varphi)$ admits the lower and upper bounds (3.6), (3.11), and that its time derivative along the solutions of system (2.19) is such that (3.15) holds. Functional (3.1) meets the conditions (2.20) and (2.21) of Theorem 2 and we conclude, as previously done in the Razumikhin framework (Theorem 1 in [13]), that the trivial solution of the homogeneous system with delay (2.19) is asymptotically stable for all delays $h > 0$ and $k > 0$. This alternative proof via Lyapunov-Krasovskii techniques allows us to present straightforwardly estimates of the domain of attraction of the trivial solution of system (2.19) on the basis of bounds (3.6) and (3.12) in the neighbourhood δ, which is any value less than $\min\{H_1, H_2\}$.

Theorem 9. *Let Δ be a positive root of equation*

$$\alpha_1 \Delta^\gamma + b_3 \Delta^{\gamma+k} = a_1 \delta^\gamma.$$

If system (2.13) is asymptotically stable, then the set of initial functions

$$\Omega = \{\varphi \in PC([-h, 0], \mathbb{R}^n) : \|\varphi\|_{\mathscr{H}} < \Delta\}, \tag{3.16}$$

is the estimate of the attraction region of the trivial solution of (2.19).

Proof. We need to prove that Ω is an invariant set such that every trajectory starting in Ω stays in Ω. Hence, we choose

$$v_0 = \inf_{\|\varphi(0)\|_{r,p}=\delta} v(\varphi) \geq a_1 \delta^\gamma, \tag{3.17}$$

where v_0 is an estimate of an inner Lyapunov surface. Using bounds (3.12) and (3.17), let $\|\varphi\|_{\mathscr{H}} < \Delta$, where Δ is a positive real root of the following equation

$$\alpha_1 \Delta^\gamma + b_3 \Delta^{\gamma+k} = a_1 \delta^\gamma.$$

Then,

$$v(\varphi) \leq \alpha_1 \|\varphi(0)\|_{r,p}^\gamma + b_3 \|\varphi\|_{\mathscr{H}}^{\gamma+k} < a_1 \delta^\gamma \leq v_0.$$

Notice that $\Delta < \delta$ and prove that $\|x_t(\varphi)\|_{\mathscr{H}} < \delta$ holds $\forall t \geq 0$. Assume that this is false, then there exits a time $\bar{t}$ such that $\|x(\bar{t}, \varphi)\|_{r,p} = \delta$, hence $v(x_{\bar{t}}(\varphi)) \geq v_0$. For $t \in [0, \bar{t})$ we have $\|x_t(\varphi)\|_{\mathscr{H}} < \delta$ and due to (3.15), then $\frac{dv(x(t))}{dt} < 0$, hence $v(x_t(\varphi))$ is decreasing, and consequently $v(x_{\bar{t}}(\varphi)) \leq v(\varphi) < v_0$. Since we consider the continuity of the solution, it is a contradiction in time $\bar{t}$. Therefore, Ω is the attraction region for (2.19) and $\|x_t(\varphi)\|_{\mathscr{H}} < \delta$ for $t \geq 0$. As a consequence, if $v(\varphi)$ is decreasing in this region, then the solution of the system is asymptotically stable. $\square$

Remark 3. *It follows from the proof of Theorem 9 that if* $\|\varphi\|_{\mathscr{H}} < \Delta$, *then* $\|x(t, \varphi)\| < \delta$ *for any* $t \geq 0$.

Now, consider a particular case of system (2.19) of the form

$$\dot{x}(t) = F(x(t)) + G(x(t-h)), \qquad (3.18)$$

where $F(x(t))$ and $G(x(t-h))$ are continuous, δ^r-homogeneous of homogeneous degree $k > 0$ and additionally, admit bounds of the form

$$|F_i(x(t))| \leq f_i \|x(t)\|_{r,p}^{k+r_i}, \quad |G_i(x(t-h))| \leq g_i \|x(t-h)\|_{r,p}^{k+r_i}. \qquad (3.19)$$

In this case, functional (3.1) can be reduced to:

$$v(\varphi) = V(\varphi(0)) + \left(\frac{\partial V(x)}{\partial x}\right)^T \Bigg|_{x=\varphi(0)} \times \int_{-h}^0 G(\varphi(\theta))d\theta$$

$$+ \int_{-h}^0 (\mathrm{w}_1 + (h+\theta)\mathrm{w}_2)\|\varphi(\theta)\|_{r,p}^{\gamma+k}d\theta. \quad (3.20)$$

This functional admits a lower and an upper bound. The fact that the second summand, $G(x(t-h))$ possess only one argument helps in reducing conservatism of the estimates.

Lemma 8. *There exist* $a_1, a_2 > 0$ *such that functional* (3.20) *admits a lower bound of the form*

$$v(\varphi) \geq a_1 \|\varphi(0)\|_{r,p}^{\gamma} + a_2 \int_{-h}^0 \|\varphi(\theta)\|_{r,p}^{\gamma+k}d\theta, \qquad (3.21)$$

in the neighbourhood $\|\varphi\|_{\mathscr{H}} \leq \delta$. *Here,* $\delta \in (0, H_1)$,

$$H_1 = \left(\frac{\alpha_0}{\sum_{i=1}^n h\beta_i g_i \chi^{-2(k+r_i)}}\right)^{\frac{1}{k}},$$

$$a_1 = \alpha_0 - \sum_{i=1}^n h\beta_i g_i \chi^{-2(k+r_i)}\delta^k,$$

$$a_2 = \mathrm{w}_1 - \sum_{i=1}^n \beta_i g_i \chi^{2(\gamma-r_i)}.$$

The constant $\chi > 0$ *is chosen in such a way that* $a_2 > 0$.

Proof. Using (2.14), (2.15), (3.1) and adding a parameter $\chi > 0$ in the second summand as in Lemma 5, functional (3.20) admits an estimate of the form

$$v(\varphi) \geq \left(\alpha_0 - \sum_{i=1}^n h\beta_i g_i \chi^{-2(k+r_i)}\|\varphi(0)\|_{r,p}^k\right)\|\varphi(0)\|_{r,p}^{\gamma}$$

$$+ \left(\mathrm{w}_1 - \sum_{i=1}^{n} \beta_i g_i \chi^{2(\gamma - r_i)} \right) \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta.$$

We choose $\chi > 0$ such that $\left(\mathrm{w}_1 - \sum_{i=1}^{n} \beta_i g_i \chi^{2(\gamma - r_i)} \right) > 0$. Next, we need to define the neighbourhood, where $v(\varphi)$ has a positive definite lower bound. Clearly, this bound is positive in the neighbourhood

$$\|\varphi\|_{\mathscr{H}} \leq \delta < \left(\frac{\alpha_0}{\sum_{i=1}^{n} h \beta_i g_i \chi^{-2(k + r_i)}} \right)^{\frac{1}{k}} = H_1.$$

It follows that

$$v(\varphi) \geq a_1 \|\varphi(0)\|_{r,p}^{\gamma} + a_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta,$$

with $a_1 = \alpha_0 - \sum_{i=1}^{n} h \beta_i g_i \chi^{-2(k + r_i)} \delta^k$ and $a_2 = \mathrm{w}_1 - \sum_{i=1}^{n} \beta_i g_i \chi^{2(\gamma - r_i)}$. $\qquad \square$

Lemma 9. *There exist $b_1, b_2 > 0$ such that functional (3.1) admits an upper bound of the form*

$$v(\varphi) \leq b_1 \|\varphi(0)\|_{r,p}^{\gamma} + b_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma} d\theta, \tag{3.22}$$

if $\|\varphi\|_{\mathscr{H}} \leq \delta$, with

$$b_1 = \alpha_1 + \sum_{i=1}^{n} h \beta_i g_i \delta^k,$$

$$b_2 = \left(\mathrm{w}_1 + h \mathrm{w}_2 + \sum_{i=1}^{n} \beta_i g_i \right) \delta^k.$$

Proof. It follows from (2.14), (2.15) and (3.1) that

$$v(\varphi) \leq \left(\alpha_1 + \sum_{i=1}^{n} h \beta_i g_i \|\varphi(0)\|_{r,p}^{k} \right) \|\varphi(0)\|_{r,p}^{\gamma} + \left(\mathrm{w}_1 + h \mathrm{w}_2 + \sum_{i=1}^{n} \beta_i g_i \right) \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma + k} d\theta.$$

Taking a neighbourhood $\|\varphi\|_{\mathscr{H}} \leq \delta$, we have

$$v(\varphi) \leq \left(\alpha_1 + \sum_{i=1}^{n} h \beta_i g_i \delta^k \right) \|\varphi(0)\|_{r,p}^{\gamma} + \left(\mathrm{w}_1 + h \mathrm{w}_2 + \sum_{i=1}^{n} \beta_i g_i \right) \delta^k \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma} d\theta,$$

or,

$$v(\varphi) \leq b_1 \|\varphi(0)\|_{r,p}^{\gamma} + b_2 \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma} d\theta,$$

where $b_1 = \alpha_1 + \sum_{i=1}^{n} h \beta_i g_i \delta^k$ and $b_2 = \left(\mathrm{w}_1 + h \mathrm{w}_2 + \sum_{i=1}^{n} \beta_i g_i \right) \delta^k$. $\qquad \square$

It also holds that

$$v(\varphi) \leq b \left(\|\varphi(0)\|_{r,p}^{\gamma} + \int_{-h}^{0} \|\varphi(\theta)\|_{r,p}^{\gamma} d\theta \right), \quad b = \max\{b_1, b_2\}. \tag{3.23}$$

Furthermore, it follows from (3.7) that $v(\varphi)$ also admits an upper bound of the form

$$v(\varphi) \leq \alpha_1 \|\varphi(0)\|^{\gamma} + b_3 \|\varphi\|_{\mathscr{H}}^{\gamma + k}, \tag{3.24}$$

where $b_3 = \left(\mathrm{w}_1 + h \mathrm{w}_2 + \sum_{i=1}^{n} h \beta_i g_i \right) h$. For the sake of satisfying Theorem 2, we differentiate functional (3.20) along the solutions of system (3.18), thus we get the following result.

Lemma 10. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional (3.20) along the solutions of system (3.18), admits a bound of the form*

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+k} - c_2 \|x(t-h)\|^{\gamma+k} - c_3 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+k} d\theta, \qquad (3.25)$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_2)$,

$$H_2 = \left(\min \left\{ \frac{w_0}{hL_1}, \frac{w_1}{hL_2}, \frac{w_2}{L_1} \right\} \right)^{\frac{1}{k}},$$

where $L_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} (f_i + g_i)$, $L_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} g_i$, $c_1 = w_0 - hL_1 \delta^k$, $c_2 = w_1 - hL_2 \delta^k$ and $c_3 = w_2 - L_1 \delta^k$.

Proof. Differentiating (3.20), we obtain

$$\frac{dv(x_t)}{dt}\bigg|_{(3.18)} = -w_0 \|x(t)\|_{r,p}^{\gamma+k} - w_1 \|x(t-h)\|_{r,p}^{\gamma+k} - w_2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta + \Lambda.$$

where

$$\Lambda = (F(x(t)) + G(x(t-h)))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2} \right) \int_{-h}^{0} G(x(t+\theta)) d\theta.$$

Using (2.14), (2.15), (2.16) and (3.1), we get the following upper bound

$$\Lambda \leq \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} (f_i \|x(t)\|_{r,p}^{k+r_i} + g_i \|x(t-h)\|_{r,p}^{k+r_i}) \|x(t)\|_{r,p}^{\gamma-r_i-r_j} \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{k+r_j} d\theta$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} (h(f_i + g_i) \|x(t)\|_{r,p}^{\gamma+2k} + hg_i \|x(t-h)\|_{r,p}^{\gamma+2k} + (f_i + g_i) \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+2k}).$$

In this way, we get the inequality

$$\frac{dv(x_t)}{dt} \leq -(w_0 - hL_1 \|x(t)\|_{r,p}^{k}) \|x(t)\|_{r,p}^{\gamma+k} - (w_1 - hL_2 \|x(t-h)\|_{r,p}^{k}) \|x(t-h)\|_{r,p}^{\gamma+k}$$

$$- (w_2 - L_1 \|x_t\|_{\mathscr{H}}^{k}) \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta, \quad (3.26)$$

where $L_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} (f_i + g_i)$ and $L_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_j \psi_{ij} g_i$. This bound is positive in the neighbourhood

$$\|x_t\|_{\mathscr{H}} \leq \delta < \left(\min \left\{ \frac{w_0}{hL_1}, \frac{w_1}{hL_2}, \frac{w_2}{L_1} \right\} \right)^{\frac{1}{k}} = H_2,$$

and the lemma is proved. $\square$

It follows from (3.25) and defining $c = \min\{c_1, c_3\}$ that

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|_{r,p}^{\gamma+k} + \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \right). \qquad (3.27)$$

The estimate of the attraction region of the trivial solution of system (3.18) uses a procedure similar to the one used to prove Theorem 9. Let Δ be a positive root of equation

$$\alpha_1 \Delta^\gamma + b_3 \Delta^{\gamma+k} = a_1 \delta^\gamma,$$

then, the set of initial functions such that $\|\varphi\|_{\mathscr{H}} \leq \Delta$ is the estimate of the attraction region of system (3.18). Here, a_1 comes from (3.21) and b_3 comes from (3.24).

3.2 Perturbed Systems

Consider the perturbed system

$$\dot{x}(t) = f(x(t), x(t-h)) + R(x(t), x(t-h)), \qquad (3.28)$$

where the vector function $f(x(t), x(t-h))$ is continuously differentiable and δ^r-homogeneous of homogeneous degree $k > 0$ and $R(x(t), x(t-h))$ is continuous. System (3.28) is viewed as a perturbation of the nominal system (2.19), where the bounded term $R(x(t), x(t-h))$ describes model errors, nonlinear approximations, uncertainties or disturbances. It is assumed that

$$|R_i(x(t), x(t-h))| \leq p_i \|x(t)\|_{r,p}^{\sigma+r_i} + q_i \|x(t-h)\|_{r,p}^{\sigma+r_i}, \qquad (3.29)$$

$$\sigma > k, \quad p_i, q_i > 0.$$

For the analysis of system (3.28), we use functional (3.1) which satisfies the lower and upper bounds of Lemma 5 and Lemma 6, respectively. We focus on finding the neighbourhood where its time derivative is negative definite, and show that this functional is suitable for the stability analysis of time-delay systems under perturbations satisfying Theorem 2.

Lemma 11. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional (2.28) along the solutions of system (3.28), admits a bound of the form*

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|_{r,p}^{\gamma+k} - c_2 \|x(t-h)\|_{r,p}^{\gamma+k} - c_3 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta, \qquad (3.30)$$

in the neighbourhood $\|x_t\|_{\mathscr{H}} \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$\begin{aligned}
\mathrm{w}_0 - 4hLH_2^k - 2hL_1H_2^\sigma - L_2H_2^{\sigma-k} &= 0, \\
\mathrm{w}_1 - 2hLH_3^k - 2hL_3H_3^\sigma - L_4H_3^{\sigma-k} &= 0, \\
\mathrm{w}_2 - 2LH_4^k - L_1H_4^\sigma &= 0.
\end{aligned} \qquad (3.31)$$

Here, $L = \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i m_j \eta_{ij} + m_i \psi_{ij} m_j)$, $L_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i \eta_{ij}(p_j + q_j) + m_j \psi_{ij}(p_i + q_i))$, $L_2 = \sum_{i=1}^{n}\beta_i(p_i + q_i)$, $L_3 = \sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i \eta_{ij} q_j + m_j \psi_{ij} q_i)$, $L_4 = \sum_{i=1}^{n}\beta_i q_i$, $c_1 = \mathrm{w}_0 - 4hL\delta^k - 2hL_1\delta^\sigma - L_2\delta^{\sigma-k}$, $c_2 = \mathrm{w}_1 - 2hL\delta^k - 2hL_3\delta^\sigma - L_4\delta^{\sigma-k}$ and $c_3 = \mathrm{w}_2 - 2L\delta^k - L_1\delta^\sigma$.

Proof. As in Lemma 10, we find estimates for each of the three summands of (2.28). We obtain

$$\begin{aligned}
\frac{dv(x_t)}{dt}\bigg|_{(3.28)} &= -\mathrm{w}_0 \|x(t)\|_{r,p}^{\gamma+k} - \mathrm{w}_1 \|x(t-h)\|_{r,p}^{\gamma+k} - \mathrm{w}_2 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \\
&\quad + \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T R(x(t), x(t-h)) + \sum_{j=1}^{2} \Lambda_j.
\end{aligned}$$

where

$$\Lambda_1 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^{t} \frac{\partial f(x(t), x(s))}{\partial x(t)} ds \times (f(x(t), x(t-h)) + R(x(t), x(t-h)))$$

$$\Lambda_2 = (f(x(t), x(t-h)) + R(x(t), x(t-h)))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2}\right) \int_{-h}^{0} f(x(t), x(t+\theta)) d\theta.$$

Using (2.6), (2.10), (2.14), (2.15), (2.16) and (3.29), we get the following upper bounds

$$\Lambda_1 \leq \sum_{i=1}^n \sum_{j=1}^n \beta_i m_j \eta_{ij} \left(4h\|x(t)\|_{r,p}^{\gamma+2k} + 2h\|x(t-h)\|_{r,p}^{\gamma+2k} + 2\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+2k}d\theta \right)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n \beta_i \eta_{ij} \|x(t)\|_{r,p}^{\gamma-r_i} (p_j\|x(t)\|_{r,p}^{\sigma+r_j} + q_j\|x(t-h)\|_{r,p}^{\sigma+r_j}) \int_{-h}^0 (\|x(t)\|_{r,p}^{k+r_i-r_j} + \|x(t+\theta)\|_{r,p}^{k+r_i-r_j})d\theta$$

$$\Lambda_1 \leq \sum_{i=1}^n \sum_{j=1}^n \beta_i m_j \eta_{ij} \left(4h\|x(t)\|_{r,p}^{\gamma+2k} + 2h\|x(t-h)\|_{r,p}^{\gamma+2k} + 2\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+2k}d\theta \right)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n \beta_i \eta_{ij} \left(2h(p_j+q_j)\|x(t)\|_{r,p}^{\gamma+k+\sigma} + 2q_jh\|x(t-h)\|_{r,p}^{\gamma+k+\sigma} + (p_j+q_j)\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+k+\sigma}d\theta \right)$$

$$\Lambda_2 \leq \sum_{i=1}^n \sum_{j=1}^n m_i \psi_{ij} m_j \left(4h\|x(t)\|_{r,p}^{\gamma+2k} + 2h\|x(t-h)\|_{r,p}^{\gamma+2k} + 2\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+2k}d\theta \right)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n m_j \psi_{ij} \|x(t)\|_{r,p}^{\gamma-r_i-r_j} (p_i\|x(t)\|_{r,p}^{\sigma+r_i} + q_i\|x(t-h)\|_{r,p}^{\sigma+r_i}) \int_{-h}^0 (\|x(t)\|_{r,p}^{k+r_j} + \|x(t+\theta)\|_{r,p}^{k+r_j})d\theta$$

$$\Lambda_2 \leq \sum_{i=1}^n \sum_{j=1}^n m_i \psi_{ij} m_j \left(4h\|x(t)\|_{r,p}^{\gamma+2k} + 2h\|x(t-h)\|_{r,p}^{\gamma+2k} + 2\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+2k}d\theta \right)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n m_j \psi_{ij} \left(2h(p_i+q_i)\|x(t)\|_{r,p}^{\gamma+k+\sigma} + 2q_ih\|x(t-h)\|_{r,p}^{\gamma+k+\sigma} + (p_i+q_i)\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+k+\sigma}d\theta \right)$$

$$\left(\frac{\partial V(x(t))}{\partial x(t)} \right)^T (R(x(t), x(t-h))) \leq \sum_{i=1}^n \beta_i(p_i+q_i)\|x(t)\|_{r,p}^{\sigma+\gamma} + \beta_i q_i\|x(t-h)\|_{r,p}^{\sigma+\gamma}.$$

Then, we obtain the inequality

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL\|x(t)\|_{r,p}^k - 2hL_1\|x(t)\|_{r,p}^\sigma - L_2\|x(t)\|_{r,p}^{\sigma-k})\|x(t)\|_{r,p}^{\gamma+k}$$

$$-(\mathrm{w}_1 - 2hL\|x(t-h)\|_{r,p}^k - 2hL_3\|x(t-h)\|_{r,p}^\sigma - L_4\|x(t-h)\|_{r,p}^{\sigma-k})\|x(t-h)\|_{r,p}^{\gamma+k} \qquad (3.32)$$

$$-(\mathrm{w}_2 - 2L\|x_t\|_{\mathscr{H}}^k - L_1\|x_t\|_{\mathscr{H}}^\sigma)\int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta$$

where $\kappa_{i,j} = \beta_i \eta_{ij} + \psi_{ij}m_i$, $L = \sum_{i=1}^n \sum_{j=1}^n \kappa_{i,j}m_j$, $L_1 = \sum_{i=1}^n \sum_{j=1}^n \kappa_{i,j}(p_j+q_j)$, $L_2 = \sum_{i=1}^n \beta_i(p_i+q_i)$, $L_3 = \sum_{i=1}^n \sum_{j=1}^n \kappa_{i,j}q_j$ and $L_4 = \sum_{i=1}^n \beta_i q_i$. This bound is positive in the neighbourhood $\|x_t\|_{\mathscr{H}} \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$\mathrm{w}_0 - 4hLH_2^k - 2hL_1H_2^\sigma - L_2H_2^{\sigma-k} = 0,$$
$$\mathrm{w}_1 - 2hLH_3^k - 2hL_3H_3^\sigma - L_4H_3^{\sigma-k} = 0,$$
$$\mathrm{w}_2 - 2LH_4^k - L_1H_4^\sigma = 0.$$

Finally, we have a negative bound of the time derivative of $v(\varphi)$

$$\frac{dv(x_t)}{dt} \leq -c_1\|x(t)\|_{r,p}^{\gamma+k} - c_2\|x(t-h)\|_{r,p}^{\gamma+k} - c_3 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta,$$

where $c_1 = \text{w}_0 - 4hL\delta^k - 2hL_1\delta^\sigma - L_2\delta^{\sigma-k}$, $c_2 = \text{w}_1 - 2hL\delta^k - 2hL_3\delta^\sigma - L_4\delta^{\sigma-k}$ and $c_3 = \text{w}_2 - 2L\delta^k - L_1\delta^\sigma$. $\qquad\square$

It follows from (3.30) and defining $c = \min\{c_1, c_3\}$, that the following bound

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|_{r,p}^{\gamma+k} + \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta \right)$$

also holds.

In Lemma 11 the neighbourhood where the time derivative is negative definite was found, thus setting conditions p_i and q_i of the perturbed term. However, we can be interested in knowing how large the perturbations can be to guarantee that all solutions are in a certain neighbourhood, i.e give conditions on p_i and q_i to ensure a negative time derivative. Thus, we take a neighbourhood such that

$$\|x_t\|_{\mathscr{H}} \leq \delta < \left(\min\left\{ \frac{\text{w}_0}{4hL}, \frac{\text{w}_1}{2hL}, \frac{\text{w}_2}{2L} \right\} \right)^{\frac{1}{k}}.$$

Reorganising (3.32) and separating terms that depend on the perturbation, we obtain

$$\frac{dv(x_t)}{dt} \leq -(\text{w}_0 - 4hL\delta^k)\|x(t)\|_{r,p}^{\gamma+k} + (2hL_1\delta^\sigma + L_2\delta^{\sigma-k})\|x(t)\|_{r,p}^{\gamma+k}$$
$$-(\text{w}_1 - 2hL\delta^k)\|x(t-h)\|_{r,p}^{\gamma+k} + (2hL_3\delta^\sigma + L_4\delta^{\sigma-k})\|x(t-h)\|_{r,p}^{\gamma+k}$$
$$-(\text{w}_2 - 2L\delta^k)\int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta + L_1\delta^\sigma \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta.$$

Now, we can establish some conditions on p_i and q_i, satisfying the following inequalities to get a negative time derivative for $v(\varphi)$

$$2h \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i\eta_{ij}(p_j+q_j) + m_j\psi_{ij}(p_i+q_i)) \right) \delta^\sigma + \left(\sum_{i=1}^{n}\beta_i(p_i+q_i) \right) \delta^{\sigma-k} < \text{w}_0 - 4hL\delta^k,$$

$$2h \left(\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i\eta_{ij}q_j + m_j\psi_{ij}q_i) \right) \delta^\sigma + \left(\sum_{i=1}^{n}\beta_iq_i \right) \delta^{\sigma-k} < \text{w}_1 - 2hL\delta^k,$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i\eta_{ij}(p_j+q_j) + m_j\psi_{ij}(p_i+q_i)) \right) \delta^\sigma < \text{w}_2 - 2L\delta^k.$$
$$(3.33)$$

Thus, we can ensure the negative time derivative of $v(\varphi)$. Hence,

$$\frac{dv(x_t)}{dt} \leq -c_1\|x(t)\|_{r,p}^{\gamma+k} - c_2\|x(t-h)\|_{r,p}^{\gamma+k} - c_3 \int_{-h}^{0} \|x(t+\theta)\|_{r,p}^{\gamma+k}d\theta,$$

where $c_1 = \text{w}_0 - 4hL\delta^k - 2hL_1\delta^\sigma - L_2\delta^{\sigma-k}$, $c_2 = \text{w}_1 - 2hL\delta^k - 2hL_3\delta^\sigma - L_4\delta^{\sigma-k}$ and $c_3 = \text{w}_2 - 2L\delta^k - L_1\delta^\sigma$.

3.3   Estimates of the solutions

The purpose of this section is to provide estimates of stable solutions, by mean of the Lyapunov-Krasovskii functional (3.1) introduced at the beginning of this chapter. To do so, we need to connect the functional $v(x_t)$ to its derivative $\frac{dv(x_t)}{dt}$ through the bounds for $v(x_t)$ and $\frac{dv(x_t)}{dt}$ found in the previous section. This is achieved with the help of the following technical results whose proofs are presented in Appendix B.

Lemma 12. *Let u and q be natural numbers such that $u, q \geq 1$ and $u > q$. Then, the following inequality is satisfied*

$$\left(\|x(t)\|_{r,p}^q + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^q d\theta \right)^{\frac{u}{q}} \leq L_1 \left(\|x(t)\|_{r,p}^u + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^u d\theta \right), \tag{3.34}$$

where $L_1 = \left(2 \max\{1, h\} \right)^{\frac{u}{q}-1}$.

Lemma 13. *Let $u \geq 2$ be an entire number. Then,*

$$\left(\|x(t)\|_{r,p}^2 + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^2 d\theta \right)^u \leq L_1 \left(\|x(t)\|_{r,p}^{2u} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{2u} d\theta \right), \tag{3.35}$$

where $L_1 = 2^{u-2}(1+h)^{u-1}$.

The connection between functional (3.1) and its derivative is as follows.

Lemma 14. *The following inequality is satisfied*

$$\frac{dv(x_t)}{dt} \leq -L_2 v(x_t)^{\frac{\gamma+k}{\gamma}}, \quad t \geq 0, \tag{3.36}$$

along the solutions with $\|x_t\|_{\mathscr{H}} \leq \delta$. Here,

$$L_2 = \frac{c}{b^{\frac{\gamma+k}{\gamma}} L_1},$$

where b is defined in (3.11), c in (3.15) and L_1 comes from Lemma 12 with $u = \gamma + k$ and $q = \gamma$.

Proof. From (3.15) we have that

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|_{r,p}^{\gamma+k} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \right).$$

Taking $u = \gamma + k$ and $q = \gamma$, Lemma 14 implies that

$$\frac{dv(x_t)}{dt} \leq -c \left(\|x(t)\|_{r,p}^{\gamma+k} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma+k} d\theta \right)$$

$$\leq -\frac{c}{L_1} \left(\|x(t)\|_{r,p}^{\gamma} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma} d\theta \right)^{\frac{\gamma+k}{\gamma}},$$

equivalently,

$$\frac{dv(x_t)}{dt} \leq -\frac{c}{L_1 b^{\frac{\gamma+k}{\gamma}}} \left[b \left(\|x(t)\|_{r,p}^{\gamma} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{\gamma} d\theta \right) \right]^{\frac{\gamma+k}{\gamma}}.$$

In view of the upper bound (3.11) for $v(x_t)$, we conclude that

$$\frac{dv(x_t)}{dt} \leq -\frac{c}{L_1 b^{\frac{\gamma+k}{\gamma}}} v(x_t)^{\frac{\gamma+k}{\gamma}}.$$

$\square$

Now, we can tackle the final step in our pursuit of estimates of the solution. Since $v(x_t)$ fulfils the differential inequality (14), we use the Comparison Lemma [16]. We define the comparison function $u(t)$ such that

$$\frac{du(t)}{dt} = -L_2 u^{\frac{\gamma+k}{\gamma}}(t), \tag{3.37}$$

and, in view of (3.12), we chose

$$u(0) = u_0 = (\alpha_1 + b_3 \Delta^k) \|\varphi\|_{\mathscr{H}}^{\gamma}.$$

The solution to this differential equation is obtained via the separation of variable technique. It follows from (3.37) that

$$u(t) = u_0 \left[1 + L_2 \left(\frac{k}{\gamma} \right) u_0^{\frac{k}{\gamma}} t \right]^{-\frac{\gamma}{k}}.$$

Then, by the Comparison Lemma,

$$v(x_t) \leq u_0 \left[1 + L_2 \left(\frac{k}{\gamma} \right) u_0^{\frac{k}{\gamma}} t \right]^{-\frac{\gamma}{k}}.$$

Finally, using the bound (3.6), we have the following estimate of the solutions of the homogeneous system with delay (2.19)

$$\|x(t,\varphi)\|_{r,p} \leq \frac{1}{a_1^{\frac{1}{\gamma}}} u_0^{\frac{1}{\gamma}} \left[1 + L_2 \left(\frac{k}{\gamma} \right) u_0^{\frac{k}{\gamma}} t \right]^{-\frac{1}{k}}$$

or,

$$\|x(t,\varphi)\|_{r,p} \leq \hat{c}_1 \|\varphi\|_{\mathscr{H}} \left[1 + \hat{c}_2 \|\varphi\|_{\mathscr{H}}^k t \right]^{-\frac{1}{k}}$$

$$\hat{c}_1 = \left(\frac{\alpha_1 + b_3 \Delta^k}{a_1} \right)^{\frac{1}{\gamma}} = \frac{\delta}{\Delta},$$

$$\hat{c}_2 = \frac{c}{b} \left(\frac{k}{\gamma} \right) \left(\frac{\alpha_1 + b_3 \Delta^k}{2b \max\{1,h\}} \right)^{\frac{k}{\gamma}}.$$

In view of the above, we can state the following result.

Theorem 10. *Let the trivial solution of system* (2.19) *be δ^r-homogeneous and asymptotically stable. The solutions of system* (2.19) *with initial functions satisfying $\|\varphi\|_{\mathscr{H}} < \Delta$, where Δ is defined in* (3.16), *admit an estimate of the form*

$$\|x(t,\varphi)\|_{r,p} \leq \hat{c}_1 \|\varphi\|_{\mathscr{H}} \left[1 + \hat{c}_2 \|\varphi\|_{\mathscr{H}}^k t \right]^{-\frac{1}{k}}$$

where

$$\hat{c}_1 = \left(\frac{\alpha_1 + b_3 \Delta^k}{a_1} \right)^{\frac{1}{\gamma}} = \frac{\delta}{\Delta},$$

$$\hat{c}_2 = \frac{c}{b} \left(\frac{k}{\gamma} \right) \left(\frac{\alpha_1 + b_3 \Delta^k}{2b \max\{1,h\}} \right)^{\frac{k}{\gamma}}.$$

Here, a_1 comes from (3.6), *b from* (3.11), *b_3 from* (3.12) *and c from* (3.15).

Remark 4. *As $\|\varphi\|_{\mathscr{H}} < \Delta$, the solution satisfies $\|x_t\|_{\mathscr{H}} \leq \delta$, hence the estimate is valid.*

Remark 5. *Repeating the steps of Theorem 10, we arrive at the same estimates for solutions of the particular case (3.18) and the perturbed system (3.28), taking their respective values a_1, b, c and β_1 and computing their estimate of the attraction region.*

3.4   Illustrative Examples

3.4.1   Example 1

Consider the following system, which is widely used to model complex interactions, either instantaneous or delayed, occurring amongst transcription factors and target genes [17]:

$$\dot{x}_1(t) = -\kappa_1 x_1(t)(x_1(t) + u_1(t)) + \rho_1 x_2(t - h) + u_2(t), \tag{3.38}$$

$$\dot{x}_2(t) = -\kappa_2 x_2^{\frac{3}{2}}(t) + (\rho_2 x_2(t) + u_3(t))x_1(t - h).$$

Here $x_1(t), x_2(t) \in \mathbb{R}^+$ represent interactions occurring in a genetic network, the inputs $u_i(t) \in \mathbb{R}^+$, $i \in \overline{1,3}$ are the model uncertainties, $h > 0$ is the transition delay in the network, and $\kappa_1, \kappa_2, \rho_1, \rho_2$ are positive parameters. For $u_i(t) = 0$, $i \in \overline{1,3}$, system (3.38) reduces to

$$\dot{x}_1(t) = -\kappa_1 x_1^2(t) + \rho_1 x_2(t - h), \tag{3.39}$$

$$\dot{x}_2(t) = -\kappa_2 x_2^{\frac{3}{2}}(t) + \rho_2 x_2(t)x_1(t - h),$$

which is δ^r-homogeneous for $(r_1, r_2) = (1, 2)$ with degree $k = 1$. For system (3.39), we set $\gamma = 4$ and consider the Lyapunov function

$$V(x) = x_1^4 + x_2^2, \tag{3.40}$$

which is positive definite. Its derivative along the trajectories of system (3.38) when $h = 0$ yields

$$\dot{V}(x) = -4\kappa_1 x_1^5 + 4\rho_1 x_1^3 x_2 - 2\kappa_2 x_2^{\frac{5}{2}} + 2\rho_2 x_2^2 x_1,$$

with homogeneous degree $\gamma + k = 5$, satisfying Theorem 1 in Section 2.1. Moreover, choosing $p = 1$ in Definition 3 in Section 2.1, we get an upper bound of the form

$$\dot{V}(x) \leq -\left(\frac{1}{8}\min\{2\kappa_1, \kappa_2\} - 4\max\{2\rho_1, \rho_2\}\right)\|x(t)\|_{r,1}^5 = -\mathrm{w}\|x(t)\|_{r,1}^5.$$

Then, we obtain the following stability condition for the delay-free system (3.38)

$$\mathrm{w} = \left(\frac{1}{8}\min\{2\kappa_1, \kappa_2\} - 4\max\{2\rho_1, \rho_2\}\right) > 0. \tag{3.41}$$

Taking $p = 5$, this condition turns out to be

$$\mathrm{w} = (2\min\{2\kappa_1, \kappa_2\} - 4\max\{2\rho_1, \rho_2\}) > 0. \tag{3.42}$$

It is worth mentioning that if the value p increases the asymptotic stability condition (3.42) is less conservative than (3.41) since the set of parameters for which it holds is wider.

The constants associated to system (3.38) and the Lyapunov function of the delay-free system (3.40) are $m_1 = \max\{|\kappa_1|, |\rho_1|\}$, $m_2 = |\kappa_2| + |\rho_2|$, $\eta_{11} = 2|\kappa_1|$, $\eta_{12} = \eta_{21} = 0$, $\eta_{22} = $

$\max\{3/2|\kappa_2|, |\rho_2|\}$, $\beta_1 = 4$, $\beta_2 = 2$, $\psi_{11} = 12$, $\psi_{12} = \psi_{21} = 0$, $\psi_{22} = 2$. For $p = 1$ we have $\alpha_0 = 1$ and $\alpha_1 = 1$, and for $p = 5$, $\alpha_0 = 1$ and $\alpha_1 = 2^{1/5}$.

For a given set of system parameters $(\kappa_1, \kappa_2, \rho_1, \rho_2) = (9, 18, 0.25, 0.5)$ satisfying (3.41) and (3.42) when $h = 0$, we compute the constants shown in Table 3.1 for the delay $h = 10$.

Table 3.1: Constants for the estimates of solutions of Example 1

p	δ	H_1	H_2	Δ	$\hat{c}_1$	$\hat{c}_2$	χ	w	w_0	w_1	w_2
1	$1 \cdot 10^{-9}$	$1.6 \cdot 10^{-8}$	$1.8 \cdot 10^{-7}$	$9.8 \cdot 10^{-10}$	1.0162	0.002	0.135	0.25	0.07	0.012	0.016
5	$1 \cdot 10^{-9}$	$1.6 \cdot 10^{-5}$	$2.5 \cdot 10^{-5}$	$9.6 \cdot 10^{-10}$	1.035	0.23	0.44	34	9.63	1.7	2.26

Then, according to Theorem 10, the solution of system (3.38) admits the estimates:

$$\|x(t, \varphi)\|_{r,1} \leq 1.0162 \|\varphi\|_{\mathscr{H}} \left[1 + 0.002 \|\varphi\|_{\mathscr{H}} t\right]^{-1}$$

and

$$\|x(t, \varphi)\|_{r,5} \leq 1.035 \|\varphi\|_{\mathscr{H}} \left[1 + 0.23 \|\varphi\|_{\mathscr{H}} t\right]^{-1}. \tag{3.43}$$

For the initial function $\varphi(\theta) = [2 \cdot 10^{-19}, 2 \cdot 10^{-19}]$, $\theta \in [-10, 0]$, the system response (continuous line) and its estimates (dashed line) for $p = 1$ and $p = 5$ are depicted in Figure 3.1 and Figure 3.2, respectively. As shown in Figure 3.2, if we take a greater value for p, we also get a less conservative bound of the estimate of the solutions since the bounds of the Lyapunov-Krasovskii depend on w, which increases if the value p increases as also shown in Table 3.1.
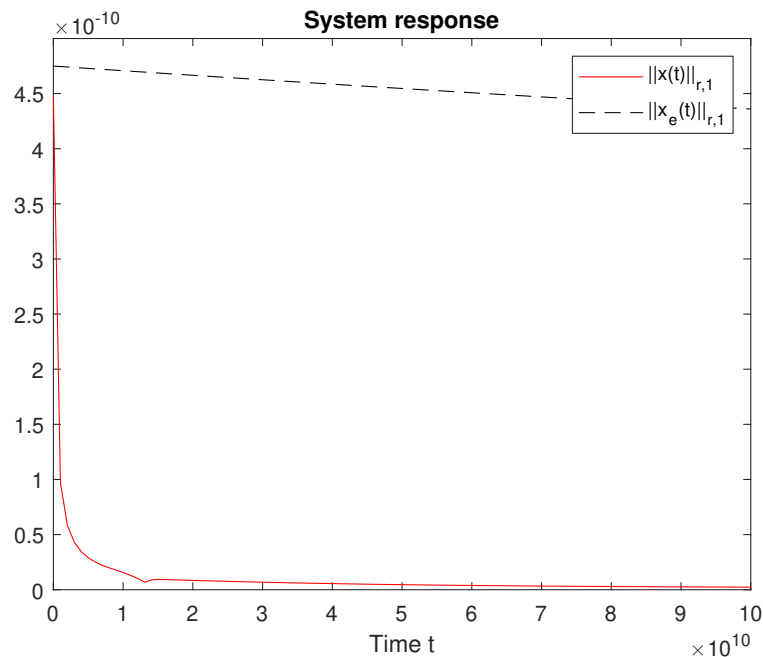

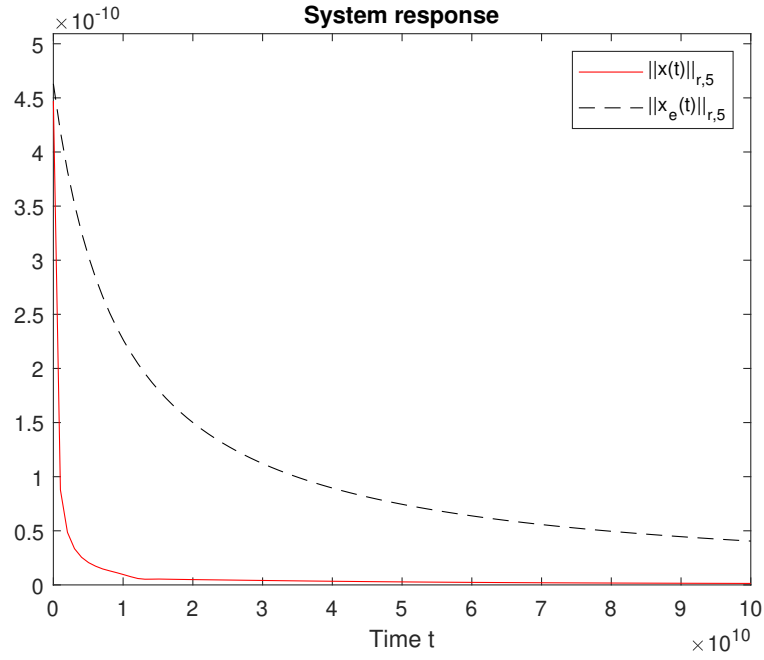Figure 3.1: Estimation of the solution of system (3.39) with $p = 1$

Figure 3.2: Estimation of the solution of system (3.39) with $p = 5$
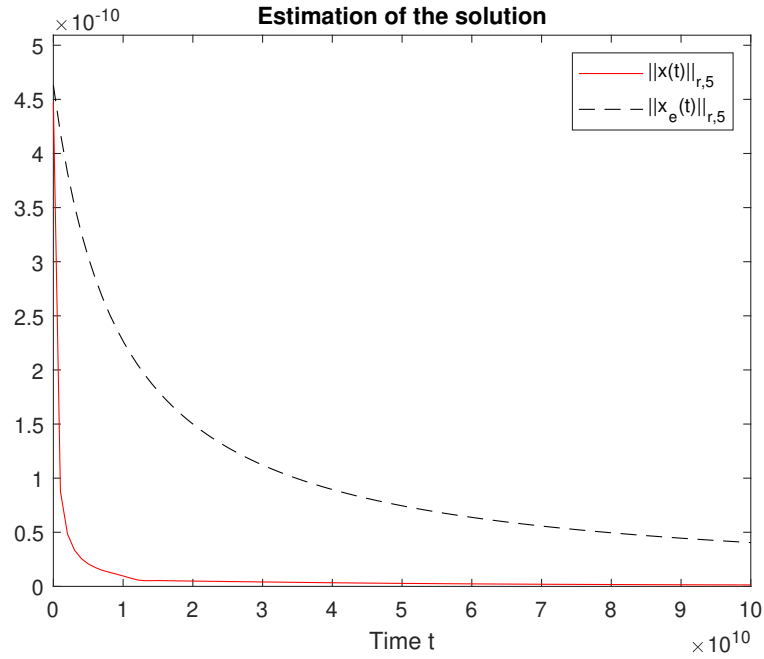
Figure 3.3: Estimation of the solution of system (3.44)

For the perturbed system (3.38), we take $u_1(t) = x_1^2(t)$, $u_2(t) = \kappa_3 x_2(t)^{\frac{3}{2}}$ and $u_3(t) = \rho x_1^3(t)$. where κ_3 and ρ_3 are positive parameters and $\kappa_3 > \kappa_1$. Thus, we get the following perturbed system:

$$\dot{x}_1(t) = -\kappa_1 x_1^2(t) + \rho_1 x_2(t-h) - \kappa_1 x_1^3(t) + \kappa_3 x_2^{\frac{3}{2}}(t),$$
$$\dot{x}_2(t) = -\kappa_2 x_2^{\frac{3}{2}}(t) + \rho_2 x_2(t)x_1(t-h) + \rho_3 x_1^3(t)x_1(t-h). \tag{3.44}$$

The perturbed terms $R_1 = -\kappa_1 x_1^3(t) + \kappa_3 x_2^{\frac{3}{2}}(t)$ and $R_2(t) = \rho_3 x_1^3(t) x_1(t-h)$ are δ^r-homogeneous of homogeneity degree $\sigma = 2$ and $\sigma > \mu$, hence they satisfy Lemma 11 in Section 3.2 and the trivial solution of perturbed system is asymptotically stable. Taking $\kappa_3 = \rho_3 = 1 \cdot 10^2$, we obtain approximate values as in Table 3.1. The response of system (3.44) and estimate (3.43) for the initial function $\varphi(\theta) = [2 \cdot 10^{-19}, 2 \cdot 10^{-19}]$, $\theta \in [-10, 0]$, are depicted in Figure 3.3 as a continuous and dashed line, respectively.

3.4.2 Example 2

Consider the system

$$\dot{x}_1(t) = -0.5x_1^3(t) + x_2(t) - x_2(t-h), \tag{3.45}$$

$$\dot{x}_2(t) = -1.5x_2^{\frac{5}{3}}(t) - x_1^2(t)x_2(t) + x_1^5(t-h) + x_1^2(t-h)x_2(t-h).$$

where $x_1(t),\ x_2(t) \in \mathbb{R}$ and $h \in \mathbb{R}^+$. It is straightforward to verify that system (3.45) is δ^r-homogeneous for $(r_1, r_2) = (1, 3)$ with degree of homogeneity $k = 2$. Set $\gamma = 6$ and apply the Lyapunov method for the delay-free system (3.45) with

$$V(x) = x_1^6 + x_2^2.$$

Its time derivative along the trajectories of system (3.45) when $h = 0$ is

$$\dot{V}(x) = -3x_1^8 - 3x_2^{\frac{8}{3}} + 2x_2 x_1^5,$$

which admits a negative bound of the form

$$\dot{V}(x) \leq -\|x(t)\|_{r,8}^8 = -\mathrm{w}\|x(t)\|_{r,8}^8.$$

Hence, the free-delay system is asymptotically stable. Due to its special structure, this can be analysed through the particular case (3.18) presented in Section 3.1.

In this example, we proceed to compute the estimate of the solutions for system (3.45) with the help of the right-hand side bounds of the general case (2.19) and the particular case (3.18). The constants involved for the general case and the particular case are $f_1 = 1.5$, $f_2 = 2.5$, $g_1 = 1$, $g_2 = 2$, $m_1 = 1.5$, $m_2 = 2.5$, $\eta_{11} = 3/2$, $\eta_{12} = 1$, $\eta_{21} = 2$, $\eta_{22} = 7/2$, $\beta_1 = 6$, $\beta_2 = 2$, $\psi_{11} = 30$, $\psi_{12} = \psi_{21} = 0$, $\psi_{22} = 2$, $\alpha_0 = 1$ and $\alpha_1 = 2^{\frac{1}{4}}$. For $h = 10$, the parameters of the estimate of the solutions and some important constants for their computations are shown in Table 3.2.

Table 3.2: Constants for the estimates of solutions of Example 2

δ	H_1	H_2	Δ	$\hat{c}_1$	$\hat{c}_2$	χ	w	w_0	w_1	w_2
Lyapunov-Krasovskii (General Case)										
0.001	0.0109	0.004	$9.69 \cdot 10^{-4}$	1.0311	0.0069	0.6143	1	0.28	0.05	0.06
Lyapunov-Krasovskii (Particular Case)										
0.001	0.0155	0.0115	$9.7 \cdot 10^{-4}$	1.0302	0.0069	0.64	1	0.28	0.05	0.06

The solution of system (3.45) admits the estimate for the general case:

$$\|x(t,\varphi)\|_{r,8} \leq 1.0311\|\varphi\|_{\mathscr{H}}\left[1 + 0.0069\|\varphi\|_{\mathscr{H}}^2 t\right]^{-\frac{1}{2}},$$

and for the particular case

$$\|x(t,\varphi)\|_{r,8} \leq 1.0302\|\varphi\|_{\mathscr{H}}\left[1 + 0.0069\|\varphi\|_{\mathscr{H}}^2 t\right]^{-\frac{1}{2}}.$$

The system response and its estimates for the general case and the particular case for the initial function $\varphi(\theta) = [6 \cdot 10^{-10}, \ 6 \cdot 10^{-10}]$, $\theta \in [-10,0]$, are depicted in Figure 3.5 as a continuous, dashed and dash-dot line, respectively.

We can see that the estimate of the attraction region is less conservative in the particular case than in the general case since a_1 is greater, and it is readily shown in Theorem 9 in Section 3.1 that $\Delta < (a_1/\alpha_1)^{\frac{1}{\gamma}}\delta$. This allows shrinking the parameter $\hat{c}_1$ and gets a better estimate for small time as shown in Figure 3.4.

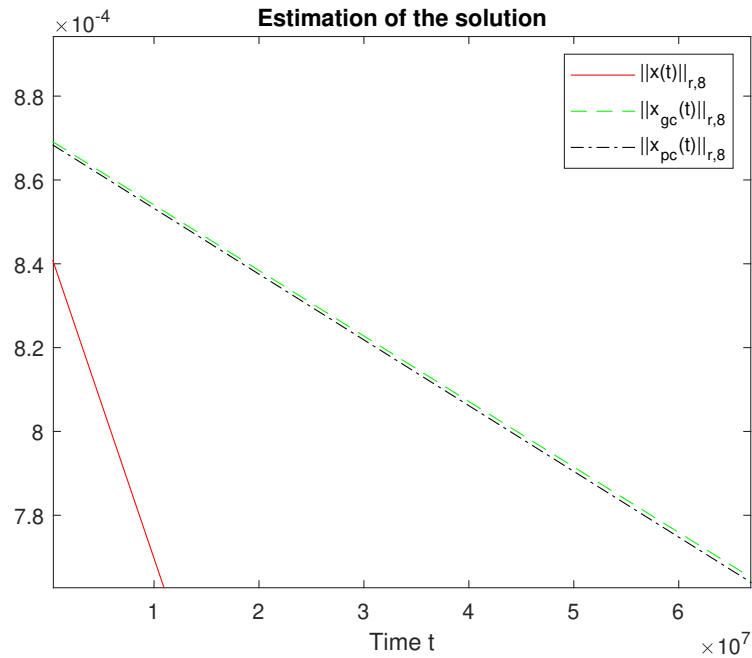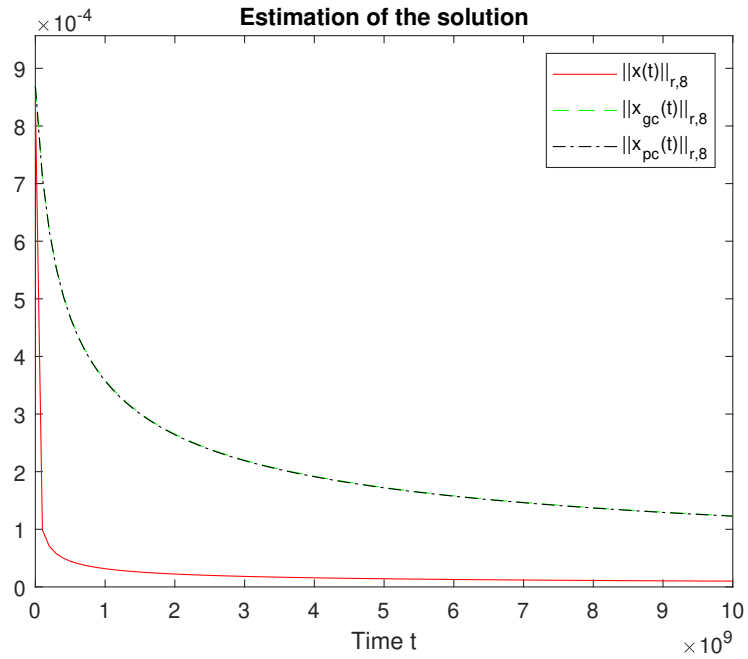Figure 3.4: Estimation of the solution of system (3.45) for small time

Figure 3.5: Estimation of the solution of system (3.45)

Chapter 4

Analysis of time-delay systems with standard dilation

Most of the results on homogeneous time-delay systems become more straightforward with standard dilation due to reduced computations. Besides, the standard dilation is a particular case of the weighted dilation when each weight is equal to one. In recent research, an analogue functional to the so-called Lyapunov-Krasovskii functional of complete type for linear systems was presented. It is necessary to remind that this type of functional allows analysing systems under perturbations. For this reason, we devote this chapter to address this functional to analyse perturbations and some variations of these for homogeneous time-delay systems. Also, we compute estimates of the solution for standard dilation.

Computations for the neighbourhood of asymptotic stability of perturbed homogeneous time-delay systems are developed in Section 4.1. Estimates of the solution for homogeneous time-delay systems are computed in Section 4.2. In the last section, some examples with and without perturbations are discussed.

4.1 Perturbed System

As shown in Section 2.2.2, for time-delay systems with right-hand side of the form $f(x(t), x(t-h))$, a novel functional was presented in [12], which admits a lower bound (Lemma 2), an upper bound (Lemma 3) and its estimate of the time derivative is negative with the assumption that $f(x(t), x(t))$ is asymptotically stable (Lemma 4). Based on the above results mentioned, we study the robustness on homogeneous time-delay systems with standard dilation with the help of functional (2.28).

Consider the perturbed system

$$\dot{x}(t) = f(x(t), x(t-h)) + R(x(t), x(t-h)), \tag{4.1}$$

where the vector function $f(x(t), x(t-h))$ is continuously differentiable and δ^1-homogeneous of homogeneous degree $\mu > 1$ and $R(x(t), x(t-h))$ is continuous. It is assumed that

$$\|R(x(t), x(t-h))\| \leq p_0\|x(t)\|^\sigma + p_1\|x(t-h)\|^\sigma, \quad p_0, p_1 > 0, \quad \sigma > \mu. \tag{4.2}$$

The stability of this system is analysed through functional (2.28). The lower and upper bound for the functional (2.28) are given in Lemma 2 and Lemma 3, respectively. We now determine the neighbourhood where we get a negative time derivative of functional (2.28) along the solutions of system (4.1).

Lemma 15. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional (2.28) along the solutions of system (4.1), admits a bound of the form*

$$\frac{dv(x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+\mu-1} - c_2 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta - c_3 \|x(t-h)\|^{\gamma+\mu-1}, \qquad (4.3)$$

in the neighbourhood $\|x_t\| \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$\begin{aligned}
w_0 - 4hLH_2^{\mu-1} - 2\beta p_0 H_2^{\sigma-\mu} - 2hL_1(p_0+p_1)H_2^{\sigma-1} &= 0, \\
w_1 - 2hLH_3^{\mu-1} - \beta p_1 H_3^{\sigma-\mu} - 2hp_1 L_1 H_3^{\sigma-1} &= 0, \\
w_2 - 2LH_4^{\mu-1} - L_1(p_0+p_1)H_4^{\sigma-1} &= 0.
\end{aligned} \qquad (4.4)$$

Here, $L = m\eta\beta + m^2\psi$, $L_1 = \beta\eta + \psi m$, $c_1 = w_0 - 4hL\delta^{\mu-1} - 2\beta p_0 \delta^{\sigma-\mu} - 2hL_1(p_0+p_1)\delta^{\sigma-1}$, $c_2 = w_1 - 2hL\delta^{\mu-1} - \beta p_1 \delta^{\sigma-\mu} - 2hp_1 L_1 \delta^{\sigma-1}$ and $c_3 = w_2 - 2L\delta^{\mu-1} - L_1(p_0+p_1)\delta^{\sigma-1}$.

Proof. Differentiating the three summands of the functional, we obtain

$$\begin{aligned}
\left.\frac{dv(x_t)}{dt}\right|_{(2.19)} &= -w_0 \|x(t)\|^{\gamma+\mu-1} - w_1 \|x(t-h)\|^{\gamma+\mu-1} - w_2 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta \\
&\quad + \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T R(x(t), x(t-h)) + \sum_{j=1}^{2} \Lambda_j.
\end{aligned}$$

Using (2.11), (2.12), (2.14), (2.17) and (2.18), we get the following upper bounds

$$\Lambda_1 \leq \beta m\eta \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^{0} \|x(t+\theta)\|^{\gamma+2\mu-2} d\theta\right)$$

$$+\beta\|x(t)\|^{\gamma-1} \int_{-h}^{0} \eta(\|x(t)\|^{\mu-1} + \|x(t+\theta)\|^{\mu-1}) d\theta \times (p_0\|x(t)\|^{\sigma} + p_1\|x(t-h)\|^{\sigma})$$

$$\Lambda_1 \leq \beta m\eta \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^{0} \|x(t+\theta)\|^{\gamma+2\mu-2} d\theta\right)$$

$$+\beta\eta \left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1 h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^{0} \|x(t+\theta)\|^{\mu+\sigma+\gamma-2} d\theta\right)$$

$$\Lambda_2 \leq \psi m^2 \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^{0} \|x(t+\theta)\|^{\gamma+2\mu-2} d\theta\right)$$

$$+(p_0\|x(t)\|^{\sigma} + p_1\|x(t-h)\|^{\sigma})\psi\|x(t)\|^{\gamma-2} \int_{-h}^{0} m(\|x(t)\|^{\mu} + \|x(t+\theta)\|^{\mu}) d\theta$$

$$\Lambda_2 \leq \psi m^2 \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^{0} \|x(t+\theta)\|^{\gamma+2\mu-2} d\theta\right)$$

$$+\psi m \left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1 h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^{0} \|x(t+\theta)\|^{\mu+\sigma+\gamma-2} d\theta\right)$$

$$\left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T (R(x(t), x(t-h))) \leq 2\beta p_0 \|x(t)\|^{\sigma+\gamma-1} + \beta p_1 \|x(t-h)\|^{\sigma+\gamma-1}.$$

In this way, we obtain the inequality

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL\|x(t)\|^{\mu-1} - 2\beta p_0\|x(t)\|^{\sigma-\mu} - 2hL_1(p_0+p_1)\|x(t)\|^{\sigma-1})\|x(t)\|^{\gamma+\mu-1}$$
$$-(\mathrm{w}_1 - 2hL\|x(t-h)\|^{\mu-1} - \beta p_1\|x(t-h)\|^{\sigma-\mu} - 2hp_1L_1\|x(t-h)\|^{\sigma-1})\|x(t-h)\|^{\gamma+\mu-1} \quad (4.5)$$
$$-(\mathrm{w}_2 - 2L\|x_t\|^{\mu-1} - L_1(p_0+p_1)\|x_t\|^{\sigma-1})\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta$$

where $L = m\eta\beta + m^2\psi$ and $L_1 = \beta\eta + \psi m$. Suppose that $\sigma > \mu > 1$, thus the bound is positive in the neighbourhood $\|x_t\|_h \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$\mathrm{w}_0 - 4hLH_2^{\mu-1} - 2\beta p_0 H_2^{\sigma-\mu} - 2hL_1(p_0+p_1)H_2^{\sigma-1} = 0,$$
$$\mathrm{w}_1 - 2hLH_3^{\mu-1} - \beta p_1 H_3^{\sigma-\mu} - 2hp_1L_1 H_3^{\sigma-1} = 0,$$
$$\mathrm{w}_2 - 2LH_4^{\mu-1} - L_1(p_0+p_1)H_4^{\sigma-1} = 0.$$

Finally, we can present a negative bound of the time derivative of $v(\varphi)$

$$\frac{dv(x_t)}{dt} \leq -c_1\|x(t)\|^{\gamma+\mu-1} - c_2\|x(t-h)\|^{\gamma+\mu-1} - c_3\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta,$$

where $c_1 = \mathrm{w}_0 - 4hL\delta^{\mu-1} - 2\beta p_0\delta^{\sigma-\mu} - 2hL_1(p_0+p_1)\delta^{\sigma-1}$, $c_2 = \mathrm{w}_1 - 2hL\delta^{\mu-1} - \beta p_1\delta^{\sigma-\mu} - 2hp_1L_1\delta^{\sigma-1}$ and $c_3 = \mathrm{w}_2 - 2L\delta^{\mu-1} - L_1(p_0+p_1)\delta^{\sigma-1}$ $\qquad\square$

It follows from (4.3) and defining $c = \min\{c_1, c_3\}$ that we can also establish the following bound

$$\frac{dv(x_t)}{dt} \leq -c\left(\|x(t)\|^{\gamma+\mu-1} + \int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta\right),$$

As in Section 3.3, we may give some conditions on p_0 and p_1 defined in (4.2) instead of giving conditions on the neighbourhood as in (4.4). Choose a neighbourhood satisfying

$$\|x_t\|_h \leq \delta < \left(\min\left\{\frac{\mathrm{w}_0}{4hL}, \frac{\mathrm{w}_1}{2hL}, \frac{\mathrm{w}_2}{2L}\right\}\right)^{\frac{1}{\mu-1}}. \quad (4.6)$$

It follows from (4.5) that if p_0 and p_1 satisfy the inequalities

$$2\beta p_0\delta^{\sigma-\mu} + 2hL_1(p_0+p_1)\delta^{\sigma-1} < \mathrm{w}_0 - 4hL\delta^{\mu-1},$$
$$\beta p_1\delta^{\sigma-\mu} + 2hp_1L_1\delta^{\sigma-1} < \mathrm{w}_1 - 2hL\delta^{\mu-1}, \quad (4.7)$$
$$L_1(p_0+p_1)\delta^{\sigma-1} < \mathrm{w}_2 - 2L\delta^{\mu-1},$$

we can get the same bound as in (4.3) in the neighbourhood (4.6).

4.1.1 Zero mean value perturbations

Let us turn now our attention to the case where we have additional conditions for the perturbed term $R(x(t), x(t-h))$. In the previous section, it was shown that if condition $\sigma > \mu$ holds, then the asymptotic stability of system (4.1) is preserved and consequently, we got bounds for the

functional (2.28) and its time derivative. However, it is proved in recent research, that under certain conditions for the perturbed term, condition $\sigma > \mu$ can be relaxed [18, 19, 20]. Thus, consider the perturbed system

$$\dot{x}(t) = f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)) \qquad (4.8)$$

where $B(t)$ is a matrix of size $n \times l$ whose entries are continuous and bounded, while the components of the l-dimensional vector $Q(x(t), x(t-h))$ are continuously differentiable. Additionally, there exist $p_0, p_1, \alpha > 0$ such that the perturbed term satisfies

$$\|Q(x,y)\| \leq p_0 \|x\|^{\sigma} + p_1 \|y\|^{\sigma}, \qquad (4.9)$$
$$\|B(t)\| \leq \hat{b} = \max \|B(t)\|.$$

Since $Q(x(t), x(t-h))$ is δ^1-homogeneous, then its derivative is also δ^1-homogeneous. Hence, there exists $p_2, p_3 > 0$ such that

$$\left\| \frac{\partial Q(x,y)}{\partial x} \right\| \leq p_2 \|x\|^{\sigma-1} + p_3 \|y\|^{\sigma-1}. \qquad (4.10)$$

Assume that the integral

$$I(t) = \int_0^{t+h} B(s) ds \qquad (4.11)$$

is bounded on $[0, \infty)$ and furthermore, there exists $\alpha_2 > 0$ such that

$$\|I(t)\| \leq \alpha_2. \qquad (4.12)$$

In particular, the entries of $B(t)$ may be describing periodic oscillations with zero mean values. Notice that the integral $I(t)$ with the upper integration limit t instead of $t + h$ was considered in [10].

To determine the conditions where the trivial solution of system (4.8) remains asymptotically stable, we use the following functional:

$$v(t, \varphi) = V(\varphi(0))$$
$$+ \left(\frac{\partial V(x)}{\partial x} \right)^T \bigg|_{x=\varphi(0)} \left(\int_{-h}^0 \left(f(\varphi(0), \varphi(\theta)) + B(t+\theta+h)Q(\varphi(0), \varphi(\theta)) \right) d\theta - I(t)Q(\varphi(0), \varphi(0)) \right)$$
$$+ \int_{-h}^0 (\mathrm{w}_1 + (h+\theta)\mathrm{w}_2) \|\varphi(\theta)\|^{\gamma+\mu-1} d\theta. \qquad (4.13)$$

Further, we set $\alpha > 1$ as in the Razumikhin condition and introduce the set

$$S_\alpha = \left\{ \|\varphi(\theta)\| \leq \alpha \|\varphi(0)\|, \quad \theta \in [-h, 0] \right\}.$$

Since the bound for the functional $v(t, \varphi)$ contains the terms the form

$$\int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\sigma-1},$$

it is questionable whether or not general lower bound of the form (2.29) exists. However, it was shown in [12] that to analyse the stability it is enough to compute the lower bound for the functional on the special set S_α. In the next section, we show that construction of the estimates for solutions by means of this lower bound is also possible. We prove next that functional (4.13) admits a lower and upper bound.

Lemma 16. *There exist $a_1(\alpha) > 0$ such that functional (4.13) admits a lower bound on the set S_α of the form*

$$v(t, \varphi) \geq a_1(\alpha)\|\varphi(0)\|^\gamma + \mathrm{w}_1 \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta, \quad \varphi \in S_\alpha, \tag{4.14}$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_1)$, where H_1 is a positive root of the following equation

$$\alpha_0 - \beta(p_0 + p_1)\alpha_2 H_1^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)H_1^{\sigma-1} - \beta mh(1 + \alpha^\mu)H_1^{\mu-1} = 0, \tag{4.15}$$

and $a_1(\alpha) = \alpha_0 - \beta(p_0 + p_1)\alpha_2\delta^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)\delta^{\sigma-1} - \beta mh(1 + \alpha^\mu)\delta^{\mu-1}$.

Proof. It follows from (2.11), (2.17), (4.9) and (4.12) that

$$v(t, \varphi) \geq \Big(\alpha_0 - (\beta(p_0 + p_1)\alpha_2 + \beta\hat{b}h(p_0 + p_1\alpha^\sigma))\|\varphi(0)\|^{\sigma-1} - \beta mh(1 + \alpha^\mu)\|\varphi(0)\|^{\mu-1}\Big)\|\varphi(0)\|^\gamma$$

$$+ \mathrm{w}_1 \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta.$$

Now, we need to define the neighbourhood where (4.13) has a positive definite lower bound. We take $\|\varphi\|_h \leq \delta$, $\delta \in (0, H_1)$, where H_1 is a positive root of the following equation

$$\alpha_0 - \beta(p_0 + p_1)\alpha_2 H_1^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)H_1^{\sigma-1} - \beta mh(1 + \alpha^\mu)H_1^{\mu-1} = 0,$$

and we arrive at the lower bound

$$v(t, \varphi) \geq \Big(\alpha_0 - \beta(p_0 + p_1)\alpha_2\delta^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)\delta^{\sigma-1} - \beta mh(1 + \alpha^\mu)\delta^{\mu-1}\Big)\|\varphi(0)\|^\gamma$$

$$+ \mathrm{w}_1 \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta.$$

$\square$

Lemma 17. *There exist $b_1, b_2 > 0$ such that functional (4.13) admits an upper bound of the form*

$$v(t, \varphi) \leq b_1\|\varphi(0)\|^\gamma + b_2 \int_{-h}^0 \|\varphi(\theta)\|^\gamma d\theta, \tag{4.16}$$

if $\|\varphi\|_h \leq \delta$, with

$$b_1 = \alpha_1 + \beta(p_0 + p_1)(\alpha_2 + h\hat{b})\delta^{\sigma-1} + 2h\beta m\delta^{\mu-1},$$
$$b_2 = (\mathrm{w}_1 + h\mathrm{w}_2 + \beta m)\delta^{\mu-1} + \beta\hat{b}p_1\delta^{\sigma-1}.$$

Proof. It follows from (2.11), (2.17), (4.9) and (4.12) that

$$v(t, \varphi) \leq (\alpha_1 + \beta(p_0 + p_1)(\alpha_2 + h\hat{b})\|\varphi(0)\|^{\sigma-1} + 2h\beta m\|\varphi(0)\|^{\mu-1})\|\varphi(0)\|^\gamma$$

$$+ (\mathrm{w}_1 + h\mathrm{w}_2 + \beta m) \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta + \beta\hat{b}p_1 \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\sigma-1}d\theta.$$

Taking a neighbourhood $\|\varphi\|_h \leq \delta$, we obtain

$$v(t, \varphi) \leq (\alpha_1 + \beta(p_0 + p_1)(\alpha_2 + h\hat{b})\delta^{\sigma-1} + 2h\beta m\delta^{\mu-1})\|\varphi(0)\|^\gamma$$

$$+ ((\mathrm{w}_1 + h\mathrm{w}_2 + \beta m)\delta^{\mu-1} + \beta\hat{b}p_1\delta^{\sigma-1}) \int_{-h}^0 \|\varphi(\theta)\|^\gamma d\theta. \tag{4.17}$$

$\square$

It follows from (4.16), defining $b = \max\{b_1, b_2\}$, that

$$v(t, \varphi) \leq b \left(\|\varphi(0)\|^{\gamma} + \int_{-h}^{0} \|\varphi(\theta)\|^{\gamma} d\theta \right). \tag{4.18}$$

Suppose $\sigma > 1$ and $\mu > 1$, then (4.13) also admits an upper bound of the form

$$v(t, \varphi) \leq \alpha_1 \|\varphi(0)\|^{\gamma} + b_3 \|\varphi\|_h^{\gamma+\sigma-1} + b_4 \|\varphi\|_h^{\gamma+\mu-1}, \tag{4.19}$$

where $b_3 = \beta(p_0 + p_1)(\alpha_2 + h\hat{b})$ and $b_4 = \left(2\beta m + w_1 + \frac{h}{2}w_2\right) h$.

To satisfy Theorem 2, we need to find a negative time derivative bound of $v(t, \varphi)$, as well as prove that functional (4.13) allows finding a neighbourhood of asymptotic stability for system (4.8). For this reason, we present the following Lemma.

Lemma 18. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional* (4.13) *along the solutions of system* (4.8), *admits a bound of the form*

$$\frac{dv(t, x_t)}{dt} \leq -c_1 \|x(t)\|^{\gamma+\mu-1} - c_2 \int_{-h}^{0} \|x(t+\theta)\|^{\gamma+\mu-1} d\theta - c_3 \|x(t-h)\|^{\gamma+\mu-1}, \tag{4.20}$$

in the neighbourhood $\|x_t\| \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$\begin{aligned}
w_0 - 4hL_1 H_2^{\mu-1} - L_2 H_2^{\sigma-1} - L_3 H_2^{2\sigma-\mu-1} &= 0, \\
w_1 - 2hL_1 H_3^{\mu-1} - L_4 H_3^{\sigma-1} - L_5 H_3^{2\sigma-\mu-1} &= 0, \\
w_2 - 2L_1 H_4^{\mu-1} - L_6 H_4^{\sigma-1} - L_7 H_3^{2\sigma-\mu-1} &= 0.
\end{aligned} \tag{4.21}$$

Here, $\kappa_1 = \hat{b}(\psi(p_0 + p_1) + \beta(p_2 + p_3))$, $\kappa_2 = \hat{b}(\beta\eta + \psi m)$, $L_1 = \beta m\eta + \psi m^2$, $L_2 = 2h\kappa_2(p_0 + p_1) + 2m\alpha_3 + 2hm\kappa_1$, $L_3 = \hat{b}(p_0 + p_1)\alpha_3 + h\hat{b}(p_0 + p_1)\kappa_1$, $L_4 = 2hp_1\kappa_2 + m\alpha_3 + hm\kappa_1$, $L_5 = \hat{b}(p_1\alpha_3 + hp_1\kappa_1)$, $L_6 = \kappa_2(p_0 + p_1) + 2\hat{b}m(\beta p_3 + \psi p_1)$, $L_7 = \hat{b}^2(p_0 + p_1)(\beta p_3 + \psi p_1)$, $c_1 = w_0 - 4hL_1\delta^{\mu-1} - L_2\delta^{\sigma-1} - L_3\delta^{2\sigma-\mu-1}$, $c_2 = w_2 - 2L_1\delta^{\mu-1} - L_6\delta^{\sigma-1} - L_7\delta^{2\sigma-\mu-1}$ and $c_3 = w_1 - 2hL_1\delta^{\mu-1} - L_4\delta^{\sigma-1} - L_5\delta^{2\sigma-\mu-1}$.

Proof. Differentiating each of the three summands of (4.13) along of solutions of (4.8), we obtain

$$\left.\frac{dv(t, x_t)}{dt}\right|_{(4.8)} = -w_0\|x(t)\|^{\gamma+\mu-1} - w_1\|x(t-h)\|^{\gamma+\mu-1} - w_2\int_{-h}^{0}\|x(t+\theta)\|^{\gamma+\mu-1}d\theta + \sum_{j=1}^{4}\Lambda_j$$

$$-(f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))^T \frac{\partial}{\partial x(t)}\left(\left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T I(t)Q(x(t), x(t))\right)$$

where

$$\Lambda_1 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^{t} \frac{\partial f(x(t), x(s))}{\partial x(t)} ds \times (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))$$

$$\Lambda_2 = (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2(t)}\right) \int_{-h}^{0} f(x(t), x(t+\theta))d\theta.$$

$$\Lambda_3 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^{t} B(s+h)\frac{\partial Q(x(t), x(s))}{\partial x(t)} ds \times (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))$$

$$\Lambda_4 = (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2(t)}\right) \int_{-h}^0 B(t+\theta+h)Q(x(t), x(t+\theta))d\theta.$$

Using (2.11), (2.12), (2.14), (2.17), (2.18) and (4.9) we get the following upper bounds

$$\Lambda_1 \leq \beta m\eta \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\mu-2}d\theta\right)$$

$$+\hat{b}\beta\eta \left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

$$\Lambda_2 \leq \psi m^2 \left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\mu-2}d\theta\right)$$

$$+\hat{b}\psi m \left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

$$\Lambda_3 \leq \beta\hat{b}^2 \left(h(p_0+p_1)(p_2+p_3)\|x(t)\|^{\gamma+2\sigma-2} + hp_1(p_2+p_3)\|x(t-h)\|^{\gamma+2\sigma-2}\right)$$

$$+\beta m\hat{b} \left(2h(p_2+p_3)\|x(t)\|^{\mu+\sigma+\gamma-2} + h(p_2+p_3)\|x(t-h)\|^{\mu+\sigma+\gamma-2} + 2p_3\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

$$+\beta\hat{b}^2 p_3(p_0+p_1)\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\sigma-2}d\theta$$

$$\Lambda_4 \leq \psi\hat{b}^2(p_0+p_1)\left(h(p_0+p_1)\|x(t)\|^{\gamma+2\sigma-2} + hp_1\|x(t-h)\|^{\gamma+2\sigma-2} + p_1\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\sigma-2}d\theta\right)$$

$$+\psi\hat{b}m \left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + h(p_0+p_1)\|x(t-h)\|^{\mu+\sigma+\gamma-2} + 2p_1\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

Since $I(t)$ is assumed to be bounded, there exist $\alpha_3 > 0$ such that

$$(f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))^T \frac{\partial}{\partial x}\left(\left(\frac{\partial V(x)}{\partial x}\right)^T I(t)Q(x(t), x(t))\right) \leq$$

$$\leq (m(\|x(t)\|^\mu + \|x(t-h)\|^\mu) + \hat{b}(p_0\|x(t)\|^\sigma + p_1\|x(t-h)\|^\sigma))\alpha_3\|x(t)\|^{\gamma+\sigma-2}$$

$$\leq 2m\alpha_3\|x(t)\|^{\gamma+\mu+\sigma-2} + \hat{b}(p_0+p_1)\alpha_3\|x(t)\|^{\gamma+2\sigma-2}$$

$$+m\alpha_3\|x(t-h)\|^{\gamma+\mu+\sigma-2} + \hat{b}p_1\alpha_3\|x(t-h)\|^{\gamma+2\sigma-2}.$$

Finally, adding each estimate, we obtain the following bound for the time derivative of (4.13) along the trajectories of system (4.8)

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL_1\|x(t)\|^{\mu-1} - L_2\|x(t)\|^{\sigma-1} - L_3\|x(t)\|^{2\sigma-\mu-1})\|x(t)\|^{\gamma+\mu-1}$$

$$-(\mathrm{w}_1 - 2hL_1\|x(t-h)\|^{\mu-1} - L_4\|x(t-h)\|^{\sigma-1} - L_5\|x(t-h)\|^{2\sigma-\mu-1})\|x(t-h)\|^{\gamma+\mu-1}$$

$$-(\mathrm{w}_2 - 2L_1\|x_t\|^{\mu-1} - L_6\|x_t\|^{\sigma-1} + L_7\|x_t\|^{2\sigma-\mu-1})\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta,$$

where, $\kappa_1 = \hat{b}(\psi(p_0+p_1) + \beta(p_2+p_3))$, $\kappa_2 = \hat{b}(\beta\eta + \psi m)$, $L_1 = \beta m\eta + \psi m^2$, $L_2 = 2h\kappa_2(p_0 + p_1) + 2m\alpha_3 + 2hm\kappa_1$, $L_3 = \hat{b}(p_0+p_1)\alpha_3 + h\hat{b}(p_0+p_1)\kappa_1$, $L_4 = 2hp_1\kappa_2 + m\alpha_3 + hm\kappa_1$, $L_5 =$

$\hat{b}(p_1\alpha_3 + hp_1\kappa_1)$, $L_6 = \kappa_2(p_0 + p_1) + 2\hat{b}m(\beta p_3 + \psi p_1)$ and $L_7 = \hat{b}^2(p_0 + p_1)(\beta p_3 + \psi p_1)$. Suppose that $2\sigma > \mu + 1$, then this bound is negative in the neighbourhood $\|x_t\|_h \leq \delta$, where $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 and H_4 are the real positive roots of the following polynomials

$$w_0 - 4hL_1H_2^{\mu-1} - L_2H_2^{\sigma-1} - L_3H_2^{2\sigma-\mu-1} = 0,$$
$$w_1 - 2hL_1H_3^{\mu-1} - L_4H_3^{\sigma-1} - L_5H_3^{2\sigma-\mu-1} = 0,$$
$$w_2 - 2L_1H_4^{\mu-1} - L_6H_4^{\sigma-1} - L_7H_3^{2\sigma-\mu-1} = 0.$$

Hence, taking the neighbourhood $\|x_t\|_h \leq \delta$, we arrive at

$$
\begin{aligned}
\frac{dv(x_t)}{dt} \leq &-(w_0 - 4hL_1\delta^{\mu-1} - L_2\delta^{\sigma-1} - L_3\delta^{2\sigma-\mu-1})\|x(t)\|^{\gamma+\mu-1} \\
&-(w_1 - 2hL_1\delta^{\mu-1} - L_4\delta^{\sigma-1} - L_5\delta^{2\sigma-\mu-1})\|x(t-h)\|^{\gamma+\mu-1} \\
&-(w_2 - 2L_1\delta^{\mu-1} - L_6\delta^{\sigma-1} - L_7\delta^{2\sigma-\mu-1})\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta,
\end{aligned}
$$

and the lemma is proved. $\qquad\square$

Defining $c = \min c_1, c_3$, it also holds that

$$\frac{dv(x_t)}{dt} \leq -c\left(\|x(t)\|^{\gamma+\mu-1} + \int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta\right),$$

Remark 6. *If*

$$\mu > 1, \quad 2\sigma > \mu + 1, \tag{4.22}$$

(4.21) *gets real positive roots and the trivial solution to* (4.8) *is asymptotically stable for all* $h > 0$.

Similarly to Theorem 8, if Δ is a positive root of equation

$$\alpha_1\Delta^\gamma + b_3\Delta^{\gamma+\sigma-1} + b_4\Delta^{\gamma+\mu-1} = a_1(\alpha)\delta^\gamma, \tag{4.23}$$

then the set of initial functions $\|\varphi\|_h < \Delta$ is the estimate of the region of attraction of the trivial solution of (4.8).

4.1.2 Almost periodic perturbations

Consider system (4.8) and assume now that (4.11) is unbounded on $[0, \infty)$. Assume again that the entries of $B(t)$ have zero mean value and that the limit relation

$$\frac{1}{T}\int_t^{t+T} B(s)ds \longrightarrow 0 \quad as \quad T \longrightarrow +\infty \tag{4.24}$$

holds uniformly in $t \geq 0$. Also, consider the integral

$$L(t,\varepsilon) = \int_0^{t+h} e^{-\varepsilon(t+h-s)}B(s)ds. \tag{4.25}$$

Notice that the integral $L(t,\varepsilon)$ with the upper integration limit t instead of $t+h$ was considered in [10]. This term is constructed under a reasonable approximation for the integral term in $B(t)$. Let us explain this approximation considering the scalar system

$$\dot{x}(\tau) = b(\tau), \tag{4.26}$$

where $b(\tau)$ satisfies (4.24). The solution of this system may be unbounded due to the assumption on (4.11). But if we introduce a second term of such a way that

$$\dot{x}(\tau) = -\varepsilon x(\tau) + b(\tau), \quad \varepsilon > 0, \tag{4.27}$$

this new system has a bounded solution. Then, we try to take $\varepsilon \to 0$ in (4.27) to study (4.26). In fact, (4.25) is a solution of (4.27) for $\tau = t + h$. As shown in [21], (4.25) also satisfies the following properties:

$$\|L(t, \varepsilon)\| \leq \frac{\alpha(\varepsilon)}{\varepsilon}$$

and

$$\alpha(\varepsilon) \to 0 \quad as \quad \varepsilon \to 0,$$

for all $t \geq 0$. In particular, the limit relation (4.24) holds uniformly in $t \geq 0$ in the case that the entries of $B(t)$ are almost periodic functions with zero mean value.

As in our previous analysis for almost periodic perturbations, let us modify functional (4.13). In this case, we will use $L(t, \varepsilon)$ instead of $I(t)$ for analysing the perturbed system (4.8):

$$v(t, \varphi) = V(\varphi(0))$$
$$+ \left(\frac{\partial V(x)}{\partial x} \right)^T \bigg|_{x=\varphi(0)} \left(\int_{-h}^{0} (f(\varphi(0), \varphi(\theta)) + B(t + \theta + h)Q(\varphi(0), \varphi(\theta))) \, d\theta - L(t, \varepsilon)Q(\varphi(0), \varphi(0)) \right)$$
$$+ \int_{-h}^{0} (\mathrm{w}_1 + (h + \theta)\mathrm{w}_2) \|\varphi(\theta)\|^{\gamma + \mu - 1} d\theta, \quad (4.28)$$

We need to find the neighbourhood where this functional satisfies the first condition of Theorem 2. It is done through the following two lemmas.

Lemma 19. *There exist $a_1(\alpha) > 0$ such that functional (4.28) admits a lower bound on the set S_α of the form*

$$v(t, \varphi) \geq a_1(\alpha) \|\varphi(0)\|^\gamma + \mathrm{w}_1 \int_{-h}^{0} \|\varphi(\theta)\|^{\gamma + \mu - 1} d\theta, \quad \varphi \in S_\alpha \tag{4.29}$$

in the neighbourhood $\|\varphi\|_h \leq \delta$. Here, $\delta \in (0, H_1)$, where H_1 is a positive root of the following equation

$$\alpha_0 - \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)H_1^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)H_1^{\sigma-1} - \beta mh(1 + \alpha^\mu)H_1^{\mu-1} = 0, \tag{4.30}$$

and $a_1(\alpha) = \alpha_0 - \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)\delta^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)\delta^{\sigma-1} - \beta mh(1 + \alpha^\mu)\delta^{\mu-1}.$

Proof. Using the estimates (2.11), (2.17) and (4.9), we obtain

$$v(t, \varphi) \geq$$
$$\left(\alpha_0 - (\beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon) + \beta\hat{b}h(p_0 + p_1\alpha^\sigma))\|\varphi(0)\|^{\sigma-1} - \beta mh(1 + \alpha^\mu)\|\varphi(0)\|^{\mu-1} \right) \|\varphi(0)\|^\gamma$$
$$+ \mathrm{w}_1 \int_{-h}^{0} \|\varphi(\theta)\|^{\gamma + \mu - 1} d\theta.$$

This lower bound is positive, if we take $\|\varphi\|_h < \delta$, where $\delta \in (0, H_1)$ and H_1 is a positive real root of the following equation

$$\alpha_0 - \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)H_1^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)H_1^{\sigma-1} - \beta mh(1 + \alpha^\mu)H_1^{\mu-1} = 0,$$

Then, we arrive at

$$v(t,\varphi) \geq \left(\alpha_0 - \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)\delta^{\sigma-1} - \beta\hat{b}h(p_0 + p_1\alpha^\sigma)\delta^{\sigma-1} - \beta mh(1 + \alpha^\mu)\delta^{\mu-1}\right) \|\varphi(0)\|^\gamma$$
$$+ \mathrm{w}_1 \int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1} d\theta.$$

$\square$

Lemma 20. *There exist $b_1, b_2 > 0$ such that functional* (4.28) *admits an upper bound of the form*

$$v(t,\varphi) \leq b_1 \|\varphi(0)\|^\gamma + b_2 \int_{-h}^0 \|\varphi(\theta)\|^\gamma d\theta, \tag{4.31}$$

if $\|\varphi\|_h \leq \delta$, with

$$b_1 = \alpha_1 + \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon + h\hat{b})\delta^{\sigma-1} + 2h\beta m\delta^{\mu-1},$$
$$b_2 = (\mathrm{w}_1 + h\mathrm{w}_2 + \beta m)\delta^{\mu-1} + \beta\hat{b}p_1\delta^{\sigma-1}.$$

Proof. According to (2.11), (2.17), (4.9), we have

$$v(t,\varphi) \leq (\alpha_1 + \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon + h\hat{b})\|\varphi(0)\|^{\sigma-1} + 2h\beta m\|\varphi(0)\|^{\mu-1})\|\varphi(0)\|^\gamma$$
$$+ (\mathrm{w}_1 + h\mathrm{w}_2 + \beta m)\int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta + \beta\hat{b}p_1\int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\sigma-1}d\theta.$$

Taking a neighbourhood $\|\varphi\|_h \leq \delta$, we obtain

$$v(t,\varphi) \leq (\alpha_1 + \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon + h\hat{b})\delta^{\sigma-1} + 2h\beta m\delta^{\mu-1})\|\varphi(0)\|^\gamma$$
$$+ (\mathrm{w}_1 + h\mathrm{w}_2 + \beta m)\int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\mu-1}d\theta + \beta\hat{b}p_1\int_{-h}^0 \|\varphi(\theta)\|^{\gamma+\sigma-1}d\theta.$$

$\square$

It also holds that

$$v(t,\varphi) \leq b\left(\|\varphi(0)\|^\gamma + \int_{-h}^0 \|\varphi(\theta)\|^\gamma d\theta\right), \tag{4.32}$$

where $b = \max\{b_1, b_2\}$. Furthermore, it follows from the proof of Lemma 20, assuming $\sigma \geq \mu > 1$, that

$$v(t,\varphi) \leq \alpha_1\|\varphi(0)\|^\gamma + b_3\|\varphi\|_h^{\gamma+\sigma-1} + b_4\|\varphi\|_h^{\gamma+\mu-1}, \tag{4.33}$$

where $b_3 = \beta(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon + h\hat{b})$ and $b_4 = \left(2\beta m + \mathrm{w}_1 + \frac{h}{2}\mathrm{w}_2\right)h$.

Next, we find the neighbourhood where functional (4.28) satisfies condition (2.21) of Theorem 2.

Lemma 21. *There exist $c_1, c_2, c_3 > 0$ such that the time derivative of functional* (4.28) *along the solutions of system* (4.8)*, admits a bound of the form*

$$\frac{dv(t,x_t)}{dt} \leq -c_1\|x(t)\|^{\gamma+\mu-1} - c_2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta - c_3\|x(t-h)\|^{\gamma+\mu-1}, \tag{4.34}$$

in the neighbourhood $\|x_t\| \leq \delta$, *where* $\delta \in (0, \min\{H_2, H_3, H_4\})$, H_2, H_3 *and* H_4 *are the real positive roots of the following polynomials*

$$\mathrm{w}_0 - 4hL_1H_2^{\mu-1} - \alpha_4\alpha(\varepsilon)H_2^{\sigma-\mu} - L_2H_2^{\sigma-1} - L_3H_2^{2\sigma-\mu-1} = 0,$$
$$\mathrm{w}_1 - 2hL_1H_3^{\mu-1} - L_4H_3^{\sigma-1} - L_5H_3^{2\sigma-\mu-1} = 0, \qquad (4.35)$$
$$\mathrm{w}_2 - 2L_1H_4^{\mu-1} - L_6H_4^{\sigma-1} - L_7H_4^{2\sigma-\mu-1} = 0.$$

Here, $\kappa_1 = \hat{b}(\psi(p_0 + p_1) + \beta(p_2 + p_3))$, $\kappa_2 = \hat{b}(\beta\eta + \psi m)$, $L_1 = \beta m\eta + \psi m^2$, $L_2 = 2h\kappa_2(p_0 + p_1) + 2m(\alpha(\varepsilon)/\varepsilon)\alpha_3 + 2hm\kappa_1$, $L_3 = \hat{b}(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)\alpha_3 + h\hat{b}(p_0 + p_1)\kappa_1$, $L_4 = 2hp_1\kappa_2 + m(\alpha(\varepsilon)/\varepsilon)\alpha_3 + hm\kappa_1$, $L_5 = \hat{b}(p_1(\alpha(\varepsilon)/\varepsilon)\alpha_3 + hp_1\kappa_1)$, $L_6 = \kappa_2(p_0 + p_1) + 2\hat{b}m(\beta p_3 + \psi p_1)$, $L_7 = \hat{b}^2(p_0 + p_1)(\beta p_3 + \psi p_1)$, $c_1 = \mathrm{w}_0 - 4hL_1\delta^{\mu-1} - \alpha_4\alpha(\varepsilon)\delta^{\sigma-\mu} - L_2\delta^{\sigma-1} - L_3\delta^{2\sigma-\mu-1}$, $c_2 = \mathrm{w}_2 - 2L_1\delta^{\mu-1} - L_6\delta^{\sigma-1} - L_7\delta^{2\sigma-\mu-1}$ *and* $c_3 = \mathrm{w}_1 - 2hL_1\delta^{\mu-1} - L_4\delta^{\sigma-1} - L_5\delta^{2\sigma-\mu-1}$.

Proof. Differentiating (4.28) along of solutions of (4.8), we get

$$\left.\frac{dv(t, x_t)}{dt}\right|_{(4.8)} = -\mathrm{w}_0\|x(t)\|^{\gamma+\mu-1} - \mathrm{w}_1\|x(t-h)\|^{\gamma+\mu-1} - \mathrm{w}_2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+\mu-1}d\theta$$

$$-(f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))^T\frac{\partial}{\partial x(t)}\left(\left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T L(t, \varepsilon)Q(x(t), x(t))\right)$$

$$+\sum_{j=1}^4 \Lambda_j + \varepsilon\left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T L(t, \varepsilon)Q(x(t), x(t)),$$

where

$$\Lambda_1 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^t \frac{\partial f(x(t), x(s))}{\partial x(t)}ds \times (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h))),$$

$$\Lambda_2 = (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2(t)}\right)\int_{-h}^0 f(x(t), x(t+\theta))d\theta.$$

$$\Lambda_3 = \left(\frac{\partial V(x(t))}{\partial x(t)}\right)^T \int_{t-h}^t B(s+h)\frac{\partial Q(x(t), x(s))}{\partial x(t)}ds \times (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))$$

$$\Lambda_4 = (f(x(t), x(t-h)) + B(t)Q(x(t), x(t-h)))^T \times \left(\frac{\partial^2 V(x(t))}{\partial x^2(t)}\right)\int_{-h}^0 B(t+\theta+h)Q(x(t), x(t+\theta))d\theta.$$

In concordance with (2.11), (2.12), (2.17) and (4.9) we get the following estimates

$$\Lambda_1 \leq \beta m\eta\left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\mu-2}d\theta\right)$$

$$+\hat{b}\beta\eta\left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

$$\Lambda_2 \leq \psi m^2\left(4h\|x(t)\|^{\gamma+2\mu-2} + 2h\|x(t-h)\|^{\gamma+2\mu-2} + 2\int_{-h}^0 \|x(t+\theta)\|^{\gamma+2\mu-2}d\theta\right)$$

$$+\hat{b}\psi m\left(2h(p_0+p_1)\|x(t)\|^{\mu+\sigma+\gamma-2} + 2p_1h\|x(t-h)\|^{\mu+\sigma+\gamma-2} + (p_0+p_1)\int_{-h}^0 \|x(t+\theta)\|^{\mu+\sigma+\gamma-2}d\theta\right)$$

$$\Lambda_3 \le \beta \hat{b}^2 \left(h(p_0 + p_1)(p_2 + p_3)\|x(t)\|^{\gamma + 2\sigma - 2} + h p_1 (p_2 + p_3)\|x(t - h)\|^{\gamma + 2\sigma - 2} \right)$$

$$+\beta m \hat{b} \left(2h(p_2 + p_3)\|x(t)\|^{\mu + \sigma + \gamma - 2} + h(p_2 + p_3)\|x(t - h)\|^{\mu + \sigma + \gamma - 2} + 2p_3 \int_{-h}^{0} \|x(t + \theta)\|^{\mu + \sigma + \gamma - 2} d\theta \right)$$

$$+\beta \hat{b}^2 p_3 (p_0 + p_1) \int_{-h}^{0} \|x(t + \theta)\|^{\gamma + 2\sigma - 2} d\theta$$

$$\Lambda_4 \le \psi \hat{b}^2 (p_0 + p_1) \left(h(p_0 + p_1)\|x(t)\|^{\gamma + 2\sigma - 2} + h p_1 \|x(t - h)\|^{\gamma + 2\sigma - 2} + p_1 \int_{-h}^{0} \|x(t + \theta)\|^{\gamma + 2\sigma - 2} d\theta \right)$$

$$+\psi \hat{b} m \left(2h(p_0 + p_1)\|x(t)\|^{\mu + \sigma + \gamma - 2} + h(p_0 + p_1)\|x(t - h)\|^{\mu + \sigma + \gamma - 2} + 2p_1 \int_{-h}^{0} \|x(t + \theta)\|^{\mu + \sigma + \gamma - 2} d\theta \right)$$

Since $Q(x, x)$ is assumed to be bounded, there exists $\alpha_3, \alpha_4 > 0$, such that

$$(f(x(t), x(t - h)) + B(t)Q(x(t), x(t - h)))^T \frac{\partial}{\partial x(t)} \left(\left(\frac{\partial V(x(t))}{\partial x(t)} \right)^T L(t, \varepsilon)Q(x(t), x(t)) \right) \le$$

$$\le (m(\|x(t)\|^{\mu} + \|x(t - h)\|^{\mu}) + \hat{p}(p_0\|x(t)\|^{\sigma} + p_1\|x(t - h)\|^{\sigma}))(\alpha(\varepsilon)/\varepsilon)\alpha_3\|x(t)\|^{\gamma + \sigma - 2}$$

$$\le 2m(\alpha(\varepsilon)/\varepsilon)\alpha_3\|x(t)\|^{\gamma + \mu + \sigma - 2} + \hat{b}(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)\alpha_3\|x(t)\|^{\gamma + 2\sigma - 2}$$

$$+m(\alpha(\varepsilon)/\varepsilon)\alpha_3\|x(t - h)\|^{\gamma + \mu + \sigma - 2} + \hat{b}p_1(\alpha(\varepsilon)/\varepsilon)\alpha_3\|x(t - h)\|^{\gamma + 2\sigma - 2},$$

$$\varepsilon \left(\frac{\partial V(x(t))}{\partial x(t)} \right)^T L(t, \varepsilon)Q(x(t), x(t)) \le \alpha_4 \alpha(\varepsilon)\|x(t)\|^{\gamma + \sigma - 1}.$$

Therefore, we get an estimate of the upper bound of the time derivative of $v(\varphi)$

$$\frac{dv(x_t)}{dt} \le -(\mathrm{w}_0 - 4hL_1\|x(t)\|^{\mu - 1} - \alpha_4\alpha(\varepsilon)\|x(t)\|^{\sigma - \mu} - L_2\|x(t)\|^{\sigma - 1} - L_3\|x(t)\|^{2\sigma - \mu - 1})\|x(t)\|^{\gamma + \mu - 1}$$

$$-(\mathrm{w}_1 - 2hL_1\|x(t - h)\|^{\mu - 1} - L_4\|x(t - h)\|^{\sigma - 1} - L_5\|x(t - h)\|^{2\sigma - \mu - 1})\|x(t - h)\|^{\gamma + \mu - 1}$$

$$-(\mathrm{w}_2 - 2L_1\|x_t\|^{\mu - 1} - L_6\|x_t\|^{\sigma - 1} + L_7\|x_t\|^{2\sigma - \mu - 1}) \int_{-h}^{0} \|x(t + \theta)\|^{\gamma + \mu - 1} d\theta,$$

where, $\kappa_1 = \hat{b}(\psi(p_0 + p_1) + \beta(p_2 + p_3))$, $\kappa_2 = \hat{b}(\beta\eta + \psi m)$, $L_1 = \beta m \eta + \psi m^2$, $L_2 = 2h\kappa_2(p_0 + p_1) + 2m(\alpha(\varepsilon)/\varepsilon)\alpha_3 + 2hm\kappa_1$, $L_3 = \hat{b}(p_0 + p_1)(\alpha(\varepsilon)/\varepsilon)\alpha_3 + h\hat{b}(p_0 + p_1)\kappa_1$, $L_4 = 2hp_1\kappa_2 + m(\alpha(\varepsilon)/\varepsilon)\alpha_3 + hm\kappa_1$, $L_5 = \hat{b}(p_1(\alpha(\varepsilon)/\varepsilon)\alpha_3 + hp_1\kappa_1)$, $L_6 = \kappa_2(p_0 + p_1) + 2\hat{b}m(\beta p_3 + \psi p_1)$ and $L_7 = \hat{b}^2(p_0 + p_1)(\beta p_3 + \psi p_1)$.

Suppose that $\sigma \ge \mu > 1$ and choose a small enough $\varepsilon > 0$. We can take the neighbourhood $\|x_t\|_h \le \delta$. Here $\delta \in (0, \min\{H_2, H_3, H_4\})$, and H_2, H_3 and H_4 are positive real roots of the following polynomials

$$\mathrm{w}_0 - 4hL_1 H_2^{\mu - 1} - \alpha_4\alpha(\varepsilon)H_2^{\sigma - \mu} - L_2 H_2^{\sigma - 1} - L_3 H_2^{2\sigma - \mu - 1} = 0,$$

$$\mathrm{w}_1 - 2hL_1 H_3^{\mu - 1} - L_4 H_3^{\sigma - 1} - L_5 H_3^{2\sigma - \mu - 1} = 0,$$

$$\mathrm{w}_2 - 2L_1 H_4^{\mu - 1} - L_6 H_4^{\sigma - 1} - L_7 H_4^{2\sigma - \mu - 1} = 0.$$

Therefore, we arrive at

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0 - 4hL_1\delta^{\mu-1} - \alpha_4\alpha(\varepsilon)\delta^{\sigma-\mu} - L_2\delta^{\sigma-1} - L_3\delta^{2\sigma-\mu-1})\|x(t)\|^{\gamma+\mu-1}$$
$$-(\mathrm{w}_1 - 2hL_1\delta^{\mu-1} - L_4\delta^{\sigma-1} - L_5\delta^{2\sigma-\mu-1})\|x(t-h)\|^{\gamma+\mu-1}$$
$$-(\mathrm{w}_2 - 2L_1\delta^{\mu-1} - L_6\delta^{\sigma-1} - L_7\delta^{2\sigma-\mu-1})\int_{-h}^{0}\|x(t+\theta)\|^{\gamma+\mu-1}d\theta,$$

It is possible to choose ε so that $\alpha_4\alpha(\varepsilon)\delta^{\sigma-\mu} < \mathrm{w}_0/2$. Hence, we get

$$\frac{dv(x_t)}{dt} \leq -(\mathrm{w}_0/2 - 4hL_1\delta^{\mu-1} - L_2\delta^{\sigma-1} - L_3\delta^{2\sigma-\mu-1})\|x(t)\|^{\gamma+\mu-1}$$
$$-(\mathrm{w}_1 - 2hL_1\delta^{\mu-1} - L_4\delta^{\sigma-1} - L_5\delta^{2\sigma-\mu-1})\|x(t-h)\|^{\gamma+\mu-1}$$
$$-(\mathrm{w}_2 - 2L_1\delta^{\mu-1} - L_6\delta^{\sigma-1} - L_7\delta^{2\sigma-\mu-1})\int_{-h}^{0}\|x(t+\theta)\|^{\gamma+\mu-1}d\theta.$$

$\square$

Remark 7. *If*

$$\sigma \geq \mu > 1, \tag{4.36}$$

(4.35) *has real positive roots and the trivial solution to* (4.8) *is asymptotically stable for all* $h > 0$.

Similarly to Theorem 8, if Δ is a positive root of equation

$$\alpha_1\Delta^{\gamma} + b_3\Delta^{\gamma+\sigma-1} + b_4\Delta^{\gamma+\mu-1} = a_1(\alpha)\delta^{\gamma}, \tag{4.37}$$

then the set of initial functions $\|\varphi\|_h < \Delta$ is the estimate of the region of attraction of the trivial solution of (4.8).

4.2 Estimates of the solution

In this section, we present estimates of the solutions for the standard dilation. These estimates are indeed a special case of the results presented in Section 3.4. The connection between functional (2.28) and its derivative is through Lemma 14. It is important to notice that Lemma 12, Lemma 13 and Lemma 14, proved in the weighted homogeneity framework in Chapter 3, are also satisfied for the standard dilation since they do not depend on the norm, only on the degrees of the inequalities. Hence, the following inequality is satisfied

$$\frac{dv(x_t)}{dt} \leq -L_2 v(x_t)^{\frac{\gamma+\mu-1}{\gamma}}, \quad t \geq 0, \tag{4.38}$$

along the solutions of (2.19) for standard dilation with $\|x_t\|_h \leq \delta$. Here,

$$L_2 = \frac{c}{b^{\frac{\gamma+\mu-1}{\gamma}}L_1},$$

where b is defined in (2.31), c in (2.34) and L_1 comes from Lemma 12 in Chapter 3 with $u = \gamma + \mu - 1$ and $q = \gamma$.

We define the comparison function $u(t)$ taking, without loss of generality, $t_0 = 0$,

$$\frac{du(t)}{dt} = -L_2 u^{\frac{\gamma+\mu-1}{\gamma}}(t), \; u(0) = u_0 = (\alpha_1 + b_3 \Delta^{\mu-1})\|\varphi\|_h^\gamma,$$

where we choose u_0 as the upper bound of $v(\varphi)$ in (2.32). The solution to this differential equation obtained by separation of variable is

$$u(t) = u_0 \left[1 + L_2 \left(\frac{\mu-1}{\gamma} \right) u_0^{\frac{\mu-1}{\gamma}} t \right]^{-\frac{\gamma}{\mu-1}}.$$

Then, by the Comparison Lemma,

$$v(x_t) \leq u_0 \left[1 + L_2 \left(\frac{\mu-1}{\gamma} \right) u_0^{\frac{\mu-1}{\gamma}} t \right]^{-\frac{\gamma}{\mu-1}}, \; u_0 = (\alpha_1 + b_3 \Delta^{\mu-1})\|\varphi\|_h^\gamma.$$

Finally, using the bound (2.29), we have the following estimate of the solutions of the homogeneous system with delay (2.19)

$$\|x(t,\varphi)\| \leq \frac{1}{a_1^{\frac{1}{\gamma}}} u_0^{\frac{1}{\gamma}} \left[1 + L_2 \left(\frac{\mu-1}{\gamma} \right) u_0^{\frac{\mu-1}{\gamma}} t \right]^{-\frac{1}{\mu-1}}.$$

In view of the above, we can state the following result:

Theorem 11. *Let the trivial solution of system (2.19) be δ^1-homogeneous and asymptotically stable. The solutions of system (2.19) with initial functions satisfying $\|\varphi\|_h < \Delta$, where Δ is defined in (2.35), admit an estimate of the form*

$$\|x(t,\varphi)\| \leq \hat{c}_1 \|\varphi\|_h \left[1 + \hat{c}_2 \|\varphi\|_h^{\mu-1} t \right]^{-\frac{1}{\mu-1}}$$

where

$$\hat{c}_1 = \left(\frac{\alpha_1 + b_3 \Delta^{\mu-1}}{a_1} \right)^{\frac{1}{\gamma}} = \frac{\delta}{\Delta},$$

$$\hat{c}_2 = \frac{c}{b} \left(\frac{\mu-1}{\gamma} \right) \left(\frac{\alpha_1 + b_3 \Delta^{\mu-1}}{2b \max\{1, h\}} \right)^{\frac{\mu-1}{\gamma}}.$$

Here, a_1, b, b_3 and c are defined in (2.29), (2.31), (2.32) and (2.34), respectively.

Remark 8. *As $\|\varphi\|_h < \Delta$, the solution satisfies $\|x_t\|_h \leq \delta$, hence the estimate is valid.*

Using Lemma 13 in Chapter 3 with $u = (\mu + 1)/2 \in \mathbb{Z}$ instead of Lemma 12 and repeating the steps of the previous section, we arrive at the estimates for the solutions of system (2.36).

Theorem 12. *If $\eta_1 + \eta_2 < 0$, then the solutions of equation (2.36) with initial functions satisfying $|\varphi|_h < \Delta$, where Δ is a positive root of (2.39), admit an estimate of the form*

$$|x(t,\varphi)| \leq \hat{c}_1 |\varphi|_h \left[1 + \hat{c}_2 |\varphi|_h^{\mu-1} t \right]^{-\frac{1}{\mu-1}}, \tag{4.39}$$

where

$$\hat{c}_1 = \left(\frac{1 + b_3 \Delta^{\mu-1}}{a_1} \right)^{\frac{1}{2}} = \frac{\delta}{\Delta},$$

$$\hat{c}_2 = \frac{c(\mu-1)}{b} \left(\frac{1 + b_3 \Delta^{\mu-1}}{2b(1+h)} \right)^{\frac{\mu-1}{2}}.$$

Now, we present the estimates for solutions of the systems with zero mean value perturbations. Peculiarity of the approach lies in the fact that the lower bound for the functional on the set S_α (see Lemma 16 and Lemma 19) instead of the set of all piece-wise continuous initial functions is used. The inequality

$$\frac{dv(t, x_t)}{dt} \leq -L_2 v(t, x_t)^{\frac{\gamma+\mu-1}{\gamma}}, \quad t \geq 0,$$

holds along all solutions and does not depend on the lower bound. Next, we take another constant $\widetilde{L} < L_2$ for the comparison equation

$$\frac{du(t)}{dt} = -\widetilde{L} u^{\frac{\gamma+\mu-1}{\gamma}}(t), \ u(0) = u_0 = (\alpha_1 + b_3 \Delta^{\sigma-1} + b_4 \Delta^{\mu-1}) \|\varphi\|_h^\gamma,$$

The solution for the comparison equation is

$$u(t) = u_0 \left[1 + \widetilde{L} \left(\frac{\mu-1}{\gamma} \right) u_0^{\frac{\mu-1}{\gamma}} t \right]^{-\frac{\gamma}{\mu-1}}.$$

Notice that $v(t, \varphi) < u_0$ due to $\|\varphi\|_h < \Delta$ and $\widetilde{L} < L_2$, hence we are able to conclude the following.

Lemma 22. *The strict inequality*

$$v(t, x_t) < u(t), \quad t \geq 0, \tag{4.40}$$

holds.

Proof. Inequality (4.40) is true for $t = 0$. Assume that there exists a point $t^\star > 0$ such that (4.40) holds for $t < t^\star$, and $v(t^\star, x_{t^\star}) = u(t^\star)$. Then,

$$v(t, x_t) - v(t^\star, x_{t^\star}) < u(t) - u(t^\star), \quad t < t^\star,$$
$$\frac{v(t, x_t) - v(t^\star, x_{t^\star})}{t - t^\star} > \frac{u(t) - u(t^\star)}{t - t^\star}, \quad t < t^\star.$$

Taking the limit with respect to $t \to t^\star - 0$, we arrive at

$$\left. \frac{dv(t, x_t)}{dt} \right|_{t=t^\star} \geq \left. \frac{du(t)}{dt} \right|_{t=t^\star}.$$

On the other hand,

$$\left. \frac{dv(t, x_t)}{dt} \right|_{t=t^\star} \leq -L_2 v(t^\star, x_{t^\star})^{\frac{\gamma+\mu-1}{\gamma}} = -L_2 u(t^\star)^{\frac{\gamma+\mu-1}{\gamma}} < -\widetilde{L} u(t^\star)^{\frac{\gamma+\mu-1}{\gamma}} = \left. \frac{du(t)}{dt} \right|_{t=t^\star}.$$

The contradiction proves the lemma. $\qquad\square$

Now, we choose $\widetilde{L}$ in accordance with the following lemma.

Lemma 23. *If*

$$1 + \widetilde{L} h \left(\frac{\mu-1}{\gamma} \right) (\alpha_1 + b_3 \Delta^{\sigma-1} + b_4 \Delta^{\mu-1})^{\frac{\mu-1}{\gamma}} \Delta^{\mu-1} \leq \alpha^{\mu-1}, \tag{4.41}$$

then

$$u(t + \theta) < \alpha^\gamma u(t) \tag{4.42}$$

for all $t \geq 0$ and $\theta \in [-h, 0]$ such that $t + \theta \geq 0$.

Proof. Denote

$$K = \widetilde{L}\left(\frac{\mu - 1}{\gamma}\right) u_0^{\frac{\mu - 1}{\gamma}}.$$

The inequality (4.42) is equivalent to

$$\frac{1 + Kt}{1 + K(t + \theta)} < \alpha^{\mu - 1}.$$

Since $t + \theta \geq 0$, we have

$$\frac{1 + Kt}{1 + K(t + \theta)} = 1 + \frac{(-K\theta)}{1 + K(t + \theta)} \leq 1 + Kh.$$

It is easy to see that (4.41) implies $1 + Kh < \alpha^{\mu - 1}$, and hence implies (4.42). $\square$

The following theorem is the key step to prove the main result. Here, the constant $a_1(\alpha)$ is taken from the lower bound for the functional on the set S_α. However, we state that the final estimate holds for all solutions.

Theorem 13. *If inequality* (4.41) *holds, then the following bound is true*

$$a_1(\alpha)\|x(t, \varphi)\|^\gamma < u(t), \quad t \geq 0.$$

Proof. Notice that $a_1(\alpha) < \alpha_1 + b_3\Delta^{\sigma - 1} + b_4\Delta^{\mu - 1}$. Hence, the required bound holds at $t = 0$. Assume that there exists a point $t^\star > 0$ such that

$$a_1(\alpha)\|x(t^\star, \varphi)\|^\gamma = u(t^\star),$$
$$a_1(\alpha)\|x(t, \varphi)\|^\gamma < u(t), \quad t < t^\star.$$

We verify that $x_{t^\star} \in S_\alpha$. Indeed, if $t^\star + \theta \geq 0$, $\theta \in [-h, 0]$, then

$$a_1(\alpha)\|x(t^\star + \theta, \varphi)\|^\gamma \leq u(t^\star + \theta) < \alpha^\gamma u(t^\star) = \alpha^\gamma a_1(\alpha)\|x(t^\star, \varphi)\|^\gamma.$$

Here, the previous step is due to Lemma 23. Hence, $\|x(t^\star + \theta, \varphi)\| < \alpha\|x(t^\star, \varphi)\|$. If $t^\star + \theta \in [-h, 0)$, then

$$a_1(\alpha)\|x(t^\star + \theta, \varphi)\|^\gamma \leq a_1(\alpha)\|\varphi\|_h^\gamma < (\alpha_1 + b_3\Delta^{\sigma - 1} + b_4\Delta^{\mu - 1})\|\varphi\|_h^\gamma = u(0)$$
$$< \alpha^\gamma u(t^\star) = \alpha^\gamma a_1(\alpha)\|x(t^\star, \varphi)\|^\gamma.$$

It follows again due to Lemma 23. Hence,

$$\|x(t^\star + \theta, \varphi)\| < \alpha\|x(t^\star, \varphi)\|, \quad \theta \in [-h, 0],$$

and we arrive at $x_{t^\star} \in S_\alpha$. This implies

$$v(t^\star, x_{t^\star}) \geq a_1(\alpha)\|x(t^\star, \varphi)\|^\gamma = u(t^\star),$$

which contradicts to Lemma 22. $\square$

The main result with the constant $a_1(\alpha)$ in the lower bound follows from here immediately.

4.3   Illustrative Examples

4.3.1   Example 1

Consider a scalar equation of the form

$$\dot{x}(t) = \kappa_1 x^3(t) + \kappa_2 x^3(t-h), \tag{4.43}$$

where the constants $\kappa_1, \kappa_2 \in \mathbb{R}$, $x \in \mathbb{R}$ and $h \in \mathbb{R}^+$. In this case the homogeneity degree is $\mu = 3$. It can be shown with the help of the Lyapunov function $V(x) = -\frac{w}{2(\kappa_1+\kappa_2)}x^2$, that when $h = 0$ the equation (4.43) is asymptotically stable for $\kappa_1 + \kappa_2 < 0$. For system parameters $(\kappa_1, \kappa_2) = (-1, 0.5)$, $w = 1$ and delay $h = 10$, the constants in the bounds for the system right-hand side and Lyapunov function are $m = \max\{|\kappa_1|, |\kappa_2|\}$, $m_1 = 3|\kappa_1|$, $m_2 = 3|\kappa_2|$, $k_0 = k_1 = -\frac{w}{2(\kappa_1+\kappa_2)}$ and $k_2 = k_3 = -\frac{w}{\kappa_1+\kappa_2}$.

First, we find the estimate of the region of attraction using both approaches. We apply Theorem 6 in Section 2.2 with $\delta = H$ for the Lyapunov-Razumikhin framework, and use inequality (2.39) and tune some parameters for the Lyapunov-Krasovskii one. The parameters and the obtained estimates for the attraction region are shown in Table 4.1, where we can observe that the attraction region described by Δ is less conservative in the Lyapunov-Krasovskii approach than in the Lyapunov-Razumikhin one.

Table 4.1: Constants for the estimates of attraction region of Example 1 for the standard dilation

Lyapunov-Krasovskii									
Δ	H_1	H_2	δ	χ	a_1	β	w_0	w_1	w_2
0.067	0.1012	0.3	0.1011	0.32	0.48	17.7	0.25	0.05	0.07
Lyapunov-Razumikhin									
Δ			H			κ			K
0.0427			0.0443			1			1.001

Next, we turn our attention to the estimates of the solutions. In spite of the fact that the estimate of the region of attraction is found to be less conservative in the Lyapunov-Krasovskii framework, for comparison of the two approaches, we take the same δ in Theorem 7 in Section 2.2 and Theorem 12 in Section 2.3. The constants characterising the estimates are shown in Table 4.2.

Table 4.2: Constants for the estimates of solutions of Example 1 for standard dilation

Lyapunov-Krasovskii									
δ	H_1	H_2	Δ	$\hat{c}_1$	$\hat{c}_2$	χ	w_0	w_1	w_2
0.01	0.015	0.26	0.0099	1.0014	$4.2 \cdot 10^{-5}$	0.015	0.33	0.5	0.017
Lyapunov-Razumikhin									
δ		H	Δ		$\tilde{c}_1$		$\tilde{c}_2$	α	ρ
0.01		0.0443	0.009		1.002		0.95	2	0.94

The estimate of the system response obtained via the Lyapunov-Krasovskii approach is then

$$|x(t,\varphi)| \leq 1.0014|\varphi|_{10} \left[1 + 4.2 \cdot 10^{-5}|\varphi|_{10}^2 t\right]^{-\frac{1}{2}},$$

whereas using the Lyapunov-Razumikhin framework we arrive at

$$|x(t,\varphi)| \leq 1.002|\varphi|_{10} \left[1 + 0.95|\varphi|_{10}^2 t\right]^{-\frac{1}{2}}.$$

For the initial condition $\varphi(\theta) = 0.009$, $\theta \in [-10, 0]$, the system response and the estimates (2.27) and (4.39) are depicted in Fig. 4.1 and Fig. 4.2 as a continuous, dashed and dashed-dot line, respectively.
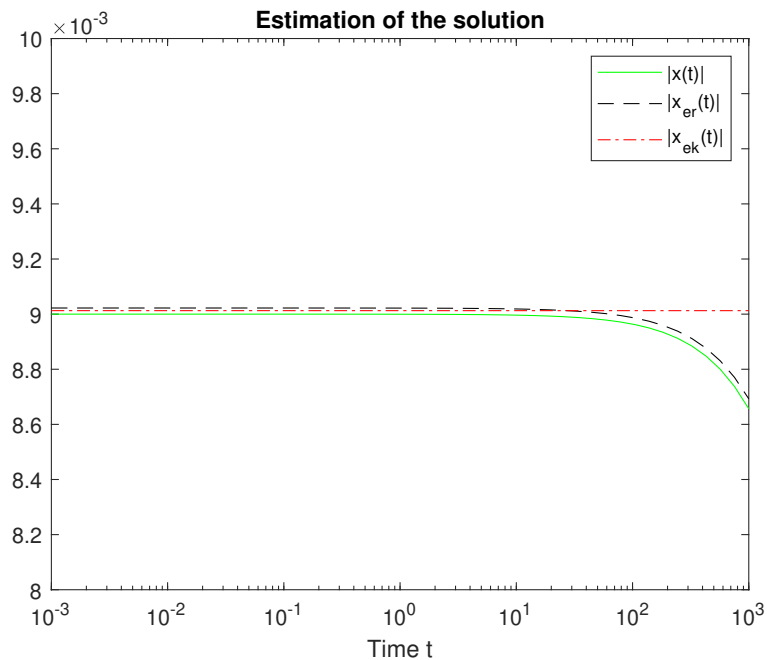

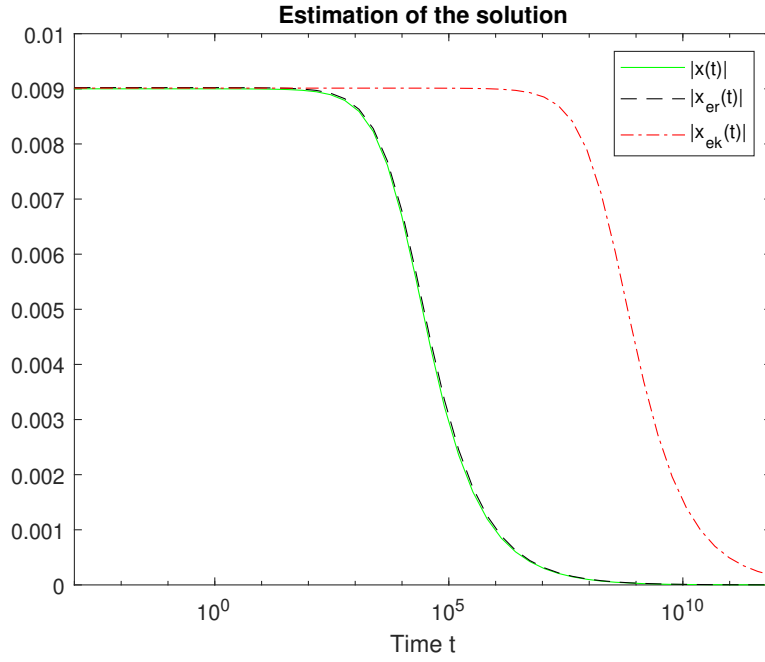Figure 4.1: Estimation of the solution of system (4.43) for small time

Figure 4.2: Estimation of the solution of system (4.43)

As shown in Figure 4.2, the estimate via the Lyapunov-Razumikhin approach, in general, is less conservative. Comparing values $\hat{c}_1$ and A both have the same structure δ/Δ, i.e., they depend on Δ, of this way we conclude that $\hat{c}_1$ is better due to the attraction region for the Lyapunov-Krasovskii is less conservative. Therefore, the estimate is better at the beginning.

Now, consider $\hat{c}_2$ and B :

$$\hat{c}_2 = \frac{c}{bL_1} \left(\frac{\mu - 1}{\gamma}\right) \left(\frac{a_1}{b}\right)^{\frac{\mu-1}{\gamma}} \left(\frac{\delta}{\Delta}\right)^{\mu-1},$$

$$B = \rho \left(\frac{\mu - 1}{\gamma}\right) k_0^{\frac{\mu-1}{\gamma}} \left(\frac{\delta}{\Delta}\right)^{\mu-1}.$$

The practical experiments show that to get a better estimate in the Lyapunov-Razumikhin approach, ρ must be close to $\bar{d}$. With such choice the value B is close to

$$\frac{k_5}{k_1} \left(\frac{\mu - 1}{\gamma}\right) \left(\frac{k_0}{k_1}\right)^{\frac{\mu-1}{\gamma}} \left(\frac{\delta}{\Delta}\right)^{\mu-1}.$$

The aim of these values is analogous to a decay constant in a function. Thus, if $\hat{c}_2$ and B increase, we get a better estimate of the system response. Clearly, the values k_5, α_1, α_0 have a similar meaning for the Lyapunov function as the values c, b, a_1 for the Lyapunov functional. Multiplier L_1 is a source of conservatism in the Lyapunov-Krasovskii approach. In other words, L_1 leads to a slow response for the estimate of the Lyapunov-Krasovskii approach.

4.3.2   Example 2

Consider the system

$$\dot{x}_1(t) = x_2^{\mu}(t), \tag{4.44}$$
$$\dot{x}_2(t) = -x_1^{\mu}(t) - x_2^{\mu}(t - h).$$

where $x_1(t)$, $x_2(t) \in \mathbb{R}$ and $h \in \mathbb{R}^+$. System (4.44) is δ^1-homogeneous of homogeneity degree μ odd. Consider the Lyapunov function introduced in [22], which is δ^1-homogeneous:

$$V(x) = \frac{1}{\mu + 1} \left(x_1^{\mu+1} + x_2^{\mu+1} \right) + \zeta x_1^\mu x_2$$

where ζ is a positive constant. For the proof of asymptotic stability of the delay-free system (4.44), condition

$$\zeta < \min \left\{ \frac{1}{\mu + 1}, \frac{4}{(\mu + 1)^2} \right\}$$

has to be satisfied. Furthermore, its derivative along the trajectories of system (4.44), when $h = 0$, is

$$\dot{V}(x) = -\zeta x_1^{2\mu} - x_2^{2\mu} + \zeta \mu x_1^{\mu-1} x_2^{\mu+1} - \zeta x_1^\mu x_2^\mu,$$

equivalently,

$$\dot{V}(x) = -x_1^{2\mu} f(z), \quad f(z) = z^{2\mu} - \zeta \mu z^{\mu+1} + \zeta z^\mu + \zeta, \quad z = \frac{x_2}{x_1}.$$

Due to the condition on ζ, $f(z)$ is a positive function. Furthermore, $f(z)$ also admits a lower bound of the form

$$f(z) \geq \kappa \left(z^{2\mu} + 1 \right), \quad \kappa = \min \left\{ 1 - \zeta(\mu + 1), \zeta, \frac{\zeta}{1 + \zeta} \left(1 - \frac{\zeta(1 + \mu)^2}{4} \right) \right\}$$

Hence, it holds that

$$\dot{V}(x) \leq -\kappa \left(x_1^{2\mu} + x_2^{2\mu} \right) \leq -\frac{\kappa}{2^{\mu-1}} \|x\|^{2\mu} = -\mathrm{w}\|x\|^{2\mu}.$$

The constants involved, characterising the bounds of the right-hand side of system (4.44) and the Lyapunov function, are $\alpha_0 = (1/2)^{\frac{\mu-1}{2}} \left(\frac{1 - \zeta(\mu+1)}{\mu+1} \right)$, $\alpha_1 = \left(\frac{1 + \zeta(\mu+1)}{\mu+1} \right)$, $m = \sqrt{2}$, $\eta = \mu$, $\beta = 2((1 + \zeta\mu)^2 + 1)^{0.5}$ and $\psi = 2(\mu + \mu^2\zeta)$. In order to validate the results achieved in Example 1, we compare the estimate of the solutions obtained via the Lyapunov-Krasovskii and Lyapunov-Razumikhin approaches, for system (4.44). For system parameters $\mu = 5$, $\zeta = 0.0001$, $\mathrm{w} = 6.2 \cdot 10^{-6}$ and $h = 10$, the constants characterising the estimates of the solutions are shown in Table 4.3.

Table 4.3: Constants for the estimates of solutions of Example 2 for the standard dilation

Lyapunov-Krasovskii									
δ	H_1	H_2	Δ	$\hat{c}_1$	$\hat{c}_2$	χ	w_0	w_1	w_2
0.001	0.0062	0.0023	$7.934 \cdot 10^{-4}$	1.2603	$2.4 \cdot 10^{-8}$	0.26	$4 \cdot 10^{-8}$	$5 \cdot 10^{-6}$	$5 \cdot 10^{-8}$
Lyapunov-Razumikhin									
δ	H		Δ		$\tilde{c}_1$	$\tilde{c}_2$		α	ρ
0.001	0.048		$7.935 \cdot 10^{-4}$		1.2602	$2.49 \cdot 10^{-5}$		2	$1.23 \cdot 10^{-4}$

Thus, the system response of system (4.44) admits an estimate of the form

$$\|x(t, \varphi)\| \leq 1.2603 \|\varphi\|_{10} \left[1 + 2.4 \cdot 10^{-8} \|\varphi\|_{10}^4 t \right]^{-\frac{1}{4}}$$

via Lyapunov-Krasovskii approach and

$$\|x(t, \varphi)\| \le 1.2602 \|\varphi\|_{10} \left[1 + 2.49 \cdot 10^{-5} \|\varphi\|_{10}^4 t \right]^{-\frac{1}{4}}$$

via Lyapunov-Razumikhin. For the initial condition $\varphi(\theta) = [5.1 \cdot 10^{-4}, \ 5.1 \cdot 10^{-4}]$, $\theta \in [-10, 0]$, the system response and the estimates via Lyapunov-Razumukhin and Lyapunov-Krasovskii approaches are depicted in Fig. 4.3 as a continuous, dashed and dashed-dot line, respectively. Observe that for the multi-variable system (4.44), the observations presented in the previous example remain valid.
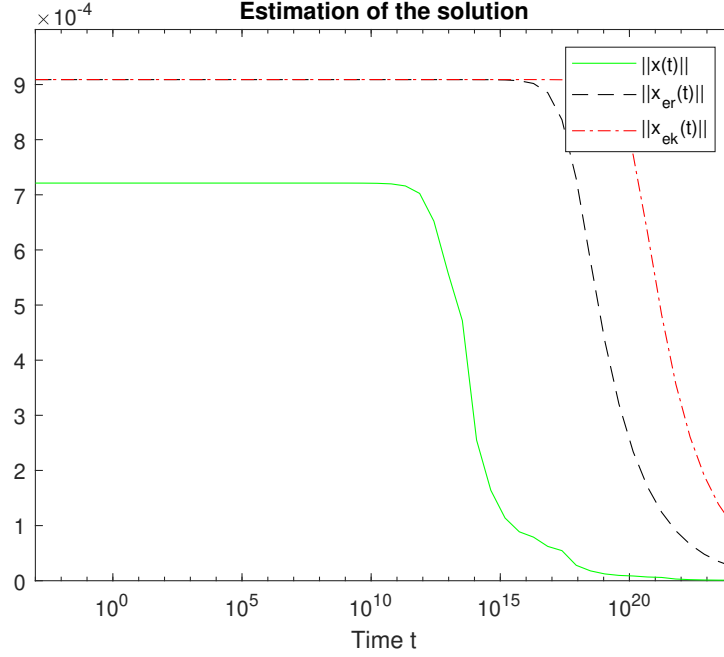

Figure 4.3: Estimation of the solution of system (4.44)

Now, consider system (4.44) with perturbations of zero mean value:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2^\mu(t) \\ -x_1^\mu(t) - x_2^\mu(t - h) \end{bmatrix} + B(t)Q(x(t), x(t - h)) \tag{4.45}$$

where

$$B(t)Q(x(t), x(t - h)) = \begin{bmatrix} cos(t) + sin(\sqrt{2}t) & 0 \\ 0 & cos(t) + cos(\sqrt{2}t) \end{bmatrix} \begin{bmatrix} x_1^\sigma(t - h) \\ x_2^\sigma(t) \end{bmatrix}$$

Suppose that $\sigma = \mu = 5$. Then, the constants characterising the perturbed terms are $p_0 = 2$, $p_1 = 1.98$, $p_2 = 5$, $\alpha_3 = \psi(p_0 + p_1) + \beta(p_2 + p_3)$, $\alpha_4 = \beta(p_0 + p_1)$ and equation (4.25) admits a bound

$$\|L(t, \varepsilon)\| \le \frac{\alpha(\varepsilon)}{\varepsilon} = \frac{3\sqrt{(\varepsilon + 1/4\varepsilon^2)} - \varepsilon/2}{\varepsilon},$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. The constants characterising the estimates of the solutions of the perturbed system (4.45) for $\zeta = 1.0001$ are shown in Table 4.4.

Table 4.4: Constants for the estimates of solutions of perturbed Example 2 for the standard dilation

δ	H_1	H_2	H_3	H_4	Δ	$\hat{c}_1$	$\hat{c}_2$	ε	$\widetilde{L}$
$1 \cdot 10^{-4}$	0.0033	$1.4 \cdot 10^{-4}$	$1 \cdot 10^{-4}$	0.0044	$7 \cdot 10^{-5}$	1.41	$6.7 \cdot 10^{-8}$	$1 \cdot 10^{-14}$	$3.3 \cdot 10^{-7}$

Then, the solutions of the perturbed system (4.45) admits an estimate of the form

$$\|x(t, \varphi)\| \leq 1.41 \|\varphi\|_{10} \left[1 + 6.7 \cdot 10^{-8} \|\varphi\|_{10}^4 t\right]^{-\frac{1}{4}} .$$

For the initial condition $\varphi(\theta) = [3.9 \cdot 10^{-5}, \ 3.9 \cdot 10^{-5}]$, $\theta \in [-10, 0]$, the system response and the estimates of the perturbed system (4.45) are depicted in Fig. 4.4 as a continuous and dashed line, respectively. Even though system (4.44) is submitted to perturbations, the new estimate manages quite well to keep itself close to the system response.
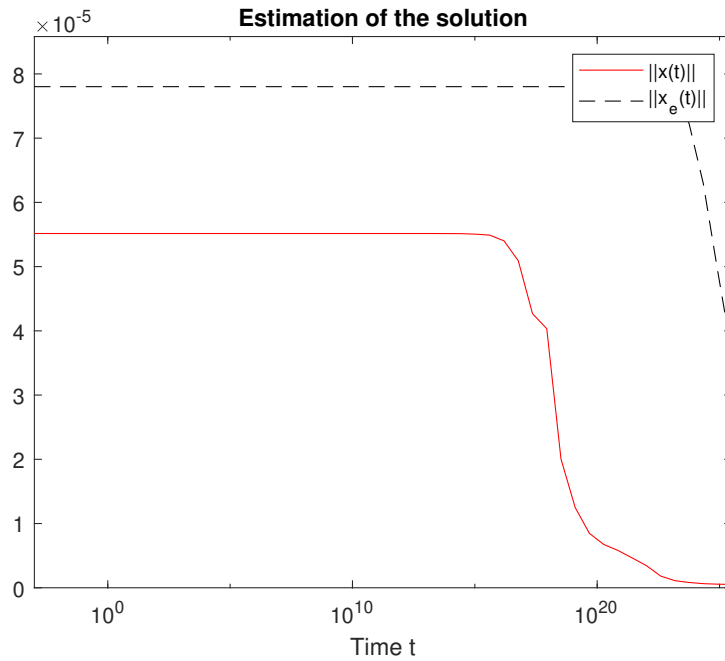

Figure 4.4: Estimation of the solution of system (4.45)

Chapter 5

Conclusions and Future Work

5.1 Conclusions

The research in this document focused on the analysis of homogeneous time-delay systems, which are a class of non-linear system, via the Lyapunov-Krasovskii approach. With the help of a novel functional introduced in recent research, we could extend some result founded via Lyapunov-Razimikhin framework for standard dilation, to the case for weighted dilation as well as the robustness analysis. The main contribution in this work was the estimate of the solutions for homogeneous time-delay systems with weighted dilation. It was showed that this estimate holds for systems submitted to perturbations.

The estimate of the solutions is based on functional bounds. For that, first and second sections in Chapter 3 were devoted to finding the upper and lower bound estimates of the functional and a negative bound of its time derivative along of the solutions of homogeneous time-delay system and this same submitted to perturbations. As an extra result, we estimate the attraction region for homogeneous time-delay systems with weighted dilation.

The robustness analysis for time-delay systems with standard dilation was studied in Chapter 4. It is presented the neighbourhood for asymptotic stability for perturbed systems and given some remarks to some modifications on this perturbations. It was carried out through a modification to the Lyapunov function for the free-delay system, which is used in the functional implemented throughout this work. Also, it obtained the estimate of the solution for the multi-variable case and the scalar case.

All the theoretical results were validated using some examples in each section, and some cases were compared with results found in the literature.

5.2 Future Work

The most evident is the necessity of a practical validation of the results, beyond the simulation. On the other hand, It is necessary to address other topics as nonlinear observers and output feedback control laws for asymptotically stabilising for homogeneous time-delay systems.

Appendix A

Proof of Theorems

In this appendix, we present the proofs of Theorems 6 and 7 from [10, 11] and [15] with a slight modification. The main feature, distinguishing the proofs from those in [10, 11, 15], is that all the constants are computed explicitly. We begin with an auxiliary lemma, which can be found in [11] in an implicit form.

Lemma 24. *If* $\|\varphi\|_h < \Delta$, *then*

$$\|x(t, \varphi)\| \le K(\|\varphi\|_h + mh\|\varphi\|_h^\mu), \quad t \in [0, h].$$

Proof. Denote $S(\varphi) = \|\varphi\|_h + mh\|\varphi\|_h^\mu$,

$$u(t) = S(\varphi) + m \int_0^t \|x(s)\|^\mu ds,$$

and observe that $\|x(t)\| \le u(t)$, $t \in [0, h]$. Further,

$$\dot{u}(t) = m\|x(t)\|^\mu \le mu^\mu(t), \quad u(0) = S(\varphi) < \frac{\kappa\delta}{K}.$$

Integrating the last inequality, we obtain

$$u(t) \le \frac{S(\varphi)}{\left(1 - (\mu - 1)mS^{\mu-1}(\varphi)t\right)^{\frac{1}{\mu-1}}}, \tag{A.1}$$

if $t < 1/\left((\mu - 1)mS^{\mu-1}(\varphi)\right)$. Now, verify that

$$\frac{1}{(\mu - 1)mS^{\mu-1}(\varphi)} > h.$$

Hence, bound (A.1) holds for $t \in [0, h]$. Next,

$$\|x(t)\| \le u(t) \le \frac{S(\varphi)}{\left(1 - (\mu - 1)mh\left(\frac{\kappa\delta}{K}\right)^{\mu-1}\right)^{\frac{1}{\mu-1}}} = KS(\varphi), \quad t \in [0, h],$$

and the lemma follows. □

Proof of Theorem 6 [11]. The Razumikhin condition (2.24) implies

$$\|x(\xi)\| < \left(\frac{\alpha\alpha_1}{\alpha_0}\right)^{\frac{1}{\gamma}}\|x(t)\|, \quad \xi \in [t-2h, t], \quad t \geq h.$$

Differentiating $V(x(t))$ along the solutions of system (2.19) satisfying the Razumikhin condition and applying the mean value theorem, one gets

$$\frac{dV(x(t))}{dt} = \left(\frac{\partial V(x)}{\partial x}\right)^T f(x(t), x(t)) - h\left(\frac{\partial V(x)}{\partial x}\right)^T$$
$$\times \int_0^1 \frac{\partial f(x(t), x(t-\theta h))}{\partial x(t-\theta h)} f(x(t-\theta h), x(t-\theta h - h))d\theta$$
$$\leq -\mathrm{w}\|x(t)\|^{\gamma+\mu-1} + k_4\|x(t)\|^{\gamma+2\mu-2},$$

where

$$k_4 = 2hm\eta\beta\left(\frac{\alpha\alpha_1}{\alpha_0}\right)^{\frac{\mu}{\gamma}}\left(1 + \left(\frac{\alpha k_1}{k_0}\right)^{\frac{\mu-1}{\gamma}}\right).$$

Taking $H = (\mathrm{w}/k_4)^{\frac{1}{\mu-1}}$ and an arbitrary $\delta \in (0, H)$, we arrive at (2.25) with $k_5 = \mathrm{w} - k_4\delta^{\mu-1}$. Consider an arbitrary solution $x(t)$ of system (2.19) with initial condition $\|\varphi\|_h < \Delta$. It follows from equation (2.26) that $\Delta < \kappa\delta$, hence

$$V(\varphi(\theta)) \leq \alpha_1\Delta^\gamma < \alpha_0\delta^\gamma, \quad \theta \in [-h, 0].$$

Lemma 24 implies $V(x(t)) \leq \alpha_1\|x(t)\|^\gamma < \alpha_0\delta^\gamma$, $t \in [0, h]$. Formula (2.25) guarantees $V(x(t)) < \alpha_0\delta^\gamma$ for any $t \geq -h$, which implies that $\|x(t)\| < \delta$ for any $t \geq -h$. The result now follows from (2.25).

Proof of Theorem 7. [10], [15] Equations (2.14) and (2.25) imply that along the solutions of system (2.19) satisfying (2.24) and $\|x_t\|_h \leq \delta$ the following differential inequality holds

$$\frac{dV(x(t))}{dt} \leq -\overline{d}V^{\frac{\gamma+\mu-1}{\gamma}}(x(t)), \quad \overline{d} = k_5\alpha_1^{-\frac{\gamma+\mu-1}{\gamma}}. \tag{A.2}$$

Lemma 24 provides an initial condition for this inequality:

$$V(x(h)) \leq \alpha_1 K^\gamma(\|\varphi\|_h + mh\|\varphi\|_h^\mu)^\gamma.$$

Introduce a parameter ρ, which satisfies the following three conditions:

$$0 < \rho < \overline{d},$$
$$1 + 2h\rho\frac{\mu-1}{\gamma}\alpha_0^{\frac{\mu-1}{\gamma}}\delta^{\mu-1} < \alpha^{\frac{\mu-1}{\gamma}}, \tag{A.3}$$
$$1 - \rho\frac{\mu-1}{\gamma}\alpha_1^{\frac{\mu-1}{\gamma}}K^{\mu-1}h\Delta^{\mu-1} > 0. \tag{A.4}$$

Then, the differential equation

$$\dot{z}(t) = -\rho z^{\frac{\gamma+\mu-1}{\gamma}}(t) \tag{A.5}$$

with the initial condition $z(h) = z_0$, where

$$z_0 = \alpha_1 K^\gamma(\|\varphi\|_h + mh\|\varphi\|_h^\mu)^\gamma, \quad z_0 < \alpha_0\delta^\gamma,$$

can be treated as a comparison equation for (A.2), if $t \geq h$. A solution to this initial-value problem is

$$z(t) = \frac{z_0}{\left(1 + \rho\dfrac{\mu - 1}{\gamma} z_0^{\frac{\mu-1}{\gamma}} (t - h)\right)^{\frac{\gamma}{\mu-1}}}.$$

Let us show that the solution $z(t)$ satisfies the Razumikhin condition (2.24). Consider $t \geq h$ and ξ such that $-2h \leq \xi \leq 0$ and $t + \xi \geq h$, then condition (2.24) requires that $z(t+\xi) < \alpha z(t)$, equivalently,

$$g(t) = \frac{1 + k(t - h)}{1 + k(t + \xi - h)} < \alpha^{\frac{\mu-1}{\gamma}}, \tag{A.6}$$

where $k = \dfrac{\rho(\mu - 1)}{\gamma} z_0^{\frac{\mu-1}{\gamma}}$. Note that

$$0 \leq g(t) = 1 - \frac{k\xi}{1 + k(t + \xi - h)} \leq 1 - k\xi \leq 1 + 2kh.$$

Since $k < \rho\dfrac{\mu - 1}{\gamma} \alpha_0^{\frac{\mu-1}{\gamma}} \delta^{\mu-1}$, we have that condition (A.3) implies (A.6). Hence, function $z(t)$ satisfies the Razumikhin condition for all $t \geq h$.

The general idea in the Razumikhin framework is that to obtain a contradiction in the proof of comparison lemma, the solution which satisfies the Razumikhin condition is taken. Hence, the fact that the solution of comparison equation also satisfies the Razumikhin condition guarantees that $V(x(t)) \leq z(t)$, $t \geq h$, for all solutions $x(t)$ with $\|\varphi\|_h < \Delta$, and not only for the solutions satisfying the Razumikhin condition. This implies that the solutions with $\|\varphi\|_h < \Delta$ admit the following bound:

$$\|x(t, \varphi)\| \leq \frac{A\|\varphi\|_h}{(1 + B\|\varphi\|_h^{\mu-1}(t - h))^{\frac{1}{\mu-1}}}, \quad t \geq h,$$

where

$$A = \frac{K(1 + mh\Delta^{\mu-1})}{\kappa}, \quad B = \rho\frac{\mu - 1}{\gamma} \alpha_1^{\frac{\mu-1}{\gamma}} K^{\mu-1}.$$

Further, since $1 - Bh\Delta^{\mu-1} > 0$ due to condition (A.4), we have

$$\|x(t, \varphi)\| \leq \frac{A\|\varphi\|_h}{(1 - B\|\varphi\|_h^{\mu-1}h + B\|\varphi\|_h^{\mu-1}t)^{\frac{1}{\mu-1}}}$$

$$\leq \frac{A_1\|\varphi\|_h}{(1 + B_1\|\varphi\|_h^{\mu-1}t)^{\frac{1}{\mu-1}}}, \quad t \geq h,$$

where

$$A_1 = \frac{A}{(1 - Bh\Delta^{\mu-1})^{\frac{1}{\mu-1}}}, \quad B_1 = \frac{B}{1 - Bh\Delta^{\mu-1}}.$$

It remains to obtain the estimate for $t \in [0, h]$. According to Lemma 24,

$$\|x(t, \varphi)\| \leq A_2\|\varphi\|_h, \quad t \in [0, h],$$

where $A_2 = K(1 + mh\Delta^{\mu-1})$. Notice that

$$\frac{(1 + B_1\Delta^{\mu-1}h)^{\frac{1}{\mu-1}}A_2}{A_1} = \kappa \leq 1.$$

Hence,

$$\|x(t,\varphi)\| \leq \frac{A_1\|\varphi\|_h}{(1+B_1\Delta^{\mu-1}h)^{\frac{1}{\mu-1}}} \leq \frac{A_1\|\varphi\|_h}{(1+B_1\|\varphi\|_h^{\mu-1}t)^{\frac{1}{\mu-1}}}$$

for $t \in [0,h]$. Hence, the required bound (2.27) holds with $\tilde{c}_1 = A_1$, $\tilde{c}_2 = B_1$ for all $t \geq 0$, and the proof is complete.

Remark 9. *In Theorem 6, the value H can be taken instead of δ in equation (2.26) and in K, as in [11], whereas in Theorem 7 it is important that the solutions satisfy $\|x(t,\varphi)\| < \delta$ for any $t \geq 0$.*

Appendix B

Inequalities

In this appendix, we present the proof of some inequalities that are instrumental in the proofs of our results.

Lemma A.1 *Let $a, b, \nu \in \mathbb{R}$, $a, b \geq 0$ and $\nu \geq 1$, then the following inequality is satisfied*

$$(a + b)^\nu \leq 2^{\nu-1}(a^\nu + b^\nu)$$

Proof. The mapping $f : x \to x^\nu$, for $x \geq 0$, is convex since $f''(x) \geq 0$, indeed

$$f''(x) = \nu(\nu - 1)x^{\nu-2} \geq 0.$$

Now, using the convexity property for $a, b > 0$, we get

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b), \ \lambda \in [0, 1].$$

For $\lambda = 1/2$

$$\left(\frac{a}{2} + \frac{b}{2}\right)^\nu \leq \frac{a^\nu}{2} + \frac{b^\nu}{2}.$$

Equivalently,

$$(a + b)^\nu \leq 2^{\nu-1}(a^\nu + b^\nu). \tag{B.1}$$

$\square$

Next, we prove Lemma 1 introduced in Chapter 3.

Proof of Lemma 12. Using (B.1) and taking $\nu = \frac{u}{q}$, $a = \|x(t)\|_{r,p}^q$, $b = \int_{-h}^0 \|x(t + \theta)\|_{r,p}^q d\theta$, we get

$$\left(\|x(t)\|_{r,p}^q + \int_{-h}^0 \|x(t + \theta)\|_{r,p}^q d\theta\right)^{\frac{u}{q}} \leq 2^{\frac{u}{q}-1}\left(\|x(t)\|_{r,p}^u + \left(\int_{-h}^0 \|x(t + \theta)\|_{r,p}^q d\theta\right)^{\frac{u}{q}}\right). \tag{B.2}$$

To conclude the proof, we use Holder's inequality,

$$\left|\int_{-h}^0 f(x_t)g(x_t)d\theta\right| \leq \left(\int_{-h}^0 |f(x_t)|^r d\theta\right)^{\frac{1}{r}} \left(\int_{-h}^0 |g(x_t)|^s d\theta\right)^{\frac{1}{s}}, \tag{B.3}$$

for $r, s \geq 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$. For $s = \frac{u}{q}$, $r = \frac{u}{u-q}$, $f(x_t) = 1$ and $g(x_t) = \|x(t + \theta)\|_{r,p}^q$, we have

$$\left(\int_{-h}^0 \|x(t + \theta)\|_{r,p}^q d\theta\right)^{\frac{u}{q}} \leq h^{\frac{u-q}{q}} \int_{-h}^0 \|x(t + \theta)\|_{r,p}^u d\theta.$$

Then, it follows from (B.2) that

$$\left(\|x(t)\|_{r,p}^q + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^q d\theta \right)^{\frac{u}{q}} \leq$$

$$2^{\frac{u}{q}-1} \left(\|x(t)\|_{r,p}^u + h^{\frac{u-q}{q}} \int_{-h}^0 \|x(t+\theta)\|_{r,p}^u d\theta \right) \leq$$

$$2^{\frac{u}{q}-1} \max\left\{1, h^{\frac{u-q}{q}}\right\} \left(\|x(t)\|_{r,p}^u + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^u d\theta \right)$$

and the lemma is proved.

Proof of Lemma 13. The proof is carried out by mathematical induction. It is easy to verify (3.35) for $k = 2$ taking into account that

$$\left(\int_{-h}^0 \|x(t+\theta)\|_{r,p}^2 d\theta \right)^2 \leq h \int_{-h}^0 \|x(t+\theta)\|_{r,p}^4 d\theta.$$

Assume that (3.35) holds for a $u \geq 2$, then

$$\left(\|x(t)\|_{r,p}^2 + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^2 d\theta \right)^{u+1} \leq 2^{u-2}(1+h)^{u-1}$$

$$\times \left(\|x(t)\|_{r,p}^{2u} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{2u} d\theta \right) \left(\|x(t)\|_{r,p}^2 + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^2 d\theta \right).$$

Transforming the right-hand side of this inequality with the help of $d^g e^z \leq d^{g+z} + e^{g+z}$, we get

$$\left(\|x(t)\|_{r,p}^2 + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^2 d\theta \right)^{u+1} \leq 2^{u-1}(1+h)^u \left(\|x(t)\|_{r,p}^{2(u+1)} + \int_{-h}^0 \|x(t+\theta)\|_{r,p}^{2(u+1)} d\theta \right).$$

The lemma is proved.

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