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"Nuevo diseño de backstepping con retardos artificiales para sistemas con retardos puntuales"

T E S I S

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JAVIER EDUARDO PEREYRA ZAMUDIO

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Notations and symbols

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}^n$	Space of n-vectors with entries in $\mathbb{R}$.
$\mathbb{R}^{n\times n}$	Space of matrices of dimension $n \times n$ with entries in $\mathbb{R}$.
I_n	Identity matrix of dimension n .
$a_{i,j}$	Matrix element located in line i and column j.
M^T	Transpose of the matrix M.
M^{-1}	Inverse of the matrix M.
$M > 0$	Matrix M: symmetric and positive definite.
$\lambda_{\text{mín}}(M)$	Minimum eigenvalue of the matrix M.
$\lambda_{\text{máx}}(M)$	Maximum eigenvalue of the matrix M.
$\lvert \cdot \rvert$	The Euclidean norm in $\mathbb{R}^n$, and the induced norm of matrices
C^n	Set of continuous functions and n times differentiable.
$C([-T,0],\mathbb{R}^n)$	Space of $\mathbb{R}^n$-valued continuous functions on $[-T,0]$, $T > 0$.
$PC([-T,0],\mathbb{R}^n)$	Space of $\mathbb{R}^n$-valued piecewise continuous functions on $[-T,0]$, $T > 0$.
C_{in}	Set of continuous functions which we call the set of initial functions.
$\Vert\varphi\Vert_h$	Uniform norm, $\varphi = \sup\limits_{-h\leq\theta\leq 0} \Vert\varphi(\theta)\Vert$.
$f^{(i)}$	Order derivative i of the function f.

x_t	Restriction of $x(t)$, $x_t : \theta \to x(t+\theta)$, $\theta \in [-h, 0]$, $h > 0$.
$\omega(\cdot)$	The Heaviside step function or the unit step function.
$\sigma_L(s)$	The symmetric saturation function, $\sigma_L(s) = min\{-L, max\{L, s\}\}$, $L > 0$

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Resumen

La estabilidad y el control de los sistemas con retardos, tanto en el estado como en la entrada, ha sido un área activa de investigación en el campo del control automático debido a que estos retardos se presentan en muchos sistemas de control derivados de una aplicación de ingeniería. Cuando estos retardos están presentes y son demasiado grandes para ser ignorados, se deben diseñar nuevas leyes de control para el modelo infinito dimensional correspondiente. El backstepping es una técnica para diseñar controles estabilizadores para una amplia clase de sistemas que incluye sistemas con no linealidades e incertidumbres. A pesar de ser una técnica tan efectiva, existen limitaciones y aspectos que pueden mejorarse. Se han obtenido nuevos avances en el enfoque de backstepping a través de una variante fundamentalmente nueva del backstepping basada en la introducción de retardos artificiales en el control o extensiones dinámicas utilizadas en el mismo. El análisis de estabilidad de estas estrategias de backstepping, aplicadas al control de clases de sistemas con retardos es el tema principal de esta tesis. También mostramos algunas de las ventajas de tales controles, en particular, las leyes de control por realimentación que usan retardos artificiales son acotadas y están dadas por fórmulas mucho más simples que las proporcionadas por el enfoque clásico de backstepping.

Abstract

Stability and control of systems whit delay, both in the state and at the input, has been an active area of research in the field of automatic control because these delays are present in many control systems arising from an engineering application. When these delays are present and are too large for being neglected, new control laws should be designed for the corresponding infinite-dimensional model. Backsteping is a technique for designing stabilizing controls for a broad class of systems that includes systems with nonlinearities and uncertainties. Despite being such an effective technique, there are both limitations and aspects that can be improved. New advances on the backstepping approach have been obtained via a fundamentally new variant of backstepping relaying on the introduction of artificial delays in the control or dynamic extensions used in it. The stability analysis of these backstepping strategies applied to the control of classes of TDS is the main topic of this thesis. We also show some of the advantages of such controls, in particular, the feedback control laws with artificial delays are bounded and are given by simpler formulas than those provided by the classical backstepping approach.

Chapter 1

Introduction

In this chapter we give a general introduction to time-delay systems, the backstepping stabilizability approach, and we present the objectives and the strategies of our work

1.1 Time-delay systems

Control systems frequently present limitations in collecting, processing and transporting information which often generates delays. When these delays are so long that they cannot be neglected, a delay system model should be used and controllers should be designed based on this infinite-dimensional model. Systems with time-delay in the state, control input, or measurements have been an important field of research in communication, physics, medicine, biology. Although in some cases the delays can have a stabilizing effect it is well known that in most cases time delays are a source of instability or bad performance. Stability analysis of TDS is significant from the applied and theoretical viewpoints.
The notion of stability of control systems, with or without delays, can be seen in two different ways. The first is the *input/output approach* where a system is stable if bounded input signals produce bounded output signals. The second and in which we will work more is *Lyapunov stability* which describe continuity properties of the solutions of the system with respect to a initial condition. If the initial condition is perturbed, then, for stability, the resulting perturbed solution is required to stay close to the solution that start in the initial condition, for all time. The direct method of Lyapunov establishes that if a positive definite functional is such that its derivative along the trajectories of the system is definite negative, then the origin of the system is asymptotically stable equilibrium point. Since the classical Lyapunov approach does not work for time-delay systems, some modifications have been proposed, one of these modification is know as the Lyapunov-Krasovskii method due to N. N. Krasovskii [8] who proposed to replace classical Lyapunov functions that depend on the instant state of the system by a functional that depend on the true state of the delay system. Motivated by this result V. Kharitonov [4] gave a new approach which consists in selecting a desired time derivative and then computing a functional whose derivative coincides with the selected one.

Another stability concept which is used throughout this work is *input to state stability* (ISS), introduced by Sontag [20]. In this approach there is a standard notion of stability, namely Lyapunov asymptotic stability of the unforced system which means that the input to state stability implies that the origin of the unforced system is globally asymptotically stable.

1.2 New backstepping approach

Backstepping is a technique for designing stabilizing controls for systems with nonlinearities and uncertainties, the key idea of backstepping is to follow a step-by-step algorithm that starts with a subsystem which is stabilizable with a know feedback control law for a know Lyapunov function and then to add to its input an integrator. For the augmented system a new stabilizing feedback control law is explicitly designed and shown to be stabilizing for a new Lyapunov function, in this way, it extends the controlled stability of this subsystem to the larger system. This simple idea has been used by many authors like J. Tsinias, V.Utkin, A. G. Loukianov, C. Praly, J. M. Coron, M. Krstic, I. Kanellakopoulos, [7], [9], [21]. But at the best of our know, was in the works of P. V. Kokotovic where it is referred by the name of backstepping [6], [7].
In this work the backstepping tool is used to study a stabilizing feedback for coupled delay systems with a triangular structure following a fundamentally new variant of backstepping. The previous works [14], [16], [11] are based in the pioneering recent work [12] that use these new backstepping results, can help overcome the obstacles encountered when using standard backstepping. The idea is to combine prediction based techniques with distributed terms and a new version of backstepping technique which uses an artificial delay in the control or dynamic extensions that are used by the control. It is reminiscent of the one introduced in the contributions [11], [13]. The control laws designed by this approach have the advantage that the feedback are given by formulas much simpler than those provided by the classical backsteppping approach. For linear time-invariant systems, our approach is constructive, as the design is based on a Lyapunov-Krasovskii functional of complete type, while for nonlinear systems the demonstrations are based on assumptions of *bounded input bounded state* (BIBS) and *converging input converging state* (CICS)

The controllers with artificial delays have the form of integral equations which contains integrals of the state, in the practical implementation, the integral terms needs to be calculated on-line, a possibility is to approximate the distributed delay by a sum of pointwise delays using a numerical approximation. This replacement changes the controller type, this new controllers can lead to an instability of the closed-loop system [22] [2]. To overcome this difficulty additional filters that can be used, this filters can be interpreted as a dynamic extension i.e. an extension of the state vector of the system by adding the control variables to the state variables. The stability of the system with the extension implies the stability of the original closed-loop system so the controllers can be designed directly for extended dynamic systems.

1.3 Objectives

The general objective of this thesis is to address the controller design in the proposed framework based on the introduction of artificial delays of time delay systems. In particular.

- We perform the stabilization of linear time delay-systems in feedback form using a backstepping approach based on functionals of complete type.
- We perform a robustness analysis of this scheme.
- We design control laws for nonlinear time delay-systems in feedback form using a backstepping approach based on BIBS and CICS assumptions.

1.4 Structure of the thesis

This work is organized as follow:

- The first chapter gives an introduction to time delay systems, and presents the general organization of the thesis.
- The second chapter is a summary of the theoretical background of the system stabilization, definitions and classical results, it also introduces the operators that will be used in the following chapters.
- The third chapter proposes the implementation of a novel version of backstepping, for linear time invariant systems, which is based on the introduction of artificial delays and whose stability analysis makes use of Lyapunnov functionals of complete type. The robustness of the scheme is also analysed.
- The fourth chapter presents a design of feedback which uses an artificial delay. Stabilization of a nonlinear system is achieved.
- The last chapter is devoted to concluding remarks and perspectives.

Chapter 2

General Theory

In this chapter we remind concepts and general results for time delay systems. In particular, we recall the main ideas of backstepping, prediction of future states, the construction of functional with prescribed derivative for time delay systems and we define some operators that will be used in the next chapters.

2.1 Classical definitions

Let us consider a system

$$\dot{x}(t) = \mathcal{A}(t, x(t), x(t-\tau), w(t)) \tag{2.1}$$

with a finite delay $\tau \geq 0$ $x \in \mathbb{R}^n$ and $w = (w_1, ..., w_m) \in \mathbb{R}^m$.

Definition 1. *[3] A function $\alpha : [0, +\infty) \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. A function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is continuous and for all $a \geq 0$, $\beta(\cdot, a)$ is of class $\mathcal{K}$ and for all $b > 0$, $\beta(b, \cdot)$ is decreasing and $\lim\limits_{c \to +\infty} \beta(b, c) = 0$.*

Definition 2. *The system (2.1) is bounded-input bounded-state (BIBS) with input w if there exist α, γ in $\mathcal{K}$ such that a solution x of (2.1) with $\phi \in C_{\text{in}}$ as initial condition and $w \in \mathcal{L}_\infty$ satisfies*

$$|x(t)| \leq \alpha(|\phi|) + \gamma\left(\sup_{m \in [0,t]} |w(m)|\right) \tag{2.2}$$

for all $t \geq 0$.

Definition 3. *Let us consider the system (2.1). We say that this system is converging-input converging-state (CICS) with input w if it is forward complete and when*

$$\lim_{t \to +\infty} |w(t)| = 0 \tag{2.3}$$

then

$$\lim_{t\to+\infty} |x(t)| = 0 \tag{2.4}$$

Definition 4. *[3]*

The system (2.1)is Input-to-State-Stable (ISS) if there exist a function β of class $\mathcal{KL}$ and a function γ of $\mathcal{K}$ such that any solution x of it satisfies:

$$|x(t)| \le \beta\left(\sup_{r\in[s-\tau,s]} |x(r)|, t-s\right) + \gamma\left(\sup_{r\in[0,t]} |w(r)|\right) \tag{2.5}$$

for all $t \ge s \ge 0$.

Remark. *The inequality (2.5) guarantees that for any bounded input $w(t)$, the state $x(t)$ remains bounded. Furthermore, as t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{r\in[0,t]} |w(r)|$. Also if $w(t)$ converges to zero as $t \to \infty$ so does $x(t)$. In other words when a system is ISS, then it is BIBS and CICS.*

Remark. *Since, with $w(t) = 0$, (2.5) reduces to*

$$|x(t)| \le \beta\left(\sup_{r\in[s-\tau,s]} |x(r)|, t-s\right)$$

input-to-state stability implies that the origin of the unforced system (2.1) is globally uniformly asymptotically stable. In the linear case the converse holds too [18].

Definition 5. *[4] System (2.1) is said to be exponentially stable if there exist $\gamma \ge 1$ and $\sigma > 0$ such that any solution $x(t)$ with initial condition ϕ of the system satisfies the inequality*

$$||x(t)|| \le \gamma e^{-\sigma t}||\phi||_\tau \quad t \ge 0 \tag{2.6}$$

Remark. *[1] The exponential stability of a linear time invariant system is equivalent to the asymptotic stability of the system.*

The following lemma provides a bound for the trajectories of the studied systems which will be useful to check conditions of ISS type.

Lemma 1. *[15] Let $T > 0$ be a constant. Let $w : [-T, \infty) \to [0, \infty)$ be a piecewise continuous locally bounded function and $d : [0, \infty) \to [0, \infty)$ be piecewise continuous.*

Assume that there exists a constant $\rho \in (0, 1)$ such that

$$w(t) \le \rho |w|_{[t-T,t]} + d(t)$$

holds for all $t \geq 0$. Then the inequality

$$w(t) \leq |w|_{[-T,0]} e^{\frac{ln(\rho)}{T} t} + \frac{1}{1-\rho} |d|_{[0,t]}$$

holds for all $t \geq 0$.

2.2 Classical Backstepping

We start reminding the classical backstepping for a system with one integrator [3] (Chapter 14, pp 589–603).

$$\begin{aligned} \dot{x} &= f(x) + g(x)y \\ \dot{y} &= u \end{aligned}$$

where $[x^T, y]^T \in \mathbb{R}^{n+1}$ is the state and $u \in \mathbb{R}$ is the input. The functions $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ are smooth in a domain $D \subset \mathbb{R}^n$ that contains $x = 0$ and $f(0) = 0$. Suppose that the component x can be stabilized by a state feedback control law $y = h(x)$, with $h(0) = 0$; that is the origin of

$$\dot{x} = f(x) + g(x)h(x)$$

is asymptotically stable. Suppose further that we know a positive definite Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)h(x)] \leq -W(x),$$

holds for all $x \in D$, where $W(x)$ is positive definite. Adding and subtracting $g(x)h(x)$ we obtain

$$\begin{aligned} \dot{x} &= [f(x) + g(x)h(x)] + g(x)[y - h(x)] \\ \dot{y} &= u \end{aligned}$$

Now a changing of variable

$$z = y - h(x)$$

results in the system

$$\begin{aligned} \dot{x} &= [f(x) + g(x)h(x)] + g(x)z \\ \dot{z} &= u - \dot{h} \end{aligned}$$

and $\dot{h} = \frac{\partial h}{\partial x}[f(x) + g(x)y]$. Taking $v = u - \dot{h}$ reduces the system to the cascade connection

$$\dot{x} = [f(x) + g(x)h(x)] + g(x)z$$
$$\dot{z} = v$$

The first component has an asymptotically stable origin when the input is zero. This feature will be exploited in the design of v to stabilize the overall system. Using

$$V_c(x, y) = V(x) + \frac{1}{2}z^2$$

as a Lyapunov function candidate, we obtain

$$\dot{V}_c \leq -W(x) + \frac{\partial V}{\partial x}g(x)z + zv$$

Choosing

$$v = -\frac{\partial V}{\partial x}g(x) - kz \quad k > 0$$

yields

$$\dot{V}_c \leq -W(x) - kz^2$$

which implies that the origin of the closed-loop system is asymptotically stale. Substituting for v, z and $\dot{h}$, we obtain the state feedback control law

$$u = \frac{\partial h(x)}{\partial x}[f(x) + g(x)y] - \frac{\partial V}{\partial x}g(x) - k[y - h(x)]]$$

If all the Assumptions hold globally and $V(x)$ is radially unbounded, we can conclude that the origin is globally asymptotically stable.

2.3 Predictor

First we introduce a predictor of future values of the state variable $\xi(t)$ of system (2.7) which will be used when we establish the stability of the system in closed-loop with the feedback we propose.

We consider the linear time-delay systems of the form

$$\dot{\xi}(t) = A_0\xi(t) + A_1\xi(t-h) + Bu(t-\tau) \tag{2.7}$$

with $\xi \in \mathbb{R}^n$, B, $A_i \in \mathbb{R}^{n\times n}$ $_{i=0,1}$, real matrices, $u \in \mathbb{R}^n$ is the input, $\tau > h \geq 0$

Lemma 2. *The solution of system (2.7) satisfies:*

$$\xi(t+\tau-h) = e^{A_0(\tau-h)}\xi(t) + \int_{t-\tau+h}^{t} e^{A_0(t-l)}[A_1\xi(l+\tau-2h) + Bu(l-h)]\ dl \qquad (2.8)$$

for all $t \geq 0$ and

$$\xi(t+\tau) = e^{A_0(\tau-h)}\xi(t+h) + \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1e^{A_0(\tau-h)}\xi(m)\ dm$$
$$+ \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1 \int_{m-\tau+h}^{m} e^{A_0(m-l)}A_1\xi(l+\tau-2h)\ dldm$$
$$+ \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1 \int_{m-\tau+h}^{m} e^{A_0(m-l)}Bu(l-h)\ dldm + \int_{t-\tau+h}^{t} e^{A_0(t-m)}Bu(m)\ dm \qquad (2.9)$$

for all $t \geq 0$

Proof. By integrating the ξ equation of the system (2.7), we obtain almost directly (2.8). Using (2.8), we obtain directly:

$$\xi(t+\tau) = e^{A_0(\tau-h)}\xi(t+h) + \int_{t-\tau+2h}^{t+h} e^{A_0(t+h-l)}[A_1\xi(l+\tau-2h) + Bu(l-h)]\ dl$$

It follows that

$$\xi(t+\tau) = e^{A_0(\tau-h)}\xi(t+h) + \int_{t-\tau+h}^{t} e^{A_0(t-m)}[A_1\xi(m+\tau-h) + Bu(m)]\ dm$$

Again by (2.8) we obtain directly:

$$\xi(t+2h) = e^{A_0(\tau-h)}\xi(t+h) + \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1e^{A_0(\tau-h)}\xi(m)\ dm + \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1\times$$
$$\int_{m-\tau+h}^{m} e^{A_0(m-l)}A_1\xi(l+\tau-2h)\ dldm + \int_{t-\tau+h}^{t} e^{A_0(t-m)}A_1 \int_{m-\tau+h}^{m} e^{A_0(m-l)}Bu(l-h)\ dldm$$
$$+ \int_{t-\tau+h}^{t} e^{A_0(t-m)}Bu(m)\ dm$$

□

2.4 Lyapunov-Krasovskii functionals of complete type

Now we recall a property of the complete type Lyapunov-Krasovskii functionals which will be used to perform a stability analysis in the next part.

Definition 6. *[4] We say that the matrix $U(\tau)$ is a Lyapunov matrix of system unforced (2.7) associated with a symmetric positive definite matrix W if it satisfies the following properties:*

1. *Dynamic property*

$$\frac{dU(\tau)}{d\tau} = U(\tau)A_0 + U(\tau - h)A_1, \qquad \tau \geq 0 \tag{2.10}$$

2. *Symmetry property*

$$U(-\tau) = U(\tau)^T \tag{2.11}$$

3. *Algebraic property*

$$-W = U(0)A_0 + A_0^T U(0) + U(-h)A_1 + A_1^T U(h) \tag{2.12}$$

The following theorem gives the form of the functional V with prescribed derivative along the trajectories of the unforced system (2.7).

Theorem 3. *[4] Given three symmetric positive definite matrices W_j, $(j = 0, 1, 2)$ let*

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^{0} \varphi^T(\theta)W_2\varphi(\theta)\ d\theta$$

If the unforced system (2.7) is exponential stable, the Lyapunov matrix function $U(\tau)$ associated with the matrix $W = W_0 + W_1 + hW_2$ exists, and the functional

$$V(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\ d\theta$$

$$+ \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1 - \theta_2)A_1\varphi(\theta_2)\right]\ d\theta_2 d\theta_1 + \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\ d\theta \tag{2.13}$$

has time derivative along the solutions of the unforced system (2.7) given by

$$\frac{d}{dt}V(\xi_t) = -w(\xi_t), \quad t \geq 0$$

2.5 Operators: definitions and lemmas

In this section, we introduce operators and establish results that will be instrumental when we establish some of the results of this work.

Let $k > 0$, $h > 0$ and $\tau > 0$ be real numbers. Let

$$\mathfrak{j} = \frac{ke^{kh}}{e^{kh} - 1} \tag{2.14}$$

Notice for later use that

$$\frac{k}{e^{kh} - 1} \int_{t-h}^{t} e^{k(s+h-t)} ds = 1 \tag{2.15}$$

for all $t \in \mathbb{R}$.

Remark. *Throughout the present work, we can let k be equal to 1, or any other positive constant. However, we keep k as a tuning parameter helping to improve the performances of the control laws we will propose.*

Let

$$\dot{x}(t) = \mathcal{R}(t, x_t) \tag{2.16}$$

be a forward complete system with $x \in \mathbb{R}^q$ and $\tau \geq 0$.

2.5.1 Operators

Let $\mathcal{U} : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ be a function such that there are a constant $L_{\mathcal{U}} > 0$ and a function $\mathcal{B}$ of class $\mathcal{K}$ such that

$$|\mathcal{U}(a, b_1) - \mathcal{U}(a, b_2)| \leq L_{\mathcal{U}} |b_1 - b_2| \tag{2.17}$$

for all $(a, b_1, b_2) \in \mathbb{R}^{1+2q}$ and

$$|\mathcal{U}(a_1, b) - \mathcal{U}(a_2, b)| \leq |a_1 - a_2| \mathcal{B}(|b|) \tag{2.18}$$

for all $(a_1, a_2, b) \in \mathbb{R}^{2+q}$.

i) Let $\Gamma_{\mathcal{U},i}$ with $i \in \mathbb{N}$ denote the operators such that along the trajectories of (2.16),

$$\Gamma_{\mathcal{U},0}(t, x_t) = \mathcal{U}(t - \tau, x(t - \tau)) \tag{2.19}$$

and, for $j \geq 1$,

$$\Gamma_{\mathcal{U},j}(t, x_t) = \mathfrak{j} \int_{t-h}^{t} e^{k(s-t)} \Gamma_{\mathcal{U},j-1}(s, x_s) ds \tag{2.20}$$

where $\mathfrak{j}$ is the constant defined in (2.14).

ii) For all $j \in \mathbb{N}$, $j > 0$, $i \in \{0, ..., j\}$, we let $\Omega_{\mathcal{U},j,i}$ denote the operators such that along the trajectories of (2.16),

$$\Omega_{\mathcal{U},j,i}(t, x_t) = \Gamma^{(i)}_{\mathcal{U},j}(t, x_t) \tag{2.21}$$

Example: for $i = 1$ and $j \geq 1$,

$$\Omega_{\mathcal{U},j,1}(t, x_t) = -k\Gamma_{\mathcal{U},j}(t, x_t) + \frac{ke^{kh}}{e^{kh} - 1}\Gamma_{\mathcal{U},j-1}(t, x_t) - \frac{k}{e^{kh} - 1}\Gamma_{\mathcal{U},j-1}(t - h, x_{t-h}) \tag{2.22}$$

iii) Let $\zeta : C_{\text{in}} \to \mathbb{R}^q$ be the operator defined by:

$$\zeta(\phi) = \frac{k}{e^{kh} - 1}\left[e^{kh}\phi(0) - \phi(-h)\right] \tag{2.23}$$

for all $\phi \in C_{\text{in}}$.

2.5.2 Estimates for the operators

The following lemma shows that the smaller the constant h is selected, the closer $\Gamma_{\mathcal{U},j}(t, x_t)$ is to $\mathcal{U}(t - \tau, x(t - \tau))$.

Lemma 4. *Along the trajectories of the system (2.16), for all $j \in \mathbb{N}$, the inequalities*

$$|\Gamma_{\mathcal{U},j}(t, x_t) - \mathcal{U}(t - \tau, x(t - \tau))| \leq L_{\mathcal{U}} \int_{t-\tau-jh}^{t-\tau} |\dot{x}(s)|ds + jh\mathcal{B}(|x(t - \tau)|) \tag{2.24}$$

hold for all $t \geq \tau + jh$.

Proof. Let us introduce the simplifying notation $\Delta(t, x_t) = |\Gamma_{\mathcal{U},j}(t, x_t) - \mathcal{U}(t-\tau, x(t-\tau))|$. The inequality (2.24) is satisfied for $j = 0$. Now, let us consider a positive integer j. From the definition of $\Gamma_{\mathcal{U},j}$ and (2.15), we deduce that

$$\begin{aligned}
&\Gamma_{\mathcal{U},j}(t, x_t) - \mathcal{U}(t - \tau, x(t - \tau)) = \\
&\mathfrak{j}\int_{t-h}^{t} e^{k(s_1-t)} ....\mathfrak{j}\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}[\mathcal{U}(s_j - \tau, x(s_j - \tau)) - \mathcal{U}(t - \tau, x(t - \tau))]ds_j...ds_1 \\
&= \mathfrak{j}\int_{t-h}^{t} e^{k(s_1-t)} ....\mathfrak{j}\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}[\mathcal{U}(s_j - \tau, x(s_j - \tau)) - \mathcal{U}(s_j - \tau, x(t - \tau))]ds_j...ds_1 \\
&+\mathfrak{j}\int_{t-h}^{t} e^{k(s_1-t)} ....\mathfrak{j}\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}[\mathcal{U}(s_j - \tau, x(t - \tau)) - \mathcal{U}(t - \tau, x(t - \tau))]ds_j...ds_1
\end{aligned} \tag{2.25}$$

It follows that

$$\begin{aligned}
&\Delta(t, x_t) \leq \\
&\mathfrak{j}\int_{t-h}^{t} e^{k(s_1-t)} ....\mathfrak{j}\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}|\mathcal{U}(s_j - \tau, x(s_j - \tau)) - \mathcal{U}(s_j - \tau, x(t - \tau))|ds_j...ds_1 \\
&+\mathfrak{j}\int_{t-h}^{t} e^{k(s_1-t)} ....\mathfrak{j}\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}|\mathcal{U}(s_j - \tau, x(t - \tau)) - \mathcal{U}(t - \tau, x(t - \tau))|ds_j...ds_1
\end{aligned}$$

From (2.17) and (2.18), we deduce that

$$\begin{aligned}&\Delta(t,x_t)\leq\\&\jmath\int_{t-h}^{t}e^{k(s_1-t)}\jmath\int_{s_1-h}^{s_1}e^{k(s_2-s_1)}....\jmath\int_{s_{j-1}-h}^{s_{j-1}}e^{k(s_j-s_{j-1})}L_{\mathcal{U}}|x(s_j-\tau)-x(t-\tau)|ds_j...ds_1\\&+\jmath\int_{t-h}^{t}e^{k(s_1-t)}\jmath\int_{s_1-h}^{s_1}e^{k(s_2-s_1)}....\jmath\int_{s_{j-1}-h}^{s_{j-1}}e^{k(s_j-s_{j-1})}|s_j-t|\mathcal{B}(|x(t-\tau)|)ds_j...ds_1\end{aligned} \tag{2.26}$$

Now, since for $s_j \geq \tau$ and $t \geq s_j$, the inequality

$$|x(s_j-\tau)-x(t-\tau)| \;\leq\; \int_{s_j-\tau}^{t-\tau}|\dot{x}(r)|dr \tag{2.27}$$

is satisfied, we obtain

$$\begin{aligned}\Delta(t,x_t)\;&\leq\;L_{\mathcal{U}}\jmath\int_{t-h}^{t}e^{k(s_1-t)}....\int_{s_{j-1}-h}^{s_{j-1}}e^{k(s_j-s_{j-1})}\int_{s_j-\tau}^{t-\tau}|\dot{x}(r)|dr ds_j...ds_1\\&\quad+\jmath\int_{t-h}^{t}e^{k(s_1-t)}....\jmath\int_{s_{j-1}-h}^{s_{j-1}}e^{k(s_j-s_{j-1})}|t+jh-t|ds_j...ds_1\mathcal{B}(|x(t-\tau))|)\\&\leq\;L_{\mathcal{U}}\int_{t-\tau-jh}^{t-\tau}|\dot{x}(r)|dr+jh\mathcal{B}(|x(t-\tau)|)\end{aligned}$$

for all $t \geq \tau + jh$.

□

Now we determine upper bounds for the operators $\Gamma_{\mathcal{U},j}$ and $\Omega_{\mathcal{U},j,i}$.

Lemma 5. *Let us consider the system (2.16). For all $j \in \mathbb{N}$, $j > 0$,*

$$|\Gamma_{\mathcal{U},j}(t,x_t)|\leq\jmath\int_{t-\tau-jh}^{t-\tau}|\mathcal{U}(s,x(s))|ds \tag{2.28}$$

for all $t \geq \tau + jh$.

Proof. We directly deduce from the definition of $\Gamma_{\mathcal{U},j}$ that:

$$|\Gamma_{\mathcal{U},j}(t,x_t)|\leq\jmath\int_{t-h}^{t}e^{k(s_1-t)}\jmath\int_{s_1-h}^{s_1}e^{k(s_2-s_1)}....\jmath\int_{s_{j-1}-h}^{s_{j-1}}e^{k(s_j-s_{j-1})}|\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j...ds_1 \tag{2.29}$$

for all $t \geq \tau + jh$. As $s_{j-1} \in [t-(j-1)h, t]$ and $s_j \in [s_{j-1}-h, s_{j-1}]$, the inequalities

$$\int_{s_{j-1}-h}^{s_{j-1}} e^{k(s_j-s_{j-1})}|\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j \leq \int_{t-jh}^{t} e^{k(s_j-s_{j-1})}|\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j$$
$$\leq \int_{t-jh}^{t} |\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j \tag{2.30}$$

are satisfied. This inequalities and (2.29) give:

$$\begin{aligned}|\Gamma_{\mathcal{U},j}(t,x_t)| &\leq \mathrm{j}\int_{t-h}^{t} e^{k(s_1-t)}\mathrm{j}\int_{s_1-h}^{s_1} e^{k(s_2-s_1)}....\mathrm{j}ds_{j-1}...ds_1\int_{t-jh}^{t}|\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j \\ &= \mathrm{j}\int_{t-jh}^{t}|\mathcal{U}(s_j-\tau,x(s_j-\tau))|ds_j\end{aligned}$$

This allows us to conclude. □

Lemma 6. *For all $j\in\mathbb{N}$, $j\geq 1$, $i\in\{0,...,j-1\}$, the inequalities*

$$|\Omega_{\mathcal{U},j,i}(t,x_t)| \leq 2^i\mathrm{j}^{i+1}\int_{t-\tau-jh}^{t-\tau}|\mathcal{U}(s,x(s))|ds \tag{2.31}$$

are satisfied for all $t\geq\tau+jh$.

Proof. Let us establish the inequality (2.31) by induction.
Induction Assumption. *For all $j\in\{1,...,p\}$, $p\in\mathbb{N}$, $p>1$, for all $i\in\{0,...,j-1\}$, the inequalities*

$$|\Omega_{\mathcal{U},j,i}(t,x_t)| \leq 2^i\mathrm{j}^{i+1}\int_{t-\tau-jh}^{t-\tau}|\mathcal{U}(s,x(s))|ds \tag{2.32}$$

are satisfied for all $t\geq\tau+jh$.

Step 1. Since

$$|\Omega_{\mathcal{U},1,0}(t,x_t)| = |\Gamma_{\mathcal{U},1}(t,x_t)| \tag{2.33}$$

we deduce from Lemma 5 that

$$|\Omega_{\mathcal{U},1,0}(t,x_t)| \leq \mathrm{j}\int_{t-\tau-h}^{t-\tau}|\mathcal{U}(s,x(s))|ds \tag{2.34}$$

for all $t\geq\tau+h$. Therefore the induction Assumption is satisfied at the step 1.

Step p**.** Let us assume that the induction Assumption is satisfied at the step p. Let us prove, by induction again that is satisfied at the step $p+1$ To do this, we proceed again by induction. Let us introduce this new induction Assumption:

Second induction Assumption. For all $a \in \{0, ..., i\}$ with $i \in \{0, ..., p-1\}$ the inequality

$$|\Omega_{\mathcal{U},p+1,a}(t, x_t)| \leq 2^a \mathfrak{j}^{a+1} \int_{t-\tau-(p+1)h}^{t-\tau} |\mathcal{U}(s, x(s))| ds \tag{2.35}$$

is satisfied.

Step 0**.** We have:

$$\begin{array}{rcl} |\Omega_{\mathcal{U},p+1,0}(t, x_t)| & = & |\Gamma_{\mathcal{U},p+1}(t, x_t)| \\ & \leq & \mathfrak{j} \int_{t-\tau-(p+1)h}^{t-\tau} |\mathcal{U}(s, x(s))| ds \end{array} \tag{2.36}$$

where the last inequality is a consequence of Lemma 5. Thus the second induction Assumption is satisfied at the step 0.

Step i**.** Let us assume that the induction Assumption is satisfied at the step $i < p$. One can easily prove that

$$\Omega_{\mathcal{U},j,i+1}(t, x_t) = k \left(-\Omega_{\mathcal{U},j,i}(t, x_t) + \frac{e^{kh}}{e^{kh} - 1} \Omega_{\mathcal{U},j-1,i}(t, x_t) - \frac{1}{e^{kh} - 1} \Omega_{\mathcal{U},j-1,i}(t - h, x_{t-h}) \right) \tag{2.37}$$

Consequently, using the two induction Assumptions, we obtain:

$$\begin{array}{rcl} |\Omega_{\mathcal{U},p+1,i+1}(t, x_t)| & \leq & k \left(|\Omega_{\mathcal{U},p+1,i}(t, x_t)| + \frac{e^{kh}}{e^{kh}-1} |\Omega_{\mathcal{U},p,i}(t, x_t)| + \frac{1}{e^{kh}-1} |\Omega_{\mathcal{U},p,i}(t - h, x_{t-h})| \right) \\ & \leq & k \left(1 + \frac{e^{kh}}{e^{kh}-1} + \frac{1}{e^{kh}-1} \right) 2^i \mathfrak{j}^{i+1} \int_{t-\tau-(p+1)h}^{t-\tau} |\mathcal{U}(s, x(s))| ds \\ & = & \frac{2ke^{kh}}{e^{kh}-1} 2^i \mathfrak{j}^{i+1} \int_{t-\tau-(p+1)h}^{t-\tau} |\mathcal{U}(s, x(s))| ds \end{array} \tag{2.38}$$

Thus the second induction Assumption is satisfied at the step $i + 1$. We deduce that the first induction Assumption is satisfied at the step $p + 1$.

□

2.5.3 Dynamic extensions to replace the operators

In this section, we show that the operators $\Gamma_{\mathcal{U},i}$ are equal along the solutions of the system (2.16), after a finite time interval, to a functional of the solutions of a dynamic extension in which are present pointwise delays only. We will use later this fact can be used to circumvent the inconvenient of using feedbacks with distributed delays.

Let us start with a well-known result:

Lemma 7. *Let*

$$\dot{w}(t) = -kw(t) + \alpha(t) \tag{2.39}$$

where $w \in \mathbb{R}$, $k > 0$ *and* $\alpha : [0, +\infty) \to \mathbb{R}$ *is a piecewise continuous function. Then for any* $T > 0$,

$$w(t) - e^{-kT} w(t-T) = \int_{t-T}^{t} e^{k(s-t)} \alpha(s) ds \tag{2.40}$$

for all $t \geq T$.

Next, let us consider the function $\mathcal{U}$ and let us introduce dynamic extensions:

$$\dot{w}_1(t) = -k w_1(t) + \mathcal{U}(t-\tau, x(t-\tau)) \tag{2.41}$$

with $w_1 \in \mathbb{R}$ and, for all $j \in \mathbb{N}$, $j \geq 2$

$$\dot{w}_j(t) = -k w_j(t) + \zeta(w_{j-1,t}) \tag{2.42}$$

with $w_j \in \mathbb{R}$, ζ defined in (2.23). We have:

Lemma 8. *Let* $m \in \mathbb{N}$, $m > 0$. *For all* $t \geq mh$, *the equality*

$$\zeta(w_{m,t}) = \Gamma_{\mathcal{U},m}(t, x_t) \tag{2.43}$$

is satisfied.

Proof. By applying Lemma 7 to (2.41), we obtain

$$\frac{k e^{kh}}{e^{kh} - 1} \left[w_1(t) - e^{-kh} w_1(t-h) \right] = \mathfrak{j} \int_{t-h}^{t} e^{k(s-t)} \mathcal{U}(s-\tau, x(s-\tau)) ds \tag{2.44}$$

Since

$$\zeta(w_{1,t}) = \mathfrak{j} \left[w_1(t) - e^{-kh} w_1(t-h) \right] \tag{2.45}$$

the equality

$$\zeta(w_{1,t}) = \Gamma_{\mathcal{U},1}(x_t) \tag{2.46}$$

holds for all $t \geq h$.

Now, we proceed by induction.

Induction Assumption. *The equality*

$$\zeta(w_{m,t}) = \Gamma_{\mathcal{U},m}(t, x_t) \tag{2.47}$$

is satisfied for all $t \geq mh$.

Step 1. We have proved that the induction Assumption is satisfied at the step 1.

Step m**.** Let us assume that the induction Assumption is satisfied for all $l \in \{1, ..., m\}$, $m \in \mathbb{N}$.

By applying Lemma 7 to (2.42) with $j = m+1$, we obtain

$$w_{m+1}(t) - e^{-kh} w_{m+1}(t-h) = \int_{t-h}^{t} e^{k(s-t)} \zeta(w_{m,s}) ds \tag{2.48}$$

for all $t \geq h$. It follows that

$$\zeta(w_{m+1,t}) = \mathfrak{j} \int_{t-h}^{t} e^{k(s-t)} \zeta(w_{m,s}) ds \tag{2.49}$$

for all $t \geq h$. From the Induction Assumption, we deduce that

$$\zeta(w_{m+1,t}) = \mathfrak{j} \int_{t-h}^{t} e^{k(s-t)} \Gamma_{\mathcal{U},m}(x_s) ds = \Gamma_{\mathcal{U},m+1}(x_t) \tag{2.50}$$

for all $t \geq (m+1)h$. □

2.6 Conclusions

In this chapter, we briefly describe the technique of classical backstepping and we have introduced important stability definitions. We have defined operators and we have introduced tools that will be useful in the following chapters.

Chapter 3

Backstepping based on artificial delays and functionals of complete type

In this chapter we present a first approach for the control of systems with delay in strict feedback form. It is based on a backstepping strategy using an artificial delay combined with the use of complete Lyapunov functional. A robustness analysis with respect to systems matrices and systems delay is carried out.

3.1 Problem Statement

We consider a coupled linear time-delay system of the form

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & A_0\xi(t) + A_1\xi(t-h) + B\eta(t-\tau) \\ \dot{\eta}(t) & = & M\eta(t) + k_1(t,\xi_t,\eta_t) + u(t) \end{array} \right. \tag{3.1}$$

with $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}\ _{i=0,1}$, $B, M \in \mathbb{R}^{n\times n}$ real matrices, $u \in \mathbb{R}^n$ is the input, $\tau > h \geq 0$ and k_1 is a continuous function.

We consider the following Assumption.

Assumption A1. *There exist two matrices K_0 and K_1 such that the origin of the system*

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) \tag{3.2}$$

with $H_0 = A_0 + BK_0$ and $H_1 = A_1 + BK_1$, is globally exponentially stable.

Problem 1. *Under Assumption A1, design a stabilizing control law for system (3.1).*

Let us introduce the notation

$$\eta_s(t,\xi_t,\eta_t) = K_0\xi(t+\tau) + K_1\xi(t+\tau-h) \tag{3.3}$$

for all $t \geq \tau$.

Theorem 9. *Let system (3.1) satisfy the Assumption A1. There is a positive real scalar* $\bar{r} > 0$ *such that the system (1) in closed loop with the feedback*

$$u(t, \xi_t, \eta_t) = N\eta(t) - \tfrac{M+N}{r}\int_{t-r}^{t} \eta_s(l, \xi_l, \eta_l)\ dl + \tfrac{\eta_s(t,\xi_t,\eta_t)-\eta_s(t-r,\xi_{t-r},\eta_{t-r})}{r} - k_1(t, \xi_t, \eta_t) \tag{3.4}$$

where N *is a matrix such that* $M + N$ *is Hurwitz, is globally exponentially stable for all* $r \in (0, \bar{r}]$.

Remark. *Even when the classical backstepping applies, it may be useful to apply the a approach which uses artificial delays because the feedback with artificial delays is given by formulas simpler than those provident by the classical backstepping approach.*

Proof. The proof starts by defining a transformation

$$\bar{\eta}(t) = \eta(t) - \frac{1}{r}\int_{t-r}^{t} \eta_s(l, \xi_l, \eta_l)\ dl \tag{3.5}$$

Then simple calculations give

$$\begin{cases} \dot{\xi}(t) &= A_0\xi(t) + A_1\xi(t-h) + B\frac{1}{r}\int_{t-r-\tau}^{t-\tau} \eta_s(l, \xi_l, \eta_l)dl + B\bar{\eta}(t-\tau) \\ \dot{\bar{\eta}}(t) &= M(\bar{\eta}(t) + \frac{1}{r}\int_{t-r}^{t} \eta_s(l, \xi_l, \eta_l)\ dl) + u(t) + \frac{\eta_s(t-r,\xi_{t-r},\eta_{t-r})-\eta_s(t,\xi_t,\eta_t)}{r} + k_1(t, \xi_t, \eta_t) \end{cases}$$

Now observe that the control law (3.4) can be rewritten as:

$$u(t, \xi_t, \eta_t) = N\bar{\eta}(t) - \frac{M}{r}\int_{t-r}^{t} \eta_s(l, \xi_l, \eta_l)\ dl + \frac{\eta_s(t, \xi_t, \eta_t) - \eta_s(t-r, \xi_{t-r}, \eta_{t-r})}{r} - k_1(t, \xi_t, \eta_t) \tag{3.6}$$

Hence the closed loop system reduces to:

$$\begin{cases} \dot{\xi}(t) &= A_0\xi(t) + A_1\xi(t-h) + B\frac{1}{r}\int_{t-r-\tau}^{t-\tau} \eta_s(l, \xi_l, \eta_l)dl + B\bar{\eta}(t-\tau) \\ \dot{\bar{\eta}}(t) &= (M+N)\bar{\eta}(t) \end{cases} \tag{3.7}$$

Since the $\bar{\eta}$-subsystem in (3.7) is exponentially stable, we try to extend its stability to the ξ-subsystem

$$\dot{\xi}(t) = A_0\xi(t) + A_1\xi(t-h) + \frac{B}{r}\int_{t-r-\tau}^{t-\tau} \eta_s(l,\xi_l,\eta_l)\ dl + \epsilon(t) \tag{3.8}$$

with $\epsilon(t) = B\bar{\eta}(t-\tau)$. We observe that

$$\begin{aligned} \dot{\xi}(t) &= A_0\xi(t) + A_1\xi(t-h) + B\eta_s(t-\tau,\xi_{t-\tau},\eta_{t-\tau}) \\ &+\tfrac{B}{r}\textstyle\int_{t-r-\tau}^{t-\tau}[\eta_s(l,\xi_l,\eta_l) - \eta_s(t-\tau,\xi_{t-\tau},\eta_{t-\tau})]\ dl + \epsilon(t) \end{aligned} \tag{3.9}$$

Thus we have

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) + \frac{B}{r}\int_{t-r-\tau}^{t-\tau}[\eta_s(l,\xi_l,\eta_l) - \eta_s(t-\tau,\xi_{t-\tau},\eta_{t-\tau})]\ dl + \epsilon(t)$$

for all $t \geq \tau$. We deduce that

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) + \frac{B}{r}\int_{t-r-\tau}^{t-\tau}[K_0(\xi(l+\tau)-\xi(t)) + K_1(\xi(l+\tau-h)-\xi(t-h))]\ dl + \epsilon(t)$$

for all $t \geq \tau$. As an immediate consequence

$$\dot{\xi}(t) = H_0\xi(t) + H_1\xi(t-h) + \zeta(t,\xi_t) \tag{3.10}$$

where

$$\zeta(t,\xi_t) = \frac{B}{r}\int_{t-r-\tau}^{t-\tau}\left[K_0\int_{t}^{l+\tau}\dot{\xi}(s)ds + K_1\int_{t-h}^{l+\tau-h}\dot{\xi}(s)ds\right] dl + \epsilon(t)$$

In view of the Assumption A1 and the results of the previous chapter, the Lyapunov matrix $U(\theta)$, $\theta \in [-h,0]$ of system (3.2) associated with $W = W_0 + W_1 + hW_2$, where W_i are symmetric positive definite matrices for $i = 0,1,2$, exist then we consider the positive defined functional

$$\begin{aligned} V(\xi_t) &= \xi^T(t)U(0)\xi(t) + 2\xi^T(t)\textstyle\int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta \\ &+ \textstyle\int_{-h}^{0}\xi^T(t+\theta_1)H_1^T\int_{-h}^{0}U(\theta_1-\theta_2)H_1\xi(t+\theta_2)\ d\theta_2 d\theta_1 \\ &+ \textstyle\int_{-h}^{0}\xi^T(t+\theta)[W_1 + (\theta+h)W_2]\xi(t+\theta)\ d\theta \end{aligned} \tag{3.11}$$

Following the same procedure as in [19] the derivative of $V(\xi_t)$ along the trajectories of system (3.10) is

$$\begin{aligned} \dot{V}(\xi_t) &= -\xi^T(t)W_0\xi(t) - \xi^T(t-h)W_1\xi(t-h) - \textstyle\int_{-h}^{0}\xi^T(t+\theta)W_2\xi(t+\theta)\ d\theta \\ &+2\zeta^T(t,\xi_t)\left[U(0)\xi(t) + \textstyle\int_{-h}^{0}U(-h-\theta)H_1\xi(t+\theta)\ d\theta\right] \end{aligned} \tag{3.12}$$

Let

$$\gamma(t,\xi_t) = U(0)\xi(t) + \int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta$$

Then

$$\dot{V}(\xi_t) = -\xi^T(t)W_0\xi(t) - \xi^T(t-h)W_1\xi(t-h) - \int_{-h}^{0} \xi^T(t+\theta)W_2\xi(t+\theta)\ d\theta + 2\zeta^T(t,\xi_t)\gamma(t,\xi_t)$$

Using the inequality $2a^Tb \leq a^TMa + b^TM^{-1}b$ which holds for an arbitrary positive definite matrix M and $a, b \in \mathbb{R}^n$, we can arrive at

$$\begin{array}{rcl} \dot{V}(\xi_t) & \leq & -\lambda_{min}(W_0)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - \lambda_{min}(W_2)\int_{-h}^{0}|\xi(t+\theta)|^2\ d\theta \\ & & +\frac{1}{\delta}|\zeta(t,\xi_t)|^2 + \delta|\gamma(t,\xi_t)|^2 \end{array} \quad (3.13)$$

for a positive real scalar $\delta > 0$. Let us define

$$\alpha_1 = \sup_{\theta\in[-h,0]} |U(-h-\theta)H_1| \quad (3.14)$$

$$\alpha_0 = |U(0)| \quad (3.15)$$

Then

$$\begin{array}{rcl} |\gamma(t,\xi_t)|^2 & = & \left|U(0)\xi(t) + \int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta\right|^2 \\ & \leq & 2\left|U(0)\xi(t)\right|^2 + 2\left|\int_{-h}^{0} U(-h-\theta)H_1\xi(t+\theta)\ d\theta\right|^2 \\ & \leq & 2\alpha_0^2|\xi(t)|^2 + 2\int_{-h}^{0}|U(-h-\theta)H_1\xi(t+\theta)|^2\ d\theta \\ & \leq & 2\alpha_0^2|\xi(t)|^2 + 2\alpha_1^2\int_{-h}^{0}|\xi(t+\theta)|^2\ d\theta \end{array} \quad (3.16)$$

and

$$|\zeta(t,\xi_t)|^2 \leq 4\left|\frac{B}{r}\int_{t-\tau-r}^{t-\tau} K_0 \int_{t}^{l+\tau}\dot{\xi}(s)\ dsdl\right|^2 + 4\left|\frac{B}{r}\int_{t-\tau-r}^{t-\tau} K_1 \int_{t-h}^{l+\tau-h}\dot{\xi}(s)\ dsdl\right|^2 + 2\left|\epsilon(t)\right|^2$$

Therefore

$$\begin{array}{l} \dot{V}(\xi_t) \leq \\ -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \\ \int_{-h}^{0}|\xi(t+\theta)|^2\ d\theta + \frac{4}{\delta}\left|\frac{B}{r}\int_{t-\tau-r}^{t-\tau} K_0 \int_{t}^{l+\tau}\dot{\xi}(s)\ dsdl\right|^2 + \frac{4}{\delta}\left|\frac{B}{r}\int_{t-\tau-r}^{t-\tau} K_1 \int_{t-h}^{l+\tau-h}\dot{\xi}(s)\ dsdl\right|^2 \\ +\frac{2}{\delta}\left|\epsilon(t)\right|^2 \end{array}$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \int_{-h}^{0} |\xi(t+\theta)|^2 \, d\theta + \frac{4}{\delta}|B|^2|K_0|^2 \left|\int_{t-r}^{t} |\dot{\xi}(s)|ds\right|^2 + \frac{4}{\delta}|B|^2|K_1|^2 \left|\int_{t-h-r}^{t-h} |\dot{\xi}(s)|ds\right|^2 + \frac{2}{\delta}\left|\epsilon(t)\right|^2$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \int_{-h}^{0} |\xi(t+\theta)|^2 \, d\theta + \frac{4}{\delta}|B|^2|K_0|^2 r \int_{t-r}^{t} |\dot{\xi}(s)|^2 ds + \frac{4}{\delta}|B|^2|K_1|^2 r \int_{t-h-r}^{t-h} |\dot{\xi}(s)|^2 ds + \frac{2}{\delta}\left|\epsilon(t)\right|^2$$

Now consider the functional

$$V_1(\xi_t) = V(\xi_t) + \frac{4}{\delta}|B|^2|K_0|^2 r \int_{t-r}^{t}\int_{m}^{t} |\dot{\xi}(s)|^2 ds dm + \frac{4}{\delta}|B|^2|K_1|^2 r \int_{t-h-r}^{t-h}\int_{m}^{t-h} |\dot{\xi}(s)|^2 ds dm + \frac{4}{\delta}|B|^2|K_1|^2 r^2 \int_{t-h}^{t} |\dot{\xi}(m)|^2 \, dm$$

Then

$$\dot{V}_1(\xi_t) \leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \int_{-h}^{0} |\xi(t+\theta)|^2 d\theta + \frac{4}{\delta}|B|^2(|K_0| + |K_1|)r^2|\dot{\xi}(t)|^2 + \frac{2}{\delta}|\epsilon(t)|^2$$

by (3.8)

$$\dot{V}_1(\xi_t) \leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2) \int_{-h}^{0} |\xi(t+\theta)|^2 d\theta + \frac{q}{\delta}r^2 \left|A_0\xi(t) + A_1\xi(t-h) + \frac{B}{r}\int_{t-\tau-r}^{t-\tau} [K_0\xi(l+\tau) + K_1\xi(l+\tau-h)]dl + \epsilon(t)\right|^2 + \frac{2}{\delta}|\epsilon(t)|^2$$

with $q = 4|B|^2(|K_0| + |K_1|)$. Thus

$$\dot{V}_1(\xi_t) \leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2) \int_{-h}^{0} |\xi(t+\theta)|^2 d\theta + \frac{2q}{\delta}r^2|A_0\xi(t)|^2 + \frac{2q}{\delta}r^2 \left|A_1\xi(t-h) + \frac{B}{r}\int_{t-\tau-r}^{t-\tau} [K_0\xi(l+\tau) + K_1\xi(l+\tau-h)] \, dl + \epsilon(t)\right|^2 + \frac{2}{\delta}|\epsilon(t)|^2$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2) \int_{-h}^{0} |\xi(t+\theta)|^2 d\theta + \frac{2q}{\delta}r^2|A_0\xi(t)|^2 + \frac{4q}{\delta}r^2|A_1\xi(t-h)|^2 + \frac{4q}{\delta}r^2 \left|\frac{B}{r}\int_{t-\tau-r}^{t-\tau} [K_0\xi(l+\tau) + K_1\xi(l+\tau-h)] \, dl + \epsilon(t)\right|^2 + \frac{2}{\delta}|\epsilon(t)|^2$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\xi(t)|^2 - \lambda_{min}(W_1)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0}|\xi(t+\theta)|^2 d\theta$$
$$+\tfrac{2q}{\delta}r^2|A_0\xi(t)|^2 + \tfrac{4q}{\delta}r^2|A_1\xi(t-h)|^2 + \tfrac{8q}{\delta}r^2\left|\tfrac{B}{r}\int_{t-\tau-r}^{t-\tau}[K_0\xi(l+\tau) + K_1\xi(l+\tau-h)]dl\right|^2$$
$$+\tfrac{8q}{\delta}r^2|\epsilon(t)|^2 + \tfrac{2}{\delta}|\epsilon(t)|^2$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \tfrac{2q}{\delta}r^2|A_0|^2)|\xi(t)|^2 - (\lambda_{min}(W_1) - \tfrac{4q}{\delta}r^2|A_1|^2)|\xi(t-h)|^2$$
$$-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times\int_{-h}^{0}|\xi(t+\theta)|^2 d\theta + \tfrac{16q}{\delta}r^2\left|\tfrac{B}{r}K_0\int_{t-r}^{t}\xi(l)dl\right|^2$$
$$+\tfrac{16q}{\delta}r^2\left|\tfrac{B}{r}K_1\int_{t-h-r}^{t-h}\xi(l)]dl\right|^2 + \left[\tfrac{8qr^2+2}{\delta}\right]|\epsilon(t)|^2$$

$$\leq -(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \tfrac{2q}{\delta}r^2|A_0|^2)|\xi(t)|^2 - (\lambda_{min}(W_1) - \tfrac{4q}{\delta}r^2|A_1|^2)|\xi(t-h)|^2$$
$$-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0}|\xi(t+\theta)|^2 d\theta + \tfrac{16qr}{\delta}|BK_0|^2\int_{t-r}^{t}|\xi(l)|^2 dl$$
$$+\tfrac{16qr}{\delta}|BK_1|^2\int_{t-h-r}^{t-h}|\xi(l)|^2 dl + \left[\tfrac{8qr^2+2}{\delta}\right]|\epsilon(t)|^2$$

Now, let

$$V_2(\xi_t) = V_1(\xi_t) + \tfrac{16qr}{\delta}|BK_0|^2\int_{t-r}^{t}\int_{m}^{t}|\xi(l)|^2\, dldm + \tfrac{16qr}{\delta}|BK_1|^2\int_{t-h-r}^{t-h}\int_{m}^{t-h}|\xi(l)|^2\, dldm$$

Then

$$\begin{aligned}\dot{V}_2(\xi_t) \;\leq\; & -(\lambda_{min}(W_0) - \tfrac{2qr^2}{\delta}|A_0|^2 - 2\delta\alpha_0^2 - \tfrac{16qr^2}{\delta}|BK_0|^2)|\xi(t)|^2 - (\lambda_{min}(W_1) \\ & -\tfrac{4qr^2}{\delta}|A_1|^2 - \tfrac{16qr^2}{\delta}|BK_1|^2)|\xi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0}|\xi(t+\theta)|^2 d\theta \\ & + \left[\tfrac{8qr^2+2}{\delta}\right]|\epsilon(t)|^2\end{aligned} \tag{3.17}$$

Recall from the previous chapter that $W_0 + W_1 + hW_2 = W$ and that $U(\theta)$, $\theta \in [-h, 0]$ depends on W. Furthermore α_0 and α_1 depend on $U(\theta)$. As W_0, W_1 and W_2 are symmetric positive definite matrices there always exist δ and r small enough so that the terms multiplied by δ and $\frac{r}{\delta}$ do not destroy the positivity of the factors of $|\xi(t)|^2$, $|\xi(t-h)|^2$ and $\int_{t-h}^{t}|\xi(\theta)|^2\, d\theta$. Hence, there exist $\bar{\delta}$ and $\bar{r}$ such that for $\delta \in (0, \bar{\delta}]$ and for $r \in (0, \bar{r}]$,

$$(\lambda_{min}(W_0) - \frac{2qr^2}{\delta}|A_0|^2 - 2\delta\alpha_0^2 - \frac{16qr^2}{\delta}|B|^2|K_0|^2) \geq \frac{\lambda_{min}(W)}{2} \tag{3.18}$$

$$(\lambda_{min}(W_1) - \frac{4qr^2}{\delta}|A_1|^2 - \frac{16qr^2}{\delta}|B|^2|K_1|^2) \geq 0 \tag{3.19}$$

$$(\lambda_{min}(W_2) - 2\delta\alpha_1^2) \geq 0 \tag{3.20}$$

and we obtain the inequality

$$\dot{V}_2(\xi_t) \leq -\frac{\lambda_{min}(W)}{2}|\xi(t)|^2 + \frac{(8qr^2+2)}{\delta}|\epsilon(t)|^2 \tag{3.21}$$

Since $\epsilon(t)$ converges exponentially to the origin, we deduce that $\xi(t)$ converges exponentially to the origin.

□

3.2 Systems with uncertainty in matrix

Now we consider additive uncertainty in the systems matrices. The perturbed system is in the form

$$\begin{cases} \dot{\chi}(t) &= (A_0+\Delta_0)\chi(t) + (A_1+\Delta_1)\chi(t-h) + B\eta(t-\tau) \\ \dot{\eta}(t) &= \eta(t) + k_1(t,\chi_t,\eta_t) + u(t) \end{cases} \tag{3.22}$$

with $\chi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, A_i, $i = 0,1, B, M \in \mathbb{R}^n \times \mathbb{R}^n$, real matrices, $u \in \mathbb{R}^n$ is the input, $\tau \geq h > 0$ and k_1 is a continuous function. Here matrices Δ_0 and Δ_1 are constants but unknown such that

$$|\Delta_k| \leq \rho_k, \ \ k = 0,1 \tag{3.23}$$

Problem 2 *Design a stabilizing control law for system (3.22).*

Theorem 10. *Let system (3.22) be such that the matrices of the Assumption A1 exist. There is a positive real scalar $\bar{r} > 0$ such that the system (3.22) in closed loop with*

$$u(t,\chi_t,\eta_t) = N\eta(t) - \tfrac{M+N}{r}\int_{t-r}^{t}\eta_s(l,\chi_l,\eta_l)\,dl + \tfrac{\eta_s(t,\chi_t,\eta_t)-\eta_s(t-r,\chi_{t-r},\eta_{t-r})}{r} - k_1(t,\chi_t,\eta_t) \tag{3.24}$$

is globally exponentially stable for $r \in (0,\bar{r}]$.

Remark. *The term η_s in control law is computed with the matrices of the nominal system (3.1) and the corresponding nominal predictors of Lemma 2*

Proof. As in the nominal case, the control law (3.24) can be rewritten using the transformation (3.5) :

$$u(t.\chi_t,\eta_t) = N\bar{\eta}(t) - \tfrac{M}{r}\int_{t-r}^{t}\eta_s(l,\chi_l,\eta_l)\ dl + \tfrac{\eta_s(t,\chi_t,\eta_t)-\eta_s(t-r,\chi_{t-r},\eta_{t-r})}{r} - k_1(t,\chi_t,\eta_t)$$

Hence the closed loop system reduces to:

$$\left\{\begin{array}{rcl} \dot{\chi}(t) & = & A_0\chi(t) + A_1\chi(t-h) + B\tfrac{1}{r}\int_{t-r-\tau}^{t-\tau}\eta_s(l,\chi_l,\eta_l)dl + B\bar{\eta}(t-\tau) \\ & & +\Delta_0\chi(t) + \Delta_1\chi(t-h) \\ \dot{\bar{\eta}}(t) & = & (M+N)\bar{\eta}(t) \end{array}\right. \quad (3.25)$$

Since the $\bar{\eta}$-subsystem in (3.25) is exponentially stable, we try to extend its stability to the χ-subsystem

$$\dot{\chi}(t) = A_0\chi(t) + A_1\chi(t-h) + \frac{B}{r}\int_{t-\tau-r}^{t-\tau}\eta_s(l,\chi_l,\eta_l)\ dl + \Delta_0\chi(t) + \Delta_1\chi(t-h) + \epsilon(t) \quad (3.26)$$

with $\epsilon(t) = B\bar{\eta}(t-\tau)$.

As in previous section

$$\dot{\chi}(t) = H_0\chi(t) + H_1\chi(t-h) + \zeta(\chi_t) \quad (3.27)$$

where

$$\zeta(t,\chi_t) = \frac{B}{r}\int_{t-\tau-r}^{t-\tau}\left[K_0\int_{t}^{l+\tau}\dot{\chi}(s)ds + K_1\int_{t-h}^{l+\tau-h}\dot{\chi}(s)ds\right]dl + [\Delta_0\chi(t) + \Delta_1\chi(t-h)] + \epsilon(t)$$

In view of Assumption A1, the functional

$$\begin{array}{rcl} V(\chi_t) & = & \chi^T(t)U(0)\chi(t) + 2\chi^T(t)\int_{-h}^{0}U(-h-\theta)H_1\chi(t+\theta)d\theta \\ & & +\int_{-h}^{0}\chi^T(t+\theta_1)H_1^T\int_{-h}^{0}U(\theta_1-\theta_2)H_1\chi(t+\theta_2)\ d\theta_2 d\theta_1 \\ & & +\int_{-h}^{0}\chi^T(t+\theta)[W_1+(\theta+h)W_2]\chi(t+\theta)\ d\theta \end{array}$$

is well defined and has as a derivative along the trajectories of (3.26)

$$\begin{array}{rcl} \dot{V}(\chi_t) & = & -\chi^T(t)W_0\chi(t) - \chi^T(t-h)W_1\chi(t-h) - \int_{-h}^{0}\chi^T(t+\theta)W_2\chi(t+\theta)d\theta \\ & & +2\zeta^T(t,\chi_t)\gamma(t,\chi_t) \\ & \leq & -\lambda_{min}(W_0)|\chi(t)|^2 - \lambda_{min}(W_1)|\chi(t-h)|^2 - \lambda_{min}(W_2)\int_{-h}^{0}|\chi(t+\theta)|^2 d\theta \\ & & +\tfrac{1}{\delta}|\zeta(t,\chi_t)|^2 + \delta|\gamma(t,\chi_t)|^2 \end{array}$$

with

$$\gamma(t,\chi_t) = U(0)\chi(t) + \int_{-h}^{0} U(-h-\theta)H_1\chi(t+\theta)\ d\theta$$

and $\delta > 0$ a positive real scalar.

Then for α_0 and α_1 defined as in (3.14)-(3.15)

$$|\gamma(t,\chi_t)|^2 \leq 2\alpha_0^2|\chi(t)|^2 + 2\alpha_1^2 \int_{-h}^{0} |\chi(t+\theta)|^2\ d\theta$$

Therefore

$$\begin{aligned}
&\dot{V}(\chi_t) \leq \\
&-(\lambda_{min}(W_0) - 2\delta\alpha_0^2)|\chi(t)|^2 - \lambda_{min}(W_1)|\chi(t-h)|^2 - (\lambda_{min}(W_2) - 2\delta\alpha_1^2)\times \\
&\int_{-h}^{0} |\chi(t+\theta)|^2 d\theta + \tfrac{2}{\delta}\left|\tfrac{B}{r}\int_{t-\tau-r}^{t-\tau} K_0 \int_{t}^{l+\tau} \dot{\chi}(s) ds dl\right|^2 + \tfrac{2^2}{\delta}\left|\tfrac{B}{r}\int_{t-\tau-r}^{t-\tau} K_1 \int_{t-h}^{l+\tau-h} \dot{\chi}(s) ds dl\right|^2 \\
&+\tfrac{2^3}{\delta}\left|\Delta_0\chi(t)\right|^2 + \tfrac{2^4}{\delta}\left|\Delta_1\chi(t-h)\right|^2 + \tfrac{2^5}{\delta}\left|\epsilon(t)\right|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \\
&-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \tfrac{2^3}{\delta}\rho_0^2)|\chi(t)|^2 - (\lambda_{min}(W_1) - \tfrac{2^4}{\delta}\rho_1^2)|\chi(t-h)|^2 \\
&-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\chi(t+\theta)|^2 d\theta + \tfrac{2}{\delta}|BK_0|^2 r \int_{t-r}^{t} |\dot{\chi}(s)|^2 ds \\
&+\tfrac{2^2}{\delta}|BK_1|^2 r \int_{t-h-r}^{t-h} |\dot{\chi}(s)|^2 ds + \tfrac{2^5}{\delta}\left|\epsilon(t)\right|^2
\end{aligned}$$

Now consider the functional

$$\begin{aligned}
&V_1(\chi_t) = \\
&V(\chi_t) + \tfrac{2}{\delta}|BK_0|^2 r \int_{t-r}^{t}\int_{m}^{t} |\dot{\chi}(s)|^2 ds dm + \tfrac{2^2}{\delta}|BK_1|^2 r \int_{t-h-r}^{t-h}\int_{m}^{t-h} |\dot{\chi}(s)|^2 ds dm \\
&+\tfrac{2^2}{\delta}|BK_1|^2 r^2 \int_{t-h}^{t} |\dot{\chi}(m)|^2 dm
\end{aligned}$$

Then

$$\begin{aligned}
&\dot{V}_1(\chi_t) \leq \\
&-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \tfrac{2^3}{\delta}\rho_0^2)|\chi(t)|^2 - (\lambda_{min}(W_1) - \tfrac{2^4}{\kappa}\rho_1^2)|\chi(t-h)|^2 \\
&-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\chi(t+\theta)|^2\ d\theta + \tfrac{2}{\delta}|B|^2(|K_0| + 2|K_1|)r^2|\dot{\chi}(t)|^2 + \tfrac{2^5}{\delta}\left|\epsilon(t)\right|^2
\end{aligned}$$

by (3.26) and defining $q = 2|B|^2(|K_0| + 2|K_1|)$

$$\begin{aligned}
&\dot{V}_1(\chi_t) \leq \\
&-(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \tfrac{2^3}{\delta}\rho_0^2 - \tfrac{2q}{\delta}r^2(|A_0|^2 + \rho_0^2))|\chi(t)|^2 \\
&-(\lambda_{min}(W_1) - \tfrac{2^4}{\delta}\rho_1^2 - \tfrac{4q}{\delta}r^2(|A_1|^2 + \rho_1^2))|\chi(t-h)|^2 \\
&-(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0} |\chi(t+\theta)|^2 d\theta \\
&+\tfrac{16qr}{\delta}|BK_0|^2 \int_{t-r}^{t} |\chi(l)|^2 dl + \tfrac{16qr}{\delta}|BK_1|^2 \int_{t-h-r}^{t-h} |\chi(l)|^2 dl + \left[\tfrac{8qr^2+2^5}{\delta}\right]\left|\epsilon(t)\right|^2
\end{aligned}$$

Now, let

$$V_2(\chi_t) = V_1(\chi_t) + \tfrac{16qr}{\delta}|BK_0|^2 \int_{t-r}^{t}\int_{m}^{t}|\chi(l)|^2 dl dm + \tfrac{16qr}{\delta}|BK_1|^2 \int_{t-h-r}^{t-h}\int_{m}^{t-h}|\chi(l)|^2 dl dm$$

Then

$$\begin{array}{rcl} \dot{V}_2(\chi_t) & \leq & -(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{2^3}{\delta}\rho_0^2 - \frac{2q}{\delta}r^2(|A_0|^2+\rho_0^2) - \frac{16qr^2}{\delta}|BK_0|^2)|\chi(t)|^2 \\ & & -(\lambda_{min}(W_1) - \frac{2^4}{\delta}\rho_1^2 - \frac{4q}{\delta}r^2(|A_1|^2+\rho_1^2) - \frac{16qr^2}{\delta}|BK_1|^2)|\chi(t-h)|^2 \\ & & -(\lambda_{min}(W_2) - 2\delta\alpha_1^2)\int_{-h}^{0}|\chi(t+\theta)|^2\, d\theta + \left[\frac{8qr^2+2^5}{\delta}\right]|\epsilon(t)|^2 \end{array}$$

As in the case of the nominal system we need find δ and r ensuring the positivity of the factors of $|\chi(t)|^2$, $|\chi(t-h)|^2$ and $\int_{t-h}^{t}|\chi(\theta)|^2\, d\theta$. Hence, there exist $\bar{\delta}$ and $\bar{r}$ such that for $\delta \in (0,\bar{\delta}]$ and $r \in (0,\bar{r}]$,

$$(\lambda_{min}(W_0) - 2\delta\alpha_0^2 - \frac{2^3}{\delta}\rho_0^2 - \frac{2q}{\delta}r^2(|A_0|^2+\rho_0^2) - \frac{16qr^2}{\delta}|BK_0|^2) \geq \frac{\lambda_{min}(W)}{2} \tag{3.28}$$

$$(\lambda_{min}(W_1) - \frac{2^4}{\delta}\rho_1^2 - \frac{4q}{\delta}r^2(|A_1|^2+\rho_1^2) - \frac{16qr^2}{\delta}|BK_1|^2) \geq 0 \tag{3.29}$$

$$(\lambda_{min}(W_2) - 2\delta\alpha_1^2) \geq 0 \tag{3.30}$$

and we obtain the inequality

$$\dot{V}_2(\chi_t) \leq -\frac{\lambda_{min}(W)}{2}|\chi(t)|^2 + \frac{(8qr^2+2^5)}{\delta}|\epsilon(t)|^2$$

Since $\epsilon(t)$ converges exponentially to the origin, we deduce that $\chi(t)$ converges exponentially to the origin.

□

3.3 Systems with uncertain delays

Now we consider additive uncertainty in the delays of the systems. The perturbed system is of the form

$$\left\{ \begin{array}{rcl} \dot{\chi}(t) & = & A_0\chi(t) + A_1\chi(t-h-x_1) + B\eta(t-\tau-x_2) \\ \dot{\eta}(t) & = & M\eta(t) + k_1(t,\chi_t,\eta_t) + u(t) \end{array} \right. \tag{3.31}$$

with $\chi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, A_i, $i = 0, 1, B, M \in \mathbb{R}^n \times \mathbb{R}^n$, real matrices, $u \in \mathbb{R}^n$ is the input, $\tau > h > 0$ and k_1 is a continuous function. Here we consider that x_1 and x_2 are constants but unknown and exist $\beta_1 \in [0, h]$ and $\beta_2 \in [0, \tau]$ such that

$$|x_1| \leq \beta_1 \tag{3.32}$$

$$|x_2| \leq \beta_2 \tag{3.33}$$

Remembering the equation (3.3) it define

$$\eta_s(t, \chi_t, \eta_t) = K_0\chi(t + \tau - \beta_2) + K_1\chi(t + \tau - \beta_2 - h) \tag{3.34}$$

Remark. *The corresponding predictors are computed with the nominal system (3.1).*

Problem 3 *Design a stabilizing control law for system (3.31).*

Theorem 11. *Let system (3.31) such that the matrices of the Assumption A1 exist. There is a positive real scalar $\bar{r} > 0$ such that the system (3.31) in closed loop with*

$$u(t, \chi_t, \eta_t) = N\eta(t) - \tfrac{M+N}{r} \int_{t-r}^{t} \eta_s(l, \chi_l, \eta_l)\ dl + \tfrac{\eta_s(t,\chi_t,\eta_t) - \eta_s(t-r,\chi_{t-r},\eta_{t-r})}{r} - k_1(t, \chi_t, \eta_t) \tag{3.35}$$

is globally exponentially stable for $r \in (0, \bar{r}]$.

Proof. Using the transformation.

$$\bar{\eta}(t) = \eta(t) - \frac{1}{r}\int_{t-r}^{t} \eta_s(l, \chi_l, \eta_l)\ dl \tag{3.36}$$

we can rewrite the control law as

$$u(t, \chi_t, \eta_t) = N\bar{\eta}(t) - \tfrac{M}{r} \int_{t-r}^{t} \eta_s(l, \chi_l, \eta_l)\ dl + \tfrac{\eta_s(t,\chi_t,\eta_t) - \eta_s(t-r,\chi_{t-r},\eta_{t-r})}{r} - k_1(t, \chi_t, \eta_t) \tag{3.37}$$

then the system (3.31) reduces to:

$$\begin{cases} \dot{\chi}(t) &= A_0\chi(t) + A_1\chi(t - x_1 - h) + B\frac{1}{r}\int_{t-\tau-x_2-r}^{t-\tau-x_2} \eta_s(l, \chi_l, \eta_l)dl + B\bar{\eta}(t - \tau - x_2) \\ \dot{\bar{\eta}}(t) &= (M + N)\bar{\eta}(t) \end{cases} \tag{3.38}$$

Since the $\bar{\eta}$-subsystem of (3.38) is exponentially stable, we try to extend its stability to the χ-subsystem

$$\dot{\chi}(t) = A_0\chi(t)+A_1\chi(t-h)+B\frac{1}{r}\int_{t-\tau-x_2-r}^{t-\tau-x_2}\eta_s(l,\chi_l,\eta_l)dl+B\bar{\eta}(t-\tau-x_2)-\int_{t-h-x_1}^{t-h}A_1\dot{\chi}(\theta)\,d\theta \tag{3.39}$$

with $\epsilon(t) = B\bar{\eta}(t-\tau-x_2)$.

Remark. *Since the definition of (3.34) is different from (3.3) then (3.36) is different from (3.5).*

As in previous section we can rewrite (3.39) as

$$\dot{\chi}(t) = H_0\chi(t) + H_1\chi(t-h) + \zeta(\chi_t) \tag{3.40}$$

where

$$\begin{aligned}\zeta(\chi_t) \;&=\; -K_0\chi(t) - K_1\chi(t-h) - \textstyle\int_{t-h-x_1}^{t-h} A_1\dot{\chi}(\theta)\,d\theta \\ &\quad+\tfrac{B}{r}\textstyle\int_{t-\tau-x_2-r}^{t-\tau-x_2}\left[K_0\chi(l+\tau-\beta_2)+K_1\chi(l+\tau-\beta_2-h)\right]dl+\epsilon(t)\end{aligned}$$

In view of Assumption A1, the functional

$$\begin{aligned}V(\chi_t) \;&=\; \chi^T(t)U(0)\chi(t) + 2\chi^T(t)\textstyle\int_{-h}^{0}U(-h-\theta)H_1\chi(t+\theta)d\theta \\ &\quad+\textstyle\int_{-h}^{0}\chi^T(t+\theta_1)H_1^T\int_{-h}^{0}U(\theta_1-\theta_2)H_1\chi(t+\theta_2)d\theta_2 d\theta_1 \\ &\quad+\textstyle\int_{-h}^{0}\chi^T(t+\theta)[W_1+(\theta+h)W_2]\chi(t+\theta)d\theta\end{aligned} \tag{3.41}$$

is well defined and admits a derivative along the trajectories of (3.40) such that

$$\begin{aligned}\dot{V}(\chi_t) \;&\leq\; -\chi^T(t)W_0\chi(t) - \chi^T(t-h)W_1\chi(t-h) - \textstyle\int_{-h}^{0}\chi^T(t+\theta)W_2\chi(t+\theta)\,d\theta \\ &\quad+|2\zeta^T(\chi_t)\gamma(\chi_t)|\end{aligned}$$

with

$$\gamma(\chi_t) = U(0)\chi(t) + \int_{-h}^{0}U(-h-\theta)H_1\chi(t+\theta)\,d\theta$$

then

$$\begin{aligned}\dot{V}(\chi_t) \;&\leq\; -\lambda_{min}(W_0)|\chi(t)|^2 - \lambda_{min}(W_1)|\chi(t-h)|^2 - \lambda_{min}(W_2)\textstyle\int_{-h}^{0}|\chi(t+\theta)|^2\,d\theta \\ &\quad+\delta|\zeta(\chi_t)|^2 + \tfrac{1}{\delta}|\gamma(\chi_t)|^2\end{aligned}$$

As we have seen in the previous section,

$$\tfrac{1}{\delta}|\gamma(\chi_t)|^2 \;\leq\; \tfrac{2}{\delta}\alpha_0^2|\chi(t)|^2 + \tfrac{2}{\delta}\alpha_1^2\textstyle\int_{-h}^{0}|\chi(t+\theta)|^2\,d\theta \tag{3.42}$$

Now we are looking for a upper bound for $\delta|\zeta^T(\chi_t)|^2$

$$\delta|\zeta^T(\chi_t)|^2 \leq$$
$$2\delta|K_0|^2\left|\chi(t)\right|^2+2^2\delta|K_1|^2\left|\chi(t-h)\right|^2+2^3\delta\frac{|B|^2}{r^2}\left|\int_{t-\tau-x_2-r}^{t-\tau-x_2}K_0\chi(l+\tau-\beta_2)\;dl\right|^2$$
$$+2^4\delta\frac{|B|^2}{r^2}\left|\int_{t-\tau-x_2-r}^{t-\tau-x_2}K_1\chi(l+\tau-\beta_2-h)\;dl\right|^2+2^5\delta\left|\int_{t-h-x_1}^{t-h}A_1\dot{\chi}(\theta)\;d\theta\right|^2+2^5\delta\left|\epsilon(t)\right|^2$$

$$\leq$$
$$2\delta|K_0|^2\left|\chi(t)\right|^2+2^2\delta|K_1|^2\left|\chi(t-h)\right|^2+2^3\delta\frac{|BK_0|^2}{r}\int_{-x_2-r}^{-x_2}\left|\chi(l+t-\beta_2)\right|^2dl$$
$$+2^4\delta\frac{|BK_1|^2}{r}\int_{-x_2-r}^{-x_2}\left|\chi(l+t-\beta_2-h)\right|^2dl+2^5\delta\left|\int_{t-h-x_1}^{t-h}A_1\dot{\chi}(\theta)\;d\theta\right|^2+2^5\delta\left|\epsilon(t)\right|^2$$

Substituting $\dot{\chi}(\theta)$ from the equation (3.38)

$$\delta|\zeta^T(\chi_t)|^2 \leq$$
$$2\delta|K_0|^2\left|\chi(t)\right|^2+2^2\delta|K_1|^2\left|\chi(t-h)\right|^2+2^3\delta\frac{|BK_0|^2}{r}\int_{-\beta_2-r}^{\beta_2}\left|\chi(l+t-\beta_2)\right|^2dl$$
$$+2^4\delta\frac{|BK_1|^2}{r}\int_{-\beta_2-r}^{\beta_2}\left|\chi(l+t-\beta_2-h)\right|^2dl$$
$$+sign(x_1)2^6\delta|x_1||A_1A_0|^2\int_{-h-x_1}^{-h}\left|\chi(t+\theta)\right|^2d\theta$$
$$+sign(x_1)2^7\delta|x_1||A_1A_1|^2\int_{-h-x_1}^{-h}\left|\chi(t+\theta-h-x_1)\right|^2\;d\theta$$
$$+sign(x_1)2^8\delta|x_1|\frac{|A_1BK_0|^2}{r}\int_{-h-x_1}^{-h}\int_{\theta-x_2-r}^{\theta-x_2}\left|\chi(s+t-\beta_2)\right|^2dsd\theta$$
$$+sign(x_1)2^9\delta|x_1|\frac{|A_1BK_1|^2}{r}\int_{-h-x_1}^{-h}\int_{\theta-x_2-r}^{\theta-x_2}\left|\chi(s+t-\beta_2-h)\right|^2dsd\theta$$
$$+sign(x_1)2^{10}\delta|x_1||A_1B|^2\int_{-h-x_1}^{-h}\left|\epsilon(\theta+t)\right|^2d\theta+2^5\delta\left|\epsilon(t)\right|^2$$

$$\leq\;2\delta|K_0|^2\left|\chi(t)\right|^2+2^2\delta|K_1|^2\left|\chi(t-h)\right|^2+2^3\delta\frac{|BK_0|^2}{r}\int_{-2\beta_2-r}^{0}\left|\chi(l+t)\right|^2dl$$
$$+2^4\delta\frac{|BK_1|^2}{r}\int_{-2\beta_2-r}^{0}\left|\chi(l+t-h)\right|^2dl$$
$$+2^6\delta|x_1||A_1A_0|^2\int_{-h-x_1\omega(x_1)}^{0}\left|\chi(t+\theta)\right|^2d\theta$$
$$+2^7\delta|x_1||A_1A_1|^2\int_{-h-2x_1\omega(x_1)}^{0}\left|\chi(t+\theta-h)\right|^2\;d\theta$$
$$+2^8\delta|x_1|\frac{|A_1BK_0|^2}{r}\int_{-h-x_1\omega(x_1)}^{0}\int_{\theta-2\beta_2-r}^{\theta}\left|\chi(s+t)\right|^2dsd\theta$$
$$+2^9\delta|x_1|\frac{|A_1BK_1|^2}{r}\int_{-h-x_1\omega(x_1)}^{0}\int_{\theta-2\beta_2-r}^{\theta}\left|\chi(s+t-h)\right|^2dsd\theta$$
$$+2^{10}\delta|x_1||A_1B|^2\int_{-h-x_1\omega(x_1)}^{0}\left|\epsilon(\theta+t)\right|^2d\theta+2^5\delta\left|\epsilon(t)\right|^2$$

$$\leq\;2\delta|K_0|^2\left|\chi(t)\right|^2+2^2\delta|K_1|^2\left|\chi(t-h)\right|^2+2^3\delta\frac{|BK_0|^2}{r}\int_{-2\beta_2-r}^{0}\left|\chi(l+t)\right|^2dl$$
$$+2^4\delta\frac{|BK_1|^2}{r}\int_{-2\beta_2-r}^{0}\left|\chi(l+t-h)\right|^2dl$$
$$+2^6\delta|x_1||A_1A_0|^2\int_{-h-x_1\omega(x_1)}^{0}\left|\chi(t+\theta)\right|^2d\theta$$
$$+2^7\delta|x_1||A_1A_1|^2\int_{-h-2x_1\omega(x_1)}^{0}\left|\chi(t+\theta-h)\right|^2\;d\theta$$
$$+(h+x_1\omega(x_1))2^8\delta|x_1|\frac{|A_1BK_0|^2}{r}\int_{-h-x_1\omega(x_1)-2\beta_2-r}^{0}\left|\chi(s+t)\right|^2ds$$
$$+(h+x_1\omega(x_1))2^9\delta|x_1|\frac{|A_1BK_1|^2}{r}\int_{-h-x_1\omega(x_1)-2\beta_2-r}^{0}\left|\chi(s+t-h)\right|^2ds$$
$$+2^{10}\delta|x_1||A_1B|^2\int_{-h-x_1\omega(x_1)}^{0}\left|\epsilon(\theta+t)\right|^2d\theta+2^5\delta\left|\epsilon(t)\right|^2$$

Now we define the positive definite functional $V_1(\chi_t)$ as

$$\begin{array}{l} V_1(\chi_t) = \\ V(\chi_t) + 2^3\delta\frac{|BK_0|^2}{r}\int_{-2\beta_2-r}^{0}(2\beta_2+r+l)\left|\chi(l+t)\right|^2 dl \\ +2^4\delta\frac{|BK_1|^2}{r}\int_{-2\beta_2-r}^{0}(2\beta_2+r+l)\left|\chi(l+t-h)\right|^2 dl \\ +2^6\delta|x_1||A_1A_0|^2\int_{-h-x_1\omega(x_1)}^{0}(h+x_1\omega(x_1)+l)\left|\chi(l+t)\right|^2 dl \\ +2^7\delta|x_1||A_1A_1|^2\int_{-h-2x_1\omega(x_1)}^{0}(h+2x_1\omega(x_1)+l)\left|\chi(l+t-h)\right|^2 dl \\ +2^8\delta|x_1|\frac{|A_1BK_0|^2}{r}(h+x_1\omega(x_1))\times \\ \int_{-h-x_1\omega(x_1)-2\beta_2-r}^{0}(h+x_1\omega(x_1)+2\beta_2+r+s)\left|\chi(s+t)\right|^2 ds \\ +2^9\delta|x_1|\frac{|A_1BK_1|^2}{r}(h+x_1\omega(x_1))\times \\ \int_{-h-x_1\omega(x_1)-2\beta_2-r}^{0}(h+x_1\omega(x_1)+2\beta_2+r+s)\left|\chi(s+t-h)\right|^2 ds \end{array} \tag{3.43}$$

Then

$$\begin{array}{rcl} \dot{V}_1(\chi_t) & \leq & -N_0|\chi(t)|^2 - N_1|\chi(t-h)|^2 - N_2\int_{-h}^{0}|\chi(t+\theta)|^2 d\theta \\ & & +2^{10}\delta|x_1||A_1|\int_{-h-x_1}^{-h}\left|\epsilon(\theta+t)\right|^2 d\theta + 2^5\delta\left|\epsilon(t)\right|^2 \end{array} \tag{3.44}$$

$$\begin{array}{rcl} N_0 & = & \lambda_{min}(W_0) - 2\delta|K_0|^2 - 2^3\delta\frac{|BK_0|^2}{r}(2\beta_2+r) - 2^6\delta|x_1||A_1A_0|^2(h+x_1\omega(x_1)) \\ & & -2^8\delta|x_1|\frac{|A_1BK_0|^2}{r}(h+x_1\omega(x_1))(h+x_1\omega(x_1)+2\beta_2+r) - \frac{2}{\delta}\alpha_0^2 \end{array} \tag{3.45}$$

$$\begin{array}{rcl} N_1 & = & \lambda_{min}(W_1) - 2\delta|K_1|^2 - 2^4\delta\frac{|BK_1|^2}{r}(2\beta_2+r) - 2^7\delta|x_1||A_1A_1|^2(h+2x_1\omega(x_1)) \\ & & -2^9\delta|x_1|\frac{|A_1BK_1|^2}{r}(h+x_1\omega(x_1))(h+x_1\omega(x_1)+2\beta_2+r) \end{array} \tag{3.46}$$

$$N_2 = \lambda_{min}(W_2) - \frac{2}{\delta}\alpha_1^2 \tag{3.47}$$

As in the case of the nominal system we need find an δ and r ensuring the positivity of the factors of $|\chi(t)|^2$, $|\chi(t-h)|^2$ and $\int_{-h}^{0}|\chi(\theta+t)|^2\, d\theta$. Hence, there exist $\bar{\delta}$ and $\bar{r}$ such that for $\delta \in (0,\bar{\delta}]$ and $r \in (0,\bar{r}]$

$$N_0 \geq \frac{\lambda_{min}(W)}{2} \tag{3.48}$$

$$N_1 \geq 0 \tag{3.49}$$

$$N_2 \geq 0 \tag{3.50}$$

and we obtain the inequality

$$\dot{V}_1(\chi_t) \leq -\frac{\lambda_{min}(W)}{2}|\chi(t)|^2 + 2^{10}\delta|x_1||A_1| \int_{-h-x_1}^{-h} |\epsilon(\theta+t)|^2 \, d\theta + 2^5\delta \, |\epsilon(t)|^2 \tag{3.51}$$

Since $\epsilon(t)$ converges exponentially to the origin, we deduce that $\chi(t)$ converges exponentially to the origin.

□

3.4 Illustrative examples

Exampe 1. Let us consider the scalar equations

$$\begin{cases} \dot{\xi}(t) &= \xi(t) + \eta(t-2) \\ \dot{\eta}(t) &= \eta(t) + u(t) \end{cases} \tag{3.52}$$

Choosing $W_0 = 1$, $W_1 = 0.02$ and $W_2 = 0.03$, to satisfy with the conditions (3.18)-(3.20) the values of $\bar{\delta}$ and $\bar{r}$ are

$$\bar{\delta} = 0.03$$

$$\bar{r} = 0.0128$$

then using the control law (3.4) we obtain

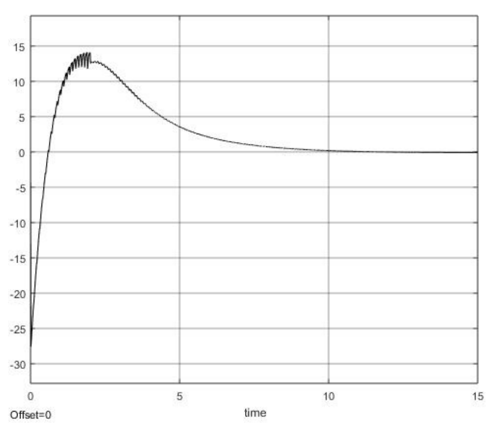

Figure 3.1: $u(t)$

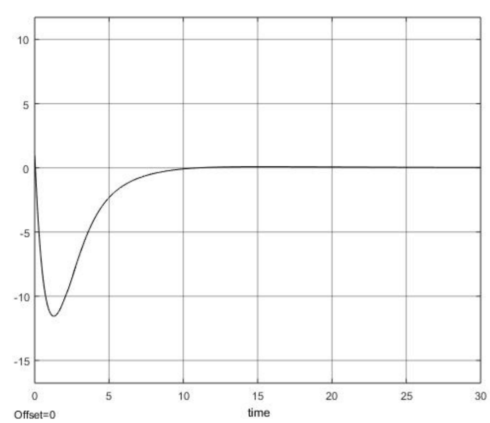

Figure 3.2: $\eta(t)$

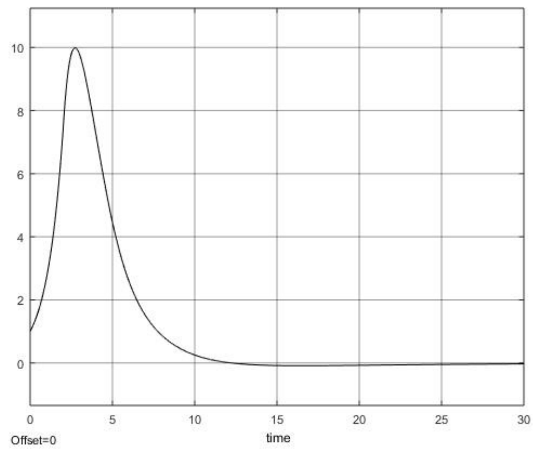

Figure 3.3: $\xi(t)$

Remark. *As we can see in [22] the semidiscretization of distributed-delay control laws may have a destabilizing effect on the systems. So for the implementation of the control law we use the filter given in [5] with $N = -1.1$ and $G = -0.7$*

Exampe 2. Consider

$$\begin{cases} \dot{\xi}(t) & = & 2\xi(t) + \xi(t - 0.25) + \eta(t - 0.5) \\ \dot{\eta}(t) & = & \eta(t) + u(t) \end{cases} \tag{3.53}$$

With $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$. If we choose $W_0 = 1$, $W_1 = 0.02$ and $W_2 = 0.03$, to satisfy with the conditions (3.18)-(3.20) the values of $\bar{\delta}$ and $\bar{r}$ are

$$\bar{\delta} = 0.01$$

$$\bar{r} = 0.0009$$

then using the control law (3.4) we obtain

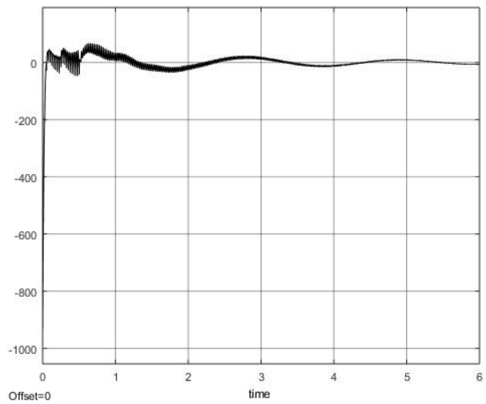

Figure 3.4: $u(t)$

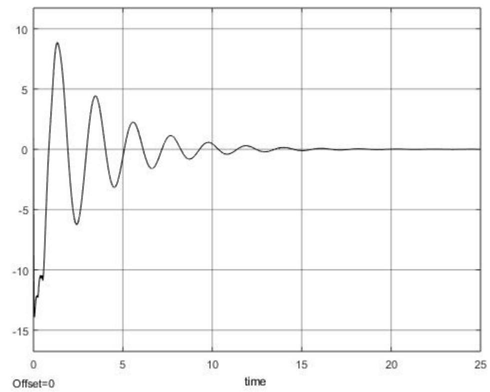

Figure 3.5: $\eta(t)$

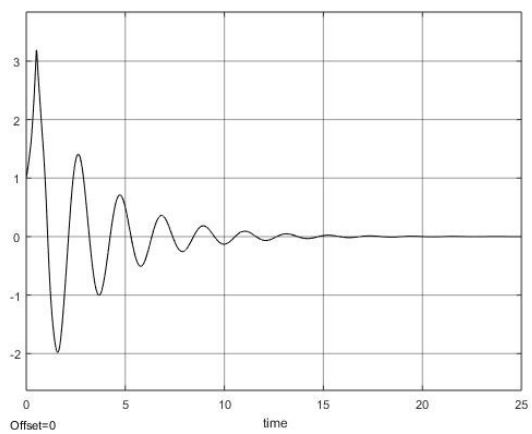

Figure 3.6: $\xi(t)$

Remark. *For the implementation of the control law using the filter previously described with* $N = -30$ *and* $G = -40$*.*

3.5 Conclusions

In this chapter we proposed a new backstepping control law based on artificial delays for time invariant linear systems and we performed a robustness analysis for disturbances in both system matrices and delays. A noticeable feature of this approach is that it is constructive. The examples presented at the end of the chapter show that control law, provides a satisfactory closed loop response.

Chapter 4

Backstepping based on artificial delays for nonlinear time delay systems

In this chapter we use artificial delays to stabilize nonlinear time invariant systems in strict feedback form with pointwise delay in the input. A remarkable aspect of this variant is that the controllers are implemented with pointwise delays.

4.1 Problem statement

We consider the system

$$\begin{cases} \dot{X}(t) = F(t, X(t), y_1(t) + r_1(t)) \\ \dot{Y}(t) = A(t)Y(t) + Bu(t-\tau) + r_2(t) \end{cases} \tag{4.1}$$

with $X \in \mathbb{R}^q$, $Y = (y_1, ..., y_n)^T \in \mathbb{R}^n$, $B = (0\ , ...\ , 0\ , 1)^\top \in \mathbb{R}^n$, $u \in \mathbb{R}$, $\tau \geq 0$, F is a nonlinear functional, locally Lipschitz with respect to its two last arguments and piecewise-continuous with respect to the first, r_1 and r_2 are disturbances, and $A : [0, +\infty) \to \mathbb{R}^{n \times n}$ is of the form:

$$A(t) = \begin{bmatrix} a_{1,1}(t) & a_{1,2}(t) & 0 & ... & 0 \\ a_{2,1}(t) & a_{2,2}(t) & a_{2,3}(t) & ... & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ \vdots & & & & \vdots \\ a_{n-1,1}(t) & a_{n-1,2}(t) & \dots & \dots & a_{n-1,n}(t) \\ a_{n,1}(t) & a_{n,2}(t) & ... & ... & a_{n,n}(t) \end{bmatrix} \tag{4.2}$$

As F does not satisfy smoothness conditions then classical backstepping approach can not be applied to the system (4.1) so we established the problem of this chapter as

Problem 4. *Design of globally asymptotically stabilizing control laws for the system (4.1) when F is not necessarily of class C^1.*

Let us introduce 3 Assumptions.

Assumption B1. *There are a functional $\mathcal{V}$, a functional φ and a constant $\overline{d} \geq 0$ such that the origin of the system*

$$\begin{cases} \dot{\zeta}(t) &= A(t)\zeta(t) + B\mathcal{V}(t-\tau, \chi_{t-\tau}, \zeta_{t-\tau}) + d(t) \\ \dot{\chi}(t) &= \varphi(t, \chi_t, \zeta_t) \end{cases} \tag{4.3}$$

with $\zeta \in \mathbb{R}^n$ and $\chi \in \mathbb{R}^r$, is BIBS and CICS with input d when

$$|d|_\infty \leq \overline{d} \tag{4.4}$$

The functional $\mathcal{V}$ is locally Lipschitz and there is a Lipschitz continuous function $\overline{\mathcal{V}}$ such that

$$|\mathcal{V}(t, \phi_1, \phi_2)| \leq \overline{\mathcal{V}}(\phi_1, \phi_2) \tag{4.5}$$

for all $t \in [0, +\infty)$ and $\phi_1 \in C_{\text{in}}$ and $\phi_2 \in \mathbb{R}$.

Assumption B2. *There are a function $\mathcal{W}$, a functional ϖ and constants $k > 0$ and $h_\star > 0$ such that when $h \in (0, h_\star]$, then the system*

$$\begin{cases} \dot{\xi}(t) &= F\left(t, \xi_t, \Gamma_{\mathcal{W},n}(t, \xi_t, \aleph_t) + r(t)\right) \\ \dot{\aleph}(t) &= \varpi(t, \xi_t, \aleph_t) \end{cases} \tag{4.6}$$

with $\xi \in \mathbb{R}^q$, $\aleph \in \mathbb{R}^u$ and $\Gamma_{\mathcal{W},n}$ defined in (2.19)-(2.20) is BIBS and CICS with input r. The function $\mathcal{W}$ is locally Lipschitz and there is a Lipschitz continuous functions $\overline{\mathcal{W}}$ such that

$$|\mathcal{W}(t, \xi, \aleph)| \leq \overline{\mathcal{W}}(\xi, \aleph) \tag{4.7}$$

for all $t \in [0, +\infty)$ and $\xi \in \mathbb{R}^q$, $\aleph \in \mathbb{R}^u$.

Remark. *In the systems (4.3) and (4.6), there are dynamic extensions. We introduce them for the sake of generality. Evidently, Assumptions B1 and B2 are satisfied in cases where these dynamic extensions are not present.*

Assumption B3. *The functions $a_{i,j}$, $i \in \{1, ..., n\}$, $j \in \{1, ..., \min\{i+1, n\}\}$ are of class C^{n-i} and there are constants $\overline{a} > 0$ and $\underline{a} > 0$ such that*

$$|A(t)| \leq \overline{a} \quad , \quad \forall t \geq 0 \tag{4.8}$$

and

$$|a_{i,j}^{(p)}(t)| \leq \overline{a} \quad , \quad \forall t \geq 0 \tag{4.9}$$

for all $i \in \{1, ..., n\}$, $j \in \{1, ..., \min\{i+1, n\}\}$, $p \in \{1, ..., n-i\}$ and for all $i \in \{1, ..., n-1\}$

$$\underline{a} \leq a_{i,i+1}(t) \quad , \quad \forall t \geq 0 \tag{4.10}$$

There is a Lipschitz continuous function $\overline{F}$ *such that*

$$|F(t, \phi_1, \phi_2)| \leq \overline{F}(\phi_1, \phi_2) \tag{4.11}$$

for all $t \in [0, +\infty)$ *and* $(\phi_1, \phi_2) \in C_{\text{in}} \times \mathbb{R}$ *and*

$$\overline{F}(0, 0) = 0 \tag{4.12}$$

4.2 Feedback stabilization

Let us define the functional

$$y_{\dagger,1}(t, X_t, \aleph_t) = \Gamma_{\mathcal{W},n}(t, X_t, \aleph_t) \tag{4.13}$$

and by induction, we define the functionals $y_{\dagger,i}$ as the functionals such that for $i \in \{1, ..., n-1\}$, along the trajectories system (4.1)

$$y_{\dagger,i+1}(t, X_t, \aleph_t) = \frac{1}{a_{i,i+1}(t)} \left[\dot{y}_{\dagger,i}(t, X_t, \aleph_t) - \sum_{l=1}^{i} a_{i,l}(t) y_{\dagger,l}(t, X_t, \aleph_t) \right] \tag{4.14}$$

Assumption B3 ensures that they are well-defined. Now, one can prove by indutions that there are continuous and bounded functions $b_{i,s}(t)$ such that, for $i \in \{1, ..., n\}$,

$$y_{\dagger,i}(t, X_t, \aleph_t) = \sum_{s=n-i+1}^{n} b_{i,s}(t) \Gamma_{\mathcal{W},s}(t, X_t, \aleph_t) \tag{4.15}$$

We deduce that there are continuous and bounded functions $c_s(t)$ such that

$$\sum_{l=1}^{n} a_{n,l}(t) y_{\dagger,l}(t, X_t, \aleph_t) - \dot{y}_{\dagger,n}(t, X_t, \aleph_t) = \sum_{s=0}^{n} c_s(t) \Gamma_{\mathcal{W},s}(t, X_t, \aleph_t) \tag{4.16}$$

Now, let us introduce:

$$\tilde{y}_i(t) = y_i(t) - y_{\dagger,i}(t, X_t, \aleph_t), \quad i = 1, ..., n \tag{4.17}$$

and

$$\tilde{Y}(t) = (\tilde{y}_1(t), \ .... \ , \tilde{y}_n(t)) \tag{4.18}$$

We are ready to state and prove the following theorem, which is the main result of this chapter.

Theorem 12. *Let the system (4.1) satisfy Assumptions B1 to B3. Then the system (4.1) in closed-loop with the dynamic feedback:*

$$\begin{aligned} u(t-\tau) &= -\sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t,X_t,\aleph_t) + \mathcal{V}\left(t-\tau,\chi_{t-\tau},\tilde{Y}_{t-\tau}\right) \\ \dot{\aleph}(t) &= \varpi(t,X_t,\aleph_t) \\ \dot{\chi}(t) &= \varphi\left(t,\chi_t,\tilde{Y}_t\right) \end{aligned} \tag{4.19}$$

with $\tilde{Y}$ defined in (4.18) is BIBS and CICS with input $(r_1(t), r_2(t))$ when

$$|r_2|_\infty \leq \overline{d} \tag{4.20}$$

where $\overline{d}$ is the constant in (4.4).

Proof. Let us consider the system (4.1) and $\tilde{Y}$ defined in (4.18). Since, according to (4.14),

$$\dot{y}_{\dagger,i}(t,X_t,\aleph_t) = \sum_{l=1}^{i+1} a_{i,l}(t) y_{\dagger,l}(t,X_t,\aleph_t) \tag{4.21}$$

for all $i = 1$ to $n-1$, we have

$$\dot{\tilde{y}}_i(t) = \sum_{l=1}^{i+1} a_{i,l}(t)\tilde{y}_l(t) \tag{4.22}$$

and

$$\begin{aligned} \dot{\tilde{y}}_n(t) &= \sum_{l=1}^{n} a_{n,l}(t) y_l(t) + u(t-\tau) - \dot{y}_{\dagger,n}(t,X_t,\aleph_t) \\ &= \sum_{l=1}^{n} a_{n,l}(t)\tilde{y}_l(t) + u(t-\tau) + \sum_{l=1}^{n} a_{n,l}(t) y_{\dagger,l}(t,X_t,\aleph_t) - \dot{y}_{\dagger,n}(t,X_t,\aleph_t) \end{aligned} \tag{4.23}$$

It follows from (4.22) and the (4.16) that

$$\dot{\tilde{Y}}(t) = A(t)\tilde{Y}(t) + B\left[u(t-\tau) + \sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t,X_t,\aleph_t)\right] \tag{4.24}$$

From (4.17), it follows that $y_1(t) = \Gamma_{\mathcal{W},n}(t,X_t,\aleph_t) + \tilde{y}_1(t)$.
Thus we have:

$$\left\{ \begin{aligned} \dot{X}(t) &= F\left(t,X_t,\Gamma_{\mathcal{W},n}(t,X_t,\aleph_t) + \tilde{y}_1(t) + r_1(t)\right) \\ \dot{\tilde{Y}}(t) &= A(t)\tilde{Y}(t) + B\left[u(t-\tau) + \sum_{s=0}^{n} c_s(t)\Gamma_{\mathcal{W},s}(t,X_t,\aleph_t)\right] + r_2(t) \end{aligned} \right. \tag{4.25}$$

Applying the feedback $u(t-\tau)$ defined in (4.19), we obtain

$$\left\{\begin{array}{lcl}\dot{X}(t) & = & F\left(t, X_t, \Gamma_{\mathcal{W},n}(t, X_t, \aleph_t) + \tilde{y}_1(t) + r_1(t)\right) \\ \dot{\aleph}(t) & = & \varpi(t, X_t, \aleph_t) \\ \dot{\tilde{Y}}(t) & = & A(t)\tilde{Y}(t) + B\mathcal{V}\left(t-\tau, \chi_{t-\tau}, \tilde{Y}_{t-\tau}\right) + r_2(t) \\ \dot{\chi}(t) & = & \varphi(t, \chi_t, \tilde{Y}_t)\end{array}\right. \tag{4.26}$$

Assumption B1 and (4.20) ensure that the $(\tilde{Y}, \chi)$-subsystem of (4.26) is BIBS and CICS with input $r_2(t)$. Next, Assumption B2 allows us to conclude.

□

4.2.1 Discussion of the main result

1) Since $\Gamma_{\mathcal{W},s}(t, X_t, \aleph_t)$ and $\mathcal{V}\left(t-\tau, \chi_{t-\tau}, \tilde{Y}_{t-\tau}\right)$ depend on values of the various variables involved at instants smaller than $t-\tau$, the feedback in (4.19) is well-defined. For any system (4.1), an explicit expression for the functions c_s in (4.16) can be determined. Thus the control in (4.19) can be used in practice.
2) When the functions $\mathcal{V}$ and $\mathcal{W}$ are bounded, then the feedback u defined in (4.19) is bounded because the functions c_s and $\Gamma_{\mathcal{W},s}$ are bounded.
4) Some considerations about the Theorem 12 are the following: (i) A depends on t, (ii) the delay τ is present in u, (iii) delays can be present in the X-subsystem.
5) Under the additional mild condition that the system (4.1) is forward complete, one can deduce from Theorem 12 and Lemma 8 the expression of a globally asymptotically stabilizing control law with pointwise delays instead of distributed delays. Indeed, let us introduce the dynamic extension:

$$\dot{w}_1(t) = -kw_1(t) + \mathcal{W}(t-\tau, X(t-\tau)) \tag{4.27}$$

and, for all $j \in \mathbb{N}$, $j \geq 2$

$$\dot{w}_j(t) = -kw_j(t) + \zeta(w_{j-1,t}) \tag{4.28}$$

with ζ defined in (2.23). Then Lemma 8 ensures that for all $m \in \mathbb{N}$, for all $t \geq mh$, the equality

$$\zeta(w_{m,t}) = \Gamma_{\mathcal{W},m}(t, X_t, \aleph_t) \tag{4.29}$$

is satisfied. Thus

$$\begin{array}{lcl} v(t-\tau) & = & -\displaystyle\sum_{s=0}^{n} c_s(t)\zeta(w_{s,t}) + \mathcal{V}\left(t-\tau, \aleph_{t-\tau}, \tilde{Y}_{t-\tau}\right) \\ \dot{\aleph}(t) & = & \varpi(t, X_t, \aleph_t) \\ \dot{\chi}(t) & = & \varphi\left(t, \chi_t, \tilde{Y}_t\right)\end{array} \tag{4.30}$$

is such that

$$v(t-\tau) \quad = \quad u(t-\tau) \tag{4.31}$$

for all $t \geq nh$ where u is the feedback defined in (4.19). From a practical point of view, implementing v may be easier than implementing u.

4.2.2 Checking the Assumptions

In general, checking that Assumption B1 is satisfied is a standard problem. Indeed, this Assumption basically means that a linear time-varying system with a delay in the input is stabilizable by a dynamic feedback. Assumption B3 is a boundedness condition on the functions of the system with respect to t. It can be easily checked. Assumption B2 is more problematic because the unusual operator $\Gamma_{\mathcal{W},n}$ is involved in it. However, one can check that it is satisfied using a strategy in 2 steps:

(i) First, one determines functions $\mathcal{W}$ and ϖ such that

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & F\left(t, \xi_t, \mathcal{W}(t-\tau, \xi(t-\tau), \aleph(t-\tau)) + r(t)\right) \\ \dot{\aleph}(t) & = & \varpi(t, \xi_t, \aleph_t) \end{array} \right. \tag{4.32}$$

is BIBS and CICS with input r.

(ii) Next, one establishes that the system (4.6) is BIBS and CICS when the tuning parameter h is sufficiently small.

To prove this we describe how the trajectory based approach presented in [15] can be applied when the particular case where the following Assumption is satisfied:

Assumption B4. There are constants $\overline{f}_i \geq 0$, $i = 1, 2$ such that

$$|F(t, \phi, z)| \leq \overline{f}_1 \sup_{s \in [-\tau, 0]} |\phi(s)| + \overline{f}_2 |z| \tag{4.33}$$

for all $t \geq 0, \phi \in C_{\text{in}}$ and $z \in \mathbb{R}$. There are a function $\mathcal{W} : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$, two constants $K_{\mathcal{W}} \geq 0$ and $B_{\mathcal{W}} \geq 0$ such that

$$|\mathcal{W}(a, b_1) - \mathcal{W}(a, b_2)| \leq K_{\mathcal{W}} |b_1 - b_2| \tag{4.34}$$

for all $a \in \mathbb{R}$, $b_1 \in \mathbb{R}^q$, $b_2 \in \mathbb{R}^q$ and

$$|\mathcal{W}(a_1, b) - \mathcal{W}(a_2, b)| \leq B_{\mathcal{W}} |a_1 - a_2| |b| \tag{4.35}$$

for all $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $b \in \mathbb{R}^q$ and constants $T > 0$ and $t_\sharp \geq T + \tau$ such that the solutions of the system:

$$\dot{\mathcal{Y}}(t) = F\left(t, \mathcal{Y}_t, \mathcal{W}(t-\tau, \mathcal{Y}(t-\tau)) + r(t)\right) \tag{4.36}$$

satisfy

$$|\mathcal{Y}(t)| \leq \iota_1 \sup_{s \in [t-T, t]} |\mathcal{Y}(s)| + \iota_2 \sup_{s \in [t-T, t]} |r(s)| \tag{4.37}$$

with

$$0 \leq \iota_1 < 1 \tag{4.38}$$

and $\iota_2 > 0$ for all $t \geq t_\sharp$.

We have the following result:

Proposition 1. *Let the system (4.1) satisfy Assumption B3. Let F be a functional and $\mathcal{W}$ be a function such that Assumption B4 is satisfied. Then Assumption B2 is satisfied.*

Proof. Under the conditions of Proposition 1, the system which corresponds to (4.6) admits the representation:

$$\dot{\xi}(t) = F\left(t, \xi_t, \Gamma_{\mathcal{W},n}(t,\xi_t) + r(t)\right) \tag{4.39}$$

Assumption B4 ensures that this system is forward complete, it admits the equivalent representation:

$$\dot{\xi}(t) \;=\; F\left(t, \xi_t, \mathcal{W}(t-\tau, \xi(t-\tau)) + \Gamma_{\mathcal{W},n}(t,\xi_t) - \mathcal{W}(t-\tau,\xi(t-\tau)) + r(t)\right) \tag{4.40}$$

The inequality (4.37) gives

$$|\xi(t)| \;\leq\; \iota_1 \sup_{m\in[t-T,t]} |\xi(m)| + \iota_2 \sup_{s\in[t-T,t]} |\Gamma_{\mathcal{W},n}(s,\xi_s) - \mathcal{W}(s-\tau,\xi(s-\tau)) + r(s)| \tag{4.41}$$

Next, from (4.34), (4.35) and Lemma 4, we deduce that

$$|\Gamma_{\mathcal{W},n}(t,\xi_t) - \mathcal{W}(t-\tau,\xi(t-\tau))| \leq K_{\mathcal{W}} \int_{t-\tau-nh}^{t-\tau} |\dot{\xi}(m)|dm + nhB_{\mathcal{W}}|\xi(t-\tau)| \tag{4.42}$$

for all $t \geq \tau + nh$. As an immediate consequence,

$$\begin{aligned}
|\xi(t)| \;&\leq\; \iota_1 \sup_{m\in[t-T,t]} |\xi(m)| + \iota_2 \sup_{s\in[t-T,t]} K_{\mathcal{W}} \int_{s-\tau-nh}^{s-\tau} |\dot{\xi}(m)|dm \\
&\quad + \iota_2 \sup_{s\in[t-T,t]} nhB_{\mathcal{W}}|\xi(s-\tau)| + \iota_2 \sup_{s\in[t-T,t]} |r(s)| \\
&\leq\; \iota_3 \sup_{m\in[t-T-\tau,t]} |\xi(m)| + \iota_4 \sup_{s\in[t-T-\tau-nh,t]} |\dot{\xi}(s)| + \iota_2 \sup_{s\in[t-T,t]} |r(s)|
\end{aligned} \tag{4.43}$$

with $\iota_3 = \iota_1 + \iota_2 nhB_{\mathcal{W}}$ and $\iota_4 = \iota_2 K_{\mathcal{W}} nh$. Thus

$$\begin{aligned}
|\xi(t)| \;&\leq\; \iota_3 \sup_{m\in[t-T-\tau,t]} |\xi(m)| + \iota_4 \sup_{s\in[t-T-\tau-nh,t]} |F\left(s, \xi_s, \Gamma_{\mathcal{W},n}(s,\xi_s) + r(s)\right)| \\
&\quad + \iota_2 \sup_{s\in[t-T,t]} |r(s)|
\end{aligned} \tag{4.44}$$

Now, from (4.33), it follows that

$$\begin{aligned}
&|\xi(t)| \leq \\
&\iota_3 \sup_{m\in[t-T-\tau,t]} |\xi(m)| \\
&+\iota_4 \sup_{s\in[t-T-\tau-nh,t]} \left(\overline{f}_1 \sup_{m\in[s-\tau,s]} |\xi(m)| + \overline{f}_2 \left|\Gamma_{\mathcal{W},n}(s,\xi_s) + r(s)\right| \right) + \iota_2 \sup_{s\in[t-T,t]} |r(s)|
\end{aligned} \tag{4.45}$$

$$\leq \iota_3 \sup_{m\in[t-T-\tau,t]} |\xi(m)| + \iota_4\overline{f}_1 \sup_{s\in[t-T-2\tau-nh,t]} |\xi(s)| + \iota_4\overline{f}_2 \sup_{s\in[t-T-\tau-nh,t]} |\Gamma_{\mathcal{W},n}(s,\xi_s)| + \iota_5 \sup_{s\in[t-T-\tau-nh,t]} |r(s)| \tag{4.46}$$

with $\iota_5 = \iota_2 + \iota_4\overline{f}_2$.

From Lemma 5, we deduce that

$$\begin{aligned}|\xi(t)| \leq\ & (\iota_3 + \iota_4\overline{f}_1) \sup_{s\in[t-T-2\tau-nh,t]} |\xi(s)| + \iota_4\overline{f}_2 \sup_{s\in[t-T-\tau-nh,t]} \mathfrak{j} \int_{s-\tau-nh}^{s-\tau} |\mathcal{W}(m,\xi(m))|dm \\ & +\iota_5 \sup_{s\in[t-T-\tau-nh,t]} |r(s)|\end{aligned} \tag{4.47}$$

with $\mathfrak{j}$ defined in (2.14). From (4.34), it follows that

$$\begin{aligned}|\xi(t)| \leq\ & (\iota_3 + \iota_4\overline{f}_1) \sup_{s\in[t-T-2\tau-nh,t]} |\xi(s)| + \iota_4\overline{f}_2 \sup_{s\in[t-T-\tau-nh,t]} \mathfrak{j} \int_{s-\tau-nh}^{s-\tau} K_{\mathcal{W}}|\xi(m)|dm \\ & +\iota_5 \sup_{s\in[t-T-\tau-nh,t]} |r(s)| \\ \leq\ & \iota_6 \sup_{s\in[t-T-2\tau-2nh,t]} |\xi(s)| + \iota_5 \sup_{s\in[t-T-\tau-nh,t]} |r(s)|\end{aligned} \tag{4.48}$$

with

$$\iota_6 = \iota_3 + \iota_4\overline{f}_1 + \iota_4 n\overline{f}_2 K_{\mathcal{W}}\mathfrak{j}h \tag{4.49}$$

Now, observe that

$$\iota_6 = \iota_1 + \iota_2 nhB_{\mathcal{W}} + \iota_2 K_{\mathcal{W}} nh\overline{f}_1 + \iota_2 K_{\mathcal{W}}^2 n^2\overline{f}_2 h\frac{khe^{kh}}{e^{kh}-1} \tag{4.50}$$

and

$$\lim_{h\to 0^+}\left(\iota_1 + \iota_2 nhB_{\mathcal{W}} + \iota_2 K_{\mathcal{W}} nh\overline{f}_1 + \iota_2 K_{\mathcal{W}}^2 n^2\overline{f}_2 h\frac{khe^{kh}}{e^{kh}-1}\right) = \iota_1 \tag{4.51}$$

Then the inequality (4.38) ensures that there is a constant $h_\natural > 0$ such that $\iota_6 < 1$ when $0 \leq h \leq h_\natural$. We deduce from Lemma 1 that Assumption B2 is satisfied. □

4.3 Illustrations of the main result

4.3.1 Benchmark system

We consider the three dimensional linear time-varying system

$$\begin{cases} \dot{X}(t) &= \gamma(t)y_1(t) \\ \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= -y_1(t) - \theta(t)y_2(t) + u(t-\tau) \end{cases} \tag{4.52}$$

where $\gamma(t) = 1 + |\sin(t)|$, $\tau \geq 0$ and where $\theta(t)$ is a continuous periodic nonnegative function not identically equal to zero. The function γ is not of class C^1 it follows that the classical backstepping approach cannot be applied to the system (4.52).

Now, to check Assumptions B1 and B3, consider the positive definite quadratic function

$$Q(Y) = \frac{1}{2}[y_1^2 + y_2^2] \tag{4.53}$$

Its derivative along the trajectories of the system (4.52) satisfies:

$$\dot{Q}(t) = -\theta(t)y_2(t)^2 \tag{4.54}$$

Since $\theta(t)$ is continuous, periodic, nonnegative and is not identically equal to zero, we deduce from the LaSalle Invariance Principle that the Y-subsystem of (4.52) is globally uniformly exponentially stable when u is identically equal to zero. Thus, with the notation of the previous section, Assumptions B1 and B3 are satisfied with $\mathcal{V} = 0$.

Now, we check that Assumption B2 is satisfied in two different cases.

First choice for the function $\mathcal{W}$

We choose a stabilizing control law (in which no dynamic extension is involved):

$$\mathcal{W}(X) = -\Upsilon X \tag{4.55}$$

where $\Upsilon > 0$ is an arbitrary constant when $\tau = 0$ and such that

$$\Upsilon < \frac{1 - e^{-1}}{8\tau} \tag{4.56}$$

when $\tau > 0$. The system (4.6) which corresponds to the choice (4.55) is

$$\dot{\xi}(t) = (1 + |\sin(t)|)[\Gamma_{\mathcal{W},2}(t, \xi_t) + r(t)] \tag{4.57}$$

We can apply Proposition 1 to it. We have: $\overline{f}_1 = 0$ and $\overline{f}_2 = 2$, $K_{\mathcal{W}} = \Upsilon$, $B_{\Upsilon} = 0$. Now, check that Assumption B4 is satisfied. To do this, let us consider:

$$\dot{\mathcal{Y}}(t) = (1 + |\sin(t)|)[\mathcal{W}(t, \mathcal{Y}(t-\tau)) + r(t)] \tag{4.58}$$

This system can be rewritten as:

$$\dot{\mathcal{Y}}(t) = (1 + |\sin(t)|)[-\Upsilon\mathcal{Y}(t-\tau) + r(t)] \tag{4.59}$$

We deduce that, for all $t \geq 2\tau$

$$\begin{array}{rcl} \dot{\mathcal{Y}}(t) & = & -(1+|\sin(t)|)\Upsilon\mathcal{Y}(t) + (1+|\sin(t)|)\Upsilon\int_{t-\tau}^{t}\dot{\mathcal{Y}}(s)ds + (1+|\sin(t)|)r(t) \\ & = & -(1+|\sin(t)|)\Upsilon\mathcal{Y}(t) + (1+|\sin(t)|)\Upsilon\int_{t-\tau}^{t}(1+|\sin(s)|)[-\Upsilon\mathcal{Y}(s-\tau)+r(s)]ds \\ & & +(1+|\sin(t)|)r(t) \end{array} \tag{4.60}$$

Now, let us introduce the positive definite quadratic function:

$$\mathcal{Q}(\mathcal{Y}) = \frac{1}{2}\mathcal{Y}^2 \tag{4.61}$$

Its derivative along the trajectories of (4.60) satisfies

$$\begin{array}{rcl} \dot{\mathcal{Q}}(t) & \leq & -(1+|\sin(t)|)\Upsilon\mathcal{Y}(t)^2 + 2(1+|\sin(t)|)\Upsilon^2|\mathcal{Y}(t)|\int_{t-2\tau}^{t-\tau}|\mathcal{Y}(s)|ds \\ & & +2(1+|\sin(t)|)\Upsilon|\mathcal{Y}(t)|\int_{t-\tau}^{t}|r(s)|ds + (1+|\sin(t)|)\mathcal{Y}(t)r(t) \end{array} \tag{4.62}$$

Using the inequalities

$$2(1+|\sin(t)|)\Upsilon|\mathcal{Y}(t)|\int_{t-\tau}^{t}|\mathfrak{s}(r)|dr \leq \frac{1}{4}(1+|\sin(t)|)\Upsilon\mathcal{Y}(t)^2+4(1+|\sin(t)|)\Upsilon\left(\int_{t-\tau}^{t}|r(s)|ds\right)^2$$

and $(1+|\sin(t)|)\mathcal{Y}(t)r(t) \leq \frac{1}{4}(1+|\sin(t)|)\Upsilon\mathcal{Y}(t)^2 + \frac{1}{\Upsilon}r(t)^2$, we obtain

$$\begin{array}{rcl} \dot{\mathcal{Q}}(t) & \leq & -\frac{1}{2}(1+|\sin(t)|)\Upsilon\mathcal{Y}(t)^2 + 2(1+|\sin(t)|)\Upsilon^2|\mathcal{Y}(t)|\int_{t-2\tau}^{t-\tau}|\mathcal{Y}(s)|ds \\ & & +4(1+|\sin(t)|)\Upsilon\left(\int_{t-\tau}^{t}|r(s)|ds\right)^2 + \frac{1}{\Upsilon}r(t)^2 \end{array} \tag{4.63}$$

As an immediate consequence,

$$\dot{\mathcal{Q}}(t) \leq -\Upsilon\mathcal{Q}(\mathcal{Y}(t)) + 8\Upsilon^2\tau\sup_{m\in[t-2\tau,t]}\mathcal{Q}(\mathcal{Y}(m)) + \varrho\sup_{m\in[t-\tau,t]}r(m)^2 \tag{4.64}$$

with $\varrho = 8\Upsilon\tau^2 + \frac{1}{\Upsilon}$. Let $T > 0$ be a positive real number to be selected later. By integrating this inequality over $[t-T, t]$ with $T > 0$, we obtain:

$$\begin{array}{rcl} \mathcal{Q}(\mathcal{Y}(t)) & \leq & e^{-\Upsilon T}\mathcal{Q}(\mathcal{Y}(t-T)) + \int_{t-T}^{t}\left[8\Upsilon^2\tau\sup_{s\in[m-2\tau,m]}\mathcal{Q}(\mathcal{Y}(s)) + \varrho\sup_{s\in[m-\tau,m]}r(s)^2\right]dm \\ & \leq & e^{-\Upsilon T}\mathcal{Q}(\mathcal{Y}(t-T)) + 8\Upsilon^2\tau T\sup_{s\in[t-T-2\tau,t]}\mathcal{Q}(\mathcal{Y}(s)) + T\varrho\sup_{s\in[t-T-\tau,t]}r(s)^2 \\ & \leq & \left[e^{-\Upsilon T} + 8\Upsilon^2\tau T\right]\sup_{s\in[t-T-2\tau,t]}\mathcal{Q}(\mathcal{Y}(s)) + T\varrho\sup_{s\in[t-T-\tau,t]}r(s)^2 \end{array} \tag{4.65}$$

The choice $T = \frac{1}{\Upsilon}$ yields $e^{-\Upsilon T} + 8\Upsilon^2\tau T = e^{-1} + 8\tau\Upsilon$. From (4.56), we deduce that $e^{-\Upsilon T} + 8\Upsilon^2\tau T < 1$. This inequality and (4.65) imply that Assumption B4 is satisfied.

It follows from Proposition 1 that Assumption B2 holds. We conclude that Theorem 12 applies to (4.52).

Now, let us determine the stabilizing control law for the system (4.52) provided by Theorem 12.

Let

$$y_{\dagger,1}(t, X_t) = \Gamma_{\mathcal{W},2}(t, X_t) \tag{4.66}$$

Then

$$y_{\dagger,2}(t, X_t) = \Omega_{\mathcal{W},2,1}(t, X_t) \tag{4.67}$$

and

$$\begin{array}{rcl} \dot{\tilde{y}}_2(t) & = & -y_1(t) - \theta(t)y_2(t) + u(t-\tau) - \Omega_{\mathcal{W},2,2}(t, X_t) \\ & = & -\tilde{y}_1(t) - \theta(t)\tilde{y}_2(t) + u(t-\tau) - \Gamma_{\mathcal{W},2}(t, X_t) - \theta(t)\Omega_{\mathcal{W},2,1}(t, X_t) - \Omega_{\mathcal{W},2,2}(t, X_t) \end{array} \tag{4.68}$$

Theorem 12 ensures that the control law:

$$u(t-\tau) = \Gamma_{\mathcal{W},2}(t, X_t) + \theta(t)\Omega_{\mathcal{W},2,1}(t, X_t) + \Omega_{\mathcal{W},2,2}(t, X_t) \tag{4.69}$$

globally asymptotically stabilizes the origin of (4.52).

Now, let us determine an explicit expression for the control defined in (4.69).

We have

$$\Gamma_{\mathcal{W},2}(t, X_t) = -\mathcal{P}\int_{t-h}^{t} e^{k(s-t)} \int_{s-h}^{s} e^{k(r-s)} X(r-\tau)drds \tag{4.70}$$

with

$$\mathcal{P} = \Upsilon \mathfrak{j}^2 \tag{4.71}$$

Elementary calculations give:

$$\begin{array}{rcl} \Omega_{\mathcal{W},2,1}(t, X_t) & = & k\mathcal{P}\int_{t-h}^{t} e^{k(s-t)} \int_{s-h}^{s} e^{k(r-s)} X(r-\tau)drds \\ & & -\mathcal{P}\int_{t-h}^{t} e^{k(r-t)} X(r-\tau)dr + \mathcal{P}\int_{t-2h}^{t-h} e^{k(r-t)} X(r-\tau)dr \end{array} \tag{4.72}$$

and

$$\begin{array}{rcl} \Omega_{\mathcal{W},2,2}(t, X_t) & = & -k^2\mathcal{P}\int_{t-h}^{t} e^{k(s-t)} \int_{s-h}^{s} e^{k(r-s)} X(r-\tau)drds \\ & & +k\mathcal{P}\int_{t-h}^{t} e^{k(r-t)} X(r-\tau)dr - k\mathcal{P}\int_{t-2h}^{t-h} e^{k(r-t)} X(r-\tau)dr \\ & & +k\mathcal{P}\int_{t-h}^{t} e^{k(r-t)} X(r-\tau)dr - k\mathcal{P}\int_{t-2h}^{t-h} e^{k(r-t)} X(r-\tau)dr \\ & & -\mathcal{P}X(t-\tau) + 2\mathcal{P}(k,h)e^{-kh}X(t-h-\tau) \\ & & -\mathcal{P}e^{-2kh}X(t-2h-\tau) \end{array} \tag{4.73}$$

Thus, finally, we obtain for the control (4.69):

$$\begin{array}{rcl} u(t-\tau) & = & (-k^2 + k\theta(t) - 1)\mathcal{P}\int_{t-h}^{t} \int_{s-h}^{s} e^{k(r-t)} X(r-\tau)drds \\ & & +(2k - \theta(t))\mathcal{P}\int_{t-h}^{t} e^{k(r-t)} X(r-\tau)dr + (\theta(t) - 2k)\mathcal{P}\int_{t-2h}^{t-h} e^{k(r-t)} X(r-\tau)dr \\ & & -\mathcal{P}X(t-\tau) + 2\mathcal{P}e^{-kh}X(t-h-\tau) - \mathcal{P}e^{-2kh}X(t-2h-\tau) \end{array} \tag{4.74}$$

For the simulation the stabilizing control law (4.69) without dynamic extension:

$$\mathcal{W}(X) = -\Upsilon X$$

where $\Upsilon = 0.06$ satisfies the equation (4.55), give us the following trajectories for the control law and states X, y_1, y_2.

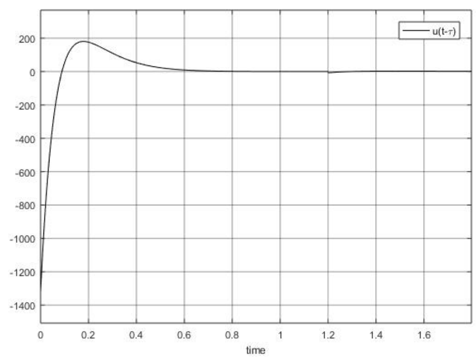

Figure 4.1: $u(t-\tau)$

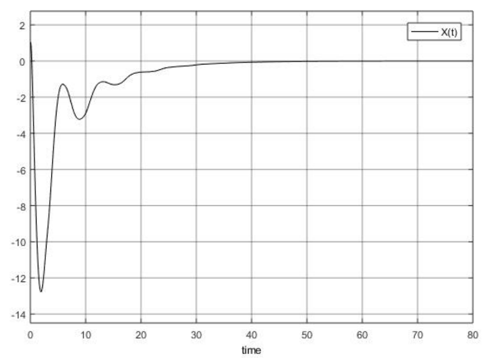

Figure 4.2: $X(t)$

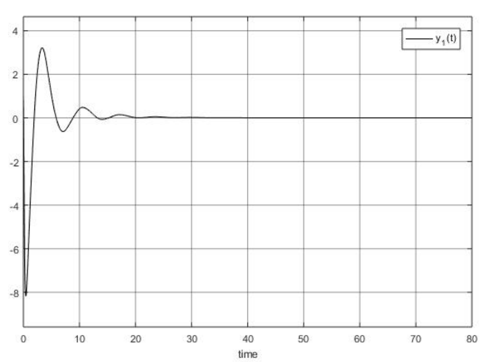

Figure 4.3: $y_1(t)$

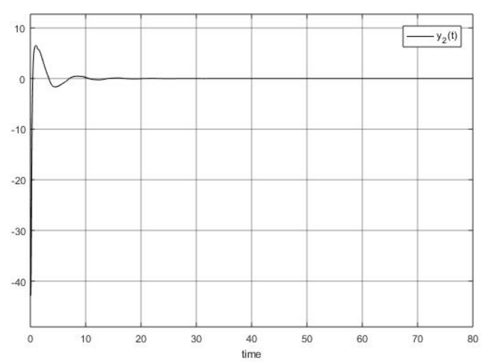

Figure 4.4: $y_2(t)$

Second choice for $\mathcal{W}$

The gain Υ in the function $\mathcal{W}$ defined in (4.55) needs to be chosen smaller than $\frac{1-e^{-1}}{8\tau}$ to ensure that the control law (4.69). This limits the rate of convergence of the solutions of the system (4.52) in closed-loop with the control law (4.74): the rate of convergence goes to zero as τ goes to $+\infty$. In order to achieve a rate of convergence independent from the size of τ, we adopt now another strategy based on the design of control laws using the dynamic extension introduced in [17].

Consider:

$$\begin{aligned} \mathcal{W}(t, \xi_t, \aleph_t) &= \aleph(t) \\ \dot{\aleph}(t) &= -\gamma(t+\tau)a_1\aleph(t) - \gamma(t+\tau)a_2\left[\xi(t) + \int_{t-\tau}^{t} \gamma(\ell+\tau)\aleph(\ell)d\ell\right] \end{aligned} \tag{4.75}$$

where $a_1 > 0$ and $a_2 > 0$ are tuning parameters. The system (4.6) corresponding to this choice is:

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & \gamma(t)\Gamma_{\mathcal{W},2}(t,\xi_t,\aleph_t) + r(t) \\ \dot{\aleph}(t) & = & -\gamma(t+\tau)a_1\aleph(t) - \gamma(t+\tau)a_2\left[\xi(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\aleph(\ell)d\ell\right] \end{array} \right. \tag{4.76}$$

Let us establish that this system is ISS with input $r(t)$ when h is sufficiently small. Let

$$\Lambda(t) = \xi(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\aleph(\ell)d\ell \tag{4.77}$$

Simple calculations give

$$\left\{ \begin{array}{rcl} \dot{\Lambda}(t) & = & \gamma(t+\tau)\aleph(t) + \gamma(t)[\Gamma_{\mathcal{W},2}(t,\xi_t,\aleph_t) - \aleph(t-\tau)] + r(t) \\ \dot{\aleph}(t) & = & -\gamma(t+\tau)a_1\aleph(t) - \gamma(t+\tau)a_2\Lambda(t) \end{array} \right. \tag{4.78}$$

Since $a_1 > 0$ and $a_2 > 0$, there are a positive time-invariant quadratic function $\mathfrak{Q}$ (for which a formula can be easily determined but which is useless of illustrative purpose) and constants $a_3 > 0$ and $a_4 > 0$ such that its derivative along the trajectories of (4.78) satisfies

$$\dot{\mathfrak{Q}}(t) \leq -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + a_3\left(\gamma(t)[\Gamma_{\mathcal{W},n}(t,\xi_t,\aleph_t) - \aleph(t-\tau)] + r(t)\right)^2 \tag{4.79}$$

Consequently,

$$\dot{\mathfrak{Q}}(t) \leq -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + 8a_3[\Gamma_{\mathcal{W},n}(t,\xi_t,\aleph_t) - \aleph(t-\tau)]^2 + 2a_3r(t)^2 \tag{4.80}$$

where the last inequality is a consequence of $|\gamma|_\infty \leq 2$. From Lemma 4, we deduce that

$$\dot{\mathfrak{Q}}(t) \;\leq\; -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + 8a_3\left[\int_{t-\tau-nh}^{t-\tau}|\dot{\aleph}(s)|ds\right]^2 + 2a_3r(t)^2 \tag{4.81}$$

Using Jensen's inequality, we obtain:

$$\dot{\mathfrak{Q}}(t) \;\leq\; -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + 8a_3nh\int_{t-\tau-nh}^{t-\tau}\dot{\aleph}(s)^2ds + 2a_3r(t)^2 \tag{4.82}$$

Using the expression of $\dot{\aleph}(s)$, we obtain

$$\begin{array}{rcl} \dot{\mathfrak{Q}}(t) & \leq & -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + 8a_3nh\int_{t-\tau-nh}^{t-\tau}\left(\gamma(s+\tau)a_1\aleph(s) + \gamma(s+\tau)a_2\Lambda(s)\right)^2ds \\ & & +2a_3r(t)^2 \\ & \leq & -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + 32a_3nh\int_{t-\tau-nh}^{t-\tau}\left(a_1\aleph(s) + a_2\Lambda(s)\right)^2ds + 2a_3r(t)^2 \end{array} \tag{4.83}$$

Since $\mathfrak{Q}$ is a positive time-invariant quadratic function, there is a constant $a_5 \geq 0$ such that

$$\dot{\mathfrak{Q}}(t) \;\leq\; -a_4\mathfrak{Q}(\Lambda(t),\aleph(t)) + a_5\int_{t-\tau-nh}^{t-\tau}\mathfrak{Q}(\Lambda(s),\aleph(s))ds + 2a_3r(t)^2 \tag{4.84}$$

As an immediate consequence,

$$\dot{\mathfrak{Q}}(t) \leq -a_4\mathfrak{Q}(\Lambda(t), \aleph(t)) + a_5 nh \sup_{s\in[t-\tau-nh,t]} \mathfrak{Q}(\Lambda(s), \aleph(s)) + 2a_3 r(t)^2 \tag{4.85}$$

It allows us to conclude that the system (4.78) is ISS with input $r(t)$ when $a_4 > a_5 nh$, which implies that (4.76) is ISS with input $r(t)$. Thus Assumption A2 is satisfied.

Now, we can apply Theorem 12 to the system (4.52). It ensures that the dynamic feedback:

$$\begin{array}{rcl} u(t-\tau) &=& \Gamma_{\mathcal{W},2}(t, X_t, \pounds_t) + \theta(t)\Omega_{\mathcal{W},2,1}(t, X_t, \pounds_t) + \Omega_{\mathcal{W},2,2}(t, X_t, \pounds_t) \\ \dot{\pounds}(t) &=& -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2[X(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\pounds(\ell)d\ell] \end{array} \tag{4.86}$$

renders the origin of (4.52) globally asymptotically stable. Through simple calculations, we obtain an explicit expression for this control law:

$$\begin{array}{rcl} u(t-\tau) &=& (k^2 - k\theta(t) + 1)\mathfrak{j}^2 \int_{t-h}^{t}\int_{s-h}^{s} e^{k(r-t)}\pounds(r-\tau)drds \\ && +(\theta(t) - 2k)\mathfrak{j}^2 \int_{t-h}^{t} e^{k(r-t)}\pounds(r-\tau)dr \\ && +(2k - \theta(t))\mathfrak{j}^2 \int_{t-2h}^{t-h} e^{k(r-t)}\pounds(r-\tau)dr \\ && +\mathfrak{j}^2\aleph(t-\tau) - 2\mathfrak{j}^2 e^{-kh}\aleph(t-h-\tau) + \mathfrak{j}^2 e^{-2kh}\pounds(t-2h-\tau) \\ \dot{\pounds}(t) &=& -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2[X(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\pounds(\ell)d\ell] \end{array} \tag{4.87}$$

Finally, let us explain why the rate of convergence of the solutions of the closed-loop system does not go to zero when τ goes to $+\infty$. This system is:

$$\left\{ \begin{array}{rcl} \dot{X}(t) &=& \gamma(t)y_1(t) \\ \dot{y}_1(t) &=& y_2(t) \\ \dot{y}_2(t) &=& -y_1(t) - \theta(t)y_2(t) + (k^2 - k\theta(t) + 1)\mathfrak{j}^2 \int_{t-h}^{t}\int_{s-h}^{s} e^{k(r-t)}\pounds(r-\tau)drds \\ && +(\theta(t) - 2k)\mathfrak{j}^2 \int_{t-h}^{t} e^{k(r-t)}\pounds(r-\tau)dr \\ && +(2k - \theta(t))\mathfrak{j}^2 \int_{t-2h}^{t-h} e^{k(r-t)}\pounds(r-\tau)dr \\ && +\mathfrak{j}^2\pounds(t-\tau) - 2\mathfrak{j}^2 e^{-kh}\pounds(t-h-\tau) + \mathfrak{j}^2 e^{-2kh}\pounds(t-2h-\tau) \\ \dot{\pounds}(t) &=& -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2[X(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\pounds(\ell)d\ell] \end{array} \right. \tag{4.88}$$

and it admits the following representation:

$$\left\{ \begin{array}{rcl} \dot{X}(t) &=& \gamma(t)\pounds(t-\tau) + \gamma(t)[\Gamma_{\mathcal{W},2}(t, X_t, \pounds_t) - \pounds(t-\tau)] + \gamma(t)\tilde{y}_1(t) \\ \dot{\pounds}(t) &=& -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2[X(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\pounds(\ell)d\ell] \\ \dot{\tilde{y}}_1(t) &=& \tilde{y}_2(t) \\ \dot{\tilde{y}}_2(t) &=& -\tilde{y}_1(t) - \theta(t)\tilde{y}_2(t) \end{array} \right. \tag{4.89}$$

Obviously, the rate of convergence of the $(\tilde{y}_1, \tilde{y}_2)$-subsystem of (4.89) is independent from τ. Now, using

$$\mathcal{G}(t) = X(t) + \int_{t-\tau}^{t}\gamma(\ell+\tau)\pounds(\ell)d\ell \tag{4.90}$$

we obtain:

$$\begin{cases} \dot{\mathcal{G}}(t) &= \gamma(t+\tau)\pounds(t) + \gamma(t)[\Gamma_{\mathcal{W},2}(t, X_t, \pounds_t) - \pounds(t-\tau)] + \gamma(t)\tilde{y}_1(t) \\ \dot{\pounds}(t) &= -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2\mathcal{G}(t) \end{cases} \tag{4.91}$$

From which through a lengthy but simple proof, one can conclude. The key ideas of the proof are the following: (i) $\gamma(t)\tilde{y}_1(t)$ goes to zero when the time goes to the infinity, thus this term can be 'forgotten', (ii) Lemma 4 ensures that the term $\gamma(t)[\Gamma_{\mathcal{W},2}(t, X_t, \pounds_t) - \pounds(t-\tau)]$ is 'small' when h is small so that it can be neglected when h is chosen sufficiently small, (iii) the rate of convergence of the solutions of

$$\begin{cases} \dot{\mathcal{G}}(t) &= \gamma(t+\tau)\pounds(t) \\ \dot{\pounds}(t) &= -\gamma(t+\tau)a_1\pounds(t) - \gamma(t+\tau)a_2\mathcal{G}(t) \end{cases} \tag{4.92}$$

is larger than a constant independent from τ, (iv) then from (4.90), we can conclude.

For the dynamic extension (4.76), using $a_1 = 0.1$ and $a_2 = 0.01$ we obtain the following trajectories for the control law (4.87) and the states X, y_1, y_2 and $\aleph$

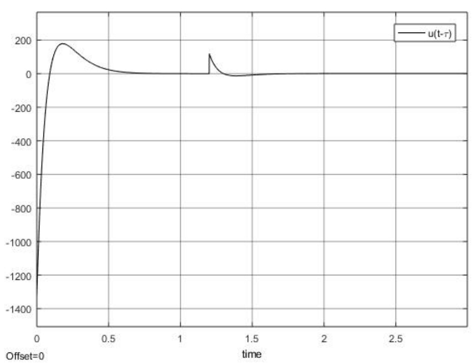

Figure 4.5: $u(t-\tau)$

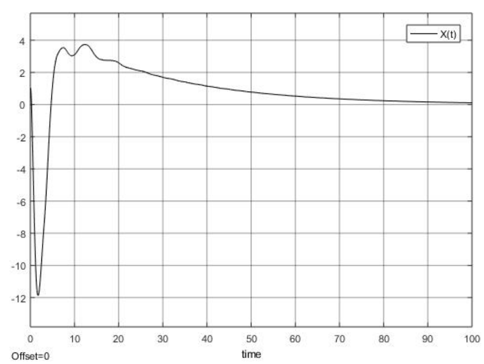

Figure 4.6: $X(t)$

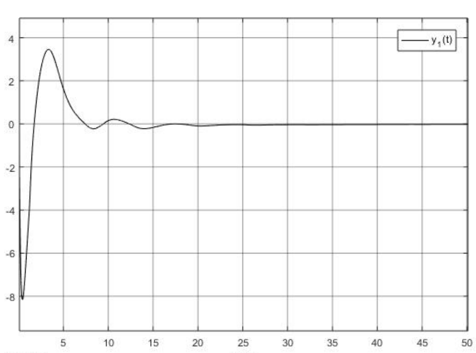

Figure 4.7: $y_1(t)$

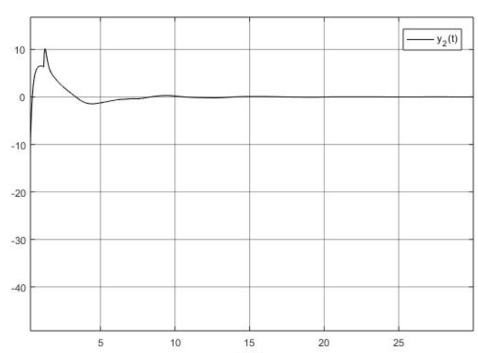

Figure 4.8: $y_2(t)$

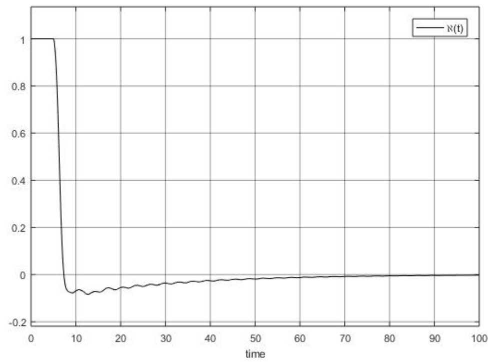

Figure 4.9: $\aleph(t)$

4.3.2 TORA system

We next illustrate our theory using the system

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + \frac{3}{4}\left[\cos\left(\frac{t}{2}\right)\sin\left(y_1(t)\right) + \sin\left(\frac{t}{2}\right)\left(\cos\left(y_1(t)\right) - 1\right)\right] \\ \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= u(t) \end{cases} \tag{4.93}$$

Now, let us check that Theorem 12 applies to this system by verifying that Assumptions B1 to B3 are satisfied by (4.93).

Through a simple proof using the Lyapunov function $\mho(\zeta) = \int_0^{\zeta_1+\zeta_2} \sigma_1(\ell)d\ell + \frac{1}{2}\zeta_2^2$, one can prove that Assumption B1 is satisfied with

$$\mathcal{V}(\zeta) = -\sigma_1(\zeta_1 + \zeta_2) - \sigma_1(\zeta_2) \tag{4.94}$$

One can check easily that Assumption B3 is satisfied. Now, let us check that Assumption B2 is satisfied with

$$\mathcal{W}(t, X) = -\varepsilon \cos\left(\frac{t}{2}\right)\sigma_1(x_2) \tag{4.95}$$

and $\varepsilon \in \left(0, \frac{1}{2}\right)$. The equation which corresponds to the system (4.6) is

$$\begin{cases} \dot{\xi}_1(t) &= \xi_2(t) \\ \dot{\xi}_2(t) &= -\xi_1(t) + \frac{3}{4}\cos\left(\frac{t}{2}\right)\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t) + r(t)\right) \\ & \quad +\frac{3}{4}\sin\left(\frac{t}{2}\right)\left(\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t) + r(t)\right) - 1\right) \end{cases} \tag{4.96}$$

It can be rewritten as:

$$\left\{\begin{array}{rcl}\dot{\xi}_1(t) & = & \xi_2(t)\\ \dot{\xi}_2(t) & = & -\xi_1(t)+\frac{3}{4}\cos\left(\frac{t}{2}\right)\sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ & & +\frac{3}{4}\sin\left(\frac{t}{2}\right)\left(\cos\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)-1\right)+\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,r(t),\xi(t))\end{array}\right. \tag{4.97}$$

with

$$\begin{array}{rcl}\mathcal{C}_1(t,\xi_t) & = & \frac{3}{4}\left[\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)\cos\left(\frac{t}{2}\right)+\left(\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)-1\right)\sin\left(\frac{t}{2}\right)\right]\\ & & -\frac{3}{4}\sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\cos\left(\frac{t}{2}\right)\\ & & -\frac{3}{4}\left(\cos\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)-1\right)\sin\left(\frac{t}{2}\right)\end{array} \tag{4.98}$$

and

$$\begin{array}{rcl}\mathcal{C}_2(t,r(t),\xi_t) & = & \frac{3}{4}\left[\left(\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)+r(t)\right)-\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)\right)\right]\cos\left(\frac{t}{2}\right)\\ & & +\frac{3}{4}\left[\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)+r(t)\right)-\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)\right]\sin\left(\frac{t}{2}\right)\end{array} \tag{4.99}$$

Let us consider the positive definite quadratic function:

$$Q(\xi_1,\xi_2)=\frac{2}{3}(\xi_1^2+\xi_2^2) \tag{4.100}$$

Its derivative along the trajectories of the system (4.97) satisfies

$$\begin{array}{rcl}\dot{Q}(t) & = & -\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\cos\left(\frac{t}{2}\right)+\xi_2(t)\left(\cos\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)-1\right)\times\\ & & \sin\left(\frac{t}{2}\right)+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,\mathfrak{s}(t),\xi(t))\right]\\ & \leq & -\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,r(t),\xi(t))\right]\\ & & +|x_2(t)|\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\end{array} \tag{4.101}$$

where the last inequality is a consequence of the fact that $1-\cos(a)\leq a\sin(a)$ for all $a\in\left[0,\frac{\pi}{2}\right]$. Since $\varepsilon\in\left(0,\frac{1}{2}\right)$, we have:

$$\begin{array}{rcl}\dot{Q}(t) & \leq & -\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,r(t),\xi(t))\right]\\ & & +\frac{1}{2}|\xi_2(t)|\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\\ & \leq & -\frac{1}{2}\cos\left(\frac{t}{2}\right)\xi_2(t)\sin\left(\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)+\xi_2(t)\left[\mathcal{C}_1(t,\xi_t)+\mathcal{C}_2(t,r(t),\xi(t))\right]\end{array} \tag{4.102}$$

Since the system (4.97) is periodic, we deduce from the LaSalle Invariance Principle that this system would be globally uniformly asymptotically stable if $\mathcal{C}_1$ and $\mathcal{C}_2$ were not present. Now, let us investigate what is the impact of these functions. The inequalities

$$\begin{array}{rcl}|\mathcal{C}_1(t,\xi_t)| & \leq & \frac{3}{4}\left|\sin\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)-\sin\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\right|\\ & & +\frac{3}{4}\left|\cos\left(\Gamma_{\mathcal{W},2}(t,\xi_t)\right)-\cos\left(-\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right)\right|\\ & \leq & \frac{3}{2}\left|\Gamma_{\mathcal{W},2}(t,\xi_t)+\varepsilon\cos\left(\frac{t}{2}\right)\sigma_1(\xi_2(t))\right|\end{array} \tag{4.103}$$

and

$$|\mathcal{C}_2(t,\mathfrak{s}(t),\xi_t)| \leq \tfrac{3}{2}|r(t)| \tag{4.104}$$

are satisfied. Then, through a lengthy but simple reasoning, one can prove that there is a constant $h_\star > 0$ such that when $h \in (0, h_\star]$ this system is ISS with restriction with input $r(t)$. Thus Assumption B2 is satisfied.

Then

$$y_{\dagger,1}(t,X_t) = \Gamma_{\mathcal{W},2}(t,X_t) \tag{4.105}$$

$$\begin{aligned} y_{\dagger,2}(t,X_t) &= \dot{y}_{\dagger,1}(t,X_t) \\ &= -k\Gamma_{\mathcal{W},2}(t,X_t) + \tfrac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) - \tfrac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \end{aligned} \tag{4.106}$$

and

$$\begin{aligned} \dot{y}_{\dagger,2}(t,X_t) &= k^2\Gamma_{\mathcal{W},2}(t,X_t) - k\tfrac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) + k\tfrac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \\ &\quad + \tfrac{ke^{kh}}{e^{kh}-1}\dot{\Gamma}_{\mathcal{W},1}(t,X_t) - \tfrac{k}{e^{kh}-1}\dot{\Gamma}_{\mathcal{W},1}(t-h,X_{t-h}) \\ &= k^2\Gamma_{\mathcal{W},2}(t,X_t) - k\tfrac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) + k\tfrac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \\ &\quad + \tfrac{ke^{kh}}{e^{kh}-1}\left[-k\Gamma_{\mathcal{W},1}(t,X_t) + \tfrac{ke^{kh}}{e^{kh}-1}\mathcal{W}(t,x_2(t)) - \tfrac{k}{e^{kh}-1}\mathcal{W}(t-h,x_2(t-h))\right] \\ &\quad - \tfrac{k}{e^{kh}-1}\Big[-k\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) + \tfrac{ke^{kh}}{e^{kh}-1}\mathcal{W}(t-h,x_2(t-h)) \\ &\quad - \tfrac{k}{e^{kh}-1}\mathcal{W}(t-2h,x_2(t-2h))\Big] \end{aligned} \tag{4.107}$$

By grouping the terms,

$$\begin{aligned} \dot{y}_{\dagger,2}(t,X_t) &= k^2\Gamma_{\mathcal{W},2}(t,X_t) - \tfrac{2k^2e^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) + \tfrac{2k^2}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \\ &\quad + \left(\tfrac{ke^{kh}}{e^{kh}-1}\right)^2\mathcal{W}(t,x_2(t)) - 2e^{kh}\left(\tfrac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-h,x_2(t-h)) \\ &\quad + \left(\tfrac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-2h,x_2(t-2h)) \end{aligned} \tag{4.108}$$

This leads us to the bounded control law:

$$\begin{aligned} u(t) &= -\sigma_1(\tilde{y}_1(t)+\tilde{y}_2(t)) - \sigma_1(\tilde{y}_2(t)) + k^2\Gamma_{\mathcal{W},2}(t,X_t) - \tfrac{2k^2e^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) \\ &\quad + \tfrac{2k^2}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \\ &\quad + \left(\tfrac{ke^{kh}}{e^{kh}-1}\right)^2\mathcal{W}(t,x_2(t)) - 2e^{kh}\left(\tfrac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-h,x_2(t-h)) \\ &\quad + \left(\tfrac{k}{e^{kh}-1}\right)^2\mathcal{W}(t-2h,x_2(t-2h)) \end{aligned} \tag{4.109}$$

with

$$\begin{aligned} \tilde{y}_1(t) &= y_1(t) - \Gamma_{\mathcal{W},2}(t,X_t) \\ \tilde{y}_2(t) &= y_2(t) + k\Gamma_{\mathcal{W},2}(t,X_t) - \tfrac{ke^{kh}}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t,X_t) + \tfrac{k}{e^{kh}-1}\Gamma_{\mathcal{W},1}(t-h,X_{t-h}) \end{aligned} \tag{4.110}$$

For the stabilizing control law (4.109) without dynamic extension with $h = 1$, $\varepsilon = 0.49$ and $k = 0.1$ gives us the following trajectories for the control law and states X, Y.

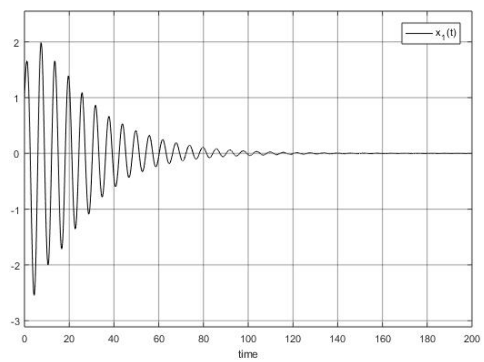

Figure 4.10: $x_1(t)$

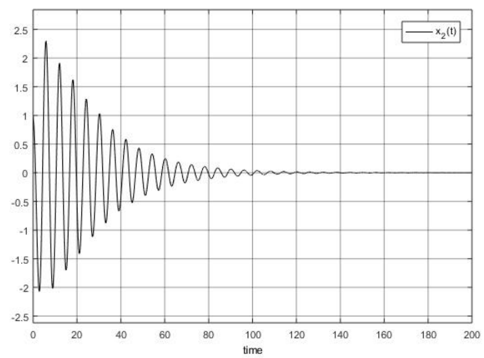

Figure 4.11: $x_2(t)$

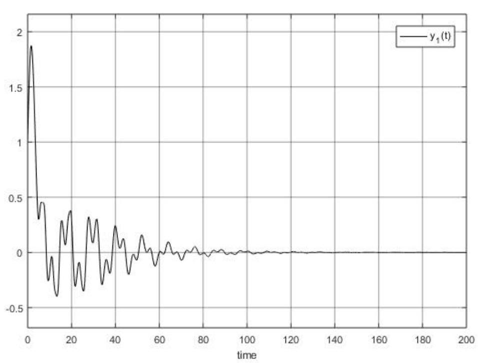

Figure 4.12: $y_1(t)$

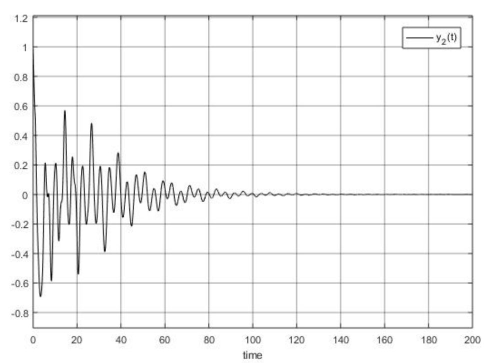

Figure 4.13: $y_2(t)$

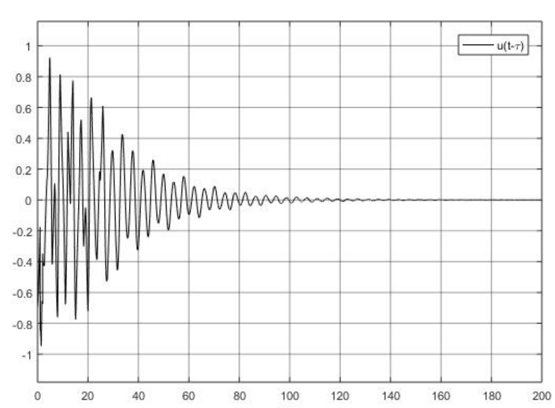

Figure 4.14: $u(t)$

4.4 Conclusions

In this chapter a new backstepping design of control laws for a family of nonlinear continuous time-varying systems with delay was introduced. This design strategy relies on a family of operators which can be replaced by terms generated by dynamic extensions with pointwise delays. This is indeed an advantage over using controllers with distributed delays. The examples presented at the end of the chapter show the successful implementation of these control laws.

Chapter 5

Conclusions

5.1 Concluding remarks

In this work we have presented new control desings for some class of systems in strict feedback form. In general, the designed control laws have the advantage of being given by simpler formulas than those provided by classical techniques. In particular for linear systems with state delay the approach is constructive, while for non-linear systems with input delay, the control laws can be implemented using pointwise delays.

5.2 Future work

Future work in this direction include in particular

- Design control laws following this backstepping approach for systems with time-varying delays, systems with distributed delays and neutral type delay systems.
- Get a better performance in the control laws given for linear systems, since as we saw in the examples they present chattering.
- Relax the conditions of the values δ and r.
- Check Assumption B2 without using other assumptions and relax the boundary conditions of the Assumptions B3.
- Extend the main result of Chapter 4 for systems with a nonlinear Y-subsystem.

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