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PRESENTA

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Resumen

Éste documento presenta el diseño de observadores para sistemas no lineales de orden entero y fraccionario, utilizando un enfoque algebraico-diferencial, con aplicación al problema de sincronización de sistemas caóticos e hipercaóticos, con la excepción de un capítulo que trata el monitoreo de bioreactores. En el capítulo 3 se propone un observador polinomial para sistemas con dinámicas Lipschitz, y para sistemas con dinámicas monótonas no crecientes y/o no decrecientes, se demuestra la convergencia exponencial del error de observación en ambos esquemas, también se incluye un observador asintótico de orden reducido. En el capítulo 4 se construye de manera natural un observador algebraico, el cual consiste en un cambio lineal de variable en términos de la coordenada a reconstruir y de la salida disponible, éste observador se aplica en el monitoreo de biorreactores con modelos de crecimiento de Monod y Haldane. En el capítulo 5 se plantea el problema de sincronización maestro–esclavo considerado como un problema de observación, donde el oscilador caótico es el sistema maestro y el observador es el sistema esclavo. Posteriormente se aplican los observadores estudiados en los capítulos previos para la sincronización de sistemas caóticos (Colpitts, Chua, Lorenz y Rössler). En el capítulo 6 se diseña un observador adaptable y se emplea en el problema de sincronización e identificación de parámetros del sistema de Rikitake, de manera simultánea. En el capítulo 7 se diseña un observador proporcional fraccionario para sistemas no lineales de orden fraccionario y se propone un esquema para la sincronización de sistemas caóticos fraccionarios, se introduce la condición de Observabilidad Algebraica Fraccionaria, la cual permite determinar si las variables desconocidas pueden ser reconstruidas mediante la salida medible y sus derivadas fraccionarias recursivas. Utilizando el observador proporcional fraccionario y la condición de observabilidad algebraica fraccionaria se obtienen los observadores para la sincronización del sistema hipercaótico fraccionario de Rössler y para la sincronización del sistema caótico fraccionario de Lorenz. En el capítulo 8 se introduce una definición de Sincronización Generalizada (SG) para sistemas no lineales diferentes en términos del elemento primitivo diferencial, el cual se expresa como una combinación lineal de los estados y de las entradas. El método consiste

en transformar el sistema original de coordenadas a otro equivalente representado por la Forma Canónica Generalizada de Observabilidad (FCGO), donde la primera componente corresponde al elemento primitivo diferencial. Lo anterior se realiza para el sistema esclavo y para el sistema maestro, en la FCGO se obtiene la sincronización completa entre ambos sistemas. Las variables originales se reconstruyen con las transformaciones inversas. Se presenta la SG entre los circuitos caóticos de Colpitts y Chua. Finalmente, en el capítulo 9, se aborda el problema de sincronización generalizada en sistemas no lineales de orden fraccionario con estructura diferente. Mediante el elemento primitivo diferencial y la condición de observabilidad algebraica fraccionaria se obtiene una Forma Canónica Generalizada de Observabilidad Fraccionaria (FCGOF). Además, se realiza la SG entre los sistemas caóticos fraccionarios de Chua y Rössler.

Abstract

This document describes the observers design for nonlinear systems of integer and fractional order, employing a differential and algebraic framework, with application to synchronization problem of chaotic and hyperchaotic systems, only one chapter deals with the monitoring of bioreactors. In chapter 3 is proposed a polynomial observer for systems with Lipschitz dynamics, and for systems with monotonic dynamics, it is shown the exponential convergence of the observation error in both schemes, as well as it is included an asymptotic observer of reduced order. In chapter 4 is constructed an algebraic observer, which consists in a linear change of variable in terms of the coordinate to be reconstructed and the available output, this observer is applied in the monitoring of bioreactors with growth models of Monod and Haldane. In chapter 5 the master–slave synchronization problem is considered as an observation problem, where the chaotic oscillator is the master system and the observer is the slave. The observers studied previously are applied in the synchronization of chaotic systems (Colpitts, Chua, Lorenz and Rössler). In chapter 6 is designed an adaptive observer, which is employed in the synchronization and parameters identification of Rikitake system. In chapter 7 is designed a fractional proportional observer for fractional order nonlinear systems and is proposed a synchronization scheme for fractional order chaotic systems, it is introduced the Fractional Algebraic Observability condition, which allows us determine if the unknown variables can be reconstructed by means of the output system and its recursive fractional derivatives. It is presented the synchronization of fractional Rössler hyperchaotic system and the synchronization of fractional Lorenz chaotic system. In chapter 8 is introduced a definition of Generalized Synchronization (GS) in different nonlinear systems in terms of the differential primitive element, which is expressed as a linear combination of states and inputs. The method consists in transforming the original coordinate system to another equivalent represented by the Generalized Observability Canonical Form, where the first component corresponds to the differential primitive element. The methodology is illustrated with the GS between Colpitts and Chua systems. Finally, chapter 9 deals with the generalized synchronization problem of nonlinear fractional order systems with different structure. By means of the differential primitive element and the fractional algebraic observability condition is obtained a Fractional Generalized Observability Canonical Form. Furthermore, is given the GS between fractional Chua and Rössler chaotic systems.

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Notación y símbolos

$\approx$	aproximadamente igual
$:=$	definido como
$<$ ($>$)	menor que (mayor que)
$\leq$ ($\geq$)	menor o igual que(mayor o igual que)
$\forall$	para todo
$\in$	pertenece a
$\subset$	subconjunto de
$\cap$	Unión
$\to$	tiende a
$\Rightarrow$	implica
$\Longleftrightarrow$	equivalente a, si y solamente si
Σ	sumatoria
$\mathbb{R}$	el conjunto de los números reales
$\mathbb{R}^+$	números reales positivos
$\mathbb{R}^{m\times n}$	el conjunto de todas las $m \times n$ matrices con elementos en $\mathbb{R}$
$\mathbb{R}_+^{m\times n}$	el conjunto de todas las $m \times n$ matrices con elementos en $\mathbb{R}^+$
$\mathbb{N}$	el conjunto de los números naturales
$\det A$	el determinante de una matriz $A \in \mathbb{R}^{n\times n}$
A^T	la transpuesta de una matriz, obtenida al intercambiar las filas y las columnas de A
rank A	el número mínimo de filas o columnas linealmente independientes de $A \in \mathbb{R}^{m\times n}$
A^{-1}	inversa de A
max	máximo
min	mínimo
sup	supremo, la menor cota superior
inf	ínfimo, la mayor cota inferior

B_R	la esfera de radio R, $\{x \in \mathbb{R}^n \mid \|x\| \leq R\}$
$\lambda_{\max}(P)$ $(\lambda_{\min}(P))$	el máximo (mínimo) valor propio de una matriz simétrica P
$P > 0$	una matriz definida positiva P
$L_f h$	la derivada de Lie de h con respecto al campo f
$sign(\cdot)$	la función signo
$f' = \frac{\partial f}{\partial x}$	la matriz Jacobiana
$\dot{y}$	la primera derivada de y con respecto al tiempo
$\ddot{y}$	la segunda derivada de y con respecto al tiempo
$\dddot{y}$	la tercera derivada de y con respecto al tiempo
$\overset{(i)}{y}$ ó $y^{(i)}$	la i-ésima derivada de y con respecto al tiempo
$\overline{\lim}$	límite superior
$\lvert a\rvert$	valor absoluto de un escalar a
$\|x\|$	La norma Euclidiana de un vector
$\square$	fin de una demostración
$K\langle u\rangle$	campo diferencial generado por el campo K, las componentes de $u(t) \in \mathbb{R}^m$ y sus componentes diferenciales.
$K\langle u, y\rangle$	campo diferencial generado por el campo K, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, y las derivadas con respecto al tiempo de u,y.
LMI	Desigualdad Matricial Lineal (Linear Matrix Inequality)
$\equiv$	idéntico a
$\hat{x}$	valor estimado de x
SG	Sincronización Generalizada
FCGO	Forma Canónica Generalizada de Observabilidad
FCGOF	Forma Canónica Generalizada de Observabilidad Fraccionaria
$\mathcal{D}^{(r\alpha)}x(t)$	derivada fraccionaria de Caputo de orden $\alpha \in \mathbb{R}^+$ aplicada $r-$veces a una variable $x(t)$

Capítulo 1

Introducción

El problema de diseño de observadores para sistemas dinámicos no lineales, es un tema activo en investigación y de gran interés para la comunidad de control automático. Desde el artículo de Luenberger (1966) se demostró que los observadores de estado no son utilizados solamente en el monitoreo y la regulación de sistemas sino que también pueden emplearse para la detección e identificación de fallas en sistemas dinámicos. Actualmente, se sabe que además de las aplicaciones anteriores, se han definido o adaptado diversos problemas para ser resueltos utilizando la teoría de observadores, por ejemplo, control basado en observación (Cheng, Xie y Sun, 2012), sincronización vista como un problema de observación (Nijmeijer y Mareels, 1997, Zhang, Shaoa, Wang y Shen, 2012) e identificación (Iqbal, Bhatti, Ayubi y Khan, 2011). Debido a que la gran mayoría de observadores estan basados en el modelo matemático del sistema, la presencia de perturbaciones, dinámicas no modeladas y no linealidades representan un reto importante.

De manera general, existen dos metodologías para el diseño de observadores no lineales. La primera esta basada en una transformación del estado mediante la cual la dinámica del error del estado es lineal de tal forma que el observador puede ser diseñado utilizando técnicas para sistemas lineales. En los artículos (Krener e Isidori, 1985) y (Xiao y Gao, 1989) intentaron determinar la transformación de estado y establecieron condiciones necesarias y suficientes para su existencia. En (Martínez-Guerra, Mata-Machuca y Rodríguez, 2013) se propuso un observador dinámico, el sistema original se transformó a una forma triangular mediante la selección de un elemento primitivo diferencial adecuado el cual se define como una combinación lineal de los estados conocidos y de las entradas. El segundo enfoque no necesita la transformación, entonces el observador se diseña *directamente* sobre el sistema original. Debido a la complejidad de los sistemas no lineales se han propuesto muchos métodos de diseño directo (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2011b), (Zhao, Tao y Shi, 2010), (Besançon, 2007). Por ejemplo, en (Raghavan y Hedrick, 1994, Rajamani, 1998, Zhu y Han, 2002) presentan el diseño de observadores para sistemas no lineales Lipschitz, los esquemas dependen de la solución de una ecuación de Riccati y se obtiene convergencia asintótica, sin embargo, en (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2010a) se presenta un observador polinomial para la misma clase de sistemas obteniéndose convergencia exponencial. En el segundo método es donde se concentra la mayor cantidad de publicaciones y es el que más se utilizará en este trabajo, con excepción de los capítulos 8 y 9.

En la literatura se puede encontrar una gran variedad de esquemas de observación, por ejemplo, en (Busawon y Khalil, 1999) se propone un observador de estado para sistemas no lineales, el cual es una extensión de la metodología empleada para sistemas lineales basándose en transformaciones relacionadas con la matriz de observabilidad lineal; en el artículo (Arcak y Ne*š*i*ć*, 2004) se presenta el diseño de un observador para sistemas no lineales con datos muestreados, el problema se aborda de dos formas: (a) se diseña un observador mediante un modelo aproximado en tiempo discreto del sistema, (b) se propone un observador basándose en el modelo en tiempo continuo del sistema y se discretiza para la implementación con datos muestreados; la referencia (Tong y Li, 2009) considera un observador adaptable difuso para la estimación de estado e identificación de sistemas no lineales; en (Hammouri, Bornard y Busawon, 2010) se emplea un observador de alta ganancia en una clase de sistemas no lineales con múltiples entradas y múltiples salidas; en (Shen y Xia, 2008) se muestra que las mismas condiciones para el diseño de un observador de alta ganancia para sistemas no lineales con una salida dadas en (Gauthier, Hammouri y Othman, 1992), es decir, que sean uniformemente observables y globalmente Lipschitz, implican la existencia de un observador con convergencia en tiempo finito; en (Ibrir, 2009) se propone un observador adaptable para sistemas no lineales con estado acotado, con retardo constante en el estado, escritos en forma triangular (forma canónica de Brunowsky); en (Cacace, Germani y Manes, 2010) se presenta un observador de estado para una clase de sistemas no lineales (representados mediante la forma canónica de Brunowsky) con retardo variable en la salida; en (Lu, 2006) se diseña un observador fraccionario de Luenberger para sistemas no lineales de orden fraccionario, la aplicación se restringe a sistemas con una salida; en (Martínez-Martínez, Mata-Machuca, Martínez-Guerra, León y Fernández-Anaya, 2011b) se propone un observador fraccionario para sistemas no lineales de orden fraccionario con múltiples salidas; entre otros. A continuación se mencionan algunos tipos de observadores: observadores por modos deslizantes (Mata-Machuca, Martínez-Guerra y Aguilar-López 2012), observadores adaptables (Mata-Machuca, Martínez-Guerra, Aguilar-López y Aguilar-Ibañez, 2010, Boizot, Busvelle y Gauthier, 2010), observadores algebraicos (Aguilar-López, Mata-Machuca, Martínez-Guerra y López-Pérez, 2009, Ibrir, 2003), observadores de alta ganancia (Zhou, Soh y Shen, 2013), observadores difusos (Roopaei, Zolghadri y Meshksar, 2009), observadores fraccionarios (Lu, 2006, Martínez-Martínez, Mata-Machuca, Martínez-Guerra, León y Fernández-Anaya, 2011a), etc. Algunos observadores se denominan de acuerdo a su convergencia, por ejemplo se tienen los observadores exponenciales (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2010a), observadores asintóticos (Mata-Machuca y Martínez-Guerra, 2012), observadores uniformemente acotados (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2011a).

Las aplicaciones de los observadores se encuentran en diversas disciplinas, por ejemplo, en el monitoreo de sistemas y procesos biológicos (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2010b), en el diagnóstico de fallas de una amplia clase de sistemas (Mata-Machuca, Martínez-Guerra y Rincón-Pasaye, 2011), en la sincronización de sistemas caóticos (Mata-Machuca, Martínez-Guerra y Aguilar-López, 2011b, Aguilar-Ibañez, Martínez-Guerra, Aguilar-López y Mata-Machuca, 2010, Martínez-Martínez, Mata-Machuca, Martínez-Guerra, León y Fernández-Anaya, 2011a), en las comunicaciones seguras (Martínez-Guerra, Mata-Machuca, Aguilar-López y Rodríguez-Bollain, 2011, Mata-Machuca, Martínez-Guerra, Aguilar-López y Aguilar-Ibañez, 2012), ésto es, la idea general de transmitir información mediante sistemas caóticos a través de una señal de información que está inmersa en el sistema transmisor el cual produce una señal caótica, sea recuperada o reconstruida posteriormente por el sistema receptor.

Una vez que se ha mencionado la tendencia del estudio de la teoría de los observadores no lineales y su campo de aplicación, se procede a plantear los temas de interés en los que se enfoca ésta tesis, los cuales se pueden resumir en los siguientes puntos:

1. El diseño de observadores para sistemas no lineales de orden entero y fraccionario.

2. La sincronización considerada como un problema de observación.

3. La sincronización completa de sistemas caóticos (de orden entero y fraccionario).

4. La sincronización generalizada en sistemas no lineales con estructura diferente (de orden entero y fraccionario).

5. Finalmente, el monitoreo de biorreactores mediante observadores.

A continuación, se describen de manera general los puntos anteriores y se mencionan las aportaciones de éste trabajo.

1.1. Caos y sincronización

Probablemente uno de los primeros estudios sobre movimiento sincronizado fue realizado por Christian Huygens, quien alrededor de 1650 describió un experimento el cual consistió en colocar dos péndulos idénticos en la misma barra (flexible). En un periodo de tiempo corto, estos péndulos exhibieron un movimiento sincronizado, aún iniciando en diferentes posiciones. A principios del siglo XX, B. Van der Pol estudió el movimiento sincronizado de ciertos sistemas (electro) mecánicos.

Un desarrollo significativo del caos (algunos dicen que su nacimiento) ocurrió tras la publicación por parte de Lorenz (1963) de un artículo sobre flujo no periódico relacionado con la turbulencia en el que se descubría la existencia de soluciones no periódicas en un modelo formado por ecuaciones diferenciales no lineales. La contribución de Lorenz proporcionó el interés en estudiar sistemas determinísticos que generan trayectorias dinámicas fuertemente influenciadas por la sensibilidad a las condiciones iniciales.

Sin embargo, fué hasta 1975 cuando se introdujo el término **caos** por (Yorke y Li, 1975), desde entonces el fenómeno caótico y el comportamiento caótico han sido observados en sistemas naturales y modelos en física, química, biología, ecología, etc. Este término permite la explicación de las propiedades inherentes de algunos sistemas naturales. Sus aplicaciones en ingeniería se han desarrollado rápidamente en áreas tales como tecnologías de plasma, ingeniería química y telecomunicaciones. Los sistemas caóticos como las ecuaciones de Lorenz y otros sistemas autónomos de orden superior se han estudiado con intensidad, y esta es una de las áreas más importantes de la investigación actual en física, matemáticas e ingeniería.

El nombre de caos y el adjetivo de caótico son usados para describir el comportamiento temporal de un sistema cuando dicho comportamiento es: no periódico y aparentemente aleatorio o ruidoso. La palabra clave aquí es “aparentemente”. Bajo esta aparente aleatoriedad caótica se encuentra un determinado orden, dado por las ecuaciones que describen el sistema.

Un sistema caótico es un sistema dinámico determinístico que exhibe un comportamiento irregular, similarmente a un comportamiento aleatorio (Kapitaniak y Bishop, 1999). Las principales características del caos son (Fradkov, 2007):

- tiene una dinámica acotada, lo que significa que en ningún momento tiende a $\pm\infty$,
- posee una extrema sensibilidad a las condiciones iniciales, lo que implica que dos puntos cercanos inicialmente, se distanciarán a medida que transcurre el tiempo.

Un aspecto muy importante en relación con la teoría del caos es la sincronización de sistemas caóticos. Después de que Pecora y Carrol (1990) demostraron que dos comportamientos caóticos aparentemente aleatorios e impredecibles pueden fundirse en una única trayectoria, nuevas expectativas surgieron en torno a la teoría del caos, buscando tanto su control en sistemas eléctricos y mecánicos, como su comprensión y predicción en sistemas geofísicos como la atmósfera o el océano.

El fenómeno de sincronización de sistemas caóticos podría considerarse como imposible debido a que las soluciones de tales sistemas con condiciones iniciales cercanas divergen rápidamente, además, no están correlacionadas. Sin embargo, en el trabajo de Pecora y Carrol se demostró que la sincronización es posible. Una de las motivaciones para sincronización es la posibilidad de enviar mensajes mediante sistemas caóticos para comunicación segura. Tales sistemas sincronizados usualmente consisten de dos partes: un generador de señales caóticas (sistema maestro) y un receptor (sistema esclavo).

El término sincronización en el lenguaje científico usualmente significa coordinación en tiempo de dos o más procesos u objetos. Por ejemplo, podría ser la coincidencia o cercanía de las variables observables de dos o más sistemas. En algunos casos la sincronía se debe a las propiedades naturales de los procesos y a su interacción natural. Un ejemplo bien conocido es la sincronización en frecuencia de cuerpos oscilando o rotando. Este proceso se conoce como sincronización natural. En otros casos, para lograr la sincronización es necesario introducir acciones especiales o imponer algunas restricciones. Entonces hablamos acerca de sincronización forzada o controlada.

El fenómeno de sincronización se presenta cuando dos o más sistemas caóticos, que inicialmente evolucionan sobre atractores diferentes, al acoplarse de algún modo, finalmente siguen una trayectoria común. Debido a que los sistemas caóticos son extremadamente sensibles a las condiciones iniciales, se podría concluir que la sincronización no es factible, ya que en sistemas reales no es posible reproducir de forma exacta condiciones iniciales idénticas para dos sistemas similares. Aunque se puede ser capaz de reproducir sistemas casi idénticos, siempre existe algún problema tecnológico inevitable y ruido que impiden la reproducción exacta de las condiciones iniciales, lo cual implicará la divergencia de las trayectorias. Por lo anterior, resultaría totalmente contranatural pensar en la posibilidad de que dos sistemas caóticos que inicialmente evolucionan sobre atractores diferentes, coincidan en una misma trayectoria. Sin embargo, desde 1990 con la comprobación experimental obtenida por Pecora y Carrol se vió que tal posibilidad era factible. Usualmente la sincronización de dos o más sistemas caóticos se determina mediante el análisis de la dinámica del error de sincronización.

Recientemente, la sincronización de sistemas caóticos ha recibido una gran atención en diversas áreas (Fradkov, 2007, Fradkov y Evans, 2006, Strogatz, 1994, Martínez-Guerra y Yu, 2008). El problema de sincronización de sistemas no lineales es muy importante en diferentes aplicaciones tales como, en la biología (Aguilar-López, Martínez-Guerra, Puebla y Hernández-Suárez, 2009), en la medicina (Andrievskii, 2004), en la criptografía (Álvarez, Montoya, Romera y Pastor, 2003, Strogatz, 1994), en la transmisión segura de datos (Feki, 2003a, Yang, 2004), en ingeniería mecánica (Fradkov y Evans, 2006) y en otros campos (Pikovsky, Rosenblum y Kurths, 2001). La existencia de métodos de la teoría de sistemas dinámicos y teoría de control puede usarse en el análisis y diseño de una variedad de sistemas de sincronización. En general la investigación en sincronización se ha enfocado en dos áreas; la primera, se relaciona con el empleo de observadores de estado, donde las principales aplicaciones se presentan para sistemas oscilatorios (Aguilar-López y Martínez-Guerra, 2008, Feki, 2003b, Feki, 2003a, **?**, Hua y Guan, 2005), mientras que la otra técnica se basa en el uso de leyes de control que permiten la sincronización de osciladores no lineales, de diferente estructura y orden (Bowong, 2004, Fradkov, 2007).

En este trabajo se considera la configuración maestro - esclavo (Pecora y Caroll, 1990). Su principal característica es que el acoplamiento es unidireccional, esto es, la señal se transmite del sistema maestro (transmisor) al sistema esclavo (receptor). Debido a lo anterior, algunos autores utilizan la terminología *transmisor/receptor*.

La sincronización de sistemas caóticos consiste en un régimen en el cual dos sistemas caóticos acoplados (maestro y esclavo), después de un tiempo de transición, exhiben oscilaciones caóticas idénticas. La sincronización puede resolverse desde el punto de vista de la teoría de control, diseñando un sistema esclavo mediante un observador de estado, el cual es capaz de estimar las variables del sistema maestro.

Otro problema de interés consiste en la sincronización entre sistemas caóticos diferentes. En éste trabajo se analiza el fenómeno de Sincronización Generalizada (SG).

La sincronización generalizada ocurre cuando las trayectorias de un sistema son iguales a las de otro mediante alguna transformación. El problema de SG se trató inicialmente en (Rulkov, Sushchik, Tsimring y Abarbanel, 1995) con la finalidad de describir la sincronización en sistemas caóticos acoplados. Actualmente la SG es un tema que se está estudiando ampliamente por su significado teórico y práctico (Dmitriev, Hramov, Koronovskii, Starodubov, Trubetskov y Zharkov, 2009, Moskalenko, Koronovskii y Hramov, 2010, Liu, Chen, Lu y Cao, 2010, Sun, Zeng y Tian, 2010, Wang y Guan, 2006, Kittel, Parisi y Pyragas, 1998).

Para sistemas iguales la transformación corresponde a la identidad (Liu, Qiana, Yang y Xiao, 2006). Para sistemas diferentes se complica la detección de la SG, ya que la transformación no coincide con la identidad, de hecho los atractores en las variables de los sistemas maestro y esclavo (x_m y x_s) pareciera que no están sincronizados.

En la SG se presentan dos problemas principales: (i) determinar si existe la transformación que relaciona a los sistemas maestro y esclavo, (ii)encontrar la forma de esa transformación. Algunos métodos requieren de su conocimiento para establecer la presencia de SG, mientras que otros no.

Las contribuciones obtenidas de ésta investigación relacionadas con el tema de sincronización se enumeran en los siguientes puntos.

1. En el capítulo 3 se presenta el diseño de un observador polinomial, donde la idea fundamental se basa en agregar como factor de corrección la suma de potencias impares del error de observación $e(t)$. El análisis se realiza para sistemas con no linealidades Lipschitz y para sistemas con no linealidades monótonas no crecientes y/o no decrecientes, en ambos casos se obtiene convergencia exponencial de $e(t)$. En el capítulo 5 se muestran aplicaciones en la sincronización de los circuitos de Colpitts y de Chua.

2. Además, en el capítulo 5 se propone un observador uniformemente acotado de orden reducido para el problema de sincronización de sistemas caóticos, el cual consiste en

un cambio lineal de variable que depende de la coordenada a reconstruir y de la salida conocida multiplicada por una constante que determina la convergencia uniforme. Se aplica este observador en la sincronización de los sistemas de Lorenz y Rössler.

3. En el capítulo 6 se trata el problema de sincronización e identificación paramétrica de un sistema caótico con incertidumbre en los parámetros denominado sistema de Rikitake. La estrategia consiste en proponer un sistema receptor (observador adaptable) que siga asintóticamente las trayectorias del sistema transmisor (oscilador de Rikitake). Los parámetros del observador se ajustan continuamente mediante una ley de adaptación conveniente, hasta que el error de sincronización converge a cero. El análisis de convergencia se realiza aplicando el lema de Barbalat. Algunos trabajos relacionados se mencionan a continuación. En (Guan, Peng, Li y Wang, 2001) aplicaron un observador para identificar el parámetro desconocido del sistema de Lorenz. El mismo método se emplea para la identificación paramétrica del sistema caótico de Chen (Lü y Zhang, 2001). El interés en la identificación de parámetros se debe a sus aplicaciones en comunicaciones, principalmente cuando la modulación de parámetros se usa para la transmisión de mensajes.

4. En el capítulo 8 se introduce una definición de sincronización generalizada en términos del elemento primitivo diferencial (ver la Definición 8.2), el cual se expresa como una combinación lineal de los estados y de las entradas. Se destaca que la SG ocurre si existe un elemento primitivo diferencial tal que genere una transformación H_{ms} de las trayectorias $x_m(t) \in M \subset \mathbb{R}^{n_m}$ del atractor del sistema maestro a las trayectorias del atractor del sistema esclavo $x_s(t) \in S \subset \mathbb{R}^{n_s}$.

5. El capítulo 8 también plantea la solución del problema de SG en sistemas no lineales diferentes. El método propuesto consiste en transformar el sistema original de coordenadas a otro equivalente representado por la Forma Canónica Generalizada de Observabilidad (FCGO), donde la primera componente corresponde al elemento primitivo diferencial. Lo anterior se realiza para el sistema esclavo y para el sistema maestro. Una vez descritos en la FCGO se obtiene la sincronización completa. Finalmente, las variables originales se reconstruyen mediante las transformaciones inversas. La metodología se explica en la SG entre los circuitos caóticos de Colpitts y Chua. Las ganancias empleadas son pequeñas comparadas con las reportadas en trabajos

existentes que son del orden de 10^3 unidades.

1.2. Sistemas de orden fraccionario

El cálculo fraccionario es tan antiguo como el cálculo convencional, sólo que el primero no es tan popular en las ciencias y en ingeniería. Recientemente se ha convertido en objeto de estudio de éstas disciplinas. Se presenta como una técnica alternativa para analizar sistemas dinámicos.

Algunas áreas que tratan a los sistemas fraccionarios son las siguientes, en sistemas lineales (Matignon y D'Andréa-Novel, 1997), dinámicas caóticas y sincronización (Grigorenko I. y Grigorenko E., 2003, Hartley, Lorenzo y Qammer, 1995, Yan y Li, 2007, Yu, Li y Su, 2008), estabilidad (Matignon, 1996, Wen, Wu y Lu, 2008), sistemas con retardos (Deng, Li y Lü, 2007), identificación de sistemas (Das, 2008), control de sistemas (Podlubny, 1999), control óptimo (Tricaud y Chen, 2009), ecuaciones Euler-Lagrange, transformada de Fourier fraccionaria, modos deslizantes, robótica, y otras (Monje, Chen, Vinagre, Xue y Feliu, 2010).

Al igual que en sistemas de orden entero, el problema de sincronización es un tópico interesante en sistemas caóticos de orden fraccionario (Zhou y Li, 2005). En el primer trabajo acerca de la sincronización de sistemas fraccionarios (Li, Liao y Yu, 2003) los autores mostraron por medio de una ley de control que los sistemas caóticos de orden fraccionario pueden ser sincronizados con un esquema similar al de sistemas de orden entero.

Una contribución de este trabajo se presenta en el capítulo 7, la cual consiste en el diseño de un observador de estado para sistemas no lineales de orden fraccionario. Asimismo, se propone un esquema para la sincronización de sistemas caóticos de orden fraccionario.

Otra aportación importante es la introducción de la noción de observabilidad algebraica denominada *Observabilidad Algebraica Fraccionaria*, la cual permitirá determinar si las variables desconocidas pueden ser reconstruidas mediante las salida medible y sus derivadas fraccionarias.

Con el observador propuesto y la condición de observabilidad algebraica fraccionaria se construyen los observadores para la sincronización del sistema hipercaótico fraccionario de Rössler y para la sincronización del sistema caótico fraccionario de Lorenz.

Otra contribución se expone en el capítulo 9, donde se trata el problema de sincronización generalizada en sistemas no lineales de orden fraccionario con estructura diferente.

En él mismo capítulo, se propone un elemento primitivo diferencial con el cual de manera natural se obtiene una forma canónica generalizada de observabilidad para sistemas fraccionarios. Finalmente, se aplica la metodología en la SG entre los sistemas caóticos fraccionarios de Chua y Rössler.

1.3. Monitoreo de biorreactores

La importancia del monitoreo en línea de procesos biotecnológicos se ha incrementado en los últimos años. Algunas de sus ventajas son: conocimiento sobre el estado del proceso y la posibilidad de detectar y de aislar procesos anormales que apenas inician. Esto reduce costos de producción, contribuye a la seguridad del proceso y ayuda en la localización del problema.

El principal problema en el monitoreo y control de un proceso de fermentación es que sus variables usualmente no pueden ser medidas en línea (Soroush, 1997), es decir, se dificulta ya que sólo están disponibles en línea medidas indirectas, y los valores calculados podrían no ser precisos. Esto sucede debido a incertidumbres en el modelo, errores de medición o ambos. Para control automático esto puede tener serias consecuencias, especialmente cuando las variables de interés no son controladas directamente. En el proceso de fermentación, las mediciones en línea y fuera de línea son la principal fuente de información acerca del estado del proceso, en combinación con los cálculos basados en el modelo se usan para producir estimaciones, ya sea para monitoreo y/o control (Diop y Martínez-Guerra, 2001).

En un biorreactor, las variables tales como concentraciones son generalmente determinadas fuera de línea, lo cual limita su uso para propósitos de monitoreo y control en línea. Sin embargo, estas variables pueden estimarse en línea usando observadores de estado (*soft sensors*).

Las técnicas de filtrado han recibido atención especial, por ejemplo, el filtro de Kalman extendido, los observadores adaptables y las redes neuronales artificiales, (Hu y Wang, 2002, Levant, 2001), sin embargo, estas técnicas presentan una estructura compleja en sus correspondientes algoritmos de estimación y el ajuste de la ganancia del estimador es difícil. Por lo anterior, resulta interesante diseñar un estimador con un algoritmo de estimación más sencillo.

En el capítulo 4 se propone un estimador de orden reducido con error de estimación uniformemente acotado. La metodología de estimación se basa en un cambio lineal de variable, que depende de la variable a estimar y de la salida, el cual permite obtener un estimador de estado algebraico-diferencial.

Para explicar el esquema propuesto, en el capítulo 4 se presenta el diseño de un estimador uniformemente acotado aplicado a un reactor biológico con modelos de crecimiento no estructurados de Monod (Bastin y Dochain, 1990) y Haldane (Vargas et. al, 2000).

Capítulo 2

Observabilidad de sistemas no lineales

El diseño de observadores de estado naturalmente implica la aproximación de un sistema, se requiere cuando necesitamos información interna (no disponible) del sistema pero se conocen algunas mediciones externas. En general, es claro que no podemos usar una gran cantidad de sensores para medir las señales de interés debido a restricciones tecnológicas, costos, etc.

Dado un sistema lineal o no lineal, la tarea del observador es estimar (reconstruir) las variables de interés desconocidas. Denotamos $x(t)$ como el estado original del sistema y $\hat{x}(t)$ como su correspondiente estimado, usualmente se requiere que $\|\hat{x}(t) - x(t)\| \to 0$, cuando $t \to \infty$, aunque en algunos casos se requiere convergencia exponencial (Gauthier, Hammouri y Othman, 1992).

La pregunta que surge inmediatamente en el diseño de observadores es la siguiente: ¿será posible reconstruir la(s) variable(s) deseada(s)? Si la respuesta es afirmativa se dice que el sistema es observable, lo cual implica que se puede proponer un observador para estimar la(s) variable(s).

La respuesta a la pregunta anterior se dará en éste capítulo a través de varios enfoques. Cuando un sistema lineal es observable, la condición de observabilidad es independiente de la entrada $u(t)$. Para sistemas no lineales esto no siempre se cumple, ya que por lo general, los sistemas no lineales tienen entradas singulares que los hacen no observables.

A continuación se aboradarán tres criterios de observabilidad.

2.1. Criterios de observabilidad

Esta sección se enfoca en la discusión de algunas condiciones requeridas para la observabilidad de sistemas no lineales, es decir, si es posible determinar las variables desconocidas mediante las salidas disponibles y entradas (ambas conocidas).

La clase de sistemas no lineales que aquí se consideran se describen mediante la siguiente representación:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{2.1}$$

donde $x(t) \in \mathbb{R}^n$ es el vector de estado, $u(t) \in \mathbb{R}^m$ denota al vector de entrada, $y(t) \in \mathbb{R}^p$ representa al vector de salidas disponibles. Se supone que las funciones $f : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ y $h : \mathbb{R}^n \to \mathbb{R}^p$ son continuamente diferenciables.

2.1.1. Observabilidad lineal

Considere el siguiente sistema lineal:[1]

$$\begin{aligned} \dot{x} &= A\,x \\ y &= C\,x \end{aligned} \tag{2.2}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $y \in \mathbb{R}$ es la salida, $A \in \mathbb{R}^{n\times n}$ y $C \in \mathbb{R}^{1\times n}$ son constantes.

Obteniendo las primeras $(n-1)$ derivadas con respecto al tiempo de la salida del sistema, se tiene lo siguiente

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \dddot{y} \\ \vdots \\ \overset{(n-1)}{y} \end{bmatrix} = \begin{bmatrix} Cx \\ CAx \\ CA^2x \\ CA^3x \\ \vdots \\ CA^{n-1}x \end{bmatrix} \tag{2.3}$$

Haciendo,

$$Y = \left[y, \dot{y}, \ddot{y}, \dddot{y}, \ldots, \overset{(n-1)}{y} \right]^T \in \mathbb{R}^n$$

[1]Para simplificar la notación se ha omitido la dependencia del tiempo

y

$$N = \left[C^T, (CA)^T, (CA^2)^T, (CA^3)^T, \ldots, (CA^{n-1})^T\right]^T \in \mathbb{R}^{n\times n}. \tag{2.4}$$

La ecuación (2.3) se expresa como,

$$Y = N\,x \tag{2.5}$$

donde N se denomina *matriz de observabilidad*.

Si N es invertible (de rango pleno), el vector de estado x puede determinarse como,

$$x = N^{-1}Y. \tag{2.6}$$

en éste caso se dice que el vector de estado x es observable con respecto al vector de salida Y.

Por otra parte, para sistemas no lineales se aplica un criterio de observabilidad equivalente (Diop y Fliess, 1991). Un sistema dinámico con salida y es localmente observable si el correspondiente sistema linealizado con salida linealizada y_L es observable.

A continuación se considera el siguiente sistema linealizado:

$$\begin{aligned} \dot{x}_L &= J(x,u)x_L \\ y_L &= H(x)x_L \end{aligned} \tag{2.7}$$

donde $J \in \mathbb{R}^{n\times n}$ y $H \in \mathbb{R}^{1\times n}$ son las matrices Jacobianas del sistema no lineal y de la salida, respectivamente.

De manera similar a los sistemas lineales, el vector de estado puede obtenerse como,

$$x_L = \Omega^{-1}Y_L \tag{2.8}$$

donde,

$$Y_L = \left[y_L, \dot{y}_L, \ddot{y}_L, \dddot{y}_L, \ldots, \overset{(n-1)}{y_L}\right]^T \in \mathbb{R}^n$$

$$\Omega = \left[H^T, (HJ)^T, (HJ^2)^T, \ldots, (HJ^{n-1})^T\right]^T \in \mathbb{R}^{n\times n}$$

y

$$\det \Omega \neq 0$$

En este caso Ω representa la matriz de observabilidad.

2.1.2. Enfoque geométrico diferencial

En ésta sección se menciona un criterio de observabilidad para sistemas no lineales empleando herramientas geométrico diferenciales.

Sea el siguiente sistema no lineal:

$$\begin{aligned} \dot{x} = &\ f(x) + g(x)u \\ y = &\ h(x) \end{aligned}$$

donde $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ y $h : \mathbb{R}^n \to \mathbb{R}$. Con f, g y h continuamente diferenciables.

Mediante derivadas de Lie se obtiene la siguiente representación:

$$\begin{aligned} y = &\ h(x) \\ \dot{y} = &\ L_f h(x) \\ \ddot{y} = &\ L_f^2 h(x) \\ \vdots & \\ \overset{(n-1)}{y} = &\ L_f^{n-1} h(x) \\ \overset{(n)}{y} = &\ L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{aligned} \tag{2.9}$$

donde L_f^r son las derivadas de Lie de orden r y $dL_f^r h$ son los diferenciales de Lie de orden r definidos recursivamente como:

$$\begin{aligned} & L_f^0 h := h, \quad dL_f^0 h := dh = \left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n} \right) \\ & L_f^1 h := \langle dh, f \rangle = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i, \ dL_f^1 h := \left(\frac{\partial}{\partial x_1} \left(\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i \right), \ldots, \frac{\partial}{\partial x_n} \left(\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i \right) \right) \\ & L_f^r h := \langle dL_f^{r-1} h, f \rangle = L_f(L_f^{r-1} h), \quad r \geq 2 \end{aligned}$$

Se define la transformación de coordenadas $T(x)$ que lleva al sistema a una forma normal tomando en cuenta las primeras $(n-1)$ derivadas de la salida. Entonces,

$$\Phi = T(x) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix} = \begin{bmatrix} y = & h(x) \\ \dot{y} = & L_f h(x) \\ \vdots & \\ \overset{(n-1)}{y} = & L_f^{n-1} h(x) \end{bmatrix} \tag{2.10}$$

transforma las coordenadas del sistema original x en un nuevo sistema con variables Φ. Por lo tanto, las dinámicas de Φ están descritas por,

$$\begin{aligned} \dot{\Phi}_1 = & \quad \Phi_2 = \quad \dot{y} \\ \dot{\Phi}_2 = & \quad \Phi_3 = \quad \ddot{y} \\ \ldots & \\ \dot{\Phi}_n = & \quad \overset{(n)}{y} = \quad L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{aligned} \tag{2.11}$$

El sistema representado por las ecuaciones (2.9) y (2.11) es, por lo menos, localmente uniformemente observable (Gauthier, Hammouri y Othman, 1992), y por lo tanto para todo $x \in \mathbb{R}^n$, se satisface la condición del rango de observabilidad:

$$\text{rango}\left\{\frac{\partial}{\partial x}\vartheta\right\} = n \tag{2.12}$$

En la ecuación (2.12) ϑ es el vector de observabilidad definido por:

$$\vartheta = \left[dL_f^0 h, dL_f^1 h, \ldots, dL_f^{n-1} h\right]^T$$

2.1.3. Observabilidad diferencial-algebraica

A mediados del siglo pasado, Ritt (1950) introdujo algunos conceptos del álgebra diferencial para analizar los sistemas de ecuaciones diferenciales mediante la teoría de sistemas de ecuaciones algebraicas, tomando en cuenta que las ecuaciones diferenciales son algebraicas en las variables dependientes. En este trabajo se determinará la condición de observabilidad de los sistemas dinámicos mediante el enfoque matemático denominado *algebraico-diferencial.*

La noción de observabilidad de un sistema, lineal o no lineal, consiste en la posibilidad de reconstruir el estado $x(t)$, teniendo el conocimiento de la salida del sistema $y(t)$, la entrada $u(t)$, y posiblemente, un número finito de sus derivadas, $y^{(k)}(t)$, $k \geq 0$ y $u^{(l)}(t)$, $l \geq 0$. Para explicar lo anterior, en este capítulo se mencionan algunas definiciones concernientes al álgebra abstracta (Fraleigh, 1987) y al álgebra diferencial (Diop y Martínez-Guerra, 2001, Kolchin, 1973, Ritt, 1950), las cuales tienen aplicación en la teoría de control.

Definición 2.1 (Extensión de campo) *Sea K un campo. Un subcampo K' de K es un subanillo de K provisto de las leyes de composición internas inducidas por aquellas de K, lo cual le da una estructura de campo y K se denomina extensión del campo K'.*

Definición 2.2 (Elemento algebraico) *Se dice que un elemento $x \in L$ es algebraico sobre K si y solamente si x satisface un polinomio con coeficientes en el campo K.*

Ejemplo 2.1 *El número $\sqrt{2}$ es una solución de la ecuación $x^2 - 2 = 0$, por lo tanto, es algebraico sobre el campo $\mathbb{Q}$.*

Ejemplo 2.2 *$i \in \mathbb{C}$ es algebraico sobre $\mathbb{R}$, al satisfacer un polinomio de la forma $P(x) = x^2 + 1 = 0$.*

Definición 2.3 (Elemento trascendente) *Un elemento $x \in L$ el cual no es algebraico sobre el campo K, se dice que es transcendente sobre el campo K.*

Ejemplo 2.3 *El número $e \in \mathbb{R}$ es trascendente sobre el campo $\mathbb{Q}$ al no encontrar una ecuación con coeficientes en el campo de los números racionales tal que e sea una solución.*

Ejemplo 2.4 *De manera análoga, el número $\pi \in \mathbb{R}$ es trascendente sobre el campo $\mathbb{Q}$.*

Definición 2.4 (Extensión algebraica) *Sean los campos K, L tales que $K \subset L$. Se dice que la extensión L del campo K denotada por L/K es algebraica si todos sus elementos son algebraicos sobre K.*

Definición 2.5 (Extensión trascendente) *Se dice que la extensión L/K es trascendente si existe al menos un elemento de L que sea trascendente sobre K.*

Definición 2.6 (Derivación) *Sea R un anillo conmutativo. Una derivación de R es una función*

$$\frac{d}{dt} \quad : \qquad R \to R$$

tal que,

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$
$$\frac{d}{dt}(xy) = \frac{dx}{dt}y + x\frac{dy}{dt}$$

$\forall \quad x, y \in R$.

Definición 2.7 (Anillo diferencial) *Un anillo diferencial es un anillo dotado con una derivación. Los elementos de R con derivada cero son llamados constantes, los cuales forman un subanillo de R, el cual es un campo si R es un campo.*

Definición 2.8 (Campo diferencial) *Sea R un campo en el cual la operación de derivación es realizable. Esta operación toma como parámetro cualquier elemento $a \in R$ y entrega su derivada, denotada por $\dot{a} \in R$. Entonces para cualesquiera elementos $a, b \in R$, se tiene lo siguiente*

(a) $\frac{d}{dt}a = \dot{a}$

(b) $\frac{d}{dt}(a + b) = \dot{a} + \dot{b}$

(c) $\frac{d}{dt}(a\,b) = \dot{a}b + a\dot{b}$

Cuando esta operación de derivación está definida para el campo R, este campo será denominado campo diferencial.

Ejemplo 2.5 $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *son campos cuyos elementos son solamente constantes.*

Ejemplo 2.6 *El campo de funciones de transferencia racionales $R(s)$, no necesariamente propias, en la variable única s es un campo diferencial respecto a la derivación d/ds.*

Definición 2.9 (Extensión de un campo diferencial) *Sean L y K campos diferenciales. Una extensión de un campo diferencial L/K está dada por K y L tal que,*

(a) *K es subcampo de L,*

(b) *la restricción a K de la derivación de L es la derivación de K.*

Ejemplo 2.7 $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *son ejemplos triviales de extensiones de campos diferenciales, donde* $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Definición 2.10 (Elemento diferencialmente algebraico) *Se dice que un elemento $x \in L$ es diferencialmente algebraico sobre K si y solamente si x satisface una ecuación diferencial polinomial con coeficientes sobre el campo K. Es decir, existe un polinomio algebraico-diferencial no cero $P(x_1, \ldots, x_{n+1}) \in K\langle x_1, \ldots, x_{n+1}\rangle$ tal que*

$$P(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}) = 0$$

con $n \in \mathbb{N}, \quad n < \infty$.

Ejemplo 2.8 $\mathbb{R}\langle \mathrm{e}^t \rangle/\mathbb{R}$ *es una extensión de un campo diferencial, donde* $\mathbb{R} \subseteq \mathbb{R}\langle \mathrm{e}^t \rangle$*. Esta extensión es diferencialmente algebraica porque* e^t *satisface* $\dot{x} - x = 0$*.*

Definición 2.11 (Elemento diferencialmente trascendente) *Un elemento* $x \in L$ *que no es diferencialmente algebraico sobre el campo* K*, se dice que es diferencialmente trascendente sobre el campo* K*.*

Ejemplo 2.9 *Sea la extensión de campo diferencial*

$$\mathbb{R}\langle \mathrm{e}^{t^2} \rangle/\mathbb{R}$$

en este caso, e^{t^2} *es trascendente sobre* $\mathbb{R}$ *ya que no existe un polinomio diferencial con coeficientes en* $\mathbb{R}$ *tal que* e^{t^2} *sea raíz.*

Definición 2.12 (Dinámica) *Sean* G *y* $K\langle u \rangle$ *campos diferenciales. Desde el punto de vista del álgebra diferencial, una dinámica es una extensión diferencialmente algebraica finitamente generada* $G/K\langle u \rangle$*,* $(G = K\langle u, \zeta \rangle, \quad \zeta \in G)$*.*

En otras palabras, todo elemento de G satisface una ecuación diferencial con coeficientes que pertenecen al campo diferencial $K\langle u \rangle$.

Un sistema entrada-salida, con entrada $u = (u_1, \ldots, u_m)$, salida $y = (y_1, \ldots, y_p)$, consiste en una extensión diferencialmente algebraica finitamente generada $K\langle u, y \rangle/K\langle u \rangle$ en la cual las componentes $y_1, \ldots, y_p$ son diferencialmente algebraicas sobre el campo $K\langle u \rangle$.

Ejemplo 2.10 *Considere la ecuación diferencial*

$$\dot{u}^2 y + 4\,\ddot{u} = 0$$

En este caso, y *es diferencialmente algebraico sobre* $K\langle u \rangle$*. Se puede expresar la dinámica de la forma* $K\langle u, y \rangle/K\langle u \rangle$*, donde* $K = \mathbb{R}$*.*

Definición 2.13 (Condición de observabilidad algebraica) *Un elemento en* $g \in G$ *se dice algebraicamente observable con respecto a* $\{u, y\}$ *si y sólo si es algebraico sobre* $K\langle u, y \rangle$*.*

Por lo tanto, un estado x *se denomina algebraicamente observable con respecto a* $\{u, y\}$ *si y solamente si es algebraico sobre* $K\langle u, y \rangle$*, esto es, satisface una ecuación diferencial polinomial en términos de* $\{u, y\}$ *, y algunas de sus derivadas con respecto al tiempo,*

$$P(x, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$$

con coeficientes en $K\langle u, y\rangle$. A esta propiedad se le conoce como la condición de observabilidad algebraica.

Ejemplo 2.11 *Sea el sistema no lineal*

$$\begin{aligned}
\dot{x}_1 &= -x_1 x_2 \\
\dot{x}_2 &= -x_2^2 + x_1 + u \\
y &= x_2
\end{aligned}$$

Podemos ver que x_1 y x_2 son algebraicamente observables, debido a que satisfacen

$$\begin{aligned}
x_1 - \dot{y} - y^2 + u &= 0 \\
x_2 - y &= 0
\end{aligned}$$

los cuales son polinomios algebraico-diferenciales con coeficientes en $\mathbb{R}\langle u, y\rangle$.

Ejemplo 2.12 *Considere el siguiente sistema no lineal,*

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{aligned} \tag{2.13}$$

Suponiendo que la salida del sistema es $y = x_2$, entonces

$$\begin{aligned}
x_2 &= y \\
x_3 &= \dot{y} \\
\dot{x}_1 &= y + \dot{y}^2
\end{aligned} \tag{2.14}$$

El sistema (2.14) no es algebraicamente observable ya que x_1 no puede expresarse como un polinomio diferencial algebraico en términos de $\{u, y\}$.

Definición 2.14 (Sistema Liouviliano) *Un sistema dinámico se dice Liouviliano si sus elementos (por ejemplo, variables de estado o parámetros) se pueden obtener por una expresión en términos de integrales o exponenciales de integrales de elementos de $\mathbb{R}$.*

Ejemplo 2.13 *Considere el mismo sistema no lineal del ejemplo 2.12. De la ecuación (2.14) se puede notar que aunque x_1 no satisface la condición de observabilidad algebraica puede obtenerse por medio de la siguiente expresión:*

$$x_1 = \int (y + \dot{y}^2)$$

Por lo tanto, el sistema no lineal (2.13) es Liouviliano. Esto significa que la variable x_1 *puede reconstruirse únicamente considerando la salida* $y = x_2$.

Capítulo 3

Diseño y análisis de algunos observadores

En el caso de sistemas lineales, una solución estándar al problema de reconstrucción del estado esta dada por el observador de Luenberger, sin embargo, para sistemas no lineales existen varios enfoques. Desde los observadores desarrollados por Kalman (1960) y Luenberger (1971) para sistemas lineales, diferentes técnicas de estimación de estado han sido propuestas para sistemas no lineales (Diop y Martínez-Guerra, 2001, Isidori, 1999, Khalil, 2002). La primera categoría de estas técnicas consiste en aplicar algoritmos lineales al sistema linealizado alrededor de la trayectoria estimada, a estas se les conoce como observadores de Kalman y Luenberger extendidos. Otra alternativa, se basa en separar el sistema en una parte lineal y otra no lineal, y diseñar el observador sobre el sistema representado en ésta forma (Aguilar-López, Martínez-Guerra y Maya-Yescas, 2003, Gauthier, Hammouri y Othman, 1992). Éstos esquemas se aplican directamente sobre el sistema original de coordenadas. En una tercera aproximación, el sistema no lineal se transforma en uno lineal mediante un cambio apropiado de coordenadas (Keller, 1987, Martínez-Guerra, Mata-Machuca y Rodríguez, 2013), los estados estimados se obtienen en estas nuevas coordenadas y las coordenadas originales se recuperan mediante una transformación inversa.

Este capítulo se enfoca en el diseño de observadores de estado para sistemas no lineales. Primero, se presentará un observador polinomial con convergencia exponencial para sistemas con dinámicas tipo Lipschitz y para sistemas con dinámicas monótonas no crecientes y no decrecientes. Enseguida, se construirá un observador de orden reducido de tipo acotado para el monitoreo de biorreactores. Finalmente, se muestra la estructura de un observador de orden reducido de tipo asintótico.

3.1. Observador polinomial exponencial (OPE)

3.1.1. OPE para sistemas Lipschitz

La principal contribución de este diseño consiste en la solución del problema de estimación mediante un observador polinomial exponencial de m-ésimo orden, aplicado a sistemas no lineales que satisfacen la condición de *Lipschitz*. La observabilidad del sistema está dada por las definiciones 2.13 y 2.14 del capítulo anterior.

Sea el sistema no lineal expresado en la siguiente forma,

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx\,, \qquad x_0 = x(t_0) \end{aligned} \tag{3.1}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $u \in \mathbb{R}^l$ es el vector de entrada, $y \in \mathbb{R}$ es la salida, $A \in \mathbb{R}^{n\times n}$ y $C \in \mathbb{R}^{1\times n}$ son constantes, $\Psi : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ es un vector no lineal que satisface la condición de *Lipschitz* con constante $L \in \mathbb{R}^+$, esto es:

$$\|\Psi(x,u) - \Psi(\widehat{x},u)\| \leq L\,\|x - \widehat{x}\| \tag{3.2}$$

Se propone el siguiente observador para el sistema (3.1):

$$\dot{\widehat{x}} = A\widehat{x} + \Psi(\widehat{x},u) + \sum_{i=1}^{m} K_i\,(y - C\widehat{x})^{2i-1} \tag{3.3}$$

donde $\hat{x} \in \mathbb{R}^n$, y $K_i \in \mathbb{R}^n$, para $1 \leq i \leq m$.

Antes de analizar la convergencia del observador, se introducen las siguientes condiciones:

H 3.1 *Para algún $\varepsilon > 0$ y $A \in \mathbb{R}^{n\times n}$ existe una solución $P \in \mathbb{R}^{n\times n}$ definida positiva, $P = P^T > 0$, para la ecuación algebraica:*

$$A^TP + PA + L^2P^2 + I + \varepsilon I = 0.$$

H 3.2 $\lambda_{min}\left(M_i + M_i^T\right) \geq 0$*, donde $M_i = PK_iC$, $1 \leq i \leq m$.*

H 3.3 *El sistema (3.1) es observable, es decir, que el vector de estado x puede ser reconstruido.*

Para demostrar el tipo de convergencia del observador se analiza el error de observación definido como $e = x - \hat{x}$. De las ecuaciones (3.1) y (3.3), la dinámica del error de observación se obtiene como:

$$\dot{e} = Ae + \Psi(x,u) - \Psi(\widehat{x},u) - \sum_{i=1}^{m} K_i\,(Ce)^{2i-1} \tag{3.4}$$

A continuación, se presenta un lema (Raghavan y Hedrick, 1994) que será utilizado en la prueba de convergencia.

Lema 3.1 *Dado el sistema (3.1) y su observador (3.3), con $e := x - \hat{x}$. Si $P = P^T > 0$ entonces:*

$$2\,e^T P\,[\Psi(x,u) - \Psi(\hat{x},u)] \leq L^2 e^T P^2\,e + e^T e \tag{3.5}$$

El siguiente teorema demuestra la convergencia del observador.

Teorema 3.1 *Sea el sistema (3.1), si se cumplen las condiciones H3.1 a H3.3, entonces, el sistema no lineal (3.3) es un observador polinomial exponencial del sistema (3.1); es decir, existen constantes $\kappa > 0$ y $\lambda > 0$ tal que*

$$\|e(t)\| \leq \kappa\ \|e_0\|\ exp(-\lambda t)$$

donde $\kappa = \sqrt{\dfrac{\beta}{\alpha}}$, $\lambda = \dfrac{\varepsilon}{2\beta}$, $\alpha = \lambda_{min}(P)$ y $\beta = \lambda_{max}(P)$.

Demostración. La prueba se basa en el segundo método de estabilidad de Lyapunov (Khalil, 2002). Sea $P = P^T > 0$, se considera la función candidata de Lyapunov $V = e^T Pe$. La derivada de V con respecto al tiempo, a lo largo de las trayectorias de (3.4) esta dada por

$$\begin{aligned}
\dot{V} &= \dot{e}^T Pe + e^T P\dot{e} \\
&= e^T\left[A^T P + PA\right]e + 2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] - 2e^T P\sum_{i=1}^{m} K_i\,(Ce)^{2i-1}
\end{aligned}$$

Mediante el lema 3.1 se obtiene,

$$\dot{V} \leq\ e^T\left[A^T P + PA + L^2 P^2 + I\right]e - 2e^T P\sum_{i=1}^{m} K_i\,(Ce)^{2i-1}$$

Haciendo algunas manipulaciones algebraicas al último término del lado derecho de la desigualdad anterior, considerando que $Ce \in \mathbb{R}$, se tiene,

$$\dot{V} \leq \; e^T \left[A^T P + PA + L^2 P^2 + I\right] e - 2\sum_{i=1}^{m} (Ce)^{2i-2} \, e^T P K_i C e$$

Por simplicidad, se define $M_i := PK_iC$, $1 \leq i \leq m$, entonces

$$\begin{aligned}\dot{V} \leq \; & e^T \left[A^T P + PA + L^2 P^2 + I\right] e - \sum_{i=1}^{m} (Ce)^{2i-2} \left[e^T M_i e + (e^T M_i e)^T\right] \\ = \; & e^T \left[A^T P + PA + L^2 P^2 + I\right] e - \sum_{i=1}^{m} (Ce)^{2i-2} \, e^T (M_i + M_i^T) e\end{aligned}$$

De la condición H3.2, el segundo término del lado derecho de la desigualdad anterior será siempre positivo o cero, por lo tanto,

$$\dot{V} \leq e^T \left[A^T P + PA + L^2 P^2 + I\right] e \tag{3.6}$$

De la condición H3.1, se obtiene

$$\dot{V} \leq -\varepsilon \|e\|^2 \tag{3.7}$$

Ahora, se expresa la función de Lyapunov como $V = \|e\|_P^2$, donde $\alpha\|e\|^2 \leq V(e) \leq \beta\|e\|^2$, con $\alpha = \lambda_{min}(P)$, $\beta = \lambda_{max}(P) \in \mathbb{R}^+$. Tomando su derivada y sustituyendo en (3.20), se tiene

$$\frac{d}{dt}\|e\|_P \leq -\frac{\varepsilon}{2\beta}\|e\|_P$$

Finalmente, se tiene el siguiente resultado,

$$\|e(t)\| \leq \kappa \, \|e_0\| \; \exp(-\lambda t) \tag{3.8}$$

donde $\kappa = \sqrt{\dfrac{\beta}{\alpha}}$ y $\lambda = \dfrac{\varepsilon}{2\beta}$. □

3.1.2. OPE para sistemas monótonos

Suponiendo que la parte no lineal del sistema dinámico (3.1) tuviera un comportamiento monótono no decreciente o monótono no creciente o una combinación de ambos, entonces se propone representar al sistema en la forma (3.9) debido a la dificultad que implica determinar la constante de Lipschitz para sistemas que contienen no linealidades de éste tipo.

Sea el siguiente sistema no lineal algebraicamente observable o Liouviliano,

$$\begin{aligned} \dot{x} = & \; A\,x + \psi(x) + \varphi(x) + \zeta(u) \\ y = & \; C\,x, \end{aligned} \tag{3.9}$$

donde $x \in \mathbb{R}^n$, es el vector de estado; $u \in \mathbb{R}^l$, es el vector de entrada, $l \leq n$; $y \in \mathbb{R}$ es la salida; $\zeta(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$ es una función que depende de la entrada, con $\|\zeta\| < \infty$; $A \in \mathbb{R}^{n\times n}$ y $C \in \mathbb{R}^{1\times n}$ son constantes; y $\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ son funciones no lineales que dependen de las variables de estado.

En este caso cada $\psi_i(\cdot)$ es una función no decreciente, es decir, para todo $a, b \in \mathbb{R}$, $a > b$, se satisface la siguiente condición,

$$0 \leq \frac{\psi_i(a) - \psi_i(b)}{a - b}, \qquad i = 1, \ldots, n. \tag{3.10}$$

De manera similar, cada $\varphi_i(\cdot)$ es una función no creciente, entonces, para todo $a, b \in \mathbb{R}$, $a > b$, se satisface

$$\frac{\varphi_i(a) - \varphi_i(b)}{a - b} \leq 0, \qquad i = 1, \ldots, n. \tag{3.11}$$

El observador polinomial para el sistema (3.9) es,

$$\dot{\widehat{x}} = A\widehat{x} + \psi(\widehat{x}) + \varphi(\widehat{x}) + \zeta(u) + \sum_{i=1}^{m} K_i \left(y - C\widehat{x}\right)^{2i-1}, \tag{3.12}$$

donde $\hat{x} \in \mathbb{R}^n$, y $K_i \in \mathbb{R}^n$, para $1 \leq i \leq m$.

Para la convergencia del observador se analizará nuevamente el error de estimación $e = x - \hat{x}$. De las ecuaciones (3.9) y (3.12), la dinámica del error está dada por,

$$\dot{e} = \bar{A}e + \phi(e) + \rho(e) - \sum_{i=2}^{m} K_i \left(Ce\right)^{2i-1}, \tag{3.13}$$

donde $\phi(e) := \psi(x) - \psi(\hat{x})$, $\rho(e) := \varphi(x) - \varphi(\hat{x})$ y $\bar{A} := A - K_1 C$.

De la condición para las funciones no decrecientes (3.10) se obtiene que cada componente de $\phi(e)$ satisface

$$0 \leq \frac{\phi_i(e_i)}{e_i}, \qquad \forall\, e_i \neq 0, \tag{3.14}$$

lo cual implica la siguiente relación entre $\phi(e)$ y e,

$$e^T\phi(e) \;\; = \;\; \sum_{i=1}^{n} e_i\phi_i(e_i) = \sum_{i=1}^{n} e_i^2 \frac{\phi_i(e_i)}{e_i}$$

De (3.14) se obtiene la siguiente condición,

$$0 \leq e^T \phi(e). \tag{3.15}$$

De manera similar, de la condición (3.11) se tiene

$$e^T \rho(e) \leq 0. \tag{3.16}$$

Las propiedades (3.15) y (3.16) se utilizarán para demostrar que el error de estimación $e(t)$ decae exponencialmente.

Proposición 3.1 *Considere el sistema no lineal (3.9) y el observador (3.12). Si existe una matriz $P = P^T > 0$, y escalares positivos ε, ϵ_1, ϵ_2 que satisfacen la LMI*

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P + \epsilon_1 I & P - \epsilon_2 I \\ P + \epsilon_1 I & 0 & 0 \\ P - \epsilon_2 I & 0 & 0 \end{bmatrix} \leq 0, \tag{3.17}$$

y

$$\lambda_{\text{mín}} \left(M_i + M_i^T \right) \geq 0 \quad , \; i = 2, \ldots, m, \tag{3.18}$$

con $M_i := PK_iC$. Entonces, existen constantes positivas κ y ξ tal que, para todo $t \geq 0$,

$$\|e(t)\| \leq \kappa \; exp(-\xi t)$$

donde $\kappa = \sqrt{\dfrac{\beta}{\alpha}} \|e(0)\|$, $\xi = \dfrac{\varepsilon}{2\beta}$, $\alpha = \lambda_{min}(P)$, y $\beta = \lambda_{max}(P)$.

Demostración. Se considera la función candidata de Lyapunov $V = e^T Pe$. La derivada de V a lo largo de las trayectorias del error de estimación (3.13) es

$$\begin{aligned} \dot{V} &= e^T \left[\bar{A}^T P + P\bar{A} \right] e \\ &\quad + 2e^T P\phi(e) + 2e^T P\rho(e) - 2\sum_{i=2}^{m} (Ce)^{2i-2} e^T M_i e \\ &= \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P + P\bar{A} & P & P \\ P & 0 & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix} \\ &\quad - \sum_{i=2}^{m} (Ce)^{2i-2} e^T (M_i + M_i^T) e \end{aligned}$$

Tomando en cuenta (3.17) y (3.18),

$$\dot{V} \leq \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix}^T \begin{bmatrix} -\varepsilon I & -\epsilon_1 I & \epsilon_2 I \\ -\epsilon_1 I & 0 & 0 \\ \epsilon_2 I & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix} \tag{3.19}$$

Desarrollando la expresión (3.19)

$$\dot{V} \leq -\varepsilon e^T e - 2\epsilon_1\, e^T \phi(e) + 2\epsilon_2\, e^T \rho(e).$$

Usando las propiedades (3.15) y (3.16) se tiene que,

$$\dot{V} \leq -\varepsilon \|e\|^2 \tag{3.20}$$

El resto de la demostración se sigue como en el Teorema 3.1. □

Corolario 3.1 *Sea $\psi(\cdot) \equiv 0$ en el sistema (3.9). Entonces el sistema (3.12) es un observador exponencial para el sistema no lineal (3.9) si existe una matriz $P = P^T > 0$, y escalares $\varepsilon > 0$, $\epsilon_2 > 0$ que satisfacen la LMI*

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P - \epsilon_2 I \\ P - \epsilon_2 I & 0 \end{bmatrix} \leq 0 \tag{3.21}$$

y

$$\lambda_{\text{mín}}\left(M_i + M_i^T\right) \geq 0 \quad , \; i = 2, \ldots, m. \tag{3.22}$$

Con κ y ξ definidos como en la Proposición 3.1.

Demostración. La prueba de este corolario se realiza como en la Proposición 3.1. □

Corolario 3.2 *Sea $\varphi(\cdot) \equiv 0$ en el sistema (3.9). Entonces el sistema (3.12) es un observador exponencial para el sistema no lineal (3.9) si existe una matriz $P = P^T > 0$, y escalares $\varepsilon > 0$, $\epsilon_1 > 0$ que satisfacen la LMI*

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P + \epsilon_1 I \\ P + \epsilon_1 I & 0 \end{bmatrix} \leq 0 \tag{3.23}$$

y

$$\lambda_{\text{mín}}\left(M_i + M_i^T\right) \geq 0 \quad , \; i = 2, \ldots, m. \tag{3.24}$$

Con κ y ξ definidos como en la Proposición 3.1.

Demostración. La prueba de este corolario se realiza como en la Proposición 3.1. □

3.2. Observador asintótico de orden reducido

Considere el siguiente sistema no lineal

$$\begin{aligned} \dot{x}(t) = & \ f(x,u) \\ y(t) = & \ h(\bar{x}) \end{aligned} \tag{3.25}$$

donde $f \in \mathbb{R}^n$ es diferenciable, con $f(0,0) = 0$, $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ es el vector de estado, $\bar{x} \in \mathbb{R}^p$ representa los estados conocidos, siendo $1 \leq p < n$, $y \in \mathbb{R}^q$ es el vector de salida, $h : \mathbb{R}^p \to \mathbb{R}^q$ es una función continua y $u \in \mathbb{R}^l$ es el vector de entrada ($l \leq n$).

Ahora se separa (3.25) en dos sistemas dinámicos con vectores de estado $\bar{x} \in \mathbb{R}^p$ y $\eta \in \mathbb{R}^{n-p}$, donde $x^T = (\bar{x}^T, \eta^T)$. El primer sistema describe los estados conocidos y el segundo representa los estados desconocidos. Por lo tanto, el sistema (3.25) se reescribe como:

$$\begin{aligned} \dot{\bar{x}}(t) = & \ \bar{f}(x,u) \\ \dot{\eta}(t) = & \ \Delta(x,u) \\ y(t) = & \ h(\bar{x}) \end{aligned} \tag{3.26}$$

donde $\bar{f} \in \mathbb{R}^p$, $f^T(x) = \left(\bar{f}^{\,T}(x,u), \Delta^T(x,u)\right)$, y $\Delta \in \mathbb{R}^{n-p}$ es una función desconocida.

El problema consiste en reconstruir el vector de estado $\eta(t) = (\eta_{(p+1)}, \ldots, \eta_n)^T$. Antes de proponer el observador se consideran la siguientes hipótesis.

H 3.4 *Cada variable de estado $\eta_i \in \mathbb{R}$ es algebraicamente observable, en el sentido de la Definición 2.13, es decir, η_i satisface un polinomio diferencial algebraico (3.27), en términos de $\{u, y\}$ y algunas de sus primeras $r_1, r_2 \in \mathbb{N}$ derivadas con respecto al tiempo,*

$$\eta_i = \phi_i(u, \dot{u}, \ldots, \overset{(r_1)}{u}, y, \dot{y}, \ldots \overset{(r_2)}{y}) \ , \quad p+1 \leq i \leq n, \tag{3.27}$$

donde $\phi_i : \mathbb{R}^{(r_1+1)l} \times \mathbb{R}^{(r_2+1)q} \to \mathbb{R}$.

H 3.5 *En el sistema dinámico:*

$$\dot{\eta}_i(t) = \Delta_i(x,u) \tag{3.28}$$

la función desconocida Δ_i es acotada, por lo tanto, $\|\Delta\| \leq M < \infty$.

Lema 3.2 *Si se cumplen las hipótesis H3.4 y H3.5, entonces el sistema*

$$\dot{\hat{\eta}}_i = K_i(\eta_i - \hat{\eta}_i) \tag{3.29}$$

es un observador asintótico del sistema (3.28), donde $\hat{\eta}_i$ es el valor estimado de η_i y $K_i \in \mathbb{R}^+$ es la ganancia del observador.

Demostración: se define el error de observación como, $e(t) = \eta_i(t) - \hat{\eta}_i(t)$. Derivando $e(t)$ con respecto al tiempo,

$$\dot{e}_i(t) = \dot{\eta}_i(t) - \dot{\hat{\eta}}_i \tag{3.30}$$

Sustituyendo (3.28) y (3.29) en (3.30), se obtiene

$$\dot{e}_i(t) + K_i e_i(t) = \Delta_i(t) \tag{3.31}$$

Resolviendo (3.31) se tiene que,

$$e_i(t) = \exp(-K_i t)\left[e_{i0} + \int_0^t \exp(K_i\tau)\Delta_i(\tau)d\tau\right], \tag{3.32}$$

donde $e_{i0} = e_i(0)$ es la condición inicial.
Aplicando las desigualdades del triángulo y de Cauchy-Schwartz en (3.32),

$$\begin{aligned} 0 \leq |e_i(t)| &\leq \exp(-K_i t)|e_{i0}| + \exp(-K_i t)\int_0^t |\exp(K_i\tau)\Delta_i(\tau)d\tau| \\ &\overset{\text{de H3.5}}{\leq} \exp(-K_i t)|e_{i0}| + M\int_0^t \exp[K_i(\tau - t)]d\tau \\ &= \exp(-K_i t)|e_{i0}| + \frac{M}{K_i}[1 - \exp(-K_i t)] \end{aligned}$$

Si $t \to \infty$,

$$0 \leq \overline{\lim_{t\to\infty}} |e_i(t)| \leq |e_{i0}| \overline{\lim_{t\to\infty}}[\exp(-K_i t)] + \overline{\lim_{t\to\infty}}\left\{\frac{M}{K_i}[1 - \exp(-K_i t)]\right\} = \frac{M}{K_i}$$

La ganancia del observador K_i determina la convergencia deseada, por lo tanto

$$\overline{\lim_{t\to\infty}} |e_i(t)| \approx 0$$

Entonces, (3.29) es un observador asintótico del sistema (3.28). □

Capítulo 4

Monitoreo de biorreactores

En éste capítulo se diseña un estimador uniformemente acotado, el cual estima la biomasa y otros productos mediante la medición de la concentración de sustrato empleando las características de los modelos de crecimiento no estructurados, los cuales son linealmente dependientes. Este estimador se propone como una versión simplificada de la metodología dada por (Lemesle y Gouzé, 2005, Veloso, Rocha y Ferreira, 2008), consiste en realizar un cambio lineal de variable, dado en forma natural el cual permite obtener un estimador de estado algebraico-diferencial.

La metodología de estimación propuesta se aplica a dos clases de modelos cinéticos no estructurados: el modelo de Haldane, el cual se considera para procesos biológicos con inhibición de sustrato y el modelo clásico de Monod. Ambos modelos se aplican a una clase de biorreactores continuos.

4.1. Planteamiento del problema

Se considera una versión simplificada del modelo de Haldane dado por (Vargas, Soto, Moreno y Buitrón, 2000). El modelo matemático es,

$$\begin{aligned} \frac{d\,S}{d\,t} &= D\,(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} + k_d X \\ \frac{d\,X}{d\,t} &= -D\,X + \mu(S)X - k_d X \end{aligned} \tag{4.1}$$

donde, $\mu(S) = \mu_{max} S/(\delta + S + S^2/\phi)$ es la rapidez de crecimiento específico, μ_{max} es la máxima rapidez de crecimiento, ϕ es la constante de inhibición y δ es la constante de saturación media.

Asimismo, se utiliza el modelo de Monod, el cual fue presentado previamente en (Vargas, Soto, Moreno y Buitrón, 2000),

$$\begin{aligned} \frac{d\,S}{d\,t} &= D\,(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} \\ \frac{d\,X}{d\,t} &= -D\,X + \mu(S)X \\ \frac{d\,P}{d\,t} &= -D\,P + \mu(S)\frac{X}{Y_{P/X}} \end{aligned} \tag{4.2}$$

donde, $\mu(S) = \mu_{max} S/(\beta + S)$.

Las concentraciones de sustrato, biomasa y producto se denotan por S, X y P, respectivamente, $D = q/V$ es la rapidez de dilución, con V el volumen del biorreactor y q la constante de flujo pasando a través de éste, S_{in} es la entrada de concentración de sustrato, $Y_{S/X}$ y $Y_{P/X}$ son los correspondientes coeficientes de crecimiento, β es la constante de saturación media. Las entradas D y S_{in} son fijas y conocidas.

En el análisis que enseguida se presenta, se considera que la salida disponible en los sistemas (4.1) y (4.2) es la concentración de sustrato S.

Antes de proponer el estimador de estado para los modelos (4.1) y (4.2), es necesario probar la condición de observabilidad algebraica (ver la Definición 2.13).

La condición de observabilidad algebraica para el modelo de Haldane se determina como sigue,

$$\dot{y} - u\,(S_{in} - y) + \left(\frac{\mu_{max}\phi\,y}{\delta\phi + \phi\,y + y^2}\;\;\frac{1}{Y_{S/X}} - k_d\right)\,X = 0 \tag{4.3}$$

donde, $u = D > 0$, $y = S$ y $\left(\frac{\mu_{max}\phi\,y}{\delta\phi + \phi\,y + y^2}\;\;\frac{1}{Y_{S/X}} - k_d\right) \neq 0$.

Nótese de la ecuación (4.3) que la variable de estado X satisface una ecuación algebraica con coeficientes en $\mathbb{R}\langle u, y\rangle$, es decir, X es algebraicamente observable.

La condición de observabilidad algebraica del modelo de Monod se calcula para las variables de estado X y P. Para X, el análisis es análogo al caso anterior,

$$-\,\dot{y}\,(\beta + y)\,Y_{S/X} + u\,(S_{in} - y)(\beta + y)\,Y_{S/X} - \mu_{max}\,y\,X = 0 \tag{4.4}$$

Para la variable P se tiene que,

$$\dot{P} + u\,P = -\frac{Y_{S/X}}{Y_{P/X}}\dot{y} + \frac{Y_{S/X}}{Y_{P/X}}u\,(S_{in} - y) \tag{4.5}$$

Como se puede notar de la ecuación (4.5), la variable P no es observable en el sentido algebraico. En la referencia (Aguilar-López, Mata-Machuca, Martínez-Guerra y López-Pérez, 2009) se muestra que P tampoco es observable en el sentido geométrico diferencial. Sin embargo, P puede ser estimada dada la estabilidad de la dinámica de la ecuación (4.5) en términos de $\{u, y\}$ y sus derivadas con respecto al tiempo.

Basándonos en lo anterior, se dice que la concentración P es detectable (Aguilar–López, Mata–Machuca y Martínez–Guerra, 2011) con respecto a $\{u, y\}$. Es importante mencionar que la estimación de P depende completamente de la estabilidad asintótica de la ecuación (4.5). En lugar de la ecuación (4.5) se plantea el siguiente sistema,

$$\left\{\begin{aligned} \dot{\overline{z}} =& \quad -u\,\overline{z} + \xi \\ \dot{z} =& \quad -u\,z + \eta \\ P =& \quad z - y \\ X =& \quad \overline{z} - y \end{aligned}\right. \tag{4.6}$$

donde, $u = D > 0$, $y = S$,

$$\eta = u\,S_{in} + \frac{\mu_{max}\,y}{\beta + y}\left[\frac{1}{Y_{P/X}} - \frac{1}{Y_{S/X}}\right]\,(\overline{z} - y)\,,$$

y

$$\xi = u\,S_{in} + \frac{\mu_{max}\,y}{\beta + y}\,(\hat{\overline{z}} - y)\left(1 - \frac{1}{Y_{S/X}}\right)$$

El esquema de estimación (4.5) requiere la diferenciabilidad de P y la diferenciación de y, mientras que el esquema (4.6) no requiere la diferenciabilidad de P ni la diferenciación de y. Esto nos permite relajar la suposición sobre la diferenciabilidad de P, la cual se necesitó para obtener el sistema (4.6).

Por lo tanto, de una forma natural se define un nuevo estimador dado por el sistema (4.6), el cual es capaz de reconstruir las variables X y P.

Entonces, la variable X para el sistema de Haldane (4.1) se determina de manera similar, es decir,

$$\begin{aligned} \dot{w} =& \quad -u\,w + \psi \\ X =& \quad w - y \end{aligned} \tag{4.7}$$

donde, $u = D > 0$, $y = S$, y $\psi = u\ S_{in} + \frac{\mu_{max} y}{(\delta + y + y^2/\phi)}\ (w - y) \left(1 - \frac{1}{Y_{S/X}}\right)$.

4.1.1. Diseño del estimador

En este caso, se supone que la salida del sistema es medida exactamente, es decir, $S = \hat{S}$. De esta manera, sólo se debe estimar la concentración de biomasa X y la concentración de producto P.

Considere el modelo de Haldane dado por el sistema (4.1). Se hace el siguiente cambio de variable,

$$\hat{w} = \hat{X} + S \tag{4.8}$$

La dinámica de $\hat{w}$ es,

$$\dot{\hat{w}} = -D\ \hat{w} + \hat{\psi} \tag{4.9}$$

donde, $D > 0$, $\hat{\psi} = D\ S_{in} + \mu(S)\ (\hat{w} - S) \left(1 - \frac{1}{Y_{S/X}}\right)$ y $\mu(S) = \mu_{max} S/(\delta + S + S^2/\phi)$.

Ahora, se considera el modelo de Monod dado por el sistema (4.2). Se introducen los siguientes cambios de variable,

$$\hat{\bar{z}} = \hat{X} + S \tag{4.10}$$

$$\hat{z} = \hat{P} + S \tag{4.11}$$

Las dinámicas de (4.10) y (4.11) son las siguientes,

$$\dot{\hat{\bar{z}}} = -D\ \hat{\bar{z}} + \hat{\xi} \tag{4.12}$$

$$\dot{\hat{z}} = -D\ \hat{z} + \hat{\eta} \tag{4.13}$$

con, $D > 0$, $\mu(S) = \mu_{max} S/(\beta + S)$,

$$\hat{\xi} = D\ S_{in} + \mu(S)\ (\hat{\bar{z}} - S) \left(1 - \frac{1}{Y_{S/X}}\right)$$

y

$$\hat{\eta} = D\ S_{in} + \mu(S)\ (\hat{\bar{z}} - S) \left(\frac{1}{Y_{S/X}} - \frac{1}{Y_{S/X}}\right)$$

Suponiendo que, $0 \leq \mu(S) \leq \mu_{max} < \infty$, para toda $S > 0$, entonces,

$$
\begin{aligned}
|\hat{\psi} - \psi| &\leq \mu_{max} \left| 1 - \frac{1}{Y_{S/X}} \right| |\delta_1| = N_1(\delta_1) < \infty \\
|\hat{\xi} - \xi| &\leq \mu_{max} \left| 1 - \frac{1}{Y_{S/X}} \right| |\delta_2| = N_2(\delta_2) < \infty \\
|\hat{\eta} - \eta| &\leq \mu_{max} \left| \frac{1}{Y_{P/X}} - \frac{1}{Y_{S/X}} \right| |\delta_2| = N_3(\delta_2) < \infty
\end{aligned}
$$

donde, $\delta_1 = \hat{w} - w$ y $\delta_2 = \hat{\overline{z}} - \overline{z}$ son errores de estimación.

Lema 4.1 *Las ecuaciones diferenciales (4.6) y (4.7) tienen la forma*

$$\dot{\vartheta} = -D\,\vartheta + \zeta \;, \tag{4.14}$$

las ecuaciones diferenciales (4.9), (4.12) y (4.13) se representan como

$$\dot{\hat{\vartheta}} = -D\,\hat{\vartheta} + \hat{\zeta} \tag{4.15}$$

y se supone que

H 4.1 $|\hat{\zeta} - \zeta| \leq N(e) < \infty$

entonces, el error de estimación es uniformemente acotado, con $D > 0$.

Demostración.. Se define el error de estimación como sigue,

$$e := \hat{\vartheta} - \vartheta \tag{4.16}$$

Derivando la ecuación (4.16), y sustituyendo las ecuaciones (4.14) y (4.15), se obtiene

$$\dot{e} + D\,e = \hat{\zeta} - \zeta \tag{4.17}$$

La solución de (4.17) está dada por la siguiente expresión

$$e = \exp^{-Dt} e_0 + \int_0^t \exp^{D(\tau - t)}(\hat{\zeta} - \zeta)\, d\,\tau \quad , \quad e(0) = e_0 \tag{4.18}$$

Aplicando las desigualdades del triángulo y de Cauchy-Schwarz a la expresión (4.18),

$$0 \leq |\,e\,| \leq \left|\,\exp^{-Dt}\,\right|\,|\,e_0\,| + \int_0^t |\,\exp^{D(\tau - t)}(\hat{\zeta} - \zeta)\,d\,\tau\,|$$

Utilizando la hipótesis H4.1,

$$0 \leq |\, e \,| \leq \left| \exp^{-Dt} \right| \,|\, e_0 \,| + \; N(e) \int_0^t |\, \exp^{D(\tau - t)} \; d\,\tau \,| \tag{4.19}$$

Cuando $t \to t_0$, t_0 suficientemente grande,

$$\begin{aligned} 0 \leq \overline{\lim_{t \to t_0}} \,|\, e \,| &\leq \overline{\lim_{t \to t_0}} \, N(e) \int_0^t |\, \exp^{D(\tau - t)} \; d\,\tau \,| \\ 0 \leq \overline{\lim_{t \to t_0}} \,|\, e \,| &\leq \overline{\lim_{t \to t_0}} \, \frac{N(e)}{D} \,|\, 1 - \exp^{-t} \,| \\ 0 \leq \overline{\lim_{t \to t_0}} \,|\, e \,| &\leq \frac{N(e)}{D} \end{aligned}$$

Se concluye que el error de estimación es uniformemente acotado. □

Corolario 4.1 *El estimador uniformemente acotado para el modelo de Haldane dado por el sistema (4.1) es,*

$$\begin{aligned} \dot{\hat{w}} &= -D\, \hat{w} + \hat{\psi} \\ \hat{X} &= \hat{w} - S \end{aligned}$$

con condiciones iniciales, $\hat{w}_0 = \hat{w}(0)$, $S_0 = S(0)$. *Además,* $\hat{\psi} = D\, S_{in} + \mu(S)\, (\hat{w} - S) \left(1 - \frac{1}{Y_{S/X}}\right)$, $D > 0$ *y* $\mu(S) = \mu_{max} S / (\delta + S + S^2/\phi)$.

Corolario 4.2 *El estimador uniformemente acotado para el modelo de Monod dado por el sistema (4.2) es el siguiente,*

$$\begin{aligned} \dot{\hat{\bar{z}}} &= -D\, \hat{\bar{z}} + \hat{\xi} \\ \dot{\hat{z}} &= -D\, \hat{z} + \hat{\eta} \\ \hat{X} &= \hat{\bar{z}} - S \\ \hat{P} &= \hat{z} - S \end{aligned}$$

con condiciones iniciales, $\hat{\bar{z}}_0 = \hat{\bar{z}}(0)$, $\hat{z}_0 = \hat{z}(0)$, $S_0 = S(0)$, $D > 0$, $\mu(S) = \mu_{max} S / (\beta + S)$ *y las funciones no lineales están dadas por,*

$$\hat{\xi} = D\, S_{in} + \mu(S)\, (\hat{\bar{z}} - S) \left(1 - \frac{1}{Y_{S/X}}\right)$$

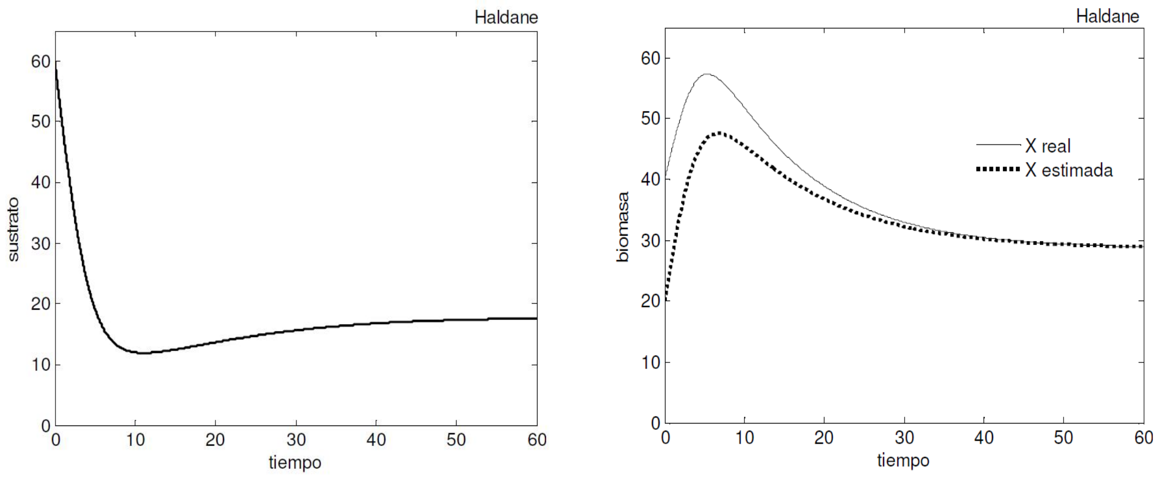
$$\hat{\eta} = D\, S_{in} + \mu(S)\, (\hat{\bar{z}} - S) \left(\frac{1}{Y_{S/X}} - \frac{1}{Y_{S/X}}\right)$$

4.2. Resultados numéricos

Se muestran algunas simulaciones; para el modelo de Monod $\mu(S) = S/(140 + S)$ y para el modelo de Haldane $\mu(S) = S/(140 + S + S^2/81.25)$.

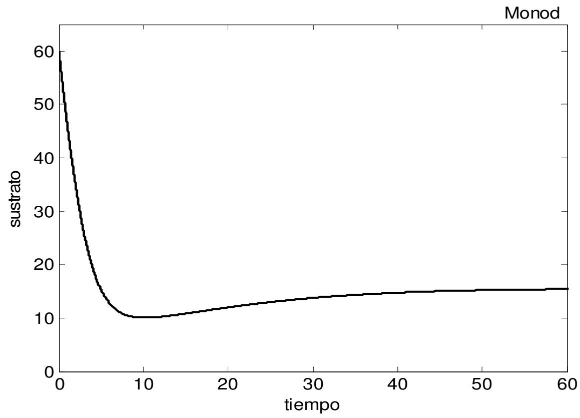
Los parámetros son, $S_{in} = 50$,$D = 0.1$, $Y_{S/X} = 0.9$, $Y_{P/X} = 0.36$, $k_d = 0.01$ y las condiciones iniciales $S(0) = 60$, $X(0) = 40$, $P(0) = 30$, $\hat{w}(0) = 80$, $\hat{\hat{z}}(0) = 80$, $\hat{z}(0) = 140$, con unidades apropiadas.

En la figura 4.1, se muestra el estado estimado para el modelo de Haldane utilizando el estimador con error uniformemente acotado dado por el corolario 4.1. Se puede notar que el observador reconstruye la variable X. Los resultados de simulación del estimador con error uniformemente acotado dado por el corolario 4.2, para el modelo de Monod se presentan en la figura 4.2. Se muestra que el estimador reconstruye las variables X y P.

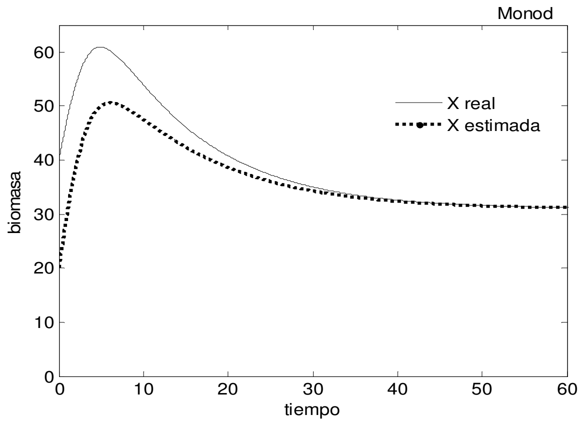


(a) (b)

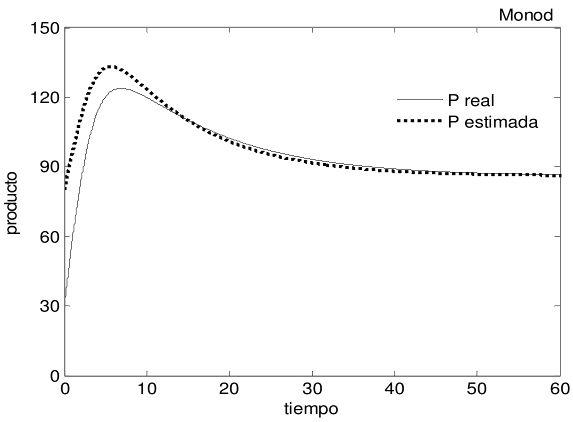
Figura 4.1: Estimación de estado para el modelo de Haldane: (a) Salida del sistema, (b) Biomasa



(a)



(b)



(c)

Figura 4.2: Estimación de estado para el modelo de Monod: (a) Salida del sistema, (b) Biomasa, (c) Producto.

Capítulo 5

Sincronización de sistemas caóticos

Sea el sistema maestro M,

$$M : \begin{cases} \dot{x} = & f(x, u) \\ y = & C\, x \end{cases} \quad , \qquad x_0 = x(t_0)$$

y el sistema esclavo E,

$$E : \begin{cases} \dot{\hat{x}} = & \hat{f}(\hat{x}, u, y) \\ \hat{y} = & C\, \hat{x} \end{cases} \quad , \qquad \hat{x}_0 = \hat{x}(t_0)$$

donde, $x, \hat{x} \in \mathbb{R}^n$; $u \in \mathbb{R}^p$, $p \leq n$; $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, $\hat{f}(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n$; $y, \hat{y} \in \mathbb{R}$.

Típicamente, el sistema esclavo E depende del sistema maestro mediante la señal y. La conexión entre los sistemas maestro y esclavo representa el esquema de sincronización, el cual se muestra en la figura 5.1.

La sincronización significa que ambos sistemas tienen un comportamiento correlacionado en el tiempo. En este sentido, la sincronización resulta de la comparación de las propiedades de los sistemas maestro y esclavo. Debido a lo anterior, se dice que la función invariante en el tiempo, $e_y = C\,(x - \hat{x})$, donde, $e_y(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, compara la medida del comportamiento de ambos sistemas.

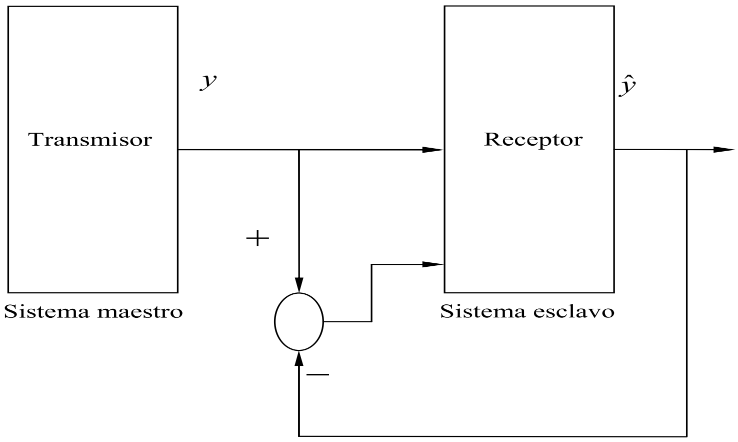


Figura 5.1: Esquema de sincronización *maestro-esclavo*

5.1. Sincronización vista como un problema de observación

La conexión entre los observadores para sistemas no lineales y la sincronización de sistemas caóticos es conocida como configuración maestro-esclavo, la cual tiene como característica principal que la señal únicamente se transmite del sistema maestro al sistema esclavo.

El problema de sincronización de sistemas caóticos se puede formular como sigue: dado un sistema caótico, el cual se considera como el sistema maestro, y otro sistema idéntico, el cual será el sistema esclavo, el objetivo es forzar la respuesta del sistema esclavo para sincronizarla con la respuesta del sistema maestro (Feki, 2003b, Nijmeijer y Mareels, 1997, Pecora y Caroll, 1990). La idea es usar la salida del sistema maestro tal que los estados del sistema esclavo sigan a los estados del sistema maestro.

Existen diferentes nociones o definiciones de sincronización de sistemas caóticos (Fradkov, 2007). La más fuerte y comúnmente extendida es la denominada sincronización idéntica completa o perfecta, donde los estados del sistema esclavo convergen de manera asintótica a los estados del sistema maestro. Por ejemplo, sean los sistemas descritos por las siguientes ecuaciones

$$\begin{cases} \dot{x} = & f(x,t) \\ y = & Cx \end{cases} \tag{5.1}$$

y

$$\dot{\hat{x}} = \hat{f}(y, \hat{x}, t) \tag{5.2}$$

donde: y es la salida del sistema maestro; x, $\hat{x}$ son los vectores de estado de los subsistemas a ser sincronizados.

Consideremos el problema de sincronización asintótica completa en la siguiente forma,

$$\lim_{t \to \infty} \|x - \hat{x}\| = 0 \tag{5.3}$$

Esto puede interpretarse como la reconstrucción del estado del sistema (5.1) mediante el estimado $\hat{x}$. A lo anterior se le conoce en teoría de control como el *problema de observación. En el problema de sincronización el sistema esclavo es considerado como el observador de estado.*

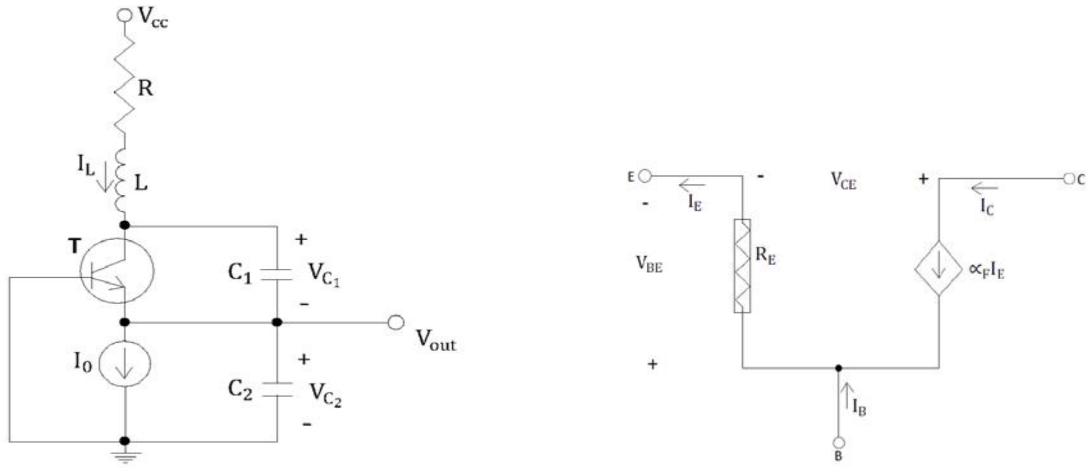
5.2. Sincronización mediante el observador polinomial exponencial

Esta sección trata sobre la sincronización de sistemas caóticos mediante el observador polinomial exponencial diseñado en el capítulo 3. Primero se presentará la sincronización del oscilador de Colpitts el cual satisface la condición de Lipschitz y posteriormente la sincronización del circuito de Chua visto como un sistema con no linealidades monótonas.

5.2.1. Sincronización del oscilador de Colpitts

Modelo

Se considera la configuración clásica del oscilador de Colpitts, el cual esta formado por un transistor de tipo BJT (Bipolar Junction Transistor), como elemento de ganancia y una red de resonancia conformada por un inductor y un par de capacitores (Figura 5.2a).



(a) (b)

Figura 5.2: Oscilador de Colpitts. (a) Circuito. (b) Modelo del transistor T. Los parámetros del circuito son: L = 100 μH; $C_1 = C_2 = 47$ nF, R = 45 Ω, $I_0 = 5$ mA.

De acuerdo a la teoría cualitativa de dinámicas no lineales, se seleccionará un modelo mínimo para este circuito. La idea es considerar un modelo simple que contenga las características más importates del oscilador de Colpitts.

Para el modelado de dicho circuito se toman las siguientes consideraciones:

1. La corriente en el emisor se generará por medio de I_0.

2. Elementos pasivos y activos ideales.

3. El transistor se modela (figura 5.2b) como una resistencia no lineal R_E controlada por medio de voltaje y como una entrada lineal controlada por corriente, además:

   (a) La característica $V - I$ de la resistencia no lineal R_E se modela con una función exponencial:

$$\begin{aligned} I_E &= I_S\left[e^{\frac{V_{BE}}{V_T}} - 1\right] \\ &\approx I_S\left[e^{\frac{V_{BE}}{V_T}}\right], \text{si } V_{BE} \gg V_T \end{aligned} \tag{5.4}$$

   donde I_S es la corriente de saturación inversa y $V_T \simeq 26$mV a temperatura ambiente.

(b) Se supone que $\alpha_F = 1$, donde α_F es la ganancia de la corriente en corto circuito en la conficuración base común. Esto corresponde a omitir la corriente de la base.

(c) Las dinámicas parásitas del transistor se omiten.

Las ecuaciones de estado del oscilador de Colpitts son las siguientes:

$$\begin{aligned} C_1\dot{V}_{C_1} =&\ \ -f(V_{C_2}) + I_L \\ C_2\dot{V}_{C_2} =&\ \ I_L - I_0 \\ L\dot{I}_L =&\ \ -V_{C_1} - V_{C_2} - RI_L + V_{CC} \end{aligned} \quad , \tag{5.5}$$

donde $f(\cdot)$ es el punto de trabajo característico de la resistencia no lineal. Esto puede ser expresado de la forma $I_E = f(V_{C_2}) = f(-V_{BE})$, entonces en particular de (5.4) se tiene:

$$f(V_{C_2}) = I_S\left[e^{-\frac{V_{C_2}}{V_T}}\right]$$

Se introducen las siguientes variables de estado adimensionales (x_1, x_2, x_3) y se escoge el punto de operación de (5.5) como el origen del nuevo sistema. En particular se normalizan voltajes, corrientes y tiempo con respecto a $V_{ref} = V_T$, $I_{ref} = I_0$, $t_{ref} = \frac{1}{\omega_0}$, respectivamente, donde $\omega_0 = \frac{1}{\sqrt{L\left(\frac{C_1C_2}{C_1+C_2}\right)}}$ es la frecuencia de resonancia del arreglo $L - C$ sin carga. Entonces las ecuaciones de estado del oscilador de Colpitts se pueden escribir de la forma:

$$\begin{aligned} \dot{x}_1 =&\ \ -a\ \exp(-x_2) +\ ax_3 + a \\ \dot{x}_2 =&\ \ bx_3 \\ \dot{x}_3 =&\ \ -cx_1 - cx_2 - dx_3 \end{aligned} \tag{5.6}$$

donde: $a = b\dfrac{C_2}{C_1}$, $b = \dfrac{I_0}{w_0C_2V_T}$, $c = \dfrac{V_T}{w_0LI_0}$, $d = \dfrac{R}{Lw_0}$.

Propiedades algebraicas del oscilador de Colpitts

De acuerdo con la Definición 2.13, es evidente que el sistema (5.6) es algebraicamente observable con respecto a la salida $y = x_2$, ya que los estados x_1 y x_3 pueden reescribirse como,

$$x_3 = \frac{\dot{x}_2}{b} = \frac{\dot{y}}{b} \tag{5.7}$$

$$x_1 = -\frac{1}{c}\left[\frac{1}{b}\ddot{y} + \frac{d}{b}\dot{y} + cy\right] \tag{5.8}$$

por lo tanto, el oscilador de Colpitts es algebraicamente observable con respecto a la salida $y = x_2$.

Diseño del observador

Primero se expresa el sistema (5.6) en la forma (3.1),

$$\dot{x} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -c & -c & -d \end{bmatrix} x + \begin{bmatrix} -a\exp(-x_2) + a \\ 0 \\ 0 \end{bmatrix} \quad , \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

De acuerdo con (3.3), el observador es

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + \Psi(\widehat{x}, u) + \sum_{i=1}^{m} K_i \left(y - C\widehat{x}\right)^{2i-1} \\ &= \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -c & -c & -d \end{bmatrix} \widehat{x} + \begin{bmatrix} -a\exp(-\widehat{x_2}) + a \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} e \right)^{2i-1} \end{aligned}$$

Por lo tanto, el observador de estado puede reescribirse como,

$$\begin{aligned} \dot{\widehat{x}}_1 &= a\widehat{x}_3 - a\exp(-\widehat{x}_2) + a + k_{1,1}e_{1,2} + k_{1,2}\left(e_{1,2}\right)^3 + ... + k_{1,m}\left(e_{1,2}\right)^{2m-1} \\ \dot{\widehat{x}}_2 &= b\widehat{x}_3 + k_{2,1}e_{1,2} + k_{2,2}\left(e_{1,2}\right)^3 + ... + k_{2,m}\left(e_{1,2}\right)^{2m-1} \\ \dot{\widehat{x}}_3 &= -c\widehat{x}_1 - c\widehat{x}_2 - d\widehat{x}_3 + k_{3,1}e_{1,2} + k_{3,2}\left(e_{1,2}\right)^3 + ... + k_{3,m}\left(e_{1,2}\right)^{2m-1} \end{aligned} \tag{5.9}$$

Resultados experimentales

En la plataforma WINCON se implementó el esquema (5.9) en la configuración maestro-esclavo. Los parámetros del circuito son: L = 100 μH; $C_1 = C_2 = 47$ nF, R = 45 Ω, $I_0 = 5$ mA. Entonces, los parámetros para el modelo del oscilador de Colpitts (5.6) son, $a = b = 6.2723$, $c = 0.0797$, y $d = 0.6898$.

El término no lineal $\Psi(x)$ satisface la condición de Lipschitz y se considera como sigue,

$$\Psi(x) = \begin{bmatrix} -a\exp(-x_2) + a \\ 0 \\ 0 \end{bmatrix}$$

Es necesario calcular la constante de Lipschitz L en el conjunto acotado,

$$\Omega = \{x \in \mathbb{R}^3 | \ \ |x_1| < M_1, \ \ |x_2| < M_2, \ \ |x_3| < M_3\} \tag{5.10}$$

Calculando el Jacobiano de $\Psi(x)$ se obtiene

$$\left[\frac{\partial \Psi(x)}{\partial x}\right] = \begin{bmatrix} 0 & a\exp(-x_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.11}$$

entonces, puede concluirse que [1]

$$\left\| \frac{\partial \Psi(x)}{\partial x} \right\|_\infty \leq 3\max\{0,\ a\exp(-x_2)\} \quad , \quad a \in \mathbb{R}^+. \tag{5.12}$$

De la expresión (5.10), las siguientes condiciones se cumplen para todos los puntos en el conjunto acotado Ω

$$a\exp(-x_2) < a\exp(M_2) = \max\{a\exp(-x_2)\} \quad , \quad a \in \mathbb{R}^+. \tag{5.13}$$

Por lo tanto,

$$\left\| \frac{\partial \Psi(x)}{\partial x} \right\|_\infty \leq 3\ a\exp(M_2) \tag{5.14}$$

De (5.10)-(5.14), la constante de Lipschitz que satisface la condición (3.2) se define como

$$L = 3\ a\exp(M_2)$$

En este caso, $M_1 = 3$, $M_2 = 0.1$, $M_3 = 6$, y $a = 6.2723$, $\Rightarrow L = 20.7959$.

Haciendo $m = 2$ en el observador (5.9), y resolviendo como una LMI la ecuación algebraica dada por la condición H3.1, las ganancias del observador K_1 y K_2, y la matriz definida positiva P son,

$$K_1 = \begin{bmatrix} 10.2130 \\ 16.1211 \\ 10.1500 \end{bmatrix},\ K_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix},\ P = \begin{bmatrix} 38.8560 & -36.7794 & 19.4606 \\ -36.7794 & 37.9331 & -20.7898 \\ 19.4606 & -20.7898 & 16.4869 \end{bmatrix} > 0\,,$$

El desempeño del observador se evaluará mediante el error cuadrático de sincronización,

$$J(t) = \frac{1}{t + 0.001}\int_0^t |e(t)|^2_{Q_0} \quad , \quad Q_0 = I \tag{5.15}$$

[1]Sea la matriz $\mathcal{A} = [a_{ij}]_{1\leq i,j\leq n}$, entonces

$$\|\mathcal{A}\|_\infty := n \max_{1\leq i,j\leq n} |a_{ij}|$$

Las figuras 5.3(a)-(c) muestran los resultados obtenidos usando el observador exponencial (5.9). Las condiciones iniciales del circuito de Colpitts son $x(0) = [0\ \ 0\ \ 0]^T$ y las condiciones iniciales arbitrarias del observador son $\hat{x}(0) = \begin{bmatrix} 1.506 & -0.1 & 2.1 \end{bmatrix}^T$. La figura 5.3-(d) presenta el índice de desempeño del observador, el cual exhibe un comportamiento exponencial.

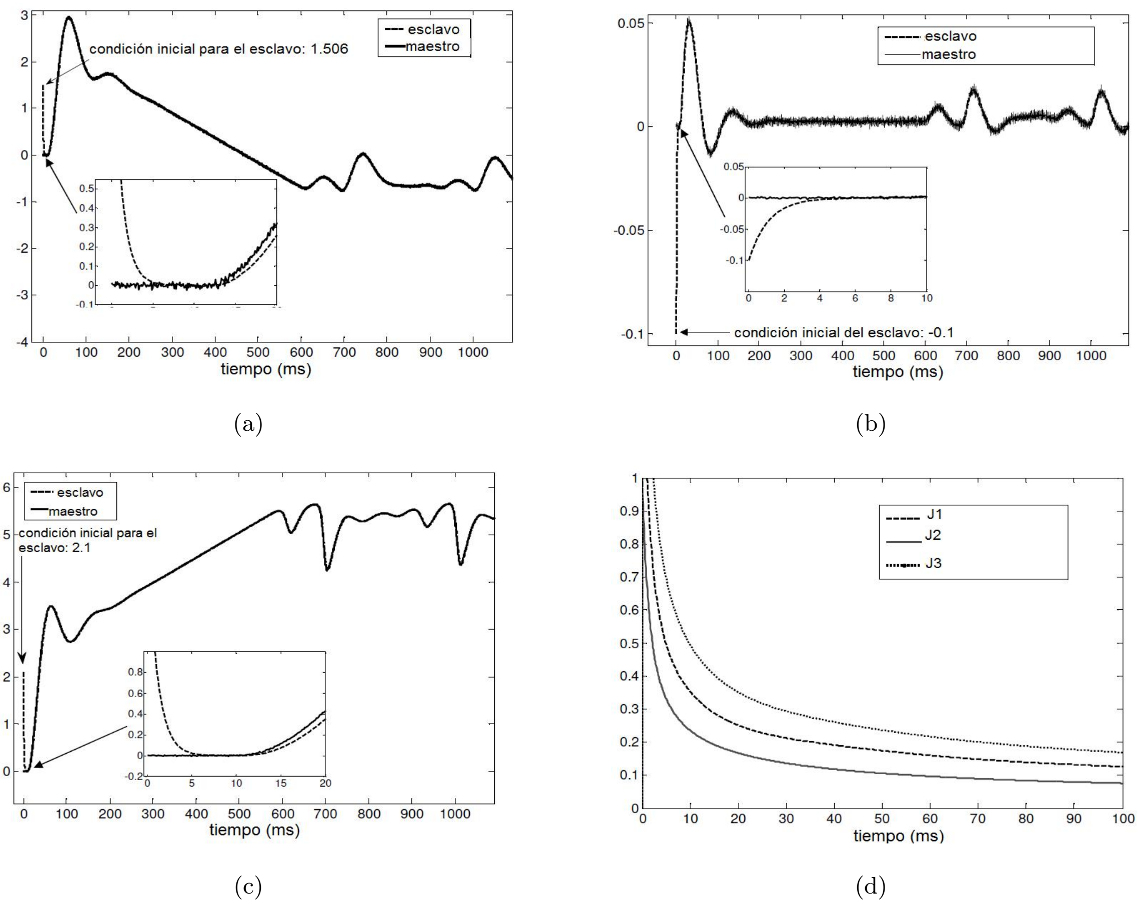


Figura 5.3: Sincronización del oscilador de Colpitts empleando el observador (5.9): (a) sincronización de x_1, (b) sincronización de x_2, (c) sincronización de x_3, (d) índice de desempeño.

5.2.2. Sincronización del circuito de Chua

Esta sección trata sobre la sincronización del oscilador de Chua considerado como un sistema caótico Liouviliano con dinámica monótona.

Modelo

El circuito de Chua (Matsumoto, 1984), mostrado en la figura 5.4, es un oscilador que presenta una variedad de bifurcaciones y caos. Básicamente, el circuito está compuesto por un inductor lineal, dos capacitores lineales , un resistor lineal, y un resistor no lineal N_R.

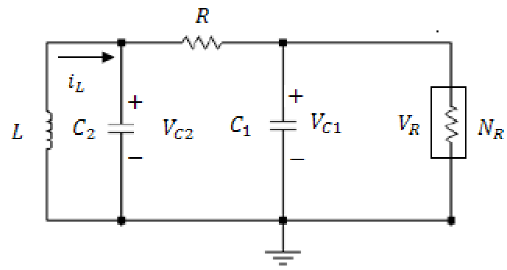


Figura 5.4: Circuito de Chua, con parámetros C_1 = 10 nF, C_2 = 100 nF, R = 1.8 kΩ, L = 18 mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS y $B_p = 1.08$ V.

Las ecuaciones de estado del circuito de Chua son:

$$\begin{aligned}
C_1 \frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\
C_2 \frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_L \\
L \frac{di_L}{dt} &= -v_{C_2}
\end{aligned} \tag{5.16}$$

donde $G = \dfrac{1}{R}$ y $g(\cdot)$ es una función no creciente definida por:

$$g(v_R) = m_0 v_R + \frac{1}{2}(m_1 - m_0)(|v_R + B_p| - |v_R - B_p|) \tag{5.17}$$

El comportamiento de (5.17) se ilustra en la figura 5.5, las pendientes en las regiones interna y externa son m_0 y m_1 respectivamente, con $m_1 < m_0 < 0$, $\pm B_p$ denota los puntos donde ocurre el cambio de pendiente. El resistor no lineal N_R está controlado por voltaje ya que su corriente es una función del voltaje en sus terminales. Para la implementación se utiliza el circuito equivalente de la figura 5.6.

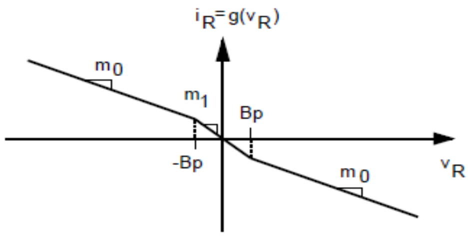


Figura 5.5: Curva característica $V - I$ del resistor lineal en el circuito de Chua.

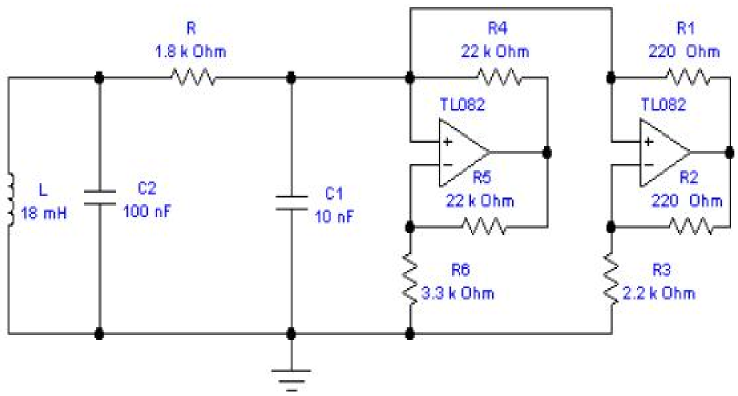


Figura 5.6: Circuito de Chua.

Observabilidad

En lo que sigue se considera como salida del sistema $y = v_{c_2}$. Del sistema (5.16) se obtiene:

$$\begin{aligned} v_{c_1} = & \frac{C_2}{G}\dot{y} + y + \frac{1}{LG}\int y dt \\ v_{c_2} = & y \\ i_L = & -\frac{1}{L}\int y dt \end{aligned} \tag{5.18}$$

De (5.18), se tiene que el circuito de Chua (5.16) es Liouviliano. Esto implica que las variables v_{C_1} e i_L pueden reconstruirse mediante la salida $y = v_{C_2}$.

Diseño del observador

Se escribe el sistema de Chua (5.16) en la forma dada por (3.9) con $\zeta(u) = 0$, $\varphi(\cdot) = 0$,

$$A = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix}, \psi(x) = \begin{bmatrix} -\frac{g(x_1)}{C_1} \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \qquad x = \begin{bmatrix} v_{C_1} & v_{C_2} & i_L \end{bmatrix}^T.$$

Debido a que $g(x_1)$ es una función no creciente y C_1 es una constante positiva, entonces $\psi_1(x) = \psi_1(x_1) = -g(x_1)/C_1$ es una función no decreciente (3.10). Además , $\psi_2(x) = \psi_3(x) = 0$ también satisfacen la propiedad (3.10), y el sistema de Chua (5.16) es Liouviliano, por lo tanto, se procede a diseñar el observador.

Puesto que $\varphi(\cdot) = 0$, se emplearán las condiciones dadas en el corolario 3.2 para obtener las ganancias del observador.

Aplicando (3.12), el observador para el sistema de Chua (5.16) tiene la siguiente estructura,

$$\dot{\hat{x}} = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -\frac{g(\hat{x}_1)}{C_1} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} e \right)^{2i-1}$$

Reescribiendo lo anterior,

$$\begin{aligned} \dot{\hat{x}}_1 &= -\frac{G}{C_1}\hat{x}_1 + \frac{G}{C_1}\hat{x}_2 - \frac{g(\hat{x}_1)}{C_1} + k_{1,1}e_{1,2} + k_{1,2}(e_{1,2})^3 + ... + k_{1,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_2 &= \frac{G}{C_2}\hat{x}_1 - \frac{G}{C_2}\hat{x}_2 + \frac{1}{C_2}\hat{x}_3 + k_{2,1}e_{1,2} + k_{2,2}(e_{1,2})^3 + ... + k_{2,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_3 &= -\frac{1}{L}\hat{x}_3 + k_{3,1}e_{1,2} + k_{3,2}(e_{1,2})^3 + ... + k_{3,m}(e_{1,2})^{2m-1} \end{aligned} \tag{5.19}$$

Resultados experimentales

Se realizaron algunos experimentos para la sincronización del circuito de Chua con el observador (5.19). Los parámetros considerados son: $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS y $B_p = 1.08$ V.

Se utilizó $m = 2$ en el observador (5.19). La LMI (3.23) con $\varepsilon = 0.001$ y $\epsilon_1 = 0.001$, tiene solución,

$$P = \begin{bmatrix} 0.0008 & -0.0006 & 0.1021 \\ -0.0006 & 0.0005 & -0.0805 \\ 0.1021 & -0.0805 & 15.0959 \end{bmatrix}, \; K_1 = \begin{bmatrix} k_{1,1} \\ k_{2,1} \\ k_{3,1} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 45 \end{bmatrix},$$

y K_2 se escoge tal que se satisface (3.24), entonces se obtiene

$$K_2 = \begin{bmatrix} k_{1,2} \\ k_{2,2} \\ k_{3,2} \end{bmatrix} = \begin{bmatrix} 7.1573 \\ 14.1040 \\ 0.0268 \end{bmatrix}.$$

La salida $y = x_2$ se obtuvó mediante el osciloscopio LeCroyWaveRunner® 104MXi. Este dispositivo soporta la plataforma de Matlab-Simulink®, por lo tanto también fué posible la implementación del observador. Las condiciones iniciales del observador son $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = 0.25$ y $\hat{x}_3(0) = 0.001$. La figura 5.7 muestra el comportamiento caótico del oscilador de Chua.

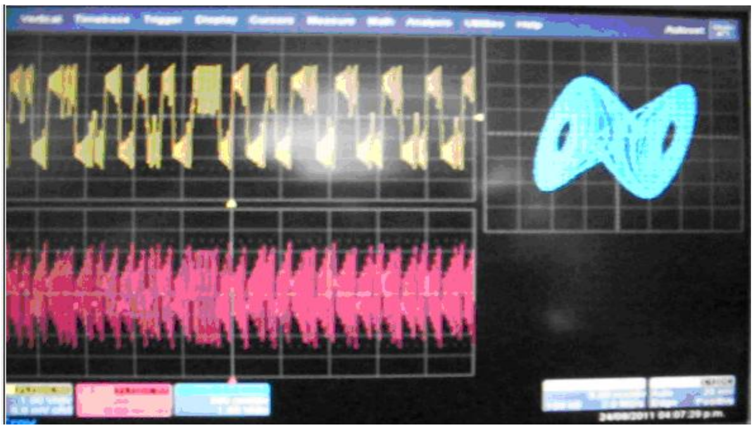


Figura 5.7: Comportamiento del oscilador de Chua.

El desempeño del observador se evalua mediante los errores relativos de la sincronización, es decir,

$$\bar{e}_i = \frac{|x_i - \hat{x}_i|}{|x_i|} \quad , \qquad i = 1, 3.$$

Las figuras 5.8-5.9 presentan los errores relativos, debe notarse que $\bar{e}_1 = 0$ y $\bar{e}_3 = 0$ para $t > 0.006s$.

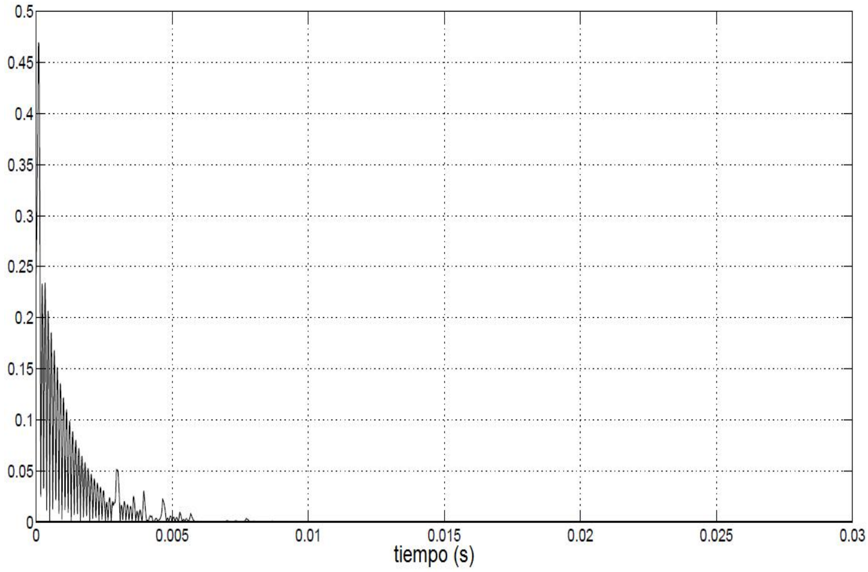


Figura 5.8: Error relativo: $\bar{e}_1 = \dfrac{|x_1 - \hat{x}_1|}{|x_1|}$.

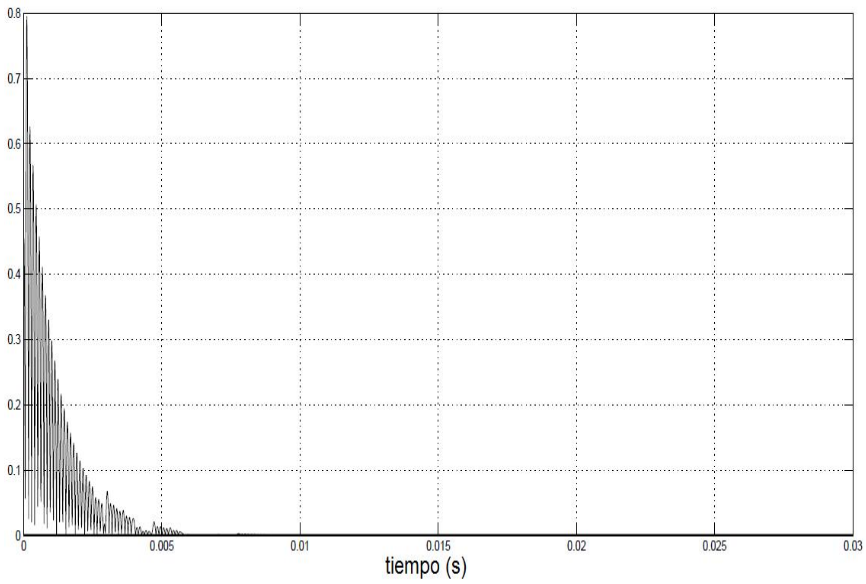


Figura 5.9: Error relativo: $\bar{e}_3 = \dfrac{|x_3 - \hat{x}_3|}{|x_3|}$.

5.3. Aplicaciones del observador uniformemente acotado

En ésta sección se presenta la sincronización de los sistemas caóticos de Rössler y Lorenz mediante el observador uniformenente acotado propuesto en el Lema 4.1.

5.3.1. Sincronización del sistema de Rössler

El sistema de Rössler es considerado como uno de los sistemas caóticos más populares, se le atribuye a Otto Rössler (1976), el cual se obtuvo de trabajos en cinéticas químicas. El modelo de Rössler se expresa por el sistema de ecuaciones siguiente,

$$\begin{cases} \dot{x}_1 = & -(x_2 + x_3) \\ \dot{x}_2 = & x_1 + a\,x_2 \\ \dot{x}_3 = & b + x_3(x_1 - c) \\ y = & x_1 \end{cases} \tag{5.20}$$

con $a, b, c \geq 0$.

El sistema de ecuaciones (5.20) representa un modelo simple de la dinámica de las reacciones químicas en un tanque de agitación.

Se sabe que en una vecindad de los parámetros $\{a = b = 0.2,\ \ c = 5$ el sistema (5.20) presenta un comportamiento caótico (Strogatz, 1994), ver figura 5.10. Nótese que sólo cuenta con un término no lineal.

A continuación se procede a comprobar si el sistema (5.20) cumple con la condición de observabilidad algebraica, la cual se obtiene expresando cada variable de estado en términos de la salida y sus derivadas.

Se expresa el sistema (5.20) en términos de la salida. Haciendo $y = x_1$, se obtiene

$$\dot{y} = -(x_2 + x_3) \tag{5.21}$$

$$\dot{x}_2 = y + a\,x_2 \tag{5.22}$$

$$\dot{x}_3 = b + x_3(y - c) \tag{5.23}$$

Derivando (5.21) con respecto al tiempo,

$$\ddot{y} = -\dot{x}_2 - \dot{x}_3 \tag{5.24}$$

De la ecuación (5.21),

$$x_3 = -(\dot{y} + x_2) \tag{5.25}$$

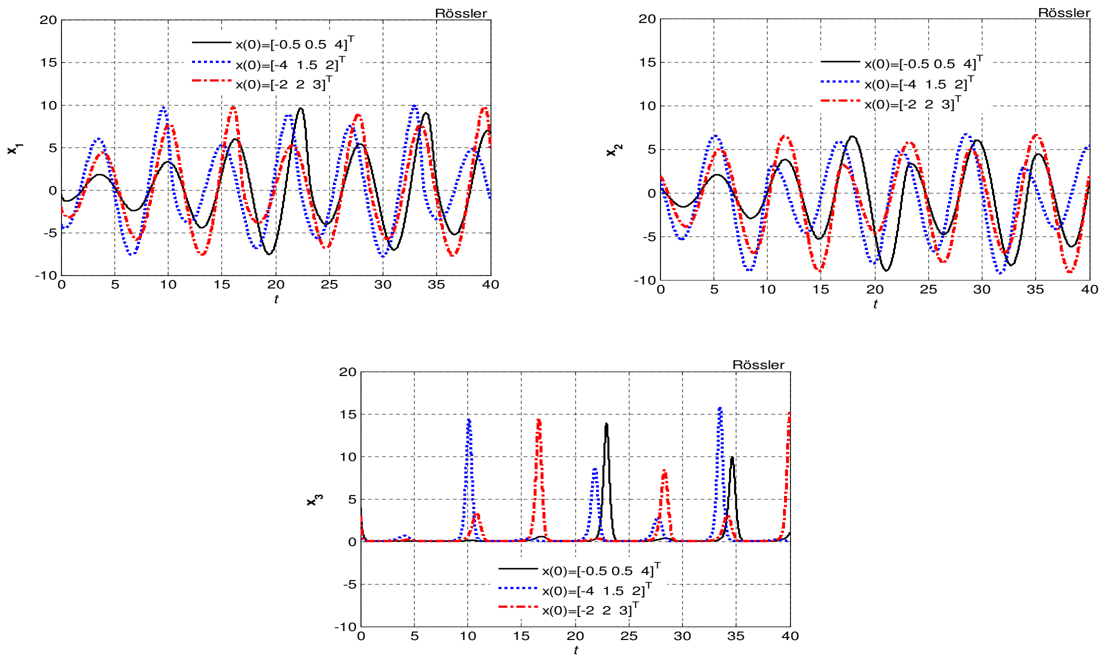


Figura 5.10: Comportamiento caótico del oscilador de Rössler

Sustituyendo las ecuaciones (5.22), (5.23) y (5.25) en la ecuación (5.24),

$$\ddot{y} - y\dot{y} + c\dot{y} + y - x_2 y + (a + c)x_2 + b = 0 \tag{5.26}$$

Para x_3 se procede de forma análoga. De la ecuación (5.21),

$$x_2 = -(\dot{y} + x_3) \tag{5.27}$$

Sustituyendo las ecuaciones (5.22), (5.23) y (5.27) en la ecuación (5.24),

$$\ddot{y} - a\dot{y} + y + x_3 y - (a + c)x_3 + b = 0 \tag{5.28}$$

De las ecuaciones (5.26) y (5.28), se tiene que x_2 y x_3 son algebraicamente observables.

En lo que sigue, se considera que la salida del sistema, $y = x_1$, se conoce exactamente, por lo que, solo se estimarán los estados x_2 y x_3, para cada sistema, respectivamente. La idea es obtener una ecuación diferencial con una estructura tal como la presentada en el Lema 4.1 mediante un cambio lineal de variable, el cual depende de la variable a estimar y de la salida del sistema.

Considere el sistema (5.20). Se introducen los siguientes cambios de variable,

$$\hat{z}_2 = \hat{x}_2 + k\,y \tag{5.29}$$

$$\hat{z}_3 = \hat{x}_3 + k\,y \tag{5.30}$$

donde $k \in \mathbb{R}$ es constante y se elige tal que el error de estimación (diferencia entre el estado real y su estimado) sea acotado.

Expresando el sistema (5.20) en términos de las nuevas variables $(\hat{z}_2, \hat{z}_3)$, se obtiene,

$$\begin{aligned}
\dot{\hat{z}}_2 &= y + a\hat{z}_2 - k[ay + \hat{z}_2 + \hat{z}_3] + 2k^2 y \\
\dot{\hat{z}}_3 &= b + \hat{z}_3[y - c] + k[-y^2 + cy - \hat{z}_2 - \hat{z}_3] + 2k^2 y
\end{aligned}$$

El estimador para el oscilador de Rössler está dado por el sistema,

$$\begin{aligned}
\dot{\hat{z}}_2 &= y + a\hat{z}_2 - k[ay + \hat{z}_2 + \hat{z}_3] + 2k^2 y \\
\dot{\hat{z}}_3 &= b + \hat{z}_3[y - c] + k[-y^2 + cy - \hat{z}_2 - \hat{z}_3] + 2k^2 y \\
\hat{x}_1 &= y \\
\hat{x}_2 &= \hat{z}_2 - ky \\
\hat{x}_3 &= \hat{z}_3 - ky
\end{aligned}$$

De manera análoga al caso anterior, se analiza la dinámica de los errores de sincronización expresando las ecuaciones anteriores en la forma general presentada en el lema 4.1,

$$\begin{aligned}
\dot{\hat{z}}_2 &= -[k - a]\hat{z}_2 + \hat{\zeta}_2 \\
\dot{\hat{z}}_3 &= -[c + k - y]\hat{z}_3 + \hat{\zeta}_3
\end{aligned}$$

donde, $\hat{\zeta}_2 = y - k[ay + \hat{z}_3] + 2k^2 y$, $\hat{\zeta}_3 = b + k[-y^2 + cy - \hat{z}_2] + 2k^2 y$, $k \geq max\{a, y - c\}$.

Se realizaron algunas simulaciones tomando los siguientes valores, $a = b = 0.2$, $c = 5$, $k = 1$, con condiciones iniciales arbitrarias $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

En la figura 5.11 se muestra la sincronización de cada variable de estado para el sistema de Rössler, se puede notar la reconstrucción de las variables x_2 y x_3.

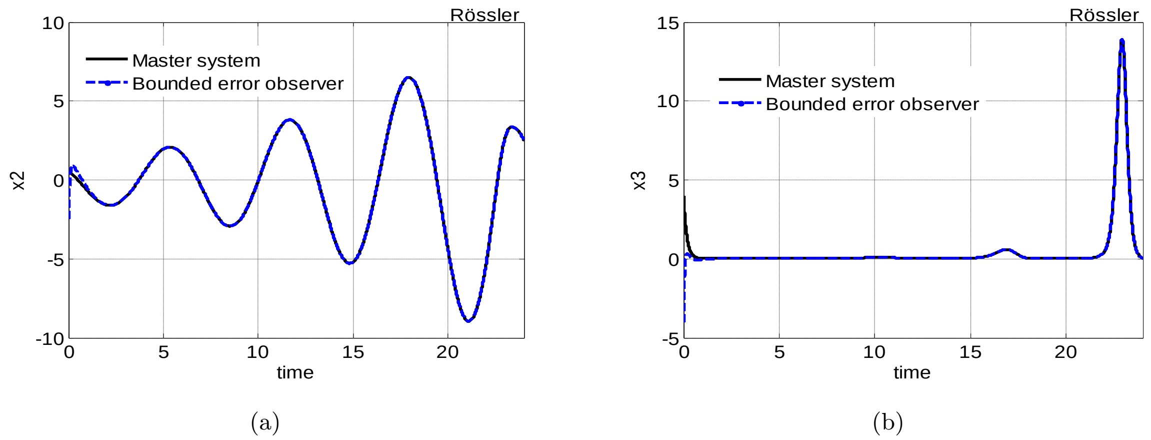


Figura 5.11: Sincronización del sistema de Rössler: (a) variable x_2, (b) variable x_3

Para el desempeño del observador se consideran los siguientes errores relativos,

$$\bar{e}_i = \frac{|x_i - \hat{x}_i|}{|x_i|} \quad , \qquad i = 2, 3.$$

Las figuras 5.12(a)-(b) muestran los errores relativos de la sincronización de las variables x_2 y x_3, se puede apreciar que para $t > 3s$ los errores relativos máximos son $\bar{e}_{2,max} \approx 0.6\,\%$ y $\bar{e}_{3,max} \approx 0.4\,\%$.

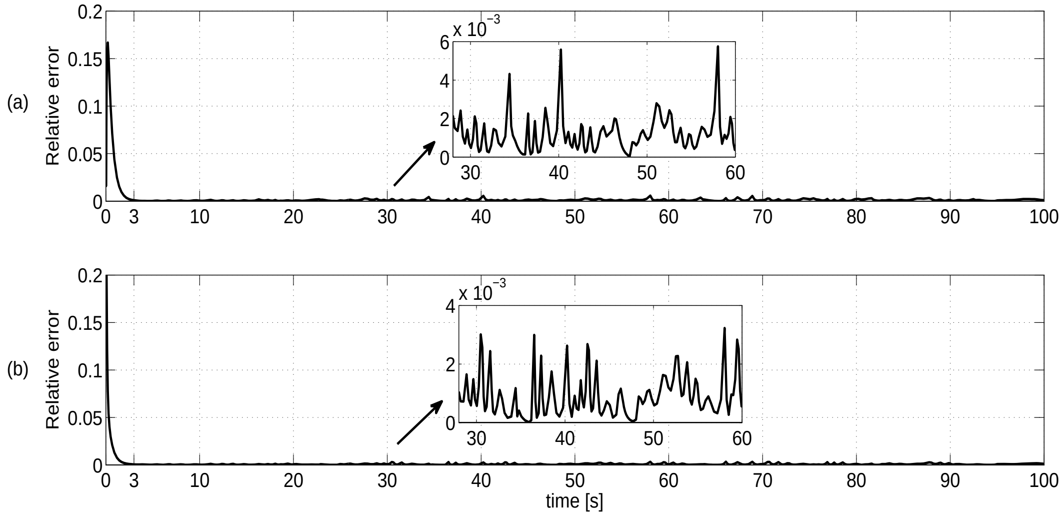


Figura 5.12: Errores relativos. (a) $\bar{e}_2 = \dfrac{|x_2 - \hat{x}_2|}{|x_2|}$; (b) $\bar{e}_3 = \dfrac{|x_3 - \hat{x}_3|}{|x_3|}$.

5.3.2. Sincronización del Sistema de Lorenz

Un problema importante en meteorología y en otras aplicaciones de la dinámica de fluidos se refiere al movimiento de una capa de fluido, como la atmósfera de la tierra que está más caliente en la parte inferior que en la superior, ver la figura 5.13. Si la diferencia vertical ΔT en las temperaturas es pequeña, existe una variación lineal de la temperatura con la altitud, pero ningún movimiento importante de la capa de fluido. Sin embargo, si ΔT es suficientemente grande, el aire más caliente sube, desplazando el aire más frío que está arriba, y se produce un movimiento de convección estable. Si la diferencia de temperaturas aumenta más entonces, el flujo estable de convección se rompe y se produce un movimiento más complejo y turbulento.

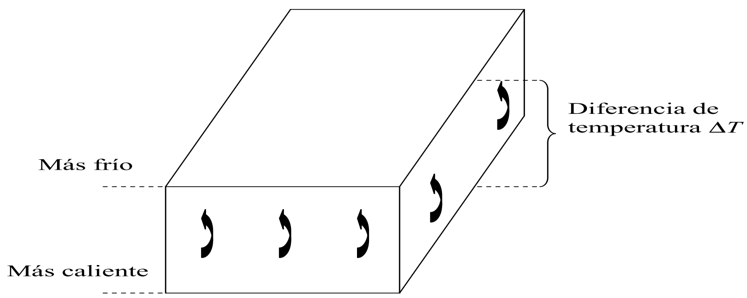


Figura 5.13: Capa de fluido calentada desde abajo

Cuando estudiaba este fenómeno, Edward N. Lorenz (1963) formuló un modelo de tercer orden que puede escribirse mediante el siguiente sistema de ecuaciones diferenciales no lineales,

$$\begin{cases} \dot{x}_1 = & \sigma(x_2 - x_1) \\ \dot{x}_2 = & \rho x_1 - x_2 - x_1\, x_3 \\ \dot{x}_3 = & x_1 x_2 - \beta x_3 \\ y = & x_1 \end{cases} \tag{5.31}$$

El sistema de ecuaciones (5.31) es conocido como sistema de Lorenz. Obsérvese que la segunda y tercera ecuaciones comprenden no linealidades cuadráticas. La variable x_1 del sistema (5.31) está relacionada con la intensidad del movimiento del fluido, mientras que las variables x_2 y x_3 están relacionadas con las variaciones en la temperatura en las direcciones horizontal y vertical. Las ecuaciones también comprenden tres parámetros σ, ρ, β los cuales son reales y positivos. Los parámetros σ y β dependen de las propiedades del material y geometría de la capa del fluido, por otra parte, el parámetro ρ es proporcional a la diferencia de temperatura ΔT.

La solución del sistema (5.31), para parámetros positivos σ, ρ, β exhibe un comportamiento caótico. La figura 5.14 muestra los estados del sistema de Lorenz para $\sigma = 10$, $\rho = 28$ y $\beta = 8/3$.

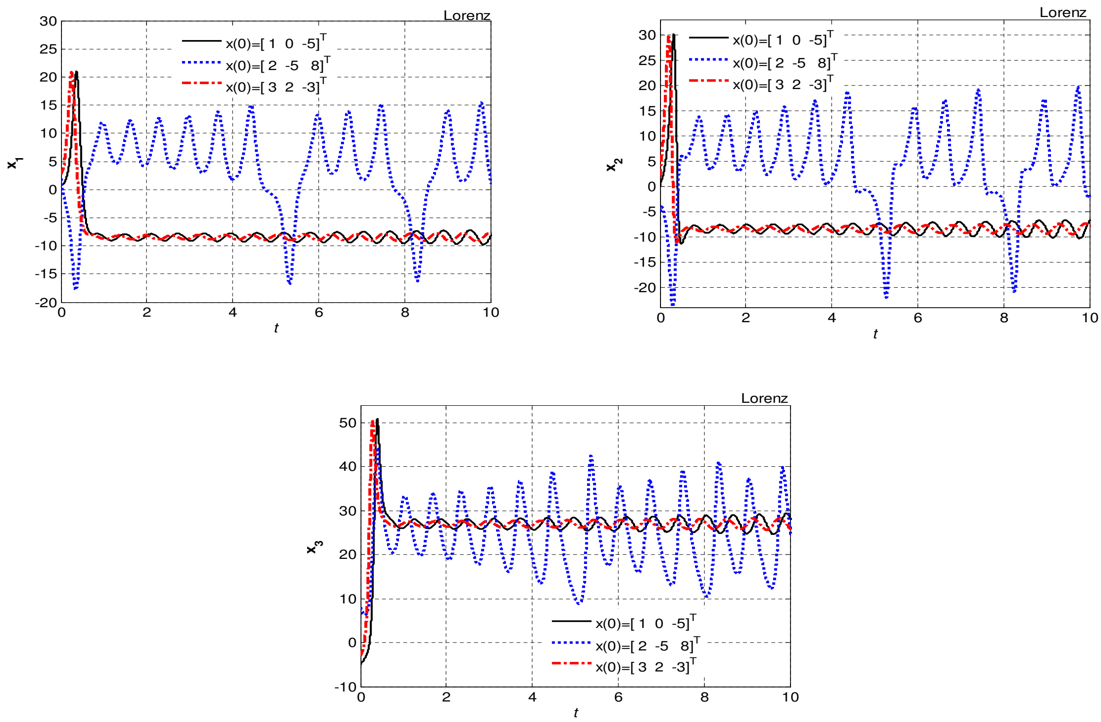


Figura 5.14: Comportamiento caótico del oscilador de Lorenz

La condición de observabilidad algebraica para el modelo de Lorenz dado por el sistema (5.31) se calcula como sigue.

Se expresa el sistema (5.31) en términos de la salida, esto es,

$$\dot{y} = \sigma(x_2 - y) \tag{5.32}$$

$$\dot{x}_2 = \rho y - x_2 - y\, x_3 \tag{5.33}$$

$$\dot{x}_3 = y x_2 - \beta x_3 \tag{5.34}$$

De la ecuación (5.32) se obtiene directamente la condición de observabilidad algebraica para la variable x_2, es decir,

$$\dot{y} + \sigma y - \sigma x_2 = 0 \tag{5.35}$$

o en otra forma,

$$x_2 = \frac{\dot{y} + \sigma y}{\sigma} \tag{5.36}$$

Ahora, se analiza la variable de estado x_3. Sustituyendo la ecuación (5.36) en (5.33) se tiene,

$$\frac{\ddot{y} + \sigma \dot{y}}{\sigma} = \rho y - \frac{\dot{y} + \sigma y}{\sigma} - y x_3$$

Simplificando términos se obtiene lo siguiente,

$$\ddot{y} + \dot{y}(\sigma + 1) - y\sigma(\rho - 1) + y\sigma x_3 = 0 \tag{5.37}$$

De las ecuaciones (5.35) y (5.37), se tiene que x_2 y x_3 son algebraicamente observables.

A continuación se procede a construir el observador, se estimarán las variables x_2 y x_3. Se consideran los cambios de variable dados por las ecuaciones (5.29) y (5.30).

Sustituyendo las ecuaciones (5.29) y (5.30) en el sistema de Lorenz (5.31) se obtiene lo siguiente,

$$\begin{aligned}
\dot{\hat{z}}_2 &= \rho y - \hat{z}_2 - y\hat{z}_3 + k\sigma\hat{z}_2 + ky(1 + y - \sigma) - k^2\sigma y \\
\dot{\hat{z}}_3 &= y\hat{z}_2 - \beta\hat{z}_3 + k(-y^2 + \beta y - \sigma\hat{z}_2 - \sigma y) - k^2\sigma y
\end{aligned}$$

El estimador uniformemente acotado que se sincroniza con el sistema de Lorenz es el siguiente,

$$\begin{aligned}
\dot{\hat{z}}_2 &= \rho y - \hat{z}_2 - y\hat{z}_3 + k\sigma\hat{z}_2 + ky(1 + y - \sigma) - k^2\sigma y \\
\dot{\hat{z}}_3 &= y\hat{z}_2 - \beta\hat{z}_3 + k(-y^2 + \beta y - \sigma\hat{z}_2 - \sigma y) - k^2\sigma y \\
\hat{x}_1 &= y \\
\hat{x}_2 &= \hat{z}_2 - ky \\
\hat{x}_3 &= \hat{z}_3 - ky
\end{aligned}$$

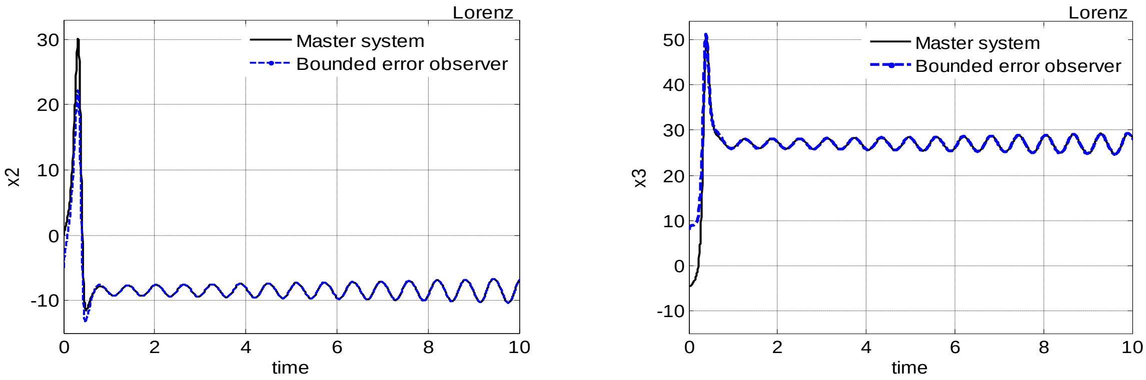
Se expresa el sistema anterior en la forma dada en el lema 4.1, lo cual permite obtener el valor de la ganancia del estimador,

$$\begin{aligned}
\dot{\hat{z}}_2 &= -[1 - k\sigma]\hat{z}_2 + \hat{\eta}_2 \\
\dot{\hat{z}}_3 &= -\beta\hat{z}_3 + \hat{\eta}_3
\end{aligned}$$

donde, $\hat{\eta}_2 = \rho y - y\hat{z}_3 + ky(1 + y - \sigma) - k^2\sigma y$, $\hat{\eta}_3 = y\hat{z}_2 + k[-y^2 + \beta y + \sigma\hat{z}_2 - \sigma y] - k^2\sigma y$, $k < 1/\sigma$

Los resultados numéricos que a continuación se presentan se obtuvieron para el sistema de Lorenz con los siguientes parámetros: $\rho = 28$, $\sigma = 10$ y $\beta = 8/3$. La ganancia del observador es $k = -1$.

En la figura 5.15 se muestran los resultados de la sincronización del sistema de Lorenz mediante el observador uniformemente acotado, tomando las siguientes condiciones iniciales: $x_1 = 1$, $x_2 = 0$, $x_3 = -5$, $\hat{x}_1 = 4$, $\hat{x}_2 = -5$, $\hat{x}_3 = 8$.



(a) (b)

Figura 5.15: Sincronización del sistema de Lorenz: (a)variable x_2, (b)variable x_3.

5.4. Sincronización mediante el observador de orden reducido

En esta sección se construirá un observador de orden reducido de tipo asintótico para la sincronización del oscilador de Colpitts (5.6).

Se sabe de la sección 5.2.1 que el oscilador de Colpitts es algebraicamente observable, por lo tanto se procede directamente con el diseño del observador utilizando el esquema (3.29) y las condiciones de observabilidad algebraica (5.7) y (5.8).

Para x_3 se propone el siguiente sistema:

$$\dot{\hat{x}}_3 = K_3(x_3 - \hat{x}_3) \tag{5.38}$$

Sustituyendo el valor de x_3 de la ecuación (5.7) en (5.38),

$$\dot{\hat{x}}_3 = \frac{K_3}{b}\dot{y} - K_3\hat{x}_3$$

de donde,

$$\dot{\hat{x}}_3 - \frac{K_3}{b}\dot{y} = -K_3\hat{x}_3$$

Se define la variable auxiliar $\gamma_3 = -\frac{K_3}{b}y + \hat{x}_3$, entonces el observador es

$$\begin{aligned} \dot{\gamma}_3 &= -\frac{K_3^2}{b}y - K_3\gamma_3 \\ \hat{x}_3 &= \frac{K_3}{b}y + \gamma_3 \end{aligned} \tag{5.39}$$

Para x_1 se considera el sistema

$$\dot{\hat{x}}_1 = K_1(x_1 - \hat{x}_1) \tag{5.40}$$

Sustituyendo (5.8) en (5.40),

$$\dot{\hat{x}}_1 = -\frac{K_1}{cb}\ddot{y} - \frac{K_1 d}{cb}\dot{y} - K_1 y - K_1\hat{x}_1 \tag{5.41}$$

La ecuación (5.41) depende de la segunda derivada de la salida. Se define el cambio de variable $x_4 = \dot{y}$ y se propone el siguiente observador para ésta variable, es decir,

$$\begin{aligned} \dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y] \\ \hat{x}_4 &= \gamma_4 + K_4 y \end{aligned}$$

Entonces,

$$\begin{aligned} \dot{\hat{x}}_1 &= -\frac{K_1}{cb}\dot{\hat{x}}_4 - \frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1 \\ \dot{\hat{x}}_1 + \frac{K_1}{cb}\dot{\hat{x}}_4 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1 \end{aligned}$$

Definiendo una nueva variable auxiliar, $\gamma_5 = \hat{x}_1 + \frac{K_1}{cb}\hat{x}_4$, se tiene

$$\begin{aligned} \dot{\gamma}_5 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y + \frac{K_1^2}{cb}\hat{x}_4 - K_1\gamma_5 \\ &= [K_1 - d]\frac{K_1}{cb}\hat{x}_4 - K_1 y - K_1\gamma_5 \\ &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5 \end{aligned}$$

Por lo tanto el observador para x_1 esta dado por

$$\begin{aligned}
\dot{\gamma}_4 = &\quad -K_4[\gamma_4 + K_4 y] \\
\dot{\gamma}_5 = &\quad [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5 \\
\hat{x}_1 = &\quad -\frac{K_1}{cb}[\gamma_4 + K_4 y] + \gamma_5
\end{aligned} \tag{5.42}$$

Entonces el sistema esclavo esta constituido por (5.39) y (5.42).

Finalmente, se presentan algunos resultados experimentales para la sincronización del oscilador de Colpitts con el observador asintótico de orden reducido dado por (5.39)–(5.42). Las figuras 5.16(a)-(b) mustran los resultados obtenidos utilizando condiciones iniciales $\hat{x}_1 = 1.506$ y $\hat{x}_3 = -2.498$ en los esquemas (5.39) y (5.42), respectivamente. La figura 5.16-(c) representa el diagrama de fase, donde claramente se observa el comportamiento caótico del oscilador de Colpitts y la sincronización con el sistema esclavo. En la figura 5.16-(d) se ilustra el índice de desempeño cuadrático (5.15).

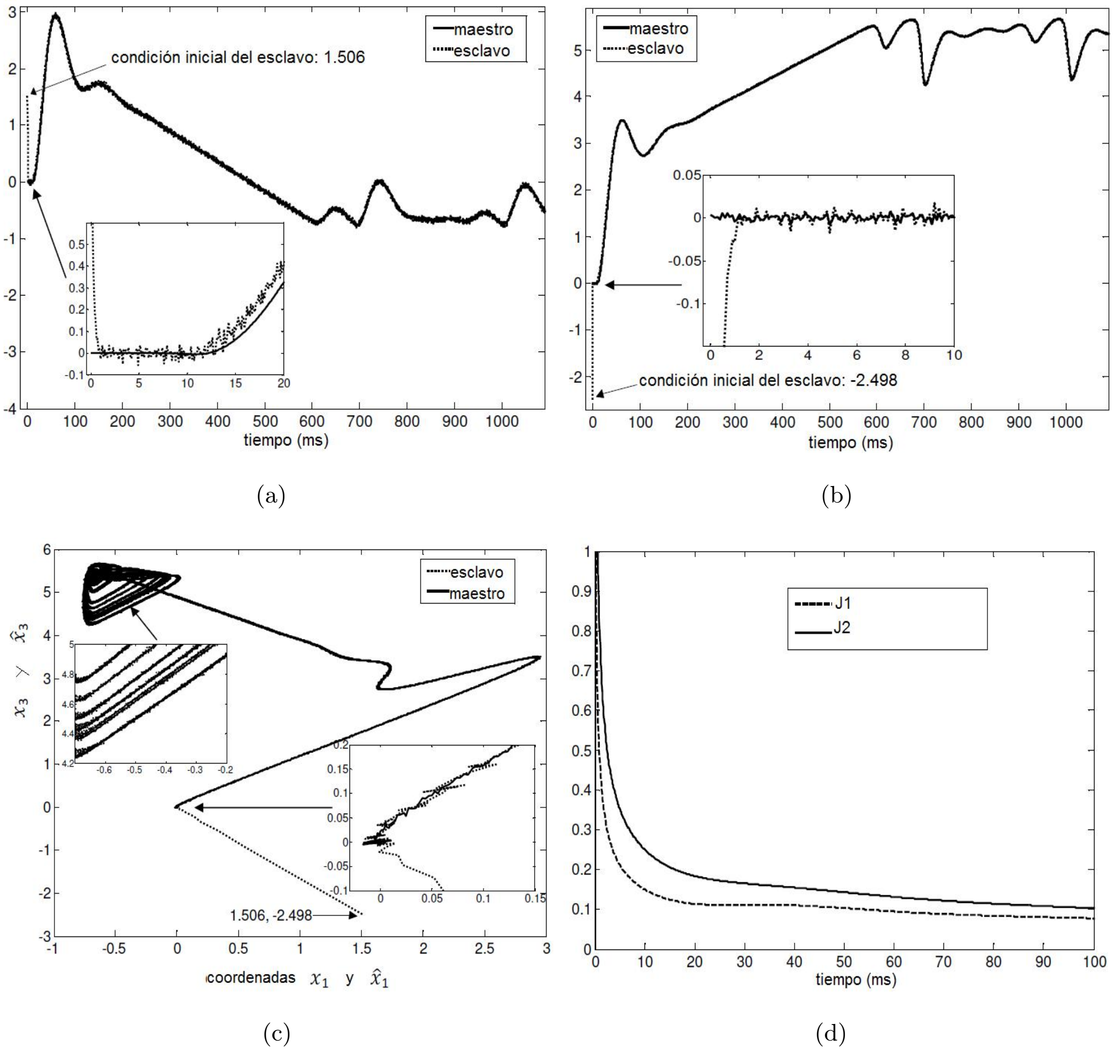


Figura 5.16: Sincronización del oscilador de Colpitts y el observador de orden reducido (5.42)-(5.39): (a) sincronización de x_1, (b) sincronización de x_3, (c) diagrama de fase del sistema maestro (x_1 y x_3) y el sistema esclavo ($\hat{x}_1$ y $\hat{x}_3$), (d) índice de desempeño.

Capítulo 6

Sincronización del sistema de Rikitake mediante un observador adaptable

En éste capítulo se presenta un método adaptable asintótico para la sincronización y la identificación del oscilador de Rikitake con parámetros desconocidos. Éste sistema modela la inversión de polaridad del campo electromagnético terrestre, y es bien sabido que tiene un comportamiento caótico para algún conjunto de condiciones iniciales y parámetros. Mediante éste método, se puede obtener la sincronización caótica y la identificación de parámetros desconocidos, simultáneamente.

En resumen, el problema consiste en lo siguiente:

- Identificación de los parámetros desconocidos y
- diseño de un observador para lograr la sincronización.

6.1. Sistema transmisor

En esta sección se presenta el modelo del oscilador de Rikitake el cual será considerado como sistema transmisor.

La teoría del dínamo es la explicación más plausible del origen del campo geomagnético. Rikitake (1958), en su afán de explicar la inversión del campo magnético terrestre, estudió el comportamiento de un sistema de dos dínamos acoplados, probó que una inversión de campo magnético puede presentarse al menos inicialmente cuando los parámetros del sistema y las condiciones iniciales son apropiados.

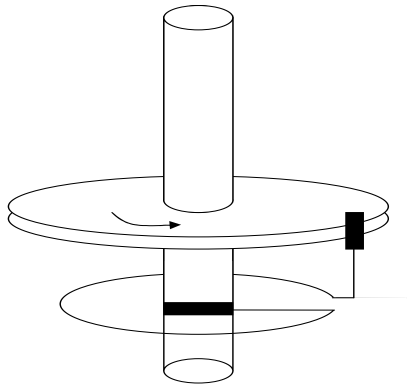


Figura 6.1: Esquema de un dínamo

La figura 6.1 muestra un dínamo de disco. Un disco de metal rota alrededor de un eje. Cuando existe un campo magnético en la dirección del eje, un campo eléctrico es excitado en el disco en direcciones radiales debido a la inducción electromagnética. De acuerdo con esto, se presenta una diferencia de potencial entre el eje y la periferia del disco. Una corriente eléctrica generada por la diferencia de potencial fluye hacia las escobillas de la bobina circular (ver figura 6.1). Puede interpretarse que la corriente que circula en el disco produce un campo magnético. *Este es el principio de trabajo de un dínamo.*

Se denota a H como el campo magnético perpendicular al disco y, w como la velocidad angular, entonces, la fuerza electromotriz en el disco está dada por $w \int_b^a H\ r\ dr$, donde a y b son los radios del disco y el eje, respectivamente. La influencia del campo magnético producido por las corrientes fluyendo en el disco es ignorada. Cuando una corriente eléctrica I fluye por la bobina, el acoplamiento mecánico de origen electromagnético actuando en el disco en dirección opuesta a la rotación es $I \int_b^a H\ r\ dr$.

El coeficiente de inducción mutua M entre la bobina y la periferia del disco esta definido por,

$$\int_b^a H\ r\ dr = M\ I$$

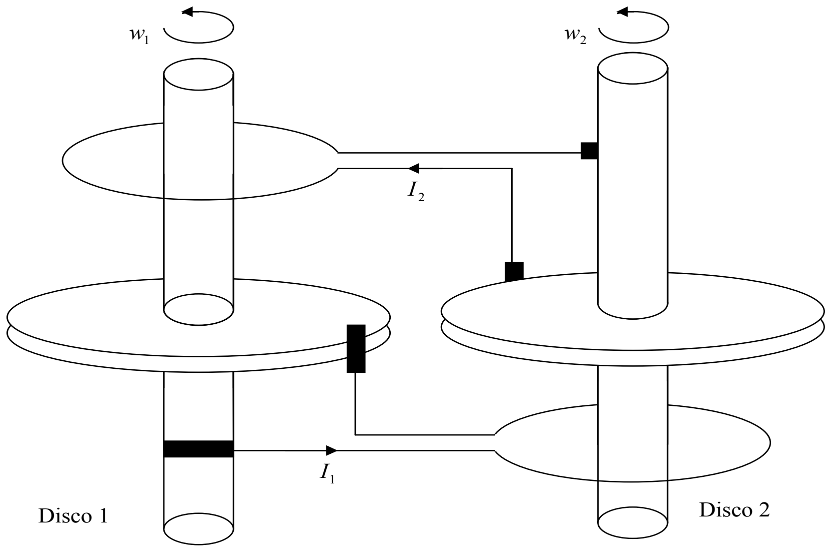


Figura 6.2: Esquema de dos dínamos acoplados

Ahora, se considera el esquema de dos dínamos acoplados como se muestra en la figura 6.2.

Antes de analizar el esquema de la figura 6.2, se define la siguiente notación,

I_i**:** corriente eléctrica fluyendo en la bobina i

L_i**:** inductancia de la bobina i

R_i**:** resistencia eléctrica de la bobina i

C_i**:** momento de inercia del disco i

G_i**:** par externo

M_i**:** coeficiente de inducción mutua

w_i**:** velocidad angular

El subíndice $i \in \{1, 2\}$, indica cantidades relacionadas con los discos 1 y 2. Las ecuaciones que describen el esquema de dos discos acoplados son,

$$\begin{aligned} L_1\frac{dI_1}{dt} + R_1 I_1 &= w_1 M_2 I_2 \\ L_2\frac{dI_2}{dt} + R_2 I_2 &= w_2 M_1 I_1 \\ C_1\frac{dw_1}{dt} &= G_1 - M_2 I_1 I_2 \\ C_2\frac{dw_2}{dt} &= G_2 - M_1 I_1 I_2 \end{aligned}$$

Se asume que los discos son idénticos, esto es, $L_1 = L_2 = L$, $R_1 = R_2 = R$, $M_1 = M_2 = M$, $C_1 = C_2 = C$, $G_1 = G_2 = G$.

A continuación, se proponen los siguientes cambios de variable,

$$I_1 = \sqrt{\frac{G}{M}}x_1 \quad , \qquad I_2 = \sqrt{\frac{G}{M}}y_1$$

$$w_1 = \sqrt{\frac{GL}{CM}}z_1 \quad , \qquad w_2 = \sqrt{\frac{GL}{CM}}(z_1 - a) \quad , \qquad \mu = \frac{R}{L}\sqrt{\frac{CL}{GM}}$$

Entonces, se obtiene el siguiente sistema,

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + z_1 y_1 \\ \dot{y}_1 &= -\mu y_1 + (z_1 - a)x_1 \\ \dot{z}_1 &= 1 - x_1 y_1 \end{aligned} \tag{6.1}$$

donde, a y μ son parámetros positivos. El sistema (6.1) representa al sistema transmisor.

6.2. Observabilidad

En ésta sección se muestran algunas propiedades algebraicas que satisface el sistema (6.1). Las salidas que se consideran disponibles son $g_1 = x_1$ y $g_2 = y_1$.

De acuerdo con la Definición 2.13 (ver capítulo 2), el sistema (6.1) es algebraicamente observable con respecto a las salidas $g_1 = x_1$ y $g_2 = y_1$, ya que el estado z_1 se puede obtener como sigue

$$z_1 = \frac{\dot{g}_2 - \mu g_2}{g_1} + a \tag{6.2}$$

por lo tanto, el oscilador de Rikitake (6.1) es algebraicamente observable con respecto a las salidas $g_1 = x_1$ y $g_2 = y_1$.

Ahora, se procede a verificar la condición de observabilidad para los parámetros, es decir, la identificabilidad (Martínez-Guerra, Ramírez-Palacios y Alvarado-Trejo, 1998). Sustituyendo la expresión (6.2) en la primera ecuación diferencial del sistema (6.1), se tiene

$$\begin{aligned} \dot{g}_1 g_1 - \dot{g}_2 g_2 = &\ -\mu\,(g_1^2 + g_2^2) + a\; g_1 g_2 \\ = &\ \begin{bmatrix} -(g_1^2 + g_2^2) & g_1 g_2 \end{bmatrix} p \end{aligned} \tag{6.3}$$

Donde p se define como el vector de parámetros $p^T := (\mu,\ a)$. Por lo tanto, se concluye que el sistema (6.1) es algebraicamente identificable con respecto a las salidas disponibles.

Entonces, la variable de estado z_1 y el vector de parámetros p pueden reconstruirse simultánemente conociendo las salidas $g_1 = x_1$ y $g_2 = y_1$.

De acuerdo con lo anterior, es posible resolver el problema de sincronización del oscilador de Rikitake con parámetros desconocidos, considerando que siempre están disponibles los estados y_1 y x_1. Además, también es posible reconstruir los parámetros desconocidos μ y a.

6.3. Sistema receptor

Considere el oscilador de Rikitake (6.1), de aquí en adelante se le llamará sistema transmisor, con las salidas disponibles y_1 y x_1. Se propone el siguiente sistema receptor controlado

$$\begin{aligned} \dot{x}_2 &= -\widehat{\mu} x_1 + z_2 y_1 + u_1, \\ \dot{y}_2 &= -\widehat{\mu} y_1 + (z_2 - \widehat{a}) x_1 + u_2, \\ \dot{z}_2 &= 1 - x_1 y_1 + u_3. \end{aligned} \tag{6.4}$$

En lo que sigue de éste capítulo, se denotan los vectores de estado relacionados con los sistemas transmisor y receptor como w_1 y w_2, respectivamente. Esto es, $w_i^T = (x_i, y_i, z_i)$, para $i \in \{1, 2\}$.

Entonces, el objetivo consiste en determinar $u = (u_1, u_2, u_3)$ y $\hat{p} = (\hat{\mu}, \hat{a})$ de manera que el sistema receptor (6.4) siga al sistema desconocido (6.1). En otras palabras, se necesita encontrar u y $\hat{p}$ del sistema (6.4) tal que $(w_2, \hat{p}) \rightarrow (w_1, p)$, cuando $t \rightarrow \infty$.

Antes de analizar la convergencia conviene definir la siguiente notación

$$\begin{aligned} &e_x = x_1 - x_2; \quad e_y = y_1 - y_2; \quad e_z = z_1 - z_2; \\ &\widetilde{\mu} = \mu - \widehat{\mu}; \qquad \widetilde{a} = a - \widehat{a}; \end{aligned}$$

de acuerdo con lo anterior, se definen los vectores

$$e^T = (e_x, e_y, e_z) \quad ; \quad \widetilde{p}^T = (\widetilde{\mu}, \widetilde{a});$$

6.4. Convergencia y Ley de Adaptación

En ésta sección se presenta la solución al problema de sincronización e identificación de parámetros desconocidos del oscilador de Rikitake, mediante el segundo método de Lyapunov. Para ésto, primero se obtiene la dinámica de los errores de sincronización entre los sistemas transmisor y receptor. Posteriormente, basándonos en una función candidata de Lyapunov definida positiva, se propone la ley de adaptación que se necesita para garantizar la sincronización de ambos sistemas.

Antes de resolver el problema se introducen las siguientes suposiciones relacionadas con las salidas del sistema maestro,

H 6.1 *Los estados $y = y_1$ y $x = x_1$ están disponibles.*

H 6.2 *Todos los estados del sistema maestro están acotados. Sin embargo, los estados estacionarios de y y x permanecen oscilando alrededor de cero.*

Nota 6.1 *Acerca de H6.1, en la mayoría de los casos todos los estados del oscilador de Rikitake están acotados, para casi todo conjunto de condiciones iniciales y casi cualesquier conjunto de parámetros positivos μ y a. De hecho, la suposición H6.2 depende del conjunto de condiciones iniciales y de los parámetros μ y a. Para aclarar ésta propiedad, se menciona un caso donde H6.2 no se cumple. Si se seleccionan los parámetros $\{\mu = 0;\ a = 0\}$ y las condiciones $\{x_1(0) = 0; y_1(0); z_1(0) = \overline{z}\}$, se tiene que $x_1(t) = 0$, $y_1(t) = 0$, $z_1(t) = t + \overline{z}$. Evidentemente, H6.2 no se satisface ya que los estados y y x permanecen en el origen y el estado z_1 no está acotado (McMillen, 1999). No se puede proponer ningún método o esquema de identificación si el sistema transmisor tiene soluciones que tiendan hacia infinito o una constante.*

Se procede en éste momento al análisis de la dinámica del error de sincronización. De las ecuaciones (6.1) y (6.4), se tiene:

$$\dot{\mathbf{e}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\widetilde{\mu}x + e_z y - \widetilde{s}_1 x - u_1 \\ -\widetilde{\mu}y + (e_z - \widetilde{a})x - \widetilde{s}_1 y - \widetilde{s}_2 x - u_2 \\ -u_3 \end{bmatrix} \tag{6.5}$$

donde por simplicidad se denota a $y = y_1$ y $x = x_1$. El sistema (6.5) puede considerarse como un problema de control donde el vector de entradas u y el vector de parámetros $\widetilde{p}$ deben ser propuestos tales que, el error de sincronización e converja asintóticamente a cero.

Proposición 6.1 *Considere las suposiciones H6.1 y H6.2. El problema de sincronización y estimación de parámetros entre los sistemas (6.1) y (6.4) puede resolverse mediante las siguientes leyes de adaptación,*

$$\dot{\hat{p}} = \begin{bmatrix} \dot{\hat{\mu}} \\ \dot{\hat{a}} \end{bmatrix} = \begin{bmatrix} -e_x x - e_y y \\ -x e_y \end{bmatrix}$$

y las señales de control

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 \ [e_x + sign(e_x) e_x^m] \\ k_2 \ [e_y + sign(e_y) e_y^m] \\ e_x y + e_y x \end{bmatrix}$$

para cualesquiera constantes estrictamente positivas $\{k_1, k_2\}$ y para todo entero positivo par m.

Demostración. Considere la función candidata de Lyapunov siguiente

$$V = \frac{1}{2} e^T e + \frac{1}{2} \widetilde{p}^T \widetilde{p} \tag{6.6}$$

La derivada de (6.6) con respecto al tiempo, a lo largo de las trayectorias de (6.5) está dada por

$$\begin{aligned} \dot{V} = \ & \widetilde{a}\dot{\widetilde{a}} + \widetilde{\mu}\dot{\widetilde{\mu}} - \widetilde{\mu} e_x x + e_x e_z y - e_x u_1 - \widetilde{\mu} e_y y \\ & + e_y e_z x - \widetilde{a} e_y x - e_y u_2 - e_z u_3 \\ = \ & -\widetilde{a}\dot{\hat{a}} - \widetilde{\mu}\dot{\hat{\mu}} - \widetilde{\mu} e_x x + e_x e_z y - e_x u_1 - \widetilde{\mu} e_y \\ & + e_y e_z x - \widetilde{a} e_y x - e_y u_2 - e_z u_3 \end{aligned} \tag{6.7}$$

Para garantizar que (6.7) sea semidefinida negativa, se proponen $\dot{\hat{p}}$ y u como

$$\dot{\hat{p}} = \begin{bmatrix} \dot{\hat{\mu}} \\ \dot{\hat{a}} \end{bmatrix} = \begin{bmatrix} -e_x x - e_y y \\ -x e_y \end{bmatrix} \tag{6.8}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 \ [e_x + \mathrm{sign}(e_x) e_x^m] \\ k_2 \ [e_y + \mathrm{sign}(e_y) e_y^m] \\ e_x y + e_y x \end{bmatrix} \tag{6.9}$$

donde k_1 y k_2 son estrictamente constantes positivas y m es cualquier entero positivo par.

Sustituyendo (6.8) y (6.9) en (6.7), se obtiene

$$\dot{V} = -\left(k_1 e_x^2 + k_2 e_y^2 + k_1 |e_x| e_x^m + k_2 |e_y| e_y^m\right). \tag{6.10}$$

Ésto implica que $\dot{V}$ es semidefinida negativa y, por lo tanto, V converge. Se tiene entonces que el conjunto de señales $\{e_x, e_y, e_z, \widetilde{\mu}, \widetilde{a}\}$ están acotadas.

Ahora, se mostrará que e converge asintóticamente a cero, cuando $t \to \infty$, aplicando el lema de Barbalat.

Integrando ambos miembros de (6.10), se tiene

$$\int_0^t \left(k_1 e_x^2(s) + k_2 e_y^2(s) + k_1 |e_x(s)| e_x^m(s) + k_2 |e_y(s)| e_y^m(s)\right) ds \quad \leq V(0) \tag{6.11}$$

Sustituyendo (6.9) en (6.5), se tiene el sistema en lazo cerrado siguiente

$$\begin{aligned} \dot{e}_x = &\quad -\widetilde{\mu}x + e_z y - \widetilde{s}_1 x - k_1 e_x - k_1 \operatorname{sign}(e_x) e_x^m \\ \dot{e}_y = &\quad -\widetilde{\mu}y + (e_z - \widetilde{a})x - \widetilde{s}_1 y - \widetilde{s}_2 x - k_2 e_y - k_2 \operatorname{sign}(e_y) e_y^m \\ \dot{e}_z = &\quad -e_x y - e_y x \end{aligned} \tag{6.12}$$

donde las dinámicas de los parámetros están dadas por

$$\dot{\widetilde{\mu}} = -e_x x - y e_y \quad ; \quad \dot{\widetilde{a}} = -x e_y. \tag{6.13}$$

De las ecuaciones de (6.12) y la hipótesis H6.2, se tiene que $\dot{e}$ es acotado lo cual implica que e es uniformemente continua. Aplicando el lema de Barbalat, se determina que $e \to 0$, cuando $t \ \to \ \infty$.

Derivando (6.12) con respecto al tiempo, se puede mostrar que $\ddot{e}$ es acotado. De ésta manera, $\dot{e}$ es uniformemente continua y también e tiene un límite finito, cuando $t \to \infty$. Por el lema de Barbalat se concluye que $\dot{e} \to 0$, cuando $t \to \infty$ (Aguilar-Ibañez, Martínez-Guerra, Aguilar-López y Mata-Machuca, 2010).

Debido a que V converge cuando $t \to \infty$, entonces de (6.13) se tiene que los errores paramétricos $\widetilde{\mu}$ y $\widetilde{a}$ convergen cuando $t \to \infty$. Además, de (6.13) se sigue que $\dot{\widetilde{\mu}}$ y $\dot{\widetilde{a}}$ convergen a cero cuando $t \to \infty$.

Cuando t es suficientemente grande, $\hat{\mu}$ y $\hat{a}$ son casi constantes y las ecuaciones diferenciales de (6.12) implican que

$$\begin{aligned} 0 = &\quad (\mu - \widehat{\mu})x \\ 0 = &\quad (a - \widehat{a})y \end{aligned} \tag{6.14}$$

De la hipótesis H6.2, se tiene que los estados estacionarios y y x se mantienen oscilando alrededor de cero. Por lo tanto, necesariamente $\mu = \hat{\mu}$ y $a = \hat{a}$. Esto es, $\widetilde{p}^T \to 0$, cuando $t \to \infty$. □

6.5. Resultados Numéricos

Se han realizado algunas simulaciones numéricas para mostrar el desempeño de la estrategia propuesta para la sincronización y reconstrucción de parámetros desconocidos del oscilador de Rikitake.

Para el sistema transmisor se escogieron los siguientes parámetros p=($\mu = 2, a = 5$); con condiciones iniciales arbitrarias $w_1(0) = (x_1(0) = 1, y_1(0) = -1,\ z_1(0) = 1)$. La figura 6.3 muestra el comportamiento caótico de las trayectorias del sistema de Rikitake. En esta figura se ilustra la propiedad descrita en la suposición H6.2. Se puede notar que los estados del sistema maestro están acotados y, que y_1 y x_1 permanecen oscilando alrededor de cero. Por lo tanto, se verifica que la suposición H6.2 se cumple plenamente.

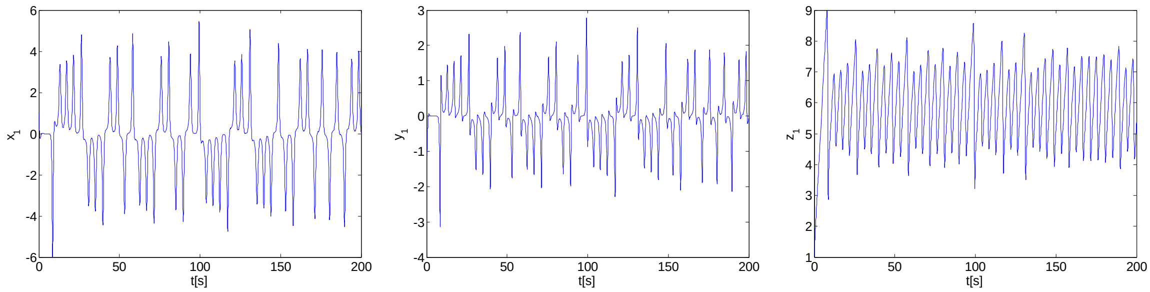


Figura 6.3: Variables de estado del sistema de Rikitake.

Para mostrar el desempeño de la metodología propuesta se realizaron algunas simulaciones, utilizando las mismas condiciones para el sistema transmisor. Las ganancias del sistema esclavo se fijan como $k_1 = k_2 = 0.8$ y $m = 4$, las condiciones iniciales del sistema receptor son $w_2(0) = 0$ y $\widehat{p}(0) = 0$. En la figura 6.4 se muestra la sincronización entre el sistema transmisor y el sistema receptor. En la figura 6.5 se observa que los errores de sincronización convergen asintóticamente a cero, esto es, el sistema esclavo sigue asintóticamente al sistema maestro. Los parámetros estimados se ilustran en la figura 6.6. Como era de esperarse, se obtiene un mejor desempeño cuando el tiempo incrementa. En éste caso, los parámetros son reconstruidos razonablemente bien después de 60 segundos.

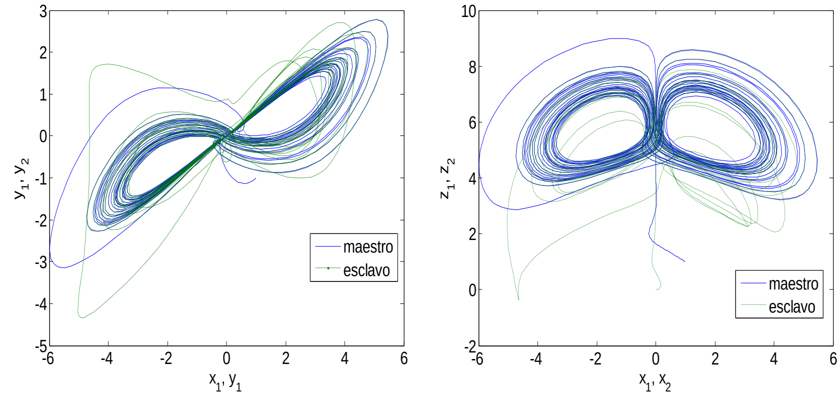


Figura 6.4: Sincronización del oscilador de Rikitake

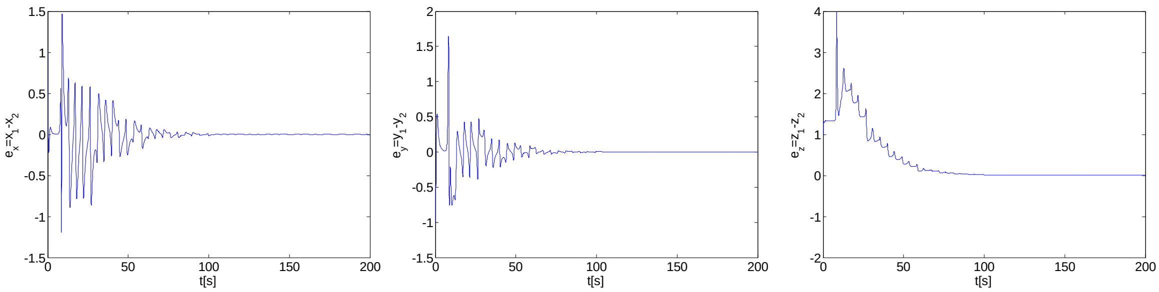
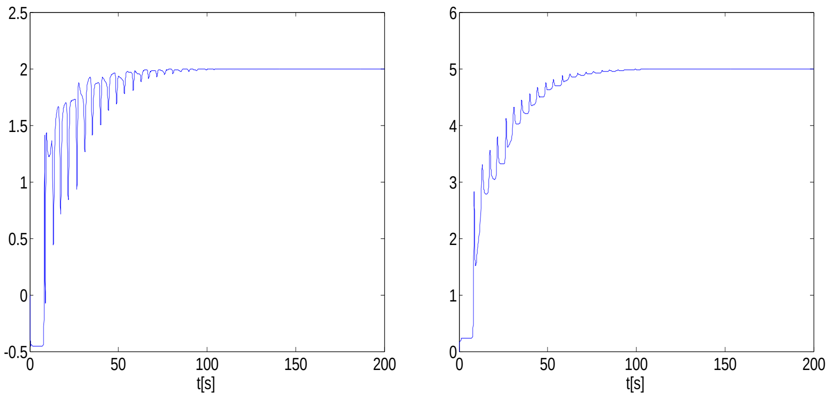


Figura 6.5: Errores de sincronización.



(a) $\hat{\mu}$ (b) $\hat{a}$

Figura 6.6: Parámetros estimados.

Capítulo 7

Sincronización de sistemas no lineales de orden fraccionario

Este capítulo se enfoca en el diseño de un observador fraccionario para una clase de sistemas no lineales de orden fraccionario. A continuación se presentarán algunos conceptos de cálculo fraccionario. Posteriormente, se introducirá la propiedad de observabilidad algebraica para sistemas fraccionarios. Después se realizará el diseño del observador. Finalmente, se aplicará el método propuesto para la sincronización del sistema hipercaótico fraccionario de Rössler y para la sincronización del sistema caótico fraccionario de Lorenz.

7.1. Derivada fraccionaria

Existen varias definiciones de derivada de orden α (Podlubny, 1999, Miller y Ross, 1993, Podlubny, 2002), sin embargo en este trabajo se usará el operador fraccionario de Caputo en la definición de sistemas fraccionarios, debido a que el significado de las condiciones iniciales para los sistemas que se describen con este operador es el mismo que para los sistemas de orden entero.

Definición 7.1 (Derivada fraccionaria de Caputo) *La derivada fraccionaria de Caputo de orden $\alpha \in \mathbb{R}^+$ de una función x esta definida como (Podlubny, 1999):*

$$x^{(\alpha)} = {}_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{d^m x(\tau)}{d\tau^m} (t-\tau)^{m-\alpha-1} \, d\tau, \tag{7.1}$$

donde: $m-1 \leq \alpha < m$, $\frac{d^m x(\tau)}{d\tau^m}$ *es la m-ésima derivada de* x *en el sentido usual,* $m \in \mathbb{N}$ *y* Γ *es la función gama. Por simplicidad se omitió la dependencia del tiempo en* $x^{(\alpha)}$*, en lo que sigue se tomará* $t_0 = 0$. □

Se define la derivada fraccionaria de Caputo de orden $\alpha \in \mathbb{R}^+$ aplicada $r-$veces a una variable $x(t)$ (Miller y Ross, 1993), como

$$\mathcal{D}^{(r\alpha)}x(t) = \underbrace{{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}\ldots{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}}_{\text{r-veces}}\,x(t) \tag{7.2}$$

donde $\mathcal{D}^{(0)}x(t) = x(t)$. Si $r = 1$ entonces $\mathcal{D}^{(\alpha)}x(t) = x^{(\alpha)}$.

7.1.1. Función de Mittag-Leffler

La función de Mittag-Leffler con dos parámetros se define como (Kilbas, Srivastava y Trujillo, 2006):

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad z, \beta \in \mathbb{C}, \ \mathbb{R}e(\alpha) > 0 \tag{7.3}$$

esta función es utilizada para resolver ecuaciones diferenciales fraccionarias como la función exponencial en sistemas de orden entero. En particular, cuando $\alpha = \beta = 1$, se tiene que $E_{1,1}(z) = e^z$. Para valores particulares de α, la función (7.3) tiene comportamiento asintótico.

Teorema 7.1 (Podlubny (1999)) *Si* $\alpha \in (0, 2)$*,* β *es un número complejo arbitrario y* μ *es un número real arbitrario tal que*

$$\frac{\pi\alpha}{2} < \mu < \text{mín}\{\pi, \pi\alpha\} \tag{7.4}$$

entonces para cualquier entero arbitrario $\kappa \geq 1$ *la siguiente expansión se cumple:*

$$E_{\alpha,\beta}(z) = -\sum_{i=1}^{\kappa} \frac{1}{\Gamma(\beta - \alpha i)z^i} + O\left(\frac{1}{|z|^{\kappa+1}}\right) \tag{7.5}$$

con $|z| \to \infty$*,* $\mu \leq |\arg(z)| \leq \pi$. □

La función de Mittag-Leffler tiene las siguientes propiedades:

Propiedad 1 (Podlubny, 1999).

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-k\tau^{\alpha})\, d\tau = t^{\beta} E_{\alpha,\beta+1}(-kt^{\alpha}) \ ,$$

con $\beta > 0$.

Propiedad 2 (Miller y Samko, 2001). $E_{\alpha,\beta}(-x)$ es completamente monótona, es decir, $(-1)^n E^{(n)}_{\alpha,\beta}(-x) \geq 0$ para $0 < \alpha \leq 1$ y $\beta \geq \alpha$, para todo $x \in (0,\infty)$ y $n \in \mathbb{N} \cup \{0\}$.

7.2. Planteamiento del problema

Considere el sistema no lineal de orden fraccionario, con $0 < \alpha < 1$:

$$\begin{aligned} x^{(\alpha)} = &\ \ f(x), \ \ x(0) = x_0 \\ y = &\ \ h(\bar{x}) \end{aligned} \tag{7.6}$$

donde $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ es una función Lipschitz continua[1], con $x_0 \in \Omega \subset \mathbb{R}^n$, en este caso y denota la salida medible del sistema, $\bar{x} \in \mathbb{R}^p$ representa los estados conocidos del sistema, $h : \mathbb{R}^p \to \mathbb{R}^q$ es una función continua y $1 \leq p < n$.

Considere el sistema dado por (7.6), se separará en dos subsistemas dinámicos con estados $\bar{x} \in \mathbb{R}^p$ y $\eta \in \mathbb{R}^{n-p}$, con $x^T = (\bar{x}^T, \eta^T)$, el primer subsistema describirá los estados conocidos y el segundo representa los estados desconocidos, entonces el sistema (7.6) puede reescribirse como:

$$\begin{aligned} \bar{x}^{(\alpha)} &= \bar{f}(\bar{x}, \eta) \\ \eta^{(\alpha)} &= \Delta(\bar{x}, \eta) \\ y &= h(\bar{x}) \end{aligned} \tag{7.7}$$

donde $f^T(x) = \left(\bar{f}^{\,T}(\bar{x},\eta), \Delta^T(\bar{x},\eta)\right)$, $\bar{f} \in \mathbb{R}^p$ y $\Delta \in \mathbb{R}^{n-p}$. El problema de observación es reconstruir las variables desconocidas η's.

Antes de pasar al diseño del observador, se introduce la propiedad de observabilidad algebraica.

Definición 7.2 (Observabilidad Algebraica Fraccionaria) *Una variable de estado $\eta_i \in \mathbb{R}$ satisface la condición de Observabilidad Algebraica Fraccionaria si esta expresada en términos de las primeras $r \in \mathbb{N}$ derivadas recursivas de la salida disponible $y_{\bar{x}}$, es decir,*

$$\eta_i = \phi_i \left(y_{\bar{x}},\, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots\,, \mathcal{D}^{(r\alpha)} y_{\bar{x}} \right) \tag{7.8}$$

[1]Esto garantiza la solución única (Kilbas, Srivastava y Trujillo, 2006)

donde $\phi_i : \mathbb{R}^{(r+1)q} \to \mathbb{R}$. □

Suponiendo que los componentes del vector de estado desconocido η satisfacen la propiedad de observalilidad algebraica fraccionaria, entonces el problema se puede describir en términos de la configuración maestro-esclavo como se muestra a continuación.

Se considera el sistema maestro:

$$\eta_i^{(\alpha)} = \Delta_i(\bar{x}, \eta) \tag{7.9}$$

$$y_{\eta_i} = \eta_i = \phi_i\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{7.10}$$

para $p+1 \leq i \leq n$, donde η_i es una componente del vector de estado η , y_{η_i} representa la salida del i-ésimo sistema maestro.

7.3. Diseño del observador

Se propone un sistema dinámico fraccionario de orden α, como el sistema esclavo (observador):

$$\hat{\eta}_i^{(\alpha)} = k_{\hat{\eta}_i}(y_{\eta_i} - \hat{\eta}_i), \tag{7.11}$$

$$y_{\hat{\eta}_i} = \hat{\eta}_i, \tag{7.12}$$

para $p+1 \leq i \leq n$, donde $\hat{\eta}_i$ es el estado, $y_{\hat{\eta}_i}$ denota la salida del sistema maestro y $k_{\hat{\eta}_i}$ es una constante positiva.

Se define el error de sincronización como:

$$e_i = y_{\eta_i} - y_{\hat{\eta}_i} = \eta_i - \hat{\eta}_i. \tag{7.13}$$

El análisis de convergencia del error de sincronización se presenta en la siguiente proposición.

Proposición 7.1 *Sea el sistema (7.6) el cual se expresa en la forma (7.7), donde se satisfacen las siguientes condiciones:*

C1: η_i *satisface la propiedad de observabilidad algebraica fraccionaria,* $p+1 \leq i \leq n$.

C2: Δ_i *es acotado, i.e.,* $\exists\, M \in \mathbb{R}^+$ *tal que* $\|\Delta(x)\| \leq M$, $\forall\, x \in \Omega$.

C3: $k_{\hat{\eta}_i} \in \mathbb{R}^+$.

Entonces, la sincronización de la salida del sistema maestro (7.10) con la salida del sistema esclavo (7.12) es posible, para condiciones iniciales globales de los estados.

Demostración. De **C1** se pueden escribir las ecuaciones (7.9)-(7.13). Tomando la derivada fraccionaria de la ecuación (7.13), se tiene

$$e_i^{(\alpha)} = \eta_i^{(\alpha)} - \hat{\eta}_i^{(\alpha)} \tag{7.14}$$

Sustituyendo (7.9) y (7.11) en (7.14), se obtiene

$$e_i^{(\alpha)} + k_{\hat{\eta}_i} e_i = \Delta_i(x) \tag{7.15}$$

Existe una solución única del sistema (7.15), debido a que $\Delta_i(x(t)) - k_{\hat{\eta}_i} e_i(t)$ es una función Lipschitz continua en e.[2]

Resolviendo la ecuación (7.15) se tiene (Kilbas, Srivastava y Trujillo, 2006),

$$\begin{aligned} e_i(t) = \; & e_0 E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(k_{\hat{\eta}_i}(t-\tau)^\alpha)\Delta_i(\tau)\mathrm{d}\tau \end{aligned} \tag{7.16}$$

donde $e_i(0) = e_{i0}$.

Usando las desigualdades del triángulo y de Cauchy-Schwarz y **C2**

$$\begin{aligned} |e_i(t)| \leq \; & |e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha)| \\ & + M \int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)|\mathrm{d}\tau \end{aligned}$$

Las funciones $(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)$ y $E_\alpha(-k_{\hat{\eta}_i}t^\alpha)$ no son negativas debido a la **Propiedad 2** de la función de Mittag-Leffler y **C3**

$$\begin{aligned} |e_i(t)| \leq \; & |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + M \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)\mathrm{d}\tau \end{aligned}$$

Usando la **Propiedad 1** de la función de Mittag-Leffler

$$|e_i(t)| \leq \; |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha)$$

[2] La ecuación (7.15) no es autónoma, aunque la condición Lipschitz garantiza la solución única (Kilbas, Srivastava y Trujillo, 2006).

Si $t \to \infty$, se utiliza la ecuación (7.5) con $\mu = 3\pi\alpha/4$, debido a **C3**.

$$\begin{aligned}\lim_{t\to\infty} |e_i(t)| \leq & \ |e_{i0}| \lim_{t\to\infty} E_{\alpha,1}(-k_{\hat{\eta}_i} t^{\alpha}) \\ & +M \lim_{t\to\infty} t^{\alpha} E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^{\alpha}) \\ = & \ \frac{M}{k_{\hat{\eta}_i}}\end{aligned}$$

□

Nota 7.1 *Debido a que la propiedad de observabilidad algebraica fraccionaria expresa una variable en términos de derivadas fraccionarias de la salida medible y (las cuales son desconocidas), entonces es necesario introducir variables artificiales (en caso de ser posible) para evitar el uso de estas variables desconocidas.*

7.4. Sincronización del sistema hipercaótico fraccionario de Rössler

La metodología propuesta se ejemplifica en ésta sección mediante la sincronización del sistema hipercaótico fraccionario de Rössler y el sistema caótico de Lorenz en la sección 7.5.

Nota 7.2 *Los sistemas caóticos se caracterizan por el acotamiento global de sus trayectorias (Fradkov, 2007). Debido a esto, C2 siempre se satisface.*

Considere el sistema hipercaótico fraccionario de Rössler (Li y Chen, 2004)

$$x^{(\alpha)} = \begin{pmatrix} x_3 + ax_1 + x_2 \\ -cx_4 + dx_2 \\ -x_1 - x_4 \\ b + x_3x_4 \end{pmatrix} \tag{7.17}$$

$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

donde $x = (x_1, x_2, x_3, x_4)^T$ es el vector de estado, $\{y_1,\ y_2\}$ son las salidas disponibles del sistema. Para $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, y $\alpha = 0.95$, el sistema de Rössler (7.17) tiene un atractor hipercaótico, como se muestra en la figura 7.1.

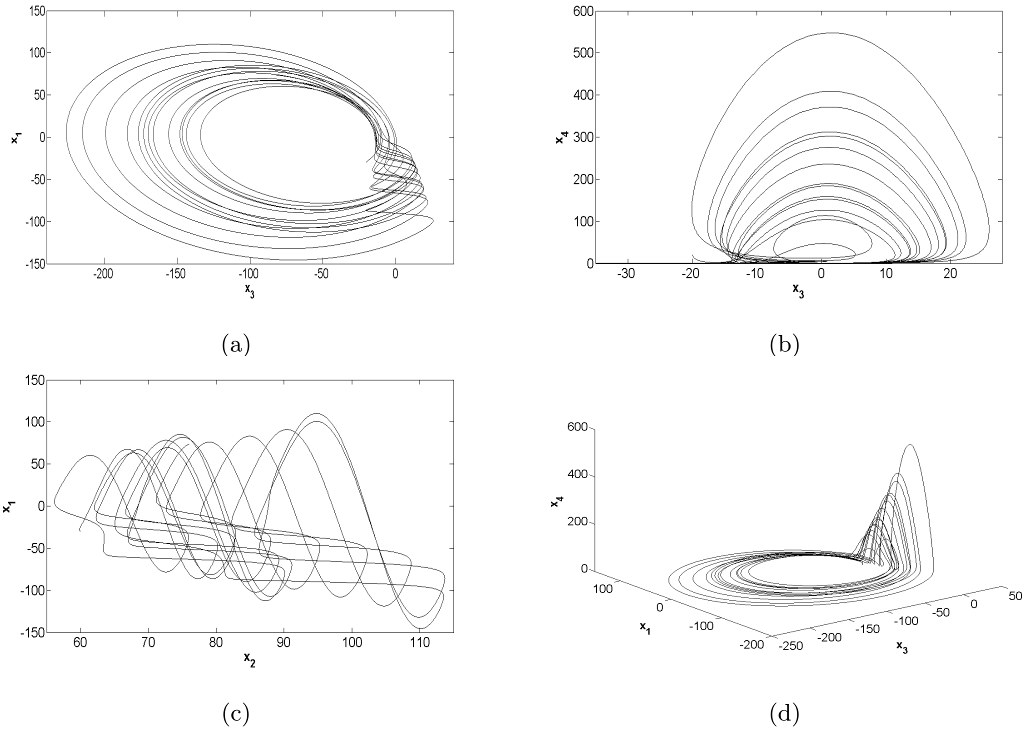


Figura 7.1: Diagrama de fase del sistema hipercaótico de Rössler, con $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, $\alpha = 0.95$, y condiciones iniciales $x_1(0) = -30$, $x_2 = 60$, $x_3(0) = -20$, y $x_4(0) = 20$: (a)Plano x_3-x_1, (b)Plano x_3-x_4, (c)Plano x_2-x_1, y (d)Espacio x_3-x_1-x_4.

Ahora, se reescribe el sistema (7.17) en la forma (7.7) como sigue

$$\begin{aligned} \bar{x}^{(\alpha)} &= \begin{pmatrix} \eta_3 + \mathrm{a}\bar{x}_1 + \bar{x}_2 \\ -\mathrm{c}\eta_4 + \mathrm{d}\bar{x}_2 \end{pmatrix} \\ \eta^{(\alpha)} &= \begin{pmatrix} -\bar{x}_1 - \eta_4 \\ \mathrm{b} + \eta_3\eta_4 \end{pmatrix} \end{aligned} \tag{7.18}$$

donde $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $\eta_3 = x_3$, $\eta_4 = x_4$, $y_{\bar{x}_1} = \bar{x}_1$, $y_{\bar{x}_2} = \bar{x}_2$. Del sistema (7.18), se determinan las siguientes relaciones

$$\eta_3 = \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2} \tag{7.19}$$

$$\eta_4 = \phi_4\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = -\frac{1}{\mathrm{c}}y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}}y_{\bar{x}_2} \tag{7.20}$$

entonces se dice $\eta_3 = x_3$ y $\eta_4 = x_4$ son algebraicamente observables y por lo tanto se satisface C1.

Los sistemas maestros están dados por,

$$\begin{cases} \eta_3^{(\alpha)} = & -\bar{x}_1 - \eta_4 \\ y_{\eta_3} = & \eta_3 = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2} \end{cases} \tag{7.21}$$

$$\begin{cases} \eta_4^{(\alpha)} = & \mathrm{b} + \eta_3\eta_4 \\ y_{\eta_4} = & \eta_4 = -\dfrac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \dfrac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2} \end{cases} \tag{7.22}$$

Ahora se diseñan los sistemas esclavos correspondientes para (7.21) y (7.22). Usando (7.11), se tiene

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}(y_{\eta_3} - \hat{\eta}_3) \tag{7.23}$$

con $k_{\hat{\eta}_3} \in \mathbb{R}^+$ (condición C3).

Sustituyendo (7.19) en (7.23) se obtiene

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}\left(y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}\hat{\eta}_3 \tag{7.24}$$

Para evitar el uso de la derivada fraccionaria $y_{\bar{x}}^{(\alpha)}$, se introduce la variable artificial $\gamma_{\hat{\eta}_3}$:

$$\gamma_{\hat{\eta}_3} = -k_{\hat{\eta}_3} y_{\bar{x}_1} + \hat{\eta}_3 \tag{7.25}$$

entonces

$$\hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3} y_{\bar{x}_1} \tag{7.26}$$

Sustituyendo (7.26) y su derivada fraccionaria de orden α en (7.24), se tiene

$$\gamma_{\hat{\eta}_3}^{(\alpha)} = -k_{\hat{\eta}_3}\gamma_{\hat{\eta}_3} - k_{\hat{\eta}_3}\left(\mathrm{a}y_{\bar{x}_1} + y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}^2 y_{\bar{x}_1} \quad , \tag{7.27}$$

con $\gamma_{\hat{\eta}_3}(0) = \gamma_{\hat{\eta}_{3_0}}$.

Entonces, el sistema esclavo de (7.21) esta dado por

$$\begin{cases} \hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3} y_{\bar{x}_1} \\ y_{\hat{\eta}_3} = \hat{\eta}_3 \end{cases} \tag{7.28}$$

Mediante un procedimiento similar se ha obtenido el sistema esclavo para (7.22)

$$\left\{ \begin{array}{l} \hat{\eta}_4 = \gamma_{\hat{\eta}_4} - \dfrac{k_{\hat{\eta}_4}}{\mathrm{c}} y_{\bar{x}_2} \\ y_{\hat{\eta}_4} = \hat{\eta}_4 \end{array} \right. \tag{7.29}$$

donde la dinámica de la variable auxiliar $\gamma_{\hat{\eta}_4}$ esta dada por

$$\gamma_{\hat{\eta}_4}^{(\alpha)} = -k_{\hat{\eta}_4}\gamma_{\hat{\eta}_4} + \frac{\mathrm{d}}{\mathrm{c}} k_{\hat{\eta}_4} y_{\bar{x}_2} + \frac{k_{\hat{\eta}_4}^2}{\mathrm{c}} y_{\bar{x}_2} \quad , \tag{7.30}$$

con $\gamma_{\hat{\eta}_4}(0) = \gamma_{\hat{\eta}_{4_0}}$ y $k_{\hat{\eta}_4} \in \mathbb{R}^+$ (condición C3).

Se realizaron algunas simulaciones tomando los siguientes valores $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, y $\alpha = 0.95$. Se consideraron las siguientes condiciones iniciales para el sistema maestro $\bar{x}_1(0) = -30$, $\bar{x}_2(0) = 60$, $\eta_3(0) = -20$, $\eta_4(0) = 20$, y para el sistema esclavo $\hat{\eta}_3(0) = -50$, $\hat{\eta}_4(0) = 10$, y las ganancias se seleccionaron como $k_{\hat{\eta}_3} = k_{\hat{\eta}_4} = 100$. La sincronización entre los sistemas maestros (7.21)-(7.22) y los sistemas esclavos (7.28)-(7.29) se muestra en la figura 7.2.

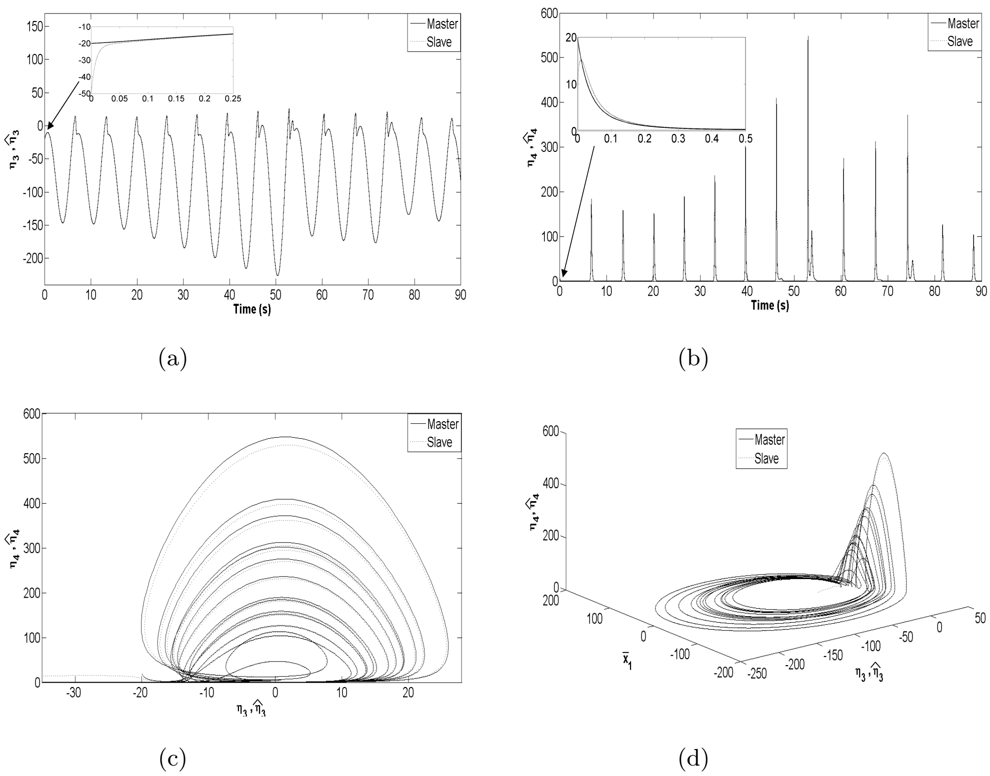


Figura 7.2: Sincronización del sistema hipercaótico fraccionario de Rössler.

7.5. Sincronización del sistema caótico fraccionario de Lorenz

Considere el sistema caótico fraccionario de Lorenz descrito por,

$$
\begin{aligned}
x^{(\alpha)} = & \begin{pmatrix} \mathrm{a}x_2 - \mathrm{a}x_1 \\ \mathrm{b}x_1 - \mathrm{c}x_2 - x_1x_3 \\ x_1x_2 - \mathrm{d}x_3 \end{pmatrix} \\
y = & \; x_1
\end{aligned}
\tag{7.31}
$$

Con a $= 10$, b $= 28$, c $= -8$, $d = 8/3$, y $\alpha = 0.8$, el sistema (7.31) presenta un comportamiento caótico (Wen, Wu y Lu, 2008). La Figura 7.3 muestra el atractor caótico del sistema (7.31).

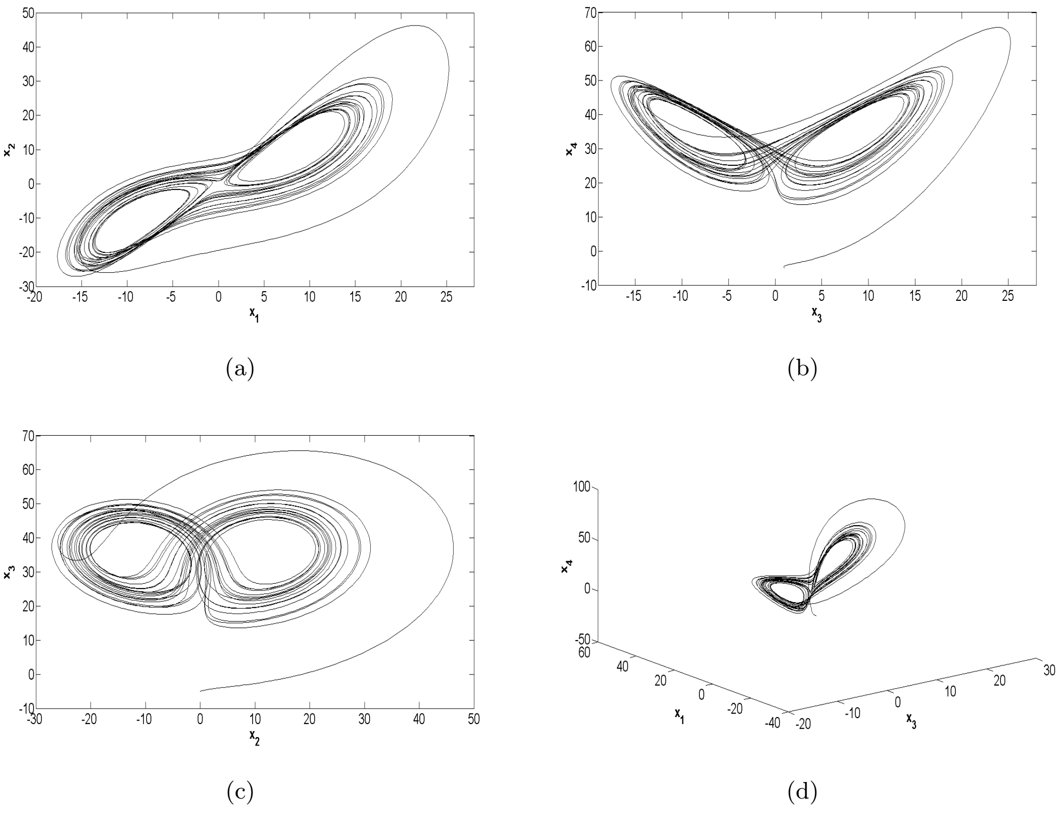


Figura 7.3: Comportamiento del sistema caótico fraccionario de Lorenz, con $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, y condiciones iniciales $x_1(0) = 1$, $x_2(0) = 0$ y $x_3(0) = -5$: (a) x_1 y x_2, (b) x_1 y x_3, (c) x_2 y x_3, (d) x_1, x_2 y x_3.

Reescribiendo el sistema (7.31) en la forma (7.7), se tiene

$$\begin{aligned} \bar{x}^{(\alpha)} &= \mathrm{a}(\eta_2 - \bar{x}_1) \\ \eta^{(\alpha)} &= \begin{pmatrix} \mathrm{b}\bar{x}_1 - \mathrm{c}\eta_2 - \bar{x}_1\eta_3 \\ \bar{x}_1\eta_2 - \mathrm{d}\eta_3 \end{pmatrix} \\ y_{\bar{x}} &= \bar{x}_1 \end{aligned} \tag{7.32}$$

donde $\bar{x}_1 = x_1$, $\eta_2 = x_2$, y $\eta_3 = x_3$.

La observabilidad algebraica fraccionaria para $\eta_2 = x_2$ y $\eta_3 = x_3$ se determina como sigue. Para η_2, se tiene que

$$y_{\bar{x}}^{(\alpha)} = \mathrm{a}(\eta_2 - y_{\bar{x}}) \quad \Rightarrow \quad \eta_2 = \phi_2\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = \frac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}} \tag{7.33}$$

Mientras que para η_3,

$$\eta_3 = -\frac{1}{y_{\bar{x}}}\left(\eta_2^{(\alpha)} + \mathrm{c}\,\eta_2 - \mathrm{b}\,y_{\bar{x}}\right) \tag{7.34}$$

Sustituyendo (7.33) en (7.34),

$$\begin{aligned} \eta_3 &= \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}\right) \\ &= -\frac{1}{\mathrm{a}\, y_{\bar{x}}}\left[\mathcal{D}^{(2\alpha)} y_{\bar{x}} + (\mathrm{a} + c) y_{\bar{x}}^{(\alpha)} + \mathrm{a}(\mathrm{c} - \mathrm{b}) y_{\bar{x}}\right] \end{aligned} \tag{7.35}$$

Con $y_{\bar{x}} = x_1 \neq 0$. Por lo tanto, η_2 y η_3 satisfacen la Definición 7.2, considerando como salida $y_{\bar{x}} = x_1$.

De la ecuación (7.33) se obtiene el siguiente sistema maestro en la forma (7.9)-(7.10), para $\eta_2 = x_2$,

$$\begin{cases} \eta_2^{(\alpha)} = \mathrm{b}\bar{x}_1 - \mathrm{c}\eta_2 - \bar{x}_1\eta_3 \\ \\ y_{\eta_2} = \eta_2 = \dfrac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}} \end{cases} \tag{7.36}$$

Utilizando la ecuación (7.11), se propone el siguiente sistema esclavo que se sincroniza con (7.36),

$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}(\eta_2 - \hat{\eta}_2) \tag{7.37}$$

Sustituyendo (7.33) en (7.37),

$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}\left(\frac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}}\right) - k_{\hat{\eta}_2}\hat{\eta}_2 \tag{7.38}$$

Debido a que la derivada fraccionaria de la salida de orden α, $y_{\bar{x}}^{(\alpha)}$, no es conocida, el sistema esclavo (7.38) no puede ejecutarse, por lo que se introduce la variable auxiliar $\gamma_{\hat{\eta}_2}$:

$$\gamma_{\hat{\eta}_2} = -\frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} + \hat{\eta}_2 \tag{7.39}$$

entonces

$$\hat{\eta}_2 = \gamma_{\hat{\eta}_2} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} \tag{7.40}$$

La derivada fraccionaria de orden α de (7.40) es

$$\hat{\eta}_2^{(\alpha)} = \gamma_{\hat{\eta}_2}^{(\alpha)} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}}^{(\alpha)} \tag{7.41}$$

Sustituyendo (7.40) y (7.41) en (7.38), se obtiene

$$\gamma_{\hat{\eta}_2}^{(\alpha)} = -k_{\hat{\eta}_2}\gamma_{\hat{\eta}_2} + \left(1 - \frac{k_{\hat{\eta}_2}}{a}\right) k_{\hat{\eta}_2} y_{\bar{x}} \quad , \qquad \gamma_{\hat{\eta}_2}(0) = \gamma_{\hat{\eta}_{2_0}} \tag{7.42}$$

Por lo tanto, el sistema esclavo de $\eta_2 = x_2$ está dado por,

$$\left\{ \begin{array}{rl} \hat{\eta}_2 = & \gamma_{\hat{\eta}_2} + \dfrac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} \\ \\ y_{\hat{\eta}_2} = & \hat{\eta}_2 \end{array} \right. \tag{7.43}$$

Para la variable η_3 no se sigue el mismo procedimiento debido a la discontinudad que presenta la ecuación (7.35) si $y_{\bar{x}} = x_1 = 0$. Se propone el sistema esclavo:

$$\left\{ \begin{array}{rl} \hat{\eta}_3^{(\alpha)} = & \bar{x}_1 \hat{\eta}_2 - d\hat{\eta}_3 \\ \\ y_{\hat{\eta}_3} = & \hat{\eta}_3 \end{array} \right. \tag{7.44}$$

Haciendo $e_3^{(\alpha)} = \eta_3^{(\alpha)} - \hat{\eta}_3^{(\alpha)}$, donde $\eta_3^{(\alpha)}$ se toma de la ecuación (7.32), entonces

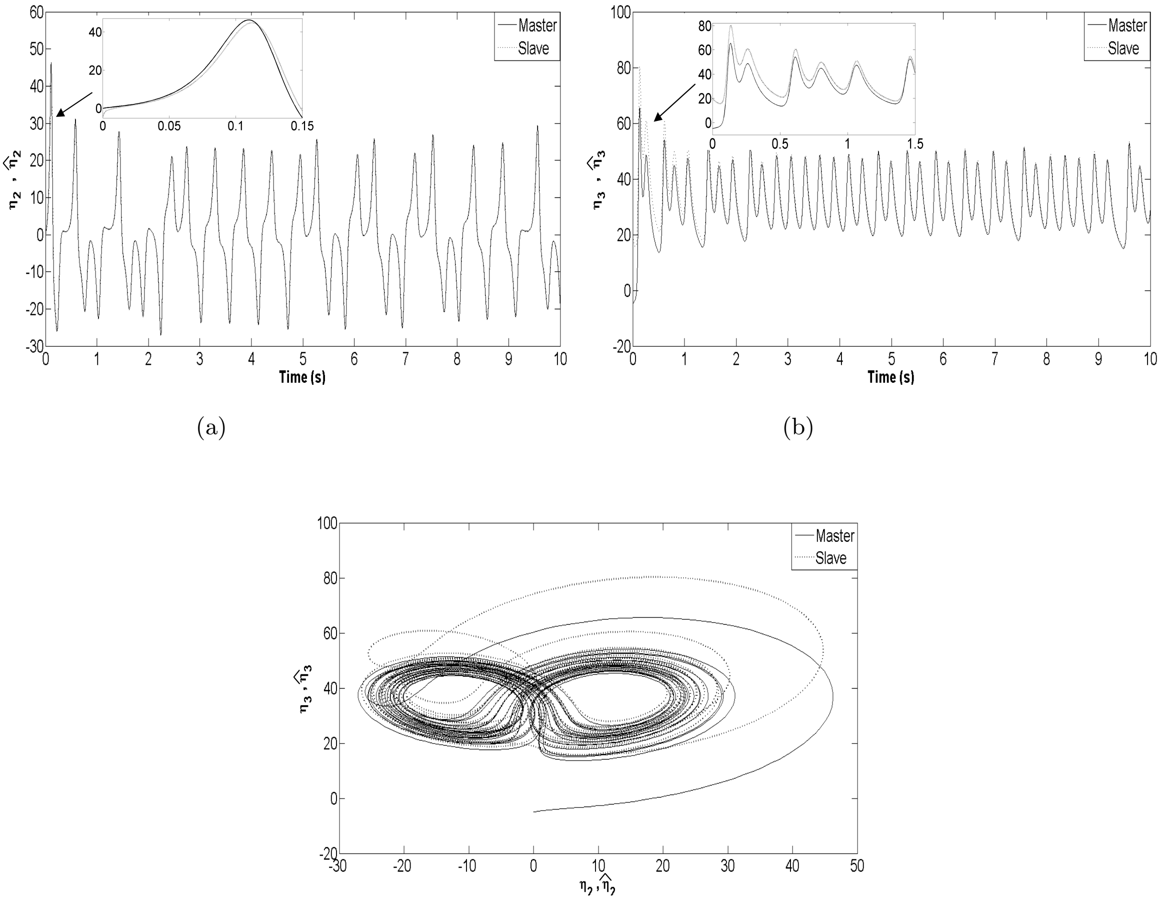
$$e_3^{(\alpha)} + de_3 = \bar{x}_1 e_2 \tag{7.45}$$

La ecuación (7.45) tiene la forma de la ecuación (7.15), por lo tanto aplicando la Proposición 7.1 se tiene que

$$|e_3| \leq \frac{mM}{dk_{\hat{\eta}_2}} \tag{7.46}$$

donde m es la cota de la variable $\bar{x}_1$.

Para las simulaciones, las condiciones iniciales arbitrarias del sistema maestro son $\bar{x}_1(0) = 1$, $\eta_2(0) = 0$, $\eta_3(0) = -5$, y para el sistema esclavo son $\hat{\eta}_2(0) = -5$, $\hat{\eta}_3(0) = 20$, con parámetros $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, la condición inicial para las variable auxiliar es $\gamma_{\hat{\eta}_2}(0) = -20$ y la ganancia del observador es $k_{\hat{\eta}_2} = 150$. La sincronización de los sistemas maestro y esclavo se muestran en la Figura 7.4.



(a) (b)

(c)

Figura 7.4: Sincronización del sistema caótico fraccionario de Lorenz, con $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, y condiciones iniciales $\bar{x}_1(0) = 1$, $\eta_2(0) = 0$, $\eta_3(0) = -5$, $\hat{\eta}_2(0) = -5$ y $\hat{\eta}_3(0) = 20$.

Capítulo 8

Sincronización generalizada en sistemas no lineales con estructura diferente

En este capítulo se propone un método para la sincronización generalizada (SG) en sistemas no lineales con estructura diferente, el esquema no requiere tener disponible el estado completo del sistema, a diferencia de algunas metodologías existentes. En este caso, es suficiente conocer la salida para generar una transformación la cual esta representada por una base de trascendencia diferencial, que lleva a un nuevo sistema de coordenadas, es decir, existe un elemento $\bar{y}$ y sea $n \geq 0$ el entero mínimo tal que $\bar{y}^{(n)}$ es analíticamente dependiente en $\bar{y}, \bar{y}^{(1)}, \ldots, \bar{y}^{(n-1)}$ entonces

$$\bar{H}(\bar{y}^{(n)}, \bar{y}^{(n-1)}, \ldots, \bar{y}, u, u^{(1)}, \ldots, u^{(\gamma)}) = 0 \tag{8.1}$$

El objetivo es determinar un control dinámico $u, u^{(1)}, \ldots, u^{(\gamma)}$ de tal manera que sea posible sincronizar el sistema transformado. Para esto, el sistema original se llevará a una forma triangular mediante la selección de un elemento primitivo diferencial adecuado definido generalmente como una combinación lineal de estados conocidos y de las entradas, donde los coeficientes pertenecen al campo diferencial generado por el campo K, la señal de control u y cantidades diferenciales de ésta. Hasta donde se sabe, en el problema de SG sólo se ha utilizado retroalimentación estática empleando herramientas geométrico diferenciales, la desventaja de esos esquemas es que requieren ganancias con valores del orden de 10000 a 90000 unidades.

8.1. Planteamiento y solución del problema

Considere la siguiente definición.

Definición 8.1 *Un sistema es Picard-Vessiot (PV) si y solamente si el espacio vectorial generado por las derivadas del conjunto $\{y^{(\mu)},\ \mu \geq 0\}$ tiene dimensión finita, con coeficientes en $k\langle u\rangle$.*

El sistema (8.1) se resuelve como

$$\bar{y}^{(n)} = -\mathcal{L}(\bar{y}^{(n-1)}, \ldots, \bar{y}, u, u^{(1)}, \ldots, u^{(\gamma-1)}) + u^{(\gamma)}$$

Definiendo $\xi_i = \bar{y}^{(i-1)}$, $1 \leq i \leq n$. Entonces, se obtiene el siguiente sistema transformado de coordenadas,

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n \\
\dot{\xi}_n &= -\mathcal{L}(\xi_1, \ldots, \xi_n, u, u^{(1)}, \ldots, u^{(\gamma-1)}) + u^{(\gamma)} \\
\bar{y} &= \xi_1
\end{aligned} \tag{8.2}$$

La representación triangular del sistema (9.2) se conoce como Forma Canónica Generalizada de Observabilidad (FCGO).

Ahora considere el sistema,

$$\begin{aligned}
\dot{x} &= F(x, u) \\
y &= Cx
\end{aligned} \tag{8.3}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $F(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ es no lineal y continuamente diferenciable, $u \in \mathbb{R}$ es la entrada, $y \in \mathbb{R}$ es la salida y $C \in \mathbb{R}^{1\times n}$ es un vector constante.

Lema 8.1 *El sistema no lineal (8.3) se puede expresar en la Forma Canónica Generalizada de Observabilidad si y solamente si es PV.*

Demostración. Sea el conjunto $\{\varepsilon, \varepsilon^{(1)}, \ldots, \varepsilon^{(n-1)}\}$ una base finita de trascendencia diferencial, con $\varepsilon^{(i-1)} = y^{(i-1)}$, $1 \leq i \leq n$, donde $n \geq 0$ es el mínimo entero tal que $y^{(n)}$ es dependiente en $y, y^{(1)}, \ldots, y^{(n-1)}, u, \ldots$.

Entonces, introduciendo el cambio de coordenadas $\xi_i = \varepsilon^{(i-1)}$, $1 \leq i \leq n$, se tiene la FCGO,

$$\begin{aligned} \dot{\xi}_j &= \xi_{j+1}, \qquad 1 \leq j \leq n-1 \\ \dot{\xi}_n &= -\mathcal{L}(\xi_1, \ldots, \xi_n, u, u^{(1)}, \ldots, u^{(\gamma-1)}) + u^{(\gamma)} \end{aligned}$$

$\square$

En éste capítulo, se discute el problema de SG en sistemas no lineales que pueden expresarse en forma triangular (FGCO). Para ésta clase de sistemas, se propone solucionar el problema de SG utilizando una retroalimentación dinámica que estabilice el error de sincronización. El problema de SG se plantea como sigue.

Considere dos sistemas no lineales en la configuración maestro–esclavo. El sistema maestro está dado por

$$\begin{aligned} \dot{x}_m &= F_m(x_m, u_m) \\ y_m &= h_m(x_m) \end{aligned} \tag{8.4}$$

y el esclavo por

$$\begin{aligned} \dot{x}_s &= F_s(x_s, u_s(x_s, y_m)) \\ y_s &= h_s(x_s) \end{aligned} \tag{8.5}$$

donde $x_s = (x_{1_s}, \ldots, x_{n_s}) \in \mathbb{R}^{n_s}$, $x_m = (x_{1_m}, \ldots, x_{n_m}) \in \mathbb{R}^{n_m}$, $h_s : \mathbb{R}^{n_s} \to \mathbb{R}$, $h_m : \mathbb{R}^{n_m} \to \mathbb{R}$, $u_m = (u_{1_m}, \ldots, u_{\bar{m}_m}) \in \mathbb{R}^{\bar{m}_m}$, $u_s : \mathbb{R}^{n_s} \times \mathbb{R} \to \mathbb{R}$, $y_m, y_s \in \mathbb{R}$, F_s, F_m, h_s, h_m son polinomios en sus argumentos.

Este es el caso general, ya que por ejemplo, los sistemas (8.4) y (8.5) no requieren ser afines en la entrada, tampoco se necesita separar la dinámica del sistema esclavo en una parte lineal y otra no lineal como en (Meng y Wang, 2008).

Se introduce la siguiente definición de SG.

Definición 8.2 (Sincronización (SG)) *Los sistemas esclavo y maestro se dicen que están en Sincronización Generalizada si existe un elemento primitivo diferencial que genere una transformación* $H_{ms} : \mathbb{R}^{n_s} \to \mathbb{R}^{n_m}$ *con* $H_{ms} = \Phi_s^{-1} \circ \Phi_m$, *donde* Φ_m *y* Φ_s *son las transformaciones que llevan a los sistemas (8.4) y (8.5) a la FCGO. Además existe una variedad algebraica* $M = \{(x_s, x_m) | x_m = H_{ms}(x_s)\}$ *y un conjunto compacto* $B \subset \mathbb{R}^{n_s} \times \mathbb{R}^{n_m}$ *con* $M \subset B$ *tal que sus trayectorias con condiciones iniciales en* B *se aproximan a* M *cuando* $t \to \infty$.

De la Definition 8.2 se tiene el siguiente criterio:

$$\lim_{t\to\infty} \|H_{ms}(x_s) - x_m\| = 0$$

Nota 8.1 *La sincronización completa o idéntica es un caso particular de SG, es decir, la transformación H_{ms} es la identidad.*

Nota 8.2 *El elemento primitivo diferencial se selecciona como*

$$y = \sum_i \alpha_i x_i + \sum_j \beta_j u_j \ , \qquad \alpha_i, \beta_j \in K\langle u\rangle$$

Teorema 8.1 *Sean los sistemas (8.4) y (8.5) transformables a la FCGO. Sean $z_m = (z_{m_1}, z_{m_2} \ldots, z_{m_n})'$ y $z_s = (z_{s_1}, z_{s_2} \ldots, z_{s_n})'$ las trayectorias de los sistemas maestro y esclavo con la transformación de coordenadas, $z_{m_i} = y_m^{(i-1)}$ y $z_{s_i} = y_s^{(i-1)}$, para $1 \leq i \leq n$. Entonces*

$$\lim_{t\to\infty} \|z_m - z_s\| = 0$$

esto es, la sincronización completa se lleva a cabo en los sistemas de coordenadas transformados y también se tiene que la SG en las coordenadas originales se satisface, esto es

$$\lim_{t\to\infty} \|H_{ms}(x_s) - x_m\| = 0$$

donde y_m y y_s son los elementos primitivos diferenciales de los sistemas maestro y esclavo, respectivamente.

Demostración. Sin pérdida de generalidad se puede hacer $u_m = 0 \in \mathbb{R}^{\bar{m}_m}$. Entonces el elemento primitivo diferencial para el sistema maestro es

$$y_m = \sum_i \alpha_{m_i} x_{m_i} = z_{m_1} \ , \qquad \alpha_{m_i} \in \mathbb{R}$$

y para el sistema esclavo

$$y_s = \sum_i \alpha_{s_i} x_{s_i} + \sum_j \beta_{s_j} u_{s_j} = z_{s_1} \ , \qquad \alpha_{s_i}, \beta_{s_j} \in \mathbb{R}\langle u_s\rangle$$

lo cual lleva a

$$\begin{aligned} \dot{z}_{m_j} &= z_{m_{j+1}}, \\ \dot{z}_{m_n} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) \end{aligned}$$

y

$$\begin{aligned}
\dot{z}_{s_j} &= z_{s_{j+1}}, \qquad 1 \leq j \leq n-1 \\
\dot{z}_{s_n} &= -\mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_s, \dot{u}_s, \ldots, u_s^{(\gamma-1)}) + u_s^{(\gamma)}
\end{aligned}$$

Se definen las señales de control como $u_1 = u_s$, $u_2 = \dot{u}_s$, …, $u_\gamma = u_s^{(\gamma-1)}$, entonces se propone el siguiente sistema dinámico

$$\begin{aligned}
\dot{u}_j &= u_{j+1}, \qquad 1 \leq j \leq \gamma-1 \\
\dot{u}_\gamma &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) + \kappa(z_m - z_s)
\end{aligned}$$

donde $z_m = (z_{m_1}, z_{m_2} \ldots, z_{m_n})'$, $z_s = (z_{s_1}, z_{s_2} \ldots, z_{s_n})'$ y $\kappa = (k_1, k_2, \ldots, k_n)$.

Entonces, el error de sincronización $e_z = z_m - z_s$ para el sistema aumentado está dado por

$$\begin{aligned}
\dot{e}_{z_j} &= e_{z_{j+1}}, \qquad 1 \leq j \leq n-1 \\
\dot{e}_{z_n} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) - \dot{u}_\gamma \\
\dot{u}_i &= u_{i+1}, \qquad 1 \leq i \leq \gamma-1 \\
\dot{u}_\gamma &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) + \kappa\ e_z
\end{aligned}$$

Por lo tanto,

$$\dot{e}_z = A\ e_z \tag{8.6}$$

con

$$A = \begin{bmatrix}
0 & 1 & 0 & & \cdots & & 0 \\
0 & 0 & 1 & 0 & \cdots & & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & & \cdots & 0 & 1 & 0 \\
0 & 0 & & \cdots & & 0 & 1 \\
-k_1 & -k_2 & & \cdots & & -k_{n-1} & -k_n
\end{bmatrix}$$

Finalmente, la ecuación (8.6) es asintóticamente estable si las ganancias $(k_1, k_2, \ldots, k_n)$ se escogen de tal manera que la matriz $A \in \mathbb{R}^{n\times n}$ sea Hurwitz. $\square$

8.2. Sincronización generalizada entre los circuitos de Colpitts y Chua

Se aplica la metodología propuesta para la sincronización generalizada entre los circuitos de Colpitts y Chua, descritos en el capítulo 5.

Como sistema maestro se considera el circuito de Colpitts, representado por:

$$\begin{aligned} \dot{x}_{1_c} =& \ -a_c \ \exp(-x_{2_c}) + \ a_c x_{3_c} + a_c \\ \dot{x}_{2_c} =& \ b_c x_{3_c} \\ \dot{x}_{3_c} =& \ -c_c x_{1_c} - c_c x_2 - d_c x_3 \end{aligned} \tag{8.7}$$

donde, $a_c = b_c \dfrac{C_2}{C_1}$, $b_c = \dfrac{I_0}{w_0 C_2 V_T}$, $c_c = \dfrac{V_T}{w_0 L I_0}$, $d_c = \dfrac{R}{L w_0}$.

Se selecciona el elemento primitivo diferencial como la salida del sistema (8.7), esto es

$$y_c = x_{2_c}$$

Ahora se propone la transformación de coordenadas del sistema,

$$\begin{pmatrix} z_{1_c} \\ z_{2_c} \\ z_{3_c} \end{pmatrix} = \begin{pmatrix} y_c \\ \dot{y}_c \\ \ddot{y}_c \end{pmatrix} = \begin{pmatrix} x_{2_c} \\ b_c x_{3_c} \\ -b_c c_c x_{1_c} - b_c c_c x_{2_c} - b_c d_c x_{3_c} \end{pmatrix} \tag{8.8}$$
$$= \Phi_c(x_c)$$

Con la transformación de coordenadas (8.8), el sistema de Colpitts (8.7) se reescribe como,

$$\begin{pmatrix} \dot{z}_{1_c} \\ \dot{z}_{2_c} \\ \dot{z}_{3_c} \end{pmatrix} = \begin{pmatrix} z_{2_c} \\ z_{3_c} \\ \Psi_c(x_c) \end{pmatrix} \tag{8.9}$$

donde

$$\Psi_c(x_c) = \ -b_c c_c \dot{x}_{1_c} - b_c c_c \dot{x}_{2_c} - b_c d_c \dot{x}_{3_c}$$

El circuito de Chua representa al sistema esclavo y está descrito por las ecuaciones,

$$\begin{aligned} \dot{x}_{1_{ch}} =& \ a_{ch}(x_{2_{ch}} - x_{1_{ch}} - \nu_x) \\ \dot{x}_{2_{ch}} =& \ x_{1_{ch}} - x_{2_{ch}} + x_{3_{ch}} \\ \dot{x}_{3_{ch}} =& \ -b_{ch} x_{2_{ch}} \\ y =& \ x_{3_{ch}} \end{aligned} \tag{8.10}$$

donde $\nu_x = m_0 x_{1_{ch}} + m_1 x_{1_{ch}}^3$ y los parámetros a_c, b_c, m_0 y m_1 se seleccionan de tal manera que el sistema (8.10) sea caótico.

Para este caso, el elemento primitivo diferencial se elige como la salida del sistema (8.10) junto con una señal de control, $y_{ch} = x_{3_{ch}} + u_1$.

Utilizando el Teorema 8.1, el sistema controlado en el cambio de coordenadas es,

$$\begin{pmatrix} z_{1_{ch}} \\ z_{2_{ch}} \\ z_{3_{ch}} \end{pmatrix} = \begin{pmatrix} y_{ch} \\ \dot{y}_{ch} \\ \ddot{y}_{ch} \end{pmatrix} = \begin{pmatrix} x_{3_{ch}} + u_1 \\ -b_{ch} x_{2_{ch}} + u_2 \\ -b_{ch}(x_{1_{ch}} - x_{2_{ch}} + x_{3_{ch}}) + u_3 \end{pmatrix} = \Phi_{ch}(x_{ch}) \tag{8.11}$$

donde $u_1 = u$, $u_2 = \dot{u}$, $u_3 = \ddot{u}$ son las señales de control que deben diseñarse para obtener la sincronización completa en los sistemas de coordenadas transformados (8.8) y (8.11).

El sistema controlado aumentado (8.11), se representa como,

$$\begin{pmatrix} \dot{z}_{1_{ch}} \\ \dot{z}_{2_{ch}} \\ \dot{z}_{3_{ch}} \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} z_{2_{ch}} \\ z_{3_{ch}} \\ \Psi_{ch}(x_{ch}) + \overline{u} \\ u_2 \\ u_3 \\ \overline{u} \end{pmatrix} \tag{8.12}$$

donde $\Psi_{ch}(x_{ch}) = \; -b_{ch}(\dot{x}_{1_{ch}} - \dot{x}_{2_{ch}} + \dot{x}_{3_{ch}})$

Entonces, el objetivo de control es determinar $\overline{u}$ tal que las trayectorias del sistema esclavo (8.12) sigan a las trayectorias del sistema maestro (8.9). En otras palabras, se requiere hallar $\overline{u}$ en el sistema (8.12), tal que $(z_{1_{ch}}, z_{2_{ch}}, z_{3_{ch}}) \to (z_{1_c}, z_{2_c}, z_{3_c})$, cuando $t \to \infty$.

Se define el error de sincronización en las coordenadas transformadas $e_z = z_c - z_{ch}$, por lo tanto

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} e_2 \\ e_3 \\ \Psi_c(x_c) - \Psi_{ch}(x_{ch}) - \bar{u}(x_{ch}, y_c) \end{pmatrix}$$

Si la entrada de control $\bar{u}$ se define como,

$$\overline{u}(x_c, y_L) = \dot{u}_3 = \Psi_c(x_c) - \Psi_{ch}(x_{ch}) + \kappa \; e_z$$

donde $\kappa = [k_1, k_2, k_3]$ es el vector de ganancia. Entonces la dinámica del error de sincronización es, $\dot{e}_z = A\ e_z$, donde

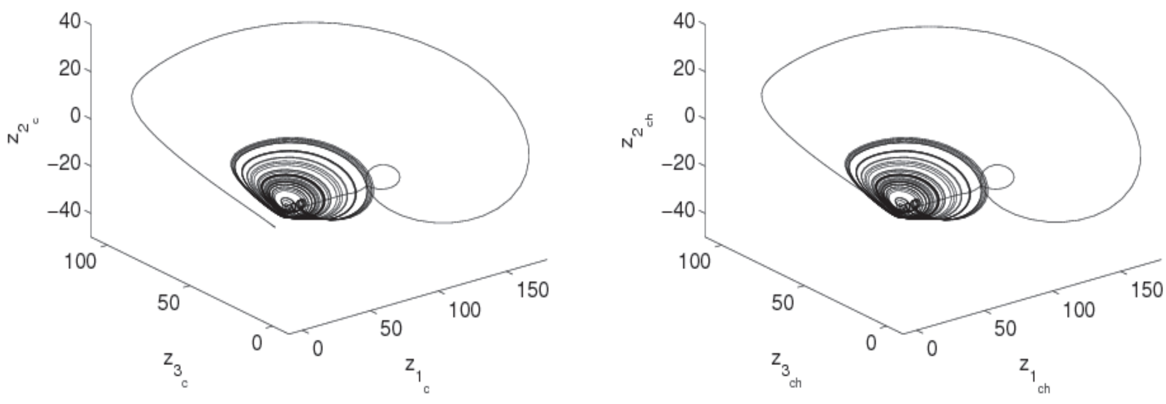
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{pmatrix}$$

Mediante el criterio de Routh-Hurwitz, se concluye que $\|e_z\| \to 0$ cuando $t \to \infty$ si $k_1 > 0$, $k_2 > \dfrac{k_1}{k_3}$ y $k_3 > 0$.

La Figura 8.1 muestra la sincronización generalizada de los sistemas maestro y esclavo en los sistemas transformados, con $a_c = b_c = 6.2723$, $c_c = 0.0797$, $d_c = 0.6898$, $a_{ch} = 9.5$, $b_c = 100/7$, $m_0 = -8/7$, $m_1 = 4/63$, $k_1 = 10$, $k_2 = 10$ y $k_3 = 10$.

Mediante la transformación inversa se obtienen las variables originales,

$$\begin{aligned} \begin{pmatrix} x_{1_c} \\ x_{2_c} \\ x_{3_c} \end{pmatrix} &= \begin{pmatrix} -\dfrac{1}{c_c}\left[\dfrac{1}{b_c} z_{3_c} + \dfrac{d_c}{b_c} z_2 + c_c z_{1_c}\right] \\ z_{1_c} \\ \dfrac{z_{2_c}}{b} \end{pmatrix} \\ &= \Phi_c^{-1}(z_c) \end{aligned} \tag{8.13}$$



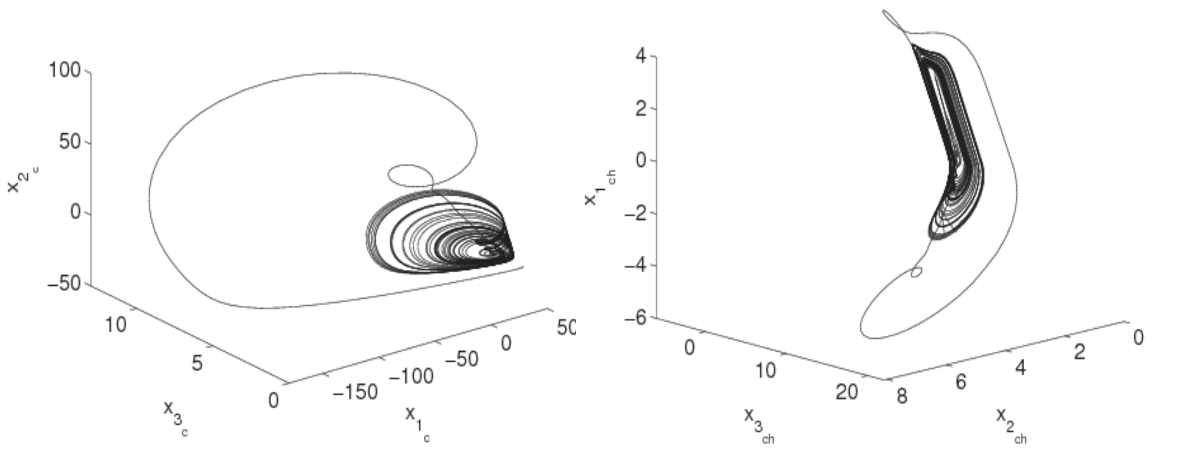
(a) (b)

Figura 8.1: Sincronización generalizada en las coordenadas $(z_{1_c}, z_{2_c}, z_{3_c})$ y $(z_{1_{ch}}, z_{2_{ch}}, z_{3_{ch}})$. (a) Sistema maestro; (b) Sistema esclavo.

y

$$
\begin{aligned}
\begin{pmatrix} x_{1_{ch}} \\ x_{2_{ch}} \\ x_{3_{ch}} \end{pmatrix} &= \begin{pmatrix} -(z_{1_{ch}} - u_1) - \dfrac{z_{2_{ch}} - u_2}{b_{ch}} - \dfrac{z_{3_{ch}} - u_3}{b_{ch}} \\ -\dfrac{z_{2_{ch}} - u_2}{b_{ch}} \\ z_{1_{ch}} - u_1 \end{pmatrix} \\
&= \Phi_{ch}^{-1}(z_{ch})
\end{aligned}
\tag{8.14}
$$

La Figura 8.2 presenta la sincronización generalizada en las coordenadas originales obtenidas mediante las transformaciones inversas (8.13) y (8.14). En la Figura 8.3 se muestran los errores de sincronización en los sistemas transformados, los cuales indican el desempeño del método propuesto.



(a) (b)

Figura 8.2: Sincronización generalizada en las variables originales $(x_{1_c}, x_{2_c}, x_{3_c})$ y $(x_{1_{ch}}, x_{2_{ch}}, x_{3_{ch}})$. (a) Sistema maestro; (b) Sistema esclavo.

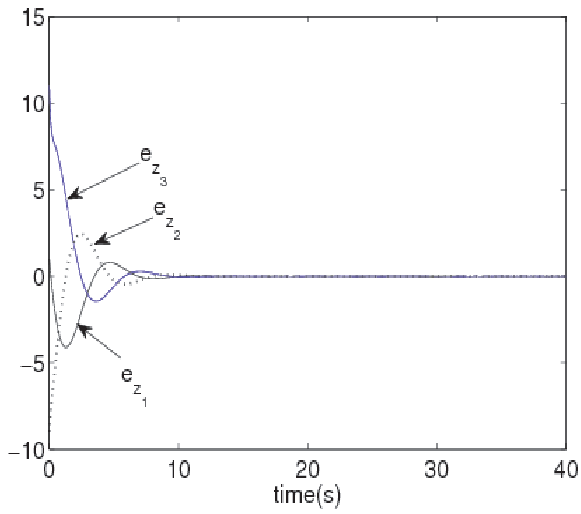


Figura 8.3: Errores de sincronización; $e_{z_1} = z_{1_c} - z_{1_{ch}}$, $e_{z_2} = z_{2_c} - z_{2_{ch}}$ and $e_{z_3} = z_{3_c} - z_{3_{ch}}$

Capítulo 9

SG en sistemas no lineales de orden fraccionario con estructura diferente

Este capítulo aborda el problema de Sincronización Generalizada (ver Definición 8.2) en sistemas no lineales de orden fraccionario con estructura diferente en la configuración maestro-esclavo. La idea es redefinir el concepto de sincronización generalizada para el caso fraccionario mediante el elemento primitivo diferencial fraccionario, el cual se expresa como una combinación lineal de los estados y de las entradas del sistema.

El esquema de sincronización que se propone consiste en transformar el sistema fraccionario original a otro equivalente representado por la Forma Canónica Generalizada de Observabilidad Fraccionaria (FCGOF), donde la primera componente corresponde al elemento primitivo diferencial. Lo anterior se realiza para el sistema esclavo y para el sistema maestro. Cuando los sistemas se encuentran representados en la FCGOF se obtiene la sincronización completa. Finalmente, las variables originales se reconstruyen usando las soluciones de los sistemas dados por la FCGOF y la condición de observabilidad algebraica fraccionaria (ver la Definición 7.2). La metodología se aplica para la SG entre los sistemas caóticos fraccionarios de Chua y Rössler.

Considere el sistema no lineal de orden fraccionario (en el sentido de la Definición 7.1), con $0 < \alpha < 1$:

$$\begin{aligned} x^{(\alpha)} = & \;\; f(x,u), \;\; x(0) = x_0 \\ y = & \;\; h(x) \end{aligned} \tag{9.1}$$

donde $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ es una función Lipschitz continua, $h : \Omega \to \mathbb{R}$ es una función continuamente diferenciable, con $x_0 \in \Omega \subset \mathbb{R}^n$, $y \in \mathbb{R}$ y $u \in \mathbb{R}^m$.

Se utiliza la misma notación definida por la ecuación (7.2) cuando el operador fraccionario se aplica recursivamente.

El método de sincronización generalizada que aquí se propone, al igual que en el capítulo anterior, se basa en la existencia de un elemento primitivo diferencial fraccionario para generar la transformación del sistema original de coordenadas a otro sistema equivalente representado por la FCGOF.

Con respecto al sistema (9.1) se considera la siguiente hipótesis, la cual justifica que pueda representarse en la FCGOF.

Definición 9.1 *El sistema de orden fraccionario (9.1) es Piccard-Vessiot, es decir, el espacio vectorial generado por las derivadas fraccionarias del conjunto* $\{\mathcal{D}^{(\mu\alpha)}\overline{y},\ \mu \in \mathbb{N} \cup \{0\}\}$ *tiene dimensión finita, donde* $\overline{y}$ *es el elemento primitivo diferencial.*

Es decir, existe un elemento $\bar{y} \in \mathbb{R}$ y sea $n \in \mathbb{N} \cup \{0\}$ el entero mínimo tal que $\mathcal{D}^{(n\alpha)}\overline{y}$ es analíticamente dependiente en

$$\left\{\overline{y}, \overline{y}^{(\alpha)}, \mathcal{D}^{(2\alpha)}\overline{y}, \ldots, \mathcal{D}^{([n-1]\alpha)}\overline{y}\right\}$$

entonces

$$\mathcal{D}^{(n\alpha)}\overline{y} = -\mathcal{L}\left(\overline{y}, \overline{y}^{(\alpha)}, \mathcal{D}^{(2\alpha)}\overline{y}, \ldots, \mathcal{D}^{([n-1]\alpha)}\overline{y}, u, u^{(\alpha)}, \mathcal{D}^{(2\alpha)}u, \ldots, \mathcal{D}^{([\gamma-1]\alpha)}u\right) + \mathcal{D}^{(\gamma\alpha)}u$$

con $n, \gamma \in \mathbb{N} \cup \{0\}$.

Haciendo el cambio de variable

$$\xi_i = \mathcal{D}^{((i-1)\alpha)}\bar{y}\ , \qquad 1 \leq i \leq n,$$

se obtiene la FCGOF del sistema (9.1)

$$\begin{aligned} \xi_1^{(\alpha)} &= \xi_2 \\ \xi_2^{(\alpha)} &= \xi_3 \\ &\vdots \\ \xi_{n-1}^{(\alpha)} &= \xi_n \\ \xi_n^{(\alpha)} &= -\mathcal{L}\left(\xi_1, \ldots, \xi_n, u, u^{(\alpha)}, \mathcal{D}^{(2\alpha)}u, \ldots, \mathcal{D}^{([\gamma-1]\alpha)}u\right) + \mathcal{D}^{(\gamma\alpha)}u \\ \bar{y} &= \xi_1 \end{aligned} \tag{9.2}$$

Para el problema de SG, sean los sistemas no lineales de orden fraccionario, con $0 < \alpha < 1$, maestro

$$\begin{aligned} x_m^{(\alpha)} &= F_m(x_m, u_m) \\ y_m &= h_m(x_m) \end{aligned} \tag{9.3}$$

y esclavo

$$\begin{aligned} x_s^{(\alpha)} &= F_s(x_s, u_s(x_s, y_m)) \\ y_s &= h_s(x_s) \end{aligned} \tag{9.4}$$

donde $x_s \in \Omega \subset \mathbb{R}^n$, $F_s \in \Omega \subset \mathbb{R}^n$, $F_m \in \Omega \subset \mathbb{R}^n$, $x_m \in \Omega \subset \mathbb{R}^n$, $h_s : \Omega \to \mathbb{R}$, $h_m : \Omega \to \mathbb{R}$, $u_m \in \mathbb{R}^{\bar{m}_m}$, $u_s \in \mathbb{R}$, $y_m, y_s \in \mathbb{R}$, F_s, F_m, h_s, h_m son polinomios en sus argumentos, con condiciones iniciales $x_{m_0} = x_m(0)$ y $x_{s_0} = x_s(0)$.

Teorema 9.1 *Sean los sistemas (9.3) y (9.4) transformables a la FCGOF. Sean* $z_m = (z_{m_1}, z_{m_2} \ldots, z_{m_n})'$ *y* $z_s = (z_{s_1}, z_{s_2} \ldots, z_{s_n})'$ *las trayectorias de los sistemas maestro y esclavo con la transformación de coordenadas,* $z_{m_i} = \mathcal{D}^{((i-1)\alpha)}\bar{y}_m$ *y* $z_{s_i} = \mathcal{D}^{((i-1)\alpha)}\bar{y}_s$*, para* $1 \leq i \leq n$*. Entonces*

$$\lim_{t\to\infty} \|z_m - z_s\| = 0$$

y por lo tanto

$$\lim_{t\to\infty} \|H_{ms}(x_s) - x_m\| = 0$$

donde $\bar{y}_m$ *y* $\bar{y}_s$ *son los elementos primitivos diferenciales de los sistemas maestro y esclavo, respectivamente.*

Demostración. Sin pérdida de generalidad se puede considerar $u_m = 0 \in \mathbb{R}^{\bar{m}_m}$. El elemento primitivo diferencial para el sistema maestro es igual a,

$$\bar{y}_m = \sum_i \alpha_{m_i} x_{m_i} = z_{m_1} \;, \qquad \alpha_{m_i} \in \mathbb{R}$$

y para el sistema esclavo

$$\bar{y}_s = \sum_i \alpha_{s_i} x_{s_i} + \sum_j \beta_{s_j} u_{s_j} = z_{s_1} \ , \qquad \alpha_{s_i}, \beta_{s_j} \in \mathbb{R}$$

de ésta manera, la FCGOF del sistema (9.3) es

$$\begin{aligned} z_{m_j}^{(\alpha)} &= z_{m_{j+1}}, \\ z_{m_n}^{(\alpha)} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) \end{aligned}$$

y la FCGOF del sistema esclavo (9.4) queda como,

$$\begin{aligned} z_{s_j}^{(\alpha)} &= z_{s_{j+1}}, \qquad 1 \leq j \leq n-1 \\ z_{s_n}^{(\alpha)} &= -\mathcal{L}_s\left(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma\right) + u_\gamma^{(\alpha)} \end{aligned}$$

donde

$$\begin{aligned} u_1 &= u_s \\ u_2 &= u_s^{(\alpha)} \\ &\vdots \\ u_\gamma &= \mathcal{D}^{((\gamma-1)\alpha)} u_s \end{aligned}$$

Se propone la ley de control dinámica como

$$\begin{aligned} u_j^{(\alpha)} &= u_{j+1}, \qquad 1 \leq j \leq \gamma - 1 \\ u_\gamma^{(\alpha)} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) + \kappa(z_m - z_s) \end{aligned}$$

donde $\kappa = (k_1, k_2, \ldots, k_n)$.

Entonces, la dinámica del error de sincronización $e_z = z_m - z_s$ para el sistema aumentado está dada por

$$\begin{aligned} e_{z_j}^{(\alpha)} &= e_{z_{j+1}}, \qquad 1 \leq j \leq n-1 \\ e_{z_n}^{(\alpha)} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) - u_\gamma^{(\alpha)} \\ u_i^{(\alpha)} &= u_{i+1}, \qquad 1 \leq i \leq \gamma - 1 \\ u_\gamma^{(\alpha)} &= -\mathcal{L}_m(z_{m_1}, \ldots, z_{m_n}) + \mathcal{L}_s(z_{s_1}, \ldots, z_{s_n}, u_1, u_2, \ldots, u_\gamma) + \kappa \ e_z \end{aligned}$$

La dinámica del error de sincronización en lazo cerrado es,

$$e_z^{(\alpha)} = A \ e_z \tag{9.5}$$

con

$$A = \begin{bmatrix} 0 & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & & \cdots & 0 & 1 & 0 \\ 0 & 0 & & \cdots & & 0 & 1 \\ -k_1 & -k_2 & & \cdots & & -k_{n-1} & -k_n \end{bmatrix}$$

La estabilidad de la ecuación (9.5) se determina utilizando los resultados existentes para ecuaciones lineales de orden fraccionario.

Teorema 9.2 (Matignon (1996)) *Sea $\alpha < 2$ y $\overline{A} \in \mathbb{C}^{n\times n}$. El sistema autónomo*

$$x^{(\alpha)} = \overline{A}x \qquad \textit{con} \quad x(0) = x_0$$

es asintóticamente estable si y solamente si $|arg(\lambda_i(\overline{A}))| > \alpha\pi/2$, donde $\lambda_i(\overline{A})$ es el i-ésimo valor propio de la matriz $\overline{A}$. □

Por lo tanto, aplicando el Teorema 9.2 se tiene que la ecuación (9.5) es asintóticamente estable si se eligen las ganancias $\kappa = (k_1, k_2, \ldots, k_n)$ tales que,

$$|\arg(\lambda_i(A))| > \frac{\alpha\pi}{2}$$

□

Nota 9.1 *Nótese que como caso particular en el Teorema 9.2, para $0 < \alpha < 1$, cualquier matriz Hurwitz satisface la condición*

$$|arg(\lambda_i(A))| > \frac{\pi}{2} > \frac{\alpha\pi}{2}$$

9.1. SG entre los sistemas caóticos fraccionarios de Chua y Rössler

Considere el sistema caótico fraccionario de Chua (Hartley, Lorenzo y Qammer, 1995),

$$
\begin{aligned}
x_{1_c}^{(\alpha)} &= a\left(x_{2_c} + \frac{x_{1_c} - 2x_{1_c}^3}{7}\right) \\
x_{2_c}^{(\alpha)} &= x_{1_c} - x_{2_c} + x_{3_c} \\
x_{3_c}^{(\alpha)} &= -\beta x_{2_c} \\
y_c &= x_{3_c}
\end{aligned}
\tag{9.6}
$$

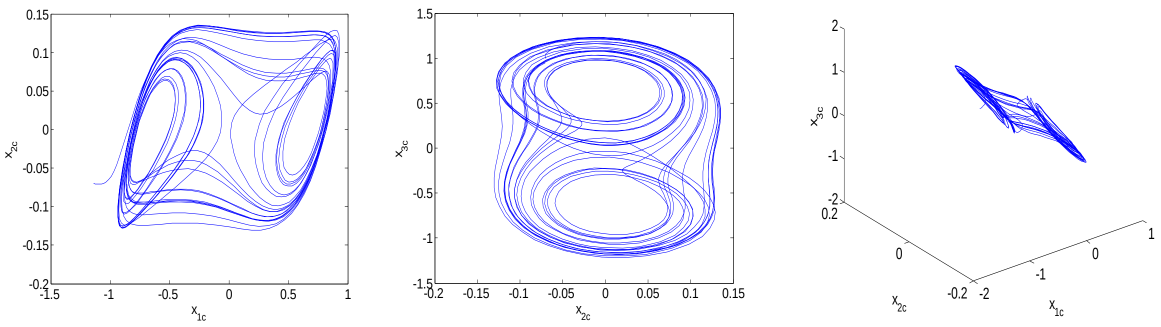
Con $a = 12.75$, $\beta = 100/7$ y $\alpha = 0.9$ el sistema (9.6) presenta un comportamiento caótico, como se ilustra en la figura 9.1.

El sistema (9.6) se considera en éste caso como el sistema maestro. Se selecciona el elemento primitivo diferencial como la salida del sistema (9.6),

$$y_c = x_{3_c}$$

Para representar el sistema (9.6) en la FCGOF se propone la transformación de coordenadas,

$$
\begin{aligned}
\begin{pmatrix} z_{1_c} \\ z_{2_c} \\ z_{3_c} \end{pmatrix} = \begin{pmatrix} y_c \\ y_c^{(\alpha)} \\ \mathcal{D}^{(2\alpha)} y_c \end{pmatrix} &= \begin{pmatrix} x_{3_c} \\ -\beta x_{2_c} \\ -\beta(x_{1_c} - x_{2_c} + x_{3_c}) \end{pmatrix} \\
&= \Phi_c(x_c)
\end{aligned}
\tag{9.7}
$$



(a) (b) (c)

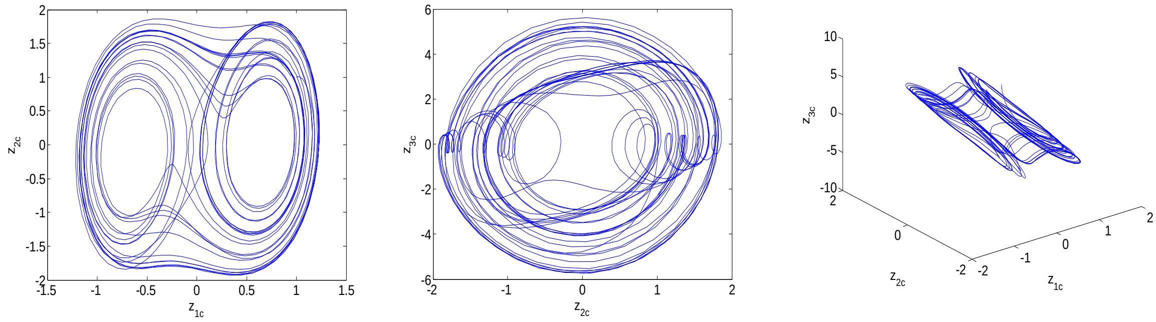
Figura 9.1: Sistema caótico fraccionario de Chua. (a) Variables x_{1_c} y x_{2_c}, (b) variables x_{2_c} y x_{3_c} y (c)variables x_{1_c}, x_{2_c} y x_{3_c}.

Con la transformación de coordenadas (9.7), el sistema de Chua (9.6) se expresa en la FCGOF,

$$\begin{pmatrix} z_{1_c}^{(\alpha)} \\ z_{2_c}^{(\alpha)} \\ z_{3_c}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} z_{2_c} \\ z_{3_c} \\ \Psi_c(x_c) \end{pmatrix} \tag{9.8}$$

donde $\Psi_c(x_c) = \;\; -\beta(x_{1_c}^* - x_{2_c}^* + x_{3_c}^*)$, $x_{1_c}^* = a\left(x_{2_c} + \dfrac{x_{1_c} - 2x_{1_c}^3}{7}\right)$, $x_{2_c}^* = x_{1_c} - x_{2_c} + x_{3_c}$, $x_{3_c}^* = -\beta x_{2_c}$, $x_{1_c} = -\frac{1}{\beta} z_{3_c} - \dfrac{z_{2_c}}{\beta} - z_{1_c}$, $x_{2_c} = -\dfrac{z_{2_c}}{\beta}$ y $x_{3_c} = z_{1_c}$.

La figura 9.2 muestra el comportamiento del sistema transformado (9.8). Las variables z_{1_c}, z_{2_c} y z_{3_c} , con respecto al tiempo, aparecen en las figuras 9.4(a), 9.4(b) y 9.4(c).



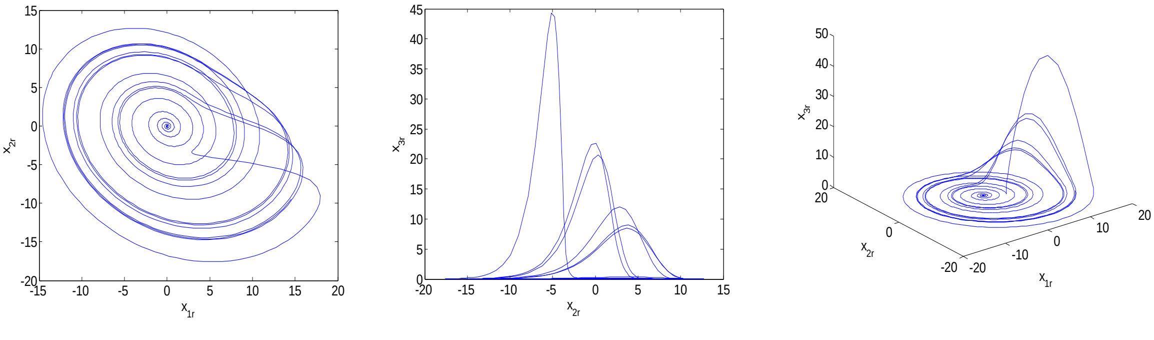
(a) (b) (c)

Figura 9.2: Sistema caótico fraccionario de Chua en la FCGOF. (a) Variables z_{1_c} y z_{2_c}, (b) variables z_{2_c} y z_{3_c} y (c)variables z_{1_c}, z_{2_c} y z_{3_c}.

Como sistema esclavo, sea el sistema caótico fraccionario de Rössler descrito por,

$$\begin{aligned} x_{1_r}^{(\alpha)} &= -(x_{2_r} + x_{3_r}) \\ x_{2_r}^{(\alpha)} &= x_{1_r} + a x_{2_r} \\ x_{3_r}^{(\alpha)} &= 0.2 + x_{3_r}(x_{1_r} - 10) \\ y_r &= x_{2_r} \end{aligned} \tag{9.9}$$

Para $a = 0.4$ y $\alpha = 0.9$ el sistema (9.9) exhibe comportamiento caótico (ver la figura 9.3).



(a) (b) (c)

Figura 9.3: Sistema caótico fraccionario de Rössler. (a) Coordenadas x_{1_r} y x_{2_r}, (b) coordenadas x_{2_r} y x_{3_r} y (c)coordenadas x_{1_r}, x_{2_r} y x_{3_r}.

Para convertir el sistema (9.9) a la FCGOF se plantean los siguientes cambios de variables,

$$\begin{pmatrix} z_{1_r} \\ z_{2_r} \\ z_{3_r} \end{pmatrix} = \begin{pmatrix} y_r \\ y_r^{(\alpha)} \\ \mathcal{D}^{(2\alpha)} y_r \end{pmatrix} = \begin{pmatrix} x_{2_r} + u_1 \\ x_{1_r} + a x_{2_r} + u_2 \\ x_{1_r}^{(\alpha)} + a x_{2_r}^{(\alpha)} + u_3 \end{pmatrix} \\ = \Phi_r(x_r) \tag{9.10}$$

donde $u_2 = u_1^{(\alpha)}$ y $u_3 = u_2^{(\alpha)}$.

Por lo tanto, la FCGOF del sistema (9.9) es,

$$\begin{pmatrix} z_{1_r}^{(\alpha)} \\ z_{2_r}^{(\alpha)} \\ z_{3_r}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} z_{2_r} \\ z_{3_r} \\ \Psi_r(x_r) \end{pmatrix} \tag{9.11}$$

donde $\Psi_r(x_r) = \ (a^2-1)x^*_{2_r} - x^*_{3_r} + a x^*_{1_r} + u_3^{(\alpha)}$, $x^*_{1_r} = -(x_{2_r} + x_{3_r})$, $x^*_{2_r} = x_{1_r} + a x_{2_r}$, $x^*_{3_r} = 0.2 + x_{3_r}(x_{1_r} - 10)$, $x_{1_r} = z_{2_r} - a z_{1_r} + a u_1 - u_2$, $x_{2_r} = z_{1_r} - u_1$ y $x_{3_r} = -z_{3_r} + a z_{2_r} - z_{1_r} + u_1 + u_3 - a(u_2 + a u_1)$.

Las trayectorias del sistema (9.11), para el caso autónomo, están dadas en las figuras 9.4(d), 9.4(e) y 9.4(f). La idea es que mediante las señales de control las variables del sistema transformado de Rössler z_{1_r}, z_{2_r} y z_{3_r} sigan a las trayectorias del sistema transformado de Chua z_{1_c}, z_{2_c} y z_{3_c}, respectivamente. Puede notarse de manera clara en las figuras 9.2 y 9.5 los atractores de los sistemas a ser sincronizados son totalmente diferentes.

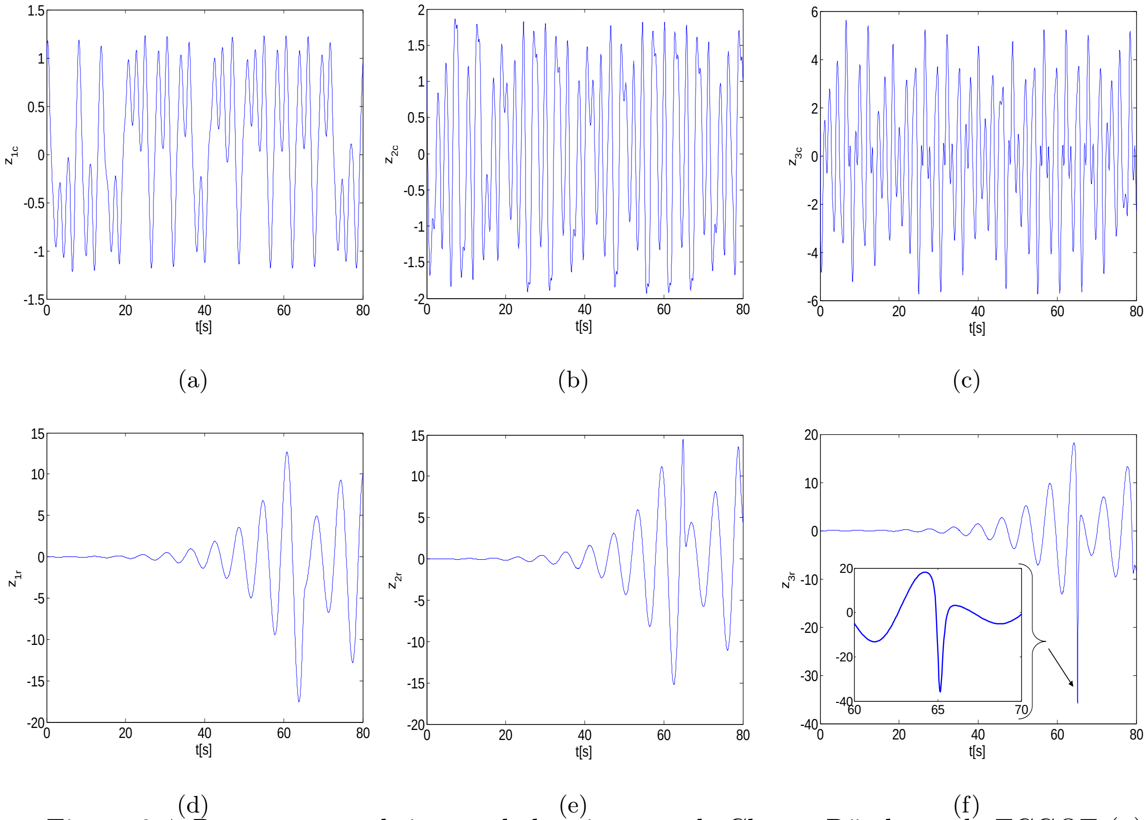


Figura 9.4: Respuesta en el tiempo de los sistemas de Chua y Rössler en la FCGOF (a) z_{1_c}, (b) z_{2_c}, (c) z_{3_c}, (d) z_{1_r}, (e) z_{2_r}, (f) z_{3_r}

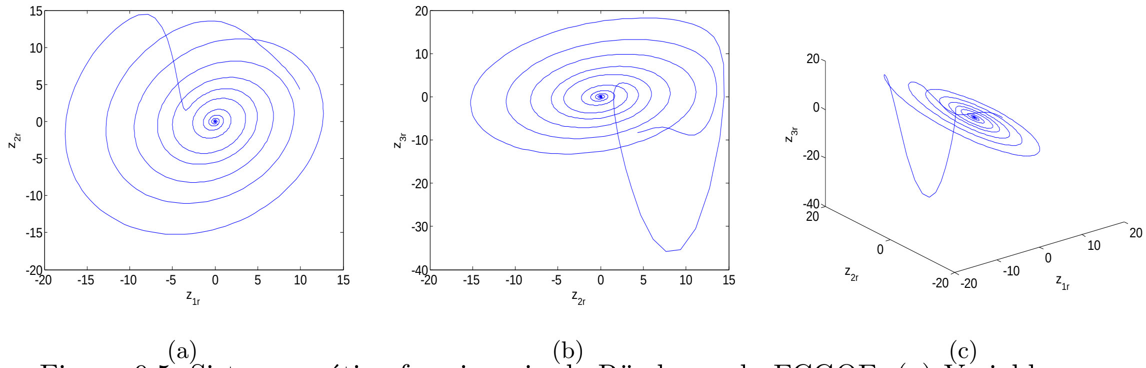


Figura 9.5: Sistema caótico fraccionario de Rössler en la FCGOF. (a) Variables z_{1_r} y z_{2_r}, (b) variables z_{2_r} y z_{3_r} y (c)variables z_{1_r}, z_{2_r} y z_{3_r}.

De acuerdo con el Teorema 9.1, se propone el siguiente sistema esclavo

$$\begin{pmatrix} z_{1_r}^{(\alpha)} \\ z_{2_r}^{(\alpha)} \\ z_{3_r}^{(\alpha)} \\ u_1^{(\alpha)} \\ u_2^{(\alpha)} \\ u_3^{(\alpha)} \end{pmatrix} = \begin{pmatrix} z_{2_r} \\ z_{3_r} \\ \Psi_r(x_r) \\ u_2 \\ u_3 \\ \Psi_c - \Psi_r + k e_z \end{pmatrix} \tag{9.12}$$

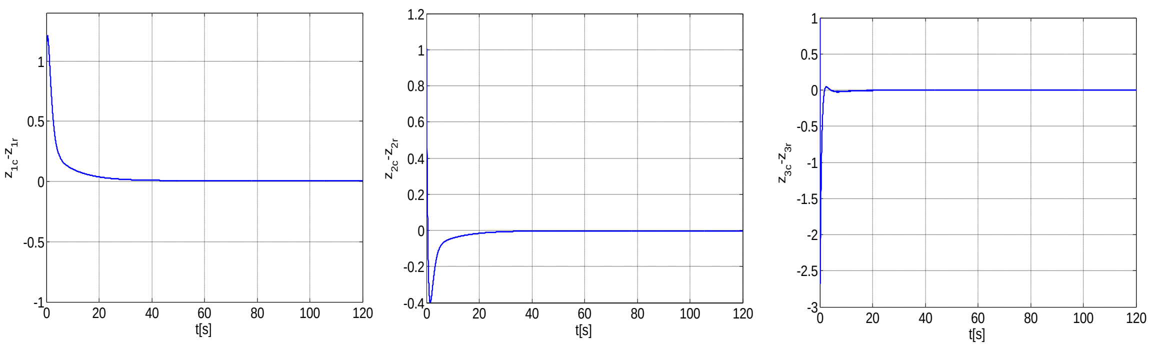
donde $k = (k_1, k_2, k_3)$ y $e_z = (z_{1_c} - z_{1_r}, z_{2_c} - z_{2_r}, z_{3_c} - z_{3_r})^T$.

La sincronización entre los sistemas de Chua y Rössler expresados en la FCGOF se presenta en la figura 9.7. Mientras que los errores de sincronización se pueden ver en la figura 9.6, con $k_1 = 1$, $k_2 = 2$ y $k_3 = 1$.

Las variables de los sistemas de coordenadas originales se obtienen con las respuestas de los sistemas transformados. Es decir

$$\left\{ \begin{array}{ll} x_{1_c} & = -\frac{1}{\beta} z_{3_c} - \dfrac{z_{2_c}}{\beta} - z_{1_c} \\ x_{2_c} & = -\dfrac{z_{2_c}}{\beta} \\ x_{3_c} & = z_{1_c} \end{array} \right. \quad \text{y} \quad \left\{ \begin{array}{ll} x_{1_r} & = z_{2_r} - a z_{1_r} + a u_1 - u_2 \\ x_{2_r} & = z_{1_r} - u_1 \\ x_{3_r} & = -z_{3_r} + a z_{2_r} - z_{1_r} + u_1 + u_3 - a(u_2 + a u_1) \end{array} \right.$$

La figura 9.8 muestra las trayectorias reconstruidas de los sistemas caóticos de Chua y Rössler.



(a) (b) (c)

Figura 9.6: Errores de sincronización: (a) $z_{1_c} - z_{1_r}$, (b) $z_{2_c} - z_{2_r}$ y (c) $z_{3_c} - z_{3_r}$.

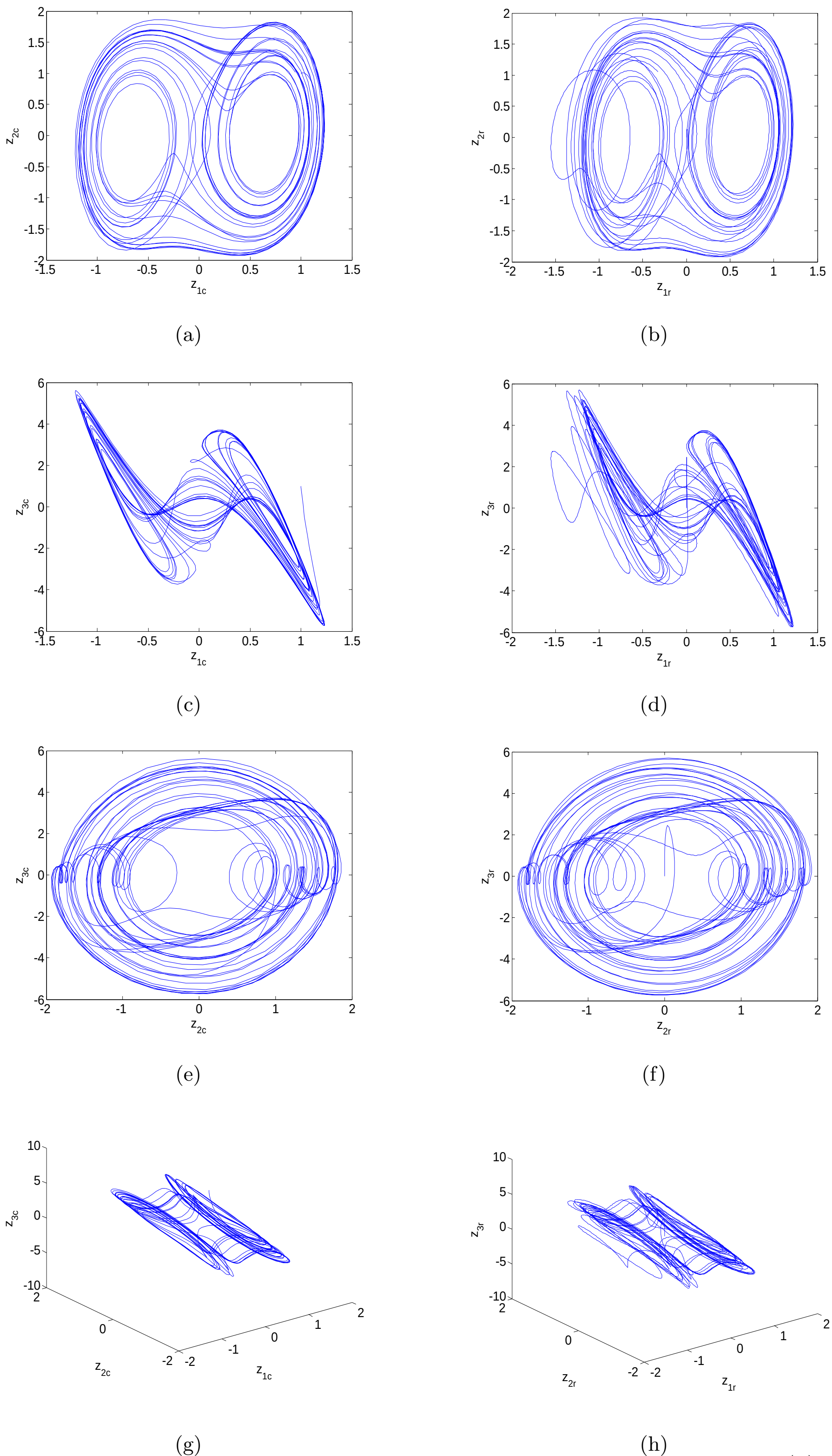


Figura 9.7: Sincronización entre los sistemas de Chua y Rössler en la FCGOF. (a) Variables z_{1_c} y z_{2_c}, (b)variables z_{1_r} y z_{2_r}, (c) variables z_{1_c} y z_{3_c}, (d)variables z_{1_r} y z_{3_r}, (e) variables z_{2_c} y z_{3_c}, (f)variables z_{2_r} y z_{3_r}, (g) variables z_{1_c}, z_{2_c} y z_{3_c}, (f)variables z_{1_r}, z_{2_r} y z_{3_r}.

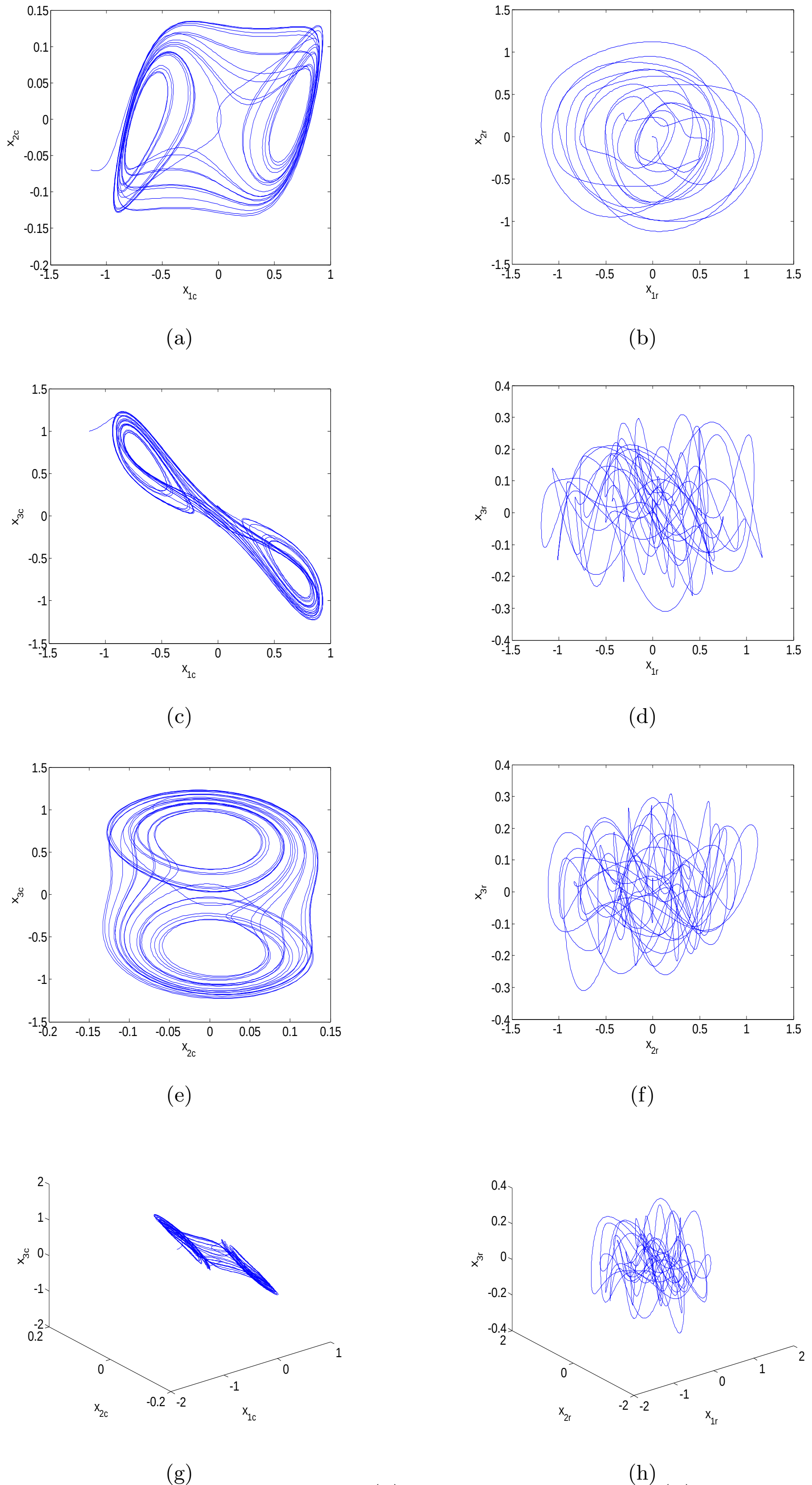


Figura 9.8: Coordenadas reconstruidas. (a) Variables x_{1_c} y x_{2_c}, (b)Variables x_{1_r} y x_{2_r}, (c)variables x_{1_c} y x_{3_c}, (d)variables x_{1_r} y x_{3_r}, (e) variables x_{2_c} y x_{3_c}, (f) variables x_{2_r} y x_{3_r}, (g)variables x_{1_c}, x_{2_c} y x_{3_c} y (h)variables x_{1_r}, x_{2_r} y x_{3_r}.

Capítulo 10

Conclusiones

En ésta tesis se propusieron diferentes tipos de observadores para sistemas no lineales de orden entero y fraccionario. En el capítulo 3 se diseñó un observador polinomial para sistemas con no linealidades Lipschitz, y para sistemas con no linealidades monótonas no crecientes y/o no decrecientes, demostrándose convergencia exponencial del error de observación en ambos casos. En el mismo capítulo, se incluyó un observador de orden reducido con convergencia asintótica. El capítulo 4 presentó un observador algebraico de orden reducido, el cual consistió básicamente en un cambio lineal de variable que depende de la coordenada a reconstruir y de la salida disponible, el algoritmo de estimación se aplicó en el monitoreo de biorreactores con modelos de crecimiento de Monod y Haldane. En el capítulo 5 se comenzó planteando el problema de sincronización maestro–esclavo considerado como un problema de observación, donde el oscilador caótico es el sistema maestro y el observador es el sistema esclavo. En el resto del capítulo 5 se aplicaron los observadores estudiados en los capítulos previos para la sincronización de sistemas caóticos, por ejemplo, se realizó la sincronización del circuito de Colpitts y la sincronización del circuito de Chua utilizando el observador polinomial exponencial, y finalmente, se empleó un observador uniformemente acotado para la sincronización del sistema caótico de Lorenz y para la sincronización del sistema caótico de Rössler. En el capítulo 6 se diseñó un observador adaptable y se aplicó al problema de sincronización e identificación de parámetros del oscilador caótico de Rikitake; la estructura propuesta es capaz de realizar ésto de manera simultánea. En el capítulo 7 se diseñó un observador proporcional fraccionario para sistemas no lineales de orden fraccionario y se propusó un esquema para la sincronización de sistemas caóticos fraccionarios. Además, se introdujo el criterio de Observabilidad Algebraica Fraccionaria, el cual permite determinar si las variables desconocidas pueden ser reconstruidas mediante la salida medible y sus derivadas fraccionarias. Con el observador proporcional fraccionario y la condición de observabilidad algebraica fraccionaria se construyen los observadores para la sincronización del sistema hipercaótico fraccionario de Rössler y para la sincronización del sistema caótico fraccionario de Lorenz. En el capítulo 8 se introdujo una definición de sincronización generalizada para sistemas no lineales diferentes en términos del elemento primitivo diferencial, el cual se expresa como una combinación lineal de los estados y de las entradas. El método que se propuso consistió en transformar el sistema original de coordenadas a otro equivalente representado por la Forma Canónica Generalizada de Observabilidad (FCGO), donde la primera componente corresponde al elemento primitivo diferencial. Lo anterior se realizó para el sistema esclavo y para el sistema maestro. Una vez descritos en la FCGO se obtiene la sincronización completa. Después, las variables originales se reconstruyeron mediante las transformaciones inversas. Se presentó la SG entre los circuitos caóticos de Colpitts y Chua. Finalmente, en el capítulo 9, se estudió el problema de sincronización generalizada en sistemas no lineales de orden fraccionario con estructura diferente. Mediante un elemento primitivo diferencial y la condición de observabilidad algebraica fraccionaria se obtuvo una forma canónica generalizada de observabilidad para sistemas fraccionarios. Finalmente, se aplicó el esquema de sincronización en la SG entre los sistemas caóticos fraccionarios de Chua y Rössler.

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Apéndice A

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Synchronization of chaotic Liouvillian systems: An application to Chua's oscillator

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ABSTRACT

In this paper we deal with the synchronization of chaotic oscillators with Liouvillian properties (chaotic Liouvillian system) based on nonlinear observer design. The strategy consists of proposing a polynomial observer (slave system) which tends to follow exponentially the chaotic oscillator (master system). The proposed technique is applied in the synchronization of Chua's circuit using Matlab-Simulink® and Matlab®-LMI programs. Simulation results are used to visualize and illustrate the effectiveness of Chua's oscillator in synchronization.

1. Introduction

Since Pecora and Carroll's observation on the possibility of synchronizing two chaotic systems [1] (so-called drive-response configuration), several synchronization schemes have been developed [2–5]. Synchronization can be classified into mutual synchronization (or bidirectional coupling) [6] and master–slave synchronization (or unidirectional coupling) [1,7].

In the past years, synchronization of chaotic systems problem has received a great deal of attention among scientists in many fields [8–11]. As it is well known, the study of the synchronization problem for nonlinear systems has been very important from the nonlinear science point of view, in particular, the applications to biology, medicine, cryptography, secure data transmission and so on [12,13]. In general, synchronization research has been focused on the following areas: nonlinear observers [2,14–17], nonlinear control [18], feedback controllers [5], nonlinear backstepping control [19], time delayed systems [12,20], directional and bidirectional linear coupling [21], adaptive control [9], adaptive observers [22,23], sliding mode observers [13,24], active control [25], among others.

This work considers the master–slave synchronization problem via an exponential polynomial observer (EPO) based on differential an algebraic techniques [26–28]. Differential and algebraic concepts allow us to establish an algebraic observability condition, and therefore they provide a first step for the construction of an algebraic observer. An observable system in this sense can be regarded as a system whose state variables can be expressed in terms of the input and output variables and a finite number of their time derivatives. Thus, chaos synchronization problem can be posed as an observer design procedure, where the coupling signal is viewed as output and the slave system is regarded as observer. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (Chua's circuit) to the slave system (EPO), the slave is requested to recover the unknown state trajectories of the master. The strategy consists of proposing an EPO which exponentially reconstructs the unknown states of Chua's system.

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Chua's circuit is a nonlinear electronic chaotic oscillator. This circuit is easily constructed [29] and has been employed in a variety of applications [30], e.g. communication systems [31]. The chaotic associative memory architecture proposed in [32] the use of a network of Chua's oscillators coupled via piecewise linear conductances.

In this paper the Chua's oscillator is viewed as a chaotic system with some Liouvillian properties [26,28], refered as Chaotic Liouvillian Oscillator (CLO). The Liouvillian character of the system (if a variable can be obtained by the adjunction of integrals or exponentials of integrals) is exploited as an observability criterion, that is to say, by this property we can know whether a variable can be reconstructed with the measurable output.

This paper is organized as follows: In Section 2 we give some definitions about differential-algebraic approach and Liouvillian systems. In Section 3 we treated the synchronization problem and its solution by means of an exponential polynomial observer. In Section 4 we presented the synchronization of the Chua's circuit [33] and we show some numerical simulations. Finally, in Section 5 we close the paper with some concluding remarks.

2. Definitions

We start with some basic definitions for the understanding of Liouvillian systems, the following definitions are presented.

Definition 1 (*Algebraic Observability Condition – AOC*). Let us consider a nonlinear dynamical system with input u, output y, and state vector $x = (x_1, x_2, \ldots, x_n)^T$. A state variable $x_i \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}\langle u, y\rangle$,[1] that is to say, x_i satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e.,

$$P_i(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0, \quad i \in \{1, 2, \ldots, n\}, \tag{1}$$

with coefficients in $\mathbb{R}\langle u, y\rangle$.

Example 1. Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \end{aligned} \tag{2}$$

If we define $y = x_2$, then

$$\begin{aligned} x_2 &= y, \\ x_3 &= \dot{y}, \\ \dot{x}_1 &= y + \dot{y}^2, \end{aligned} \tag{3}$$

The above system is not algebraically observable since x_1 cannot be expressed as a differential algebraic polynomial in terms of $\{u, y\}$.

Motivated by this fact, we present the next definition.

Definition 2 (*Liouvillian system*). A dynamical system is said to be Liouvillian if the elements (for example, state variables or parameters) can be obtained by an adjunction of integrals or exponentials of integrals of elements of $\mathbb{R}$.

Example 2. We consider the nonlinear system as in Example 1. From (3) we can observe that, although x_1 does not satisfy the AOC we can obtain it by means of the integral

$$x_1 = \int (y + \dot{y}^2),$$

Therefore the nonlinear system (2) is Liouvillian.

Example 3. Consider the following nonlinear second order system that models a prey–predator situation,

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 - x_1 + k, \\ \dot{x}_2 &= -x_1 x_2 - x_2, \\ y &= x_1, \end{aligned} \tag{4}$$

so, we have:

[1] $\mathbb{R}\langle u, y\rangle$ denotes the differential field generated by the field $\mathbb{R}$, the input u, the measurable output y, and the time derivatives of u and y.

$$\begin{aligned}\dot{y} &= yx_2 - y + k,\\ \dot{x}_2 &= -yx_2 - x_2,\end{aligned} \tag{5}$$

we can solve directly (5), and obtain:

$$\begin{aligned}\dot{y} &= yx_2 - y + k,\\ x_2 &= \exp\left(-\int (y+1)dt\right),\end{aligned}$$

we obtain the parameter k to be:

$$k = \dot{y} - \exp\left(-\int (y+1)dt\right) + y$$

and we say that system (4) is Liouvillian.

For further information we recommend to see [26,28].

3. Problem formulation and main result

Let us consider the following chaotic Liouvillian system,

$$\begin{aligned}\dot{x} &= Ax + \psi(x) + \varphi(x) + \zeta(u),\\ y &= Cx,\end{aligned} \tag{6}$$

where $x \in \mathbb{R}^n$, is the state vector[2]; $u \in \mathbb{R}^l$, is the input vector, $l \leqslant n$; $y \in \mathbb{R}$ is the measured output; $\zeta(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$ is an input dependent vector function; $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$ are constants; and $\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are state dependent nonlinear vector functions.

We restrict each $\psi_i(\cdot)$ to be nondecreasing, that is, for all $a, b \in \mathbb{R},\ a > b$, it satisfies the following monotone sector condition

$$0 \leqslant \frac{\psi_i(a) - \psi_i(b)}{a - b}, \quad i = 1, \dots, n. \tag{7}$$

In the same manner, we restrict each $\varphi_i(\cdot)$ to be nonincreasing, that is, for all $a, b \in \mathbb{R},\ a > b$, it satisfies

$$\frac{\varphi_i(a) - \varphi_i(b)}{a - b} \leqslant 0, \quad i = 1, \dots, n. \tag{8}$$

To show the relation between the observers for nonlinear systems and chaos synchronization we give the observer's definition.

Definition 3 (*Exponential Observer*). An exponential observer for (6) is a system with state $\hat{x}$ such that

$$\|x - \hat{x}\| \leqslant \kappa \exp(-\xi t),$$

where κ and ξ are positive constants.

In the context of master–slave synchronization, x can be considered as the state variable of the master system, and $\hat{x}$ can be viewed as the state variable of the slave system. Hence, the master–slave synchronization problem can be solved by designing an observer for (6).

In what follows, we will solve the synchronization problem by using a exponential polynomial observer based upon the Lyapunov method [34]. To this end, we first compute the dynamics of the synchronization error (difference between the master and the slave systems). Next, by means of a simple quadratic Lyapunov function, we prove the exponential convergence.

System (6) is assumed to be a chaotic Liouvillian system, then by Definition 2 all states of (6) can be reconstructed. In this sense, we will propose an observer scheme.

The observer structure. The observer for system (6) has the next form

$$\dot{\hat{x}} = A\hat{x} + \psi(\hat{x}) + \varphi(\hat{x}) + \zeta(u) + \sum_{i=1}^{m} K_i\left(y - C\hat{x}\right)^{2i-1}, \tag{9}$$

where $\hat{x} \in \mathbb{R}^n$, and $K_i \in \mathbb{R}^n$, for $1 \leqslant i \leqslant m$.

[2] Mathematically, chaotic systems are characterized by local instability and *global boundedness of the trajectories*, i.e. $\|x(t)\|$ is bounded for all $t \geqslant 0$.

Remark 1. The meaning of m can be understood as follows. As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance high order terms are included in the observer structure. In other words, the rate of convergence can be increased by injecting additional terms with increasing powers of the output error.

Observer convergence analysis. In order to prove the observer convergence, we analyze the observer error which is defined as $e = x - \hat{x}$. From Eqs. (6) and (9), the dynamics of the state estimation error is given by

$$\dot{e} = (A - K_1C)e + \phi(e) + \rho(e) - \sum_{i=2}^{m} K_i(Ce)^{2i-1}, \tag{10}$$

where $\phi(e) := \psi(x) - \psi(\hat{x})$ and $\rho(e) := \varphi(x) - \varphi(\hat{x})$.

It follows from (7) that each component of $\phi(e)$ satisfies

$$0 \leqslant \frac{\phi_i(e_i)}{e_i}, \quad \forall e_i \neq 0, \tag{11}$$

which implies a relationship between $\phi(e)$ and e as follows,

$$e^T\phi(e) = \sum_{i=1}^{n} e_i\phi_i(e_i) = \sum_{i=1}^{n} e_i^2 \frac{\phi_i(e_i)}{e_i},$$

by using (11) we have the following condition

$$0 \leqslant e^T\phi(e). \tag{12}$$

By a similar analysis, from (8) we have

$$e^T\rho(e) \leqslant 0. \tag{13}$$

Properties (12) and (13) will allow us prove that the state estimation error $e(t)$ decays exponentially.

We have the main result.

Proposition 1. *Consider the chaotic Liouvillian system (6) and the observer (9). If there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$ satisfying the linear matrix inequality (LMI)*

$$\begin{bmatrix} (A - K_1C)^TP + P(A - K_1C) + \varepsilon I & P + \epsilon_1 I & P - \epsilon_2 I \\ P + \epsilon_1 I & 0 & 0 \\ P - \epsilon_2 I & 0 & 0 \end{bmatrix} \leqslant 0 \tag{14}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geqslant 0, \quad i = 2, \ldots, m, \tag{15}$$

with $M_i := PK_iC$. Then, there exist positive constants κ and ξ such that, for all $t \geqslant 0$,

$$\|e(t)\| \leqslant \kappa \exp(-\xi t),$$

where $\kappa = \sqrt{\frac{\beta}{\alpha}}\|e(0)\|$, $\xi = \frac{\varepsilon}{2\beta}\alpha = \lambda_{min}(P)$, and $\beta = \lambda_{max}(P)$.

Proof. We use the following Lyapunov function candidate $V = e^TPe$. From (10), the time derivative of V is

$$\begin{aligned} \dot{V} &= e^T\left[(A - K_1C)^TP + P(A - K_1C)\right]e + 2e^TP\phi(e) + 2e^TP\rho(e) - 2\sum_{i=2}^{m}(Ce)^{2i-2}e^TM_ie \\ &= e^T\left[(A - K_1C)^TP + P(A - K_1C)\right]e + 2e^TP\phi(e) + 2e^TP\rho(e) - \sum_{i=2}^{m}(Ce)^{2i-2}e^T(M_i + M_i^T)e \end{aligned}$$

and, in view of (14) and (15),

$$\dot{V} \leqslant -\varepsilon e^Te - 2\epsilon_1 e^T\phi(e) + 2\epsilon_2 e^T\rho(e).$$

By properties (12) and (13) we have

$$\dot{V} \leqslant -\varepsilon\|e\|^2. \tag{16}$$

We write the Lyapunov function as $V = \|e\|_P^2$, then by Rayleigh–Ritz inequality we have that

$$\alpha\|e\|^2 \leqslant \|e\|_P^2 \leqslant \beta\|e\|^2, \tag{17}$$

where $\alpha := \lambda_{min}(P)$, and $\beta := \lambda_{max}(P) \in \mathbb{R}^+$ (because P is positive definite).

By using (17) we obtain the following upper bound of (16)

$$\dot{V} \leqslant -\frac{\varepsilon}{\beta}\|e\|_P^2. \tag{18}$$

Taking the time derivative of $V = \|e\|_P^2$ and replacing in inequality (18), we obtain

$$\frac{d}{dt}\|e\|_P \leqslant -\frac{\varepsilon}{2\beta}\|e\|_P.$$

Finally, the result follows with

$$\|e(t)\| \leqslant \kappa \exp(-\xi t), \tag{19}$$

where $\kappa = \sqrt{\frac{\beta}{\alpha}}\|e(0)\|$, and $\xi = \frac{\varepsilon}{2\beta}$. □

Corollary 1. *Let us consider $\psi(\cdot) \equiv 0$. Then system (9) is an exponential observer of system (6) if there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_2 > 0$ satisfying*

$$\begin{bmatrix} (A - K_1C)^T P + P(A - K_1C) + \varepsilon I & P - \epsilon_2 I \\ P - \epsilon_2 I & 0 \end{bmatrix} \leqslant 0 \tag{20}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geqslant 0, \quad i = 2, \ldots, m, \tag{21}$$

With κ and ξ defined as in Proposition 1.

Proof. The result is proven as in Proposition 1. □

Corollary 2. *Let us consider $\varphi(\cdot) \equiv 0$. Then system (9) is an exponential observer of system (6) if there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_1 > 0$ satisfying the LMI*

$$\begin{bmatrix} (A - K_1C)^T P + P(A - K_1C) + \varepsilon I & P + \epsilon_1 I \\ P + \epsilon_1 I & 0 \end{bmatrix} \leqslant 0 \tag{22}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geqslant 0, \quad i = 2, \ldots, m. \tag{23}$$

With κ and ξ defined as in Proposition 1.

Proof. It follows directly from the procedure in Proposition 1. □

4. Numerical results

In this section we consider the synchronization of a Chua's system considered as a chaotic Liouvillian oscillator. Chua's circuit [29], shown in Fig. 1, is a simple oscillator circuit which exhibits a variety of bifurcations and chaos. The circuit contains three linear energy-storage elements (an inductor, and two capacitors), a linear resistor, and a single nonlinear resistor N_R.

The state equations for the Chua's circuit are as follows:

$$\begin{aligned} C_1\frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}), \\ C_2\frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_L, \\ L\frac{di_L}{dt} &= -v_{C_2}, \end{aligned} \tag{24}$$

where $G = \frac{1}{R}$ and $g(\cdot)$ is a nonincreasing function defined by:

$$g(v_R) = m_0 v_R + \frac{1}{2}(m_1 - m_0)\left(\left|v_R + B_p\right| - \left|v_R - B_p\right|\right). \tag{25}$$

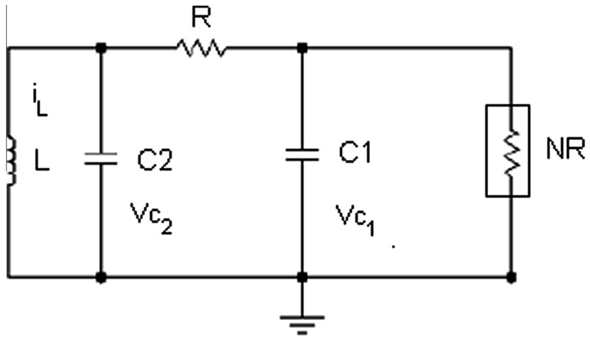

Fig. 1. Chua's circuit, with parameter values $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS and $B_p = 1.08$ V.

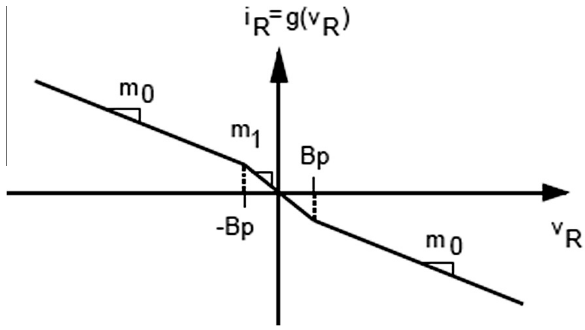

Fig. 2. Three-segment piecewise-linear v-i characteristic of the linear resistor in Chua's circuit. The outer regions have slopes m_0, the inner region has slope m_1. There are two breakpoints at $\pm B_p$.

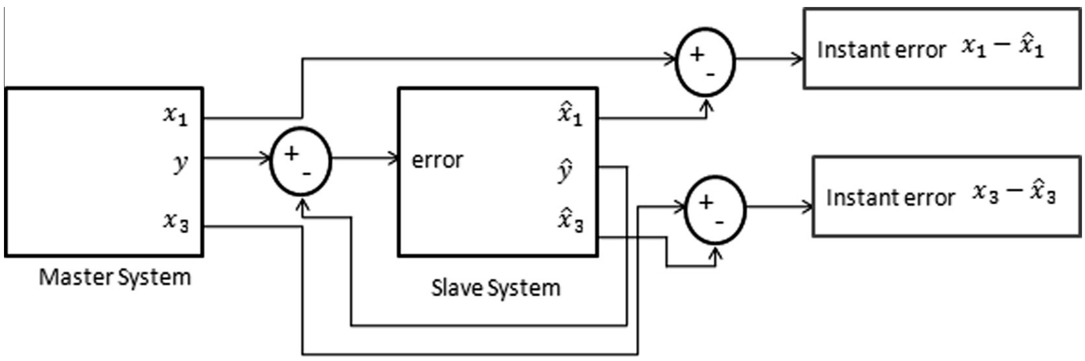

Fig. 3. Synchronization diagram.

This relation is shown graphically in Fig. 2, the slopes in the inner and outer regions are m_0 and m_1 respectively, with $m_1 < m_0 < 0$, $\pm B_p$ denote the breakpoints. The nonlinear resistor N_R is termed voltage-controlled because the current in the element is a function of the voltage across its terminals.

In the first reported study of this circuit, Matsumoto [29] showed by computer simulation that the system possesses a strange attractor called the Double Scroll. Experimental confirmation of the presence of this attractor was made shortly afterwards in [35]. Since then, the system has been studied extensively; a variety of bifurcation phenomena and chaotic attractors in the circuit have been discovered experimentally and confirmed mathematically [36].

In what follows we consider the measured output $y = v_{c_2}$. From equations of (24) we obtain:

$$\begin{aligned} v_{c_1} &= \frac{C_2}{G}\dot{y} + y + \frac{1}{LG}\int y dt, \\ v_{c_2} &= y, \\ i_L &= -\frac{1}{L}\int y dt, \end{aligned} \tag{26}$$

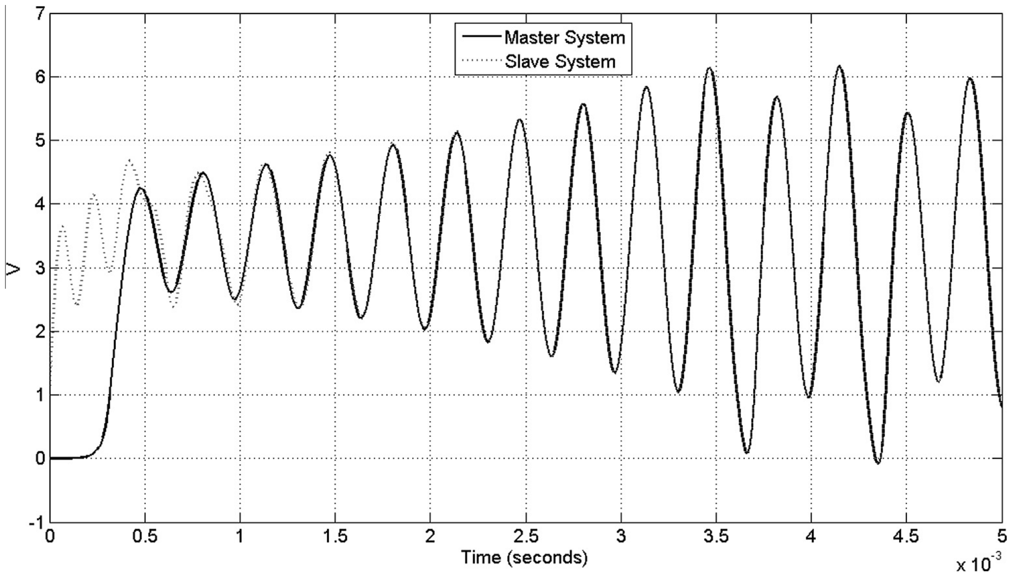

Fig. 4. Synchronization of v_{c_1}.

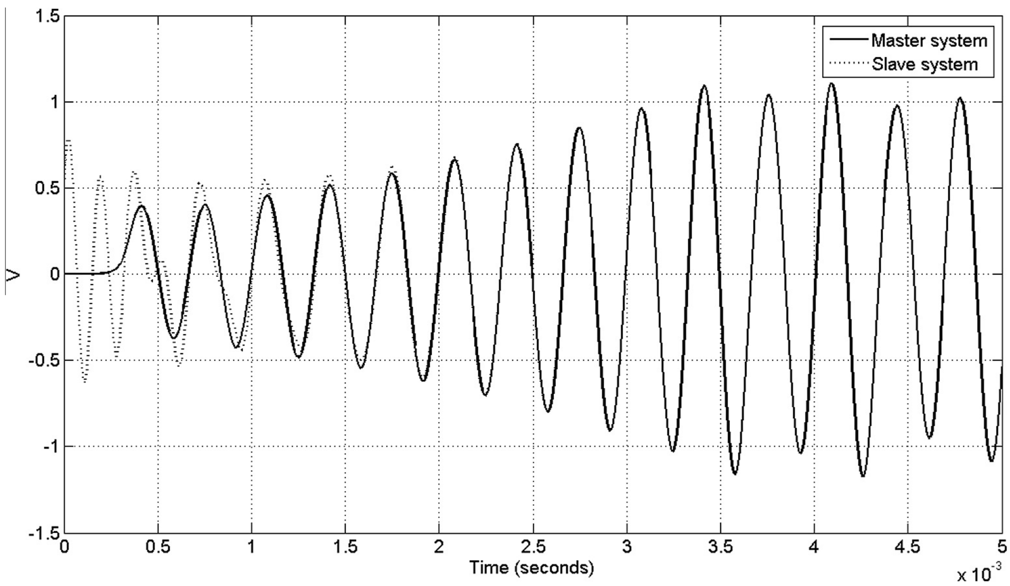

Fig. 5. Synchronization of v_{c_2}.

From (26), the Chua's system (24) is Liouvillian. This implies that unknown variables v_{C_1} and i_L can be reconstructed with the selected output $y = v_{C_2}$.

Chua's system (24) is of the form (6) with $\zeta(u) = 0,\ \varphi(\cdot) = 0$,

$$A = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix}, \quad \psi(x) = \begin{bmatrix} -\frac{g(x_1)}{C_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 1 \quad 0], \quad x = \begin{bmatrix} v_{C_1} & v_{C_2} & i_L \end{bmatrix}^T.$$

Since $g(x_1)$ is nonincreasing and C_1 is a positive constant, then $\psi_1(x) = \psi_1(x_1) = -g(x_1)/C_1$ is nondecreasing as in (7). Indeed, $\psi_2(x) = \psi_3(x) = 0$ also satisfy property (7), and Chua's system (24) is Liouvillian, so that, we proceed with the observer design.

Taking into account that $\varphi(\cdot) = 0$, we will use conditions in Corollary 2 to obtain the observer gains. Using LMI software, observer gains are computed to drive the estimation error to zero.

Applying (9), we have the observer for Chua's system (24),

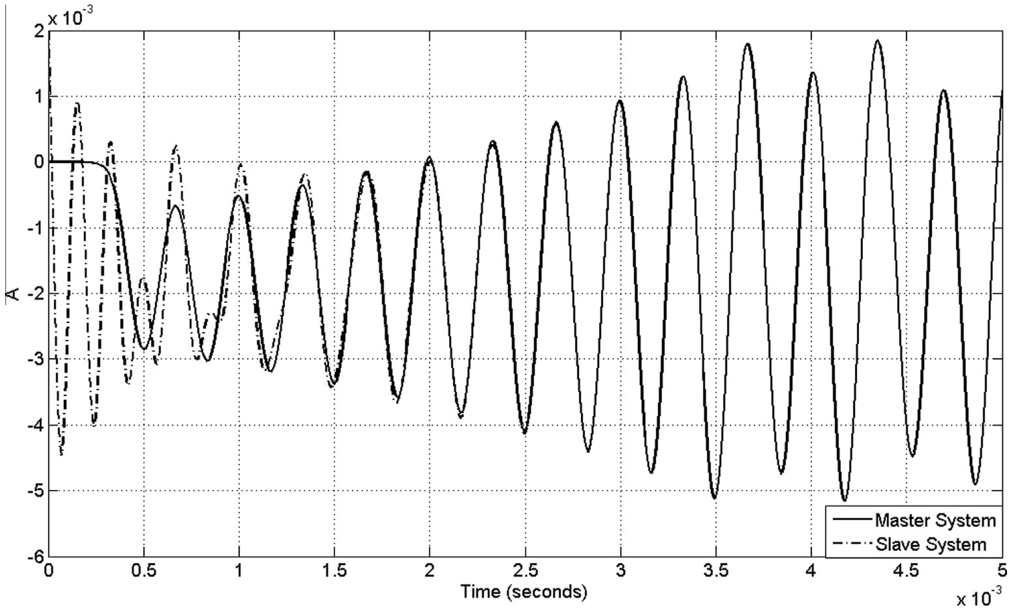

Fig. 6. Synchronization of i_L.

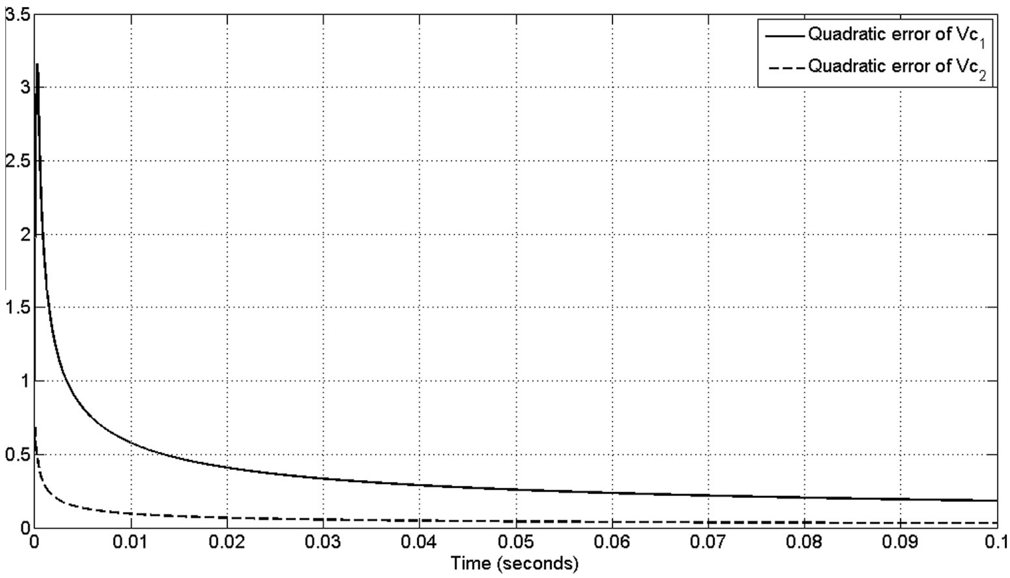

Fig. 7. Quadratic synchronization errors of v_{c_1} and v_{c_2}.

$$\dot{\hat{x}} = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -\frac{g(\hat{x}_1)}{C_1} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} ([0 \quad 1 \quad 0]e)^{2i-1}.$$

Hence, the state observer is rewritten as,

$$\begin{aligned}
\dot{\hat{x}}_1 &= -\frac{G}{C_1}\hat{x}_1 + \frac{G}{C_1}\hat{x}_2 - \frac{g(\hat{x}_1)}{C_1} + k_{1,1}e_{1,2} + k_{1,2}(e_{1,2})^3 + \cdots + k_{1,m}(e_{1,2})^{2m-1}, \\
\dot{\hat{x}}_2 &= \frac{G}{C_2}\hat{x}_1 - \frac{G}{C_2}\hat{x}_2 + \frac{1}{C_2}\hat{x}_3 + k_{2,1}e_{1,2} + k_{2,2}(e_{1,2})^3 + \cdots + k_{2,m}(e_{1,2})^{2m-1}, \\
\dot{\hat{x}}_3 &= -\frac{1}{L}\hat{x}_3 + k_{3,1}e_{1,2} + k_{3,2}(e_{1,2})^3 + \cdots + k_{3,m}(e_{1,2})^{2m-1},
\end{aligned} \tag{27}$$

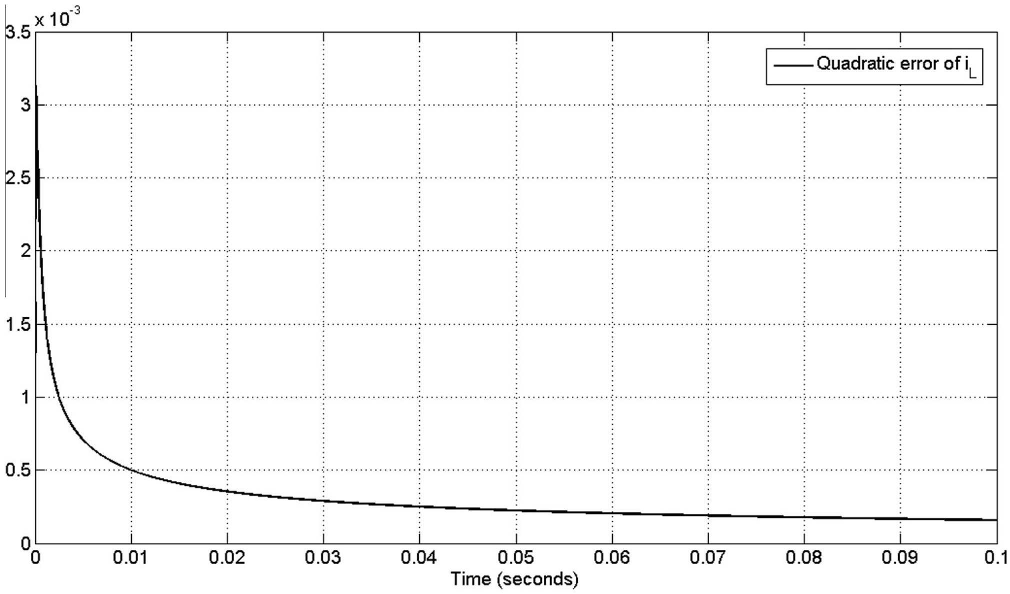

Fig. 8. Quadratic synchronization error of i_L.

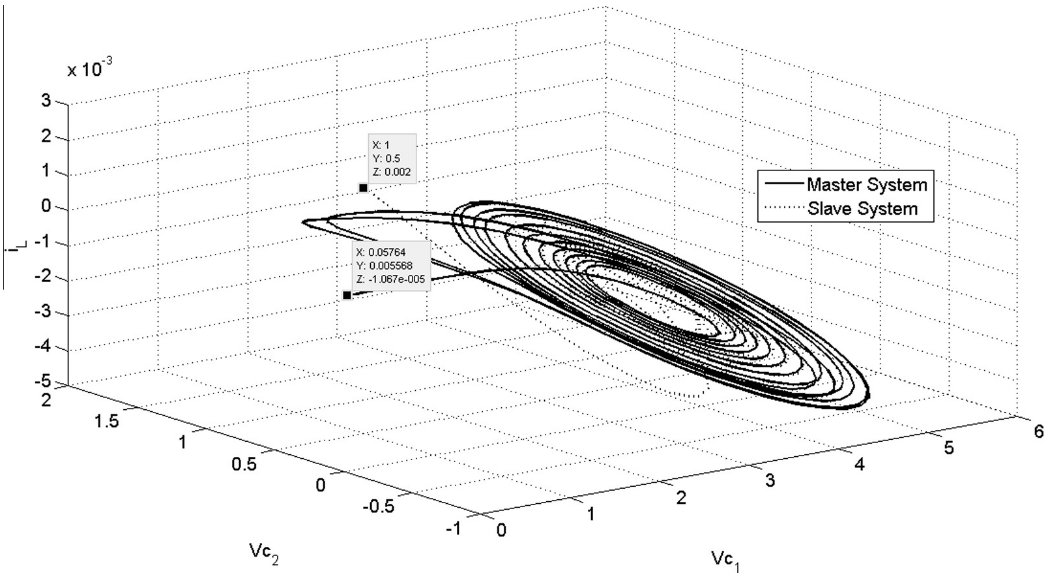

Fig. 9. Phase Diagram.

Fig. 3 shows the general diagram of the synchronization of Chua's circuit (24) and the exponential observer (27) in master–slave configuration.

Numerical simulations for the synchronization of Chua's system are carried out in order to show the performance of the exponential observer. The parameter values considered in the numerical simulations correspond to chaotic behavior [33] and these are: $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS and $B_p = 1.08$ V. The Matlab-Simulink® program uses the Dormand–Prince integration algorithm, with the integration step set to 1×10^{-5}.

We fix $m = 2$ in the observer (27). The LMI (22) is feasible with $\varepsilon = 0.001$ and $\epsilon_1 = 0.001$, a solution is

$$P = \begin{bmatrix} 0.0008 & -0.0006 & 0.1021 \\ -0.0006 & 0.0005 & -0.0805 \\ 0.1021 & -0.0805 & 15.0959 \end{bmatrix}, \quad K_1 = \begin{bmatrix} k_{1,1} \\ k_{2,1} \\ k_{3,1} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 45 \end{bmatrix}$$

and K_2 is chosen such that (23) is satisfied, then we obtain

$$K_2 = \begin{bmatrix} k_{1,2} \\ k_{2,2} \\ k_{3,2} \end{bmatrix} = \begin{bmatrix} 7.1573 \\ 14.1040 \\ 0.0268 \end{bmatrix}.$$

Figs. 4–6 show the obtained results for the initial conditions $x_1 = x_2 = x_3 = 0,\ \hat{x}_1 = 1,\ \hat{x}_2 = 0.5$ and $\hat{x}_3 = 0.002$, the synchronization results achieved with the polynomial observer are good.

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as [37]

$$J(t) = \frac{1}{t + 0.001} \int_0^t |e(t)|^2_{Q_0}, \quad Q_0 = I,$$

Figs. 7 and 8 illustrate the performance index, which has a tendency to decrease. Finally, Fig. 9 presents the synchronization in a phase diagram, where clearly is observed the chaotic behavior of the Chua's circuit.

5. Concluding remarks

The synchronization problem of chaotic Liouvillian systems has been treated by using differential and algebraic techniques. We proposed a polynomial observer, and by means of properties of nondecreasing and nonincreasing functions, linear matrix inequalities and with the help of the Lyapunov method we proved that the estimation error exponentially converges to zero. This observer has been used as a slave system whose states are exponentially synchronized with the chaotic system (Chua's circuit). A reduced set of measurable state variables were needed to achieve the synchronization with this approach. The effectiveness of the suggested methodology was shown by means of numerical simulations.

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Fault diagnosis viewed as a left invertibility problem

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ABSTRACT

This work deals with the fault diagnosis problem, some new properties are found using the *left invertibility condition* through the concept of differential output rank. Two schemes of nonlinear observers are used to estimate the fault signals for comparison purposes, one of these is a proportional reduced order observer and the other is a sliding mode observer. The methodology is tested in a real time implementation of a three-tank system.

1. Introduction

A fault can be considered as a process degradation or degradation of the equipment performance caused by the change in the physical characteristic of the process, the input process or the external conditions. Industrial control systems have to deal with faults, therefore, fault diagnosis is a very important subject in control theory. System diagnosis helps us to detect and estimate faults in a process. In other words, the task of diagnosis is, from measurements of outputs and inputs, to reconstruct the fault vector.

The fault detection and isolation problem have been studied for more than three decades, many papers dealing with this problem can be found, see for instance the surveys [1–4] and the books [5–7]. For the case of nonlinear systems a variety of approaches have been proposed [8–13], such as those based upon differential geometric methods [14,15], and on the other hand, alternative approaches based on an algebraic and differential framework can be found in [16–20].

Currently, the diagnosis problem is playing an important role in modern industrial processes. This has led control theory into a wide variety of model-based approaches which rely on descriptions via differential and/or difference equations, contrary to other standpoints developed mainly among computer scientist (see [18,19] and references therein). The primary objectives of fault diagnosis are fault detectability and isolability, i.e., the possible location and determination of the faults present in a system and the time of their occurrences. The tasks of fault detection and isolation are to be accomplished by measuring only the input and the output variables.

This paper focuses on the diagnosis of nonlinear systems and the goal is to determine malfunctions in the dynamics. In this communication, the outputs are mainly signals obtained from the sensors. Their number is important to know whether a system is diagnosable or not.

In this paper, the diagnosis problem is tackled as a left invertibility problem throughout the concept of differential output rank ρ. Two schemes of observers are proposed in order to estimate the fault signals, one of them is a reduced-order observer based on a free-model approach and another is a sliding-mode observer based on a Generalized Observability Canonical Form (GOCF) [18]. Both schemes are proved to possess asymptotic convergence properties.

The class of systems for which this methodology can be applied contains systems that depend on the inputs and their time derivatives in a polynomial form. The type of faults considered in this work is additive and bounded, however, the algebraic approach can also be used to deal with multiplicative faults.

These proposals are applied in this paper to a three-tank system [21]. The Amira DTS200 three-tank system [22] has been widely considered for the experimental fault diagnosis study, see for instance [15,17,23], even recently, one work based on the geometric approach has been reported [15]. We can also mention one previous work with the three-tank system using the differential algebraic

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approach [17]; in that work the authors only report a numerical simulation study and not a real-time experiment, also they only solve the simplest case in which three measured outputs are available to estimate two faults, that is to say, they do not analyze the minimal number of measurements to attack the diagnosis problem as we do in the present work. The intention of choosing the three-tank system example is to clarify the proposed methodology and to highlight the simplicity and flexibility of the present approach. The three-tank system is known as a system with parameter uncertainties, so this work also deals with these uncertainties by means of algebraic parameter estimation, considering that there is no simultaneous presence of uncertainty and faults, in the same way as it is considered in [17].

This paper is organized as follows. In Section 2, some definitions of differential algebra are given. In Section 3, we discuss the left invertibility condition and we present some examples. In Sections 4 and 5 we give a brief description of the proposed observers. In Section 6 the three-tank system is analyzed. Finally, in Section 7 we illustrate this methodology with some experimental results.

2. Some definitions

Some basic definitions are introduced. Further details can be found in [17,18] and references therein.

Definition 1. Let $\mathcal{L}$ and $\mathcal{K}$ be differential fields. A differential field extension $\mathcal{L}/\mathcal{K}$ is given by $\mathcal{K}$ and $\mathcal{L}$ such that: (1) $\mathcal{K}$ is a subfield of $\mathcal{L}$ and (2) the derivation of $\mathcal{K}$ is the restriction to $\mathcal{K}$ of the derivation of $\mathcal{L}$.

Example 1. $\mathbb{R}\langle e^t\rangle/\mathbb{R}$ is a differential field extension, where $\mathbb{R}\subseteq\mathbb{R}\langle e^t\rangle$. e^t being a solution of $\dot{x}-x=0$.

Definition 2. Let $\xi=(\xi_1,\xi_2,\dots,\xi_n)$ be a set of elements of $\mathcal{L}$. If it satisfies an algebraic differential equation $P(\xi,\dot{\xi},\ddot{\xi}\dots)=0$ with coefficients in $\mathcal{K}$ it is called differentially $\mathcal{K}-$algebraically dependent, otherwise ξ is called differentially $\mathcal{K}-$algebraically independent.

Definition 3. Any set of elements of $\mathcal{L}$ which is differentially $\mathcal{K}-$algebraically independent and maximal with respect to inclusion forms is a differential transcendence basis of $\mathcal{L}/\mathcal{K}$. Two such bases have the same cardinality. This is called the *differential transcendence degree* of $\mathcal{L}/\mathcal{K}$ and denoted by *diff tr* $d^\circ\mathcal{L}/\mathcal{K}$.

Definition 4. Let $\mathcal{G}$, $\mathcal{K}\langle u\rangle$ be differential fields. A nominal dynamic consists of a finitely generated differential algebraic extension $\mathcal{G}/\mathcal{K}\langle u\rangle$, $(\mathcal{G}=\mathcal{K}\langle u,\xi\rangle,\xi\in\mathcal{G})$. Any element of $\mathcal{G}$ satisfies an algebraic differential equation with coefficients over $\mathcal{K}$ in the components of u and their time derivatives.

Example 2. Consider the following differential equation:

$$\dot{u}^2y+4\ddot{u}=0$$

In this case, y is algebraic over $\mathcal{K}\langle u\rangle$, therefore, it can be seen as a dynamic of the form $\mathcal{K}\langle u,y\rangle/\mathcal{K}\langle u\rangle$ where $\mathcal{K}=\mathbb{R}$ and $y\in\mathcal{K}\langle u,y\rangle$.

Definition 5. Any unknown variable x in a dynamic is said to be *algebraically observable* with respect to $\mathcal{K}\langle u,y\rangle$ if x satisfies a differential algebraic equation with coefficients over $\mathcal{K}$ in the components of u, y and a finite number of their derivatives. Any dynamic with output y is said to be algebraically observable if, and only if, any state variable has this property.

Example 3. Let us consider the following system:

$$\begin{cases}\dot{x}_1=3x_1x_2^2+u_1\\ \dot{x}_2=x_1+x_2^3u_2\\ y=x_2,\end{cases}\tag{1}$$

since x_1 and x_2 satisfy the polynomials $x_1+y^3u_2-\dot{y}=0$ and $x_2-y=0$, respectively, then x_1,x_2 are algebraically observable over $\mathbb{R}\langle u,y\rangle$ and by applying Definition 5, system (1) is algebraically observable.

Definition 6. A fault is not a permitted deviation of at least one characteristic property or parameter of any process in relation to the development of the same parameter under normal conditions. Faults are defined as transcendent elements over $\mathcal{K}\langle u\rangle$, therefore, a system with the presence of faults is a differential transcendental extension, denoted as $\mathcal{K}\langle u,f,y\rangle/\mathcal{K}\langle u\rangle$, where f is a vector that includes the faults and their time derivatives.

Definition 7. Let $\mathcal{G}$, $\mathcal{K}\langle u\rangle$ be differential fields. A fault dynamic consists of a finitely generated differential algebraic extension $\mathcal{G}/\mathcal{K}\langle u,f\rangle$, $\mathcal{G}=\mathcal{K}\langle u,f,\xi\rangle$, $\xi\in\mathcal{G}$. Any element of $\mathcal{G}$ satisfies an algebraic differential equation with coefficients over $\mathcal{K}$ in the components of u,f and their time derivatives.

Definition 8 (*Algebraic observability condition*). A fault $f\in\mathcal{G}$ is said to be diagnosable if it is algebraically observable over $R\langle u,y\rangle$, i.e., if it is possible to estimate the fault from the available measurements of the system.

Let us consider the class of nonlinear systems with faults described by the following equation:

$$\begin{cases}\dot{x}(t)=A(x,\overline{u})\\ y(t)=h(x,\overline{u}),\end{cases}\tag{2}$$

where $x=(x_1,\dots,x_n)^T\in\mathbb{R}^n$ is a state vector, $u=(u_1,\dots,u_m)\in\mathbb{R}^m$ is a known input vector, $f=(f_1,\dots,f_\mu)\in\mathbb{R}^\mu$ is an unknown input vector, $\overline{u}=(u,f)\in\mathbb{R}^{m+\mu}$, $y(t)\in\mathbb{R}^p$ is the output vector. A and h are assumed to be analytical vector functions.

Example 4. Let us consider the nonlinear system with one fault (f_1) on the actuator and one fault (f_2) on the sensor of output y_1:

$$\begin{cases}\dot{x}_1=x_1x_2+f_1+u\\ \dot{x}_2=x_1\\ y_1=x_1+f_2\\ y_2=x_2.\end{cases}\tag{3}$$

Since f_1,f_2 satisfy the differential algebraic equations

$$\begin{aligned}&f_1-\ddot{y}_2+y_2\dot{y}_2+u=0\\ &f_2-y_1+\dot{y}_2=0\end{aligned}\tag{4}$$

the system (3) is diagnosable and the faults can be reconstructed from the knowledge of u, y and their time derivatives.

Remark 1. The diagnosability condition is independent of the observability of a system.

Example 5. Let us consider the system

$$\begin{cases}\dot{x}_1=x_1x_2+f+u\\ \dot{x}_2=x_1\\ \dot{x}_3=x_3f+u\\ y=x_2.\end{cases}\tag{5}$$

In this case f is diagnosable. However, x_3 is not algebraically observable.

3. On the left invertibility condition

We have some definitions concerning the differential output rank of a system.

Definition 9. The differential output rank ρ of a system is equal to the differential transcendence degree of the differential extension $\mathcal{K}\langle y\rangle$ over the differential field $\mathcal{K}$, i.e.,

$$\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K}.$$

Property 1 (*Kolchin [24]*). *Let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be the differential fields such that $\mathcal{K}\subset\mathcal{L}\subset\mathcal{M}$. Then*

$$diff\ tr\ d^o(\mathcal{M}/\mathcal{K}) = diff\ tr\ d^o(\mathcal{M}/\mathcal{L}) + diff\ tr\ d^o(\mathcal{L}/\mathcal{K}) \quad \square \tag{6}$$

Property 2. *The differential output rank ρ of a system is smaller than or equal to* $\min(m,p)$:

$$\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \le \min(m,p),$$

where m and p are the total number of inputs and outputs, respectively.

A proof of Property 2 can be given in the following manner: an input–output system, with input $u=(u_1,\dots,u_m)$ and output $y=(y_1,.,y_p)$, is defined by the next conditions:

- $(u_1,\dots,u_m)$ are differentially $\mathcal{K}-$algebraically independent, i.e.,
$$diff\ tr\ d^o \mathcal{K}\langle u\rangle/\mathcal{K} = m \tag{7}$$
- $(y_1,\dots,y_p)$ are differentially algebraic over $\mathcal{K}\langle u\rangle$, i.e., $\mathcal{K}\langle u,y\rangle/\mathcal{K}\langle u\rangle$ is differentially algebraic or
$$diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle u\rangle = 0 \tag{8}$$

Consider the field tower

$$\mathcal{K}\subset\mathcal{K}\langle u\rangle\subset\mathcal{K}\langle u,y\rangle \tag{9}$$

By Property 1,

$$diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K} = diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle u\rangle + diff\ tr\ d^o \mathcal{K}\langle u\rangle/\mathcal{K} \tag{10}$$

Replacing (7) and (8) into (10), we obtain

$$diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K} = m \tag{11}$$

Now, let us consider the field tower

$$\mathcal{K}\subset\mathcal{K}\langle y\rangle\subset\mathcal{K}\langle u,y\rangle \tag{12}$$

By using Property 1,

$$diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K} = diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle + diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \tag{13}$$

Substituting (11) into (13),

$$m = diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle + diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K}$$

Since the differential transcendence degree is not negative, we have that

$$\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \le m$$

In a similar manner, $y=(y_1,.,y_p)$ and $\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \le p$.

Finally,

$$\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \le \min(m,p). \quad \square$$

The differential output rank ρ is also the maximum number of outputs that are related by a differential polynomial equation with coefficients over $\mathcal{K}$ (independent of x and u).

A practical way, for certain simple cases, to determine the differential output rank is by taking into account all possible differential polynomials of the form

$$h_r(y_1,\dots,y_p) = 0 \tag{14}$$

and if it is possible to find r independent relations of the form (14), then the differential output rank is given by $\rho = p-r$, that is to say, there exist only $p-r$ independent outputs.

Example 6. Consider the following system with one input and two outputs:

$$\left.\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 \quad \text{with} \quad y_1 = x_1 u \\ \dot{x}_3 &= x_1 u \qquad\qquad y_2 = x_3 \end{aligned}\right\} \tag{15}$$

Replacing y_1 and y_2,

$$\left.\begin{aligned} \frac{\dot{y}_1}{u} - \frac{\dot{u}}{u^2}y_1 &= u \\ \dot{x}_2 &= \frac{y_1}{u} \\ \dot{y}_2 &= y_1 \end{aligned}\right\} \tag{16}$$

Only one equation of (16) can be written in the form of (14). Then we have the relation

$$y_1 - \dot{y}_2 = 0$$

where only one output is differentially independent and then $\rho = 1$.

Example 7. Consider the following system with three inputs and two outputs:

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \quad \text{with} \quad y_1 = x_1 \\ \dot{x}_3 &= u_3 \qquad\qquad y_2 = x_2 x_3 \end{aligned}\right\} \tag{17}$$

Substituting y_1 and y_2,

$$\left.\begin{aligned} \dot{y}_1 &= u_1 \\ \frac{\dot{y}_2}{x_3} - \frac{u_3}{x_3^2}y_2 &= u_2 \\ \frac{\dot{y}_2}{x_2} - \frac{u_2}{x_2^2}y_2 &= u_3 \end{aligned}\right\} \tag{18}$$

By (18), we have that in this case there exists no any differential equation in which the outputs appear (independent of x and u). We conclude that the two outputs are differentially dependent and therefore $\rho = 2$.

Definition 10. A system is left-invertible if, and only if, the differential output rank is equal to the total number of inputs, i.e., $\rho = m$.

Example 8. Let us consider once again the system (15). In this case we have that $m=1$, $p=2$, and $\rho=1$, then the system is left invertible ($\rho = m$), this implies that it will be possible to recover the input by means of the available outputs. In fact, the input can be found since the following differential equation is satisfied:

$$u\dot{y}_1 = u^3 + \dot{y}_2\dot{u}$$

which is a Bernoulli equation and by means of a change of variable $z = u^{-2}$ it is transformed in the linear differential equation

$$\dot{z} + 2\frac{\dot{y}_1}{\dot{y}_2}z = \frac{2}{\dot{y}_2},$$

where $\dot{y}_1$ and $\dot{y}_2$ are known functions of t.

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Example 9. Now, let us consider the system with two inputs and one output:

$$\left.\begin{array}{l}\dot{x}_1 = u_1\\ \dot{x}_2 = u_2 \quad \text{with} \quad y_1 = x_1 x_2\end{array}\right\} \tag{19}$$

we have that $\rho = p = 1$, $m=2$, then the system (19) is not left invertible.

Proposition 1 (Fliess [25]). *Let us consider a class of systems given by (2). A system is said to be left invertible if and only if*

$$\rho = diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} = diff\ tr\ d^o \mathcal{K}\langle u,f\rangle/\mathcal{K}. \qquad \square$$

Property 1 is the main tool used to prove the following theorem that looks quite natural. The theorem shows the relationship between the diagnosability and the left invertibility condition.

Theorem 1. *If system (2) is left invertible, then the fault vector f can be obtained by means of the output vector.*

Proof. Let us consider the following field towers:

$$\mathcal{K} \subset \mathcal{K}\langle u\rangle \subset \mathcal{K}\langle u,f\rangle \subset \mathcal{K}\langle u,y,f\rangle, \tag{20}$$

$$\mathcal{K} \subset \mathcal{K}\langle y\rangle \subset \mathcal{K}\langle u,y\rangle \subset \mathcal{K}\langle u,y,f\rangle, \tag{21}$$

From (20) and Property 1, we have

$$\begin{aligned} diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K} &= diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,f\rangle \\ &\quad + diff\ tr\ d^o \mathcal{K}\langle u,f\rangle/\mathcal{K}\langle u\rangle + diff\ tr\ d^o \mathcal{K}\langle u\rangle/\mathcal{K} \\ &= 0 + \mu + m \\ &= m + \mu \end{aligned} \tag{22}$$

From Proposition 1, $diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} = m + \mu$. By using this fact in (21) we obtain

$$\begin{aligned} diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K} &= diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,y\rangle \\ &\quad + diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle + diff\ tr\ d^o \mathcal{K}\langle y\rangle/\mathcal{K} \\ &= diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,y\rangle \\ &\quad + diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle + m + \mu \end{aligned} \tag{23}$$

From (22) and (23) we have

$$diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,y\rangle + diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle + m + \mu = m + \mu.$$

This implies that

$$diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,y\rangle = -diff\ tr\ d^o \mathcal{K}\langle u,y\rangle/\mathcal{K}\langle y\rangle \tag{24}$$

Since the transcendence degree is always positive, we have the following:

$$diff\ tr\ d^o \mathcal{K}\langle u,y,f\rangle/\mathcal{K}\langle u,y\rangle = 0 \tag{25}$$

This means that f is differentially algebraic over $\mathcal{K}\langle u,y\rangle$. Thus, the diagnosability condition is satisfied and the theorem is proven. □

3.1. Illustrative examples

In this section we present some academic examples in which the left invertibility condition is applied.

Example 10. Let us consider the system

$$\left.\begin{array}{ll}\dot{x}_1 = x_2 + f_1 + f_2 \quad \text{with} & y_1 = x_1\\ \dot{x}_2 = x_1 + f_1 & y_2 = x_2\end{array}\right\} \tag{26}$$

The differential output rank of (26) is 2 since there exists no relation h_r such that

$h_r(y_1, y_2) = 0.$

In fact, because ρ is equal to the number of faults, we conclude that the system (26) is left invertible, in other words, f_1 and f_2 are diagnosable.

To verify the result, we substitute y_1 and y_2 in (26), then

$$\begin{array}{l}\dot{y}_1 = y_2 + f_1 + f_2\\ \dot{y}_2 = y_1 + f_1\end{array} \tag{27}$$

From (27),

$$\begin{array}{l}f_1 = \dot{y}_2 - y_1\\ f_2 = \dot{y}_1 - \dot{y}_2 + y_1 - y_2\end{array}$$

Example 11. Consider the system that describes the growth of methanol in a bioreactor, where f represents the presence of an unexpected catalyst in the substrate concentration that produces wrong measures of methanol:

$$\left.\begin{array}{ll}\dot{x}_1 = f\mu(x_2)x_1 + ux_1 \quad & \text{with} \quad y = x_2\\ \dot{x}_2 = -\sigma(x_2)x_1 + u[B - x_2] & \end{array}\right\} \tag{28}$$

where

x_1	represents the density of the methylomonas
x_2	represents the methanol concentration
u	constant of dilution rate
$\mu(\cdot)$	specific growth rate of substrate, $\mu(\cdot) > 0$
$\sigma(\cdot)$	consumption rate of substrate, $\sigma(\cdot) > 0$

In this case the differential output rank is equal to 1 since there does not exist any relation h_r (independent of x_1, x_2, u and f) such that

$h_r(y) = 0.$

The system (28) is left invertible because the differential output rank is equal to the number of faults and inputs. This implies that system (28) is diagnosable, where the fault can be expressed as follows:

$$f = \frac{\frac{[\dot{u}(B-y) - u\dot{y} - \ddot{y}]\sigma(y) - \dot{\sigma}(y)[u(B-y) - \dot{y}]}{\sigma^2(y)} - u\frac{u(B-y) - \dot{y}}{\sigma(y)}}{\mu(y)\frac{u(B-y) - \dot{y}}{\sigma(y)}} \tag{29}$$

From (29), we have that the fault is diagnosable if $[u(B-y) - \dot{y}] \neq 0$.

Moreover, the unknown state x_1 is algebraically observable since it satisfies an equation in terms of y, i.e.,

$$x_1 = \frac{u(B-y) - \dot{y}}{\sigma(y)}.$$

4. Reduced-order observer

Let us consider system (2). The fault vector f is unknown and it can be assimilated as a state with uncertain dynamics. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector. The new extended system is given by

$$\dot{x}(t) = A(x, \overline{u})$$
$$\dot{f} = \Omega(x, \overline{u})$$
$$y(t) = h(x, u) \tag{30}$$

where $\Omega(x, \overline{u}) = (\Omega_1(x, \overline{u}), \ldots, \Omega_\mu(x, \overline{u}))^T : \mathbb{R}^{n+m+\mu} \to \mathbb{R}^\mu$ is an uncertain function.

Note that a classic Luenberger observer cannot be constructed because the term $\Omega(x, \overline{u})$ is unknown. This problem is overcome by using a reduced order uncertainty observer in order to estimate the failure variable f. Next lemma describes the construction of a proportional reduced order observer for (30).

Lemma 1. *If the following hypotheses are satisfied*:

H1: $\Omega(x, \overline{u})$ *is bounded, i.e.*, $|\Omega_i(x, \overline{u})| \le N \in \mathbb{R}^+ \ \forall 1 \le i \le \mu$.

H2: $f(t)$ *is algebraically observable over* $\mathbb{R}\langle u, y\rangle$.

Then the system

$$\dot{\hat{f}}_i = k_i(f_i - \hat{f}_i), \quad 1 \le i \le \mu \tag{31}$$

is a reduced order observer for system (30), *where* $\hat{f}_i$ *denotes the estimate of fault* f_i *and* $k_i \in \mathbb{R}^+ \ \forall i = 1, \ldots, \mu$ *are positive real coefficients that determine the desired convergence rate of the observer.*

Proof. Let us define the estimation error $\varepsilon_i(t)$ as

$$\varepsilon_i(t) = f_i - \hat{f}_i$$

The dynamics of the error $\varepsilon_i(t)$ can be expressed as

$$\dot{\varepsilon}_i(t) + k_i\varepsilon(t) = \Omega_i(x, \overline{u}) \tag{32}$$

The solution of Eq. (32) is given by

$$\varepsilon_i(t) = \exp(-k_i t)\left[\varepsilon_{i_0} + \int_0^t \exp(k_i\tau)\Omega_i(\tau)\, d\tau\right] \tag{33}$$

where ε_{i_0} is the initial condition. Then Eq. (33) yields

$$|\varepsilon_i(t)| = \left|\exp(-k_i t)\left[\varepsilon_{i_0} + \int_0^t \exp(k_i\tau)\Omega_i(\tau)\, d\tau\right]\right| \tag{34}$$

then, by applying the triangle and Schwarz inequalities, the following is obtained:

$$|\varepsilon_i(t)| \le \exp(-k_i t)|\varepsilon_{i_0}| + \exp(-k_i t)\left|\int_0^t \exp(k_i\tau)\Omega_i(\tau)\, d\tau\right|$$

From H1,

$$0 \le |\varepsilon_i(t)| \le \exp(-k_i t)|\varepsilon_{i_0}| + \left|N\int_0^t \exp[-k_i(t-\tau)]\, d\tau\right|$$

by solving the integral we have

$$0 \le |\varepsilon_i(t)| \le \exp(-k_i t)|\varepsilon_{i_0}| + \left|\frac{N}{k_i}[1-\exp(-k_i t)]\right|$$

When $t \to \infty$,

$$0 \le \limsup_{t\to\infty}|\varepsilon_i(t)| \le \lim\sup_{t\to\infty}\exp(-k_i t)|\varepsilon_{i_0}| + \limsup_{t\to\infty}\left|\frac{N}{k_i}[1-\exp(-k_i t)]\right|$$

simplifying,

$$0 \le \limsup_{t\to\infty}|\varepsilon_i(t)| \le \frac{N}{k_i}$$

and the proof is completed. □

Remark 2. Sometimes the output time derivatives (which are unknown) appear in the algebraic equation of the fault, then it is necessary to use an auxiliary variable to avoid using them as is described in the next lemma.

Lemma 2. *If a fault signal* f_i, $i \in \{1, \ldots, \mu\}$, *of system* (2) *is algebraically observable and can be written in the following form*:

$$f_i = a_i\dot{y} + b_i(u, y) \tag{35}$$

where $a_i = [a_{i1} \ldots, a_{im}] \in \mathbb{R}^m$ *is a constant vector and* $b_i(u, y)$ *is a bounded function, then there exists a function* $\gamma_i \in C^1$, *such that the reduced order observer* (31) *can be written as the following asymptotically stable system*:

$$\dot{\gamma}_i = -k_i\gamma_i + k_i b_i(u, y) - k_i^2 a_i y, \quad \gamma_i(0) = \gamma_{i0} \in \mathbb{R}$$
$$\hat{f}_i = \gamma_i + k_i a_i y \tag{36}$$

Proof. From (31) and (35) we obtain

$$\dot{\hat{f}}_i = k_i a_i\dot{y} + k_i b_i(u, y) - k_i\hat{f}_i. \tag{37}$$

Let us define

$$\gamma_i \triangleq \hat{f}_i - k_i a_i y, \tag{38}$$

we get

$$\dot{\gamma}_i \triangleq \dot{\hat{f}}_i - k_i a_i\dot{y}. \tag{39}$$

From (37) and (39) we have

$$\dot{\gamma}_i = -k_i\gamma_i + k_i b_i(u, y) - k_i^2 a_i y, \tag{40}$$

where $\gamma_i \in C^1$. □

5. Sliding-mode observer

Consider the nonlinear system with faults given by (2), assuming that the fault vector f is algebraically observable over $\mathbb{R}\langle u, y\rangle$ and therefore it satisfies a differential algebraic polynomial

$$\overline{\psi}(f, y, \dot{y}, \ddot{y} \ldots, \overset{(r)}{y}, u, \dot{u} \ldots) = 0 \tag{41}$$

where r is the maximum order of the output time derivatives.

Introducing the following change of coordinates:

$$\eta_1 = y, \quad \eta_2 = \dot{y}, \ldots, \eta_r = \overset{(r-1)}{y} \tag{42}$$

we obtain the following representation of (41) which is the so-called Generalized Observability Canonical Form [18]:

$$\dot{\eta}_1 = \eta_2$$
$$\dot{\eta}_2 = \eta_3$$
$$\vdots$$
$$\dot{\eta}_r = \Phi(f, \eta_1, \eta_2 \ldots \eta_r, u, \dot{u}, \ldots \overset{(r-1)}{u})$$
$$y = \eta_1, \tag{43}$$

where $\Phi(\cdot)$ is considered as an unmodeled dynamic.

The observer structure: The following system is a sliding-mode observer for the system (43):

$$\dot{\hat{\eta}}_i = \hat{\eta}_2 + m_1 sign(y - \hat{y})$$
$$\vdots$$
$$\dot{\hat{\eta}}_{r-1} = \hat{\eta}_r + m_{r-1} sign(y - \hat{y})$$
$$\dot{\hat{\eta}}_r = m_r\ \mathrm{sign}(y - \hat{y})$$
$$\text{with } \hat{y} = \hat{\eta}_1 \tag{44}$$

where $m_j > 0$, $\forall\ 1 \le j \le r$, and

$$\mathrm{sign}(y - \hat{y}) = \begin{cases} 1 & \text{if } (y - \hat{y}) > 0 \\ -1 & \text{if } (y - \hat{y}) < 0 \\ \text{undefined} & \text{if } (y - \hat{y}) = 0. \end{cases}$$

Then returning to the original coordinates and taking into account (41), the fault can be estimated from the following relationship:

$$\overline{\psi}(\hat{f}, \hat{\eta}, \dot{\hat{\eta}}, \ddot{\hat{\eta}} \ldots, \overset{(r)}{\hat{\eta}}, u, \dot{u} \ldots) = 0 \tag{45}$$

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Observer convergence analysis: We will analyze the convergence properties of the proposed observer considering the presence of a noise signal δ contaminating the output measurements, such that

$$y = \eta_1 + \delta. \tag{46}$$

Let us define the state estimation errors as

$$e_1 = \eta_1 - \hat{\eta}_1, \quad e_i = (\eta_i - \hat{\eta}_i)/m, \; i = 2\ldots, r, \tag{47}$$

where $m > 0$, it follows that the estimation error vector $e = [e_1 \ldots e_r]^T$ verifies the relationship

$$\dot{e} = A_{\overline{\mu}} e - K \operatorname{sign}(Ce + \delta) + \Delta s \tag{48}$$

where $\overline{\mu} > 0$ is a regularizing parameter,

$$A_{\overline{\mu}} = \begin{bmatrix} -\overline{\mu} & m & 0 & \ldots & 0 \\ 0 & -\overline{\mu} & m & & 0 \\ 0 & 0 & -\overline{\mu} & & \vdots \\ & & & \ddots & m \\ 0 & 0 & 0 & \ldots & -\overline{\mu} \end{bmatrix}, \quad K = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{bmatrix},$$

$$C = [1 \; 0 \; \ldots \; 0] \quad \text{and} \quad \Delta s = \begin{bmatrix} \overline{\mu} e_1 \\ \vdots \\ \overline{\mu} e_{r-1} \\ \Phi + \overline{\mu} e_r \end{bmatrix}$$

are uncertainty terms.

Assumption 1. There exist nonnegative constants L_{0s}, L_{1s}, such that the following generalized quasi-Lipschitz condition holds:

$$\|\Delta s\| \le L_{0s} + (L_{1s} + \|A_{\overline{\mu}}\|)\|e\|. \tag{49}$$

Assumption 2. The additive output noise δ is bounded, namely

$$|\delta| \le \delta^+ < \infty, \tag{50}$$

Assumption 3. There exists a positive definite matrix $Q_0 = Q_0^T > 0$, such that the following matrix Riccati equation:

$$PA_{\overline{\mu}} + A_{\overline{\mu}}^T P + PRP + Q = 0 \tag{51}$$

with

$$R{:=}\Lambda_s^{-1} + 2\|\Lambda_s\| L_{1s} I, \quad \Lambda_s = \Lambda_s^T > 0,$$

$$Q = Q_0 + 2(L_{1s} + \|A_{\overline{\mu}}\|^2) I$$

has a positive definite solution $P = P^T > 0$.

Remark 3. A1 only limits the maximum slope present in the uncertainty term Δs which depends on the Lipschitz properties of η_r. A2 is a standard assumption that allows us to avoid involving the statistic behavior of the noise signal. The expression (51) from A3 has a positive definite solution if the matrix $A_{\overline{\mu}}$ is stable, which is true for any $\overline{\mu} > 0$. Since $P > 0$, there exists $k > 0$ such that $K = kP^{-1}C^T$, then A3 provides an additional degree of freedom to choose the gain k which can be used to establish the size of the region defined by $\tilde{\mu}$.

Theorem 2. *If assumptions from* A1 *to* A3 *are satisfied, then*

$$[V - V^*]_+ \to 0 \tag{52}$$

where

$$V = V(e) = \|e\|_P^2 {:=} e^T P e,$$

$$V^* {:=} \frac{2\|\Lambda_s\| L_{0s}^2 + 4k\delta^+}{\lambda_{\min}(P^{-1/2} Q^T Q P^{-1/2})},$$

and the function $[\cdot]_+$ *is defined as follows*:

$$[x]_+ = \begin{cases} x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{53}$$

Proof. Let $V(e)$ be the following Lyapunov candidate function:

$$V(e) \triangleq e^T P e = \|e\|_P^2 \tag{54}$$

where $0 < P = P^T \in R^{r \times r}$ is the solution of the Riccati equation (51). By taking the time derivative of (54) and taking into account (48) it yields

$$\dot{V}(e) = 2e^T P \dot{e} = 2e^T P [A_{\overline{\mu}} e - K \operatorname{sign}(Ce + \delta) + \Delta s],$$

according to assumption A3, $K = kP^{-1}C^T$, then the previous equation can be written as

$$\dot{V}(e) = 2e^T P A_{\overline{\mu}} e - 2ke^T C^T \operatorname{sign}(Ce + \delta) + 2e^T P \Delta s.$$

By using the following matrix inequality:

$$X^T Y + Y^T X \le X^T \Lambda_s X + Y^T \Lambda_s^{-1} Y$$

which is valid for any $X, Y \in R^{r \times m}$, $0 < \Lambda_s = \Lambda_s^T \in R^{r \times r}$, then it follows that

$$\dot{V}(e) \le e^T (PA_{\overline{\mu}} + A_{\overline{\mu}}^T P) e - 2ke^T C^T \operatorname{sign}(Ce + \delta) + e^T P \Lambda_s^{-1} P e + (\Delta s)^T \Lambda_s \Delta_s,$$

from assumption A1 the following is obtained:

$$(\Delta s)^T \Lambda_s \Delta_s \le \|\Delta s\|^2 \|\Lambda_s\| \le [L_{0s} + (L_{1s} + \|A_{\overline{\mu}}\|)\|e\|]^2 \|\Lambda_s\|,$$

then

$$\begin{aligned} \dot{V}(e) \le\; & e^T (PA_{\overline{\mu}} + A_{\overline{\mu}}^T P + P\Lambda_s^{-1} P + Q) e - e^T Q e \\ & -2ke^T C^T \operatorname{sign}(Ce + \delta) + 2[L_{0s}^2 + (L_{1s} + \|A_{\overline{\mu}}\|)^2 \|e\|^2] \|\Lambda_s\|, \end{aligned}$$

from the definition of matrix R in assumption A3, the previous expression can be rewritten as

$$\begin{aligned} \dot{V}(e) \le\; & e^T (PA_{\overline{\mu}} + A_{\overline{\mu}}^T P + PRP + Q) e - e^T Q e - 2ke^T C^T \operatorname{sign}(Ce + \delta) \\ & + 2L_{0s}^2 \|\Lambda_s\|, \end{aligned}$$

and taking (51) into account, it follows that

$$\dot{V}(e) \le -e^T Q e - 2ke^T C^T \operatorname{sign}(Ce + \delta) + 2L_{0s}^2 \|\Lambda_s\|. \tag{55}$$

In order to eliminate the discontinuity contained in the function $\operatorname{sign}(\cdot)$ in (55) the following inequality, valid for any $x, y \in \mathbb{R}$, is considered:

$$\begin{aligned} x \operatorname{sign}(x + y) &= (x + y)\operatorname{sign}(x + y) - y \operatorname{sign}(x + y) \\ &\ge |x + y| - |y| \end{aligned}$$

and furthermore $|x + y| \ge |x| - |y|$, then

$$x \operatorname{sign}(x + y) \ge |x| - 2|y|. \tag{56}$$

Now using (56) in (55) the following inequality is obtained:

$$\dot{V}(e) \le -e^T Q e - 2k|Ce| + 2L_{0s}^2 \|\Lambda_s\| + 4k\delta^+, \tag{57}$$

which can be rewritten as

$$\dot{V}(e) \le -\|e\|_Q^2 + 2L_{0s}^2 \|\Lambda_s\| + 4k\delta^+,$$

that is to say,

$$\dot{V}(e) \le -\alpha_Q V(e) + \beta, \tag{58}$$

where

$$\alpha \triangleq \lambda_{\min}(P^{-1/2} Q^T Q P^{-1/2}) > 0,$$

$$\beta = 2L_{0s}^2 \|\Lambda_s\| + 4k\delta^+.$$

Now, considering the following differential equation related to (58):

$$\dot{V}(e) = -\alpha V + \beta, \tag{59}$$

which is linear and stable and such that $V \to V^*$ as $t \to \infty$, where V^* is the single equilibrium point of Eq. (59):

$$V^* = \frac{\beta}{\alpha} \geq 0,$$

it follows that the function

$$G_t \triangleq [V - V^*]_+^2$$

where $[\cdot]_+$ is defined as in (53), according to (58) satisfies (for any $V \neq V^*$)

$$\dot{G}_t \leq -2[V - V^*]_+[-\alpha V + \beta] \leq 0$$

subtracting $-\alpha V^* + \beta = 0$, it yields

$$\dot{G}_t \leq -2\alpha(V - V^*)[V - V^*]_+ \leq 0$$

that is to say,

$$\dot{G}_t \leq -2\alpha G_t \leq 0.$$

Integrating the last inequality it follows that

$$G_t - G_0 \leq -2\alpha \int_0^t G_\tau \, d\tau,$$

in other words

$$2\alpha \int_0^t G_\tau \, d\tau \leq G_0 - G_t \leq G_0$$

then

$$\lim_{t \to \infty} 2\alpha \int_0^t G_\tau \, d\tau \leq G_0.$$

From Barbalat Lemma [26], it follows $G_t \to 0$, which is equivalent to say $[V - V^*]_+ \to 0$. □

Remark 4. Theorem 2 states that the weighted estimation error norm $V(e)$ asymptotically converges to the zone bounded by V^*. In other words, it is ultimately bounded.

6. Application to the three-tank system

6.1. Description of the three-tank system

The Amira DTS200 is described in Fig. 1. The corresponding model with faults is given by the following equations [22]:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{A}(u_1 - q_{13} + f_1) \\ \dot{x}_2 &= \frac{1}{A}(u_2 + q_{32} - q_{20} + f_2) \\ \dot{x}_3 &= \frac{1}{A}(q_{13} - q_{32}) \end{aligned} \tag{60}$$

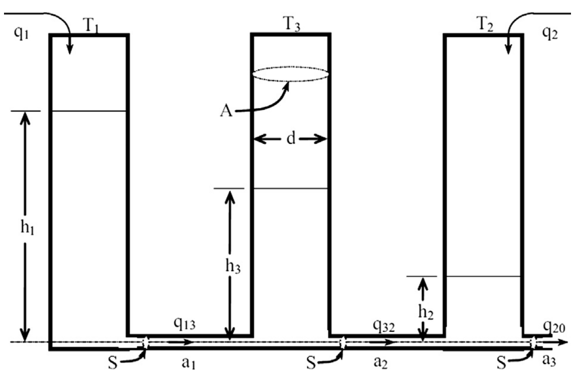

Fig. 1. Schematic diagram of the three-tank system.

where $u_1 = q_1$ and $u_2 = q_2$ are the manipulable input flows, $x_i = h_i$ is the level in the tank i. A is the transversal constant section of any of the identical tanks, and q_{ij} represents the water flow from tank i to tank j $(1 \leq i, j \leq 3)$ which according to the generalized Torricelli's rule, is valid for laminar flow

$$q_{ij} = a_i S \,\mathrm{sign}(h_i - h_j)\sqrt{2g|h_i - h_j|} \tag{61}$$

with $q_{20} = a_2 S\sqrt{2gh_2}$, where S is the transversal area of the pipe that interconnects the tanks (see Fig. 1) and a_i are the output flow coefficients, which are not exactly known, so they are considered as uncertain parameters. We assume the existence of actuator faults denoted by f_1 and f_2 $(\mu = 2)$, each one of these faults represents a variation in the respective pump driver gain, which can be originated by an electronic component malfunction, or even by a leakage or an obstruction in the pump pipes.

The system (60) has four state regions in which the corresponding model is differentiable [17], any of these regions can be chosen to do the analysis, just avoiding the loss of differentiability by crossing from one to another. In this work $x_1 > x_3 > x_2 > 0$ is the only considered region of operation, which experimentally is easy to operate.

6.2. Diagnosability analysis

According to Theorem 1 we need two or more measured outputs, this can only happen in the following cases:

- Case 0: $p = 3$ (h_1, h_2, and h_3 measurable).
- Case 1: $p = 2$ (h_1 not measurable, h_2 and h_3 measurable).
- Case 2: $p = 2$ (h_2 not measurable, h_1 and h_3 measurable).
- Case 3. $p = 2$ (h_3 not measurable, h_1 and h_2 measurable).

6.2.1. Case 0

The simplest case (and the only one reported in previous works [17], with numerical results) takes place when we can measure the full state vector, that is to say, we have three outputs: $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$; in this case, from (60) we have

$$f_1 = A\dot{y}_1 + a_1 S\sqrt{2g(y_1 - y_3)} - u_1 \tag{62}$$

$$f_2 = A\dot{y}_2 - a_3 S\sqrt{2g(y_3 - y_2)} + a_2 S\sqrt{2gy_2} - u_2 \tag{63}$$

System (60) is left invertible because the differential output rank is equal to 2. This means that faults f_1 and f_2 are diagnosable.

6.2.2. Case 1

We consider only the outputs $y_2 = x_2$ and $y_3 = x_3$. By taking into account (60) we have

$$A\dot{y}_3 = a_1 S\sqrt{2g(x_1 - y_3)} - a_3 S\sqrt{2g(y_3 - y_2)}, \tag{64}$$

we get

$$x_1 = y_3 + \frac{1}{2ga_1^2 S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g(y_3 - y_2)}\right)^2 \tag{65}$$

Then, by replacing x_1 in (62) we obtain a set of two differential equations with coefficients in $\mathbb{R}\langle u, y\rangle$ with two unknowns f_1 and f_2, this means that system (60) is left invertible (i.e., faults f_1 and f_2 are diagnosable) with the two considered outputs.

6.2.3. Case 2

We consider only the outputs: $y_1 = x_1$ and $y_3 = x_3$. By taking into account (64) we obtain

$$x_2 = y_3 - \frac{1}{2ga_3^2 S^2}\left(-A\dot{y}_3 + a_1 S\sqrt{2g(y_1 - y_3)}\right)^2. \tag{66}$$

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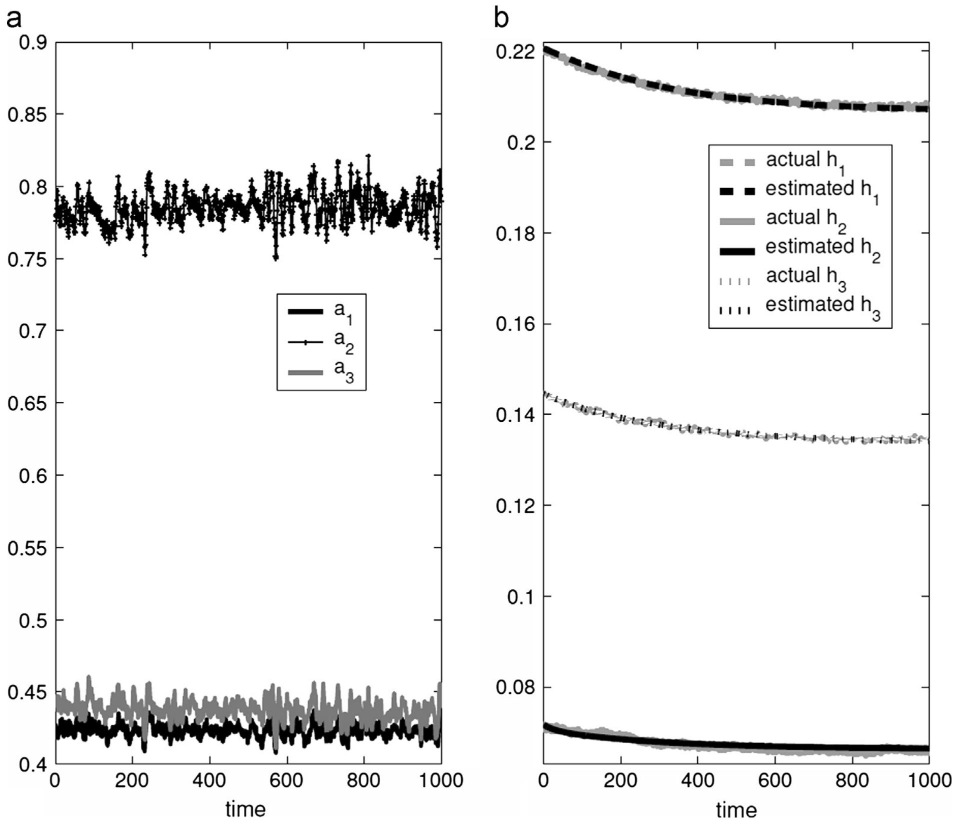

Fig. 2. (a) Evolution of the parameter identification. Coefficients a_1, a_2, a_3. (b) Validation of the estimated model. Levels h_1, h_2, h_3.

From (63) in a similar way we can obtain system (60) that is left invertible (i.e., faults f_1 and f_2 are diagnosable) with the two considered outputs.

6.2.4. Case 3

We consider only the outputs $y_1 = x_1$ and $y_2 = x_2$. By taking into account (62) we get

$$x_3 = y_1 - \frac{1}{2ga_1^2S^2}(-A\,\dot{y}_1 + f_1 + u_1)^2. \tag{67}$$

From (63) we can only obtain one differential equation involving the two faults, therefore, system (60) is not left invertible, i.e., faults f_1 and f_2 are not diagnosable with the two considered outputs.

6.3. Fault reconstruction

We present two novel observers to obtain effective fault estimations, as well as they can be used to estimate time derivatives as follows.

Reduced order observer: Let us consider the following time derivative to be estimated:

$$\eta = \dot{y}. \tag{68}$$

According to (31), we propose the observer structure

$$\dot{\hat{\eta}} = k_3(\eta - \hat{\eta}) \tag{69}$$

introducing the change of variable

$$\hat{\eta} = \gamma + k_3 y \tag{70}$$

and from (69) and (70) we can get $\dot{\gamma} = -k_3\hat{\eta}$, then again from (70)

$$\dot{\gamma} = -k_3\gamma - k_3^2 y \tag{71}$$

then (71) together with (70) constitutes an asymptotic estimator for η.

Sliding-mode observer: We introduce the following change of variables: $\eta_1 = y$, $\eta_2 = \dot{\eta}_1$, then we obtain the following observer:

$$\left.\begin{aligned} \dot{\hat{\eta}}_1 &= \hat{\eta}_2 + m_1 \operatorname{sign}(y - \hat{\eta}_1) \\ \dot{\hat{\eta}}_2 &= m_2 \operatorname{sign}(y - \hat{\eta}_1) \end{aligned}\right\} \tag{72}$$

which can be used to estimate η_2 from the knowledge of y.

7. Experimental results

We verified the real time performance of the proposed estimators in a laboratory setting of the Amira DTS200 system. The known parameter values for the utilized system are $A = 0.0149$ m^2, $S = 5 \times 10^{-5}$ m^2 and the unknown parameters a_1, a_2, and a_3. The sample time in all the experiments was 0.001 s, this was chosen to be so small in order to get the best performance from the sliding-mode observer. The experimental results are described as follows.

7.1. Identification results

With no presence of faults, the unknown parameters a_1, a_2, and a_3 are algebraically observable, in other words, they can be expressed in terms of inputs, outputs and time derivatives of these variables, respectively, as follows:

$$a_1 = \frac{q_1 - A\dot{y}_1}{S\sqrt{2g(y_1 - y_3)}} \tag{73}$$

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$$a_2 = \frac{q_1 + q_2 - A(\dot{y}_1 + \dot{y}_2 + \dot{y}_3)}{S\sqrt{2gy_2}} \tag{74}$$

$$a_3 = \frac{q_1 - A(\dot{y}_1 + \dot{y}_3)}{S\sqrt{2g(y_3 - y_2)}} \tag{75}$$

Estimations of these parameters were obtained by estimating the time derivatives involved in the expressions (73)–(75); meanwhile the values for the input flows were $q_1 = 0.000025\ \mathrm{m}^3/\mathrm{s}$ and $q_2 = 0.000020\ \mathrm{m}^3/\mathrm{s}$, along 1000 s in these conditions the evolution of the estimated values for the unknown coefficients is shown in Fig. 2a.

At the end of the identification process the estimated values for the flow parameters were obtained. These are the values that were used in the fault estimation schemes (62) and (63) as they were implemented in the conditions described in the next section. These values were the following:

$$a_1 = 0.418, \quad a_2 = 0.789, \quad a_3 = 0.435 \tag{76}$$

In Fig. 2b the simulated and the measured actual levels are shown in order to give a visual comparison between the actual and the estimated model, the actual level measurements are drawn in a gray color, while the levels obtained by simulating the model using the estimated values given by (76) for the flow coefficients are shown in black color.

7.2. Fault estimation results

In all the experiments described in this subsection the input flows were maintained constant as $q_1 = 0.00002\ \mathrm{m}^3/\mathrm{s}$ and $q_2 = 0.000015\ \mathrm{m}^3/\mathrm{s}$, also two faults were artificially generated through the following expressions: $f_1 = 0.00005[1 + \sin(0.2te^{-0.01t})]\mathcal{U}(t-220)$, $f_2 = 0.00005[1 + \sin(0.05te^{-0.001t})]\mathcal{U}(t-300)$, where $\mathcal{U}(t)$ is the unit step function.

As we do not know the dynamics Φ, we can take as a reference the Lipschitz constants of the fault signals, which are 10.6×10^{-7} and 11.25×10^{-7}, then we choose L_{1s} bigger enough, for example $L_{1s} = 0.001$, in a similar way, we choose $m = 0.1$, $\overline{\mu} = 1$, $\Lambda_s = 20$, $Q_0 = I$, then $R = 0.09I$, $Q = 3.2122I$, with these parameters we obtain

$$P = \begin{bmatrix} 20.4009 & -1.2107 \\ -1.2107 & 20.5446 \end{bmatrix} > 0$$

The two proposed schemes for fault estimation were evaluated in case 1 (x_1 not measurable), the results are described as follows.

Only the two outputs $y_2 = x_2$ and $y_3 = x_3$ were taken into account; an estimation for the unknown state x_1 was necessary to be obtained. In Fig. 3 we show the resulting estimations achieved with the reduced-order observer. A low-pass filter was necessary in order to reduce the effect of the measurement noise, we chose a second-order Butterwort filter whose transfer function is given by $G_f(s) = 1/(32s^2 + 8s + 1)$. The gain values chosen for both the fault observers were $k_1 = k_2 = 2$, and for the state observer x_1, $k_{x_1} = 0.3$. As we can observe the estimation results with this scheme are good (Fig. 3).

A sliding-mode observer was also tested in this case. In Fig. 4 the corresponding results achieved with the sliding-mode observer are shown. It is worth to mention that with this observer it was not necessary to include the reducing noise filter providing the inherent robustness of this observer. The gain values chosen for the fault and state observers were $m_1 = 0.1$ and $m_2 = 0.01$. As we can observe from Fig. 4, this scheme also provides good estimation results.

8. Concluding remarks

We have tackled the fault diagnosis problem in nonlinear systems using the condition of left invertibility through the

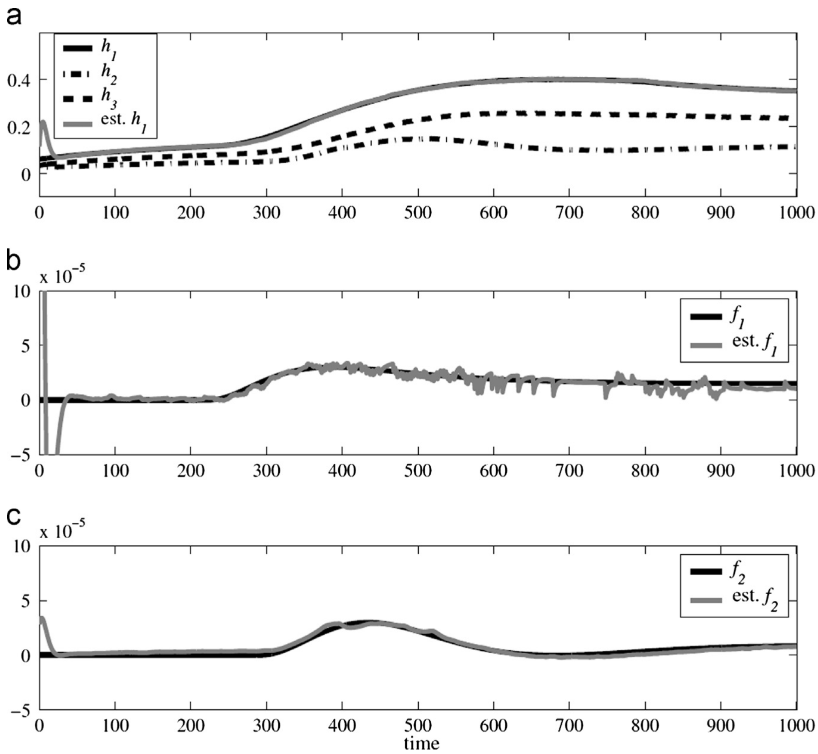

Fig. 3. Fault diagnosis for unknown h_1 using the reduced order observer: (a) Levels. (b) Actual and estimated f_1. (c) Actual and estimated f_2.

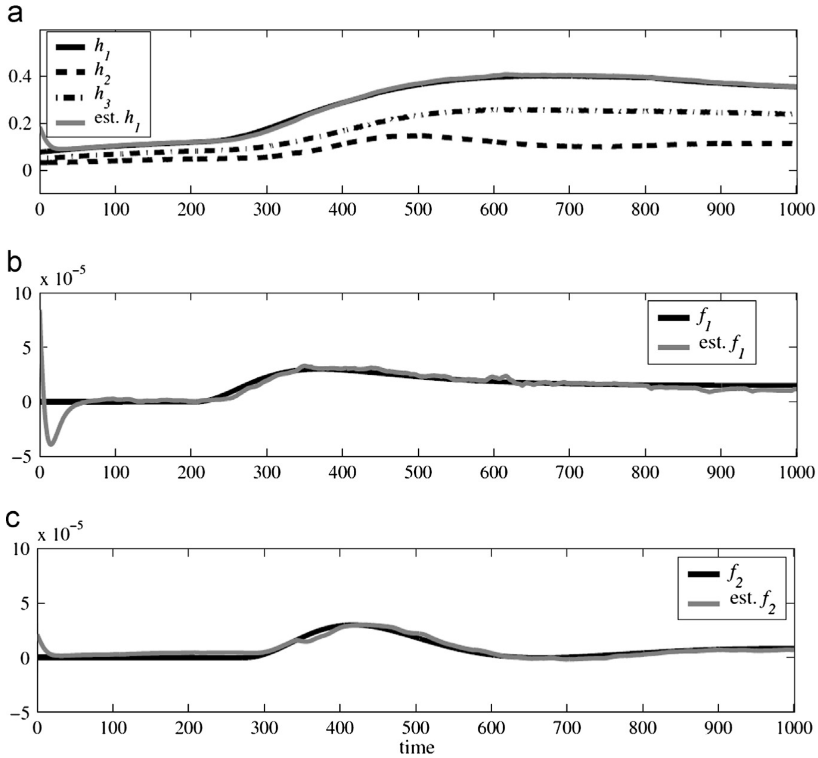

Fig. 4. Fault diagnosis for unknown h_1 using the sliding mode observer: (a) Levels. (b) Actual and estimated f_1. (c) Actual and estimated f_2.

concept of differential output rank. The usefulness of Theorems 1, 2 and Lemma 2 was shown; this allowed the estimation of two simultaneous faults with less measurements. The theoretical and simulation results were tested in a real-time implementation (three-tank system). The experimental results for the two observers showed similar performance, however the proposed sliding-mode observer is more robust against measurement noise, as it was expected.

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Asymptotic synchronization of the Colpitts oscillator

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ABSTRACT

In this paper we deal with the observer-based asymptotic synchronization problem for a class of chaotic oscillators. Some results based on a differential algebraic approach are used in order to determine the algebraic observability of unknown variables. The strategy consists of proposing a slave system (observer) which tends to follow asymptotically the master system. The methodology is tested in the real-time asymptotic synchronization of the Colpitts oscillator by means of a proportional reduced order observer (PROO) of free-model type.

1. Introduction

Chaotic systems synchronization has been investigated since its introduction in the paper [1]. Among the publications dedicated to chaos synchronization, many different approaches can be found [2–4]. We cite the papers [4–7] which propose the use of state observers to synchronize chaotic systems; in references [8–10] use feedback controllers; in [11,12] use nonlinear backstepping control; in papers [13,14] consider synchronization time delayed systems; in works [15,16] consider directional and bidirectional linear coupling; papers [17,18] use nonlinear control; in [10,19] use adaptive control; in [20] apply sliding-mode techniques; in [21] attack the anti-synchronization problem; in [22] a fuzzy sliding-mode controller is applied; in [23] the synchronization in time delay systems is given, and so on.

Synchronization of the chaotic systems problem has received a great deal of attention among scientists in many fields due to its potential applications, such as: secure communications, biological systems, chemical reactions, etc., [4,8,24–26].

As we can note, there exist several methods to solve the synchronization problem since from the control theory perspective in this work, we study the asymptotic synchronization by means of state observers.

The method is based on a *master–slave* configuration [1]. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (transmitter) to the slave system (receiver), the receiver is requested to recover the unknown (or full) state trajectories of the transmitter. By this fact, the terminology *transmitter–receiver* is also used. Thus, the chaos synchronization problem can be regarded as an observer design procedure, where the coupling signal is viewed as an output and the slave system is the observer [27–29].

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances [30,31].

As we know, it is almost impossible to measure all the elements of the state vector in practice (e.g., the unknown state variables, fault signals, etc.). Here arises a basic practical question: would it be possible to reconstruct these unknown

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signals? We give an answer to this question by introducing a basic definition related with the estimation (reconstruction) of the states, the algebraic observability property (AOP).

In this work the observability property for a class of systems (oscillators, chaotic systems, and systems with bounded dynamics) is determined by means of a relatively new approach which is related with the differential-algebra framework. This mathematical approach has been recently shown to be a very effective tool for understanding basic questions such as input–output inversions and observer realizations.

The main contributions consist of the following. (i) An observer as a numerical technique to reconstruct unknown variables is designed. Before proposing the observer structure we should verify whether the signal to be estimated satisfies the AOP. Then we design a PROO which is based on the free-model approach. The main advantage of the PROO is that the free-model quality of its structure allows us to reconstruct the unknown variables in spite of model uncertainties. (ii) The suggested approach is implemented in the real-time asymptotic synchronization of the Colpitts oscillator via the PROO.

The paper is organized as follows. In Section 2 the synchronization problem and its solution by means of a reduced order observer are treated. Section 3 presents the procedure for the synchronization in real-time of the Colpitts oscillator [32]. Section 4 illustrates the obtained experimental results and shows the performance of the reduced order observer. Finally, in Section 5 we close the paper with some concluding remarks.

2. Observer design

Let us consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(x, u) \\ y(t) &= h(\bar{x}) \end{aligned} \tag{1}$$

where $f \in \mathbb{R}^n$ is differentiable and satisfies $f(0,0) = 0$, $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the state vector, $\bar{x} \in \mathbb{R}^p$ represents the known states (with $1 \le p < n$), $y \in \mathbb{R}^q$ denotes the output of the system, $h : \mathbb{R}^p \to \mathbb{R}^q$ is a continuous function and $u \in \mathbb{R}^l$ is the known input ($l \le n$).

Now, let us consider the nonlinear system described by (1). We separate (1) into two dynamical systems with states $\bar{x} \in \mathbb{R}^p$ and $\eta \in \mathbb{R}^{n-p}$ respectively with $x^T = (\bar{x}^T, \eta^T)$. The first system describes the known states and the second represents unknown states, then the system (1) can be rewritten as:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{f}(x, u) \\ \dot{\eta}(t) &= \Delta(x, u) \\ y(t) &= h(\bar{x}) \end{aligned} \tag{2}$$

where $\bar{f} \in \mathbb{R}^p$, $f^T(x) = \left(\bar{f}^T(x,u), \Delta^T(x,u)\right)$, and $\Delta \in \mathbb{R}^{n-p}$ is an uncertain function. The problem is to estimate the state variable $\eta(t) = (\eta_{(p+1)}, \ldots, \eta_n)^T$.[1] In order to solve this observation problem let us introduce the following property.

Definition 1 (*Algebraic Observability Property-AOP*)**.** Let us consider the nonlinear dynamical system (2). A state variable $\eta_i \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}\langle u, y\rangle$,[2] that is to say, η_i satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their first $r_1, r_2 \in \mathbb{N}$ time derivatives, respectively, i.e.,

$$\eta_i = \phi_i(u, \dot{u}, \ldots, \overset{(r_1)}{u}, y, \dot{y}, \ldots, \overset{(r_2)}{y}), \quad i \in \{p+1, \ldots, n\}, \tag{3}$$

where $\phi_i : \mathbb{R}^{(r_1+1)l} \times \mathbb{R}^{(r_2+1)q} \to \mathbb{R}$.

If any unknown variable satisfies the AOP, then a numerical technique from the so-called observer can be used to reconstruct the required signal.

The next system represents the dynamics of the unknown states:

$$\dot{\eta}_i(t) = \Delta_i(x, u). \tag{4}$$

Lemma 1. *If the following hypotheses are satisfied:*

H1: *$\eta_i(t)$ satisfies the AOP (Definition 1), for $i \in \{p+1, \ldots, n\}$.*
H2: *γ_i is a C^1 real-valued function.*
H3: *Δ_i is bounded, i.e., $|\Delta| \le M < \infty$.*
H4: *For t_0, sufficiently large, there exists $K_i > 0$, such that,* $\limsup_{t \to t_0} \frac{M}{K_i} = 0$.

Then, the system

$$\dot{\hat{\eta}}_i = K_i(\eta_i - \hat{\eta}_i) \tag{5}$$

is an asymptotic reduced order observer of free-model type for system (4), where $\hat{\eta}_i$ denotes the estimate of η_i and $K_i \in \mathbb{R}^+$ determines the desired convergence rate of the observer.

[1] In practice, identification of the variable η depends on the variable choice to be estimated.

[2] $\mathbb{R}\langle u, y\rangle$ denotes the differential field generated by the field $\mathbb{R}$, the input u, the measurable output y, and the time derivatives of u and y.

Proof. Let us define the estimation error as $e(t) = \eta_i(t) - \hat{\eta}_i(t)$. The dynamics of the error is given by

$$\dot{e}_i(t) = \dot{\eta}_i(t) - \dot{\hat{\eta}}_i$$

then

$$\dot{e}_i(t) + K_i e_i(t) = \Delta_i(t). \tag{6}$$

The solution of (6) is given by,

$$e_i(t) = \exp(-K_i t)\left[e_{i0} + \int_0^t \exp(K_i\tau)\Delta_i(\tau)d\tau\right], \tag{7}$$

where $e_{i0} = e_i(0)$ denotes the initial condition.

Then, using Cauchy–Schwartz and triangle inequalities in the expression (7), we have,

$$0 \le |e_i(t)| \le \exp(-K_i t)|e_{i0}| + \exp(-K_i t)\int_0^t |\exp(K_i\tau)\Delta_i(\tau)d\tau|.$$

If H3 is satisfied, we obtain,

$$\begin{aligned} 0 \le |e_i(t)| &\le \exp(-K_i t)|e_{i0}| + M\int_0^t \exp[K_i(\tau - t)]d\tau \\ &= \exp(-K_i t)|e_{i0}| + \frac{M}{K_i}[1 - \exp(-K_i t)]. \end{aligned}$$

If $t \to t_0$, for t_0 sufficiently large, then

$$\begin{aligned} 0 &\le \limsup_{t\to t_0} |e_i(t)| \\ &\le |e_{i0}| \limsup_{t\to t_0}[\exp(-K_i t)] + \limsup_{t\to t_0}\left\{\frac{M}{K_i}[1 - \exp(-K_i t)]\right\} \\ &= \limsup_{t\to t_0} \frac{M}{K_i}. \end{aligned}$$

Finally, by taking into account H4, we have

$$0 \le \limsup_{t\to t_0} |e_i(t)| \le \limsup_{t\to t_0} \frac{M}{K_i} = 0.$$

Then,

$$\limsup_{t\to t_0} |e_i(t)| = 0.$$

Therefore, (5) is an asymptotic reduced order observer for (4). □

Sometimes the output time derivatives (which are unknown), appear in the algebraic equation of the state variable, then it is necessary to use an auxiliary variable to avoid using them.

Corollary 1. *The dynamic system* (5) *along with*

$$\dot{\gamma} = \psi(x, u, \gamma), \quad \text{with } \gamma_0 = \gamma(0) \text{ and } \gamma \in C^1$$

constitute a proportional asymptotic reduced order observer for system (4)*, where* γ *is a change of variable which depends on the estimated state* $\hat{\eta}$*, and the state variables.*

3. Synchronization of the Colpitts oscillator

In 1994, chaotic oscillations in the Colpitts circuit with a generic transistor 2N2222A were reported [33]. In the work [34], the author refers to the multi-oscillations phenomenon in the RF bipolar oscillators, which are parasitic oscillations or not desired that coexist with a main oscillation. Due to these parasitic oscillations the resultant signal in steady state is severely affected, in this sense, those circuits have few applications. The paper [33] shows the parasitic oscillations are not present in the Colpitts oscillator, and moreover, the chaotic behavior of the circuit is presented, in fact, the Colpitts oscillator is widely used in electronic devices and communication systems. Fig. 1(a) shows the circuit configuration [32], as well as some values for which the Colpitts oscillator has chaotic behavior.

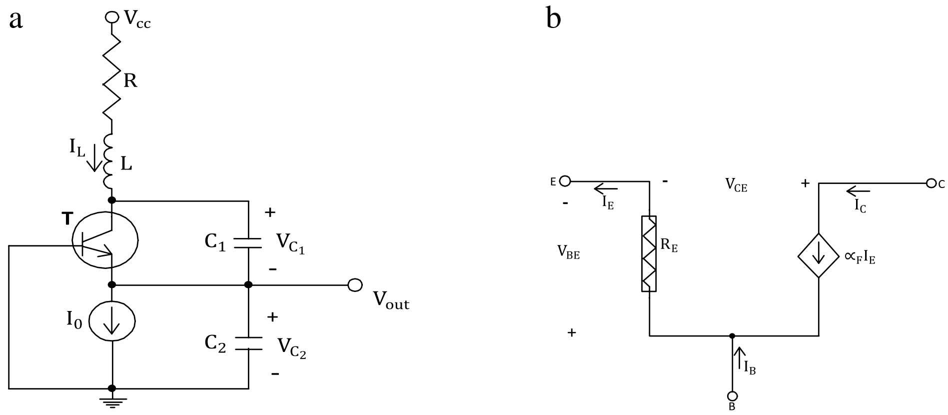

Fig. 1. Colpitts oscillator. (a) Circuit configuration. (b) Model of the bipolar junction transistor (BJT). The circuit parameters are: $L = 100\ \mu H$; $C_1 = C_2 = 47$ nF, $R = 45\Omega$, $I_0 = 5$ mA.

3.1. Modeling

In this work we consider the classical configuration of the Colpitts oscillator. The circuit contains a bipolar junction transistor (BJT) as the gain element, two resistors, and a resonant network consisting of an inductor and two capacitors (Fig. 1(a)).

By taking into account the qualitative theory in nonlinear dynamics, we select a minimum model for the circuit. The idea is to consider as simple a circuit model as possible that describes the essential features exhibited by the real Colpitts oscillator.

For the modeling of such circuit the following assumptions are considered:

1. The emitter current is generated by means of I_0.
2. The passive and active elements are ideals.
3. The transistor is modeled as a nonlinear resistor R_E controlled by voltage and a linear input controlled by current (Fig. 1(b)), indeed,
   (a) We model V–I characteristic of R_E with an exponential function, as follows
   $$\begin{aligned} I_E &= I_S\left[\exp\left(\frac{V_{BE}}{V_T}\right) - 1\right] \\ &\approx I_S \exp\left(\frac{V_{BE}}{V_T}\right), \quad \text{if } V_{BE} \gg V_T, \end{aligned} \tag{8}$$
   where I_S is the inverse saturation current and $V_T \simeq 26$ mV at room temperature.
   (b) We assume that $\alpha_F = 1$, where α_F is the common–base forward short–circuit current gain. This corresponds to neglecting the base current.
   (c) The parasitic dynamics of the transistor are omitted.

The Colpitts circuit is described by a system of three nonlinear differential equations, as follows:

$$\begin{aligned} C_1\dot{V}_{C_1} &= -f(V_{C_2}) + I_L \\ C_2\dot{V}_{C_2} &= I_L - I_0 \\ L\dot{I}_L &= -V_{C_1} - V_{C_2} - RI_L + V_{CC} \end{aligned} \tag{9}$$

where $f(\cdot)$ is the driving-point characteristic of the nonlinear resistor. This can be expressed in the form $I_E = f(V_{C_2}) = f(-V_{BE})$. In particular, we have $f(V_{C_2}) = I_S \exp\left(-V_{C_2}/V_T\right)$.

3.2. Parameter normalization

We introduce the dimensionless state variables (x_1, x_2, x_3), and we choose the operating point of (9) to be the origin of the new coordinate system. In particular, we normalize voltages, currents and time with respect to $V_{\text{ref}} = V_T$, $I_{\text{ref}} = I_0$ and $t_{\text{ref}} = 1/w_0$, respectively, where $w_0 = 1/\sqrt{LC_1C_2/(C_1 + C_2)}$, is the resonant frequency of the unloaded L–C tank circuit. Then, the state equations for the Colpitts oscillator can be rewritten in the next form [35]:

$$\begin{aligned} \dot{x}_1 &= -a\exp(-x_2) + ax_3 + a \\ \dot{x}_2 &= bx_3 \\ \dot{x}_3 &= -cx_1 - cx_2 - dx_3 \end{aligned} \tag{10}$$

where, $a = b\frac{C_2}{C_1}$, $b = \frac{I_0}{w_0C_2V_T}$, $c = \frac{V_T}{w_0LI_0}$, $d = \frac{R}{Lw_0}$.

3.3. *Observer for the Colpitts oscillator (slave system)*

Let us consider the normalized system of the Colpitts oscillator. Throughout this paper we assume that the output system is $y = x_2$ and the unknown variables are $\eta = (x_1, x_3)^T$. Therefore, the slave system consists of two estimation structures to achieve synchronization with the master system. Such structures are obtained as follows. Firstly, verify that the master system (Colpitts oscillator) fulfills the AOP, and then, by using (5), construct the observer for the unknown states.

The AOP for x_3 is given by,

$$x_3 = \frac{\dot{x}_2}{b} = \frac{\dot{y}}{b} = \phi_3(\dot{y}). \tag{11}$$

For x_1, we have

$$x_1 = -\frac{1}{c}\left[\frac{1}{b}\ddot{y} + \frac{d}{b}\dot{y} + cy\right] \Rightarrow x_1 = \phi_1(y, \dot{y}, \ddot{y}). \tag{12}$$

Then, both unknown states of the master system satisfies the AOP, therefore, we can construct the observers based on (5) and Corollary 1.

Observer for x_3:

$$\begin{aligned}\dot{\hat{x}}_3 &= K_3(x_3 - \hat{x}_3)\\ &= \frac{K_3}{b}\dot{y} - K_3\hat{x}_3\end{aligned}$$

$$\dot{\hat{x}}_3 - \frac{K_3}{b}\dot{y} = -K_3\hat{x}_3.$$

If we define $\gamma_3 = -\frac{K_3}{b}y + \hat{x}_3$, then

$$\begin{aligned}\dot{\gamma}_3 &= -\frac{K_3^2}{b}y - K_3\gamma_3\\ \hat{x}_3 &= \frac{K_3}{b}y + \gamma_3.\end{aligned} \tag{13}$$

Observer for x_1:

$$\begin{aligned}\dot{\hat{x}}_1 &= K_1(x_1 - \hat{x}_1)\\ &= -\frac{K_1}{cb}\ddot{y} - \frac{K_1 d}{cb}\dot{y} - K_1 y - K_1\hat{x}_1.\end{aligned}$$

Now, we consider the change of variable $x_4 = \dot{y}$ and we design an observer for this new variable according to (5),

$$\begin{aligned}\dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y]\\ \hat{x}_4 &= \gamma_4 + K_4 y.\end{aligned}$$

Then,

$$\begin{aligned}\dot{\hat{x}}_1 &= -\frac{K_1}{cb}\dot{\hat{x}}_4 - \frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1\\ \dot{\hat{x}}_1 + \frac{K_1}{cb}\dot{\hat{x}}_4 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1.\end{aligned}$$

If we define $\gamma_5 = \hat{x}_1 + \frac{K_1}{cb}\hat{x}_4$, then

$$\begin{aligned}\dot{\gamma}_5 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y + \frac{K_1^2}{cb}\hat{x}_4 - K_1\gamma_5\\ &= [K_1 - d]\frac{K_1}{cb}\hat{x}_4 - K_1 y - K_1\gamma_5\\ &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5.\end{aligned}$$

Finally, the observer scheme for x_1 is given by

$$\begin{aligned}\dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y]\\ \dot{\gamma}_5 &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5\\ \hat{x}_1 &= -\frac{K_1}{cb}[\gamma_4 + K_4 y] + \gamma_5.\end{aligned} \tag{14}$$

Therefore, (13) and (14) constitute the slave system.

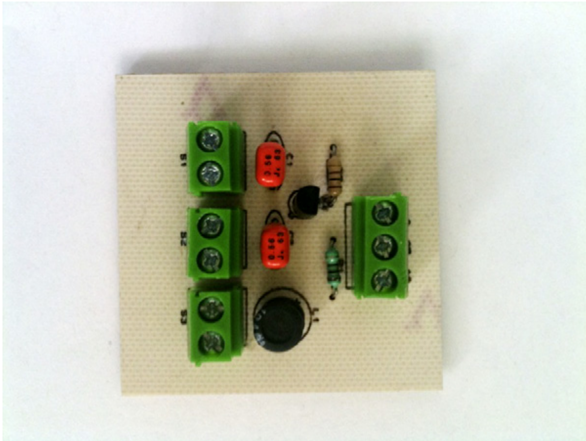

Fig. 2. Implementation of the Colpitts circuit (master system).

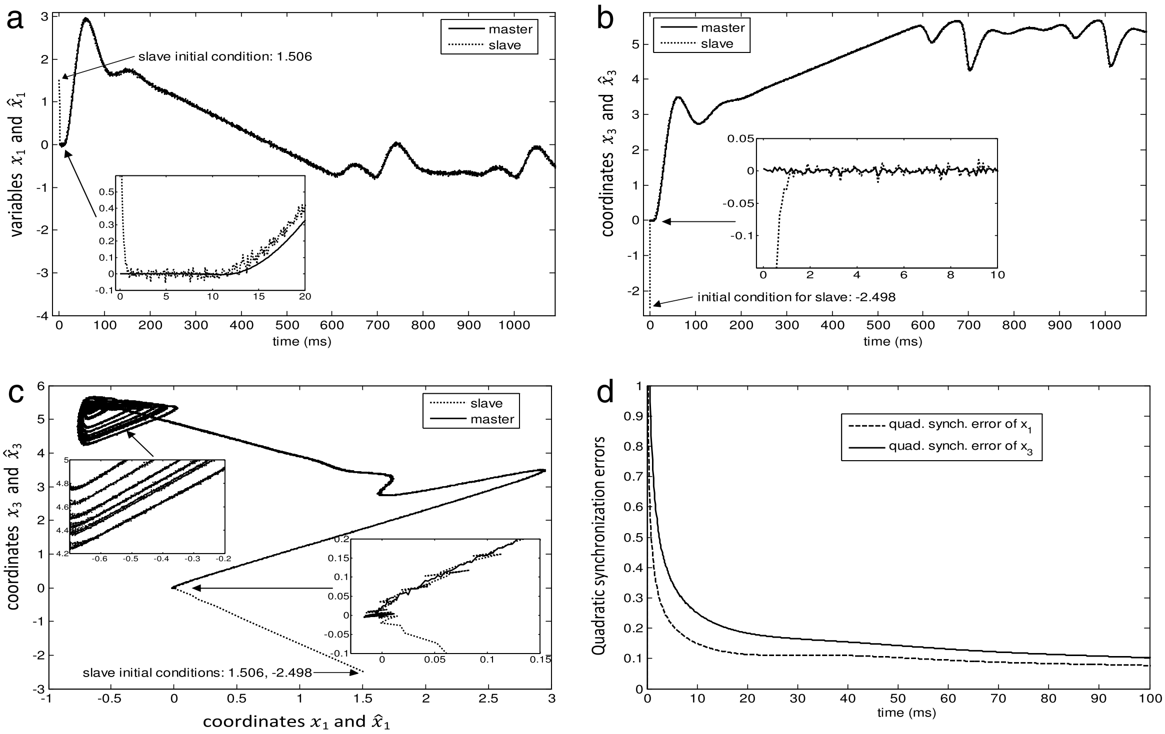

Fig. 3. Real-time synchronization of Colpitts oscillator using reduced-order observer (14)–(13): (a) synchronization of x_1, (b) synchronization of x_3, (c) Phase portrait of the master system (x_1 versus x_3) and the slave system ($\hat{x}_1$ versus $\hat{x}_3$), and (d) performance index.

4. Experimental results

We verified the real time performance of the proposed observers by using the WINCON platform. To achieve the synchronization in real time, in WINCON were implemented the schemes (13) and (14) in the master–slave configuration. The circuit parameters are: $L = 100\ \mu\text{H}$; $C_1 = C_2 = 47$ nF, $R = 45\ \Omega$, $I_0 = 5$ mA. Using the circuit parameters we obtain $a = b = 6.2723$, $c = 0.0797$, and $d = 0.6898$. In Fig. 2 the real Colpitts circuit (master system) is shown.

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as [36].

$$J(t) = \frac{1}{t + 0.001} \int_0^t |e(t)|^2_{Q_0}, \quad Q_0 = I$$

where $e(t)$ denotes the synchronization error.

Fig. 3(a)–(b) show the obtained results for the initial conditions $\hat{x}_1 = 1.506$ and $\hat{x}_3 = -2.498$ in the schemes (13) and (14), respectively. As we can note, the synchronization results achieved with the reduced order observer are good. Fig. 3(c) presents the phase portrait, where clearly the chaotic behavior of the Colpitts oscillator is observed. Finally, Fig. 3(d) illustrates the performance index, which has a tendency to decrease.

5. Conclusions

In this paper we study the asymptotic synchronization problem in chaotic systems based on the observer theory. We also show the real-time synchronization in the Colpitts oscillator by using a reduced order observer of free-model type, which has asymptotic convergence. Some experimental results show the effectiveness of the proposed methodology.

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Temperature Control of Continuous Chemical Reactors Under Noisy Measurements and Model Uncertainties

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ABSTRACT
The aim of this paper is to present the synthesis of a robust control law for the control of a class of nonlinear systems named Liouvillian. The control design is based on a sliding-mode uncertainty estimator developed under the framework of algebraic-differential concepts. The estimation convergence is done by the Lyapunov-type analysis and the closed-loop system stability is shown by means of the regulation error dynamics. Robustness of the proposed control scheme is tested in the face of noise output measurements and model uncertainties. The performance of the proposed control law is illustrated with numerical simulations in which a class of oscillatory chemical system is used as application example.

Keywords: I/O linearizing controller, sliding-mode observer, uncertainty estimation, noisy measurements.

RESUMEN
El objetivo de este artículo es presentar la síntesis de una ley de control robusta para una clase de sistemas no lineales denominados Liouvilianos. El diseño de control esta basado en un estimador de incertidumbres de modos deslizantes, desarrollado bajo el enfoque de conceptos algebraico-diferenciales. La convergencia del estimador se realiza mediante el método de Lyapunov y la estabilidad del sistema en lazo cerrado se demuestra mediante la dinámica del error de regulación. La robustez del esquema de control propuesto se determina tomando en cuenta la presencia de ruido en la salida del sistema e incertidumbres en el modelo. El desempeño de la ley de control propuesta se ilustra con simulaciones numéricas, donde se considera una clase de sistema químico oscilatorio como ejemplo.

1. Introduction

Since the early 1990s, some papers have been related with the dynamic characterization of a particular class of nonlinear systems named differentially flat [1,2] and Liouvillian systems [3], based on the frame of differential algebra. One of the most important aspects of this approach for this kind of systems is the explicit relationship that can be obtained for particular state variables; it is an advantage for a class of observation and control problems. Differential-algebra based techniques have been employed for differential algebraic as well as ordinary differential equations systems.

On the other hand, control of non-linear systems has been widely studied during the last 20 years, specially the characterization of input/output (I/O) and exact linearizable systems. This corresponds to systems that can be fully or partially linearized by a change of coordinates and/or state feedback [4], [5]. Such class of non-linear systems can be linearized by a state feedback control, which cancels all the nonlinearities assuming perfect knowledge of the mathematical model, producing global asymptotic stability [5]. A drawback of exact linearization technique is that it relies on complete cancellation of nonlinearities. In practice, precise knowledge of system dynamics is not possible. A more realistic situation is to know some nominal functions of the corresponding nonlinearities, which are employed in the control design. However, the use of nominal model nonlinearities can lead to performance degradation and even to closed-loop instability. In fact, when the systems posses strong nonlinearities, the standard linearizing controller cannot cancel completely such nonlinearities and instabilities can be induced. The worst case is when the knowledge of the nonlinearities is very poor or null such that conventional linearizing techniques are inadequate. In the face of these events, the robust stability problem for uncertain systems arises as a necessary control design approach to supply the controller with the corresponding on-line information and try to realize a satisfactory closed-loop performance. Research on robust control design for linearizable nonlinear systems has been done considering observer-based controllers [6] where peaking phenomena, stability issues and robust performance are still topics that deserve further study.

In recent works, it has been employed Luenberger-type observer structures to obtain on-line estimates of uncertain signals [7]. However, the resulting schemes become sensitive to measurement noise. Since measurement noise is propagated through the control loop, high frequency chattering can induce premature degradation of actuator (***e.g.***, valves) components. In this paper, a design of robust control law based on on-line uncertainty estimation is addressed; the robustness is referred to model uncertainties and noisy output measurements. The uncertainty estimator contains a sliding-mode structure and it is designed within the framework of algebraic theory. Subsequently, the uncertainty estimator is coupled with an input-output linearizing controller, which produces practical stability (*i.e.*, the closed-loop trajectories are forced to remain in a neighborhood of the operating equilibrium point). The performance of the robust control design is illustrated via numerical simulations.

2. Main definitions

The framework of the observer design for control purposes is based on capturing the input-output behavior of the system employing a set of equations generated by the system under study. The definitions presented in this section have been discussed previously in [8] and are summarized below for completeness.

Definition 1. A dynamics is defined as a finitely generated differentially algebraic extension $H/k\langle u\rangle$ of the differential field $k\langle u\rangle$, where $k\langle u\rangle$ denotes the differential field generated by k and elements of a finite set $u = (u_1, u_2, \ldots, u_n)$ of differential quantities.

Definition 2. A differential transcendence basis $y = (y_1, y_2, \ldots, y_m)$ of H/k such that $H = k\langle y\rangle$ is called linearizing or flat output of the system H/k.

Definition 3. The number of state variables, not permissible in terms of the flat outputs, is known as the defect of the non-flat system, that is, the integer number, which does the differential transcendence degree of H/k is minimal, is called algebraic defect of the system.

Definition 4. A system H/k is differentially flat if and only if its defect is zero. If its algebraic defect is non-zero, then the system H/k is said to be differentially non flat.

Definition 5. Let H/k be a given system and let M be such that $k \subset M \subset H$. Moreover, it is assumed that M/k is a flat subsystem of H/k, and then it can be said that H/k is Liouvillian if the elements of $H - M$ can be obtained by an adjunction of integrals or exponential of integrals of elements of the flat field M.

2.1 Example

Now, consider the following generic mathematical model of a class of continuous chemical reactors, wher the following chemical reaction is considered: $x_1 + x_2 \rightarrow$ Prod

- Mass Balance for reactive 1 (x_1):

$$\dot{x}_1 = f_1(x_1, x_2, x_3) \equiv a(\alpha - x_1) - x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) \qquad \text{(a)}$$

- Mass Balance for reactive 2 (x_2):

$$\dot{x}_2 = f_2(x_1, x_2, x_3) \equiv a(\delta - x_2) - x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) \tag{b}$$

- Energy Balance (x_3):

$$\dot{x}_3 = f_3(x_1, x_2, x_3) \equiv b(\rho - x_3) - d\, x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) - \gamma(u - x_3) \tag{c}$$

- Measured output:

$$y = C\,x \quad where \quad C = [0 \quad 0 \quad 1] \tag{d}$$

The above system will be expressed via differential-algebraic tools, based on the definitions given in Section 2, as a set of mapping in the variables x_i, y and u, which will be considered to describe the input-output behavior in the system and it is below used in the observer design procedure.

From system (a)-(d), and after algebraic manipulation, the following expressions are generated:

- Reactive 1

$$x_1 = g_1(y,u) \equiv \int \exp(a(t-\sigma))\, h_1(y(t),u)\, dt \tag{e}$$

Where:

$$h_1(y,u) = -\left[\frac{\dot{y} + \gamma(u-y) - b(\rho - y)}{d} - a\alpha\right]$$

- Reactive 2:

$$x_2 = g_2(y,u) \equiv \int \left(\left(\dot{y} + \gamma(u-y) - b(\rho - y)\right) \frac{a + g_1(y,u)\exp\left(\frac{-\beta}{y}\right)}{d\, g_1(y,u)\exp\left(\frac{-\beta}{y}\right)} - a\delta \right) dt \tag{f}$$

- Temperature:

$$x_3 = g_3(y,u) \equiv y \tag{g}$$

As can be observed $H = k\langle x_1, x_2, x_3, u\rangle$, $k = \Re$, the reactor model (a)-(d) is a nonlinear Liouvillian system, besides note that the state variables of the reactor x_1 and x_2 are observables from temperature measurements.

3. Problem statement

Non-linear approaches to design control laws have been tested successfully in theoretical research. In particular, the I/O linearizing technique shows attractive characteristics for the control of the non-linear systems.

To motivate the control problem, consider the following non-linear Liouvillian system, which represents the general mathematical model of a continuous stirred tank reactor (CSTR):

$$\dot{x}_1 = \theta\left(x_{1e} - x_1\right) - ER(x_1, x_2) \quad (1)$$

$$\dot{x}_2 = \theta\left(x_{2e} - x_2\right) + \Delta HR(x_1, x_2) + \gamma\left(u - x_2\right) \quad (2)$$

where x_1 is a n-dimensional vector of chemical species, $R(x_0, x_1)$ is a m-dimensional vector of reaction kinetics, ΔH is a m-dimensional vector of reaction enthalpies, ***E*** is the stoichiometric matrix, x_2 is the reactor temperature, ***u*** is the cooling jacket temperature, and *1/θ* and γ are the residence time and the heat-transfer global coefficient, respectively. If the reactor temperature x_2 is the controlled output, in compact form, the Liouvillian system (1)-(2) can be rewritten as follows:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + B(x_2)u \\ y &= h(x) = x_2 \end{aligned} \quad (3)$$

The ***zero-dynamics*** are given by the n-dimensional dynamics of the chemical species concentration at a constant temperature, which are assumed to be locally stable [9]. The study of relative-degree one systems is very important for many control applications, since the dynamics of a wide class of chemical reactors can be described in this form. Such systems are mathematically modeled as ***affine*** systems with respect to the control input [9]. Systems that present relative-degree one display some interesting features, such as the ***equivalent dissipativeness*** by means of state or output feedback. In general, it is easier to stabilize ***dissipative*** systems than ***non-dissipative*** ones [10].

In what follows, non-linear systems of the form (3) will be considered. In order to stabilize the system defined by Equation (3) via regulation of x_2, the following nominal I/O linearizing feedback control is proposed:

$$u = B(x_2)^{-1}\left[-\tau_g^{-1}e_3 - f_2\left(x_1, x_2\right)\right] \quad (4)$$

where $\tau_g > 0$ is a prescribed time-constant. As usual, $e_3 = y - y_{sp}$ and y_{sp} are tracking error and set point, respectively. The controller defined by Equation (4) guarantees asymptotic stability of non-linear systems (3) with no uncertainties and perfect measurements. Moreover, it imposes a linear behavior to the system I/O dynamics by canceling the nonlinearities.

4. Feedback controller design

As it can be noticed, the synthesis of the ideal control law requires accurate knowledge of the mathematical model of the process to be realizable. However, a perfect model is difficult or even impossible to be obtained in practice and, consequently, for uncertain systems, a conventional I/O linearizing controller design is not adequate.

Let us assume that x_1 and x_2 trajectories are bounded for all $t \geq 0$ (*i.e.*, the system is bounded input to bounded output state). The basis of the non-ideal controller design is the nominal control law (4). In order to design the practical robust control law, let us propose the following non-linear dynamic system representation:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + (\overline{B}(x_2) + \Delta B(x_2))u \\ y &= h(x) = x_2 \end{aligned} \quad (5)$$

The functions $f_2(x_1, x_2)$ and $\Delta B(x_2)$ are model uncertainties related to the non-linear system, and $\overline{B}(x)$ is a nominal value of the control input coefficient. In the most general case, the functions $f_2(x_1, x_2)$ and $\Delta B(x_2)$ are assumed to be unknown. Now, introduce the following function, which corresponds to the I/O modeling error:

$$\zeta(x,u) = f_2(x_1, x_2) + \Delta B(x_2)u \qquad (6)$$

By using (6) into (5), a new representation of the system is obtained:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= \zeta(x,u) + \overline{B}(x_2)u \\ y &= h(x) = x_2 \end{aligned} \qquad (7)$$

Since the uncertainty term, $\zeta(x,u)$, is an unknown function of the states and the control input, the ideal control law for the regulation of x_2 is not causal and therefore, it cannot be implemented in practice. Nevertheless, there is another way to develop an input-output linearizing controller that is robust against uncertainties. The procedure described below provides a method to estimate the uncertainty term, $\zeta(x,u)$. Estimators or observers for states and uncertainties can play a key role during the early detection of hazardous and unsafe operating conditions. Motivated by this, much research has focused on the proposition of estimation methodologies for states and uncertainties for monitoring and control purposes [11], [12].

4.1 The uncertainty estimation methodology

Consider the following dynamic subsystem:

$$\begin{aligned} \dot{x}_2 &= \zeta + \overline{B}(x_2)u \\ \dot{\zeta} &= \Phi(x,u) \\ y &= h(x) = x_2 \end{aligned} \qquad (8)$$

The uncertain term, $\zeta(x,u)$, is considered as a new state and $\Phi(x,u)$ is a non-linear unknown function that describes the ζ -dynamics.

In order to provide a background previous to the proposed estimation methodology, the following definitions are considered [8]:

Definition 6. Let $\{u, y\}$ be a subset of $\Im$ in a dynamics $\Im / k\langle u \rangle$. An element in $\Im$ is said to be observable with respect to $\{u, y\}$ if it is algebraic over $k\langle u, y \rangle$. Therefore, a state **x** is said observable if, and only if, it is observable with respect to $\{u, y\}$.

Definition 7. - An element $\mathbf{X_u}$ in $\Im$ is said to be an algebraically observable uncertainty if $\mathbf{X_u}$ satisfies a differential algebraic equation with coefficients over $k\langle u, y \rangle$.

Now, consider the system (3), which according to ***Definition 1*** defines an algebraic-differential dynamic system. From this subsystem, the following algebraic-differential equations can be obtained:

$$x_2 - y = 0 \qquad (9)$$

$$\dot{y} - \zeta(x,u) - \overline{B}(x_2)u = 0 \qquad (10)$$

Remark 1. From Definitions 6 and 7, it follows that the pair *(x_2,ζ)* is universally observable in the Diop-Fliess sense [8].

The corresponding Input-Output representation of Equations (9) and (10) can be rewritten in new coordinates as follows:

$$\eta_i = \frac{d^{i-1}Y}{dt^{i-1}} \qquad (i=1,2) \qquad (11)$$

$$\dot{\eta}_1 = \eta_2$$
$$\dot{\eta}_2 = \Phi\left(\eta_1, \eta_2, \dot{u}\right) \qquad (12)$$
$$Y = \eta_1$$

It should be noted that a partial change of coordinate enables us to estimate $\eta_1 = y$ and $\dot{\eta}_1 = \dot{y} = \eta_2$ (or, equivalently, x_2 and $\dot{x}_2$).

4.2 Measurement output noise considerations

Now, considering the noise case presence:

$$Y = \eta_1 + \delta$$

where δ is an additive bounded noise. Our aim is to design an observer to obtain η_2 (the uncertainty term in the transformed space). However, as it can be seen from the nature of the system given by Equation (8), a standard structure of an observer, based on a copy of the system plus measurement error correction is not realizable in this case since the term Φ is a priory unknown.

4.3 Sliding-Mode Observer

4.3.1 Observer Structure

Proposition 1. The following dynamic system is a *Sliding-mode asymptotic type observer* of the system (12) to estimate the variables η_1 and η_2, respectively:

$$\dot{\hat{\eta}}_1 = \hat{\eta}_2 + m\tau^{-1} sign\left(Y - \hat{Y}\right),\ m>0, \qquad (13)$$

$$\dot{\hat{\eta}}_2 = m^2 \tau^{-2} sign\left(Y - \hat{Y}\right), \qquad (14)$$

$$\hat{Y} = \hat{\eta}_1 + \delta \qquad (15)$$

and

$$sign(Y - \hat{Y}) := \begin{cases} 1 & if\ (Y - \hat{Y}) > 0 \\ -1 & if\ (Y - \hat{Y}) < 0 \\ undefined & if\ (Y - \hat{Y}) = 0 \end{cases}$$

Now, returning back to the original state space, in view of (6), the heat of reaction can be evaluated as:

$$\hat{\zeta} = -\hat{\eta}_2 - \theta\left(X_{2e} - \hat{\eta}_1\right) - \gamma\left(u - \hat{\eta}_1\right)$$

According to the variable change given by (7), the variable η_1 is the thermodynamic reactor temperature (system output). From the above equation for $\hat{\zeta}$, if temperature measurements are noisy, the noise would be transmitted to the estimation of the heat of reaction that may lead to poor performance in the estimation procedure. That is because it is necessary to filter the temperature measurements. This is the main reason why the structure of the proposed observer (13) - (14) makes sense.

4.3.2 Errors estimation dynamics

Now, let us define the following estimation errors:

$$e_1 = \eta_1 - \hat{\eta}_1 \qquad (16)$$

$$e_2 = \frac{\eta_2 - \hat{\eta}_2}{m} \qquad (17)$$

By (12) and (13-14), it follows that the estimation errors $\boldsymbol{e} = (e_1, e_2)^T$ verify the following ordinary differential equation:

$$\dot{e} = A_\mu e - Ksign\left(Ce + \delta\right) + \Delta f \qquad (18)$$

where:

$A_\mu = \begin{bmatrix} -\mu & m \\ 0 & -\mu \end{bmatrix}$, $\mu > 0$ is a regularizing parameter,

$K = m\tau^{-1}\begin{bmatrix} 1 \\ m\tau^{-1} \end{bmatrix}$, C=[1, 0] and $\Delta f = \begin{bmatrix} \mu e_1 \\ \frac{1}{m}\Phi + \mu e_2 \end{bmatrix}$ is an uncertainty term (or unmodelled dynamics term).

4.3.3 Main assumptions

A1. There exist nonnegative constants L_{0f}, L_{1f} such that for any e the following generalized quasi-Lipschitz (strip-bound) condition holds:

$$\|\Delta f\| \le L_{0f} + (L_{1f} + \|A_\mu\|)\|e\|$$

A2. The output noise is assumed to be bounded as $\|\delta\|_\Lambda^2 := \delta^T \Lambda \delta \le (\delta^+)^2 < \infty$ where Λ is a symmetric definite positive matrix playing role of a normalizing matrix (since different components of the output measurements may have a different physical nature).

A3. There exits a positive definite matrix $Q_0 = Q_0^T > 0$ such that the following matrix Riccati equation:

$$PA_\mu + A_\mu^T P + PRP + Q = 0$$

with $R := \Lambda_f^{-1} + 2\|\Lambda_f\|L_{1f} I$, $0 < \Lambda_f = \Lambda_f^T$ and $Q = Q_0 + 2(L_{1f} + \|A_\mu\|)^2 I$, has a positive definite solution $P = P^T > 0$.

A4. The gain matrix K is selected as $K = kP^{-1}C^T$ where k is a positive constant.

Comment 1. The algebraic Riccati equation in A3 has a positive definite solution if the matrix A_μ is stable (that is valid for any positive μ) and the following matrix inequality is fulfilled:

$$A_\mu^T R^{-1} A_\mu - Q > \frac{1}{4}\left[A_\mu^T R^{-1} - R^{-1} A_\mu\right] R \left[A_\mu^T R^{-1} - R^{-1} A_\mu\right]^T$$

In our case this inequality may be transformed to the following one:

$$G_\mu := A_\mu^T R^{-1} A_\mu - 2(L_{1f} + \|A_\mu\|)^2 I - \frac{1}{4}\left[A_\mu^T R^{-1} - R^{-1} A_\mu\right] R \left[A_\mu^T R^{-1} - R^{-1} A_\mu\right]^T > Q_0$$

Hence, the matrix Q_0 providing the existence of the solution to the Riccati equation, always exists if

$G_\mu > 0$.

4.3.4 Lyapunov-type Analysis

Let us define the Lyapunov function candidate $V(e)$ as

$$V(e) = \|e\|_P^2 := e^T Pe, \quad 0 < P = P^T \in \Re^{nxn} \quad \dot{e} = A_\mu e - Ksign(Ce + \delta) + \Delta f \tag{19}$$

From (15) and using the matrix inequality

$$X^T Y + Y^T X \le X^T \Lambda_f X + Y^T \Lambda^{-1}{}_f Y$$

valid for any $X, Y \in R^{n\times m}$, $0 < \Lambda_f = \Lambda^T{}_f$, it follows:

$$\dot{V}(e) = 2e^T P\dot{e} = 2e^T P\left[A_\mu e - Ksign\left(Ce+\delta\right)+\Delta f\right]$$

$$\begin{aligned}
&= 2e^T PA_\mu e - 2Ke^T C^T sign\left(Ce+\delta\right)+2e^T P\Delta f \\
&\le e^T\left(PA_\mu + A_\mu^T P\right)e - 2Ke^T C^T sign\left(Ce+\delta\right)+e^T P\Lambda_f^{-1}Pe+\left(\Delta f\right)^T \Lambda_f \Delta f \\
&\le e^T\left(PA_\mu + A_\mu^T P + P\Lambda_f P + Q\right)e - e^T Qe \\
&\quad + 2\left(L_{0f}^2 + \left[L_{1f} + \left\|A_\mu\right\|\right]^2 \left\|e\right\|^2\right)\left\|\Lambda_f\right\| - 2Ke^T C^T sign\left(Ce+\delta\right) \\
&= e^T\left(PA_\mu + A_\mu^T P + PRP + Q\right)e - e^T Qe \\
&\quad + 2\left\|\Lambda_f\right\| L_{0f}^2 - 2K\left(Ce\right)^T sign\left(Ce+\delta\right)
\end{aligned} \tag{20}$$

The main assumption consists in the implementation of the following inequalities valid for each component:

$$\begin{aligned}
&x^T \; sign\left(x+z\right) = \left(x+z\right)^T sign\left(x+z\right) - z^T sign\left(x+z\right) \\
&\ge \sum_{i=1}^{n}\left|(x+z)_i\right| - \sum_{i=1}^{n}\left|z_i\right| \ge \sum_{i=1}^{n}\left|x_i\right| - 2\sum_{i=1}^{n}\left|z_i\right| \ge \sum_{i=1}^{n}\left|x_i\right| - 2\sqrt{n}\left\|z\right\|
\end{aligned} \tag{21}$$

Here:

$$\left|\left(x+z\right)_i\right| \ge \left|x_i\right| - \left|z_i\right| \tag{22}$$

and

$$\sum_{i=1}^{n}\left|z_i\right| \le \sqrt{n}\left\|z\right\| \tag{23}$$

The last results from the Cauchy-Bounyakowski-Schwarz inequality:

$$\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \tag{24}$$

for $a_i := n^{-1}$, $b_i := \left|z_i\right|$. Applying (21) to (20) and taking into account A3, it follows:

$$\begin{aligned}
\dot{V}(e) &= -e^T Qe + 2\left\|\Lambda_f\right\| L_{0f}^2 - 2K\left(Ce\right)^T sign\left(Ce+\delta\right) \\
&\le -e^T Qe + 2\left\|\Lambda_f\right\| L_{0f}^2 - 2K\left(\sum_{i=1}^{n}\left|\left(Ce\right)_i\right| - 2\sqrt{n}\left\|\delta\right\|\right) \\
&\le -e^T Qe - 2K\sum_{i=1}^{n}\left|\left(Ce\right)_i\right| + \rho(K)
\end{aligned} \tag{25}$$

where $\rho(K) := 2\left\|\Lambda_f\right\| L_{0f}^2 + 4K\sqrt{n\left\|\Lambda^{-1}\right\|}\,\delta^+$. (26)

Since $\|\delta\| = \sqrt{\delta^T \Lambda^{1/2}\Lambda^{-1}\Lambda^{1/2}\delta} \le \sqrt{\left\|\Lambda^{-1}\right\|\delta^T\Lambda\delta} = \sqrt{\left\|\Lambda^{-1}\right\|\,\|\delta\|_\Lambda^2}$ and

$$\left(\sum_{i=1}^{n}\left|(Ce)_i\right|\right)^2 \ge \sum_{i=1}^{n}\left[\left|(Ce)\right|^2\right] = \|Ce\|^2 = \left\|CP^{-1/2}P^{1/2}e\right\|^2 \ge \alpha_P e^T Q e \tag{27}$$

with $\alpha_P := \lambda_{\min}\left(P^{-\frac{1}{2}}C^T C P^{-\frac{1}{2}}\right) \ge 0$.

Then, (25) implies

$$\begin{aligned}\dot V(e) &= \frac{d}{dt}\left(\|e\|_P^2\right) \le -\|e\|_Q^2 - 2K\alpha_P\|e\| + \rho(K)\\ &\le -\alpha_Q V(e) - \vartheta\sqrt{V(e)} + \beta\end{aligned} \tag{28}$$

where $\alpha_Q := \lambda_{\min}\left(P^{-\frac{1}{2}}Q^T Q P^{-\frac{1}{2}}\right) > 0$, $\vartheta := 2K\alpha_P$, and $\beta := \rho(K)$.

At this point we are ready to formulate the main result.

Theorem 1. If the assumptions ***A1*** - ***A3*** are satisfied then

$$\left[1 - \frac{\tilde\mu}{V}\right]_+ \to 0 \tag{29}$$

where $\tilde\mu = \tilde\mu(K) := \left(\frac{\rho(K)}{\sqrt{(K\alpha_P)^2 + \rho(K)\alpha_Q} + K\alpha_P}\right)^2$, and the function $[\bullet]_+$ is defined as

$$[Z]_+ := \begin{cases} Z & if \quad Z \ge 0 \\ 0 & if \quad Z < 0 \end{cases}$$

♦

4.3.5 Proof of Theorem 1

Consider the Lyapunov function *V(e)* verifying the differential equation:

$\dot V = -\alpha V - \vartheta\sqrt{V} + \beta$

The equilibrium point *V** of this equation, satisfying

$-\alpha V - \vartheta\sqrt{V} + \beta = 0$, Defining $\Delta := (V - V^*)^2$. The time derivative is given by:

$$\begin{aligned}\dot{\Delta} = 2(V - V^*)\dot{V} &\leq 2(V - V^*)\left(-\alpha V - \vartheta\sqrt{V} + \beta\right) \\ &= 2(V - V^*)\left(-\alpha V - \vartheta\sqrt{V} + \beta + \left(\alpha V^* + \vartheta\sqrt{V^*} - \beta\right)\right) \\ &= 2(V - V^*)\left(-\alpha(V - V^*) - \vartheta\left(\sqrt{V} - \sqrt{V^*}\right)\right) \\ &= -2\alpha(V - V^*)^2 - 2\vartheta\left(\sqrt{V} + \sqrt{V^*}\right)\left(\sqrt{V} - \sqrt{V^*}\right)^2 \leq 0\end{aligned}$$

for any $V \neq V^*$, that implies: $\lim_{t\to\infty} V \to V^*$. For

$$G := [V - \tilde{\mu}]_+^2 = V^2\left[1 - \frac{\tilde{\mu}}{V}\right]_+^2$$

we obtain:

$$\dot{G}_t := 2[V - \tilde{\mu}]_+\dot{V} = 2V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\dot{V} \leq$$

$$2V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\left(-\alpha V - \vartheta\sqrt{V} + \beta\right) =$$

$$-2V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\left[\alpha(V - V^*) + \vartheta\left(\sqrt{V} - \sqrt{V^*}\right)\right] \leq 0$$

The last inequality implies that G_t converges, that is,
$G_t \to G^* < \infty$

The integration of the last inequality from 0 to t yields

$$G_t - G_0 \leq$$

$$-2\int_0^t V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\left[\alpha(V - V^*) + \vartheta\left(\sqrt{V} - \sqrt{V^*}\right)\right]d\tau$$

That leads to the following inequality

$$2\int_0^t V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\left[\alpha(V - V^*) + \vartheta\left(\sqrt{V} - \sqrt{V^*}\right)\right]d\tau \leq G_0 - G_t \leq G_0$$

Dividing by t and taking the upper limits of both sides, we obtain:

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t V\left[1 - \frac{\tilde{\mu}}{V}\right]_+\left[\alpha(V - V^*) + \vartheta\left(\sqrt{V} - \sqrt{V^*}\right)\right]d\tau \leq 0$$

and there exists a subsequence t_k such that:

$$V_{t_k}\left[1-\frac{\tilde{\mu}}{V_{t_k}}\right]_+ \left[\alpha\left(V_{t_k}-V^*\right)+\vartheta\left(\sqrt{V_{t_k}}-\sqrt{V^*}\right)\right]\to 0$$
$$G_{t_k}\underset{k\to\infty}{\to} 0$$

Hence, it follows that $G^* = 0$; that is equivalent to the fact:

$$\left[1-\frac{\tilde{\mu}}{V_t}\right]_+ \to 0$$

The theorem is proven.

Remark 2. Theorem 1 actually states that the weighted estimation error $V = e^T Pe$ converges to the zone $\tilde{\mu}$ asymptotically, that is, it is ultimately bounded, such that $\|V\| \le \Omega$.

The final expression for the input-output non-ideal linearizing controller with uncertainty estimation can be obtained, introducing the estimate of the uncertain term in Eq. (4), to generate:

$$u = \bar{B}(x_2)^{-1}\left[-\tau_g^{-1}e_3 - \hat{\zeta}\right] \qquad (30)$$

Since the proposed controller uses estimated values of the uncertainty, it cannot cancel the system nonlinearities completely. Thus, the system trajectory remains inside a neighborhood close to the set point. Practical stability is achieved as long as the uncertainty estimation error is bounded. The restraint of the boundeness of the heat of reaction (uncertain term) is common for a wide class of chemical reactions and is consequence of characteristics of the mathematical modeling commonly employed; chemical reactions are usually Lipschitz with respect to temperature. It is not hard to see that global Lipschitz property of $\Delta HR(x_0, x_1)$ is found if the functionality $R(x_0, x_1)$ with respect to temperature is of Arrhenius-type.

Notice that it is not hard to implement in standard technology (***e.g.***, PLCs) the practical controller given by Eqs. (14) to (17). In fact, the implementation only requires output measurements and the on-line solution of two quite simple dynamical systems (14) and (15). Moreover, the implementation effort is equivalent to other control strategies, such as PI and predictive control. As a matter of fact, standard (industrial) predictive control is more complex than the proposed one, since the former requires implementation of a non-linear optimization method.

4.4 Closed-loop stability analysis

In order to analyze the closed-loop stability of the reactor temperature trajectories in the reactor, the closed-loop dynamic equation of the energy balance should be used.

$$\dot{e}_3 = ge_3 + \left(\zeta - \hat{\zeta}\right) \qquad (31)$$

If $\hat{\zeta} \to \zeta$ then $\zeta - \hat{\zeta} \to 0$, the ideal control law is recovered together with its stability properties; otherwise, the estimation error is limited as $\left\|\zeta - \hat{\zeta}\right\| \le \alpha\sqrt{\Omega} = \Pi$, accordingly with the above development.

A4. - If λ_1, λ_2,.., λ_k are the distinct eigenvalues of the matrix *A,* where λ_j has multiplicity n_j and $n_1 + n_2 + \ldots + n_k = n$ and ρ is any number larger than the real part of λ_1, λ_2,.., λ_k, that is $\rho > \max(\Re e(\lambda_j))$, then there exists a constant $j > 0$ that satisfies:

$$\left\|\exp(mAt)e_3\right\| \le j\exp(-m\rho t)\left\|e_3\right\|$$

Solving Eq. (16), the error can be expressed as:

$$e_3 = \exp(mAt)e_{30} + \int_0^t \exp\{mA(t-s)\}\left(\zeta - \hat{\zeta}\right)ds$$

$$e_3 = \exp(mAt)e_{30} + \int_0^t \exp\{mA(t-s)\}\left(\zeta - \hat{\zeta}\right)ds \qquad (32)$$

Considering the assumptions ***A1*** and ***A2***, it is possible to find a bound for the closed-loop system (Eq. (18)).

$$\|e_3\| \leq j\,\exp(-m\rho t)\left[\|e_{30}\| - \frac{j\Pi}{m^2\rho}\right] + \frac{j\Pi}{m^2\rho} \tag{33}$$

Taking the limit when $t \to \infty$:

$$\|e_3\| \leq \frac{j\Pi}{m^2\rho} \tag{34}$$

The above inequality implies that the closed-loop error can be made as small as desired, if the observer parameter *m* is chosen large enough.

5. Application example

The chemical reactor model proposed as application example shows periodic or even chaotic dynamic behavior depending on the set of parameters employed [13]. The reactor temperature is regulated by means of water flowing through a cooling jacket. A consecutive chemical reaction scheme is considered here, where a stream with a reactive *A* enters into the continuous reactor and it is converted to an intermediate product *B*, with *rate*$_1$, which reacts to transforming to the final product *C*, with *rate*$_2$, such that:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

where: a is the reactant *A* concentration, b is the reactant *B* concentration and k_i is the rate constant for reaction *i*.

Both consecutive reactions are of first order with exothermic chemical reactions and the kinetic constant is modeled by the classical Arrhenius model to include the temperature dependence, as follows:

$$k_i = A_i \exp\left(-\frac{Ea_i}{RT}\right) \quad for \quad i = 1,2$$

Via a standard mass and energy balances the following ordinary differential equations are obtained:

$$\frac{da}{dt} = \frac{1}{t_{res}}(a_0 - a) - k_1 a \tag{35}$$

$$\frac{db}{dt} = \frac{1}{t_{res}}(b_0 - b) + k_1 a - k_2 b \tag{36}$$

$$Cp\rho\frac{dT}{dt} = \frac{1}{t_{res}}Cp\rho(T_0 - T) + (-\Delta H_1)k_1 a + (-\Delta H_2)k_2 b - \frac{1}{V}UA_c(T - T_c) \tag{37}$$

The following conditions of the reactor model are imposed; there is not inflow in *B* and *C*, both reactions have the same reaction heats, the same activation energy and the inflow reactor temperature is the same as the cooling jacket device. In accordance with the model structure proposed above, the following system representation is done:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \zeta_1(X) \\ \zeta_2(X) \\ \zeta_3(X) \end{bmatrix} + \begin{bmatrix} \ell_1 & 0 & 0 \\ 0 & \ell_2 & 0 \\ 0 & 0 & \ell_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{38}$$

where:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} a & b & T \end{bmatrix}$$

$$\begin{bmatrix} \zeta_1(X) \\ \zeta_2(X) \\ \zeta_2(X) \end{bmatrix} = \begin{bmatrix} k_1 a \\ k_1 a - k_2 b \\ \frac{1}{t_{res}} Cp\rho(T_0 - T) + (-\Delta H_1)k_1 a + (-\Delta H_2)k_2 b - \frac{1}{V} UA_c T \end{bmatrix}$$

$$\ell_1 = \frac{1}{t_{res}}(a_0 - a)$$

$$\ell_2 = \frac{1}{t_{res}}(b_0 - b)$$

$$\ell_3 = \frac{1}{V} UA_c$$

and finally,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ T_c \end{bmatrix}$$

Now, applying the mean residence time and the reactive *A* concentration as the time a concentration scales, the following set of dimensionless mass and energy balance equations is obtained, as presented in [13]:

$$\frac{d\alpha}{d\tau} = 1 - \alpha - \frac{1}{\tau_{ch}} \alpha \exp(\theta) \tag{39}$$

$$\frac{d\beta}{d\tau} = \frac{1}{\tau_{ch}} \alpha \exp(\theta) - \frac{1}{\tau_{ch}} \phi \beta \exp(\theta) - \beta \tag{40}$$

$$\frac{d\theta}{d\tau} = \frac{1}{\tau_{ch}} \theta_j \alpha \exp(\theta) + \frac{1}{\tau_{ch}} \theta_j \phi \beta \exp(\theta) - \left(1 + \tau_N^{-1}\right)\theta \tag{41}$$

The corresponding dimensionless concentrations and temperature are the follows:

$$\alpha = \frac{a}{a_0};\ \beta = \frac{b}{a_0}$$

$$\theta = \frac{E_a(T - T_c)}{RT_c^2};\ \tau = \frac{t}{t_{res}}$$

The parameters related with the named *chemical* time, reaction rate ratio, dimensionless temperature of the cooling jacket and the Newtonian cooling time are $\tau_{ch} = 1/(k_1 t_{res})$, $\phi = A_2 / A_1$, $\theta_j = -\Delta H_1 a_0 Ea_1 / \left(Cp\rho RT_c^2\right)$ and $\tau_N = t_N / t_{res} = Cp\rho V / (UA_c t_{res})$, respectively:

An important characteristic of this reactor model is its minimum phase behavior, i.e., the corresponding inner dynamic when the temperature of the reactor is regulated is stable as is proved below.

As mentioned above, given that the controller regulates only x_3 , the analysis of the inner dynamics is related with closed-loop behavior of x_1 and x_2 while x_3 is kept constant. Therefore, the system (39)-(41) is reduced to:

$$\dot{x}_1^* = 1 - (1+\delta_1)x_1^*$$
$$\dot{x}_2^* = \delta_1 x_1^* - (1+\delta_2)x_2^* \tag{42}$$

where:

$$\delta_1 = \frac{\exp(\theta_{sp})}{\tau_{ch}}; \quad \delta_2 = \frac{\phi \exp(\theta_{sp})}{\tau_{ch}}$$

now, solving (42) for x_1 and x_2:

$$x_1^* = \left[x_{10}^* - \left(\frac{1}{1+\delta_1}\right)\right]\exp(-\{1+\delta_1\}\tau) + \left(\frac{1}{1+\delta_1}\right)$$
$$x_2^* = \left[x_{20}^* - (\delta_3+\delta_4)\right]\exp(-\{1+\delta_1\}\tau) + \delta_3 \exp(-\{1+\delta_1\}\tau) + \delta_4 \tag{43}$$

where:

$$\delta_3 = \left(\frac{1}{\phi - 1}\right)\left(x_{10}^* - \left[\frac{1}{1+\exp(\theta_{sp})/\tau_{ch}}\right]\right)$$
$$\delta_4 = \frac{\exp(\theta_{sp})/\tau_{ch}}{(1+\exp(\theta_{sp})/\tau_{ch})(1+\phi\exp(\theta_{sp})/\tau_{ch})}$$

from Equations (43), the reactor inner dynamic is asymptotically stable such that:

$$\lim_{\tau\to\infty} x_1^* = \frac{1}{(1+\exp(\theta_{sp}))/\tau_{ch}} = \alpha_{eq}$$
$$\lim_{\tau\to\infty} x_2^* = \frac{\exp(\theta_{sp})/\tau_{ch}}{(1+\exp(\theta_{sp})/\tau_{ch})(1+\phi\exp(\theta_{sp})/\tau_{ch})} = \beta_{eq}$$

Numerical simulations for the closed-loop system were performed in order to show the properties of the control scheme proposed. The set of parameters of the chemical reactor are chosen as in [14], and initial conditions of the differential Equations (9-11) are $[x_1 = 0.45, x_2 = 0.1, x_3 = 0.9]$. For comparison purposes, an ideal I/O linearizing control, standard sliding-mode and high order sliding-mode controllers are implemented too. The temperature set point is $\theta_{sp} = 3$, and the nominal value of the control input is $u_0 = 17.5$; the controller is tuned-on at $t = 15$, the order of the high order sliding-mode controller is considered as $p = 3$. The temperature measurements are corrupted with a ± 5% around the current temperature value.

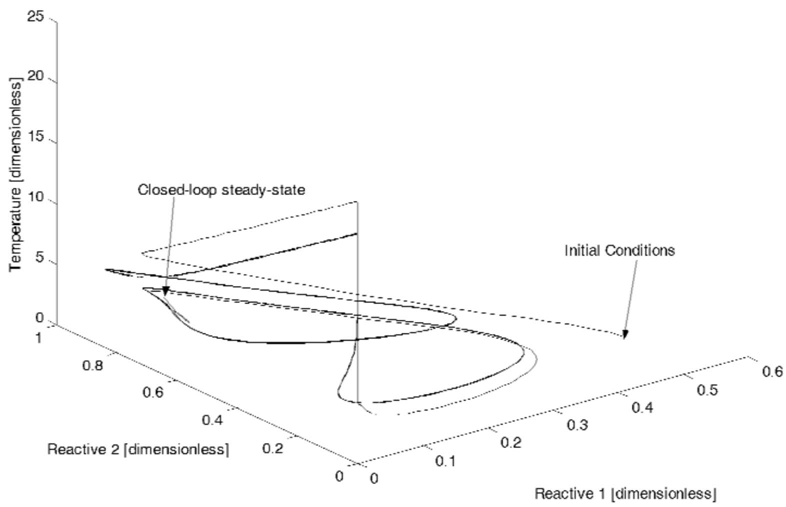

Figure 1. Closed-loop space portrait for the dimensionless state variables, a (Reactive 1), b (Reactive 2) and temperature.

Figure 1 shows the closed-loop behavior in steady state of the corresponding space portrait; note the oscillatory behavior. All the variables in the figures in this work are dimensionless.

Closed loop performance of temperature trajectories show that the ideal I/O linearizing controller shows the best performance so that it cancels the nonlinearities, imposing a desired linear behavior, with a satisfactory performance. The proposed controller tries to compensate the nonlinear terms via the integral high-order sliding-mode contribution, besides it is able to reach the set point value required (Figure 2), exhibiting smaller oscillations around the regulated point ($\theta_{sp} = 3$) than the other sliding-mode controllers. As predicted by the theoretical frame presented, sliding-mode and high order sliding-mode controllers can suppress nonlinear oscillations; however, both controllers exhibit a considerable off-set from the corresponding set point.

Figure 3 shows the corresponding estimation of the uncertainty. Another important difference is that effort performed by the manipulate variable (Figure 4) is very different for each controller. As it can be noticed, the I/O linearizing controller posseses the best performance, the sliding-mode controller exhibits the second smoothest behavior, followed by the high-order sliding-mode control, which exhibits more demanding effort at the start up of the regulation task. Finally, the proposed methodology shows the more demanding control action, where small oscillations are presented.

Comparing the performance and control effort, it is possible to note that the high-order sliding mode control is not very efficient because the regulated variable (temperature) exhibits the largest off-set, even when the effort is higher than in the case of the sliding-mode controller; nevertheless, the performance of the sliding-mode controller is not satisfactory because the set point are not reached. an be notice that the ideal I/O linearizing controller and the proposed controller are able to reach the corresponding set point, in some sense this is an advantage for the proposed methodology because in the case of this controller, the perfect knowledge of the model of the process is nor considere and its possible implementation looks more feasible than the ideal I/O controller.

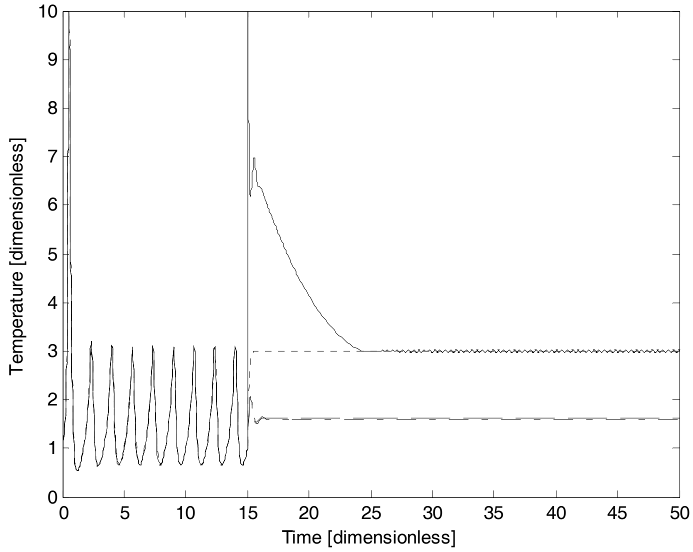

Figure 2. Closed-loop performance of the dimensionless temperature trajectories.
(___ Proposed Controller; ideal I/O Linearizing Controller; _ _ _ _ High-order Sliding-mode Controller; .... Sliding-mode controller).

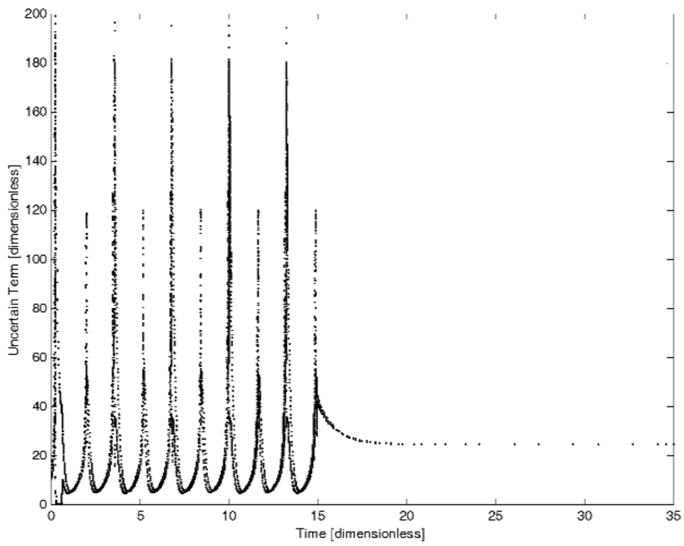

Figure 3. Estimation of the uncertain term (dimensionless reaction heat).

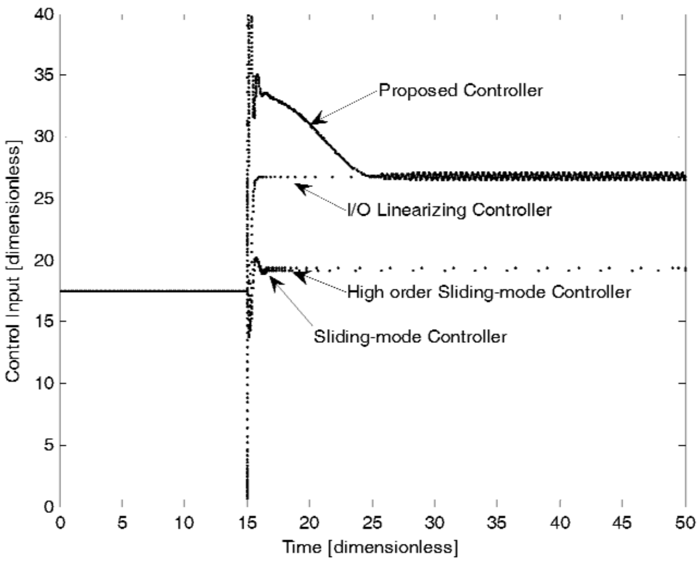

Figure 4. Practical effort of the controllers considered.

However, it is important to mention that the value of the control gains has to be chosen very carefully for the proposed methodology, such that with smaller values of control gains it is not possible to stabilize the oscillatory behavior of the chemical reactor, whereas for large values of these parameters, it is possible to lead to unacceptable control efforts or, even worse, to provoke additional closed-loop instabilities.

Besides, in order to measure the impact of the error, the "Integral Time-Weighted Squared Error" (ITSE) defined by (44) suggested by Ogunnaike and Ray [15] is employed. ITSE exhibits the advantage of heavy penalization of large errors at long time; therefore, is a good measure of resilience of the controller.

$$ITSE = \int_0^\infty t\,\varepsilon^2\,dt \tag{44}$$

In order to compare the resilience of the controllers simulated, the ITSE was evaluated for the dynamic system under the influence of four controllers

(Figure 5). As it is possible to note, and confirming the findings from Figure 2, the I/O linearizing and the proposed controller are the only two able to stabilize the system in the long time ($t > 20$), whereas for sliding-mode and high order sliding- mode controllers this error increases in an unlimited way. This result is due to the ability of the controller proposed to eliminate offset, property that is not exhibited by the other controllers (Figure 2).

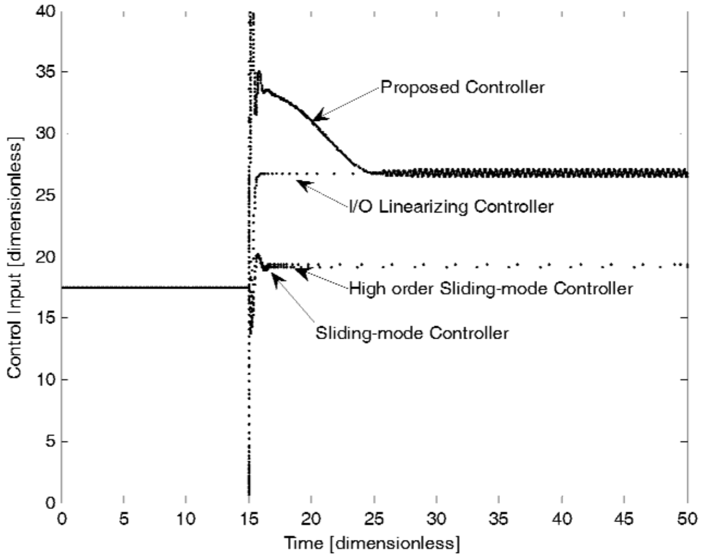

Figure 5. Evaluation of the ITSE for the controllers considered.
(___ Proposed Controller; …… ideal I/O Linearizing Controller;
High-order Sliding-mode Controller; Sliding-mode controller).

6. Conclusions

In this paper, a sliding-mode observer based I/O linearizing control law is designed to regulate the temperature of a class of chemical reactor. It is considered that the reaction heat is unknown and noisy measurements exists such that an uncertainty estimation methodology is based on algebraic-differential concepts and a sliding-mode frame; it is proposed to be coupled with the considered controller to induce robust properties, against the model's uncertainties and output measurements disturbances. The proposed controller is very simple since it is composed by a linearizing feedback coupled with a first-order sliding-mode observer. Besides, its implementation only requires measurements of the system output. The performance of the proposed methodology is adequate in comparison with an ideal I/O linearizing controller, standard sliding-mode controller and high order sliding-mode controller. Numerical simulation was carried out showing the above mentioned.

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A chaotic system in synchronization and secure communications

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ABSTRACT

In this paper we deal with the synchronization and parameter estimations of an uncertain Rikitake system and its application in secure communications employing chaotic parameter modulation. The strategy consists of proposing a receiver system which tends to follow asymptotically the unknown Rikitake system, refereed as transmitter system. The gains of the receiver system are adjusted continually according to a convenient high order sliding-mode adaptative controller (HOSMAC), until the measurable output errors converge to zero. By using HOSMAC, synchronization between transmitter and receiver is achieved and message signals are recovered. The convergence analysis is carried out by using Barbalat's Lemma.

1. Introduction

The synchronization of chaotic systems has been investigated since its introduction in the paper by Pecora and Carrol in 1990 [1]. Synchronization of chaotic systems has attracted much attention due to its potential applications in secure communications, chemical reactions, biological systems and so on [2–5]. Several synchronization schemes have been proposed to tackle the problem [2,6]. In the last years, some methods to achieve synchronization have been proposed from the control theory perspective such as the observer-based approach [7,8], the so-called adaptive synchronization method [9–11] and so on [12–14]. For the synchronization problem, we consider a chaotic system, the master (or transmitter), together with the slave (or slave). The goal is to synchronize the complete response of the slave system to the master system by driving the slave with a signal derived from the master. The problem can now be easily tackled when the parameters of the master system are known. The aforementioned methods and many others are valid for chaotic systems only when the system's parameters are known. However, to achieve synchronization between two chaotic systems is far from being straightforward, in fact there is no much work about this challenging problem because it consists of both identification of the unknown parameters and the design of a controller to achieve synchronization. Guan et al. applied an observer to identify the unknown parameter of the Lorenz system [15]. Lü et al. studied the same problem for Chen's chaotic system with the same method [16]. The interest in parameters identification lies on its potential applications in communications, essentially when *parameter modulation* is used for message transmission. This is an important issue in this work since we will apply chaotic parameter modulation [17] when parameters of Rikitake system[1] are used to transmit message signals.

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[1] This system resembles the reversal of polarity of the Earth's electromagnetic field, and it is well-known that it has a chaotic behavior for some set of initial conditions and some set of parameter values.

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The general idea for transmitting information via chaotic systems is that, an information signal is embedded in the transmitter system which produces a chaotic signal, the information signal is recovered by the receiver. Synchronization can be classified into mutual synchronization (or bidirectional coupling) [18] and master–slave synchronization (or unidirectional coupling) [1]. The chaos-based secure communications have updated their fourth generation [19]. The continuous synchronization is adopted in the first three generations while the impulsive synchronization is used in the fourth generation. Less than 94 Hz of bandwidth is needed to transmit the synchronization signal for a third-order chaotic transmitter in the fourth generation while 30-kHz bandwidth is needed to transmit the synchronization signals in the other three generations [20].

There are many applications to chaotic communication [21,22] and chaotic network synchronization [23]. The techniques of chaotic communication can be divided into three categories (a) chaos masking [24], the information signal is added directly to the transmitter; (b) chaos modulation [25], it is based on the master–slave synchronization, where the information signal is injected into the transmitter as a nonlinear filter; (c) chaos shift keying [26], the information signal is supposed to be binary, and it is mapped into the transmitter and the receiver. In these three cases, the information signal can be recovered by a receiver if the transmitter and the receiver are synchronized.

In this paper is presented an adaptative asymptotic method for the synchronization, the identification of the Rikitake system with several unknown parameters and an application in secure communications via chaotic parameter modulation. By this method, we can achieve chaos synchronization, identify the unknown parameters, and recover message signals simultaneously. Roughly speaking the suggested approach consists of designing a controlled slave system by means of a high order sliding mode adaptive controller (HOSMAC), whose adaptive parameters are adjusted accordingly to a proposed adaptive algorithm. It is done in such a way that the synchronization errors between the outputs of both systems, the uncertain Rikitake and the slave, asymptotically converge to zero. The convergence analysis of the proposed scheme is carried out using the Lyapunov method in conjunction with the Barbalat's Lemma.

The remaining of this work is organized as follows. In Section 2 we introduce the problem statement. In Section 3 we develop our solution to synchronize, identify the constant unknown parameters of the Rikitake system and recover message signals. To assess the effectiveness of our method we present some numerical simulations in Section 4. Finally, we present the conclusions in Section 5.

2. Problem statement

In normal chaotic communication, the transmitter and the receiver are chaotic systems. We discuss a case, the transmitter and the receiver are third order chaotic oscillators.

2.1. Transmitter

In this paper, all results are based on Rikitake system [27], however, this technique can be applied to any chaotic systems such as Chua's circuit, Chen's circuit, and so on, satisfying Definitions 1 and 2 given in sub Section 2.2. Rikitake system is a simple mechanical model used to study the reversals of the magnetic field of the Earth, idealized by the Japanese geophysicist Rikitake [27], consists of two identical single Faraday-disk dynamos of the Bullard type coupled together. The dynamics of this system is governed by the following three dimensional system of nonlinear differential equations:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + z_1 y_1, \\ \dot{y}_1 &= -\mu y_1 + (z_1 - a)x_1, \\ \dot{z}_1 &= 1 - x_1 y_1, \end{aligned} \tag{1}$$

where the parameters μ and a have some physical meaning when they are positive. For a physical meaning of the states x_1, y_1 and z_1 we recommend to see [27]. However, the states x_1 and y_1 are directly related to the currents through each disc of the dynamo system, and z_1 is related to the angular velocity of one of the discs. This system displays a chaotic behavior for the parameters values in a neighborhood $\{\mu = 2, a = 5\}$ and for a large enough set of initial conditions.

2.2. Some algebraic properties

In this section we present some algebraic properties that the Rikitake System satisfies. To this end we introduce the following definitions.

Definition 1. Consider a smooth nonlinear system, described by a state vector $X = \{x_i\}_1^{i=n} \in R^n$ and by the output vector $G = \{g_i\}_1^{i=m} \in R^m$, of the form:

$$\dot{X} = f(X, P), \quad G = h(X), \tag{2}$$

where $h(\cdot)$ is a smooth vector function and $P \in R^l$ is a constant parameters vector, with $l < n$. Let $G^{(j)}$ denote the jth time derivative of the vector G. We say that the vector state X is algebraically observable, if it can be uniquely expressed as

$$X = \Phi(G, G^{(1)}, \ldots, G^{(j)})$$

for some integer j and for some smooth function Φ.

Definition 2. Under same conditions as in Definition 1. If the vector of parameters, P satisfies the following relation

$$\Omega_1(G,..,G^{(j)}) = \Omega_2(Y,\ldots,Y^{(j)})P, \tag{3}$$

where $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are, respectively, $n \times 1$ and $n \times n$ smooth matrices, then P is said to be algebraically identifiable with respect to the output vector G [28].

According to the previous definitions, it is evident that system (1) is algebraically observable with respect to the outputs $g_1 = x_1$ and $g_2 = y_1$, since the state, z_1, can be rewritten, as

$$z_1 = \frac{\dot{g}_2 - \mu g_2}{g_1} + a, \tag{4}$$

hence, Rikitake system is algebraically observable with respect to the selected outputs, $g_1 = x_1$ and $g_2 = y_1$. Moreover, substituting the above expression into the first differential equation of (1), we have

$$\dot{g}_1 g_1 - \dot{g}_2 g_2 = -\mu(g_1^2 + g_2^2) + a g_1 g_2. \tag{5}$$

Therefore, we conclude that system (1) vector of parameters $\mathbf{p} = (\mu, a)$ is algebraically identifiable with respect to the available outputs. That is, the non-available state z_1 and the vector of parameters $\mathbf{p}$ can be simultaneously recovered, from the knowledge of the outputs $g_1 = x_1$ and $g_2 = y_1$.

From the above definitions, it is possible to solve the synchronization problem of the uncertain Rikitake system, provided that the states x_1 and y_1 are always available. Moreover, it is also possible to recover the unknown parameters μ and a. Thus, we are ready to establish the main control problem of this work.

2.3. Receiver

Consider the uncertain Rikitake system (1), referred as the transmitter system, with the available output states $x_1$and y_1. And let us propose the following receiver controlled system:

$$\begin{aligned} \dot{x}_2 &= -\hat{\mu}x_1 + z_2 y_1 + u_1, \\ \dot{y}_2 &= -\hat{\mu}y_1 + (z_2 - \hat{a})x_1 + u_2, \\ \dot{z}_2 &= 1 - x_1 y_1 + u_3. \end{aligned} \tag{6}$$

Then, the control objective is to find $\mathbf{u} = (u_1, u_2, u_3)$ and $\hat{\mathbf{p}} = (\hat{\mu}, \hat{a})$ such that the slave system (6) follows to the unknown Rikitake system (6); with $\hat{\mathbf{p}}$ converging to the actual values of (μ, a). In other words, we need to find $\mathbf{u}$ and $\hat{\mathbf{p}}$ of system (6), such that $(\mathbf{w}_1, \hat{\mathbf{p}}) \to (\mathbf{w}_2, \mathbf{p})$, as long as $t \to \infty$.[2]

2.4. Transmission of message signals by chaotic parameter modulation

In this paper we discuss the case when both parameters of system (1) are used to transmit message signals $s_1(t)$ and $s_2(t)$. We use *modulation rules* to modulate $s_1(t)$ and $s_2(t)$ in parameters of the transmitter in (1).

The modulation rules are given by

$$\mu(t) = \mu + s_1(t), \quad \hat{\mu}(t) = \hat{\mu} + \hat{s}_1(t), \tag{7}$$

$$a(t) = a + s_2(t), \quad \hat{a}(t) = \hat{a} + \hat{s}_2(t), \tag{8}$$

where $\hat{s}_1(t)$ and $\hat{s}_2(t)$ are the recovered message signals.

Now let us introduce the following errors:

$$\begin{aligned} &e_x = x_1 - x_2; \quad e_y = y_1 - y_2; \quad e_z = z_1 - z_2; \\ &\tilde{\mu} = \mu - \hat{\mu}; \quad \tilde{a} = a - \hat{a}; \quad \tilde{s}_1 = s_1 - \hat{s}_1; \quad \tilde{s}_2 = s_2 - \hat{s}_2. \end{aligned}$$

and according to them, we define the following vectors:

$$\mathbf{e}^T = (e_x, e_y, e_z); \quad \tilde{\mathbf{p}}^T = (\tilde{\mu}, \tilde{a}); \quad \tilde{\mathbf{s}}^T = (\tilde{s}_1, \tilde{s}_2).$$

From Eqs. (1)–(6), and taking into account the modulation rules (7), (8) we have:

$$\dot{\mathbf{e}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\tilde{\mu}x + e_z y - \tilde{s}_1 x - u_1 \\ -\tilde{\mu}y + (e_z - \tilde{a})x - \tilde{s}_1 y - \tilde{s}_2 x - u_2 \\ -u_3 \end{bmatrix}, \tag{9}$$

where for simplicity, we stand for $y = y_1$ and $x = x_1$. As we can see, the above system can be considered as a control problem where the vector inputs $\mathbf{u}$ and $\tilde{\mathbf{p}}$ must be proposed such that $\mathbf{e}$ asymptotically converges to zero.

[2] Here we denote the vector state related with the master and slave system, as $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. That is, $\mathbf{w}_i^T = (x_i, y_i, z_i)$; for $i = \{1,2\}$.

3. High order sliding-mode adaptative controller (HOSMAC)

In this section a HOSMAC is used at the receiver to maintain synchronization by continuously tracking the changes in the modulated parameters. Then, $s_1(t)$ and $s_2(t)$ can be recovered by using this controller.

We solve the control problem (9) by means of the Lyapunov method. To this end, based on a simple quadratic Lyapunov function, we propose the needed HOSMAC and the needed estimator that assure the synchronization of both systems.

Before solving the control problem we introduce the following assumptions related with the selected outputs of the transmitter system and the transmitted signals:

(A1) The states $x = x_1$ and $y = y_1$ are available, for measurement.

(A2) All the states of the transmitter system are bounded, with the generic property that the steady solution x and y, continues oscillating around zero.

Comment 1. Assumption A2 is a realistic because in most case all the states of Rikitake system are bounded; for a large set of initial conditions and for a large set of positive parameters μ and a. In fact Assumption A2 depends on the set of initial conditions and the values of the parameter vector q. To clarify the meaning of this property, we present a case where Assumption A2 does not hold. Selecting the parameters values as $\{\mu > 0; a > 0\}$, and the initial condition as $w_1(0) = (x_1(0) = 0, x_1(0) = 0, z_1(0) = \bar{z})$; we have that $x_1(t) = 0$, $y_1(t) = 0$ and $z_1(t) = t + \bar{z}$. Evidently, Assumption A2 cannot be fulfilled because the states x and y remain fixed at the origin and, the state z_1 is not bounded [29]. In fact, no identification method or scheme can be proposed if the transmitter system has solutions that tend either to infinity or to a constant.

Consider a Lyapunov function

$$V = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2}\tilde{\mathbf{p}}^T\tilde{\mathbf{p}} + \frac{1}{2}\tilde{\mathbf{s}}^T\tilde{\mathbf{s}}. \tag{10}$$

The time derivative of V along the trajectories of (9) is then given by

$$\dot{V} = \tilde{a}\dot{\tilde{a}} + \tilde{\mu}\dot{\tilde{\mu}} - \tilde{\mu}e_x x + e_x e_z y - e_x u_1 - \tilde{\mu}e_y y + e_y e_z x - \tilde{a}e_y x - e_y u_2 - e_z u_3 - \tilde{s}_1 x e_x - \tilde{s}_1 y e_y - \tilde{s}_2 x e_y + \tilde{s}_1\dot{\tilde{s}}_1 + \tilde{s}_2\dot{\tilde{s}}_2. \tag{11}$$

Now, in order to make V semi-definite negative, we propose $\dot{\tilde{\mathbf{p}}}$, $\mathbf{u}$, and $\dot{\tilde{\mathbf{s}}}$ as

$$\dot{\tilde{\mathbf{p}}} = \begin{bmatrix} \dot{\tilde{\mu}} \\ \dot{\tilde{a}} \end{bmatrix} = \begin{bmatrix} e_x x + e_y y \\ x e_y \end{bmatrix}, \tag{12}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 \ \mathrm{sign}(e_x)e_x^m \\ k_2 \ \mathrm{sign}(e_y)e_y^m \\ e_x y + e_y x \end{bmatrix}, \tag{13}$$

$$\dot{\tilde{\mathbf{s}}} = \begin{bmatrix} \dot{\tilde{s}}_1 \\ \dot{\tilde{s}}_2 \end{bmatrix} = \begin{bmatrix} e_x x + e_y y \\ x e_y \end{bmatrix}, \tag{14}$$

where k_1 and k_2 are strictly positive constants and m is any positive even integer.

Substituting (12)–(14) into (11), we obtain

$$\dot{V} = -\left(k_1|e_x|e_x^m + k_2|e_y|e_y^m\right). \tag{15}$$

This implies that $\dot{V}$ is semi-definite negative and so V converges. Hence, the set of signals $\{e_x, e_y, e_z, \tilde{\mu}, \tilde{a}, \tilde{s}_1, \tilde{s}_2\}$ are bounded. Let us proceed to show that $\mathbf{e}$ converges to zero as long as $t \to \infty$, by applying Barbalat's Lemma [30].[3] Integrating both sides of (15), we obtain

$$\int_0^t \left(k_1|e_x(s)|e_x^m(s) + k_2|e_y(s)|e_y^m(s)\right)ds \leqslant V(0). \tag{16}$$

Substituting the control law (13) in (9), we have that the closed-loop system can be read as

$$\begin{aligned} \dot{e}_x &= -\tilde{\mu}x + e_z y - \tilde{s}_1 x - k_1 \ \mathrm{sign}(e_x)e_x^m, \\ \dot{e}_y &= -\tilde{\mu}y + (e_z - \tilde{a})x - \tilde{s}_1 y - \tilde{s}_2 x - k_2 \ \mathrm{sign}(e_y)e_y^m, \\ \dot{e}_z &= -e_x y - e_y x, \end{aligned} \tag{17}$$

where the parameter dynamics are give by

$$\dot{\tilde{\mu}} = -e_x x - y e_y; \quad \dot{\tilde{a}} = -x e_y. \tag{18}$$

[3] Lemma (Barbalat): If the differential function $f(t)$ has a finite limit as $t \to \infty$, and if df/dt is uniformly continuous, then $df/dt \to 0$ as $t \to \infty$. A consequence of this Lemma is that if $f \in L_2$ and df/dt is bounded then $f \to 0$ as $t \to \infty$.

From the equations of (17) and **A2**, it follows that $\dot{\mathbf{e}}$ is bounded which implies that $\mathbf{e}$ is uniformly continuous. Using Barbalat's Lemma, it follows that vector states $\mathbf{e} \to 0$ as $t \to \infty$. Once again, differentiating (17), it is easily to show that $\ddot{\mathbf{e}}$ is bounded. Thus, $\dot{\mathbf{e}}$ is uniformly continuous and also $\mathbf{e}$ has a finite limit, as $t \to \infty$, from Barbalat's Lemma, we conclude that $\dot{\mathbf{e}} \to 0$ as $t \to \infty$. Since V converges, as $t \to \infty$, then we have that the two parameter errors $\tilde{\mu}$ and $\tilde{a}$ converge as $t \to \infty$ (10). Besides, from (18), it follows that $\dot{\hat{\mu}}$ and $\dot{\hat{a}}$ converge to zero, as $t \to \infty$. Roughly speaking, when t is large enough $\hat{\mu}$ and $\hat{a}$ are almost constant, and the differential equations of (17) implies that

$$0 = (\mu - \hat{\mu})x \quad 0 = (a - \hat{a})y \tag{19}$$

However once again from Assumption A2, we have that the steady states x and y remains oscillating around zero. Therefore necessarily $\mu = \hat{\mu}$ and $a = \hat{a}$. That is, $\tilde{\mathbf{p}}^T \to 0$, as $t \to \infty$.

Considering that $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{p}}$ have the same dynamics, defined by Eqs. (12) and (14), then the signal recovery errors $\tilde{s}_1 = s_1 - \hat{s}_1$ and $\tilde{s}_2 = s_2 - \hat{s}_2$ asymptotically converge to zero.

We summarize the previous discussion in the following proposition:

Proposition 1. *Under the Assumptions A1 and A2 the synchronization and the parameters estimation problem between systems (1) and (6), can be achieved for any strictly positive constants k_1,k_2 and for any even integer m. Furthermore, the receiver system (6) can recover the information signals s_1 and s_2 which are embedded in the chaotic transmitter (1) via the modulation rules (7) and (8), respectively.* □

4. Numerical results

Computer simulations have been carried out in order to test the effectiveness of the proposed asymptotic control strategy. The program uses the Runge–Kutta integration algorithm, with the integration step set to 0.001.

The information signals s_1 and s_2 are chosen as sinusoidal signals with frequency of 100 Hz as in [4,25], i.e.

$$s_1(t) = 0.05\sin(200\pi t),$$
$$s_2(t) = 0.02\sin(200\pi t).$$

In the first simulation we illustrate the qualitative property described in the Assumption A2. For this end, we fixed the transmitter system parameter as $\mathbf{p} = (\mu = 2, a = 5)$; while the arbitrary initial conditions were selected as $\mathbf{w}_1(0) = (x_1(0) = 1, y_1(0) = -1, z_1(0) = 1)$. Fig. 1 shows the behavior of the whole state of the Rikitake system.

To show the performance of the proposed control strategy we carried out a second simulation using the same set-up as above, and fixing the receiver system gains as $k_1 = k_2 = 0.8$ and $m = 4$; with the receiver system initialized at $\mathbf{w}_2(0) = 0$, $\hat{\mathbf{p}}(0) = 0$, $\hat{s}_1 = -2$ and $\hat{s}_2 = -5$. In Fig. 2 we can see that the synchronization errors asymptotically converge to zero. That is, the slave system follows almost perfectly the uncertain master system. The estimated parameters are shown in Fig. 3. As we expect, a better performance can be obtained as long as the time is increased. From this simulations we can concluded that the proposed estimator reconstructs reasonably well the parameters after elapsed 60 s.

Figs. 4 and 5 show the communication process, here the waveforms of the modulated parameters are shown in subplots (a), the convergence behavior of the information recovery errors is shown in subplots (b).

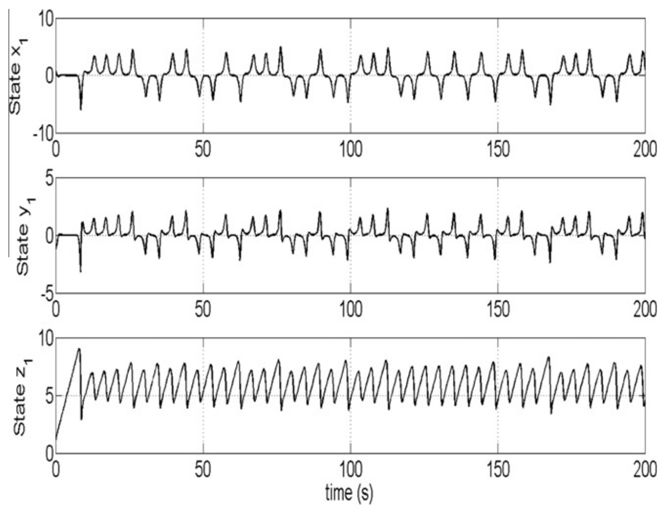

Fig. 1. Qualitative behavior of the Rikitake system using modulation rules (7), (8) when is initialized at $\mathbf{w}_1(0) = (1, -1, 1)$ and the parameters vector is fixed as $\mathbf{p} = (2, 5)$. Clearly, the Assumption A2 is satisfied.

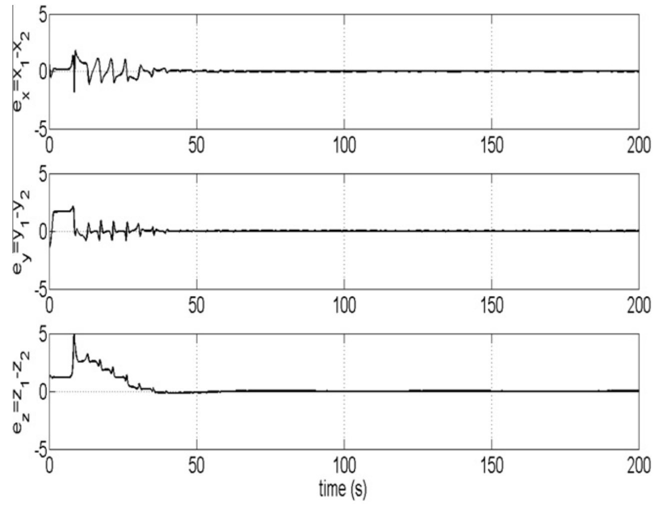

Fig. 2. Synchronization errors, when the transmitter system is initialized at $\mathbf{w}_1(0) = (1, -1, 1)$ and its parameters vector is fixed as $\mathbf{p} = (2, 5)$.

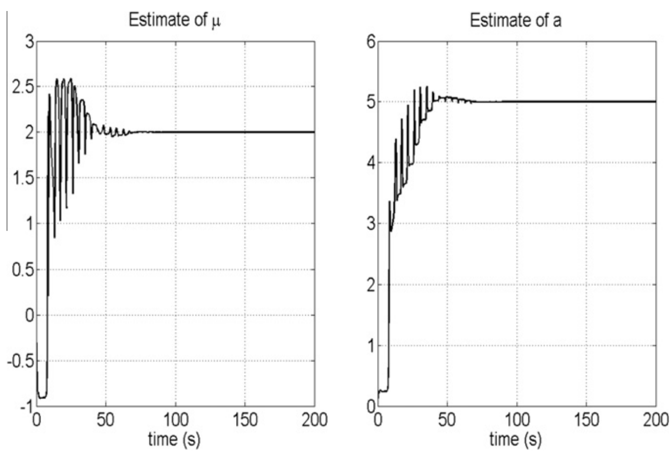

Fig. 3. Parameters estimation, when the master system is initialized at $\mathbf{w}_1(0) = (1, -1, 1)$; and the actual parameters vector is fixed as $\mathbf{p} = (2, 5)$.

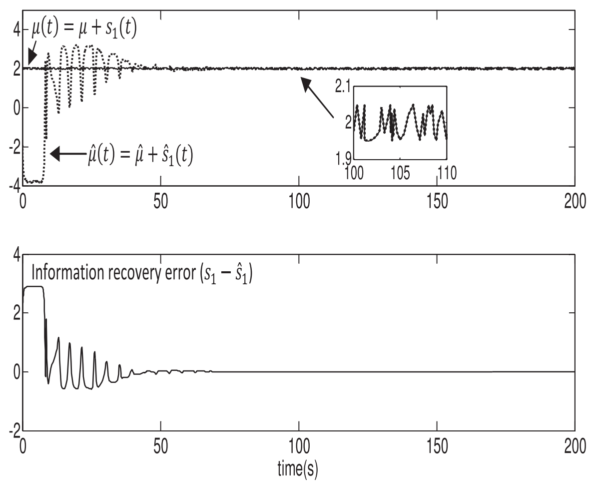

Fig. 4. Rikitake system for chaotic communication. Numerical results for message signal s_1.

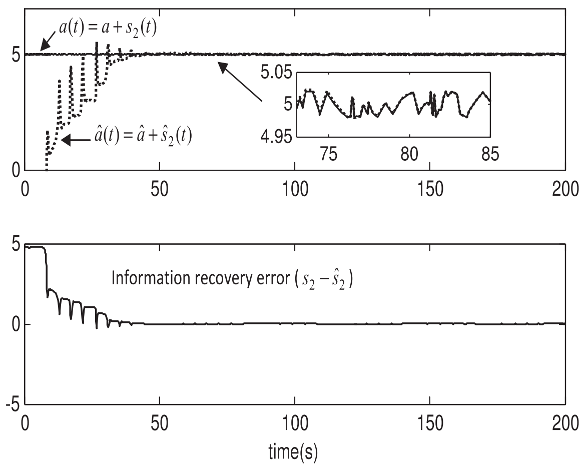

Fig. 5. Rikitake system for chaotic communication. Numerical results for message signal s_2.

5. Conclusions

A Lyapunov based approach for the synchronization and parameters identification of the constant unknown parameters of Rikitake system was presented. Indeed, we propose a chaotic communication approach via parameter modulation, where the receiver is controlled by a high order sliding mode adaptive controller, under the Assumption A1 that the two output states x and y are available for measurement. To accomplish this task, we first shown that the system is observable and algebraically identifiable, with respect to the available outputs. Then, we propose a receiver controlled system; where the controllers were proposed such that the vector synchronization error, the vector parameter error and the vector information recovery error, asymptotically converge to zero. The convergence proof was carried out by using the traditional Lyapunov method in combination with the Lemma of Barbalat and the Assumption A2. Finally, numerical simulations were carried out to evaluate the performance of the proposed solution.

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Synchronization of nonlinear fractional order systems

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ABSTRACT

This paper deals with the master–slave synchronization scheme for partially known nonlinear fractional order systems, where the unknown dynamics is considered as the master system and we propose the slave system structure which estimates the unknown state variables. For solving this problem we introduce a Fractional Algebraic Observability (FAO) property which is used as a main tool in the design of the master system. As numerical examples we consider a fractional order Rössler hyperchaotic system and a fractional order Lorenz chaotic system and by means of some simulations we show the effectiveness of the suggested approach.

1. Introduction

Fractional calculus is as old as conventional calculus, but is not as popular in science and engineering as conventional calculus. In the last three centuries this subject was studied only in mathematics, but in recent years it has been used in many fields of engineering and science [1]. "It might be that this mathematical tool help us modelling the reality in a better way and also, might be, that this is the calculus of the XXI century" [2].

Fractional calculus has a history almost as large as the one of the ordinary calculus, however the applications in physics and engineering have been recently appearing [1,3]. There are many known systems where the modelling with fractional operators has turned out to be useful, for example systems involved with phenomena such as: viscoelasticity [4], dielectric polarization, electrode-electrolite polarization and electromagnetic waves [5].

In fractional systems there are also studied subjects as: chaotic dynamics and its synchronization [6–10], stability [11–13], delayed systems [14,15], systems identification [2], control systems [2,3], numerical methods [16], quantitative finances [17], quantic evolution of complex systems [18], digital image processing, variational principles and its applications, Euler–Lagrange equations, applications in finances and economy, bioengineering applications, fractional Fourier transform, fractional sliding modes, robotics, among others [19].

In [20] the authors claimed that many systems exhibit the fractional phenomena, such as motion in complex media/environments, random walk of bacteria in a fractal substance and the chemotaxi behavior and food seeking of microbes [21], these phenomena are always related to the complexity and heredity of systems due to the fractional properties of systems components, such as fractional viscoelastic material, the fractional circuit element and the fractal structure [22].

The fractional order chaotic systems has more adjustable variables than integer-order chaotic systems, it is widely believed that fractional-order chaotic systems can be applied in encryption efficiently and thus enlarge the key space [23,24].

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The synchronization problem is an interesting topic in fractional chaotic systems [25]. The synchronization of integer order chaotic systems has been extensively investigated since its introduction by Pecora and Carroll [26]. On the other hand, [27] is the first work concerning on synchronization of fractional systems, the authors showed by means of a control law that fractional order chaotic systems can be synchronized by using the similar scheme as that of their integer counterparts.

Some techniques related to chaos synchronization in fractional systems have been proposed. For instance, we mention the works [8,28,29] in which the authors propose the employment of feedback controllers, which allows to achieve the synchronization between two identical fractional-order chaotic systems, the theoretical analysis is derived by utilizing the stability criterion of linear fractional systems; in [30] is studied the synchronization of fractional order chaotic systems with unidirectional linear error feedback coupling; in [31] is presented a classical Luenberger observer design for the synchronization of fractional-order chaotic systems, i.e., the observer structure needs a copy of the system and a linear output error feedback, the application is restricted to scalar coupling signals; in [32,33] are given sufficient conditions for the synchronization between two identical fractional systems by using the Laplace transform theory.

The main contribution in this work is to show a novel technique for the synchronization problem in partially known nonlinear fractional-order systems, we propose a new procedure using the master–slave synchronization scheme for estimating the unknown state variables based on Fractional Algebraic Observability (FAO) property (is not necessary the system's copy). As far as we know in the literature this class of estimation scheme has not been used.

The rest of this paper is organized as follows. In Section 2 is given a brief note about fractional derivatives and Mittag–Leffler type function. Section 3 presents the problem statement and its solution, based on FAO and the master–slave synchronization scheme. In Section 4 we apply the methodology presented in Section 3 to the fractional order Rössler hyperchaotic system and the fractional order Lorenz chaotic system, also some numerical results are shown. The intention of choosing these systems is to clarify the proposed methodology and to highlight the simplicity and flexibility of the suggested approach. Finally, we conclude with some remarks in Section 5.

2. On fractional derivatives

There are several definitions of a fractional derivative of order α [3,34,35], we will use the Caputo fractional operator in the definition of fractional order systems, because the meaning of the initial conditions for systems described using this operator is the same as for integer order systems.

Definition 1 (*Caputo Fractional derivative).* The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function x is defined as: see [3]

$$x^{(\alpha)} = {}_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{d^m x(\tau)}{d\tau^m}(t-\tau)^{m-\alpha-1}d\tau, \tag{1}$$

where: $m-1 \leqslant \alpha < m$, $\frac{d^m x(\tau)}{d\tau^m}$ is the mth derivative of x in the usual sense, $m \in \mathbb{N}$ and Γ is the gamma function.[1] □

Now we define a sequential operator, see [34], as follows

$$\mathcal{D}^{(r\alpha)}x(t) = \underbrace{{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}\cdots{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}}_{\text{r-times}}x(t) \tag{2}$$

i.e., it is the Caputo fractional derivative of order α applied $r \in \mathbb{N}$ times sequentially, with $\mathcal{D}^{(0)}x(t) = x(t)$, we can note that if $r = 1$ then $\mathcal{D}^{(\alpha)}x(t) = x^{(\alpha)}$.

2.1. Mittag–Leffler type function

The Mittag–Leffler function with two parameters is defined as [36]:

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}, \quad z,\ \beta \in \mathbb{C},\ \mathbb{R}e(\alpha) > 0 \tag{3}$$

this function is used to solve fractional differential equations as the exponential function in integer order systems. In the particular case when $\alpha = \beta = 1$, we have that $E_{1,1}(z) = e^z$. Now if we have particular values of α, the function (3) has asymptotic behavior at infinity.

Theorem 1 [3]. *If $\alpha \in (0,2)$, β is an arbitrary complex number and μ is an arbitrary real number such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{4}$$

then for an arbitrary integer $\kappa \geqslant 1$ the following expansion holds:

[1] To simplify the notation we omitted the time dependence in $x^{(\alpha)}$, in what follows we take $t_0 = 0$.

$$E_{\alpha,\beta}(z) = -\sum_{i=1}^{\kappa} \frac{1}{\Gamma(\beta - \alpha i) z^i} + O\left(\frac{1}{|z|^{\kappa+1}}\right), \tag{5}$$

with $|z| \to \infty, \mu \leqslant |arg\ (z)| \leqslant \pi$. $\square$

The Mittag–Leffler function has the following properties:

Property 1. $\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-k\tau^\alpha) d\tau = t^\beta E_{\alpha,\beta+1}(-kt^\alpha), \beta > 0$, see [3].

Property 2. $E_{\alpha,\beta}(-x)$, is completely monotonic, i.e., $(-1)^n E^{(n)}_{\alpha,\beta}(-x) \geqslant 0$ for $0 < \alpha \leqslant 1$ and $\beta \geqslant \alpha$, for all $x \in (0,\infty)$ and $n \in \mathbb{N} \cup \{0\}$, see [37].

We will use these facts in the following problem.

3. Problem statement and main result

We take the initial condition problem for an autonomous fractional order nonlinear system, with $0 < \alpha < 1$:

$$\begin{aligned} x^{(\alpha)} &= f(x), \quad x(0) = x_0, \\ y &= h(\bar{x}), \end{aligned} \tag{6}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is a Lipschitz continuous function,[2] with $x_0 \in \Omega \subset \mathbb{R}^n$, in this case y denotes the output of the system (the measure that we can obtain), $\bar{x} \in \mathbb{R}^p$ represents the states that we can observe (known states), $h : \mathbb{R}^p \to \mathbb{R}^q$ is a continuous function and $1 \leqslant p < n$.

Consider the system given by (6), we will separate in two dynamical systems with states $\bar{x} \in \mathbb{R}^p$ and $\eta \in \mathbb{R}^{n-p}$, respectively with $x^T = (\bar{x}^T, \eta^T)$, the first system will describe the known states and the second represents unknown states, then the system (6) can be written as:

$$\begin{aligned} \bar{x}^{(\alpha)} &= \bar{f}(\bar{x}, \eta), \\ \eta^{(\alpha)} &= \Delta(\bar{x}, \eta), \\ y &= h(\bar{x}), \end{aligned} \tag{7}$$

where $f^T(x) = (\bar{f}^T(\bar{x}, \eta), \Delta^T(\bar{x}, \eta)), \bar{f} \in \mathbb{R}^p$ and $\Delta \in \mathbb{R}^{n-p}$. Now the problem is: how can we estimate the η's states? this question arises because if we know the η's states we can use these signals to generate measuring depending on them. In order to solve this observation problem let us introduce the following observability property.

Definition 2 (*FAO*). A state variable $\eta_i \in \mathbb{R}$ satisfies the Fractional Algebraic Observability (FAO) if it is a function of the first $r \in \mathbb{N}$ sequential derivatives of the available output $y_{\bar{x}}$, i.e.,

$$\eta_i = \phi_i\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right), \tag{8}$$

where $\phi_i : \mathbb{R}^{(r+1)p} \to \mathbb{R}$. $\square$

If we assume that the components of unknown state vector η are FAO, then we can describe our problem in terms of the master–slave synchronization scheme, which is defined in the following way.

Let us consider the master system:

$$\eta_i^{(\alpha)} = \Delta_i(\bar{x}, \eta), \tag{9}$$

$$y_{\eta_i} = \eta_i = \phi_i\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{10}$$

for $i \in \{p+1, \ldots, n\}$, where η_i is a component of the state vector η and y_{η_i} denotes the output of the ith master system.

Now let us propose a fractional dynamical system with the same order α, which will be the slave system:

$$\hat{\eta}_i^{(\alpha)} = k_{\hat{\eta}_i}(y_{\eta_i} - \hat{\eta}_i), \tag{11}$$

$$y_{\hat{\eta}_i} = \hat{\eta}_i \tag{12}$$

for $i \in \{p+1, \ldots, n\}$, where $\hat{\eta}_i$ is the state, and $y_{\hat{\eta}_i}$ denotes the output of the slave system and $k_{\hat{\eta}_i}$ is a positive constant.

In the master–slave synchronization scheme, the output of the master system (10) describes the target signal, while (12) represents the response signal. Therefore the synchronization problem can be established as follows.

Given the master system (9) and our slave system (11), it should be determined some conditions, such that the output of the slave system (12) synchronizes with the output of the master system (10).

[2] This assures the unique solution [36].

Let us define the synchronization error as:

$$e_i = y_{\eta_i} - y_{\hat{\eta}_i} = \eta_i - \hat{\eta}_i. \tag{13}$$

Now we establish a convergence analysis of the synchronization error.

Proposition 1. *Let the system (6) which can be expressed in the form (7), where the following conditions are fulfilled:*

H1: η_i *satisfies the FAO property for* $i \in \{p+1, \ldots, n\}$.
H2: Δ_i *is bounded, i.e.,* $\exists M \in \mathbb{R}^+$ *such that* $\|\Delta(x)\| \leqslant M, \forall x \in \Omega$.
H3: $k_{\hat{\eta}_i} \in \mathbb{R}^+$.

Then, the synchronization of the master output (10) with the slave output (12) is achieved, for global initial condition of the states.

Proof. From **H1** we can write Eqs. (9)–(13). Taking the fractional derivative of the Eq. (13), we have

$$e_i^{(\alpha)} = \eta_i^{(\alpha)} - \hat{\eta}_i^{(\alpha)}. \tag{14}$$

Substituting the fractional dynamics (9) and (11) into (14), we obtain

$$e_i^{(\alpha)} + k_{\hat{\eta}_i} e_i = \Delta_i(x). \tag{15}$$

There exists a unique solution for the system (15), due to $\Delta_i(x(t)) - k_{\hat{\eta}_i} e_i(t)$ is a Lipschitz continuous function on e.[3] Solving the above equation [36], we have

$$e_i(t) = e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(k_{\hat{\eta}_i}(t-\tau)^\alpha)\Delta_i(\tau)\mathrm{d}\tau, \tag{16}$$

where $e_i(0) = e_{i0}$.

Using Triangle and Cauchy–Schwarz inequalities and **H2**

$$|e_i(t)| \leqslant |e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha)| + M \int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)|\mathrm{d}\tau$$

The functions $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)$ and $E_\alpha(-k_{\hat{\eta}_i} t^\alpha)$ are not negative due to Property 2 of Mittag–Leffler function and **H3**

$$|e_i(t)| \leqslant |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)\mathrm{d}\tau$$

Using Property 1 of Mittag–Leffler function

$$|e_i(t)| \leqslant |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha)$$

If $t \to \infty$, we use the Eq. (5) with $\mu = 3\pi\alpha/4$ due to **H3**.

$$\lim_{t\to\infty} |e_i(t)| \leqslant |e_{i0}| \lim_{t\to\infty} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M \lim_{t\to\infty} t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha) = \frac{M}{k_{\hat{\eta}_i}} \qquad \square$$

Remark 1. If the FAO of a state variable is expressed in terms of the fractional sequential derivatives of the output y, which are unknown, then is necessary to introduce an artificial variable (if it is possible) in order to avoid the use of these unknown derivatives.

4. Numerical examples

In this section we study the synchronization of nonlinear fractional order systems by numerical simulations. The fractional order Rössler hyperchaotic system and the fractional order Lorenz chaotic system are examples.

Remark 2. Chaotic systems are characterized by global boundedness of the trajectories [38]. By this fact, H2 is always satisfied.

First, consider the fractional order Rössler hyperchaotic system [39]

[3] Eq. (15) is non-autonomous, but the Lipschitz condition assures a unique solution [36].

$$x^{(\alpha)} = \begin{pmatrix} x_3 + ax_1 + x_2 \\ -cx_4 + dx_2 \\ -x_1 - x_4 \\ b + x_3x_4 \end{pmatrix}, \qquad (17)$$
$$y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $x = (x_1,x_2,x_3,x_4)^T$ is the state vector, y_1 and y_2 are the considered outputs. When $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$, the Rössler equations (17) has a hyperchaotic attractor (see Fig. 1).

Now, we rewrite system (17) in the form (7) as follows

$$\bar{x}^{(\alpha)} = \begin{pmatrix} \eta_3 + \mathrm{a}\bar{x}_1 + \bar{x}_2 \\ -\mathrm{c}\eta_4 + \mathrm{d}\bar{x}_2 \end{pmatrix},$$
$$\eta^{(\alpha)} = \begin{pmatrix} -\bar{x}_1 - \eta_4 \\ \mathrm{b} + \eta_3\eta_4 \end{pmatrix}, \qquad (18)$$
$$y_{\bar{x}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix},$$

where $x_1 = \bar{x}_1,\ x_2 = \bar{x}_2,\ \eta_3 = x_3,\ \eta_4 = x_4,\ y_{\bar{x}_1} = \bar{x}_1$ and $y_{\bar{x}_2} = \bar{x}_2$. From (18), it is not difficult to find the following relations

$$\eta_3 = \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2}, \qquad (19)$$
$$\eta_4 = \phi_4\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = -\frac{1}{\mathrm{c}}y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}}y_{\bar{x}_2}, \qquad (20)$$

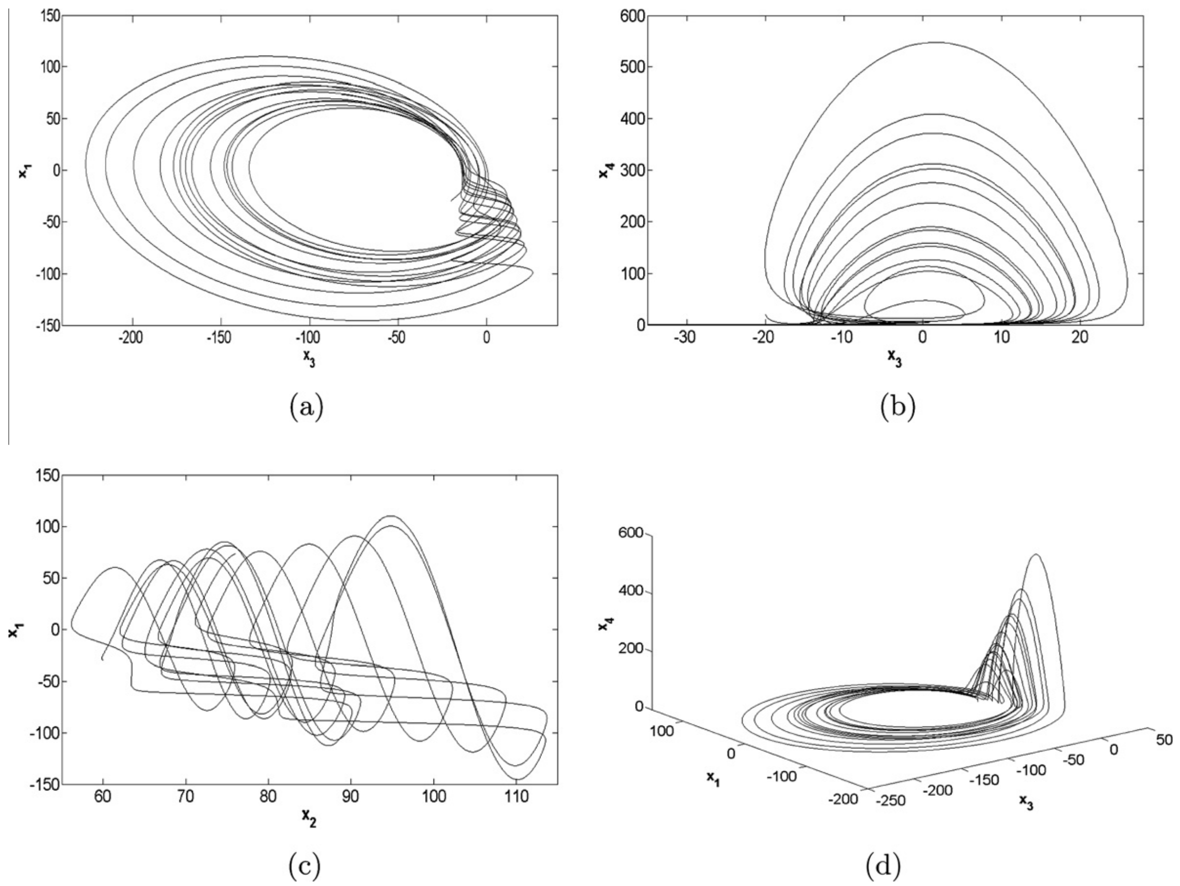

Fig. 1. Phase plot of the fractional-order Rössler hyperchaotic system with $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, $\alpha = 0.95$, and initial conditions $x_1(0) = -30$, $x_2 = 60$, $x_3(0) = -20$, and $x_4(0) = 20$: (a) $x_3 - x_1$ plane, (b) $x_3 - x_4$ plane, (c) $x_2 - x_1$ plane, and (d) $x_3 - x_1 - x_4$ space.

then we say that $\eta_3 = x_3$ and $\eta_4 = x_4$ are FAO and therefore H1 is fulfilled.

From above, the master systems are given by

$$\begin{cases} \eta_3^{(\alpha)} = -\bar{x}_1 - \eta_4, \\ y_{\eta_3} = \eta_3 = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a} y_{\bar{x}_1} - y_{\bar{x}_2}, \end{cases} \tag{21}$$

$$\begin{cases} \eta_4^{(\alpha)} = \mathrm{b} + \eta_3 \eta_4, \\ y_{\eta_4} = \eta_4 = -\frac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2}. \end{cases} \tag{22}$$

Now we design the corresponding slave systems for (21) and (22). By using (11), we have

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}(y_{\eta_3} - \hat{\eta}_3), \tag{23}$$

with $k_{\hat{\eta}_3} \in \mathbb{R}^+$ (condition H3).

Replacing (19) into (23) leads to

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}\left(y_{\bar{x}_1}^{(\alpha)} - \mathrm{a} y_{\bar{x}_1} - y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}\hat{\eta}_3 \tag{24}$$

In order to avoid the use of the fractional derivative $y_{\bar{x}}^{(\alpha)}$, we introduce an auxiliary variable $\gamma_{\hat{\eta}_3}$:

$$\gamma_{\hat{\eta}_3} = -k_{\hat{\eta}_3} y_{\bar{x}_1} + \hat{\eta}_3, \tag{25}$$

then

$$\hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3} y_{\bar{x}_1}. \tag{26}$$

Substituting (26) and its fractional derivative of order α into (24), we obtain

$$\gamma_{\hat{\eta}_3}^{(\alpha)} = -k_{\hat{\eta}_3}\gamma_{\hat{\eta}_3} - k_{\hat{\eta}_3}(\mathrm{a} y_{\bar{x}_1} + y_{\bar{x}_2}) - k_{\hat{\eta}_3}^2 y_{\bar{x}_1}, \quad \gamma_{\hat{\eta}_3}(0) = \gamma_{\hat{\eta}_{3_0}}. \tag{27}$$

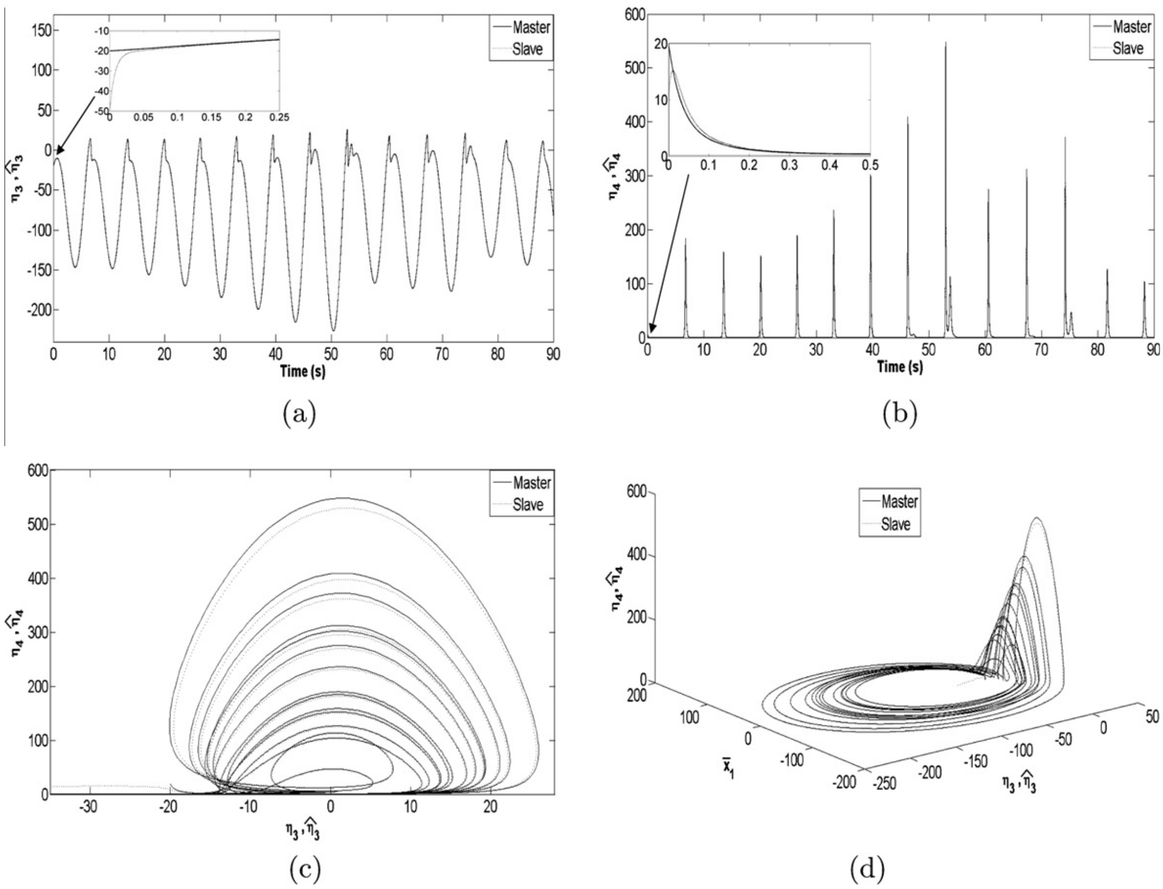

Fig. 2. Synchronization of the fractional-order Rössler hyperchaotic system.

Then, the corresponding slave system of (21) is given by

$$\begin{cases} \hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3} y_{\bar{x}_1}, \\ y_{\hat{\eta}_3} = \hat{\eta}_3. \end{cases} \tag{28}$$

By means of the same procedure we have obtained the following slave system for (22)

$$\begin{cases} \hat{\eta}_4 = \gamma_{\hat{\eta}_4} - \frac{k_{\hat{\eta}_4}}{c} y_{\bar{x}_2}, \\ y_{\hat{\eta}_4} = \hat{\eta}_4, \end{cases} \tag{29}$$

where the dynamics of the auxiliary variable $\gamma_{\hat{\eta}_4}$ is given by

$$\gamma_{\hat{\eta}_4}^{(\alpha)} = -k_{\hat{\eta}_4}\gamma_{\hat{\eta}_4} + \frac{d}{c} k_{\hat{\eta}_4} y_{\bar{x}_2} + \frac{k_{\hat{\eta}_4}^2}{c} y_{\bar{x}_2}, \quad \gamma_{\hat{\eta}_4}(0) = \gamma_{\hat{\eta}_{4_0}}, \tag{30}$$

with $k_{\hat{\eta}_4} \in \mathbb{R}^+$ (condition H3).

Numerical simulations are performed for $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$. We consider the following initial conditions to the master system $\bar{x}_1(0) = -30$, $\bar{x}_2(0) = 60, \eta_3(0) = -20, \eta_4(0) = 20$, the initial conditions to the slave system $\hat{\eta}_3(0) = -50$, $\hat{\eta}_4(0) = 10$, and the gain parameters are taken as $k_{\hat{\eta}_3} = k_{\hat{\eta}_4} = 100$. The synchronization between masters (21) and (22) and slaves (28) and (29) is shown in Fig. 2.

Now, we consider the fractional-order Lorenz system described by a set of three fractional differential equations, as follows:

$$x^{(\alpha)} = \begin{pmatrix} ax_2 - ax_1 \\ bx_1 - cx_2 - x_1x_3 \\ x_1x_2 - dx_3 \end{pmatrix}, \tag{31}$$

$$y = x_1,$$

with parameters $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, and $\alpha = 0.8$, the system (31) exhibits chaotic behavior [40]. Fig. 3 depicts the chaotic attractor of (31).

Now, we will apply the proposed methodology. Firstly, we rewrite system (31) in the form (7)

$$\bar{x}^{(\alpha)} = a(\eta_2 - \bar{x}_1),$$

$$\eta^{(\alpha)} = \begin{pmatrix} b\bar{x}_1 - c\eta_2 - \bar{x}_1\eta_3 \\ \bar{x}_1\eta_2 - d\eta_3 \end{pmatrix}, \tag{32}$$

$$y_{\bar{x}} = \bar{x}_1,$$

where $\bar{x}_1 = x_1$, $\eta_2 = x_2$, and $\eta_3 = x_3$.

Before proposing a master system in the form (9) and (10) we need to determine if $\eta_2 = x_2$ and $\eta_3 = x_3$ satisfy the FAO property.

For η_2, we have

$$y_{\bar{x}}^{(\alpha)} = a(\eta_2 - y_{\bar{x}}) \Rightarrow \eta_2 = \phi_2\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = \frac{1}{a} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}}. \tag{33}$$

In the same manner for η_3, we obtain

$$\eta_3 = -\frac{1}{y_{\bar{x}}}\left(\eta_2^{(\alpha)} + c\eta_2 - by_{\bar{x}}\right). \tag{34}$$

Substituting (33) into (34) we have

$$\eta_3 = \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}\right) = -\frac{1}{ay_{\bar{x}}}\left[\mathcal{D}^{(2\alpha)} y_{\bar{x}} + (a + c) y_{\bar{x}}^{(\alpha)} + a(c - b) y_{\bar{x}}\right]. \tag{35}$$

Note that $\eta_3 = x_3$ looses algebraic observability property when $y_{\bar{x}} = x_1 = 0$, hence, only Eq. (33) satisfies FAO with respect to the selected output $y_{\bar{x}} = x_1$.

Then from (33) we obtain the following master system in the form (9) and (10), for $\eta_2 = x_2$,

$$\begin{cases} \eta_2^{(\alpha)} = b\bar{x}_1 - c\eta_2 - \bar{x}_1\eta_3, \\ y_{\eta_2} = \eta_2 = \frac{1}{a} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}}. \end{cases} \tag{36}$$

The next step is the design of the slave system that synchronizes with (36).

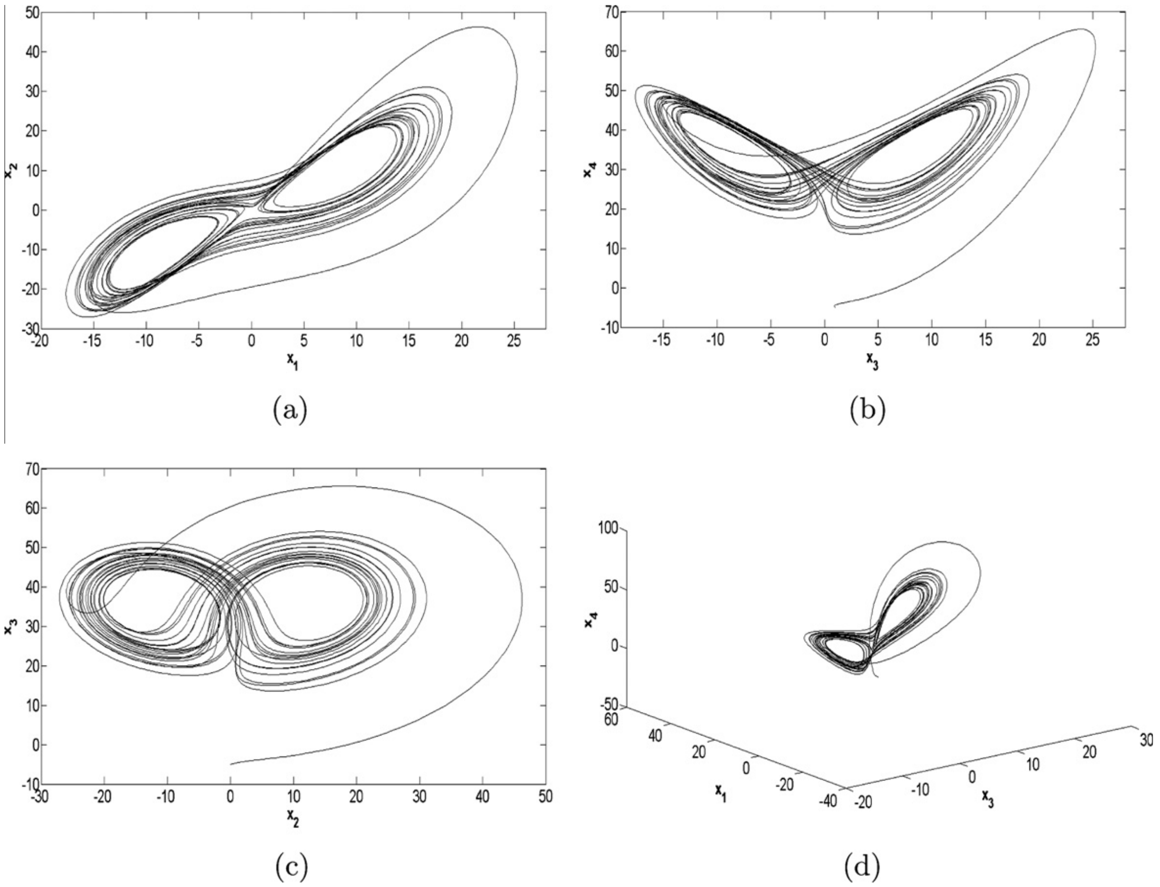

Fig. 3. Phase portrait of the fractional-order Lorenz system with $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, and initial conditions $x_1(0) = 1$, $x_2(0) = 0$ and $x_3(0) = -5$: (a) x_1 versus x_2, (b) x_1 versus x_3, (c) x_2 versus x_3, and (d) x_1 vs x_2 vs x_3.

Using the Eq. (11), we obtain

$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}(\eta_2 - \hat{\eta}_2). \tag{37}$$

Replacing (33) into (37) leads to

$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}\left(\frac{1}{a} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}}\right) - k_{\hat{\eta}_2}\hat{\eta}_2. \tag{38}$$

Since $y_{\bar{x}}^{(\alpha)}$ is not available, slave (38) cannot be implemented. In order to overcome this problem let us consider the following auxiliary variable $\gamma_{\hat{\eta}_2}$:

$$\gamma_{\hat{\eta}_2} = -\frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} + \hat{\eta}_2, \tag{39}$$

then

$$\hat{\eta}_2 = \gamma_{\hat{\eta}_2} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}}. \tag{40}$$

The fractional derivative of order α of (40) is

$$\hat{\eta}_2^{(\alpha)} = \gamma_{\hat{\eta}_2}^{(\alpha)} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}}^{(\alpha)}. \tag{41}$$

Substituting (40) and (41) into (38), we obtain

$$\gamma_{\hat{\eta}_2}^{(\alpha)} = -k_{\hat{\eta}_2}\gamma_{\hat{\eta}_2} + \left(1 - \frac{k_{\hat{\eta}_2}}{a}\right) k_{\hat{\eta}_2} y_{\bar{x}}, \quad \gamma_{\hat{\eta}_2}(0) = \gamma_{\hat{\eta}_{2_0}}. \tag{42}$$

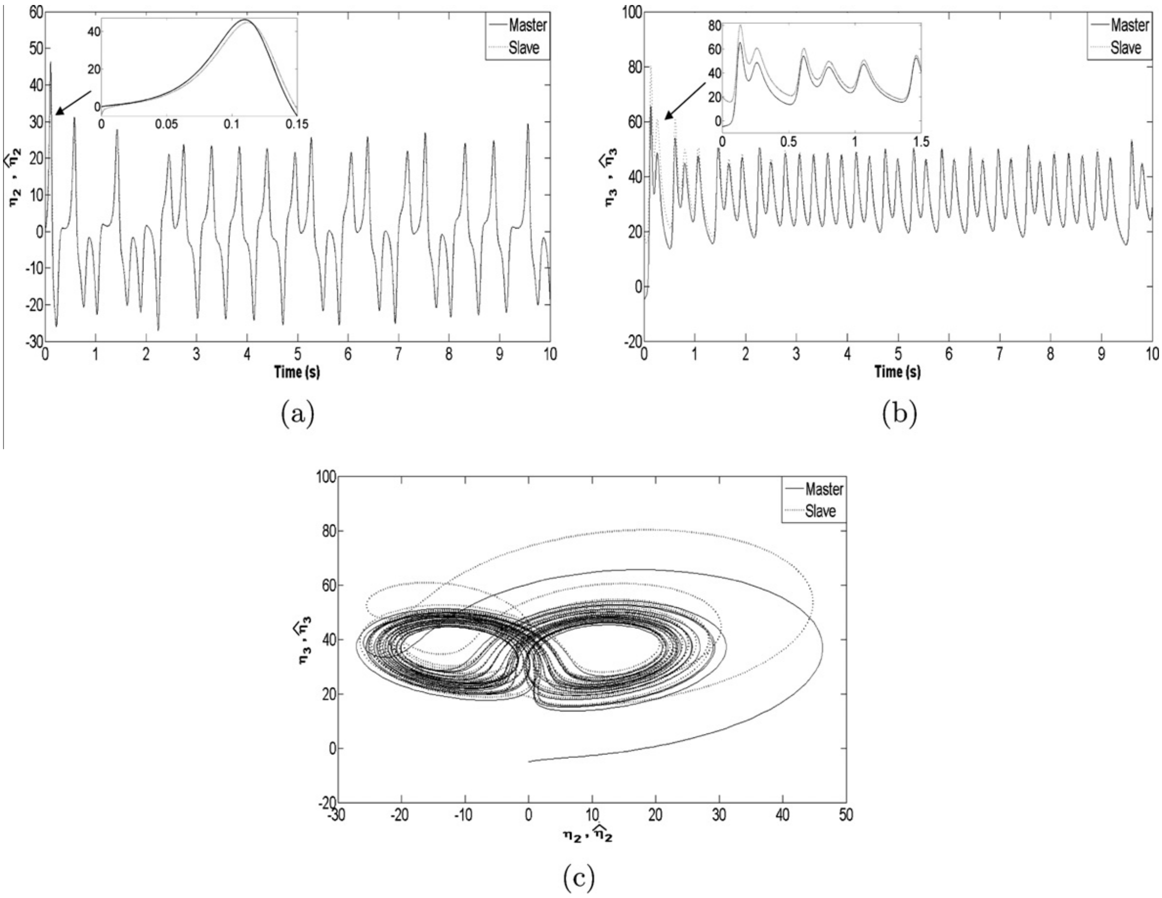

Fig. 4. Synchronization of the fractional-order Lorenz system with $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, and initial conditions for master $\bar{x}_1(0) = 1, \eta_2(0) = 0, \eta_3(0) = -5$ and for slave $\hat{\eta}_2(0) = -5, \hat{\eta}_3(0) = 20$.

Then, the slave system of $\eta_2 = x_2$ is given by

$$\begin{cases} \hat{\eta}_2 = \gamma_{\hat{\eta}_2} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}}, \\ y_{\hat{\eta}_2} = \hat{\eta}_2. \end{cases} \tag{43}$$

Now, since the discontinuity of Eq. (35) for $y_{\bar{x}} = x_1 = 0$ we cannot construct a slave system for η_3 in the proposed form, to overcome this problem we propose the following slave system

$$\begin{cases} \hat{\eta}_3^{(\alpha)} = \bar{x}_1 \hat{\eta}_2 - d\hat{\eta}_3, \\ y_{\hat{\eta}_3} = \hat{\eta}_3, \end{cases} \tag{44}$$

due to the Eq. (13) and $\eta_3^{(\alpha)}$ from (32)[4] we have

$$e_3^{(\alpha)} + de_3 = \bar{x}_1 e_2. \tag{45}$$

It should be noted that Eq. (45) has the same form of (15), thus in this case by using Proposition 1 we obtain

$$|e_3| \leqslant \frac{mM}{dk_{\hat{\eta}_2}}, \tag{46}$$

where m is the bound of $\bar{x}_1$.

We present the corresponding simulation, we consider the following initial conditions to the master system $\bar{x}_1(0) = 1,\ \eta_2(0) = 0,\ \eta_3(0) = -5$, the initial conditions to the slave system $\hat{\eta}_2(0) = -5,\ \hat{\eta}_3(0) = 20$, the parameters are $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, the initial conditions of the auxiliary functions are $\gamma_{\hat{\eta}_2}(0) = -20$ and finally, the gain parameter is $k_{\hat{\eta}_2} = 150$. The convergence of the estimates to the true signals is shown in Fig. 4.

[4] This equation is used instead of FAO.

5. Conclusions

It was introduced a new concept so-called Fractional Algebraic Observability (FAO) which is a fundamental issue to determinate the unknown dynamics of fractional order nonlinear systems by means of the master–slave synchronization scheme, in particular we applied the results to chaotic and hyperchaotic fractional order systems, however this technique can be applied to others classes of systems which satisfy the properties of Proposition 1. As well as FAO can be combined with other techniques as was done in the Lorenz's example. Some numerical simulations have illustrated the effectiveness of the suggested approach.

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Chaotic Systems Synchronization Via High Order Observer Design

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ABSTRACT
In this paper, we consider the synchronization problem via a nonlinear observer design. A new exponential polynomial observer for a class of nonlinear oscillators is proposed, which is robust against output noises. A sufficient condition for synchronization is derived analytically with the help of the Lyapunov stability theory. The proposed technique has been applied to synchronize chaotic systems (Rikitake and Rössler systems) by numerical simulation.

Keywords: synchronization, polynomial observer, Lipschitz system, algebraic observability condition.

RESUMEN
En este trabajo se considera el problema de sincronización por medio del diseño de un observador no lineal. Se propone un nuevo observador polinomial exponencial para una clase de osciladores no lineales. La condición suficiente para lograr la sincronización es desarrollada analíticamente con la ayuda de la teoría de estabilidad de Lyapunov. La técnica propuesta ha sido aplicada para sincronizar sistemas caóticos (los sistemas de Rikitake y Rössler) empleando simulaciones numéricas.

1. Introduction

In the last years, the problem of synchronization of chaotic systems has received a great deal of attention among scientist in many fields [1], [2], [3], [4], [5]. In general, the synchronization research has been focused on two areas. The first one relates with the employ of state observers, where the main application is the synchronization of nonlinear oscillators [6], [7], [8], [9], [10]. The second one is the use of control laws, which allows achieving the synchronization with different structure and order between nonlinear oscillators [11], [12]. A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master- slave configuration [15]; thus, the chaos synchronization problem can be regarded as an observer design procedure, where the coupling signal is viewed as output and the slave system is the observer [4], [9], [13]. In this configuration, the two coupled systems are identical and, therefore, identical synchronization occurs, which means that the difference of master and slave state vectors converges to zero for $t \to \infty$.

In this paper the synchronization scheme is proposed for a class of Lipschitz nonlinear systems. Many problems in engineering and other applications are globally Lipschitz, for instance the sinusoidal terms in robotics. Nonlinearities which are square or cubic in nature are not globally Lipschitz, however, they are locally so; moreover, when such functions occur in physical systems, they frequently have a saturation in their growth rate, making them globally Lipschitz functions [14]. Thus, this class of systems covered by this note is fairly general.

The main contribution of this paper consists in the solution of the synchronization problem via an exponential polynomial observer. In [14], [15], [16], existence conditions of the full-order observers for Lipschitz nonlinear systems were established. The main purpose in this work is to extend those results by showing that the conditions given in [16] also guarantee the existence of a full-order observer with a high-order correction term. The reason is very simple, as it is well known an extended

Luenberger observer can be seen as a first order Taylor series around the observed state, therefore, to improve the estimation performance, a high-order term is now included in the observer structure.

The intention of choosing two examples as the Rössler and Rikitake systems is to clarify the proposed methodology; however, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

2. Exponential polynomial observer

A. Problem statement

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \quad (1a)$$

Where $x \in \Re^n$ the vector of the state variables; $u \in \Re^l$, $(l \leq n)$, is the control input; $f(\circ): \Re^n \times \Re^l \to \Re^l$, is a nonlinear smooth vector function and Lipschitz in *x* and uniformly bounded in *u*; $y \in \Re$ is the vector of measured states, with $C \in \Re^{1\times n}$.

Any nonlinear system of the form of Eq. (1a) can be expressed in the form of Eq. (1b) as long as $f(x,u)$ is differentiable with respect to x.

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx \quad , \quad x_0 = x(t_0) \end{aligned} \quad (1b)$$

In system (1b), $\Psi(x,u)$ is a nonlinear vector function which satisfies the Lipschitz condition with a Lipschitz constant L, i.e,

$$\|\Psi(x,u) - \Psi(\hat{x},u)\| \leq L\|x - \hat{x}\| \quad (2)$$

B. A note on the Algebraic Observability Condition (AOC)

Before proposing the exponential polynomial observer, a definition concerning an algebraic observability condition is given (for more details see [17]).

Definition 1: Consider the system described by systems (1b), where $x \in \Re^n$. A state x_i, is said to be algebraically observable with respect to $\{u, y\}$ if it satisfies a differential polynomial in terms of u, y and some of their time derivatives, i. e., $P(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$, $1 \leq i \leq n$.

C. Observer design

We consider system (1b), the observer has the next form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \Psi(\hat{x},u) + K_1 C(x - \hat{x}) + K_2 [C(x - \hat{x})]^m \\ \hat{x}_0 &= \hat{x}(t_0) \end{aligned} \quad (3)$$

Throughout this paper, the following assumptions are considered:

Assumption 1: $m \in Z^+, m \ odd, m > 1$ (3a)

Assumption 2: K_1 can be chosen such as the following Algebraic Riccati Equation (ARE) which has a symmetric positive-definite solution P for some $\varepsilon > 0$

$$(A - K_1 C)^T P + P(A - K_1 C) + L^2 PP + I + \varepsilon I = 0 \quad (3b)$$

Assumption 3: K_2 can be chosen such as the following relation which holds

$$\lambda_{\min}(N + N^T) \geq 0 \text{, with } N := PK_2C. \quad (3c)$$

In (3), $\hat{x} \in \Re^n$, $K_1 = [k_{1,1} \quad k_{2,1} \quad \ldots \quad k_{n,1}]^T \in \Re^n$ and $K_2 = [k_{1,2} \quad k_{2,2} \quad \ldots \quad k_{n,2}]^T \in \Re^n$.

We analyze the observer error which is defined as $e = x - \hat{x}$. From (1b) and (3), the dynamics of the observer error $e = x - \hat{x}$ are given by

$$\dot{e} = (A - K_1 C)e - K_2 [Ce]^m + [\Psi(x,u) - \Psi(\hat{x},u)] \quad (4)$$

The following theorem proves observer convergence.

Theorem 1: For the nonlinear system (1b), suppose $x(t)$ exists for all $t \geq 0$ and the nonlinear vector function $\Psi(x,u)$ satisfies the Lipschitz condition (2). If a matrix $0 < P = P^T$ and observer gains K_1 and K_2 can be found such that (3) is an observer for (1b), then the observer error converges to zero exponentially; that is, there exist constants $\kappa > 0$ and $\lambda > 0$ such that

$$\| e(t) \| \leq \kappa \| e(0) \| \exp(-\lambda t)$$

where $\kappa = \sqrt{\dfrac{\beta}{\alpha}}$, $\lambda = \dfrac{\varepsilon}{2\beta}$, $\alpha = \lambda_{\min}(P)$, and $\beta = \lambda_{\max}(P)$.

Proof: Consider the Lyapunov function candidate, $V = e^T P e$, where $0 < P = P^T$ and satisfies Eq. (3b).

Its derivative is

$$\begin{aligned}\dot{V} &= \dot{e}^T P e + e^T P \dot{e} \\ &= e^T \left[(A - K_1 C)^T P + P(A - K_1 C) \right] e \\ &\quad - 2(Ce)^{m-1} e^T P K_2 C e + 2 e^T P [\Psi(x,u) - \Psi(\hat{x},u)]\end{aligned}$$

In [14] the next inequality is presented based on (2) as a lemma which is useful for this proof,

$$2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \leq L^2 e^T P P e + e^T e$$

From Rayleigh inequality [18], and taking into account inequality (3c), we have

$$-2e^T P K_2 C e = -e^T (N + N^T) e \leq -\lambda_{\min}\left(N + N^T\right) \| e \|^2$$

where $N := P K_2 C$.

Eq. (5) leads to

$$\begin{aligned}\dot{V} &\leq e^T \left[(A - K_1 C)^T P + P(A - K_1 C) \right] e \\ &\quad - (Ce)^{m-1} \lambda_{\min}\left(N + N^T\right) \| e \|^2 + L^2 e^T P P e + e^T e \\ &= e^T \left[(A - K_1 C)^T P + P(A - K_1 C) + L^2 P P + I \right] e \\ &\quad - 2(Ce)^{m-1} \lambda_{\min}\left(N + N^T\right) \| e \|^2 \end{aligned} \tag{5}$$

From assumption 1, the second term in the right hand side of the inequality (5) always will be positive or zero,

$$\begin{aligned}\dot{V} &\leq e^T \left[(A - K_1 C)^T P + P(A - K_1 C) + L^2 P P + I \right] e \\ &= -\varepsilon \| e \|^2\end{aligned} \tag{6}$$

We write the Lyapunov function as $V = \| e \|_P^2$, where $\alpha \| e \|^2 \leq V(e) \leq \beta \| e \|^2$, with $\alpha = \lambda_{\min}(P) \in \Re^+$, and $\beta = \lambda_{\max}(P) \in \Re^+$. Taking its derivative and replacing in inequality (6), we obtain

$$\frac{d}{dt} \| e \|_P \leq -\frac{\varepsilon}{2\beta} \| e \|_P$$

Finally, we have the next result $\| e \| \leq \kappa \| e(0) \| \exp(-\lambda t)$, where $\kappa = \sqrt{\dfrac{\beta}{\alpha}}$, and $\lambda = \dfrac{\varepsilon}{2\beta}$.

This implies that system (3) is an observer for system (1b) and the corresponding dynamics of the observer error (4) is exponentially stable.

3. Application to sincronization of chaotic systems

To illustrate the effectiveness of the obtained results, we give two applications to chaotic systems. The former is an application to the denominated Rössler system which presents a chaotic behavior and exhibits the simplest possible strange attractor. Originally, the Rössler system is credited to Otto Rössler, and it is said to be originated from work on chemical kinetics [21] and the second one is the so-called Rikitake system, a model which attempts to explain the reversal of the earth's magnetic field [22].

A. Example 1: Rössler system

We consider the popular nonlinear Rössler's System [19], which is described by

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3) \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \\ y &= x_1 \end{aligned} \tag{7}$$

It is well known that in a large neighborhood of {a= b=0.2, c=5} this system has a chaotic behavior.

Remark 1: It is not difficult to prove that system (7) is Lipschitz.

Before proposing the state observer, we prove the algebraic observability condition (see definition 1) for system (7). Replacing $y = x_1$ into system (7), we obtain

$$\dot{y} = -x_2 - x_3 \tag{8}$$

$$\dot{x}_2 = y + a\, x_2 \tag{9}$$

$$\dot{x}_3 = b + x_3(y - c) \tag{10}$$

Taking the time derivative from Eq. (8)

$$\ddot{y} = -\dot{x}_2 - \dot{x}_3 \tag{11}$$

From Eq. (8), we get

$$x_3 = -\dot{y} - x_2 \tag{12}$$

Replacing Eqs. (9), (10) and (12) into Eq. (11)

$$\ddot{y} - y\,\dot{y} + c\,\dot{y} + y - x_2 y + (a + c)x_2 + b = 0 \tag{13}$$

In the same manner for x_3, we have from Eq. (8)

$$x_2 = -\dot{y} - x_3 \tag{14}$$

substituting Eqs. (9), (10) and (14) into Eq. (11)

$$\ddot{y} - a\,\dot{y} + y + x_3 y - (a + c)x_3 + b = 0 \tag{15}$$

Remark 2: From Eqs. (13) and (15), is clear that x_2 and x_3 are algebraically observable.

According to Theorem 1, we get the following system (slave system) for the observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -(\hat{x}_2 + \hat{x}_3) + k_{1,1}(x_1 - \hat{x}_1) + k_{1,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_2 &= \hat{x}_1 + a\,\hat{x}_2 + k_{2,1}(x_1 - \hat{x}_1) + k_{2,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_3 &= b + \hat{x}_3(\hat{x}_1 - c) + k_{3,1}(x_1 - \hat{x}_1) + k_{3,2}(x_1 - \hat{x}_1)^m \end{aligned} \tag{16}$$

We show some simulations for the Rössler system (7) and its observer given by system (16), we have taken for the parameter values $a = b = 0.2$, $c = 5$, $K_1 = [k_{1,1} \quad k_{2,1} \quad k_{3,1}]^T = [5 \quad -5 \quad 5]^T$, $K_2 = [k_{1,2} \quad k_{2,2} \quad k_{3,2}]^T = [10 \quad 10 \quad 10]^T$, $m = 3$. All simulation results in this paper were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox. The design of the exponential observer presented in this paper is based on the solution of the Riccati Equation which can be obtained by using the Matlab function ARE.

The performance index of the corresponding synchronization process is calculated as [20]

$$J(t) = \frac{1}{t + 0.001} \int_0^t \| e(t) \|_{Q_0}^2 \, d\tau \quad , \quad Q_0 = I$$

where $e(t)$ denotes the estimation error.

Figures 1(a)-(c) show the convergence of the estimated states (slave system) to the real states (master system), without any noise in the system output. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = 3$, $\hat{x}_3 = -4$.

Figures 2(a)-(b) show the chaotic behavior of system (7) and the observer given by system (16), and also show the convergence of the state estimates to the real states, without any noise in the system output.

Furthermore, Figures 3(a)-(c) show the effect of noise in the estimation process. A white noise is added to the measurement ($\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). We can see that the exponential polynomial observer is robust against noisy measurement.

Finally, Figure 4 illustrates the performance index for the corresponding estimation processes. It should be noted that the quadratic estimation error (performance index) is bounded and has a tendency to decrease.

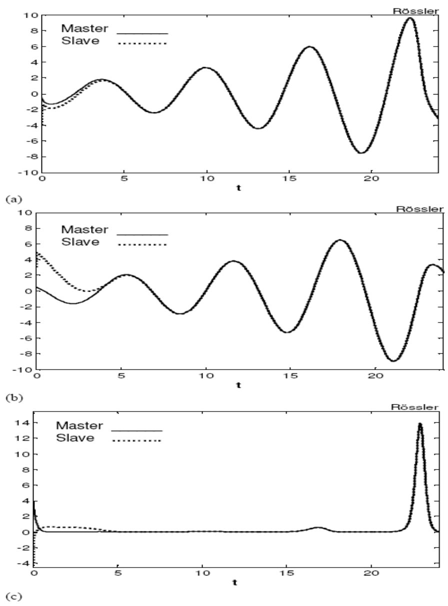

Figure 1. Synchronization between drive system (7) and response system (16), without any noise in the system output, (a) – signals x_1 and $\hat{x}_1$; (b) – signals and x_2; (c) – signals x_3 and $\hat{x}_3$.

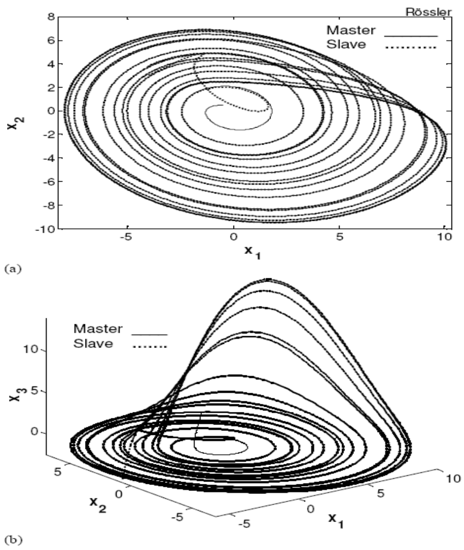

Figure 2. Chaotic behavior of drive system (7) and response system (16), without any noise in the system output, (a) – signals x_1, x_3 and $\hat{x}_1, \hat{x}_3$; (b) – signals x_1, x_2, x_3 and $\hat{x}_1, \hat{x}_2, \hat{x}_3$.

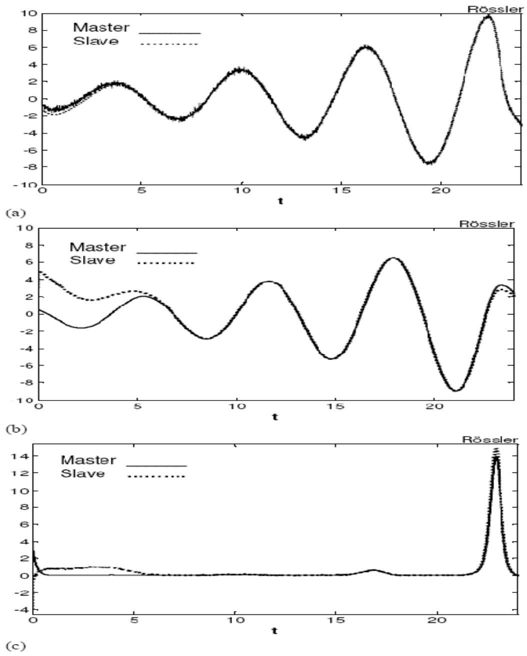

Figure 3. Synchronization between drive system (7) and response system (16), with white noise in the system output $(\sigma = 0.1)$, (a) – signals x_1 and $\hat{x}_1$; (b) – signals x_2 and $\hat{x}_2$; (c) – signals x_3 and $\hat{x}_3$.

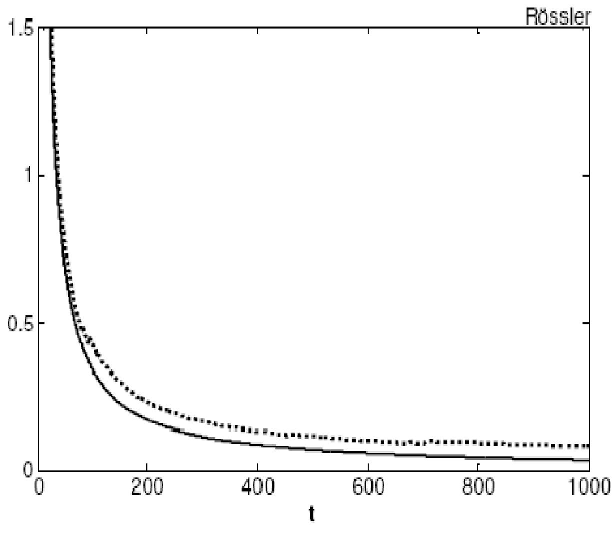

Figure 3. Quadratic estimation error, (a) – without any noise in the system output (solid line); (b) – with white noise $(\sigma = 0.1)$ in the system output (dotted line).

B. Example 2: Rikitake oscillator

This system describes the currents of two coupled dynamo disks [21]. The governing equations are

$$\begin{aligned} \dot{x}_1 &= -\mu\, x_1 + x_2 x_3 \\ \dot{x}_2 &= -\mu\, x_2 + (x_3 - a)x_1 \\ \dot{x}_3 &= 1 - x_2 x_1 \\ y &= x_1 \end{aligned} \tag{17}$$

here a and μ are parameters which we will assume to be nonnegative.

Remark 3: It is not hard to see that the above system is Lipschitz.

Before proposing the exponential polynomial observer, we prove the algebraic observability condition (see definition 1) for system (17).

Replacing $y = x_1$ in system (17), we obtain:

$$\dot{y} = -\mu\, y + x_2 x_3 \tag{18}$$

$$\dot{x}_2 = -\mu\, x_2 + (x_3 - a)y \tag{19}$$

$$\dot{x}_3 = 1 - x_2 y \tag{20}$$

Taking the derivative with respect to time from Eq. (18), we have

$$\ddot{y} = -\mu\, \dot{y} + \dot{x}_2 x_3 + x_2 \dot{x}_3 \tag{21}$$

from Eq. (18), we get

$$x_3 = \frac{1}{x_2}[\, \dot{y} + \mu\, y] \tag{22}$$

substituting Eqs. (19), (20) and (22) into Eq. (21), we obtain

$$\begin{aligned} 0 = x_2^4 - \frac{1}{y}x_2^3 + \left[\frac{\ddot{y}}{y} + 2\mu\frac{\dot{y}}{y} + \mu^2\right]x_2^2 \\ + a\,[\dot{y} + \mu\, y]x_2 - [\dot{y} + \mu\, y]^2 \end{aligned} \tag{23}$$

In the same manner, for x_3 we have from Eq. (18)

$$x_2 = \frac{1}{x_3}[\, \dot{y} + \mu\, y] \tag{24}$$

substituting Eqs. (19), (20) and (24) into Eq. (21)

$$\begin{aligned} 0 = x_3^4 - a\, x_3^3 - \left(\frac{\ddot{y}}{y} + 2\,\mu\frac{\dot{y}}{y} + \mu^2\right)x_3^2 \\ + \left(\frac{\dot{y}}{y} + \mu\right)x_3 - (\dot{y} + \mu\, y)^2 \end{aligned} \tag{25}$$

Remark 4: From Eqs. (23) and (25), x_2 and x_3 are algebraically observable.

Going back to the original coordinate system, we get the following system (slave system) for the observer:

$$\begin{aligned} \dot{\hat{x}}_1 &= -\mu\, \hat{x}_1 + \hat{x}_2\hat{x}_3 + k_{1,1}(x_1 - \hat{x}_1) + k_{1,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_2 &= -\mu\, \hat{x}_2 + (\hat{x}_3 - a)\hat{x}_1 + k_{2,1}(x_1 - \hat{x}_1) + k_{2,2}(x_1 - \hat{x}_1)^m \\ \dot{\hat{x}}_3 &= 1 - \hat{x}_1\hat{x}_2 + k_{3,1}(x_1 - \hat{x}_1) + k_{3,2}(x_1 - \hat{x}_1)^m \end{aligned} \tag{26}$$

Now, some numerical results for the Rikitake system (17) and its observer given by system (26) are presented. System (17) is chaotic with the set of parameter values $\mu = 1$, $a = 0.375$.

We have chosen the parameter values for systems (17) and (26) as $\mu = 1$, $a = 0.375$, $m = 3$,

$$K_1 = [k_{1,1} \quad k_{2,1} \quad k_{3,1}]^T = [2 \quad 2 \quad 2]^T,$$
$$K_2 = [k_{1,2} \quad k_{2,2} \quad k_{3,2}]^T = [3 \quad 3 \quad 3]^T.$$

Figures 5(a)-(c) show the convergence of the estimated states (slave system) to the real states (master system), without any noise in the system output. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -1$, $\hat{x}_3 = 2$.

Figures 6(a)-(c) show the chaotic behavior of the master system (17) and the slave system (26) and also show the convergence of the estimated states (slave system) to the real states (master system) without any noise in the system output.

Figures 7(a)-(c) show the estimated states with the presence of noise in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). It should be noted that the proposed observer is robust against noisy measurements.

Figure 8 illustrates the performance index for the corresponding synchronization process without any noise in the system output and with noise in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). In both cases, the corresponding performance index has a tendency to decrease.

4. Conclusion

In this paper, we have designed a new exponential polynomial observer (high order polynomial type) for a class of nonlinear oscillators to attack the synchronization problem. Also, we have proven the exponential stability of the resulting state estimation error and by means of simple algebraic manipulations we construct the observer (slave system). Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

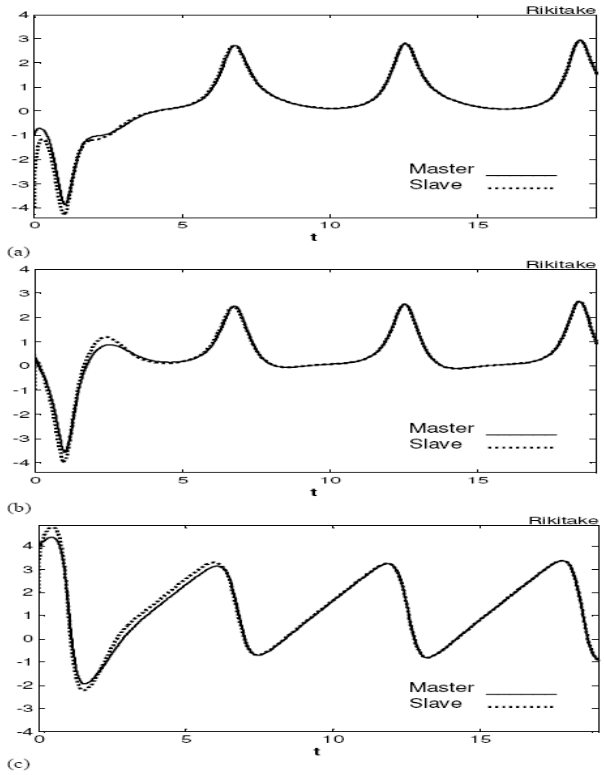

Figure 5. Synchronization between drive system (17) and response system (26), without any noise in the system output, (a) – signals x_1 and $\hat{x}_1$; (b) – signals x_2 and $\hat{x}_2$; (c) – signals x_3 and $\hat{x}_3$.

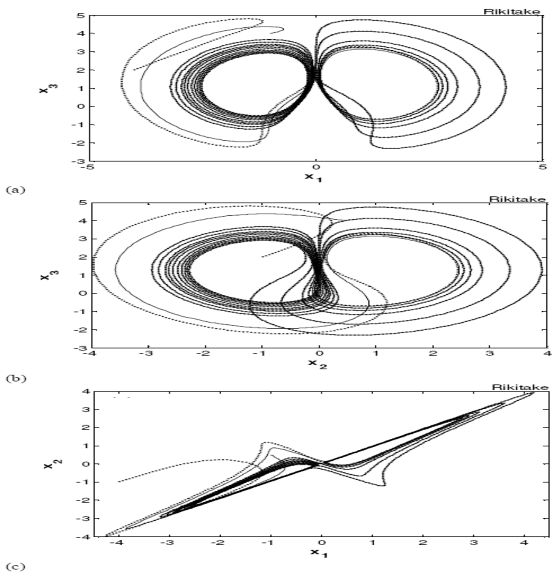

Figure 6. Chaotic behavior of drive system (17) and response system (26), without any noise in the system output, (a) – signals x_1, x_3 and $\hat{x}_1, \hat{x}_3$; (b) – signals x_2, x_3 and $\hat{x}_2, \hat{x}_3$; (c) – signals x_1, x_2 and $\hat{x}_1, \hat{x}_2$.

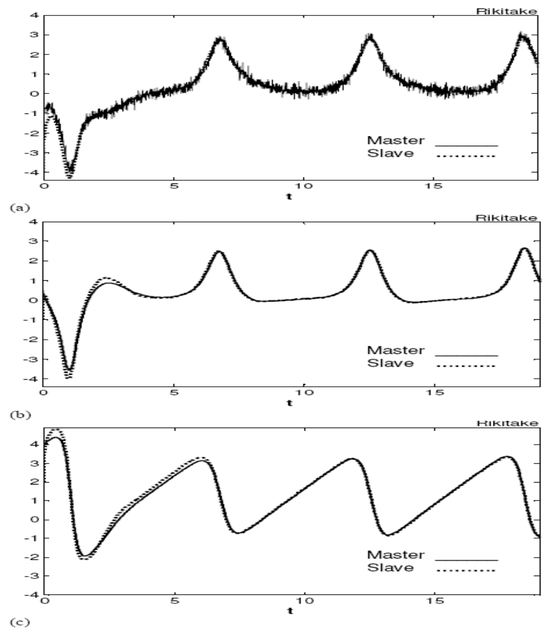

Figure 7. Synchronization between drive system (17) and response system (26), with white noise in the system output $(\sigma = 0.1)$, (a) – signals x_1 and $\hat{x}_1$; (b) – signals x_2 and $\hat{x}_2$; (c) – signals x_3 and $\hat{x}_3$.

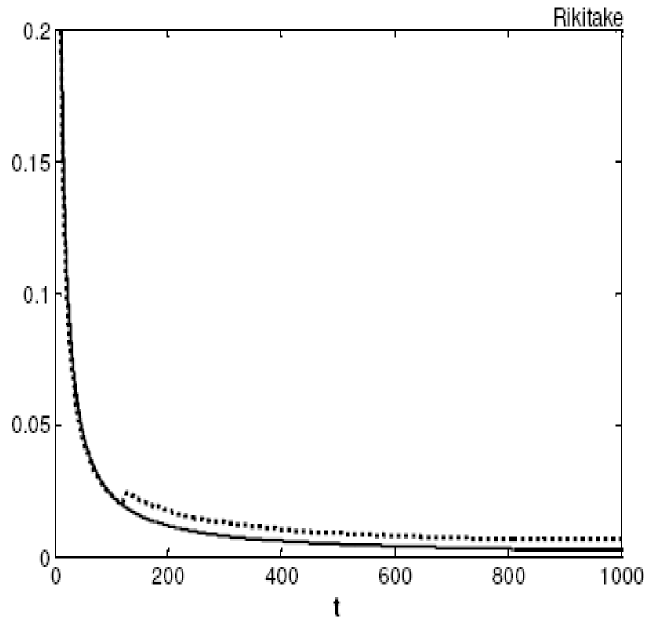

Figure 8. Performance index, (a) – without any noise in the system output (solid line); (b) – with white noise $(\sigma = 0.1)$ in the system output (dotted line).

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DIFFERENTIAL ALGEBRAIC ESTIMATOR FOR THE MONITORING OF A CLASS OF PARTIALLY KNOWN BIOREACTOR MODELS

ESTIMADOR ALGEBRAICO DIFERENCIAL PARA EL MONITOREO DE UNA CLASE DE BIORREACTORES CON MODELOS PARCIALMENTE CONOCIDOS

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Abstract

The problem of monitoring in a common class of partially known bioreactor models is addressed. A reduced order observer namely differential algebraic estimator is proposed. The biomass is estimated by means of substrate concentration measurements. The estimation methodology is based on a suitable change of variable which allows generating artificial variables to infer the remaining mass concentrations constructing a differential-algebraic structure. The proposed methodology is applied to a class of Haldane unstructured kinetic model with success. Stability analysis in a Lyapunov sense for the estimation error is performed. Some remarks about the convergence characteristics of the proposed estimator are given and numerical simulations show its satisfactory performance. Finally, for comparison purposes, a high gain observer is presented: the convergence is possible only when the model is perfectly known.

Keywords: differential-algebraic estimator, state variable estimation, continuous bioreactor, Haldane kinetics.

Resumen

En el presente trabajo se considera el problema del monitoreo de una clase de biorreactores con modelos parcialmente conocidos. Se propone un tipo de observador de orden reducido denominado estimador diferencial algebraico. La metodología de estimación se basa en un cambio de variables que permite generar variables artificiales para inferir las concentraciones no medibles. La metodología propuesta es aplicada a un modelo cinético no estructurado de Haldane con éxito. Se efectúa un análisis de Lyapunov para demostrar la estabilidad de la metodología considerada. Algunos comentarios sobre las características de la convergencia del estimador son proporcionados y simulaciones numéricas muestran un desempeño satisfactorio. Finalmente, con propósitos de comparación, un observador de alta ganancia se presenta en donde su convergencia se garantiza solo cuando se conoce perfectamente el modelo del sistema bajo estudio.

Palabras clave: estimator algebraico-diferencial, estimación de variables de estado, biorreactor continuo, cinética de Haldane.

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1 Introduction

Operating a bioreactor is not a simple task, as during a bioreacting process, variables such as concentrations are generally determined by off-line laboratory analysis, making this set of variables of limited use for control purposes and on-line monitoring. However, these variables can be on-line estimated using *soft sensors*.

Over the last few years, the importance of on-line monitoring of biotechnological processes has increased. A first step to efficient bioreactor operation is the adequate implementation of online measurements of essential variables such as substrate and biomass concentrations. Advantages of continuous monitoring of key variables include gaining knowledge about the state of the process and the possibility of detecting and isolating abnormal process developments at early stages. This reduces process costs, contributes to process safety and helps in trouble-shooting and process accommodation. The main problem in fermentation monitoring and control is the fact that process variables usually cannot be measured on-line. Monitoring and controlling these processes can therefore be difficult because only indirect measurements are available online, while calculated values may be rather uncertain. This can be due to uncertainty with respect to the equations used, measurement errors or both. For automatic control this may have serious consequences, especially as the actual variables of interest often cannot be directly controlled and related variables are controlled instead. In fermentation processes, on-line and off-line measurements are the main source of information about the state of the process. In combination with model-based calculations, they are used to produce estimations for monitoring purposes as well as for automatic and manual process control (Bastin and Dochain, 1990), (Masoud, 1997).

Observation schemes are widely used for reconstructing states of dynamical systems (Aguilar-López *et al.*, 2006). Most of the contributions are related to asymptotic observers for monitoring, fault detections and control issues whereas the real necessities of industrial plants are related to a fast response of the monitoring and regulation methodologies.

Special attention was given to filtering techniques, namely extended Kalman filter, adaptive observers, and artificial neural networks (ANN), (Dávila and Fridman, 2005), (Hu and Wang, 2002), (Levant, 2001), however for these techniques the right tuning of the estimators gains is difficult. It is shown that software based state estimation is a powerful technique that can be successfully used to enhance automatic control performance of biological systems as well as in system monitoring and on-line optimization.

In this paper we consider the growth rate partially known. Following this idea, the necessity to adapt an observation scheme to the available knowledge of the growth rate immediately arises. The main contribution in this work is to show a state estimator which is a simplified version of the methodology given by (Lemesle and Gouzé, 2005) where a simple linear change of variable given in a natural manner allows to develop a differential-algebraic state estimator. Results show an adequate performance of the considered methodology. The technique is not the same as (Alvarez-Ramirez *et al.*, 1999) since we do not have derivators. The proposed estimation methodology is applied to a kind of unstructured kinetic model: the Haldane model, which is considered for biological process with substrate inhibition. The above mentioned kinetic model is applied to a class of continuous stirred bioreactors.

In what follows, the statement of the problem is presented; an observability condition is given in the differential-algebraic setting. In section 3, the bounded error estimator is designed. Section 4 shows a high gain observer as a comparison with the proposed methodology. Finally, we give some concluding remarks.

2 Problem statement

2.1 The model

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where, $x \in R^n$, $u \in R^m$, $m \leq n$, $y \in R^p$.

Let us recall the classical observer definition. An observer for system (1) is a dynamical system $\dot{\hat{x}} = \hat{f}(\hat{x}, u, y)$, whose task is state estimation. Usually is required at least that $\|\hat{x} - x\| \rightarrow 0$ as $t \rightarrow \infty$. Although in some cases, exponentially convergence is also required (Gauthier *et al.*, 1992).

Definition 1: an estimator is said to be bounded if the estimation error ($\|\hat{x} - x\|$) belongs to an open ball with radius proportional to some value that depends on its estimation error.

In all paper, we will consider a class of bioreactor model. The simplified Haldane model taken from (Vargas *et al*., 2000), is described by

$$\frac{dS}{dt} = D\,(S_{in} - S) - \mu(S)\,\frac{X}{Y_{S/X}} + k_d X \quad \text{(2a)}$$

$$\frac{dX}{dt} = -DX + \mu(S)\,X - k_d X \quad \text{(2b)}$$

where $\mu(S) = \mu_{\max} S \big/ \left(\delta + S + S^2 \big/ \varphi\right)$ is the specific growth rate and μ_{max} is the maximum growth rate.

We assume that $\mu(S)$ is partially known, which is common in biology (Gouzé and Lemesle, 2001). Generally, $\mu(S)$ is between two bounds meaning that we know a function $\hat{\mu}(S)$ such that $|\mu(S) - \hat{\mu}(S)| < a$, where $a \in R^+$, and $\mu(0) = \hat{\mu}(0) = 0$. We introduce an important lemma about lower bounded properties of $\mu(S)$.

Lema 1 (Hadj-Sadok, 1999): there exists a constant $\varepsilon \in R$, such that $S(0) > \varepsilon$ implies $S(t) > \varepsilon$ for all t. Thus, for any smooth function $\mu(S)$, $\mu(S(t)) > \mu(\varepsilon)$ for all t.

From lemma 1, we could always choose ε such that $\hat{\mu}(S(t)) > \hat{\mu}(\varepsilon) = r$, where $r \in R^+$.

The state variables S, X are the substrate and biomass concentrations, respectively, $D = q/V$ is the dilution rate with V the volume of the bioreactor and q the constant flow passing through the bioreactor, S_{in} is the input substrate concentration, $Y_{S/X}$ is the corresponding yield coefficient. Let us notice that the inputs $D = u$ and S_{in} are fixed. Moreover, we assume that the measured output is,

$$y = S \quad (3)$$

2.2 Algebraic Observability Condition (AOC)

Before proposing the bounded error estimator, a definition concerning on *algebraic observability condition* is given, for more details see (Diop and Martínez-Guerra, 2001).

Definition 2: consider the system described by (1), where $x = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^T$. A state x_i, $i = \{1, 2, \cdots, n\}$, is said to be algebraically observable with respect to $\{u, y\}$ if it satisfies a differential polynomial in terms of u, y and some of their time derivatives, i.e., $P(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$, $i = \{1, 2, \cdots, n\}$.

Replacing $y = S$ into Eq. (2a), the algebraic observability condition for Haldane model is calculated as follows,

$$\dot{y} - u\,(S_{in} - y) + \left(\frac{\mu_{\max}\varphi\, y}{\delta\varphi + \varphi\, y + y^2}\,\frac{1}{Y_{S/X}} - k_d\right) X = 0 \quad (4)$$

From Eq. (4), it is clear that the state variable X satisfies the AOC thus, X is algebraically observable.

3 Bounded error estimator

3.1 Estimator design

In what follows, the corresponding estimated concentration is denoted by $\hat{}$, and we assume that S is measured exactly, i.e., $S = \hat{S}$. Then, we only reconstruct the biomass variable X.

Consider the Haldane's model given by system (2), and make the change of variable

$$z = X + kS \quad (5)$$

where $k \in R$ is fixed.

The dynamics of z is

$$\begin{aligned} \dot{z} = &-\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right] z + \\ &+ (1-k)\,k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1\right) k\,\mu(S)\,S + k\,D\,S_{in} \end{aligned} \quad (6)$$

Proposition 1: if we choose the estimator's gain such that $Y_{S/X} < k \leqslant 1 + D/k_d$ and $|\mu(S) - \hat{\mu}(S)| < a$, $a \in R^+$. Then, the system (7) is a bounded error estimator of (6).

$$\begin{aligned} \dot{\hat{z}} = &-\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\hat{\mu}(S)\right] \hat{z} + \\ &+ (1-k)\,k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1\right) k\,\hat{\mu}(S)\,S + k\,D\,S_{in} \end{aligned} \quad (7)$$

For the proof, define the estimation error,

$$e = z - \hat{z} \quad (8)$$

Then, using eqs. (6) and (7) the estimation error dynamic is obtained as

$$\begin{aligned} \dot{e} = &-\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\hat{\mu}(S)\right] e + \\ &+ \left(\frac{k}{Y_{S/X}} - 1\right)[\mu(S) - \hat{\mu}(S)]\,k\,S - \left(\frac{k}{Y_{S/X}} - 1\right)[\mu(S) - \hat{\mu}(S)]\,z \end{aligned} \quad (9)$$

To analyze the stability of Eq. (9) we consider the following Lyapunov function candidate

$$V = \frac{1}{2}e^2 \quad (10)$$

The time derivative of Eq. (10) is

$$\dot{V} = e\,\dot{e} \tag{11}$$

Replacing (9) into (11) yields

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\hat{\mu}(S)\right] e^2 + \\ + \left(\frac{k}{Y_{S/X}} - 1\right)[\mu(S) - \hat{\mu}(S)]\,k\,S\;e - \left(\frac{k}{Y_{S/X}} - 1\right) \\ [\mu(S) - \hat{\mu}(S)]\,z\,e \tag{12}$$

Equation (12) is written alternatively as

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\hat{\mu}(S)\right] e^2 - \\ -\left(\frac{k}{Y_{S/X}} - 1\right)[\mu(S) - \hat{\mu}(S)]\;X\,e \tag{13}$$

Now, from lemma 1 and taking into account that $Y_{S/X} < k \leqslant 1 + D/k_d$, $|\mu(S) - \hat{\mu}(S)| < a$, and X is bounded, Eq. (13) leads to,

$$\dot{V} \leqslant -\left[D + k_d - kk_d + \left(\frac{k}{Y_{S/X}} - 1\right)r\right] e^2 \\ + \left(\frac{k}{Y_{S/X}} - 1\right)aX_{\max}\;|e| = -\lambda\,e^2 + w\;|\,e\,|$$

where,

$$\lambda = D + k_d - kk_d + \left(\frac{k}{Y_{S/X}} - 1\right)r$$

and

$$w = \left(\frac{k}{Y_{S/X}} - 1\right)aX_{\max}$$

The right-hand side of the foregoing inequality is not negative since near the origin, the positive linear term $w\,|\,e\,|$ dominates the negative quadratic term $-\lambda e^2$. However, $\dot{V}$ is negative outside the set $\{|\,e\,| \leqslant w/\lambda\}$. Let c, ε be some upper bounds for $V(e)$. With $c > w^2\big/2\lambda^2$, solutions starting in the set $\{V(e) \leqslant c\}$ will remain therein for all time because $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions of Eq. (9) are uniformly bounded (Khalil, 2002). Moreover, if $\left(w^2\big/2\lambda^2\right) < \varepsilon < c$, then $\dot{V}$ will be negative in the set $\{\varepsilon \leqslant V \leqslant c\}$, which shows that, in this set V will decrease monotonically until the solutions enters the set $\{V \leqslant \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \leqslant \varepsilon\}$ since $\dot{V}$ is negative on the boundary $V = \varepsilon$. According to (Khalil, 2002), the solution is uniformly ultimately bounded with the ultimate bound $|\,e\,| \leqslant \sqrt{2\varepsilon}$. For instance, defining c and ε as follows

$$c = \left(\frac{k}{Y_{S/X}}\frac{aX_{\max}}{\lambda}\right)^2, \quad \varepsilon = \left(k\frac{aX_{\max}}{\lambda}\right)^2$$

the ultimate bound is, $|\,e\,| \leqslant \sqrt{2}\,k\frac{aX_{\max}}{\lambda}$.

Corollary 1: if the growth rate is perfectly known, i.e., $\mu(S) = \hat{\mu}(S)$, and we choose the estimator's gain such that $Y_{S/X} < k \leqslant 1 + D/k_d$. Then, the system (14) is an asymptotic estimator of (6).

$$\dot{\hat{z}} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]\hat{z} + \\ + (1-k)\,k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1\right)k\;\mu(S)\;S + k\,D\,S_{in} \tag{14}$$

Indeed, the dynamics of the error in this case is

$$\dot{e} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]e$$

and the corresponding time derivative of Lyapunov function candidate (10) is

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]e^2 < 0$$

Moreover, X can be reconstructed considering

$$\hat{X} = \hat{z} - k\,S \tag{15}$$

3.2 Numerical simulations

For all simulations in this paper we take $S_{in} = 50$, $D = 0.1$, $Y_{S/X} = 0.9$, $k_d = 0.01$ and the initial conditions $S(0) = 60$, $X(0) = 40$, $\hat{X}(0) = 30$, $\hat{z}(0) = 90$, with appropriate units, these values are taken from Vargas *et al.*, (2000). The estimator's gain is $k = 1$. The growth rates are chosen as

$$\mu(S) = \frac{S}{140 + S + S^2\big/81.25}$$

and

$$\hat{\mu}(S) = \frac{0.8\,S}{140 + S + S^2\big/81.25}$$

when the model is well known for the asymptotic estimator and when the model is partially known for the bounded error estimator, respectively. The simulations results were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox.

The performance index of the corresponding estimation process is calculated as (Martínez-Guerra, *et al.*, 2000)

$$J = \frac{1}{t + 0.001} \int_0^t \|e(\tau)\|^2 d\tau \qquad (16)$$

where $e(t)$ is the corresponding state estimation error (the difference between the actual observed signal and its estimate).

First, in Fig. 1 we show the simulation results for the bounded error estimator given by proposition 1, and the corresponding results for the asymptotic estimator given by corollary 1 (without any noise in the system output). Furthermore, in Fig. 2 is shown the effect of noise in the estimation process. A white noise is added in the measurement ($\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output) this is considering the corresponding measurement error of the corresponding sensor and/or experimental measurement technique. We can observe that the bounded error estimator is robust against noisy measurement. Finally, in Fig. 3 is illustrated the performance index given by (16) for the corresponding estimation process. It should be noted that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease.

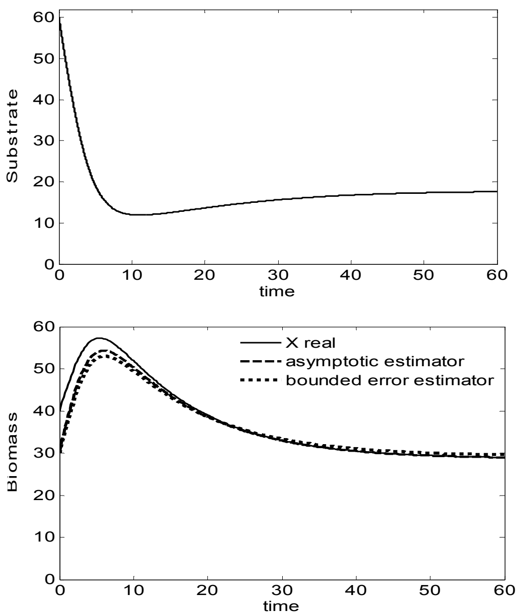


Fig. 1. State Variables.

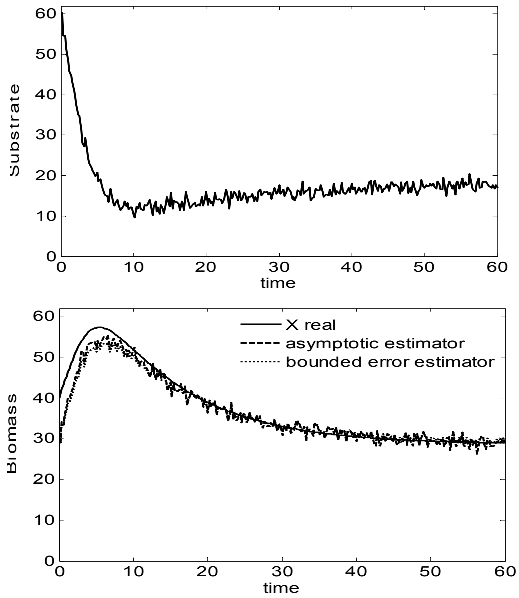
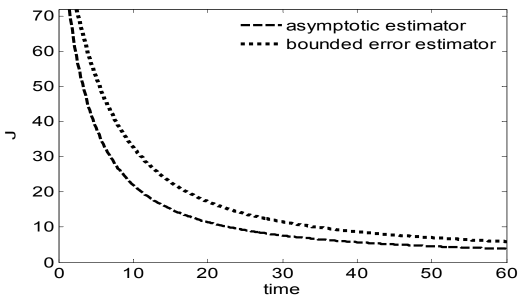
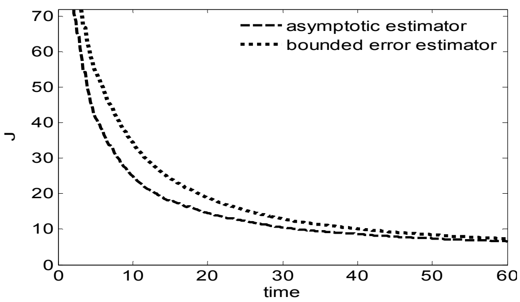


Fig. 2. State Variables (with noise in the system output).



a)



b)

Fig. 3. Quadratic estimation error. (a) Without any noise, (b) with white noise; in the system output.

4 A note on full-order observers: The high gain observer

4.1 Observer design

Consider that system (1) satisfies the AOC. In this case to estimate the state-space vector x, we can suggest a nonlinear high gain observer (Gauthier *et al.*, 1992), (Martínez-Guerra *et al.*, 2000) with the following structure,

$$\dot{\hat{x}} = f(\hat{x}, u) + K(y - C\hat{x}), \quad \hat{x} \in R^n, \quad \hat{x}_0 = \hat{x}(t_0) \tag{17}$$

where the observer's gain matrix is given by,

$$K = S_\theta^{-1} C^T, S_\theta = \left(\frac{1}{\theta^{i+j-1}} S_{ij}\right)_{i,j=1,\ldots,n}$$

and the positive parameter θ determines the desired convergence velocity. Moreover, $S_\theta > 0$, $S_\theta = S_\theta^T$ should be a solution of the algebraic equation,

$$S_\theta\left(E + \frac{\theta}{2}I\right) + \left(E^T + \frac{\theta}{2}I\right) S_\theta = C^T C, \quad E = \begin{pmatrix} 0 & I_{n-1,n-1} \\ 0 & 0 \end{pmatrix}$$

As shown by (Gauthier *et al.*, 1992), (Martínez Guerra and de Leon-Morales, 1996), under certain technical a assumptions (Lipschitz conditions for nonlinear functions under consideration) this nonlinear observer has an arbitrary exponential decay for any initial conditions. We obtain the following high order observer for the system (2) applying the observation scheme (17),

$$\dot{\hat{S}} = D\left(S_{in} - \hat{S}\right) - \frac{\mu_{\max}\hat{S}}{\delta + \hat{S} + \hat{S}^2/\varphi}\frac{\hat{X}}{Y_{S/X}} + k_d\hat{X} - 2\theta\left(\hat{S} - y\right)$$

$$\begin{aligned}\dot{\hat{X}} = {} & -D\hat{X} + \frac{\mu_{\max}\hat{S}}{\delta + \hat{S} + \hat{S}^2/\varphi}\hat{X} - k_d\hat{X} - \\ & - \frac{1}{-\mu_{\max}\hat{S} + Y_{S/X}\left(\delta + \hat{S} + \frac{\hat{S}^2}{\varphi}\right)k_d} \\ & \left\{2\theta\frac{\mu_{\max}\hat{X}\left(\delta - S^2/\varphi\right)}{\left(\delta + \hat{S} + \hat{S}^2/\varphi\right)} + \right. \\ & \left. + \theta^2 Y_{S/X}\left(\delta + \hat{S} + \hat{S}^2/\varphi\right)\right\}\left(\hat{S} - y\right)\end{aligned}$$

4.2 Simulations

In the same way, we show two simulations: when the model is well known and when the model is partially known. The initial conditions for the observer are $\hat{S}(0) = 40$, $\hat{X}(0) = 30$, with appropriate units. The estimator's gain is $\theta = 2$. The simulations results of high gain observer are presented in figs. 4 and 5. In Fig. 4, without any noise in the system output, when the model is perfectly known the rate of convergence is fast, on the other hand, when the model is partially known the observer does not reconstruct the state variables. In Fig. 5, we studied the effect of noise in the measurement (white noise with $\sigma = 0.1$, ±5% around the current value of the measured output), we can see that the high gain observer is very sensitive to the noise in the system output. Fig. 6 shows the performance index. It should be noted that this observer only reconstruct the state variables when the model is well known.

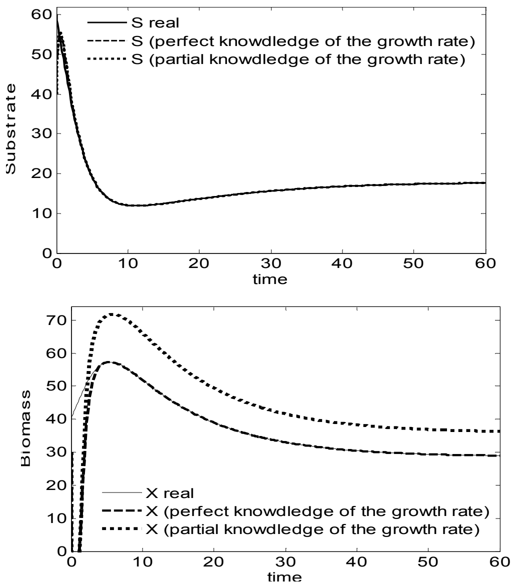


Fig. 4. State variables.

Conclusions

In this paper we have presented a bounded error estimator for bioprocess with unstructured growth models. We have proven the stability of the corresponding estimation error in a Lyapunov sense. By means of a linear change of variable given in a natural manner and with some algebraic manipulations have been constructed the state estimator, which converges to the current states of the reference model given. We have demonstrated that the bounded error estimator under consideration provides good enough

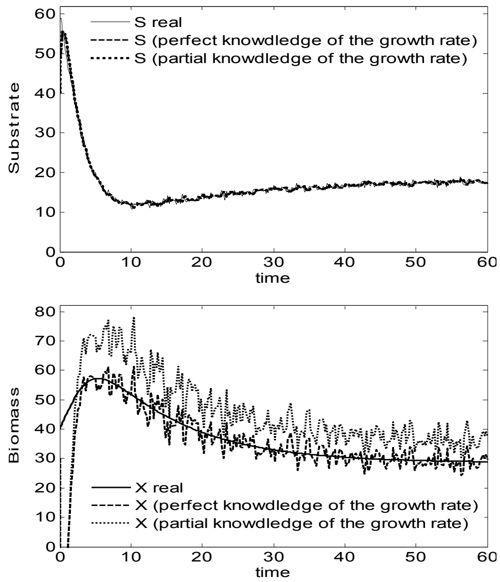
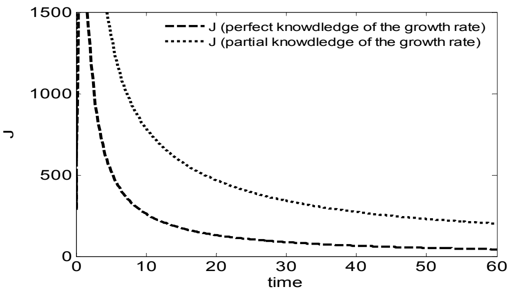
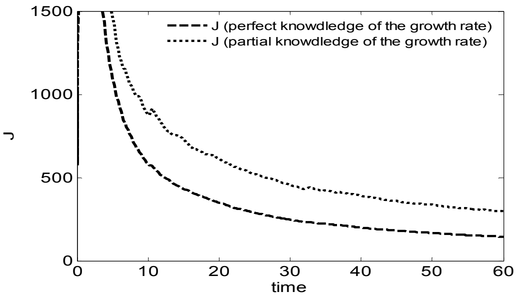


Fig. 5. State variables (with noise in the system output).



a)



b)

Fig. 6. Quadratic estimation error. (a) Without any noise, (b) with white noise; in the system output.

state-space estimates which were bounded on average, besides the proposed state estimator does not depend of a particular set of initial conditions or specific model structure. Moreover, we have constructed a high gain observer in which the convergence is fast only if the model is well known, but does not exists convergence if the model is partially known. Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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Monitoring in a predator–prey systems via a class of high order observer design

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ABSTRACT

The goal of this work is the monitoring of the corresponding species in a class of predator–prey systems, this issue is important from the ecology point of view to analyze the population dynamics. The above is done via a nonlinear observer design which contains on its structure a high order polynomial form of the estimation error. A theoretical frame is provided in order to show the convergence characteristics of the proposed observer, where it can be concluded that the performance of the observer is improved as the order of the polynomial is high. The proposed methodology is applied to a class of Lotka–Volterra systems with two and three species. Finally, numerical simulations present the performance of the proposed observer.

1. Introduction

Ecological systems and their component biological populations exhibit a broad spectrum of non-equilibrium dynamics ranging from characteristic natural cycles to more complex chaotic oscillations (May, 1973; Ranta and Kaitala, 1997; Royama, 1992), a diversity of abiotic variables, spatial and temporal heterogeneity, and most importantly, the presence of other species (as food, as competitors, and as predators) all affect the population dynamics of every species. The monitoring in an ecological system with several populations is generally a difficult task, because only the density of certain populations can be observed or measured.

System analysis, either static or dynamic, frequently involves uncertain parameters and inputs. Propagating these uncertainties through a complex model to determine their effect on system states and outputs can be a challenging problem, especially for dynamic models. From the above, the uncertainties presents on the ecological modeling, together with the corresponding variable measured for these kinds of systems can be very important; in consequence the modeling tasks can be difficult (Gámez et al., 2008a).

On other hand, control theory provides a useful tool to design mathematical algorithms to infer unmeasured variables from the corresponding measured ones, these algorithms are called state observers, a state observer is a system that models a real system in order to provide an estimate of its internal state, given measurements of the input and output of the real system. It is typically a computer-implemented mathematical model.

Knowing the state system is necessary to solve many control theory problems; for example, stabilizing a system using state feedback. In most practical cases, the physical state of the system cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs. If a system is observable, it is possible to reconstruct the system state from its output measurements using the state observer. In particular the local observability conditions for several Lotka–Volterra models were analyzed in López et al. (2007a) and Shamandy (2005).

The application of the concept of observability to the monitoring of population systems goes back to Varga (1992) where, concerning frequency-dependent population models, a general sufficient condition for local observability of nonlinear dynamic systems with invariant manifold was developed and applied. Later this method was applied to different models of population genetics and evolutionary dynamics in Gámez et al. (2003) and López et al. (2004, 2005). Observer design in frequency-dependent population models was studied in López et al. (2008). Different Lotka–Volterra models were considered for observability and observer design in Gámez et al. (2008a,b), López et al. (2007a,b) and Varga et al. (2003). Monitoring problems of non-Lotka–Volterra type ecological and cell population models were studied in Gámez et al. (2009), Shamandy (2005) and Varga et al. (2009). An approach based on a new system inversion method was applied in Szigeti et al. (2002) to monitoring of a five-species predator–prey system.

Following these ideas, the estimation theory deserves an interesting research field, because the estimation methodolo-

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gies developed are widely employed in online monitoring, fault detection, and control for a wide class of systems. Some of the most important estimation methodologies are related with the observers design, where nonlinear techniques (Jarre-Teichmann, 1998; Kalman, 1960; Keller, 1987; Levant, 2001; Rapaport and Harmand, 2002) have been presented in the open literature, these nonlinear approaches provides robustness against modeling errors and noisy measurements. On the other hand some techniques as neural-networks have been successfully used too (Spitz and Lek, 1999).

2. Predator–Prey Mathematical Models

As is it well known, the Lotka–Volterra model describes interactions between several species in an ecosystem, predators and preys (Freedman, 1980; Hinrichsen and Pritchard, 2005; May, 1973). This represents our first multi-species model under study. Firstly, let us consider a two-species model, this model involves two equations, one which describes how the prey population changes and the second which describes how the predator population changes.

If we let x_1 and x_2 represent the number of preys and predators, respectively, that are alive at time t, then the Lotka–Volterra model is:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(a - bx_2)\\ \frac{dx_2}{dt} &= x_2(-c + ex_1)\end{aligned} \tag{1}$$

where the parameters are defined by: a is the natural growth rate of prey in the absence of predation, c is the natural death rate of predator in the absence of food, b is the death rate per encounter of prey due to predation, e is the efficiency of turning predated prey into predator. Let us consider $y = x_1$ as the measured output, that is, we suppose that we observe the total quantity of population preys.

Many population cycles have the unusual property that their period length remains remarkably constant while their abundance levels are highly erratic. In López et al. (2007a) and May and Leonard (1975) is shown that such more complex oscillations can be achieved in simple predator–prey models by including more tropic levels. As a second case, we consider a three-species predator–prey model of two preys and one predator of the form $dx/dt = f(x)$, determined by the following system:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(a_1 - b_1x_1 - c_1x_2)\\ \frac{dx_2}{dt} &= x_2(-a_2 + b_2x_1 + c_2x_3)\\ \frac{dx_3}{dt} &= x_3(a_3 - b_3x_2 - c_3x_3)\end{aligned} \tag{2}$$

Here x_1, x_3 are prey populations, x_2 is the predator population, and $a_i, b_i, c_i > 0$ for all $i = 1, 2, 3$. Moreover, $y = x_1$ is the system measured output.

Since populations are nonnegative, we will restrict our attention to the nonnegative orthants $\{(x_1, x_2, x_3) \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\} \subset R^3$ and the positive orthants $R^3 = \{(x_1, x_2, x_3) \mid x_1 > 0, x_2 > 0, x_3 > 0\} \subset R^3$.

It is worthwhile to mention that the both the nonnegative and the positive quadrants are positively invariant for the general Lotka–Volterra model.

3. Proposed Observer

Now, consider a general representation of the Lotka–Volterra models, which can be generally described by the following lumped parameter model (Eqs. (3a) and (3b)):

$$\frac{dx}{dt} = f(x) \tag{3a}$$

$$y = h(x) = Cx \tag{3b}$$

Here, $C = (\,1 \quad 0 \ldots \quad 0\,)$, $x \in \Re^n$ is the vector of states; $f : \Re^n \to \Re^n$ is a nonlinear vector field and $y \in \Re^m$ is the system measured output, with $m < n$.

It is necessary the design of an auxiliary system so called *observer system* to reconstruct the unknown states or unmeasurables. Firstly, we give necessary and sufficient conditions to establish whether the system (3a) and (3b) is observable. Now, consider the following assumptions:

A1. The system given in Eqs. (3a) and (3b) is locally uniformly observable (Gauthier et al., 1992), hence for all $x \in \Re^n$, satisfies the observability rank condition

$$rank\ \left\{\frac{\partial}{\partial x}\vartheta\right\} = n \tag{4}$$

Here ϑ is the observability vector function defined as $\vartheta = (dL_f^0 h, dL_f^1 h, \ldots, dL_f^{n-1} h)^T$, being $L_f^r h$ the r-order Lie derivatives, which are the directional derivatives of the corresponding state variables along the measured output trajectory. And $dL_f^r h$ are the differentials of the rth-order Lie derivatives defined recursively as follows:

$$L_f^0 h{:=}h, \quad dL_f^0 h{:=}dh = \left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right)$$

$$L_f^1 h := \langle dh, f\rangle = \sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i,$$

$$dL_f^1 h := \left(\frac{\partial}{\partial x_1}\left(\sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i\right), \ldots, \frac{\partial}{\partial x_n}\left(\sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i\right)\right)$$

$$L_f^r h{:=}\left\langle dL_f^{r-1} h, f\right\rangle = L_f(L_f^{r-1} h), \quad r \geq 2$$

A2. All the trajectories $x(t, x_0)$, $x_0 \in \Re^n$ of the system ((3a) and (3b)) are bounded.

Considering the set $\Omega \subset \Re^n$ as the corresponding physically realizable domain, such that: $\Omega = \{(x_i)_{i=1}^n \in \Re_+^n / 0 \leq x_i \leq x_{\max}\}$.

In most practical cases, Ω will be an open connected relatively compact subset of $\Re^n$, and in the ideal cases, Ω will be positively invariant under the dynamics (3a) and (3b).

In order to analyze the estimation error $\xi = x - \hat{x}$ we consider the next assumption.

A3. The nonlinear difference vector function $\Delta\Phi = f(x) - f(\hat{x})$ is Lipschitz bounded, i.e. $\left|\Delta\Phi\right| \leq \Lambda \left|\xi\right|$

Condition A3 can be fulfilled satisfied if the following supremum is finite:

$$\Lambda{:=}\sup_{x \in \Omega} \left\|f'(x)\right\|$$

where $f'(x)$ is the Jacobian and $\|\cdot\|$ is the matrix norm associated with the Euclidian vector norm. This is the case in our models (1) and (2), when considered in a bounded part of the positive orthant.

Proposition 1. *Consider that system (3a) and (3b) satisfies* A1, A2 and A3 (*with these assumptions is possible to construct an observer*) *then, there exist* k_1, k_2 *such that for any positive odd integer* $w > 1$, system (5) *is an asymptotic observer for system* (3a) and (3b)

$$\frac{d\hat{x}}{dt} = f(\hat{x}) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \tag{5}$$

See the appendix for the corresponding proof.

Before designing the asymptotic observer for systems (1) and (2), assumptions A1–A3 are checked.

For two-species predator–prey system (1) the observability condition given by assumption A1 is proven. The observability matrix is

$$\frac{\partial \vartheta}{\partial x} = \begin{bmatrix} 1 & 0 \\ a - bx_2 & -bx_1 \end{bmatrix}$$

Its determinant is:

$$\det\left(\frac{\partial \vartheta}{\partial x}\right) = -bx_1$$

The observability rank condition is satisfied if and only if $x_1 \neq 0$. Then by assumption A1 the two-species system (1) is locally uniformly observable, that is, the whole system can be monitored observing only the prey population ($y = x_1$), without any further condition.

As will be shown in Section 4, for systems (1) and (2), is not difficult to provide a simple algebraic condition for the existence of an equilibrium in mathematical sense, however its positivity depends on the model parameters, that means, for any set of initial conditions the trajectories for $t \geq 0$ tend or cycle around this equilibrium point in the positive quadrant, in this form, condition A2 is satisfied.

In the same manner the three-dimensional predator–prey system (2), the observability matrix is given by:

$$\frac{\partial \vartheta}{\partial x} = \begin{pmatrix} 1 & 0 & 0 \\ a_1 - 2b_1x_1 - c_1x_2 & -c_1x_1 & 0 \\ \dfrac{\partial \vartheta_3}{\partial x_1} & \dfrac{\partial \vartheta_3}{\partial x_2} & -c_1c_2x_1x_2 \end{pmatrix}$$

where

$$\frac{\partial \vartheta_3}{\partial x_1} = (a_1 - 2b_1x_1 - c_1x_2)^2 - 2b_1x_1(a_1 - b_1x_1 - c_1x_2) - c_1x_2(-a_2 + 2b_2x_1 + c_2x_3)$$

$$\frac{\partial \vartheta_3}{\partial x_2} = -c_1x_1(2a_1 - a_2 + (b_2 - 3b_1)x_1 - 2c_1x_2 + c_2x_3)$$

Then, $\det\left(\frac{\partial \vartheta}{\partial x}\right) = c_1^2c_2x_1^2x_2$

If populations x_1, x_2 are nonzero, then the observability rank condition is fulfilled. By using assumption A1 the three-species model (2) is locally uniformly observable, in this sense, all state variables can be estimated if we only observe the prey population x_1.

Notice also that, if we take the initial condition such that $x_1(0) = 0$, then for all $t \geq 0$, the output is zero. In these systems, preys populations cannot grow if they are not present at the beginning of the story.

Corollary 1. *The proposed observer for Lotka–Volterra model given by system* (1) *is as follows*:

$$\begin{aligned} \frac{d\hat{x}_1}{dt} &= \hat{x}_1(a - b\hat{x}_2) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \\ \frac{d\hat{x}_2}{dt} &= \hat{x}_2(-c + e\hat{x}_1) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \end{aligned} \tag{6}$$

where $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

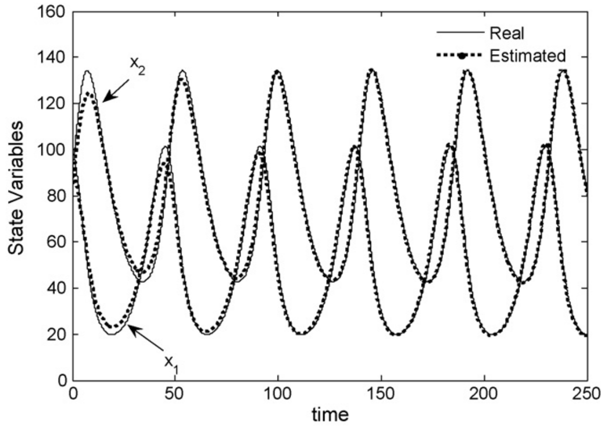

Fig. 1. Estimated states for Lotka–Volterra model.

Corollary 2. *The proposed observer for the three order predator–prey model given by system* (2) *is*:

$$\begin{aligned} \frac{d\hat{x}_1}{dt} &= \hat{x}_1(a_1 - b_1\hat{x}_1 - c_1\hat{x}_2) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \\ \frac{d\hat{x}_2}{dt} &= \hat{x}_2(-a_2 + b_2\hat{x}_1 + c_2\hat{x}_3) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \\ \frac{d\hat{x}_3}{dt} &= \hat{x}_3(a_3 - b_3\hat{x}_2 - c_3\hat{x}_3) + k_1(y - C\hat{x}) + k_2(y - C\hat{x})^w \end{aligned} \tag{7}$$

with $C = (1 \quad 0 \quad 0)$.

4. Numerical Examples and Results

In order to measure the performance of the proposed observer under different polynomial degrees is employing the measure the impact of the error, suggested in Ogunnaike and Ray (1994), is the "Integral Time-Weighted Squared Error" (ITSE) defined in (8). ITSE exhibits the advantage of heavy penalization of large errors at long time; therefore is a good measure of resilience of the observer.

$$\text{ITSE} = \int_0^\infty t\varepsilon^2 \, dt \tag{8}$$

Firstly, we present some simulations for Lotka–Volterra model given in (1) and its corresponding observer (6). We have taken the parameters values as: $a = 0.2$, $b = 0.0025$, $e = 0.002$, $c = 0.1$, $k_1 = 1$, $k_2 = 1$, and the initial conditions $x_1 = 100$, $x_2 = 90$, $\hat{x}_1 = 90$, $\hat{x}_2 = 85$.

In Fig. 1 are shown the state variables and their corresponding estimated states, using $w = 3$ in system (6). The model mirrors a qualitative feature which has been observed in many real predator–prey systems: *the periodic fluctuations*. This is illustrated in Fig. 2, where $\bar{x} = (c/e, a/b) = (50, 80)$ is an equilibrium point of (1) and any initial state $x(0) \neq \bar{x}$, $x_1(0) > 0$, $x_2(0) > 0$ leads to a periodic trajectory cycling around this equilibrium point in the positive quadrant.

Moreover, in Fig. 3, is presented the performance index (ITSE) given in (8). We have taken $w = 3, 5, 7$, for the observer (6). It should be noted that the value of the corresponding performance index decrease as w increase.

Now, we present some simulations for three order predator–prey model given in (2) and its corresponding observer (7). We have chosen the parameters values as: $a_1 = 1$, $a_2 = 0.5$, $a_3 = 3.2$, $b_1 = 0.01$, $b_2 = 0.01$, $b_3 = 0.03$, $c_1 = 0.01$, $c_2 = 0.02$, $c_3 = 0.04$, $k_1 = 5$, $k_2 = 10$, and the initial conditions $x_1 = 90$, $x_2 = 100$, $x_3 = 80$, $\hat{x}_1 = 105$, $\hat{x}_2 = 85$, $\hat{x}_3 = 89$.

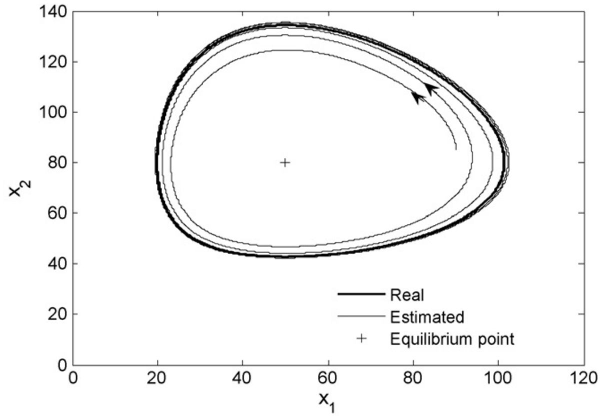

Fig. 2. Lotka–Volterra trajectories.

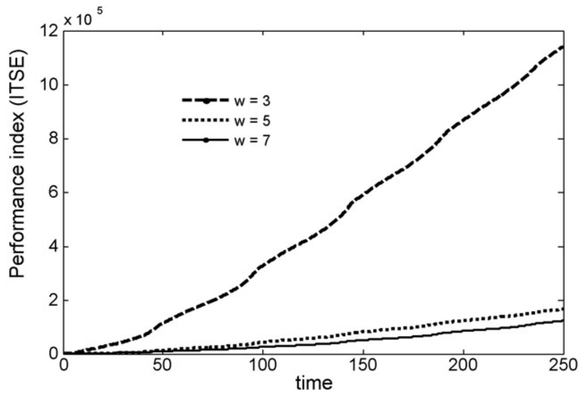

Fig. 3. Performance index (Lotka–Volterra model).

In Fig. 4 are shown the state variables and their corresponding estimated states, using $w = 3$ in system (7). The model has no periodic fluctuations. This is illustrated in Fig. 5. By some algebraic manipulations an asymptotically stable equilibrium point of (2) is obtained as:

$$\bar{x}_1 = \frac{a_2c_1c_3 + a_1b_3c_2 - a_3c_1c_2}{b_1b_3c_2 + b_2c_1c_3} = 16$$

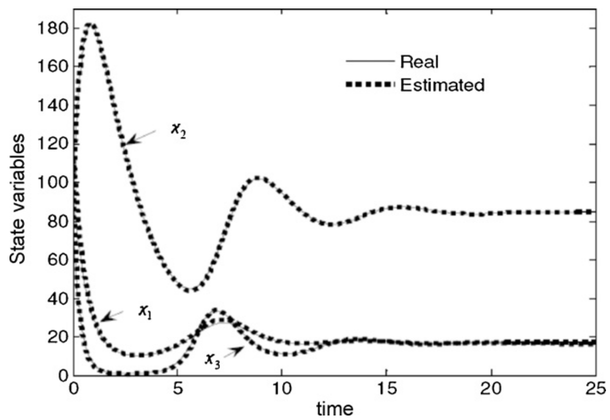

Fig. 4. Estimated states for three order predator–prey model.

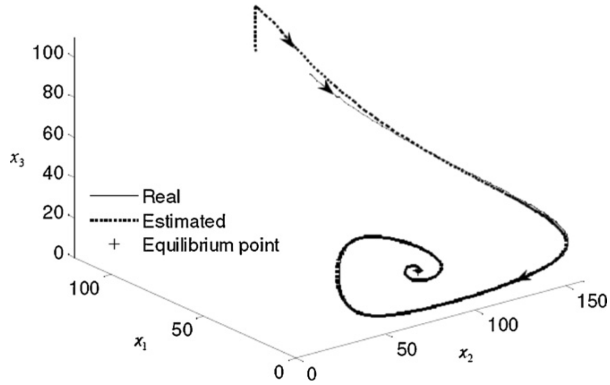

Fig. 5. Three order predator–prey model trajectories.

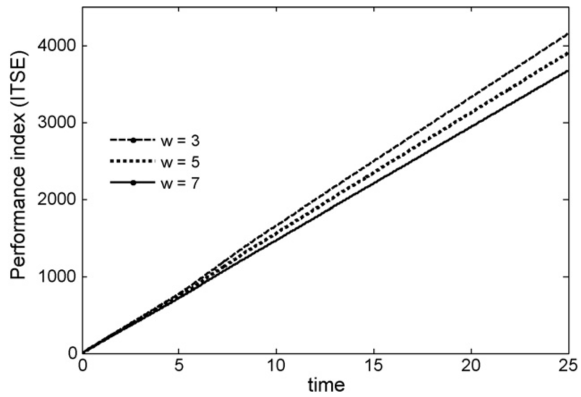

Fig. 6. Performance index for predator–prey model.

$$\bar{x}_2 = \frac{a_1b_2c_3 - a_2b_1c_3 + a_3b_1c_2}{b_1b_3c_2 + b_2c_1c_3} = 84$$

$$\bar{x}_3 = \frac{a_2b_1b_3 + a_3b_2c_1 - a_1b_2b_3}{b_1b_3c_2 + b_2c_1c_3} = 17$$

Any initial state $0 < x_1(0) \neq \bar{x}_1$, $0 < x_2(0) \neq \bar{x}_2$, $0 < x_3(0) \neq \bar{x}_3$ leads to this equilibrium point in the positive quadrant.

Furthermore, in Fig. 6, is presented the performance index (ITSE), for three order predator–prey model (2). We have taken $w = 3,\ 5,\ 7$, for the observer (7). It should be noted that the value of the corresponding performance index decrease as w increase.

5. Concluding Remarks

In this work we have presented a high order polynomial observer to attack the problem of monitoring in predator–prey systems. By some algebraic manipulations the convergence of the corresponding estimation error was proven. The estimation error depends on the observer gains and a Lipschitz constant. The proposed methodology was applied to a class of Lotka–Volterra model with two and three species with success.

Appendix A.

Proof of Proposition 1. Defining the observation error as $\xi = x - \hat{x}$ the corresponding dynamic of the estimation error is:

$$\dot{\xi} = \Delta\Phi(\xi) + k_1C\xi + k_2C^w\xi^w \tag{9}$$

Considering assumption A3:

$$\left|\dot{\xi}\right| \le \Lambda\left|\xi\right| + k_1 C\left|\xi\right| + k_2 C^w\left|\xi^w\right| = (\Lambda + k_1 C)\left|\xi\right| + k_2 C^w\left|\xi^w\right| \quad (10)$$

Considering the i-th coordinate in the above vector inequality and employing the equality:

$$\left|\xi\right| = \mathrm{sign}(\xi)\xi,$$

the following equation is obtained:

$$sign(\dot{\xi}_i)\dot{\xi}_i \le (\Lambda + k_1 C)_i sign(\xi_i)\xi_i + (k_2 C^w)_i sign({\xi_i}^w){\xi_i}^w \quad (11)$$

i.e.

$$\dot{\xi}_i \le \frac{(\Lambda + k_1 C)_i\, sign(\xi_i)}{sign(\dot{\xi}_i)} e_i + \frac{(k_2 C^w)_i sign({\xi_i}^w)}{sign(\dot{\xi}_i)} {e_i}^w; \quad \text{if } sign(\dot{\xi}_i) > 0 \quad (12)$$

defining:

$$\pi_{1i} = \frac{(\Lambda + k_1 C)_i sign(\xi_i)}{sign(\dot{\xi}_i)}; \quad \pi_{2i} = \frac{(k_2 C^w)_i sign({\xi_i}^w)}{sign(\dot{\xi}_i)}$$

The following inequality is obtained:

$$\dot{\xi}_i - \pi_{1i}\xi_i \le \pi_{2i}{\xi_i}^w \quad (13)$$

To solve the above inequality, consider the change of variable:

$$\gamma_i = {\xi_i}^{1-w}, \quad w > 1 \quad (14)$$

Thus

$$\xi_i = 0 \Rightarrow \gamma_i = 0$$

and

$$\xi_i \ne 0 \Rightarrow \gamma_i \ne 0, \quad \text{i.e.}$$

$$\left\{\begin{array}{l} \xi_i > 0 \\ \xi_i < 0, \quad w\,\text{odd} \end{array}\right\} \Rightarrow \gamma_i > 0$$

Hereafter we consider $\xi_i \ne 0$ with $w \in Z^+$, w odd, $w > 1$.

Therefore the following first order ordinary differential inequality is generated:

$$\dot{\gamma}_i - (1-w)\pi_{1i}\gamma_i \ge (1-w)\pi_{2i} \quad (15)$$

By solving the inequality above:

$$\gamma_i \ge \gamma_{0i}\exp(-\{w-1\}\pi_{1i}t) + \frac{\pi_{2i}}{\pi_{1i}}(1 - \exp(-\{w-1\}\pi_{1i}t)) \quad (16)$$

For $t \to \infty$ we get:

$$\gamma_i \ge \frac{\pi_{2i}}{\pi_{1i}} \quad (17)$$

In terms of the original variable (observation error):

$$\xi_i \le \left(\frac{\pi_{1i}}{\pi_{2i}}\right)^{1/w-1} = \left(\frac{(\Lambda + k_1 C)_i\, sign(\xi_i)}{(k_2 C^w)_i\, sign({\xi_i}^w)}\right)^{1/w-1}, \quad w > 1 \quad (18)$$

Remark 1. Note that the estimation error can be diminished arbitrarily, considering k_2 large enough or considering k_1 small enough, besides as w increases the estimation error is diminished as can be noticed from Eq. (18).

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An exponential polynomial observer for synchronization of chaotic systems

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ABSTRACT

In this paper, we consider the synchronization problem via nonlinear observer design. A new exponential polynomial observer for a class of nonlinear oscillators is proposed, which is robust against output noises. A sufficient condition for synchronization is derived analytically with the help of Lyapunov stability theory. The proposed technique has been applied to synchronize chaotic systems (Rikitake and Rössler systems) by means of numerical simulation.

1. Introduction

In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields [1–5]. Among the publications dedicated to chaos synchronization, many different approaches can be found. We cite the papers [6–10] which propose the use of state observers to synchronize chaotic systems; in Refs. [11–13] use feedback controllers; in [14,15] use nonlinear backstepping control; in papers [16,17] consider synchronization time delayed systems; in works [18,19] consider directional and bidirectional linear coupling; papers [20,21] use nonlinear control; in papers [11,12] use active control, in [13,22] use adaptive control and so on. A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master–slave configuration [23,24]. Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer [4,9,24]. In this configuration, the two coupled systems are identical and therefore identical synchronization occurs which means that the difference of master and slave state vectors converges to zero for $t \to \infty$.

In this paper, the synchronization scheme is proposed for a class of Lipschitz nonlinear systems. Many problems in engineering and other applications are globally Lipschitz for instance the sinusoidal terms in robotics. Nonlinearities which are square or cubic in nature are not globally Lipschitz, however, they are locally so, moreover when such functions occur in physical systems, they frequently have a saturation in their growth rate, making them globally Lipschitz functions [25]. In this work, we have considered the trajectories are defined in the same invariant set, i.e., we consider an invariant set which contains the globally Lipschitz functions in this region. Based on this fact, we state that trajectories remain in an invariant set. Thus, this class of systems covered by this note is fairly general.

The main contribution of this paper consists in the solution of the synchronization problem via an exponential polynomial observer. In [25–27] established existence conditions of the full-order observers for Lipschitz nonlinear systems. The main

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purpose in this work is to extend those results by showing that the conditions given in [27], also guarantee the existence of a full-order observer with a high-order correction term. The reason is very simple, as it is well known an extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance a high-order term is now included in the observer structure.

The intention of choosing two examples as the Rössler and Rikitake systems is to clarify the proposed methodology. However, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

2. Exponential polynomial observer

2.1. Problem statement

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables; $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$, $(l \leqslant n)$ is a nonlinear smooth vector function Lipschitz in x and uniformly bounded in u in an invariant set, $u \in \mathbb{R}^l$ is the control input, $y \in \mathbb{R}$ is the measured output.

Any nonlinear system of the form of Eq. (1) can be expressed in the form of Eq. (2) as long as $f(x,u)$ is differentiable with respect to x.

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx, \quad x_0 = x(t_0) \end{aligned} \tag{2}$$

In system (2), $\Psi(x,u)$ is a nonlinear vector function Lipschitz in x and uniformly bounded in u in an invariant set, with a Lipschitz constant L, i.e.,

$$\|\Psi(x,u) - \Psi(\hat{x},u)\| \leqslant L\|x - \hat{x}\| \tag{3}$$

2.2. A note on algebraic observability condition (AOC)

Before proposing the exponential polynomial observer, a definition concerning on algebraic observability condition is given (for more details see [28]).

Definition 1. Consider the system described by Eq. (2), where $x \in \mathbb{R}^n$. A state x_i, is said to be algebraically observable with respect to $\{u, y\}$ if it satisfies a differential polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e., $P(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0,\ 1 \leqslant i \leqslant n$.

2.3. Observer design

We consider the system (2), the observer has the next form

$$\dot{\hat{x}} = A\hat{x} + \Psi(\hat{x},u) + K_1C(x - \hat{x}) + K_2[C(x - \hat{x})]^m, \qquad \hat{x}_0 = \hat{x}(t_0) \tag{4}$$

where $x \in \mathbb{R}^n$, $K_i = \begin{bmatrix} k_{1,i} & k_{2,i} & \cdots & k_{n,i} \end{bmatrix}^T \in \mathbb{R}^n,\ i = 1,2$.

We make the following assumptions throughout this paper:

Assumption 1. $m \in \mathbb{Z}^+,\ m$ odd, $m > 1$.

Assumption 2. K_1 can be chosen such as the following Algebraic Riccati Equation (ARE) has a symmetric positive-definite solution P for some $\epsilon > 0$

$$(A - K_1C)^T P + P(A - K_1C) + L^2PP + I + \epsilon I = 0$$

Assumption 3. K_2 can be chosen such as the following relation holds:

$$\lambda_{min}(PK_2C) \geqslant 0$$

We analyze the observer error which is defined as $e = x - \hat{x}$. From Eqs. (2) and (4), the dynamics of the observer error are given by

$$\dot{e} = (A - K_1C)e - K_2[Ce]^m + [\Psi(x,u) - \Psi(\hat{x},u)] \tag{5}$$

The following theorem proves observer convergence.

Theorem 1. *For the nonlinear system* (2)*, suppose* $x(t)$ *exists for all* $t \geqslant 0$ *and the nonlinear vector function* $\Psi(x,u)$ *satisfies the Lipschitz condition* (3)*. If a matrix* $0 < P = P^T$ *and observer gains* K_1 and K_2 *can be found such that* Eq. (4) *is an observer for system* (2) *then, the observer error converges to zero exponentially; that is, there exists constants* $\kappa > 0$ and $\lambda > 0$ *such that*

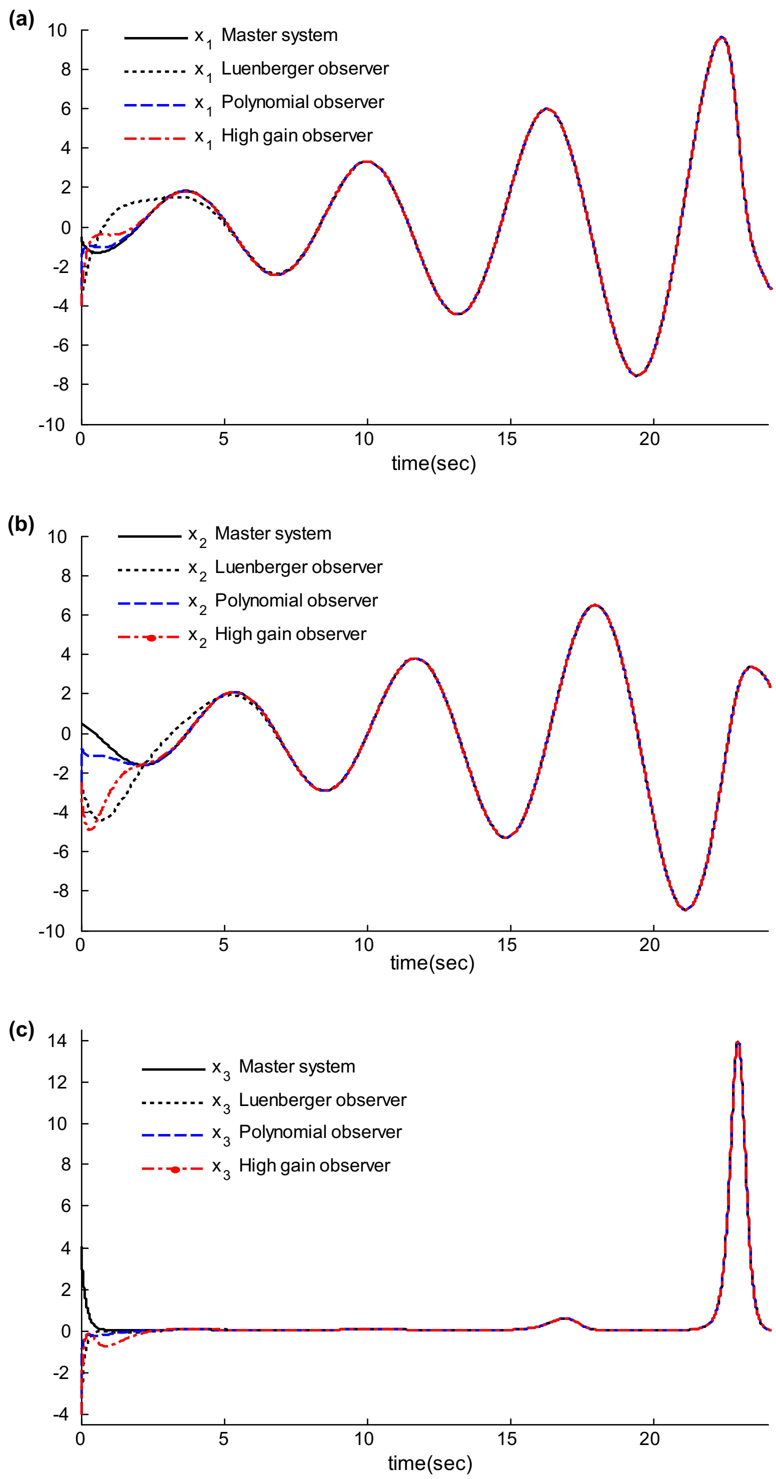

Fig. 1. Synchronization between master system (12) and its observers: exponential polynomial observer (21) and high gain observer (22), without any noise in the system output: (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

$$\|e(t)\| \leqslant \kappa \exp(-\lambda t)$$

where $\kappa = \frac{\|e_0\|_P}{\alpha}$ *and* $\lambda = \frac{\epsilon}{2}$.

Proof. Consider the Lyapunov function candidate,

$$V = e^T P e$$

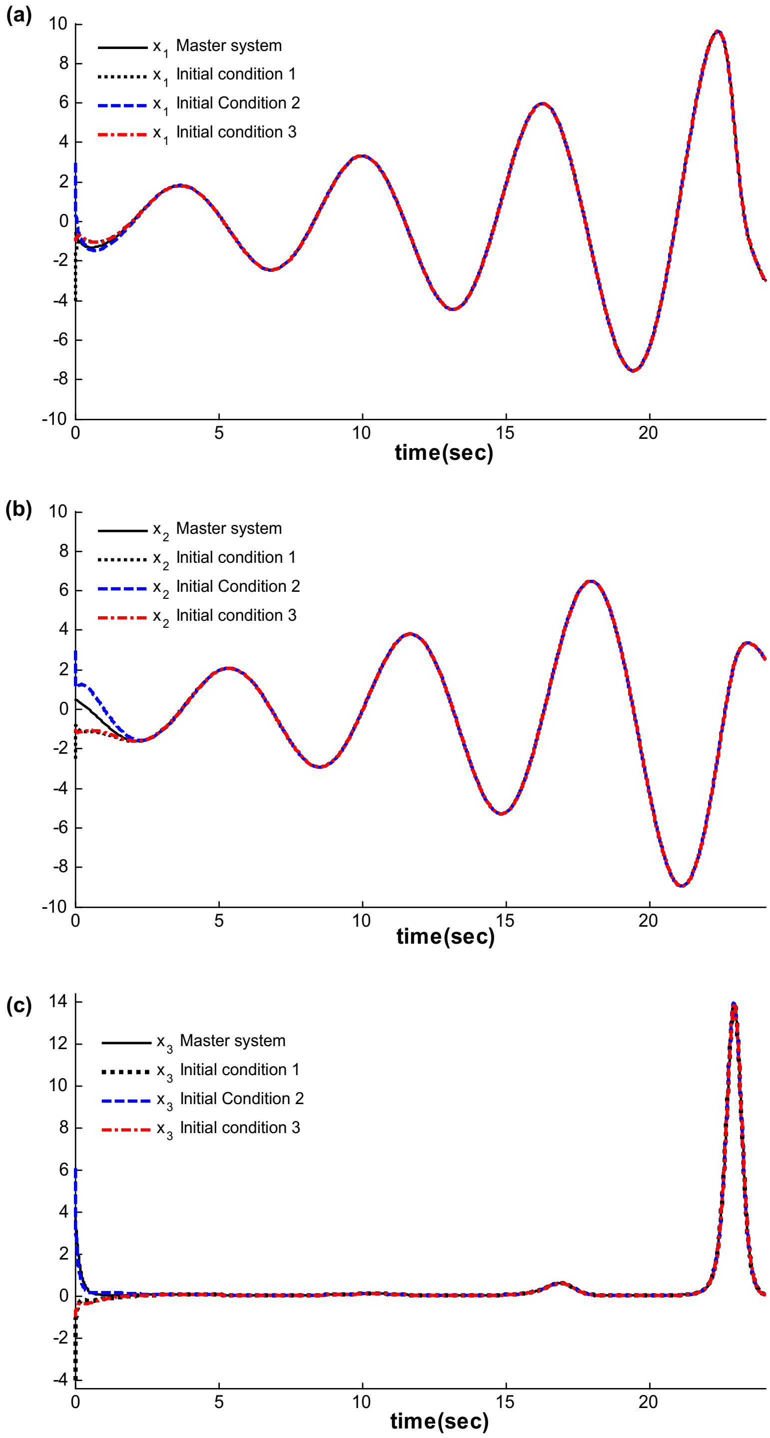

Fig. 2. Synchronization between master system (12) and the exponential polynomial observer (21), without any noise in the system output: (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$; initial condition 1: $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$; initial condition 2: $\hat{x}_1 = 3$, $\hat{x}_2 = 3$, $\hat{x}_3 = 6$; and initial condition 3: $\hat{x}_1 = -1$, $\hat{x}_2 = -1$, $\hat{x}_3 = -1$.

where $0 < P = P^T$ and satisfies Assumption 2. Its derivative is

$$\dot{V} = \dot{e}^T Pe + e^T P\dot{e} = e^T\left[(A - K_1C)^T P + P(A - K_1C)\right]e - 2[Ce]^{m-1}e^T PK_2Ce + 2e^T P[\Psi(x,u) - \Psi(\hat{x},u)]$$

In [25] is presented the next inequality as a lemma based on Eq. (3) which is useful for this proof, it is supposed the nonlinear vector function satisfies inequality (3)

$$2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \leqslant L^2 e^T PPe + e^T e \tag{6}$$

From Rayleigh inequality [29], and taking into account Assumption 3, we have,

$$-e^T PK_2Ce \leqslant -\lambda_{min}(PK_2C)\|e\|^2 \tag{7}$$

Then, combining inequalities (6) and (7) we obtain,

$$\begin{aligned} \dot{V} &\leqslant e^T\left[(A - K_1C)^T P + P(A - K_1C)\right]e - 2[Ce]^{m-1}\lambda_{min}(PK_2C)\|e\|^2 + L^2 e^T PPe + e^T e \\ \dot{V} &\leqslant e^T\left[(A - K_1C)^T P + P(A - K_1C)L^2 PP + I\right]e - 2[Ce]^{m-1}\lambda_{min}(PK_2C)\|e\|^2 \end{aligned} \tag{8}$$

From Assumption 1, the second term in the right-hand side of the inequality (8) always will be positive or zero, therefore,

$$\dot{V} \leqslant e^T\left[(A - K_1C)^T P + P(A - K_1C)L^2 PP + I\right]e = -\epsilon\|e\|^2 \tag{9}$$

We write the Lyapunov function as $V = \|e\|_P^2$, where $\alpha\|e\|^2 \leqslant V(e) \leqslant \beta\|e\|^2$, with $\alpha, \beta \in \mathbb{R}$. Taking its derivative and replacing in inequality (9), we obtain

$$\frac{d}{dt}\|e\|_P \leqslant -\frac{\epsilon}{2}\|e\|_P$$

Finally, we have the next result

$$\|e(t)\| \leqslant \kappa \exp(-\lambda t)$$

where $\kappa = \frac{\|e_0\|_P}{\alpha}$ and $\lambda = \frac{\epsilon}{2}$.

This implies that system (4) is an observer for system (2) and the corresponding dynamics of the observer error (5) is exponentially stable. □

Remark 1. If $K_2 = 0$, the observer is called an "Extended Luenberger Observer".

3. High gain observer

In this section, we present a well-known estimation structure (high gain observer) as a comparison with our proposed methodology.

Table 1
Synchronization error and performance index (J) for Rössler system (12) and its observers, without any noise in the system output. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

Time (s)	High gain observer		Luenberger observer		Polynomial observer	
	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J
0	9.2331	1000	9.2331	1000	9.2331	1000
0.1	5.7457	62.5	5.7369	59.7	3.1343	32.2
0.2	4.5347	44.0	5.2891	44.6	2.0965	19.4
0.3	4.2553	35.7	5.2256	39.0	1.6553	14.1
0.4	4.2537	31.2749	5.0407	35.8683	1.4150	11.1777
0.5	4.2932	28.6791	4.7276	33.4891	1.2389	9.2970
0.6	4.3037	26.9867	4.3358	31.3433	1.0884	7.9751
0.7	4.2673	25.7619	3.9046	29.2983	0.9529	6.9860
0.8	4.1821	24.7783	3.4612	27.3362	0.8293	6.2129
0.9	4.0517	23.9128	3.0239	25.4701	0.7162	5.5896
1.0	3.8812	23.0983	2.6054	23.7175	0.6134	5.0753
2.0	1.3709	15.3352	0.2686	12.7588	0.0694	2.5892
3.0	0.8714	10.5007	0.1129	8.5150	0.0227	1.7268
4.0	0.6187	8.0438	0.0310	6.3881	0.0073	1.2953
5.0	0.1681	6.4645	0.0095	5.1108	0.0017	1.0363
10	0.0098	3.2379	7.7×10^{-6}	2.5557	1.59×10^{-6}	0.5182
100	17.4×10^{-6}	0.3238	0	0.2556	0	0.0518
1000	0	0.0324	0	0.0256	0	0.005

Consider the class of nonlinear systems given by Eq. (1). In this case, to estimate the state-space vector x, we can suggest a nonlinear high gain observer of the following structure:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) + K(y - C\hat{x}) \\ \hat{x} &\in \mathbb{R}^n, \quad \hat{x}_0 = \hat{x}(t_0) \end{aligned} \tag{10}$$

where the observer high gain matrix is given by

$$K = S_\theta^{-1} C^T, \qquad S_\theta = \left(\frac{1}{\theta^{i+j-1}} S_{ij} \right)_{i,j=1,\dots,n}$$

and the positive parameter θ determines the desired convergence velocity. Moreover, $S_\theta > 0$, $S_\theta = S_\theta^T$, should be a positive solution of the algebraic equation

$$\begin{aligned} & S_\theta \left(E + \frac{\theta}{2} I \right) + \left(E^T + \frac{\theta}{2} I \right) S_\theta = C^T C \\ & E = \begin{pmatrix} 0 & I_{n-1,n-1} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{11}$$

As is shown in [30], under certain technical assumptions (Lipschitz conditions in an invariant set for nonlinear functions under consideration) this nonlinear observer has an arbitrary exponential decay for any initial conditions.

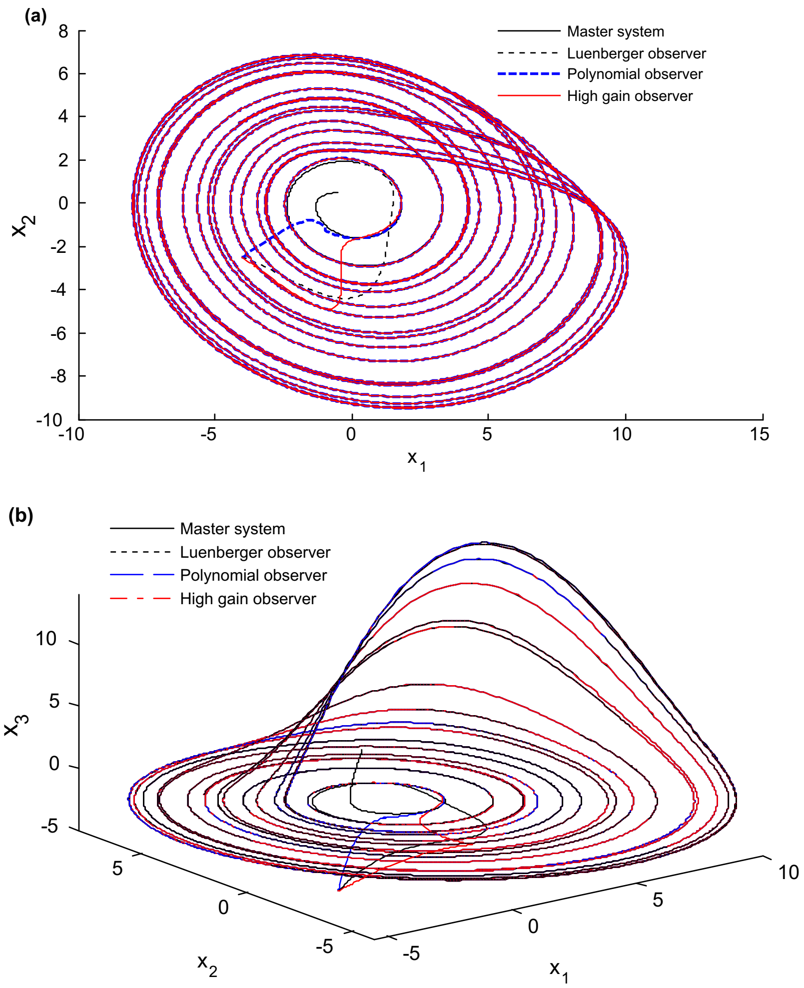

Fig. 3. Chaotic behavior of master system (12) and its observers: exponential polynomial observer (21) and high gain observer (22), without any noise in the system output: (a) signals x_1, x_2 and $\hat{x}_1$, $\hat{x}_2$; (b) signals x_1, x_2, x_3 and $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

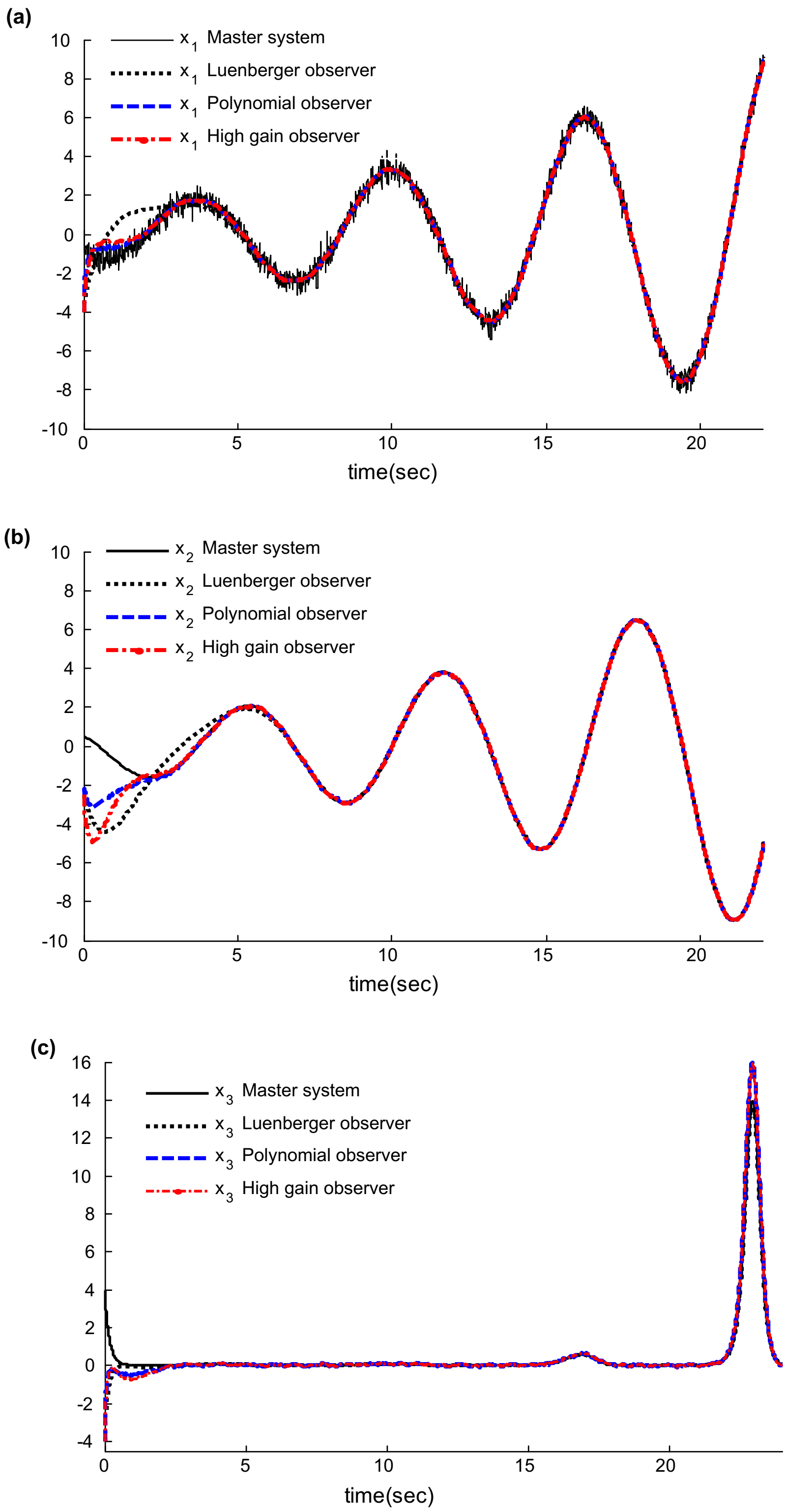

Fig. 4. Synchronization between master system (12) and its observers: exponential polynomial observer (21) and high gain observer (22), with white noise in the system output ($\sigma = 0.1$): (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

4. Application to synchronization of chaotic systems

To illustrate the effectiveness of the obtained results, we give two applications to chaotic systems. The former is an application to the denominated Rössler system which presents a chaotic behavior and exhibits the simplest possible strange attractor. Originally, the Rössler system is credited to Otto Rössler, arose from work in chemical kinetics [31] and the second one is the so-called Rikitake system one model which attempts to explain the reversal of the earth's magnetic field [32].

4.1. Example 1: Rössler system

We consider the popular nonlinear Rössler system [31], which is described by

$$\begin{aligned}
\dot{x}_1 &= -(x_2 + x_3) \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= b + x_3(x_1 - c) \\
y &= x_1
\end{aligned} \tag{12}$$

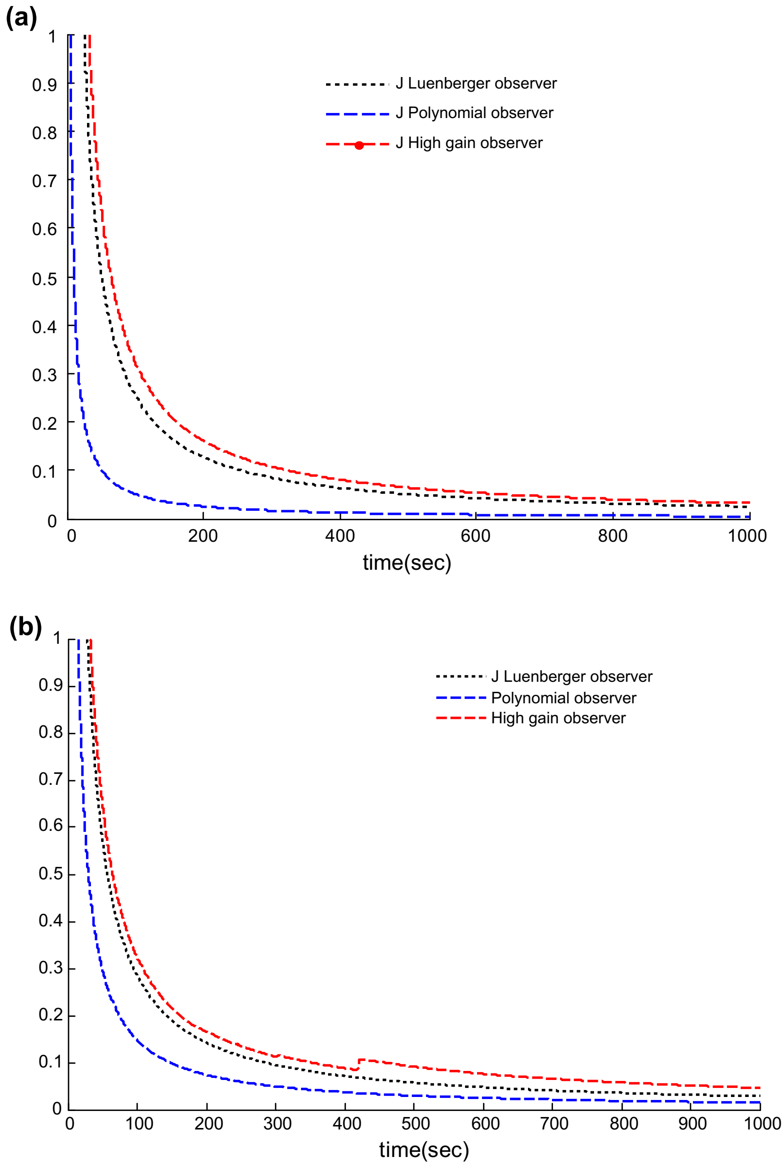

Fig. 5. Quadratic estimation error: (a) without any noise in the system output; (b) with white noise in the system output ($\sigma = 0.1$). The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

It is well known that in a large neighborhood of $\{a = b = 0.2, c = 5\}$ this system has a chaotic behavior.

Remark 2. It is not difficult to prove that system (12) is Lipschitz.

Before proposing the state observer, we prove the algebraic observability condition (see Definition 1) for system (12). Replacing $y = x_1$ into system (12), we obtain

$$\dot{y} = -(x_2 + x_3) \tag{13}$$

$$\dot{x}_2 = y + ax_2 \tag{14}$$

$$\dot{x}_3 = b + x_3(y - c) \tag{15}$$

Taking the time derivative from Eq. (13)

$$\ddot{y} = -\dot{x}_2 - \dot{x}_3 \tag{16}$$

From Eq. (13), we obtain

$$x_3 = -(\dot{y} + x_2) \tag{17}$$

Replacing Eqs. (14), (15) and (17) into Eq. (16)

$$\ddot{y} + y\dot{y} - c\dot{y} + y - x_2 y + (a + c)x_2 = 0 \tag{18}$$

In the same manner for x_3, we have from Eq. (13)

$$x_2 = -(\dot{y} + x_3) \tag{19}$$

Substituting Eqs. (14), (15) and (19) into Eq. (16)

$$\ddot{y} - a\dot{y} + y + x_3 y - (a + c)x_3 + b = 0 \tag{20}$$

Remark 3. From Eqs. (18) and (20), is clear that x_2 and x_3 are algebraically observable.

According to Theorem 1, we obtain the following exponential polynomial observer (slave system):

$$\begin{aligned}
\dot{\hat{x}}_1 &= -(\hat{x}_2 + \hat{x}_3) + k_{1,1}(x_1 - \hat{x}_1) + k_{1,2}(x_1 - \hat{x}_1)^m \\
\dot{\hat{x}}_2 &= \hat{x}_1 + a\hat{x}_2 + k_{2,1}(x_1 - \hat{x}_1) + k_{2,2}(x_1 - \hat{x}_1)^m \\
\dot{\hat{x}}_3 &= b + \hat{x}_3(\hat{x}_1 - c) + k_{3,1}(x_1 - \hat{x}_1) + k_{3,2}(x_1 - \hat{x}_1)^m
\end{aligned} \tag{21}$$

The corresponding high gain observer is given by

Table 2
Synchronization error and performance index (J) for Rössler system (12) and its observers, with white noise in the system output ($\sigma = 0.1$). The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$.

Time (s)	High gain observer		Luenberger observer		Polynomial observer	
	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J
0	9.2331	1000	9.2331	1000	9.2331	1000
0.1	5.7457	62.4974	5.7456	59.6396	4.4000	46.0293
0.2	4.5347	43.9955	5.2889	44.6271	3.6826	30.9327
0.3	4.2554	35.6779	5.2395	39.3133	3.4879	24.8657
0.4	4.2533	31.2725	5.0444	35.8412	3.2927	21.5264
0.5	4.2925	28.6760	4.7205	33.4402	3.0569	19.2354
0.6	4.3024	26.9820	4.3091	31.2379	2.8069	17.4537
0.7	4.2664	25.7574	3.8936	29.2123	2.5753	16.0117
0.8	4.1836	24.7741	3.5285	27.2713	2.3942	14.7763
0.9	4.0538	23.9107	3.0935	25.4614	2.1574	13.7100
1.0	3.8816	23.0972	2.6137	23.7251	1.8693	12.7413
2.0	1.3631	15.3343	0.2821	12.7746	0.2323	7.2693
3.0	0.8703	10.4999	0.0894	8.5158	0.0448	4.8506
4.0	0.6186	8.0430	0.0632	6.3953	0.0279	3.6389
5.0	0.1704	6.4620	0.0533	5.1172	0.0160	2.9114
10	0.0098	3.2379	0.0097	2.5562	0.0090	1.4561
100	0.0041	0.3323	0.0125	0.3648	0.0110	0.2566
1000	0.0042	0.0672	0.0042	0.0366	0.003	0.0258

$$
\begin{aligned}
\dot{\hat{x}}_1 &= -(\hat{x}_2 + \hat{x}_3) - 3\theta(\hat{x}_1 - y) \\
\dot{\hat{x}}_2 &= \hat{x}_2 + a\hat{x}_2 - \frac{1}{\hat{x}_1 - c - a}\left[3\theta(1 + \hat{x}_3) - 3\theta^2(\hat{x}_1 - c) + \theta^3\right](\hat{x}_1 - y) \\
\dot{\hat{x}}_3 &= b + \hat{x}_3(\hat{x}_1 - c) - \frac{1}{\hat{x}_1 - c - a}\left[-3\theta(1 + \hat{x}_3) + 3\theta^2 a - \theta^3\right](\hat{x}_1 - y)
\end{aligned}
\tag{22}
$$

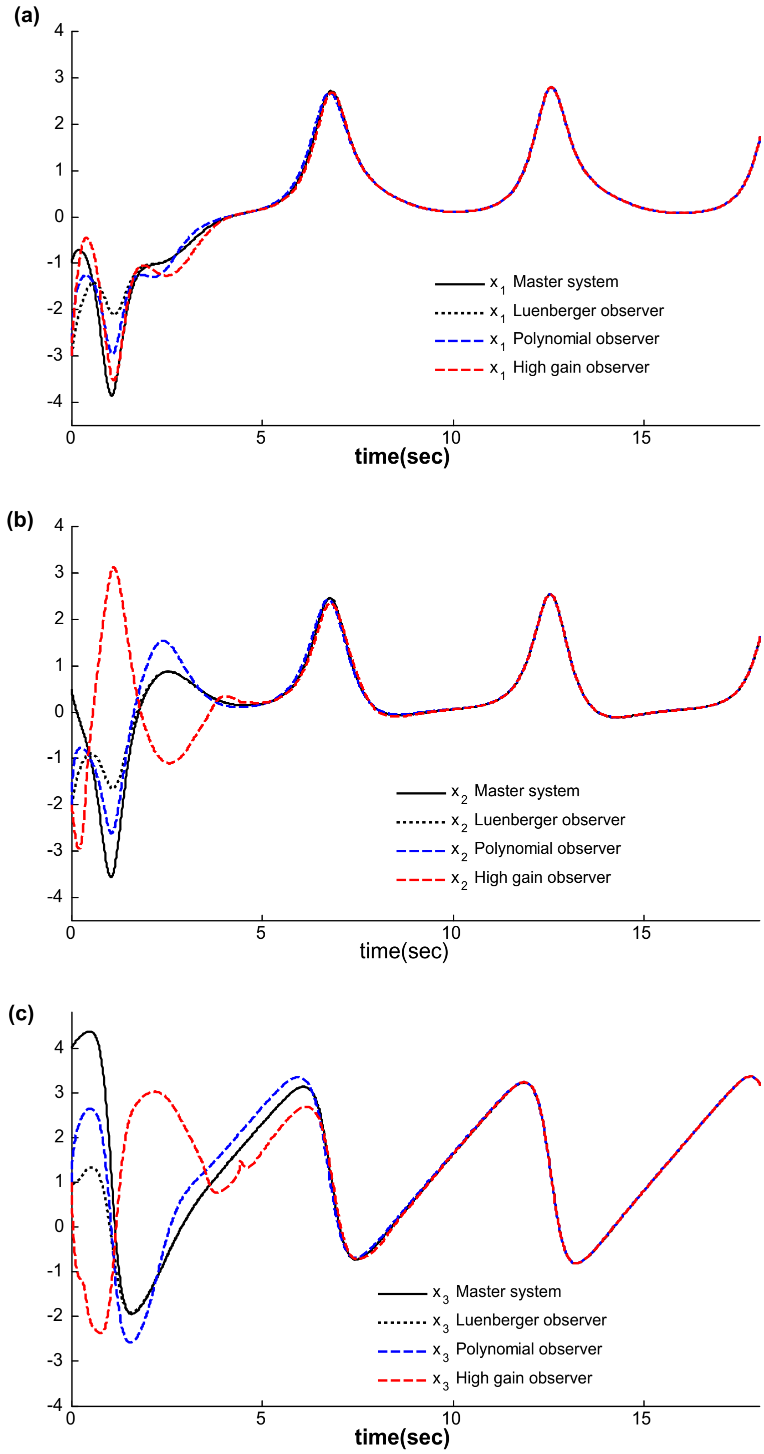

Fig. 6. Synchronization between master system (23) and its observers: exponential polynomial observer (32) and high gain observer (33), without any noise in the system output: (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -3$, $\hat{x}_2 = -2$, $\hat{x}_3 = -1$.

We show some simulations for the Rössler system (12) and its observers given by systems (21) and (22), we have taken for the parameter values $a = b = 0.2$, $c = 5$, $m = 3$, $K_1 = [k_{1,1}\ \ k_{2,1}\ \ k_{3,1}]^T = [5\ \ -5\ \ 5]^T$, $K_2 = [k_{1,2}\ \ k_{2,2}\ \ k_{3,2}]^T = [10\ \ 10\ \ 10]^T$, $\theta = 2$. All simulation results in this paper were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox. The design of the exponential observer presented in this paper is based on the solution of the Riccati equation which can be obtained by using the Matlab function ARE.

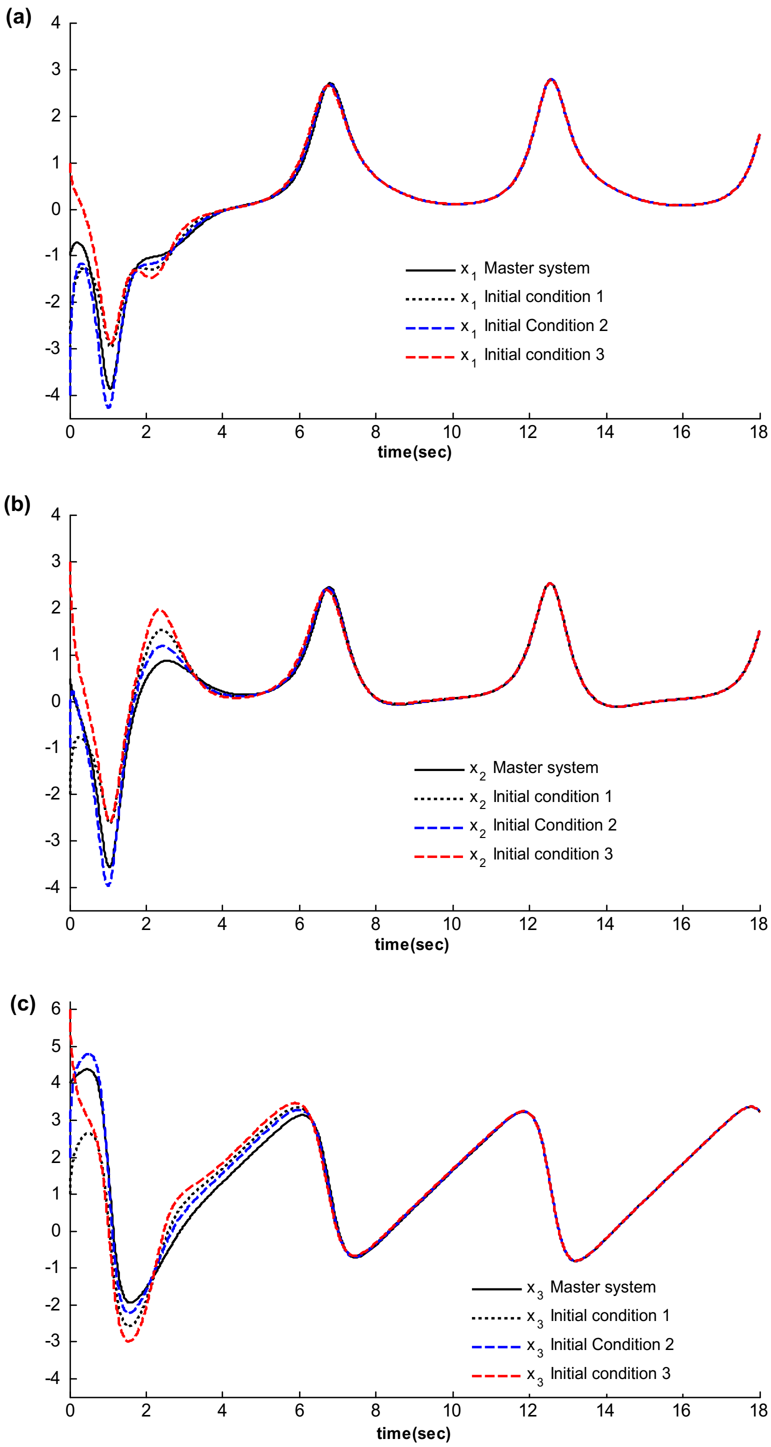

Fig. 7. Synchronization between master system (23) and the exponential polynomial observer (32), without any noise in the system output: (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$; initial condition 1: $\hat{x}_1 = -3$, $\hat{x}_2 = -2$, $\hat{x}_3 = 1$; initial condition 2: $\hat{x}_1 = -4$, $\hat{x}_2 = -1$, $\hat{x}_3 = 2$; and initial condition 3: $\hat{x}_1 = 1$, $\hat{x}_2 = 3$, $\hat{x}_3 = 6$.

The performance index of the corresponding synchronization process is calculated as [33]

$$J(t) = \frac{1}{t + 0.001} \int_0^t \|e(t)\|_{Q_0}^2 \, d\tau, \qquad Q_0 = I$$

where $e(t)$ denotes the estimation error.

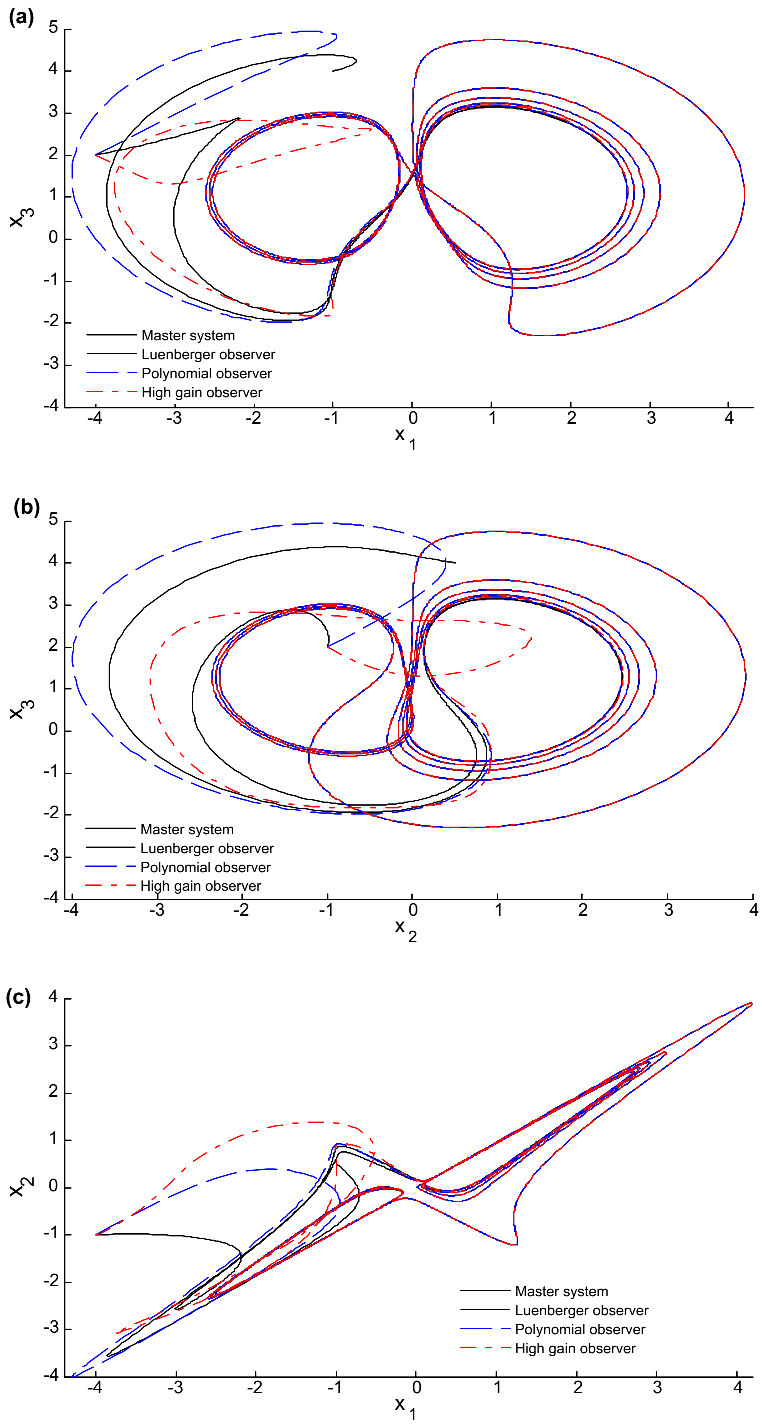

Fig. 8. Chaotic behavior of master system (23) and its observers: exponential polynomial observer (32) and high gain observer (33), without any noise in the system output: (a) signals x_1, x_3 and $\hat{x}_1$, $\hat{x}_3$; (b) signals x_2, x_3 and $\hat{x}_2$, $\hat{x}_3$; and (c) signals x_1, x_2 and $\hat{x}_1$, $\hat{x}_2$. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = -1$, $\hat{x}_3 = 2$.

Figs. 1(a)–(c) and 2(a)–(c) show the convergence of the estimated states (slave system) to the real states (master system), for different sets of initial conditions, without any noise in the system output. Table 1 shows the estimation error and the performance index for some values.

Fig. 3(a) and (b) shows the chaotic behavior of system (12) and the observer given by system (21), and also show the convergence of the state estimates to the real states, without any noise in the system output.

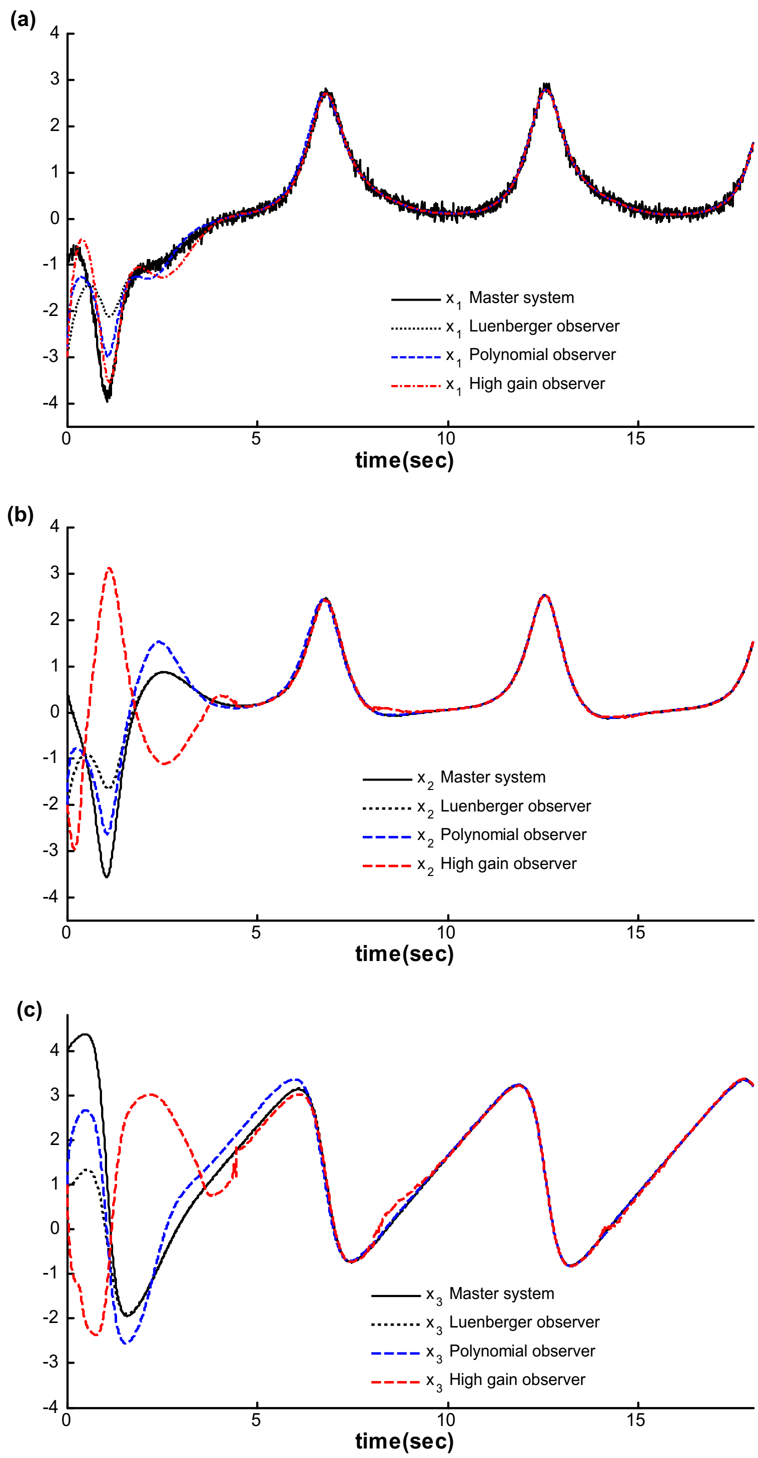

Fig. 9. Synchronization between master system (23) and its observers: exponential polynomial observer (32) and high gain observer (33), with white noise in the system output ($\sigma = 0.1$): (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; and (c) signals x_3 and $\hat{x}_3$. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -3$, $\hat{x}_2 = -2$, $\hat{x}_3 = 1$.

Moreover, in Fig. 4(a)–(c) is shown the effect of noise in the estimation process. A white noise is added in the measurement ($\sigma = 0.1$, ±10% around the current value of the measured output). We can see that the exponential polynomial observer is robust against noisy measurement.

Finally, in Fig. 5 is illustrated the performance index for the corresponding estimation processes. It should be noted that the quadratic estimation error (performance index) is bounded and has a tendency to decrease. Table 2 contains some values for the estimation error and the performance index when there exist noise in the system output.

4.2. Example 2: Rikitake oscillator

This system describes the currents of two coupled dynamo disks [32]. The governing equations are

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + x_2 x_3 \\ \dot{x}_2 &= -\mu x_2 + (x_3 - a)x_1 \\ \dot{x}_3 &= 1 - x_2 x_1 \\ y &= x_1 \end{aligned} \tag{23}$$

here a and μ are nonnegative parameters.

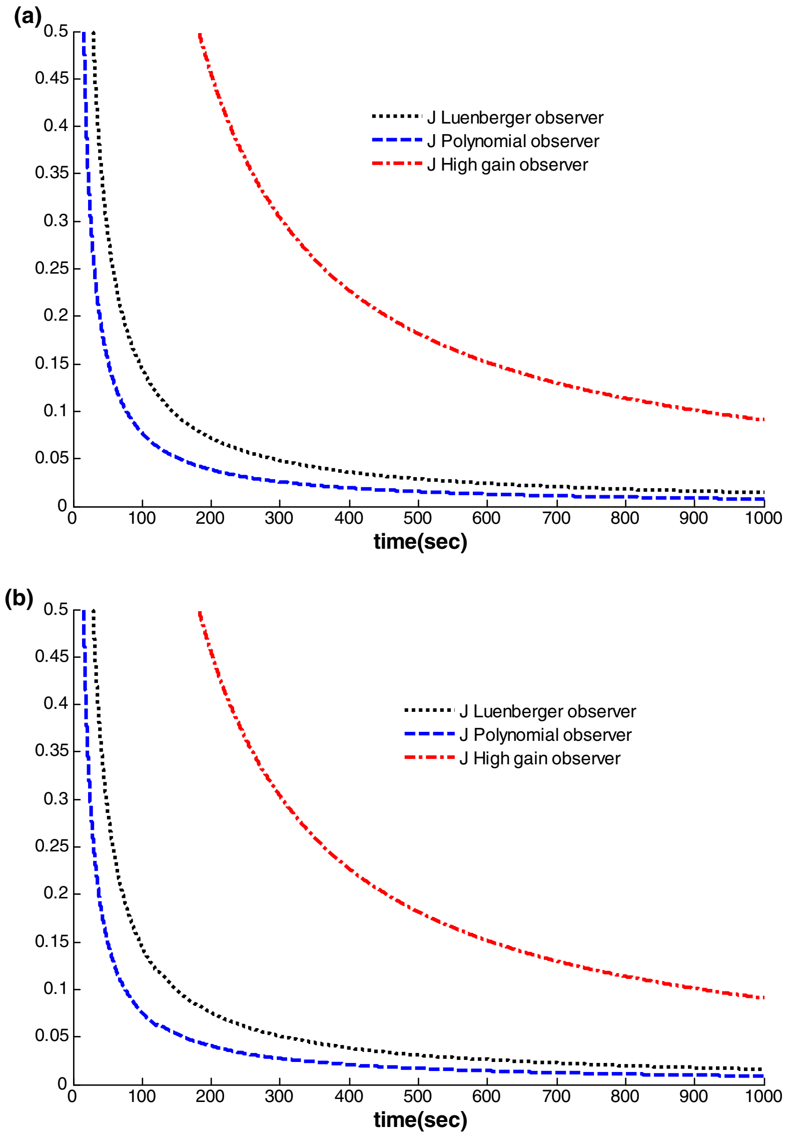

Fig. 10. Performance index: (a) without any noise in the system output; (b) with white noise in the system output ($\sigma = 0.1$). The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -3$, $\hat{x}_2 = -2$, $\hat{x}_3 = 1$.

Remark 4. It is not hard to see that above system is Lipschitz.

Before proposing the state estimator, we prove the algebraic observability condition (see Definition 1) for system (23). Replacing $y = x_1$ in system (23), we obtain,

$$\dot{y} = -\mu y + x_2 x_3 \tag{24}$$

$$\dot{x}_2 = -\mu x_2 + (x_3 - a)y \tag{25}$$

$$\dot{x}_3 = 1 - x_2 y \tag{26}$$

Taking the derivative with respect to time from Eq. (24), we have

$$\ddot{y} = -\mu\dot{y} + \dot{x}_2 x_3 + x_2\dot{x}_3 \tag{27}$$

From Eq. (24) we obtain x_3,

$$x_3 = \frac{1}{x_2}[\dot{y} + \mu y] \tag{28}$$

Substituting Eqs. (25), (26) and (28) into Eq. (27), we obtain

$$x_2^4 - \frac{1}{y}x_2^3 + \left[\frac{\ddot{y}}{y} + 2\mu\frac{\dot{y}}{y} + \mu^2\right]x_2^2 + a[\dot{y} + \mu y]x_2 - [\dot{y} + \mu y]^2 = 0 \tag{29}$$

In the same manner for x_3, we have from Eq. (24)

$$x_2 = \frac{1}{x_3}[\dot{y} + \mu y] \tag{30}$$

Replacing Eqs. (25), (26) and (30) into Eq. (27),

$$x_3^4 - ax_3^3 + \left[\frac{\ddot{y}}{y} + 2\mu\frac{\dot{y}}{y} + \mu^2\right]x_3^2 + \left[\frac{\dot{y}}{y} + \mu\right]x_3 - [\dot{y} + \mu y]^2 = 0 \tag{31}$$

Remark 5. From Eqs. (29) and (31), x_2 and x_3 are algebraically observable.

Going back to the original coordinate system, we propose the following system (slave system) for the exponential polynomial observer:

$$\begin{aligned}
\dot{\hat{x}}_1 &= -\mu\hat{x}_1 + \hat{x}_2\hat{x}_3 + k_{1,1}(x_1 - \hat{x}_1) + k_{1,2}(x_1 - \hat{x}_1)^m \\
\dot{\hat{x}}_2 &= -\mu\hat{x}_2 + (\hat{x}_3 - a)\hat{x}_1 + k_{2,1}(x_1 - \hat{x}_1) + k_{2,2}(x_1 - \hat{x}_1)^m \\
\dot{\hat{x}}_3 &= 1 - \hat{x}_2\hat{x}_1 + k_{3,1}(x_1 - \hat{x}_1) + k_{3,2}(x_1 - \hat{x}_1)^m
\end{aligned} \tag{32}$$

The high gain observer is given by the next system

Table 3
Synchronization between Rikitake system and its observers, without any noise in the system output. The initial conditions are $x_1 = -1$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -3$, $\hat{x}_2 = -2$, $\hat{x}_3 = 1$.

Time (s)	High gain observer		Luenberger observer		Polynomial observer	
	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J	$\|x - \hat{x}\|$	J
0	4.3875	1000	4.3875	1000	4.3875	1000
0.1	5.6258	35.3879	4.0039	27.3276	2.6769	20.7849
0.2	6.0493	35.0586	3.6601	21.0248	2.1820	13.3122
0.3	5.9907	35.5570	3.3803	18.1450	1.9358	10.2823
0.4	6.0856	35.5976	3.1769	16.2904	1.8026	8.5834
0.5	6.5267	36.4802	3.0590	14.9703	1.7376	7.4922
0.6	6.8091	37.8132	3.0334	14.0154	1.7268	6.7423
0.7	7.1264	39.3383	3.0975	13.3507	1.7771	6.2155
0.8	7.4279	41.0532	3.2173	12.9273	1.8826	5.8566
1.0	7.3296	44.0661	3.1652	12.4671	1.8972	5.4341
2.0	4.6204	36.4837	0.6943	7.2268	0.0477	3.3145
6.0	0.4824	15.1579	0.2766	2.4098	0.0003	1.2781
50	0	1.8214	0	0.2892	0	0.1547
100	0	0.9107	0	0.1446	0	0.0773
1000	0	0.0911	0	0.0145	0	0.0077

Table 4
Synchronization between Rikitake system and its observers, with white noise in the system output ($\sigma = 0.1$). The initial conditions are $x_1 = -1,\ x_2 = 0.5,\ x_3 = 4,\ \hat{x}_1 = -3,\ \hat{x}_2 = -2,\ \hat{x}_3 = 1$.

Time (s)	High gain observer		Luenberger observer		Polynomial observer	
	$\lVert x-\hat{x}\rVert$	J	$\lVert x-\hat{x}\rVert$	J	$\lVert x-\hat{x}\rVert$	J
0	4.3875	1000	4.3875	1000	4.3875	1000
0.1	5.6260	35.3898	3.9968	27.2802	2.6400	20.4772
0.2	6.0492	35.0602	3.6581	20.9740	2.1672	13.0785
0.3	5.9909	35.5572	3.3754	18.1122	1.9122	10.1153
0.4	6.0855	35.5975	3.1732	16.2608	1.7799	8.4391
0.5	6.5267	36.4801	3.0568	14.9449	1.7171	7.3632
0.6	6.8090	37.8133	3.0343	13.9963	1.7091	6.6254
0.7	7.1264	39.3381	3.0963	13.3328	1.7572	6.1052
0.8	7.4275	41.0527	3.2145	12.9104	1.8584	5.7503
1.0	7.3297	44.0652	3.1601	12.4490	1.8679	5.3274
2.0	4.6202	36.4833	0.6734	7.2137	0.0233	3.2335
6.0	0.3621	15.1035	0.2756	2.4055	0.0221	1.2413
50	0.0873	2.4182	0.0281	0.2893	0.0273	0.1506
100	0.0724	0.9098	0.0143	0.1449	0.0142	0.0756
1000	0.0097	0.0915	0.0098	0.0160	0.0009	0.0089

$$
\begin{aligned}
\dot{\hat{x}}_1 &= -\mu\hat{x}_1 + \hat{x}_2\hat{x}_3 - 3\theta(\hat{x}_1 - y) \\
\dot{\hat{x}}_2 &= -\mu\hat{x}_2 + (\hat{x}_3 - a)\hat{x}_1 - \frac{1}{-2\hat{x}_1\hat{x}_3^2 + a\hat{x}_1\hat{x}_3 + \hat{x}_2 - 2\hat{x}_1\hat{x}_2^2}\left[3\theta z_1 + 3\theta^2 z_2 + \theta^3\hat{x}_2\right](\hat{x}_1 - y) \\
\dot{\hat{x}}_3 &= 1 - \hat{x}_2\hat{x}_1 + \frac{1}{-2\hat{x}_1\hat{x}_3^2 + a\hat{x}_1\hat{x}_3 + \hat{x}_2 - 2\hat{x}_1\hat{x}_2^2}\left[3\theta z_3 + 3\theta^2 z_4 + \theta^3\hat{x}_3\right](\hat{x}_1 - y)
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
z_1 &= \mu^2\hat{x}_2 - 2\mu\hat{x}_1\hat{x}_3 + a\mu\hat{x}_1 - \hat{x}_2\hat{x}_3^2 + a\hat{x}_2\hat{x}_3 + \hat{x}_2^3 \\
z_2 &= 2\mu\hat{x}_2 - 2\hat{x}_1\hat{x}_3 + a\hat{x}_1 \\
z_3 &= \mu^2\hat{x}_3 - \mu + 2\mu\hat{x}_1\hat{x}_2 - \hat{x}_3^3 + a\hat{x}_3^2 + \hat{x}_2^2\hat{x}_3 \\
z_4 &= 2\mu\hat{x}_3 - 1 + 2\hat{x}_1\hat{x}_2
\end{aligned}
$$

Now, some numerical results for Rikitake system (23) and its observers given by systems (32) and (33) are presented. System (23) is chaotic with the set of parameter values $\mu = 1,\ a = 0.375$.

We have chosen the parameter values for systems (23), (32) and (33) as $\mu = 1,\ a = 0.375,\ m = 3,\ K_1 = [k_{1,1}\quad k_{2,1}\quad k_{3,1}]^T = [2\quad 2\quad 2]^T,\ K_2 = [k_{1,2}\quad k_{2,2}\quad k_{3,2}]^T = [3\quad 3\quad 3]^T,\ \theta = 2.$

Figs. 6(a)–(c) and 7(a)–(c) show the convergence of the estimated states (slave system) to the real states (master system), for different sets of initial conditions, without any noise in the system output.

Fig. 8(a)–(c) shows the chaotic behavior of the master system (23) and the slave system (32), and also show the convergence of the estimated states (slave system) to the real states (master system), without any noise in the system output.

In Fig. 9(a)–(c) are shown the estimated states with the presence of noise in the system output (white noise with σ = 0.1, ±10% around the current value of the system output). It should be noted that the proposed observer is robust against noisy measurements.

In Fig. 10 is illustrated the performance index for the corresponding synchronization process without any noise in the system output and with noise in the system output (white noise with σ = 0.1, ±10% around the current value of the system output). In both cases, the corresponding performance index has a tendency to decrease. Tables 3 and 4 show the estimation error and the performance index in order to clarify the results.

5. Conclusions

In this paper, we have designed a new exponential polynomial observer (high-order polynomial type) for a class of nonlinear oscillators to attack the synchronization problem. Also, we have proven the exponential stability of the resulting state estimation error and by means of simple algebraic manipulations we construct the observer (slave system). Some comparisons between exponential polynomial, high gain and extended Luenberger observers, were presented. Finally, we have shown some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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Synchronization and parameter estimations of an uncertain Rikitake system ☆

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ABSTRACT

In this Letter we address the synchronization and parameter estimation of the uncertain Rikitake system, under the assumption the state is partially known. To this end we use the master/slave scheme in conjunction with the adaptive control technique. Our control approach consists of proposing a slave system which has to follow asymptotically the uncertain Rikitake system, refereed as the master system. The gains of the slave system are adjusted continually according to a convenient adaptation control law, until the measurable output errors converge to zero. The convergence analysis is carried out by using the Barbalat's Lemma. Under this context, uncertainty means that although the system structure is known, only a partial knowledge of the corresponding parameter values is available.

1. Introduction

The synchronization of chaotic systems has been investigated since its introduction in the seminal paper by Pecora and Carrol in 1990 [1]. This topic has attracted the attention of many researchers due to its potential applications in secure communications, chemical reactions and biological systems, just to mention a few [2–5]. Until now several synchronization schemes can be found in the literature, like mutual, master–slave, weak, strong, phase and generalized [6,4,7]. It is well known that two identical chaotic systems can be synchronized. However, in actual applications there always exists some level of uncertainty in the parameter values, due to its physical realization, or some signals are not available. To overcome this inconvenient and accomplish synchronization, even if the constant parameters are unknown and/or some signals are not available, some parameter estimation techniques borrowed from the modern control theory have recently used [8,9]. It is worth mentioning that from the point of view of control theory, synchronization corresponds either to the tracking or stabilization problems [10]. Basically, there are three different approaches to solve the synchronization. The first one consists of reconstructing the unknown system by using the inverse identification procedure, which can be summarized as follows: the unknown parameters are seen as an external input, the available measurable signal is considered as the system output and the goal is to find an asymptotic inverse of this mapping [11,12]. The second approach is based on adaptive control and, identification and state observer design [9,3,13]. Most recently a third approach, based on algebraic methods in signal processing and state estimation has been developed by Fliess and his co-workers [14–17]. This approach straightforwardly provides a manner to design algebraic estimators that are able to recover the unknown parameter values and the underlaying dynamic.

In this Letter we present a master/slave scheme in conjunction with an adaptive control technique for synchronization and identification of the uncertain chaotic Rikitake system, where only a partial knowledge of the state is available. That is, we assume that the state initial condition and the unknown parameters belong to some region where the system exhibits chaotic behavior. Roughly speaking, the suggested approach consists of designing a controlled slave system, whose controllers and adaptive parameters are adjusted accordingly to a proposed adaptive algorithm. It is done in such a way that the synchronization errors between the outputs of both systems, the uncertain Rikitake and the slave, asymptotically converge to zero. The corresponding convergence analysis of our scheme is done applying the Lyapunov method and the Barbalat's Lemma [9]. It is important to emphasize that the robustness of our control strategy allows to effectively detect piecewise constant variations on the parameter values of the uncertain Rikitake system.

The remaining of this work is organized as follows. In Section 2 we introduce the problem statement. In Section 3 we develop our

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solution to synchronize and identify the constant unknown parameters of the Rikitake system by means of the Lyapunov method. To assess the effectiveness of our method we present some numerical simulations in Section 4. Finally, we present the conclusions in Section 5.

2. Problem statement

2.1. Rikitake model system

A simple mechanical model used to study the reversals of the magnetic field of the Earth, idealized by the Japanese geophysicist Rikitake [18], consists of two identical single Faraday-disk dynamos of the Bullard type coupled together. The dynamics of this system is governed by the following three-dimensional system of nonlinear differential equations:

$$\begin{aligned}\dot{x}_1 &= -\mu x_1 + z_1 y_1,\\ \dot{y}_1 &= -\mu y_1 + (z_1 - a)x_1,\\ \dot{z}_1 &= 1 - x_1 y_1\end{aligned} \tag{1}$$

where the parameters μ and a have some physical meaning when they are positive. For a physical meaning of the states x_1, y_1 and z_1 we recommend to see [18]. However, the states x_1 and y_1 are directly related to the currents through each disc of the dynamo system, and z_1 is related to the angular velocity of one of the discs. This system displays a chaotic behavior for the parameters values in a neighborhood $\{\mu = 5,\ a = 2\}$ and for a large enough set of initial conditions.

2.2. Some algebraic properties and problem formulation

In this section we present some algebraic properties that the Rikitake System satisfies. To this end we introduce the following definitions.

Definition 1. Consider a smooth nonlinear system, described by a state vector $X = \{x_i\}_1^{i=n} \in R^n$ and by the output vector $G = \{g_i\}_1^{i=m} \in R^m$, of the form:

$$\dot{X} = f(X, P), \qquad G = h(X), \tag{2}$$

where $h(.)$ is a smooth vector function and $P \in R^l$ is a constant parameters vector, with $l < n$. Let $G^{(j)}$ denote the j-th time derivative of the vector G. We say that the vector state X is algebraically observable, if it can be uniquely expressed as

$$X = \Phi\big(G, G^{(1)}, \ldots, G^{(j)}\big)$$

for some integer j and for some smooth function Φ.

Definition 2. Under same conditions as in Definition 1. If the vector of parameters, P satisfies the following relation

$$\Omega_1\big(G, \ldots, G^{(j)}\big) = \Omega_2\big(Y, \ldots, Y^{(j)}\big)P, \tag{3}$$

where $\Omega_1(.)$ and $\Omega_2(.)$ are, respectively, $n \times 1$ and $n \times n$ smooth matrices, then P is said to be algebraically linearly identifiable with respect to the output vector G [14].

According to the previous definitions, it is evident that system (1) is algebraically observable with respect to the outputs $g_1 = x_1$ and $g_2 = y_1$, since the state, z_1, can be rewritten, as

$$z_1 = \frac{\dot{g}_2 - \mu g_2}{g_1} + a, \tag{4}$$

hence, Rikitake system is algebraically observable with respect to the selected outputs, $g_1 = x_1$ and $g_2 = y_1$. Moreover, substituting the above expression into the first differential equation of (1), we have

$$\dot{g}_1 - \dot{g}_2 = -\mu(g_1 + g_2) + a g_1. \tag{5}$$

Therefore, we conclude that system (1) vector of parameters $\mathbf{p} = (\mu, a)$ is algebraically identifiable with respect to the available outputs. That is, the non-available state z_1 and the vector of parameters $\mathbf{p}$ can be simultaneously recovered, from the knowledge of the outputs $g_1 = x_1$ and $g_2 = y_1$.

From the above definitions, it is possible to solve the synchronization problem of the uncertain Rikitake system, provided that the states x_1 and y_1 are always available. Moreover, it is also possible to recover the unknown parameters μ and a. Thus, we are ready to establish the main control problem of this work.

2.3. Problem statement

Consider the uncertain Rikitake system (1), referred as the master system, with the available output states x_1 and y_1. And let us propose the following slave controlled system:

$$\begin{aligned}\dot{x}_2 &= -\hat{\mu} x_1 + z_2 y_1 - u_1,\\ \dot{y}_2 &= -\hat{\mu} y_2 + (z_2 - \hat{a})x_1 - u_2,\\ \dot{z}_2 &= 1 - x_1 y_1 - u_3.\end{aligned} \tag{6}$$

Then, the control objective is to find $\mathbf{u} = (u_1, u_2, u_3)$ and $\hat{\mathbf{p}} = (\hat{\mu}, \hat{a})$ such that the slave system (6) follows to the unknown Rikitake system (6); with $\hat{\mathbf{p}}$ converging to the actual values of (μ, a). In other words, we need to find $\mathbf{u}$ and $\hat{\mathbf{p}}$ of system (6), such that $(\mathbf{w}_1, \hat{\mathbf{p}}) \to (\mathbf{w}_2, \mathbf{p})$, as long as $t \to \infty$.[1]

We finish this section by introducing the following errors:

$$e_x = x_1 - x_2, \qquad e_y = y_1 - y_2, \qquad e_z = z_1 - z_2,$$
$$\tilde{\mu} = \mu - \hat{\mu}, \qquad \tilde{a} = a - \hat{a},$$

and according to them, we define the following vectors:

$$\mathbf{e}^T = (e_x, e_y, e_z); \qquad \tilde{\mathbf{p}}^T = (\tilde{\mu}, \tilde{a}).$$

3. Lyapunov based formulation

In this section we solve the synchronization and parameters identification of the constant unknown parameters of Rikitake system by means of the Lyapunov method. To this end, we first compute the dynamics of the synchronization errors, between the master and the slave systems. Next, based on a simple quadratic Lyapunov function, we propose the needed controller and the needed estimator that assure the synchronization of both systems.

Before to solve the control problem we introduce the following assumptions related with the selected outputs of the master system:

A1) The states $x = x_1$ and $y = y_1$ are available, for measurement.
A2) All the states of the master system are bounded, with the generic property that the steady solution x and y, continues oscillating around zero.

Comment 1. Assumption A2 is a realistic because in most case all the states of Rikitake system are bounded; for a large set of initial conditions and for a large set of positive parameters μ and

[1] Here we denote the vector state related with the master and slave system, as $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. That is, $\mathbf{w}_i^T = (x_i, y_i, z_i)$; for $i = \{1, 2\}$.

a. In fact assumption A2 depends on the set of initial conditions and the values of the parameter vector q. To clarify the meaning of this property, we present a case where assumption A2 does not hold. Selecting the parameters values as $\{\mu > 0;\ a > 0\}$, and the initial condition as $w_1(0) = (x_1(0) = 0, x_1(0) = 0, z_1(0) = \bar{z})$; we have that $x_1(t) = 0$, $y_1(t) = 0$ and $z_1(t) = t + \bar{z}$. Evidently, assumption A2 cannot be fulfilled because the states x and y remain fixed at the origin and, the state z_1 is not bounded [19]. In fact, no identification method or scheme can be proposed if the master system have solutions that tend either to infinity or to a constant.

Comment 2. In this work, as a fundamental assumption, we suppose that all states of the master system are bounded with the property that the steady states x and y continues oscillating around zero, *i.e.*, these steady solutions should not be fixed at zero. In this manner, the results are not affected when any additional terms are included. For bounded perturbations, if the trajectories remain on the belong to the same manifold or an equivalent space state region of the unperturbed system, all the assumptions considered are adequate.

Now, from Eqs. (1) and (6), we have:

$$\dot{\mathbf{e}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\tilde{\mu}x + e_z y + u_1 \\ -\tilde{\mu}y + (e_z - \tilde{a})x + u_2 \\ u_3 \end{bmatrix} \tag{7}$$

where for simplicity, we stand for $y = y_1$ and $x = x_1$. As we can see, the above system can be considered as a control problem where the vector inputs $\mathbf{u}$ and $\tilde{\mathbf{p}}$ must be proposed such, the state $\mathbf{e}$ asymptotically converges to zero.

Consider a Lyapunov function

$$V = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2}\tilde{\mathbf{p}}^T\tilde{\mathbf{p}}. \tag{8}$$

The time derivative of V along the trajectories of (7) is then given by

$$\begin{aligned} \dot{V} = {} & -\tilde{\mu}\dot{\hat{\mu}} - \tilde{a}\dot{\hat{a}} - \tilde{\mu}e_x^2 + e_x e_z y + e_x u_1 - \tilde{\mu} y e_y \\ & + e_z x e_y - \tilde{a} x e_y + u_2 e_y + e_z u_3. \end{aligned} \tag{9}$$

Now, in order to make V semi-definite negative, we propose $\dot{\hat{\mathbf{p}}}$, as

$$\dot{\hat{\mathbf{p}}} = \begin{bmatrix} \dot{\hat{\mu}} \\ \dot{\hat{a}} \end{bmatrix} = \begin{bmatrix} -e_x x - e_y y \\ -x e_y \end{bmatrix} \tag{10}$$

and $\mathbf{u}$, as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -k_1 e_x - k_{11} e_x^k \\ -k_2 e_y - k_{21} e_y^k \\ -e_y x \end{bmatrix}, \tag{11}$$

where k_1, k_2, k_{11} and k_{21} are strictly positive constants and k is any odd integer.

Substituting the above $\mathbf{u}$ in (7), we have that the closed-loop system can be read as

$$\begin{aligned} \dot{e}_x &= -\tilde{\mu}x + e_z y - k_1 e_x - k_{11} e_x^{2k+1}, \\ \dot{e}_y &= -\tilde{\mu}y + (e_z - \tilde{a})x - k_2 e_y - k_{21} e_y^{2k+1}, \\ \dot{e}_z &= -e_x y \end{aligned} \tag{12}$$

where the parameter dynamics are give by

$$\dot{\hat{\mu}} = -e_x x - y e_y; \qquad \dot{\hat{a}} = -x e_y. \tag{13}$$

Substituting (12) and (13) into (9), we obtain

$$\dot{V} = -\left(k_1 e_x^2 + k_2 e_y^2 + k_{11} e_x^{k+1} + k_{22} e_y^{k+1}\right). \tag{14}$$

This implies that $\dot{V}$ is semi-definite negative and so V converges. Hence, the set of signals $\{e_x, e_y, e_z, \tilde{\mu}, \tilde{a}\}$ are bounded.

Before proceeding with the error convergence analysis, we introduce the needful Barbalat's Lemma: [9]

Barbalat's Lemma. *If the differential function $f(t)$ has a finite limit as $t \to \infty$, and if df/dt is uniformly continuous, then $df/dt \to 0$ as $t \to \infty$.*

A useful consequence of this lemma is that if $f \in L_2$ and df/dt is bounded then $f \to 0$ as $t \to \infty$.

Now, to show that $\mathbf{e}$ converges to zero as long as $t \to \infty$, we integrate both sides of (14), obtaining

$$\int_0^t \left(k_1 e_x^2(s) + k_2 e_y^2(s) + k_{11} e_x^{k+1}(s) + k_{22} e_y^{k+1}(s)\right) ds \leqslant V(0). \tag{15}$$

Then, from (15) it can be seen that $\mathbf{e}$ is L_2 and, from (12) and A2, it follows that $\dot{\mathbf{e}}$ is bounded. Finally, invoking Barbalat's Lemma, we have that $\mathbf{e} \to 0$ as $t \to \infty$. Once again, differentiating (12), it is easily to show that $\ddot{\mathbf{e}}$ is bounded. Thus, $\dot{\mathbf{e}}$ is uniformly continuous and also $\mathbf{e}$ has a finite limit, as $t \to \infty$, from Barbalat's Lemma, we conclude that $\dot{\mathbf{e}} \to 0$ as $t \to \infty$. Since V converges, as $t \to \infty$, then we have that the two parameter errors $\tilde{\mu}$ and $\tilde{a}$ converge as $t \to \infty$ (8). Besides, from (13), it follows that $\dot{\hat{\mu}}$ and $\dot{\hat{a}}$ converge to zero, as $t \to \infty$. Roughly speaking, when t is large enough $\hat{\mu}$ and $\hat{a}$ are almost constant, and the differential equations of (12) implies that

$$0 = (\mu - \hat{\mu})x, \qquad 0 = (a - \hat{a})y. \tag{16}$$

However once again from assumption A2, we have that the steady states x and y remains oscillating around zero. Therefore necessarily $\mu = \hat{\mu}$ and $a = \hat{a}$. That is, $\tilde{\mathbf{p}}^T \to 0$, as $t \to \infty$.

We summarize the previous discussion in the following proposition:

Proposition 1. *Under the assumptions* A1 *and* A2 *the synchronization and the parameters estimation problem between systems* (1) *and* (6)*, can be achieved for any strictly positive constants $\{k_1, k_2, k_{11}, k_{22}\}$ and for any odd integer k.*

4. Numerical results

Computer simulations have been carried out in order to test the effectiveness of the proposed asymptotic control strategy for the synchronization and for the recovery of the unknown parameters of an uncertain Rikitake system. The program uses the Runge–Kutta integration algorithm, with the integration step set to 0.001.

In the first simulation we illustrate the qualitative property describe in the assumption A2. For this end, we fixed the master system parameter as $\mathbf{q} = (\mu = 2,\ a = 5)$; while the initial conditions were selected as $\mathbf{w}_1(0) = (x_1(0) = 1,\ y_1(0) = -1,\ z_1(0) = 0)$. Fig. 1 shows the behavior of the whole state of the Rikitake system. We can see from this figure that the whole state solution of the master system is bounded and the states x_1 and y_1 remain oscillating around zero; therefore, we can claim that the assumption A2 is completely fulfilled.

To show the performance of the proposed control strategy we carried out a second simulation using the same set-up as above, and fixing the slave system gains as $k_1 = k_2 = 0.8$ and $k_{12} = k_{22} = 0.266$; with the slave system initialized at zero, *i.e.*, $\mathbf{w}_2(0) = 0$ and $\hat{\mathbf{p}}(0) = 0$. In Fig. 2 we can see that the synchronization errors asymptotically converge to zero. That is, the slave system follows

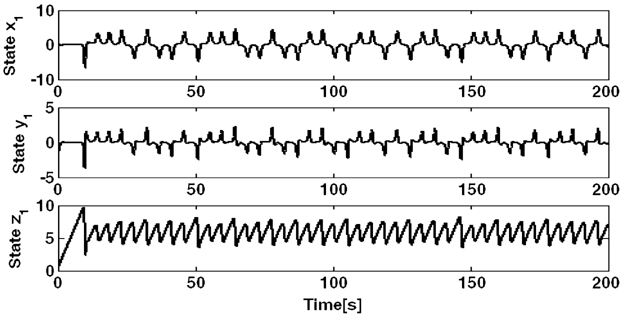

Fig. 1. This figure depicts the qualitative behavior of the Rikitake system when is initialized at $\mathbf{w}_1(0) = (1, -1, 0)$ and the parameters vector is fixed as $\mathbf{q} = (2, 5)$. Clearly, the assumption A2 is satisfied.

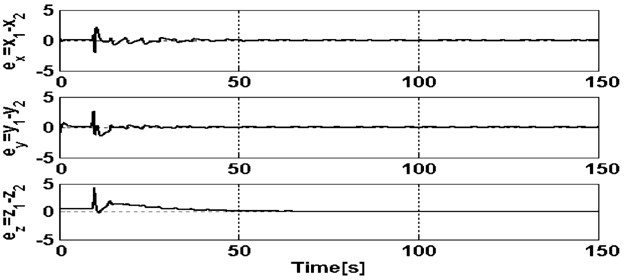

Fig. 2. This figure shows the convergence to zero of the master–slave synchronization error, when the master system is initialized at $\mathbf{w}_1(0) = (1, -, 0)$ and its parameters vector is fixed as $\mathbf{q} = (2, 5)$.

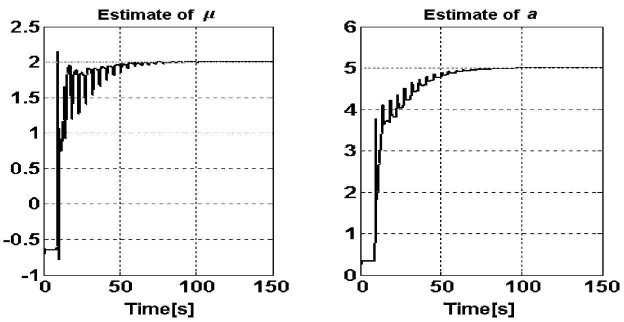

Fig. 3. This figure shows the parameters estimation, when the master system is initialized at $\mathbf{w}_1(0) = (1, -1, 0)$; and the actual parameters vector is fixed as $\mathbf{q} = (2, 5)$.

almost perfectly the uncertain master system. The estimated parameters are shown in Fig. 3.

As we expect, a better performance can be obtained as long as the time is increased. From this simulations we can concluded that the proposed estimator reconstructs reasonably well the parameters after elapsed 100 seconds.

Finally, to show the robustness of the proposed asymptotic parameters estimation method, we have subjected the parameters values of the master system to piecewise constant variations, as follows:

$$\text{if } t \leqslant 150 \text{ then } \{\mu = 1,\ a = 1.5\} \text{ else } \{\mu = 0.5,\ a = 1.0\}.$$

For this case, the initial conditions and the control gains of the slave system were set as before, while the master system was initialized at $\mathbf{w}_1(0) = (0.5, -0.3, 0.2)$. Fig. 4 shows the numerical simulation of the corresponding parameters estimation process; from $t = 0$ [s] to $t = 300$ [s]. From this simulation we can see that even when the values of both parameters were abruptly change

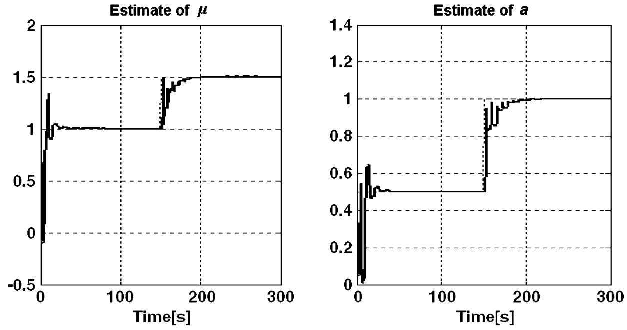

Fig. 4. Parameter Estimates when abrupt parametric variations in μ and a are presented in the master system.

at $t = 150$, the slave is able to detect it and estimate once again the new unknown parameters values, rather accurately and needing only 60 seconds.

5. Conclusions

A Lyapunov based approach for the synchronization and parameters identification of the constant unknown parameters of Rikitake system was presented. Under the assumptions that the two output states x and y are available for measurement. To accomplish this task, we first shown that the system is observable and linearly algebraically identifiable, with respect to the available outputs. Then, we propose a slave controlled system; where theirs controllers were proposed such that the vector synchronization error and the vector parameter error, among the master and slave system, asymptotically converge to zero. The convergence proof was carried out by using the traditional Lyapunov method in combination with the Lemma of Barbalat and the assumption A2. Also, it is worth to mentioning that our parameters identification algorithm does not requires that the Rikitake system always exhibits a chaotic behavior. Finally, numerical simulations were carried out to evaluate the performance of the proposed solution.

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On the Observability for a Class of Nonlinear (Bio)chemical Systems

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On the Observability for a Class of Nonlinear (Bio)chemical Systems*

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Abstract

In this work the observability properties for a class of nonlinear systems is presented, by considering linear, geometric and algebraic approaches. The observability conditions for state variables, unstructured uncertainties and detectable states are considered for a class of nonlinear systems related with several (bio)-chemical reacting processes. The considered examples are related with (bio)-chemical continuous reactors and a metabolic model, where their observability properties are analyzed. A comparison of the corresponding results is done, showing the suitability of each approach.

KEYWORDS: observability approaches, (bio) chemical process

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1. INTRODUCTION

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general indeed, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances.

However, the preceding problem to observer design is to analyze the observability conditions of the nonlinear systems under study. For linear systems, classical observability index as observability matrix and observability Grammian (Moore, 1981) for observability analysis and the estimator design have been extensively studied, and have proven extremely useful, especially for on-line monitoring and control applications such as observer based control design. However, for nonlinear systems, the theory of observers is not nearly as neither complete nor successful as it is for linear systems.

The design of observability conditions for nonlinear systems is a challenging problem (even for accurately known systems) that has received a considerable amount of attention. A first category of observability techniques consists in applying linear algorithms to the system, which is linearized around the estimated trajectory. In a second approach the nonlinear system is transformed into a linear one by an appropriate change of coordinates (Keller, 1987). The corresponding analysis is computed in these new coordinates and the original coordinates are retrieved through the inverse transformation. In most approaches, nonlinear coordinate transformations are employed to transform the nonlinear system into block triangular observer canonical forms (Aguilar-López, et. al., 2003; Tornambé, 1992).

The third methodology is relatively new and it is related to the differential-algebra framework, which is a mathematical approach that has been recently shown to be a very effective tool for understanding basic questions such as input-output inversions and observer realizations (Martínez-Guerra, Aguilar-Lopez and Poznyak, 2004). Commutative algebra, which is mainly concerned with the study of commutative rings and fields, provides the right tools for understanding algebraic equations (Zariski, 1975). Differential algebra, which was mainly founded by Ritt (1950) and Kolchin (1973), extends to differential equations concepts and results from commutative algebra; this approach allows finding explicit relationships, which are very useful for observer design applications.

From the above, in this work the observability properties of several types of continuous (bio)chemical reactors are analyzed, firstly a non-isothermal, non-adiabatic continuous stirred chemical reactor is considered, employing the three considered observability approaches, where the reactor temperature is considered as the measured output. Other case is related with a non-isothermal aerobic wastewater bioreactor model (Olsson et al 2001), which is analyzed via (local) linear observalility approach; besides, both two continuous bioreactors with Monod´s and Haldane kinetics models are also analyzed, considering a three state and two state dynamic models, respectively, their observalility properties are studied via geometric and algebraic frameworks. Finally a class of metabolic model related with a human glucose metabolic dynamics is considered too.

From the algebraic approach the concepts of observable uncertainty and detectable state are presented.

2. OBSERVABILITY CONDITIONS

The purpose of this section is to discuss some conditions required on the system for possible solutions to the above mentioned observability issues. Such conditions above all correspond to what are usually called observability conditions. A question fundamental to the analysis of physical systems is whether the state of the system can be uniquely determined from its output data. Specifically, given the dynamic description of the system and the observation process, we can ask under what conditions can the initial state of the system be determined uniquely on the basis of the observed output on a given time interval.

This problem is called the *inverse* or *observability* problem. The test of a system's observability is a necessary prerequisite to the estimation of states and parameters from the output of the system. In short, they must express that there indeed is a possibility that the purpose of the observer can be achieved, namely that it might be possible to recover the state variables $x(t)$ from the only knowledge of the inputs u and the measured outputs y up to time t: at a first glance, this will be possible only if $y(t)$ bears the information on the full state vector when considered over some time interval: this roughly corresponds to the notion of observability. Observability, in control theory, is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and controllability of a system are mathematical duals.

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs (this definition is slanted towards the state space representation). Less formally, this means that from the system outputs it is possible to determine the behavior of the entire system. If a system is not observable, this means the current values of some of its states cannot be determined through output sensors: this implies that their value is unknown to the controller and, consequently, that it will be unable to fulfill the control specifications referred to these outputs.

The class of systems under consideration is described by a state-space representation generally of the following form:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \\
y &= h(x(t))
\end{aligned}
\tag{1}
$$

where x denotes the state vector, taking values in X a connected manifold of dimension n, u denotes the vector of known external inputs, taking values in some open subset U, and y denotes the vector of measured outputs taking values in some open subset Y. Functions f and h will in general be assumed to be C^{∞} of their arguments, and input functions $u(\circ)$ to be locally essentially bounded and measurable functions in a set U. The system will be assumed to be forward complete.

2.1 Linear observability approach

As a background, consider the following lineal system representation of the system (1):

$$
\begin{aligned}
\dot{x} &= Ax \\
y &= Cx
\end{aligned}
\tag{2}
$$

where x is the state vector and y is the measured vector.

Now, from a general framework, let us to consider the following, be the finite sets $\boldsymbol{Y} = (y, y', y'', y''', \ldots)^{T} \in \Re^{n}$ y $\chi = (x_1, x_2, x_3, x_4, \ldots)^{T} \in \Re^{n}$, which are related with the system output vector and a finite time derivatives, therefore it is possible to construct the following linear dynamic system:

$$
\begin{bmatrix} y \\ y' \\ y'' \\ y''' \\ \vdots \\ y^{n} \end{bmatrix} = \begin{bmatrix} Cx \\ CAx \\ CA^{2}x \\ CA^{3}x \\ \vdots \\ CA^{n}x \end{bmatrix}
\tag{3}
$$

or in vector form:

$$
\boldsymbol{Y} = \mathbf{N}\,\chi
\tag{4}
$$

where:

$$\mathbf{N} = [C, CA, CA^2, CA^3, \ldots\ldots, CA^n]^{\mathrm{T}} \tag{5}$$

As can be seen, if the state vector can be determinate, the matrix **N** (named as the observability matrix) must be invertible (full rank) in order to obtain:

$$\chi = \mathbf{N}^{-1}\, \boldsymbol{Y} \tag{6}$$

Such that, the state vector χ is observable respect to the measures output $\boldsymbol{Y}$.
For nonlinear systems an equivalent observability criteria can be applied (Diop and Fliess, 1991). A dynamic system with measured output $\boldsymbol{Y}$ is locally observable if the corresponding linearized systems with linearized output $\boldsymbol{Y_L}$ is observable.
Therefore, consider the following linearized system:

$$\begin{aligned} \dot{X}_L &= J(X,u)X_L \\ Y_L &= H(X)\,X_L \end{aligned} \tag{7}$$

where $J(X,u)$ and $H(X)$ are the Jacobian matrices of the nonlinear system and the nonlinear output vector, respectively. From the above a type of observability matrix Ω can be constructed too, which must be non singular in order to assure full observability conditions, in accordance with the following structure:

$$X = \Omega^{1} Y \tag{8}$$

where: $\Omega = [H, HJ, HJ^2, \ldots, HJ^n]^{\mathrm{T}}$ (9)

and det $\Omega \neq 0$.

2.2 Differential-Geometric observability approach

The approach involves coming up with a diffeomorfic transformation of the nonlinear system into an equivalent linear system through a change of variables and a suitable control input. The corresponding transformation may be applied to nonlinear systems of the form:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where via a diffeomorfic tranformation employing Lie derivatives:

$$
\begin{aligned}
y &= h(x) \\
\dot{y} &= L_f h(x) \\
\ddot{y} &= L_f^2 h(x) \\
&\bullet \\
&\bullet \\
&\bullet \\
y^{n-1} &= L_f^{n-1} h(x) \\
y^{n} &= L_f^{n} h(x) + L_g L_f^{n-1} h(x) u
\end{aligned}
\tag{10}
$$

being $L_f^r h$ the r-order Lie derivatives, which are the directional derivatives of the corresponding state variables along the measured output trajectory. $dL_f^r h$ are the differentials of the rth-order Lie derivatives defined recursively as follows:

$$L_f^0 h := h\,, \quad dL_f^0 h := dh = \left(\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n} \right)$$
$$L_f^1 h := \left\langle dh, f \right\rangle = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i\,, \quad dL_f^1 h := \left(\frac{\partial}{\partial x_1} \left(\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i \right), \ldots, \frac{\partial}{\partial x_n} \left(\sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i \right) \right)$$

$$L_f^r h := \left\langle dL_f^{r-1} h, f \right\rangle = L_f \left(L_f^{r-1} h \right), \quad r \geq 2$$

The coordinate transformation $T(x)$ that carried out the system into normal form comes from the first $(n-1)$ derivatives. In particular,

$$\Phi = T(X) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \bullet \\ \bullet \\ \Phi_n \end{bmatrix} = \begin{bmatrix} y = h(x) \\ \dot{y} = L_f h(x) \\ \bullet \\ \bullet \\ y^n = L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{bmatrix} \tag{11}$$

transforms trajectories from the original x coordinate system into the new Φ coordinate system. So long as this transformation is a diffeomorphism, smooth trajectories in the original coordinate system will have unique counterparts in the Φ coordinate system that are also smooth. Those Φ trajectories will be described by the new system,

$$\begin{aligned} \dot{\Phi}_1 &= \Phi_2 = \dot{y} \\ \dot{\Phi}_2 &= \Phi_3 = \ddot{y} \\ &\bullet \\ &\bullet \\ &\bullet \\ \dot{\Phi}_n &= L_f^n h(x) + L_g L_f^{n-1} h(x) u \end{aligned} \tag{12}$$

The system given by Equations (10) and (12) is, at least, locally uniformly observable (Gauthier et al., 1992), hence for all $x \in \Re^n$, satisfies the observability rank condition:

$$rank \left\{ \frac{\partial}{\partial x} \vartheta \right\} = n \tag{13}$$

Here ϑ is the observability vector function defined as $\vartheta = \left(dL_f^0 h, dL_f^1 h, \ldots, dL_f^{n-1} h \right)^T$,

2.3 Differential-Algebraic observability approach

Since early of 1990s some papers have been related with the dynamic characterization of a particular class of nonlinear systems named differentially flat

(Fliess et al, 1999; Fliess, et al., 1995) and Liouvillian systems (Chelouah, 1997), based on the frame of differential algebra. One of the most important useful of this approach for this kind of systems is the explicit relationships that can be obtained for particular state variables; it is an advantage for a class of observation and control problems. Differential-algebra based techniques have been employed for differential algebraic as well as ordinary differential equations systems.

Fliess and coworkers (Diop, 2001; Fliess, 1990) develop much of the work in the control literature on the employ of differential-algebra. Important contributions of these techniques have been used for model identifiably (Fliess, et al., 1999).

In order to give a background previous to the estimation methodology proposed under the differential algebraic frame, the following definitions are considered:

Definition 1 Let L and K be differential fields. A differential field extension L/K is given by K and L such that: 1) K is subfield of L and; 2) the derivation of K is the restriction to K of the derivation of L.

Example 1 Q, $\Re$, C are trivial examples of differential field extensions, where $\mathrm{Q} \subset \Re \subset \mathrm{C}$.

Definition 2 An element of a differential field K is said to be *algebraic* if it satisfies a differential polynomial equation with coefficients over K. If an element is not algebraic then it is said to be *transcendent*.

Example 2 $R < e^t > / R$ is a differential field extension, where $R \subseteq R < e^t >$. This extension is *algebraic* because e^t satisfies $\dot{x} - x = 0$.

Definition 3 Let G, $K < u >$ be differential fields. A *dynamics* consists in a finitely generated differential *algebraic* extension $G/K < u >$. $(G = K < u, \xi >, \xi \in G)$

Example 3 Let us consider the following differential equation:

$$\dot{u}^2 y + 4\ddot{u} = 0$$

In this case, y is algebraic over $K < u >$, therefore, it can be seen as a dynamics of the form $K < u, y > / K < u >$ where $K = \Re$.

Definition 4 An element in G is said to be *algebraically observable* with respect to $\{u, y\}$ if it is algebraic over $K < u, y >$.

So that a state x is said to be algebraically observable if it is algebraic over $K < u, y >$, that is to say, it satisfies a differential polynomial in terms of u, y and some of their time derivatives, i. e.

$$P(x, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$$

with coefficients over $K < u, y >$. Above condition is called *the algebraic observability condition (AOC).*

For further information of these definitions see (Martínez-Guerra, et al., 2004 and Kolchin, 1973).

3. APPLICATION EXAMPLES

In order to apply the above presented observability approaches, several examples are considered;

3.1 Local linear observability approach

Case 1

Consider the following mathematical model structure of a class of non-isothermal, non-adiabatic stirred chemical reactor, where a second order chemical reaction, $A + B \rightarrow P$, take place:

- Mass balance for reactive *A* (x_1):

$$\dot{x}_1 = f_1(x_1, x_2, x_3) \equiv a(\alpha - x_1) - x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) \tag{14}$$

- Mass balance for reactive *B* (x_2):

$$\dot{x}_2 = f_2(x_1, x_2, x_3) \equiv a(\delta - x_2) - x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) \tag{15}$$

- Energy balance (x_3):

$$\dot{x}_3 = f_3(x_1, x_2, x_3) \equiv b(\rho - x_3) - d\,x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right) - \gamma(u - x_3) \tag{16}$$

- Measured output:

$$y = x_3 \qquad u = \text{cte.} \tag{17}$$

Now, consider the following linearized version of system (14-17)

$$\dot{X} = AX \tag{18}$$

where:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix};$$

$$X = [x_1 \quad x_2 \quad x_3]$$

A is the Jacobian matrix of the system (14-16), X is the state vector, and

$$\frac{\partial f_1}{\partial x_1} = -a - x_2 \exp\left(-\frac{\beta}{x_3}\right);$$

$$\frac{\partial f_1}{\partial x_2} = -x_1 \exp\left(\frac{-\beta}{x_3}\right); \frac{\partial f_1}{\partial x_3} = x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right)\frac{\beta}{x_3^2}\ \frac{\partial f_2}{\partial x_1} = -x_2 \exp\left(\frac{-\beta}{x_3}\right);$$

$$\frac{\partial f_2}{\partial x_2} = -a - x_1 \exp\left(\frac{-\beta}{x_3}\right); \frac{\partial f_2}{\partial x_3} = x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right)\frac{\beta}{x_3^2}; \frac{\partial f_3}{\partial x_1} = -d\,x_2 \exp\left(\frac{-\beta}{x_3}\right);$$

$$\frac{\partial f_3}{\partial x_2} = -d\,x_1 \exp\left(\frac{-\beta}{x_3}\right); \frac{\partial f_3}{\partial x_3} = -b + d\,x_1 x_2 \exp\left(\frac{-\beta}{x_3}\right)\frac{\beta}{x_3^2} + \gamma$$

and the corresponding observability matrix is defined as:

$$\aleph = \begin{bmatrix} C & CA & CA^2 \end{bmatrix}^T \tag{19}$$

Now, for our system:

$$\aleph = \begin{bmatrix} 0 & \frac{\partial f_3}{\partial x_1} & \chi_{13} \\ 0 & \frac{\partial f_3}{\partial x_2} & \chi_{23} \\ 1 & \frac{\partial f_3}{\partial x_3} & \chi_{33} \end{bmatrix} \tag{20}$$

where:

$$\chi_{13} = \frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_1} + \frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_3}{\partial x_1}\frac{\partial f_3}{\partial x_3}; \quad \chi_{23} = \frac{\partial f_1}{\partial x_2}\frac{\partial f_3}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_3}{\partial x_2}\frac{\partial f_3}{\partial x_3};$$

$$\chi_{33} = \frac{\partial f_1}{\partial x_3}\frac{\partial f_3}{\partial x_1} + \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\frac{\partial f_3}{\partial x_3}$$

In order to assure observability conditions, the observability matrix (20) must be invertible, i. e. $\det(\aleph) \neq 0$, therefore:

$$\det(\aleph) = \chi_{23}\frac{\partial f_3}{\partial x_1} - \chi_{13}\frac{\partial f_3}{\partial x_2} \neq 0 \tag{21}$$

From this, can be concluded that the system (14-17) is locally observable.

Case 2.

Another application case is related with the mathematical model of a complex non-isothermal wastewater plant; the process is described by classic mass balance equations (Aguilar-López, et al, 2006); temperature effect on different parameters is considered introducing energy balance, neglecting metabolic heat generation in comparison to other energy flows. The bioreactor was assumed to be a perfect mixed. The temperature effect on the maximum specific growth rate was evaluated by equation (29); the mass transfer coefficient for the oxygen ($k_L a$) in equation (31) is an empirical function of the airflow rate (Bastin, et Al. 1990, Dochain et Al 2001), the death coefficient (k_d) in equation (30) and the evaporation flux ($K_{ev}S$) of volatile organic compounds ($VOCs$) is also considered in the COD balance, together with the inactivate biomass $(1-f_n)X$ which contributes to increase the substrate concentration in the bioreactor. In all balances, $V = 15{,}000m^3$ and $V_S = 750m^3$.

Model for the bioreactor:

Substrate (*S*) concentration (COD) mass balance:

$$\frac{dS}{dt} = \frac{Q_f}{V} S_r - \frac{Q_O}{V} S - \frac{\mu_{\max}}{Y_{X/S}} \left(\frac{S}{K_S + S} \right) \left(\frac{C_{O_2}}{K_{OH} + C_{O_2}} \right) X + (1 - f_n) X - K_{ev} S \qquad (22)$$

Here, $Y_{X/S} = 0.67$.

Biomass (*X*) concentration mass balance:

$$\frac{dX}{dt} = \frac{Q_f}{V} X_r - \frac{Q_O}{V} X - \mu_{\max} \left(\frac{S}{K_S + S} \right) \left(\frac{C_{O_2}}{K_{OH} + C_{O_2}} \right) X - k_d X \qquad (23)$$

Oxygen (C_{O2}) concentration mass balance:

$$\frac{dC_{O_2}}{dt} = \frac{Q_f}{V} C_{O_2 f} - \frac{Q_O}{V} C_{O_2} - \frac{\mu_{\max}}{Y_{O_2}} \left(\frac{S}{K_S + S} \right) \left(\frac{C_{O_2}}{K_{OH} + C_{O_2}} \right) X + k_L a (C_{O_2 Sat} - C_{O_2}) \qquad (24)$$

Here, $Y_{O_2} = 2.03$.

Temperature (T) dynamics via the energy balance:

$$\frac{dT}{dt} = \frac{Q_O}{V} \left(T_{in} - T \right) + \frac{Q_{air} \rho_{air} Cp_{air}}{V \rho Cp} T_{air} + \frac{h_c A}{V \rho Cp} \left(T - T_\infty \right) \qquad (25)$$

Model for the biosettler:

Biomass balance, assuming that there is not biomass in the overflow from the settler (Rahmalo, 1999):

$$\frac{dX_r}{dt} = \frac{Q_U}{V_S} X_r - \frac{Q_O}{V_S} X \tag{26}$$

Here,

$$Q_O = Q_f + Q_r \tag{27}$$

$$Q_U = Q_W + Q_r \tag{28}$$

Temperature effect on the maximum specific growth rate:

$$\mu_{\max} = b^2 \cdot (T - 285K)^2 \{1 - \exp[c \cdot (T - 330.5)]\}^2 \tag{29}$$

Here, $b = 0.05K^{-1}h^{-0.5}$ and $c = 5*10^{-3}K^{-1}$ (fitted to experimental data).
Temperature effect on the death coefficient:

$$k_d(T_W) = k_d(20°C) \cdot 1.05^{(T_W - 20°C)} \tag{30}$$

Here, $k_d(20°C) = 0.03d^{-1}$.

Temperature effect on the mass transfer coefficient:

$$k_L a(T_W) = k_L a(20°C) \cdot 1.02^{(T_W - 20°C)} \tag{31}$$

Here, $$k_L a(20°C) = \left(\alpha + \exp\left(\frac{Q_{air}}{\beta}\right)\right) \tag{32}$$

As can be seen this model is highly nonlinear, therefore a numerical local linear observability test based on the observability matrix is done, previously two optimal steady states were determinated, which are related with a feasible minimum COD concentration in the bioreactor, the corresponding values are given in table 1, they are considered to evaluate the Jacobian matrix of the system (22-26).

State Variable	Optimum Point 1	Optimum Point 2
COD ($mg \cdot L^{-1}$)	168.8	150.19
Biomass ($mg \cdot L^{-1}$)	2988	3033
Biomass Settler ($mg \cdot L^{-1}$)	11020	10356
Dissolved Oxygen ($mg \cdot L^{-1}$)	4.2	4.28
Temperature [ºC]	35	35

Table 1. Optimum steady states (for minimum COD)

Now, several calculations were done considering different measured outputs as bioreactor temperature (T), dissolved oxygen (DO) and chemical oxygen demand (COD), in order to obtain observability condition for the both two steady state points and the considered measures, the following tables show the corresponding characteristics of the observability matrix for the several cases:

Optimum Point 1	y = COD; [1 0 0 0 0]
1.0e+003 *	Determinant of the observability matrix = 0.9929

$$\begin{pmatrix} 0.0010 & 0 & 0 & 0 & 0 \\ -0.0061 & -0.0005 & 0 & -0.0339 & 0.0188 \\ 0.0398 & 0.0036 & -0.0001 & 0.2158 & -0.1283 \\ -0.2574 & -0.0241 & 0.0006 & -1.3907 & 0.8241 \\ 1.6602 & 0.1597 & -0.0052 & 8.9602 & -5.2885 \end{pmatrix}$$

Optimum Point 1	y = COD-DO; [1 0 0 1 0]
1.0e+003 *	Determinant of the observability matrix = -1.3518

$$\begin{pmatrix} 0.0010 & 0 & 0 & 0.0010 & 0 \\ -0.0062 & -0.0005 & 0 & -0.0345 & 0.0192 \\ 0.0405 & 0.0036 & -0.0001 & 0.2195 & -0.1305 \\ -0.2619 & -0.0245 & 0.0006 & -1.4150 & 0.8384 \\ 1.6892 & 0.1625 & -0.0053 & 9.1170 & -5.3808 \end{pmatrix}$$

Optimum Point 1	y = COD-T; [1 0 0 0 1]
1.0e+003 *	Determinant of the observability matrix = -22.2930

$$\begin{pmatrix} 0.0010 & 0 & 0 & 0 & 0.0010 \\ -0.0061 & -0.0005 & 0 & -0.0339 & 0.0181 \\ 0.0398 & 0.0036 & -0.0001 & 0.2158 & -0.1278 \\ -0.2574 & -0.0241 & 0.0006 & -1.3907 & 0.8237 \\ 1.6602 & 0.1597 & -0.0052 & 8.9602 & -5.2882 \end{pmatrix}$$

Table 2. Full observability condition for the steady state 1

Optimum Point 1	y = DO; [0 0 0 1 0]
Determinant of the observability matrix = -8.4358e-006	

0	0	0	1.000	0
-0.110	-0.013	0	-0.572	0.400
0.7001	0.065	-0.002	3.756	-2.204
-4.512	-0.432	0.014	-24.352	14.341
29.068	2.838	-0.103	156.804	-92.314

Optimum Point 1	y = DO-T; [0 0 0 1 1]
Determinant of the observability matrix = -8.2174e-005	

0	0	0	1	1
-0.1103	-0.013	0	-0.5718	-0.3313
0.7001	0.0656	-0.0016	3.7556	-1.6691
-4.5120	-0.432	0.0137	-24.351	13.9499
29.068	2.8380	-0.1025	156.804	-92.028

Optimum Point 1	y = T; [0 0 0 0 1]
Determinant of the observability matrix = 0	

0	0	0	0	1.0000
0	0	0	0	-0.7313
0	0	0	0	0.5348
0	0	0	0	-0.3912
0	0	0	0	0.2861

Table 3. Null observability condition for the steady state 1

Optimum Point 2	y = DO; [0 0 0 1 0]			
Determinant of the observability matrix = -1.3530e-005				
0	0	0	1.0000	0
-0.124	-0.012	0	-0.891	0.358
0.905	0.072	-0.007	4.463	-2.494
-6.408	-0.515	0.015	-31.162	17.909
45.308	3.701	-0.124	219.947	-126.69

Optimum Point 2	y = DO-T; [0 0 0 1 1]			
Determinant of the observability matrix = -2.8267e-005				
0	0	0	1	1
- 0.107	-0.010	0	-0.783	-0.2398
0.778	0.062	-0.001	3.826	-1.8409
- 5.506	-0.443	0.013	-26.735	15.2233
38.927	3.180	-0.107	188.966	-108.760

Optimum Point 2	y = T; [0 0 0 0 1]			
Determinant of the observability matrix = -4.2807e-009				
0	0	0	0	1.0000
0.0162	0.0016	0	0.1085	-0.5983
- 0.1265	-0.0103	0.0002	-0.6372	0.6537
0.9018	0.0726	-0.0021	4.3928	-2.6863
-6.3805	-0.5206	0.0173	-30.9819	17.9321

Table 4. Null observability condition for the steady state 2

Note that the temperature-dissolved oxygen measurements does not provide full observability conditions, which is unfortunate, such that these both two measures can be easily on-line obtained and as conclusion the substrate concentration measurement is needed to provide full observability conditions, remembering that these conclusions are only locally valid under the considered process conditions.

3.2 Differential-geometric observability approach

Case 3

Let us consider again the non-isothermal, non-adiabatic continuous chemical reactor model, given by equations (14)-(17). Applying the issues contained in section 2.2 the following observability matrix for the transformed system is:

$\vartheta = \left(dL_f^0 h, dL_f^1 h, dL_f^2 h\right)^T$, considering as measured output the reactor temperature, i.e. $y = x_3$, and employing the nomenclature presented in section 3.1, the following is defined:

$$\Phi_{1r} = dL_f^0 = x_3$$

$$\Phi_{2r} = dL_f^1 = b(\rho - x_3) - dx_1 x_2 \exp\left(-\frac{\beta}{x_3}\right) - \gamma(u - x_3)$$

$$\Phi_{3r} = dL_f^2 = \left[-dx_1 \exp\left(-\frac{\beta}{x_3}\right), -dx_2 \exp\left(-\frac{\beta}{x_3}\right), -b + dx_1 \exp\left(-\frac{\beta}{x_3}\right)\frac{\beta}{{x_3}^2} + \gamma\right] \bullet \begin{bmatrix} a(\alpha - x_1) - x_1 x_2 \exp\left(-\frac{\beta}{x_3}\right) \\ a(\delta - x_2) - x_1 x_2 \exp\left(-\frac{\beta}{x_3}\right) \\ b(\rho - x_3) - dx_1 x_2 \exp\left(-\frac{\beta}{x_3}\right) - \gamma(u - x_3) \end{bmatrix} \quad (33)$$

$$\vartheta = \left[\frac{\partial \Phi_r}{\partial x}\right] = \begin{pmatrix} \frac{\partial \Phi_{1r}}{\partial x_1} & \frac{\partial \Phi_{1r}}{\partial x_2} & \frac{\partial \Phi_{1r}}{\partial x_3} \\ \frac{\partial \Phi_{2r}}{\partial x_1} & \frac{\partial \Phi_{2r}}{\partial x_2} & \frac{\partial \Phi_{2r}}{\partial x_3} \\ \frac{\partial \Phi_{3r}}{\partial x_1} & \frac{\partial \Phi_{3r}}{\partial x_2} & \frac{\partial \Phi_{3r}}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\partial \Phi_{2r}}{\partial x_1} & \frac{\partial \Phi_{2r}}{\partial x_2} & \frac{\partial \Phi_{2r}}{\partial x_3} \\ \frac{\partial \Phi_{3r}}{\partial x_1} & \frac{\partial \Phi_{3r}}{\partial x_2} & \frac{\partial \Phi_{3r}}{\partial x_3} \end{pmatrix} \tag{34}$$

$$\det \vartheta \neq 0 \Rightarrow rank\left[\frac{\partial \Phi_r}{\partial x}\right] = 3 \tag{35}$$

where $\det[\vartheta] = \frac{\partial \Phi_{2r}}{\partial x_1}\frac{\partial \Phi_{3r}}{\partial x_2} - \frac{\partial \Phi_{2r}}{\partial x_2}\frac{\partial \Phi_{3r}}{\partial x_1}$

The algebraic definitions of some of the above derivatives are omitted to save space on the manuscript, however is easy to see from the presented definitions, that the full rank of the observability matrix obey the corresponding condition. Note that this result is in agreement with the first presented in section 3.1.

Case 4

For this approach, let us consider the following three state nonlinear mathematical model of a class of continuous stirred bioreactor, described by a Monod´s kinetic model (Neria-González and Aguilar-López, 2007):

$$\begin{aligned} \frac{dS}{dt} &= D(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} \\ \frac{dX}{dt} &= -DX + \mu(S)X \\ \frac{dP}{dt} &= -DP + \mu(S)\frac{X}{Y_{P/X}} \end{aligned} \tag{36}$$

where $\mu(S) = \mu_{\max} S/(\beta + S)$ is considered that obey a Monod's model. S, X and P are substrate, biomass and product concentrations, respectively, $D = q/V$ is the dilution rate with V the volume of the bioreactor and q the constant flow passing through the bioreactor, S_{in} is the input substrate concentration, $Y_{S/X}$ and $Y_{P/X}$ are the corresponding coefficients yield, β is the half saturation constant. Let us notice that the inputs $D = u$ and S_{in} are fixed.

Moreover, we assume that the measured output is:

$$y = S$$

The differential system described by (36) can be represented as follows,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{37}$$

where $x = \begin{pmatrix} S & X & P \end{pmatrix}^T \in R^3$, f and g are both smooth functions $R^3 \to R^3$ and the observation function h is also smooth $R^3 \to R$.

The observation space of (36) is given by

$$\Phi_b : \begin{pmatrix} S \\ X \\ P \end{pmatrix} \to \begin{pmatrix} h \\ L_f h \\ L_f^2 h \end{pmatrix} = \begin{pmatrix} \Phi_{1b} \\ \Phi_{2b} \\ \Phi_{3b} \end{pmatrix} \tag{38}$$

where,

$$\Phi_{1b} = S \qquad , \qquad \Phi_{2b} = -\frac{\mu_{\max} S}{\beta + S} \frac{X}{Y_{S/X}} \qquad ,$$

$$\Phi_{3b} = \frac{X^2}{Y_{S/X}^2} \frac{\mu_{\max}^2 \beta S}{(\beta + S)^3} - \frac{X}{Y_{S/X}} \frac{\mu_{\max}^2 S^2}{(\beta + S)^2}$$

In (38), Φ_b denotes the diffeomorphism defined on R^3 and $L_f h$ denotes the Lie derivative of h along the field f.

from (38) we obtain,

$$\vartheta_b = \left[\frac{\partial \Phi_b}{\partial x} \right] = \begin{pmatrix} \frac{\partial \Phi_{1b}}{\partial S} & \frac{\partial \Phi_{1b}}{\partial X} & \frac{\partial \Phi_{1b}}{\partial P} \\ \frac{\partial \Phi_{2b}}{\partial S} & \frac{\partial \Phi_{2b}}{\partial X} & \frac{\partial \Phi_{2b}}{\partial P} \\ \frac{\partial \Phi_{3b}}{\partial S} & \frac{\partial \Phi_{3b}}{\partial X} & \frac{\partial \Phi_{3b}}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial \Phi_{2b}}{\partial S} & \frac{\partial \Phi_{2b}}{\partial X} & 0 \\ \frac{\partial \Phi_{3b}}{\partial S} & \frac{\partial \Phi_{3b}}{\partial X} & 0 \end{pmatrix} \tag{39}$$

where:

$$\frac{\partial \Phi_{2b}}{\partial S} = \frac{\mu_{\max}}{\beta + S}\frac{X}{Y_{S/X}}\left(\frac{S}{\beta + S} - 1\right) \quad , \quad \frac{\partial \Phi_{2b}}{\partial X} = -\frac{\mu_{\max} S}{\beta + S}\frac{1}{Y_{S/X}}$$

$$\frac{\partial \Phi_{3b}}{\partial S} = \frac{X^2}{Y_{S/X}^2}\frac{\mu_{\max}\beta}{(\beta + S)^3}\left(1 - 3\frac{S}{\beta + S}\right) - 2\frac{X}{Y_{S/X}}\frac{\mu_{\max}^2 S}{(\beta + S)^2}\left(1 + \frac{S}{\beta + S}\right)$$

$$\frac{\partial \Phi_{3b}}{\partial X} = \frac{2X}{Y_{S/X}^2}\frac{\mu_{\max}^2 \beta S}{(\beta + S)^3} - \frac{1}{Y_{S/X}}\frac{\mu_{\max}^2 S^2}{(\beta + S)^2}$$

from (39) we obtain that, $rank[\vartheta_b] < 3$, that is to say, the system (36) is not completely observable. (see Gauthier, et. Al. 1992 for more details)

3.3 Differential-algebraic observability approach

Case 5

As a first case of study, it is considered a nonlinear mathematical model for a human glucose metabolism, given by (Parisa, et al. 2008), in accordance with the following equations, in order to analyze state observability:

- Glucose mass balance (x_1):

$$\dot{x}_1 = -P_1(x_1 - G_b) - x_1 x_2 \tag{40}$$

- Insulin for Glucose consumption mass balance (x_2):

$$\dot{x}_2 = -P_2 x_2 + P_3(x_3 - I_b) \tag{41}$$

- Insulin in blood mass balance (x_3):

$$\dot{x}_3 = -n(x_3 - I_b) + \gamma(x_1 - h)t \tag{42}$$

- Measurement output:

$$y = x_1$$

From definition ***1*** can be noted that equations (40)-(42) are a dynamical model. Now, from definitions ***2 – 4*** and after an algebraic manipulation of the state equation the following is obtained:

Glucose:

$$x_1 = y \tag{43}$$

- Insulin for glucose consumption rate:

$$x_2 = -y^{-1}\left(\dot{y} + P_1(y - G_b)\right) \tag{44}$$

- Insulin in blood:

$$x_3 = P_3^{-1}\left[-y^{-1}\left(\ddot{y}+P_1\dot{y}\right)+y^{-2}\left(\dot{y}+P_1(y-G_b)\right)-P_2y^{-1}\left(\dot{y}+P_1(y-G_b)\right)\right]+I_b \tag{45}$$

Note that equation (45) is obtained by substituting (43) into (40) and solving for x_3 from equation (41).

As can be observed $H = k<x_1, x_2, x_3>$, $k = \Re$, the flat output is $y = x_1$, therefore, the glucose model is a nonlinear flat system, beside note that the state variables of the model x_2 and x_3 are observables from Glucose concentration measurements. It means in plain words that observability is equivalent to the following fact: any system variable, a state variable for instance, may be expressed as a function of the input and output variables and of their derivatives up to some finite order.

Case 6

The following example considers a simplified Haldane model which is taken from Vargas et al, (2000). The mathematical model is

$$\frac{dS}{dt} = D(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} + k_d X \tag{46}$$

$$\frac{dX}{dt} = -DX + \mu(S)X - k_d X \tag{47}$$

where $\mu(S) = \mu_{\max} S/\left(\delta + S + S^2/\phi\right)$ is the specific growth rate which is considered that obey a Haldane's model and $\mu_{\max}$ is the maximum growth rate.

The algebraic observability condition for Haldane's model is computed as follows

$$\dot{y} - u(S_{in} - y) + \left(\frac{\mu_{\max}\phi\, y}{\delta\phi + \phi\, y + y^2}\frac{1}{Y_{S/X}} - k_d\right)X = 0 \tag{48}$$

hence, X satisfies an algebraic equation with coefficients in $K < u, y >$, that is to say, the biomass concentration X is algebraically observable from substrate concentrations measurements, i. e. $y = S$.

Case 7

Now, let us to consider again the model described by equations (36). The algebraic observability condition for the Monod's model is proven as follows, considering again as measured output the substrate concentration ($y = S$).

Biomass concentration:

$$-\dot{y}(\beta + y)Y_{S/X} + u\left(S_{in} - y\right)(\beta + y)Y_{S/X} - \mu_{\max} yX = 0 \tag{49}$$

in the same way given above, X is algebraically observable.

Product concentration:

The variable P is not observable in the algebraic sense, which is in accordance with the geometric-differential approach result, but, it is easy to see that a differential equation of lowest order of P is the following:

$$\dot{P} + u\,P = -\frac{Y_{S/X}}{Y_{P/X}}\dot{y} + \frac{Y_{S/X}}{Y_{P/X}}u\left(S_{in} - y\right) \tag{50}$$

which shows that we still can estimate P given the stability of its dynamics in terms of u, y and their time derivatives.

We would say here that the concentration P is detectable with respect to u, y.

Case 8

Let us to consider the system (14-17) corresponding to the non-isothermal, non-adiabatic continuous chemical reactor, which is take in account again in order to show its state observability properties, from definitions 2 to 4.

$$x_1 = g_1(y,u) \equiv \int \exp(a(t-\sigma))h_1(y(t),u(t))dt \tag{51}$$

where:

$$h_1(y,u) = -\left[\frac{\dot{y} + \gamma(u-y) - b(\rho - y)}{d} - a\alpha\right] \tag{52}$$

$$x_2 = g_2(y,u) \equiv \int \left(\left(\dot{y} + \gamma(u-y) - b(\rho - y) \right) \frac{a + g_1(y,u)\exp\left(\frac{-\beta}{y}\right)}{d\, g_1(y,u)\exp\left(\frac{-\beta}{y}\right)} - a\delta \right) dt \tag{53}$$

$$x_3 = g_3(y,u) \equiv y \tag{54}$$

Note that this result is in agreement with the previous results of section 3.1 and 3.2.

Case 9

Finally, consider the following nonlinear dynamic system as a last example to study uncertainty observability conditions; the model represents a standard continuous stirred tank reactor where a second order reaction $(2A \to P)$ takes place, in this case is considered that the corresponding reaction heat, $X_3 = \Delta HKX_1^2$, is unknown and it is considered as a new state variable with unknown dynamic:

Energy Balance:

- $\dot{X}_2 = \theta\left(X_{2,in} - X_2\right) + X_3 + \gamma\left(u - X_2\right)$ (55)

Mass Balance:

- $\dot{X}_1 = \theta\left(X_{1,in} - X_1\right) - KX_1^2$ (56)

Uncertainty dynamics:

- $\dot{X}_3 = f\left(X_1, X_2\right)$ (57)

System output:

- $Y = X_2$ (58)

The following concept is introduced in order to define an algebraically observable uncertainty condition:

Definition 5. An element $\mathbf{X_u}$ in G is said to be an algebraically observable uncertainty if $\mathbf{X_u}$ satisfies a differential algebraic equation with coefficients over $K < u, y >$.

Now, consider equations (55) and (58), which according to definition 2 define an algebraic differential dynamic system. From this subsystem, the following differential-algebraic equations can be obtained:

$$X_2 - Y = 0 \tag{59}$$

$$\dot{Y} + (\theta + \gamma)Y - \theta X_{2,in} - \gamma u - X_3 = 0 \tag{60}$$

From definitions 4 and 5, the pair $\mathbf{X_2}$, $\mathbf{X_3}$ is universally observable in the Diop-Fliess (1991) sense.

CONCLUDING REMARKS

In this paper the observability conditions were analyzed for a wide class of nonlinear (bio)chemical systems employing different approaches: linear, geometric and algebraic frameworks. The application examples consider several types of continuous chemical (bio)reactors, which can be of industrial interest. For a class of non-isothermal, non-adiabatic continuous stirred chemical reactor, the three observability frameworks were applied; all of them conclude that the respective concentrations can be observed from reactor temperature measurements. The (local) linear observability approach was applied to a class of non-isothermal aerobic wastewater bioreactor, where is concluded that the substrate concentration must be included as output measurement in order to provide full order observability condition. The geometric-differential and algebraic-differential approaches were applied to a class of three states continuous bioreactor with a Monod´s kinetic model, both two approaches predict an unobservability condition considering substrate concentration as measured output, however the algebraic differential approach provides a detectability condition for the product concentration. The algebraic-differential approach was also applied to similar bioreactor, now with two states and Haldane kinetic model, predicting full order observability condition from substrate concentration measurements. Finally a metabolic model related to human glucose metabolism is also analyzed, considering the algebraic-differential observability framework, where full order observability is proven via glucose measurements; the uncertainty observability condition is proven from Case 9, where is concluded that the reaction heat in a class of continuous stirred reactor is observable from temperature measurements. Although with these approaches the observability condition was proven for this kind of systems it should be noted that in the algebraic framework an explicit relationship for each state variable is obtained.

NOMENCLATURE

D.- Dilution rate [1/hour]
K.- Kinetic constant
N.- Lineal Observability matrix
n.- Control´s parameter
u.- Control input [1/hour]
t.- time [hours]
X.- Biomass concentration [g/L]
S.- Sulfate concentration [g/L]
P.- Sulfide concentration [g/L]
x.- state variables vector
y.- measured output
$Y_{S/X}$.- Sulfate yield coefficient
$Y_{P/X}$.- Sulfide yield coefficient

Greek Letters :

α .- Inlet concentration reactive A
β.- Activation Energy
δ.- Inlet concentration reactive B
ρ.- Inlet entalpy
ΔH.- Reaction Heat
Ω.-Linearized observability matrix
μ.- specific growth rate [1/hour]
Φ.- Transformed states

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Uniformly Bounded Error Estimator for Bioprocess with Unstructured Cell Growth Models

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Uniformly Bounded Error Estimator for Bioprocess with Unstructured Cell Growth Models*

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Abstract

In this paper, we proposed a uniformly bounded error estimator for a common class of bioreactor models. The biomass and other products are estimated by means of substrate concentration measurements employing the characteristics of the unstructured cell growth models, which are linearly dependent. The estimation methodology is based on a suitable change of variable which allows generating artificial variables to infer the remaining mass concentrations constructing a differential-algebraic structure. The proposed methodology is applied to a class of Monod unstructured kinetic model with success. Some remarks about the convergence characteristics of the proposed estimator are given and numerical simulations show its satisfactory performance.

KEYWORDS: differential-algebraic estimator, unstructured kinetic models, biomass, products estimation

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1. Introduction

The importance of on-line monitoring of biotechnological processes has increased during the last years. Advantages include gaining knowledge about the state of the process and the possibility of detecting and isolating abnormal process developments at early stages. This reduces process costs, contributes to process safety and helps in trouble-shooting and process accommodation. The main problem in fermentation monitoring and control is the fact that process variables usually cannot be measured on-line (Soroush, 1997). Monitoring and controlling these processes can therefore be difficult because only indirect measurements are available online, while calculated values may be rather uncertain. This can be due to uncertainty with respect to the equations used, measurement errors or both. For automatic control this may have serious consequences, especially as the actual variables of interest often cannot be directly controlled and related variables are controlled instead. In fermentation processes, on-line and off-line measurements are the main source of information about the state of the process. In combination with model-based calculations, they are used to produce estimations for monitoring purposes as well as for automatic and manual process control (Bastin and Dochain, 1990; Soroush, 1997).

However, during a bioreacting process the variables such as concentrations can be on-line estimated using *soft sensors*. Special attention was given to filtering techniques, namely extended Kalman filter, adaptive observers, and artificial neural networks (ANN), (Dávila, et al., 2005; Hu and Wang, 2002; Levant, 2001) however these techniques lead to over-parameterization of the corresponding estimation algorithms, where the right tuning of the estimators gains is difficult. From the above the research on low parameter estimators looks adequate. It is shown that software based state estimation is a powerful technique that can be successfully used to enhance automatic control performance of biological systems as well as in system monitoring and on-line optimization. The main contribution in this work is to show a state estimator which is a simplified version of the methodology given by Lemesle and Gouzé (2005) where a simple linear change of variable given in a natural manner allows to develop a differential-algebraic state estimator avoiding the over parameterization and the corresponding tuning, numerical results show an adequate performance of the considered methodology. The proposed estimation methodology is applied to the classical Monod model, which is a kind of unstructured kinetic models; this kinetic model is applied to a class of continuous stirred bioreactors.

The remainder of this paper is organized as follows: in section 2, we present some definitions about differential algebra to establish the Algebraic Observability Condition; in section 3, we show the uniformly bounded error estimator applied to the unstructured kinetic model mentioned above, also the dynamic of its corresponding estimation error is analyzed; in section 4, the performance of the proposed methodology is illustrated by numerical simulations.

2. Statement of the problem

Let us recall the classical observer definition. Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

with $x \in R^n$, $u \in R^m$, $m \le n$, $y \in R$.

An observer for system (1) is a dynamical system:

$$\dot{\hat{x}} = \hat{f}(\hat{x}, u, y)$$

whose task is state estimation. Usually is required at least that $\|\hat{x} - x\| \to 0$, as $t \to \infty$. Although, in some cases, exponentially convergence is also required (Gauthier, et. al., 1992).

We define a bounded uniformly error estimator giving $\hat{x}$ with $\|\hat{x} - x\|$ which belongs to open ball with radius proportional to some value that depends on its error estimation.

In all paper, we will consider the Monod's continuous stirred bioreactor model that was previously presented in Bastin and Dochain (1990):

$$\begin{aligned} \frac{dS}{dt} &= D(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} \\ \frac{dX}{dt} &= -DX + \mu(S)X \\ \frac{dP}{dt} &= -DP + \mu(S)\frac{X}{Y_{P/X}} \end{aligned} \tag{2}$$

where $\mu(S) = \mu_{\max} S/(\beta + S)$ is the specific growth rate and $\mu_{\max}$ is the maximum growth rate. The state variables S, X and P, are substrate, biomass and product concentrations, respectively, $D = q/V$ is the dilution rate with V the volume of the bioreactor and q the constant flow passing through the bioreactor, S_{in} is the input substrate concentration, $Y_{S/X}$ and $Y_{P/X}$ are the corresponding yield coefficients, β is the half saturation constant. Let us notice that the inputs $D = u$ and S_{in} are fixed.

Moreover, we assume that the measured output is:

$$y = S$$

Remark 1. Before proposing the state estimator, the algebraic observability condition (AOC), given by definition 4 in the appendix, is proven for model (2) as follows,
Replacing $S = y$ and $D = u$ in the system (2), we obtain

$$\dot{y} = u\left(S_{in} - y\right) - \frac{\mu_{\max} y}{\beta + y}\frac{X}{Y_{S/X}} \tag{3}$$

$$\dot{X} = -DX + \frac{\mu_{\max} y}{\beta + y} X \tag{4}$$

$$\dot{P} = -DP + \frac{\mu_{\max} y}{\beta + y}\frac{X}{Y_{P/X}} \tag{5}$$

From equation (3) we obtain the AOC for biomass concentration X, i.e,

$$-\dot{y}(\beta + y)Y_{S/X} + u\left(S_{in} - y\right)(\beta + y)Y_{S/X} - \mu_{\max} yX = 0 \tag{6}$$

It should be noted in equation (6) that X satisfies an algebraic equation with coefficients in $K < u, y >$, that is to say, X is algebraically observable.

Now, we analyze the AOC for product concentration using equation (5). The variable P is not observable in the algebraic sense, but, it is easy to see that a differential equation of lowest order of P is the following

$$\dot{P} + u\,P = -\frac{Y_{S/X}}{Y_{P/X}}\dot{y} + \frac{Y_{S/X}}{Y_{P/X}} u\left(S_{in} - y\right) \tag{7}$$

which shows that we still can estimate P given the stability of its dynamics in terms of u, y and their time derivatives.

We would say here that the concentration P is detectable with respect to u, y. We entirely rely on the asymptotic stability of equation (7) for the estimation of P. Instead of its form (7) we may realize equation (7) in the following form

$$\begin{cases} \dot{z} = -u\,z + \eta \\ P = z - y \\ X = \bar{z} - y \end{cases} \tag{8}$$

where, $\eta = u\,S_{in} + \dfrac{\mu_{\max} y}{\beta + y}\left[\dfrac{1}{Y_{P/X}} - \dfrac{1}{Y_{S/X}}\right](\bar{z} - y) \;<\; \infty\;,\; u > 0$

The estimation scheme (7) requires the differentiability of P and the differentiation of y at order 1, while the realization scheme (8) requires the non differentiability of P and the non differentiation of y (order only zero). This remark allows us to relax the assumption on the differentiability of P which was earlier needed to derive equation (8). Then, of a natural manner we can define a new estimator given by (8), which will be able of reconstruct the two variables X and P.

Remark 2. It should be noted that the propose estimator does not need adjustable gain being a robust estimator in spite of the presence of noise in the measurement.

Remark 3. We observe that the system (2) is unobservable using differential-geometric techniques (Gauthier, et al., 1992).

The differential system (2) can be represented as follows,

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u \\ y &= h(x) \end{aligned} \tag{9}$$

where $x = (S \quad X \quad P)^T \in R^3$, f and g are both smooth functions $R^3 \to R^3$ and the observation function h is also smooth $R^3 \to R$.

The observation space of (9) is given by

$$\Phi : \begin{pmatrix} S \\ X \\ P \end{pmatrix} \quad \to \quad \begin{pmatrix} h \\ L_f h \\ L_f^2 h \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \tag{10}$$

where,

$$\Phi_1 = S \ , \quad \Phi_2 = -\frac{\mu_{\max} S}{\beta + S}\frac{X}{Y_{S/X}} \quad , \quad \Phi_3 = \frac{X^2}{Y_{S/X}^2}\frac{\mu_{\max}^2 \beta\, S}{(\beta + S)^3} - \frac{X}{Y_{S/X}}\frac{\mu_{\max}^2 S^2}{(\beta + S)^2}$$

In (10), Φ denotes the diffeomorphism, defined on R^3 and $L_f h$ denotes the Lie derivative of h along the field f.

From (10) we obtain

$$\left[\frac{\partial \Phi}{\partial x}\right] = \begin{pmatrix} \frac{\partial \Phi_1}{\partial S} & \frac{\partial \Phi_1}{\partial X} & \frac{\partial \Phi_1}{\partial P} \\ \frac{\partial \Phi_2}{\partial S} & \frac{\partial \Phi_2}{\partial X} & \frac{\partial \Phi_2}{\partial P} \\ \frac{\partial \Phi_3}{\partial S} & \frac{\partial \Phi_3}{\partial X} & \frac{\partial \Phi_3}{\partial P} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial \Phi_2}{\partial S} & \frac{\partial \Phi_2}{\partial X} & 0 \\ \frac{\partial \Phi_3}{\partial S} & \frac{\partial \Phi_3}{\partial X} & 0 \end{pmatrix} \tag{11}$$

where,

$$\frac{\partial \Phi_2}{\partial S} = \frac{\mu_{\max}}{\beta + S}\frac{X}{Y_{S/X}}\left(\frac{S}{\beta + S} - 1\right) \quad , \quad \frac{\partial \Phi_2}{\partial X} = -\frac{\mu_{\max} S}{\beta + S}\frac{1}{Y_{S/X}}$$

$$\frac{\partial \Phi_3}{\partial S} = \frac{X^2}{Y_{S/X}^2}\frac{\mu_{\max}\beta}{(\beta + S)^3}\left(1 - 3\frac{S}{\beta + S}\right) - 2\frac{X}{Y_{S/X}}\frac{\mu_{\max}^2 S}{(\beta + S)^2}\left(1 + \frac{S}{\beta + S}\right)$$

$$\frac{\partial \Phi_3}{\partial X} = \frac{2X}{Y_{S/X}^2}\frac{\mu_{\max}^2 \beta S}{(\beta + S)^3} - \frac{1}{Y_{S/X}}\frac{\mu_{\max}^2 S^2}{(\beta + S)^2}$$

From (11) we obtain that, $rank[\partial \Phi / \partial x] < 3$, that is to say, the system (2) is not completely observable (see Gauthier et. al., 1992, for more details).

According to equation (6) the biomass concentration X is algebraically observable as fulfils an algebraic equation, on the other hand, from equation (7), the product concentration P is only detectable, that is to say, Monod's system (2) is not completely observable in the differential algebraic sense (Diop and Martínez-Guerra, 2001). Moreover, from observability matrix (11), system (2) is not completely observable in a strict differential-algebraic way (Gauthier et al., 1992). Based on these facts, is not possible to construct some estimation schemes such as, a finite-time convergence observer (Aguilar-López et al., 2006) or a classical high gain observer (Gauthier et al., 1992).

2.1 Estimator Design

In what follows, the corresponding estimated concentration is denoted by ^, and we assume that S is measured exactly, i.e., $S = \hat{S}$. Then, we only reconstruct the biomass variable X and product variable P.

Let us consider the Monod's model given by (2), and make two changes of variables

$$\hat{\bar{z}} = \hat{X} + S \tag{12}$$

and

$$\hat{z} = \hat{P} + S \tag{13}$$

The dynamics of (12) and (13) are given by

$$\dot{\hat{\bar{z}}} = -D\,\hat{\bar{z}} + \hat{\xi} \tag{14}$$

$$\dot{\hat{z}} = -D\,\hat{z} + \hat{\eta} \tag{15}$$

with $\hat{\xi} = D\,S_{in} + \mu(S)\left(\hat{\bar{z}} - S\right)\left(1 - \dfrac{1}{Y_{S/X}}\right)$, $\hat{\eta} = D\,S_{in} + \mu(S)\left(\hat{\bar{z}} - S\right)\left(\dfrac{1}{Y_{P/X}} - \dfrac{1}{Y_{S/X}}\right)$

If we assume that, $0 \le \mu(S) \le \mu_{\max} < \infty$, for all $S \ge 0$, then

$$\left|\hat{\xi} - \xi\right| \le \mu_{\max}\left|1 - \frac{1}{Y_{S/X}}\right|\left|\delta_2\right| \;\le\; N_2(\delta_2) < \infty$$

$$\left|\hat{\eta} - \eta\right| \le \mu_{\max}\left|\frac{1}{Y_{P/X}} - \frac{1}{Y_{S/X}}\right|\left|\delta_2\right| \;\le\; N_3(\delta_2) < \infty$$

with $\delta_1 = \hat{w} - w$ and $\delta_2 = \hat{\bar{z}} - \bar{z}$

Lemma 1. In accordance with equations (14) and (15) the general differential equation is given by

$$\dot{\hat{\vartheta}} = -D\,\hat{\vartheta} + \hat{\zeta} \tag{16}$$

which is uniformly bounded, with $D > 0$, if the following assumptions are considered

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H1: $\left|\hat{\zeta}-\zeta\right|<N(\delta)<\infty$.

H2: For t_0 sufficiently large, $\lim\sup_{t\to t_0}\dfrac{N(\delta)}{D}=0$.

Proof. Let us define the estimation error (the difference between the actual observed signal and its estimate) as follows:

$$e := \hat{\vartheta}-\vartheta \tag{17}$$

applying the time derivative to (17), and taking $\Psi=\hat{\zeta}-\zeta$, we obtain

$$\dot{e}+D\,e=\Psi \tag{18}$$

we obtain the solution from (18)

$$e=\exp^{-Dt}e_0+\int_0^t \exp^{D(\tau-t)}\Psi\,d\tau \tag{19}$$

where e_0 is an initial condition.

Using Triangle and Cauchy-Schwarz inequalities from expression (19):

$$0\le\left|\,e\,\right|\le\left|\,\exp^{-Dt}\,\right|\left|e_0\right|+\int_0^t\left|\,\exp^{D(\tau-t)}\,\Psi\,d\tau\right|$$

from H1,

$$0\le\left|\,e\,\right|\le\left|\,\exp^{-Dt}\,\right|\left|e_0\right|+N(\delta)\int_0^t\left|\,\exp^{D(\tau-t)}\;d\tau\right| \tag{20}$$

Thus, as $t\to t_0$, t_0 sufficiently large,

$$\begin{aligned}
&0\le\lim_{t\to t_0}\sup\left|\,e\,\right|\le\lim_{t\to t_0}\sup N(\delta)\int_0^t\left|\,\exp^{D(\tau-t)}\;d\tau\right|\\
&0\le\lim_{t\to t_0}\sup\left|\,e\,\right|\le\lim_{t\to t_0}\sup\frac{N(\delta)}{D}\left|\,1-\exp^{-t}\,\right|\\
&0\le\lim_{t\to t_0}\sup\left|\,e\,\right|\le\lim_{t\to t_0}\sup\frac{N(\delta)}{D}
\end{aligned} \tag{21}$$

From H2,

$$\lim_{t \to t_0} |\, e \,| = 0 \quad \text{, for } t_0 \text{ sufficiently large}$$

Remark 4. From inequality (21) is clear that the convergence of the estimation error only depends on the rate of convergence (D) and the upper bound given by H1. Moreover, we can see from inequality (20) that the convergence of the estimation error is independent of initial conditions since the first term of the right-hand side contains an exponential expression which converges to zero as $t \to t_0$, for t_0 sufficiently large.

Corollary 1. For any initial conditions, the proposed uniformly bounded error estimator for Monod's model given by system (2) is:

$$\begin{cases} \dot{\hat{\hat{z}}} = -D\,\hat{\hat{z}} + \hat{\xi} \\ \dot{\hat{z}} = -D\,\hat{z} + \hat{\eta} \\ \hat{X} = \hat{\hat{z}} - S \\ \hat{P} = \hat{z} - S \end{cases} \quad , \quad \hat{\hat{z}}_0 = \hat{\hat{z}}(0),\ \hat{z}_0 = z(0),\ S_0 = S(0)$$

with $\hat{\xi} = D\,S_{in} + \dfrac{\mu_{\max} S}{\beta + S}\left(\hat{\hat{z}} - S\right)\left(1 - \dfrac{1}{Y_{S/X}}\right)$, $\hat{\eta} = D\,S_{in} + \dfrac{\mu_{\max} S}{\beta + S}\left(\hat{\hat{z}} - S\right)\left(\dfrac{1}{Y_{P/X}} - \dfrac{1}{Y_{S/X}}\right)$

Remark 5. It should be noted the estimation error belongs to open ball with radius proportional to $\dfrac{N(\delta)}{D}$.

Although in this paper we only consider a three dimensional system, a generalization to higher dynamical system is evidently possible, application to a wide class of unstructured kinetic models as Haldane, Contois, Andrew, Moser, etc. would be considered. In the case of a state variable was not algebraically observable, it should be detectable and display some asymptotic stability conditions.

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3. Simulations results

We show some simulations for Monod's model. For all simulations in this paper, we take $\mu(S) = 0.035\,S/(0.9+S)$, $S_{in} = 10$, $D = 0.01$, $Y_{S/X} = 0.25$, $Y_{P/X} = 0.36$ and for initial conditions $S(0) = 6$, $X(0) = 2$, $P(0) = 2$, $\hat{\hat{z}}(0) = 10$, $\hat{z}(0) = 10$, with the appropriate units. The simulations results were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox.

The performance index of the corresponding estimation process is calculated as (see Levant (2001))

$$J = \frac{1}{t+0.001}\int_0^t \|e(\tau)\|^2\, d\tau \qquad (22)$$

where $e(\tau)$ is the corresponding state estimate error (the difference between the actual observed signal and its estimated).

The simulation results of the proposed estimator for Monod's model, without any noise in the system output, are presented in Figure 1. It is shown the satisfactory performance of this estimator, i.e., the state variables X and P are reconstructed (approximately in, 50 time units for biomass concentration and 250 times units for product concentration).

The performance of the proposed estimator is fast for biomass concentration X because its observability property. However, as expected the performance of the estimator is poorer in the case of product concentration P.

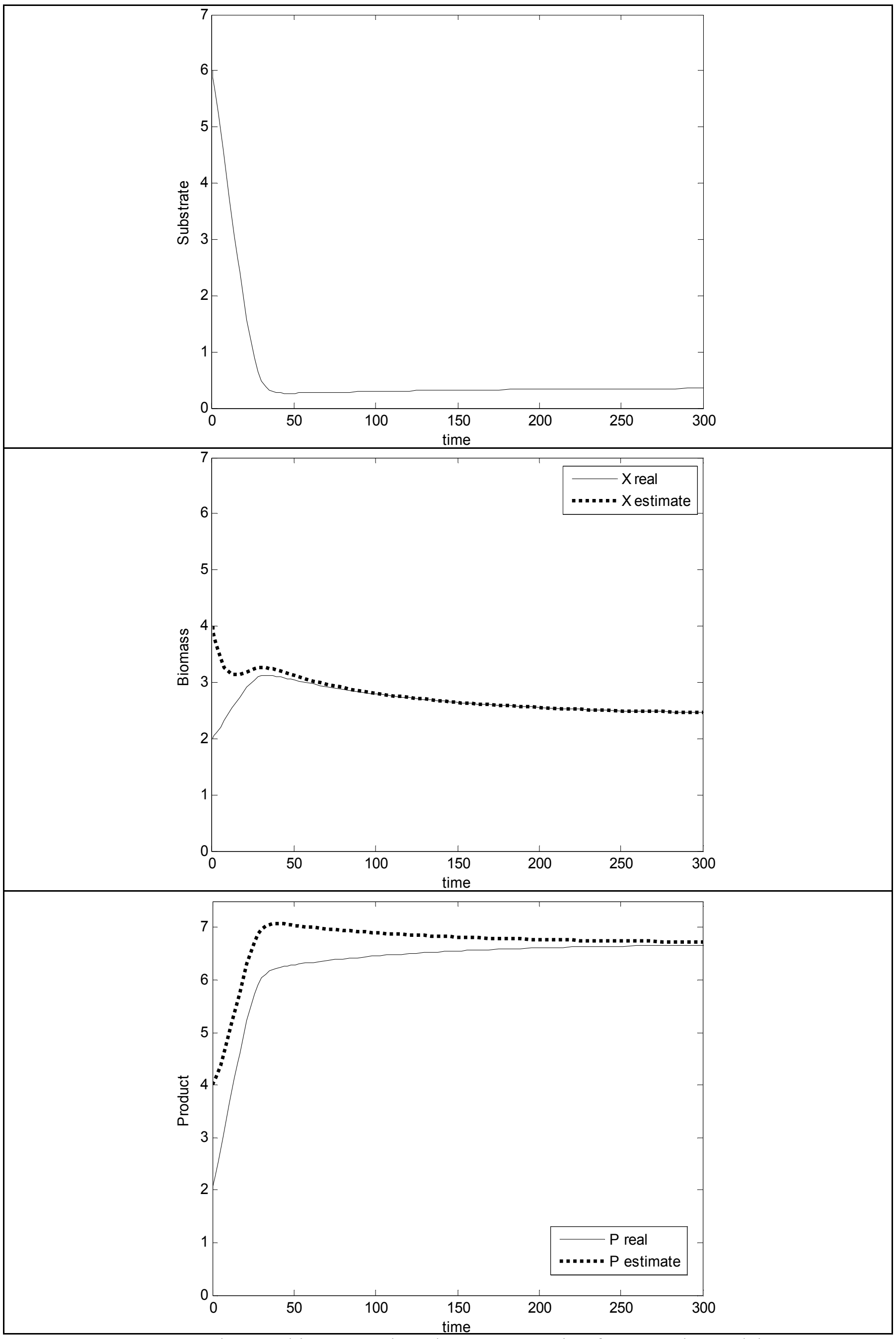

Fig. 1 Substrate, biomass and product concentration for Monod's model.

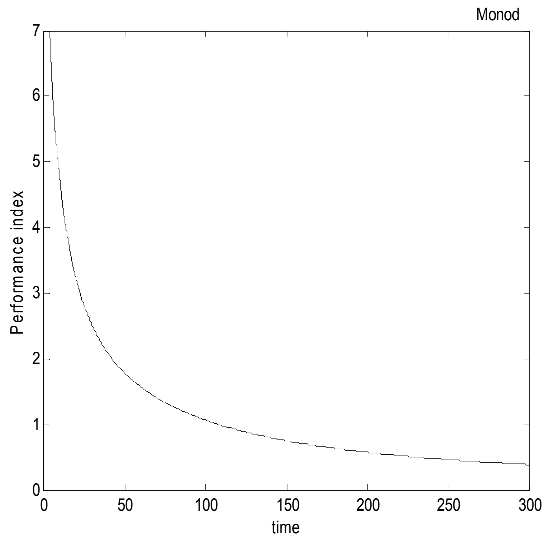

Fig. 2 Average estimate error (without any noise in the system output)

Figure 2 shows that the quadratic estimation error is bounded on average (the performance index) in t and has a tendency to decrease.

Furthermore, in Figure 3 is shown the performance when a white noise is added in the measurement ($\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). Note that the proposed methodology constructs the states X and P in spite of the presence of noise in the measurement.

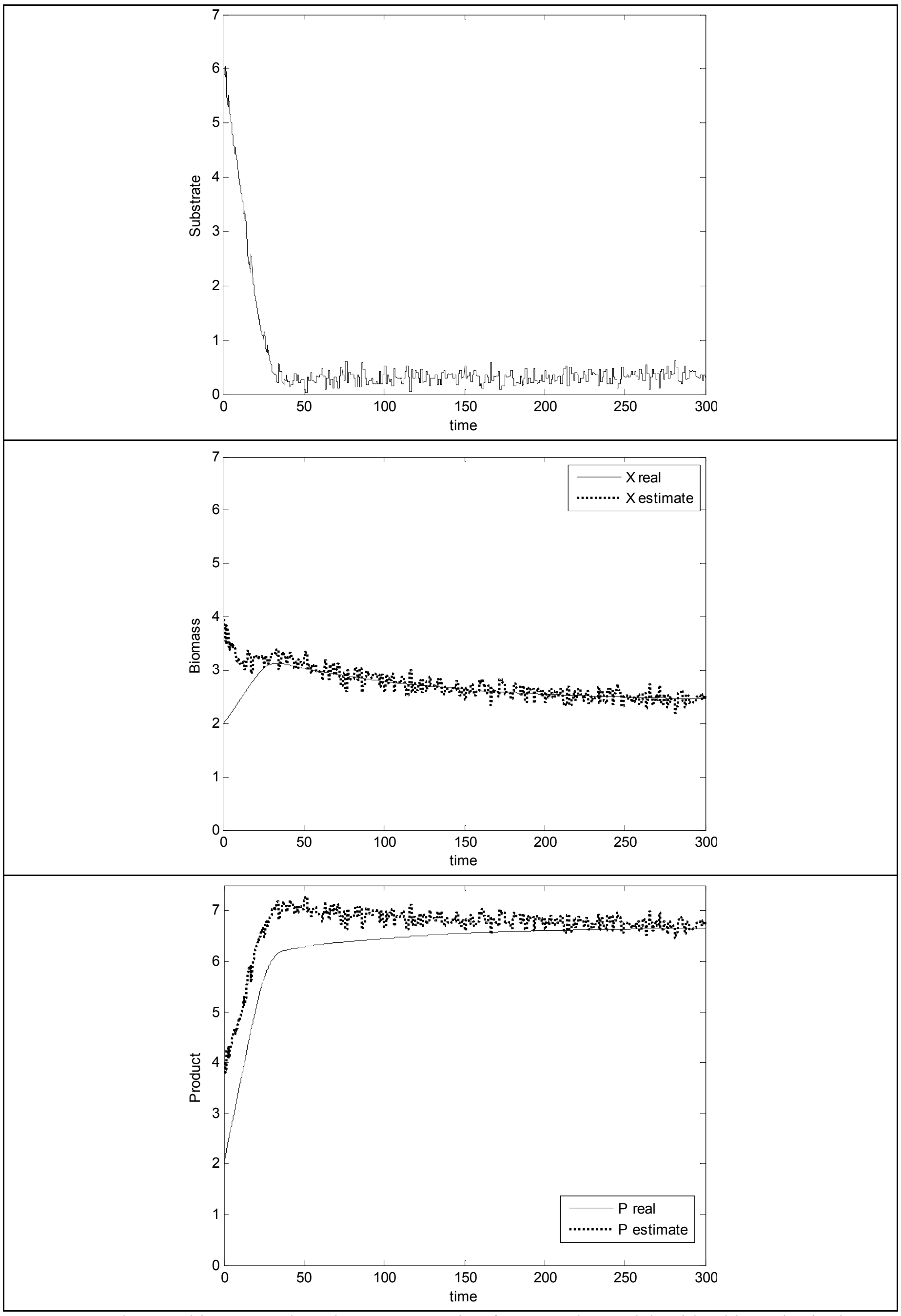

Fig. 3 Substrate, biomass and product concentration for Monod's model, with white noise in the measurement

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We can observe that the uniformly bounded error estimator is robust against noisy measurement, furthermore, does not need adjustable gain which avoids the noise propagation. Finally, Figure 4 shows that the quadratic estimation error (when exits white noise in the system output) is bounded on average (the performance index) in t.

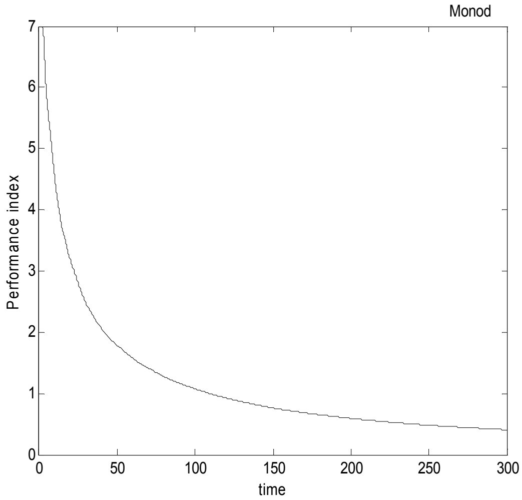

Fig. 4 Average estimate error, with white noise in the system output

4. Concluding Remarks

In this paper we have presented a uniformly bounded error estimator for bioprocess with unstructured growth models. We have proven the uniformly bounded error and by means of a linear change of variable given in a natural manner and with some algebraic manipulations have been constructed the state estimator, which converges to the current states of the reference model given, without dependence of the initial conditions. We have demonstrated that the uniformly bounded error estimator under consideration provides good enough state-space estimates which were bounded on average. Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

5. Appendix: Definitions

We give some basic definitions to establish the differential algebraic setting on which the proposed approach is based. Further details can be found in Diop and Martínez-Guerra (2001).

Definition 1 Let L and K be differential fields. A differential field extension L/K is given by K and L such that: 1) K is subfield of L and; 2) the derivation of K is the restriction to K of the derivation of L.

Example 1 $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are trivial examples of differential field extensions, where $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Definition 2 An element of a differential field K is said to be *algebraic* if it satisfies a differential polynomial equation with coefficients over K. If an element is not algebraic then it is said to be *transcendent*.

Example 2 $R < e^t > / R$ is a differential field extension, where $R \subseteq R < e^t >$. This extension is *algebraic* because e^t satisfies $\dot{x} - x = 0$.

Definition 3 Let G, $K < u >$ be differential fields. A *dynamics* consists in a finitely generated differential *algebraic* extension $G / K < u >$. $(G = K < u, \xi >, \xi \in G)$

Example 3 Let us consider the following differential equation

$$\dot{u}^2 y + 4\ddot{u} = 0$$

In this case, y is algebraic over $K < u >$, therefore, it can be seen as a dynamics of the form $K < u, y > / K < u >$ where $K = \mathbb{R}$.

Definition 4 An element in G is said to be *algebraically observable* with respect to $\{u, y\}$ if it is algebraic over $K < u, y >$.

So that a state x is said to be algebraically observable if it is algebraic over $K < u, y >$, that is to say, it satisfies a differential polynomial in terms of u, y and some of their time derivatives, i. e.

$$P(x, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$$

with coefficients over $K < u, y >$. Above condition is called *the algebraic observability condition (AOC).*

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Understanding Complex Systems

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Applications of Chaos and Nonlinear Dynamics in Engineering – Vol. 1

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Preface

In the past 60 years, the terms nonlinear dynamics and chaos have become familiar in the technical vocabulary of most sciences and technology. Indeed, the mathematical formulation of the vast majority of phenomena evolving in time, which has been given so far, consists of nonlinear ordinary or partial differential equations, or of nonlinear space and time discrete iterative processes. The typical situation, then, presents a kind of propagation of uncertainties which is exponential in time, is known as sensitive dependence on initial conditions, and is concisely and suggestively called chaos. Nonlinearities appear in feedback phenomena and generically in the evolution equations of most systems consisting of interacting parts or interacting with an external environment, which is itself affected by the interaction. Because of the mutual interactions, a given perturbation or action performed on the system of interest is shared by its elementary constituents in different ways, and the resulting effect is hardly predictable, especially in the long term, when the concatenation of causes and effects amounts to a long list. This concatenation may have constructive or destructive outcomes, with respect to the initial perturbations, and, typically, the consequent response of the object under investigation to the external actions will not be merely proportional to the size of the imposed perturbations. This is what nonlinearities mean, in general terms, and it is evident that observable natural phenomena and human artifacts commonly behave in nonlinear fashions.

Long-term predictions of the response of nonlinear dynamics to perturbations are usually problematic, but unpredictability is not an exclusive feature of nonlinear dynamics; even the simplest idealization of motion in space, the uniform motion of a single body subjected to no forces, enhances in time the initial uncertainty on its initial condition. Therefore, it is impossible to even predict whether an ideal arrow, which moves strictly along a straight line, will hit its target, if the direction of the motion is affected by some uncertainty and the target is sufficiently far. In this case, uncertainties grow linearly in time: to double the accuracy of predictions, it suffices to halve the uncertainty on the initial conditions. Beyond a certain limit, however, that may be practically impossible to achieve. Then, given the fact that any measurement one may perform, like any estimate of the initial state of any material object, is bound to be affected by uncertainties, one concludes that some degree of unpredictability is intrinsic, in practice as well as in principle, to our descriptions of all time dependent phenomena.

Nonlinearities commonly result in more serious difficulties as far as predictions, and hence control of the phenomena of interest is concerned. The limiting situation, known as chaos, is obtained when uncertainties are enhanced at an exponential rate. This situation is qualitatively, not simply quantitatively, different from those that enjoy linear or polynomial growths of perturbations. The striking fact is that even simple devices, such as the double pendulum, and not only very complex phenomena, like the climate, follow this kind of dynamics, in which there are no chances of reliable predictions beyond quite short times. This has then pushed researchers to develop the statistical approach to nonlinear dynamics, an approach that performs surprisingly well, in many circumstances, particularly when the degree of chaos, measured, e.g., by Lyapunov exponents, is quite high, or when the number of interacting elementary constituents of the system of interest is very large.

In the past decades, the growing understanding of these concepts has turned in practical applications, of particular interest in the development of present day technology. Time is therefore ripe for a review of the applications of chaos and nonlinear dynamics in engineering. As a matter of fact, numerous books are devoted to this purpose, but they are typically quite theoretical in nature. Therefore, the present collection of articles takes a more practical stand by addressing the various issues in the form of self-contained tutorials, which will guide step by step even the inexperienced reader to an informed use of the existing mathematical tools and softwares.

The first contribution, by S. Lynch, introduces the very popular and powerful Matlab software, by considering examples drawn from mechanical and electrical engineering applications, like Leon Chua's circuit, relevant for the problem of synchronization. M.F. Alves and Z.M. Assis Peixoto then tackle the important phenomenon of voltage flicker in electrical networks, which may be produced by an arc furnace operation. A case study concerning a 30 MVA arc furnace plant is considered in detail. Chapter 3, by S. Lynch and A. Steel, concerns the nonlinear dynamics and the stability properties of optical resonators, which are, for instance, one fundamental component of lasers. Chapter 4 is devoted to the problem of turbulence, and to the control of some of its aspects which are of primary technological importance, like turbulent mixing, which concerns the emissions of carbon dioxide and the spreading of pollutants, among its very many manifestations. This chapter is coauthored by B.S.V. Patnaik and S. Muddada. Vibration-based damage detection methods are widely used to identify hidden damages in beam and structural components. This application of nonlinear dynamics is considered by C.D. Dubey and V. Kapila in Chap. 5. The authors of Chap. 6, B. Kaygisiz, M. Karahan, A.M. Erkmen, and I. Erkmen, address another aspect of modern technology, that of the design of robots. This technology faces strong challenges, including those posed by vibrations, noisy sensing, and robot/irregular-environment interactions. The last four chapters are devoted to the evergrowing field of telecommunications and communications protocols, a number of which are based on chaos techniques. Chapter 7, by Jose M.V. Grzybowski, M. Eisencraft, and E.E.N. Macau, addresses, among other issues, the possibility of chaos-based ultra-fast communication. Chapter 8, by R. Martínez-Guerra, J.L. Mata-Machuca, A. Rodríguez-Bollain, and R. Aguilar-López, concerns secure transmission of information. In Chap. 9, Yu. Andreyev, A. Dmitriev, A.N. Miliou, and A.N. Anagnostopoulos describe an efficient method for storing, retrieving and processing information. Chapter 10, by S. Banerjee and S. Mukhopadhyay, concludes this book with another presentation of chaos-based secure communication methods.

This volume is the first of two, devoted to applications of chaos and nonlinear dynamics in engineering. The ten chapters present are organized in five parts, each concerning one of the most active fields of present-day engineering:

I. Nonlinearity and Computer simulations
II. Chaos and Nonlinear Dynamics in Electrical Engineering
III. Chaos and Nonlinear Dynamics in Building Mechanism and Fluid Dynamics
IV. Chaos in Robotics
V. Chaos and Nonlinear Dynamics in Communication.

We wish that this collection of essays, with their tutorial character, provide valuable help to those who intend to familiarize with both the theoretical and mathematical aspects of engineering applications of the fascinating modern theory of dynamical systems. As proved by the applications discussed in the following ten chapters, and those of the second volume which will follow, many applications of this theory, which are of high technological interest, have already been realized. But given the pace at which new ideas and new techniques are being developed, we anticipate that many more applications will be developed in the coming years, which makes of tantamount importance for scientists as well as engineers to familiarize with issues such as those addresses in this book.

Torino

Santo Banerjee
Mala Mitra
Lamberto Rondoni

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Chapter 8
Chaotic Synchronization and Its Applications in Secure Communications

Rafael Martínez-Guerra, Juan L. Mata-Machuca, Ricardo Aguilar-López, and Andrés Rodríguez-Bollain

8.1 Introduction

Synchronization in chaotic systems has been investigated since its introduction in [58]. This research area has received a great deal of attention among scientist in many fields due to its potential applications mainly in secure communications [2, 6, 11, 21, 30, 52, 74]. However, a certain number of drawbacks have been revealed in the practical implementation of most chaos-based secure communications algorithms [7, 17, 59].

During the last years (almost two decades), many different approaches related to chaos synchronization have been proposed. For instance, we mention the works [3, 10, 31, 47, 53, 56] in which the authors propose the employment of state observers, where the main applications pertain to the synchronization of nonlinear oscillators; in references [19, 71] use feedback controllers, which allow to achieve the synchronization between nonlinear oscillators, with different structure and order; in [29, 70] use nonlinear backstepping control; in papers [25, 26] consider synchronization time delayed systems; in works [22, 41] consider directional and bidirectional linear coupling; papers [14, 15] use nonlinear control; in [1, 54] use adaptive control; in [18, 20] employ adaptive observers; [5] considers an adaptive sliding mode observer and so on.

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As we have mentioned there are many applications to chaotic communications. The techniques can be divided into three categories:

1. *Chaos masking*, the information signal is added directly to the transmitter.
2. *Chaos modulation*, it is based on the master-slave synchronization, where the information signal is injected into the transmitter as a nonlinear filter.
3. *Chaos shift keying*, the information signal is supposed to be binary, and it is mapped into the transmitter and the receiver. In these three cases, the information signal can be recovered by a receiver if the transmitter and the receiver are synchronized.

From the above, in this book chapter, there are considered several cases for chaotic synchronization and some applications, in particular:

- *Observer-based synchronization*. Of particular interest is the connection between the observers for nonlinear systems and the chaos synchronization, which is also known as master-slave configuration. Thus, chaos synchronization problem can be posed as an observer design procedure, where the coupling signal is viewed as output and the slave system is regarded as observer.
- *Control of chaotic Liouvillian systems*. Application to a class of chemical reacting system via sliding-mode observer based feedback.
- *Synchronization of chaotic Liouvillian oscillators*. Application to a Colpitts chaotic oscillator by means of observers: exponential observer of polynomial type, and asymptotic observer of reduced order.
- *Application to secure communications*. The general idea for transmitting information via chaotic systems is that, an information signal is embedded in the transmitter system which produces a chaotic signal, the information signal is recovered when the transmitter and the receiver are identical. In this work, a novel design approach for chaotic communication is proposed, where the receiver is a pure sliding-mode observer.

8.2 Observer-Based Synchronization

Now, we mention a brief note about observers theory. The design of observers for nonlinear systems is a challenging problem, that has received a considerable amount of attention. Since the observers developed by Kalman [35] and Luenberger [44], several years ago for linear systems, different state observation techniques have been propose to handle the systems nonlinearities. A first category of techniques consists in applying linear algorithms to the system linearized around the estimated trajectory. These are known as the extended Kalman and Luenberger observers. Alternatively, the nonlinear dynamics are split into a linear part and a nonlinear one. The observer gains then are chosen large enough so that the linear part dominates over the nonlinear one. Such observers are known as, high gain observers [23, 62]. And many other approaches such as [36, 40, 73].

In this work the synchronization method is based on a *master–slave* configuration [58]. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (transmitter) to the slave system (receiver), the receiver is requested to recover the unknown (or full) state trajectories of the transmitter. By this fact, the terminology *transmitter–receiver* is also used. Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer [16, 55, 57].

Let us consider the following nonlinear system,

$$\begin{aligned} \dot{x} &= f\ (x,u) \\ y &= C\,x, \qquad x_0 = x(t_0) \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$, is the state vector; $u \in \mathbb{R}^p$, is the input vector, $p \leq n$; $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ is locally Lipschitz on x and uniformly bounded on u; $y \in \mathbb{R}$ is the output of the system. To show the relation between the observers for nonlinear systems and chaos synchronization we recall the observer definition.

Definition 8.1 (Observer). An observer system for (8.1) is a system with state $\hat{x}$ such that $\|x - \hat{x}\| \rightarrow 0$ as $t \rightarrow \infty$.

In the context of master-slave synchronization, $\hat{x}$ can be viewed as the state variable of the slave system. Hence, the master-slave synchronization problem can be solved by designing an observer for (8.1).

8.2.1 Some Definitions

As we can note, there exists several methods to solve the synchronization problem since the control theory perspective, in this work, we study the synchronization in master-slave configuration [58] by means of state observers based on the differential algebraic approach [48]. In order to solve the synchronization problem as an observation problem we introduce the following observability property.

Definition 8.2 (Algebraic Observability Condition–AOC). A state variable $x \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}\langle u,y\rangle$,[1] that is, x satisfies a differential algebraic polynomial in terms of $\{u,y\}$ and some of their time derivatives, i.e.,

$$P(x,u,\dot{u},\ldots,y,\dot{y},\ldots) = 0 \tag{8.2}$$

with coefficients in $\mathbb{R}\langle u,y\rangle$.

[1] $\mathbb{R}\langle u,y\rangle$ denotes the differential field generated by the field $\mathbb{R}$, the input u, the measurable output y, and the time derivatives of u and y.

Example 8.1. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 x_2 \\ \dot{x}_2 &= -x_2^2 + x_1 + u \\ y &= x_2\end{aligned} \tag{8.3}$$

System (8.3) is algebraically observable since $x_1 - \dot{y} - y^2 + u = 0$ and $x_2 - y = 0$. That is to say, x_1 and x_2 can be reconstructed knowing $\{u, y\}$ and their time derivatives.

Example 8.2. Let us consider the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u\end{aligned} \tag{8.4}$$

If we define $y = x_2$, then

$$\begin{aligned}x_2 &= y \\ x_3 &= \dot{y} \\ \dot{x}_1 &= y + \dot{y}^2\end{aligned} \tag{8.5}$$

The above system is not algebraically observable since x_1 cannot be expressed as a differential algebraic polynomial in terms of $\{u, y\}$.

Motivated by this fact, we present the next definition.

Definition 8.3 (Liouvillian System). A dynamical system is said to be Liouvillian if the elements (for example, state variables or parameters) can be obtained by an adjunction of integrals or exponentials of integrals of elements of $\mathbb{R}$.

Example 8.3. We consider the nonlinear system as in Example 8.2. From (8.5) we can observe that, although x_1 does not satisfy the AOC we can obtain it by means of the integral

$$x_1 = \int (y + \dot{y}^2)$$

Therefore the nonlinear system (8.4) is Liouvillian.

For further information we recommend to see [8, 49].

8.3 Control of a Class of Chaotic Liouvillian Chemical Systems

The aim of this section is the synthesis of a robust control law for the control of a class of nonlinear systems named Liouvillian (Definition 8.3). The control design is based on sliding-mode uncertainty estimator, developed under the framework of algebraic-differential concepts. The estimation convergence is done via Lyapunov-type analysis and the closed-loop system stability is shown via the regulation error dynamics. Robustness of the proposed control scheme is proven in the face of noise output measurements and model uncertainties. The performance of the proposed control law is illustrated with numerical simulations, when a class of chaotic chemical system is used as application example.

Non-linear approaches to design control laws have been tested successfully in theoretical research. In particular, the Input/Output linearizing technique shows attractive characteristics for the control of the non-linear systems.

To motivate the control problem, consider the following non-linear Liouvillian system, which represents the mathematical model of a continuous stirred tank reactor (CSTR):

$$\begin{aligned} \dot{x}_1 &= \theta(x_{1e} - x_1) - E\ R(x_1, x_2) \\ \dot{x}_2 &= \theta(x_{2e} - x_2) + \Delta H\ R(x_1, x_2) + \gamma(u - x_2) \end{aligned} \tag{8.6}$$

where x_1 is a n-dimensional vector of chemical species, $R(x_1, x_2)$ is a m-dimensional vector of reaction kinetics, ΔH is a m-dimensional vector of reaction enthalpies, E is the stoichiometric matrix, x_2 is the reactor temperature, u is the cooling jacket temperature, $1/\theta$ and γ are the residence time and the heat-transfer global coefficient, respectively. If the reactor temperature x_2 is the controlled output, in compact form, the Liouvillian system (8.6) can be rewritten as follows:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + B(x_2)u \\ y &= h(x) = x_2 \end{aligned} \tag{8.7}$$

The *zero-dynamics* are given by the n-dimensional dynamics of the chemical species concentration at a constant temperature, which are assumed to be locally stable [24]. The study of relative-degree one systems is very important for many control applications, since the dynamics of a wide class of chemical reactors can be described in this form. Such systems are mathematically modeled as *affine* systems with respect to the control input. Systems that present relative-degree one display some interesting features, such as the *equivalent dissipativeness* by means of state or output feedback. In general, it is easier to stabilize dissipative systems than *non-dissipative* ones [65].

In what follows, non-linear systems of the form (8.7) will be considered. In order to stabilize the system defined by (8.7) via regulation of x_2, the following nominal I/O linearizing feedback control is proposed:

$$u = B^{-1}(x_2)\left[-\tau_g^{-1} e_y - f_2(x_1, x_2)\right] \tag{8.8}$$

where $\tau_g > 0$ is a prescribed time-constant. As usual, $e_y = y - y_{sp}$ and y_{sp} are tracking error and set point, respectively. The controller defined by (8.8) guarantees asymptotic stability of non-linear system (8.7) with no uncertainties and perfect measurements [33]. Moreover, it imposes a linear behavior to the system I/O dynamics by canceling the nonlinearities.

8.3.1 Feedback Controller Design

As it can be noticed, the synthesis of the ideal control law requires accurate knowledge of the mathematical model of the process to be realizable. However, a perfect model is difficult or even impossible to be obtained in practice and, consequently, for uncertain systems a conventional I/O linearizing controller design is not adequate.

Let us assume that trajectories x_1 and x_2 are bounded for all $t > 0$ (i.e., the system is bounded input to bounded output state). The basis of the non-ideal controller design is the nominal control law (8.8). In order to design the practical robust control law, let us propose the following non-linear dynamic system representation:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + (\bar{B}(x_2) + \Delta B(x_2))\, u \\ y &= h(x) = x_2 \end{aligned} \tag{8.9}$$

The functions $f_2(x_1, x_2)$ and $\Delta B(x_2)$ are model uncertainties related to the non-linear system, and $\bar{B}(x_2)$ is a nominal value of the control input coefficient. In the most general case, the functions $f_2(x_1, x_2)$ and $\Delta B(x_2)$ are assumed to be unknown. Now, the following function is introduced, which corresponds to the I/O modeling error:

$$\zeta(x, u) = f_2(x_1, x_2) + \Delta B(x_2) u \tag{8.10}$$

By using (8.10) into (8.9), a new representation of the system is obtained:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= \zeta(x, u) + \bar{B}(x_2) u \\ y &= h(x) = x_2 \end{aligned} \tag{8.11}$$

Since the uncertainty term, $\zeta(x, u)$, is an unknown function of the states and the control input, the ideal control law for the regulation of x_2 is not causal and therefore it can not be implemented in practice. Nevertheless, there is another way to develop an input-output linearizing controller that is robust against uncertainties. The procedure described below provides a method to estimate the uncertainty term, $\zeta(x, u)$. Estimators or observers for states and uncertainties can play a key role during the early detection of hazardous and unsafe operating conditions. Following this spirit, several researches have been focused in the proposition of estimation methodologies for states and uncertainties for monitoring and control purposes [39, 63].

In order to estimate the uncertain term $\zeta(x,u)$ let us consider the following dynamic subsystem:

$$\begin{aligned}\dot{x}_2 &= \zeta + \bar{B}(x_2)u\\ \dot{\zeta} &= \Phi(x,u)\\ y &= h(x) = x_2\end{aligned} \tag{8.12}$$

From this subsystem, the following algebraic-differential equation can be obtained:

$$\dot{y} - \zeta(x,u) - \bar{B}(y)u = 0 \tag{8.13}$$

Remark 8.1. From Definition 8.2 it follows that ζ is algebraically observable, however, it is necessary the estimation of $\dot{y}$.

The corresponding Input-Output representation of (8.13) can be rewritten in new coordinates as follows:

$$\eta_i = \frac{\mathrm{d}^{i-1}y}{\mathrm{d}t^{i-1}} \qquad (i = 1,2) \tag{8.14}$$

This implies that

$$\begin{aligned}\dot{\eta}_1 &= \eta_2\\ \dot{\eta}_2 &= \Phi(\eta_1,\eta_2,\dot{u})\\ y &= \eta_1\end{aligned} \tag{8.15}$$

It should be noted that a partial change of coordinate enables us to estimate $\eta_1 = y$ and $\dot{\eta}_1 = \dot{y} = \eta_2$ (or, equivalently, x_2 and $\dot{x}_2$).

Now, considering the noise case presence:

$$y = \eta_1 + \delta$$

where δ is an additive bounded noise. Our aim is to design an observer to obtain η_2 (the uncertainty term in the transformed space). However, as it can be seen from the nature of the system given by (8.12), a standard structure of an observer, based on a copy of the system plus measurement error correction is not realizable in this case since the term Φ is a priori unknown.

The following dynamic system is a *sliding-mode asymptotic type observer* of the system (8.15) to estimate the variables η_1 and η_2, respectively:

$$\begin{aligned}\dot{\hat{\eta}}_1 &= \hat{\eta}_2 + m\tau^{-1}\mathrm{sign}(y-\hat{y}), \qquad m > 0,\\ \dot{\hat{\eta}}_2 &= m^2\tau^{-2}\mathrm{sign}(y-\hat{y})\end{aligned} \tag{8.16}$$

where

$$y = \eta_1 + \delta, \;\; \hat{y} = \hat{\eta}_1, \;\; \text{sign} := \begin{cases} 1 & \text{if } (y - \hat{y}) > 0 \\ -1 & \text{if } (y - \hat{y}) < 0 \\ \text{undefined} & \text{if } (y - \hat{y}) = 0 \end{cases}$$

Now, returning back to the original state space, in view of (8.10), the heat of reaction can be evaluated as:

$$\zeta = -\hat{\eta}_2 - \theta(x_{2e} - \hat{\eta}_1) - \gamma(u - \hat{\eta}_1)$$

According to the variable change given by (8.11), the variable η_1 is the thermodynamic reactor temperature (system output). From the above equation for $\hat{\zeta}$, if temperature measurements are noisy, the noise would be transmitted to the estimation of the heat of reaction that may lead to poor performance in the estimation procedure. That is because it is necessary to filter the temperature measurements. This is the main reason that the structure of the proposed observer (8.16) makes sense.

Convergence analysis. Let us define the following estimation errors:

$$e_1 = \eta_1 - \hat{\eta}_1 \tag{8.17}$$

$$e_2 = \frac{\eta_2 - \hat{\eta}_2}{m} \tag{8.18}$$

By (8.15) and (8.16), it follows that the estimation errors $e = (e_1, e_2)^T$ verify the following ordinary differential equation:

$$\dot{e} = A_{\bar{\mu}} e - K sign(Ce + \delta) + \Delta s \tag{8.19}$$

where $\bar{\mu} > 0$ is a regularizing parameter, $A_{\bar{\mu}} = \begin{bmatrix} -\bar{\mu} & m \\ 0 & -\bar{\mu} \end{bmatrix}$, $K = m\tau^{-1} \begin{bmatrix} 1 \\ m\tau^{-1} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\Delta s = \begin{bmatrix} \bar{\mu} e_1 \\ \frac{1}{m}\Phi + \bar{\mu} e_2 \end{bmatrix}$ is an uncertainty term (or unmodelled dynamics term).

Assumption 8.1. There exist nonnegative constants L_{0s}, L_{1s}, such that the following generalized quasi-Lipschitz condition holds

$$\|\Delta s\| \leq L_{0s} + \left(L_{1s} + \left\|A_{\bar{\mu}}\right\|\right) \|e\| . \tag{8.20}$$

Assumption 8.2. The additive output noise δ, is bounded, namely

$$|\delta| \leq \delta^+ < \infty, \tag{8.21}$$

Assumption 8.3. There exists a positive definite matrix $Q_0 = Q_0^T > 0$, such that the following matrix Riccati equation

$$PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + PRP + Q = 0 \tag{8.22}$$

with

$$R := \Lambda_s^{-1} + 2\|\Lambda_s\| L_{1s} I, \ \Lambda_s = \Lambda_s^T > 0,$$
$$Q = Q_0 + 2(L_{1s} + \|A_{\bar{\mu}}\|^2) I$$

has a positive definite solution $P = P^T > 0$.

Remark 8.2. Assumption 8.1 only limits the maximum slope present in the uncertainty term Δs which depends on the Lipschitz properties of η_2. Assumption 8.2 is a standard assumption that allows us to avoid involving the statistic behavior of the noise signal. The expression (8.22) from Assumption 8.3 has a positive definite solution if the matrix $A_{\bar{\mu}}$ is stable, which is true for any $\bar{\mu} > 0$. Since $P > 0$, there exists $k > 0$ such that $K = kP^{-1}C^T$, then Assumption 8.3 provides an additional degree of freedom to choose the gain k which can be used to establish the size of the region defined by $\tilde{\mu}$.

Theorem 8.1. *If Assumptions 8.1 to 8.3 are satisfied, then*

$$[V - V^*]_+ \to 0 \tag{8.23}$$

where

$$V = V(e) = \|e\|_P^2 := e^T Pe,$$
$$V^* = \frac{2\|\Lambda_s\| L_{0s}^2 + 4k\delta^+}{\lambda_{\min}(P^{-1/2}Q^T QP^{-1/2})},$$

and the function $[\cdot]_+$ *is defined as follows*

$$[x]_+ = \begin{cases} x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}. \tag{8.24}$$

Proof. Let $V(e)$ be the following Lyapunov candidate function

$$V(e) \stackrel{\triangle}{=} e^T Pe = \|e\|_P^2 \tag{8.25}$$

where $0 < P = P^T \in R^{r \times r}$ is the solution of the Riccati equation (8.22). By taking the time derivative of (8.25) and taking into account (8.19) it yields

$$\dot{V}(e) = 2e^T P\dot{e} = 2e^T P\left[A_{\bar{\mu}} e - K\text{sign}\,(Ce + \delta) + \Delta s\right],$$

according to Assumption 8.3, $K = kP^{-1}C^T$, then the previous equation can be written as

$$\dot{V}(e) = 2e^T PA_{\bar{\mu}} e - 2ke^T C^T \text{sign}\,(Ce + \delta) + 2e^T P\Delta s.$$

By using the following matrix inequality

$$X^T Y + Y^T X \leq X^T \Lambda_s X + Y^T \Lambda_s^{-1} Y$$

which is valid for any $X, Y \in R^{r \times m}$, $0 < \Lambda_s = \Lambda_s^T \in R^{r \times r}$ [60], then it follows that

$$\dot{V}(e) \leq e^T (PA_{\bar{\mu}} + A_{\bar{\mu}}^T P)e - 2ke^T C^T \text{sign}\,(Ce + \delta) + e^T P\Lambda_s^{-1} Pe + (\Delta s)^T \Lambda_s \Delta_s,$$

from Assumption 8.1 the following is obtained

$$(\Delta s)^T \Lambda_s \Delta_s \leq \|\Delta s\|^2 \|\Lambda_s\| \leq \left[L_{0s} + (L_{1s} + \|A_{\bar{\mu}}\|) \|e\|\right]^2 \|\Lambda_s\|,$$

then

$$\begin{aligned}\dot{V}(e) \leq\; & e^T (PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + P\Lambda_s^{-1} P + Q)e - e^T Qe \\ & -2ke^T C^T \text{sign}\,(Ce + \delta) + 2\left[L_{0s}^2 + (L_{1s} + \|A_{\bar{\mu}}\|)^2 \|e\|^2\right] \|\Lambda_s\|,\end{aligned}$$

from the definition of matrix R in Assumption 8.3, the previous expression can be rewritten as

$$\dot{V}(e) \leq e^T (PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + PRP + Q)e - e^T Qe - 2ke^T C^T \text{sign}\,(Ce + \delta) + 2L_{0s}^2 \|\Lambda_s\|,$$

and taking into account (8.22), it follows that

$$\dot{V}(e) \leq -e^T Qe - 2ke^T C^T \text{sign}\,(Ce + \delta) + 2L_{0s}^2 \|\Lambda_s\|. \tag{8.26}$$

In order to eliminate the discontinuity contained in the function sign$(\cdot)$ in (8.26) the following inequality, valid for any $x, y \in \mathbb{R}$, is considered

$$\begin{aligned} x\text{sign}(x + y) &= (x + y)\text{sign}(x + y) - y\text{sign}(x + y) \\ &\geq |x + y| - |y| \end{aligned}$$

and furthermore $|x + y| \geq |x| - |y|$, then

$$x\text{sign}(x + y) \geq |x| - 2|y|. \tag{8.27}$$

Now using (8.27) in (8.26) the following inequality is obtained

$$\dot{V}(e) \leq -e^T Qe - 2k\,|Ce| + 2L_{0s}^2\,\|\Lambda_s\| + 4k\delta^+, \tag{8.28}$$

which can be rewritten as

$$\dot{V}(e) \leq -\|e\|_Q^2 + 2L_{0s}^2\,\|\Lambda_s\| + 4k\delta^+,$$

that is to say,

$$\dot{V}(e) \leq -\alpha_Q V(e) + \beta, \tag{8.29}$$

where

$$\alpha \triangleq \lambda_{\min}\left(P^{-1/2}Q^T QP^{-1/2}\right) > 0,$$
$$\beta = 2L_{0s}^2\,\|\Lambda_s\| + 4k\delta^+.$$

Now, considering the following differential equation related to (8.29)

$$\dot{V}(e) = -\alpha V + \beta, \tag{8.30}$$

which is linear and stable and such that $V \to V^*$ as $t \to \infty$, where V^* is the single equilibrium point of equation (8.30)

$$V^* = \frac{\beta}{\alpha} \geq 0,$$

it follows that the function

$$G_t \triangleq [V - V^*]_+^2$$

where $[\cdot]_+$ is defined as in (8.24), according to (8.29) satisfies (for any $V \neq V^*$)

$$\dot{G}_t \leq -2\,[V - V^*]_+\,[-\alpha V + \beta] \leq 0$$

subtracting $-\alpha V^* + \beta = 0$, it yields

$$\dot{G}_t \leq -2\alpha(V - V^*)\,[V - V^*]_+ \leq 0$$

that is to say,

$$\dot{G}_t \leq -2\alpha G_t \leq 0.$$

Integrating the last inequality it follows that

$$G_t - G_0 \leq -2\alpha \int_0^t G_\tau d\tau,$$

in other words

$$2\alpha \int_0^t G_\tau d\tau \leq G_0 - G_t \leq G_0$$

then

$$\lim_{t\to\infty} 2\alpha \int_0^t G_\tau d\tau \leq G_0.$$

From Barbalat Lemma [38], it follows that $G_t \to 0$, which is equivalent to say that $[V - V^*]_+ \to 0$. □

Remark 8.3. Theorem 8.1 states that the weighted estimation error norm $V(e)$ asymptotically converges to the zone bounded by β/α. In other words, it is ultimately bounded.

The final expression for the input-output non-ideal linearizing controller with uncertainty estimation can be obtained, introducing the estimate of the uncertain term in (8.8), to generate:

$$u = B^{-1}(x_2)\left[-\tau_g^{-1} e_y - \hat{\zeta}\right] \tag{8.31}$$

Since the proposed controller uses estimated values of the uncertainty, it cannot cancel the system nonlinearities completely. Thus, the system trajectory remains inside a neighborhood close to the set point. Practical stability is achieved as long as the uncertainty estimation error is bounded. The restraint of the boundedness of the heat of reaction (uncertain term) is common for a wide class of chemical reactions and is consequence of characteristics of the mathematical modeling commonly employed; chemical reactions are usually Lipschitz with respect to temperature. It is not hard to see that global Lipschitz property of $\Delta H\ R(x_1,x_2)$ is found if the functionality $R(x_1,x_2)$ with respect to temperature is of Arrhenius-type [49].

Notice that it is not hard to implement in standard technology (e.g., PLCs) the practical controller given by Eqs. (8.16) to (8.18). In fact, the implementation only requires output measurements and the on-line solution of the dynamical system (8.16). Moreover, the implementation effort is equivalent to other control strategies, such as PI and predictive control. As a matter of fact, standard (industrial) predictive control is more complex than the proposed one, since the former requires implementation of a non-linear optimization method.

In order to analyze the closed-loop stability of the reactor temperature trajectories in the reactor, the closed-loop dynamic equation of the energy balance should be used.

$$e_y = g\, e_y + \left(\zeta - \hat{\zeta}\right) \tag{8.32}$$

If $\zeta \to \hat{\zeta}$ then $\zeta - \hat{\zeta} \to 0$, the ideal control law is recovered together with its stability properties; otherwise, the estimation error is limited as $\|\zeta - \hat{\zeta}\| \leq \alpha\sqrt{\Omega} = \Pi$, accordingly with the above development.

Assumption 8.4. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of the matrix A, where λ_j has multiplicity n_j and $n_1 + n_2 + \ldots + n_k = n$ and ρ is any number larger than the real part of $\lambda_1, \lambda_2, \ldots, \lambda_k$ that is $\rho > max(\mathscr{R}e(\lambda_j))$, then there exists a constant $j > 0$ that satisfies:

$$\|\exp(mAt)e_y\| \leq j\exp(-m\rho t)\|e_y\|$$

Solving (8.17), the error can be expressed as:

$$e_y = \exp(mAt)e_{y0} + \int_0^t \exp\left[mA(t-s)\right](\zeta - \hat{\zeta})\mathrm{d}s \tag{8.33}$$

Considering the Assumptions 8.1 and 8.2, it is possible to find a bound for (8.33),

$$\|e_y\| \leq j\exp(-m\rho t)\left[\|e_{y0}\| - \frac{j\Pi}{m^2\rho}\right] + \frac{j\Pi}{m^2\rho} \tag{8.34}$$

Taking the limit when $t \to \infty$:

$$\|e_y\| \leq \frac{j\Pi}{m^2\rho} \tag{8.35}$$

The above inequality implies that the closed-loop error can be made as small as desired, if the observer parameter m is chosen large enough.

8.3.2 *Application Example*

The chemical reactor proposed as application example has been studied previously in [27, 34], the reactors model shows periodic or even chaotic dynamic behavior depending of the set of parameters employed [27]. The reactor temperature is regulated by means of water flowing through a cooling jacket. A stream with a reactive A enters into the continuous reactor and it is converted to an intermediate product B, which reacts to transforming to the final product C, such that

$$A \to B, \quad rate_1 = k_1 a$$
$$B \to C, \quad rate_2 = k_2 b$$

A description of symbols introduced in this section is given in Table 8.1. Both two reactions are first order process with exothermic chemical reactions and the kinetic constant is modeled by the classical Arrhenius model to include the temperature dependence, as follows:

$$k_i = A_i \exp\left(-\frac{E_{ai}}{RT}\right), \quad \text{for} \quad i = 1, 2.$$

Table 8.1 Notation

Symbol	Description	Symbol	Description
a	Reactant A concentration	α	Dimensionless concentration of reactant A
a_0	Concentration of A in feed	A_c	Effective jacket heat transfer area
b	Reactant B concentration	C_p	Specific heat of reacting mixture
b_0	Concentration of B in feed	E_i	Activation energy for reaction i
k_i	Rate constant for reaction i	U	Overall heat transfer coefficient
t_N	Newtonian cooling time	β	Dimensionless concentration of reactant B
t_{res}	Mean residence time	θ_c	Dimensionless temperature of cooling water
T	Reactor temperature	θ_{sp}	Set point of reactor dimensionless temperature
T_c	Cooling water temperature	θ	Dimensionless temperature
V	Reactor volume	ρ	Density of reacting mixture
ΔH_i	Heat of i-th reaction	τ	Dimensionless time
ϕ	Arrhenius constants ratio	τ_N	Dimensionless Newtonian cooling time
t	Time	τ_{ch}	Dimensionless chemical time

Via the standard mass and energy balances the following reactor's governing equations are presented:

$$\begin{aligned} \frac{\mathrm{d}a}{dt} &= \frac{1}{t_{res}}(a_0 - a) - k_1 a \\ \frac{\mathrm{d}b}{dt} &= \frac{1}{t_{res}}(b_0 - b) - k_1 a - k_2 b \\ C_p\rho\frac{\mathrm{d}T}{dt} &= \frac{1}{t_{res}}C_p\rho(T_0 - T) + (-\Delta H_1)k_1 a + (-\Delta H_2)k_2 b - \frac{1}{V}UA_c(T - T_c) \end{aligned} \tag{8.36}$$

In accordance with [27] the following condition of the reactor's model are imposed; There is not inflow of the compounds B and C, two both reactions have the same reaction heats, the same activation energy and the inflow reactor temperature is the same as the cooling jacket device. Based on the model structure proposed above, the following system representation is done:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \zeta_1(x) \\ \zeta_2(x) \\ \zeta_3(x) \end{bmatrix} + \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{8.37}$$

where:

$$l_1 = \frac{1}{t_{res}}(a_0 - a), \qquad l_2 = \frac{1}{t_{res}}(b_0 - b), \qquad l_3 = \frac{1}{V}UA_c,$$

$$\begin{bmatrix} \zeta_1(x) \\ \zeta_2(x) \\ \zeta_3(x) \end{bmatrix} = \begin{bmatrix} k_1 a \\ k_1 a - k_2 b \\ \frac{1}{t_{res}} C_p \rho (T_0 - T) + (-\Delta H_1) k_1 a + (-\Delta H_2) k_2 b - \frac{1}{V} U A_c T \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ T \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ T_c \end{bmatrix}$$

Now, applying the mean residence time and the reactive A concentration as the time a concentration scales, it is obtained the following set of dimensionless mass and energy balance equations

$$\begin{aligned} \frac{d\alpha}{d\tau} &= 1 - \alpha - \frac{1}{\tau_{ch}} \alpha \exp(\theta) \\ \frac{d\beta}{d\tau} &= \frac{1}{\tau_{ch}} \alpha \exp(\theta) - \frac{1}{\tau_{ch}} \phi \ \beta \exp(\theta) - \beta \\ \frac{d\theta}{d\tau} &= \frac{1}{\tau_{ch}} \theta_c \alpha \exp(\theta) + \frac{1}{\tau_{ch}} \theta_c \phi \ \beta \exp(\theta) - (1 + \tau_N^{-1}) \theta \end{aligned} \tag{8.38}$$

The corresponding dimensionless concentrations and temperature are the follows

$$\alpha = \frac{a}{a_0}; \ \theta = \frac{E_a (T - T_c)}{R T_c^2}; \ \beta = \frac{b}{a_0}; \ \tau = \frac{t}{t_{res}}.$$

The parameters related with the named *chemical* time, dimensionless temperature of the cooling jacket and the Newtonian cooling time, are respectively

$$\tau_{ch} = \frac{1}{k_1 t_{res}}; \ \theta_c = -\frac{\Delta H_1 a_0 E_{a1}}{C_p \rho R T_c^2}; \ \tau_N = \frac{t_N}{t_{res}} = \frac{C_p \rho V}{U A_c t_{res}}, \ \phi = \frac{A_2}{A_1}.$$

An important characteristic of this reactor's model is its minimum phase behavior, i.e. the corresponding internal dynamic when the temperature of the reactor is regulated is stable.

As mentioned above, given that the controller regulates only x_3 the analysis of the inner dynamics is related with closed-loop behavior of x_1 and x_2 while x_3 is kept constant. Therefore the system (8.38) is reduced to:

$$\begin{aligned} \dot{x}_1^* &= 1 - (1 + \delta_1)_1^* \\ \dot{x}_2^* &= \delta_1 x_1^* - (1 + \delta_2)_2^* \end{aligned} \tag{8.39}$$

where:

$$\delta_1 = \frac{\exp(\theta_{sp})}{\tau_{ch}}, \qquad \delta_2 = \frac{\phi \exp(\theta_{sp})}{\tau_{ch}}$$

Now, solving (8.39) we have

$$x_1^* = \left[x_{10}^* - \frac{1}{1+\delta_1}\right] \exp\left(-\{1+\delta_1\}\tau\right) + \frac{1}{1+\delta_1}$$

$$x_2^* = [x_{20}^* - (\delta_3 + \delta_4)] \exp\left(-\{1+\delta_1\}\tau\right) + \delta_3 \exp\left(-\{1+\delta_1\}\tau\right) + \delta_4 \qquad (8.40)$$

with

$$\delta_3 = \left(\frac{1}{\phi - 1}\right)\left(x_{10}^* - \frac{1}{1+\exp(\theta_{sp})/\tau_{ch}}\right),$$

$$\delta_4 = \frac{\exp(\theta_{sp})/\tau_{ch}}{[1+\exp(\theta_{sp})/\tau_{ch}]\,[1+\phi\exp(\theta_{sp})/\tau_{ch}]}$$

From (8.40) the reactor's inner dynamic is asymptotically stable such that:

$$\lim_{\tau\to\infty} x_1^* = \frac{1}{1+\exp(\theta_{sp})/\tau_{ch}} = \alpha_{eq}$$

$$\lim_{\tau\to\infty} x_2^* = \frac{\exp(\theta_{sp})/\tau_{ch}}{[1+\exp(\theta_{sp})/\tau_{ch}]\,[1+\phi\exp(\theta_{sp})/\tau_{ch}]} = \beta_{eq}$$

Numerical simulations for the closed-loop system were performed in order to show the properties of the control scheme proposed. The set of parameters of the chemical reactor are chosen as in [34], and initial conditions of system (8.37) are $x_1 = 0.45$, $x_2 = 0.1$, and $x_3 = 0.9$. For comparison purposes, an ideal I/O linearizing control, standard sliding-mode and a high order sliding-mode controllers are implemented too. The temperature set point is $\theta_{sp} = 3$, and the nominal value of the control input is $u_0 = 17.5$, the controller is tuned-on at $t = 15$, the order of the high order sliding-mode controller is considered as $p = 3$. The temperature measurements are corrupted with a $\pm 5\%$ around the current temperature value.

Figure 8.1 shows the open-loop behavior of the corresponding space portrait; note that oscillatory behavior of the corresponding trajectories makes the open-loop mode inadequate for industrial operation.

Closed loop performance of temperature trajectories show that the ideal I/O linearizing controller shows the better performance, so that it cancels the nonlinearities, imposing a desired linear behavior, with a satisfactory performance. The proposed controller tries to compensate the nonlinear terms via the integral high order sliding-mode contribution, besides it is able to reach the set point value required (Fig. 8.2), exhibiting smaller oscillations around the regulated point ($\theta_{sp} = 3$) than the other sliding-mode controllers. As predicted by the theoretical frame presented; sliding-mode and high order sliding-mode controllers can suppress nonlinear oscillations; however both controllers exhibit a considerable off-set from the corresponding set point. The estimation of the uncertain term is depicted in Fig. 8.3.

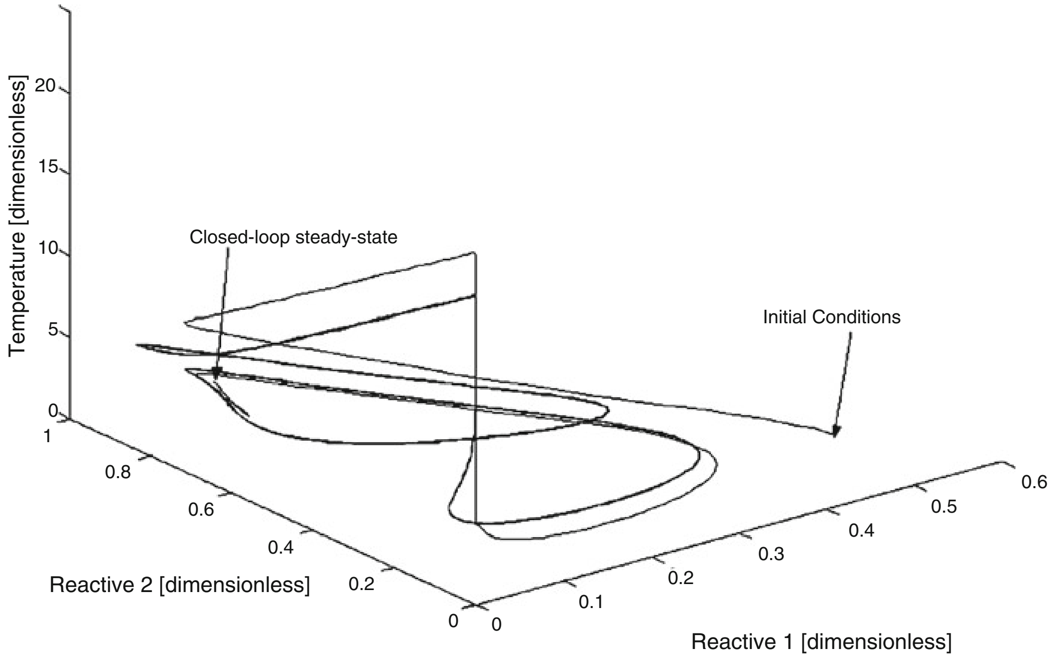

Fig. 8.1 Open-loop space portrait

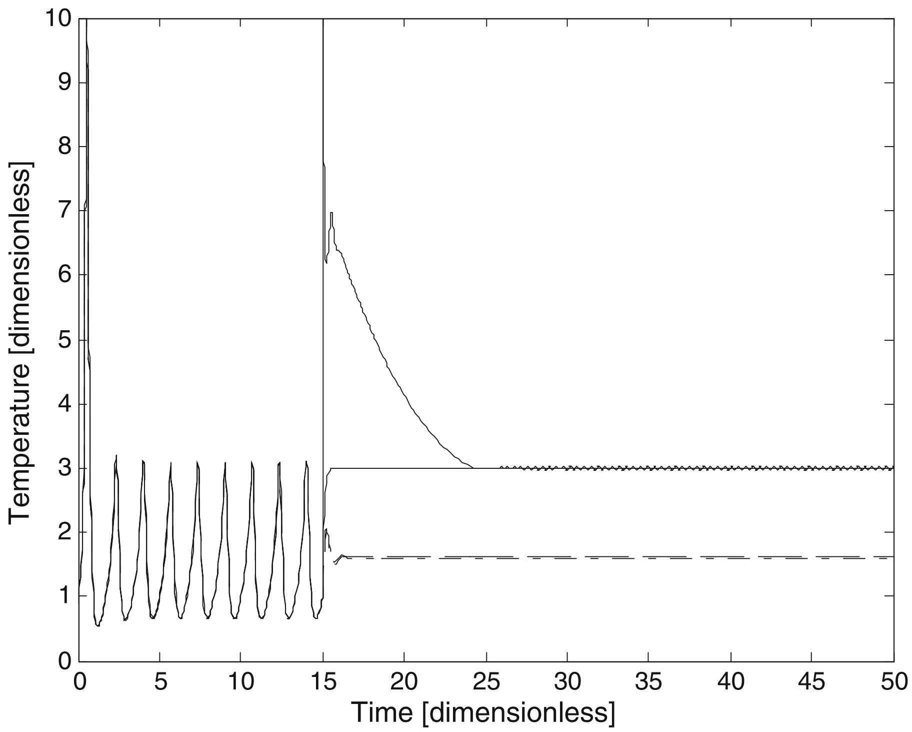

Fig. 8.2 Temperature synchronization. (*Dashed lines*: Proposed controller; *Dotted lines*: Ideal I/O linearizing controller); *Solid lines*: High-order sliding-mode controller; *Dashed dotted lines*: Sliding mode controller

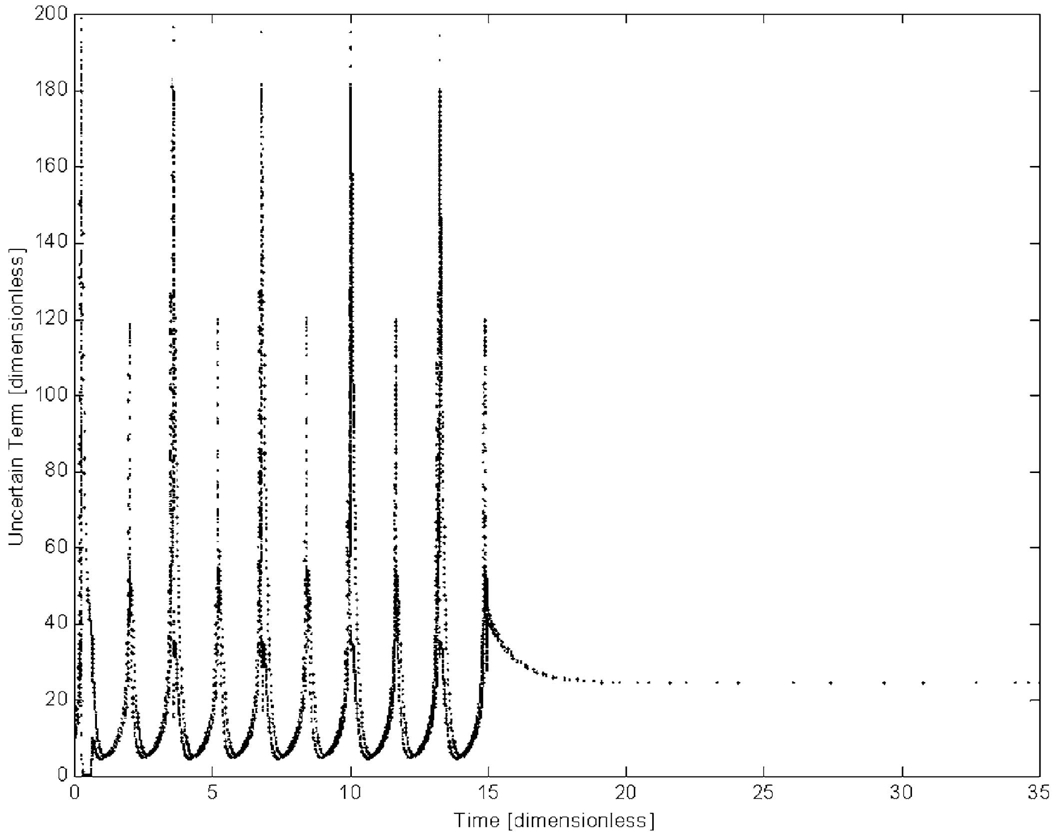

Fig. 8.3 Estimation of the uncertainty

Another important difference is that effort performed by the manipulate variable (Fig. 8.4) is very different for each controller. As it can be noticed, the I/O linearizing controller posses the best performance, sliding-mode controller exhibits the second smoothest behavior, followed by high-order sliding-mode control, which exhibits more demanding effort at the start up of the regulation task. Finally, the proposed methodology presents the more demanding control action, where small oscillations are present.

Comparing performance and control effort it is possible to note that high-order sliding-mode control is not very efficient because the regulated variable, temperature, exhibits the largest off-set, even when the effort is higher than in the case of the sliding-mode controller; nevertheless, performance of the sliding-mode controller is not satisfactory because the set point is not reached. At the expense of higher control effort, the I/O linearizing controller and the proposed controller are able to reach the set point; this is not a disadvantage because this controller is able to force to the temperature trajectory through the desired value. This is a great advantage, especially because this controller is not considering the model of the process.

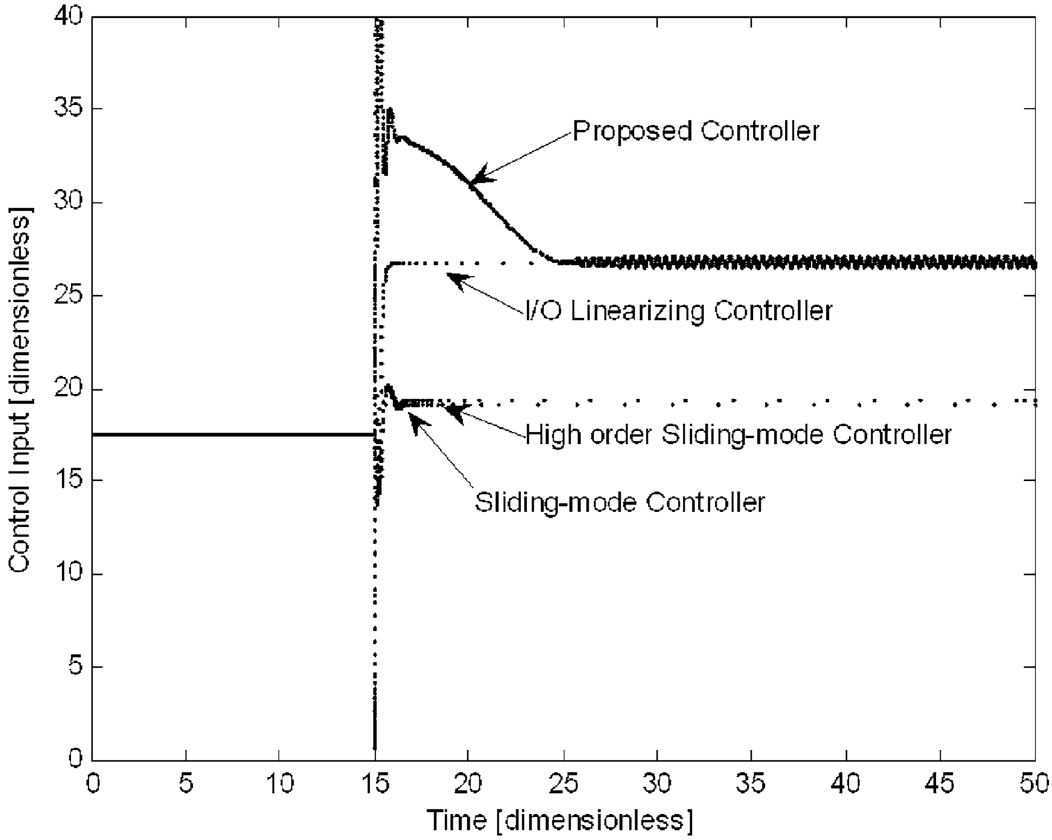

Fig. 8.4 Practical effort of the controllers

It is important to mention that the value of the control gains, has to be chosen very carefully; other numerical simulations (not presented here) showed that, sometimes (smaller values of control gains), it is not possible to stabilize the oscillatory behavior of the chemical reactor; whereas for large values of these parameters, it is possible to lead to unacceptable control efforts or, even worse, to provoke additional closed-loop instabilities.

8.4 Synchronization of Chaotic Liouvillian Oscillators

This section deals with the synchronization and parameter estimation of the Colpitts oscillator considered as a *Chaotic Liouvillian System* (CLS) in a real-time implementation. Even though the algebraic approach has been applied to the synchronization problem for almost one decade [1,47], there are not reported works containing real-time applications based on this theoretical framework. A polynomial observer is used for synchronizing the Colpitts oscillator. A comparison with a reduced order observer is given to asses the performance of the proposed observer.

8.4.1 Exponential Polynomial Observer

In this section we solve the synchronization problem by using a polynomial observer based upon the Lyapunov method [38]. To this end, we first compute the dynamics of the synchronization error (difference between the master and the slave systems). Next, by means of a simple quadratic Lyapunov function, we prove the exponential convergence. The system (8.1) can be expressed in the following form,

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx \qquad x_0 = x(t_0) \end{aligned} \tag{8.41}$$

where $\Psi(x,u)$ is a nonlinear vector that satisfies the Lipschitz condition with constant φ, that is:

$$\|\Psi(x,u) - \Psi(\widehat{x},u)\| \leq \varphi \|x - \widehat{x}\| \tag{8.42}$$

The observer structure. The observer for system (8.41) has the next form

$$\dot{\widehat{x}} = A\widehat{x} + \Psi(\widehat{x},u) + \sum_{i=1}^{m} K_i (y - C\widehat{x})^{2i-1} \tag{8.43}$$

where $\hat{x} \in \mathbb{R}^n$, and $K_i \in \mathbb{R}^n$, for $1 \leq i \leq m$.

Remark 8.4. The meaning of m can be understood as follows. As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance high order terms are included in the observer structure. In other words, the rate of convergence can be increased by injecting additional terms with increasing powers of the output error.

Let us consider the following assumptions:

Assumption 8.5. For $\overline{A} := A - K_1C$, there exist a unique symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ which satisfies the following linear matrix inequality (LMI)

$$\begin{bmatrix} -\overline{A}^T P - P\overline{A} - I & \varphi P \\ \varphi P & I \end{bmatrix} > 0, \qquad \text{where } \varphi \text{ is the Lipschitz constant.}$$

Assumption 8.6. Let us define $M_i := PK_iC$, then

$$\lambda_{\min}\left(M_i + M_i^T\right) \geq 0, \quad \text{for } i \in \{2,\ldots,m\}.$$

Remark 8.5. By using Schur complement (see Chap. 11 in [60]) the LMI in Assumption 8.5 can be represented as an algebraic Riccati equation:

$$\overline{A}^T P + P\overline{A} + \varphi^2 PP + I < 0,$$

or for some $\varepsilon > 0$

$$\overline{A}^T P + P\overline{A} + \varphi^2 PP + I + \varepsilon I = 0.$$

Remark 8.6. Assumption 8.6 is used to improve the rate of convergence of the estimation error by injecting additional terms (from 2 to m) which depend upon odd powers of the output error.

Observer Convergence Analysis. In order to prove the observer convergence, we analyze the observer error which is defined as $e = x - \hat{x}$. From Eqs. (8.41) and (8.43), the dynamics of the observer is given by

$$\dot{e} = \overline{A}e + F - \sum_{i=2}^{m} K_i\,(Ce)^{2i-1}$$

where $\overline{A} := A - K_1 C$, and $F := \Psi(x,u) - \Psi(\widehat{x},u)$.

Now, we present a lemma which will be useful in the convergence analysis.

Lemma 8.1 ([61]). *Given the system (8.41) and its observer (8.43), with the error given by $e := x - \hat{x}$. If $P = P^T > 0$ then:*

$$2\,e^T P\,[\Psi(x,u) - \Psi(\hat{x},u)] \le \varphi^2 e^T PP\,e + e^T e \qquad \square$$

The following proposition proves the observer convergence.

Proposition 8.1. *Let the system (8.41) be algebraically observable and Assumption 8.1 and Assumption 8.2 hold. The nonlinear system (8.43) is an exponential polynomial observer of the system (8.41); that is to say, there exist constants $\kappa > 0$ and $\xi > 0$ such that*

$$\|e(t)\| \le \kappa\ exp(-\xi t)$$

where $\kappa = \dfrac{\|e_0\|_P}{\sqrt{\alpha}}$, $\xi = \dfrac{\varepsilon}{2\beta}$, $\alpha = \lambda_{min}(P)$, *and* $\beta = \lambda_{max}(P)$.

Proof. We use the following Lyapunov function candidate $V = e^T Pe$,

$$\Rightarrow \dot{V} = \dot{e}^T Pe + e^T P\dot{e} = e^T \left[\overline{A}^T P + P\overline{A}\right] e + 2e^T PF - 2e^T P \sum_{i=2}^{m} K_i\,(Ce)^{2i-1}$$

Using Lemma 8.1 we obtain,

$$\dot{V} \le e^T \left[\overline{A}^T P + P\overline{A} + \varphi^2 PP + I\right] e - 2e^T P \sum_{i=2}^{m} K_i\,(Ce)^{2i-1}$$

Making some algebraic manipulations on the last term of the above inequality, and taking into account that $Ce \in \mathbb{R}$, we obtain,

$$\dot{V} \leq e^T \left[\overline{A}^T P + P\overline{A} + \varphi^2 PP + I\right] e - 2\sum_{i=2}^{m} (Ce)^{2i-2} e^T PK_iCe$$

For simplicity, we define $M_i = PK_iC$, $i \in \{2, \ldots, m\}$, then we have

$$\begin{aligned}\dot{V} &\leq e^T \left[\overline{A}^T P + P\overline{A} + \varphi^2 PP + I\right] e - \Big\{(Ce)^2 \left[e^T M_2 e + (e^T M_2 e)^T\right] + \\ &+ (Ce)^4 \left[e^T M_3 e + (e^T M_3 e)^T\right] + \ \ldots + (Ce)^{2m-2} \left[e^T M_m e + (e^T M_m e)^T\right]\Big\}\end{aligned}$$

Above expression can be rewritten in a simplified form

$$\dot{V} \leq e^T \left[\overline{A}^T P + P\overline{A} + \varphi^2 PP + I\right] e - \sum_{i=2}^{m} (Ce)^{2i-2} e^T (M_i + M_i^T) e$$

From Assumption 8.2, the second term in the right hand side of the above inequality always will be positive or zero, therefore

$$\dot{V} \leq e^T \left[\overline{A}^T P + P\overline{A} + \varphi^2 PP + I\right] e \tag{8.44}$$

By Assumption 1 (and Remark 8.5), we have

$$\dot{V} \leq -\varepsilon \|e\|^2 \tag{8.45}$$

We write the Lyapunov function as $V = \|e\|_P^2$, then by Rayleigh-Ritz inequality we have that

$$\alpha \|e\|^2 \leq \|e\|_P^2 \leq \beta \|e\|^2 \tag{8.46}$$

where $\alpha := \lambda_{min}(P)$, and $\beta := \lambda_{max}(P) \in \mathbb{R}^+$ (because P is positive definite).

By using (8.46) we obtain the following upper bound of (8.45)

$$\dot{V} \leq -\frac{\varepsilon}{\beta} \|e\|_P^2 \tag{8.47}$$

Taking the time derivative of $V = \|e\|_P^2$ and replacing in inequality (8.47), we obtain

$$\frac{d}{dt} \|e\|_P \leq -\frac{\varepsilon}{2\beta} \|e\|_P$$

Finally, the result follows with

$$\|e(t)\| \leq \kappa \exp(-\xi \ t) \tag{8.48}$$

where $\kappa = \|e_0\|_P / \sqrt{\alpha}$, and $\xi = \varepsilon / 2\beta$. □

8.4.2 Asymptotic Reduced Order Observer

Now, let us consider the nonlinear system described by (8.1). The unknown states of the system can be included in a new variable $\eta(t)$ and the following new augmented system is considered

$$\begin{aligned} \dot{x}(t) &= f(x,u,\eta) \\ \dot{\eta}(t) &= \Delta(x,u) \\ y(t) &= h(x) \end{aligned} \tag{8.49}$$

where $\Delta(x,u)$ is a bounded uncertain function. The problem is to reconstruct the variable $\eta(t)$. This problem is overcome by using a reduced order observer [47]. Before proposing the corresponding observer for reconstructing the variable $\eta(t)$ we introduce some hypotheses:

Assumption 8.7. $\eta(t)$ satisfies the AOC (Definition 8.2).

Assumption 8.8. γ is a C^1 real-valued function.

Assumption 8.9. Δ is bounded, i.e., $|\Delta| \leq M < \infty$.

Assumption 8.10. For t_0, sufficiently large, there exists $K > 0$, such that, $\lim\sup_{t\to t_0} \frac{M}{K} = 0$.

Next Lemma describes the design of a proportional reduced order observer for system (8.49).

Lemma 8.2 ([47]). *If Assumptions 8.7 to 8.10 are satisfied, then the system*

$$\dot{\hat{\eta}} = K(\eta - \hat{\eta}) \tag{8.50}$$

is an asymptotic reduced order observer of free-model type for system (8.49), where $\hat{\eta}$ denotes the estimate of η and $K \in \mathbb{R}^+$ determines the desired convergence rate of the observer. □

Remark 8.7. To reconstruct $\eta(t)$ by using an auxiliary state $\hat{\eta}(t)$ sometimes we need to use the output time derivatives, but these may be unavailable. To overcome this fact, an auxiliary function completely artificial γ is defined in such away that it cancels out all nonmeasurable terms. This action defines a differential equation for γ. This equation is solved, then, γ is substituted in the differential equation of the estimated state and finally the estimate of η is obtained.

We give the following immediate corollary.

Corollary 8.1 ([47]). *The dynamic system (8.50) along with*

$$\dot{\gamma} = \psi(x,u,\gamma), \quad \textit{with} \quad \gamma_0 = \gamma(0) \quad \textit{and} \quad \gamma \in C^1$$

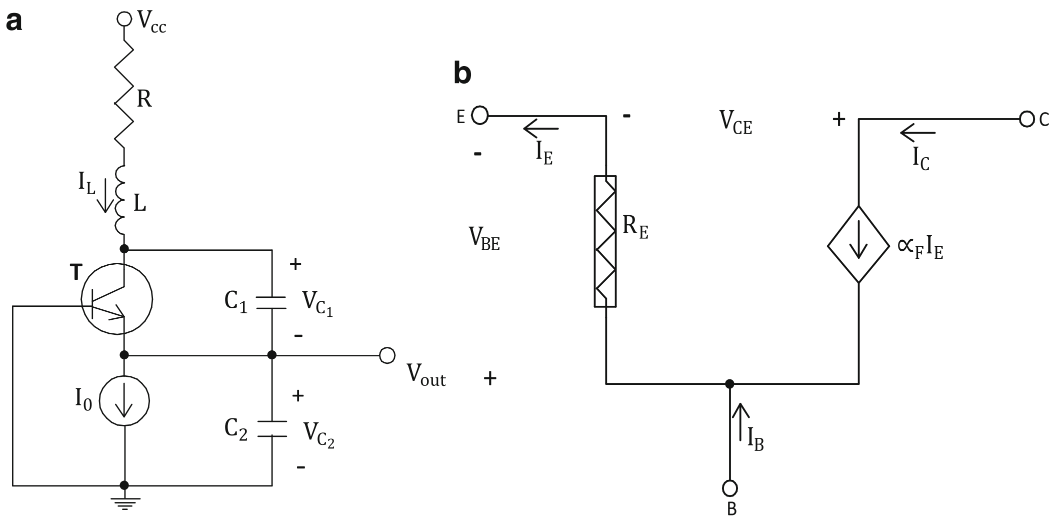

Fig. 8.5 Colpitts oscillator. (**a**) Circuit configuration. (**b**) Model of the Bipolar Junction Transistor (BJT)

constitute a proportional asymptotic reduced order observer for system (8.49), where γ is a change of variable which depends on the estimated state $\hat{\eta}$, and the state variables. □

8.4.3 Experimental Results

These proposals are applied to a Colpitts oscillator [37]. The Colpitts oscillator has been widely considered for the synchronization problem, see for instance [18, 28]. We can also mention one previous work with the Colpitts system using the differential algebraic approach [51], in that work the authors only report a numerical simulation study and not a real-time experiment. The intention of choosing the Colpitts system example is to clarify the proposed methodology and to highlight the simplicity and flexibility of the present approach.

In this work we consider the classical configuration of the Colpitts oscillator [45]. The circuit contains a bipolar junction transistor 2N2222A as the gain element (Fig. 8.5b), and a resonant network consisting of an inductor and two capacitors (Fig. 8.5a).

The Colpitts circuit is described by a system of three nonlinear differential equations, as follows:

$$\begin{aligned} C_1\dot{V}_{C_1} &= -f(V_{C_2}) + I_L \\ C_2\dot{V}_{C_2} &= I_L - I_0, \\ L\dot{I}_L &= -V_{C_1} - V_{C_2} - RI_L + V_{CC} \end{aligned} \tag{8.51}$$

where $f(\cdot)$ is the driving–point characteristic of the nonlinear resistor. This can be expressed in the form $I_E = f(V_{C_2}) = f(-V_{BE})$. In particular, we have $f(V_{C_2}) = I_S\ \exp(-V_{C_2}/V_T)$.

We introduce the dimensionless state variables (x_1, x_2, x_3), and we choose the operating point of (8.51) to be the origin of the new coordinate system. In particular, we normalize voltages, currents and time with respect to $V_{ref} = V_T$, $I_{ref} = I_0$ and $t_{ref} = 1/w_0$, respectively, where $w_0 = 1/\sqrt{LC_1C_2/(C_1+C_2)}$, is the resonant frequency of the unloaded L–C tank circuit. Then, the state equations for the Colpitts oscillator can be rewritten in the next form:

$$
\begin{aligned}
\dot{x}_1 &= -a\ \exp(-x_2) + ax_3 + a \\
\dot{x}_2 &= bx_3 \\
\dot{x}_3 &= -cx_1 - cx_2 - dx_3
\end{aligned}
\tag{8.52}
$$

where, $a = b\frac{C_2}{C_1}$, $b = \frac{I_0}{w_0C_2V_T}$, $C = \frac{V_T}{w_0LI_0}$, $d = \frac{R}{Lw_0}$.

According to Definition 8.2, it is evident that system (8.52) is algebraically observable with respect to the output $y = x_2$, since the unknown states x_1 and x_3, can be rewritten as

$$x_3 = \frac{\dot{x}_2}{b} = \frac{\dot{y}}{b} \tag{8.53}$$

$$x_1 = -\frac{1}{c}\left[\frac{1}{b}\ddot{y} + \frac{d}{b}\dot{y} + cy\right] \tag{8.54}$$

hence, Colpitts oscillator is algebraically observable with respect to the selected output $y = x_2$.

The system parameters (a, b, c, d) are not exactly known, but as it can be easily verified, they can be reconstructible or identifiable in the sense of Definition 8.2, that is to say, they satisfy differential equations in $\mathbb{R}\langle u, y\rangle$. Indeed by considering available the complete state vector[2] $(y_1 = x_1, y_2 = x_2, y_3 = x_3)$, we can obtain the following relationships from (8.52):

$$
\begin{aligned}
a &= \frac{\dot{y}_1}{-\exp(-y_2) + y_3 + 1}; \qquad & b &= \frac{\dot{y}_2}{y_3} \\
c &= \frac{-\dot{y}_3^2 + y_3\ddot{y}_3}{\dot{y}_3(y_1+y_2) - (\dot{y}_1+\dot{y}_2)}; \qquad & d &= -\frac{(\dot{y}_1+\dot{y}_2)\dot{y}_3 - (y_1+y_2)\ddot{y}_3}{(\dot{y}_1+\dot{y}_2)y_3 - (y_1+y_2)\dot{y}_3}
\end{aligned}
\tag{8.55}
$$

with $(-\exp(-y_2) + y_3 + 1) \neq 0$, $y_3 \neq 0$, $[\dot{y}_3(y_1+y_2) - (\dot{y}_1+\dot{y}_2)] \neq 0$ and $[(\dot{y}_1+\dot{y}_2)y_3 - (y_1+y_2)\dot{y}_3] \neq 0$.

[2]This is a realistic assumption taking into account that in the physical circuit these variables are related to currents and voltages, which can be measurable.

The Colpitts oscillator (8.52) is Liouvillian (Definition 8.3), since the parameter a is expressed in terms of an exponential of the output $y_2 = x_2$, or equivalently, $\exp(-\int \dot{y}_2) = \exp(-y_2)$.

8.4.3.1 Synchronization of the Colpitts Oscillator Employing the Exponential Polynomial Observer

For the implementation of the observer we first rewrite (8.52) in the form (8.41),

$$\dot{x} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -c & -c & -d \end{bmatrix} x + \begin{bmatrix} -a\exp(-\widehat{x}_2) + a \\ 0 \\ 0 \end{bmatrix}$$
$$y = \begin{bmatrix} 0 \; 1 \; 0 \end{bmatrix} x \tag{8.56}$$

Applying Proposition 8.1, we have

$$\dot{\widehat{x}} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -c & -c & -d \end{bmatrix} \widehat{x} + \begin{bmatrix} -a\exp(-\widehat{x}_2) + a \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} e \right)^{2i-1}$$

Hence, the state observer is rewritten as,

$$\begin{aligned} \dot{\widehat{x}}_1 &= a\widehat{x}_3 - a\exp(-\widehat{x}_2) + a + k_{1,1}e_{1,2} + k_{1,2}(e_{1,2})^3 + \ldots + k_{1,m}(e_{1,2})^{2m-1} \\ \dot{\widehat{x}}_2 &= b\widehat{x}_3 + k_{2,1}e_{1,2} + k_{2,2}(e_{1,2})^3 + \ldots + k_{2,m}(e_{1,2})^{2m-1} \\ \dot{\widehat{x}}_3 &= -c\widehat{x}_1 - c\widehat{x}_2 - d\widehat{x}_3 + k_{3,1}e_{1,2} + k_{3,2}(e_{1,2})^3 + \ldots + k_{3,m}(e_{1,2})^{2m-1} \end{aligned} \tag{8.57}$$

It is clear that in order to have a useful implementation of (8.57) we need a fairly good knowledge of the model parameters, (a,b,c,d). In theory we know this parameters from the modeling and the relations of the circuit components, but for some experimental results using data from the Colpitts oscillator is not enough. System parameters could be estimated by using (8.55), however, we can not guarantee that the denominator in each relation is different to zero. In this context, relations given in (8.55) are not defined for all time.

In order to estimate (a,b,c,d) in any time, we define

$$\hat{a} = \frac{\dot{y}_1}{\chi_1(0.5 - \psi_1) + \psi_1}; \quad \hat{c} = \frac{-\dot{y}_3^2 + y_3\ddot{y}_3}{\chi_3(0.5 - \psi_3) + \psi_3}$$
$$\hat{b} = \frac{\dot{y}_2}{\chi_2(0.5 - \psi_2) + \psi_2}; \quad \hat{d} = -\frac{(\dot{y}_1 + \dot{y}_2)\dot{y}_3 - (y_1 + y_2)\ddot{y}_3}{\chi_4(0.5 - \psi_4) + \psi_4} \tag{8.58}$$

where: $\psi_1 = -\exp(-y_2) + y_3 + 1$, $\psi_2 = y_3$, $\psi_3 = \dot{y}_3(y_1 + y_2) - (\dot{y}_1 + \dot{y}_2)$, $\psi_4 = (\dot{y}_1 + \dot{y}_2)y_3 - (y_1 + y_2)\dot{y}_3$, and χ_i is the characteristic function defined by

$$\chi_i = \begin{cases} 1 & \text{if} \ \ |\psi_i| \leq 0.5 \\ 0 & \text{if} \ \ |\psi_i| > 0.5, \quad \text{for} \ \ i \in \{1,\ldots,4\}. \end{cases}$$

The system (8.58) requires the knowledge of the time derivatives of the measurement y_j, $1 \leq j \leq 3$. Let us consider the following time derivatives to be estimated

$$\begin{aligned} \eta_j &= \dot{y}_j \\ \overline{\eta}_j &= \ddot{y}_j, \quad \text{for} \ \ 1 \leq j \leq 3 \end{aligned}$$

Using the reduced order observer proposed in [51],

$$\dot{\hat{\eta}}_j = K_j(\eta_j - \hat{\eta}_j) \tag{8.59}$$

introducing the change of variable γ_j

$$\hat{\eta}_j = \gamma_j + K_j y_j \tag{8.60}$$

and from (8.59) and (8.60) we can get $\dot{\gamma}_j = -K_j\hat{\eta}_j$, then again from (8.60)

$$\dot{\gamma}_j = -K_j\gamma_j - K_j^2 y_j \tag{8.61}$$

then, (8.61) together with (8.60) constitute an asymptotic observer for $\eta_j = \dot{y}_j$.

In the same manner, the corresponding asymptotic observer for $\overline{\eta}_j = \ddot{y}_j$ is given by

$$\begin{aligned} \dot{\overline{\gamma}}_j &= -\overline{K}_j\overline{\gamma}_j - \overline{K}_j^2\hat{\eta}_j \\ \hat{\overline{\eta}}_j &= \overline{\gamma}_j + \overline{K}_j\hat{\eta}_j \end{aligned}$$

We verified the real time performance of the exponential observer by using the WINCON platform. To achieve the synchronization in real time, in WINCON was implemented the scheme (8.57) in the master–slave configuration. Figure 8.6 shows the real implementation of the Colpitts circuit. The circuit parameters are: L = 100 μH; $C_1 = C_2 = 47$ nF, R = 45 Ω, $I_0 = 5$ mA. Using the circuit parameters we obtain $a = b = 6.2723$, $c = 0.0797$, and $d = 0.6898$.

The nonlinear term $\Psi(x)$ in (8.56), satisfies the Lipschitz condition and is considered as follows

$$\Psi(x) = \begin{bmatrix} -a\exp(-x_2) + a \\ 0 \\ 0 \end{bmatrix}$$

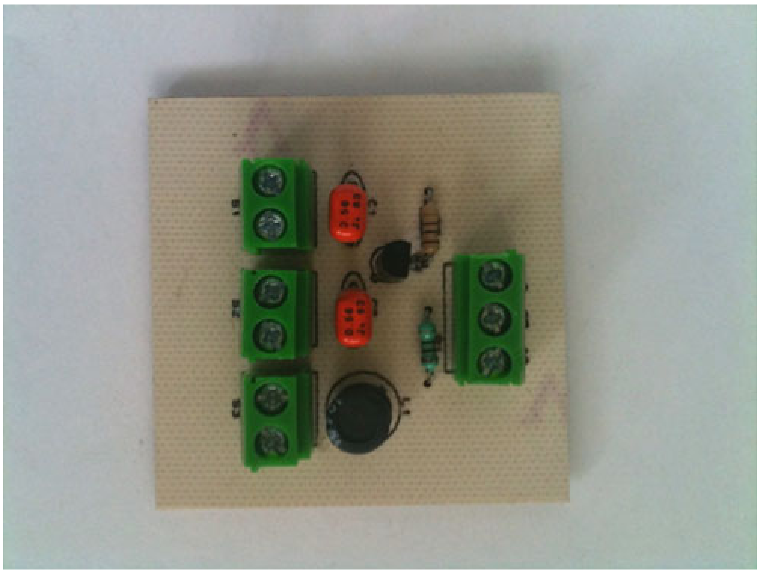

Fig. 8.6 Implementation of the Colpitts circuit (master system)

It is necessary to calculate the Lipschitz constant φ introduced in (8.42) over the bounded set

$$\Omega = \{x \in \mathbb{R}^3 | \;\; |x_1| < M_1, \;\; |x_2| < M_2, \;\; |x_3| < M_3\} \tag{8.62}$$

Considering the Jacobian of $\Psi(x)$ as

$$\left[\frac{\partial \Psi(x)}{\partial x}\right] = \begin{bmatrix} 0 & a\,\exp(-x_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.63}$$

it can be concluded that[3]

$$\left\| \frac{\partial \Psi(x)}{\partial x} \right\|_\infty \leq 3\,\max\{0,\; a\,\exp(-x_2)\}, \quad a \in \mathbb{R}^+. \tag{8.64}$$

From (8.62), it is obvious that the following conditions hold for all the points in the bounded set Ω

$$a\,\exp(-x_2) < a\,\exp(M_2) = \max\{a\,\exp(-x_2)\}, \quad a \in \mathbb{R}^+. \tag{8.65}$$

[3]Let us consider the matrix $\mathscr{A} = [a_{ij}]_{1 \leq i,j \leq n}$, then (see Chap. 5 in [60])

$$\|\mathscr{A}\|_\infty := n \max_{1 \leq i,j \leq n} |a_{ij}|$$

Hence

$$\left\| \frac{\partial \Psi(x)}{\partial x} \right\|_\infty \leq 3\ a \exp(M_2) \tag{8.66}$$

With (8.62)–(8.66) a Lipschitz constant that satisfies the Lipschitz condition (8.42) is defined as follows

$$\varphi = 3\ a \exp(M_2)$$

In this case $M_1 = 3$, $M_2 = 0.1$, $M_3 = 6$, and $a = 6.2723$, $\Rightarrow \varphi = 20.7959$.

Following the observer 8.57, for $m = 2$, and solving the LMI given by Assumption 8.5, the observer gains K_1 and K_2, and a positive definite matrix P are as follows

$$K_1 = \begin{bmatrix} 10.2130 \\ 16.1211 \\ 10.1500 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad P = \begin{bmatrix} 38.8560 & -36.7794 & 19.4606 \\ -36.7794 & 37.9331 & -20.7898 \\ 19.4606 & -20.7898 & 16.4869 \end{bmatrix} > 0,$$

with eigenvalues $\lambda_1(P) = 1.3151$, $\lambda_2(P) = 5.2514$, and $\lambda_3(P) = 86.7095$.

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as [50]

$$J(t) = \frac{1}{t + 0.001} \int_0^t |e(t)|^2_{Q_0}, \quad Q_0 = I$$

Figures 8.7a–c show the obtained results by using the exponential polynomial observer (8.57), it is clear that the synchronization is achieved fairly acceptable even with the noisy measurements. The Colpitts circuit starts in $x(0) = [0\ \ 0\ \ 0]^T$ and the arbitrary initial conditions for the observer are $\hat{x}(0) = \left[1.506\ -0.1\ 2.1\right]^T$. Figure 8.7d shows the performance index of the synchronization, which depicts an exponential behavior.

8.4.3.2 Synchronization of the Colpitts Oscillator by Means of the Asymptotic Reduced Order Observer

Let us consider the normalized system of the Colpitts oscillator. We assume that the output system is $y = x_2$. Therefore, the slave system consists in two estimation structures to achieve synchronization with the master system. Such structures are obtained as follows. Firstly, verify that the master system (Colpitts oscillator) is algebraically observable, and then, by using (8.50), construct the observer for the unknown states. Previously, we have verified that master system (Colpitts oscillator) is algebraically observable – see Eqs. (8.53) and (8.54). Then, both unknown states of the master system are algebraically observable, and, therefore, we can construct the observers based on Lemma 8.2 and Corollary 8.1.

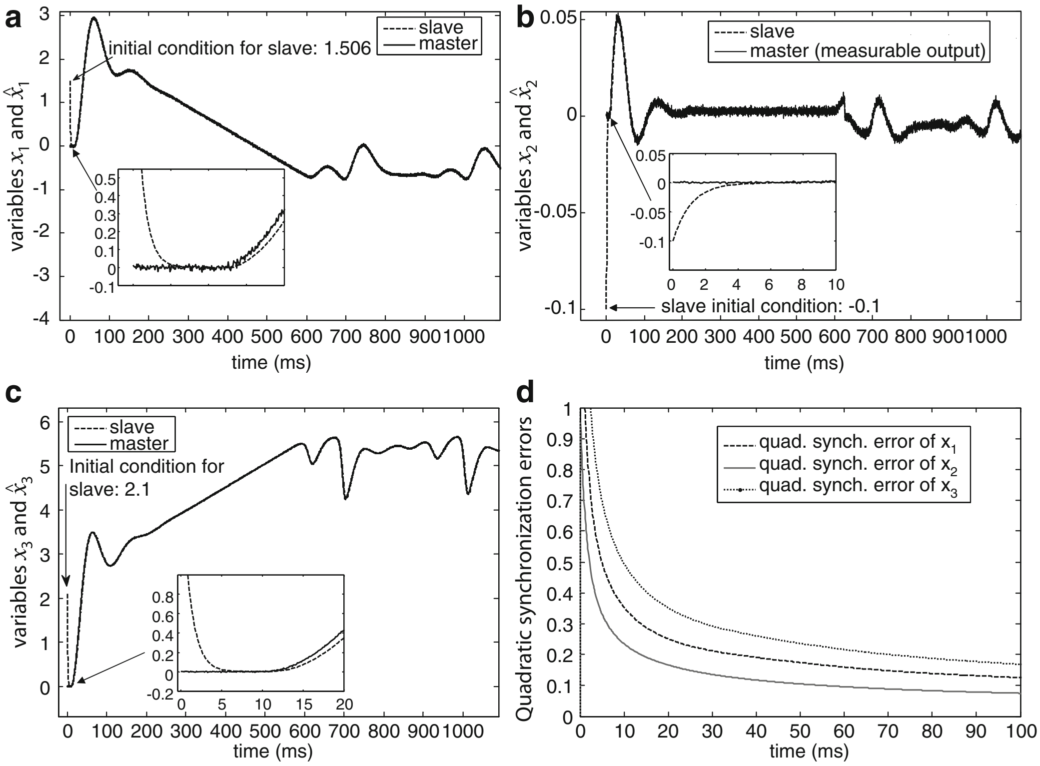

Fig. 8.7 Real-time synchronization of Colpitts oscillator employing observer (8.57): (**a**) synchronization of x_1, (**b**) synchronization of x_2, (**c**) synchronization of x_3, and (**d**) performance index

For x_3 the observer is given by,

$$\begin{aligned} \dot{\gamma}_3 &= -\frac{K_3^2}{b}y - K_3\gamma_3 \\ \hat{x}_3 &= \frac{K_3}{b}y + \gamma_3 \end{aligned} \tag{8.67}$$

and for x_1, we have

$$\begin{aligned} \dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y] \\ \dot{\gamma}_5 &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5 \\ \hat{x}_1 &= -\frac{K_1}{cb}[\gamma_4 + K_4 y] + \gamma_5 \end{aligned} \tag{8.68}$$

Therefore, (8.67) and (8.68) constitute the slave system. Now, we present some experimental results for the synchronization of the Colpitts oscillator by using the asymptotic reduced order observer (8.67)–(8.68). Figure 8.8a, b show the obtained

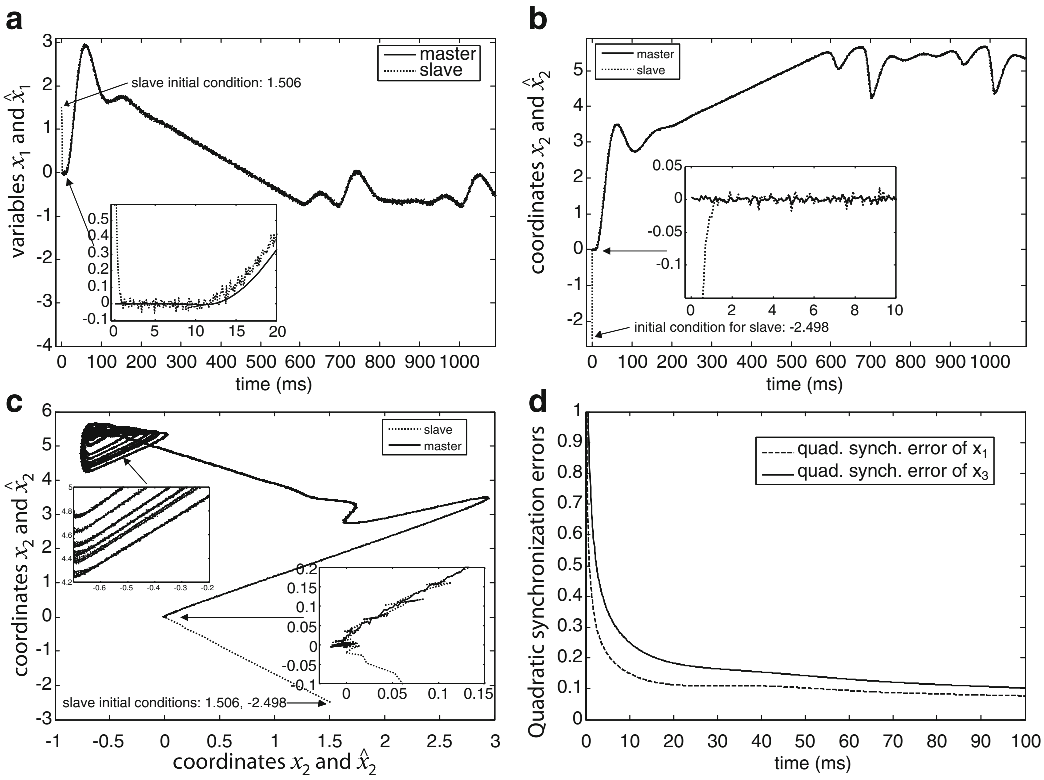

Fig. 8.8 Real-time synchronization of Colpitts oscillator using reduced-order observer (8.68)–(8.67): (**a**) synchronization of x_1, (**b**) synchronization of x_3, (**c**) Phase portrait of the master system (x_1 versus x_3) and the slave system ($\hat{x}_1$ versus $\hat{x}_3$), and (**d**) performance index

results for the initial conditions $\hat{x}_1 = 1.506$ and $\hat{x}_3 = -2.498$ in the schemes (8.67) and (8.68), respectively. As we can note, the synchronization results achieved with the reduced order observer are good. Figure 8.8c presents the phase portrait, where clearly is observed the chaotic behavior of the Colpitts oscillator. Finally, Fig. 8.8d illustrates the performance index, which has a tendency to decrease.

8.5 Application to Secure Communications

Synchronization can be classified into mutual synchronization (or bidirectional coupling) [69] and master-slave synchronization (or unidirectional coupling) [58]. The chaos-based secure communications have updated their fourth generation [68]. The continuous synchronization is adopted in the first three generations while the impulsive synchronization is used in the fourth generation. Less than 94 Hz of bandwidth is needed to transmit the synchronization signal for a third-order chaotic transmitter in the fourth generation while 30 kHz bandwidth is needed to transmit the synchronization signals in the other three generations [72].

Information signal embedded in a chaotic transmitter can be recovered by a receiver if it is a replica of the transmitter. In this work, a new aspect of chaotic communication is introduced. A sliding-mode observer replaces the conventional chaotic system at the receiver side, which does not need information from the transmitter. So the uncertainties in the transmitter and the transmission line do not affect the synchronization, the proposed communication scheme is robust with respect to some disturbances and uncertainties. Duffing equation is provided to illustrate the effectiveness of the chaotic communication.

Linear and nonlinear observers in control theory literatures can be applied to design receivers. The receiver is regarded as a chaotic observer, which has two parts a duplicated chaotic system of the transmitter and an adjustable observer gain [43]. Some modifications were made when it is difficult to obtain a replica of the synchronization. For example, the transmitter and the receiver are set into the same chaotic structures, parameter identification methods can be used to construct the chaotic receiver [32]; when there are uncertainties in synchronization (the transmitter is not known exactly, there is noise in the transmission line, etc.), the transmitter and the receiver could be established in the same fuzzy models, fuzzy model-based design method was applied to reach synchronization [42]; stability analysis of observer-based chaotic communication with respect to uncertainties can be found in [6, 47].

Robust control techniques and many traditional schemes have been applied in robust synthesis for chaotic synchronization, e.g. robust observer and H_∞ technique are used in [66, 67]. Since sliding-mode observer contains a sliding-mode term, it provides the robustness against an inaccurate modeling of measurements and output noises. The early works dealing with sliding mode observers which consider measurement noise were proposed in [13], where is discussed the state estimation using sliding mode technique. In [12] is discussed the variable structure control as a high-speed switched feedback control resulting in a sliding mode. In [4] is treated an analysis of systems with sliding mode in the presence of noises. In [64], successfully designed, so named, sliding-mode approach to construct observers which are highly robust with respect to noises in the input of the system. But, it turns out that the corresponding stability analysis cannot be directly applied in the situations with the output noise (or, mixed uncertainty) presence. So, it is still a challenge to suggest a workable technique to analyze the stability of identification error generated by sliding-mode (discontinuous nonlinearity) type observers [46].

In this work, a novel design approach for chaotic communication is proposed, where the receiver is a pure sliding-mode observer. The main difference with the above methods is that the receiver is no longer a chaotic system. The uncertainty of the transmitter will not affect the synchronization. The proposed communication scheme can be more robust than both transmitter and receiver employed in chaotic systems. But the information may be recovered by the observer who does not have knowledge about the transmitter, this is a big challenge to secure communication by means of chaos.

8.5.1 Chaotic Communication Based on Sliding-Mode Observer

In normal chaotic communication, the transmitter and the receiver are chaotic systems. They can be described in the form of the following nonlinear system

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)u \\ y &= h(\xi) \end{aligned} \tag{8.69}$$

where $\xi \in \mathbb{R}^n$ is the state of the plant, $u \in \mathbb{R}$ is a control input, $y \in \mathbb{R}$ is a measurable output, f, g and h are smooth nonlinear functions. Most chaotic systems have uniform relative degree n, i.e.

$$L_g h(\xi) = \cdots = L_g L_f^{n-2} h(\xi) = 0, \;\; L_g L_f^{n-1} h(\xi) \neq 0$$

So there exists a mapping $\eta = T(\xi)$ which can transform the system (8.69) into the following normal form [33]

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1}, \quad i = 1,2,n-1 \\ \dot{\eta}_n &= \Phi(\eta, u) \\ y &= \eta_1 \end{aligned} \tag{8.70}$$

where $\Phi(\cdot)$ is a continuous nonlinear function.

In this paper, *chaos modulation* [6, 43] is used for communication, where the information signal s is embedded into the output of the chaotic transmitter. The transmitter is a slight modification of the normal chaotic systems (8.70) as follows:

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1}, \quad i = 1,2,n-1 \\ \dot{\eta}_n &= \Phi(\eta, u) \\ y &= \eta_1 + s \end{aligned} \tag{8.71}$$

where the output $y = \eta_1 + s$ is chaotic masking.

Now, we discuss a new observer-based receiver, we propose the following sliding-mode observer for the receiver

$$\begin{aligned} \dot{\hat{\eta}}_i &= \hat{\eta}_{i+1} + m_i sign(y - \hat{y}), \quad i = 1,2,n-1 \\ \dot{\hat{\eta}}_n &= m_n sign(y - \hat{y}) \end{aligned} \tag{8.72}$$

where $\hat{\eta}_j$ are the states on the receiver side and $\hat{y} = \hat{\eta}_1$ the estimate of the output y, $m_j = m^j \; \tau^{-j}$, $\forall\, 1 \le j \le n$, m and τ are small positive parameters, $m > 0$, $0 < \tau < 1$ and

$$sign(y - \hat{y}) = \begin{cases} 1 & \text{if } (y - \hat{y}) > 0 \\ -1 & \text{if } (y - \hat{y}) < 0 \\ \text{undefined} & \text{if } (y - \hat{y}) = 0 \end{cases}.$$

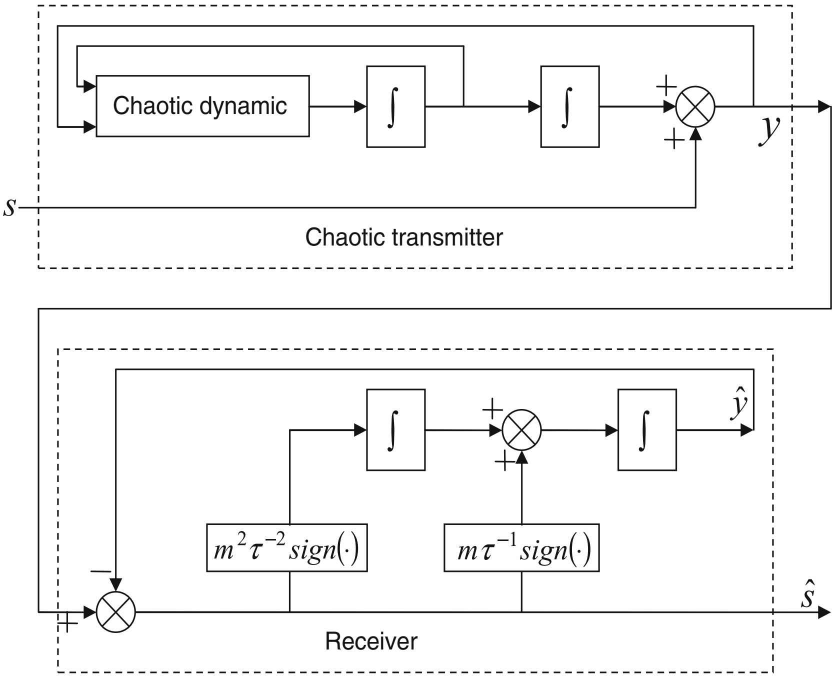

Fig. 8.9 Sliding-mode chaotic communication

The schematic diagram of the chaotic communication based on sliding-mode observer for $n = 2$ is shown in Fig. 8.9.

Let us define the synchronization errors as

$$e_1 = \eta_1 - \hat{\eta}_1,\ e_i = (\eta_i - \hat{\eta}_i)/m,\ i = 2, \ldots, n, \tag{8.73}$$

The recovered signal at receiver is

$$\hat{s} = y - \hat{y} = e_1 + s$$

By (8.71) and (8.72) the synchronization error $e = \begin{bmatrix} e_1 & \ldots & e_n \end{bmatrix}^T$ can be formed as

$$\dot{e} = A_{\bar{\mu}}e - Ksign(Ce + s) + \Delta f \tag{8.74}$$

where $\bar{\mu} > 0$ is a regularizing parameter, $C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$, $A_{\bar{\mu}} = \begin{bmatrix} -\bar{\mu} & m & 0 & \ldots & 0 \\ 0 & -\bar{\mu} & m & & 0 \\ 0 & 0 & -\bar{\mu} & & \vdots \\ & & & \ddots & m \\ 0 & 0 & 0 & \ldots & -\bar{\mu} \end{bmatrix}$,

$K = \begin{bmatrix} m_1 \\ m_2 \\ \ldots \\ m_n \end{bmatrix}$, and $\Delta f = \begin{bmatrix} \bar{\mu}e_1 \\ \ldots \\ \bar{\mu}e_{n-1} \\ \Phi + \bar{\mu}e_n \end{bmatrix}$ is an uncertainty term.

The following assumptions are used for our theoretical result:

Assumption 8.11. There exist nonnegative constants L_{0f}, L_{1f}, such that the following generalized quasi-Lipschitz condition holds

$$\|\Delta f\| \leq L_{0f} + (L_{1f} + \|A_{\bar{\mu}}\|)\|e\|. \tag{8.75}$$

Assumption 8.12. Information signal is assumed to be bounded as $\|s\|_{\Lambda}^2 = s^T \Lambda s \leq (\bar{s})^2 < \infty$ where Λ is a symmetric definite positive matrix.

Assumption 8.13. There exists a positive definite matrix $Q_0 = Q_0^T > 0$, such that the following matrix Riccati equation

$$PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + PRP + Q = 0 \tag{8.76}$$

with

$$R := \Lambda_f^{-1} + 2\|\Lambda_f\| L_{1f} I,\ \Lambda_f = \Lambda_f^T > 0,$$
$$Q := Q_0 + 2(L_{1f} + \|A_{\bar{\mu}}\|^2)I$$

has a positive definite solution $P = P^T > 0$. Since $P > 0$, there exists $k > 0$ such that $K = kP^{-1}C^T$.

To calculate the solution to the Riccati equation (8.76), the following parameters have been selected

$$\Lambda_f = \lambda_f I,\ L_{1f} = \bar{\mu},\ R = (\lambda_f^{-1} + 2\lambda_f \mu)I,\ Q_0 = q_0 I,\ Q = (q_0 + 8\bar{\mu}^2)I. \tag{8.77}$$

Theorem 8.2. *The sliding-mode observer-based receiver (8.72) can recover the information signal s which is embedded in the chaotic transmitter (8.71), the signal recovery error $\tilde{s} = s - \hat{s}$ converges to the following residual set*

$$D_{\varepsilon} = \{\tilde{s} \mid \|\tilde{s}\|_P \leq \tilde{\mu}(k)\}$$

where P is a solution of the Riccati equation (8.76)

$$\tilde{\mu}(k) = \left(\frac{\rho(k)}{\sqrt{(k\alpha_P)^2 + \rho(k)\alpha_Q} + k\alpha_P} \right)^2$$

with

$$\rho(k) = 2\|\Lambda_f\| L_{0f}^2 + 4k\left(\sqrt(n\Lambda_f^{-1})\right)\bar{s}$$

$$k\alpha_P = k\left(\lambda_{min}(P^{-1/2}C^T CP^{-1/2})\right)$$

$$\alpha_Q = \lambda_{min}(P^{-1/2}C^T CP^{-1/2}) > 0$$

and n is the dimension of the chaotic system.

Proof. Taking the Lyapunov function candidate V (e) as $V(e) \overset{\triangle}{=} e^T Pe = \|e\|_P^2$, $0 < P = P^T \in R^{r\times r}$, and using the matrix inequality

$$X^T Y + Y^T X \leq X^T \Lambda_f X + Y^T \Lambda_f^{-1} Y$$

which is valid for any $X, Y \in R^{n\times m}$, $0 < \Lambda_f = \Lambda_f^T \in R^{r\times r}$ [60], then it follows that

$$\dot{V}(e) \leq e^T\left(PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + PRP + Q\right)e - e^T Qe + 2\|\Lambda_f\| L_{0f}^2 - 2k(Ce)^T \operatorname{sign}(Ce + s)$$

by using

$$x^T \operatorname{sign}[x+z] \geq \sum_{i=1}^{n} |x_i| - 2\sqrt{n}\|z_i\|$$

Then

$$\dot{V}(e) \leq -e^T Qe + 2\|\Lambda_f\| L_{0f}^2 - 2k\left(\sum_{i=1}^{n} |(Ce)_i| - 2\sqrt{n}\|s\|\right)$$

$$\leq -e^T Qe - 2k\sum_{i=1}^{n} |(Ce)_i| + \rho(k)$$

where

$$\rho(k) = 2\|\Lambda_f\| L_{0f}^2 + 4k\left(\sqrt{n\Lambda_f^{-1}}\right)\bar{s}$$

Thus

$$\dot{V}(e) \leq -\|e\|_Q - 2k\alpha_P\|e\|_P + \rho(k)$$

where

$$\left(\sum_{i=1}^{n} |(Ce)_i|\right)^2 \geq \sum_{i=1}^{n} (|(Ce)_i|)^2 = \|Ce\|^2 = \|CP^{-1/2}P^{-1/2}e\|^2 \geq \alpha_P e^T Qe$$

with

$$\alpha_P = \lambda_{min}(P^{-1/2}C^TCP^{-1/2}) \geq 0$$

So that, from (8.77) we obtain

$$\dot{V}(e) = -\alpha_Q V(e) - \vartheta\sqrt{(V(e))} + \beta$$

where

$$\alpha_Q = \lambda_{min}(P^{-1/2}QQ^TP^{-1/2}) > 0, \quad \vartheta = 2k\alpha_P, \quad \beta = \rho(k)$$

By the theorem proposed in [46] and Assumptions 8.11–8.13, we can formulate the following

$$\left[1 - \frac{\tilde{\mu}}{V}\right]_+ \to 0$$

where the function $[\cdot]_+$ is defined as in Theorem 8.1. The other part of the proof is similar as [46]. □

8.5.2 Numerical Simulation

We use the Duffing chaotic system as transmitter, the information signal s is embedded in the output of the transmitter. The Duffing equation is [9]

$$\begin{aligned}
\dot{\eta}_1 &= \eta_2 + \frac{1}{\tau}s \\
\dot{\eta}_2 &= -1.1y - y^3 - 0.4\eta_2 + 2.1\cos(1.8t) + \frac{1}{\tau^2}s \\
y &= \eta_1 + s, \qquad \eta(0) = [0,0]^T
\end{aligned} \tag{8.78}$$

Now we design the sliding-mode receiver as (8.72). We choose $m = 0.1$, $\tau = 0.01$. The sliding-mode observer-based receiver is

$$\begin{aligned}
\dot{\hat{\eta}}_1 &= \hat{\eta}_2 + 10\ \mathrm{sign}(y - \hat{y}) \\
\dot{\hat{\eta}}_2 &= 10^2\ \mathrm{sign}(y - \hat{y}) \\
\hat{y} &= \hat{\eta}_1, \qquad \hat{\eta}(0) = [1,1]^T
\end{aligned} \tag{8.79}$$

The information signal s is chosen as sinusoidal signal with frequency of 100 Hz as in [6, 43], i.e.

$$s = 0.05\ \sin(200\pi t)$$

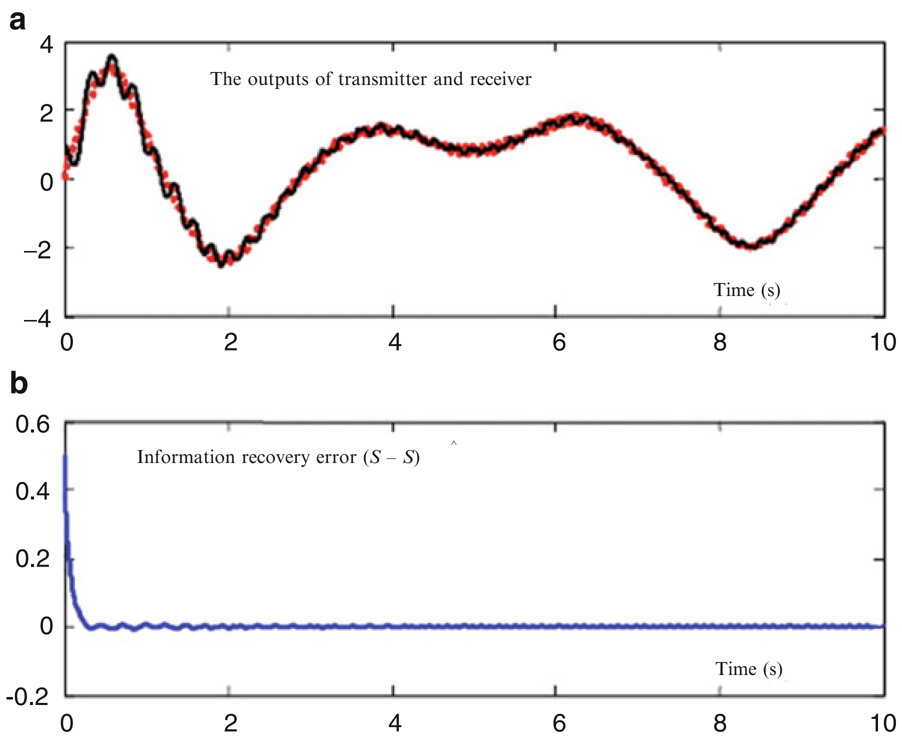

Fig. 8.10 Duffing equation for chaotic communication

Figure 8.10 shows the communication process with Duffing chaotic transmitter and the receiver, here the waveform of the transmitted signal y is shown in subplot (a), the convergence behavior of $s - \hat{s}$ is shown in subplot (b).

After transient process ($t > 0.1$ s), the maximum relative error is defined as

$$e_{\max} = \frac{\max(|s - \hat{s}|)}{\max(|s|)}$$

In this case, $e_{\max} \approx 1.5\%$ which is acceptable for signal communication.

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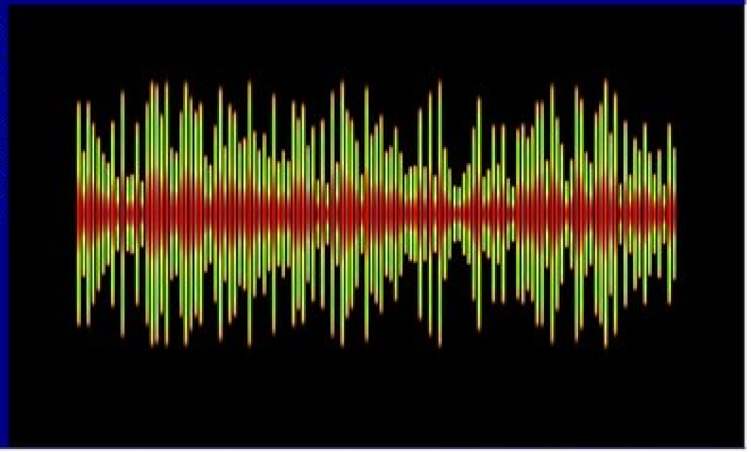
Nonlinear Estimation Topics



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Observability and Observers for Nonlinear Dynamical Sysyems

Nonlinear Systems Analysis

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Preface

Control theory provides a useful tool to design mathematical algorithms to infer unmeasured variables from the corresponding measured ones, these algorithms are called state observers, a state observer is a system that models a real system in order to provide an estimate of its internal state, given measurements of the input and output of the real system. It is typically a computer-implemented mathematical model.

Knowing the state system is necessary to solve many control theory problems; for example, stabilizing a system using state feedback. In most practical cases, the physical state of the system cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs. If a system is observable, it is possible to reconstruct the system state from its output measurements using the state observer.

As we know, it is almost impossible to measure all the elements of the state vector in practice (e.g., the unknown state variables, fault signals, etc.). Here arises a basic practical question: would it be possible to reconstruct these unknown signals? We give an answer to this question by introducing a basic definition related with the estimation (reconstruction) of the states, the algebraic observability condition (AOC).

In this book the observability property for a class of systems (oscillators, chaotic systems, (bio)chemical systems) is determined by means of a relatively new approach which is related with the differential-algebraic framework, this mathematical approach has been recently shown to be a very effective tool for understanding basic questions such as input-output inversions and observer realizations.

This text is designed to be extremely flexible. In Chapter 1 the observability properties for a class of nonlinear systems are presented, by considering geometric and algebraic approaches. The observability conditions for state variables, unstructured uncertainties and detectable states are considered for a class of nonlinear systems related with several (bio)-chemical reacting processes. The considered examples are related with (bio)- chemical continuous reactors and a metabolic model, where their observability properties are an-

alyzed. A comparison of the corresponding results is done, showing the suitability of each approach. Chapter 2 covers the design of some state observers. Section 2.1 contains an observer structure of polynomial type which has exponential convergence. Section 2.2 shows a high gain observer which will be used for comparison purposes. Section 2.3 deals with a bounded observer including an application to synchronization problem. Chapter 3 deals with the monitoring of a class of Continuous Stirred Tank Reactors (CSTR) via observers theory. The importance of on-line monitoring of biotechnological processes has increased during the last years. Advantages include gaining knowledge about the state of the process and the possibility of detecting and isolating abnormal process developments at early stages. This reduces process costs, contributes to process safety and helps in trouble-shooting and process accommodation. Chapter 4 attacks some control problems such as *stabilization of CSTR* and *synchronization of an uncertain chaotic system* by means of adaptive techniques. In section 4.1 a class of nonlinear controller with hyperbolic tangent output injection which provides semi-global stabilization under the considered assumptions is proposed. No perfect system knowledge is needed to design the controller, actions based on hyperbolic tangent functions work satisfactorily, which is a valuable feature for process application; the proposed controller is able to provide adequate performance for regulation and tracking purposes in a bioreactor, despite the presence of non-minimum phase behavior in the system. Although the drastic change to the set point, control effort remains moderate, this also favors the industrial implementation of this controller. Section 4.2 presents a master/slave scheme in conjunction with an adaptive control technique for synchronization and identification of the uncertain chaotic Rikitake system, where only a partial knowledge of the state is available. That is, we assume that the state initial condition and the unknown parameters belong to some region where the system exhibits chaotic behavior. Roughly speaking, the suggested approach consists of designing a controlled slave system, whose controllers and adaptive parameters are adjusted accordingly to a proposed adaptive algorithm. It is done in such a way that the synchronization errors between the outputs of both systems, the uncertain Rikitake and the slave, asymptotically converge to zero. It is important to emphasize that the robustness of our control strategy allows to effectively detect piecewise constant variations on the parameter values of the uncertain Rikitake system. Finally, the issue of Chapter 5 is the synchronization of a class of chaotic continuous stirred reactors to regulate the temperature of each one of the considered chemical reactors all of them at the same set point. This problem can be seen as synchronization one, where the master reactor is controlled at the former set point and the others reactors (slaves) are forced to follows the temperature trajectory of the controlled (master) reactor. It can be done via a robust active control applied to each one of the chaotic stirred reactors. The proposed control strategy considers a linear stabilizing term plus a high order sliding-mode contribution to compensate the unknown nonlinearities of the system which is robust against model uncertainties, avoiding the use of complete model information. Theoretical analysis is done to prove semi-global closed-loop stability. Numerical simulations related with chemical reactors with complex behavior are employed to illustrate the performance of the proposed methodology.

Intended audience

The aim of this book is to present a modern viewpoint of differential and algebraic techniques applied to the control theory in a framework that requires no prerequisites other than a good differential equations and algebra course.

One objective is to give the student, college senior, graduate level or practicing engineers, with little background in control theory the tools to design practical control systems and the confidence to tackle the more advanced literature in areas of particular interest and provide a ready reference for those who have attained working knowledge of the subject.

Certain concepts from algebra and differential equations are used throughout, but the necessary background material is given in Chapter 1.

The book is intended to introduce control theory concepts for application to real problems, with sufficient theoretical background to justify their use.

Rafael Martínez-Guerra
Ricardo Aguilar-López
Juan L. Mata-Machuca

Mexico City, Mexico, 2011

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Generalized Synchronization between Colpitts and Chua Circuits

Rafael Martínez-Guerra, Juan L. Mata-Machuca*, Aurora Rodríguez-Martínez

Abstract— **In this contribution, we investigate the Generalized synchronization (GS) problem when we have strictly different nonlinear systems and we consider that for both the slave and master systems only some states are available from measurements. GS in nonlinear systems appears when the states of one system, through a functional mapping are equal to states of another. This mapping can be obtained if there exist a differential primitive element which generates a differential transcendence basis. The first component of the differential transcendence basis is called differential primitive element and, in general, is defined by means of a linear combination of the known states and the control input. Furthermore, we construct a dynamical feedback controller able to achieve complete synchronization in the coordinate transformation systems and GS in the original coordinates. These particular forms of GS are illustrated with numerical results of Colpitts and Chua circuits.**

I. Introduction

One of the current challenges is to achieve and explain synchronization of strictly different chaotic systems. The generalized synchronization (GS) concept was introduced in [1] and it is used to describe the onset of synchronization in directionally coupled chaotic systems. GS is one of the fundamental phenomena, widely studied recently, having both theoretical and applied significance [2]–[6]. GS occurs when the trajectories of one system, through a functional mapping are equal to trajectories of another. GS was taken to occur if there exists a mapping H_{ms} from the trajectories $x_m(t)$ of the attractor in the master algebraic manifold M to the trajectories $x_s(t)$ in the slave space S, i.e., $H_{ms}(x_s(t)) = x_m(t)$ [7]. For identical systems the functional mapping corresponds to the identity [8]. For nonidentical master and slave systems, the map differs from identity, which complicates the detection of the GS, in fact, the attractors in the variables x_m and x_s seems to be unsynchronized.

In this paper we propose a method for GS in nonlinear systems, where it is sufficient to know the output of the system to generate this transformation which is represented by means of differential transcendence basis, that is to say there exists an element $\bar{y}$ and let $n \geq 0$ be the minimum integer such that $\bar{y}^{(n)}$ is analytically dependent on $\bar{y}, \bar{y}^{(1)}, \ldots, \bar{y}^{(n-1)}$ such that

$$\bar{H}(\bar{y}^{(n)}, \bar{y}^{(n-1)}, \ldots, \bar{y}, u, u^{(1)}, \ldots, u^{(\gamma)}) = 0 \qquad (1)$$

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The main idea is to find a synchronization dynamical control signal such that it is possible to synchronize the coordinate transformation system, that is to say, the original system is carried out to a triangular form trough an adequate choice of the differential primitive element given normally as a linear combination of the known states and the inputs of the system where the coefficients belongs to the differential field generated by the field K and the control input u.

II. Statement of the Problem and Main Result

Let us consider the following basic definitions.

Definition 1: A differential field extension L/K is given by two differential fields K, L such that:

(i) K is a subfield of L,

(ii) the derivation (usual rules) of K is the restriction to K of the derivation of L.

Definition 2: An element $a \in L$ is said to be differentially algebraic over K if and only if it satisfies a differential equation $P(a, \dot{a}, \ldots, a^{(\alpha)}) = 0$, where P is a polynomial over K in a and its time derivatives.

The theorem of differential primitive element [9] states that there exists a single element $\delta \in L$, which is a differential primitive element, such that $L = K\langle\delta\rangle$, that is to say L is differentially generated by K and δ.

Definition 3: A dynamics is defined as a finitely generated differentially algebraic extension $L/K\langle u\rangle$ of the differential field $K\langle u\rangle$, where $K\langle u\rangle$ denotes the differential field generated by K and the elements of a finite set $u = (u_1, u_2, \ldots, u_m)$ of differential quantities.

Further details can be found in [10].

Definition 4: A system is Picard-Vessiot (PV) if and only if the $k\langle u\rangle$–vector space generated by the derivatives of the set $\{y^{(\mu)},\ \mu \geq 0\}$ has finite dimension.

System (1) can be solved locally as

$$\bar{y}^{(n)} = -\mathcal{L}(\bar{y}^{(n-1)}, \ldots, \bar{y}, u, u^{(1)}, \ldots, u^{(\gamma-1)}) + u^{(\gamma)}$$

and recalling $\xi_i = \bar{y}^{(i-1)}$, $1 \leq i \leq n$. Then a local form is obtained which can be seen as a GOCF (Generalized Observability Canonical Form),

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n \\ \dot{\xi}_n &= -\mathcal{L}(\xi_1, \ldots, \xi_n, u, u^{(1)}, \ldots, u^{(\gamma-1)}) + u^{(\gamma)} \\ \bar{y} &= \xi_1 \end{aligned} \qquad (2)$$

Let us now consider the following nonlinear systems class:

$$\begin{aligned} \dot{x} &= F(x, u) \\ y &= Cx \end{aligned} \qquad (3)$$

where $x \in \mathbb{R}^n$ is the state vector, $F(\cdot)$ is a nonlinear vector function, u is the input, y is the output and C is a real matrix of appropriate size.

The next lemma is an important technical result.

Lemma 1: A nonlinear system (3) is transformable to a GOCF if and only if it is PV.

Proof. Let the set $\{\varepsilon, \varepsilon^{(1)}, \dots, \varepsilon^{(n-1)}\}$ be a finite differential transcendence basis with $\varepsilon^{(i-1)} = y^{(i-1)}$, $1 \leq i \leq n$, where $n \geq 0$ is the minimum integer such that $y^{(n)}$ is dependent of $y, y^{(1)}, \dots, y^{(n-1)}, u, \dots$

Redefining $\xi_i = \varepsilon^{i-1}$, $1 \leq i \leq n$, yields to

$$\begin{array}{rcl} \dot{\xi}_j & = & \xi_{j+1}, \qquad 1 \leq j \leq n-1 \\ \dot{\xi}_n & = & -\mathcal{L}(\xi_1, \dots, \xi_n, u, u^{(1)}, \dots, u^{(\gamma-1)}) + u^{(\gamma)} \end{array}$$

□

We discuss the GS of nonlinear systems which are completely triangularizable. For this class of systems, the GS problem is solvable in the sense of [1], [13] using a dynamic feedback that stabilizes the synchronization error dynamics. The GS problem is stated as follows:

Let us consider two nonlinear systems in a master–slave configuration, where the master system is given by

$$\begin{array}{rl} \dot{x}_m = & F_m(x_m, u_m) \\ y_m = & h_m(x_m) \end{array} \tag{4}$$

and the slave by

$$\begin{array}{rl} \dot{x}_s = & F_s(x_s, u_s(x_s, y_m)) \\ y_s = & h_s(x_s) \end{array} \tag{5}$$

where $x_s = (x_{1_s}, \dots, x_{n_s}) \in \mathbb{R}^{n_s}$, $x_m = (x_{1_m}, \dots, x_{n_m}) \in \mathbb{R}^{n_m}$, $h_s : \mathbb{R}^{n_s} \to \mathbb{R}$, $h_m : \mathbb{R}^{n_m} \to \mathbb{R}$, $u_m = (u_{1_m}, \dots, u_{\bar{m}_m}) \in \mathbb{R}^{\bar{m}_m}$, $u_s : \mathbb{R}^{n_s} \times \mathbb{R} \to \mathbb{R}$, $y_m, y_s \in \mathbb{R}$, F_s, F_m, h_s, h_m are assumed to be polynomial in their arguments.

It is the more general case because systems (4) and (5) are not necessarily affine nonlinear systems. Indeed, the dynamics of the slave system does not need to be expressed as a linear part and a nonlinear part as in [14], where the nonlinear vector function is restricted to satisfy a Lipschitz condition.

Definition 5 (Generalized synchronization (GS)): The slave and master systems are said to be in a state of GS if there exists a differential primitive element such that generates a transformation $H_{ms} : \mathbb{R}^{n_s} \to \mathbb{R}^{n_m}$ with $H_{ms} = \Phi_s^{-1} \circ \Phi_m$ as well as there exists an algebraic manifold $M = \{(x_s, x_m) | x_m = H_{ms}(x_s)\}$ and a compact set $B \subset \mathbb{R}^{n_m} \times \mathbb{R}^{n_s}$ with $M \subset B$ such that their trajectories with initial conditions in B approach M as $t \to \infty$.

Definition 5 lead us to the following criterion: $\lim_{t\to\infty} \|H_{ms}(x_s) - x_m\| = 0$. It should be noted that identical or complete synchronization is a particular case of GS, that is to say the transformation H_{ms} is the identity.

Remark 1: The differential primitive element is chosen as

$$y = \sum_i \alpha_i x_i + \sum_j \beta_j u_j \ , \qquad \alpha_i, \beta_j \in K\langle u \rangle$$

where $K\langle u \rangle$ is a differential field generated by K and u, and differential quantities.

Proposition 1: Let systems (4) and (5) be transformable to a GOCF. In this case we suppose $u_m = 0$ however it can have any value. Then $\lim_{t\to\infty} \|z_m - z_s\| = 0$, which implies that $\lim_{t\to\infty} \|H_{ms}(x_s) - x_m\| = 0$, where z_m and z_s are the trajectories of master and slave systems in the coordinate transformation.

Proof. The differential primitive element for the master is taken as

$$y_m = \sum_i \alpha_{m_i} x_{m_i} = z_{m_1} \ , \qquad \alpha_{m_i} \in \mathbb{R}$$

and for the slave

$$y_s = \sum_i \alpha_{s_i} x_{s_i} + \sum_j \beta_{s_j} u_{s_j} = z_{s_1} \ , \qquad \alpha_{s_i}, \beta_{s_j} \in \mathbb{R}\langle u_s \rangle$$

which lead us to

$$\begin{array}{rcl} \dot{z}_{mj} & = & z_{m_{j+1}}, \\ \dot{z}_{m_n} & = & -\mathcal{L}_m(z_{m_1}, \dots, z_{m_n}) \end{array}$$

and

$$\begin{array}{rcl} \dot{z}_{sj} & = & z_{s_{j+1}}, \qquad 1 \leq j \leq n-1 \\ \dot{z}_{s_n} & = & -\mathcal{L}_s(z_{s_1}, \dots, z_{s_n}, u_s, \dot{u}_s, \dots, u_s^{(\gamma-1)}) + u_s^{(\gamma)} \end{array}$$

Let us define de control signals as $u_1 = u_s$, $u_2 = \dot{u}_s$, $\dots$, $u_\gamma = u_s^{(\gamma-1)}$, then we propose the following dynamical system

$$\begin{array}{rcl} \dot{u}_j & = & u_{j+1}, \qquad 1 \leq j \leq \gamma-1 \\ \dot{u}_\gamma & = & -\mathcal{L}_m(z_{m_1}, \dots, z_{m_n}) \\ & & +\mathcal{L}_s(z_{s_1}, \dots, z_{s_n}, u_1, u_2, \dots, u_\gamma) + \kappa(z_m - z_s) \end{array}$$

where $z_m = (z_{m_1}, z_{m_2} \dots, z_{m_n})'$, $z_s = (z_{s_1}, z_{s_2} \dots, z_{s_n})'$ and $\kappa = (k_1, k_2, \dots, k_n)$.

Then the closed loop dynamics of the synchronization error $e_z = z_m - z_s$ is given by the augmented system

$$\begin{array}{rcl} \dot{e}_{z_j} & = & e_{z_{j+1}}, \qquad 1 \leq j \leq n-1 \\ \dot{e}_{z_n} & = & -\mathcal{L}_m(z_{m_1}, \dots, z_{m_n}) \\ & & +\mathcal{L}_s(z_{s_1}, \dots, z_{s_n}, u_1, u_2, \dots, u_\gamma) - \dot{u}_\gamma \\ \dot{u}_i & = & u_{i+1}, \qquad 1 \leq i \leq \gamma-1 \\ \dot{u}_\gamma & = & -\mathcal{L}_m(z_{m_1}, \dots, z_{m_n}) \\ & & +\mathcal{L}_s(z_{s_1}, \dots, z_{s_n}, u_1, u_2, \dots, u_\gamma) + \kappa \ e_z \end{array}$$

Finally, we have that $\dot{e}_z = A \ e_z$, with

$$A = \begin{bmatrix} 0 & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & & \cdots & 0 & 1 & 0 \\ 0 & 0 & & \cdots & & 0 & 1 \\ -k_1 & -k_2 & & \cdots & & -k_{n-1} & -k_n \end{bmatrix}$$

where the control gains $(k_1, k_2, \dots, k_n)$ are chosen such that the spectrum of $A \in \mathbb{R}^{n \times n}$ has only negative real parts. □

III. GS of Colpitts and Chua Circuits

In this work we consider the classical configuration of the Colpitts oscillator (Fig. 1).

The state equations for the Colpitts oscillator are [11]:

$$\begin{aligned} \dot{x}_{1_c} =& \;\; -a_c \;\exp(-x_{2_c}) + \; a_c x_{3_c} + a_c \\ \dot{x}_{2_c} =& \;\; b_c x_{3_c} \\ \dot{x}_{3_c} =& \;\; -c_c x_{1_c} - c_c x_2 - d_c x_3 \end{aligned} \tag{6}$$

where, $a_c = b_c \dfrac{C_2}{C_1}$, $b_c = \dfrac{I_0}{w_0 C_2 V_T}$, $c_c = \dfrac{V_T}{w_0 L I_0}$, $d_c = \dfrac{R}{L w_0}$.

In what follows, system (6) is considered as the master.

Let the differential primitive element be the output of system (6) as, $y_c = x_{2_c}$. Then we propose a coordinate transformation system as,

$$\begin{pmatrix} z_{1_c} \\ z_{2_c} \\ z_{3_c} \end{pmatrix} = \begin{pmatrix} y_c \\ \dot{y}_c \\ \ddot{y}_c \end{pmatrix} = \begin{pmatrix} x_{2_c} \\ b_c x_{3_c} \\ -b_c c_c x_{1_c} - b_c c_c x_{2_c} - b_c d_c x_{3_c} \end{pmatrix} = \Phi_c(x_c) \tag{7}$$

In the transformed coordinates (7), the Colpitts system (6) is rewritten as,

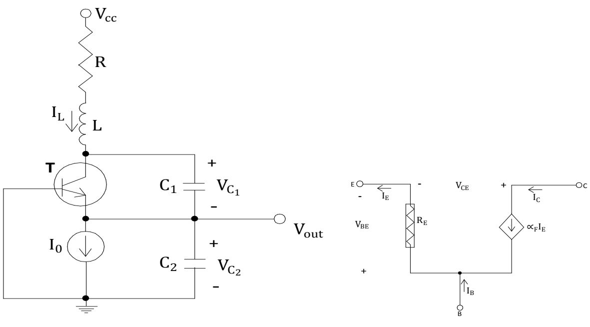
$$\begin{pmatrix} \dot{z}_{1_c} \\ \dot{z}_{2_c} \\ \dot{z}_{3_c} \end{pmatrix} = \begin{pmatrix} z_{2_c} \\ z_{3_c} \\ \Psi_c(x_c) \end{pmatrix} \tag{8}$$

where $\Psi_c(x_c) = -b_c c_c \dot{x}_{1_c} - b_c c_c \dot{x}_{2_c} - b_c d_c \dot{x}_{3_c}$ Suppose the Chua chaotic system as the controlled slave system. Chua's circuit [12], shown in Fig. 2, is a simple oscillator circuit which exhibits a variety of bifurcations and chaos. The circuit contains three linear energy-storage elements (an inductor, and two capacitors), a linear resistor, and a single nonlinear resistor N_R.

The state equations for the Chua's circuit are as follows:

$$\begin{aligned} \dot{x}_{1_{ch}} =& \;\; a_{ch}(x_{2_{ch}} - x_{1_{ch}} - \nu_x) \\ \dot{x}_{2_{ch}} =& \;\; x_{1_{ch}} - x_{2_{ch}} + x_{3_{ch}} \\ \dot{x}_{3_{ch}} =& \;\; -b_{ch} x_{2_{ch}} \end{aligned} \tag{9}$$

where $\nu_x = m_0 x_{1_{ch}} + m_1 x_{1_{ch}}^3$ and the parameters a_c, b_c, m_0 and m_1 are chosen such that (9) is chaotic.



(a) (b)

Fig. 1. Colpitts oscillator. (a) Circuit configuration. (b) Model of the Bipolar Junction Transistor (BJT). The circuit parameters are: L = 100 μH; $C_1 = C_2 = 47$ nF, R = 45 Ω, I_0 = 5 mA.

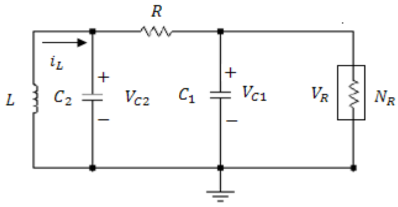


Fig. 2. Chua's circuit.

In this case we assume the differential primitive element as the output of system (9) along with the control input, $y_{ch} = x_{3_{ch}} + u_1$.

Using proposition 1, the controlled coordinate transformation system as,

$$\begin{pmatrix} z_{1_{ch}} \\ z_{2_{ch}} \\ z_{3_{ch}} \end{pmatrix} = \begin{pmatrix} y_{ch} \\ \dot{y}_{ch} \\ \ddot{y}_{ch} \end{pmatrix} = \begin{pmatrix} x_{3_{ch}} + u_1 \\ -b_{ch} x_{2_{ch}} + u_2 \\ -b_{ch}(x_{1_{ch}} - x_{2_{ch}} + x_{3_{ch}}) + u_3 \end{pmatrix} = \Phi_{ch}(x_{ch}) \tag{10}$$

where $u_1 = u$, $u_2 = \dot{u}$, $u_3 = \ddot{u}$ are control signals which need to be designed in order to achieve synchronization between trajectories of coordinate transformation systems (7) and (10).

The augmented controlled system in the transformed coordinates (10), is represented as,

$$\begin{pmatrix} \dot{z}_{1_{ch}} \\ \dot{z}_{2_{ch}} \\ \dot{z}_{3_{ch}} \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} z_{2_{ch}} \\ z_{3_{ch}} \\ \Psi_{ch}(x_{ch}) + \overline{u} \\ u_2 \\ u_3 \\ \overline{u} \end{pmatrix} \tag{11}$$

where $\Psi_{ch}(x_{ch}) = -b_{ch}(\dot{x}_{1_{ch}} - \dot{x}_{2_{ch}} + \dot{x}_{3_{ch}})$

Then, the control objective is to find $\overline{u}$ such that the trajectories of slave system (11) follows the trajectories of master system (8). In other words, we need to find $\overline{u}$ of system (11), such that $(z_{1_{ch}}, z_{2_{ch}}, z_{3_{ch}}) \to (z_{1_c}, z_{2_c}, z_{3_c})$, as long as $t \to \infty$.

Defining the synchronization error in the transformed coordinates as $e_z = z_c - z_{ch}$, the dynamics are given by

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} e_2 \\ e_3 \\ \Psi_c(x_c) - \Psi_{ch}(x_{ch}) - \bar{u}(x_{ch}, y_c) \end{pmatrix}$$

If the control input $\bar{u}$ is defined as,

$$\overline{u}(x_c, y_L) = \dot{u}_3 = \Psi_c(x_c) - \Psi_{ch}(x_{ch}) + \kappa \; e_z$$

where vector gain $\kappa = [k_1, k_2, k_3]$. Then the dynamics of the closed synchronization error is given by, $\dot{e}_z = A \; e_z$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{pmatrix}$.

By means of the Routh-Hurwitz criterion, we conclude that $\|e_z\| \to 0$ as $t \to \infty$ if $k_1 > 0$, $k_2 > \dfrac{k_1}{k_3}$ and $k_3 > 0$.

Figure 3 shows the generalized synchronization of master and slave systems in the coordinate transformation, with $a_c = b_c = 6.2723$, $c_c = 0.0797$, $d_c = 0.6898$, $a_{ch} = 9.5$, $b_c = 100/7$, $m_0 = -8/7$, $m_1 = 4/63$, $k_1 = 10$, $k_2 = 10$ and $k_3 = 10$.

The inverse transformations lead us to

$$\begin{pmatrix} x_{1_c} \\ x_{2_c} \\ x_{3_c} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{c_c}\left[\dfrac{1}{b_c} z_{3_c} + \dfrac{d_c}{b_c} z_2 + c_c z_{1_c}\right] \\ z_{1_c} \\ \dfrac{z_{2_c}}{b} \end{pmatrix} = \Phi_c^{-1}(z_c) \tag{12}$$

and

$$\begin{pmatrix} x_{1_{ch}} \\ x_{2_{ch}} \\ x_{3_{ch}} \end{pmatrix} = \begin{pmatrix} -(z_{1_{ch}} - u_1) - \dfrac{z_{2_{ch}} - u_2}{b_{ch}} - \dfrac{z_{3_{ch}} - u_3}{b_{ch}} \\ -\dfrac{z_{2_{ch}} - u_2}{b_{ch}} \\ z_{1_{ch}} - u_1 \end{pmatrix} = \Phi_{ch}^{-1}(z_{ch}) \tag{13}$$

Figure 4 presents the generalized synchronization in the original coordinates obtained by means of the inverse transformations (12) and (13). In figure 5 are given the corresponding synchronization errors in the transformed coordinates, which illustrates the performance of the proposed approach.

IV. Concluding remarks

This paper tackled the generalized synchronization problem in strictly different nonlinear systems by means of dynamical feedback control signals. To overcome this problem we have employed the differential primitive element given in general form as a linear combination of the measurements and the control input. Indeed, we have designed a dynamical feedback controller able to achieve identical synchronization in the coordinate transformation systems and then GS in the original coordinates was obtained. The GS of Colpitts and Chua circuits was presented, with relatively small gains in the dynamical controller. Finally, some numerical results illustrated the effectiveness of the proposed approach.

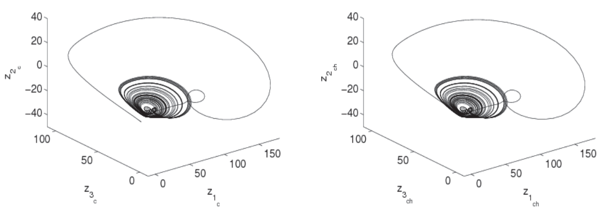


Fig. 3. Generalized synchronization in the transformed variables $(z_{1_c}, z_{2_c}, z_{3_c})$ and $(z_{1_{ch}}, z_{2_{ch}}, z_{3_{ch}})$. (a) Master system; (b) slave system.

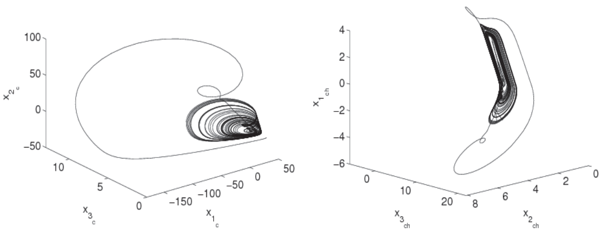


Fig. 4. Generalized synchronization in the original variables $(x_{1_c}, x_{2_c}, x_{3_c})$ and $(x_{1_{ch}}, x_{2_{ch}}, x_{3_{ch}})$. (a) Master system; (b) slave system.

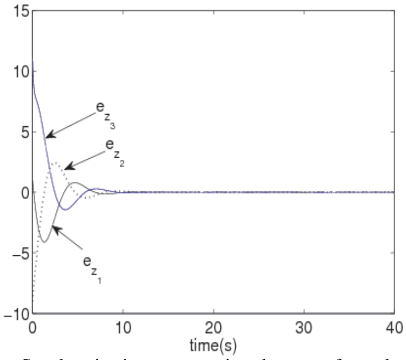


Fig. 5. Synchronization errors in the transformed coordinates; $e_{z_1} = z_{1_c} - z_{1_{ch}}$, $e_{z_2} = z_{2_c} - z_{2_{ch}}$ and $e_{z_3} = z_{3_c} - z_{3_{ch}}$

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An Observer for the Synchronization of Chaotic Liouvillian Systems: A Real-time Application to Chua's Oscillator

Rafael Martínez-Guerra and Juan L. Mata-Machuca

Abstract— **This work deals with the master-slave synchronization of chaotic oscillators with Liouvillian properties (chaotic Liouvillian system) based on nonlinear observer design. The strategy consists of proposing a polynomial observer (slave system) which tends to follow exponentially the chaotic oscillator (master system). The proposed technique is applied in the synchronization of Chua's circuit using Matlab-Simulink® and Matlab®-LMI programs. Experimental results are shown to visualize and illustrate the effectiveness of the proposed observer.**

I. INTRODUCTION.

Since Pecora and Carroll's observation on the possibility of synchronizing two chaotic systems [1](so-called drive-response configuration), several synchronization schemes have been developed [2]-[4]. In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientists in many fields [5]-[7]. As it is well known, the study of the synchronization problem for nonlinear systems has been very important from the nonlinear science point of view, in particular, the applications to biology, medicine, cryptography, secure data transmission and so on [8]-[9]. In general, synchronization research has been focused on the following areas: nonlinear observers [2], [10], nonlinear control [11], feedback controllers [4], [12], nonlinear backstepping control [13], time delayed systems [14], [15], directional and bidirectional coupling [16], [17], adaptive control [18], adaptive observers [19], [20], sliding mode observers [9], [21], impulsive control [22], active control [23], among others.

This work considers the master-slave synchronization problem via an exponential polynomial observer (EPO) based on differential an algebraic techniques [24]-[26]. Differential and algebraic concepts allow us to establish an algebraic observability condition, and therefore they provide a first step for the construction of an algebraic observer. An observable system in this sense can be regarded as a system whose state variables can be expressed in terms of the input and output variables and a finite number of their time derivatives. The Liouvillian character of the system (if a variable can be obtained by the adjunction of integrals or exponentials of integrals) is used as an observability criterion. Thus, chaos synchronization problem can be posed as an observer design procedure, where the coupling signal is viewed as output and the slave system is regarded as observer. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system to the slave system (EPO), the slave is requested to recover the unknown state trajectories of the master. The strategy consists of proposing an EPO which exponentially reconstructs the unknown states of master system.

As experimental example we consider the Chua's oscillator as the master system. Chua's circuit is a nonlinear electronic chaotic oscillator. This circuit is easily constructed [27] and has been employed in a variety of applications [28], e.g. communication systems [29].

In this framework the Chua's oscillator is viewed as a chaotic Liouvillian system with respect to the selected output. The intention of choosing this system is to clarify the proposed methodology and to highlight the simplicity and flexibility of the suggested approach, however, this technique can be applied to a wide variety of chaotic systems.

This paper is organized as follows, in section II we give some definitions about differential-algebraic approach and Liouvillian systems. In section III is treated the synchronization problem and its solution by means of an exponential polynomial observer. In section IV is presented the synchronization of the Chua's circuit [30] and we show some experimental results. Finally, in section V we close the paper with some concluding remarks.

II. DEFINITIONS

We start with some basic definitions for the understanding of Liouvillian systems, the following definitions are presented.

Definition 1 (Algebraic Observability Condition–AOC): Let us consider a nonlinear dynamical system with input u, output y, and state vector $x = (x_1, x_2, \ldots, x_n)^T$. A state variable $x_i \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}\langle u, y\rangle$[1], that is to say, x_i satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e.,

$$P_i(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0 \ , \quad i \in \{1, 2, \ldots, n\}, \quad (1)$$

with coefficients in $\mathbb{R}\langle u, y\rangle$.

Example 1: Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \quad (2)$$

[1] $\mathbb{R}\langle u, y\rangle$ denotes the differential field generated by the field $\mathbb{R}$, the input u, the measurable output y, and the time derivatives of u and y.

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If we define $y = x_2$, then

$$\begin{aligned} x_2 =& \quad y \\ x_3 =& \quad \dot{y} \\ \dot{x}_1 =& \quad y + \dot{y}^2 \end{aligned} \tag{3}$$

The above system is not algebraically observable since x_1 cannot be expressed as a differential algebraic polynomial in terms of $\{u, y\}$.

Motivated by this fact, we present the next definition.

Definition 2 (Liouvillian system): A dynamical system is said to be Liouvillian if the elements (for example, state variables or parameters) can be obtained by an adjunction of integrals or exponentials of integrals of elements of $\mathbb{R}$.

Example 2: We consider the nonlinear system as in example 1. From (3) we can observe that, although x_1 does not satisfy the AOC we can obtain it by means of the integral

$$x_1 = \int (y + \dot{y}^2)$$

Therefore the nonlinear system (2) is Liouvillian.

For further information we recommend to see [24], [26].

III. Problem formulation and main result

Let us consider the following chaotic Liouvillian system,

$$\begin{aligned} \dot{x} =& \quad A\,x + \psi(x) + \varphi(x) + \zeta(u) \\ y =& \quad C\,x, \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, is the state vector[2]; $u \in \mathbb{R}^l$, is the input vector, $l \leq n$; $y \in \mathbb{R}$ is the measured output; $\zeta(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$ is an input dependent vector function; $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{1\times n}$ are constants; and $\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are state dependent nonlinear vector functions.

We restrict each $\psi_i(\cdot)$ to be nondecreasing, that is, for all $a, b \in \mathbb{R}$, $a > b$, it satisfies the following monotone sector condition

$$0 \leq \frac{\psi_i(a) - \psi_i(b)}{a - b}, \qquad i = 1, \ldots, n. \tag{5}$$

In the same manner, we restrict each $\varphi_i(\cdot)$ to be nonincreasing, that is, for all $a, b \in \mathbb{R}$, $a > b$, it satisfies

$$\frac{\varphi_i(a) - \varphi_i(b)}{a - b} \leq 0, \qquad i = 1, \ldots, n. \tag{6}$$

To show the relation between the observers for nonlinear systems and chaos synchronization we give the observer's definition.

Definition 3 (Exponential Observer): An exponential observer for (4) is a system with state $\hat{x}$ such that

$$\|x - \hat{x}\| \leq \kappa \exp(-\xi t)$$

where κ and β are positive constants.

In the context of master-slave synchronization, x can be considered as the state variable of the master system, and $\hat{x}$ can be viewed as the state variable of the slave system. Hence, the master-slave synchronization problem can be solved by designing an observer for (4).

[2]Chaotic systems are characterized by local instability and *global boundedness of the trajectories*, i.e. $\|x(t)\|$ is bounded for all $t \geq 0$.

In what follows, we will solve the synchronization problem by using a exponential polynomial observer based upon the Lyapunov method [31]. To this end, we first compute the dynamics of the synchronization error (difference between the master and the slave systems). Next, by means of a simple quadratic Lyapunov function, we prove the exponential convergence.

System (4) is assumed to be a chaotic Liouvillian system, then by definition 2 all states of (4) can be reconstructed. In this sense, we propose an observer scheme.

The observer structure. The observer for system (4) has the next form

$$\dot{\widehat{x}} = A\widehat{x} + \psi(\widehat{x}) + \varphi(\widehat{x}) + \zeta(u) + \sum_{i=1}^{m} K_i \left(y - C\widehat{x}\right)^{2i-1}, \tag{7}$$

where $\hat{x} \in \mathbb{R}^n$, and $K_i \in \mathbb{R}^n$, for $1 \leq i \leq m$.

Remark 1: The meaning of m can be understood as follows. As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance high order terms are included in the observer structure. In other words, the rate of convergence can be increased by injecting additional terms with increasing powers of the output error.

Observer Convergence Analysis. In order to prove the observer convergence, we analyze the observer error which is defined as $e = x - \hat{x}$. From Eqs. (4) and (7), the dynamics of the state estimation error is given by

$$\dot{e} = \bar{A}e + \phi(e) + \rho(e) - \sum_{i=2}^{m} K_i \left(Ce\right)^{2i-1}, \tag{8}$$

where $\phi(e) := \psi(x) - \psi(\hat{x})$, $\rho(e) := \varphi(x) - \varphi(\hat{x})$ and $\bar{A} := A - K_1 C$.

It follows from (5) that each component of $\phi(e)$ satisfies

$$0 \leq \frac{\phi_i(e_i)}{e_i}, \qquad \forall\, e_i \neq 0, \tag{9}$$

which implies a relationship between $\phi(e)$ and e as follows,

$$e^T \phi(e) \quad = \quad \sum_{i=1}^{n} e_i \phi_i(e_i) = \sum_{i=1}^{n} e_i^2 \frac{\phi_i(e_i)}{e_i}$$

by using (9) we have the following condition

$$0 \leq e^T \phi(e). \tag{10}$$

By a similar analysis, from (6) we have

$$e^T \rho(e) \leq 0. \tag{11}$$

Properties (10) and (11) will allow us to prove that the state estimation error $e(t)$ decays exponentially.

Proposition 1: Consider the chaotic Liouvillian system (4) and the observer (7). If there exists a matrix $P = P^T > 0$, and positive scalars ε, ϵ_1, ϵ_2 satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P + \epsilon_1 I & P - \epsilon_2 I \\ P + \epsilon_1 I & 0 & 0 \\ P - \epsilon_2 I & 0 & 0 \end{bmatrix} \leq 0, \tag{12}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geq 0 \quad , \; i = 2, \ldots, m, \tag{13}$$

with $M_i := PK_iC$. Then, there exist positive constants κ and ξ such that, for all $t \geq 0$,

$$\|e(t)\| \leq \kappa \exp(-\xi t)$$

where $\kappa = \sqrt{\frac{\beta}{\alpha}}\,\|e(0)\|$, $\xi = \frac{\varepsilon}{2\beta}$, $\alpha = \lambda_{min}(P)$, and $\beta = \lambda_{max}(P)$.

Proof: We use the following Lyapunov function candidate $V = e^T P e$. From (8), the time derivative of V is

$$\begin{aligned}
\dot{V} &= e^T\left[\bar{A}^T P + P\bar{A}\right]e \\
&\quad + 2e^T P\phi(e) + 2e^T P\rho(e) - 2\sum_{i=2}^{m}(Ce)^{2i-2}e^T M_i e \\
&= \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P + P\bar{A} & P & P \\ P & 0 & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix} \\
&\quad - \sum_{i=2}^{m}(Ce)^{2i-2}\, e^T(M_i + M_i^T)e
\end{aligned}$$

and, in view of (12) and (13),

$$\dot{V} \leq \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix}^T \begin{bmatrix} -\varepsilon I & -\epsilon_1 I & \epsilon_2 I \\ -\epsilon_1 I & 0 & 0 \\ \epsilon_2 I & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi(e) \\ \rho(e) \end{bmatrix} \tag{14}$$

Equation (14) can be rewritten as

$$\dot{V} \leq -\varepsilon e^T e - 2\epsilon_1\, e^T\phi(e) + 2\epsilon_2\, e^T\rho(e).$$

By properties (10) and (11) we have

$$\dot{V} \leq -\varepsilon\|e\|^2 \tag{15}$$

We write the Lyapunov function as $V = \|e\|_P^2$, then by Rayleigh-Ritz inequality we have that

$$\alpha\|e\|^2 \leq \|e\|_P^2 \leq \beta\|e\|^2 \tag{16}$$

where $\alpha := \lambda_{min}(P)$, and $\beta := \lambda_{max}(P) \in \mathbb{R}^+$ (because P is positive definite).

By using (16) we obtain the following upper bound of (15)

$$\dot{V} \leq -\frac{\varepsilon}{\beta}\|e\|_P^2 \tag{17}$$

Taking the time derivative of $V = \|e\|_P^2$ and replacing in inequality (17), we obtain

$$\frac{d}{dt}\|e\|_P \leq -\frac{\varepsilon}{2\beta}\|e\|_P$$

Finally, the result follows with

$$\|e(t)\| \leq \kappa \exp(-\xi\, t) \tag{18}$$

where $\kappa = \sqrt{\frac{\beta}{\alpha}}\|e(0)\|$, and $\xi = \frac{\varepsilon}{2\beta}$. ■

Corollary 1: Let us consider $\psi(\cdot) \equiv 0$. Then system (7) is an exponential observer of system (4) if there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_2 > 0$ satisfying

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P - \epsilon_2 I \\ P - \epsilon_2 I & 0 \end{bmatrix} \leq 0 \tag{19}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geq 0 \quad , \; i = 2, \ldots, m. \tag{20}$$

With κ and ξ defined as in proposition 1.

Proof: The result is proven as in proposition 1. ■

Corollary 2: Let us consider $\varphi(\cdot) \equiv 0$. Then system (7) is an exponential observer of system (4) if there exists a matrix $P = P^T > 0$, and scalars $\varepsilon > 0$, $\epsilon_1 > 0$ satisfying the LMI

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \varepsilon I & P + \epsilon_1 I \\ P + \epsilon_1 I & 0 \end{bmatrix} \leq 0 \tag{21}$$

and

$$\lambda_{\min}\left(M_i + M_i^T\right) \geq 0 \quad , \; i = 2, \ldots, m. \tag{22}$$

With κ and ξ defined as in proposition 1.

Proof: It follows directly from the procedure in proposition 1. ■

IV. Experimental results

In this section we consider the synchronization of a Chua's system viewed as a chaotic Liouvillian oscillator. Chua's circuit [27], see Fig. 1, it is a simple oscillator circuit which exhibits a variety of bifurcations and chaos. The circuit contains three linear energy-storage elements (an inductor, and two capacitors), a linear resistor, and a single nonlinear resistor N_R.

The state equations for the Chua's circuit are as follows:

$$\begin{aligned}
C_1\frac{dv_{C_1}}{dt} &= G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\
C_2\frac{dv_{C_2}}{dt} &= G(v_{C_1} - v_{C_2}) + i_L \\
L\frac{di_L}{dt} &= -v_{C_2}
\end{aligned} \tag{23}$$

where $G = \frac{1}{R}$ and $g(\cdot)$ is a nonincreasing function defined by:

$$g(v_R) = m_0 v_R + \frac{1}{2}(m_1 - m_0)(|v_R + B_p| - |v_R - B_p|) \tag{24}$$

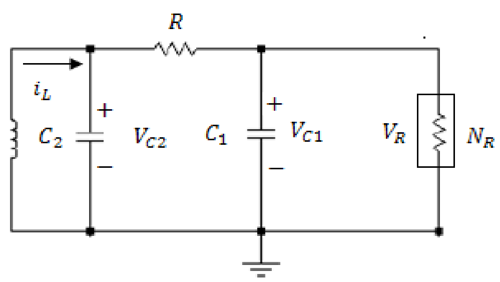

Fig. 1. Chua's circuit, with parameter values $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS and $B_p = 1.08$ V.

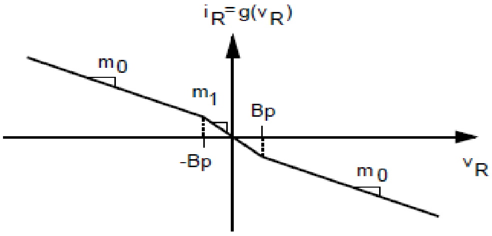

Fig. 2. Three-segment piecewise-linear v-i characteristic of the linear resistor in Chua's circuit. The outer regions have slopes m_0, the inner region has slope m_1. There are two breakpoints at $\pm B_p$.

It is not difficult to show that $g(v_R)$ is a nonincreasing function and therefore condition (6) is fulfilled.

Relation (24) is shown graphically in Fig. 2, the slopes in the inner and outer regions are m_0 and m_1 respectively, with $m_1 < m_0 < 0$, $\pm B_p$ denote the breakpoints. The nonlinear resistor N_R is termed voltage-controlled because the current in the element is a function of the voltage across its terminals.

The implementation of the Chua's circuit is shown in Figs. 3 and 4.

In what follows we consider the measured output $y = v_{c_2}$. From equations of (23) we obtain:

$$\begin{aligned} v_{c_1} &= \frac{C_2}{G}\dot{y} + y + \frac{1}{LG}\int y dt \\ v_{c_2} &= y \\ i_L &= -\frac{1}{L}\int y dt \end{aligned} \tag{25}$$

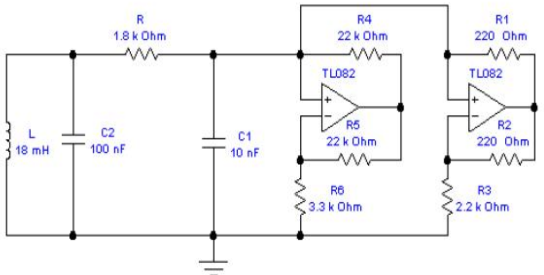

Fig. 3. Chua's circuit.

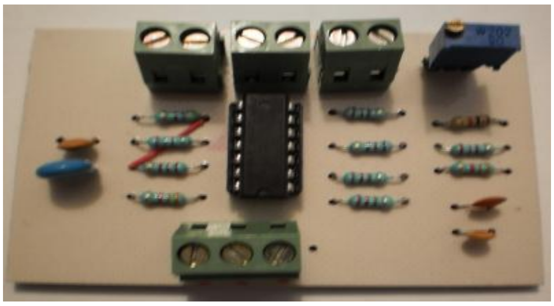

Fig. 4. Implementation of the Chua's circuit.

From (25), the Chua's system (23) is Liouvillian. This implies that unknown variables v_{C_1} and i_L can be reconstructed with the selected output $y = v_{C_2}$.

Chua's system (23) is of the form (4) with $\zeta(u) = 0$, $\varphi(\cdot) = 0$,

$$A = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix}, \psi(x) = \begin{bmatrix} -\frac{g(x_1)}{C_1} \\ 0 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} v_{C_1} & v_{C_2} & i_L \end{bmatrix}^T.$$

Since $g(x_1)$ is nonincreasing and C_1 is a positive constant, then $\psi_1(x) = \psi_1(x_1) = -g(x_1)/C_1$ is nondecreasing as in (5). Indeed, $\psi_2(x) = \psi_3(x) = 0$ also satisfy property (5), and Chua's system (23) is Liouvillian, so that, we proceed with the observer design.

Taking into account that $\varphi(\cdot) = 0$, we will use conditions in Corollary 2 to obtain the observer gains. Using LMI software, observer gains are computed to drive the estimation error to zero.

Applying (7), we have the observer for Chua's system (23),

$$\begin{aligned} \dot{\hat{x}} = & \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -\frac{g(\hat{x}_1)}{C_1} \\ 0 \\ 0 \end{bmatrix} \\ & + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} e \right)^{2i-1} \end{aligned}$$

Hence, the state observer is rewritten as,

$$\begin{aligned} \dot{\hat{x}}_1 = & -\frac{G}{C_1}\hat{x}_1 + \frac{G}{C_1}\hat{x}_2 - \frac{g(\hat{x}_1)}{C_1} + k_{1,1}e_{1,2} \\ & + k_{1,2}(e_{1,2})^3 + ... + k_{1,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_2 = & \frac{G}{C_2}\hat{x}_1 - \frac{G}{C_2}\hat{x}_2 + k_{2,1}e_{1,2} \\ & + k_{2,2}(e_{1,2})^3 + ... + k_{2,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_3 = & -\frac{1}{L}\hat{x}_3 + k_{3,1}e_{1,2} \\ & + k_{3,2}(e_{1,2})^3 + ... + k_{3,m}(e_{1,2})^{2m-1} \end{aligned} \tag{26}$$

Some experimental studies are carried out for the synchronization of Chua's system in order to show the performance of the exponential observer. The parameter values considered in the experimental implementation correspond to chaotic behavior [30] and these are: $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS and $B_p = 1.08$ V.

We fix $m = 2$ in the observer (26). The LMI (21) is feasible with $\varepsilon = 0.001$ and $\epsilon_1 = 0.001$, a solution is

$$P = \begin{bmatrix} 0.0008 & -0.0006 & 0.1021 \\ -0.0006 & 0.0005 & -0.0805 \\ 0.1021 & -0.0805 & 15.0959 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} k_{1,1} \\ k_{2,1} \\ k_{3,1} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 45 \end{bmatrix},$$

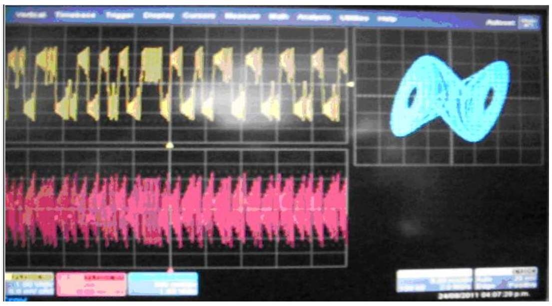

Fig. 5. Chaotic behavior of the Chua's oscillator.

and K_2 is chosen such that (22) is satisfied, then we obtain

$$K_2 = \begin{bmatrix} k_{1,2} \\ k_{2,2} \\ k_{3,2} \end{bmatrix} = \begin{bmatrix} 7.1573 \\ 14.1040 \\ 0.0268 \end{bmatrix}.$$

The measured output was obtained by means of the oscilloscope LeCroyWaveRunner® 104MXi. This device supports Matlab-Simulink® platform, hence was also possible the implementation of the observer. The initial conditions of the observer were chosen as $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = 0.25$ and $\hat{x}_3(0) = 0.001$. Figure 5 shows the behavior of the Chua's oscillator.

The performance of the proposed observer is evaluated by means of the relative error which in this case is defined as

$$\bar{e}_i = \frac{|x_i - \hat{x}_i|}{|x_i|} \quad , \qquad i = 1, 3.$$

Figs. 6-7 illustrate the corresponding relative errors, it should be noted that $\bar{e}_1 = 0$ and $\bar{e}_3 = 0$ for $t > 0.006s$, as was expected.

V. Concluding Remarks

The synchronization problem of chaotic Liouvillian systems has been treated by using differential and algebraic techniques. We proposed a polynomial observer, and by means of properties of nondecreasing and nonincreasing functions, linear matrix inequalities and with the help of the Lyapunov method we proved that the estimation error exponentially converges to zero. This observer has been used as a slave system whose states are exponentially synchronized with the chaotic system (Chua's circuit). A reduced set of measurable state variables were needed to achieve the synchronization with this approach. The effectiveness of the suggested methodology was shown by means of a real-time implementation.

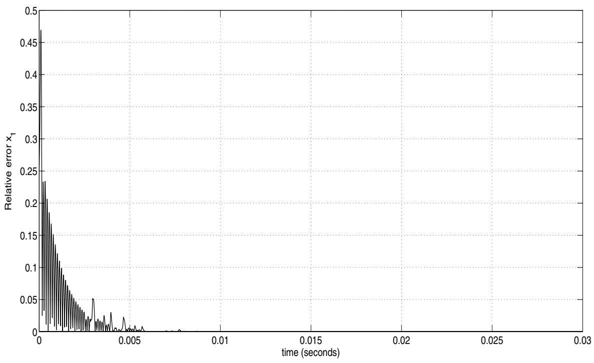

Fig. 6. Relative error: $\bar{e}_1 = \frac{|x_1 - \hat{x}_1|}{|x_1|}$.

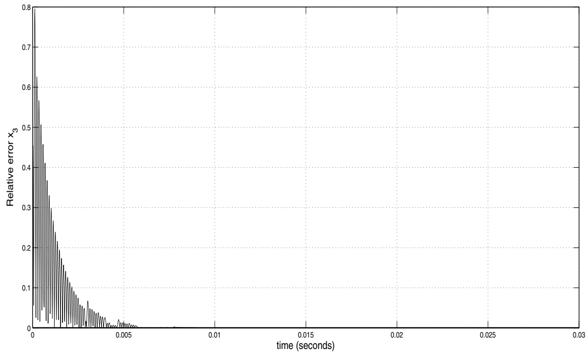

Fig. 7. Relative error: $\bar{e}_3 = \frac{|x_3 - \hat{x}_3|}{|x_3|}$.

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A Synchronization Scheme for Partially Known Nonlinear Fractional Order Systems

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Abstract: In this work a synchronization scheme for partially known nonlinear fractional order systems is proposed. In this approach the unknown dynamics is considered as the master system and the slave system estimates the unknown variables. For solving this problem we introduce a Fractional Algebraic Observability (FAO) property which is used as a main tool in the design of the master system. As numerical example we consider a fractional order Lorenz chaotic system and by means of some simulations we show the effectiveness of the suggested approach.

Keywords: Fractional Algebraic Observability, fractional chaotic systems, fractional order Lorenz chaotic system, nonlinear fractional order systems, synchronization

1. INTRODUCTION

In the first work concerning on synchronization of fractional systems, Li et al. (2003) showed by means of a control law that fractional order chaotic systems can be synchronized by using the similar scheme as that of their integer counterparts.

We mention some works related to synchronization in fractional systems, for example, Li and Yan (2007) and Zhu et al. (2009) proposed the employment of feedback controllers, which allows to achieve the synchronization between two identical fractional-order chaotic systems, the theoretical analysis is derived by utilizing the stability criterion of linear fractional systems; Peng (2007) studied the synchronization of fractional order chaotic systems with unidirectional linear error feedback coupling; Lu (2006) presented a classical Luenberger observer design for the synchronization of fractional-order chaotic systems, i.e., the observer structure needs a copy of the system and a linear output error feedback, the application is restricted to scalar coupling signals; Deng and Li (2005) and Li and Deng (2006) gave sufficient conditions for the synchronization between two identical fractional systems by using the Laplace transform theory.

The main contribution in this paper is to show a novel technique for the synchronization problem in partially known nonlinear fractional-order systems, we propose a new procedure using the master-slave synchronization scheme for estimating the unknown state variables based on Fractional Algebraic Observability (FAO) property (is not necessary the system's copy).

2. ON FRACTIONAL DERIVATIVES

There are several definitions of a fractional derivative of order α, for example in Podlubny (1999), Miller and Ross (1993) and Podlubny (2002), we will use the Caputo fractional operator in the definition of fractional order systems, because the meaning of the initial conditions for systems described using this operator is the same as for integer order systems.

Definition 1. (Caputo Fractional derivative). The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function x is defined as: see Podlubny (1999)

$$x^{(\alpha)} = {}_{t_0}D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{d^m x(\tau)}{d\tau^m}(t-\tau)^{m-\alpha-1}\,d\tau, \quad (1)$$

where: $m-1 \leq \alpha < m$, $\frac{d^m x(\tau)}{d\tau^m}$ is the m-th derivative of x in the usual sense, $m \in \mathbb{N}$ and Γ is the gamma function [1]. $\square$

Now we define a sequential operator, see Miller and Ross (1993), as follows

$$\mathcal{D}^{(r\alpha)}x(t) = \underbrace{{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}\cdots{}_{t_0}D_t^{\alpha}\,{}_{t_0}D_t^{\alpha}}_{\text{r-times}}\,x(t) \quad (2)$$

i. e., it is the Caputo fractional derivative of order α applied $r \in \mathbb{N}$ times sequentially, with $\mathcal{D}^{(0)}x(t) = x(t)$, we can note that if $r = 1$ then $\mathcal{D}^{(\alpha)}x(t) = x^{(\alpha)}$.

2.1 Mittag-Leffler type function

The Mittag-Leffler function with two parameters is defined as Kilbas et al. (2006):

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}, \quad z,\,\beta \in \mathbb{C},\ \mathbb{R}e(\alpha) > 0 \quad (3)$$

[1] To simplify the notation we omitted the time dependence in $x^{(\alpha)}$, in what follows we take $t_0 = 0$

this function is used to solve fractional differential equations as the exponential function in integer order systems. In the particular case when $\alpha = \beta = 1$, we have that $E_{1,1}(z) = e^z$. Now if we have particular values of α, the function (3) has asymptotic behavior at infinity.

Theorem 1. (Podlubny (1999)). If $\alpha \in (0, 2)$, β is an arbitrary complex number and μ is an arbitrary real number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\} \tag{4}$$

then for an arbitrary integer $\kappa \geq 1$ the following expansion holds:

$$E_{\alpha,\beta}(z) = -\sum_{i=1}^{\kappa} \frac{1}{\Gamma(\beta - \alpha i) z^i} + O\left(\frac{1}{|z|^{\kappa+1}}\right) \tag{5}$$

with $|z| \to \infty$, $\mu \leq |\arg(z)| \leq \pi$. ■

The Mittag-Leffler function has the following properties:

Property 1. $\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-k\tau^\alpha)\, d\tau = t^\beta E_{\alpha,\beta+1}(-kt^\alpha)$, $\beta > 0$, see Podlubny (1999).

Property 2. $E_{\alpha,\beta}(-x)$, is completely monotonic, i. e., $(-1)^n E_{\alpha,\beta}^{(n)}(-x) \geq 0$ for $0 < \alpha \leq 1$ and $\beta \geq \alpha$, for all $x \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$, see Miller and Samko (2001).

We will use these facts in the following problem.

3. PROBLEM STATEMENT AND MAIN RESULT

We take the initial condition problem for an autonomous fractional order nonlinear system, with $0 < \alpha < 1$:

$$\begin{array}{l} x^{(\alpha)} = f(x), \;\; x(0) = x_0 \\ y = h(\bar{x}) \end{array} \tag{6}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is a Lipschitz continuous function [2], with $x_0 \in \Omega \subset \mathbb{R}^n$, in this case y denotes the output of the system (the measure that we can obtain), $\bar{x} \in \mathbb{R}^p$ represents the states that we can observe (known states), $h : \mathbb{R}^p \to \mathbb{R}^q$ is a continuous function and $1 \leq p < n$.

Consider the system given by (6), we will separate in two dynamical systems with states $\bar{x} \in \mathbb{R}^p$ and $\eta \in \mathbb{R}^{n-p}$ respectively with $x^T = (\bar{x}^T, \eta^T)$, the first system will describe the known states and the second represents unknown states, then the system (6) can be written as:

$$\begin{array}{l} \bar{x}^{(\alpha)} = \bar{f}(\bar{x}, \eta) \\ \eta^{(\alpha)} = \Delta(\bar{x}, \eta) \\ y = h(\bar{x}) \end{array} \tag{7}$$

where $f^T(x) = \left(\bar{f}^{\,T}(\bar{x}, \eta), \Delta^T(\bar{x}, \eta)\right)$, $\bar{f} \in \mathbb{R}^p$ and $\Delta \in \mathbb{R}^{n-p}$. Now the problem is: How can we estimate the η's states? this question arises because if we know the η's states we can use these signals to generate measuring depending on them. In order to solve this observation problem let us introduce the following observability property (similar to the differential and algebraic approach used in nonlinear integer order systems, see Aguilar-Lopez et al. (2011))

[2] This assures the unique solution, see Kilbas et al. (2006)

Definition 2. (FAO). A state variable $\eta_i \in \mathbb{R}$ satisfies the Fractional Algebraic Observability (FAO) if it is a function of the first $r \in \mathbb{N}$ sequential derivatives of the available output $y_{\bar{x}}$, i.e.,

$$\eta_i = \phi_i\left(y_{\bar{x}},\, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots,\, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{8}$$

where $\phi_i : \mathbb{R}^{(r+1)p} \to \mathbb{R}$. □

If we assume that the components of unknown state vector η are FAO, then we can describe our problem in terms of the master-slave synchronization scheme, which is defined in the following way.

Let us consider the master system:

$$\eta_i^{(\alpha)} = \Delta_i(\bar{x}, \eta) \tag{9}$$

$$y_{\eta_i} = \eta_i = \phi_i\left(y_{\bar{x}},\, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots,\, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{10}$$

for $i \in \{p+1, \ldots, n\}$, where η_i is a component of the state vector η and y_{η_i} denotes the output of the i-th master system.

Now let us propose a fractional dynamical system with the same order α, which will be the slave system:

$$\hat{\eta}_i^{(\alpha)} = k_{\hat{\eta}_i}(y_{\eta_i} - \hat{\eta}_i), \tag{11}$$

$$y_{\hat{\eta}_i} = \hat{\eta}_i, \tag{12}$$

for $i \in \{p+1, \ldots, n\}$, where $\hat{\eta}_i$ is the state, and $y_{\hat{\eta}_i}$ denotes the output of the slave system and $k_{\hat{\eta}_i}$ is a positive constant.

In the master-slave synchronization scheme, the output of the master system (10) describes the target signal, while (12) represents the response signal. Therefore the synchronization problem can be established as:
Given the master system (9) and our slave system (11), it should be determined some conditions, such that the output of the slave system (12) synchronizes with the output of the master system (10).

Let us define the synchronization error as:

$$e_i = y_{\eta_i} - y_{\hat{\eta}_i} = \eta_i - \hat{\eta}_i. \tag{13}$$

Now we establish a convergence analysis of the synchronization error.

Proposition 1. Let the system (6) which can be expressed in the form (7), where the following conditions are fulfilled:

H1: η_i satisfies the FAO property for $i \in \{p+1, \ldots, n\}$.
H2: Δ_i is bounded, i.e., $\exists\, M \in \mathbb{R}^+$ such that $\|\Delta(x)\| \leq M$, $\forall\, x \in \Omega$.
H3: $k_{\hat{\eta}_i} \in \mathbb{R}^+$.

Then, the synchronization of the master output (10) with the slave output (12) is achieved, for global initial condition of the states.

Proof. From **H1** we can write equations (9)-(13). Taking the fractional derivative of the equation (13), we have

$$e_i^{(\alpha)} = \eta_i^{(\alpha)} - \hat{\eta}_i^{(\alpha)} \tag{14}$$

Substituting the fractional dynamics (9) and (11) into (14), we obtain

$$e_i^{(\alpha)} + k_{\hat{\eta}_i} e_i = \Delta_i(x) \tag{15}$$

There exists a unique solution for the system (15), due to $\Delta_i(x(t)) - k_{\hat{\eta}_i} e_i(t)$ is a Lipschitz continuous function on e.[3]

The solution of 15 is taken from Kilbas et al. (2006), then we have

$$\begin{aligned} e_i(t) =& \, e_0 E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(k_{\hat{\eta}_i}(t-\tau)^\alpha)\Delta_i(\tau)\mathrm{d}\tau \end{aligned} \tag{16}$$

where $e_i(0) = e_{i0}$.

Using Triangle and Cauchy-Schwarz inequalities and **H2**

$$\begin{aligned} |e_i(t)| \leq& \, |e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha)| \\ &+ M \int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)| \mathrm{d}\tau \end{aligned}$$

The functions $(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)$ and $E_\alpha(-k_{\hat{\eta}_i} t^\alpha)$ are not negative due to **Property 2** of Mittag-Leffler function and **H3**

$$\begin{aligned} |e_i(t)| \leq& \, |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ &+ M \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha) \mathrm{d}\tau \end{aligned}$$

Using **Property 1** of Mittag-Leffler function

$$|e_i(t)| \leq |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha)$$

If $t \to \infty$, we use the equation (5) with $\mu = 3\pi\alpha/4$ due to **H3**.

$$\begin{aligned} \lim_{t\to\infty} |e_i(t)| \leq& \, |e_{i0}| \lim_{t\to\infty} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ &+ M \lim_{t\to\infty} t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha) = \frac{M}{k_{\hat{\eta}_i}} \end{aligned}$$

■

Remark 1. If the FAO of a state variable is expressed in terms of the fractional sequential derivatives of the output y, which are unknown, then is necessary to introduce an artificial variable (if it is possible) in order to avoid the use of these unknown derivatives.

4. NUMERICAL EXAMPLE

In this section we study the synchronization of nonlinear fractional order systems by numerical simulations.

Remark 2. Chaotic systems are characterized by global boundedness of the trajectories, see Fradkov (2007). By this fact, H2 is always satisfied.

We consider the fractional-order Lorenz system described by a set of three fractional differential equations, as follows:

$$\begin{aligned} x^{(\alpha)} &= \begin{pmatrix} \mathrm{a}x_2 - \mathrm{a}x_1 \\ \mathrm{b}x_1 - \mathrm{c}x_2 - x_1x_3 \\ x_1x_2 - \mathrm{d}x_3 \end{pmatrix} \\ y &= x_1 \end{aligned} \tag{17}$$

[3] Equation (15) is non-autonomous, but the Lipschitz condition assures a unique solution Kilbas et al. (2006).

with parameters a = 10, b = 28, c = −8, $d = 8/3$, and $\alpha = 0.8$, the system (17) exhibits chaotic behavior Wen et al. (2008).

Now, we will apply the proposed methodology. Firstly, we rewrite system (17) in the form (7)

$$\begin{aligned} \bar{x}^{(\alpha)} &= \mathrm{a}(\eta_2 - \bar{x}_1) \\ \eta^{(\alpha)} &= \begin{pmatrix} \mathrm{b}\bar{x}_1 - \mathrm{c}\eta_2 - \bar{x}_1\eta_3 \\ \bar{x}_1\eta_2 - \mathrm{d}\eta_3 \end{pmatrix} \\ y_{\bar{x}} &= \bar{x}_1 \end{aligned} \tag{18}$$

where $\bar{x}_1 = x_1$, $\eta_2 = x_2$, and $\eta_3 = x_3$.

Before proposing a master system in the form (9)-(10) we need to determine if $\eta_2 = x_2$ and $\eta_3 = x_3$ satisfy the FAO property.

For η_2, we have

$$y_{\bar{x}}^{(\alpha)} = \mathrm{a}(\eta_2 - y_{\bar{x}}) \quad \Rightarrow \quad \eta_2 = \phi_2\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = \frac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}} \tag{19}$$

In the same manner for η_3, we obtain

$$\eta_3 = -\frac{1}{y_{\bar{x}}}\left(\eta_2^{(\alpha)} + \mathrm{c}\,\eta_2 - \mathrm{b}\,y_{\bar{x}}\right) \tag{20}$$

Substituting (19) into (20) we have

$$\begin{aligned} \eta_3 &= \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}\right) \\ &= -\frac{1}{\mathrm{a}\, y_{\bar{x}}}\left[\mathcal{D}^{(2\alpha)} y_{\bar{x}} + (\mathrm{a} + c) y_{\bar{x}}^{(\alpha)} + \mathrm{a}(\mathrm{c} - \mathrm{b}) y_{\bar{x}}\right] \end{aligned} \tag{21}$$

Note that $\eta_3 = x_3$ looses algebraic observability property when $y_{\bar{x}} = x_1 = 0$, hence, only Equation (19) satisfies FAO with respect to the selected output $y_{\bar{x}} = x_1$.

Then from (19) we obtain the following master system in the form (9)-(10), for $\eta_2 = x_2$,

$$\begin{cases} \eta_2^{(\alpha)} = \mathrm{b}\bar{x}_1 - \mathrm{c}\eta_2 - \bar{x}_1\eta_3 \\ y_{\eta_2} = \eta_2 = \dfrac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}} \end{cases} \tag{22}$$

The next step is the design of the slave system that synchronizes with (22).

Using the equation (11), we obtain

$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}(\eta_2 - \hat{\eta}_2) \tag{23}$$

Replacing (19) into (23) leads to

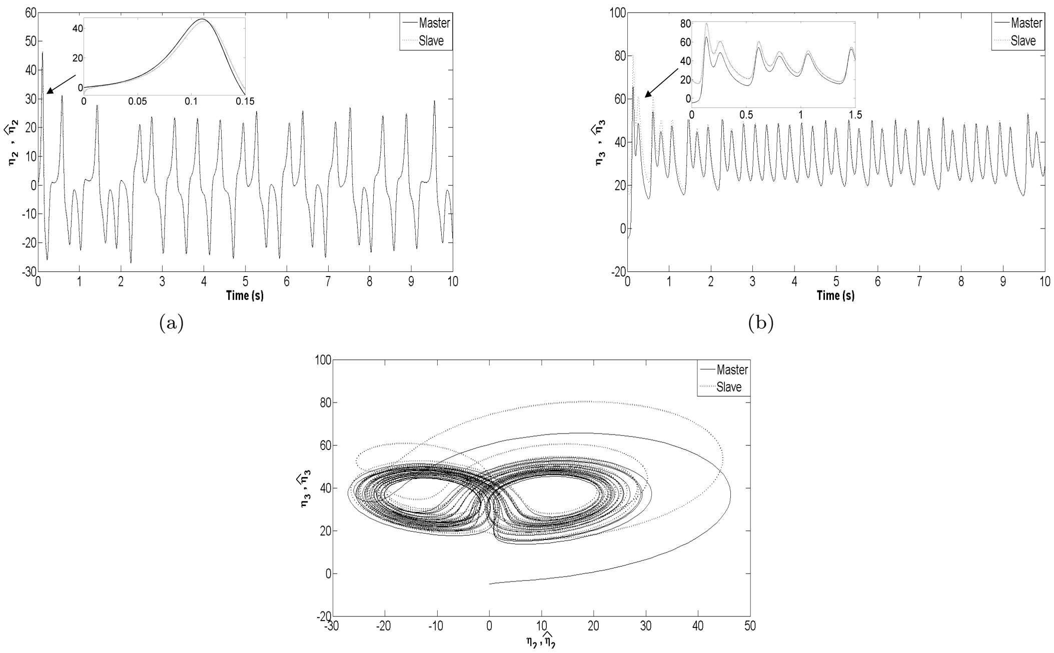
$$\hat{\eta}_2^{(\alpha)} = k_{\hat{\eta}_2}\left(\frac{1}{\mathrm{a}} y_{\bar{x}}^{(\alpha)} + y_{\bar{x}}\right) - k_{\hat{\eta}_2}\hat{\eta}_2 \tag{24}$$

Since $y_{\bar{x}}^{(\alpha)}$ is not available, slave (24) cannot be implemented. In order to overcome this problem let us consider the following auxiliary variable $\gamma_{\hat{\eta}_2}$:

$$\gamma_{\hat{\eta}_2} = -\frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} + \hat{\eta}_2 \tag{25}$$

then

$$\hat{\eta}_2 = \gamma_{\hat{\eta}_2} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} \tag{26}$$



(a) (b) (c)

Fig. 1. Synchronization of the fractional-order Lorenz system with $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, and initial conditions for master $\bar{x}_1(0) = 1$, $\eta_2(0) = 0$, $\eta_3(0) = -5$ and for slave $\hat{\eta}_2(0) = -5$, $\hat{\eta}_3(0) = 20$.

The fractional derivative of order α of (26) is

$$\hat{\eta}_2^{(\alpha)} = \gamma_{\hat{\eta}_2}^{(\alpha)} + \frac{k_{\hat{\eta}_2}}{a} y_{\bar{x}}^{(\alpha)} \tag{27}$$

Substituting (26) and (27) into (24), we obtain

$$\gamma_{\hat{\eta}_2}^{(\alpha)} = -k_{\hat{\eta}_2}\gamma_{\hat{\eta}_2} + \left(1 - \frac{k_{\hat{\eta}_2}}{a}\right) k_{\hat{\eta}_2} y_{\bar{x}} \quad , \qquad \gamma_{\hat{\eta}_2}(0) = \gamma_{\hat{\eta}_{2_0}} \tag{28}$$

Then, the slave system of $\eta_2 = x_2$ is given by

$$\begin{cases} \hat{\eta}_2 = \gamma_{\hat{\eta}_2} + \dfrac{k_{\hat{\eta}_2}}{a} y_{\bar{x}} \\ y_{\hat{\eta}_2} = \hat{\eta}_2 \end{cases} \tag{29}$$

Now, since the discontinuity of equation (21) for $y_{\bar{x}} = x_1 = 0$ we cannot construct a slave system for η_3 in the proposed form, to overcome this problem we propose the following slave system

$$\begin{cases} \hat{\eta}_3^{(\alpha)} = \bar{x}_1\hat{\eta}_2 - d\hat{\eta}_3 \\ y_{\hat{\eta}_3} = \hat{\eta}_3 \end{cases} \tag{30}$$

due to the equation (13) and $\eta_3^{(\alpha)}$ from (18) [4] we have

$$e_3^{(\alpha)} + de_3 = \bar{x}_1 e_2 \tag{31}$$

it should be noted that equation (31) has the same form of (15), thus in this case by using Proposition 1 we obtain

$$|e_3| \leq \frac{mM}{dk_{\hat{\eta}_2}} \tag{32}$$

where m is the bound of $\bar{x}_1$.

We present the corresponding simulation, we consider the following initial conditions to the master system $\bar{x}_1(0) = 1$, $\eta_2(0) = 0$, $\eta_3(0) = -5$, the initial conditions to the slave system $\hat{\eta}_2(0) = -5$, $\hat{\eta}_3(0) = 20$, the parameters are $a = 10$, $b = 28$, $c = -8$, $d = 8/3$, $\alpha = 0.8$, the initial conditions of the auxiliary functions are $\gamma_{\hat{\eta}_2}(0) = -20$ and finally, the gain parameter is $k_{\hat{\eta}_2} = 150$. The convergence of the estimates to the true signals is shown in Fig. 1.

5. CONCLUSIONS

We have designed a fractional observer for partially known nonlinear fractional order systems. It was introduced a new concept so-called Fractional Algebraic Observability (FAO) which is a fundamental issue to determine the unknown dynamics of fractional order nonlinear systems by means of the master-slave synchronization scheme, in particular we applied the results to chaotic fractional order systems, however this technique can be applied to others classes of systems which satisfy the properties of Proposition 1.

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Fault estimation using a polynomial observer: A real-time application

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Abstract: This work has been developed in the context of the faults diagnosis problem for nonlinear systems. The problem is viewed as the estimation of fault signals using extended state observers theory. A differential algebra approach is proposed to determine the observability and diagnosability of the system. A polynomial observer is used, to estimate the faults, for a multiple available outputs system. Other two schemes of nonlinear observers are used in the faults reconstruction process, only for comparison purposes. The first one is a reduced order observer while the second one is a sliding mode observer. The results of a real-time application are shown to illustrate these methodologies. The approaches were tested in the experimental setting Amira DTS200.

Keywords: Fault diagnosis, nonlinear systems, observability, observers, real time operating systems.

1. INTRODUCTION

The problem of fault diagnosis can be considered as the problem of identifying and reconstructing unknown inputs. As control systems have to deal with faults, fault diagnosis is a very important subject in control theory. A fault can be considered as a degradation of the equipment performance caused by the change in the physical characteristics of the process, the input process or the external conditions. Fault diagnosis is an important problem in process engineering. The fault diagnosis problem has been studied for more than three decades, many approaches and papers regarding this problem can be found, even some surveys as Isermann (1984); Alcorta-García and Frank (1997), and some reviews as Katipamula and Brambley (2005); Venkatasubramanian et al. (2003). For the approaches based on quantitative models, can be found approaches based upon differential geometric methods as Join et al. (2005), Massoumnia (1986). On the other hand, alternative approaches has been proposed based on an algebraic and differential framework as Martínez-Guerra et al. (2004); Cruz-Victoria et al. (2008); Fliess et al. (2004).

This paper deals with nonlinear systems diagnosis and the goal is to find malfunctions in the system, based on input/output measurements. The outputs are mainly measured signals obtained from sensors, their number is important in order to know if the system is diagnosable or not.

The fault diagnosis problem is considered as the problem of observing the fault signals. So a system is called diagnosable if the faults satisfy the so-called algebraic observability condition The main contribution of this article consists of the solution of the fault diagnosis problem by means of a polynomial observer for the case of multiples available measurements. In addition, another two schemes of observer are proposed in order to estimate the fault signals for comparison purposes, one of them is a reduced order observer and the other is a sliding mode observer based on partial change of coordinates.

This paper is organized as follows: In section 2 a definition related with Observability and Diagnosability is given. In section 3 the relation between the number of faults and the number of available measurements in terms of the differential transcendence degree concept is given. An asymptotic polynomial observer for the fault signals is presented in section 4. In section 5 the Amira DTS200 three-tank system model is described, and the diagnosability condition is evaluated. In section 6 the experimental results are shown for the fault and state estimation with the three different observers. Finally, in section 7 the paper is closed with some concluding remarks.

2. OBSERVABILITY AND DIAGNOSABILITY CONDITION

The observability and diagnosability notion of a system, linear or nonlinear in the differential algebra approach needs a basic definition. Further details can be found in Cruz-Victoria et al. (2008), Martínez-Guerra et al. (2004) and Aguilar-Lopez, Mata-Machuca, and Martinez-Guerra (2011).

Definition 1. Algebraic observability and diagnosability condition. A fault $f \in \mathcal{G}$ is said to be diagnosable if it is possible to estimate the fault from the available measurements of the system, i.e., f is diagnosable if it is algebraically observable over the differential field $\mathbb{R}\langle u, y\rangle$.

Let us consider the class of nonlinear systems with faults described by the following equation

$$\left\{\begin{array}{l} \dot{x}(t) = g(x,u,f) \\ y(t) = h(x,u) \end{array}\right. \tag{1}$$

Where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is a state vector, $u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m$ is a know input vector, $f = (f_1, f_2, ..., f_\mu) \in \mathbb{R}^\mu$ is a unknown input vector, $y(t) \in \mathbb{R}^p$ is the output vector (available measurements), g and h are assumed to be analytical vector functions.

Remark 1. The diagnosability condition is independent of the observability of a system.

Example 1. Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 + e^{x_1} + u \\ \dot{x}_2 = x_3^3 + 2u + f \\ \dot{x}_3 = x_2 - x_3 \\ y_3 = x_3 \end{cases} \tag{2}$$

Since f satisfies the differential algebraic equation $f + y_3^3 - \dot{y}_3 - \ddot{y}_3 + 2u = 0$, then the system (2) is diagnosable and the fault can be reconstruct knowing u, y and their time derivatives. However the state x_1 is not algebraically observable.

3. NUMBER OF FAULTS AND NUMBER OF AVAILABLE MEASUREMENTS

Let consider the system (1). The fault vector f is unknown and it can be assimilated as a state with uncertain dynamics. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector. The new extended system is given by

$$\begin{cases} \dot{x}(t) = g(x, u, f) \\ \dot{f}_j = \Omega_j(x, u, f) \quad 1 \leq j \leq \mu \\ y(t) = h(x, u) \end{cases} \tag{3}$$

The following results from the theory of differential algebraic field extensions are an useful tool to determine whether a fault can be reconstructed from the know inputs and available measurements.

Definition 2. A maximal family of elements of the field $\mathcal{K}$ that are $\mathcal{K}$-differentially independent is denominated differential transcendence basis of extension $\mathcal{L}/\mathcal{K}$ and the cardinality of this base is called the differential transcendence degree denoted by: $diff\ tr\ d^o\ \mathcal{L}/\mathcal{K}$.

Property 1. Let $\mathcal{K}$,$\mathcal{L}$,$\mathcal{M}$ be differential fields such that $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$. Then

$$diff\ tr\ d^o(\mathcal{M}/\mathcal{K}) = diff\ tr\ d^o(\mathcal{M}/\mathcal{L}) + diff\ tr\ d^o(\mathcal{L}/\mathcal{K}) \tag{4}$$

This property is an important tool to proof the theorems 1 and 2, see Cruz-Victoria et al. (2008).

Theorem 1. Assume that the system (1) is diagnosable, then the number of faults is less or equal to the number of available measurements (outputs), i.e.

$$\mu \leq p \tag{5}$$

Theorem 2. The system (1) is diagnosable if and only if

$$diff\ tr\ d^o\mathcal{K}\langle u, y\rangle / \mathcal{K}\langle u\rangle = \mu \tag{6}$$

4. OBSERVER SYNTHESIS

4.1 Polynomial Observer

The system (3) can be expressed in the following form

$$\begin{cases} \dot{x}(t) = Ax + \Psi(x, u, f) \\ \dot{f}_k = \Omega_k(x, u, f), \quad 1 \leq k \leq \mu \\ y(t) = Cx \end{cases} \tag{7}$$

where $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\mu) \in \mathbb{R}^\mu$ is a unknown bounded function i.e., $\|\Omega_j\| \leq N < \infty$, $N \in \mathbb{R}^+$ and $\Psi(x, \bar{u})$ is a nonlinear function that satisfies the Lipschitz condition with $\bar{u} = (u, f)$, i.e.,

$$\|\Psi(x, \bar{u}) - \Psi(\hat{x}, \bar{u})\| \leq L\|x - \hat{x}\| \tag{8}$$

and $\bar{u}$ uniformly bounded.

The following system is an observer for the system (7)

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x} + \Psi(x, \bar{u}) + \\ \qquad + \sum_{i=1}^{p}\sum_{j=1}^{m} K_{ij}(y_i - C_i\hat{x})^{2j-1} \\ \dot{\hat{f}}_k(t) = \sum_{l=1}^{m} \bar{K}_{kl}\left(f_k - \hat{f}_k\right)^{2l-1} \\ y(t) = C_i x \end{cases} \tag{9}$$

where, $\hat{x} \in \mathbb{R}^n$, $[K_{ij}]_{\substack{1\leq i\leq p \\ 1\leq j\leq m}}$, $\left[\bar{K}_{kl}\right]_{\substack{1\leq k\leq \mu \\ 1\leq l\leq m}}$, K_{ij}, $\bar{K}_{kl} > 0$, $\hat{x}_0 = \hat{x}(t_0)$ and $\hat{f}_0 = \hat{f}(t_0)$ are arbitrarily initial conditions.

4.2 Main Assumptions.

For this observer the following assumptions are considered:

A1: $f(t)$ is algebraically observable on $\mathbb{R}\langle u, y\rangle$

A2: The gains $[K_{i1}]_{1\leq i\leq p}$ can be chosen such that algebraic Riccati equation has a symmetric and positive definite solution P for some $\epsilon > 0$.

$$\left(A - \sum_{i=1}^{p} K_{i1}C_i\right)^T P + P\left(A - \sum_{i=1}^{p} K_{i1}C_i\right) + L^2PP + I + \varepsilon I = 0$$

A3: $[K_{ij}]_{\substack{1\leq i\leq p \\ 2\leq j\leq m}}$ is chosen such that

$$\lambda_{\min}\left((PK_{ij}C_i)^T + (PK_{ij}C_i)\right) \geq 0$$

For analyzing the estimation error, we define $e = [e_x, e_k]^T$, where $e_x = x - \hat{x}$ and $e_k = f_k - \hat{f}_k$. From systems (7) and (9), the dynamics of estimation error is given by

$$\begin{cases} \dot{e}_x = & \left(A - \sum_{i=1}^{p} K_{i1}C_i\right)e_x - \\ & - \sum_{i=1}^{p}\sum_{j=2}^{m} K_{ij}(C_ie_x)^{2j-1} + \\ & + [\Psi(x, \bar{u}) - \Psi(\hat{x}, \bar{u})] \\ \dot{e}_k = & \Omega_k - \bar{K}_{k1}e_k - \sum_{j=2}^{m} \bar{K}_{kj}(e_k)^{2l-1} \end{cases} \tag{10}$$

4.3 Lyapunov-Like Analysis.

The following theorem proves the observer convergence.

Theorem 3. For the nonlinear system (7), suppose that $x(t)$ exists for all $t \geq 0$, the nonlinear function $\Psi(x, u, f)$ satisfies de Lipschitz condition (8), $x(t)$ and $f(t)$ are algebraically observable. If there exists a matrix P positive definite and observer gains K_{ij} and $\bar{K}_{kl} > 0$ such that the system (9) is an observer of the system (7), then the estimator error converges asymptotically to zero.

Proof 1. Consider the following Lyapunov function candidate

$$\begin{gathered} V = V_1 + V_2 \\ V_1 = e_x^T P e_x \quad V_2 = \frac{1}{2}e_k^2 \end{gathered} \tag{11}$$

where P satisfies the assumption A2.

The proof of the theorem is given in two parts, as follows:

i) The time derivate of V_1 along the trajectories of (11) is

$$\begin{aligned}\dot{V}_1 &= \dot{e}_x^T P e_x + e_x^T P \dot{e}_x \\ &= e_x^T \left(\left(A - \sum_{i=1}^{p} K_{i1} C_i \right)^T P + P \left(A - \sum_{i=1}^{p} K_{i1} C_i \right) \right) e_x + \\ &\quad + 2 e_x^T P \left[\Psi(x, \bar{u}) - \Psi(\hat{x}, \bar{u}) \right] - \\ &\quad - 2 \sum_{i=1}^{p} \sum_{j=2}^{m} K_{ij} (C_i e_x)^{2j-2} e_x^T \left((P K_{i,j} C_i)^T + (P K_{ij} C_i) \right) e_x\end{aligned}$$

The next inequality (12), which is based on the Lipschitz condition (8)

$$2 e_x^T P \left[\Psi(x, \bar{u}) - \Psi(\hat{x}, \bar{u}) \right] \le L^2 e_x^T P P e_x + e_x^T e_x \tag{12}$$

Now, applying the Rayleigh's inequality and consider the assumption A3, we obtain

$$-e_x^T P K_{ij} C_i e_x \le -\lambda_{\min} \left(P K_{ij} C_i + (P K_{ij} C_i)^T \right) \|e_x\|^2 \tag{13}$$

for $2 \le i \le m$.

Hence, combining the inequalities (12) and (13)

$$\begin{aligned}\dot{V}_1 &\le e_x^T \left[\left(A - \sum_{i=1}^{p} K_{i1} C_i \right)^T P + P \left(A - \sum_{i=1}^{p} K_{i1} C_i \right) + \right. \\ &\quad \left. + L^2 P P + I \right] e_x - \\ &\quad - 2 \sum_{i=1}^{p} \sum_{j=2}^{m} K_{ij} (C_i e_x)^{2j-2} \lambda_{\min} \left(P K_{ij} C_i + (P K_{ij} C_i)^T \right) \|e_x\|^2 \\ &\le e_x^T \left[\left(A - \sum_{i=1}^{p} K_{i1} C_i \right)^T P + P \left(A - \sum_{i=1}^{p} K_{i1} C_i \right) + \right. \\ &\quad \left. + L^2 P P + I \right] e_x \\ &= -\epsilon \|e_x\|^2\end{aligned}$$

(ii) On the other hand, considering the second term of the Lyapunov function candidate, we have

$$\begin{aligned}\dot{V}_2 &= e_k \dot{e}_k \\ &= e_k \left(\Omega_k - \bar{K}_{k1} e_k - \sum_{l=2}^{m} \bar{K}_{kl} (e_k)^{2l-1} \right) \\ &= e_k \Omega_k - \bar{K}_{k1} e_k^2 - \sum_{l=2}^{m} \bar{K}_{kl} (e_k)^{2l} \\ &\le |e_k| |\Omega_k| - \bar{K}_{k1} e_k^2 \\ &\le |e_k| N - \bar{K}_{k1} |e_k|^2 \\ &= - \left[\bar{K}_{k1} |e_k| - N \right] |e_k|\end{aligned}$$

$\dot{V}_2$ is negative inside the set $\left\{ |e_k| > N/\bar{K}_{k1} \right\}$, i.e., exists $\bar{\epsilon} > 0$ such that $\bar{K}_{k1} |e_k| - N = \bar{\epsilon} > 0$.

Now we proves that $|e_k|$ is upper bounded. Let α, β upper bounds of $V_2(e_k)$. With $\beta > N^2/2\bar{K}_{k1}^2$, the solution in the set $\{V_2(e_k) \le \beta\}$ will remain there for all $t \ge 0$, due to that $\dot{V}_2$ is negative in $V_2 = \beta$. Hence, the solution of $\dot{e}_k$ are uniformly bounded. Furthermore, if $N^2/2\bar{K}_{l1}^2 < \alpha < \beta$, then $\dot{V}_2$ will be negative in the set$\{\alpha \le V_2 \le \beta\}$. In this set V_2 will decrease monotonously until the solution of the set $\{V_2 \le \alpha\}$. According to the solution is ultimately bounded with bound $|e_k| \le \sqrt{2\alpha}$. For example, if we defined α and β as $\alpha = N^2/2\bar{K}_{k1}^2$ and $\beta = 2N^2/2\bar{K}_{k1}^2$. Then, the ultimate bound is

$$|e_k| \le \frac{N}{\bar{K}_{k1}}$$

Hence

$$\dot{V}_2 \le -\bar{\epsilon} |e_k|$$

Finally, from (i) and (ii), we conclude that

$$\dot{V} \le -\epsilon \|e_x\|^2 - \bar{\epsilon} |e_k| < 0$$

and the theorem is proven. ∎

5. APPLICATION OF THE THREE-TANK SYSTEM

5.1 Description of the three-tank system

A complete description of the three-tank system can be found on Amira DTS (1996), the system consists of three cylinders (T_1, T_2 and T_3) with transversal constant section A, which are connected in series one to another one by means cylindrical tubes with transversal section S (Figure 1). The corresponding nominal model is given by the following system

$$\begin{cases} \dfrac{dh_1}{dt} = \dfrac{1}{A} (q_1 - q_{13} + u_1) \\ \dfrac{dh_2}{dt} = \dfrac{1}{A} (q_2 + q_{32} - q_{20} + u_2) \\ \dfrac{dh_3}{dt} = \dfrac{1}{A} (q_{13} - q_{32}) \end{cases} \tag{14}$$

According to the generalized rule of Torricelli, this is valid for laminar flow

$$\begin{aligned} q_{13} &= a_1 S \text{sign}(h_1 - h_3) \sqrt{2g |h_1 - h_3|} \\ q_{32} &= a_3 S \text{sign}(h_3 - h_2) \sqrt{2g |h_3 - h_2|} \\ q_{20} &= a_2 S \sqrt{2 g h_2} \end{aligned} \tag{15}$$

A more detailed description of parameters of the system (14) is given in table 1.

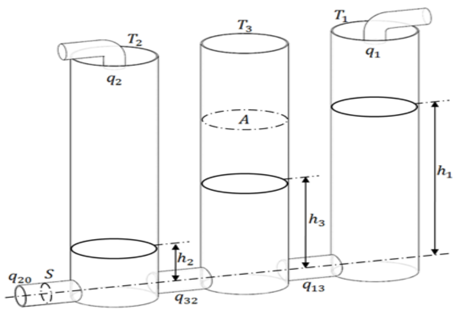

Fig. 1. Diagram of the Three-Tank System

Table 1. Variables and parameters

Symbols	Description
h_i	Level of the liquid of i-th tank (m)
A	Transversal section of each tank (m^2)
S	Transversal section of the interconnection tubes (m^2)
q_{13}	Water flow of tank 1 to tank 3 (m^3/s)
q_{32}	Water flow of tank 3 to tank 2 (m^3/s)
q_{20}	Output flow (m^3/s)
q_1	Input flow to tank 1 (m^3/s)
q_2	Input flow to tank 2 (m^3/s)
a_i	Output flow coefficients

In what follows, it is considered that the state vector is $x = [x_1\ x_2\ x_3]^T = [h_1\ h_2\ h_3]^T$ and the input vector is $u = [u_1\ u_2]^T = [q_1\ q_2]^T$.

The system (14) has four regions of operations in which the corresponding model is differentiable, in this paper we only consider the operation region $x_1 \geq x_3 \geq x_2$.

5.2 Considered faults and measurements of the system

The nominal model (14) is transformed into the following, where the additive faults f_1 and f_2 ($\mu = 2$) are considered in the actuators that control the input flows $u_1 = q_1$, $u_2 = q_2$.

$$\begin{cases} \dot{x}_1 = \dfrac{1}{A}\left(u_1 - q_{13} + u_1 + f_1\right) \\ \dot{x}_2 = \dfrac{1}{A}\left(u_2 + q_{32} - q_{20} + u_2 + f_2\right) \\ \dot{x}_3 = \dfrac{1}{A}\left(q_{13} - q_{32}\right) \end{cases} \tag{16}$$

5.3 Diagnosability analysis

For the theorem 1, for the diagnosability is required that the number of faults be less or equal that the number or available measurements. The system (16) consists of two faults. In order to determine observability and the diagnosability of the system (16), we have to evaluate the algebraic observability condition from definition 5 for f_1 and f_2. and the unknown state x_1. We only consider the region of operation $x_1 \geq x_3 \geq x_2$.

For this case we only consider two outputs $y_2 = x_2$, and $y_3 = x_3$. In this case from (16) we have

$$f_1 = A\dot{x}_1 + a_1 S\sqrt{2g\left(x_1 - y_3\right)} - u_1 \tag{17}$$

$$f_2 = A\dot{y}_2 - a_3 S\sqrt{2g\left(y_3 - y_2\right)} + a_2 S\sqrt{2gy_2} - u_2 \tag{18}$$

$$x_1 = y_3 + \frac{1}{2ga_1^2 S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g\left(y_3 - y_2\right)}\right)^2 \tag{19}$$

The algebraic observability condition for x_1 is deduced directly from (19), which is an equation with coefficients in $\mathbb{R}\langle u, y\rangle$.

Replacing (19) into (17) and (18), we have

$$\begin{aligned} f_1 = {}& A\dot{x}_1 + \\ & +a_1 S\sqrt{\left(2g-1\right)y_3 + \tfrac{1}{a_1^2 S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g\left(y_3 - y_2\right)}\right)^2} \\ & -u_1 \\ f_2 = {}& A\dot{y}_2 - a_3 S\sqrt{2g\left(y_3 - y_2\right)} + a_2 S\sqrt{2gy_2} - u_2 \end{aligned} \tag{20}$$

Then, combining (19) and (20) we conclude that the system (16) with the available outputs is diagnosable and observable.

5.4 Fault reconstruction

We consider the outputs $y_2 = x_2$ and $y_3 = x_3$, then for this case we need to estimate the time derivatives from outputs y_2 and y_3. In this section a methodology appears to reconstruct first $r - 1$ time derivatives from the available output y.

Reduced Order Observer We propose a reduced order observer as in Rincón-Pasaye et al (2008). Let us consider the following time derivative to be estimated

$$\eta = \dot{y} \tag{21}$$

we propose the observer structure

$$\dot{\hat{\eta}} = K\left(\eta - \hat{\eta}\right) \tag{22}$$

now, applying the next change of coordinates

$$\hat{\eta} = \gamma + Ky \tag{23}$$

and from (21) and (22) we can get $\dot{\gamma} = -K\hat{\eta}$, then again from (23)

$$\dot{\gamma} = -K\gamma - K^2 y \tag{24}$$

Hence (24) and (23) constitute an asymptotic estimator of η.

Sliding Mode Observer We propose a sliding mode observer as in Martínez-Guerra et al. (2004), then we introduce the following change of variables: $\eta_1 = y$, $\eta_2 = \dot{\eta}_1$, then we obtain the following observer

$$\begin{aligned} \dot{\hat{\eta}}_1 &= \hat{\eta}_2 + m_1 \mathrm{sign}\left(y - \hat{y}\right) \\ \dot{\hat{\eta}}_2 &= m_2 \mathrm{sign}\left(y - \hat{y}\right) \end{aligned} \tag{25}$$

which can be used to estimate η_2 from the output y.

Polynomial Observer We introduce the following change of variables

$$\eta_1 = y, \eta_2 = \dot{y}, \ldots, \eta_r = y^{(r-1)} \tag{26}$$

Then the system (1) can be expressed as follows

$$\begin{aligned} \dot{\eta} &= A\eta + \bar{\phi}\left(\eta, u\right) \\ y &= \eta_1 \end{aligned} \tag{27}$$

So, by the theorem 3, the observer has the following structure

$$\dot{\hat{\eta}} = A\hat{\eta} + \bar{\phi}\left(\hat{\eta}, u\right) + \sum_{i=1}^{p}\sum_{j=1}^{m} K_{ij}\left(y - C_i\hat{x}\right)^{2j-1} \tag{28}$$

Then, the polynomial observer (28) is used to estimate the variables $\eta_1 = y$, $\eta_2 = \dot{y}$, ... , $\eta_r = y^{(r-1)}$ by means of the available output y.

6. FAULT ESTIMATION RESULTS

For the uncertain parameters a_i from the system (14), we consider the following identification results

$$a_1 = 0.4663,\ a_2 = 0.7654,\ a_3 = 0.4616$$

For all the experiments reported in this section the input flows were maintained constant as $u_1 = q_1 = 0.00002\frac{m^3}{s}$ and $u_2 = q_2 = 0.000015\frac{m^3}{s}$, for 3000 seconds. The two faults were artificially generated through the following equations:

$$f_1 = 5 \times 10^{-6}\left[1 + \sin\left(0.2te^{-0.01t}\right)\right]\mathcal{U}\left(t - 220\right)$$
$$f_2 = 5 \times 10^{-6}\left[1 + \sin\left(0.05te^{-0.001t}\right)\right]\mathcal{U}\left(t - 330\right)$$

where $\mathcal{U}\left(t\right)$ is the unit step function.

The three proposed schemes for fault diagnosis were evaluated for the case when $x_2 = y_2$ and $x_3 = y_3$ are available, for this reason an estimation for the unknown state x_1 was necessary to be obtained. In the Figure 2 we show the estimation of unknown state x_1 and the fault estimation results and as we can observe, the reconstruction results of signals are good. The gain values for the fault observers were $K_1 = 0.045$ and $K_2 = 1.25$ In the same way, in the Figure 3 we show the state and fault estimation results and also the reconstruction results of signals are good. The gain values for the fault observers were $m_{11} = 0.0195$, $m_{12} = 0.00055$ and $m_{21} = 0.0165$, $m_{22} = 0.0005$. Finally in the Figure 4 the corresponding results achieved wiht the polynomial observer are shown. The gain values for the fault observers were $\bar{K}_{11} = 0.045$, $\bar{K}_{12} = 23$, $\bar{K}_{13} = 19$ and $\bar{K}_{21} = 0.196$, $\bar{K}_{22} = 25$, $\bar{K}_{23} = 15$. It is worth to mention that with this observer we estimated the full state vector and the both fault signals. As we can observe, this scheme also provides good results.

The performance of the proposed observer is evaluated using the following cost function, with $\varepsilon' = 0.0001$, in this case is defined as

$$J_t = \frac{1}{t + \varepsilon'}\int_0^t \left\|e_k\left(\tau\right)\right\|^2 d\tau \qquad (29)$$

Figure 5 shows the performance of three observers proposed to estimate the fault f_1. We observe that the polynomial observer converges much faster than the other two. In the same way, in Figure 6 we observe that the performance of the three observers for the estimation of the fault f_2 is similar.

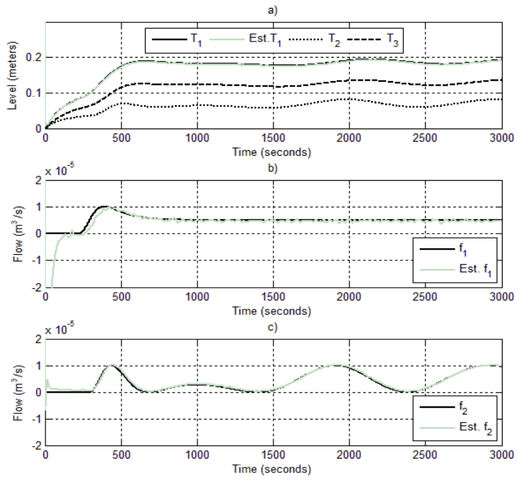

Fig. 2. Reduce order observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

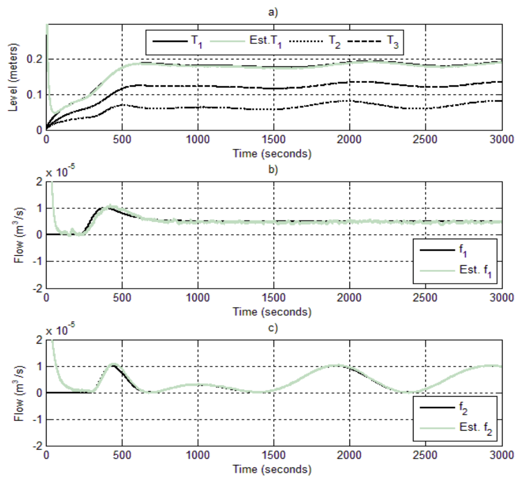

Fig. 3. Sliding mode observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

7. CONCLUDING REMARKS

In this work the fault diagnosis problem for a class of nonlinear systems using the differential algebraic approach was used. The method consists of considering to the fault as an augmented state of the system. To achieve the reconstruction of the unknown states of the extended system, we proposed a polynomial observer and another two schemes of nonlinear observers for comparison purposes. The approach was tested in a real-time experimental setting Amira DTS-200. From the fault reconstruction result we can see that for the three different observers a similar performance and estimation results. Finally the proposed polynomial observer presents a good performance.

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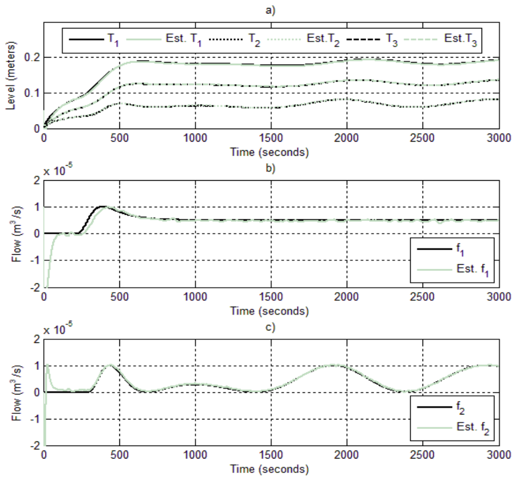

Fig. 4. Polynomial observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

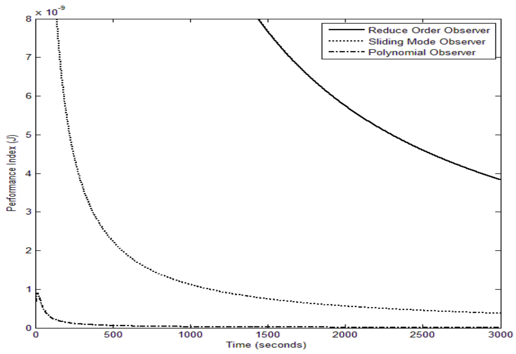

Fig. 5. Performance evaluation of observers for the estimation error of fault f_1

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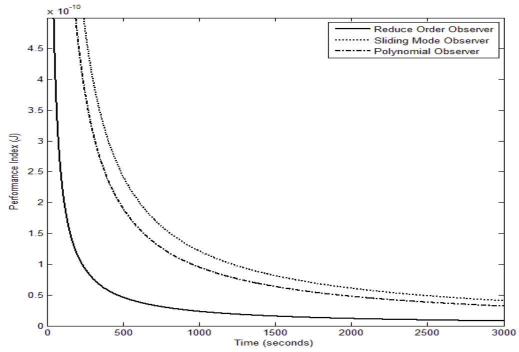

Fig. 6. Performance evaluation of observers for the estimation error of fault f_2

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A New Observer for Nonlinear Fractional Order Systems

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and Guillermo Fernández-Anaya

***Abstract*— In this work an observer structure for a certain class of nonlinear fractional order systems is proposed. For solving this task we introduce a Fractional Algebraic Observability (FAO) property which is used as a main tool in the design of the observer system. We apply our proposals in the master-slave synchronization problem, where the coupling signal is viewed as output and the slave system is regarded as observer (the slave is requested to recover the unknown state trajectories of the master). Finally, as numerical example we consider a fractional order Rössler hyperchaotic system and by means of some simulations we show the effectiveness of the suggested approach.**

I. INTRODUCTION

Fractional calculus is as old as conventional calculus, but is not as popular in science and engineering as conventional calculus. In the last three centuries this subject was studied only in mathematics, but in recent years it has been used in many fields of engineering and science [1]. "It might be that this mathematical tool help us modelling the reality in a better way and also, might be, that this is the calculus of the XXI century" [2].

Among the publications dedicated to fractional order systems, some subjects have been studied, e.g., linear systems [3], chaotic dynamics and its synchronization [4]–[7], stability [8], [9], delayed systems [10], systems identification [2], control systems [11], optimal control [12], quantitative finances [13], quantic evolution of complex systems [14], digital image processing, variational principles and its applications, Euler-Lagrange equations, applications in finances and economy, bioengineering applications, fractional Fourier transform, sliding modes, robotics, among others [15].

The synchronization problem is an interesting topic in fractional chaotic systems [16]. The synchronization of integer order chaotic systems has been extensively investigated since its introduction by Pecora and Carroll [17]. On the other hand, [18] is the first work concerning on synchronization of fractional systems, the authors showed by means of a control law that fractional order chaotic systems can be synchronized by using the similar scheme as that of their integer counterparts.

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Some techniques related to chaos synchronization in fractional systems have been proposed. For instance, we mention the works [6], [19], [20] in which the authors propose the employment of feedback controllers, which allows to achieve the synchronization between two identical fractional-order chaotic systems, the theoretical analysis is derived by utilizing the stability criterion of linear fractional systems; in [21] is studied the synchronization of fractional order chaotic systems with unidirectional linear error feedback coupling; in [22] is presented a classical Luenberger observer design for the synchronization of fractional-order chaotic systems, i.e., the observer structure needs a copy of the system and a linear output error feedback, the application is restricted to scalar coupling signals; in [23], [24] are given sufficient conditions for the synchronization between two identical fractional systems by using the Laplace transform theory.

The main contribution in this work is to show a novel technique for the synchronization problem in nonlinear fractional-order systems via observer design. Here arises a basic practical question: would it be possible to reconstruct the unknown signals? We give an answer to this question by introducing a basic definition (similar to the differential and algebraic approach used in nonlinear integer order systems [25]) related with the estimation (reconstruction) of the unknown variables, so-called Fractional Algebraic Observability (FAO) property[1]. As far as we know in the literature this class of observer structure has not been used in fractional order systems.

The rest of this paper is organized as follows. In Section II is given a brief note about fractional derivatives and Mittag-Leffler type function. Section III presents the problem statement and its solution, based on FAO and the master-slave synchronization scheme. In Section IV we apply the methodology presented in Section III to the fractional order Rössler hyperchaotic system, also some numerical results are shown. The intention of choosing this system is to clarify the proposed methodology and to highlight the simplicity and flexibility of the suggested approach. Finally, we conclude with some remarks in Section V.

II. ON FRACTIONAL DERIVATIVES

There are several definitions of a fractional derivative of order α [11], [26], [27], we will use the Caputo fractional operator in the definition of fractional order systems, because

[1]An observable system in this sense can be regarded as a system in which the unknown variables can be expressed in terms of the output signal and a finite number of its fractional derivatives.

the meaning of the initial conditions for systems described using this operator is the same as for integer order systems.

Definition 1 (Caputo Fractional derivative): The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function x is defined as: see [11]

$$x^{(\alpha)} = {}_{t_0}D_t^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{d^m x(\tau)}{d\tau^m} (t-\tau)^{m-\alpha-1} \, d\tau, \tag{1}$$

where: $m-1 \leq \alpha < m$, $\frac{d^m x(\tau)}{d\tau^m}$ is the m-th derivative of x in the usual sense, $m \in \mathbb{N}$ and Γ is the gamma function[2].□

Now we define the following notation

$$\mathcal{D}^{(r\alpha)} x(t) = \underbrace{{}_{t_0}D_t^{\alpha} \, {}_{t_0}D_t^{\alpha} \ldots {}_{t_0}D_t^{\alpha} \, {}_{t_0}D_t^{\alpha}}_{\text{r-times}} \, x(t) \tag{2}$$

i. e., it is the Caputo fractional derivative of order α applied $r \in \mathbb{N}$ times sequentially, with $\mathcal{D}^{(0)}x(t) = x(t)$, we can note that if $r = 1$ then $\mathcal{D}^{(\alpha)}x(t) = x^{(\alpha)}$.

A. Mittag-Leffler type function

The Mittag-Leffler function with two parameters is defined as [28]:

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}, \quad z, \beta \in \mathbb{C}, \ \mathbb{R}e(\alpha) > 0 \tag{3}$$

this function is used to solve fractional differential equations as the exponential function in integer order systems. In the particular case when $\alpha = \beta = 1$, we have that $E_{1,1}(z) = e^z$. Now if we have particular values of α, the function (3) has asymptotic behavior at infinity.

Theorem 1 ([11]): If $\alpha \in (0, 2)$, β is an arbitrary complex number and μ is an arbitrary real number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\} \tag{4}$$

then for an arbitrary integer $\kappa \geq 1$ the following expansion holds:

$$E_{\alpha,\beta}(z) = -\sum_{i=1}^{\kappa} \frac{1}{\Gamma(\beta - \alpha i) z^i} + O\left(\frac{1}{|z|^{\kappa+1}}\right) \tag{5}$$

with $|z| \to \infty$, $\mu \leq |\arg(z)| \leq \pi$. ■

The Mittag-Leffler function has the following properties:

Property 1 [11].

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-k\tau^{\alpha}) \, d\tau = t^{\beta} E_{\alpha,\beta+1}(-kt^{\alpha}) \,,$$

with $\beta > 0$.

Property 2 [29]. $E_{\alpha,\beta}(-x)$ is completely monotonic, i. e., $(-1)^n E_{\alpha,\beta}^{(n)}(-x) \geq 0$ for $0 < \alpha \leq 1$ and $\beta \geq \alpha$, for all $x \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$.

We will use these facts in the following problem.

[2]To simplify the notation we omitted the time dependence in $x^{(\alpha)}$, in what follows we take $t_0 = 0$

III. Problem Statement and Main Result

We take the initial condition problem for an autonomous fractional order nonlinear system, with $0 < \alpha < 1$,

$$\begin{aligned} x^{(\alpha)} &= f(x), \ \ x(0) = x_0 \\ y &= h(\bar{x}) \end{aligned} \tag{6}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is a Lipschitz continuous function[3], with $x_0 \in \Omega \subset \mathbb{R}^n$, in this case y denotes the output of the system (the measure that we can obtain), $\bar{x} \in \mathbb{R}^p$ represents the states that we can observe (known states), $h : \mathbb{R}^p \to \mathbb{R}^q$ is a continuous function and $1 \leq p < n$.

Consider the system given by (6), we will separate in two dynamical systems with states $\bar{x} \in \mathbb{R}^p$ and $\eta \in \mathbb{R}^{n-p}$ respectively with $x^T = (\bar{x}^T, \eta^T)$, the first system will describe the known states and the second represents unknown states, then the system (6) can be written as:

$$\begin{aligned} \bar{x}^{(\alpha)} &= \bar{f}(\bar{x}, \eta) \\ \eta^{(\alpha)} &= \Delta(\bar{x}, \eta) \\ y_{\bar{x}} &= y = h(\bar{x}) \end{aligned} \tag{7}$$

where $f^T(x) = \left(\bar{f}^{\,T}(\bar{x}, \eta), \Delta^T(\bar{x}, \eta)\right)$, $\bar{f} \in \mathbb{R}^p$ and $\Delta \in \mathbb{R}^{n-p}$. Now the problem is: How can we estimate the η's states? this question arises because if we know the η's states we can use these signals to generate measuring depending on them. In order to solve this observation problem let us introduce the following observability property.

Definition 2 (FAO): A state variable $\eta_i \in \mathbb{R}$ satisfies the Fractional Algebraic Observability (FAO) if it is a function of the first $r \in \mathbb{N}$ sequential derivatives (in the sense of the equation (2)) of the available output $y_{\bar{x}}$, i.e.,

$$\eta_i = \phi_i\left(y_{\bar{x}}, \, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots, \, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{8}$$

where $\phi_i : \mathbb{R}^{(r+1)q} \to \mathbb{R}$. □

If we assume that the components of unknown state vector η are FAO, then we can describe our problem in terms of the master-slave synchronization scheme, which is defined in the following way.

Let us consider the master system:

$$\eta_i^{(\alpha)} = \Delta_i(\bar{x}, \eta) \tag{9}$$

$$y_{\eta_i} = \phi_i\left(y_{\bar{x}}, \, y_{\bar{x}}^{(\alpha)}, \mathcal{D}^{(2\alpha)} y_{\bar{x}}, \ldots, \, \mathcal{D}^{(r\alpha)} y_{\bar{x}}\right) \tag{10}$$

for $i \in \{p+1, \ldots, n\}$, where η_i is a component of the state vector η and y_{η_i} denotes the output of the i-th master system.

Now let us propose a fractional dynamical system with the same order α, which will be the slave system (observer):

$$\hat{\eta}_i^{(\alpha)} = k_{\hat{\eta}_i}(y_{\eta_i} - \hat{\eta}_i), \tag{11}$$

$$y_{\hat{\eta}_i} = \hat{\eta}_i, \tag{12}$$

for $i \in \{p+1, \ldots, n\}$, where $\hat{\eta}_i$ is the state, and $y_{\hat{\eta}_i}$ denotes the output of the slave system and $k_{\hat{\eta}_i}$ is a positive constant.

In the master-slave synchronization scheme, the output of the master system (10) describes the target signal, while (12)

[3]This assures the unique solution [28]

represents the response signal. Therefore the synchronization problem can be established as:
Given the master system (9) and our slave system (11), it should be determined some conditions, such that the output of the slave system (12) synchronizes with the output of the master system (10).

Let us define the synchronization error as:

$$e_i = y_{\eta_i} - y_{\hat{\eta}_i} = \eta_i - \hat{\eta}_i. \tag{13}$$

Now we establish a convergence analysis of the synchronization error.

Proposition 1: Let the system (6) which can be expressed in the form (7), where the following conditions are fulfilled:

H1: η_i satisfies the FAO property for $i \in \{p+1, \dots, n\}$.
H2: Δ_i is bounded, i.e., $\exists M \in \mathbb{R}^+$ such that $\|\Delta(x)\| \leq M$, $\forall x \in \Omega$.
H3: $k_{\hat{\eta}_i} \in \mathbb{R}^+$.

Then, the synchronization of the master output (10) with the slave output (12) is achieved, for global initial condition of the states.

Proof. From **H1** we can write equations (9)-(13). Taking the fractional derivative of the equation (13), we have

$$e_i^{(\alpha)} = \eta_i^{(\alpha)} - \hat{\eta}_i^{(\alpha)} \tag{14}$$

Substituting the fractional dynamics (9) and (11) into (14), we obtain

$$e_i^{(\alpha)} + k_{\hat{\eta}_i} e_i = \Delta_i(x) \tag{15}$$

There exists a unique solution for the system (15), due to $\Delta_i(x(t)) - k_{\hat{\eta}_i} e_i(t)$ is a Lipschitz continuous function on e.[4]

Solving the above equation [28], we have

$$\begin{aligned} e_i(t) = \;& e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(k_{\hat{\eta}_i}(t-\tau)^\alpha)\Delta_i(x(\tau))\mathrm{d}\tau \end{aligned} \tag{16}$$

where $e_i(0) = e_{i0}$.

Using Triangle and Cauchy-Schwarz inequalities and **H2**

$$\begin{aligned} |e_i(t)| \leq \;& |e_{i0} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha)| \\ & + M \int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)|\mathrm{d}\tau \end{aligned}$$

The functions $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)$ and $E_\alpha(-k_{\hat{\eta}_i} t^\alpha)$ are not negative due to **Property 2** of Mittag-Leffler function and **H3**

$$\begin{aligned} |e_i(t)| \leq \;& |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + M \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-k_{\hat{\eta}_i}(t-\tau)^\alpha)\mathrm{d}\tau \end{aligned}$$

Using **Property 1** of Mittag-Leffler function

$$|e_i(t)| \leq \; |e_{i0}| E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) + M t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha)$$

[4]Equation (15) is non-autonomous, but the Lipschitz condition assures a unique solution [28].

If $t \to \infty$, we use the equation (5) with $\mu = 3\pi\alpha/4$ due to **H3**.

$$\begin{aligned} \lim_{t\to\infty} |e_i(t)| \leq \;& |e_{i0}| \lim_{t\to\infty} E_{\alpha,1}(-k_{\hat{\eta}_i} t^\alpha) \\ & + M \lim_{t\to\infty} t^\alpha E_{\alpha,\alpha+1}(-k_{\hat{\eta}_i} t^\alpha) \\ = \;& \frac{M}{k_{\hat{\eta}_i}} \end{aligned} \qquad \blacksquare$$

Remark 1: If the FAO of a state variable is expressed in terms of the fractional sequential derivatives of the output y, which are unknown, then is necessary to introduce an artificial variable (if it is possible) in order to avoid the use of these unknown derivatives.

IV. Numerical Example

In this section, the synchronization of the fractional order Rössler hyperchaotic system is treated.

Remark 2: Chaotic systems are characterized by global boundedness of the trajectories [30]. By this fact, H2 is always satisfied.

First, consider the fractional order Rössler hyperchaotic system [31]

$$x^{(\alpha)} = \begin{pmatrix} x_3 + a x_1 + x_2 \\ -c x_4 + d x_2 \\ -x_1 - x_4 \\ b + x_3 x_4 \end{pmatrix} \qquad y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

where $x = (x_1, x_2, x_3, x_4)^T$ is the state vector, y_1 and y_2 are the considered outputs. When $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$, the Rössler equations (17) has a hyperchaotic attractor (see Fig. 1).

Now, we rewrite system (17) in the form (7) as follows

$$\bar{x}^{(\alpha)} = \begin{pmatrix} \eta_3 + \mathrm{a}\bar{x}_1 + \bar{x}_2 \\ -\mathrm{c}\eta_4 + \mathrm{d}\bar{x}_2 \end{pmatrix} \qquad \eta^{(\alpha)} = \begin{pmatrix} -\bar{x}_1 - \eta_4 \\ \mathrm{b} + \eta_3\eta_4 \end{pmatrix} \tag{18}$$

where $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $\eta_3 = x_3$, $\eta_4 = x_4$, $y_{\bar{x}_1} = \bar{x}_1$ and $y_{\bar{x}_2} = \bar{x}_2$. From (18), it is not difficult to find the following relations

$$\eta_3 = \phi_3\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2} \tag{19}$$

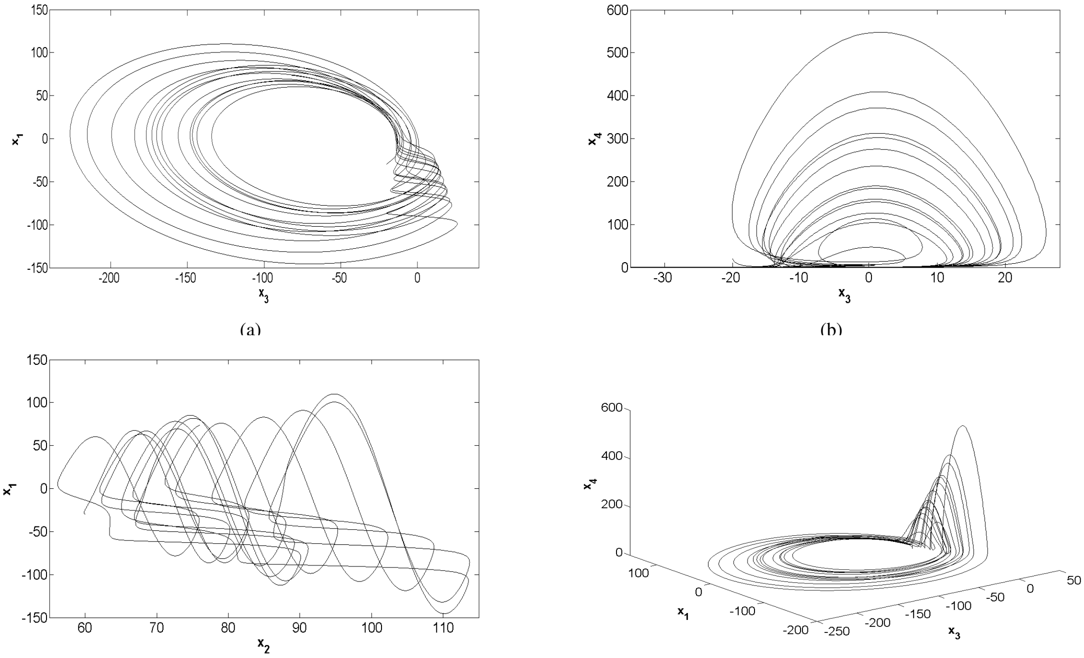
$$\eta_4 = \phi_4\left(y_{\bar{x}}, y_{\bar{x}}^{(\alpha)}\right) = -\frac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2} \tag{20}$$

then we say that $\eta_3 = x_3$ and $\eta_4 = x_4$ are FAO and therefore H1 is fulfilled.

From above, the master systems are given by

$$\begin{cases} \eta_3^{(\alpha)} = & -\bar{x}_1 - \eta_4 \\ y_{\eta_3} = & \eta_3 = y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2} \end{cases} \tag{21}$$

$$\begin{cases} \eta_4^{(\alpha)} = & \mathrm{b} + \eta_3\eta_4 \\ y_{\eta_4} = & \eta_4 = -\frac{1}{\mathrm{c}} y_{\bar{x}_2}^{(\alpha)} + \frac{\mathrm{d}}{\mathrm{c}} y_{\bar{x}_2} \end{cases} \tag{22}$$



(a) (b)

(c) (d)

Fig. 1. Phase plot of the fractional-order Rössler hyperchaotic system with $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, $\alpha = 0.95$, and initial conditions $x_1(0) = -30$, $x_2 = 60$, $x_3(0) = -20$, and $x_4(0) = 20$: (a) x_3-x_1 plane, (b) x_3-x_4 plane, (c) x_2-x_1 plane, and (d) x_3-x_1-x_4 space.

Now we design the corresponding slave systems for (21) and (22). By using (11), we have

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}(y_{\eta_3} - \hat{\eta}_3) \tag{23}$$

with $k_{\hat{\eta}_3} \in \mathbb{R}^+$ (condition H3).

Replacing (19) into (23) leads to

$$\hat{\eta}_3^{(\alpha)} = k_{\hat{\eta}_3}\left(y_{\bar{x}_1}^{(\alpha)} - \mathrm{a}y_{\bar{x}_1} - y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}\hat{\eta}_3 \tag{24}$$

In order to avoid the use of the fractional derivative $y_{\bar{x}}^{(\alpha)}$, we introduce an auxiliary variable $\gamma_{\hat{\eta}_3}$:

$$\gamma_{\hat{\eta}_3} = -k_{\hat{\eta}_3}y_{\bar{x}_1} + \hat{\eta}_3 \tag{25}$$

then

$$\hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3}y_{\bar{x}_1} \tag{26}$$

Substituting (26) and its fractional derivative of order α into (24), we obtain

$$\gamma_{\hat{\eta}_3}^{(\alpha)} = -k_{\hat{\eta}_3}\gamma_{\hat{\eta}_3} - k_{\hat{\eta}_3}\left(\mathrm{a}y_{\bar{x}_1} + y_{\bar{x}_2}\right) - k_{\hat{\eta}_3}^2 y_{\bar{x}_1} \quad , \tag{27}$$

with $\gamma_{\hat{\eta}_3}(0) = \gamma_{\hat{\eta}_{3_0}}$.

Then, the corresponding slave system of (21) is given by

$$\begin{cases} \hat{\eta}_3 = \gamma_{\hat{\eta}_3} + k_{\hat{\eta}_3}y_{\bar{x}_1} \\ y_{\hat{\eta}_3} = \hat{\eta}_3 \end{cases} \tag{28}$$

By means of the same procedure we have obtained the following slave system for (22)

$$\begin{cases} \hat{\eta}_4 = \gamma_{\hat{\eta}_4} - \dfrac{k_{\hat{\eta}_4}}{\mathrm{c}}y_{\bar{x}_2} \\ y_{\hat{\eta}_4} = \hat{\eta}_4 \end{cases} \tag{29}$$

where the dynamics of the auxiliary variable $\gamma_{\hat{\eta}_4}$ is given by

$$\gamma_{\hat{\eta}_4}^{(\alpha)} = -k_{\hat{\eta}_4}\gamma_{\hat{\eta}_4} + \frac{\mathrm{d}}{\mathrm{c}}k_{\hat{\eta}_4}y_{\bar{x}_2} + \frac{k_{\hat{\eta}_4}^2}{\mathrm{c}}y_{\bar{x}_2} \quad , \tag{30}$$

with $\gamma_{\hat{\eta}_4}(0) = \gamma_{\hat{\eta}_{4_0}}$ and $k_{\hat{\eta}_4} \in \mathbb{R}^+$ (condition H3).

Numerical simulations are performed for $a = 0.32$, $b = 3$, $c = -0.5$, $d = 0.05$, and $\alpha = 0.95$. We consider the following initial conditions to the master system $\bar{x}_1(0) = -30$, $\bar{x}_2(0) = 60$, $\eta_3(0) = -20$, $\eta_4(0) = 20$, the initial conditions to the slave system $\hat{\eta}_3(0) = -50$, $\hat{\eta}_4(0) = 10$, and the gain parameters are taken as $k_{\hat{\eta}_3} = k_{\hat{\eta}_4} = 100$. The synchronization between masters (21)-(22) and slaves (28)-(29) is shown in Fig. 2.

V. Conclusions

It was introduced a new concept so-called Fractional Algebraic Observability (FAO) which is a fundamental issue to determinate the unknown variables of nonlinear fractional order systems by means of the master-slave synchronization scheme, in particular we applied the results to a hyperchaotic fractional order system with success, however this technique can be applied to other class of systems which satisfy the properties of Proposition 1. Some numerical simulations have illustrated the effectiveness of the suggested approach.

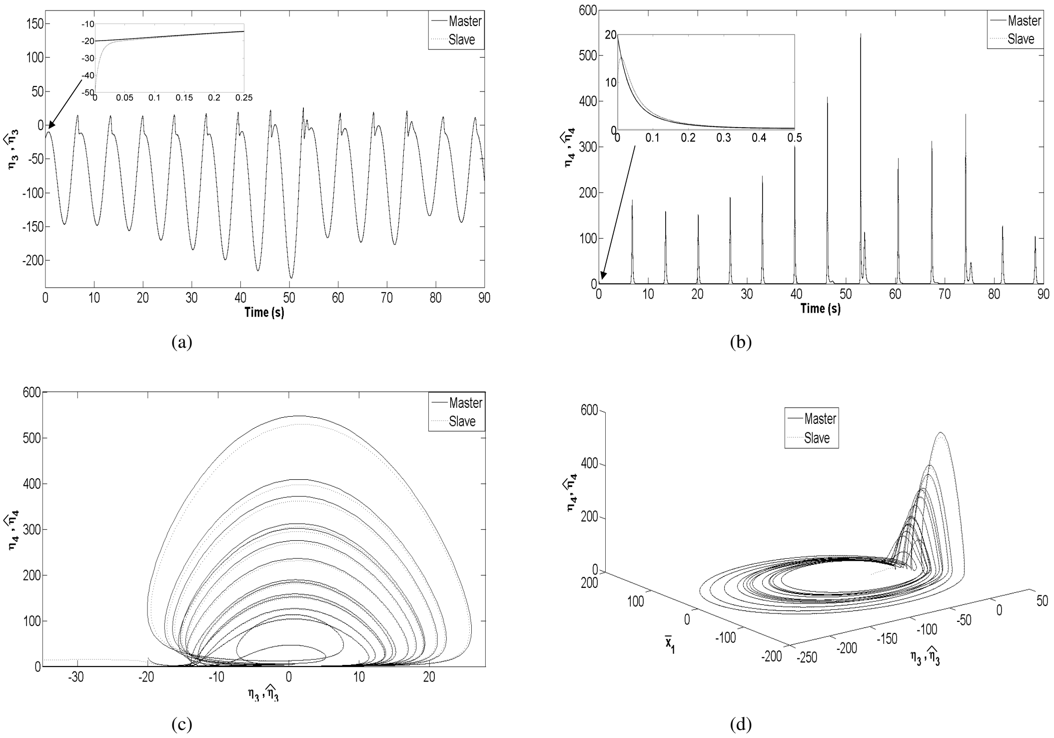


Fig. 2. Synchronization of the fractional-order Rössler hyperchaotic system.

VI. ACKNOWLEDGMENTS

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Fault diagnosis in nonlinear dynamical systems based on left invertibility condition: a real–time application to three-tank system

Juan L. Mata–Machuca, Rafael Martínez–Guerra, José J. Rincón–Pasaye

Abstract— **This work deals with the fault diagnosis problem, some new properties are found using the *left invertibility condition* through the concept of differential output rank. Two schemes of nonlinear observers are used to estimate the fault signals for comparison purposes, one of these is a reduced order observer and the other is a sliding mode observer. The methodology is tested in a real time implementation of a three-tank system.**

I. INTRODUCTION

A fault can be considered as a process degradation or degradation of the equipment performance caused by the change in the physical characteristic of the process, the input process or the external conditions.

The fault detection and isolation problem has been studied for more than three decades, many papers dealing with this problem can be found, see for instance the surveys [1]-[4] and the books [5]-[7]. For the case of nonlinear systems a variety of approaches have been proposed [1]. Some model-based approaches can be found, such as those based upon differential geometric methods [8], [9]. On the other hand, for the fault diagnosis problem, alternative approaches have been proposed based on an algebraic and differential framework [10]-[19]. These approaches consist in the estimation of the fault variables, which are defined as uncertain inputs [10].

Currently, the diagnosis problem is playing an important role in modern industrial processes. This has led control theory into a wide variety of model–based approaches which rely on descriptions via differential and/or difference equations, contrary to other standpoints developed mainly among computer scientist (see [16],[17] and references therein). The primary objectives of fault diagnosis are fault detectability and isolability, i.e., the possible location and determination of the faults present in a system and the time of their occurrences. The tasks of fault detection and isolation are to be accomplished by measuring only the input and the output variables.

This paper focuses on diagnosis of nonlinear systems and the goal is to determine malfunctions in the dynamics. In this communication, the outputs are mainly signals obtained from the sensors. Their number is important to know whether a system is diagnosable or not.

In this article, the diagnosis problem is tackled as a left invertibility problem throughout the concept of differential output rank ρ. Two schemes of observers are proposed in order to estimate the fault signals, one of them is a reduced-order observer based on a free-model approach and another is a sliding-mode observer based on a Generalized Observability Canonical Form (GOCF) [16]. Both schemes are proved to possess asymptotic convergence properties.

The type of faults considered in this work are additive and bounded, however, the algebraic approach can also be used to deal with multiplicative faults (see [11]).

This paper is organized as follows. In section II, some definitions of differential algebra are given. In section III, we discuss the left invertibility condition. In sections IV and V we give a description of the proposed observers. In section VI the three-tank system is analyzed. Finally, in section VII we illustrate this methodology with some experimental results to the three-tank system Amira DTS200 three-tank system [20],[21].

II. SOME DEFINITIONS

Some basic definitions are introduced. Further details can be found in [10]-[13], [22] and references therein.

Definition 1: Let $\mathcal{L}$ and $\mathcal{K}$ be differential fields. A differential field extension $\mathcal{L}/\mathcal{K}$ is given by $\mathcal{K}$ and $\mathcal{L}$ such that: 1) $\mathcal{K}$ is a subfield of $\mathcal{L}$ and; 2) the derivation of $\mathcal{K}$ is the restriction to $\mathcal{K}$ of the derivation of $\mathcal{L}$.

Definition 2: Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a set of elements of $\mathcal{L}$. If it satisfies an algebraic differential equation $P\left(\xi, \dot{\xi}, \ddot{\xi} \ldots\right) = 0$ with coefficients in $\mathcal{K}$ it is called differentially $\mathcal{K}$-algebraically dependent, otherwise, ξ is called differentially $\mathcal{K}$-algebraically independent.

Definition 3: Any set of elements of $\mathcal{L}$ which is differentially $\mathcal{K}$-algebraically independent and maximal with respect to inclusion forms is a differential transcendence basis of $\mathcal{L}/\mathcal{K}$. Two such basis have the same cardinality. This is called the *differential transcendence degree* of $\mathcal{L}/\mathcal{K}$ and denoted by $\mathit{diff}\ tr\ d^\circ\ \mathcal{L}/\mathcal{K}$.

Definition 4: Let $\mathcal{G}$, $\mathcal{K}\langle u\rangle$ be differential fields. A nominal dynamic consists in a finitely generated differential algebraic extension $\mathcal{G}/\mathcal{K}\langle u\rangle$, $(\mathcal{G} = \mathcal{K}\langle u, \xi\rangle,\ \xi \in \mathcal{G})$. Any element of $\mathcal{G}$ satisfies an algebraic differential equation with coefficients over $\mathcal{K}$ in the components of u and their time derivatives.

Definition 5: Any unknown variable x in a dynamic is said to be *algebraically observable* with respect to $\mathcal{K}\langle u, y\rangle$ if x satisfies a differential algebraic equation with coefficients over $\mathcal{K}$ in the components of u, y and a finite number of their derivatives. Any dynamic with output y is said to be

J. L. Mata-Machuca and R. Martínez-Guerra are with Department of Automatic Control, Center for Advanced Studies and Research of the National Polytechnic Institute (CINVESTAV-IPN), 07360 Mexico DF, Mexico {jmata, rguerra}@ctrl.cinvestav.mx

J. J. Rincón–Pasaye is with Facultad de Ingeniería Eléctrica, Universidad Michoacana de San Nicolás de Hidalgo, Ciudad Universitaria, Morelia Michoacan, Mexico jrincon@zeus.umich.mx algebraically observable if, and only if any state variable has this property.

Definition 6: Let $\mathcal{G}$, $\mathcal{K}\langle u\rangle$ be differential fields. A fault dynamics consists in a finitely generated differential algebraic extension $\mathcal{G}/\mathcal{K}\langle u, f\rangle$, $\mathcal{G} = \mathcal{K}\langle u, f, \xi\rangle$, $\xi \in \mathcal{G}$. Any element of $\mathcal{G}$ satisfies an algebraic differential equation with coefficients over $\mathcal{K}$ in the components of u, f and their time derivatives.

Definition 7: A fault $f \in \mathcal{G}$ is said to be diagnosable if it is algebraically observable over $R\langle u, y\rangle$, i.e. if it is possible to estimate the fault from the available measurements of the system.

Let us consider the class of nonlinear systems with faults described by the following equation

$$\left\{ \begin{array}{l} \dot{x}(t) = A(x, \bar{u}) \\ y(t) = h(x, \bar{u}) \end{array} \right. . \tag{1}$$

Where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a state vector, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is a known input vector, $f = (f_1, \ldots, f_\mu) \in \mathbb{R}^\mu$ is an unknown input vector, $\bar{u} = (u, f) \in \mathbb{R}^{m+\mu}$, $y(t) \in \mathbb{R}^p$ is the output vector. A and h are assumed to be analytical vector functions.

Example 1: Let us consider the nonlinear system with one fault (f_1) on the actuator and one fault (f_2) on the sensor of output y_1.

$$\left\{ \begin{array}{ll} \dot{x}_1 = & x_1 x_2 + f_1 + u \\ \dot{x}_2 = & x_1 \\ y_1 = & x_1 + f_2 \\ y_2 = & x_2 \end{array} \right. . \tag{2}$$

Since f_1, f_2 satisfy the differential algebraic equations

$$\begin{array}{rcl} f_1 - \ddot{y}_2 + y_2 \dot{y}_2 + u & = & 0 \\ f_2 - y_1 + \dot{y}_2 & = & 0 \end{array} \tag{3}$$

the system (2) is diagnosable and the faults can be reconstructed from the knowledge of u, y and their time derivatives.

Remark 1: The diagnosability condition is independent of the observability of a system.

III. On the Left Invertibility Condition

We have some definitions concerning on the differential output rank of a system.

Definition 8: The differential output rank ρ of a system is equal to the differential transcendence degree of the differential extension $\mathcal{K}\langle y\rangle$ over the differential field $\mathcal{K}$, i.e.,

$$\rho = diff\ tr\ d^o\mathcal{K}\langle y\rangle/\mathcal{K}.$$

Property 1 [23]: Let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, be differential fields such that $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$. Then

$$\begin{array}{rl} diff\ tr\ d^o(\mathcal{M}/\mathcal{K}) = & diff\ tr\ d^o(\mathcal{M}/\mathcal{L}) \\ & +diff\ tr\ d^o(\mathcal{L}/\mathcal{K}) \end{array} \tag{4}$$

■

Property 2: The differential output rank ρ of a system is smaller or equal to $\min(m, p)$, i.e., $\rho = diff\ tr\ d^o\mathcal{K}\langle y\rangle/\mathcal{K} \leq \min(m, p)$, where m and p are the total number of inputs and outputs, respectively. ■

The differential output rank ρ is also the maximum number of outputs that are related by a differential polynomial equation with coefficients over $\mathcal{K}$ (independent of x and u).

A practical way to determinate the differential output rank is by taking into account all possible differential polynomials of the form

$$h_r(y_1, \ldots, y_p) = 0 \tag{5}$$

and if is possible to find r independent relations of the form (5), then the differential output rank is given by $\rho = p - r$, that is to say, there exists only $p - r$ independent outputs.

Proposition 1 [24]: Let consider a class of systems given by (1). A system is said to be left invertible if and only if

$$\rho = diff\ tr\ d^o\mathcal{K}\langle y\rangle/\mathcal{K} = diff\ tr\ d^o\mathcal{K}\langle u, f\rangle/\mathcal{K}. \quad ■$$

Property 1 is the main tool used to prove the following theorem that looks quite natural. The theorem shows the relationship between the diagnosability and the left invertibility condition.

Theorem 1: If system (1) is left invertible, then the fault vector f can be obtained by means of the output vector.

Proof: let us consider the following field towers:

$$\mathcal{K} \subset \mathcal{K}\langle u\rangle \subset \mathcal{K}\langle u, f\rangle \subset \mathcal{K}\langle u, y, f\rangle, \tag{6}$$

$$\mathcal{K} \subset \mathcal{K}\langle y\rangle \subset \mathcal{K}\langle u, y\rangle \subset \mathcal{K}\langle u, y, f\rangle, \tag{7}$$

From (6) and property 1, we have:

$$\begin{array}{rl} diff\ tr\ d^o\mathcal{K}\langle u, y, f\rangle/\mathcal{K} = & diff\ tr\ d^o\mathcal{K}\langle u, y, f\rangle/\mathcal{K}\langle u, f\rangle \\ & +diff\ tr\ d^o\mathcal{K}\langle u, f\rangle/\mathcal{K}\langle u\rangle \\ & +diff\ tr\ d^o\mathcal{K}\langle u\rangle/\mathcal{K} \\ = & 0 + m + \mu \end{array} \tag{8}$$

From proposition 1, $diff\ tr\ d^o\mathcal{K}\langle y\rangle/\mathcal{K} = m + \mu$. From (7) we obtain

$$diff\ tr\ d^o\mathcal{K}\langle u, y, f\rangle/\mathcal{K}\langle u, y\rangle = -diff\ tr\ d^o\mathcal{K}\langle u, y\rangle/\mathcal{K}\langle y\rangle \tag{9}$$

Since the transcendence degree is always positive, we have the following:

$$diff\ tr\ d^o\mathcal{K}\langle u, y, f\rangle/\mathcal{K}\langle u, y\rangle = 0 \tag{10}$$

This means that f is differentially algebraic over $\mathcal{K}\langle u, y\rangle$. Thus, the diagnosability condition is satisfied and the theorem is proven. ■

IV. Reduced-Order Observer

Let consider system (1). The fault vector f is unknown and it can be assimilated as a state with uncertain dynamics. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector. The new extended system is given by

$$\begin{array}{c} \dot{x}(t) = A(x, \bar{u}) \\ \dot{f} = \Omega(x, \bar{u}) \\ y(t) = h(x, u) \end{array} \tag{11}$$

where $\Omega(x, \bar{u}) = [\Omega_1(x, \bar{u}), ..., \Omega_\mu(x, \bar{u})]^T : \mathbb{R}^{n+m+\mu} \rightarrow \mathbb{R}^\mu$ is an uncertain function. Note that a classic Luenberger observer can not be constructed because the term $\Omega(x, \bar{u})$ is unknown. This problem is overcome by using a reduced order uncertainty observer in order to estimate the failure variable f. Next Lemma describes the construction of a proportional reduced order observer for (11).

Lemma 1 [22]: If the following hypotheses are satisfied:

H1: $\Omega(x, \bar{u})$ is bounded, i.e., $|\Omega_i(x, \bar{u})| \leq N \in \mathbb{R}^+ \ \forall\ 1 \leq i \leq \mu$.

H2: $f(t)$ is algebraically observable over $\mathbb{R}\langle u, y\rangle$.

Then the system

$$\dot{\hat{f}}_i = k_i\left(f_i - \hat{f}_i\right), \ 1 \leq i \leq \mu \tag{12}$$

is a reduced order observer for system (11), where $\hat{f}_i$ denotes the estimate of fault f_i and $k_i \in \mathbb{R}^+ \ \forall i = 1, .., \mu$ are positive real coefficients that determine the desired convergence rate of the observer. ■

Lemma 2: If a fault signal $f_i, i \in \{1, ..., \mu\}$ of system (1) is algebraically observable and can be written in the following form

$$f_i = a_i \dot{y} + b_i(u, y) \tag{13}$$

where $a_i = [a_{i1}, ..., a_{im}] \in \mathbb{R}^m$ is a constant vector and $b_i(u, y)$ is a bounded function, then there exists a function $\gamma_i \in C^1$, such that the reduced order observer (12) can be written as the following asymptotically stable system

$$\begin{aligned} \dot{\gamma}_i &= -k_i\gamma_i + k_i b_i(u, y) - k_i^2 a_i y \\ \hat{f}_i &= \gamma_i + k_i a_i y, \end{aligned} \tag{14}$$

with $\gamma_i(0) = \gamma_{i0} \in \mathbb{R}$ ■

V. SLIDING-MODE OBSERVER

Consider the nonlinear system with faults given by (1), assuming that the fault vector f is algebraically observable over $\mathbb{R}\langle u, y\rangle$ and therefore it satisfies a differential algebraic polynomial

$$\bar{\psi}(f, y, \dot{y}, \ddot{y}, ..., \overset{(r)}{y}, u, \dot{u}, ...) = 0 \tag{15}$$

Where r is the maximum order of the output time derivatives.

Introducing the following change of coordinates

$$\eta_1 = y, \ \eta_2 = \dot{y}, \ ..., \eta_r = \overset{(r-1)}{y} \tag{16}$$

we obtain the following representation of (15) which is the so-called Generalized Observability Canonical Form [16].

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \eta_3 \\ &\cdots \\ \dot{\eta}_r &= \Phi(f, \eta_1, \eta_2, ...\eta_r, u, \dot{u}, .. \overset{(r-1)}{u}) \\ y &= \eta_1 \end{aligned} \tag{17}$$

Where $\Phi(\cdot)$ is considered as an unmodeled dynamics.

The observer structure. The following system is a sliding-mode observer for the system (17).

$$\begin{aligned} \dot{\hat{\eta}}_1 &= \hat{\eta}_2 + m_1 sign(y - \hat{y}) \\ &\cdots \\ \dot{\hat{\eta}}_{r-1} &= \hat{\eta}_r + m_{r-1} sign(y - \hat{y}) \\ \dot{\hat{\eta}}_r &= m_r sign(y - \hat{y}) \\ &\text{with } \hat{y} = \hat{\eta}_1 \end{aligned} \tag{18}$$

where $m_j > 0, \forall\ 1 \leq j \leq r$, and

$$sign(y - \hat{y}) = \begin{cases} 1 & if\ (y - \hat{y}) > 0 \\ -1 & if\ (y - \hat{y}) < 0 \\ undefined & if\ (y - \hat{y}) = 0 \end{cases}.$$

Then returning to the original coordinates and taking into account (15), the fault can be estimated from the following relationship

$$\bar{\psi}(\hat{f}, \hat{\eta}, \dot{\hat{\eta}}, \ddot{\hat{\eta}}, ..., \overset{(r)}{\hat{\eta}}, u, \dot{u}, ...) = 0 \tag{19}$$

Observer Convergence Analysis.

We will analyze the convergence properties of the proposed observer considering the presence of a noise signal δ contaminating the output measurements, such that

$$y = \eta_1 + \delta. \tag{20}$$

Let us define the state estimation errors as

$$e_1 = \eta_1 - \hat{\eta}_1, \quad e_i = (\eta_i - \hat{\eta}_i)/m, \quad i = 2, ..., r\ , \tag{21}$$

where $m > 0$, it follows that the estimation error vector $e = \begin{bmatrix} e_1 & ... & e_r \end{bmatrix}^T$ verifies the relationship

$$\dot{e} = A_{\bar{\mu}} e - K sign(Ce + \delta) + \Delta s \tag{22}$$

where $\bar{\mu} > 0$ is a regularizing parameter, $A_{\bar{\mu}} = \begin{bmatrix} -\bar{\mu} & m & 0 & ... & 0 \\ 0 & -\bar{\mu} & m & & 0 \\ 0 & 0 & -\bar{\mu} & & \vdots \\ & & & \ddots & m \\ 0 & 0 & 0 & ... & -\bar{\mu} \end{bmatrix}$, $K = \begin{bmatrix} m_1 \\ m_2 \\ ... \\ m_r \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix}$ and $\Delta s = \begin{bmatrix} \bar{\mu} e_1 \\ ... \\ \bar{\mu} e_{r-1} \\ \Phi + \bar{\mu} e_r \end{bmatrix}$ is an uncertainty term.

Assumption A1. There exist nonnegative constants L_{0s}, L_{1s}, such that the following generalized quasi-Lipschitz condition holds

$$\|\Delta s\| \leq L_{0s} + (L_{1s} + \|A_{\bar{\mu}}\|)\|e\|. \tag{23}$$

Assumption A2. The additive output noise δ, is bounded, namely

$$|\delta| \leq \delta^+ < \infty, \tag{24}$$

Assumption A3. There exists a positive definite matrix $Q_0 = Q_0^T > 0$, such that the following matrix Riccati equation

$$PA_{\bar{\mu}} + A_{\bar{\mu}}^T P + PRP + Q = 0 \tag{25}$$

with $R := \Lambda_s^{-1} + 2\|\Lambda_s\| L_{1s} I,\ \Lambda_s = \Lambda_s^T > 0,$

$$Q = Q_0 + 2(L_{1s} + \|A_{\bar{\mu}}\|^2) I$$

has a positive definite solution $P = P^T > 0$.

Theorem 2: If assumptions from A1 to A3 are satisfied, then

$$[V - V^*]_+ \to 0 \tag{26}$$

where

$$V = V(e) = \|e\|_P^2 := e^T P e,$$

$$V^* := \frac{2\|\Lambda_s\| L_{0s}^2 + 4k\delta^+}{\lambda_{\text{mín}}(P^{-1/2} Q^T Q P^{-1/2})},$$

and the function $[\cdot]_+$ is defined as follows

$$[x]_+ = \begin{cases} x & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}. \tag{27}$$

■

Remark 2: Theorem 2 states that the weighted estimation error norm $V(e)$ asymptotically converges to the zone bounded by V^*. In other words, it is ultimately bounded.

VI. Application to the three-tank system

VI-A. Description of the three-tank system

The Amira DTS200 is described in figure 1. The corresponding model with faults is given by the following equations [21]

$$\begin{aligned} \dot{x}_1 &= \frac{1}{A}(u_1 - q_{13} + f_1) \\ \dot{x}_2 &= \frac{1}{A}(u_2 + q_{32} - q_{20} + f_2) \\ \dot{x}_3 &= \frac{1}{A}(q_{13} - q_{32}) \end{aligned} \tag{28}$$

where $u_1 = q_1$ and $u_2 = q_2$ are the manipulable input flows, $x_i = h_i =$ level in the tank i. A is the transversal constant section of any of the identical tanks, and q_{ij} represents the water flow from tank i to tank j, $(1 \leq i, j \leq 3)$ which according to the generalized Torricelli's rule, valid for laminar flow

$$q_{ij} = a_i S\ sign(h_i - h_j)\sqrt{2g\,|h_i - h_j|} \tag{29}$$

with $q_{20} = a_2 S\ \sqrt{2gh_2}$. Where S is the transversal area of the pipe that interconnects the tanks (see figure 1) and a_i are the output flow coefficients, which are not exactly known, so they are considered as uncertain parameters. We assume the existence of actuator faults denoted by f_1 and f_2 ($\mu = 2$), each one of these faults represents a variation in the respective pump driver gain, which can be originated by an electronic component malfunction, or even by a leakage or an obstruction in the pump pipes.

The system (28) has four state regions in which the corresponding model is differentiable [15], any of these regions can be chosen to do the analysis, just avoiding loss of differentiability by crossing from one to another. In this work $x_1 > x_3 > x_2 > 0$ is the only considered region of operation, which experimentally is easy to operate.

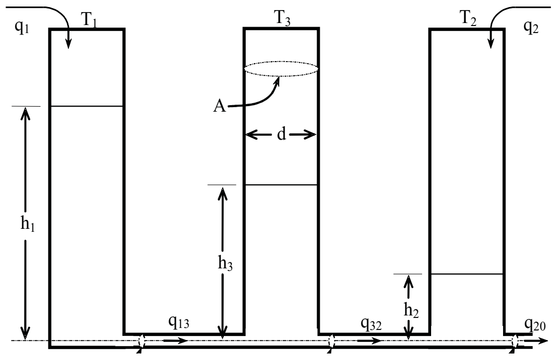

Fig. 1. Schematic diagram of the three-tank system.

VI-B. Diagnosability analysis

According to theorem 1 we need two or more measured outputs, this can only happen in the following cases:

- Case 0. $p = 3$ (h_1, h_2, and h_3 measurable)
- Case 1. $p = 2$ (h_1 not measurable, h_2, and h_3 measurable)
- Case 2. $p = 2$ (h_2 not measurable, h_1, and h_3 measurable)
- Case 3. $p = 2$ (h_3 not measurable, h_1, and h_2 measurable)

VI-B.1. Case 0: The simplest case (and the only one reported in previous works [15], with numerical results) takes place when we can measure the full state vector, that is to say, we have three outputs: $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$; in this case, from (28) we have

$$f_1 = A\dot{y}_1 + a_1 S\sqrt{2g\,(y_1 - y_3)} - u_1 \tag{30}$$

$$f_2 = A\ \dot{y}_2 - a_3 S\sqrt{2g\,(y_3 - y_2)} + a_2 S\sqrt{2gy_2} - u_2 \tag{31}$$

System (28) is left invertible because the differential output rank is equal to 2. This means that faults f_1 and f_2 are diagnosable.

VI-B.2. Case 1: We consider only the outputs: $y_2 = x_2$ and $y_3 = x_3$. By taking into account (28) we have

$$A\dot{y}_3 = a_1 S\sqrt{2g\,(x_1 - y_3)} - a_3 S\sqrt{2g\,(y_3 - y_2)}, \tag{32}$$

we get

$$x_1 = y_3 + \frac{1}{2ga_1^2 S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g\,(y_3 - y_2)}\right)^2 \tag{33}$$

Then, by replacing x_1 in (30) we obtain a set of two differential equations with coefficients in $\mathbb{R}\langle u, y\rangle$ with two unknowns f_1 and f_2, this means system (28) is left invertible (i.e., faults f_1 and f_2 are diagnosable) with the two considered outputs.

VI-B.3. Case 2: We consider only the outputs: $y_1 = x_1$ and $y_3 = x_3$. By taking into account (32) we obtain

$$x_2 = y_3 - \frac{1}{2ga_3^2 S^2}\left(-A\ \dot{y}_3 + a_1 S\sqrt{2g\,(y_1 - y_3)}\right)^2. \tag{34}$$

From (31) in a similar way we can obtain system (28) is left invertible (i.e., faults f_1 and f_2 are diagnosable) with the two considered outputs.

VI-B.4. Case 3: We consider only the outputs: $y_1 = x_1$ and $y_2 = x_2$. By taking into account (30) we get

$$x_3 = y_1 - \frac{1}{2ga_1^2 S^2} \left(-A\, \dot{y}_1 + f_1 + u_1\right)^2. \qquad (35)$$

From (31) we only can obtain one differential equation involving the two faults, therefore, system (28) is not left invertible, i.e., faults f_1 and f_2 are not diagnosable with the two considered outputs.

VII. Experimental Results

We verified the real time performance of the proposed estimators in a laboratory setting of the Amira DTS200 system. The known parameter values for the utilized system are: $A = 0.0149\ m^2$, $S = 5 \times 10^{-5}\ m^2$ and the unknown parameters: a_1, a_2, and a_3. The sample time in all the experiments was $0.001\ s$, this was chosen so small in order to get the best performance from the sliding-mode observer. The experimental results are described as follows

VII-A. Identification results

With no presence of faults, the unknown parameters a_1, a_2, and a_3 were estimated meanwhile the values for the input flows were: $q_1 = 0.000025\ m^3/s$ and $q_2 = 0.000020 m^3/s$, along $1000\ s$ in these conditions the evolution of the estimated values for the unknown coefficients is shown in figure 2a.

At the end of the identification process the estimated values for the flow parameters were obtained:

$$a_1 = 0.418, a_2 = 0.789, a_3 = 0.435. \qquad (36)$$

In figure 2b the simulated and the measured actual levels are shown in order to give a visual comparison between the actual and the estimated model, the actual level measurements are drawn in a gray color, while the levels obtained by simulating the model using the estimated values given by (36) for the flow coefficients are shown in black color.

VII-B. Fault estimation results

In all the experiments described in this subsection the input flows were maintained constant as $q_1 = 0.00002\ m^3/s$ and $q_2 = 0.000015\ m^3/s$, also two faults were artificially generated through the following expressions: $f_1 = 0.00005\ \left[1 + \sin\left(0.2te^{-0.01t}\right)\right] \mathcal{U}(t - 220)$, $f_2 = 0.00005\ \left[1 + \sin\left(0.05te^{-0.001t}\right)\right] \mathcal{U}(t - 300)$, where $\mathcal{U}(t)$ is the unit step function.

As we do not know the dynamics Φ, we can take as a reference the Lipschitz constants of the fault signals, which are 10.6×10^{-7} and 11.25×10^{-7} respectively, then we choose L_{1s} bigger enough, for example $L_{1s} = 0.001$, in a similar way, we choose $m = 0.1$, $\bar{\mu} = 1$, $\Lambda_s = 20$, $Q_0 = I$, then $R = 0.09I$, $Q = 3.2122I$, with these parameters we obtain

$$P = \begin{bmatrix} 20.4009 & -1.2107 \\ -1.2107 & 20.5446 \end{bmatrix} > 0$$

The two proposed schemes for fault estimation were evaluated in case 1 (x_1 not measurable), the results are described as follows.

Only the two outputs $y_2 = x_2$ and $y_3 = x_3$ were taken into account, an estimation for the unknown state x_1 was necessary to be obtained. In figure 3 we show the resulting estimations achieved with the reduced-order observer. A low-pass filter was necessary in order to reduce the effect of the measurement noise, we chose a second-order Butterwort filter whose transfer function is given by $G_f(s) = 1/\left(32s^2 + 8s + 1\right)$. The gain values chosen for both fault observers were $k = 2$, and for the state observer x_1, $k_{x_1} = 0.3$. As we can observe the estimation results with this scheme are good (figure 3).

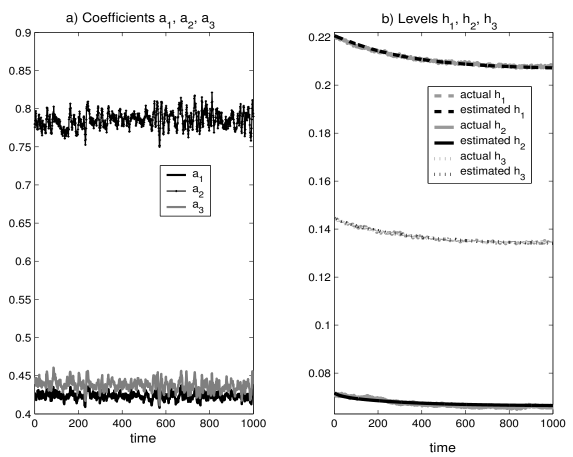

Fig. 2. a) Parameter identification. b) Validation of the estimated model.

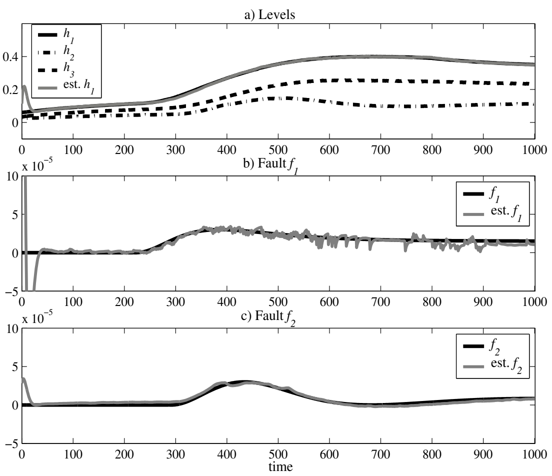

Fig. 3. Fault diagnosis for unknown h_1 using the reduced order observer.

A sliding-mode observer was also tested in this case. In figure 4 the corresponding results achieved with the sliding-mode observer are shown. It is worth to mention that with this observer it was not necessary to include the reducing noise filter providing the inherent robustness of this observer. The gain values chosen for the fault and state observers were $m_1 = 0.1$, $m_2 = 0.01$. As we can observe from figure 4, this scheme also provides good estimation results.

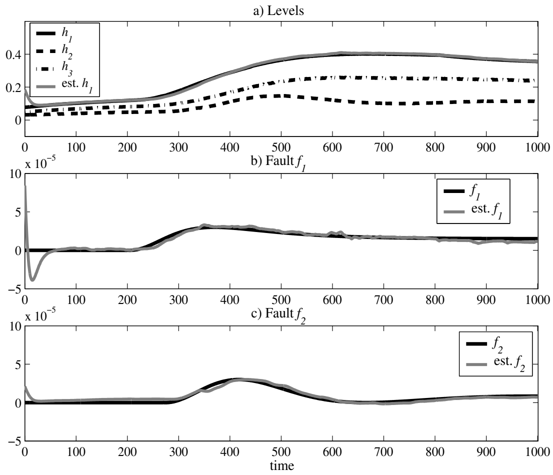

Fig. 4. Fault diagnosis for unknown h_1 using the sliding mode observer.

VIII. Concluding Remarks

We have tackled the fault diagnosis problem in nonlinear systems using the condition of left invertibility through the concept of differential output rank. The usefulness of theorem 1, theorem 2, and lemma 2 was shown, this allowed the estimation of two simultaneous faults with less measurements. The theoretical and simulation results were tested in a real-time implementation (three-tank system). The experimental results for the two observers showed similar performance, however the proposed sliding-mode observer is more robust against measurement noise, as it was expected.

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Fault diagnosis via a polynomial observer

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***Abstract*— The fault diagnosis problem of a class of nonlinear systems based on a differential approach is used to determine fault diagnosability with the minimum number of measurements from the system. In order to reconstruct the faults on the system, a polynomial observer is proposed, which includes in its structure corrections terms of high order. Another two schemes of nonlinear observers are used for reconstructing the faults for comparison purposes, one of them being a reduced order observer and the other a sliding mode observer.The approach was tested in a real-time experimental setting Amira DTS-200.**

***Index Terms*— Fault diagnosis, polynomial observer, reduced order observer, sliding mode observer, differential algebra.**

I. Introduction

Fault diagnosis is an important problem in process engineering. The fault diagnosis problem has been studied for more than three decades, there are many papers on process of fault diagnosis ranging from approaches based on quantitative models, qualitative models and approaches based on process history data [1]. For the approaches based on quantitative models, can be found approaches based upon differential geometric methods [2], [3]. On the other hand, alternative approaches has been proposed based on an algebraic and differential framework [4] and [5].

This paper deals with nonlinear systems diagnosis and the goal is to find malfunctions in the system, based on input/output measurements. The outputs are mainly measured signals obtained from sensors, their number is important in order to know if the system is diagnosable or not.

The fault diagnosis problem is considered as the problem of observing the fault signals. So a system is called diagnosable if the faults satisfy the so-called algebraic observability condition [10]. The main contribution of this article consists of the solution of the fault diagnosis problem by means of a polynomial observer for the case of multiples available measurements. In addition, another two schemes of observer are proposed in order to estimate the fault signals for comparison purposes, one of them is a reduced order observer and the other is a sliding mode observer based on partial change of coordinates.

This paper is organized as follows: In section II A definition related on Observability and Diagnosability is given. In section III the relation between the number of faults and the number of available measurements in terms of the differential transcendence degree concept is given. An asymptotic polynomial for the fault signals is presented in section IV. In section V the Amira DTS200 three-tank system model is described, and the diagnosability condition is evaluated. In section VI the experimental results are shown for the fault and state estimation with the three different observers. Finally, in section VII the paper is closed with some concluding remarks.

II. Observability and Diagnosability Condition

The observability and diagnosability notion of a system, linear or nonlinear in the differential algebra approach need a basic definition. Further details can be found in [4] and [10].

Definition 1: Algebraic observability and diagnosability condition. A fault $f \in \mathcal{G}$ is said to be diagnosable if it is possible to estimate the fault from the available measurements of the system, i.e., f is diagnosable if it is algebraically observable over the differential field $\mathbb{R}\langle u, y\rangle$[4].

Let us consider the class of nonlinear systems with faults described by the following equation

$$\begin{cases} \dot{x}(t) = g(x, u, f) \\ y(t) = h(x, u) \end{cases} \tag{1}$$

Where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is a state vector, $u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m$ is a know input vector, $f = (f_1, f_2, ..., f_\mu) \in \mathbb{R}^\mu$ is a unknown input vector, $y(t) \in \mathbb{R}^p$ is the output vector (available measurements), g and h are assumed to be analytical vector functions.

Remark 1: The diagnosability condition is independent of the observability of a system.

Example : Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = & x_1 x_2 + f \\ \dot{x}_2 = & -x_2^2 + x_1 + u \\ \dot{x}_3 = & x_3 x_2 \\ y = & x_2 \end{cases} \tag{2}$$

Since f satisfies the differential algebraic equation $f - \ddot{y} - y\dot{y} + y^3 - uy + \dot{u} = 0$, then the system (2) is diagnosable and the fault can be reconstruct knowing u, y and their time derivatives. However the state x_3 is not algebraically observable.

III. Relation between number of faults and the number of available measurements

Let consider the system (1). The fault vector f is unknown and it can be assimilated as a state with uncertain dynamics. Then, in order to estimate it, the state vector is extended to deal with the unknown fault vector. The new extended system is given by

$$\begin{cases} \dot{x}(t) = & g(x, u, f) \\ \dot{f}_j = & \Omega_j(x, u, f) \qquad 1 \le j \le \mu \\ y(t) = & h(x, u) \end{cases} \tag{3}$$

The following results from the theory of differential algebraic field extensions are an useful tool to determine whether a

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fault can be reconstructed from the know inputs and available measurements.

Definition 2: A maximal family of elements of the field $\mathcal{K}$ that are $\mathcal{K}$-differentially independent is denominated differential transcendence basis of extension $\mathcal{L}/\mathcal{K}$ and the cardinality of this base is called the differential transcendence degree denoted by: $diff\ tr\ d^o\ \mathcal{L}/\mathcal{K}$.

Property 1: Let $\mathcal{K}$,$\mathcal{L}$,$\mathcal{M}$ be differential fields such that $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$. Then

$$diff\ tr\ d^o\left(\mathcal{M}/\mathcal{K}\right) = diff\ tr\ d^o\left(\mathcal{M}/\mathcal{L}\right) + diff\ tr\ d^o\left(\mathcal{L}/\mathcal{K}\right) \quad (4)$$

This property is an important tool to proof the theorems 1 and 2, see [4].

Theorem 1:.Assume that the system (1) is diagnosable, then the number of faults is less or equal to the number of available measurements (outputs), i.e.

$$\mu \leq p$$

Theorem 2: The system (1) is diagnosable if and only if

$$diff\ tr\ d^o\mathcal{K}\left\langle u,y\right\rangle/\mathcal{K}\left\langle u\right\rangle = \mu \quad (5)$$

IV. Polynomial Observer

The system (3) can be expressed in the following form

$$\begin{cases} \dot{x}(t) = & Ax + \Psi(x,u,f) \\ \dot{f}_k = & \Omega_k(x,u,f), \qquad 1 \leq k \leq \mu \\ y(t) = & Cx \end{cases} \quad (6)$$

where $\Omega = (\Omega_1, \Omega_2, ..., \Omega_\mu) \in \mathbb{R}^\mu$ is a unknown bounded function i.e., $\|\Omega_j\| \leq N < \infty$, $N \in \mathbb{R}^+$ and $\Psi(x,\bar{u})$ is a nonlinear function that satisfies the Lipschitz condition with $\bar{u} = (u,f)$, i.e.,

$$\|\Psi(x,\bar{u}) - \Psi(\hat{x},\bar{u})\| \leq L\|x - \hat{x}\| \quad (7)$$

and $\bar{u}$ uniformly bounded.

The following system is an observer for the system (6)

$$\begin{cases} \dot{\hat{x}}(t) = & A\hat{x} + \Psi(x,\bar{u}) + \\ & +\sum_{i=1}^{p}\sum_{j=1}^{m} K_{ij}\left(y_i - C_i\hat{x}\right)^{2j-1} \\ \dot{\hat{f}}_k(t) = & \sum_{l=1}^{m}\bar{K}_{kl}\left(f_k - \hat{f}_k\right)^{2l-1} \\ y(t) = & C_i x \end{cases} \quad (8)$$

where, $\hat{x} \in \mathbb{R}^n$, $[K_{ij}]_{\substack{1\leq i\leq p \\ 1\leq j\leq m}}$, $[\bar{K}_{kl}]_{\substack{1\leq k\leq \mu \\ 1\leq l\leq m}}$, K_{ij}, $\bar{K}_{kl} > 0$, $\hat{x}_0 = \hat{x}(t_0)$ and $\hat{f}_0 = \hat{f}(t_0)$ are arbitrarily initial conditions.

For this observer the following hypotheses are considered:

H1: $f(t)$ is algebraically observable on $\mathbb{R}\left\langle u,y\right\rangle$

H2 The gains $[K_{i1}]_{1\leq i\leq p}$ can be chosen such that algebraic Riccati equation has a symmetric and positive definite solution P for some $\epsilon > 0$.

$$\left(A - \sum_{i=1}^{p} K_{i1}C_i\right)^T P + P\left(A - \sum_{i=1}^{p} K_{i1}C_i\right) + L^2PP + I + \varepsilon I = 0$$

H3 $[K_{ij}]_{\substack{1\leq i\leq p \\ 2\leq j\leq m}}$ is chosen such that

$$\lambda_{\min}\left((PK_{ij}C_i)^T + (PK_{ij}C_i)\right) \geq 0$$

For analyzing the estimation error, we define $e = [e_x, e_k]^T$, where $e_x = x - \hat{x}$ and $e_k = f_k - \hat{f}_k$. From systems (6) and (8), the dynamics of estimation error is given by

$$\begin{cases} \dot{e}_x = & \left(A - \sum_{i=1}^{p} K_{i1}C_i\right)e_x - \\ & -\sum_{i=1}^{p}\sum_{j=2}^{m} K_{ij}\left(C_ie_x\right)^{2j-1} + \\ & +\left[\Psi(x,\bar{u}) - \Psi(\hat{x},\bar{u})\right] \\ \dot{e}_k = & \Omega_k - \bar{K}_{k1}e_k - \sum_{j=2}^{m}\bar{K}_{kj}\left(e_k\right)^{2l-1} \end{cases} \quad (9)$$

A. Observer convergence analysis

The following theorem proves the observer convergence.

Theorem 3: For the nonlinear system (6), suppose that $x(t)$ exists for all $t \geq 0$, the nonlinear function $\Psi(x,u,f)$ satisfies de Lipschitz condition (7), $x(t)$ and $f(t)$ are algebraically observable. If there exists a matrix P positive definite and observer gains K_{ij} and $\bar{K}_{kl} > 0$ such that the system (8) is an observer of the system (6), then the estimator error converges asymptotically to zero.

Proof: Consider the following Lyapunov function candidate

$$\begin{matrix} V = V_1 + V_2 \\ V_1 = e_x^T P e_x \quad V_2 = \frac{1}{2}e_k^2 \end{matrix} \quad (10)$$

where P satisfies the assumption H2.

The proof of the theorem is given in two parts, as follows:

(i) The time derivate of V_1 along the trajectories of (9) is

$$\begin{aligned} \dot{V}_1 &= \dot{e}_x^T P e_x + e_x^T P \dot{e}_x \\ &= e_x^T\left(\left(A - \sum_{i=1}^{p} K_{i1}C_i\right)^T P + P\left(A - \sum_{i=1}^{p} K_{i1}C_i\right)\right)e_x + \\ &\quad +2e_x^T P\left[\Psi(x,\bar{u}) - \Psi(\hat{x},\bar{u})\right] - \\ &\quad -2\sum_{i=1}^{p}\sum_{j=2}^{m} K_{ij}\left(C_ie_x\right)^{2j-2} e_x^T\left((PK_{i,j}C_i)^T + (PK_{ij}C_i)\right)e_x \end{aligned}$$

The next inequality [7], which is based on the Lipschitz condition (7)

$$2e_x^T P\left[\Psi(x,\bar{u}) - \Psi(\hat{x},\bar{u})\right] \leq L^2 e_x^T PPe_x + e_x^T e_x \quad (11)$$

Now, applying the Rayleigh's inequality [8] and consider the assumption H3, we obtain

$$-e_x^T PK_{ij}C_ie_x \leq -\lambda_{\min}\left(PK_{ij}C_i + (PK_{ij}C_i)^T\right)\|e_x\|^2 \quad (12)$$

for $2 \leq i \leq m$.

Hence, combining the inequalities (11) and (12)

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$$\begin{aligned}\dot{V}_1 &\leq e_x^T\left[\left(A-\sum_{i=1}^{p}K_{i1}C_i\right)^T P + P\left(A-\sum_{i=1}^{p}K_{i1}C_i\right)+\right.\\ &\quad \left. + L^2PP+I\right]e_x - \\ &\quad -2\sum_{i=1}^{p}\sum_{j=2}^{m}K_{ij}\left(C_ie_x\right)^{2j-2}\lambda_{\min}\left(PK_{ij}C_i+\left(PK_{ij}C_i\right)^T\right)\|e_x\|^2\\ &\leq e_x^T\left[\left(A-\sum_{i=1}^{p}K_{i1}C_i\right)^T P + P\left(A-\sum_{i=1}^{p}K_{i1}C_i\right)+\right.\\ &\quad \left. + L^2PP+I\right]e_x\\ &= -\epsilon\|e_x\|^2\end{aligned}$$

(ii) On the other hand, considering the second term of the Lyapunov function candidate, we have

$$\begin{aligned}\dot{V}_2 &= e_k\dot{e}_k\\ &= e_k\left(\Omega_k - \bar{K}_{k1}e_k - \sum_{l=2}^{m}\bar{K}_{kl}\left(e_k\right)^{2l-1}\right)\\ &= e_k\Omega_k - \bar{K}_{k1}e_k^2 - \sum_{l=2}^{m}\bar{K}_{kl}\left(e_k\right)^{2l}\\ &\leq |e_k|\,|\Omega_k| - \bar{K}_{k1}e_k^2\\ &\leq |e_k|\,N - \bar{K}_{k1}|e_k|^2\\ &= -\left[\bar{K}_{k1}|e_k| - N\right]|e_k|\end{aligned}$$

$\dot{V}_2$ is negative inside the set $\{|e_k| > N/\bar{K}_{k1}\}$, i.e., exists $\bar{\epsilon} > 0$ such that $\bar{K}_{k1}|e_k| - N = \bar{\epsilon} > 0$.

Now we proves that $|e_k|$ is upper bounded. Let α, β upper bounds of $V_2(e_k)$. With $\beta > N^2/2\bar{K}_{k1}^2$, the solution in the set $\{V_2(e_k) \leq \beta\}$ will remain there for all $t \geq 0$, due to that $\dot{V}_2$ is negative in $V_2 = \beta$. Hence, the solution of $\dot{e}_k$ are uniformly bounded [9]. Furthermore, if $N^2/2\bar{K}_{l1}^2 < \alpha < \beta$, then $\dot{V}_2$ will be negative in the set$\{\alpha \leq V_2 \leq \beta\}$. In this set V_2 will decrease monotonously until the solution of the set $\{V_2 \leq \alpha\}$. According to [9] the solution is ultimately bounded with bound $|e_k| \leq \sqrt{2\alpha}$. For example, if we defined α and β as $\alpha = N^2/2\bar{K}_{k1}^2$ and $\beta = 2N^2/2\bar{K}_{k1}^2$. Then, the ultimate bound is

$$|e_k| \leq \frac{N}{\bar{K}_{k1}}$$

Hence

$$\dot{V}_2 \leq -\bar{\epsilon}\,|e_k|$$

Finally, from (i) and (ii), we conclude that

$$\dot{V} \leq -\epsilon\|e_x\|^2 - \bar{\epsilon}\,|e_k| < 0$$

and the theorem is proven. ■

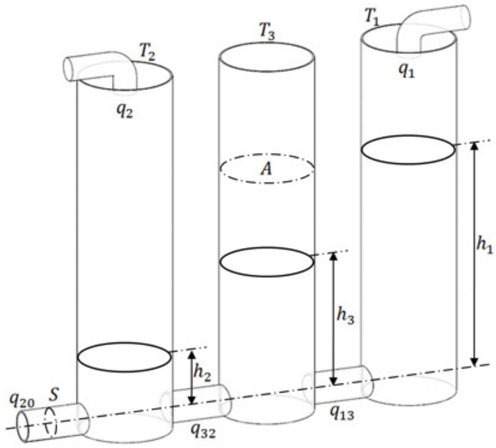

Fig 1. Diagram of the Three-Tank System

V. Application of the Three-Tank System

A. Description of the three-tank system

The Amira DTS200 [6] system consists of three cylinders (T_1, T_2 y T_3) with transversal constant section A, which are connected in series one to another one by means cylindrical tubes with transversal section S (Figure 1). The corresponding nominal model is given by the following system

$$\left\{\begin{aligned}\frac{dh_1}{dt} &= \frac{1}{A}\left(q_1 - q_{13} + u_1\right)\\ \frac{dh_2}{dt} &= \frac{1}{A}\left(q_2 + q_{32} - q_{20} + u_2\right)\\ \frac{dh_3}{dt} &= \frac{1}{A}\left(q_{13} - q_{32}\right)\end{aligned}\right. \tag{13}$$

According to the generalized rule of Torricelli, this is valid for laminar flow

$$\begin{aligned}q_{13} &= a_1 S\,\mathrm{sign}\left(h_1 - h_3\right)\sqrt{2g\,|h_1 - h_3|}\\ q_{32} &= a_3 S\,\mathrm{sign}\left(h_3 - h_2\right)\sqrt{2g\,|h_3 - h_2|}\\ q_{20} &= a_2 S\sqrt{2gh_2}\end{aligned} \tag{14}$$

A completed description of the system (13) is given in the following table.

TABLE 1
VARIABLES AND PARAMETERS OF THE AMIRA DTS 200

Symbols	Description
h_i	Level of the liquid of i-th tank $i \in \{1,2,3\}$.
A	Transversal section of each tank (m^2).
S	Transversal section of the interconnection tubes (m^2).
q_{13}	Water flow of T_1 to T_3 (m^3/s).
q_{32}	Water flow of T_3 to T_2 (m^3/s).
q_{20}	Output flow (m^3/s).
q_1	Input flow to T_1 (m^3/s).
q_2	Input flow to T_2 (m^3/s).
a_i	Output flow coefficients.

In what follows, it is considered that the state vector is $x = [x_1\ x_2\ x_3]^T = [h_1\ h_2\ h_3]^T$ and the input vector is $u = [u_1\ u_2]^T = [q_1\ q_2]^T$.

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The system (13) has four regions of operations in which the corresponding model is differentiable, in this paper we only consider the operation region $x_1 \geq x_3 \geq x_2$.

B. Considered faults and measurements of the system

The nominal model (13) is transformed into the following, where the additive faults f_1 and f_2 $(\mu = 2)$ are considered in the actuators that control the input flows $u_1 = q_1$, $u_2 = q_2$.

$$\begin{cases} \dot{x}_1 = & \frac{1}{A}(u_1 - q_{13} + u_1 + f_1) \\ \dot{x}_2 = & \frac{1}{A}(u_2 + q_{32} - q_{20} + u_2 + f_2) \\ \dot{x}_3 = & \frac{1}{A}(q_{13} - q_{32}) \end{cases} \tag{15}$$

C. Diagnosability analysis

For the theorem 1, for the diagnosability is required that the number of faults be less or equal that the number or available measurements. The system (15) consists of two faults. In order to determine observability and the diagnosability of the system (15), we have to evaluate the algebraic observability condition from definition 5 for f_1 and f_2. and the unknown state x_1. We only consider the region of operation $x_1 \geq x_3 \geq x_2$.

For this case we only consider two outputs $y_2 = x_2$, and $y_3 = x_3$. In this case from (15) we have

$$f_1 = A\dot{x}_1 + a_1 S\sqrt{2g(x_1 - y_3)} - u_1 \tag{16}$$

$$f_2 = A\dot{y}_2 - a_3 S\sqrt{2g(y_3 - y_2)} + a_2 S\sqrt{2gy_2} - u_2 \tag{17}$$

$$x_1 = y_3 + \frac{1}{2ga_1^2S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g(y_3 - y_2)}\right)^2 \tag{18}$$

The algebraic observability condition for x_1 is deduced directly from (18), which is an equation with coefficients in $\mathbb{R}\langle u, y\rangle$.

Replacing (18) into (16) and (17), we have

$$\begin{aligned} f_1 &= A\dot{x}_1 + \\ &\quad + a_1 S\sqrt{(2g-1)y_3 + \frac{1}{a_1^2S^2}\left(A\dot{y}_3 + a_3 S\sqrt{2g(y_3 - y_2)}\right)^2} \\ &\quad - u_1 \\ f_2 &= A\dot{y}_2 - a_3 S\sqrt{2g(y_3 - y_2)} + a_2 S\sqrt{2gy_2} - u_2 \end{aligned} \tag{19}$$

It is easy to check from (19) that $diff\ \ tr\ d^o\mathcal{K}\langle u, y\rangle / \mathcal{K}\langle u\rangle = 2$ and for the theorem 2 that the system is diagnosable with the two considered outputs.

D. Fault reconstruction

We consider the outputs $y_2 = x_2$ and $y_3 = x_3$, then for this case we need to estimate the time derivatives from outputs y_2 and y_3. In this section a methodology appears to reconstruct first $r-1$ time derivatives from the available output y.

1) Reduced Order Observer: We propose a reduced order observer [10]. Let us consider the following time derivative to be estimated

$$\eta = \dot{y} \tag{20}$$

we propose the observer structure

$$\dot{\hat{\eta}} = K(\eta - \hat{\eta}) \tag{21}$$

now, applying the next change of coordinates

$$\hat{\eta} = \gamma + Ky \tag{22}$$

and from (20) and (21) we can get $\dot{\gamma} = -K\hat{\eta}$, then again from (22)

$$\dot{\gamma} = -K\gamma - K^2 y \tag{23}$$

Hence (23) and (22) constitute an asymptotic estimator of η.

2) Sliding Mode Observer: We propose a sliding mode observer [10], then we introduce the following change of variables: $\eta_1 = y$, $\eta_2 = \dot{\eta}_1$, then we obtain the following observer

$$\begin{aligned} \dot{\hat{\eta}}_1 &= \hat{\eta}_2 + m_1 \mathrm{sign}(y - \hat{y}) \\ \dot{\hat{\eta}}_2 &= m_2 \mathrm{sign}(y - \hat{y}) \end{aligned} \tag{24}$$

which can be used to estimate η_2 from the output y.

3) Polynomial Observer: We introduce the following change of variables

$$\eta_1 = y,\ \eta_2 = \dot{y},\ \ldots,\ \eta_r = y^{(r-1)} \tag{25}$$

Then the system (1) can be expressed as follows

$$\begin{aligned} \dot{\eta} &= A\eta + \bar{\phi}(\eta, u) \\ y &= \eta_1 \end{aligned} \tag{26}$$

So, by the theorem 3, the observer has the following structure

$$\dot{\hat{\eta}} = A\hat{\eta} + \bar{\phi}(\hat{\eta}, u) + \sum_{i=1}^{p}\sum_{j=1}^{m} K_{ij}(y - C_i\hat{x})^{2j-1} \tag{27}$$

Then, the polynomial observer (27) is used to estimate the variables $\eta_1 = y$, $\eta_2 = \dot{y}$, ... , $\eta_r = y^{(r-1)}$ by means of the available output y.

VI. Fault Estimation Results

For the uncertain parameters a_i from the system (13), we consider the following identification results

$$a_1 = 0.4663,\ a_2 = 0.7654,\ a_3 = 0.4616$$

For all the experiments reported in this section the input flows were maintained constant as $u_1 = q_1 = 0.00002\frac{m^3}{s}$ and $u_2 = q_2 = 0.000015\frac{m^3}{s}$, for 3000 seconds. The two faults were artificially generated through the following equations:

$$\begin{aligned} f_1 &= 5\times10^{-6}\left[1 + \sin\left(0.2te^{-0.01t}\right)\right]\mathcal{U}(t - 220) \\ f_2 &= 5\times10^{-6}\left[1 + \sin\left(0.05te^{-0.001t}\right)\right]\mathcal{U}(t - 330) \end{aligned}$$

where $\mathcal{U}(t)$ is the unit step function.

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The three proposed schemes for fault diagnosis were evaluated for the case when $x_2 = y_2$ and $x_3 = y_3$ are available, for this reason an estimation for the unknown state x_1 was necessary to be obtained. In the Figure 2 we show the estimation of unknown state x_1 and the fault estimation results and as we can observe, the reconstruction results of signals are good. The gain values for the fault observers were $K_1 = 0.045$ and $K_2 = 1.25$ In the same way, in the Figure 3 we show the state and fault estimation results and also the reconstruction results of signals are good. The gain values for the fault observers were $m_{11} = 0.0195$, $m_{12} = 0.00055$ and $m_{21} = 0.0165$, $m_{22} = 0.0005$. Finally in the Figure 4 the corresponding results achieved wiht the polynomial observer are shown. The gain values for the fault observers were $\bar{K}_{11} = 0.045$, $\bar{K}_{12} = 23$, $\bar{K}_{13} = 19$ and $\bar{K}_{21} = 0.196$, $\bar{K}_{22} = 25$, $\bar{K}_{23} = 15$. It is worth to mention that with this observer we estimated the full state vector and the both fault signals. As we can observe, this scheme also provides good results.

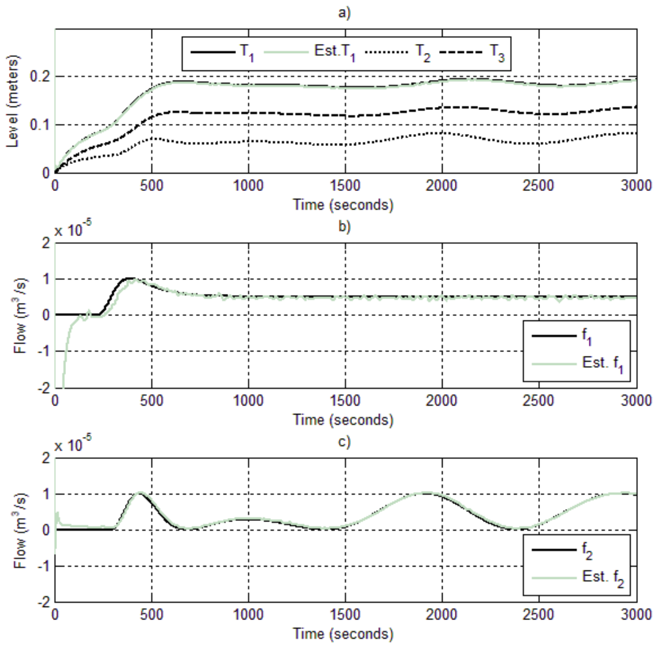

Fig 2. Reduce order observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

The performance of the proposed observer is evaluated using the following cost function, with $\varepsilon' = 0.0001$, in this case is defined as

$$J_t = \frac{1}{t + \varepsilon'} \int_0^t \|e_k(\tau)\|^2 \, d\tau \tag{28}$$

Figure 5 shows the performance of three observers proposed to estimate the fault f_1. We observe that the polynomial observer converges much faster than the other two. In the same way, in Figure 6 we observe that the performance of the three observers for the estimation of the fault f_2 is similar.

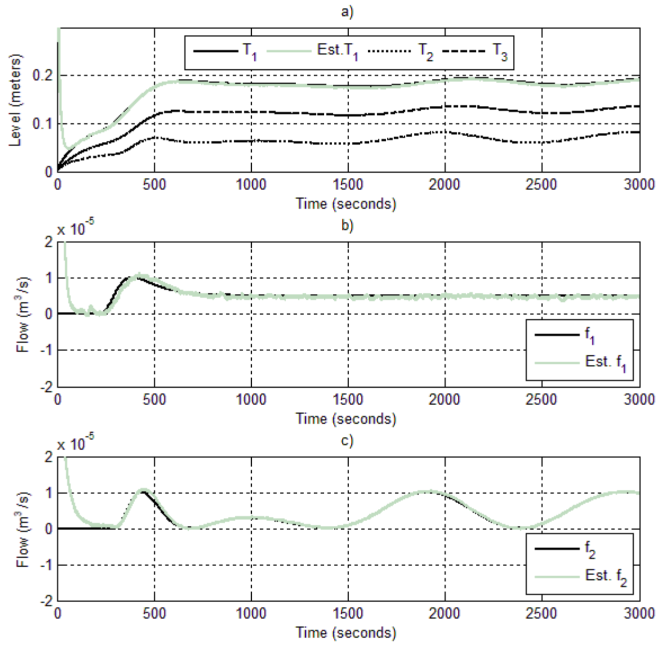

Fig 3. Sliding mode observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

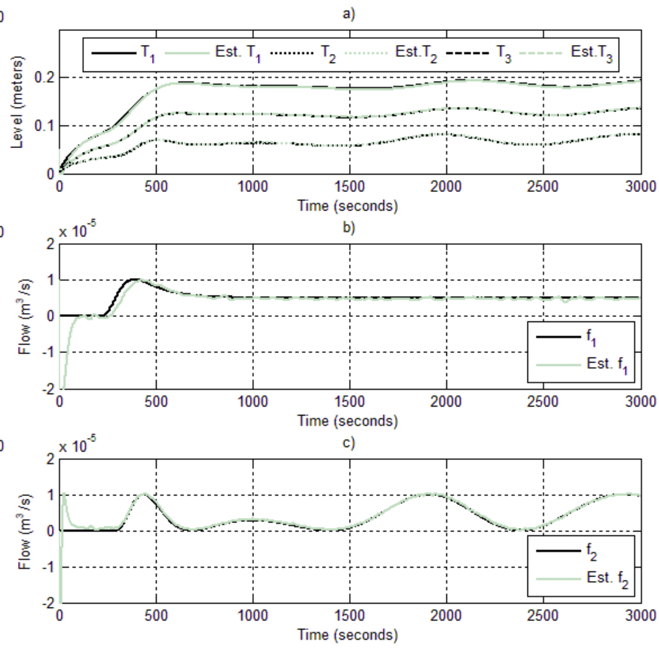

Fig 4. Polynomial observer a) Level estimation b) Fault reconstruction f_1 c) Fault reconstruction f_2

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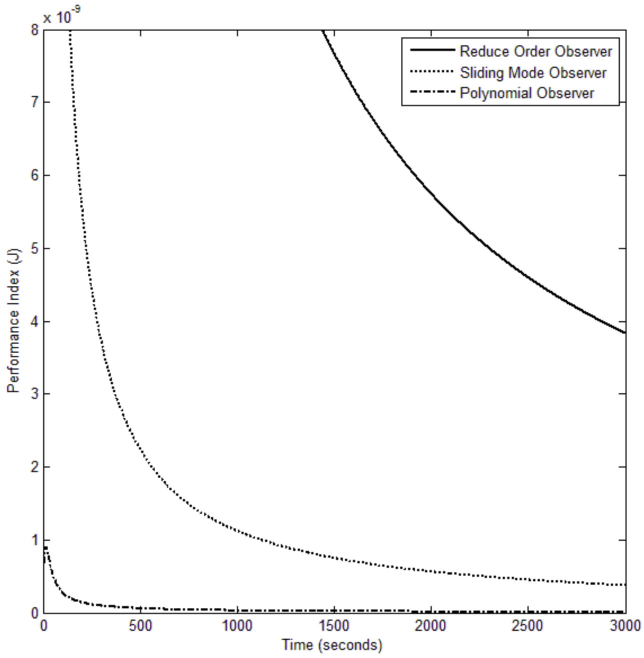

Fig 5. Performance evaluation of observers for the estimation error of fault f_1

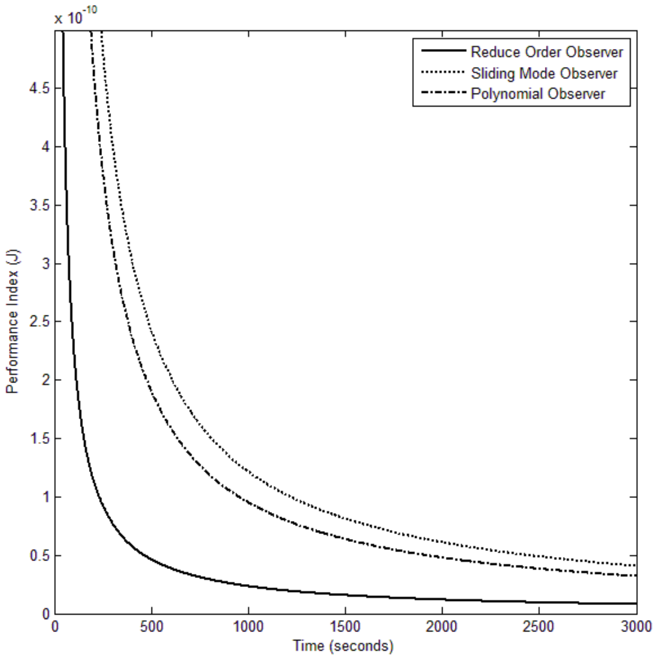

Fig 6. Performance evaluation of observers for the estimation error of fault f_2

VII. Concluding Remarks

In this work the fault diagnosis problem for a class of nonlinear systems using the differential algebraic approach was used. The method consists of considering to the fault as an augmented state of the system. To achieve the reconstruction of the unknown states of the extended system, we proposed a polynomial observer and another two schemes of nonlinear observers for comparison purposes. The approach was tested in a real-time experimental setting Amira DTS-200. From the fault reconstruction result we can see that for the three different observers a similar performance and estimation results. Finally the proposed polynomial observer presents a good performance.

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Synchronization of chaotic Liouvillian systems

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***Abstract*— In this paper we deal with the synchronization of the Chua oscillator, which is considered as a chaotic Liouvillian system. The synchronization problem is treated as an observation problem. The results of this work are based on a differential algebraic approach, which are used in order to determine observability with the measurements from the system, this strategy consists of proposing a polynomial observer (slave system) which tends to follow exponentially the chaotic oscillator (master system).**

***Index Terms*— Synchronization, chaotic Liouvillian systems, exponential polynomial observer, Chua's oscillator.**

I. Introduction

In the past years, synchronization of chaotic systems problem has received a great deal of attention among scientists in many fields [1]-[3], in particular, secure communications, biological systems, chemical reactions, etc., [6], [9], [10].In general, synchronization research has been focused on the following areas: nonlinear observers [4], [11], feedback controllers [12], [13], nonlinear backstepping control [14], time delayed systems [15], [16], directional and bidirectional linear coupling [17], [18], adaptive control [19], [20], adaptive observers [21], [22], sliding mode observers [23], impulsive control [24], active control [25], among others.

Since Pecora and Carroll's observation on the possibility of synchronizing two chaotic systems [1] (so-called drive-response configuration), several synchronization schemes have been developed [4]-[6]. Synchronization can be classified into mutual synchronization (or bidirectional coupling) [7] and master-slave synchronization (or unidirectional coupling) [1], [8].

There are many methods to solve the synchronization problem since the control theory perspective. This work considers the master-slave synchronization problem via an exponential polynomial observer (EPO) based on differential an algebraic techniques [26]-[28]. Differential and algebraic concepts allow us to establish an algebraic observability condition, and therefore they provide a first step for the construction of an algebraic observer. An observable system in this sense can be regarded as a system whose state variables can be expressed in terms of the input and output variables and a finite number of their time derivatives. Thus, chaos synchronization problem can be posed as an observer design procedure, where the coupling signal is viewed as output and the slave system is regarded as observer. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (Chua's circuit) to the slave system (EPO), the slave is requested to recover the unknown state trajectories of the master. The strategy consists of proposing an EPO which exponentially reconstructs the unknown states of Chua's system.

In this paper the Chua's oscillator is viewed as a chaotic system with some Liouvillian properties [28], [26], refereed as Chaotic Liouvillian Oscillator (CLO). The Liouvillian character of the system (if a variable can be obtained by the adjunction of integrals or exponentials of integrals) is used as an observability criterion, that is to say, by this property we can know whether a variable can be reconstructed with the measurable output.

The present paper is organized as follows: in section II we give some definitions about differential-algebraic approach and Liouvillian systems. In section III is treated the synchronization problem and its solution by means of a polynomial observer.In section IV presents the numerical results applied to the oscillator chua. Finally, section V presents the conclusions of this work.

II. Definitions

Some basic definitions are introduced in this section.

Definition 1: (Algebraic Observability Condition–AOC). Let us consider a nonlinear dynamical system with input u, output y, and state vector $x = (x_1, x_2, ..., x_n)^T$. A state variable $x_i \in \mathbb{R}$ is said to be algebraically observable if it is algebraic over $\mathbb{R}\langle u, y\rangle$[1],that is to say, x_i satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e.,

$$P_i(x_i, u, \dot{u}, ..., y, \dot{y}, ...) = 0, \quad i \in \{1, 2, ..., n\} \tag{1}$$

with coefficients in $\mathbb{R}\langle u, y\rangle$

Example 1: Considering the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= u + x_2 x_1 \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 + a x_1 \end{aligned} \tag{2}$$

If we define $y = x_2$, then

$$\begin{aligned} x_1 &= \dot{y} \\ x_2 &= y \\ \dot{x}_3 &= y + a\dot{y} \end{aligned} \tag{3}$$

The previous system is not algebraically observable since x_3 cannot be expressed as a differential algebraic polynomial in terms of $\{u, y\}$. For that reason, we present the next definition.

Definition 2 (Liouvillian system): A dynamical system is said to be Liouvillian if the elements (for example, state variables or parameters) can be obtained by an adjunction of integrals or exponentials of integrals of elements of $\mathbb{R}$.

[1] $\mathbb{R}\langle u, y\rangle$ denotes the differential field generated by the field $\mathbb{R}$, the input u, the measurable output y, and the time derivatives of u and y.

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Example 2: We consider the nonlinear system (2). From (3) we can observe that, although x_3 does not satisfy the AOC we can obtain it by means of the integral

$$x_3 = \int (y + a\dot{y})$$

then the nonlinear system (2) is Liouvillian.

III. Problem formulation and main result

If we consider the following chaotic Liouvillian system

$$\begin{aligned} \dot{x} &= Ax + \psi(x) + \zeta(u) \\ y &= Cx \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^n$, is the state vector; $u \in \mathbb{R}^l$, is the input vector, $l \leq n$; $y \in \mathbb{R}$ is the measured output; $\zeta(\cdot) : \mathbb{R}^l \to \mathbb{R}^n$is an input dependent vector function; $A \in \mathbb{R}^{nxn}$ and $C \in \mathbb{R}^{1xn}$ are constants; and $\psi(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is an state dependent nonlinear vector function.

We restrict each $\psi(\cdot)$ to be nondecreasing, that is, for all $a, b \in \mathbb{R}$, $a > b$, it satisfies the following monotone sector condition

$$0 \leq \frac{\psi_i(a) - \psi_i(b)}{a - b}, \quad i = 1, ..., n \tag{5}$$

The following observer's definition shows the relation between the observers for nonlinear systems and chaos synchronization.

Definition 3: (Exponential Observer). An exponential observer for (4) is a system with state $\hat{x}$ such that

$$\|x - \hat{x}\| \leq k \exp(-\xi t) \tag{6}$$

where k and ξ are positive constants.

In this work we will use the master-slave synchronization, in this type of synchronization x can be considered as the state variable of the master system, and $\hat{x}$ can be considered as the state variable of the slave system. the master-slave synchronization problem can be solved by designing an observer for (4)

We will solve the synchronization problem by using a exponential polynomial observer based upon the Lyapunov method [35]. To this end, we first compute the dynamics of the synchronization error (difference between the master and the slave systems). Next, by means of a simple quadratic Lyapunov function, we prove the exponential convergence.

System (4) is assumed to be a chaotic Liouvillian system, then by definition 2 all states of (4) can be reconstructed. In this sense, we will propose the following observation scheme.

The observer structure. The observer for system (4) has the next form

$$\dot{\widehat{x}} = A\hat{x} + \psi(\widehat{x}) + \zeta(u) + \sum_{i=1}^{m} K_i(y - C\widehat{x})^{2i-1} \tag{7}$$

where $\widehat{x} \in \mathbb{R}^n$,and $K_i \in \mathbb{R}^n$, for $1 \leq i \leq m$.

Remark 1: The meaning of m can be understood as follows. As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the observed state, therefore to improve the estimation performance high order terms are included in the observer structure. In other words, the rate of convergence can be increased by injecting additional terms with increasing powers of the output error.

Observer Convergence Analysis. In order to prove the observer convergence, we analyze the observer error which is defined as $e = x - \widehat{x}$. From eqs. (4) and (7), the dynamics of the state estimation error is given by

$$\dot{e} = (A - K_1C)e + \phi(e) - \sum_{i=1}^{m} K_i(y - C\widehat{x})^{2i-1} \tag{8}$$

where $\phi(e) = \psi(x) - \psi(\widehat{x})$.

It follows from (5) that each component of $\phi(e)$ satisfies

$$0 \leq \frac{\phi_i(e_i)}{e_i}, \quad \forall e_i \neq 0 \tag{9}$$

this implies a relationship between $\phi(e)$ and e as follows,

$$e^T\phi(e) = \sum_{i=1}^{n} e_i\phi_i(e_i) = \sum_{i=1}^{n} e_i^2 \frac{\phi_i(e_i)}{e_i}$$

and using (9) we have the following condition

$$0 \leq e^T\phi(e) \tag{10}$$

Properties (9) and (10) will allow us prove that the state estimation error $e(t)$ decays exponentially.

We have the main result.

Proposition 1: Consider the chaotic Liouvillian system (4) and the observer (7). If there exists a matrix $P = P^T > 0$,and scalars $\varepsilon > 0$, $\varepsilon_1 > 0$ satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} (A - K_1C)^TP + P(A - K_1C) + \varepsilon I & P - \varepsilon_1 I \\ P - \varepsilon_1 I & 0 \end{bmatrix} \leq 0 \tag{11}$$

and

$$\lambda_{\min}(M_i + M_i^T) \geq 0, \quad i = 2, ..., m \tag{12}$$

with $M_i := PK_iC$.Then, there exist positive constants k and ξ such that, for all $t \geq 0$

$$\|e(t)\| \leq k \exp(-\xi t)$$

with $k = \sqrt{\frac{\beta}{\alpha}}\|e(0)\|, \xi = \frac{\varepsilon}{2\beta}$, $\alpha = \lambda_{\min}(P)$, and $\beta = \lambda_{\max}(P)$

Proof: We proposed the following Lyapunov function candidate $V = e^TPe$. From (8), the time derivative of V is

$$\begin{aligned} \dot{V} &= e^T[(A - K_1C)P + P(A - K_1C)]e + 2e^TP\phi(e) \\ &\quad -2\sum_{i=2}^{m}(Ce)^{2i-2}e^TM_ie \\ &= e^T[(A - K_1C)P + P(A - K_1C)]e + 2e^TP\phi(e) \\ &\quad -\sum_{i=2}^{m}(Ce)^{2i-2}e^T\left(M_i + M_i^T\right)e \end{aligned}$$

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From (11) and (12) we have

$$\dot{V} \leq -\varepsilon e^T e - 2\varepsilon_1 e^T \phi(e)$$

by properties (9) and (10) we have

$$\dot{V} \leq -\varepsilon \|e\|^2 \tag{13}$$

We write the Lyapunov function as $V = \|e\|_p^2$, then by Rayleigh-Ritz inequality we have that

$$\alpha \|e\|^2 \leq \|e\|_p^2 \leq \beta \|e\|^2 \tag{14}$$

where $\alpha = \lambda_{\min}(P)$ and $\beta = \lambda_{\max}(P) \in \mathbb{R}^+$ (because P is positive definite). By using (14) we obtain the following upper bound of (13)

$$\dot{V} \leq -\frac{\varepsilon}{\beta} \|e\|_p^2 \tag{15}$$

Taking the time derivative of $V = \|e\|_p^2$ and replacing in inequality (15), we obtain

$$\frac{d}{dt} \|e\|_p \leq -\frac{\varepsilon}{2\beta} \|e\|_p$$

Finally, the result follows with

$$\|e(t)\| \leq k \exp(-\xi t)$$

where $k = \sqrt{\frac{\beta}{\alpha}} \|e(0)\|$, and $\xi = \frac{\varepsilon}{2\beta}$.

■

IV. Numerical results

Chua's circuit has become a paradigm for learning, understanding and studying nonlinear dynamics and chaos [29]. In recent years, a lot of modified Chua's circuits were developed [30], [31]; the applications of Chua's circuit were investigated well, especially in utilizing synchronization of Chua's circuit to realize secure communication and utilizing chaos in Chua's circuit to describe practical chaotic systems.

The circuit shown in Fig 1 contains three linear energy-storage elements (an inductor, and two capacitors), a linear resistor, and a single nonlinear resistor N_R.

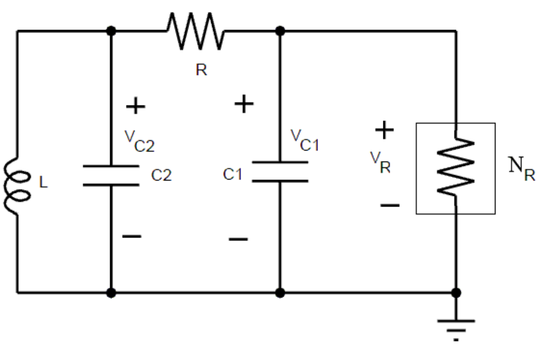

Fig. 1: Chua's circuit

Using KVL and KCL, the equations that describe the nonlinear dynamics of Chua's circuit are as follows:

$$\begin{aligned} c_1 \frac{dv_{c_1}}{dt} &= G(v_{c_2} - v_{c_1}) - g(v_{c_1}) \\ c_2 \frac{dv_{c_2}}{dt} &= G(v_{c_1} - v_{c_2}) + i_L \\ L\frac{di_L}{dt} &= -v_{c_2} \end{aligned} \tag{16}$$

where $G = \dfrac{1}{R}$ and $g(\cdot)$ is a nonincreasing function defined by:

$$g(v_{c_1}) = m_0 v_{c_1} + \frac{1}{2}(m_1 - m_0)\left\{|v_{c_1} + B_p| - |v_{c_1} - B_p|\right\} \tag{17}$$

This function is shown graphically in Fig. 2 the slopes in the inner and outer regions are m_0 and m_1 respectively; $\pm B_p$ denote the breakpoints. The nonlinear resistor N_R is termed voltage-controlled because the current in the element is a function of the voltage across its terminals.

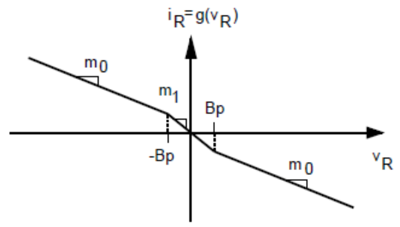

Fig. 2:Three-segment piecewise-linear v-i characteristic of the resistor in Chua's circuit. The outer regions have slopes m_0, the inner region has slope m_1. There are two breakpoints at $\pm B_p$.

In the first reported study of this circuit, Matsumoto [33] showed by computer simulation that the system possesses a strange attractor called the Double Scroll. Experimental confirmation of the presence of this atractor was made shortly afterwards in [34].

If we consider the measured output $y = v_{c_2}$ from equations of (16) we obtain:

$$\begin{aligned} v_{c_1} &= \frac{C_2}{G}\dot{y} + y + \frac{1}{LG}\int y dt \\ v_{c_2} &= y \\ i_L &= -\frac{1}{L}\int y dt \end{aligned} \tag{18}$$

From (18), the Chua's system (16) is Liouvillian. This implies that unknown variables v_{c_1} and i_L can be reconstructed with the selected output $y = v_{c_2}$.

Chua's system (16) is of the form (4) with $\zeta(u) = 0$,

$$A = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \quad \psi(x) = \begin{bmatrix} -\frac{g(x_1)}{C_1} \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} v_{c_1} & v_{c_2} & i_L \end{bmatrix}^T$$

Since $g(x_1)$ is nonincreasing and C_1 is a positive constant, then $\psi_1(x) = -g(x_1)/C_1$ is nondecreasing as in (5) and

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$\psi_2(x) = \psi_3(x) = 0$ are nondecreasing as in (5) and Chua's system is Liouvillian, so that, we proceed with the observer design.

If we apply (9) to the system (16) we get the following

$$\dot{\hat{x}} = \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ \frac{G}{C_2} & -\frac{G}{C_2} & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} -\frac{g(v_{c_1})}{C_1} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} k_{1,i} \\ k_{2,i} \\ k_{3,i} \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} e \right)^{2i-1}$$

Hence, the state observer is rewritten as,

$$\begin{aligned} \dot{\hat{x}}_1 = & -\tfrac{G}{C_1}\hat{x}_1 + \tfrac{G}{C_1}\hat{x}_2 - \tfrac{g(\hat{x}_1)}{C_1} + k_{1,1}e_{1,2} \\ & + k_{1,2}(e_{1,2})^3 + ... + k_{1,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_2 = & \tfrac{G}{C_2}\hat{x}_1 - \tfrac{G}{C_2}\hat{x}_2 + \tfrac{1}{C_2} + k_{2,1}e_{1,2} \\ & + k_{2,2}(e_{1,2})^3 + ... + k_{2,m}(e_{1,2})^{2m-1} \\ \dot{\hat{x}}_3 = & -\tfrac{1}{L}\hat{x}_3 + k_{3,1}e_{1,2} \\ & + k_{3,2}(e_{1,2})^3 + ... + k_{3,m}(e_{1,2})^{2m-1} \end{aligned} \qquad (19)$$

Figure 3 shows the general diagram of the synchronization of Chua's circuit (18) and the exponential observer (19) in master-slave configuration.

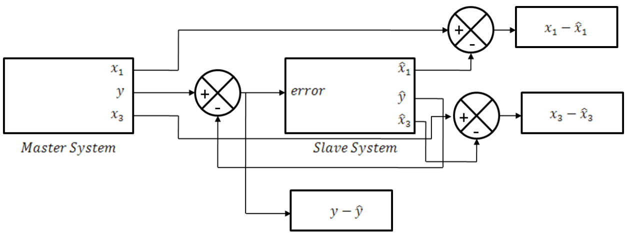

Fig. 3: Synchronization diagram.

Numerical simulations for the synchronization of Chua's system are carried out in order to show the performance of the exponential observer. The parameter values considered in the numerical simulations correspond to chaotic behavior [32] and these are: $C_1 = 10$ nF, $C_2 = 100$ nF, $R = 1.8$ kΩ, $L = 18$ mH, $m_0 = -0.409$ mS, $m_1 = -0.756$ mS and $B_p = 1.08$ V. The Matlab-Simulink® program uses the Dormand–Prince integration algorithm, with the integration step set to 1×10^{-5}.

We fix $m = 2$ in the observer (19). The LMI (11) is feasible with $\varepsilon = 0.001$ and $\epsilon_1 = 0.001$, a solution is

$$P = \begin{bmatrix} 0.0008 & -0.0006 & 0.1021 \\ -0.0006 & 0.0005 & -0.0805 \\ 0.1021 & -0.0805 & 15.0959 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} k_{1,1} \\ k_{2,1} \\ k_{3,1} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 45 \end{bmatrix},$$

and K_2 is chosen such that (12) is satisfied, then we obtain

$$K_2 = \begin{bmatrix} k_{1,2} \\ k_{2,2} \\ k_{3,2} \end{bmatrix} = \begin{bmatrix} 7.1573 \\ 14.1040 \\ 0.0268 \end{bmatrix}.$$

Figs. 4-5 show the obtained results for the initial conditions $x_1 = -0.5, x_2 = 1, x_3 = 0, \hat{x}_1 = 1, \hat{x}_2 = 0.5$ and $\hat{x}_3 = 0.002$.

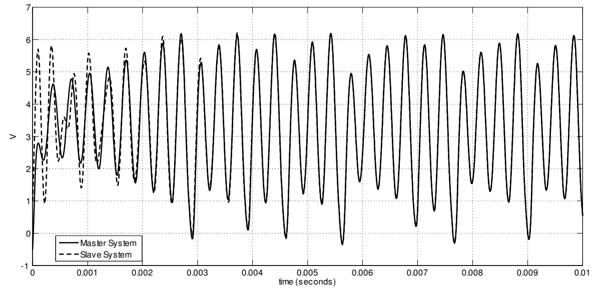

Fig. 4: Synchronization of x_1.

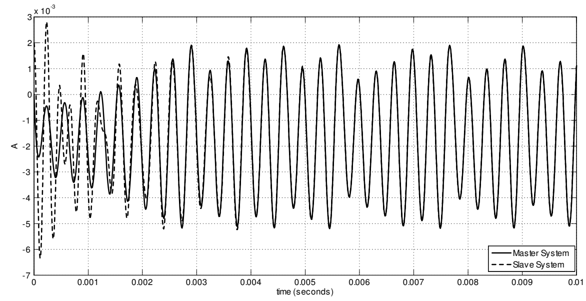

Fig. 5: Synchronization of x_3.

The performance of the proposed observer is evaluated by means of the relative error which in this case is defined as

$$\bar{e}_i = \frac{|x_i - \hat{x}_i|}{|x_i|}, \qquad i = 1, 3.$$

Figs. 6-7 illustrate the corresponding relative errors, it should be noted that $\bar{e}_1 = 0$ and $\bar{e}_3 = 0$ for $t > 0.006s$, as was expected. Finally.

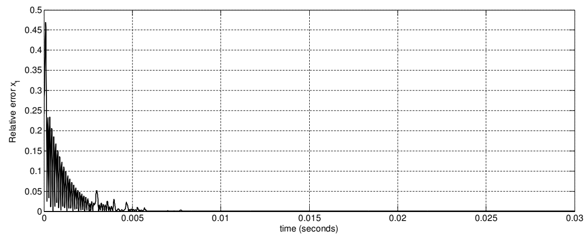

fig. 6: Relative error e_1

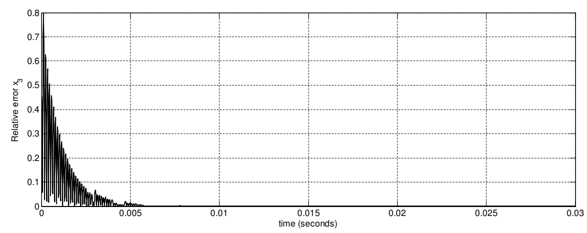

fig. 7: Relative error e_3

Fig. 9 presents the synchronization in a phase diagram, where clearly is observed the chaotic behavior of the Chua's circuit.

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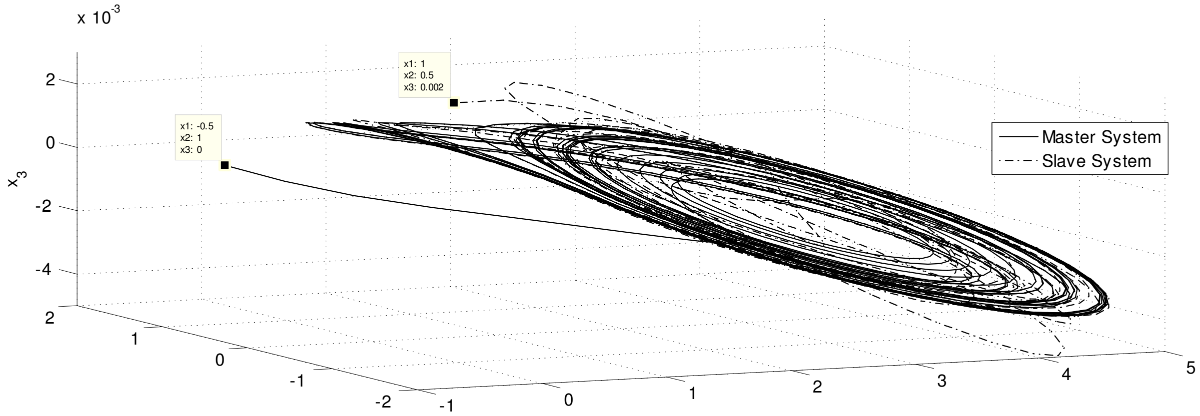

Fig. 8: Phase diagram

V. Conclusions

The synchronization problem of chaotic Liouvillian systems has been treated by using differential and algebraic techniques. We proposed a polynomial observer, and by means of properties of nondecreasing functions, linear matrix inequalities and with the help of the Lyapunov method we proved that the estimation error exponentially converges to zero.

This observer has been used as a slave system whose states are synchronized with the chaotic system (Chua's circuit). A reduced set of measurable state variables were needed to achieve the synchronization with this approach.

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Synchronization of an Uncertain Chaotic System Based on Sliding Mode Control

Juan L. Mata, Rafael Martínez–Guerra, Ricardo Aguilar, and Carlos Aguilar

Abstract— **In this paper we deal with the synchronization and parameter estimations of an uncertain Rikitake system. The strategy consists of proposing a slave system which has to follow asymptotically the unknown Rikitake system, refereed as master system. The gains of the slave system are adjusted continually according to a convenient adaptation control law, until the measurable output errors converge to zero. The convergence analysis is carried out by using Barbalat's Lemma.**

I. INTRODUCTION

The synchronization of chaotic systems has been investigated since its introduction in the paper by Pecora and Carrol in 1990 [1]. Synchronization of chaotic systems has attracted much attention due to its potential applications in secure communications, chemical reactions, biological systems and so on [2], [3], [4] . Several synchronization schemes have been proposed to tackle the problem [2], [5]. In the last years, some methods to achieve synchronization have been proposed from the control theory perspective such as the famous observer-based approach [6], [7], and the so-called adaptive synchronization method [8]. For the synchronization problem, we consider a chaotic system, the master (or drive), together with the slave (or response). The goal is to synchronize the complete response of the slave system to the master system by driving the slave with a signal derived from the master. The problem can now be easily tackled when the parameters of the master system are known. The aforementioned methods and many others are valid for chaotic systems only when the system's parameters are known. However, to achieve synchronization between two chaotic systems is far from being straightforward, in fact there is no much work about this challenging problem because it consists of both identification of the unknown parameters and the design of a controller to achieve synchronization. Guan *et al.* applied an observer to identify the unknown parameter of the Lorenz system [9]. Lü *et al.* studied the same problem for Chen's chaotic system with the same method [10]. The interest in parameters identification lies on its potential applications in communications, essentially when parameter modulation is used for message transmission.

In this paper an adaptative asymptotic method for the synchronization and the identification of the Rikitake system with several unknown parameters is presented. This system resembles the reversal of polarity of the Earth's electromagnetic field, and it is well-known that it has a chaotic behaviour for some set of inital conditions and some set of parameter values. By this method, we can achieve chaos synchronization and identify the unknown parameters simultaneously. Roughly speaking the suggested approach consists of desinging a controlled slave system, whose controllers and adaptive parameters are adjusted accordingly to a proposed adaptive algorithm. It is done in such a way that the synchronization errors between the outputs of both systems, the uncertain Rikitake and the slave, asymptotically converge to zero. The convergence analysis of the proposed scheme is carried out using the Lyapunov method in conjunction with the Barbalat's Lemma. It is important to emphasize that the robustness of our control strategy allows to effectively detect piecewise constant variations on the parameter values of the uncertain Rikitake system.

The remaining of this work is organized as follows. In Section II we introduce the problem statement. In Section III we develope our solution to synchronize and identify the constant unknown parameters of the Rikitake system by means of the Lyapunov method. To assess the effectiveness of our method we present some numerical simulations in Section IV. Finally, we present the conclusions in Section V.

II. PROBLEM STATEMENT

II-A. Rikitake model system

A simple mechanical model used to study the reversals of the magnetic field of the Earth, idealized by the Japanese geophysicist Rikitake [11], consists of two identical single Faraday-disk dynamos of the Bullard type coupled together. The dynamics of this system is governed by the following three dimensional system of nonlinear differential equations:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + z_1 y_1 \\ \dot{y}_1 &= -\mu y_1 + (z_1 - a)x_1 \\ \dot{z}_1 &= 1 - x_1 y_1 \end{aligned} \tag{1}$$

where the parameters μ and a have some physical meaning when they are positive. For a physical meaning of the states x_1, y_1 and z_1 we recommend to see [11]. However, the states $x_1$and y_1 are directly related to the currents through each disc of the dynamo system, and z_1 is related to the angular

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velocity of one of the discs. This system displays a chaotic behavior for the parameters values in a neighborhood $\{\mu = 5, a = 2\}$ and for a large enough set of initial conditions.

II-B. *Some Algebraic properties and Problem formulation*

In this section we present some algebraic properties that the Rikitake System satisfies. To this end we introduce the following definitions.

Definition 1: Consider a smooth nonlinear system, described by a state vector $X = \{x_i\}_1^{i=n} \in R^n$ and by the output vector $G = \{g_i\}_1^{i=m} \in R^m$, of the form:

$$\dot{X} = f(X, P) \quad , \quad G = h(X), \tag{2}$$

where $h(.)$ is a smooth vector function and $P \in R^l$ is a constant parameters vector, with $l < n$. Let $G^{(j)}$ denote the j-th time derivative of the vector G. We say that the vector state X is algebraically observable, if it can be uniquely expressed as

$$X = \Phi(G, G^{(1)}, .., G^{(j)})$$

for some integer j and for some smooth function Φ.

Definition 2: Under same conditions as in *definition 1*. If the vector of parameters, P satisfies the following relation

$$\Omega_1(G, .., G^{(j)}) = \Omega_2(Y, .., Y^{(j)})P, \tag{3}$$

where $\Omega_1(.)$ and $\Omega_2(.)$ are, respectively, $n \times 1$ and $n \times n$ smooth matrices, then P is said to be algebraically linearly identifiable with respect to the output vector G [12].

According to the previous definitions, it is evident that system (1) is algebraically observable with respect to the outputs $g_1 = x_1$ and $g_2 = y_1$, since the state, z_1, can be rewritten, as

$$z_1 = \frac{\dot{g}_2 - \mu g_2}{g_1} + a, \tag{4}$$

hence, Rikitake system is algebraically observable with respect to the selected outputs, $g_1 = x_1$ and $g_2 = y_1$. Moreover, substituting the above expression into the first differential equation of (1), we have

$$\dot{g}_1 - \dot{g}_2 = -\mu(g_1 + g_2) + a g_1. \tag{5}$$

Therefore, we conclude that system (1) vector of parameters $\mathbf{p}$=(μ, a) is algebraically identifiable with respect to the available outputs. That is, the non-available state z_1 and the vector of parameters $\mathbf{p}$ can be simultaneously recovered, from the knowledge of the the outputs $g_1 = x_1$ and $g_2 = y_1$.

From the above definitions, it is possible to solve the synchronization problem of the uncertain Rikitake system, provided that the states x_1 and y_1 are always available. Moreover, it is also possible to recover the unknown parameters μ and a. Thus, we are ready to establish the main control problem of this work.

II-C. *Problem Statement*

Consider the uncertain Rikitake system (1), referred as the master system, with the available output states x_1 and y_1. And let us propose the following slave controlled system:

$$\begin{aligned} \dot{x}_2 &= -\widehat{\mu}x_1 + z_2 y_1 + u_1, \\ \dot{y}_2 &= -\widehat{\mu}y_2 + (z_2 - \widehat{a})x_1 + u_2, \\ \dot{z}_2 &= 1 - x_1 y_1 + u_3. \end{aligned} \tag{6}$$

Then, the control objective is to find $\mathbf{u}$=$(u_1,\ u_2, u_3)$ and $\widehat{\mathbf{p}}$=$(\widehat{\mu}, \widehat{a})$ such that the slave system (6) follows to the unknown Rikitake system (6); with $\widehat{\mathbf{p}}$ converging to the actual values of (μ, a). In other words, we need to find $\mathbf{u}$ and $\widehat{\mathbf{p}}$ of system (6), such that $(\mathbf{w}_1, \widehat{\mathbf{p}}) \to (\mathbf{w}_2, \mathbf{p})$, as long as $t \to \infty$.[1]

We finish this section by introducing the following errors:

$$\begin{aligned} &e_x = x_1 - x_2; \quad e_y = y_1 - y_2; \quad e_z = z_1 - z_2; \\ &\widetilde{\mu} = \mu - \widehat{\mu}; \qquad \widetilde{a} = a - \widehat{a}, \end{aligned}$$

and according to them, we define the following vectors:

$$\mathbf{e}^T = (e_x, e_y, e_z) \quad ; \quad \widetilde{\mathbf{p}}^T = (\widetilde{\mu}, \widetilde{a}).$$

III. LYAPUNOV BASED FORMULATION

In this section we solve the synchronization and parameters identification of the constant unknown parameters of Rikitake system by means of the Lyapunov method. To this end, we first compute the dynamics of the synchronization errors, between the master and the slave systems. Next, based on a simple quadratic Lyapunov function, we propose the needed controller and the needed estimator that assure the synchronization of both systems.

Before to solve the control problem we introduce the following assumptions related with the selected outputs of the master system:

A1) *The states x=x_1 and y=y_1 are available, for measurement.*

A2) *All the states of the master system are bounded, with the generic property that the steady solution x and y, continues oscillating around zero.*

Comment 1: *Assumption A2 is a realistic because in most case all the states of Rikitake system are bounded; for a large set of initial conditions and for a large set of positive parameters μ and a. In fact assumption A2 depends on the set of initial conditions and the values of the parameter vector q. To clarify the meaning of this property, we present a case where assumption A2 does not hold. Selecting the parameters values as $\{\mu > 0; a > 0\}$, and the initial condition as $w_1(0) = (x_1(0) = 0, x_1(0) = 0, z_1(0) = \overline{z})$; we have that $x_1(t) = 0$, $y_1(t) = 0$ and $z_1(t) = t + \overline{z}$. Evidently, assumption A2 can not be fulfilled because the states x and y remain fixed at the origin and, the state z_1 is not bounded [13]. In fact, no identification method or scheme can be proposed if the master system have solutions that tend either to infinity or to a constant.*

[1]Here we denote the vector state related with the master and slave system, as $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. That is, $\mathbf{w}_i^T = (x_i, y_i, z_i)$; for $i = \{1, 2\}$.

Now, from equations (1) and (6), we have:

$$\dot{\mathbf{e}}= \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\widetilde{\mu}x + e_z y - u_1 \\ -\widetilde{\mu}y + (e_z - \widetilde{a})x - u_2 \\ -u_3 \end{bmatrix} \tag{7}$$

where for simplicity, we stand for $y = y_1$ and $x = x_1$. As we can see, the above system can be considered as a control problem where the vector inputs $\mathbf{u}$ and $\widetilde{\mathbf{p}}$ must be proposed such, the state $\mathbf{e}$ asymptotically converges to zero.

Consider a Lyapunov function

$$V = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2}\widetilde{\mathbf{p}}^T\widetilde{\mathbf{p}}. \tag{8}$$

The time derivative of V along the trajectories of (7) is then given by

$$\begin{aligned} \dot{V} = \;& \widetilde{a}\dot{\widetilde{a}} + \widetilde{\mu}\dot{\widetilde{\mu}} - \tilde{\mu}xe_x + e_x e_z y - e_x u_1 - \tilde{\mu}ye_y \\ & + e_z x e_y - \widetilde{a}xe_y - u_2 e_y - e_z u_3 \end{aligned} \tag{9}$$

Now, in order to make $\dot{V}$ semi-definite negative, we propose $\dot{\widehat{\mathbf{p}}}$, as

$$\dot{\widehat{\mathbf{p}}} = \begin{bmatrix} \dot{\widehat{\mu}} \\ \dot{\widehat{a}} \end{bmatrix} = \begin{bmatrix} e_x x + e_y y \\ x e_y \end{bmatrix} \tag{10}$$

and $\mathbf{u}$, as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_{11}\mathrm{sign}(e_x)e_x^m \\ k_{21}\mathrm{sign}(e_y)e_y^m \\ e_x y + e_y x \end{bmatrix}. \tag{11}$$

where k_{11} and k_{21} are strictly positive constants and m is any positive even integer.

Substituting the above $\mathbf{u}$ in (7), we have that the closed-loop system can be read as

$$\begin{aligned} \dot{e}_x &= -\widetilde{\mu}x + e_z y - k_{11}\mathrm{sign}(e_x)e_x^m \\ \dot{e}_y &= -\widetilde{\mu}y + (e_z - \widetilde{a})x - k_{21}\mathrm{sign}(e_y)e_y^m \\ \dot{e}_z &= -e_x y - e_y x \end{aligned} \tag{12}$$

where the parameter dynamics are give by

$$\dot{\widehat{\mu}} = e_x x + y e_y \quad ; \quad \dot{\widehat{a}} = x e_y. \tag{13}$$

Substituting (12) and (13) into (9), we obtain

$$\dot{V} = -\left(k_{11}|e_x|e_x^m + k_{21}|e_y|e_y^m\right). \tag{14}$$

This implies that $\dot{V}$ is semi-definite negative and so V converges. Hence, the set of signals $\{e_x, e_y, e_z, \widetilde{\mu}, \widetilde{a}\}$ are bounded. Let us proceed to show that $\mathbf{e}$ converges to zero as long as $t \to \infty$, by applying Barbalat's Lemma [14].[2] Integrating both sides of (14), we obtain

$$\int_0^t \left[k_{11}|e_x(s)|e_x^m(s) + k_{21}|e_y(s)|e_y^m(s)\right] ds \quad \leq V(0) \tag{15}$$

From the equations of (12) and **A2**, it follows that $\dot{\mathbf{e}}$ is bounded which implies that $\mathbf{e}$ is uniformly continuous, Using Barbalat's Lemma, it follows that vector states $\mathbf{e} \to 0$ as $t \to \infty$. Once again, differentiating (12), it is easily to show that $\ddot{\mathbf{e}}$ is bounded. Thus, $\dot{\mathbf{e}}$ is uniformly continuous and also $\mathbf{e}$ has a finite limit, as $t \to \infty$, from Barbalat's Lemma, we conclude that $\dot{\mathbf{e}} \to 0$ as $t \to \infty$. Since V converges, as $t \to \infty$, then we have that the two parameter errors $\widetilde{\mu}$ and $\widetilde{a}$ converge as $t \to \infty$ (8). Besides, from (13), it follows that $\dot{\widehat{\mu}}$ and $\dot{\widehat{a}}$ converge to zero, as $t \to \infty$. Roughly speaking, when t is large enough $\widehat{\mu}$ and $\widehat{a}$ are almost constant, and the differential equations of (12) implies that

$$0 = (\mu - \widehat{\mu})x \quad 0 = (a - \widehat{a})y \tag{16}$$

However once again from assumption **A2,** we have that the steady states x and y remains oscillating around zero. Therefore necessarily $\mu = \widehat{\mu}$ and $a = \widehat{a}$. That is, $\widetilde{\mathbf{p}}^T \to 0$, as $t \to \infty$.

We summarize the previous discussion in the following proposition:

Proposition 1: *Under the assumptions* ***A1*** *and* ***A2*** *the synchronization and the parameters estimation problem between systems (1) and (6), can be achieved for any strictly positive constants*$\{k_{11}, k_{22}\}$ *and for any positive even integer* m. ■

IV. NUMERICAL RESULTS

Computer simulations have been carried out in order to test the effectiveness of the proposed asymptotic control strategy for the synchronization and for the recovery of the unknown parameters of an uncertain Rikitake system. The program uses the Runge–Kutta integration algorithm, with the integration step set to 0.001.

In the first simulation we illustrate the qualitative property describe in the assumption **A2**. For this end, we fixed the master system parameter as $\mathbf{q}$=($\mu = 2, a = 5$); while the initial conditions were selected as $\mathbf{w}_1(0) = (x_1(0) = 1, y_1(0) = -1,\ z_1(0) = 0)$. Figure 1 shows the behavior of the whole state of the Rikitake system. We can see from this figure that the whole state solution of the master system is bounded and the states x_1 and y_1 remain oscillating around zero; therefore, we can claim that the assumption **A2** is completely fulfilled.

To show the performance of the proposed control strategy we carried out a second simulation using the same set-up as above, and fixing the slave system gains as $k_1 = k_2 = 0{,}8$ and $k_{12} = k_{22} = 0{,}266$; with the slave system initialized at zero, *i.e.*, $\mathbf{w}_2(0) = 0$ and $\widehat{\mathbf{p}}(0) = 0$. In Figure 2 we can see that the synchronization errors asymptotically converge to zero. That is, the slave system follows almost perfectly the uncertain master system. The estimated parameters are shown in Figure 3. As we expect, a better performance can be obtained as long as the time is increased. From this simulations we can concluded that the proposed estimator reconstructs reasonably well the parameters after elapsed 100 seconds.

V. CONCLUSIONS

A Lyapunov based approach for the synchronization and parameters identification of the constant unknown parameters of Rikitake system was presented. Under the assumptions

[2] Lemma (Barbalat): If the differential function $f(t)$ has a finite limit as $t \to \infty$, and if df/dt is uniformly continuous, then $df/dt \to 0$ as $t \to \infty$. A consequence of this Lemma is that if $f \in L_2$ and df/dt is bounded then $f \to 0$ as $t \to \infty$.

that the two output states x and y are available for measurement. To accomplish this task, we first shown that the system is observable and linearly algebraically identifiable, with respect to the available outputs. Then, we propose a slave controlled system; where theirs controllers were proposed such that the vector synchronization error and the vector parameter error, among the master and slave system, asymptotically converge to zero. The convergence proof was carried out by using the traditional Lyapunov method in combination with the Lemma of Barbalat and the assumption **A2**.

Also, it is worth to mentioning that our parameters identification algorithm does not requires that the Rikitake system always exhibits a chaotic behavior. Finally, numerical simulations were carried out to evaluate the performance of the proposed solution.

VI. ACKNOWLEDGMENTS

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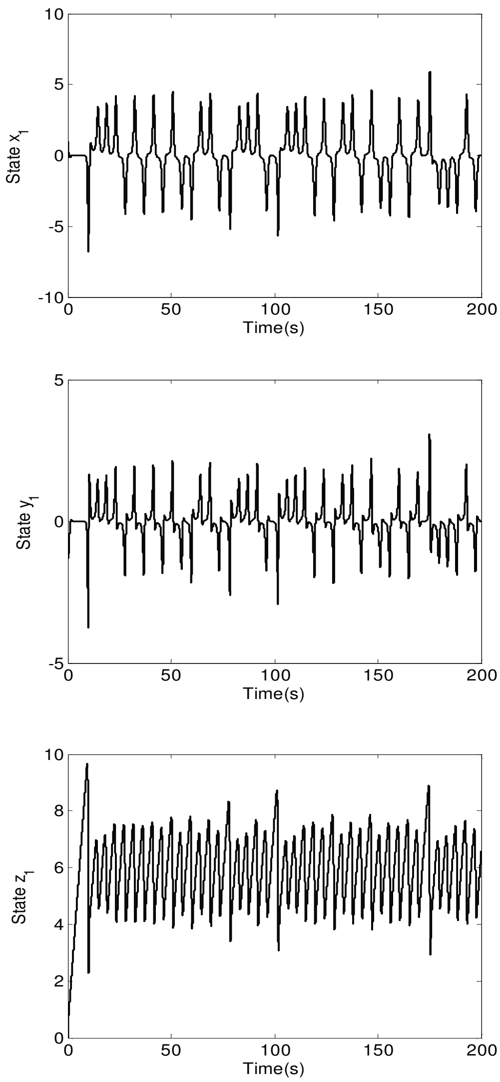

Fig. 1. This figure depicts the qualitative behavior of the Rikitake system when is initialized at $\mathbf{w}_1(0) = (1, -1,0)$ and the parameters vector is fixed as $\mathbf{q} = (2, 5)$. Clearly, the assumption **A2** is satisfied.

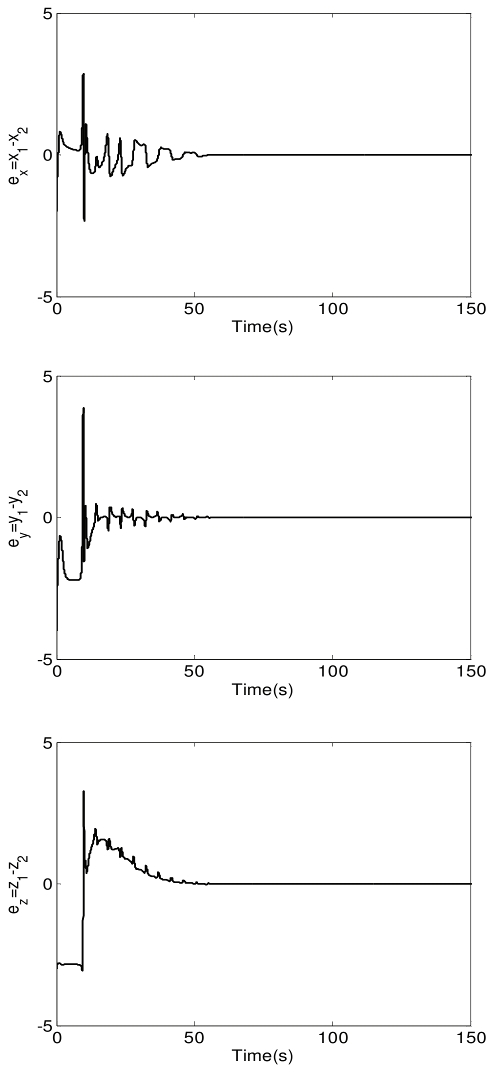

Fig. 2. This figure shows the convergence to zero of the master-slave synchronization error, when the master system is initialized at $\mathbf{w}_1(0) = (1, -, 0)$ and its parameters vector is fixed as $\mathbf{q} = (2, 5)$.

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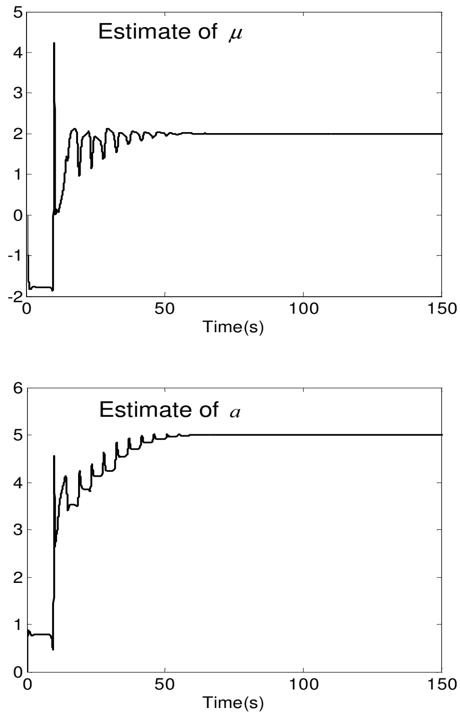

Fig. 3. This figure shows the parameters estimation, when the master system is initialized at $\mathbf{w}_1(0) = (1, -1,0)$; and the actual parameters vector is fixed as $\mathbf{q} = (2, 5)$.

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Synchronization of Chaotic Systems: A Real–Time Application to Colpitts Oscillator

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***Abstract*— In this paper we deal with the synchronization of the Colpitts oscillator. The synchronization problem for chaotic systems is treated as an observation problem, some results based on a differential algebraic approach are used in order to determine observability with the measurements from the system. The strategy consists of proposing a slave system (observer) which tends to follow asymptotically the Colpitts oscillator, refereed as master system. The methodology was tested in a real–time implementation by means of a reduced order observer.**

***Index Terms*— Colpitts oscillator, reduced order observer, synchronization.**

I. INTRODUCTION

I-A. *Chaos in the Colpitts oscillator*

In 1994, chaotic oscillations in the Colpitts circuit with a generic transistor 2N2222A were reported [1]. In the work [2], the author refers to the multi–oscillations phenomenon in the RF bipolar oscillators, which are parasitic oscillations or not desired that coexist with a main oscillation. Due to these parasitic oscillations the resultant signal in steady state is severely affected, in this sense, those circuits have few applications. In the paper [1] is shown the parasitic oscillations are not present in the Colpitts oscillator, and moreover, the chaotic behavior of the circuit is presented, in fact, Colpitts oscillator is widely used in electronic devices and communication systems. Fig. 1 shows the circuit configuration [3], as well as some values for which the Colpitts oscillator has chaotic behavior.

I-B. *Synchronization in chaotic systems*

Chaotic systems synchronization has been investigated since its introduction in the paper [4]. Among the publications dedicated to chaos synchronization, many different approaches can be found. We cite the papers [5]–[10] which propose the use of state observers to synchronize chaotic systems; in references [11]–[14] use feedback controllers; in [15], [16] use nonlinear backstepping control; in papers [17], [18] consider synchronization time delayed systems; in works [19], [20] consider directional and bidirectional linear coupling; papers [21], [22] use nonlinear control; in papers [12], [13] use active control, in [14], [23], [24] use adaptive control and so on.

Synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields due to its potential applications, such as: secure communications, biological systems, chemical reactions, etc., [11], [25]–[27].

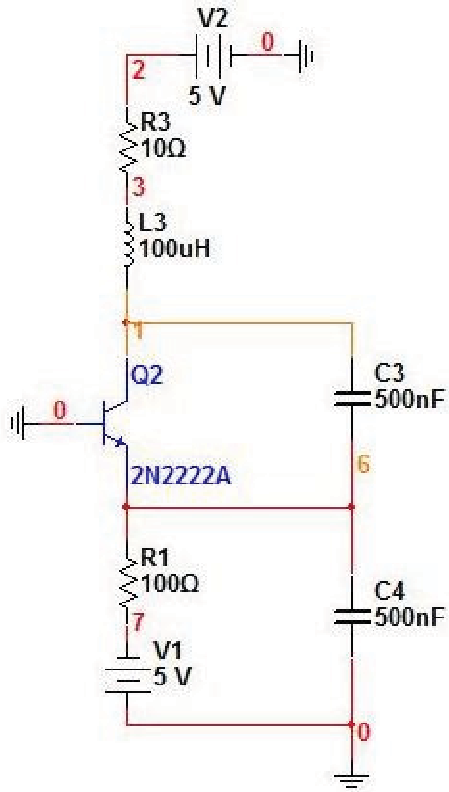

Fig. 1. Colpitts circuit.

As we can note, there exists several methods to solve the synchronization problem since the control theory perspective, in this work, we study the synchronization by means of state observers based on the differential algebraic approach.

The main contributions consist in the following: (i) the design of a reduced order observer for the synchronization of chaotic systems, which has asymptotic convergence; (ii) real–time implementation employing the Colpitts oscillator. Even though, the differential algebraic approach has been applied to the synchronization problem in the last years, there are not reported works containing real–time applications using this theoretical framework.

The method is based in a *master–slave* configuration [4]. The main characteristic is that the coupling signal is

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unidirectional, that is, the signal is transmitted from the master system (transmitter) to the slave system (receiver), the receiver is requested to recover the unknown (or full) state trajectories of the transmitter. By this fact, the terminology *transmitter–receiver* is also used. Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer [28]–[30].

The paper is organized as follows. In section II is presented the modeling of the Colpitts oscillator [3]. In section III is treated the synchronization problem and its solution by means of a reduced order observer. Section IV presents the procedure for the synchronization in real–time and the obtained results, as wells as, is shown the performance of the reduced order observer. Finally, in section V we close the paper with some concluding remarks.

II. THE MODEL OF THE COLPITTS OSCILLATOR

II-A. Modeling

In this work we consider the classical configuration of the Colpitts oscillator. The circuit contains a bipolar junction transistor (BJT) as the gain element, two resistors, and a resonant network consisting of an inductor and two capacitors (Fig. 1).

By taking into account the qualitative theory in nonlinear dynamics, we select a minimum model for the circuit. The idea is to consider as simple a circuit model as possible that describes the essential features exhibited by the real Colpitts oscillator. For the modeling of such circuit the following assumptions are considered:

1. The emitter current is generated by means of I_0.
2. The passive and active elements are ideals.
3. The transistor is modeled as a nonlinear resistor R_E controlled by voltage and a linear input controlled by current (Fig. 2), indeed,
   (a) We model V – I characteristic of R_E with an exponential function, as follows
   $$\begin{aligned} I_E = \; & I_S \left[\exp\left(\frac{V_{BE}}{VT}\right) - 1\right] \\ \approx \; & I_S \exp\left(\frac{V_{BE}}{VT}\right), \text{ if } V_{BE} \gg VT, \end{aligned} \tag{1}$$
   where I_S is the inverse saturation current and $VT \simeq 26$ mV at room temperature.
   (b) We assume that $\alpha_F = 1$, where α_F is the common–base forward short–circuit current gain. This corresponds to neglecting the base current.
   (c) The parasitic dynamics of the transistor are omitted.

The Colpitts circuit is described by a system of three nonlinear differential equations, as follows:

$$\begin{aligned} C_3\dot{V}_{C_3} = \; & -f(V_{C_4}) + I_{L_3} \\ C_4\dot{V}_{C_4} = \; & I_{L_3} - I_0 \\ L_3\dot{I}_{L_3} = \; & -V_{C_3} - VC_4 - R_3 I_{L_3} + V_{CC} \end{aligned} \quad , \tag{2}$$

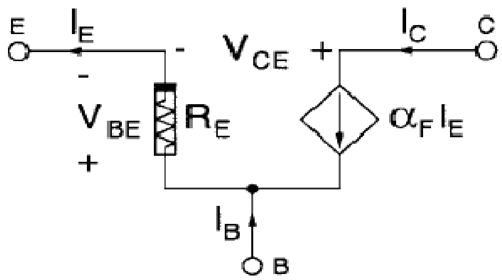

Fig. 2. BJT Transistor 2N2222A in common base configuration

where $f(\cdot)$ is the driving–point characteristic of the nonlinear resistor. This can be expressed in the form $I_E = f(V_{C_4}) = f(-V_{BE})$. In particular, from (1), we have

$$f(V_{C_4}) = I_S \;\exp\left(-\frac{V_{C_4}}{VT}\right).$$

II-B. Parameter normalization

We introduce the dimensionless state variables (x_1, x_2, x_3), and we choose the operating point of (2) to be the origin of the new coordinate system. In particular, we normalize voltages, currents and time with respect to $V_{ref} = VT$, $I_{ref} = I_0$ and $t_{ref} = 1/w_0$, respectively, where

$$w_0 = \frac{1}{\sqrt{L_3\left(\dfrac{C_3C_4}{C_3+C_4}\right)}} \quad ,$$

is the resonant frequency of the unloaded L–C tank circuit. Then, the state equations for the Colpitts oscillator can be rewritten in the next form:

$$\begin{aligned} \dot{x}_1 &= -a\;\exp(-x_2) + \; ax_3 + a \\ \dot{x}_2 &= bx_3 \\ \dot{x}_3 &= -cx_1 - cx_2 - dx_3 \end{aligned} \quad ,$$

where, $a = \frac{g}{Q(1-k)}$, $b = \frac{g}{Qk}$, $c = \frac{Qk(1-k)}{g}$, $d = \frac{1}{Q}$, $k = \frac{C_4}{C_3+C_4}$ and $Q = \frac{w_0 L_3}{R_3}$.

It should be noted that the dynamical behavior of the new coordinate system depends only on the parameters Q and g, which are described by:

- Q is the quality factor of the unloaded tank circuit.
- g is the open loop gain of the oscillator when the phase conditions of the Barkhausen criterion [31] are fulfilled.

On the other hand, the parameter k has just a scaling effect on the state variables.

III. SYNCHRONIZATION VIA A REDUCED ORDER OBSERVER

III-A. Observer design

Let us consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) = \; & f(x, u) \\ y(t) = \; & h(x) \end{aligned} \tag{3}$$

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where $f \in \mathbb{R}^n$ is differentiable and satisfies $f(0,0) = 0$, $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ is the output system (nonsingular function) and $u \in \mathbb{R}^l$ is the known input ($l \leq n$).

Definition 1 [8]: a state variable $x \in \mathbb{R}$ is said to be algebraically observable if is algebraic over $\mathbb{R}\langle u, y\rangle$, that is, x satisfies a differential algebraic polynomial in terms of $\{u, y\}$ and some of their time derivatives, i.e.,

$$P(x, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0 \tag{4}$$

with coefficients in $\mathbb{R}\langle u, y\rangle$.

In this paper the above definition is refereed as Algebraic Observability Condition (AOC).

Now, let us consider the nonlinear system described by (3). The unknown states of the system can be included in a new variable $\eta(t)$ and the following new extended system is considered

$$\begin{aligned} \dot{x}(t) &= f(x, u, \eta) \\ \dot{\eta}(t) &= \Delta(x, u, \eta) \\ y(t) &= h(x) \end{aligned}$$

where $\Delta(x, u, \eta)$ is a bounded uncertain function. The problem is to construct the variable $\eta(t)$.

The next system represents the dynamics of the unknown states:

$$\dot{\eta}(t) = \Delta(x, u, \eta) \tag{5}$$

Lemma 1: if the following hypotheses are satisfied:

H1: $\eta(t)$ is algebraically observable (definition 1).

H2: γ is a C^1 real-valued function.

H3: Δ is bounded, i.e., $|\Delta| \leq M < \infty$.

H4: For t_0, sufficiently large, there exists $K > 0$, such that, $\lim\sup_{t\to t_0} \frac{M}{K} = 0$.

Then, the system

$$\dot{\hat{\eta}} = K(\eta - \hat{\eta}) \tag{6}$$

is an asymptotic reduced order observer of free-model type for system (5), where $\hat{\eta}$ denotes the estimate of η and $K \in \mathbb{R}^+$ determines the desired convergence rate of the observer.

Proof: let us define the estimation error as $e(t) = \eta(t) - \hat{\eta}(t)$. The dynamics of the error is given by

$$\dot{e}(t) = \dot{\eta}(t) - \dot{\hat{\eta}}$$

then

$$\dot{e}(t) + Ke(t) = \Delta(t) \tag{7}$$

The solution of (7) is given by,

$$e(t) = \exp(-Kt)\left[e_0 + \int_0^t \exp(K\tau)\Delta(\tau)d\tau\right], \tag{8}$$

where $e_0 = e(0)$ denotes the initial condition.

Then, using Cauchy-Schwartz and triangle inequalities in the expression (8), we have,

$$\begin{aligned} 0 \leq |e(t)| &\leq \exp(-Kt)|e_0| \\ &\quad + \exp(-Kt)\int_0^t |\exp(K\tau)\Delta(\tau)d\tau| \end{aligned}$$

If H3 is satisfied, we obtain,

$$\begin{aligned} 0 \leq |e(t)| &\leq \exp(-Kt)|e_0| \\ &\quad + M\int_0^t \exp[K(\tau - t)]d\tau \\ &= \exp(-Kt)|e_0| \\ &\quad + \frac{M}{K}[1 - \exp(-Kt)] \end{aligned}$$

If $t \to t_0$, for t_0 sufficiently large, then

$$\begin{aligned} 0 &\leq \lim\sup_{t\to t_0}|e(t)| \\ &\leq |e_0|\lim\sup_{t\to t_0}[\exp(-Kt)] \\ &\quad + \lim\sup_{t\to t_0}\left\{\frac{M}{K}[1 - \exp(-Kt)]\right\} \\ &= \lim\sup_{t\to t_0}\frac{M}{K} \end{aligned}$$

Finally, by taking into account H4, we have

$$0 \leq \lim\sup_{t\to t_0}|e(t)| \leq \lim\sup_{t\to t_0}\frac{M}{K} = 0$$

Then,

$$\lim\sup_{t\to t_0}|e(t)| = 0$$

Therefore, (6) is an asymptotic reduced order observer for (5). ■

Sometimes the output time derivatives (which are unknown), appear in the algebraic equation of the state variable, then it is necessary to use an auxiliary variable to avoid using them.

Corollary 1: The dynamic system (6) along with

$$\dot{\gamma} = \psi(x, u, \gamma), \quad \text{with} \quad \gamma_0 = \gamma(0) \quad \text{and} \quad \gamma \in C^1$$

constitute a proportional asymptotic reduced order observer for system (5), where γ is a change of variable which depends on the estimated state $\hat{\eta}$, and the state variables.

III-B. *Observer for the Colpitts oscillator (slave system)*

Let us consider the normalized system of the Colpitts oscillator. In all this paper we assume that the output system is $y = x_2$. Therefore, the slave system consists in two estimation structures to achieve synchronization with the master system. Such structures are obtained as follows. Firstly, verify that the master system (Colpitts oscillator) is algebraically observable, and then, by using (6), construct the observer for the unknown states.

The AOC for x_3 is given by,

$$x_3 = \frac{\dot{x}_2}{b} = \frac{\dot{y}}{b} = \phi(\dot{y}) \tag{9}$$

For x_1, we have

$$x_1 = -\frac{1}{c}\left[\frac{1}{b}\ddot{y} + \frac{d}{b}\dot{y} + cy\right] \Rightarrow x_1 = \overline{\phi}(y, \dot{y}, \ddot{y}) \tag{10}$$

Then, both unknown states of the master system are algebraically observable, then we said that the master system is algebraically observable and, therefore, we can construct the observers based on (6) and corollary 1.

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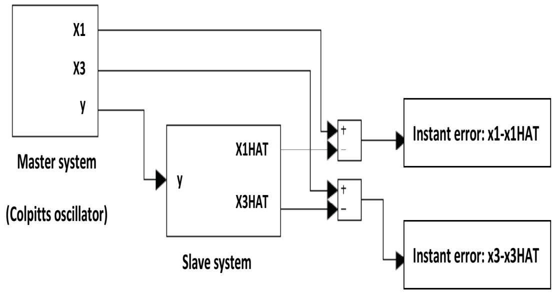

Fig. 3. General scheme for the synchronization in real–time.

Observer for x_3:

$$\begin{aligned}\dot{\hat{x}}_3 &= K_3(x_3 - \hat{x}_3)\\ &= \frac{K_3}{b}\dot{y} - K_3\hat{x}_3\end{aligned}$$

$$\dot{\hat{x}}_3 - \frac{K_3}{b}\dot{y} = -K_3\hat{x}_3$$

If we define $\dot{\gamma}_3 = -\frac{K_3}{b}\dot{y} + \dot{\hat{x}}_3$, then

$$\begin{aligned}\dot{\gamma}_3 &= -\frac{K_3^2}{b}y - K_3\gamma_3\\ \hat{x}_3 &= \frac{K_3}{b}y + \gamma_3\end{aligned} \qquad (11)$$

Observer for x_1:

$$\begin{aligned}\dot{\hat{x}}_1 &= K_1(x_1 - \hat{x}_1)\\ &= -\frac{K_1}{cb}\ddot{y} - \frac{K_1 d}{cb}\dot{y} - K_1 y - K_1\hat{x}_1\end{aligned}$$

Now, we consider the change of variable $x_4 = \dot{y}$ and we design an observer for this new variable according to (6),

$$\begin{aligned}\dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y]\\ \hat{x}_4 &= \gamma_4 + K_4 y\end{aligned}$$

Then,

$$\begin{aligned}\dot{\hat{x}}_1 &= -\frac{K_1}{cb}\dot{\hat{x}}_4 - \frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1\\ \dot{\hat{x}}_1 + \frac{K_1}{cb}\dot{\hat{x}}_4 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y - K_1\hat{x}_1\end{aligned}$$

If we define $\gamma_5 = \hat{x}_1 + \frac{K_1}{cb}\hat{x}_4$, then

$$\begin{aligned}\dot{\gamma}_5 &= -\frac{K_1 d}{cb}\hat{x}_4 - K_1 y + \frac{K_1^2}{cb}\hat{x}_4 - K_1\gamma_5\\ &= [K_1 - d]\frac{K_1}{cb}\hat{x}_4 - K_1 y - K_1\gamma_5\\ &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5\end{aligned}$$

Finally, the observer scheme for x_1 is given by

$$\begin{aligned}\dot{\gamma}_4 &= -K_4[\gamma_4 + K_4 y]\\ \dot{\gamma}_5 &= [K_1 - d]\frac{K_1}{cb}[\gamma_4 + K_4 y] - K_1 y - K_1\gamma_5\\ \hat{x}_1 &= -\frac{K_1}{cb}[\gamma_4 + K_4 y] + \gamma_5\end{aligned} \qquad (12)$$

Therefore, (11) and (12) constitute the slave system.

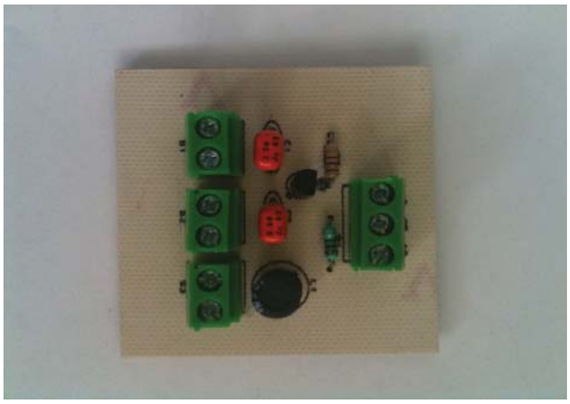

Fig. 4. Master system implementation.

III-C. *Parameter identification*

The system parameters (a, b, c, d) are not exactly known, but as it can be easily verified, they are algebraically identifiable [25], that is to say, they satisfy differential algebraic equations in $\mathbb{R}\langle u, y\rangle$. Indeed by considering available the complete state vector $(y_1 = x_1, y_2 = x_2, y_3 = x_3)$, we can obtain the following relationships:

$$\begin{aligned}a &= \frac{\dot{y}_1}{-exp(-y_2) + y_3 + 1}\\ b &= \frac{\dot{y}_2}{y_3}\\ c &= \frac{-\dot{y}_3^2 + y_3\ddot{y}_3}{\dot{y}_3(y_1 + y_2) - (\dot{y}_1 + \dot{y}_2)}\\ d &= -\frac{(\dot{y}_1 + \dot{y}_2)\dot{y}_3 - (y_1 + y_2)\ddot{y}_3}{(\dot{y}_1 + \dot{y}_2)y_3 - (y_1 + y_2)\dot{y}_3}\end{aligned}$$

IV. EXPERIMENTAL RESULTS

We verified the real time performance of the proposed observers by using the WINCON platform. To achieve the synchronization in real time, in WINCON were implemented the schemes (11) and (12) in the master–slave configuration (Fig. 3). In Fig. 4 is shown the real Colpitts circuit (master system).

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as [32]

$$J(t) = \frac{1}{t + 0.001}\int_0^t |e(t)|^2_{Q_0} \quad , \quad Q_0 = I$$

where $e(t)$ denotes the synchronization error.

Figs. 5 and 6 show the obtained results for the initial conditions $\hat{x}_1 = -0.5$ and $\hat{x}_3 = 0.9$ in the schemes (11) and (12), respectively. As we can note, the synchronization results achieved with the reduced order observer are good.

Fig. 7 illustrates the performance index, which has a tendency to decrease. Finally, Fig. 8 presents the phase

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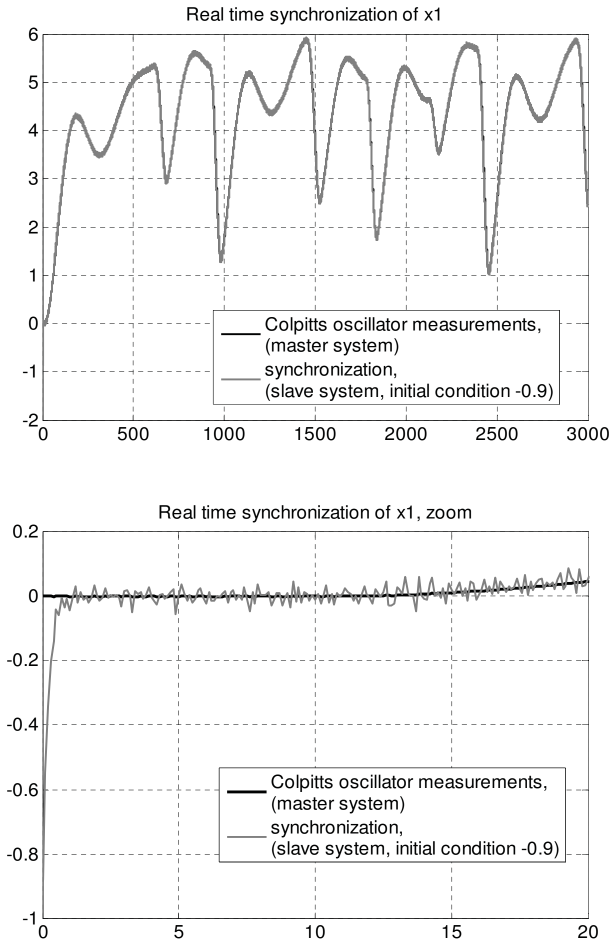

Fig. 5. Synchronization of x_1 by using the reduced order observer (12).

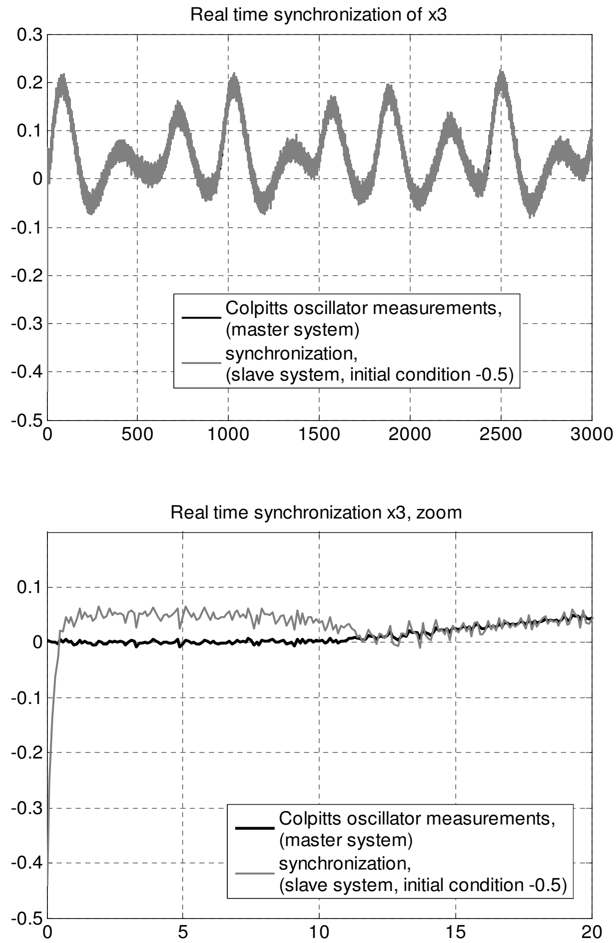

Fig. 6. Synchronization of x_3 by using the reduced order observer (11).

diagram, where clearly is observed the chaotic behavior of the Colpitts oscillator.

V. CONCLUSIONS

In this paper we study the synchronization problem in chaotic systems based on the observers theory. As well as, as a main contribution, we show the real–time synchronization in the Colpitts oscillator by using a reduced order observer of free–model type, which has asymptotic convergence. Some experimental results show the effectiveness of the proposed methodology.

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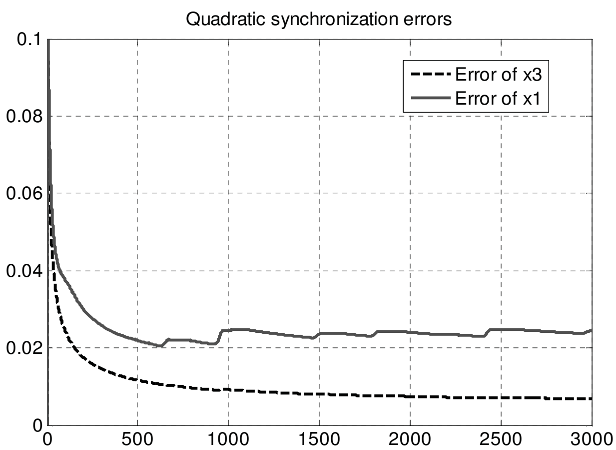

Fig. 7. Performance index of the corresponding synchronization process.

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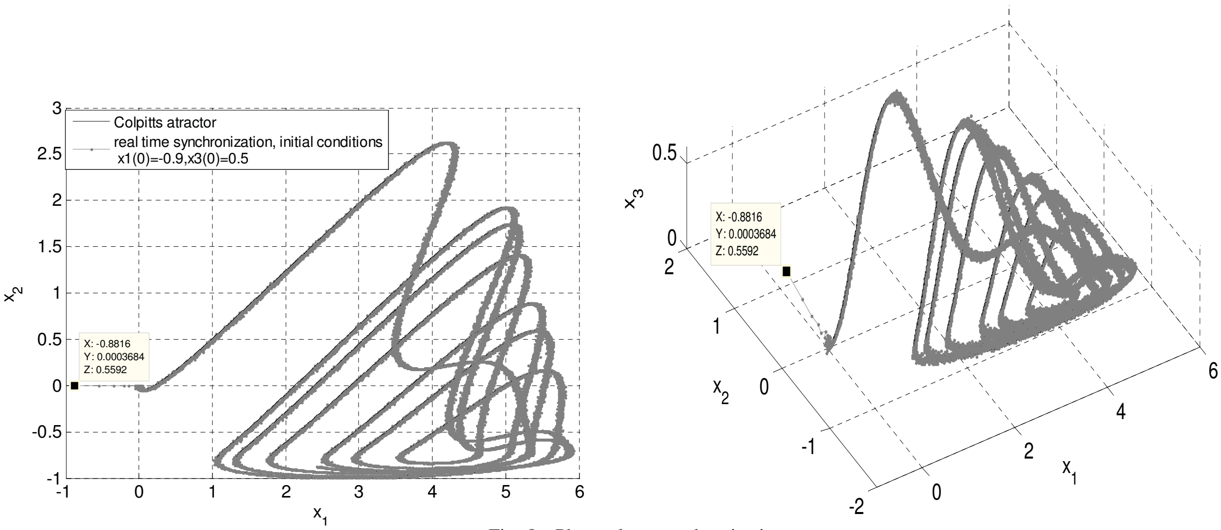

Fig. 8. Phase-plane synchronization.

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Fault Diagnosis for a Class of Nonlinear Systems by means of a Polynomial Observer

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***Abstract*— This paper deals about the problem of fault diagnosis of a class of nonlinear systems using the theory of state observers. In order to reconstruct the faults of the system, a polynomial observer is proposed, which includes in its structure correction terms of high order. The methodology consists of add to the original system the faults as new states, which increase the order of the observer. This scheme reconstructs simultaneously faults and state variables. In addition, as a comparative study, it has been designed an observer of reduced order. Both techniques are applied to fault diagnosis of a three-tank system.**

***Index Terms*— fault diagnosis, observers, nonlinear system.**

I. INTRODUCTION

A fault is a not allowed deviation of at least one characteristic property or parameter of the system with respect to its usual, normal or acceptable condition.

In the last three decades a considerable number of papers related to the problem of fault diagnosis have been reported [1]–[4]. Among the publications around the diagnosis of faults have different approaches, for example, in [5]–[7] used differential geometric tools; papers [8]–[10] are based on an alternative method using differential algebraic techniques.

This work considers the problem of fault diagnosis of a class of nonlinear systems. The outputs are mainly obtained by sensors measurements. The outputs and the number of faults determine if the system is diagnosable or not [11], [12]. The fault diagnosis problem is considered as a problem of fault signals observation. In this sense, the diagnosability of a system is given by the algebraic observability condition of the fault [8]. The main contribution of this article consists of the solution of the fault diagnosis problem by means of a polynomial observer. This work combines the ideas of [13] on the design of observers for nonlinear Lipschitz systems with the method introduced in [11], adding high order correction terms[1], such that the asymptotic convergence of the observer is guaranteed. In addition, is designed an observer of reduced order based on the free model approach, which converges asymptotically. The proposed schemes to the fault diagnosis and reconstructions of states are applied to a three-tank system.

[1]As it is well known, an Extended Luenberger observer can be seen as a first order Taylor series around the estimated state, therefore to improve the estimation performance high order terms are included in the observer structure.

II. PROBLEM STATEMENT

Let us consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) = &\ g(x,u,f) \\ y(t) = &\ h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^l$ is the known input vector, $f = (f_1, f_2, ..., f_\mu) \in \mathbb{R}^\mu$ is the faults vector (unknown inputs), and $y \in \mathbb{R}^p$ is the output vector. With g, h analytic functions.

The fault vector f is unknown, which can be interpreted as a state with uncertainties. The faults estimation is obtained by an extended state. Consider the following extended system:

$$\begin{aligned} \dot{x}(t) = &\ g(x,u,f) \\ \dot{f}_j(t) = &\ \Omega_j(x,u,f) \quad , \quad 1 \le j \le \mu \\ y(t) = &\ h(x,u) \end{aligned} \tag{2}$$

where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_\mu) \in \mathbb{R}^\mu$ is an unknown bounded function, that is to say,

$$\|\Omega(x,u,f)\| \le N < \infty \tag{3}$$

The system (2) can be expressed in the following form,

$$\begin{aligned} \dot{x}(t) = &\ A\,x + \Psi\,(x,u,f) \\ \dot{f}_j(t) = &\ \Omega_j(x,u,f) \quad , \quad 1 \le j \le \mu \\ y(t) = &\ C\,x \end{aligned} \tag{4}$$

where $\Psi(x,u,f)$ is a nonlinear function that satisfies the Lipschitz condition, i.e.,

$$\|\Psi(x,u,f) - \Psi(\hat{x},u,f)\| \le L\|x - \hat{x}\| \tag{5}$$

III. OBSERVABILITY AND DIAGNOSABILITY: DIFFERENTIAL ALGEBRAIC APPROACH

Before proposing an observer for the extended system (4), in this section are presented some definitions about observability and diagnosability.

III-A. Definitions

The observability notion (diagosability) of a system, linear or nonlinear, consists of the possibility of reconstructing the state x (fault f), having the knowledge of the output y and, the input u, and possibly a finite number of their derivatives, ($y^{(k)}$, $k \ge 0$ and $u^{(l)}(t)$, $l \ge 0$). Next some definitions concerning differential algebra [10], [11], [14] appear.

Definition 1: Let $\mathcal{L}$ and $\mathcal{K}$ be differentials fields. A differential field extension $\mathcal{L}/\mathcal{K}$ is given by two differentials fields $\mathcal{K}$, $\mathcal{L}$, such that:

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1. $\mathcal{K} \subset \mathcal{L}$.
2. The derivation of $\mathcal{K}$ is the restriction to $\mathcal{K}$ of the derivation of $\mathcal{L}$.

Example 1: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are trivial examples of extensions of differentials fields, where $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Definition 2: A variable $x \in \mathcal{L}$ is said to be differentially algebraic on $\mathcal{K}$ if and only if x satisfies a differential algebraic equation with coefficients in $\mathcal{K}$. That is to say, exists a polynomial P different to zero, such that: $P(x, \dot{x}, \ddot{x}, ..., x^{(n)}) = 0$, then x is called $\mathcal{K}$ algebraically differentially dependent ; otherwise, one says that x is $\mathcal{K}$-algebraically differentially independent.

Example 2: $\mathbb{R}\langle e^t \rangle / \mathbb{R}$ is an extension of a differential field where $\mathbb{R} \subseteq \mathbb{R}\langle e^t \rangle$. This extension is differentially algebraic because e^t satisfies $\dot{x} - x = 0$.

Remark 1: Let $u = (u_1, ..., u_l)$, and $y = (y_1, ..., y_p)$ be two finite families of differentials elements. $\mathcal{K}\langle u, y\rangle$ denotes the differential field generated by field $\mathcal{K}$, the components of u, the components of y, and, the components of the differentials elements of the families $\{u, y\}$.

Definition 3: Faults are defined as transcendent elements on the field $\mathcal{K}\langle u\rangle$, therefore, a system with faults is a differential transcendental extension, denoted by $\mathcal{K}\langle u, f, y\rangle / \mathcal{K}\langle u, y\rangle$, where $f = (f_1, ..., f_\mu)$ is the fault vector and their derivatives with respect to the time.

Definition 4: Let $\mathcal{G}$ and $\mathcal{K}\langle u\rangle$ be differentials fields. A dynamics with faults is an algebraic extension differentially finitely generated $\mathcal{G}/\mathcal{K}\langle u, f\rangle$. Therefore, all element of $\mathcal{G}$ satisfies an algebraic differentially equation with coefficients that belong to the differential field $\mathcal{K}\langle u, f\rangle$.

Example 3: Consider the differential equation $\dot{u}^2 f + 4\ddot{u} = 0$. In this case, f is differentially algebraic on $\mathcal{K}\langle u\rangle$. We can express the dynamics of the form $\mathcal{K}\langle u, y\rangle / \mathcal{K}\langle u\rangle$, where $\mathcal{K} = \mathbb{R}$.

Definition 5: {Algebraic observability (diagnosability) Condition}. Given the system (4), a state $x_i \in \mathbb{R}$ (a fault $f_j \in \mathbb{R}$) is said algebraically observable (diagnosable) if is differentially algebraic on the differential field $\mathbb{R}\langle u, y\rangle$, $1 \le i \le n$, $1 \le j \le \mu$.

Remark 2: The properties of observability and diagnosability imply that the states and the faults of the system can be reconstructed by means of the available outputs. A fault is diagnosable if it is algebraically observable.

Example 4: Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + f \\ \dot{x}_2 &= -x_2^2 + x_1 + u \\ y &= x_2 \end{aligned} \tag{6}$$

System (6) is algebraically observable since $x_1 - \dot{y} - y^2 + u = 0$ and $x_2 - y = 0$, which are polynomials with coefficients in $\mathbb{R}\langle u, y\rangle$. Analogously, because f satisfies the equation $f - \ddot{y} - y\dot{y} - y^3 + uy + \dot{u} = 0$, then, the system (6) is diagnosable, i.e., the fault can be reconstructed knowing $\{u, y\}$ and their time derivatives.

III-B. Relation between the number of faults and the number of available outputs

The following results of the theory of extensions of differentiable algebraic fields are used to determine if a fault can be reconstructed from the known inputs and outputs.

Definition 6: A maximal family of elements of the field $\mathcal{K}$ that are $\mathcal{K}$- differentially algebraically independent is denominated differential trascendence basis of extension $\mathcal{L}/\mathcal{K}$ and the cardinality of this base is called the differential trascendence degree denoted by: $diff\ tr\ d^\circ\ \mathcal{L}/\mathcal{K}$.

Property 1 [14]: Let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be differentials fields such that $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$. Then,

$$diff\ tr\ d^\circ\ \mathcal{M}/\mathcal{K} = diff\ tr\ d^\circ\ \mathcal{M}/\mathcal{L} + diff\ tr\ d^\circ\ \mathcal{L}/\mathcal{K}$$

Remark 3: The property 1 is an indispensable tool for the test of theorems 1 and 2.

Theorem 1: [8] Suppose that the system (4) is diagnosable, then the number of faults is less or equal to the number of outputs, that is to say, $\mu \le p$. ■

Theorem 2: [11] The system (4) is diagnosable if and only if $diff\ tr\ d^\circ\ \mathcal{K}\langle u, y\rangle / \mathcal{K}\langle u\rangle = \mu$. ■

IV. POLYNOMIAL OBSERVER

Let us consider the system (4), the observer has the following form

$$\begin{aligned} \dot{\hat{x}}(t) = &\ A\hat{x} + \Psi(\hat{x}, u, f) + \sum_{i=0}^{m} K_i (y - C\hat{x})^{2i-1} \\ \dot{\hat{f}}_j(t) = &\ \sum_{i=1}^{m} \overline{K}_{ji} (f_j - \hat{f}_j)^{2i-1} \end{aligned} \tag{7}$$

where, $\hat{x} \in \mathbb{R}^n$, $K_i = \begin{bmatrix} k_{1,i} & k_{2,i} & \dots & k_{n,i} \end{bmatrix}^T \in \mathbb{R}^n$, $1 \le i \le n$, $\hat{x}_0 = \hat{x}(t_0)$, $\overline{K}_{ji} > 0$ y $\hat{f}_{j0} = \hat{f}_j(t_0)$, $1 \le j \le \mu$.

For this observer the following hypotheses are considered:

H1: $f(t)$ is algebraically observable on $\mathbb{R}\langle u, y\rangle$.

H2: K_1 can be chosen such as the following algebraic Riccati equation has a symmetric and positive definite solution P for some $\epsilon > 0$,

$$(A - K_1\ C)^T P + P\ (A - K_1 C) + L^2 P\ P + I + \epsilon I = 0$$

H3: K_i is chosen such that $\lambda_{min}(PK_iC) \ge 0$, $2 \le i \le m$.

We analyze the estimation error, which is defined as $e^T = (e_x, e_y)$, where, $e_x = x - \hat{x}$ and $e_f = f_j - \hat{f}_j$. From systems (4) and (7), the dynamics of estimation error is given by

$$\begin{aligned} \dot{e}_x = &\ (A - K_1 C)e_x - \sum_{i=2}^{m} K_i [Ce_x]^{2i-1} \\ &\ + [\Psi(x, u, f) - \Psi(\hat{x}, u, f)] \\ \dot{e}_f = &\ \Omega_j - \overline{K}_{j1} e_f - \sum_{i=2}^{m} \overline{K}_{ji} (e_f)^{2i-1} \end{aligned} \tag{8}$$

The next theorem proves the observer convergence.

Theorem 3: For the nonlinear system (4), supposes that $x(t)$ exists for all $t \ge 0$, the nonlinear function $\Psi(x, u, f)$

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satisfies the Lipschitz condition (5), $x(t)$ and $f(t)$ are algebraically observable. If there exists $0 < P = P^T$ and observer gains K_i and $\bar{K}_j > 0$ such that the system (7) is an observer of the system (4), then the estimation error converges asymptotically to zero.

Proof: Consider the Lyapunov fuction candidate

$$\begin{aligned} V =&\ V_1 + V_2 \\ V_1 =&\ e_x^T P\, e_x \ \ , \ \ V_2 = \frac{1}{2} e_f^2 \end{aligned} \tag{9}$$

where $0 < P = P^T$ and satisfies H2.

The proof is divided into two parts, as follows:
(a). The time derivative of V_1 along the trajectories of (8) is

$$\begin{aligned} \dot{V}_1 &= \dot{e_x}^T P e_x + e_x^T P \dot{e}_x \\ &= e_x^T [(A - K_1 C)^T P + P(A - K_1 C)] e_x \\ &\quad + 2 e_x^T P[\Psi(x, u, f) - \Psi(\hat{x}, u, f)] \\ &\quad - 2 \sum_{i=2}^{m} [C e_x]^{2i-2} e_x^T P K_i C e_x \end{aligned}$$

The following inequality is used [15], which is based on the Lipschitz condition (5)

$$2 e_x^T P[\Psi(x, u) - \Psi(\hat{x}, u)] \leq L^2 e_x^T P P e_x + e_x^T e_x \tag{10}$$

Applying the inequality of Rayleigh [16], and taking into account H3, we have

$$-e_x^T P K_i C e_x \leq -\lambda_{min}(P K_i C) \|e_x\|^2 \tag{11}$$

for $2 \leq i \leq m$.

Therefore, combining the inequalities (10) and (11)

$$\begin{aligned} \dot{V}_1 &\leq e_x^T [(A - K_1 C)^T P + P(A - K_1 C) + L^2 P P + I] e_x \\ &\quad - 2 \sum_{i=2}^{m} [C e_x]^{2i-2} \lambda_{min}(P K_i C) \|e_x\|^2 \\ &\leq e_x^T [(A - K_1 C)^T P + P(A - K_1 C) + L^2 P P + I] e_x \\ &= -\epsilon \|e_x\|^2 \end{aligned}$$

(b). In the same way, considering the second term of the Lyapunov function candidate, we obtain

$$\begin{aligned} \dot{V}_2 &= e_f \dot{e}_f \\ &= e_f \left(\Omega_j - \overline{K}_{j1} e_f - \sum_{i=2}^{m} \overline{K}_{ji} (e_f)^{2i-1} \right) \\ &= e_f \Omega_j - \overline{K}_{j1} e_f^2 - \sum_{i=2}^{m} \overline{K}_{ji} (e_f)^{2i} \\ &\leq |e_f||\Omega_j| - \overline{K}_{j1} e_f^2 \\ &\leq |e_f| N - \overline{K}_{j1} |e_f|^2 \\ &= -[\overline{K}_{j1} |e_f| - N] |e_f| \end{aligned}$$

$\dot{V}_2$ is negative inside the set $\{|e_f| > N/\bar{K}_{j1}\}$, i.e., exists $\bar{\epsilon} > 0$ such that $\bar{K}_{j1}|e_f| - N = \bar{\epsilon} > 0$.

The following analysis shows that $|e_f|$ is upper bounded. Let α, β upper bounds of $V_2(e_f)$. With $\beta > N^2/2\bar{K}_{ij}^2$, the solutions that initiate in the set $\{V_2(e_f) \leq \beta\}$ will remain there for all $t \geq 0$, due to that $\dot{V}_2$ is negative in $V_2 = \beta$. Therefore, the solutions of $\dot{e}_f$ are uniformly bounded [17]. Furthermore, if $N^2/2\bar{K}_{ij}^2 < \alpha < \beta$, then $\dot{V}_2$ will be negative in the set $\{\alpha \leq V_2 \leq \beta\}$. In this set V_2 will decrease monotonously until the solutions of the set $\{V_2 \leq \alpha\}$. According to [17], the solution is uniformly ultimately bounded with ultimate bound $|e_f| \leq \sqrt{2\alpha}$. For example, if we define α and β as $\alpha = N^2/2\bar{K}_{j1}^2$ and $\beta = 3N^2/2\bar{K}_{j1}^2$. Then, the ultimate bound is

$$|e_f| \leq \frac{N}{\bar{K}_{j1}}$$

Therefore

$$\dot{V}_2 \leq -\bar{\epsilon}|e_f|$$

From (a) and (b), it concludes that,

$$\dot{V} \leq -\epsilon \|e_x\|^2 - \bar{\epsilon}|e_f| \leq 0$$

The theorem is proven. ■

V. REDUCED ORDER OBSERVER

In this section we present a reduced order observer, which can estimate the faults of the system (4). Before proposing the observer we introduce the following assumption:

H4: γ is a real valued function of class C^1.

Lemma 1: Assume that (3), H1, and H4, are satisfied then the system

$$\dot{\hat{f}} = K(f - \hat{f}) \tag{12}$$

is a reduced order observer of the system (4), where $\hat{f}$ denotes the fault estimation, $K > 0$ is the gain of the observer and it determines the convergence rate of the observer.

The proof of the lemma is given in [12]. ■

In case the condition of algebraic observability of the fault is expressed in terms of the outputs derivatives with respect to the time, which are not known, then, is necessary to use the following estimation method [12].

Corollary 1: The system (13) along with (12), constitute an asymptotic proportional observer of reduced order for the faults of the system (4)

$$\dot{\gamma} = \ \phi(x, u, f, \gamma) \ \ , \ \ \gamma_0 = \gamma(0), \ \ \gamma \in C^1 \tag{13}$$

where γ is a change of variable that depends on the estimate fault $\hat{f}$ and the state variables. ■

VI. APPLICATION TO A THREE-TANK MODEL

VI-A. Model

The description of the system of three tanks that is analyzed in this work is the following. The plant consists of three cylinders (T1, T2 and T3) with transversal constant section A, which are connected in series one to another one by means of cylindrical tubes of cross-sectional section S (Fig. 1). In the T2 tank is located the valve of exit, also with transversal constant section S. The flows q_1 and q_2 denote the well-known input signals. Next, the mathematical model of a system of three identical tanks interconnected [18] appears. The corresponding nominal model is expressed by the following system of differential equations

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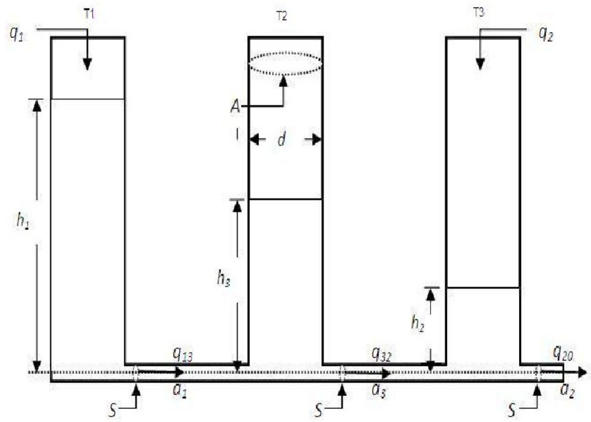

Fig. 1. Three-tank system

$$\begin{aligned}\frac{dh_1}{dt} &= \frac{1}{A}(q_1 - q_{13})\\ \frac{dh_2}{dt} &= \frac{1}{A}(q_2 + q_{32} - q_{20})\\ \frac{dh_3}{dt} &= \frac{1}{A}(q_{13} - q_{32})\end{aligned} \tag{14}$$

According to the generalized rule of Torricelli, which is valid for laminar flow, the following is obtained

$$\begin{aligned}q_{13} &= a_1 S\ sign(h_1 - h_3)\sqrt{2g|h_1 - h_3|}\\ q_{32} &= a_3 S\ sign(h_3 - h_2)\sqrt{2g|h_3 - h_2|}\\ q_{20} &= a_2 S\sqrt{2gh_2}\end{aligned} \tag{15}$$

The complete description of the system (14) is given in Table I.

In what follows, it is considered that the state vector is $x = [x_1\ x_2\ x_3]^T = [h_1\ h_2\ h_3]^T$ and the input vector is $u = [u_1\ u_2]^T = [q_1\ q_2]^T$.

The system (14) has four regions of operation in which the corresponding model is differentiable. In this paper one focuses along the operation region $x_1 \geq x_3 \geq x_2$.

VI-B. *Consideration of faults and outputs of the system*

The model (14) is expressed in terms of faults and outputs as follows

$$\begin{aligned}\dot{x_1} &= \tfrac{1}{A}(u_1 - q_{13} + f_1)\\ \dot{x_2} &= \tfrac{1}{A}(u_2 + q_{32} - q_{20} + f_2)\\ \dot{x_3} &= \tfrac{1}{A}(q_{13} - q_{32})\\ y_2 &= x_2\\ y_3 &= x_3\end{aligned} \tag{16}$$

where f_1 and f_2 are additives faults associated with the actuators that control the input flows u_1 and u_2,respectively. In addition, y_1, y_2 are the available outputs.

TABLE I

VARIABLES AND PARAMETERS OF THE THREE-TANK SYSTEM

Symbols	Description
h_i	Level of the liquid of i-th tank (μ) $i \in \{1,2,3\}$.
A	Transversal section of each tank (m^2).
S	Transversal section of the interconnection tubes (m^2).
q_{13}	Water flow of T1 to T3 (m^3/s).
q_{32}	Water flow of T3 to T2 (m^3/s).
q_{20}	Output flow (m^3/s).
q_1	Input flow to T1 (m^3/s).
q_2	Input flow to T2 (m^3/s).
a_i	Output flow coefficients.

VI-C. *Observability and diagnosability analysis*

For the diagnosability is required that the number of faults be less or equal that the number of available outputs (Theorem 1). The system (16) consists of two faults $\mu = 2$ and two outputs ($p = 2$).

In order to guarantee observability and diagnosability of (16) we have to determine the algebraic observability condition (definition 5) for f_1, f_2 and the unknown state x_1. From (16) we obtain

$$\begin{aligned}f_1 &= A\dot{x}_1 + a_1 S\sqrt{2g(x_1 - y_3)} - u_1\\ f_2 &= A\dot{y}_2 - a_3 S\sqrt{2g(y_3 - y_2)}\\ &\qquad + a_2 S\sqrt{2gy_2} - u_2\end{aligned} \tag{17}$$

and
$$x_1 = y_3 + \tfrac{1}{2ga_1^2 S^2}(A\dot{y}_3 + a_3 S\sqrt{2g(y_3 - y_2)})^2 \tag{18}$$

The algebraic observability condition for x_1 is deduced directly of (18), which is an equation with coefficients in $\mathbb{R}\langle u, y\rangle$.

Replacing (18) into (17) a system of two differential equations is obtained with coefficients in $\mathbb{R}\langle u, y\rangle$, with unknowns f_1 and f_2,

$$\begin{aligned}f_1 = & A\Big[\dot{y}_3 + \tfrac{1}{a_1^2 S^2}\Big(A\dot{y}_3 + a_3 S\sqrt{2g(y_3 - y_2)}\Big)\\ & \cdot\Big(A\ddot{y}_3 + \tfrac{1}{2}a_3 S\tfrac{\dot{y}_3 - \dot{y}_2}{\sqrt{y_3 - y_2}}\Big)\Big] + A\dot{y}_3\\ & + a_3 S\sqrt{2g(y_3 - y_2)} - u_1\\ f_2 = & A\dot{y}_2 - a_3 S\sqrt{2g(y_3 - y_2)}\\ & + a_2 S\sqrt{2gy_2} - u_2\end{aligned} \tag{19}$$

which implies that $diff\ tr\ d^\circ\,\mathbb{R}\langle u, y\rangle / \mathbb{R}\langle u\rangle = 2$. Applying theorem 2, we conclude that the faults are diagnosable with the considered outputs.

VI-D. *Reconstruction of faults and states*

Equations (18) and (19) require the estimation of the time derivatives from outputs y_2, y_3. In this one section a methodology appears to reconstruct first $r - 1$ derivatives with respect to the time from the output y.

Polynomial observer. Consider the nonlinear system with faults (1). Suppose that this system satisfies definition 5, that is to say, the fault vector f is algebraically observable in $\mathbb{R}\langle u, y\rangle$. Therefore, f satisfies a differential polynomial

$$\psi(f, y, \dot{y}, \ldots, y^{(r)}, u, \dot{u}, \ldots) = 0 \tag{20}$$

where r is the maximum order of the derivatives from the input with respect to the time.

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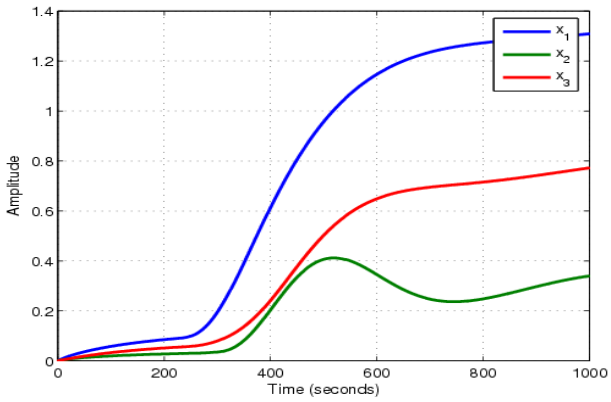

Fig. 2. Region of operation.

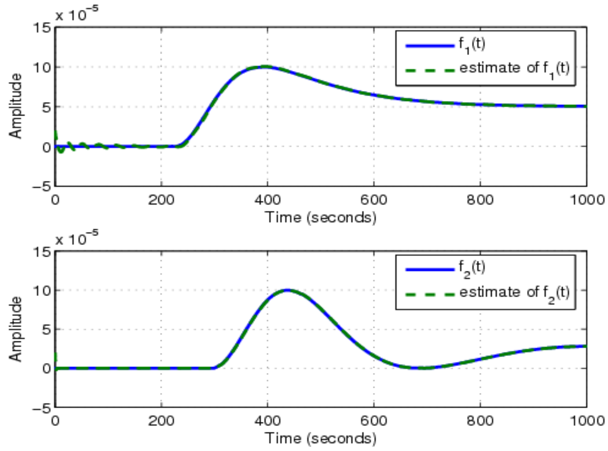

Fig. 3. Fault diagnosis by using polynomial observer.

Introducing the following changes of variables

$$\eta_1 = y, \quad \eta_2 = \dot{y}, \quad \ldots, \quad \eta_r = y^{(r-1)} \tag{21}$$

Then, in a domain D where $\partial\varphi/\partial y^{(r)}$ are invertible, the representation input-output of (1) and (20) can be expressed as follows

$$\begin{aligned} \dot{\eta} &= A\eta + \overline{\phi}(\eta, u) \\ y &= \eta_1 \end{aligned}$$

where, $\overline{\phi}(\eta, u) = \left[\begin{array}{cccc} 0 & 0 & \ldots & \phi(\eta, u) \end{array}\right]^T \in \mathbb{R}^r$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\ldots & 0 \\ 0 & 0 & 1 & 0\ldots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 0 & 0 & 0 & 0\ldots & 0 \end{bmatrix} \in \mathbb{R}^{r\times r},\ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{bmatrix} \in \mathbb{R}^r.$$

The nonlinear function $\bar{\phi}(\eta, u)$ (respectively $\phi(\eta, u)$) is Lipschitz in η and uniformly bounded in u.

By theorem 3, the observer has the following structure

$$\dot{\hat{\eta}} = \quad A\hat{\eta} + \overline{\phi}(\hat{\eta}, u) + \sum_{i=0}^{m} K_i (y - \hat{\eta}_1)^{2i-1} \tag{22}$$

The polynomial observer (22) is used to estimate the variable $\eta_2 = \dot{y}, ..., \eta_r = y^{r-1}$ by means of the available output $y = \eta_1$.

Reduced order observer. Let us consider the following time derivative to be estimated

$$\eta = \dot{y} \tag{23}$$

According to Lemma 1, we propose the next observer

$$\dot{\hat{\eta}} = K(\eta - \hat{\eta}) \tag{24}$$

The following change of variable is introduced (Corollary 1)

$$\hat{\eta} = \gamma + Ky \tag{25}$$

From (24) and (25) we obtain

$$\dot{\gamma} = -K\hat{\eta} \tag{26}$$

Replacing (25) into (26)

$$\dot{\gamma} = -K\gamma - K^2 y \tag{27}$$

Equations (25) and (27) constitute an asymptotic estimator of η.

VII. Numerical results

In this section there are some simulations to illustrate the performance of the proposed estimation schemes, which are applied to the system (16). The parameter values are.

$$\begin{array}{ll} A = 0.149\ \mathrm{m}^2, & \\ S = 5\times 10^{-5}\ \mathrm{m}^2, & q_{1max} = q_{2max} = 0.1\ \mathrm{m}^3/\mathrm{s}, \\ g = 9.81\ \mathrm{m/s}^2, & a_1 = 0.0418, \\ a_2 = 0.789, & a_3 = 0.435 \end{array}$$

In all the simulations, the input flows q_1 and q_2 are constants, that is to say, $q_1 = 2\times 10^{-5} m^3/s$ and $q_2 = 1.5\times 10^{-5} m^3/s$, for all $t \geq 0$. The faults additives are

$$f_1 = (5\times 10^{-5})[1 + \sin(0.2te^{-0.01t})]\mathcal{U}(t - 220)$$

$$f_2 = (5\times 10^{-5})[1 + \sin(0.5te^{-0.01t})]\mathcal{U}(t - 300)$$

where $\mathcal{U}$ is the unit step function.

Fig. 2 shows the considered region of operation ($x_1 \geq x_3 \geq x_2$).

To the polynomial observer $m = 3$ is fixed, the gains used for the observer of states are $K_1 = [2\ 3.2\ 3.7]^T$, $K_2 = [2\ 2.2\ 3.7]^T$ and $K_3 = [2\ 3.2\ 4.7]^T$. The gains of the observers of faults were taken as $\bar{K}_{11} = 2$, $\bar{K}_{12} = 4.2$, $\bar{K}_{13} = 5.7$, $\bar{K}_{21} = 2$,$\bar{K}_{22} = 3.2$ and $\bar{K}_{23} = 3.7$. In Figs. 3 and 4 are the obtained results employing the polynomial observer.

The observer of reduced order does not reconstruct the complete state of the system. Since $y_2 = x_2$, $y_3 = x_3$ are considered available, it is just necessary reconstruct the state x_1, as well as faults f_1 and f_2.

The gains of the reduced order observer for the faults are $k_1 = 2$, $k_2 = 2$ and for the state x_1 we choose $k = 0.3$.

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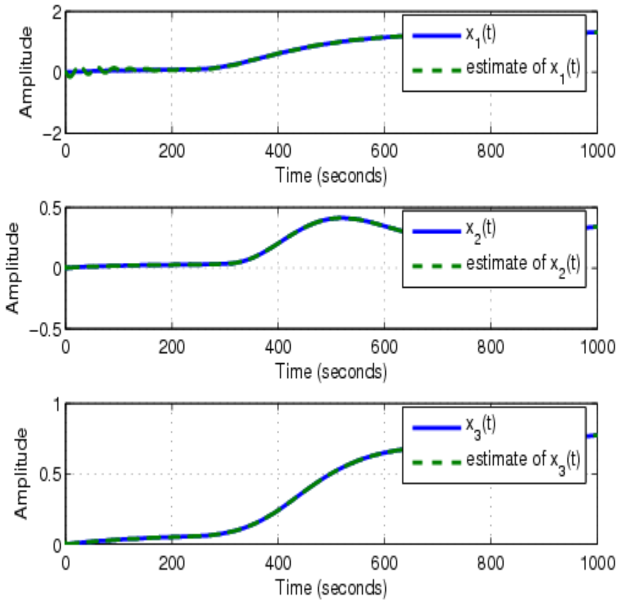

Fig. 4. State estimation via polynomial observer.

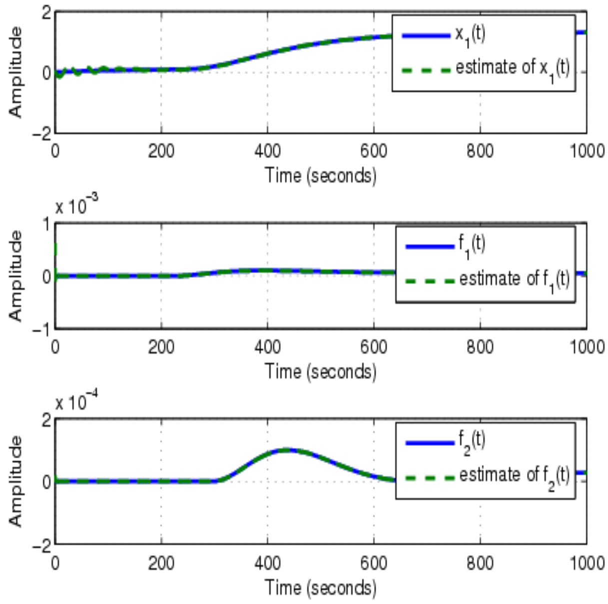

Fig. 5. Estimation results via the reduced-order observer.

The reconstructed faults of f_1 and f_2, and the corresponding estimate of x_1 are shown in Fig. 5.

VIII. Conclusions

In this work the fault diagnosis problem for a class of nonlinear systems using the theory of observers was treated. Some results based upon the differential algebraic approach were used as a criterion for the estimation of states and faults. The method consists of considering to the fault as an augmented state of the system. To achieve the reconstruction of the unknown states of the extended system, we proposed a polynomial observer, which estimates faults and state variables simultaneously. In addition, we design an observer of reduced order to compare the obtained results. Both techniques were applied to the fault diagnosis of a system of three-tank. The numerical results show the performance of the observers.

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A New Asymptotic Polynomial Observer to Synchronization Problem

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Abstract — **In this paper, we consider the synchronization problem via nonlinear observer design. A new asymptotic polynomial observer for a class of nonlinear oscillators is proposed, which is robust against output noises. A sufficient condition for synchronization is derived analytically with the help of Lyapunov stability theory. The proposed technique has been applied to synchronize chaotic systems (Lorenz and Rössler systems) by numerical simulation.**

Keywords — **Nonlinear systems, Riccati equation, state observers, synchronization.**

I. INTRODUCTION

The growing interest in synchronization was probably caused in 1990 by the paper by Pecora and Carrol [1] where, among others, secure communication as a potential application has been indicated. In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields, for instance in [2], [3]. It is well known that study of the synchronization problem for nonlinear systems has been very important for nonlinear science, in particular the applications to biology, medicine, cryptography, secure data transmission and so on. In general, the synchronization research has been focused onto two areas. The first one relates with the employ of state observers, where the main applications lies on the synchronization of nonlinear oscillators [4], [5], [6], [7]. The second one is the use of control laws, which allows achieve the synchronization with different structure and order between nonlinear oscillators [2], [8]. A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master- slave configuration [1]. Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer.

The purpose of this paper is to address the synchronization problem from a control theory perspective. More specifically, the paper addresses the synchronization problem from the perspective of nonlinear observer design. The phenomenon of synchronization also occurs for uni-directionally coupled systems and in this case the driven system (slave or receiver system) may be viewed as a nonlinear observer of the driving system (master or transmitter system). In this configuration, the two coupled systems are (almost) identical and therefore identical synchronization occurs which means that the difference of master and slave state vectors converges to zero for $t \to \infty$.

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general indeed, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances.

The design of observers for nonlinear systems is a challenging problem (even for accurately known systems) that has received a considerable amount of attention. Since the observers developed by Kalman [9] and Luenberger [10] several years ago for linear systems, different state observation techniques have been proposed to handle the systems nonlinearities. A first category of techniques consists in applying linear algorithms to the system linearized around the estimated trajectory. These are known as the extended Kalman and Luenberger observers. Alternatively, the nonlinear dynamics are split into a linear part and a nonlinear one. The observer gains are then chosen large enough so that the linear part dominates the nonlinear one. Such observers are known as high-gain observers [11], [12]. In a third approach the nonlinear system is transformed into a linear one by an appropriate change of coordinates [13]. The estimate is computed in these new coordinates and the original coordinates are recovered through the inverse transformation. In most approaches, nonlinear coordinate transformations are employed to transform the nonlinear system into a block triangular observer canonical form. Then, high gain [14] or sliding mode observers [15], [16] can be designed.

In this paper the synchronization scheme is proposed for a class of Lipschitz nonlinear systems. Many problems in engineering and other applications are globally Lipschitz for instance the sinusoidal terms in robotics. Nonlinearities which are square or cubic in nature are not globally Lipschitz, however, they are locally so, moreover when such functions occur in physical systems, they frequently have a saturation in their growth rate, making them globally Lipschitz functions [17]. Thus, this class of systems covered by this note is fairly general. References [17], [18], [19] established existence conditions of the full-order observers for Lipschitz nonlinear systems. The main purpose in this work is to extend these results by showing that the stability

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conditions given in [19] also guarantee the existence of a full-order observer with a high-order correction term.

The main contribution of this paper consists in the solution of the synchronization problem via an asymptotic polynomial observer. The obtained state space estimation error is shown to be bounded, and this bound depends on observer's gain and a Lipchitz constant. This communication presents some fundamental insights into polynomial observer design for the class of Lipschitz nonlinear systems, that it means, that any autonomous nonlinear system of the form $\dot{x} = f(x,u)$ can be regarded as Lipschitz continuous system with respect to x, with a Lipchitz constant L.

The intention of choosing examples as Lorenz and Rössler system is to clarify the proposed methodology. However, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

In what follows, an asymptotic polynomial observer is proposed as well as an easy numerical design is given. Numerical results show its satisfactory performance. Finally we close this paper with some concluding remarks.

II. ASYMPTOTIC POLYNOMIAL OBSERVER

A. Problem Statement

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \tag{1a}$$

where $x \in \Re^n$ is the vector of the state variables; $f(\circ): \Re^n \times \Re^m \to \Re^n$, $(m \le n)$ is a nonlinear smooth vector function and Lipschitz in x and uniformly bounded in u, $y \in \Re$ is the vector of measured states.

Any nonlinear system of the form (1a) can be expressed in the form (1b) as long as $f(x,u)$ is differentiable with respect to x.

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx \quad , \quad x_0 = x(t_0) \end{aligned} \tag{1b}$$

In this paper, we always assume that the pair (A,C) is observable. In system (1b), $\Psi(x,u)$ is a nonlinear vector function which satisfies the Lipschitz condition with a Lipschitz constant L, i.e,

$$\|\Psi(x,u) - \Psi(\hat{x},u)\| \le L\|x - \hat{x}\|$$

B. Observer design

Proposition 1. For any initial conditions, the following nonlinear dynamic system is a full order state observer of the system (1b)

$$\dot{\hat{x}} = A\hat{x} + \Psi(\hat{x},u) + \sum_{i=1}^{m} K_i \left[y - C\hat{x}\right]^{2i-1} \tag{2}$$

In proposition 1 the following assumptions should be considered

A1. K_1 *can be chosen such as the following Algebraic Riccati Equation (ARE) has a symmetric positive-definite solution* $P \in \Re^{n\times n}$ *for some* $\varepsilon > 0$

$$(A - K_1C)^T P + P(A - K_1C) + L^2 PP + I + \varepsilon I = 0 \tag{3}$$

A2. $\lambda_{\min}(P K_i C) \ge 0$, $i \in \{2,3,\ldots,m\}$ (4)

In (2), $\hat{x} \in \Re^n$, $K_i = \begin{bmatrix} k_{1,i} & k_{2,i} & \ldots & k_{n,i} \end{bmatrix}^T \in \Re^n$, $i \in \{1,2,\ldots,m\}$.

C. Stability analysis

To prove that system (2) is an observer for (1b) we analyze the estimation error (difference between real states and their estimated) employing Lyapunov stability theory.

Let us define the estimation error as $e := x - \hat{x}$. The corresponding dynamic of the estimation error is

$$\dot{e} = Ae - \sum_{i=1}^{m}\left[K_i\,(Ce)^{2i-1}\right] + \left[\Psi(x,u) - \Psi(\hat{x},u)\right] \tag{5}$$

Equation (5) is written alternatively as

$$\dot{e} = (A - K_1C)e - \sum_{i=2}^{m}\left[K_i\,(Ce)^{2i-1}\right] + \left[\Psi(x,u) - \Psi(\hat{x},u)\right] \tag{6}$$

Consider the Lyapunov function candidate $V = e^T Pe$, where $0 < P = P^T$ and satisfies (3). Its derivative is

$$\begin{aligned} \dot{V} &= \dot{e}^T Pe + e^T P\dot{e} \\ &= e^T\left[(A - K_1C)^T P + P(A - K_1C)\right]e \\ &\quad - 2\sum_{i=2}^{m}\left[(Ce)^{2i-2} e^T PK_i\, Ce\right] + 2e^T P\left[\Psi(x,u) - \Psi(\hat{x},u)\right] \end{aligned} \tag{7}$$

In [17] is presented the next inequality as a lemma which is useful for this proof,

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$$2e^T P[\Psi(x,u)-\Psi(\hat{x},u)] \le L^2 e^T PPe + e^T e$$

From Rayleigh inequality [20], and taking into account *A2*, we have

$$-e^T PK_i\, Ce \le -\lambda_{\min}(PK_i\, C)\|e\|^2$$

where $i \in \{2, 3, \ldots, m\}$. Equation (7) leads to

$$\begin{aligned}\dot{V} \le\ & e^T\left[(A-K_1C)^T P + P(A-K_1C)\right]e \\ & -2\sum_{i=2}^{m}\left[(Ce)^{2i-2}\lambda_{\min}(PK_i\, C)\right]\|e\|^2 + L^2 e^T PPe + e^T e \\ =\ & e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2 PP + I\right]e \\ & -2\sum_{i=2}^{m}\left[(Ce)^{2i-2}\lambda_{\min}(PK_i\, C)\right]\|e\|^2 \qquad (8)\end{aligned}$$

From *A2*, the second term in the right hand side of (8) always will be positive or zero, then

$$\dot{V} \le e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2 PP + I\right]e \qquad (9)$$

According to *A1*, since $\varepsilon > 0$, it is clear that $(A-K_1C)^T P + P(A-K_1C) + L^2 PP + I < 0$. Hence, $\dot{V} < 0$. This implies that system (2) is an observer for system (1b) and the corresponding dynamic of the estimation error (5) is asymptotically stable.

III. APLICATION TO SYNCHRONIZATION OF CHAOTIC SYSTEMS

To illustrate our methodology, we give two applications to chaotic systems. In fact, these are applications to the so-called Rössler system [21] which presents a chaotic behavior and exhibits the simplest possible strange attractor, and the well known Lorenz chaotic system [22].

A. Example 1: Rössler system

We consider the popular nonlinear Rössler system, which is described by

$$\begin{aligned}\dot{x}_1 &= -(x_2 + x_3) \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \\ y &= x_1\end{aligned} \qquad (10)$$

It is well known that in a large neighborhood of $\{a=b=0.2,\ c=5\}$ this system has a chaotic behavior.

Remark 1. It is not difficult to prove that above system is Lipschitz.

The system (10) may be written in the form given by (1b), where $x = [x_1 \quad x_2 \quad x_3]^T$,

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix},\ \Psi = \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix},\quad C = [1 \quad 0 \quad 0]$$

It can easily be shown that with the selection of $y = x_1$, the corresponding pair (A, C) is observable. By choosing the observer's gains appropriately, the observer given by (2) may achieve local synchronization.

According to proposition 1, we get the following equations system (slave system) as the observer,

$$\begin{aligned}\dot{\hat{x}}_1 &= -(\hat{x}_2 + \hat{x}_3) + \sum_{i=1}^{m} k_{1,i}\,[y - C\hat{x}]^{2i-1} \\ \dot{\hat{x}}_2 &= \hat{x}_1 + a\hat{x}_2 + \sum_{i=1}^{m} k_{2,i}\,[y - C\hat{x}]^{2i-1} \\ \dot{\hat{x}}_3 &= b + \hat{x}_3(\hat{x}_1 - c) + \sum_{i=1}^{m} k_{3,i}\,[y - C\hat{x}]^{2i-1}\end{aligned} \qquad (11)$$

where, $K_i = [k_{1,i} \quad k_{2,i} \quad k_{3,i}]^T \in \Re^3$, $i \in \{1, 2, \ldots, m\}$ and $a, b, c > 0$.

Now, to illustrate the effectiveness of the proposed approach, some numerical simulations are presented.

The design of the full-order observer presented in this paper is based on the solution of the Riccati equation which can be obtained by using the Matlab function ARE.

We have chosen the values for the Rössler system (10) and the observer (11) as $a = b = 0.2$, $c = 5$, and the observer's gains have been taken as $K_1 = [5 \quad -5 \quad 5]^T$ and $K_2 = [10 \quad 10 \quad 10]^T$, $K_3 = [10 \quad 10 \quad 10]^T$. All simulations results in this paper were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox. In this work, the performance index of the corresponding synchronization process was calculated as [12]

$$J(t) = \frac{1}{t + 0.001}\int_0^t \|e(t)\|_{Q_0}^2\, d\tau \qquad (12)$$

where $e(t)$ denotes the estimation error and $Q_0 = I$.

Fig. 1 shows the convergence of the estimated states (slave system) to the real states (master system), without any noise in the system output. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = 3$, $\hat{x}_3 = -4$. It should be noted that the rate of convergence is faster for $m = 3$.

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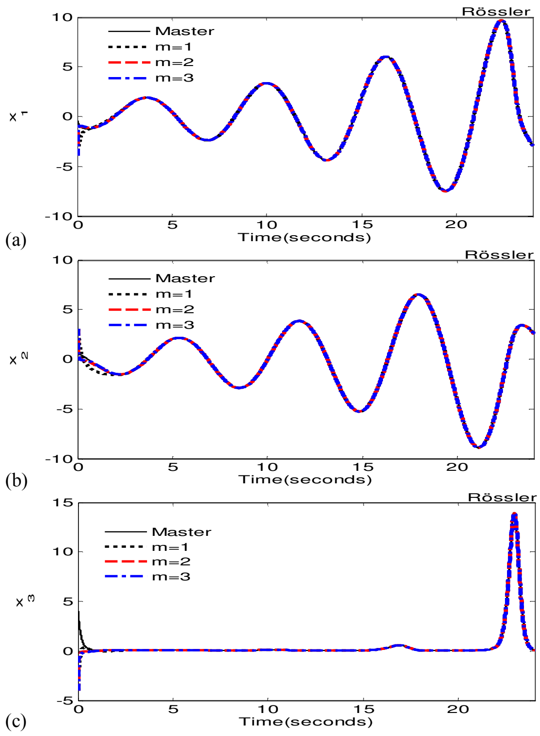

Fig. 1. Synchronization between system (10) and its observer (11), without any noise in the system output, (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

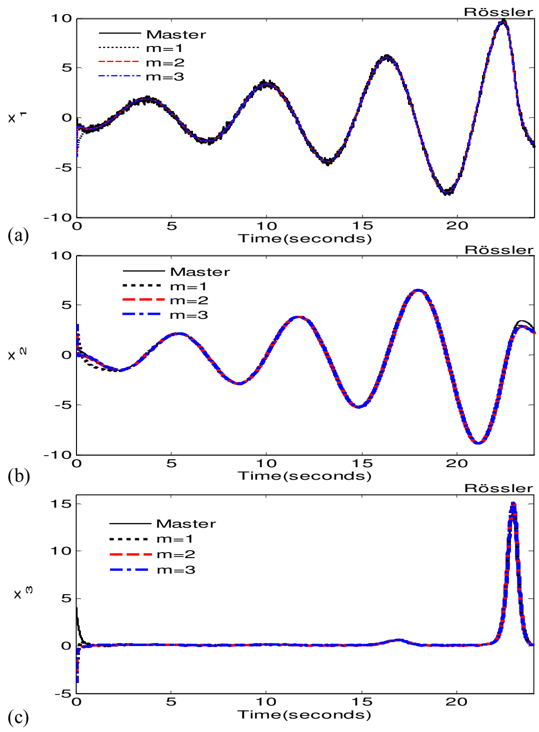

Fig. 2. Synchronization between system (10) and its observer (11), with white noise in the system output, (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

Now, the effect of noise in the measurements is analyzed. In Fig. 2 are presented the numerical results when a noise is added in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat{x}_1 = -4$, $\hat{x}_2 = 3$, $\hat{x}_3 = -4$. We can see that synchronization is possible, i.e., the estimated states tend to the real states. In Fig. 3 is illustrated the performance index given by (12) for the corresponding synchronization process. It should be noted that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease. Clearly, we can see that the proposed observer is robust against noisy measurements.

B. Example 2: Lorenz system

Let us consider the Lorenz chaotic system described by the following set of differential equations,

$$\begin{aligned} \dot{x}_1 &= \alpha\left(x_2 - x_1\right) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ y &= x_1 \end{aligned} \tag{13}$$

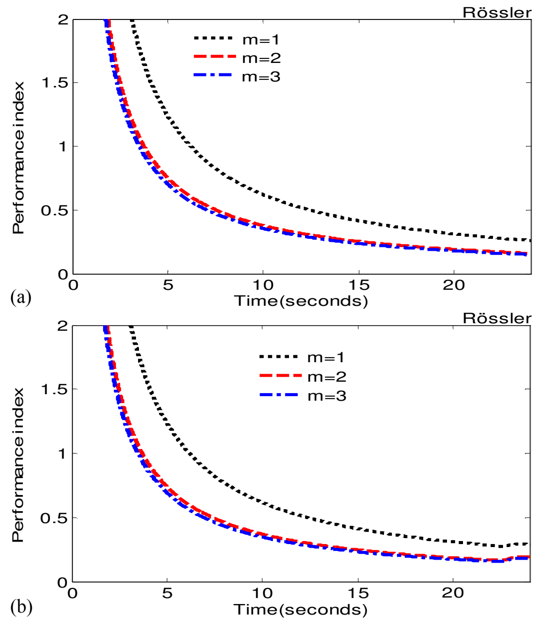

Fig. 3. Quadratic estimation error, (a) without any noise in the system output; (b) with white noise in the system output.

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with positive parameters ($\alpha, \rho, \beta > 0$) system (13) exhibits chaotic behaviour.

Remark 2. It is not difficult to prove that system (13) is Lipschitz.

The system (13) may be written in the form given by (1b), where $x = [x_1 \quad x_2 \quad x_3]^T$,

$$A = \begin{bmatrix} -\alpha & \alpha & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, \ \Psi = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}, \ C = [1 \quad 0 \quad 0]$$

With the selection of $y = x_1$, the corresponding pair (A, C) is detectable, hence, by an appropriate choice of K_1, one may obtain a stable matrix $(A - K_1C)$ and use the observer given by (2) for synchronization of chaos.

According to proposition 1, we get the following equations system (slave system) as the observer,

$$\begin{aligned} \dot{\hat{x}}_1 &= \alpha(\hat{x}_2 - \hat{x}_1) + \sum_{i=1}^{m} k_{1,i}\,[y - C\,\hat{x}\,]^{2i-1} \\ \dot{\hat{x}}_2 &= \rho\,\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3 + \sum_{i=1}^{m} k_{2,i}\,[y - C\,\hat{x}\,]^{2i-1} \qquad (14) \\ \dot{\hat{x}}_3 &= \hat{x}_1\hat{x}_2 - \beta\,\hat{x}_3 + \sum_{i=1}^{m} k_{3,i}\,[y - C\,\hat{x}\,]^{2i-1} \end{aligned}$$

where, $K_i = [k_{1,i} \quad k_{2,i} \quad k_{3,i}]^T \in \Re^3$, $i \in \{1, 2, \ldots, m\}$ and $\alpha, \rho, \beta > 0$.

We show some simulations for the Lorenz system. The parameter values for system (13) and its observer (14) are taken as $\alpha = 10, \rho = 28, \beta = 8/3$. The observer's gains have been fixed as $K_1 = [10 \quad 10 \quad 10]^T$, $K_2 = [10 \quad 10 \quad 10]^T$, and $K_3 = [\,5 \quad 5 \quad 5]^T$. Fig. 4 shows the convergence of the estimated states (slave system) to the real states (master system), without any noise in the system output. The initial conditions are $x_1 = 1$, $x_2 = 0$, $x_3 = -5$, $\hat{x}_1 = 4$, $\hat{x}_2 = -5$, $\hat{x}_3 = 8$.

It should be noted that the rate of convergence is faster for $m = 3$.

To analyze the effect of noise in the measurements a noise is added in the system output as in example 1. In Fig. 5 are presented the numerical results when a noise is added in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). The initial conditions are $x_1 = 1$, $x_2 = 0$, $x_3 = -5$, $\hat{x}_1 = 4$, $\hat{x}_2 = -5$, $\hat{x}_3 = 8$. We can see that synchronization is still possible, i.e., the estimated states tend to the real states.

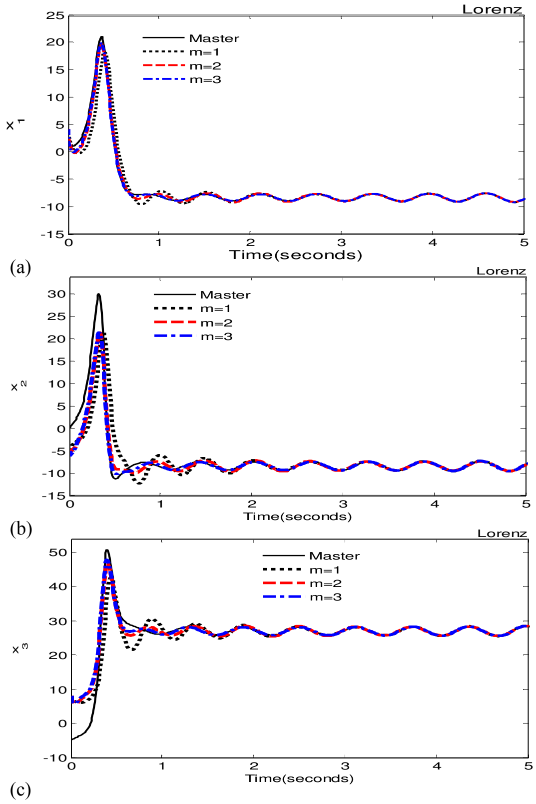

Fig. 4. Synchronization between system (13) and its observer (14), without any noise in the system output, (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

In Fig. 6 is illustrated the performance index given by (12) for the corresponding synchronization process, without any noise system output and with noise in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). We can see that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease. Clearly, we can see that the proposed observer is robust against noisy measurements.

IV. CONCLUSION

In this paper, we have designed a new asymptotic polynomial observer (high order polynomial type) for a class of nonlinear oscillators to attack the synchronization problem. Also, we have proven the asymptotic stability of the resulting state estimation error and by means of simple algebraic manipulations we construct the observer (slave system). Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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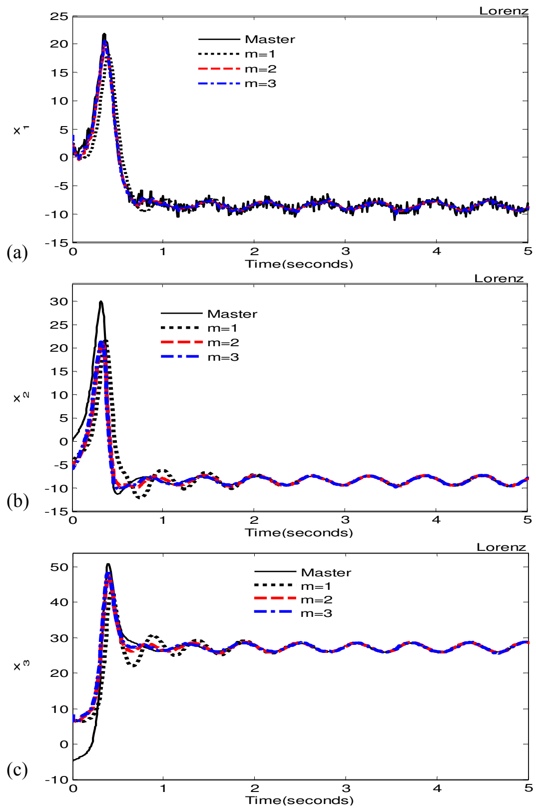

Fig. 5. Synchronization between system (13) and its observer (14), with white noise in the system output, (a) signals x_1 and $\hat{x}_1$; (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

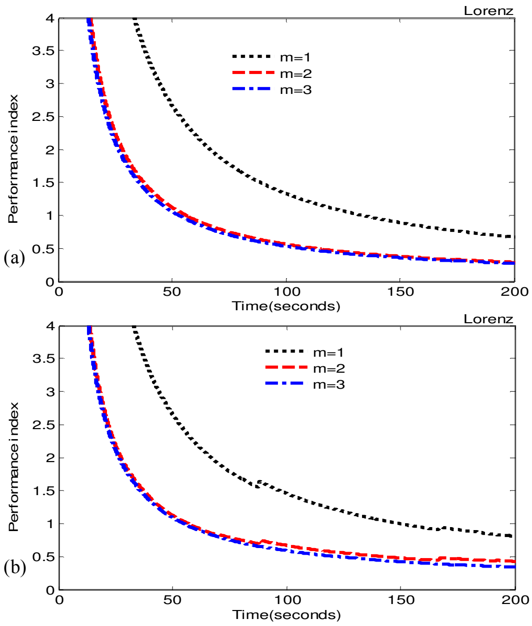

Fig. 6. Quadratic estimation error, (a) without any noise in the system output; (b) with white noise in the system output.

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A Bounded Error Observer for Synchronization of Chaotic Systems

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Abstract — **In this paper, we propose a bounded error observer of reduced order for a class of chaotic systems. The state variables are estimated by means of output system, which is supposed to be exactly known. The estimation methodology is based on a suitable change of variable which allows generating artificial variables to infer the remaining states constructing a differential-algebraic structure. The proposed methodology is applied to a class of Lipschitz nonlinear systems with success. Some remarks about the convergence characteristics of the proposed estimator are given and numerical simulations illustrate the effectiveness of the suggested approach.**

Keywords — **Nonlinear systems, state observers, synchronization.**

I. Introduction

In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields, for instance in [1], [2]. The growing interest in synchronization was probably caused by the paper by Pecora and Carrol [3] where, among others, secure communication as a potential application has been indicated. Although sofar it is still questionable whether this application can be fully realized, the Pecora and Carrol paper has formed an impulse for much research along these lines. It is well known that study of the synchronization problem for nonlinear systems has been very important for nonlinear science, in particular the applications to biology, medicine, cryptography, secure data transmission and so on.

In general, the synchronization research has been focused onto two areas. The first one relates with the employ of state observers, where the main applications lies on the synchronization of nonlinear oscillators [4], [5], [6], [7]. The second one is the use of control laws, which allows achieve the synchronization with different structure and order between nonlinear oscillators [1], [8], [9]. A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master- slave configuration [3].

The purpose of this paper is to address the synchronization problem from a control theory perspective. More specifically, the paper addresses the synchronization problem from the perspective of nonlinear observer design. Thus, chaos synchronization problem can be posed as an observer design where the coupling signal is viewed as the output and the slave system as the (reduced-order) observer. The phenomenon of synchronization also occurs for uni-directionally coupled systems and in this case the driven system (slave or receiver system) may be viewed as a nonlinear observer of the driving system (master or transmitter system). In this configuration, the two coupled systems are (almost) identical and therefore identical synchronization occurs which means that the difference of master and slave state vectors converges to zero for $t \to \infty$. In other words, the idea is to use the output of the master system so that the states of the slave system follow the states of the master system asymptotically.

In our procedure is not necessary the construction of a full order observer, that is to say, we construct a reduced order observer using the algebraic observability condition (AOC) applied to the estimation problem. We propose a bounded error observer to synchronize with a chaotic system.

The main contribution in this work is to show a bounded error observer where a simple linear change of variable allows develop a differential-algebraic state observer which is a simplified version of the observer presented in [10], numerical results show an adequate performance of the considered methodology. As far as we know in the literature this class of observers have not been used in the synchronization problem.

To illustrate our methodology, we give two applications to chaotic systems. The former is an application to the denominated Rössler system, this system presents a chaotic behavior and exhibits the simplest possible strange attractor. Originally, the Rössler system is credited to Otto Rössler, arose from work in chemical kinetics [11]; the second one, is an application to the well known Lorenz system [12], given by E. Lorenz, who studied the turbulent dynamics of the thermally induced fluid convection in the atmosphere described by the Navier-Stocks partial-differential equations.

The intention of choosing examples as Lorenz and Rössler systems is to clarify the proposed methodology. However, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

The paper is organized as follows. In the following section, the algebraic observability condition (AOC) will be presented. In section III will be shown that under certain circumstances the synchronization problem can be viewed as an observer problem. Section IV explores the AOC for Rössler and Lorenz systems. In section V will be presented a bounded error observer for synchronization problem, indeed, will be shown some numerical simulations. Section VI contains some conclusions.

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II. A NOTE ON ALGEBRAIC OBSERVABILITY CONDITION (AOC)

Before proposing the bounded error observer, a definition concerning on algebraic observability condition is given, for more details see [13].

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \Re^n$ the vector of the state variables; $f(\circ): \Re^n \times \Re^m \to \Re^n$, $(m \le n)$ is a nonlinear smooth vector function and Lipschitz in x and uniformly bounded in u, $y \in \Re^p$ is the vector of measured states.

Definition 1. Consider the system described by systems (1), where $x \in \Re^n$. A state x_i, is said to be algebraically observable with respect to $\{u, y\}$ if it satisfies a differential polynomial in terms of $\{u, y\}$ and some of their time derivatives, i. e., $P(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$, $i \in \{1, 2, \cdots, n\}$.

III. PROBLEM STATEMENT

A. The model

Consider that the system (1) satisfies the algebraic observability condition (AOC).

Let us recall the classical observer definition. An observer for system (1) is a dynamical system $\dot{\hat{x}} = \hat{f}(\hat{x}, u, y)$, whose task is state estimation. Usually is required at least that $\|\hat{x} - x\| \to 0$ as $t \to \infty$. Although in some cases, exponentially convergence is also required [14].

Definition 2. An estimator is said to be bounded if the estimation error ($\|\hat{x} - x\|$) belongs to an open ball with radius proportional to some value that depends on its estimation error.

In all paper, we will consider two classes of chaotic systems.

Firstly, we consider the popular nonlinear Rössler system, which is described by

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3) \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \\ y &= x_1 \end{aligned} \tag{2}$$

It is well known that in a large neighborhood of $\{a = b = 0.2,\ c = 5\}$ this system has a chaotic behavior.

Now, we consider the *Lorenz chaotic system* described by the following set of differential equations,

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ y &= x_1 \end{aligned} \tag{3}$$

with positive parameters ($\sigma, \rho, \beta > 0$) the system (3) exhibits chaotic behaviour.

Remark 1. It is not difficult to prove that above systems (2) and (3) are Lipschitz.

B. Synchronization as an observer problem

We consider that the system (1) satisfies the algebraic observability condition (AOC).

We also consider a second system that is driven by (1),

$$\dot{\hat{x}} = g(y, \hat{x}, u) \tag{4}$$

where $\hat{x} \in \Re^n$ is the vector of estimated states; and $g(\circ): \Re^n \times \Re^m \times \Re \to \Re^n$ is smooth.

The system (1) is the so-called *master* (*or transmitter*) and the system (4) is the *slave* (*or receiver*). Synchronization is understood as a coincidence of the variables of two or several systems; or as a concurring change of some quantitative characteristics of the systems. In the classical synchronization scheme we assume that the system (1) runs at the transmitter end and the output system is sent to the receiver (4) via communication channel as the synchronization signal [3], [6], [15], [16].

Receiver's task is to construct a dynamical system to estimate or construct the unknown states using the available signal output and the known parameters.

Synchronization of (1) and (4) occurs if, no matter how (1) and (4) are initialized, we have that asymptotically their states will match, i.e.,

$$\lim_{t \to \infty} \|\hat{x} - x\| = 0$$

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IV. ALGEBRAIC OBSERVABILITY CONDITION (AOC) OF RÖSSLER AND LORENZ SYSTEMS

A. Observability of Rössler system

Before proposing the state observer, we prove the algebraic observability condition (see definition 1) for system (2).

Replacing $y = x_1$ in the system (2), we obtain

$$\dot{y} = -(x_2 + x_3) \tag{5}$$

$$\dot{x}_2 = y + a\,x_2 \tag{6}$$

$$\dot{x}_3 = b + x_3(y - c) \tag{7}$$

Taking the time derivative of (5)

$$\ddot{y} = -\dot{x}_2 - \dot{x}_3 \tag{8}$$

From (5),

$$x_3 = -(\dot{y} + x_2) \tag{9}$$

Substituting (6), (7) and (9) into (8),

$$\ddot{y} + y\dot{y} - c\,\dot{y} + y - x_2 y + (a + c)x_2 = 0 \tag{10}$$

In the same manner for x_3, we have

$$\ddot{y} - a\,\dot{y} + y + x_3 y - (a + c)x_3 + b = 0 \tag{11}$$

Remark 2. From (10) and (11), is clear that x_2 and x_3 are algebraically observable.

B. Observability of Lorenz system

In the same form, we prove the algebraic observability condition for system (3). After algebraic manipulations we obtain

$$\dot{y} + \sigma y + \sigma x_2 = 0 \tag{12}$$

$$-\ddot{y} + \dot{y}(\sigma + 1) + y\sigma(\rho + 1) + y\sigma x_3 = 0 \tag{13}$$

Remark 3. From (12) and (13), is clear that x_2 and x_3 satisfy the AOC, thus, x_2 and x_3 are algebraically observable.

V. SYNCHRONIZATION OF CHAOTIC SYSTEMS

In this section, we assume that the output system is measured exactly. Then we only reconstruct the remaining variables.

A. Example 1: synchronization of Rössler system via bounded error observer

Consider the system (2) and make the change of variables

$$z_2 = x_2 + k\,y$$
$$z_3 = x_3 + k\,y$$

where k is a fixed real constant. The dynamics of z_2 and z_3 are given by

$$\dot{z}_2 = -[k - a]z_2 + \zeta_2 \tag{14}$$

$$\dot{z}_3 = -[c + k - y]\hat{z}_3 + \zeta_3 \tag{15}$$

with
$$\zeta_2 = y - k[a\,y + z_3] + 2k^2 y$$
$$\zeta_3 = b + k[-y^2 + c\,y - z_2] + 2k^2 y\,.$$

Proposition 1. The system

$$\dot{\hat{z}}_2 = -[k - a]\hat{z}_2 + \hat{\zeta}_2 \tag{16}$$

$$\dot{\hat{z}}_3 = -[c + k - y]\hat{z}_3 + \hat{\zeta}_3 \tag{17}$$

is a bounded error observer of (14) and (15), where k is a gain ($k \geq \max\{a, y - c\}$), $\hat{\zeta}_2 = y - k[a\,y + \hat{z}_3] + 2k^2 y$, and $\hat{\zeta}_3 = b + k[-y^2 + c\,y - \hat{z}_2] + 2k^2 y$.

A proof of convergence is given in the next lemma.

Lemma 1. According to equations (16) and (17) the general differential equation is given by the next form

$$\dot{\hat{\vartheta}} = -d\,\hat{\vartheta} + \hat{\zeta} \tag{18}$$

which is uniformly bounded, with $d > 0$, if the following assumptions are considered

A1. $\left|\hat{\zeta} - \zeta\right| < N < \infty$.

A2. For t_0 sufficiently large, $\lim\sup_{t \to t_0} \dfrac{N}{d} = 0$.

The proof is given as follows. Let us define the estimation error (the difference between the actual observed signal and its estimate) as

$$e := \hat{\vartheta} - \vartheta \tag{19}$$

The dynamic of (19) is

$$\dot{e} + d\,e = \Psi \tag{20}$$

where $\Psi = \hat{\zeta} - \zeta$.

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We obtain the solution from (20)

$$e = \exp^{-d\,t}\, e_0 + \int_0^t \exp^{d(\tau - t)}\, \Psi \, d\tau \qquad (21)$$

where e_0 is an initial condition.

Using Triangle and Cauchy-Schwarz inequalities from expression (21)

$$0 \le |\, e \,| \le \left| \exp^{-d\,t} \right| |\, e_0 \,| + \int_0^t \left| \exp^{d(\tau - t)}\, \Psi \, d\tau \right| .$$

From A1

$$0 \le |\, e \,| \le \left| \exp^{-d\,t} \right| |\, e_0 \,| + N \int_0^t \left| \exp^{d(\tau - t)} \; d\tau \right| \qquad (22)$$

Thus, as $t \to t_0$, t_0 sufficiently large,

$$0 \le \limsup_{t \to t_0} |\, e \,| \le \limsup_{t \to t_0} N \int_0^t \left| \exp^{d(\tau - t)} \; d\tau \right|$$

$$0 \le \limsup_{t \to t_0} |\, e \,| \le \limsup_{t \to t_0} \frac{N}{d} \left| 1 - \exp^{-t} \right|$$

$$0 \le \limsup_{t \to t_0} |\, e \,| \le \limsup_{t \to t_0} \frac{N}{d} \qquad (23)$$

From A2

$$\lim_{t \to t_0} |\, e \,| = 0 \quad \text{, for } t_0 \text{ sufficiently large}$$

Corollary 1. The bounded error observer for Rössler system (2) is given by

$$\begin{aligned} \dot{\hat{z}}_2 &= -[k - a]\hat{z}_2 + \hat{\zeta}_2 \\ \dot{\hat{z}}_3 &= -[c + k - y]\hat{z}_3 + \hat{\zeta}_3 \\ \hat{x}_2 &= \hat{z}_2 - k\, y \\ \hat{x}_3 &= \hat{z}_3 - k\, y \end{aligned} \qquad (24)$$

with $k \ge \max\{a, y - c\}$, $\hat{\zeta}_2 = y - k[a\,y + \hat{z}_3] + 2k^2 y$, and $\hat{\zeta}_3 = b + k[-y^2 + c\,y - \hat{z}_2] + 2k^2 y$.

Now, to illustrate the effectiveness of the proposed approach, some numerical simulations are presented.

We have considered the initial conditions to the master system (2) as $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$ and the initial conditions to the slave system – observer given by system (24) – as $\hat{x}_2 = -2.5$, $\hat{x}_3 = -4$, the parameter values are $a = b = 0.2$, $c = 5$ and observer gain is fixed as $k = 5$.

Fig. 1 shows the convergence of the estimated states (slave system) to the real states (master system), in other words, the synchronization between Rössler system (2) and its observer (24).

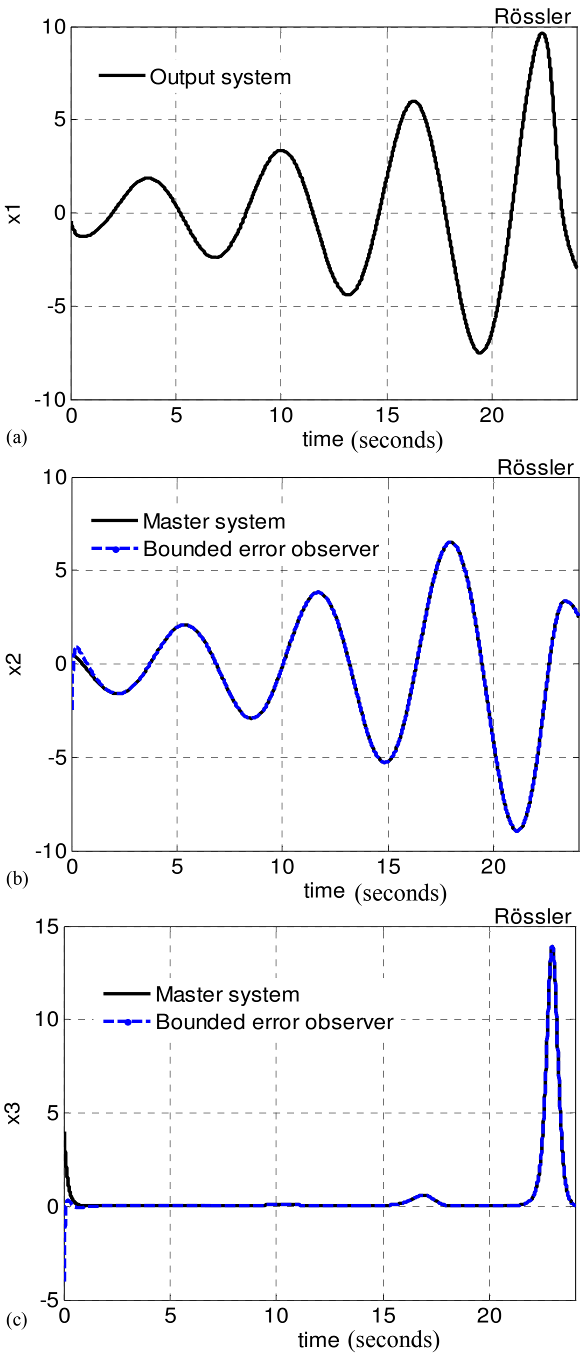

Fig. 1. Synchronization between system (2) and its observer (24), (a) signal x_1 (output system); (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

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B. Example 2: synchronization of Lorenz system via bounded error observer

Consider the system (3) and make the change of variables

$$z_2 = x_2 + k\,y$$
$$z_3 = x_3 + k\,y$$

where k is a fixed real constant. The dynamics of z_2 and z_3 are given by

$$\dot{z}_2 = -[1-k\,\sigma]z_2 + \eta_2 \quad (25)$$
$$\dot{z}_3 = -\beta\,z_3 + \eta_3 \quad (26)$$

with
$$\eta_2 = \rho\,y - y\,z_3 + k\,y(1+y-\sigma) - k^2\sigma\,y$$
$$\eta_3 = y\,z_2 + k\left[-y^2 + \beta\,y + \sigma\,z_2 - \sigma\,y\right] - k^2\sigma\,y\,.$$

Proposition 2. The system

$$\dot{\hat{z}}_2 = -[1-k\,\sigma]\hat{z}_2 + \hat{\eta}_2 \quad (27)$$
$$\dot{\hat{z}}_3 = -\beta\,\hat{z}_3 + \hat{\eta}_3 \quad (28)$$

is a bounded error observer of (25) and (26), where k *is a gain* ($k < 1/\sigma$), $\hat{\eta}_2 = \rho\,y - y\,\hat{z}_3 + k\,y(1+y-\sigma) - k^2\sigma\,y$, *and* $\hat{\eta}_3 = y\,\hat{z}_2 + k\left[-y^2 + \beta\,y + \sigma\,\hat{z}_2 - \sigma\,y\right] - k^2\sigma\,y\,.$

A proof of convergence was given in lemma 1.

Corollary 2. The bounded error observer for Lorenz system (3) is given by

$$\dot{\hat{z}}_2 = -[1-k\,\sigma]\hat{z}_2 + \hat{\eta}_2$$
$$\dot{\hat{z}}_3 = -\beta\,\hat{z}_3 + \hat{\eta}_3$$
$$\hat{x}_2 = \hat{z}_2 - k\;y \quad (29)$$
$$\hat{x}_3 = \hat{z}_3 - k\;y$$

with $k < 1/\sigma$, $\hat{\eta}_2 = \rho\,y - y\,\hat{z}_3 + k\,y(1+y-\sigma) - k^2\sigma\,y$, and $\hat{\eta}_3 = y\,\hat{z}_2 + k\left[-y^2 + \beta\,y + \sigma\,\hat{z}_2 - \sigma\,y\right] - k^2\sigma\,y\,.$

We show some numerical results for Lorenz system (3) and its observer (29). We have taken the parameter values as $\sigma = 10$, $\beta = 8/3$, $\rho = 28$, and observer gain is fixed as $k = -1$. The initial conditions to the master system are $x_1 = 1$, $x_2 = 0$, $x_3 = -5$ and the initial conditions to the slave system (observer) $\hat{x}_2 = -5$, $\hat{x}_3 = 8$. Figure 2 shows the synchronization of Lorenz system.

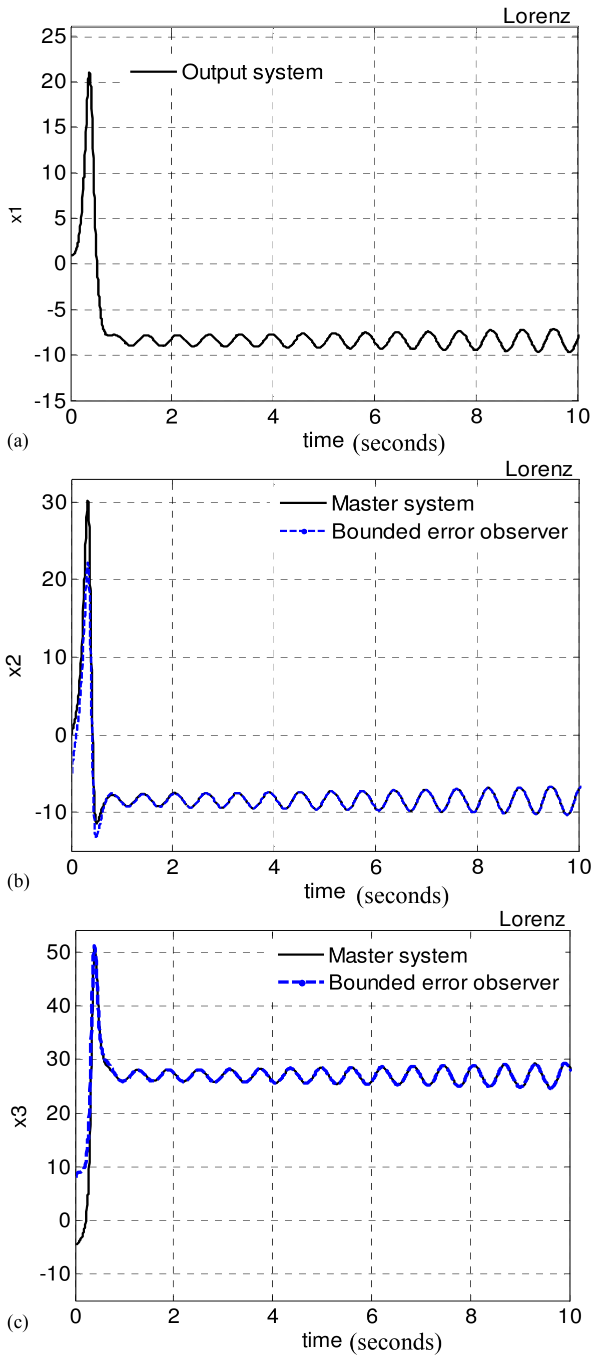

Fig. 2. Synchronization between system (3) and its observer (29), (a) signal x_1 (output system); (b) signals x_2 and $\hat{x}_2$; (c) signals x_3 and $\hat{x}_3$.

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VI. CONCLUSION

In this paper, we have attacked the synchronization problem from the perspective of nonlinear observer design, that is to say, *the synchronization as an observer problem*. We have presented a bounded (reduced order) error observer to synchronize with a chaotic system and we have proven the convergence of the resulting error system. Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach.

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Synchronization problem via an asymptotic polynomial observer

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Abstract— **In this paper a new observer is proposed for the synchronization problem, this new observer is an asymptotic polynomial observer for a class of nonlinear oscillators which turns out be robust against output noises. Furthermore this observer is of high order polynomial type. Stability analysis in a Lyapunov sense for the synchronization error is performed. The proposed methodology is applied to synchronization of chaotic systems with success: the performance of this observer is shown by using Rössler system.**

Keywords: Nonlinear systems, state observers, Riccati equation, synchronization.

I. INTRODUCTION

In the last years, synchronization of chaotic systems problem has received a great deal of attention among scientist in many fields, for instance in (Fradkov, 2007) and (Chen et al., 2005). It is well known that study of the synchronization problem for nonlinear systems has been very important for nonlinear science, in particular the applications to biology, medicine, cryptography, secure data transmission and so on. In general, the synchronization research has been focused onto two areas. The first one relates with the employ of state observers, where the main applications lies on the synchronization of nonlinear oscillators (Hua and Guan, 2005) and (Martínez-Guerra et al., 2006). The second one is the use of control laws, which allows achieve the synchronization with different structure and order between nonlinear oscillators (Femat and Solis-Perales, 2008). A particular interest is the connection between the observers for nonlinear systems and chaos synchronization, which is also known as master- slave configuration (Pecora and Caroll, 1990). Thus, chaos synchronization problem can be regarded as observer design procedure, where the coupling signal is viewed as output and the slave system is the observer.

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general indeed, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances.

The design of observers for nonlinear systems is a challenging problem (even for accurately known systems) that has received a considerable amount of attention. Since the observers developed by Kalman and Luenberger several years ago for linear systems, different state observation techniques have been proposed to handle the systems nonlinearities. A first category of techniques consists in applying linear algorithms to the system linearized around the estimated trajectory. These are known as the extended Kalman and Luenberger observers. Alternatively, the nonlinear dynamics are split into a linear part and a nonlinear one. The observer gains are then chosen large enough so that the linear part dominates the nonlinear one. Such observers are known as high-gain observers (Aguilar et al., 2003), (Martínez-Guerra et al., 2000). In a third approach the nonlinear system is transformed into a linear one by an appropriate change of coordinates (Keller, 1987). The estimate is computed in these new coordinates and the original coordinates are recovered through the inverse transformation. In most approaches, nonlinear coordinate transformations are employed to transform the nonlinear system into a block triangular observer canonical form. Then, high gain (Gauthier et al, 1992), backstepping (Young and Farrel, 2000), or sliding mode observers (Levant, 2001) can be designed.

Many problems in engineering and other applications are globally Lipschitz for instance the sinusoidal terms in robotics. Nonlinearities which are square or cubic in nature are not globally Lipschitz, however, they are locally so, moreover when such functions occur in physical systems, they frequently have a saturation in their growth rate, making them globally Lipschitz functions (Raghavan, 1994). Thus, this class of systems covered by this note is fairly general.

The main contribution of this paper consists in the solution of the synchronization problem via an asymptotic polynomial observer. The obtained state space estimation error is shown to be bounded, and this bound depends on observer's gain and a Lipchitz constant. This communication presents some fundamental insights into polynomial observer design for the class of Lipschitz nonlinear systems, that it means, that any autonomous nonlinear system of the form $\dot{x} = f(x,u)$ can be regarded as Lipschitz continuous system with respect to x, with a Lipchitz constant L.

The intention of choosing an example as the Rössler system is to clarify the proposed methodology. However, it is worth to mention that this technique can be applied to almost any chaotic synchronization problem.

In what follows, an asymptotic polynomial observer is proposed as well as an easy numerical design is given. Numerical results show its satisfactory performance. Finally we close this paper with some concluding remarks.

II. MAIN RESULT

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= Cx \end{aligned} \tag{1a}$$

where $x \in \Re^n$ is the vector of the state variables; $f(\circ): \Re^n \times \Re^l \to \Re^n$, $(l \le n)$ is a nonlinear smooth vector function and Lipschitz in x and uniformly bounded in u, $y \in \Re$ is the vector of measured states.

Any nonlinear system of the form (1a) can be expressed in the form (1b) as long as $f(x,u)$ is differentiable with respect to x.

$$\begin{aligned} \dot{x} &= Ax + \Psi(x,u) \\ y &= Cx \quad , \quad x_0 = x(t_0) \end{aligned} \tag{1b}$$

In system (1b), $\Psi(x,u)$ is a nonlinear vector function which satisfies the Lipschitz condition with a Lipschitz constant L, i.e,

$$\|\Psi(x,u) - \Psi(\hat{x},u)\| \le L\|x - \hat{x}\|$$

In this paper, we always assume that the pair (A,C) is observable.

We have the main result.

Proposition 1: the following nonlinear dynamic system is a full order state observer of the system (1b):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \Psi(\hat{x},u) + K_1C(x-\hat{x}) + K_2[C(x-\hat{x})]^m \\ \hat{x}_0 &= \hat{x}(t_0) \end{aligned} \tag{2}$$

If the following assumptions are considered,

- $m \in Z^+, m \;\; odd, m > 1$ (3)
- K_1 can be chosen such as the following Algebraic Riccati Equation (ARE) has a symmetric positive-definite solution P for some $\varepsilon > 0$

$$(A - K_1C)^T P + P(A - K_1C) + L^2PP + I + \varepsilon I = 0 \tag{4}$$

- $\lambda_{\min}(PK_2C) \ge 0$ (5)

In (2), $\hat{x} \in \Re^n$, $K_1 = [k_{1,1} \quad k_{2,1} \quad \ldots \quad k_{n,1}]^T \in \Re^n$, $K_2 = [k_{1,2} \quad k_{2,2} \quad \ldots \quad k_{n,2}]^T \in \Re^n$.

Proof. Defining the estimation error as $e = x - \hat{x}$, the corresponding dynamic of the estimation error is:

$$\dot{e} = (A - K_1C)e - K_2[Ce]^m + [\Psi(x,u) - \Psi(\hat{x},u)] \tag{6}$$

Consider the Lyapunov function candidate,

$$V = e^T Pe$$

Its derivative is

$$\begin{aligned} \dot{V} &= \dot{e}^T Pe + e^T P\dot{e} \\ &= e^T\left[(A - K_1C)^T P + P(A - K_1C)\right]e - \\ &\quad - 2(Ce)^{m-1} e^T PK_2Ce + 2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \end{aligned} \tag{7}$$

In (Raghavan and Hedrick, 1994) is presented the next inequality as a lemma which is useful for this proof,

$$2e^T P[\Psi(x,u) - \Psi(\hat{x},u)] \le L^2 e^T PPe + e^T e$$

From Rayleigh inequality, and taking into account (5), we have,

$$-e^T PK_2Ce \le -\lambda_{\min}(PK_2C)\|e\|^2$$

Equation (7) leads to

$$\begin{aligned}\dot V \le\ & e^T\left[(A-K_1C)^T P + P(A-K_1C)\right]e - \\ & -2(Ce)^{m-1}\lambda_{\min}(PK_2C)\|e\|^2 + L^2 e^T PPe + e^T e \\ =\ & e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2PP + I\right]e - \\ & -2(Ce)^{m-1}\lambda_{\min}(PK_2C)\|e\|^2 \end{aligned} \tag{8}$$

From assumption (3), the second term in the right hand side of the inequality (8) always will be positive or zero,

$$\dot V \le e^T\left[(A-K_1C)^T P + P(A-K_1C) + L^2PP + I\right]e \tag{9}$$

According with assumption (4), since $\varepsilon > 0$, it is clear that

$$(A-K_1C)^T P + P(A-K_1C) + L^2PP + I < 0$$

Hence, $\dot V < 0$ .This implies that system (2) is an observer for system (1b) and the corresponding dynamic of the estimation error (6) is asymptotically stable.

♦

III. APPLICATION TO SINCRONIZATION OF CHAOTIC SYSTEMS

To illustrate our methodology, we give an application to chaotic systems. In fact, this is an application to the so-called Rössler's system (Rössler, 1976) which presents a chaotic behavior and exhibits the simplest possible strange attractor, arose from work in chemical kinetics.

A. The model

We consider the popular nonlinear Rössler's System, which is described by

$$\begin{aligned}\dot x_1 &= -(x_2 + x_3)\\ \dot x_2 &= x_1 + a x_2\\ \dot x_3 &= b + x_3(x_1 - c)\\ y &= x_1\end{aligned} \tag{10}$$

It is well known that in a large neighborhood of $\{a = b = 0.2,\ c = 5\}$ this system has a chaotic behavior.

Remark 1: it is not difficult to prove that above system is Lipschitz.

B. Observability condition

The system (10) may be written in the form given by (1b), where $x = [x_1 \quad x_2 \quad x_3]^T$,

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix},\ \Psi = \begin{bmatrix} 0 \\ 0 \\ b + x_1 x_3 \end{bmatrix},\quad C = [1 \quad 0 \quad 0]$$

It can easily be shown that with the selection of $y = x_1$, the corresponding pair (A, C) is observable. By choosing the observer's gain K_1 and K_2 appropriately, the observer given by (2) may achieve local synchronization.

C. The observer for the Rössler's System

According with proposition 1, we get the following equations system (slave system) as the observer,

$$\begin{aligned}\dot{\hat x}_1 &= -(\hat x_2 + \hat x_3) + k_{1,1}(x_1 - \hat x_1) + k_{1,2}(x_1 - \hat x_1)^m\\ \dot{\hat x}_2 &= \hat x_1 + a\hat x_2 + k_{2,1}(x_1 - \hat x_1) + k_{2,2}(x_1 - \hat x_1)^m\\ \dot{\hat x}_3 &= b + \hat x_3(\hat x_1 - c) + k_{3,1}(x_1 - \hat x_1) + k_{3,2}(x_1 - \hat x_1)^m\end{aligned} \tag{11}$$

where, $K_1 = [k_{1,1} \quad k_{2,1} \quad k_{3,1}]^T$, $K_2 = [k_{1,2} \quad k_{2,2} \quad k_{3,2}]^T$ and $a, b, c > 0$.

D. Numerical simulations

The design of the full-order observer presented in this paper is based on the solution of the Riccati equation which can be obtained by using the Matlab function ARE.

We have chosen the values for the Rössler's system (10) and the observer (11) as $a = b = 0.2$, $c = 5$, $m = 3$, and the observer's gain have been taken as $K_1 = [5 \quad -5 \quad 5]^T$ and $K_2 = [10 \quad 10 \quad 10]^T$. All the simulations results in this paper were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox.

In this work, the performance index of the corresponding synchronization process was calculated as,

$$J(t) = \frac{1}{t + 0.001}\int_0^t \|e(t)\|_{Q_0}^2 \, d\tau \tag{12}$$

where $e(t)$ denotes the estimation error and $Q_0 = I$.

Figure 1 shows the convergence of the estimated states (slave system) to the real states (master system), without noise in the system output. The initial conditions are $x_1 = -0.5$, $x_2 = 0.5$, $x_3 = 4$, $\hat x_1 = -4$, $\hat x_2 = 3$, $\hat x_3 = -4$.

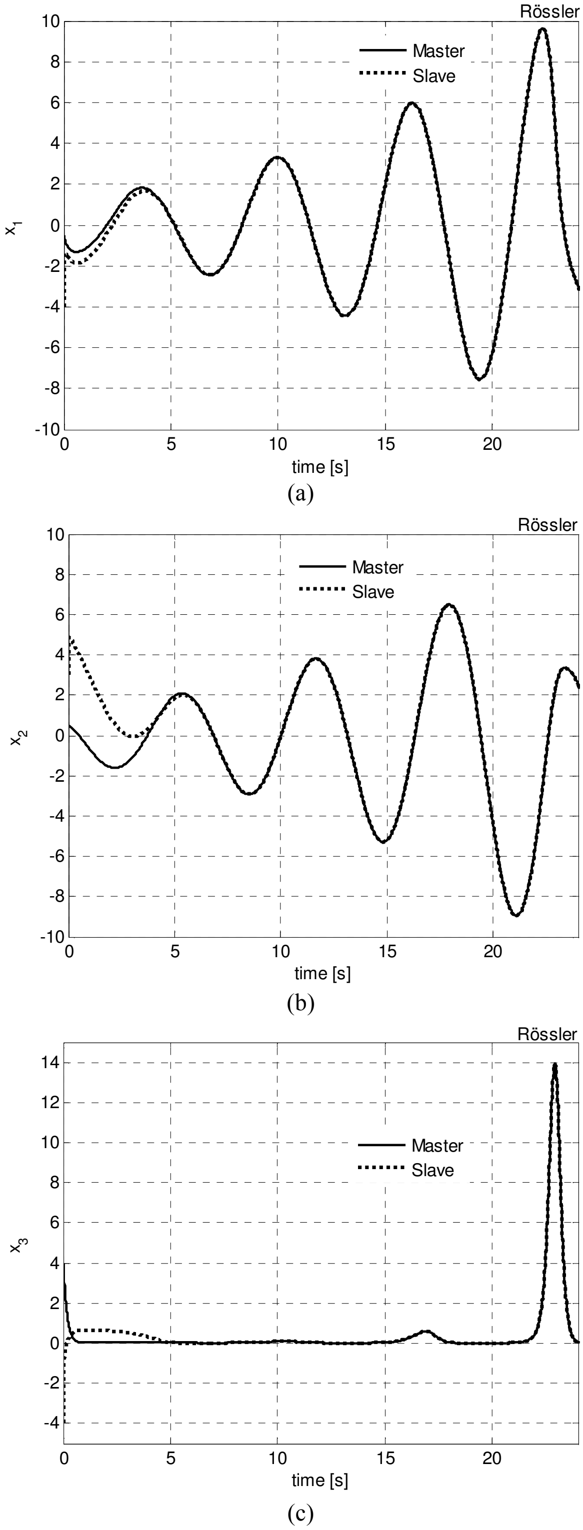


Figure 1. State estimation, without any noise in the system output.

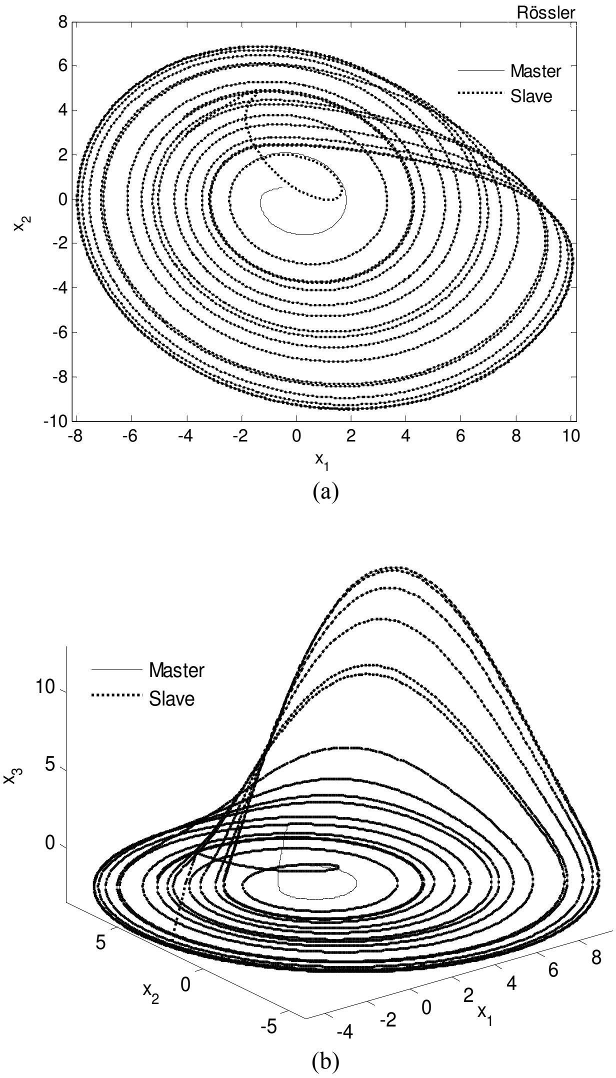
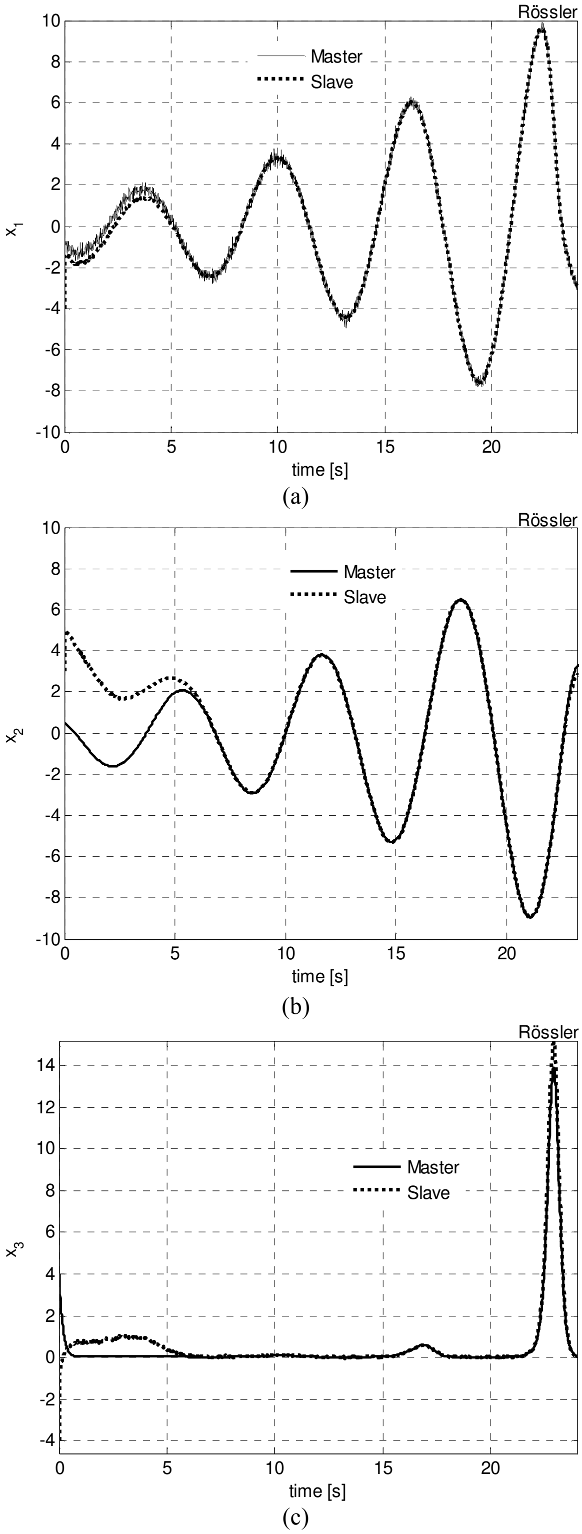


Figure 2. Rössler's system trajectories

Figure 2 shows the chaotic behaviour to the Rössler's System (master system) and its observer (slave system), and also shows the convergence of the estimated states to the real states, without noise in the system output.

Now, we analyze the effect of noise in the measurements. In figure 3 are presented the numerical results when a noise is added in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the system output). We can see that synchronization is possible, i. e, the estimated states tend to the real states.



(a)

(b)

(c)

Figure 3. State estimation, with white noise in the system output.

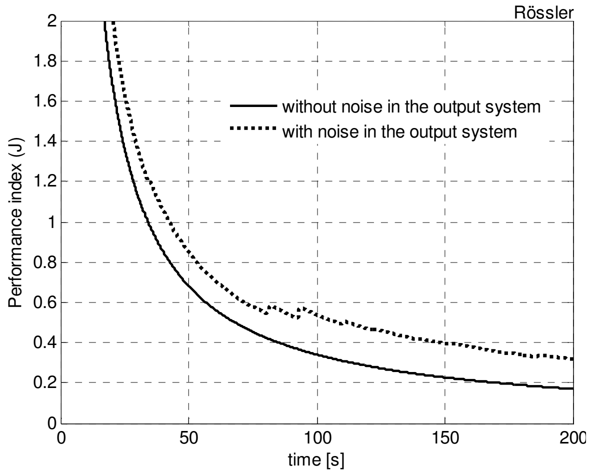


Figure 4. Quadratic estimation error.

In figure 4 is illustrated the performance index given by (12) for the corresponding synchronization process, without any noise system output and with noise in the system output (white noise with $\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). It should be noted that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease. Clearly, we can see that the proposed observer is robust against noisy measurements.

IV. Conclusion

In this paper, we have designed a new asymptotic polynomial observer (high order polynomial type) for a class of nonlinear oscillators to attack the synchronization problem. Also, we have proven the asymptotic stability of the resulting state estimation error and by means of simple algebraic manipulations we construct the observer (slave system). Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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Monitoring of a class of partially known bioreactor models employing a bounded error estimator

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Abstract— **The problem of monitoring in a common class of partially known bioreactor models is addressed. A reduced order observer namely bounded error estimator is proposed. The biomass is estimated by means of substrate concentration measurements. The estimation methodology is based on a suitable change of variable which allows generating artificial variables to infer the remaining mass concentrations constructing a differential-algebraic structure. The proposed methodology is applied to a class of Haldane unstructured kinetic model with success. Stability analysis in a Lyapunov sense for the estimation error is performed. Some remarks about the convergence characteristics of the proposed estimator are given and numerical simulations show its satisfactory performance. Finally, a high gain observer is presented: the convergence is possible only when the model is perfectly known.**

Keywords: Observability, nonlinear systems, Lyapunov stability, state observers.

I. INTRODUCTION

Operating a bioreactor is not a simple task, as during a bioreacting process, variables such as concentrations are generally determined by off-line laboratory analysis, making this set of variables of limited use for control purposes and on-line monitoring. However, these variables can be on-line estimated using *soft sensors*.

Over the last few years, the importance of on-line monitoring of biotechnological processes has increased. A first step to efficient bioreactor operation is the adequate implementation of online measurements of essential variables such as substrate and biomass concentrations. Advantages of continuous monitoring of key variables include gaining knowledge about the state of the process and the possibility of detecting and isolating abnormal process developments at early stages. This reduces process costs, contributes to process safety and helps in trouble-shooting and process accommodation. The main problem in fermentation monitoring and control is the fact that process variables usually cannot be measured on-line. Monitoring and controlling these processes can therefore be difficult because only indirect measurements are available online, while calculated values may be rather uncertain. This can be due to uncertainty with respect to the equations used, measurement errors or both. For automatic control this may have serious consequences, especially as the actual variables of interest often cannot be directly controlled and related variables are controlled instead. In fermentation processes, on-line and off-line measurements are the main source of information about the state of the process. In combination with model-based calculations, they are used to produce estimations for monitoring purposes as well as for automatic and manual process control (Bastin and Dochain, 1990), (Masoud, 1997).

Observation schemes are widely used for reconstructing states of dynamical systems (Aguilar-López et. al, 2006). Most of the contributions are related to asymptotic observers for monitoring, fault detections and control issues whereas the real necessities of industrial plants are related to a fast response of the monitoring and regulation methodologies.

Special attention was given to filtering techniques, namely extended Kalman filter, adaptive observers, and artificial neural networks (ANN), (Dávila and Fridman, 2005), (Hu and Wang, 2002), (Levant, 2001), however for these techniques the right tuning of the estimators gains is difficult. It is shown that software based state estimation is a powerful technique that can be successfully used to enhance automatic control performance of biological systems as well as in system monitoring and on-line optimization.

In this paper we consider the growth rate partially known. Following this idea, the necessity to adapt an observation scheme to the available knowledge of the growth rate immediately arises. The main contribution in this work is to show a state estimator which is a simplified version of the methodology given by (Lemesle and Gouzé, 2005) where a simple linear change of variable given in a natural manner allows to develop a differential-algebraic state estimator. Results show an adequate performance of the considered methodology. The technique is not the same as (Alvarez-Ramirez et. al, 1999) since we do not have derivators. The proposed estimation methodology is applied to a kind of unstructured kinetic model: the Haldane model, which is considered for biological process with substrate inhibition. The above mentioned kinetic model is applied to a class of continuous stirred bioreactors.

In what follows, the statement of the problem is presented; an observability condition is given in the differential-algebraic setting. In section III, the bounded error estimator is designed. Section IV shows a high gain observer as a comparison with the proposed methodology. Finally, we give some concluding remarks.

II. Problem Statement

A. The model

Consider the following nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $m \leq n$, $y \in R^p$.

Let us recall the classical observer definition. An observer for system (1) is a dynamical system $\dot{\hat{x}} = \hat{f}(\hat{x}, u, y)$, whose task is state estimation. Usually is required at least that $\|\hat{x} - x\| \to 0$ as $t \to \infty$. Although in some cases, exponentially convergence is also required (Gauthier et al., 1992).

Definition 1: an estimator is said to be bounded if the estimation error ($\|\hat{x} - x\|$) belongs to an open ball with radius proportional to some value that depends on its estimation error.

In all paper, we will consider a class of bioreactor model. The simplified Haldane model taken from (Vargas et al., 2000), is describe by

$$\frac{dS}{dt} = D(S_{in} - S) - \mu(S)\frac{X}{Y_{S/X}} + k_d X \tag{2a}$$

$$\frac{dX}{dt} = -DX + \mu(S)X - k_d X \tag{2b}$$

where $\mu(S) = \mu_{\max} S / \left(\delta + S + S^2/\phi\right)$ is the specific growth rate and $\mu_{\max}$ is the maximum growth rate.

We assume that $\mu(S)$ is partially known, which is common in biology (Gouzé and Lemesle, 2001). Generally, $\mu(S)$ is between two bounds meaning that we know a function $\hat{\mu}(S)$ such that $|\mu(S) - \hat{\mu}(S)| < a$, where $a \in R^+$, and $\mu(0) = \hat{\mu}(0) = 0$. We introduce an important lemma about lower bounded properties of $\mu(S)$.

Lema 1 (Hadj-Sadok, 1999): there exists a constant $\varepsilon \in R$, such that $S(0) > \varepsilon$ implies $S(t) > \varepsilon$ for all t . Thus, for any smooth function $\mu(S)$, $\mu(S(t)) > \mu(\varepsilon)$ for all t .

From lemma 1, we could always choose ε such that $\hat{\mu}(S(t)) > \hat{\mu}(\varepsilon) = r$, where $r \in R^+$.

The state variables S , X are substrate and biomass concentrations, respectively, $D = q/V$ is the dilution rate with V the volume of the bioreactor and q the constant flow passing through the bioreactor, S_{in} is the input substrate concentration, $Y_{S/X}$ is the corresponding yield coefficient. Let us notice that the inputs $D = u$ and S_{in} are fixed. Moreover, we assume that the measured output is,

$$y = S \tag{3}$$

B. Algebraic Observability Condition (AOC)

Before proposing the bounded error estimator, a definition concerning on *algebraic observability condition* is given, for more details see (Diop and Martínez-Guerra, 2001).

Definition 2: consider the system described by (1), where $x = (x_1 \quad x_2 \quad \ldots \quad x_n)^T$. A state x_i , $i = \{1, 2, \cdots, n\}$, is said to be algebraically observable with respect to $\{u, y\}$ if it satisfies a differential polynomial in terms of u , y and some of their time derivatives, i. e., $P(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots) = 0$, $i = \{1, 2, \cdots, n\}$.

Replacing $y = S$ into equation (2a), the algebraic observability condition for Haldane model is calculated as follows,

$$\dot{y} - u(S_{in} - y) + \left(\frac{\mu_{\max}\phi\, y}{\delta\phi + \phi\, y + y^2} \frac{1}{Y_{S/X}} - k_d \right) X = 0 \tag{4}$$

From equation (4), it is clear that the state variable X satisfies the AOC thus, X is algebraically observable.

III. BOUNDED ERROR ESTIMATOR

A. Estimator design

In what follows, the corresponding estimated concentration is denoted by ^, and we assume that S is measured exactly, i.e., $S = \hat{S}$. Then, we only reconstruct the biomass variable X .

Consider the Haldane's model given by system (2), and make the change of variable

$$z = X + k\,S \tag{5}$$

where $k \in R$ is fixed.

The dynamics of z is,

$$\dot{z} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1 \right) \mu(S) \right] z + \\ + (1-k) k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1 \right) k\,\mu(S) S + k\,D\,S_{in} \quad (6)$$

Proposition 1: if we choose the estimator's gain such that $Y_{S/X} < k \le 1 + D / k_d$ and $|\mu(S) - \hat{\mu}(S)| < a$, $a \in R^+$. Then, the system (7) is a bounded error estimator of (6).

$$\dot{\hat{z}} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1 \right) \hat{\mu}(S) \right] \hat{z} + \\ + (1-k) k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1 \right) k\,\hat{\mu}(S) S + k\,D\,S_{in} \quad (7)$$

For the proof, define the estimation error,

$$e = z - \hat{z} \quad (8)$$

Then, using equations (6) and (7) the estimation error dynamic is obtained as

$$\dot{e} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1 \right) \hat{\mu}(S) \right] e + \\ + \left(\frac{k}{Y_{S/X}} - 1 \right) [\mu(S) - \hat{\mu}(S)] k\,S - \left(\frac{k}{Y_{S/X}} - 1 \right) [\mu(S) - \hat{\mu}(S)] z \quad (9)$$

To analyze the stability of equation (9) we consider the following Lyapunov function candidate

$$V = \frac{1}{2} e^2 \quad (10)$$

The time derivative of equation (10) is

$$\dot{V} = e\,\dot{e} \quad (11)$$

Replacing (9) into (11) yields

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1 \right) \hat{\mu}(S) \right] e^2 + \\ + \left(\frac{k}{Y_{S/X}} - 1 \right) [\mu(S) - \hat{\mu}(S)] k\,S\,e - \left(\frac{k}{Y_{S/X}} - 1 \right) [\mu(S) - \hat{\mu}(S)] z\,e \quad (12)$$

Equation (12) is written alternatively as

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1 \right) \hat{\mu}(S) \right] e^2 - \\ - \left(\frac{k}{Y_{S/X}} - 1 \right) [\mu(S) - \hat{\mu}(S)] X\,e \quad (13)$$

Now, from lemma 1 and taking into account that $Y_{S/X} < k \le 1 + D / k_d$, $|\mu(S) - \hat{\mu}(S)| < a$, and X is bounded, equation (13) leads to,

$$\dot{V} \le -\left[D + k_d - kk_d + \left(\frac{k}{Y_{S/X}} - 1 \right) r \right] e^2 + \left(\frac{k}{Y_{S/X}} - 1 \right) aX_{\max} |e| \\ = -\lambda\,e^2 + w|\,e\,|$$

where,

$\lambda = D + k_d - kk_d + \left(\frac{k}{Y_{S/X}} - 1 \right) r$ and $w = \left(\frac{k}{Y_{S/X}} - 1 \right) aX_{\max}$

The right-hand side of the foregoing inequality is not negative since near the origin, the positive linear term $w|\,e\,|$ dominates the negative quadratic term $-\lambda\,e^2$. However, $\dot{V}$ is negative outside the set $\{|\,e\,| \le w/\lambda\}$. Let c, ε be some upper bounds for $V(e)$. With $c > w^2/2\lambda^2$, solutions starting in the set $\{V(e) \le c\}$ will remain therein for all time because $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions of equation (9) are uniformly bounded (Khalil, 2002). Moreover, if $(w^2/2\lambda^2) < \varepsilon < c$, then $\dot{V}$ will be negative in the set $\{\varepsilon \le V \le c\}$, which shows that, in this set V will decrease monotonically until the solutions enters the set $\{V \le \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \le \varepsilon\}$ since $\dot{V}$ is negative on the boundary $V = \varepsilon$. According to (Khalil, 2002), the solution is uniformly ultimately bounded with the ultimate bound $|\,e\,| \le \sqrt{2\varepsilon}$. For instance, defining c and ε as follows

$$c = \left(\frac{k}{Y_{S/X}} \frac{aX_{\max}}{\lambda} \right)^2, \quad \varepsilon = \left(k \frac{aX_{\max}}{\lambda} \right)^2$$

the ultimate bound is, $|\,e\,| \le \sqrt{2}\,k \frac{aX_{\max}}{\lambda}$

Corollary 1: if the growth rate is perfectly known, i. e., $\mu(S) = \hat{\mu}(S)$, and we choose the estimator's gain such that

$Y_{S/X} < k \le 1 + D/k_d$. Then, the system (14) is an asymptotic estimator of (6).

$$\dot{\hat{z}} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]\hat{z} + $$
$$+ (1-k)k\,k_d S + \left(\frac{k}{Y_{S/X}} - 1\right)k\,\mu(S)S + k\,D\,S_{in} \quad (14)$$

Indeed, the dynamics of the error in this case is

$$\dot{e} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]e$$

and the corresponding time derivative of Lyapunov function candidate (10) is

$$\dot{V} = -\left[D + k_d - k\,k_d + \left(\frac{k}{Y_{S/X}} - 1\right)\mu(S)\right]e^2 < 0$$

Moreover, X can be reconstructed considering

$$\hat{X} = \hat{z} - k\,S \quad (15)$$

B. Numerical simulations

For all simulations in this paper we take $S_{in} = 50$, $D = 0.1$, $Y_{S/X} = 0.9$, $k_d = 0.01$ and the initial conditions $S(0) = 60$, $X(0) = 40$, $\hat{X}(0) = 30$, $\hat{z}(0) = 90$, with appropriate units. The estimator's gain is $k = 1$. The growth rates are chosen as

$$\mu(S) = \frac{S}{140 + S + S^2/81.25} \text{ and } \hat{\mu}(S) = \frac{0.8\,S}{140 + S + S^2/81.25}$$

when the model is well known for the asymptotic estimator and when the model is partially known for the bounded error estimator, respectively. The simulations results were carried out with the help of Matlab 7.1 Software with Simulink 6.3 as the toolbox.

The performance index of the corresponding estimation process is calculated as (Martínez-Guerra, et. al, 2000)

$$J = \frac{1}{t + 0.001}\int_0^t \|e(\tau)\|^2\,d\tau \quad (16)$$

where $e(t)$ is the corresponding state estimation error (the difference between the actual observed signal and its estimate).

First, in figure 1 we show the simulation results for the bounded error estimator given by proposition 1, and the corresponding results for the asymptotic estimator given by corollary 1 (without any noise in the system output). Furthermore, in figure 2 is shown the effect of noise in the estimation process. A white noise is added in the measurement ($\sigma = 0.1$, $\pm 10\%$ around the current value of the measured output). We can observe that the bounded error estimator is robust against noisy measurement. Finally, in figure 3 is illustrated the performance index given by (16) for the corresponding estimation process. It should be noted that the quadratic estimation error (performance index) is bounded on average and has a tendency to decrease.

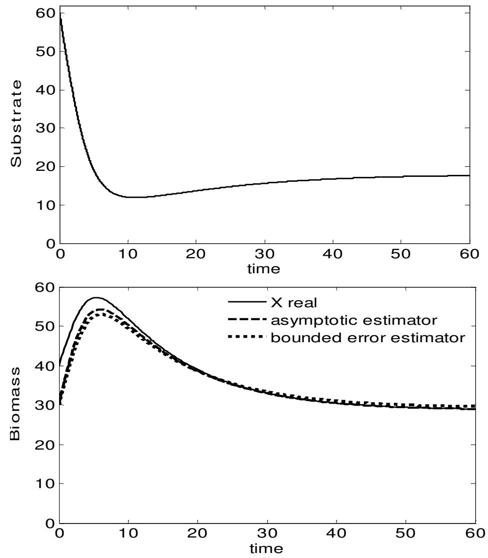


Figure 1. State Variables.

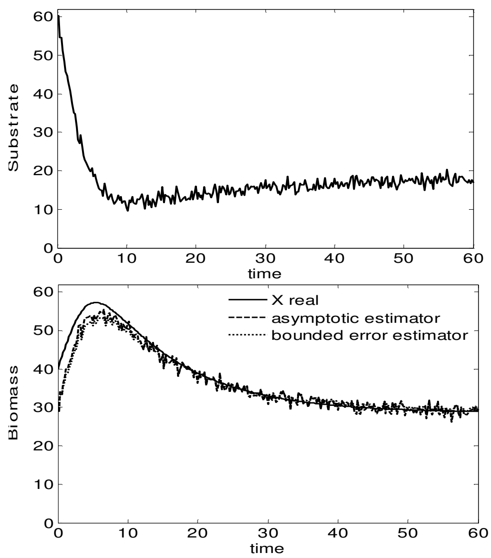
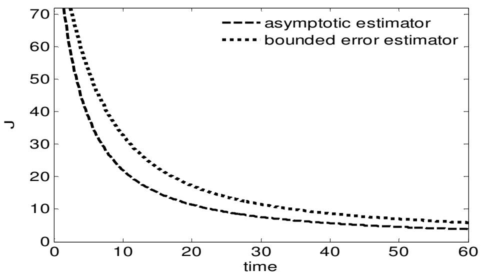
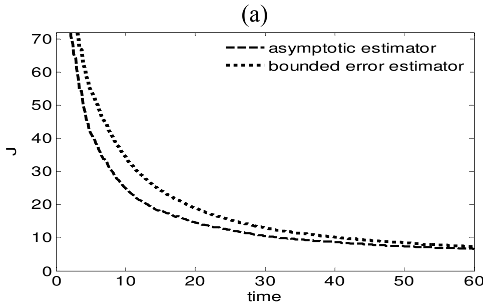


Figure 2. State Variables (with noise in the system output).



(a)



(b)

Figure 3. Quadratic estimation error. (a) Without any noise, (b) with white noise; in the system output.

IV. A NOTE ON FULL-ORDER OBSERVERS: THE HIGH GAIN OBSERVER

A. Observer design

Consider that system (1) satisfies the AOC. In this case to estimate the state-space vector x, we can suggest a nonlinear high gain observer (Gauthier et. al, 1992), (Martínez-Guerra et. al, 2000) with the following structure,

$$\dot{\hat{x}} = f(\hat{x}, u) + K(y - C\hat{x}) \qquad (17)$$
$$\hat{x} \in R^n , \qquad \hat{x}_0 = \hat{x}(t_0)$$

where the observer's gain matrix is given by,

$$K = S_\theta^{-1} C^T , \qquad S_\theta = \left(\frac{1}{\theta^{i+j-1}} S_{ij} \right)_{i,j=1,\dots,n}$$

and the positive parameter θ determines the desired convergence velocity. Moreover, $S_\theta > 0$, $S_\theta = S_\theta^T$ should be a solution of the algebraic equation,

$$S_\theta\left(E + \frac{\theta}{2} I\right) + \left(E^T + \frac{\theta}{2} I\right) S_\theta = C^T C , \qquad E = \begin{pmatrix} 0 & I_{n-1,n-1} \\ 0 & 0 \end{pmatrix}$$

As shown by (Gauthier et. al, 1992), (Martínez Guerra and de Leon-Morales, 1996), under certain technical a assumptions (Lipschitz conditions for nonlinear functions under consideration) this nonlinear observer has an arbitrary exponential decay for any initial conditions. We obtain the following high order observer for the system (2) applying the observation scheme (17),

$$\dot{\hat{S}} = D\left(S_{in} - \hat{S}\right) - \frac{\mu_{\max}\hat{S}}{\delta + \hat{S} + \hat{S}^2/\phi} \frac{\hat{X}}{Y_{S/X}} + k_d \hat{X} - 2\theta\left(\hat{S} - y\right)$$

$$\dot{\hat{X}} = -D\hat{X} + \frac{\mu_{\max}\hat{S}}{\delta + \hat{S} + \hat{S}^2/\phi} \hat{X} - k_d \hat{X} -$$
$$- \frac{1}{-\mu_{\max}\hat{S} + Y_{S/X}\left(\delta + \hat{S} + \frac{\hat{S}^2}{\phi}\right) k_d} \left\{ 2\theta \frac{\mu_{\max}\hat{X}\left(\delta - S^2/\phi\right)}{\left(\delta + \hat{S} + \hat{S}^2/\phi\right)} + \right.$$
$$\left. + \theta^2 Y_{S/X}\left(\delta + \hat{S} + \hat{S}^2/\phi\right) \right\} \left(\hat{S} - y\right)$$

B. Simulations

In the same way, we show two simulations: when the model is well known and when the model is partially known. The initial conditions for the observer are $\hat{S}(0) = 40$, $\hat{X}(0) = 30$, with appropriate units. The estimator's gain is $\theta = 2$. The simulations results of high gain observer are presented in figure 4 and figure 5. In figure 4, without any noise in the system output, when the model is perfectly known the rate of convergence is fast, on the other hand, when the model is partially known the observer does not reconstruct the state variables. In figure 5, we studied the effect of noise in the measurement (white noise with $\sigma = 0.1$, $\pm 5\%$ around the current value of the measured output), we can see that the high gain observer is very sensitive to the noise in the system output. Figure 6 shows the performance index. It should be noted that this observer only reconstruct the state variables when the model is well known.

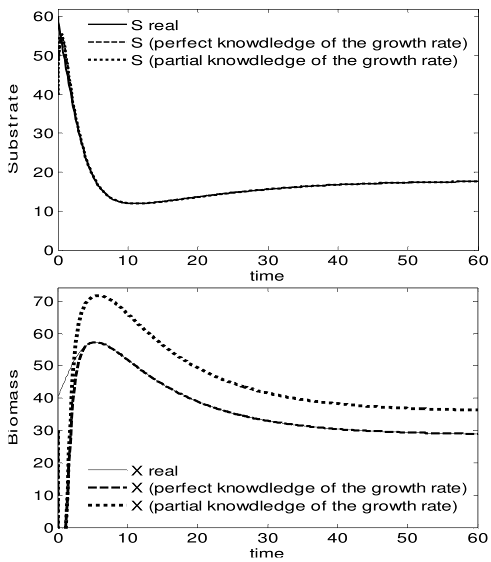


Figure 4. State Variables

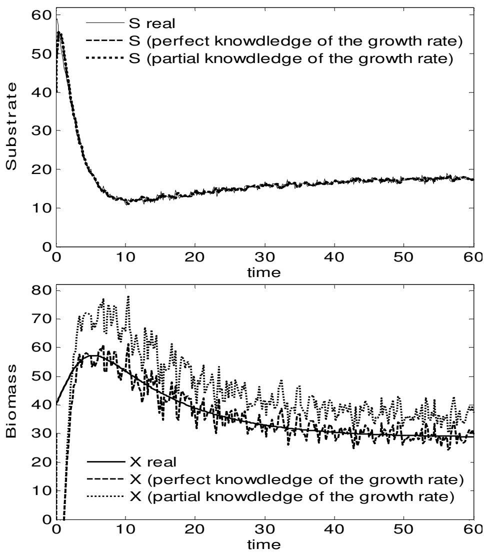
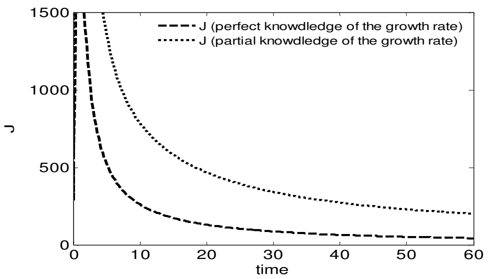
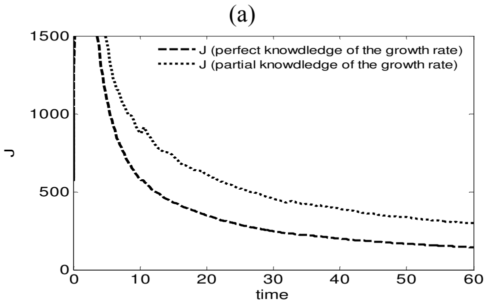


Figure 5. State Variables (with noise in the system output).



(a)



(b)

Figure 6. Quadratic estimation error. (a) Without any noise, (b) with white noise; in the system output.

V. Conclusion

In this paper we have presented a bounded error estimator for bioprocess with unstructured growth models. We have proven the stability of the corresponding estimation error in a Lyapunov sense. By means of a linear change of variable given in a natural manner and with some algebraic manipulations have been constructed the state estimator, which converges to the current states of the reference model given. We have demonstrated that the bounded error estimator under consideration provides good enough state-space estimates which were bounded on average. Moreover, we have constructed a high gain observer in which the convergence is fast only if the model is well known, but does not exists convergence if the model is partially known. Finally, we have presented some simulations to illustrate the effectiveness of the suggested approach, which shows some robustness properties against noisy measurements.

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Sincronización e Identificación Paramétrica del Oscilador de Rikitake

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***Resumen*— En este artículo se trata el problema de sincronización e identificación paramétrica de un sistema caótico incierto (oscilador de Rikitake). La estrategia consiste en proponer un sistema esclavo que siga asintóticamente al oscilador de Rikitake, el cual es denominado sistema maestro. Las ganancias del sistema esclavo se ajustan continuamente mediante una ley de adaptación conveniente. El análisis de convergencia se realiza aplicando el lema de Barbalat.**

Palabras clave: Identificación paramétrica, sistema caótico, sincronización.

I. INTRODUCCIÓN

La sincronización de sistemas caóticos ha sido investigada desde su introducción en el artículo por (Pecora y Carrol, 1990). Desde entonces, varios esquemas de sincronización han sido propuestos (Martínez-Guerra et al., 2006), (Morgül y Solak, 1996). El fenómeno de sincronización de sistemas caóticos ha recibido atención importante debido a sus aplicaciones en diversas áreas, como son: comunicaciones seguras, sistemas biológicos, reacciones químicas, entre otras (Fradkov, 2007), (Chen y Dong, 1995), (Martínez-Guerra y Wen Yu, 2008), (Mata-Machuca, 2009).

Existen algunos métodos para resolver el problema de sincronización desde la perspectiva de la Teoría de Control tal como son:

- el enfoque basado en observadores de estado (Morgül y Solak 1996), (Mata-Machuca et al., 2010), y
- el denominado método de sincronización adaptable (Femat et al., 2000).

Para el problema de sincronización, en éste trabajo se considera un sistema caótico, denominado sistema maestro ”SM”(o impulso), acoplado con el sistema esclavo ”SE”(o respuesta). El objetivo es sincronizar la respuesta completa del SE con la del SM empleando la señal de salida del SM.

El problema puede resolverse cuando los parámetros del sistema maestro son conocidos. Los métodos mencionados anteriormente y algunos otros, son válidos para sistemas caóticos sólo cuando los parámetros del sistema son conocidos.

Sin embargo, cuando los parámetros del sistema son desconocidos, la sincronización entre dos sistemas caóticos se complica. De hecho, el problema consiste en lo siguiente:

- Identificación de los parámetros desconocidos, y
- diseño de un controlador para lograr la sincronización.

Algunos trabajos relacionados se mencionan a continuación. En (Guan et al., 2001) aplicaron un observador para identificar el parámetro desconocido del sistema de Lorenz. El mismo método se emplea en (Lü y Zhang, 2001) para la identificación paramétrica del sistema caótico de Chen. El interés en la identificación de parámetros se debe a sus aplicaciones en comunicaciones, principalmente cuando la modulación de parámetros se usa para la transmisión de mensajes.

En éste artículo se presenta un método adaptable asintótico para la sincronización y la identificación del oscilador de Rikitake con varios parámetros desconocidos. Éste sistema modela la inversión de polaridad del campo electromagnético terrestre, y es bien sabido que tiene un comportamiento caótico para algún conjunto de condiciones iniciales y parámetros. Mediante éste método, se puede obtener la sincronización caótica y la identificación de parámetros desconocidos, simultáneamente.

En general, la aproximación sugerida consiste en el diseño de un sistema esclavo controlado, en el cual los controladores y los parámetros se ajustan de acuerdo con un algoritmo adaptable propuesto. Lo anterior se realiza hasta que los errores de sincronización entre las salidas de ambos sistemas (el oscilador de Rikitake y el esclavo) converjan asintóticamente a cero. El análisis de convergencia del esquema propuesto se realiza mediante el método de Lyapunov y el Lema de Barbalat.

El artículo está organizado de la siguiente manera. En la Sección II se presenta el planteamiento del problema, así como las definiciones de observabilidad e identificabilidad de

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un sistema dinámico desde el punto de vista del álgebra diferencial. En la Sección III se propone la solución para la sincronización e identificación de parámetros desconocidos del oscilador de Rikitake empleando el segundo método de Lyapunov. Posteriormente, se muestra el desempeño de la metodología propuesta mediante algunos resultados numéricos.

II. Planteamiento del Problema

II-A. *Modelo del Oscilador de Rikitake*

La teoría del dínamo es la explicación más plausible del origen del campo geomagnético. El oscilador de Rikitake es un modelo mecánico de tercer orden utilizado para estudiar la inversión de polaridad del campo magnético terrestre, el sistema fué presentado por el geofísico japonés (Rikitake, 1958), consiste de dos discos idénticos acoplados. La dinámica del oscilador de Rikitake se describe por el siguiente sistema de ecuaciones diferenciales no lineales:

$$\begin{aligned} \dot{x}_1 &= -\mu x_1 + z_1 y_1 \\ \dot{y}_1 &= -\mu y_1 + (z_1 - a)x_1 \\ \dot{z}_1 &= 1 - x_1 y_1 \end{aligned} \tag{1}$$

donde los parámetros a y μ son parámetros no negativos. Los estados y_1 y x_1 están directamente relacionados con las corrientes a través de cada disco, y z_1 se asocia con la velocidad angular de uno de los discos.

Para más detalles del significado físico de los estados x_1, y_1, z_1 y de los parámetros a, μ los autores recomiendan consultar (Rikitake, 1958).

II-B. *Definiciones*

En ésta sección se muestran algunas propiedades algebraicas que satisface el sistema (1). A continuación se presentan las definiciones de observabilidad algebraica y detectabilidad algebraica.

Definición 1: considere un sistema no lineal suave, descrito por el vector de estado $X = \{x_i\}_1^{i=n} \in \mathbb{R}^n$ y por el vector de salida $G = \{g_i\}_1^{i=m} \in \mathbb{R}^m$, de la forma

$$\dot{X} = f(X, P) \quad , \quad G = h(X), \tag{2}$$

donde $h(\cdot)$ es una función vectorial suave y $P \in \mathbb{R}^l$ es un vector de parámetros constantes, con $l \leq n$. Sea G^j la j-ésima derivada del vector G. Se dice que el vector de estado es ***algebraicamente observable*** si puede expresarse únicamente como

$$X = \Phi(G, G^{(1)}, .., G^{(j)})$$

para algún entero j, y para alguna función suave Φ.

Definición 2: Sean las mismas condiciones que en la Definición 1. Si el vector de parámetros P satisface la siguiente relación

$$\Omega_1(G, .., G^{(j)}) = \Omega_2(Y, .., Y^{(j)})P, \tag{3}$$

donde $\Omega_1(\cdot) \in \mathbb{R}^{n\times l}$ and $\Omega_2(\cdot) \in \mathbb{R}^{n\times n}$ son suaves, entonces P se dice ***linealmente algebraicamente identificable*** con respecto al vector de salida (Fliess y Sira-Ramirez, 2003).

De acuerdo con la Definición 1, es evidente que el sistema (1) es algebraicamente observable con respecto a las salidas $g_1 = x_1$ y $g_2 = y_1$, ya que el estado z_1 se puede obtener como sigue

$$z_1 = \frac{\dot{g}_2 - \mu g_2}{g_1} + a \tag{4}$$

por lo tanto, el oscilador de Rikitake (1) es algebraicamente observable con respecto a las salidas $g_1 = x_1$ y $g_2 = y_1$.

Ahora, se procede a verificar la identificabilidad del sistema aplicando la Definición 2. Sustituyendo la expresión (4) en la primera ecuación diferencial del sistema (1), tenemos

$$\dot{g}_1 g_1 - \dot{g}_2 g_2 = -\mu\,(g_1^2 + g_2^2) + a\; g_1 g_2 \tag{5}$$

Se define el vector de parámetros $p := (\mu,\, a)$. Por lo tanto, se concluye que el sistema (1) es algebraicamente identificable con respecto a las salidas disponibles. Esto es, la variable de estado z_1 y el vector de parámetros p pueden reconstruirse simultáneamente conociendo las salidas $g_1 = x_1$ y $g_2 = y_1$.

Con las definiciones anteriores, es posible resolver el problema de sincronización del oscilador de Rikitake incierto, considerando que siempre están disponibles los estados y_1 y x_1. Además, también es posible reconstruir los parámetros desconocidos μ y a. En éste momento, estamos listos para establecer el principal problema de control de éste trabajo.

II-C. *Sistema esclavo*

Considere el oscilador de Rikitake incierto (1), de aquí en adenlante se le llamará sistema maestro, con las salidas disponibles y_1 y x_1. Proponemos el siguiente sistema esclavo controlado

$$\begin{aligned} \dot{x}_2 &= -\hat{\mu} x_1 + z_2 y_1 + u_1 \\ \dot{y}_2 &= -\hat{\mu} y_2 + (z_2 - \hat{a})x_1 + u_2 \\ \dot{z}_2 &= 1 - x_1 y_1 + u_3 \end{aligned} \tag{6}$$

En lo que sigue de éste artículo, se denotan los vectores de estado relacionados con los sistemas maestro y esclavo como w_1 y w_2, respectivamente. Esto es, $w_i^T = (x_i, y_i, z_i)$, para $i = \{1, 2\}$.

Entonces, el objetivo de control consiste en encontrar $u = (u_1, u_2, u_3)$ y $\hat{p} = (\hat{\mu}, \hat{a})$ de manera que el sistema esclavo (6) siga al sistema desconocido (1). En otras palabras, se necesita encontrar u y $\hat{p}$ del sistema (6) tales que $(w_2, \hat{p}) \;\rightarrow\; (w_1, p)$, cuando $t \rightarrow \infty$.

Finalizamos ésta sección introduciendo la notación siguiente

$$e_x = x_1 - x_2; \quad e_y = y_1 - y_2; \quad e_z = z_1 - z_2;$$

$$\widetilde{\mu} = \mu - \widehat{\mu}; \qquad \widetilde{a} = a - \widehat{a}$$

de acuerdo con lo anterior, se definen los vectores

$$e^T = (e_x, e_y, e_z) \quad ; \quad \widetilde{p}^T = (\widetilde{\mu}, \widetilde{a})$$

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III. Diseño del Controlador y Ley de Adaptación

En ésta sección se presenta la solución al problema de sincronización e identificación de parámetros desconocidos del oscilador de Rikitake mediante el segundo método de Lyapunov. Para ésto, primero se obtiene la dinámica de los errores de sincronización entre los sistemas maestro y esclavo. Posteriormente, basándonos en una función candidata de Lyapunov definida positiva, se proponen el controlador y la ley de adaptación que se necesitan para asegurar la sincronización de ambos sistemas.

Antes de resolver el problema de control se introducen las siguientes suposiciones relacionadas con las salidas del sistema maestro

- ***A*1)** Los estados $y = y_1$ y $x = x_1$ están disponibles.
- ***A*2)** Todos los estados del sistema maestro están acotados. Sin embargo, los estados estacionarios de y y x permanecen oscilando alrededor de cero.

Nota 1. Consideramos que **A2** es realista ya que en la mayoría de los casos todos los estados del oscilador de Rikitake están acotados, para casi todo conjunto de condiciones iniciales y casi cualesquiera conjunto de parámetros positivos μ y a. De hecho, la suposición **A2** depende del conjunto de condiciones iniciales y de los parámetros μ y a. Para clarificar ésta propiedad, presentamos un caso donde **A2** no se cumple. Si seleccionamos los parámetros $\{\mu = 0;\ a = 0\}$ y las condiciones $\{x_1(0) = 0; y_1(0); z_1(0) = \overline{z}\}$, se tiene que $x_1(t) = 0$, $y_1(t) = 0$, $z_1(t) = t + \overline{z}$. Evidentemente, **A2** no se satisface ya que los estados y y x permanecen en el origen y el estado z_1 no está acotado (McMillen, 1999). No se puede proponer ningún método o esquema de identificación si el sistema maestro tiene soluciones que tiendan hacia infinito o una constante.

Procedemos en éste momento al análisis de la dinámica del error de sincronización. De las ecuaciones (1) y (6), tenemos

$$\dot{e} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\widetilde{\mu}x + e_z y - u_1 \\ -\widetilde{\mu}y + (e_z - \widetilde{a})x - u_2 \\ -u_3 \end{bmatrix} \tag{7}$$

donde por simplicidad denotamos a $y = y_1$ y $x = x_1$. El sistema (7) puede considerarse como un problema de control donde el vector de entradas u y el vector de parámetros $\widetilde{p}$ deben ser propuestos tales que, el error de sincronización e converja asintóticamente a cero.

Considere la función candidata de Lyapunov siguiente

$$V = \frac{1}{2}e^T e + \frac{1}{2}\widetilde{p}^T\widetilde{p} \tag{8}$$

La derivada de (8) con respecto al tiempo, a lo largo de las trayectorias de (7) está dada por

$$\begin{aligned} \dot{V} = \;& \widetilde{a}\dot{\widetilde{a}} + \widetilde{\mu}\dot{\widetilde{\mu}} - \widetilde{\mu}e_x x + e_x e_z y - e_x u - \widetilde{\mu}e_y y \\ & + e_y e_z x - \widetilde{a}e_y x - e_y u_2 - e_z u_3 \end{aligned} \tag{9}$$

Para garantizar que (9) sea semidefinida negativa, proponemos $\dot{\widetilde{p}}$ como

$$\dot{\widetilde{p}} = \begin{bmatrix} \dot{\widetilde{\mu}} \\ \dot{\widetilde{a}} \end{bmatrix} = \begin{bmatrix} e_x x + e_y y \\ e_y x \end{bmatrix} \tag{10}$$

y la entrada de control como

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1 e_x + k_{11}e_x^k \\ k_2 e_y + k_{21}e_y^k \\ e_x y + e_y x \end{bmatrix} \tag{11}$$

donde k_1, k_2, k_{11} y k_{21} son constantes estrictamente positivas y k es cualesquiera entero positivo impar.

Sustituyendo (11) en (7), se tiene el sistema en lazo cerrado siguiente

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -\widetilde{\mu}x + e_z y - k_1 e_x - k_{11}e_x^k \\ -\widetilde{\mu}y + (e_z - \widetilde{a})x - k_2 e_y - k_{21}e_y^k \\ -e_x y - e_y x \end{bmatrix} \tag{12}$$

donde las dinámicas de los parámetros están dadas por

$$\begin{aligned} \dot{\widetilde{\mu}} &= e_x x + e_y y \\ \dot{\widetilde{a}} &= e_y x \end{aligned} \tag{13}$$

Sustituyendo (12) y (13) en (9), obtenemos

$$\dot{V} = -\left(k_1 e_x^2 + k_2 e_y^2 + k_{11}e_x^{k+1} + k_{22}e_y^{k+1}\right) \tag{14}$$

Ésto implica que $\dot{V}$ es semidefinida negativa y, por lo tanto, V converge. Tenemos entonces que el conjunto de señales $\{e_x, e_y, e_z, \widetilde{\mu}, \widetilde{a}\}$ están acotadas.

Ahora, mostraremos que e converge asintóticamente a cero, cuando $t \to \infty$, aplicando el lema de Barbalat (Åström y Wittenmark, 1995).

Integrando ambos miembros de (14), se tiene

$$\begin{aligned} &\int_0^t \left[k_1 e_x^2(s) + k_2 e_y^2(s)\right. \\ &\quad \left. + k_{11}e_x^{k+1}(s) + k_{22}e_y^{k+1}(s)\right] ds \quad \leq V(0) \end{aligned} \tag{15}$$

De las ecuaciones de (12) y **A2**, se tiene que $\dot{e}$ es acotado lo cual implica que e es uniformemente continuo. Aplicando el lema de Barbalat, se determina que $e \to 0$, cuando $t \to \infty$.

Derivando (12) con respecto al tiempo, no es difícil mostrar que $\ddot{e}$ es acotado. De ésta manera, $\dot{e}$ es uniformemente continuo y también e tiene un límite finito, cuando $t \to \infty$. Por el lema de Barbalat concluimos que $\dot{e} \to 0$, cuando $t \to \infty$.

Debido a que V converge cuando $t \to \infty$, entonces de (8) se tiene que los errores paramétricos $\widetilde{\mu}$ y $\widetilde{a}$ convergen cuando $t \to \infty$. Además, de (13) obtenemos que $\dot{\widetilde{\mu}}$ y $\dot{\widetilde{a}}$ convergen a cero cuando $t \to \infty$.

Cuando t es suficientemente grande, $\hat{\mu}$ y $\hat{a}$ son casi constantes. Las ecuaciones diferenciales de (12) implican que

$$\begin{aligned} 0 &= (\mu - \hat{\mu})x \\ 0 &= (a - \hat{a})y \end{aligned} \tag{16}$$

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De **A2**, tenemos que los estados estacionarios y y x se mantienen oscilando alrededor de cero. Por lo tanto, necesariamente $\mu = \hat{\mu}$ y $a = \hat{a}$. Esto es, $\widetilde{p}^T \to 0$, cuando $t \to \infty$.

La discusión de ésta sección se resume en la proposición siguiente.

Proposición 1: Considere las suposiciones **A1** y **A2**. El problema de sincronización y estimación de parámetros entre los sistemas (1) y (6) puede resolverse para cualesquiera constantes estrictamente positivas $\{k_1, k_2, k_{11}, k_{2,1}\}$ y para todo entero positivo impar k.

■

IV. Resultados Numéricos

Se han realizado algunas simulaciones numéricas para mostrar el desempeño de la estrategia de control asintótico propuesta para la sincronización y reconstrucción de parámetros desconocidos del oscilador de Rikitake. El programa de cómputo utiliza el algoritmo de integración Runge–Kutta, el paso de integración es 0.001.

En la primera simulación se ilustra la propiedad cuantitativa descrita en la suposición **A2**. Los parámetros del sistema maestro son $p = (\mu = 2, a = 5)$, mientras que las condiciones iniciales fueron seleccionadas como $w_1(0) = (x_1(0) = 1\ , y_1(0) = -1\ , z_1(0) = 0)$. La Figura 1 muestra el comportamiento del estado completo del oscilador de Rikitake. Podemos notar de ésta figura que los estados del sistema maestro están acotados y, que y_1 y x_1 permanecen oscilando alrededor de cero. Por lo tanto, se verifica que la suposición **A2** se cumple plenamente.

Para mostrar el desempeño de la metodología propuesta realizamos otra simulación, utilizando las mismas condiciones para el sistema maestro. Las ganancias del sistema esclavo se fijan como $k_1 = k_2 = 0.8$ y $k_{11} = k_{21} = 0.6$, el sistema esclavo parte de condiciones iniciales cero, es decir, $w_2(0) = 0$ y $\hat{p}(0) = 0$.

En la Figura 2 podemos ver que los errores de sincronización convergen asintóticamente a cero, esto es, el sistema esclavo sigue casi perfectamente al sistema maestro. Los parámetros estimados se ilustran en la Figura 3. Como era de esperarse, se obtiene un mejor desempeño cuando el tiempo incrementa. En éste caso, los parámetros son reconstruidos razonablemente bien después de 50 segundos.

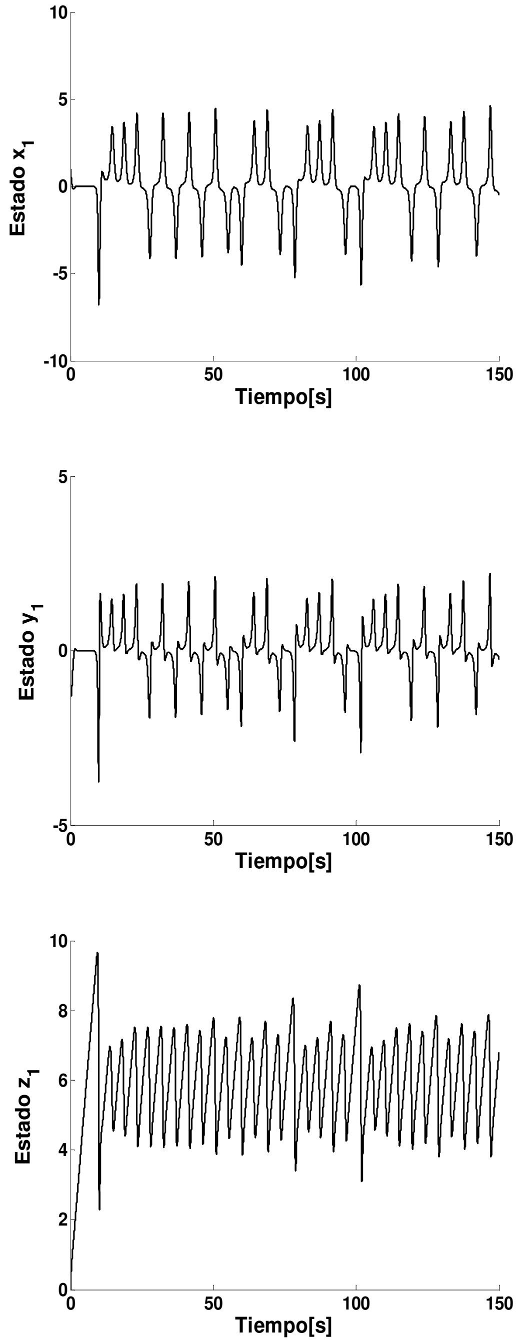

Figura 1. Comportamiento cuantitativo del oscilador de Rikitake.

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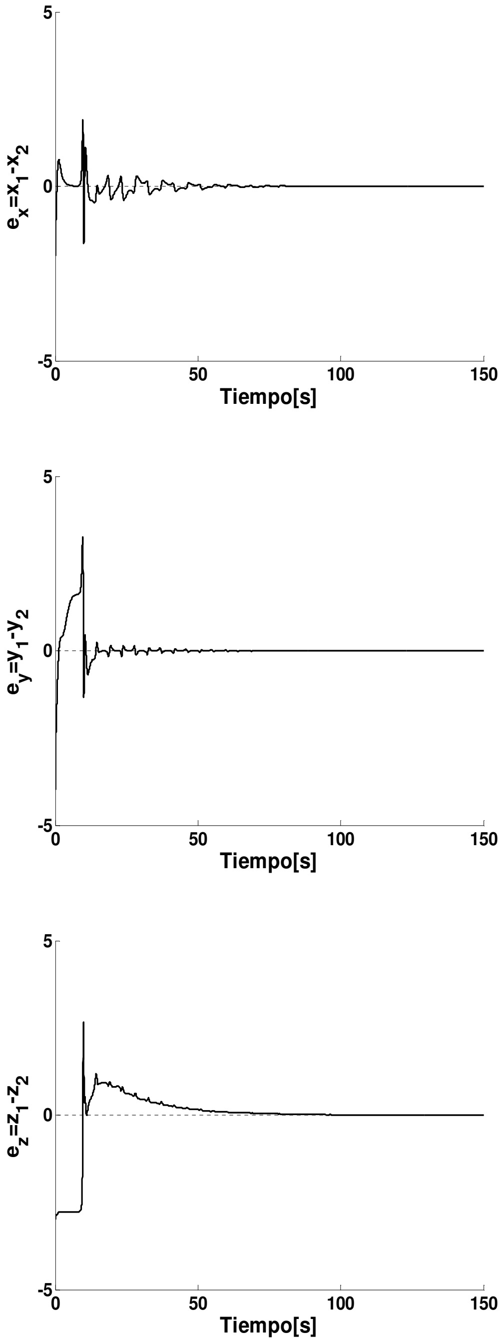

Figura 2. Errores de sincronización.

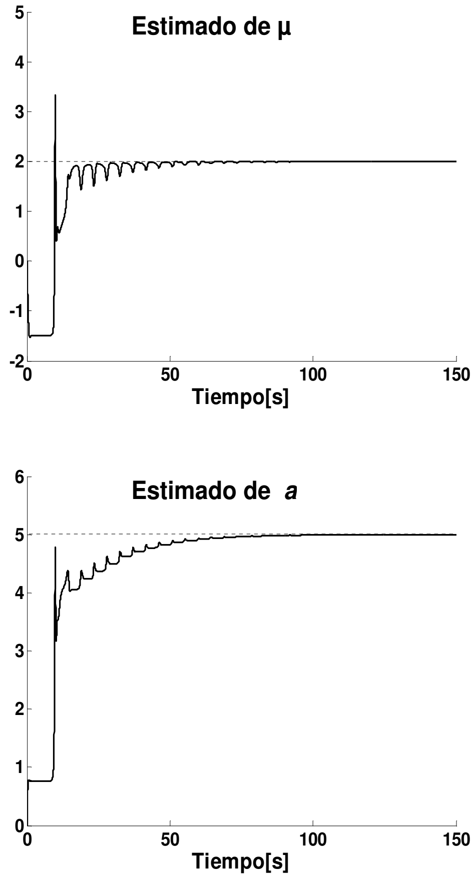

Figura 3. Parámetros estimados.

V. Conclusiones

En éste artículo se presentó una estrategia de control para la sincronización e identificación de parámetros del oscilador de Rikitake, la cual está basada en el segundo método de Lyapunov.

Mediante herramientas del álgebra diferencial se comprobó la observabilidad e identificabilidad del sistema, con respecto a las salidas disponibles y y x. Después, se propuso un sistema esclavo controlado, donde los controladores y las leyes de adaptación se diseñaron de tal forma que los errores de sincronización y los errores paramétricos, entre el sistema esclavo y el sistema maestro, tuvieran convergencia asintótica a cero. Para la prueba de convergencia se utilizaron el método tradicional de Lyapunov y el lema de Barbalat. Es importante mencionar que el algoritmo de identificación de parámetros presentado en éste trabajo no requiere que el oscilador de Rikitake exiba siempre un comportamiento caótico. Finalmente, se presentaron algunas simulaciones para evaluar el desempeño del método propuesto.

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Sincronización por medio de un observador de orden reducido: aplicación en tiempo real al oscilador de Colpitts

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***Resumen*— En este artículo se pretende mostrar el fenómeno de la sincronización en tiempo real por medio de un observador de orden reducido aplicado al oscilador de Colpitts. Dicho oscilador presenta comportamiento caótico como se mostró en (Kennedy, 1994). Esta propiedad para sincronizarse en tiempo real a un sistema caótico, da la pauta para aplicaciones con mucho interés para los sistemas de comunicaciones seguras.**

Palabras clave: Observador de orden reducido, oscilador de Colpitts, sincronización.

I. INTRODUCCIÓN

I-A. Caos en el oscilador de Colpitts

En 1995 oscilaciones caóticas en el circuito de Colpitts con un transistor genérico 2N2222A, fueron reportados en (Kennedy, 1994). En la tesis doctoral "Monolithic microwave oscillators and amplifiers", Nguyen se refiere al fenónemo de multi-oscilaciones en los osciladores bipolares de RF, que son oscilaciones parasíticas o no deseadas que coexisten con una oscilación principal (Nguyen, 1991). Debido a estas oscilaciones parasíticas la señal resultante en estado estacionario se distorciona severamente, por lo tanto tienen pocas aplicaciones. En (Kennedy, 1994) se muestra que el oscilador de Colpitts no tiene estas oscilaciones parasíticas y además presenta comportamiento caótico, lo cual se puede aplicar como transmisores en señales caóticas transportadoras para la comunicación. La figura 1, muestra algunos valores para los cuales el oscilador de Colpitts presenta comportamiento caótico.

I-B. Sincronización

La sincronización de sistemas caóticos ha sido investigada desde su introducción en el artículo por (Pecora y Carrol, 1990). Desde entonces, varios esquemas de sincronización han sido propuestos (Martínez-Guerra et al., 2006), (Martínez-Guerra y Rincón Pasaye, 2009), (Morgul y Solak, 1996). El fenómeno de sincronización de sistemas caóticos ha recibido atención importante debido a sus aplicaciones en diversas áreas, como son: comunicaciones seguras, sistemas biológicos, reacciones químicas, etc., (Martinez-Guerra y Wen Yu, 2008), (Mata-Machuca, 2009).

Existen algunos métodos para resolver el problema de sincronización desde la perspectiva de la Teoría de Control, el que se presenta en este trabajo es utilizando observadores de estado.

Las contribuciones principales consisten en dos aspectos: (i) el diseño de un obervador de orden reducido para la sincronización de sistemas caóticos, el cual tiene convergencia asintótica, (ii) implementación en tiempo real empleando el oscilador de Colpitts.

El método se basa en la configuración maestro - esclavo (Pecora y Carrol 1990). Su principal característica es que el acoplamiento es unidireccional, esto es, la señal se transmite del sistema maestro (transmisor) al sistema esclavo (receptor). Debido a lo anterior, algunos autores utilizan la terminología *transmisor/receptor*. La sincronización de sistemas caóticos consiste en un régimen en el cual dos sistemas caóticos acoplados (maestro y esclavo), después de un tiempo de transición, exhiben oscilaciones caóticas idénticas (Nijmeijer y Mareels, 1997). El principal objetivo en este trabajo es mostrar que la teoría de observadores puede aplicarse en la sincronización de sistemas caóticos. La sincronización puede resolverse desde el punto de vista de la teoría de control, diseñando un sistema esclavo mediante un observador de estado, el cual es capaz de estimar las variables del sistema maestro.

La estructura de este artículo es la siguiente. En la sección II se presenta el modelado del oscillador de Colpitts (Maggio et al., 1999). En la seccion III se plantea el problema de la sincronización y su solución por medio del observador de orden reducido. En la sección IV se presenta el procedimiento para la sincronización en tiempo real y los resultados, mostrando la efectividad del observador de orden reducido en tiempo real. Por último, en la seccion V son dadas algunas conclusiones.

II. MODELO DEL OSCILADOR DE COLPITTS

En este trabajo consideraremos la configuración clásica del oscilador de Colpitts, el cual esta formado por un transistor de tipo BJT, como elemento de ganancia y una red de resonancia conformada por un inductor y un par de capacitores (Figura 1).

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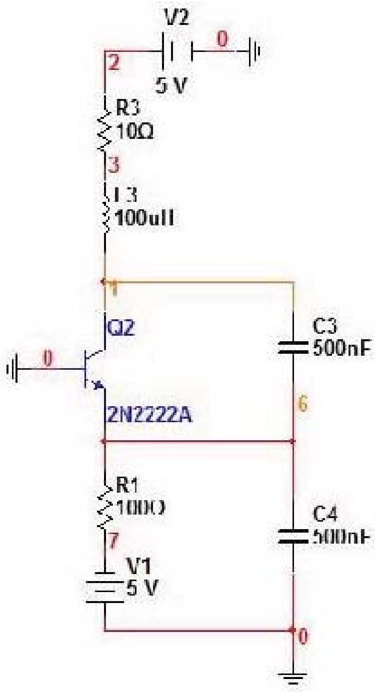

Figura 1. Oscilador de Colpitts

De acuerdo a la teoría cualitativa de dinámicas no lineales, se seleccionará un modelo mínimo para este circuito. La idea es considerar un modelo muy simple que contenga las características más importes del verdadero oscilador de Colpitts. Para el modelado de dicho circuito tomaremos las siguientes consideraciones:

1. La corriente en el emisor se generará por medio de I_0.
2. Elementos pasivos y activos ideales.
3. El transistor se modela (Figura 2) como una resistencia no lineal R_E controlada por medio de voltaje y como una entrada lineal controlada por corriente, además:
   a) Modelamos $V-I$, característico de R_e con una función exponencial:

$$\begin{aligned} I_E &= I_S\left[e^{\frac{V_{BE}}{VT}}-1\right] \\ &\approx I_S\left[e^{\frac{V_{BE}}{VT}}\right], \text{si } V_{BE} \gg VT \qquad (1) \end{aligned}$$

   donde I_S es la corriente de saturación inversa y $VT \simeq 26mV$ a temperatura ambiente.

   b) Asumimos que $\alpha F = 1$ donde αF es la ganancia de la corriente en corto circuito la conficuración base comun. Esto corresponde a omitir la corriente de la base.

   c) Las dinámicas parasíticas del transistor se omiten.

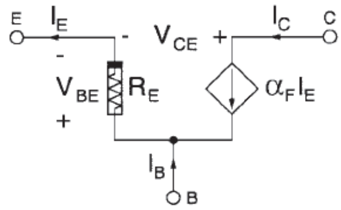

Figura 2. Modelo del transistor BJT 2N2222A en configuración base común

Ahora bien, las ecuaciones de estado del oscilador de Colpitts son las siguientes:

$$\begin{aligned} C_1\dot{V}_{C_1} &= -f(V_{C_2})+I_L \\ C_2\dot{V}_{C_2} &= I_L-I_0 \\ L\dot{I}_L &= -V_{C_1}-V_{C_2}-RI_L+V_{CC} \qquad (2) \end{aligned}$$

donde $f(\cdot)$ es el punto de trabajo característico de la resistencia no lineal. Esto puede ser expresado de la forma $I_E = f(V_{C_2}) = f(-V_{BE})$, entonces en particular de (1) tenemos:

$$f(V_{C_2}) = I_S\left[e^{-\frac{V_{C_2}}{VT}}\right]$$

II-A. Normalización de los parámetros

Ahora introducimos variables de estado sin dimensiones (x_1, x_2, x_3) y elegimos el punto de operación de (2) como el origen del nuevo sistema. En particular normalizamos voltajes, corrientes y tiempo con respecto a $V_{ref} = VT$, $I_{ref} = I_0$, $t_{ref} = \frac{1}{\omega_0}$, respectivamente, donde $\omega_0 = \frac{1}{\sqrt{L\left(\frac{C_1C_2}{C_1+C_2}\right)}}$ es la frecuencia de resonancia del tanque $L-C$ sin carga. Entonces las ecuaciones de estado del oscilador de Colpitts se pueden escribir de la forma:

$$\begin{aligned} \dot{x}_1 &= -ae^{-x_2}+a+ax_3 \\ \dot{x}_2 &= bx_3 \\ \dot{x}_3 &= -cx_1-cx_2-dx_3 \end{aligned}$$

donde: $a = \frac{g}{Q(1-k)}$, $b = \frac{g}{Qk}$, $c = \frac{Qk(1-k)}{g}$, $d = \frac{1}{Q}$, $k = \frac{C_2}{C_1+C_2}$, $Q = \frac{\omega_0 L}{R}$

- Q es el factor de calidad del tanque del circuito sin carga.
- g es la ganancia en lazo abierto del oscilador.

III. Sincronización por medio del observador de orden reducido

Consideremos el siguiente sistema no-lineal descrito por:

$$\begin{aligned} \dot{x}(t) &= f(x,u) \\ y(t) &= h(x) \qquad (3) \end{aligned}$$

donde $f \in \mathbb{R}^n$ es diferenciable y satisface $f(0) = 0$, $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ es el vector de estados, $y \in \mathbb{R}$ es una salida sin singularidades y $u \in \mathbb{R}^l$ es la entrada de control, $(l \leq n)$.

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Definición 1: un estado x es algebraicamente observable si es algebraico sobre $\mathbb{R}\langle u, y\rangle$, esto es que satisface un polinomio diferencial en términos de u, y y algunas de sus derivadas:

$$P(x, u, \dot{u}, ..., y, \dot{y}, ...) = 0 \tag{4}$$

con coeficientes en $\mathbb{R}\langle u, y\rangle$.

Ahora bien, consideremos el sistema descrito por (3). El estado desconocido del sistema puede ser incluido en una nueva variable $\eta(t)$ y un sistema aumentado puede ser considerado en lugar del original:

$$\begin{aligned} \dot{x}(t) &= f(x, u, \eta) \\ \dot{\eta}(t) &= \Delta(x, u, \eta) \\ y(t) &= h(x) \end{aligned}$$

donde $\Delta(x, u, \eta)$ es una función en términos de los estados y las entradas conocidas. El asunto es ahora construir una variable $\eta(t)$ y una vez que se conoce, determinar los valores de los estados deseados. Para poder lograr esto debemos imponer algunas condiciones sobre $\Delta(x, u, \eta)$ y $\eta(t)$ y posteriormente proponer algún procedimiento para encontrar dicha función.

Condiciones:

1. $\eta(t)$ es algebraicamente observable (definición 1).
2. La variable auxiliar γ es una función real, continua y diferenciable.
3. $\Delta(x, u, \eta)$ es acotada, esto es: $\|\Delta(x, u, \eta)\| \leq M, \quad 0 < M < \infty$.

La siguiente ecuación representa la dinámica de los estados desconocidos:

$$\dot{\eta}(t) = \Delta(x, u, \eta) \tag{5}$$

Lema 1: El sistema:

$$\dot{\widehat{\eta}} = K(\eta - \widehat{\eta}) \tag{6}$$

es un observador de orden reducido asintótico para el sistema (5), donde $\widehat{\eta}$ es el estimado de η y K es una constante positiva que determina la tasa de convergencia del estimado con el real. Si la siguiente condición se satisface:

4. $\left|\exp\left(-\int Kdt\right)\right| = 0$ con t_0 suficientemente grande y $\text{lím sup}_{t\to t_0} \frac{M}{|K|} = 0$

Demostración: El error esta dado por: $e(t) = \eta(x) - \widehat{\eta}(x)$, lo cual implica una dinámica no lineal del error:

$$\begin{aligned} \dot{e}(t) &= \dot{\eta}(t) - \dot{\widehat{\eta}}(t) \Rightarrow \\ \dot{e}(t) + ke(t) &= \Delta(t) \end{aligned} \tag{7}$$

La solución de (7) esta dada por:

$$e(t) = \exp\left(-\int Kdt\right)\left[e_0 + \int_0^t \exp\left(\int Kdt\right)\Delta(\tau)d\tau\right] \tag{8}$$

Entonces, si las condiciones 1-4 se cumplen y aplicando la desigualdad del triangulo y Cauchy-Schwartz a la expresión (8) tenemos:

$$\begin{aligned} 0 \leq\ & |e(t)| \leq \exp\left(-\int Kdt\right)|e_0| \\ & + \exp\left(-\int Kdt\right)\left[\int_0^t \left|\exp\left(\int Kdt\right)\Delta(\tau)d\tau\right|\right] \end{aligned}$$

y si $t \to t_o$ con t_o suficientemente grande:

$$\begin{aligned} 0 &\leq \underset{t\to t_0}{\text{lím sup}}\ |e(t)| \\ &\leq |e_0| \underset{t\to t_0}{\text{lím sup}} \left[\exp\left(-\int Kdt\right)\right] \\ &+ \frac{\text{lím sup}_{t\to t_0}\left[\int_0^t \left|\exp\left(\int Kdt\right)\Delta(\tau)d\tau\right|\right]}{\left|\exp\left(\int Kdt\right)\right|} \end{aligned}$$

de las condiciones 1 a 4:

$$\begin{aligned} 0 &\leq \underset{t\to t_0}{\text{lím sup}}\ |e(t)| \\ &\leq \frac{\text{lím sup}_{t\to t_0}\left[M * \int_0^t \left|\exp\left(\int Kdt\right)d\tau\right|\right]}{\left|\exp\left(\int Kdt\right)\right|} \end{aligned}$$

Aplicando del caso $\frac{\infty}{\infty}$ de la regla uniforme de l'H$\widehat{o}$pital´s:

$$\begin{aligned} 0 &\leq \underset{t\to t_0}{\text{lím sup}}\ |e(t)| \\ &\leq \underset{t\to t_0}{\text{lím sup}}\left[\frac{M * \left|\exp\left(\int Kdt\right)\right|}{\left|\exp\left(\int Kdt\right)\right||K|}\right] \\ &= \underset{t\to t_0}{\text{lím sup}}\ \frac{M}{|K|} \end{aligned}$$

finalmente de la condición 4 :

$$0 \leq \underset{t\to t_0}{\text{lím sup}}\ |e(t)| \leq \underset{t\to t_0}{\text{lím sup}}\ \frac{M}{|K|} = 0$$

entonces:

$$\lim_{t\to t_0} |e(t)| = 0$$

por lo tanto el observador de orden reducido descrito por (6) es asintótico. ■

III-A. Construcción de los observadores (sistema esclavo)

El sistema esclavo en este caso estará formado por dos estructuras de observación para poder alcanzar la sincronización con el sistema maestro. Dichas estructuras se obtienen de la siguiente manera. Primero, verificar que el sistema maestro sea algebraicamente observable y posteriormente, mediante algun procedimiento, construir los esquemas de observación. En todo el artículo se considera que la salida del sistema es $y = x_2$.

Para x_3 tenemos:

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$$x_3 = \frac{\dot{x}_2}{b} = \frac{\dot{y}}{b} \Rightarrow x_3 = \phi(\dot{y}) \tag{9}$$

Para x_1 tenemos:

$$x_1 = -\frac{1}{c}\left[\frac{1}{b}\ddot{y} + \frac{d}{b}\dot{y} + cy\right] \Rightarrow x_1 = \phi(y, \dot{y}, \ddot{y}) \tag{10}$$

Entonces, ambos estados del sistema maestro son algebraicamente observables, por lo tanto decimos que el sistema maestro es algebraicamente observable y procedemos a construir los observadores utilizando el esquema (6) haciendo uso de (10) y (9).

Para x_3 se tiene:

$$\begin{aligned} \dot{\widehat{x}}_3 &= K_3(x_3 - \widehat{x}_3) \\ &= \frac{K_3}{b}\dot{y} - K_3\widehat{x}_3 \end{aligned}$$

$$\dot{\widehat{x}}_3 - \frac{K_3}{b}\dot{y} = -K_3\widehat{x}_3$$

si elegimos: $\gamma_3 = -\frac{K_3}{b}y + x_3$ entonces:

$$\begin{aligned} \dot{\gamma}_3 &= -\frac{K_3^2}{b}y - K_3\gamma_3 \\ \widehat{x}_3 &= \frac{K_3}{b}y + \gamma_3 \end{aligned} \tag{11}$$

Para x_1 se procede similarmente:

$$\begin{aligned} \dot{\widehat{x}}_1 &= K_1(x_1 - \widehat{x}_1) \\ &= -\frac{K_1}{cb}\ddot{y} - \frac{K_1 d}{cb}\dot{y} - K_1 y - K_1\widehat{x}_1 \end{aligned}$$

Ahora definimos $x_4 := \dot{y}$ y diseñamos un observador para esta nueva variable haciendo uso del esquema(6):

$$\begin{aligned} \dot{\gamma}_4 &= -K_4\left[\gamma_4 + K_4 y\right] \\ \widehat{x}_4 &= \gamma_4 + K_4 y \end{aligned}$$

Entonces:

$$\dot{\widehat{x}}_1 = -\frac{K_1}{cb}\dot{\widehat{x}}_4 - \frac{K_1 d}{cb}\widehat{x}_4 - K_1 y - K_1\widehat{x}_1$$

$$\dot{\widehat{x}}_1 + \frac{K_1}{cb}\dot{\widehat{x}}_4 = -\frac{K_1 d}{cb}\widehat{x}_4 - K_1 y - K_1\widehat{x}_1$$

si elegimos : $\widehat{x}_1 = -\frac{K_1}{cb}\widehat{x}_4 + \gamma_5$, entonces:

$$\begin{aligned} \dot{\gamma}_5 &= -\frac{K_1 d}{cb}\widehat{x}_4 - K_1 y + \frac{K_1^2}{cb}\widehat{x}_4 - K_1\gamma_5 \\ &= [K_1 - d]\frac{K_1}{cb}\widehat{x}_4 - K_1 y - K_1\gamma_5 \\ &= [K_1 - d]\frac{K_1}{cb}\left[\gamma_4 + K_4 y\right] - K_1 y - K_1\gamma_5 \end{aligned}$$

y el esquema del observador para x_1 es:

$$\begin{aligned} \dot{\gamma}_4 &= -K_4\left[\gamma_4 + K_4 y\right] \\ \dot{\gamma}_5 &= [K_1 - d]\frac{K_1}{cb}\left[\gamma_4 + K_4 y\right] - K_1 y - K_1\gamma_5 \\ \widehat{x}_1 &= -\frac{K_1}{cb}\left[\gamma_4 + K_4 y\right] + \gamma_5 \end{aligned} \tag{12}$$

Ahora conocemos el sistema esclavo, el cual estará formado por (12) y (11).

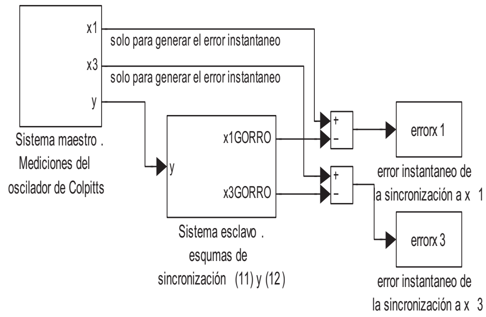

Figura 3. Esquema general de la sincronización en tiempo real

IV. Sincronización en tiempo real

Para alcanzar la sincronización en tiempo real se utilizó la plataforma WINCON con la tarjeta de adquisición de datos ISA Bus Servo I/O Card, junto con un reloj de tiempo real para alcanzar un tiempo de muestreo apropiado para el oscilador de Colpitts. En WINCON se implementaron los esquemas (11) y (12) obtenidos en la sección anterior y se procedió a obtener los datos mediante la tarjeta y notar la sincronización (figura 3). En la figura 4 se muestra el circuito de Colpitts (sistema maestro).

El índice de desempeño del correspondiente proceso de sincronización se calcula como

$$J(t) = \frac{1}{t + 0{,}001}\int_0^t \| e(t) \|_{Q_0}^2 d\tau \ , \quad Q_0 = I$$

donde $e(t)$ denota el error de sincronización.

IV-A. *Resultados*

En las figuras 5 y 6 se muestran los resultados obtenidos para algunas condiciones iniciales en los esquemas (11) y (12).

En la figura 7 se ilustra el índice de desempeño, el cual tiene tendencia decreciente. Finalmente, en la figura 7 se presenta el diagrama de fase, donde claramente se observa el comportamiento caótico del oscilador de Colpitts.

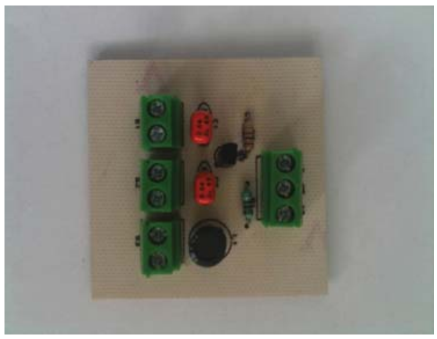

Figura 4. Sistema Maestro

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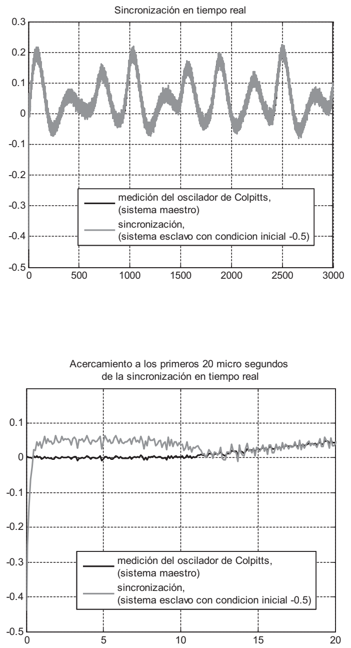

Figura 5. Sincronizaciòn a x1

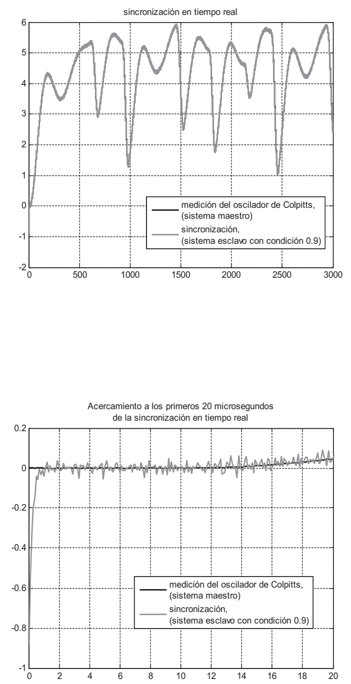

Figura 6. Sincronizaciòn a x3

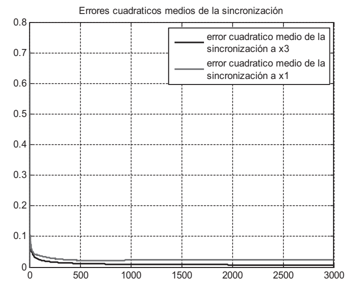

Figura 7. Errores cuadraticos medio de la sincronización

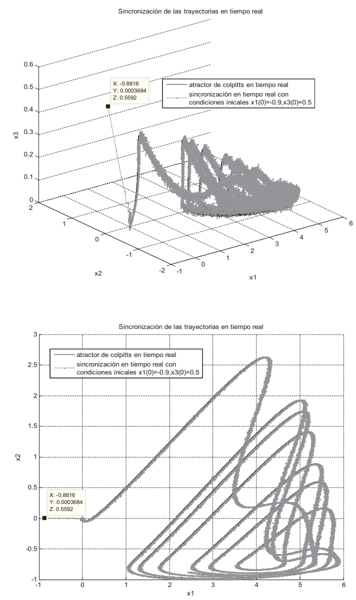

Figura 8. Sincronizaciòn del sistema

V. Conclusiones

Se mostró que los esquemas (11) y (12) en tiempo real son muy efectivos aún con la gran deficiencia en la medición de la salida disponible para realizar la sincronización. Esto podría darnos algunos indicios de que ademas dichos esquemas son robustos. También es importante notar que las mediciones en tiempo real del oscilador de Colpitts no son tan sencillas de obtener, debido a las altas frecuencias, en las cuales presenta el comportamiento caótico que se busca para las aplicaciones en las comunicaciones seguras. Las mediciones en este trabajo se realizaron haciendo ayuda de un reloj en tiempo real, el cual aumentaba la capacidad de muestreo de nuestra tarjeta, pero en aplicaciones prácticas es claro que la implementación de los esquemas de medición serían mas elaborados.

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Diagnóstico de fallas de una clase de sistemas no lineales mediante observadores algebraicos

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***Resumen*— Este artículo se concentra en el problema de diagnóstico de fallas de una clase de sistemas no lineales empleando la teoría de observadores de estado. Con el objetivo de reconstruir las fallas del sistema, se propone un observador polinomial. ”Polinomial”significa la inclusión de términos de corrección de alto orden en la estructura del observador. La metodología consiste en la estimación de las fallas mediante un estado aumentado del sistema, lo cual resulta en el incremento del orden del observador. Este esquema es capaz de reconstruir fallas y variables de estado simultáneamente. Además, como un estudio comparativo, se diseña un observador de orden reducido. Ambas técnicas se aplican al diagnóstico de fallas de un sistema de tres tanques.**

Palabras clave: diagnóstico de fallas, observadores, sistemas no lineales.

I. Introducción

Una falla es una desviación no permitida de al menos una propiedad característica o parámetro del sistema con respecto a su condición usual, nominal o aceptable.

En las últimas tres décadas un número considerable de artículos relacionados con el problema de diagnóstico de fallas han sido reportados (Alcorta y Frank, 1997), (Frank y Ding, 1977), (Willsky, 1976), (Massoumnia et al., 1989). Entre las publicaciones alrededor del diagnóstico de fallas se tienen diferentes enfoques, por ejemplo: en (Massoumnia, 1986), (De Persis e Isidori, 2001), (Join et al., 2005), emplean herramientas geométrico diferenciales; en (Diop y Martínez-Guerra, 2001), (Fraleigh, 1987), (Ritt, 1950) se basan en un método alternativo aplicando técnicas algebraico diferenciales.

Éste trabajo se concentra en el problema de diagnóstico de fallas de una clase de sistemas no lineales. El diagnóstico de fallas busca por medio de mediciones de variables conocidas (salidas y entradas) lograr reconstruir la dinámica de una entrada desconocida que efecta el desempeño de una dinámica. Las salidas son principalmente mediciones obtenidas por medio de sensores. Las salidas y el número de fallas determinan si el sistema es diagnosticable o no (Cruz y Martinez-Guerra, 2005), (Martínez-Guerra et al., 2007). El problema de diagnóstico de fallas es considerado como un problema de observación de las señales de falla. En éste sentido, la diagnosticabilidad de un sistema ésta dada por la denominada *condición de observabilidad algebraica* de la falla (Diop y Martínez-Guerra, 2001). La principal contribución de éste artículo consiste en la solución del problema de diagnóstico de fallas mediante un observador polinomial. Éste trabajo combina las ideas de (Rajamani, 1998) sobre el diseño de observadores para sistemas no lineales Lipschitz con el método introducido en (Cruz y Martinez-Guerra, 2005), agregando términos de corrección de alto orden, tales que se garantice la convergencia asintótica del observador. Además, se diseña un observador de orden reducido basado en modelo libre, el cual posee convergencia asintótica. Se presenta la aplicación de los esquemas propuestos al diagnóstico de fallas y reconstrucción de estados de un sistema de tres tanques.

II. Planteamiento del problema

Considere el siguiente sistema no lineal

$$\begin{aligned} \dot{x}(t) &= g(x,u,f) \\ y(t) &= h(x,u) \end{aligned} \tag{1}$$

donde $x \in \mathbb{R}^n$ es el vector de estado, $u \in \mathbb{R}^l$ es el vector de entradas conocidas, $f = (f_1, f_2, \ldots, f_\mu) \in \mathbb{R}^\mu$ es el vector de fallas (entradas desconocidas), $y \in \mathbb{R}^p$ es el vector de salida. Con g, h siendo funciones analíticas.

El vector de fallas f es desconocido, lo cual puede interpretarse como un estado con incertidumbres. La estimación de las fallas es obtenida mediante un estado aumentado. Considere el siguiente sistema extendido

$$\begin{aligned} \dot{x}(t) &= g(x,u,f) \\ \dot{f}_j(t) &= \Omega_j(x,u,f) \quad , \quad 1 \le j \le \mu \\ y(t) &= h(x,u) \end{aligned} \tag{2}$$

donde $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_\mu) \in \mathbb{R}^\mu$ es una función desconocida acotada, es decir

$$\|\Omega(x,u,f)\| \le N < \infty \tag{3}$$

El sistema (2) puede expresarse en la siguiente forma,

$$\begin{aligned} \dot{x}(t) &= A\,x + \Psi\,(x,u,f) \\ \dot{f}_j(t) &= \Omega_j(x,u,f) \quad , \quad 1 \le j \le \mu \\ y(t) &= C\,x \end{aligned} \tag{4}$$

donde $\Psi(x,u,f)$ es una función no lineal que satisface la condición de Lipschitz, es decir,

$$\|\Psi(x,u,f) - \Psi(\hat{x},u,f)\| \le L\|x - \hat{x}\| \tag{5}$$

III. Observabilidad y Diagnosticabilidad: Enfoque Algebraico Diferencial

Antes de proponer un observador para el sistema extendido (4), en ésta sección se presentan las nociones de observabilidad y diagnosticabilidad.

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III-A. *Definiciones*

A mediados del siglo pasado Ritt (1950) introdujo algunos conceptos del álgebra diferencial para analizar los sistemas de ecuaciones diferenciales mediante la teoría de sistemas de ecuaciones algebraicas, tomando en cuenta que las ecuaciones diferenciales son algebraicas en las variables dependientes.

La noción de observabilidad (diagnosticabilidad) de un sistema, lineal o no lineal, consiste en la posibilidad de reconstruir el estado x (falla f), teniendo el conocimiento de la salida del sistema y, la entrada u, y posiblemente, un número finito de sus derivadas, $y^{(k)}$, $k \geq 0$ y $u^{(l)}(t)$, $l \geq 0$. A continuación, se presentan algunas definiciones concernientes al álgebra abstracta (Fraleigh 1987) y al álgebra diferencial (Cruz y Martinez-Guerra, 2005), (Kolchin, 1973), (Ritt, 1950).

Definición 1: Sean $\mathcal{L}$ y $\mathcal{K}$ campos diferenciales. Una extensión de campos diferenciales $\mathcal{L}/\mathcal{K}$ esta dada por dos campos diferenciales $\mathcal{K}$, $\mathcal{L}$, tales que:

1. $\mathcal{K} \subset \mathcal{L}$
2. La restricción a $\mathcal{K}$ de la derivación de $\mathcal{L}$ es la derivación de $\mathcal{K}$.

Ejemplo 1: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ son ejemplos triviales de extensiones de campos diferenciales, donde $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Definición 2: Se dice que un elemento $x \in \mathcal{L}$ es diferencialmente algebraico sobre $\mathcal{K}$ si y solamente si x satisface una ecuación diferencial algebraica con coeficientes sobre el campo $\mathcal{K}$. Es decir, existe un polinomio P diferente de cero, tal que: $P(x, \dot{x}, \ddot{x}, ..., x^{(n)}) = 0$, entonces a x se llama $\mathcal{K}$-algebraicamente diferencialmente dependiente; en otro caso, se dice que x es $\mathcal{K}$-algebraicamente diferencialmente independiente.

Ejemplo 2: $\mathbb{R}\langle e^t\rangle/\mathbb{R}$ es una extensión de un campo diferencial, donde $\mathbb{R} \subseteq \mathbb{R}\langle e^t\rangle$. Esta extensión es diferencialmente algebraica porque e^t satisface $\dot{x} - x = 0$.

Nota 1: Sean $u = (u_1, \dots, u_l)$, $y = (y_1, \dots, y_p)$ dos familias finitas de elementos diferenciales. Se denota por $\mathcal{K}\langle u, y\rangle$ al campo diferencial generado por el campo $\mathcal{K}$, las componentes de u, las componentes de y, así como las componentes de los elementos diferenciales de las familias $\{u, y\}$.

Definición 3: Las fallas son definidas como elementos trascendentes sobre el campo $\mathcal{K}\langle u\rangle$, por lo tanto, un sistema con fallas es una extensión diferencial trascendente, denotado por $\mathcal{K}\langle u, f, y\rangle/\mathcal{K}\langle u, y\rangle$, donde $f = (f_1, ..., f_\mu)$ es el vector de fallas y sus derivadas con respecto al tiempo.

Definición 4: Sean $\mathcal{G}$ y $\mathcal{K}\langle u\rangle$ campos diferenciales. Una dinámica con fallas es una extensión diferencialmente algebraica finitamente generada $\mathcal{G}/\mathcal{K}\langle u, f\rangle$. Por lo tanto, todo elemento de $\mathcal{G}$ satisface una ecuación diferencialmente algebraica con coeficientes que pertenecen al campo diferencial $\mathcal{K}\langle u, f\rangle$.

Ejemplo 3: Considere la ecuación diferencial $\dot{u}^2 f + 4\ddot{u} = 0$. En este caso, f es diferencialmente algebraico sobre $\mathcal{K}\langle u\rangle$. Podemos expresar la dinámica de la forma $\mathcal{K}\langle u, y\rangle/\mathcal{K}\langle u\rangle$, donde $\mathcal{K} = \mathbb{R}$.

Definición 5: {Condición de observabilidad (diagnosticabilidad) algebraica} Dado el sistema (4), un estado $x_i \in \mathbb{R}$ (una falla $f_j \in \mathbb{R}$) se dice algebraicamente observable (diagnosticable) si es diferencialmente algebraico sobre el campo diferencial $\mathbb{R}\langle u, y\rangle$, $1 \leq i \leq n$, $1 \leq j \leq \mu$.

Nota 2: Las propiedades de observabilidad y diagnosticabilidad implican que los estados y las fallas del sistema pueden reconstruirse mediante las salidas disponibles. Una falla es diagnosticable si es algebraicamente observable.

Ejemplo 4: Considere el siguiente sistema no lineal

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + f \\ \dot{x}_2 &= -x_2^2 + x_1 + u \\ y &= x_2 \end{aligned} \tag{6}$$

El sistema (6) es algebraicamente observable puesto que $x_1 - \dot{y} - y^2 + u = 0$ y $x_2 - y = 0$, los cuales son polinomios con coeficientes en $\mathbb{R}\langle u, y\rangle$.

Análogamente, debido a que f satisface la ecuación $f - \ddot{y} - y\dot{y} - y^3 + uy + \dot{u} = 0$, entonces, el sistema (6) es diagnosticable, i.e., la falla puede reconstruirse conociendo $\{u, y\}$ y sus derivadas con respecto al tiempo.

III-B. *Relación entre el número de fallas y el número de salidas disponibles*

Los siguientes resultados de la teoría de extensiones de campos diferencialmente algebraicos son de utilidad para determinar si una falla puede reconstruirse a partir de las entradas y salidas del sistema.

Definición 6: Una familia maximal de elementos del campo $\mathcal{L}$ que son $\mathcal{K}$-algebraicamente diferencialmente independientes se denomina base de trascendencia diferencial de la extension $\mathcal{L}/\mathcal{K}$ y la cardinalidad de esta base es llamada el grado de trascendencia diferencial denotado por: $d^o tr\text{diff}\ \mathcal{L}/\mathcal{K}$.

Propiedad 1 (Kolchin, 1973): Sean $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ campos diferenciales tales que $\mathcal{K} \subset \mathcal{L} \subset \mathcal{M}$. Entonces,

$$d^o tr\text{diff}\ \mathcal{M}/\mathcal{K} = d^o tr\text{diff}\ \mathcal{M}/\mathcal{L} + d^o tr\text{diff}\ \mathcal{L}/\mathcal{K}$$

Nota 3: La propiedad 1 es una herramienta indispensable para la prueba de los teoremas 1 y 2.

Teorema 1: (Diop y Martínez-Guerra, 2001) Suponga que el sistema (4) es diagnosticable, entonces el numero de fallas es menor o igual al número de salidas, es decir, $\mu \leq p$. ■

Teorema 2: (Cruz y Martinez-Guerra, 2005) El sistema (4) es diagnosticable sí y sólo sí, $d^o tr\text{diff}\ \mathcal{K}\langle u, y\rangle/\mathcal{K}\langle u\rangle = \mu$. ■

IV. OBSERVADOR POLINOMIAL

Considere el sistema (4), el observador tiene la siguiente forma

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x} + \Psi(\hat{x}, u, f) + \sum_{i=1}^{m} K_i (y - C\hat{x})^{2i-1} \\ \dot{\hat{f}}_j(t) &= \sum_{i=1}^{m} \overline{K}_{ji} (f_j - \hat{f}_j)^{2i-1} \end{aligned} \tag{7}$$

donde, $\hat{x} \in \mathbb{R}^n$, $K_i = \begin{bmatrix} k_{1,i} & k_{2,i} & \dots & k_{n,i} \end{bmatrix}^T \in \mathbb{R}^n$, $1 \leq i \leq m$, $\hat{x}_0 = \hat{x}(t_0)$, $\overline{K}_{ji} > 0$ y $\hat{f}_{j0} = \hat{f}_j(t_0)$, $1 \leq j \leq \mu$.

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En éste artículo las siguientes hipótesis son consideradas

H1: $f(t)$ es algebraicamente observable sobre $\mathbb{R}\langle u, y\rangle$

H2: K_1 puede elegirse tal que la siguiente ecuación algebraica de Riccati tiene una solución simétrica y definida positiva P para algún $\epsilon > 0$,

$$(A-K_1\ C)^T P + P\ (A-K_1C) + L^2 P\ P + I + \epsilon I = 0$$

H3: Se elige K_i tal que $\lambda_{min}(M_i + M_i^T) \geq 0$, donde $M_i = PK_iC$, para $2 \leq i \leq m$.

H4: γ es una función valuada-real de clase C^1.

Analizamos el error de estimación, el cual está definido como $e^T = (e_x, e_y)$, donde, $e_x = x - \hat{x}$ y $e_f = f_j - \hat{f}_j$. De los sistemas (4) y (7), la dinámica del error del estimación está dada por

$$\begin{aligned} \dot{e}_x = &\ (A - K_1C)e_x - \sum_{i=2}^{m} K_i[Ce_x]^{2i-1} \\ & + [\Psi(x,u,f) - \Psi(\hat{x},u,f)] \quad (8) \\ \dot{e}_f = &\ \Omega_j - \overline{K}_{j1}e_f - \sum_{i=2}^{m} \overline{K}_{ji}(e_f)^{2i-1} \end{aligned}$$

El siguiente teorema demuestra la convergencia del observador.

Teorema 3: Sea el sistema extendido no lineal (4), suponga que $x(t)$ existe para todo $t \geq 0$, la función no lineal $\Psi(x,u,f)$ satisface la condición de Lipschitz (5), $x(t)$ y $f(t)$ son algebraicamente observables. Si existe $0 < P = P^T$ y ganancias del observador K_i y $\overline{K}_j > 0$ tales que el sistema (7) es un observador del sistema (4), entonces, el error de estimación converge asintóticamente a cero.

Demostración: Considere la función candidata de Lyapunov

$$\begin{aligned} V = &\ V_1 + V_2 \\ V_1 = &\ e_x^T P\ e_x \ \ , \ \ V_2 = \frac{1}{2}e_f^2 \end{aligned} \quad (9)$$

donde $0 < P = P^T$ y satisface H2.

La prueba se divide en dos partes, como sigue

(a)

La derivada de V_1 con respecto al tiempo, a lo largo de las trayectorias de (8) es

$$\begin{aligned} \dot{V}_1 = &\ \dot{e_x}^T Pe_x + e_x^T P\dot{e}_x \\ = &\ e_x^T[(A - K_1C)^T P + P(A - K_1C)]e_x \\ & + 2e_x^T P[\Psi(x,u,f) - \Psi(\hat{x},u,f)] \\ & - \sum_{i=2}^{m}[Ce_x]^{2i-2}e_x^T(M_i + M_i^T)e_x \end{aligned}$$

Se utiliza la siguiente desigualdad (Rajamani, 1998), la cual se basa en la condición Lipschitz (5)

$$2e_x^T P[\Psi(x,u) - \Psi(\hat{x},u)] \leq L^2 e_x^T PPe_x + e_x^T e_x \quad (10)$$

Aplicando la desigualdad de Rayleigh (Horn y Johnson, 1985), y tomando en cuenta H3, se tiene

$$-e_x^T(M_i + M_i^T)e_x \leq -\lambda_{min}(M_i + M_i^T)\|e_x\|^2 \quad (11)$$

para $2 \leq i \leq m$.

Por lo tanto, combinando las desigualdades (10) y (11)

$$\begin{aligned} \dot{V}_1 \leq &\ e_x^T[(A - K_1C)^T P + P(A - K_1C) + L^2PP + I]e_x \\ & - \sum_{i=2}^{m}[Ce_x]^{2i-2}\lambda_{min}(M_i + M_i^T)\|e_x\|^2 \\ \leq &\ e_x^T[(A - K_1C)^T P + P(A - K_1C) + L^2PP + I]e_x \\ = &\ -\epsilon\|e_x\|^2 \end{aligned}$$

(b)

En la misma forma, considerando el segundo término de la función candidata de Lyapunov, obtenemos

$$\begin{aligned} \dot{V}_2 = &\ e_f\dot{e}_f \\ = &\ e_f\left(\Omega_j - \overline{K}_{j1}e_f - \sum_{i=2}^{m}\overline{K}_{ji}(e_f)^{2i-1}\right) \\ = &\ e_f\Omega_j - \overline{K}_{j1}e_f^2 - \sum_{i=2}^{m}\overline{K}_{ji}(e_f)^{2i} \\ \leq &\ |e_f||\Omega_j| - \overline{K}_{j1}e_f^2 \\ \leq &\ |e_f|N - \overline{K}_{j1}|e_f|^2 \\ = &\ -[\overline{K}_{j1}|e_f| - N]|e_f| \end{aligned}$$

$\dot{V}_2$ es negativa dentro del conjunto $\{|e_f| > N/\overline{K}_{j1}\}$, i.e., existe $\bar{\epsilon} > 0$ tal que $\overline{K}_{j1}|e_f| - N = \bar{\epsilon} > 0$.

El siguiente análisis muestra que $|e_f|$ está acotado superiomente. Sean α, β cotas superiores de $V_2(e_f)$. Con $\beta > N^2/2\overline{K}_{j1}^2$, las soluciones que inician en el conjunto $\{V_2(e_f) \leq \beta\}$ permanecerán ahí para todo $t \geq 0$, debido a que $\dot{V}_2$ es negativo en $V_2 = \beta$. Por lo tanto, las soluciones de $\dot{e}_f$ son uniformemente acotadas (Khalil, 2002). Más aún, si $N^2/2\overline{K}_{j1}^2 < \alpha < \beta$, entonces $\dot{V}_2$ será negativa en el conjunto $\{\alpha \leq V_2 \leq \beta\}$. En éste conjunto V_2 decrecerá monótonamente hasta las soluciones del conjunto $\{V_2 \leq \alpha\}$. De acuerdo a (Khalil, 2002), la solución es uniformemente últimamente acotada con cota última $|e_f| \leq \sqrt{2\alpha}$. Por ejemplo, si definimos α y β como $\alpha = 2N^2/\overline{K}_{j1}^2$ y $\beta = 3N^2/\overline{K}_{j1}^2$. Entonces, la cota última es

$$|e_f| \leq 2\frac{N}{\overline{K}_{j1}}.$$

Por lo tanto,

$$\dot{V}_2 \leq -\bar{\epsilon}|e_f|$$

De (a) y (b), concluimos que

$$\dot{V} \leq -\epsilon\|e_x\|^2 - \bar{\epsilon}|e_f| \leq 0$$

Ésto finaliza la demostración del teorema. ■

V. OBSERVADOR DE ORDEN REDUCIDO

En ésta sección se presenta un observador de orden reducido, el cual es capaz de estimar las fallas del sistema (4).

Lema 1: Si se satisfacen (3), H1 y H4, entonces, el sistema

$$\dot{\hat{f}} = K(f - \hat{f}) \quad (12)$$

es un observador de orden reducido del sistema (4), donde $\hat{f}$ denota el estimado de la falla, $K > 0$ es la ganancia del observador y determina la rapidez de convergencia.

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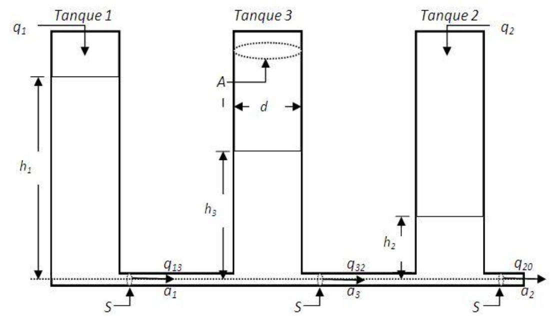

Figura 1. Sistema de tres tanques interconectados.

La demostración del lema es dada en (Martínez-Guerra et al., 2007). ■

En caso de que la condición de observabilidad algebraica de la falla quede expresada en términos de las derivadas de la salida con respecto al tiempo, las cuales son desconocidas, entonces, es necesario usar el siguiente método de estimación (Martínez-Guerra et al., 2007).

Corolario 1: El siguiente sistema (13), junto con (12), constituyen un observador proporcional asintótico de orden reducido para las fallas del sistema (4)

$$\dot{\gamma} = \phi(x, u, f, \gamma) \ , \ \gamma_0 = \gamma(0), \ \ \gamma \in C^1 \tag{13}$$

donde γ es un cambio de variable que depende de la falla estimada $\hat{f}$ y de las variables de estado. ■

VI. Aplicación a un Modelo de Tres Tanques

VI-A. Modelo

La descripción del sistema de tres tanques que se analiza en éste trabajo es la siguiente. La planta consiste de tres cilindros (T1, T2 y T3) con sección transversal A, los cuales están conectados en serie uno a otro mediante tubos cilíndricos de sección transversal S (figura 1). En el tanque T2, se encuentra localizada la denominada válvula de salida, también con sección transversal S. Los flujos q_1 y q_2 denotan las señales de entrada conocidas.

A continuación, se presenta el modelo matemático de un sistema de tres tanques idénticos interconectados (Amira DTS200, 1996). El modelo nominal correspondiente está expresado por el siguiente sistema de ecuaciones diferenciales

$$\begin{aligned} \frac{dh_1}{dt} &= \frac{1}{A}(q_1 - q_{13}) \\ \frac{dh_2}{dt} &= \frac{1}{A}(q_2 + q_{32} - q_{20}) \\ \frac{dh_3}{dt} &= \frac{1}{A}(q_{13} - q_{32}) \end{aligned} \tag{14}$$

De acuerdo a la regla generalizada de Torricelli, la cual es válida para flujo laminar, se obtiene lo siguiente

$$\begin{aligned} q_{13} &= a_1 S\, sign(h_1 - h_3)\sqrt{2g|h_1 - h_3|} \\ q_{32} &= a_3 S\, sign(h_3 - h_2)\sqrt{2g|h_3 - h_2|} \\ q_{20} &= a_2 S\sqrt{2gh_2} \end{aligned} \tag{15}$$

La descripción completa del sistema (14) se muestra en la Tabla I.

TABLA I

Variables y parámetros del sistema de tres tanques

Símbolo	Descripción
h_i	Nivel del líquido del i-ésimo tanque (m), $i \in \{1, 2, 3\}$.
A	Sección transversal constante de cada tanque (m^2).
S	Área tranversal de los tubos de interconexión (m^2).
q_{13}	Flujo de agua del tanque 1 al tanque 3 (m^3/s).
q_{32}	Flujo de agua del tanque 3 al tanque 2 (m^3/s).
q_{20}	Flujo de salida (m^3/s).
q_1	Flujo de entrada al tanque 1 (m^3/s).
q_2	Flujo de entrada al tanque 2 (m^3/s).
a_i	Coeficientes de flujo de salida.

En lo que sigue, se considera que el vector de estado es $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}^T$ y el vector de entrada es $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$.

El sistema (14) tiene cuatro regiones de operación en las cuales el modelo correspondiente es diferenciable. En éste artículo se enfoca a la región de operación $x_1 \geq x_3 \geq x_2$.

VI-B. Consideración de fallas y salidas del sistema

El modelo (14) se expresa en términos de fallas y salidas como sigue

$$\begin{aligned} \dot{x_1} &= \tfrac{1}{A}(u_1 - q_{13} + f_1) \\ \dot{x_2} &= \tfrac{1}{A}(u_2 + q_{32} - q_{20} + f_2) \\ \dot{x_3} &= \tfrac{1}{A}(q_{13} - q_{32}) \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned} \tag{16}$$

donde f_1 y f_2 son fallas aditivas asociadas con los actuadores que controlan los flujos de entrada u_1 y u_2, respectivamente. Además, y_1, y_2 son las salidas disponibles.

VI-C. Análisis de observabilidad y diagnosticabilidad

Para la diagnosticabilidad se requiere que el número de fallas no sea mayor que el número de salidas disponibles (Teorema 1). El sistema (16) consiste de dos fallas ($\mu = 2$) y dos salidas ($p = 2$).

Para garantizar observabilidad y diagnosticabilidad de (16) se debe determinar la condición de observabilidad algebraica (definición 5) para f_1, f_2 y el estado desconocido x_1. De (16) se obtiene

$$\begin{aligned} f_1 &= A\dot{x_1} + a_1 S\sqrt{2g(x_1 - y_3)} - u_1 \\ f_2 &= A\dot{y_2} - a_3 S\sqrt{2g(y_3 - y_2)} \\ &\qquad + a_2 S\sqrt{2gy_2} - u_2 \end{aligned} \tag{17}$$

y
$$x_1 = y_3 + \tfrac{1}{2ga_1^2 S^2}(A\dot{y_3} + a_3 S\sqrt{2g(y_3 - y_2)})^2 \tag{18}$$

La condición de observabilidad algebraica para x_1 se deduce directamente de (18), la cual es una ecuación en $\mathbb{R}\langle u, y\rangle$.

Sustituyendo (18) en (17) se obtiene un sistema de dos ecuaciones diferenciales con coeficientes en $\mathbb{R}\langle u, y\rangle$, con incógnitas f_1 y f_2,

$$\begin{aligned} f_1 &= A\Big[\dot{y_3} + \tfrac{1}{a_1^2 S^2}\Big(A\dot{y_3} + a_3 S\sqrt{2g(y_3 - y_2)}\Big) \\ &\qquad \cdot \Big(A\ddot{y_3} + \tfrac{1}{2}a_3 S\tfrac{\dot{y_3} - \dot{y_2}}{\sqrt{y_3 - y_2}}\Big)\Big] + A\dot{y_3} \\ &\qquad + a_3 S\sqrt{2g(y_3 - y_2)} - u_1 \\ f_2 &= A\dot{y_2} - a_3 S\sqrt{2g(y_3 - y_2)} \\ &\qquad + a_2 S\sqrt{2gy_2} - u_2 \end{aligned} \tag{19}$$

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lo cual implica que, $d^o tr\text{diff}\ \mathbb{R}\langle u,y\rangle/\mathbb{R}\langle u\rangle = 2$. Aplicando el teorema 2, se concluye que las fallas son diagnosticables con las salidas consideradas.

VI-D. Reconstrucción de fallas y estados

Las ecuaciones (19)y (18) requieren de la estimación de las derivadas de las salidas y_2, y_3. En ésta sección se presenta una metodología para reconstruir las primeras $r-1$ derivadas con respecto al tiempo de la salida y.

Observador polinomial. Considere el sistema no lineal con fallas (1). Suponga que éste sistema satisface la definición 5, es decir, el vector de falla f es algebraicamente observable en $\mathbb{R}\langle u,y\rangle$. Por lo tanto, f satisface un polinomio diferencial

$$\psi(f, y, \dot{y}, \ldots, y^{(r)}, u, \dot{u}, \ldots) = 0 \tag{20}$$

donde r es el orden máximo de las derivadas de la salida con respecto al tiempo.

Introduciendo los siguientes cambios de variables

$$\eta_1 = y, \quad \eta_2 = \dot{y}, \quad \ldots, \quad \eta_r = y^{(r-1)} \tag{21}$$

Entonces, en un dominio D donde $\partial\psi/\partial y^{(r)}$ es invertible, la representación entrada–salida de (1) y (20) puede expresarse como sigue

$$\begin{aligned} \dot{\eta} &= A\eta + \overline{\phi}(\eta, u) \\ y &= \eta_1 \end{aligned}$$

donde, $\overline{\phi}(\eta,u) = \begin{bmatrix} 0 & 0 & \ldots & \phi(\eta,u) \end{bmatrix}^T \in \mathbb{R}^r$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\ldots & 0 \\ 0 & 0 & 1 & 0\ldots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 0 & 0 & 0 & 0\ldots & 0 \end{bmatrix} \in \mathbb{R}^{r\times r},\ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{bmatrix} \in \mathbb{R}^r.$$

La función no lineal $\overline{\phi}(\eta,u)$ (respectivamente $\phi(\eta,u)$) es Lipschitz en η y uniformemente acotada en u.

Por el teorema 3, el observador tiene la siguiente estructura

$$\dot{\hat{\eta}} = \quad A\hat{\eta} + \overline{\phi}(\hat{\eta}, u) + \sum_{i=0}^{m} K_i (y - \hat{\eta}_1)^{2i-1} \tag{22}$$

El observador polinomial (22) es usado para estimar las variables $\eta_2 = \dot{y}, \ldots, \eta_r = y^{(r-1)}$ mediante la salida disponible $y = \eta_1$.

Observador de orden reducido. Considere la siguiente derivada a ser estimada

$$\eta = \dot{y} \tag{23}$$

Del acuerdo al lema 1, se propone el siguiente observador

$$\dot{\hat{\eta}} = K(\eta - \hat{\eta}) \tag{24}$$

Se introduce el siguiente cambio de variable (corolario 1)

$$\hat{\eta} = \gamma + Ky \tag{25}$$

De (24) y (25) se obtiene que

$$\dot{\gamma} = -K\hat{\eta} \tag{26}$$

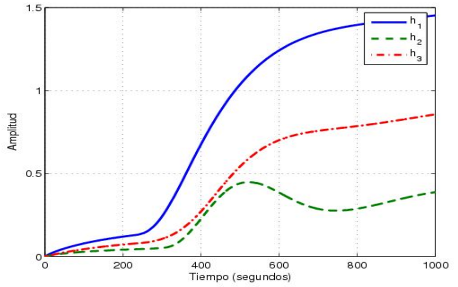

Figura 2. Región de operación.

Sustituyendo (25) en (26)

$$\dot{\gamma} = -K\gamma - K^2 y \tag{27}$$

Las ecuaciones (27) y (25) constituyen un estimador asintótico de η.

VII. RESULTADOS NUMÉRICOS

En ésta sección se muestran algunas simulaciones para ilustrar el desempeño de los esquemas de estimación propuestos, los cuales son aplicados al sistema (16). Los parámetros utilizados son

$$\begin{array}{ll} A = 0.149 \text{ m}^2, & \\ S = 5\times 10^{-5} \text{ m}^2, & q_{1max} = q_{2max} = 0.1 \text{ m}^3/\text{s}, \\ g = 9.81 \text{ m/s}^2, & a_1 = 0.0418, \\ a_2 = 0.789, & a_3 = 0.435 \end{array}$$

En todas las simulaciones, los flujos de entrada q_1 y q_2 se mantienen constantes, es decir, $q_1 = 2\times 10^{-5}$ m^3/s y $q_2 = 1.5\times 10^{-5}$ m^3/s, para todo $t \geq 0$. Las fallas aditivas son

$$\begin{aligned} f_1 &= \left(5\times 10^{-5}\right)\left[1 + \sin(0.2\ t\ e^{-0.01\ t})\right]\mathcal{U}(t-220) \\ f_2 &= \left(5\times 10^{-5}\right)\left[1 + \sin(0.5\ t\ e^{-0.01\ t})\right]\mathcal{U}(t-300) \end{aligned}$$

donde $\mathcal{U}(t)$ es la función escalón unitario.

La figura 2 muestra la región de operación considerada $(x_1 \geq x_3 \geq x_2)$.

Para el observador polinomial se fijo $m = 3$, las ganancias utilizadas para el observador de estados son $K_1 = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}^T$, $K_2 = \begin{bmatrix} 0.8 & 0.8 & 0.8 \end{bmatrix}^T$, $K_3 = \begin{bmatrix} 1.2 & 1.2 & 1.2 \end{bmatrix}^T$. Las ganancias de los observadores de fallas se tomaron como $\overline{K}_{11} = 0.5$, $\overline{K}_{12} = 0.5$, $\overline{K}_{13} = 0.8$, $\overline{K}_{21} = 0.8$, $\overline{K}_{22} = 1.2$ y $\overline{K}_{23} = 1.2$. En las figuras 3 y 4 se muestran los resultados obtenidos empleando el observador polinomial.

El observador de orden reducido no reconstruye el estado completo del sistema. Puesto que $y_2 = x_2$, $y_3 = x_3$ se consideran disponibles, únicamente es necesario reconstruir el estado x_1, así como las fallas f_1 y f_2.

Las ganancias del observador de orden reducido para las fallas son $k_1 = 2$, $k_2 = 2$ y para el estado x_1 se tiene $K = 0.3$.

Las fallas reconstruidas de f_1 y f_2, y el correspondiente estimado de x_1 se ilustran en la figura 5.

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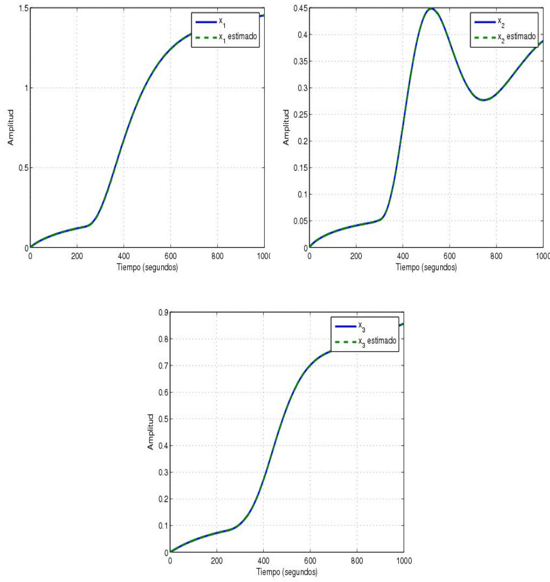

Figura 3. Estados estimados mediante el observador polinomial.

VIII. Conclusiones

En éste trabajo se trató el problema de diagnóstico de fallas de una clase de sistemas no lineales empleando la teoría de observadores. Se utilizó el enfoque algebraico–diferencial como un criterio para la estimación de estados y fallas. El método consiste en considerar a la falla como un estado aumentado del sistema, de ésta manera se propusó el observador de tipo polinomial, el cual reconstruye fallas y variables de estado simultáneamente. Además, se diseñó un observador de orden reducido para comparar los resultados obtenidos. Ambas técnicas se aplicaron al diagnóstico de fallas de un sistema de tres tanques. Los resultados numéricos muestran el desempeño de los observadores.

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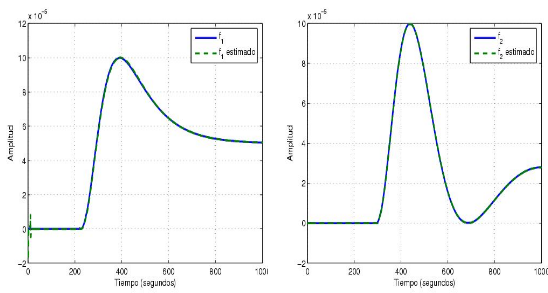

Figura 4. Fallas estimadas empleando el observador polinomial.

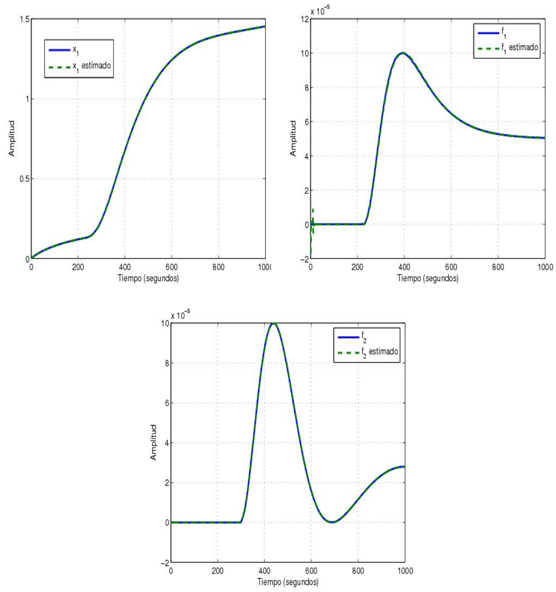

Figura 5. Resultados obtenidos con el observador de orden reducido.

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