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Procesos de difusion en una dimension y polinomios ortogonales

T E S I S

Que presenta

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One dimensional diffusion processes and orthogonal polynomials

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Dedicado a mi familia,
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Abstract

We describe the general theory of diffusion processes, which contains as a particular case the solutions of stochastic differential equations. The idea of the theory is to construct explicitly the generator of the Markov process using the so-called scale function and the speed measure. We also explain how the theory of orthogonal polynomials help to study some diffusions. In addition, using the theory of diffusions, we present the Brox model, which is a process in a random environment.

Resumen

Describimos la teoría general de procesos de difusion, que contiene como caso particular las soluciones de ecuaciones diferenciales estocásticas. La idea de la teoría es construir explícitamente el generador de los procesos de Markov usando la suncion de escala y la medida de velocidad. Tambien explicamos cómo la teoría de polinomios ortogonales ayuda a estudiar algunas difusiones. Además, usando la teoria de difusiones, presentamos el modelo de Brox, que es un proceso con ambiente aleatorio.

Introduction

In the world where we live there are many natural phenomena that have a random character, and the theory of probability is a tool that permits us to explain some of these events.

In this work, we will study the so-called **diffusion processes** in one dimension. The diffusion processes allow us to model some natural phenomena, such as random motion of particles. A diffusion is a random process that has two important features: **continuous paths** and the **Markov property**. This last property is characterized by the loss of memory, which means that one can make estimates of the future of the process based solely on its present state just as one could know the full history of the process. In other words, by conditioning on the present state of the system, its future and past are independent.

It turns out that a process X_t with the Markov property, is characterized by its **infinitesimal generator**, which is specified by the following operator

$$Af(x) := \lim_{t \to 0^+} \frac{E(f(X_t)|X_0 = x) - f(x)}{t} \quad \text{for some functions } f.$$

In this work we construct the infinitesimal generator using the **scale function** and the **speed measure**; these two objects characterize the infinitesimal generator. If (l, r), an interval in $\mathbb{R}$, is the state space of the process X_t, then the scale function is the probability that the process first reaches r before l. On the other hand, the speed measure can be written in terms of the expectation of the first time the process reaches either l or r.

Under appropriate assumptions on the scale function and the speed measure one can obtain that the infinitesimal generator can be written as a differential operator of order 2, also called a Sturm-Liouville operator, given by

$$Af(x) = \mu(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x), \quad \text{where } \mu \text{ and } \sigma \text{ are certain functions.}$$

This type of operators, with suitable conditions, are associated with diffusion processes that are solutions of a **stochastic differential equations**. This differential operator has particular interest because it is related to some **orthogonal polynomials**. More precisely, if we consider the functions $\mu(x)$ and $\sigma^2(x)$ to be polynomials of degree at most 1 and 2 respectively, then there exist orthogonal polynomials that are eigenfunctions of the operator, and the process associated with this operator has a density probability function that can be written in terms of these orthogonal polynomials.

In this work, we also study an example of a diffusion whose infinitesimal generator is not a Sturm-Liouville operator. What one has to do is to construct explicitly the scale function and the speed measure. The process we consider is called the **Brox diffusion** [11], and it is an example of a process in a **random environment**. The importance of this type of processes is that it considers the random medium.

This thesis is mainly based on Chapter VII of D. Revuz, and M. Yor. [3]. In chapter 1, we show in detail the results that allow us to write the infinitesimal generator of a diffusion process X in terms of the scale function s and the speed measure m. We prove the following result

$$\frac{d}{dm}\frac{d}{ds}f_{+} = \frac{d}{dm}\frac{d}{ds}f_{-} = \lim_{t\to 0^{+}} \frac{E(f(X_t)|X_0 = x) - f(x)}{t},$$

where on the left hand side we have derivatives with respect to the function s and the measure m.

In chapter 2 we present the main result on orthogonal polynomials that we use in chapter 3. In chapter 3 we characterize the probability density of the Ornstein-Uhlenbeck, the Cox-Ingersoll-Ross and Jacobi diffusions through Hermite, Laguerre and Jacobi orthogonal polynomials respectively. At the end of chapter 3 we give an example of a diffusion that does not fit in the classical context: the Brox diffusion, which is a diffusion in a random environment.

Contents

Chapter 1

Diffusion processes in one dimension

1.1 Notation

1. The symbol θ_t represents a transformation such that the effect on a path $\omega := \{\omega(s)\}_{s\geq 0}$ is the following $\theta_t(\omega) = \{\omega(s)\}_{s\geq t}$. This transformation is called the **shift operator** and has the properties
   (a) $X_t(\theta_s(\omega)) = X_{t+s}(\omega)$ for all stochastic process $X := \{X_t : t \geq 0\}$ and
   (b) $\theta_t \circ \theta_s = \theta_{t+s}$.
2. The expression $a_n \nearrow a$ means that the sequence a_n converges to a, and $a_n \leq a_{n+1}$ for all n. Similarly $a_n \searrow a$ means that the sequence a_n converges to a, and $a_n \geq a_{n+1}$ for all n.
3. Let $X := \{X_t : t \geq 0\}$ be a Markov process with state space E, and $A \subseteq E$ measurable, then $P_x(X_t \in A) := P(X_t \in A \mid X_0 = x)$ and $E_x(X_t) := E(X_t \mid X_0 = x)$. Also, $p(t, x, dy)$ represents the transition probability distribution, i.e. $P(X_t \in A|X_0 = x) = \displaystyle\int_A p(t, x, dy)$.

1.2 Diffusion process

In this section we will look at some basic concepts about diffusions.

Definition 1.2.1. *Let $E = (l, r)$ be an interval of $\mathbb{R}$ which may be closed, open, semi open, bounded or unbounded.*

1. *The **hitting time** of a one-point set $\{x\}$ is denote by T_x, and defined by*

$$T_x := \inf\{t > 0 : X_t = x\}.$$

2. *A stochastic process $X = \{X(t), t \geq 0\}$ with state space E is called a **diffusion process** if it satisfies the following:*
   (a) The paths of X are continuous.

(b) X enjoys the strong Markov property, i.e. for $F : \Omega \to \mathbb{R}$ measurable and τ a stopping time, then

$$E_x(F(X \circ \theta_\tau)|\mathfrak{F}_\tau) = E_{X_\tau}(F(X)).$$

(c) X is regular, i.e. for any $x \in int(E)$ and $y \in E$, $P_x(T_y < \infty) > 0$.

3. For any interval $I = (a, b)$ such that $[a, b] \subseteq E$, we denote by σ_I the exit time of I, i.e. $\sigma_I := T_a \wedge T_b$. We also write $m_I(x) := E_x(\sigma_I)$.

Theorem 1.2.2. *If I is bounded, the function m_I is bounded on I.*

Proof. Note that for any fixed y in I, we may pick $\alpha < 1$ and $t > 0$ such that

$$\max\{P_y(T_a > t), P_y(T_b > t)\} = \alpha. \tag{1.1}$$

From the regularity of X, $P_y(T_a < \infty) > 0$, then

$$0 < P_y(T_a < \infty) = P_y\left(\bigcup_{n=0}^{\infty}\{T_a < n\}\right) \leq \sum_{n=0}^{\infty} P_y\left(T_a < n\right).$$

Consequently there exists an n_0 such that $P_y(T_a < n_0) > 0$. Thus $P_y(T_a > n_0) < 1$, and by the same procedure we obtain an m_0 such that $P_y(T_b > m_0) < 1$. Now, define $t := \max\{n_0, m_0\}$ and $\alpha := \max\{P_y(T_a > n_0), P_y(T_b > m_0)\}$.

We will prove that

$$\sup_{x \in I} P_x(\sigma_I > t) \leq \alpha. \tag{1.2}$$

Let y be a fixed point in I. If $y < x < b$ then

$$P_x(\sigma_I > t) \leq P_x(T_b > t) \leq P_y(T_b > t) \leq \alpha < 1.$$

The second inequality is because any path that starts in y has to pass through x.

Similarly if $a < x < y$, then

$$P_x(\sigma_I > t) \leq P_x(T_a > t) \leq P_y(T_a > t) \leq \alpha < 1,$$

which yields (1.2).

Now, since $\sigma_I = u + \sigma_I \circ \theta_u$ on $\{\sigma_I > u\}$, we have

$$\begin{aligned}
P_x(\sigma_I > nt) &= P_x(\sigma_I > (n-1)t, \sigma_I > nt) \\
&= P_x(\sigma_I > (n-1)t, (n-1)t + \sigma_I \circ \theta_{(n-1)t} > nt) \\
&= E_x\left(1_{\{\sigma_I > (n-1)t\}} \cdot 1_{\{(n-1)t + \sigma_I \circ \theta_{(n-1)t} > nt\}}\right) \\
&= E_x\left(1_{\{\sigma_I > (n-1)t\}} \cdot 1_{\{\sigma_I \circ \theta_{(n-1)t} > t\}}\right) \\
&= E_x\left(1_{\{\sigma_I > (n-1)t\}} \cdot 1_{\{\sigma_I > t\}} \circ \theta_{(n-1)t}\right) \\
&\overset{\binom{Markov}{property}}{=} E_x\left(1_{\{\sigma_I > (n-1)t\}} \cdot E_{X_{(n-1)t}}{(1_{\{\sigma_I > t\}})}\right).
\end{aligned}$$

Since $\sigma_I > (n-1)t$, then $X_{(n-1)t} \in I$ and, by (1.2), we obtain

$$E_{X_{(n-1)t}}\left(1_{\{\sigma_I > t\}}\right) \leq \alpha.$$

Therefore $P_x(\sigma_I > nt) \leq \alpha\, P_x\left(\sigma_I > (n-1)t\right)$.

Recursively it follows that

$$P_x(\sigma_I > nt) \leq \alpha^n. \tag{1.3}$$

On the other hand, we know that $E_x(\sigma_I) \leq \sum_{n=0}^{\infty} t\, P_x(\sigma_I > nt)$. Hence

$$E_x(\sigma_I) \leq \sum_{n=0}^{\infty} t\alpha^n = t(1-\alpha)^{-1} < \infty.$$

□

Remark: For a and b in E and $l \leq a < x < b \leq r$, the probability $P_x(T_b < T_a)$ is the probability that the process started at x exits (a, b) by its right end. We also have that

$$P_x(T_b < T_a) + P_x(T_a < T_b) = 1.$$

1.3 The scale function

Theorem 1.3.1. *There exists a continuous, strictly increasing function s on E such that*

$$P_x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}. \tag{1.4}$$

for any a, b, x in E with $l \leq a < x < b \leq r$. Also if $\tilde{s}$ is another function with the same properties, then $\tilde{s} = \alpha s + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$.

Proof. The event $\{T_r < T_l\}$ is equal to the disjoint union

$$\{T_r < T_l, T_a < T_b\} \bigcup \{T_r < T_l, T_b < T_a\}.$$

Note that $T_l = T_a + T_l \circ \theta_{T_a}$ always, and $T_r = T_a + T_r \circ \theta_{T_a}$ on the set $\{T_a < T_b\}$. Then by the Markov property we have

$$\begin{aligned}
P_x\left(T_r < T_l, T_a < T_b\right) &= E_x\left(1_{\{T_r < T_l\}} \cdot 1_{\{T_a < T_b\}}\right) \\
&= E_x\left(1_{\{T_a + T_r \circ \theta_{T_a} < T_a + T_l \circ \theta_{T_a}\}} \cdot 1_{\{T_a < T_b\}}\right) \\
&= E_x\left(1_{\{T_r < T_l\}} \circ \theta_{T_a} \cdot 1_{\{T_a < T_b\}}\right) \\
&\overset{\binom{Markov}{property}}{=} E_x\left(E_{X_{T_a}}(1_{\{T_r < T_l\}}) \cdot 1_{\{T_a < T_b\}}\right).
\end{aligned}$$

Since $X_{T_a} = a$, then $E_{X_{T_a}}(1_{\{T_r<T_l\}}) = E_a(1_{\{T_r<T_l\}})$, thus

$$\begin{aligned} P_x\left(T_r < T_l, T_a < T_b\right) &= E_x\left(E_a(1_{\{T_r<T_l\}}) \cdot 1_{\{T_a<T_b\}}\right) \\ &= E_a(1_{\{T_r<T_l\}}) \cdot E_x(1_{\{T_a<T_b\}}) \\ &= P_a(T_r < T_l) \cdot P_x(T_a < T_b). \end{aligned}$$

We finally obtain

$$P_x(T_r < T_l) = P_a(T_r < T_l) \cdot P_x(T_a < T_b) + P_b(T_r < T_l) \cdot P_x(T_b < T_a). \tag{1.5}$$

Define $s(x) := P_x(T_r < T_l)$, and solving for $P_x(T_b < T_a)$ we obtain the formula in the statement.

Let us now verify first s is increasing. Indeed, because the paths of X are continuous

$$P_x(T_r < T_l) \leq P_y(T_r < T_l) \text{ if } x < y.$$

We now see that s is strictly increasing. Suppose there exists $x < y$ such that $s(x) = s(y)$. Then for any $b > y$, by (1.4) we obtain

$$P_y(T_b < T_x) = \frac{s(y) - s(x)}{s(b) - s(x)} = 0.$$

This contradicts the regularity of the process. Therefore s is strictly increasing.

We now will prove that s is continuous. To this end, we first prove that

$$\lim_{n\to\infty} P_x(T_{a_n} < T_b) = P_x(T_a < T_b), \text{ if } a_n \searrow a. \tag{1.6}$$

Note that if $a < x$ and $a_n \searrow a$ then $T_{a_n} \nearrow T_a$ if $X(0) = x$.

We know that $X_{T_a} = a$ if $X(0) = x$. Then

$$P_x(T_a < T_b) = P_x\left(\bigcap_{n=1}^{\infty} \{T_{a_n} < T_b\}\right) = \lim_{n\to\infty} P_x(T_{a_n} < T_b),$$

because $\{T_{a_{n+1}} < T_b\} \subseteq \{T_{a_n} < T_b\}$.

On the other hand, if $l \leq a < x < b \leq r$ we know that it holds the equality (1.5).

Suppose that $y \in E$, and consider $y_n \searrow y$ (the case $y_n \nearrow y$ is similar).

We will see that $s(y_n) \to s(y)$. Take $b \in E$ such that $l \leq y < y_n < b \leq r$. By (1.5) we know that

$$P_y(T_b < T_{y_n}) = \frac{s(y) - s(y_n)}{s(b) - s(y_n)}.$$

Then

$$0 = \lim_{n\to\infty} P_y(T_b < T_{y_n}) = \lim_{n\to\infty} \frac{s(y) - s(y_n)}{s(b) - s(y_n)}. \tag{1.7}$$

Since $y_n \searrow y$ and s is strictly increasing, $s(y_n) > s(y_{n+1}) > ... > s(y)$, so the sequence $s(y_n)$ is decreasing and bounded by $s(y)$. Therefore there exists α such that $s(y_n) \to \alpha$ and $s(b) > \alpha$. Hence by (1.7) we obtain that $s(y_n) \to s(y)$.

Suppose now that $\tilde{s}$ is another function such that $\tilde{s}$ is continuous, strictly increasing, and

$$P_x(T_b < T_a) = \frac{\tilde{s}(x) - \tilde{s}(a)}{\tilde{s}(b) - \tilde{s}(a)},$$

then we will prove $\tilde{s} = \alpha s + \beta$, for some $\alpha > 0$, $\beta \in \mathbb{R}$.

Let $l \leq a < x < b \leq r$, from (1.5)

$$P_x(T_r < T_l) = P_x(T_a < T_b)P_a(T_r < T_l) + P_x(T_b < T_a)P_b(T_r < T_l), \tag{1.8}$$

where, by definition, $s(x) = P_x(T_r < T_l)$ and $P_x(T_b < T_a) = \frac{\tilde{s}(x) - \tilde{s}(a)}{\tilde{s}(b) - \tilde{s}(a)}$. Then by (1.8) we obtain

$$s(x) = \frac{\tilde{s}(b) - \tilde{s}(x)}{\tilde{s}(b) - \tilde{s}(a)} \cdot P_a(T_r < T_l) + \frac{\tilde{s}(x) - \tilde{s}(a)}{\tilde{s}(b) - \tilde{s}(a)} \cdot P_b(T_r < T_l).$$

Therefore $\tilde{s} = \alpha s + \beta$, with

$$\alpha := \frac{\tilde{s}(b) - \tilde{s}(a)}{P_b(T_r < T_l) - P_a(T_r < T_l)}, \quad \beta := \frac{\tilde{s}(a)P_b(T_r < T_l) - \tilde{s}(b)P_a(T_r < T_l)}{P_b(T_r < T_l) - P_a(T_r < T_l)}.$$

□

Definition 1.3.2. *The function s of the Theorem 1.3.1 is called the* ***scale function*** *of the process X. In particular we say that X is on its natural or standard scale if $s(x) = x$.*

Note that if a process X has scale function s, then the process $Y := s(X)$ is on its natural scale.

1.4 The speed measure

Definition 1.4.1. *Let s be a continuous function and strictly increasing on $[l, r]$. A function f is called s-convex if for $l \leq a < x < b \leq r$*

$$(s(b) - s(a))f(x) \leq (s(b) - s(x))f(a) + (s(x) - s(a))f(b). \tag{1.9}$$

Define also the right and left s-derivatives $\frac{df_+}{ds}$ and $\frac{df_-}{ds}$ as

$$\frac{df_+}{ds}(x) := \lim_{y \to x^+} \frac{f(y) - f(x)}{s(y) - s(x)}, \quad \text{and} \quad \frac{df_-}{ds}(x) := \lim_{y \to x^-} \frac{f(x) - f(y)}{s(x) - s(y)}. \tag{1.10}$$

At the points where they coincide we say that f has an s-derivative.

If m is a measure, then we define the m-derivative of f at x as

$$\frac{df}{dm}(x) := \lim_{y \to x^+} \frac{f(y) - f(x)}{m((x, y])}.$$

Lemma 1.4.2. *Let s be a continuous function and strictly increasing on $[l, r]$. If f is s-convex, then f is continuous on $[l, r]$.*

Proof. Since f is s-convex, if $l \leq u < w < v \leq r$, then

$$(s(v) - s(u))f(w) \leq (s(w) - s(u))f(v) + (s(v) - s(w))f(u).$$

This implies

$$\frac{f(w) - f(u)}{s(w) - s(u)} \leq \frac{f(v) - f(u)}{s(v) - s(u)} \leq \frac{f(v) - f(w)}{s(v) - s(w)}. \tag{1.11}$$

Now, let $x \in (l, r)$ and pick $v \in (l, r)$ such that $l \leq x < v \leq r$, and apply (1.11) to obtain

$$\frac{f(x) - f(l)}{s(x) - s(l)} \leq \frac{f(v) - f(x)}{s(v) - s(x)} \leq \frac{f(r) - f(x)}{s(r) - s(x)}. \tag{1.12}$$

By (1.12) we obtain

$$\frac{f(x) - f(l)}{s(x) - s(l)} \cdot (s(v) - s(x)) + f(x) \leq f(v) \leq \frac{f(r) - f(x)}{s(r) - s(x)} \cdot (s(v) - s(x)) + f(x). \tag{1.13}$$

Take $v \searrow x$ in (1.13) to see that $\lim_{v \to x^+} f(v) = f(x)$. Therefore f is right continuous. Left continuity is shown in exactly the same way. □

Lemma 1.4.3. *Let s be a continuous function and strictly increasing on $[l, r]$. If f is a function s-convex on $[l, r]$, then $\frac{df_+}{ds}$ and $\frac{df_-}{ds}$ are increasing, thus of bounded variation.*

Proof. Let $l \leq x < y \leq r$, and pick x_n, y_n such that $l \leq x < x_n < y < y_n \leq r$.

Since f is s-convex then

$$\frac{f(x_n) - f(x)}{s(x_n) - s(x)} \leq \frac{f(y) - f(x_n)}{s(y) - s(x_n)} \leq \frac{f(y_n) - f(y)}{s(y_n) - s(y)}. \tag{1.14}$$

From (1.14) we obtain

$$\frac{f(x_n) - f(x)}{s(x_n) - s(x)} \leq \frac{f(y_n) - f(y)}{s(y_n) - s(y)}.$$

Finally take $x_n \to x$ and $y_n \to y$. □

Lemma 1.4.4. *Let f be a function s-convex on $I = [a, b]$ and $f(a) = f(b) = 0$. Then there exists a measure μ such that $\mu((c, d]) = \frac{df+}{ds}(d) - \frac{df+}{ds}(c)$, and*

$$f(x) = -\int_a^b G_I(x, y)\ \mu(dy),$$

where

$$G_I(x, y) := \begin{cases} \dfrac{(s(x) - s(a))(s(b) - s(y))}{s(b) - s(a)}, & a \leq x \leq y \leq b, \\ \dfrac{(s(y) - s(a))(s(b) - s(x))}{s(b) - s(a)}, & a \leq y \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

G_I is called the Green function.

Proof. Using the integration by parts formula for Stieltjes integrals we obtain

$$\begin{aligned}
\int_a^b G_I(x,y)\,\mu(dy) &= \int_a^b G_I(x,y)\,d\left(\frac{df+}{ds}(y)\right)\\
&= \frac{s(b)-s(x)}{s(b)-s(a)}\int_a^x (s(y)-s(a))\,d\left(\frac{df+}{ds}(y)\right)\\
&+ \frac{s(x)-s(a)}{s(b)-s(a)}\int_x^b (s(b)-s(y))\,d\left(\frac{df+}{ds}(y)\right)\\
&= \frac{s(b)-s(x)}{s(b)-s(a)}\left((s(x)-s(a))\frac{df+}{ds}(x)-\int_a^x \frac{df+}{ds}(y)\,ds(y)\right)\\
&+ \frac{s(x)-s(a)}{s(b)-s(a)}\left((s(x)-s(b))\frac{df+}{ds}(x)-\int_x^b \frac{df+}{ds}(y)\,ds(y)\right)\\
&= \frac{s(b)-s(x)}{s(b)-s(a)}\,(f(a)-f(x))+\frac{s(x)-s(a)}{s(b)-s(a)}\,(f(x)-f(b))\\
&= -f(x).
\end{aligned}$$

□

Theorem 1.4.5. *There exists a* ***unique measure*** *m such that for all $I=(a,b)$*

$$m_I(x)=\int_a^b G_I(x,y)\,m(dy)=\int_a^b G_I(x,y)\,d\left(\frac{-d(m_I(y))_+}{ds}\right),$$

with $[a,b]\subseteq E$, and where G_I is the Green function.

Proof. We apply the Lemma 1.4.4. We will first show that $-m_I$ is s-convex. Let $a<c<x<d<b$, and $J:=(c,d)$, $I:=(a,b)$. Then by the Markov property we obtain

$$\begin{aligned}
m_I(x) &= E_x(\sigma_I)\\
&\overset{(\sigma_I=\sigma_J+\sigma_I\circ\theta_{\sigma_J})}{=} E_x(\sigma_J+\sigma_I\circ\theta_{\sigma_J})\\
&= m_J(x)+E_x(\sigma_I\circ\theta_{\sigma_J})\\
&= m_J(x)+E_x(\sigma_I\circ\theta_{\sigma_J})\\
&\overset{\binom{Markov}{property}}{=} m_J(x)+E_x\left(E_{X_{\sigma_J}}(\sigma_I)\right)\\
&= m_J(x)+E_x\left(E_{X_{\sigma_J}}(\sigma_I)\cdot 1_{\{T_c<T_d\}}\right)+E_x\left(E_{X_{\sigma_J}}(\sigma_I)\cdot 1_{\{T_d<T_c\}}\right)\\
&= m_J(x)+E_x\left(E_{X_{T_c}}(\sigma_I)\cdot 1_{\{T_c<T_d\}}\right)+E_x\left(E_{X_{T_d}}(\sigma_I)\cdot 1_{\{T_d<T_c\}}\right)\\
&= m_J(x)+E_x\left(E_c(\sigma_I)\cdot 1_{\{T_c<T_d\}}\right)+E_x\left(E_d(\sigma_I)\cdot 1_{\{T_d<T_c\}}\right)\\
&= m_J(x)+E_c(\sigma_I)\cdot E_x\left(1_{\{T_c<T_d\}}\right)+E_d(\sigma_I)\cdot E_x\left(1_{\{T_d<T_c\}}\right)\\
&= m_J(x)+E_c(\sigma_I)\cdot P_x(T_c<T_d)+E_d(\sigma_I)\cdot P_x(T_d<T_c)\\
&= m_J(x)+m_I(c)\cdot\left(\frac{s(d)-s(x)}{s(d)-s(c)}\right)+m_I(d)\cdot\left(\frac{s(x)-s(c)}{s(d)-s(c)}\right). \qquad (1.15)
\end{aligned}$$

Hence if follows that $-m_I$ is s-convex. Also, $m_I(a) = m_I(b) = 0$. Then by Lemma 1.4.4 there exists a measure m with $m(dy) := d\left(\frac{-d(m_I(y))_+}{ds}\right)$, such that

$$m_I(x) = \int_a^b G_I(x,y)\, m(dy) = \int_a^b G_I(x,y)\, d\left(\frac{-d(m_I(y))_+}{ds}\right).$$

Let us check the uniqueness of m. Let $J := (c,d) \subseteq I := (a,b)$. If we consider the functions $m_I(x)$ and $m_J(x)$, and we apply the Theorem 1.4.5, then there exist measures m and $\tilde{m}$ such that

$$m(dy) = d\left(\frac{-d(m_I(y))_+}{ds}\right), \quad \text{and} \quad \tilde{m}(dy) = d\left(\frac{-d(m_J(y))_+}{ds}\right).$$

However, if we calculate the s-derivative in both side of (1.15), we obtain

$$\frac{d(m_I(y))_+}{ds} = \frac{d(m_J(y))_+}{ds} + C,$$

where C is a constant, in particular this holds for $J = I$. Therefore $m = \tilde{m}$. □

Corollary 1.4.6. *The measure m satisfies that*

$$m((w,z]) = \left(\frac{-d(m_I(z))_+}{ds}\right) - \left(\frac{-d(m_I(w))_+}{ds}\right).$$

Definition 1.4.7. *The measure m of the Theorem 1.4.5 is called the* ***speed measure*** *of the process X.*

Lemma 1.4.8. *If $v(x) := E_x\left(\int_0^{T_a \wedge T_b} 1_{(c,b)} \circ X_u du\right)$, then*

$$v(x) = \begin{cases} E_x(T_c \wedge T_b) + P_x(T_c < T_b) \cdot v(c), & x \in (c,b), \\ P_x(T_c < T_a) \cdot v(c), & x \in (a,c]. \end{cases}$$

In particular if $x \in (c,b)$ we have $\dfrac{dv_+}{ds} = \dfrac{dE_x(T_c \wedge T_b)_+}{ds} + C$, where C is a constant.

Proof. Let $x \in (c,b)$, and define now $\sigma_J := T_c \wedge T_b$. By the Markov property we obtain

$$\begin{aligned}
P_x(T_c < T_b) \cdot v(c) &= E_x\left(1_{\{T_c<T_b\}}\right) \cdot E_c\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right) \\
&= E_x\left(1_{\{T_c<T_b\}} \cdot E_c\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right)\right) \\
&\overset{(c=X_{T_c}=X_{\sigma_J})}{=} E_x\left(1_{\{T_c<T_b\}} \cdot E_{X_{\sigma_J}}\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right)\right) \\
&\overset{\binom{Markov}{property}}{=} E_x\left(1_{\{T_c<T_b\}} \cdot \left\{\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right) \circ \theta_{\sigma_J}\right\}\right) \\
&= E_x\left(1_{\{T_c<T_b\}} \cdot \left(\int_0^{\sigma_I \circ \theta_{\sigma_J}} 1_{(c,b)} \circ X_u \circ \theta_{\sigma_J}\, du\right)\right) \\
&= E_x\left(1_{\{T_c<T_b\}} \cdot \int_{\sigma_J}^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right).
\end{aligned}$$

On the other hand, since $x \in (c,b)$ we have

$$\begin{aligned}
E_x(T_c \wedge T_b) &= E_x\left(1_{\{T_c<T_b\}} \cdot (T_c \wedge T_b)\right) + E_x\left(1_{\{T_b<T_c\}} \cdot (T_c \wedge T_b)\right) \\
&= E_x\left(1_{\{T_c<T_b\}} \cdot \int_0^{\sigma_J} du\right) + E_x\left(1_{\{T_b<T_c\}} \cdot \int_0^{\sigma_J} du\right) \\
&\overset{\left(\substack{\sigma_J=\sigma_I \\ on\ \{T_b<T_c\}}\right)}{=} E_x\left(1_{\{T_c<T_b\}} \cdot \int_0^{\sigma_J} 1_{(c,b)} \circ X_u\, du\right) + E_x\left(1_{\{T_b<T_c\}} \cdot \int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right).
\end{aligned}$$

Therefore if $x \in (c,b)$, then $E_x(T_c \wedge T_b) + P_x(T_c < T_b) \cdot v(c)$ equals

$$E_x\left(1_{\{T_c<T_b\}} \cdot \int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right) + E_x\left(1_{\{T_b<T_c\}} \cdot \int_0^{\sigma_I} 1_{(c,b)} \circ X_u\, du\right) = v(x). \tag{1.16}$$

In the same way we obtain that, if $x \in (a,c]$ then

$$v(x) = P_x(T_c < T_a) \cdot v(c). \tag{1.17}$$

Combining (1.16) and (1.17), we arrive at the equality

$$\begin{aligned}
v(x) &= [E_x(T_c \wedge T_b) + P_x(T_c < T_b) \cdot v(c)] \cdot 1_{(c,b)} + P_x(T_c < T_a) \cdot v(c) \cdot 1_{(a,c]} \\
&= \left[E_x(T_c \wedge T_b) + \frac{s(b)-s(x)}{s(b)-s(c)} \cdot v(c)\right] \cdot 1_{(c,b)} + \frac{s(x)-s(a)}{s(c)-s(a)} \cdot v(c) \cdot 1_{(a,c]} \\
&= E_x(T_c \wedge T_b) \cdot 1_{(c,b)} + \frac{s(b)-s(x)}{s(b)-s(c)} \cdot v(c) \cdot 1_{(c,b)} + \frac{s(x)-s(a)}{s(c)-s(a)} \cdot v(c) \cdot 1_{(a,c]}.
\end{aligned}$$

□

In the following, $C_0(\mathbb{R})$ will denote the space of the continuous functions that vanish at infinity.

Corollary 1.4.9. *If $I = (a,b)$, $x \in I$, $f \in C_0(\mathbb{R})$, then*

$$E_x\left(\int_0^{T_a \wedge T_b} f(X_s)\, ds\right) = \int_a^b G_I(x,y) f(y) m(dy). \tag{1.18}$$

Proof. Let $c \in (a,b)$. We will now prove that the function $v(x) := E_x\left(\int_0^{T_a \wedge T_b} 1_{(c,b)} \circ X_u du\right)$ is s-concave on (a,b). So $-v(x)$ is s-convex.

We will show that

$$P_x(T_m < T_n) \cdot v(m) + P_x(T_n < T_m) \cdot v(n) \le v(x),$$

where $a < m < x < n < b$, which means that v is s-concave. Let $\sigma_J := T_m \wedge T_m$ and $\sigma_I := T_a \wedge T_b$, then $\sigma_I = \sigma_J + \sigma_I \circ \theta_{\sigma_J}$. Note that by the Markov property

$$\begin{aligned}
P_x(T_m < T_n) \cdot v(m) &= E_x\left(1_{\{T_m<T_n\}}\right) \cdot E_m\left(\int_0^{T_a\wedge T_b} 1_{(c,b)} \circ X_u \, du\right) \\
&= E_x\left(1_{\{T_m<T_n\}} \cdot E_m\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right)\right) \\
&\overset{(m=X_{T_m}=X_{\sigma_J})}{=} E_x\left(1_{\{T_m<T_n\}} \cdot E_{X_{\sigma_J}}\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right)\right) \\
&\overset{\binom{Markov}{property}}{=} E_x\left(1_{\{T_m<T_n\}} \cdot \left\{\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right) \circ \theta_{\sigma_J}\right\}\right) \\
&= E_x\left(1_{\{T_m<T_n\}} \cdot \int_0^{\sigma_I\circ\theta_{\sigma_J}} 1_{(c,b)} \circ X_u \circ \theta_{\sigma_J} \, du\right) \\
&= E_x\left(1_{\{T_m<T_n\}} \cdot \int_{\sigma_J}^{\sigma_J+\sigma_I\circ\theta_{\sigma_J}} 1_{(c,b)} \circ X_u \, du\right) \\
&\overset{(\sigma_J+\sigma_I\circ\theta_{\sigma_J}=\sigma_I)}{=} E_x\left(1_{\{T_m<T_n\}} \cdot \int_{\sigma_J}^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right).
\end{aligned}$$

In the same way we obtain that

$$P_x(T_n < T_m) \cdot v(n) = E_x\left(1_{\{T_n<T_m\}} \cdot \int_{\sigma_J}^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right).$$

Then

$$\begin{aligned}
P_x(T_m < T_n) \cdot v(m) + P_x(T_n < T_m) \cdot v(n) &= E_x\left(\int_{\sigma_J}^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right) \\
&\leq E_x\left(\int_0^{\sigma_I} 1_{(c,b)} \circ X_u \, du\right) \\
&= v(x).
\end{aligned}$$

So $-v$ is s-convex and $v(a) = v(b) = 0$. Therefore, applying the Lemma 1.4.4, there exists a measure μ such that $\mu(dy) = -d\left(\frac{dv_+}{ds}(y)\right)$ and

$$v(x) = \int_a^b G_I(x,y) \, \mu(dy) = -\int_a^b G_I(x,y) \, d\left(\frac{dv_+}{ds}(y)\right). \tag{1.19}$$

Then by Lemma 1.4.8 we arrive to

$$v(x) = -\int G_I(x,y) \, 1_{(c,b)} \, d\left(\frac{dE_y(T_c \wedge T_b)_+}{ds}\right) = \int G_I(x,y) \, 1_{(c,b)} \, m(dy),$$

where m is the speed measure. □

1.5 Infinitesimal operator

The purpose of this section is to prove that the scale function and the speed measure characterize the infinitesimal operator of the diffusion process X. We will appeal to some standard results of the theory of semigroups (see e.g. [3]).

In the following, P_t will denote the semigroup $P_t(f)(x) := E(f(X_t)|X_0 = x)$.

Definition 1.5.1. *Let X be a diffusion process. A function $f \in C_0(\mathbb{R})$ is said to belong to the domain $\mathfrak{D}_A$ of the infinitesimal operator of X if the limit*

$$Af = \lim_{t \to 0^+} \frac{P_t f - f}{t} \tag{1.20}$$

*exists in $C_0(\mathbb{R})$. This operator $A : \mathfrak{D}_A \to C_0(\mathbb{R})$ is called the **infinitesimal operator** of the process X or of the semigroup P_t.*

By the properties of the Markov process, if $f \in C_0(\mathbb{R})$ then

$$E(f(X_{t+h}) \mid \mathfrak{F}_t) = P_h(f(X_t)), \tag{1.21}$$

where $\mathfrak{F}_t := \sigma(X_s : \ s \leq t)$.

The following result which is useful for our purposes can be found in [3, p.282].

Lemma 1.5.2. *If $f \in \mathfrak{D}_A$, then:*

1. *The function $t \to P_t f$ is differentiable in $C_0(\mathbb{R})$ and*

$$\frac{d}{dt} P_t f = A P_t f = P_t A f.$$

2. $P_t f - f = \int_0^t P_s A f \, ds.$

Lemma 1.5.3. *For all $y \in E$ we have that*

$$\alpha := E_y \left(f(X_{t-s}) - f(X_0) - \int_0^{t-s} Af(X_u) du \right) = 0, \tag{1.22}$$

Proof.

$$\begin{aligned}
\alpha &= E_y(f(X_{t-s})) - f(y) - E_y \left(\int_0^{t-s} Af(X_u) du \right) \\
&= E_y \left(E(f(X_{t-s}) | \, \mathfrak{F}_0) \right) - f(y) - E_y \left(E \left(\int_0^{t-s} Af(X_u) du \,\middle|\, \mathfrak{F}_0 \right) \right) \\
&\overset{\binom{(1.21)\ and}{Fubini}}{=} E_y(P_{t-s} f(X_0)) - f(y) - E_y \left(\int_0^{t-s} E(Af(X_u) \, | \mathfrak{F}_0) \, du \right) \\
&\overset{(1.21)}{=} P_{t-s} f(y) - f(y) - E_y \left(\int_0^{t-s} P_u A f(X_0) \, du \right) \\
&= P_{t-s} f(y) - f(y) - \int_0^{t-s} P_u A f(y) \, du \\
&\overset{(Lemma\ 1.5.2)}{=} 0.
\end{aligned}$$

□

Theorem 1.5.4. *Let $X_0 := x$. If $f \in \mathfrak{D}_A$, then the process*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds,$$

is a martingale with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$ for every x.

Proof. Since f and Af are bounded, one can check that M_t^f is integrable for each t. Now, let $s < t$, by the Markov property we have

$$\begin{aligned}
E_x(M_t^f \mid \mathfrak{F}_s) &= E_x\left(f(X_t) - f(X_0) - \int_0^t Af(X_u)\, du \,\middle|\, \mathfrak{F}_s \right) \\
&\overset{(adding\ \pm f(X_s))}{=} M_s^f + E_x\left(f(X_t) - f(X_s) - \int_s^t Af(X_u) du \,\middle|\, \mathfrak{F}_s \right) \\
&= M_s^f + E_x\left(\left\{ f(X_{t-s}) - f(X_0) - \int_0^{t-s} Af(X_u) du \right\} \circ \theta_s \,\middle|\, \mathfrak{F}_s \right) \\
&\overset{\binom{Markov}{property}}{=} M_s^f + E_{X_s}\left(f(X_{t-s}) - f(X_0) - \int_0^{t-s} Af(X_u) du \right) \\
&\overset{(Lemma\ 1.5.3)}{=} M_s^f .
\end{aligned}$$

Therefore M_t^f is a martingale with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$. □

Theorem 1.5.5. *Let $f \in \mathfrak{D}_A$ and x in the interior of E. Then the s-derivative of f exists except possibly on the set $\{x : m(\{x\}) > 0\}$.*

Proof. Let $f \in \mathfrak{D}_A$, $\sigma_I := T_a \wedge T_b$ and $x \in (a, b)$ such that $[a, b] \subseteq E$.

By Theorem 1.5.4, we know that M_t^f is a martingale with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$. Also $|M_t^f| \leq \alpha + t \cdot \beta$. This implies that $|M_{t\wedge\sigma_I}^f| \leq \alpha + \sigma_I \cdot \beta$.

Define $S := \alpha + \sigma_I \cdot \beta$. By Theorem 1.2.2 we have that $E_x(\sigma_I) < \infty$, which implies $E_x(S) < \infty$.

Let us prove that $M_{t\wedge\sigma_I}^f$ is uniformly integrable. Indeed

$$\begin{aligned}
\lim_{c\to\infty} \sup_t \int_{\{|M_{t\wedge\sigma_I}^f| > c\}} \left| M_{t\wedge\sigma_I}^f \right| dP &\leq \lim_{c\to\infty} \sup_t \int_{\{|S|>c\}} \left| S \right| dP \\
&= \lim_{c\to\infty} \int_{\{|S|>c\}} \left| S \right| dP \\
&= 0.
\end{aligned}$$

By Doob optional stopping theorem (see e.g. [2, p.87]) , we have that $E_x(M_{\sigma_I}^f) = E_x(M_0^f) = 0$, so

$$E_x(f(X_{\sigma_I})) - f(x) = E_x\left(\int_0^{\sigma_I} Af(X_s)\, ds \right).$$

Now, by Corollary 1.4.9 we obtain

$$E_x(f(X_{\sigma_I})) - f(x) = E_x\left(\int_0^{\sigma_I} Af(X_s)\, ds \right) = \int_a^b G_I(x, y) Af(y)\, m(dy). \tag{1.23}$$

On the other hand, we have that

$$\begin{aligned} E_x\left(f(X_{\sigma_I})\right) &= E_x\left(f(X_{\sigma_I})\cdot 1_{\{T_a<T_b\}}\right) + E_x(f(X_{\sigma_I})\cdot 1_{\{T_b<T_a\}}) \\ &= E_x(f(X_{T_a})\cdot 1_{\{T_a<T_b\}}) + E_x(f(X_{T_b})\cdot 1_{\{T_b<T_a\}}) \\ &= E_x(f(a)\cdot 1_{\{T_a<T_b\}}) + E_x(f(b)\cdot 1_{\{T_b<T_a\}}) \\ &= f(a)\cdot E_x(1_{\{T_a<T_b\}}) + f(b)\cdot E_x(1_{\{T_b<T_a\}}) \\ &= f(a)\cdot P_x(T_a<T_b) + f(b)\cdot P_x(T_b<T_a) \\ &= f(a)\cdot\frac{s(b)-s(x)}{s(b)-s(a)} + f(b)\cdot\frac{s(x)-s(a)}{s(b)-s(a)}. \end{aligned} \tag{1.24}$$

Then substituting in (1.24) in the equation (1.23) we arrive at

$$f(a)\cdot\frac{s(b)-s(x)}{s(b)-s(a)} + f(b)\cdot\frac{s(x)-s(a)}{s(b)-s(a)} - f(x) = \int_a^b G_I(x,y)Af(y)m(dy). \tag{1.25}$$

By the definition of Green function we have for every $x\in(a,b)$ that

$$\begin{aligned} \int_a^b G_I(x,y)Af(y)\,m(dy) &= \frac{s(b)-s(x)}{s(b)-s(a)}\cdot\int_a^x (s(y)-s(a))Af(y)m(dy) \\ &+ \frac{s(x)-s(a)}{s(b)-s(a)}\cdot\int_x^b (s(b)-s(y))Af(y)m(dy). \end{aligned} \tag{1.26}$$

Replacing (1.26) in the equation (1.25) and simplifying we obtain

$$\frac{f(b)-f(x)}{s(b)-s(x)} - \frac{f(x)-f(a)}{s(x)-s(a)} = J_1 + J_2, \tag{1.27}$$

where

$$\begin{aligned} J_1 &:= \int_a^x \frac{s(y)-s(a)}{s(x)-s(a)}\,Af(y)m(dy). \\ J_2 &:= \int_x^b \frac{s(b)-s(y)}{s(b)-s(x)}\,Af(y)m(dy). \end{aligned}$$

If we take $b\searrow x$ in the equality (1.27) and we apply the theorem of differentiation of Lebesgue to $J_2\cdot\dfrac{m((x,b])}{m((x,b])}$, then

$$\frac{df_+}{ds}(x) - \frac{f(x)-f(a)}{s(x)-s(a)} = \int_a^x \frac{s(y)-s(a)}{s(x)-s(a)}\,Af(y)m(dy) + m(\{x\})Af(x).$$

In the same manner, in this last equality we take $a\nearrow x$, which yields

$$\frac{df_+}{ds}(x) - \frac{df_-}{ds}(x) = 2m(\{x\})Af(x).$$

Therefore the s-derivative of f exists except possibly on the set $\{x : m(\{x\})>0\}$. □

Corollary 1.5.6. *The s-derivative of $f \in \mathfrak{D}_A$ exists for almost every $x \in \mathbb{R}$.*

Proof. Note that by construction of the speed measure m we known that

$$m((w,z]) = -\frac{dE_z(\sigma_I)_+}{ds} + \frac{dE_w(\sigma_I)_+}{ds},$$

Since $-m_I(x) := -E_x(\sigma_I)$ is s-convex, then $\frac{-E_x(\sigma_I)_+}{ds}$ is increasing, and so $\frac{-E_x(\sigma_I)_+}{ds}$ is continuous except possibly in a countable set. Therefore $m(\{x\}) = 0$ for almost every $x \in \mathbb{R}$. By the Theorem 1.5.5we obtain that the s-derivative of $f \in \mathfrak{D}_A$ exists for almost every $x \in \mathbb{R}$. □

The following result resembles the fundamental theorem of calculus.

Theorem 1.5.7. *Let $f \in \mathfrak{D}_A$, then*

$$\int_x^{x+h} Af(y)m(dy) = \frac{df_+}{ds}(x+h) - \frac{df_+}{ds}(x). \tag{1.28}$$

Proof. Since the formula (1.25) is valid for all $x \in (a,b)$, pick $h > 0$ such that $x + h \in (a,b)$, and apply the formula to obtain

$$\int_a^b G_I(x+h,y)Af(y)m(dy) = f(a)\cdot\frac{s(b)-s(x+h)}{s(b)-s(a)} + f(b)\cdot\frac{s(x+h)-s(a)}{s(b)-s(a)} - f(x+h). \tag{1.29}$$

By substraction (1.29) and (1.25) and simplifying, we obtain

$$\frac{f(b)-f(a)}{s(b)-s(a)} - \frac{f(x+h)-f(x)}{s(x+h)-s(x)} = \int_a^b \frac{G_I(x+h,y)-G_I(x,y)}{s(x+h)-s(x)}Af(y)m(dy). \tag{1.30}$$

By definition of the Green function is easy check that

$$\left|\frac{G_I(x+h,y)-G_I(x,y)}{s(x+h)-s(x)}\right| \leq 2.$$

Note that by definition of the Green function, we have that for every y, the s-derivative of G_I exists on x.

Then take $h \searrow 0$ in (1.30), and apply the convergence dominated theorem

$$\frac{f(b)-f(a)}{s(b)-s(a)} - \frac{df_+}{ds}(x) = \int_a^b \frac{dG_I(x,y)}{ds}Af(y)m(dy).$$

Apply the definition of G_I we arrive at

$$\frac{f(b)-f(a)}{s(b)-s(a)} - \frac{df_+}{ds}(x) = \int_x^b \frac{s(b)-s(y)}{s(b)-s(a)}Af(y)m(dy) - \int_a^x \frac{s(y)-s(a)}{s(b)-s(a)}Af(y)m(dy) \tag{1.31}$$

This formula is true for any point $x \in (a,b)$. Hence we can pick $h > 0$ such that $x + h \in (a,b)$, and then apply the formula. Indeed, by substituting $x + h$ in lieu of x in (1.31) we obtain

$$\frac{f(b)-f(a)}{s(b)-s(a)} - \frac{df_+}{ds}(x+h) = \int_{x+h}^{b} \frac{s(b)-s(y)}{s(b)-s(a)} Af(y)m(dy) - \int_{a}^{x+h} \frac{s(y)-s(a)}{s(b)-s(a)} Af(y)m(dy). \quad (1.32)$$

Finally, subtracting (1.32) and (1.31) and simplifying, we obtain (1.28)

$$\int_x^{x+h} Af(y)m(dy) = \frac{df_+}{ds}(x+h) - \frac{df_+}{ds}(x).$$

□

Corollary 1.5.8. *Let $f \in \mathfrak{D}_A$, then*

$$\int_x^{x+h} Af(y)m(dy) = \frac{df_-}{ds}(x+h) - \frac{df_-}{ds}(x).$$

However, if the s-derivative of f exists in x and $x+h$, then

$$\int_x^{x+h} Af(y)m(dy) = \frac{df}{ds}(x+h) - \frac{df}{ds}(x).$$

Theorem 1.5.9. *Let $f \in \mathfrak{D}_A$, and $x \in (a,b)$, then*

$$Af(x) = \frac{d}{dm}\frac{d}{ds}f_+(x) = \frac{d}{dm}\frac{d}{ds}f_-(x). \quad (1.33)$$

Proof. Let x_n be a sequence such that $x_n \searrow x$. By the Theorem 1.5.7 we have

$$\int_x^{x_n} Af(y)m(dy) = \frac{df_+}{ds}(x_n) - \frac{df_+}{ds}(x).$$

Dividing both sides by $m((x,x_n])$, we have

$$\frac{1}{m((x,x_n])}\int_x^{x_n} Af(y)m(dy) = \frac{\frac{df_+}{ds}(x_n) - \frac{df_+}{ds}(x)}{m((x,x_n])}.$$

If we take $x_n \searrow x$ and apply Lebesgues theorem of differentiation, then

$$Af(x) = \frac{d}{dm}\frac{d}{ds}f_+(x).$$

Similarly $Af(x) = \frac{d}{dm}\frac{d}{ds}f_-(x)$. □

Corollary 1.5.10. *Let $f \in \mathfrak{D}_A$, then for almost every $x \in (a,b)$*

$$Af(x) = \frac{d}{dm}\frac{d}{ds}f(x). \quad (1.34)$$

1.6 A particular classic case

In this section we will see a particular class of diffusions, namely those for which the infinitesimal operator is of the form $Lf(x) := \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$. This operator L is associated with a stochastic process that is solution of the stochastic differential equation $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$, where B_t is the Brownian motion. (see e.g. [1]).

Let us see how L is a special case of the infinitesimal operator $Af(x) := \frac{d}{dm}\frac{d}{ds}f(x)$. Suppose that the following limits exist

$$\mu(x) := \lim_{h\to 0} \frac{E(X_h - X_0|X_0 = x)}{h}, \text{ and } \sigma^2(x) := \lim_{h\to 0} \frac{E\left((X_h - X_0)^2|X_0 = x\right)}{h}. \tag{1.35}$$

We first show that the scale function $s(x) := P_x(T_r < T_l)$, where $l < x < r$, is solution of the equation

$$\mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) = 0, \quad f(l) = 0, \; f(r) = 1. \tag{1.36}$$

Note that the boundary conditions $s(l) = 0$ and $s(r) = 1$ are obvious, since $s(x)$ is the probability of reaching r before l, starting the process from x.

Since $l < X_0 = x < r$, then by the continuity of the trajectory there exists $h > 0$ such that $l < X_h < r$. At time h, conditioning on the position of X_h, the probability of reaching r before l is $s(X_h)$. Denote $E(\cdot|X_0 = x) = E_x(\cdot)$. According to ([10, p.193]), by the law of total probabilities

$$s(x) = E_x(s(X_h)) + o(h). \tag{1.37}$$

On the other hand, let $\triangle X := X_h - X_0$. If s has continuous second derivative, then using Taylors formula

$$\begin{aligned} s(X_h) &= s(X_0 + \triangle X) \\ &= s(X_0) + \triangle X \cdot s'(X_0) + \frac{(\triangle X)^2}{2} \cdot s''(\xi), \end{aligned} \tag{1.38}$$

where ξ is between x and X_h. Taking expectation of (1.38) and dividing by h we arrive to

$$\frac{1}{h}E_x(s(X_h)) = \frac{1}{h}s(x) + \frac{1}{h}E_x(\triangle X) \cdot s'(x) + \frac{1}{h}E_x((\triangle X)^2) \cdot \frac{s''(\xi)}{2}. \tag{1.39}$$

By substituting (1.37) in (1.39) one has

$$\frac{1}{h}s(x) + \frac{o(h)}{h} = \frac{1}{h}s(x) + \frac{1}{h}E_x(\triangle X) \cdot s'(x) + \frac{1}{h}E_x((\triangle X)^2) \cdot \frac{s''(\xi)}{2}. \tag{1.40}$$

Adding $\pm\frac{1}{h}E_x((\triangle X)^2) \cdot \frac{s''(x)}{2}$ we have that (1.40) becomes

$$\frac{o(h)}{h} = \frac{1}{h}E_x(\triangle X) \cdot s'(x) + \frac{1}{h}E_x((\triangle X)^2) \cdot \frac{s''(x)}{2} + \frac{1}{h}E_x((\triangle X)^2) \cdot \left(\frac{s''(\xi)}{2} - \frac{s''(x)}{2}\right). \tag{1.41}$$

Finally, if s'' is continuous, by taking $h \searrow 0$ we obtain

$$0 = \mu(x) \cdot s'(x) + \sigma^2(x) \cdot \frac{s''(x)}{2}. \tag{1.42}$$

Therefore we see that the function s satisfies the equation (1.36), thus by solving (1.36) for s

$$s'(x) := \tilde{s}(x), \text{ where } \tilde{s}(x) := e^{-\int_l^x \frac{2\mu(z)}{\sigma^2(z)} dz}. \tag{1.43}$$

Also, it is easy to check that

$$\frac{\tilde{s}'(x)}{\tilde{s}(x)} = \frac{-2\mu(x)}{\sigma^2(x)}. \tag{1.44}$$

Now we will prove that the speed measure m can be obtained from the solution of the equation

$$f'(x) \cdot \mu(x) + \frac{\sigma^2(x)}{2} \cdot f''(x) = -1, \quad f(l) = 0, \; f(r) = 0. \tag{1.45}$$

This is so because, by construction, the speed measure m is in term of $\frac{d(m_I)_+}{ds}$, and we show that $m_I(x)$ is solution of the equation (1.45).

Remember that $m_I(x) := E_x(\sigma_I)$ and $\sigma_I := T_l \wedge T_r$. Note that

$$w(x) := E\left(\int_0^{\sigma_I} 1du \middle| X_0 = x\right) = E_x(\sigma_I) = m_I(x).$$

Thus we study the following. Let g be a continuous and bounded function, and define

$$Z := \int_0^{\sigma_I} g(X_u)du, \text{ and } w(x) := E\left(\int_0^{\sigma_I} g(X_u)du \middle| X_0 = x\right).$$

Now, choose $h > 0$ small enough such that $\sigma_I = h + \sigma_I \circ \theta_h$, and note that

$$Z \circ \theta_h = \left(\int_0^{\sigma_I} g(X_u)du\right) \circ \theta_h = \int_0^{\sigma_I \circ \theta_h} g(X_u \circ \theta_h)du = \int_h^{h+\sigma_I \circ \theta_h} g(X_u)du = \int_h^{\sigma_I} g(X_u)du. \tag{1.46}$$

On the other hand, if we denote $E(\cdot|X_0 = x) = E_x(\cdot)$, by the Markov property

$$\begin{aligned} E(w(X_h)|X_0 = x) &= E_x\left[E\left(\int_0^{\sigma_I} g(X_u)du \middle| X_0 = X_h\right)\right] \\ &= E_x[E_{X_h}(Z)] \\ &\overset{\binom{Markov}{property}}{=} E_x(Z \circ \theta_h) \\ &= E\left(\int_h^{\sigma_I} g(X_u)du \middle| X_0 = x\right). \end{aligned} \tag{1.47}$$

Again, with h small enough

$$
\begin{aligned}
E\left(\int_0^h g(X_u)du \middle| X_0 = x\right) &= h\cdot E\left(\frac{1}{h}\int_0^h g(X_u)du \middle| X_0 = x\right)\\
&= h\cdot E(g(X_0)|X_0 = x) + o(h)\\
&= h\cdot g(x) + o(h). \qquad (1.48)
\end{aligned}
$$

If $\triangle X := X_h - X_0$, then

$$
\begin{aligned}
w(x) &:= E_x\left(\int_0^{\sigma_I} g(X_u)du\right)\\
&= E_x\left(\int_0^{h} g(X_u)du\right) + E_x\left(\int_h^{\sigma_I} g(X_u)du\right)\\
&\overset{\binom{by\ 1.48}{and\ 1.47}}{=} h\cdot g(x) + o(h) + E_x(w(x+\triangle X))\\
&\overset{\binom{byTaylors}{formula}}{=} h\cdot g(x) + o(h) + E_x\left(w(x) + w'(x)\triangle X + w''(\xi)\frac{(\triangle X)^2}{2}\right)\\
&= h\cdot g(x) + o(h) + w(x) + w'(x)\cdot E_x(\triangle X) + \frac{w''(\xi)}{2}E_x\left((\triangle X)^2\right),
\end{aligned}
$$

where ξ is between x and X_h. From the previous equality adding $\pm\frac{w''(x)}{2}E_x((\triangle X)^2)$ we obtain

$$
w'(x)\cdot\frac{1}{h}E_x(\triangle X) + \frac{w''(x)}{2}\cdot\frac{1}{h}E_x\left((\triangle X)^2\right) + \left(\frac{w''(\xi)}{2} - \frac{w''(x)}{2}\right)\cdot\frac{1}{h}E_x\left((\triangle X)^2\right) + \frac{o(h)}{h} = -g(x). \qquad (1.49)
$$

Now, suppose that w'' is continuous, and by (1.35), then by taking $h \searrow 0$ in the equation (1.49), we arrive to

$$
\mu(x)w'(x) + \frac{\sigma^2(x)}{2}w''(x) = -g(x). \qquad (1.50)
$$

The equality in (1.50) is true for any continuous and bounded function g, then if $g = 1$, we have by definition of w that

$$
w(x) := E\left(\int_0^{\sigma_I} 1du \middle| X_0 = x\right) = E_x(\sigma_I) = m_I(x).
$$

Then m_I is solution of the equation

$$
\mu(x)w'(x) + \frac{\sigma^2(x)}{2}w''(x) = -1. \qquad (1.51)
$$

Solving (1.51) for m_I we obtain

$$
m_I(x) = -\int_l^x e^{\int_l^y \frac{-2\mu(z)}{\sigma^2(z)}dz}\cdot H(y)dy + C, \qquad (1.52)
$$

where $H(x) := \int_l^x \frac{2}{\sigma^2(y)}e^{\int_l^y \frac{2\mu(z)}{\sigma^2(z)}dz}dy$, and C is a constant.

Using (1.44) we have that (1.52) becomes

$$m_I(x) = -\int_l^x \left[\tilde{s}(y)\left(\int_l^y \frac{2}{\sigma^2(z)\tilde{s}(z)}dz\right)dy\right] + C. \tag{1.53}$$

Now, define $M(x) := \int_l^x \frac{2dz}{\sigma^2(z)\tilde{s}(z)}$. Since $s'(x) = \tilde{s}(x)$, then the previous equation becomes

$$m_I(x) = \int_l^x -M(y)s'(y)dy + C = \int_l^x -M(y)s(dy) + C. \tag{1.54}$$

Calculating the s-derivative in both sides of (1.54), we arrive to

$$-\frac{dm_I(x)_+}{ds} = M(x). \tag{1.55}$$

By Corollary 1.4.6 we have that the measure defined by $m((a,b]) := M(b) - M(a)$ is the speed measure. Also, we have an explicit formula for the speed measure:

$$m((a,b]) := M(b) - M(a) = \int_a^b M'(x)dx = \int_a^b \frac{2}{s'(x)\sigma^2(x)}dx. \tag{1.56}$$

If we consider this measure m given in (1.56) and the scale function (1.43), it turns out that the infinitesimal operator $\frac{d}{dm}\frac{d}{ds}f(x)$ is equal to $Lf(x) := \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$. Let us see this fact.

First note that

$$\begin{aligned}
\frac{1}{\tilde{m}(x)} \cdot \frac{d}{dx}\left(\frac{1}{\tilde{s}(x)}\frac{df(x)}{dx}\right) &= \frac{\tilde{s}(x)\sigma^2(x)}{2} \cdot \left(\frac{-\tilde{s}'(x)}{\tilde{s}(x)} \cdot \frac{f'(x)}{\tilde{s}(x)} + \frac{1}{\tilde{s}(x)} \cdot f''(x)\right) \\
&\overset{(1.44)}{=} \frac{\tilde{s}(x)\sigma^2(x)}{2} \cdot \left(\frac{2\mu(x)}{\sigma^2(x)} \cdot \frac{f'(x)}{\tilde{s}(x)} + \frac{1}{\tilde{s}(x)} \cdot f''(x)\right) \\
&= \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\
&= Lf(x).
\end{aligned} \tag{1.57}$$

Also, we have

$$\begin{aligned}
\frac{d}{dm}\frac{d}{ds}f(x) &= \frac{d}{dm}\left(\lim_{y\to x}\frac{f(y)-f(x)}{s(y)-s(x)}\right) \\
&= \frac{d}{dm}\left(\lim_{y\to x}\frac{f(y)-f(x)}{y-x}\cdot\lim_{y\to x}\frac{y-x}{s(y)-s(x)}\right) \\
&= \frac{d}{dm}\left(f'(x)\cdot\frac{1}{s'(x)}\right) \\
&\overset{(s'(x)=\tilde{s}(x))}{=} \frac{d}{dm}\left(\frac{f'(x)}{\tilde{s}(x)}\right) \\
&= \lim_{y\to x}\frac{\left(\frac{f'(y)}{\tilde{s}(y)}-\frac{f'(x)}{\tilde{s}(x)}\right)}{m((x,y])} \\
&= \lim_{y\to x}\frac{\left(\frac{f'(y)}{\tilde{s}(y)}-\frac{f'(x)}{\tilde{s}(x)}\right)}{y-x}\cdot\lim_{y\to x}\frac{y-x}{M(y)-M(x)} \\
&= \frac{d}{dx}\left(\frac{1}{\tilde{s}(x)}f'(x)\right)\cdot\frac{1}{M'(x)} \\
&\overset{(M'(x)=\tilde{m}(x))}{=} \frac{d}{dx}\left(\frac{1}{\tilde{s}(x)}f'(x)\right)\cdot\frac{1}{\tilde{m}(x)} \\
&\overset{(1.57)}{=} Lf(x).
\end{aligned} \tag{1.58}$$

Therefore the differential operator L is a special case of the operaror $A := \frac{d}{dm}\frac{d}{ds}$.

Chapter 2

Orthogonal polynomials in one dimension

2.1 Orthogonal polynomials

In this section we will look at some basic concepts on orthogonal polynomials, and some important theorems in relation with this topic.

Definition 2.1.1. *Let $(a,b) \subseteq \mathbb{R}$ and $w : (a,b) \to \mathbb{R}$ a positive function. We say that a set of polynomials $p_0, p_1, ...$ (with p_n of degree n) is orthogonal respect to w, if*

$$\langle p_n, p_m \rangle := \langle p_n, p_m \rangle_w := \int_a^b p_n(x) p_m(x)\ w(x) dx = d_n \delta_{nm},$$

where $d_n \neq 0$, and δ_{nm} denotes the Kronecker delta, that is, $\delta_{nm} = 0$ if $n \neq m$, and $\delta_{nm} = 1$ if $n = m$. If, in addition, $d_n = 1$ we say that the polynomials are orthonormal.

Theorem 2.1.2. *Let $w : (a,b) \to \mathbb{R}$ be a positive function. Then there exists a sequence of orthogonal polynomials $\{p_n\}$ with respect to w such that $\langle p_n, p_m \rangle_w = d_n \delta_{nm}$.*

Proof. Use the Gram-Schmidt process in the sequence of polynomials $p_n(x) := x^n$ with the inner product $\langle f, g \rangle := \int_a^b f(x) g(x) w(x) dx$. □

Theorem 2.1.3. *Let p_n be a sequence of orthogonal polynomials (of degree n) with respect to w. Then $\langle p_n, q \rangle_w = 0$ for any polynomial q of degree $m < n$; that is, the polinomial p_n is orthogonal to any polynomial of degree less that n.*

Proof. Note that any polynomial of degree m, where $m < n$ can be written as a linear combination of the polynomials $\{p_0, p_1, ..., p_m\}$. Let q be a polynomial of degree m. Then there exist $\alpha_1, \alpha_2, ..., \alpha_m$ such that $q = \sum_{k=0}^{m} \alpha_k p_k$. Then $\langle p_n, q \rangle_w = \sum_{k=0}^{m} \alpha_k \langle p_n, p_k \rangle_w = 0$. □

Theorem 2.1.4. *If p_n and q_n are two orthogonal polynomials (of degree n) with respect to w, then there exists $\lambda \neq 0$ such that $p_n = \lambda q_n$.*

Proof. Pick λ such that the degree of $p_n - \lambda q_n$ is $n-1$. By the previous theorem we know that $\langle p_n, p_n - \lambda q_n \rangle_w = 0$ and $\langle q_n, p_n - \lambda q_n \rangle_w = 0$. This implies that $\langle p_n, p_n \rangle_w = \lambda \langle p_n, q_n \rangle_w$ and $\langle q_n, p_n \rangle_w = \lambda \langle q_n, q_n \rangle_w$.

Then $\|p_n - \lambda q_n\|^2 = \langle p_n - \lambda q_n, p_n - \lambda q_n \rangle_w = 0$, therefore $p_n - \lambda q_n = 0$. □

Note 1: In this chapter, A always represents a polynomial of degree at most 2 and B a polynomial of degree at most 1. This is so is because if A or B do not satisfy this condition, then there does not exist a polynomial P that is solution of the differential equation $A(x)P''(x)+B(x)P'(x)+\lambda P(x) = 0$ (see e.g. [12]). Also, $w : (a, b) \to \mathbb{R}$ is a positive function.

Theorem 2.1.5. *Let A, B and w be defined as in Note 1. Assume that*

$$\frac{d}{dx}(A(x)w(x)) = B(x)w(x).$$

Then

$$\lim_{x \to a^+} x^n A(x)w(x) = 0, \quad n = 0, 1, 2... \tag{2.1}$$

Proof. We have that $(x^n A(x)w(x))' = x^n(A(x)w(x))' + nx^{n-1}A(x)w(x)$, $n = 0, 1, 2...$, and by assumption we know that $(A(x)w(x))' = B(x)w(x)$, then

$$(x^n A(x)w(x))' = x^n B(x)w(x) + nx^{n-1}A(x)w(x).$$

By integrating both sides we obtain

$$x^n A(x)w(x) = \int_a^x [s^n B(s)w(s) + ns^{n-1}A(s)w(s)]ds.$$

Since $s^n B(s)w(s) + ns^{n-1}A(s)w(s)$ is continuous, the integral is continuous. Therefore we arrive to $\lim\limits_{x \to a^+} x^n A(x)w(x) = 0$. □

Similarly we obtain that

$$\lim_{x \to b^-} x^n A(x)w(x) = 0, \quad n = 0, 1, 2... \tag{2.2}$$

2.2 The derivative of orthogonal polynomials

Theorem 2.2.1. *Let A and B be polynomials as in Note 1, and let $w : (a, b) \to \mathbb{R}$ be a positive function. Let $\{q_n\}$ be a sequence of orthogonal polynomials with respect to w, such that $(A(x)w(x))' = B(x)w(x)$. Then $\{q_n'\}$ is a sequence of orthogonal polynomials with respect to $w_1 := Aw$.*

Proof. Let $n > m$, and define $\phi(x) := x^{m-1}B(x)$. We know that

$$\langle \phi, q_n \rangle_w = \int_a^b x^{m-1}B(x)q_n(x)w(x)dx = 0, \tag{2.3}$$

because the degree of $x^{m-1}B(x)$ is less than n.

On the other hand, by using $(A(x)w(x))' = B(x)w(x)$

$$\langle \phi, q_n \rangle_w = \int_a^b x^{m-1} q_n(x)[B(x)w(x)]dx = \int_a^b x^{m-1} q_n(x)(A(x)w(x))'dx.$$

Using (2.3), integration by parts and Theorem 2.1.5 we arrive to

$$0 = \langle \phi, q_n \rangle_w = -(m-1)\int_a^b x^{m-2} A(x) q_n(x) w(x)dx - \int_a^b x^{m-1} A(x) q_n'(x) w(x) dx.$$

Also, we know that

$$\int_a^b x^{m-2} A(x) q_n(x) w(x) dx = 0,$$

because the degree of $x^{m-2}A(x)$ is less than n.

Hence

$$\int_a^b x^{m-1} q_n'(x)[A(x)w(x)]dx = 0.$$

Therefore, the sequence $\{q_n'\}$ is orthogonal with respect to $w_1 = Aw$. □

Corollary 2.2.2. *Let A and B as in the note 1. Let $\{q_n\}$ be a sequence of orthogonal polynomials with respect to w, such that they satisfy the equation $(A(x)w(x))' = B(x)w(x)$. Then the sequence $\{q_n^{(m)}\}$ of polynomials is orthogonal with respect to $w_m := A^m w$.*

Also $(A(x)w_m(x))' = B_m(x)w_m(x)$, with $B_m(x) = A'(x)m + B(x)$.

Proof. One can repeat the process in the proof of Theorem 2.2.1. □

2.3 A differential equation for the orthogonal polynomials

Theorem 2.3.1. *Let A, B and w as in note 1, and such that $(A(x)w(x))' = B(x)w(x)$. Then the set of orthogonal polynomials with respect to w are solution of the differential equation*

$$A(x)f''(x) + B(x)f'(x) + \lambda_n f(x) = 0,$$

where $\lambda_n = -n[\frac{(n-1)}{2}A'' + B']$.

Proof. By Theorem 2.2.1 we know that q_n' is a sequence of orthogonal polynomials with respect to Aw. Also, abusing of the notation we have

$$\langle q_n'(x), x^{m-1} \rangle_{Aw} = 0 \text{ if } m < n. \tag{2.4}$$

Using integration by parts, (2.1) and (2.2) we obtain

$$0 = \langle q_n'(x), x^{m-1} \rangle_{Aw} = \frac{-1}{m}\int_a^b x^m [A(x) q_n'(x) w(x)]' dx. \tag{2.5}$$

Note that

$$[A(x)q_n'(x)w(x)]' = A'(x)q_n'(x)w(x) + A(x)q_n''(x)w(x) + A(x)q_n'(x)w'(x),$$

but

$$A'(x)q_n'(x)w(x) + A(x)q_n'(x)w'(x) = q_n'(x)[A'(x)w(x) + A(x)(x)w'(x)] = q_n'(x)B(x)w(x).$$

Then

$$[A(x)q_n'(x)w(x)]' = A(x)q_n''(x)w(x) + q_n'(x)B(x)w(x),$$

And by (2.5)

$$0 = \int_a^b x^m[A(x)q_n''(x) + q_n'(x)B(x)]w(x)dx.$$

This says that the polynomial $A(x)q_n''(x)+q_n'(x)B(x)$ of degree n is orthogonal to the polynomials of degree m with respect to w, with $m < n$. Therefore by Theorem 2.1.4 there exists $-\lambda_n \neq 0$ such that

$$A(x)q_n''(x) + B(x)q_n'(x) = -\lambda_n q_n(x).$$

Then, comparing coefficients in

$$A(x)q_n''(x) + B(x)q_n'(x) + \lambda_n q_n(x) = 0, \tag{2.6}$$

we obtain $\lambda_n = -n[\frac{(n-1)}{2}A'' + B']$. □

2.4 Classical orthogonal polynomials

Some classical orthogonal polynomials are specified in the following table:

Polynomial	Interval	$A(x)$	$w(x)$	$B(x)$
Jacobi	$(-1,1)$	$1-x^2$	$(1-x)^\alpha(1+x)^\beta$	$\beta-\alpha-(\alpha+\beta+2)x$
Laguerre	$(0,\infty)$	x	$x^\alpha e^{-x}$	$\alpha+1-x$
Hermite	$(-\infty,\infty)$	1	e^{-x^2}	$-2x$

By Theorem 2.3.1 we know that the Jacobi polynomials are solution of the differential equation

$$(1-x^2)J_n''(x) + [\beta-\alpha-(\alpha+\beta+2)x]J_n'(x) + n(n+\alpha+\beta+1)J_n(x) = 0. \tag{2.7}$$

The Laguerre polynomials are solution of

$$xL_n''(x) + [\alpha+1-x]L_n'(x) + nL_n(x) = 0. \tag{2.8}$$

The Hermite polynomials are solution of

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{2.9}$$

Note that the general equation is of the form

$$A(x)p_n''(x) + B(x)p_n'(x) = -\lambda_n p_n(x),$$

and we see that the orthogonal polynomials are eigenfunctions of the operator $Lf := Af'' + Bf'$ associated with the eigenvalues given by the Theorem 2.3.1.

2.5 The formula of Rodrigues type

The solutions of the equations (2.7), (2.8) and (2.9) can be expressed compactly using the formula of Rodrigues type which is derived using the following lemma.

Lemma 2.5.1. *For $m = 0, 1, 2...$, the sequence of orthogonal polynomials $\{q_n^{(m)}\}$ satisfies*

$$\frac{d}{dx}\left(A(x)w_m(x)\frac{d}{dx}(q_n^{(m)}(x))\right) = \lambda_{n,m}w_m(x)q_n^{(m)}(x), \quad n = 0, 1, 2..., \tag{2.10}$$

where $\lambda_{n,m} = (n-m)\left(\frac{1}{2}(n+m-1)A''(0) + B'(0)\right)$ and $w_m(x)$ is as in Corollary 2.2.2.

Proof. Note that

$$\begin{aligned}\frac{d}{dx}\left(A(x)w_m(x)\frac{d}{dx}(q_n^{(m)}(x))\right) &= w_m(x)[B_m(x)q_n^{(m+1)}(x) + A(x)q_n^{(m+2)}(x)] \\ &= w_m(x)[B_m(x)(q_n^{(m)}(x))' + A(x)(q_n^{(m)}(x))'].\end{aligned}$$

because $(A(x)w_m(x))' = B_m(x)w_m(x)$ by Corollary 2.2.2. Also, again by Corollary 2.2.2 and Theorem 2.3.1, there exist $\lambda_{n,m}$ such that the sequence of orthogonal polynomials $q_n^{(m)}$ are solution of

$$A(x)(q_n^{(m)}(x))'' + B_m(x)(q_n^{(m)}(x))' = \lambda_{n,m}q_n^{(m)}(x). \tag{2.11}$$

Then

$$\frac{d}{dx}\left(A(x)w_m(x)\frac{d}{dx}(q_n^{(m)}(x))\right) = w_m(x)\lambda_{n,m}q_n^{(m)}(x).$$

Comparing coefficients in the equation (2.11) we obtain

$$\lambda_{n,m} = (n-m)\left(\frac{1}{2}(n+m-1)A''(0) + B'(0)\right).$$

□

Theorem 2.5.2 (Formula of Rodrigues type). *The orthogonal polynomials q_n with respect to w can be written in the form*

$$q_n(x) = \frac{c_n}{w(x)}\frac{d^n}{dx^n}\left(A^n(x)w(x)\right), \tag{2.12}$$

where c_n is a constant.

Proof. We apply Lemma 2.5.1 several times. If $m = 0$ in (2.10), we have

$$(A(x)w(x)q_n'(x))' = \lambda_{n,0} w(x) q_n(x). \tag{2.13}$$

Note that $w_1(x)q_n'(x) = A(x)w(x)q_n'(x)$. Then by substituting in (2.13) we obtain

$$\lambda_{n,0} w(x) q_n(x) = (w_1(x)q_n'(x))' = \frac{(\lambda_{n,1} w_1(x) q_n'(x))'}{\lambda_{n,1}}. \tag{2.14}$$

Applying again (2.10) in (2.14) with $m = 1$ we obtain

$$\frac{(\lambda_{n,1} w_1(x) q_n'(x))'}{\lambda_{n,1}} = \frac{(A(x) w_1(x) q_n''(x))''}{\lambda_{n,1}}.$$

Now, since $w_2(x) = A^2(x)w(x) = A(x)w_1(x)$, by applying again (2.10) to the above equation we arrive at

$$\lambda_{n,0} w(x) q_n(x) = \frac{(w_2(x)q_n''(x))''}{\lambda_{n,1}} = \frac{(\lambda_{n,2} w_2(x) q_n''(x))''}{\lambda_{n,2}\lambda_{n,1}} = \frac{(A(x) w_2(x) q_n'''(x))'''}{\lambda_{n,2}\lambda_{n,1}}.$$

Continuing with this process we obtain

$$\lambda_{n,0} w(x) q_n(x) = \frac{(A(x) w_{n-1}(x) q_n^{(n)}(x))^{(n)}}{\lambda_{n,n-1} \cdots \lambda_{n,1}}. \tag{2.15}$$

Note that $q_n^{(n)}$ is constant. Then from (2.15) we arrive to

$$q_n(x) = \frac{c_n}{w(x)} (A(x) w_{n-1}(x))^{(n)}, \tag{2.16}$$

where $c_n = \dfrac{q_n^{(n)}(x)}{\lambda_{n,n-1} \cdots \lambda_{n,1}\lambda_{n,0}}$. Since $w_{n-1}(x) = A^{n-1}(x)w(x)$, substituting in (2.16) we obtain the result. $\square$

2.6 Completeness of the orthogonal polynomials

Other important result that we will prove is that the orthogonal polynomials with respect to the function w form a base of $L^2((a,b), w(x)dx) := \left\{ f : \int_a^b f^2(x) w(x) dx < \infty \right\}$. For this purpose, we use the Fourier transform.

2.6.1 Hermite polynomials

Note that the space generated by the Hermite polynomials coincides with the space generated by the polynomials $\{1, x, x^2, x^3, ...\}$. Let E be the closure of the space generated by these monomials.

Suppose that $E \neq L^2(\mathbb{R}, e^{-x^2}dx)$. Then there exists $f \in L^2(\mathbb{R}, e^{-x^2}dx) - E$ such that $f - P_E(f) \neq 0$, where P_E is the orthogonal projection on E.

Then, by definition of P_E, we have that for all $e \in E$

$$\langle e, f - P_E(f)\rangle_w = 0, \tag{2.17}$$

In particular for $e := x^n$, for all $n = 0, 1, 2...$

On the other hand, let $g \in L^2(\mathbb{R}, e^{-x^2}dx)$. If $\langle x^n, g\rangle_w = 0$ for all n, then the function G defined by

$$G(z) := \int_{-\infty}^{\infty} e^{zx} g(x) e^{-x^2} dx,$$

is 0 for every z, because

$$G(z) = \int_{-\infty}^{\infty} e^{zx} g(x) e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-\infty}^{\infty} x^n g(x) e^{-x^2} dx = 0.$$

If we pick $z = it$, we have $\displaystyle\int_{-\infty}^{\infty} e^{itx} g(x) e^{-x^2} dx = 0$. This implies that the Fourier transform of $g(x)e^{-x^2}$ is 0. Then $g(x)e^{-x^2} = 0$ is the function 0 in $L^2(\mathbb{R}, e^{-x^2}dx)$, and thus so is g. In particular, this holds for $g := f - P_E(f)$, which implies that $P_E(f) = f$. This contradicts the fact that $f - P_E(f) \neq 0$.

Therefore the Hermite polynomials form a orthogonal basis for the space $L^2(\mathbb{R}, e^{-x^2}dx)$.

2.6.2 Jacobi polynomials

We use the same idea of the previous proof. Note that the space generated by the Jacobi polynomials equals the space generated by the polynomials $\{1, x, x^2, x^3, ...\}$. Let E be the closure of the space generated by these monomials.

Suppose that $E \neq L^2((-1, 1), (1 - x)^\alpha (1 + x)^\beta dx)$. Then there exists $f \in L^2(\mathbb{R}, e^{-x^2}dx) - E$ such that $f - P_E(f) \neq 0$, where P_E is the orthogonal projection on E.

Then, by definition of P_E, we have that for all $e \in E$

$$\langle e, f - P_E(f)\rangle_w = 0, \tag{2.18}$$

In particular for $e := x^n$, for all $n = 0, 1, 2...$

On the other hand, let $g \in L^2((-1, 1), (1 - x)^\alpha (1 + x)^\beta dx)$. If $\langle x^n, g\rangle_w = 0$ for all n, then the function G defined by

$$G(z) := \int_{-1}^{1} e^{zx} g(x)(1 - x)^\alpha (1 + x)^\beta dx,$$

is 0 for every z, because

$$G(z) = \int_{-1}^{1} e^{zx} g(x)(1-x)^\alpha (1+x)^\beta dx = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-1}^{1} x^n g(x)(1-x)^\alpha (1+x)^\beta dx = 0.$$

If we pick $z = it$, we have $\int_{-\infty}^{\infty} e^{itx} g(x) 1_{(-1,1)}^{(x)} (1-x)^\alpha (1+x)^\beta dx = 0$. This implies that the Fourier transform of $g(x)1_{(-1,1)}^{(x)}(1-x)^\alpha(1+x)^\beta$ is 0. Then $g(x)1_{(-1,1)}^{(x)}(1-x)^\alpha(1+x)^\beta$ is the function 0 in $L^2((-1,1),(1-x)^\alpha(1+x)^\beta dx)$, and thus so is g. In particular, this holds for $g := f - P_E(f)$, which implies that $P_E(f) = f$. This contradicts the fact that $f - P_E(f) \neq 0$.

Therefore the Jacobi polynomials form an orthogonal basis for the space $L^2((-1,1),(1-x)^\alpha(1+x)^\beta dx)$.

Note: The proof that the Laguerre polynomials form an orthogonal basis for the Hilbert space $L^2((0,\infty), x^\alpha e^{-x} dx)$ is similar.

2.7 Example: Hermite polynomials

2.7.1 The three terms recurrence relation

The Hermite polynomials can be defined applying the Rodrigues formula, with $A(x) := 1$, $B(x) := -2x$, and $w(x) := e^{-x^2}$. Then one arrives at

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{2.19}$$

By induction it is easy to check that

$$\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = -2x \frac{d^n}{dx^n} e^{-x^2} - 2n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}. \tag{2.20}$$

Multiplying both sides of (2.20) by $(-1)^n e^{x^2}$ we arrive at the following recurrence relation for the Hermite polynomials:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{2.21}$$

On the other hand, it is also easy to verify

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x), \tag{2.22}$$

and also that

$$H_n'(x) = 2nH_{n-1}(x). \tag{2.23}$$

2.7.2 Orthogonality of the Hermite polynomials

The Hermite polynomials are orthogonal with respect to the function $w(x) = e^{-x^2}$. We will show that

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0 \text{ if } n \neq m.$$

Suppose that $n > m$. Then using integration by parts m times we have

$$\begin{aligned}
\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx &= (-1)^n \int_{-\infty}^{\infty} H_m(x)\frac{d^n}{dx^n}e^{-x^2}dx \\
&= (-1)^{n+1} \int_{-\infty}^{\infty} H_m'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}dx \\
&= (-1)^{n+2} \int_{-\infty}^{\infty} H_m''(x)\frac{d^{n-2}}{dx^{n-2}}e^{-x^2}dx \\
&\vdots \\
&= (-1)^{n+m} \int_{-\infty}^{\infty} H_m^{(m)}(x)\frac{d^{n-m}}{dx^{n-m}}e^{-x^2}dx \\
&= (-1)^{n+m} H_m^{(m)}(x) \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}}e^{-x^2}dx \\
&= (-1)^{n+m} H_m^{(m)}(x)\frac{d^{n-m-1}}{dx^{n-m-1}}e^{-x^2}\Big|_{-\infty}^{\infty} \\
&= 0. \qquad (2.24)
\end{aligned}$$

Now, we will show that

$$\int_{-\infty}^{\infty} (H_n(x))^2 e^{-x^2}dx = n!2^n\sqrt{\pi}.$$

By (2.24), if $n = m$ we arrive at

$$\int_{-\infty}^{\infty} (H_n(x))^2 e^{-x^2}dx = \int_{-\infty}^{\infty} H_n^{(n)}(x)e^{-x^2}dx. \qquad (2.25)$$

By (2.23) we have that $H_n'(x) = 2nH_{n-1}(x)$, and differentiating both side of this equality we obtain $H_n''(x) = 2n \cdot 2(n-1)H_{n-2}(x)$. Applying this process n times we arrive at

$$H_n^{(n)}(x) = 2n \cdot 2(n-1) \cdots 2(1)H_0(x).$$

Since $H_0(x) = 1$, then $H_n^{(n)}(x) = n!2^n$. Substituting in (2.25) we obtain

$$\int_{-\infty}^{\infty} (H_n(x))^2 e^{-x^2}dx = n!2^n \int_{-\infty}^{\infty} e^{-x^2}dx = n!2^n\sqrt{\pi}.$$

Therefore the Hermite polynomials are orthogonal on $L^2(\mathbb{R}, e^{-x^2}dx)$.

Chapter 3

Diffusions and orthogonal polynomials

In this chapter we characterize the density probability function of the diffusion processes: Jacobi, Ornstein-Uhlenbeck and Cox-Ingersoll-Ross. We do so by using orthogonal polynomials. At the end we give an example of a diffusion that does not fit as a classical example: the Brox diffusion.

3.1 The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is solution of the stochastic differential equation

$$dX_t = -X_t dt + dB_t. \tag{3.1}$$

It is known that the infinitesimal operator associated with this process is

$$Lf(x) := \frac{1}{2}f''(x) - xf'(x). \tag{3.2}$$

Note that the Hermite polynomials are solution of $LH_n(x) = -nH_n(x)$. Remember that the domain of L are the functions f such that f and Lf vanish at infinity. Then the Hermite polynomials do not belong to the domain of L.

But if we consider the space $L^2(\mathbb{R}, e^{-x^2}dx) := \left\{ f : \int_{-\infty}^{\infty} f^2(x)e^{-x^2}dx < \infty \right\}$, then the Hermite polynomials are functions vanishing at infinity on $L^2(\mathbb{R}, e^{-x^2}dx)$. Therefore on $L^2(\mathbb{R}, e^{-x^2}dx)$ it makes sense to consider $LH_n(x) = -nH_n(x)$.

We known that the Hermite polynomials form an orthogonal basis of the Hilbert space $L^2(\mathbb{R}, e^{-x^2}dx)$ with the inner product given by $\langle f, g\rangle := \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx$. Also these polynomials satisfy $\int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}dx = n!2^n\sqrt{\pi}$. See chapter 2.

Then the polynomials $\frac{H_n}{\sqrt{n!2^n\sqrt{\pi}}}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}, e^{-x^2}dx)$ and they satisfy $L\left(\frac{H_n(x)}{\sqrt{n!2^n\sqrt{\pi}}}\right) = -n\left(\frac{H_n(x)}{\sqrt{n!2^n\sqrt{\pi}}}\right)$.

We want to express the transition function of the Ornstein-Uhlenbeck process in terms of the Hermite polynomials. To this end, note that $C_0(\mathbb{R}) \subseteq L^2(\mathbb{R}, e^{-x^2}dx)$, and so any $f \in C_0(\mathbb{R})$ can be written as a linear combination of the orthonormal basis of $L^2(\mathbb{R}, e^{-x^2}dx)$. Therefore we arrive to

$$f = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n, \tag{3.3}$$

where $\phi_n := \dfrac{H_n}{\sqrt{n! 2^n \sqrt{\pi}}}$, and $\langle f, \phi_n \rangle := \displaystyle\int_{-\infty}^{\infty} f(x)\phi_n(x)e^{-x^2}dx$.

Since ϕ_n is an eigenfunction of L corresponding to the eigenvalue $\lambda_n := -n$, i.e. $L\phi_n = -n\phi_n$, then one can check that $u_f(t,x) := e^{-nt}\langle f, \phi_n \rangle \phi_n(x)$ solves the equation

$$\frac{\partial u_f(t,x)}{\partial t} = Lu_f(t,x), \;\; u(0,x) = \langle f, \phi_n \rangle \phi_n(x). \tag{3.4}$$

If $u_f(t,x)$ is a superposition of such functions, i.e. $u_f(t,x) := \displaystyle\sum_{n=0}^{\infty} e^{-nt}\langle f, \phi_n \rangle \phi_n(x)$, then $u_f(t,x)$ solves the equation

$$\frac{\partial u_f(t,x)}{\partial t} = Lu_f(t,x), \;\; u_f(0,x) = f(x). \tag{3.5}$$

On the other hand, applying the properties of semigroups we obtain that $P_t f(x)$ also satisfies the equation (3.5). Hence $P_t f(x) = u_f(t,x)$, because the solution is unique. (See e.g [9, p.500]).

Since $P_t f(x) := E(f(X_t)|X_0 = x) = \displaystyle\int_{-\infty}^{\infty} f(y)p(t,x,dy)$, using the dominated convergence theorem we arrive at

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)p(t,x,dy) &= \sum_{n=0}^{\infty} e^{-nt}\langle f, \phi_n \rangle \phi_n(x) \\ &= \sum_{n=0}^{\infty} e^{-nt} \left(\int_{-\infty}^{\infty} f(y)\phi_n(y)e^{-y^2}dy \right) \phi_n(x) \\ &= \int_{-\infty}^{\infty} f(y) \left(\sum_{n=0}^{\infty} e^{-nt}\phi_n(y)\phi_n(x)e^{-y^2} \right) dy. \end{aligned}$$

Then $p(t,x,dy)$ has a density $p(t,x,y)$, and it is given by

$$p(t,x,y) = e^{-y^2} \sum_{n=0}^{\infty} \frac{e^{-nt}}{n! 2^n \sqrt{\pi}} H_n(y)H_n(x). \tag{3.6}$$

These same arguments can be applied to diffusions where other orthonormal polynomials are eigenfunctions of the infinitesimal operator. This can be done by taking into consideration a suitable Hilbert space where the polynomials form an orthonormal basis.

3.2 The Cox-Ingersoll-Ross process

The Cox-Ingersoll-Ross process is solution of the stochastic differential equation

$$dX_t = (1 - X_t)dt + \sqrt{2X_t}dB_t.$$

The infinitesimal operator associated with this process is

$$Lf(x) := xf''(x) + (1 - x)f'(x).$$

Also, the Laguerre polynomials $L_n,\ n = 0, 1, 2...$ in (2.8), are eigenfunctions of L associated with the eigenvalues $\lambda_n := -n$.

If we consider the space $L^2((0, \infty), e^{-x}dx)$, with the inner product $\langle f, g\rangle := \int_0^{\infty} f(x)g(x)e^{-x}dx$, then the Laguerre polynomials form an orthogonal basis, and multiplying by a suitable constant C_n we obtain that $\varphi_n := C_nL_n,\ n = 0, 1, 2...$, form an orthonormal basis of the Hilbert space $L^2((0, \infty), e^{-x}dx)$.

Applying the same arguments used in section 3.1, we obtain that the Cox-Ingersoll-Ross process has a density function, which is given by

$$p(t, x, y) = e^{-y} \sum_{n=0}^{\infty} e^{-nt}C_n^2 L_n(y)L_n(x).$$

3.3 The Jacobi process

The Jacobi process is solution of the stochastic differential equation

$$dX_t = [(\beta - \alpha) - (\alpha + \beta + 2)X_t]dt + \sqrt{2(1 - X_t^2)}\, dB_t.$$

In this case, the associated infinitesimal operator is

$$Lf(x) := (1 - x^2)f''(x) + [(\beta - \alpha) - (\alpha + \beta + 2)x]f'(x).$$

And the eigenfunctions of L are the Jacobi polynomials $J_n, n = 0, 1, 2...$ in (2.7), associated with the eigenvalues $\lambda_n := -n(n + \alpha + \beta + 1)$.

Consider $L^2\left((-1, 1), (1 - x)^{\alpha}(1 + x)^{\beta}dx\right)$, with $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)(1 - x)^{\alpha}(1 + x)^{\beta}dx$. Then for suitable constants K_n, the functions $\psi_n := K_nJ_n, n = 0, 1, 2...$ form an orthonormal basis of the Hilbert space $L^2\left((-1, 1), (1 - x)^{\alpha}(1 + x)^{\beta}dx\right)$.

Now, apply the same arguments as in the section 3.1, we obtain that the Jacobi process has a density function given by

$$p(t, x, y) = (1 - y)^{\alpha}(1 + y)^{\beta} \sum_{n=0}^{\infty} e^{-n(n+\alpha+\beta+1)t}K_n^2 J_n(y)J_n(x).$$

3.4 The Brox process

As part of the motivation of this thesis, we will show an example of one diffusion whose associated infinitesimal operator is not of the classical form $Lf(x) := \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$. This diffusion is called the Brox process (see e.g. [11]).

Consider informally the equation

$$dX_t = -\frac{1}{2}W'(X_t)dt + dB_t, \tag{3.7}$$

where $B := \{B_t : t \geq 0\}$ is the standard Brownian motion, and $W := \{W(x) : x \in \mathbb{R}\}$ is a two sided Brownian motion, and they are both independent of each other. Here W' denotes the derivative of W, sometimes called the white noise.

When leaving fixed a trajectory of W, the equation (3.7) can be interpreted as a stochastic differential equation. This way of thinking corresponds to considering the process $X := \{X_t : t \geq 0\}$ associated with the infinitesimal operator

$$Lf(x) := \frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{df(x)}{dx}\right). \tag{3.8}$$

This corresponds to considering the scale function

$$s(x) := \int_0^x e^{W(y)}dy, \tag{3.9}$$

and the speed measure

$$m(A) := \int_A 2e^{-W(y)}dy, \quad \text{for Borel sets } A \subseteq \mathbb{R}. \tag{3.10}$$

Let us check so:

$$\begin{aligned}
\frac{d}{dm}\frac{d}{ds}f(x) &= \frac{d}{dm}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{s(x+h)-s(x)}\right) \\
&= \frac{d}{dm}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0}\frac{1}{\frac{1}{h}\int_x^{x+h}e^{W(y)}dy}\right) \\
&= \frac{d}{dm}\left(e^{-W(x)}f'(x)\right) \\
&= \lim_{h\to 0}\frac{f'(x+h)e^{-W(x+h)}-f'(x)e^{-W(x)}}{m((x,x+h])} \\
&= \lim_{h\to 0}\frac{f'(x+h)e^{-W(x+h)}-f'(x)e^{-W(x)}}{\int_x^{x+h}2e^{-W(y)}dy} \\
&= \lim_{h\to 0}\frac{f'(x+h)e^{-W(x+h)}-f'(x)e^{-W(x)}}{h}\cdot\lim_{h\to 0}\frac{1}{\frac{1}{h}\int_x^{x+h}2e^{-W(y)}dy} \\
&= \frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}f(x)\right)\cdot\frac{e^{W(x)}}{2} \\
&= Lf(x).
\end{aligned} \tag{3.11}$$

Then one considers rigourously the operator $\frac{d}{dm}\frac{d}{ds}f$, where s is the scale function defined in (3.9), and m is the speed measure defined in (3.10). We want to see that the process X associated with this operator is a diffusion, when leaving fixed W.

To this end, we use a result of K. Itô and H.P.McKean (see e.g. [5, p.165]), where we can reconstruct a process Y in natural scale through the speed measure m_Y and the local time. The procedure is done in the following way. Observe that

$$X_t = B_{T_t^{-1}}, \tag{3.12}$$

where

$$T_t := \int_{-\infty}^{\infty} L_t(x) m_Y(dx), \tag{3.13}$$

and $L_t(y)$ is the so-called local time (see e.g. [7, p.32]), which can be calculated as

$$L_t(y) := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{x-\epsilon < B_s < x+\epsilon\}} ds. \tag{3.14}$$

In our case, the process X is not in natural scale, but the new process $Y_t := s(X_t)$ is in natural scale [3]. Now, by applying the Itô formula (see e.g. [4, p.149]), we find that $s(X_t)$ satisfies the equation

$$ds(X_t) = e^{W(X_t)} dB_t = e^{W(s^{-1}(s(X_t)))} dB_t.$$

Thus

$$dY_t = 0dt + e^{W(s^{-1}(Y_t))} dB_t. \tag{3.15}$$

Leaving fixed a trajectory of W, the equation (3.15) can be interpreted as a stochastic differential equation. Then the process Y is associated with the infinitesimal operator

$$L_Y f(x) = e^{W(s^{-1}(x))} f''(x). \tag{3.16}$$

By the formula (1.56) and (3.16), we obtain that the speed measure associated with the process Y_t is

$$m_Y(A) = \int_A 2e^{-W(s^{-1}(x))} dx,$$

for any Borel set A.

Then applying the reconstruction of K. Itô and and H.P.McKean one has

$$s(X_t) = B_{T_t^{-1}},$$

with T_t as in (3.13) .

But $B_{T_t^{-1}}$ is a diffusion (see e.g. [6, p.277]), and so $s(X_t)$ is a diffusion. Hence since s is continuous and strictly increasing, $X_t = s^{-1}(B_{T_t^{-1}})$ is a diffusion process for each trajectory W. One could write X^W to emphasize that the diffusion process X is conditioned to W.

Let us define rigourously the process X when it is not conditioned to W. We leave fixed a trajectory W, then we denote by P_W the corresponding probability measure of X^W over $C([0,\infty))$, and if μ

is the probability measure over $C(\mathbb{R})$ associated to the Brownian motion $W : \Omega \to C(\mathbb{R})$, then by the law of total probability we obtain the formula for the corresponding probability measure of X without fixing W:

$$P(C) = \int_{\Omega} P_{W(\omega)}(C)\mu(d\omega),$$

for any measurable set C in $C([0, \infty))$.

Bibliography

[1] B. Oksendal. (2003). *Stochastic Differential Equations*. Springer-Verlag.

[2] C. Tudor. (2002). *Procesos Estocasticos*. Sociedad Matemática Mexicana.

[3] D. Revuz, M. Yor. (1999). *Continuous Martingales and Brownian Motion*. Springer-Verlag.

[4] F.C. Klebaner. (2005). *Introduction to Stochastic Calculus with Applications*. Imperial College Press.

[5] K. Ito, H.P. Mckean. (1965). *Diffusion Processes and their Sample Paths*. Springer-Verlag

[6] L.Rogers, D. Williams. (1987). *Diffusions, Markov Processes, and Martingales*. Wiley.

[7] M.B. Marcus, J. Rosen. (2006). *Markov Processes, Gaussian Processes, and Local Times*. Cambridge University Press.

[8] P. Mandl. (1968). *Analytical Treatment of one Dimensional Markov Processes*. Academia, Publishing House of the Czechoslovak Academy of Sciences.

[9] R.N. Bhattacharya, E. C. Waymire. (1990). *Stochastic Processes with Applications*. Wiley.

[10] S. Karlin, H.M. Taylor. (1981). *A Second Course in Sthochastic Processes*. Academic Press.

[11] Th. Brox. (1986). A one dimensional diffusion process in a Wiener medium. *The Annals of Probability*, Vol.14. No.4, 1206-1218.

[12] T.S. Chihara (1978). *An Introduction to Orthogonal Polynomials*. Gordon and Breach.

[13] W. Schoutens. (2000). *Stochastic Processes and Orthogonal Polynomials*. Springer-Verlag.