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Arithmetical Structures of Graphs and the Algebraic co-rank of Threshold Graphs

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Estructuras Aritméticas de Gráficas y el corrango Algebraico de Gráficas Umbral

Tesis que presenta

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Dedicado a mi familia, por todo su apoyo siempre

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Resumen

La matriz Laplaciana de una gráfica G se define como $L(G) = D(G) - A(G)$ donde $A(G)$ es la matriz de adyacencia de G y $D(G)$ su matriz de grados. El grupo crítico de G, denotado por $K(G)$, es la parte de torsión del cokernel de $L(G)$. En ésta tesis tratamos con dos generalizaciones de dicha definición. Definimos la Laplaciana generalizada de una gráfica G con n vertices como la matriz $L(G, X_G) = D(X_G) - A(G)$ donde X_G es un conjunto de n variables indeterminadas indexadas por los vertices de G. Como $L(G, X_G)$ es una matriz con entradas en $\mathbb{Z}[X_G]$, los ideales determinantes de $L(G, X_G)$ son ideales sobre $\mathbb{Z}[X_G]$ los que llamamos ideales críticos de G, nuestra primera generalización de grupos críticos. En ésta tesis estudiamos los ideales críticos de las gráficas umbral.

Ahora definamos la segunda generalización, una gráfica aritmética es una tripleta $(G, \mathbf{d}, \mathbf{r})$, dada por una gráfica G y un par de vectores $\mathbf{d}, \mathbf{r} \in \mathbb{N}^V$ tales que

$$\gcd(\mathbf{d}_v \in V(G)) = 1 \text{ y } L(G, \mathbf{d})\mathbf{r}^t = \mathbf{0}^t$$

Dada una gráfica aritmética $(G, \mathbf{d}, \mathbf{r})$, decimos que el par $(\mathbf{d}, \mathbf{r})$ es una *estructura aritmética* de G, nuestra segunda generalización de un grupo crítico de una gráfica, y en ésta tesis de maestría estudiamos este concepto y como calcular las estructuras aritméticas para una gráfica dada.

Para resumir, en esta tesis, se realiza un primer acercamiento al estudio de ideales críticos de gráficas umbrales usando, más precisamente, una familia base de gráficas umbrales $\{T_n\}_{n\geq 1}$. Por otro lado, trabajaremos con el cáclculo de las estructuras aritméticas de una gráfica conexa en general.

Abstract

The Laplacian matrix of a graph G is defined as $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix of G and $D(G)$ his degree matrix. The critical group of G, denoted $K(G)$, is the torsion part of the cokernel of $L(G)$. In this thesis we deal with two generalizations of this definition. We define the generalized Laplacian matrix of a graph G with n vertices as the matrix $L(G, X_G) = D(X_G) - A(G)$ where X_G is the set of n undetermined variables indexed by the vertices of G. Since $L(G, X_G)$ is a matrix with entries over $\mathbb{Z}[X_G]$, the determinantal ideals of $L(G, X_G)$ are ideals on $\mathbb{Z}[X_G]$ which we call critical ideals of G, our first generalization of critical groups. In this master thesis we study the critical ideals of Threshold graphs.

Now let us define the second generalization, an arithmetical graph is a triplet $(G, \mathbf{d}, \mathbf{r})$, given by a graph G and a pair of vectors $\mathbf{d}, \mathbf{r} \in \mathbb{N}^V$ such that

$$\gcd(\mathbf{d}_v \in V(G)) = 1 \text{ and } L(G, \mathbf{d})\mathbf{r}^t = \mathbf{0}^t$$

Given an arithmetical graph $(G, \mathbf{d}, \mathbf{r})$, we say that the pair $(\mathbf{d}, \mathbf{r})$ is an *arithmetical structure* of G, our second generalizing of a critical group of a graph, and in this master thesis we study this concept and how to compute the arithmetical structures for a given graph.

For summarize, in this work, we do a first approach to the study of critical ideals of Threshold Graphs using, more precisely, a base family of Threshold Graphs $\{T_n\}_{n \geq 1}$. On the other hand we work with the calculation of arithmetic structures of a general connected graph.

Contents

Chapter 1

Introduction

There are two main purposes on this master thesis. One is the study of arithmetic structures of graphs, their properties and how to compute them for a given graph. The other purpose is to give a first approach to the critical ideals of threshold graphs.

Given a multigraph $G = (V, E)$, its generalized Laplacian matrix is given by

$$L(G, X_G)_{u,v} = \begin{cases} x_u & \text{if } u = v, \\ -m_{u,v} & \text{if } u \neq v. \end{cases}$$

where $m_{u,v}$ is the number of arcs between the vertices u and v and

$$X_G = \{x_u \,|\, u \in V(G)\}$$

is a set of undetermined variables indexed by the vertices of G. For any $d \in \mathbb{Z}^V$, let $L(G, d)$ be the integer matrix that result by setting $x_u = \mathbf{d}_u$ on $L(G, X_G)$.

Definition 1.0.1. *An arithmetical graph is a triplet* $(G, \mathbf{d}, \mathbf{r})$*, given by a graph* G *and a pair of vectors* $\mathbf{d}, \mathbf{r} \in \mathbb{N}^V$ *such that* $\gcd(\mathbf{d}_v \in V(G)) = 1$ *and*

$$L(G, \mathbf{d})\, \mathbf{r}^t = \mathbf{0}^t$$

Given an arithmetical graph $(G, \mathbf{d}, \mathbf{r})$ *we say that the pair* $(\mathbf{d}, \mathbf{r})$ *is an arithmetical structure of* G*.*

This concept was introduced by Lorenzini [11].

Theorem 1.0.2 (Lemma 1.6 [11]). *There exist only finitely many arithmetical structures on any connected simple graph.*

Given the theorem above, it makes sense to ask about the description of the arithmetical structures of graphs. Consider

$$\mathcal{A}(G) = \left\{ (\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^{V(G)} \times \mathbb{N}_+^{V(G)} \;\middle|\; (\mathbf{d}, \mathbf{r}) \text{ be an arithmetical structure of } G \right\}.$$

Given $(\mathbf{d}, \mathbf{r}) \in \mathcal{A}(G)$, let

$$K(G, \mathbf{d}, \mathbf{r}) = \ker(\mathbf{r}^t) \big/ \mathrm{Im} L(G, \mathbf{d})^t$$

be the critical group of $(G, \mathbf{d}, \mathbf{r})$, which generalize the concept of critical group of G introduced on [3]. On the other side, we can define our second object of interest.

Definition 1.0.3. *Given a graph G with n vertices and $1 \leq i \leq n$, let*

$$I_i(G, X_G) = \langle \mathrm{minors}_i(L(G, X_G)) \rangle \subseteq \mathcal{P}[X_G]$$

be the i-th critical ideal of G.

Note that in general the critical ideals depend on the base ring $\mathcal{P}$, in this work we are mainly interested on the case when $\mathcal{P} = \mathbb{Z}$. By convention,

$$I_i(G, X_G) = \langle 1 \rangle \text{ if } i \leq 0 \text{ and } I_i(G, X_G) = \langle 0 \rangle \text{ if } i > n.$$

Clearly $I_n(G, X_G)$ is a principal ideal generated by the determinant of the generalized Laplacian matrix. The following is an important invariant that comes from the definition of critical ideals.

Definition 1.0.4. *Let $G = (V, E)$ be a graph. The algebraic co-rank, denoted by $\gamma(G)$ of G is the maximum integer i such that $I_i(G, X_G)$ is trivial.*

Note that for every H induced subgraph of G we have that $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $1 \leq i \leq |V(H)|$, and in consequence $\gamma(G) \leq \gamma(H)$. In particular we are interested on the critical ideals of the class of *threshold graphs.*

Definition 1.0.5. *A graph $T = (V, E)$ is called a threshold graph if it can be constructed from K_1 (the trivial graph) by any finite sequence of the following two graph operations:*

i) adding an isolated vertex v' to the graph.

ii) add a dominant vertex v', i.e., a vertex adjacent to every other vertex of the graph.

To be more specific we work with a special class of threshold graphs.

Definition 1.0.6. *Let T_n be the threshold graph define as follows*

$$T_n = \begin{cases} K_2, & \text{if } n = 1, \\ c(T_{n-1} + v), & \text{if } n > 1 \ . \end{cases}$$

where $c(G)$ is the cone graph (see definition 2.2.4) of G.

The family of graphs $\{T_n\}_{n\geq 1}$ described above is a type of "basis" for all threshold graphs, in the sense that every such graph can be built from some graph T^n by exploding vertices, this is, by adding weak and strong twin vertices to it.
Threshold graphs have an important role in graph theory and in several applied areas such as psychology, computer science, scheduling theory, etc [12], They where studied and "discovered" simultaneously by Chavatal and Hammer (whom coined the name "threshold graphs"); Henderson and Zalcstein, discovered the same graphs and called them **PV-chunk definable graphs**, motivated by applications in synchronizing parallel processes [9]. Ecker and Zaks discovered these graphs independently and investigated them for their use in graph labeling as applied to open shop scheduling.

Chapter 2

Preliminars

2.1 Commutative Algebra

First of all, we need to introduce the ideas and concepts of commutative algebra that its use on this work.

2.1.1 Gröbner Basis

The Theory of Gröbner basis deals, generally, with ideals in a polynomial ring over a field. However, in this work we manage with a polynomial ring over the integers.
We recall some basic concepts on Gröbner basis, for more details see [12]. First, let $\mathcal{P}$ be a principal ideal domain. A *monomial order* or *order term* in the polynomial ring $R = \mathcal{P}[x_1, ..., x_n]$ is a total order $\prec$ in the set of monomials of R such that:

i) $1 \prec \mathbf{x}^\alpha$ for all $\mathbf{0} \neq \alpha \in \mathbb{N}^n$, and

ii) if $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$, then $\mathbf{x}^{\alpha+\gamma} \prec \mathbf{x}^{\beta+\gamma}$, for all $\gamma \in \mathbb{N}^n$,

where $\mathbf{x}^\alpha = x_1^{\alpha_1}, \cdots x_n^{\alpha_n}$.
Now, given an order term $\prec$ and $p \in \mathcal{P}[X]$, let $\mathrm{lt}(p)$, $\mathrm{lp}(p)$, $\mathrm{lc}(p)$ be the *leading term*, the *leading power*, and *the leading coefficient* of p, respectively. Given a subset S of $\mathcal{P}[X]$ its leading term ideal of S is the ideal

$$Lt(S) = \langle\, lt(s) | s \in S \rangle.$$

A finite set of nonzero polynomials $B = \{\, b_1, ..., b_s\}$ of an ideal I is called a Gröbner basis of I with respect to an order term $\prec$ if $Lt(B) = Lt(I)$. Moreover, it is called

reduced if $\mathrm{lc}(b_i) = 1$ for all $1 \leq i \leq s$ and no nonzero term in b_i is divisible by any $\mathrm{lp}(b_j)$ for all $1 \leq i \neq j \leq s$.
A good characterization of Gröbner basis is given in terms of the so called S-polynomials.

Definition 2.1.1. *Let $f, f^{'}$ be polynomials in $\mathcal{P}[X]$ and B be a set of polynomials in $\mathcal{P}[X]$. We say that f reduces strongly to f' modulo B if*

- $\mathrm{lt}(f') \prec \mathrm{lt}(f)$, *and*
- *there exists $b \in B$ and $h \in \mathcal{P}[X]$ such that $f' = f - hb$.*

Moreover, if $f^* \in \mathcal{P}[X]$ can be obtained from f in a finite number of reductions, we write $f \to_B f^*$.

That is, if $f = \sum_{j=1}^{t} p_{i_j} b_{i_j} + f^*$ with $p_{i_j} \in \mathcal{P}[X]$ and $\mathrm{lt}(p_{i_j} b_{i_j}) \neq \mathrm{lt}(p_{i_k} b_{i_k})$ for all $j \neq k$, then $f \to_B f^*$.
Now, given f and g polynomials in $\mathcal{P}[X]$, their S-polynomial, denoted by $S(f, g)$, is given by

$$S(f,g) = \frac{c}{c_f}\frac{X}{X_f} f - \frac{c}{c_g}\frac{X}{X_g} g,$$

where $X_f = \mathrm{lt}(f)$, $c_f = \mathrm{lc}(f)$, $X_g = \mathrm{lt}(g)$, $c_g = \mathrm{lc}(g)$, $X = \mathrm{lcm}(X_f, X_g)$, and $c = \mathrm{lcm}(c_f, c_g)$.
The next Lemma, known as Buchberger's criterion, gives us a useful criterion for checking whether a set of generators of an ideal is a Gröbner basis.

Lemma 2.1.2. *Let I be an ideal of polynomials over a PID and B be a generating set of I. Then B is a Gröbner basis for I if and only if $S(f, g) \to_B 0$ for all $f \neq g \in B$.*

The *degree lexicographic order* is defined as follows.

Definition 2.1.3. *Let $P[x_1, ..., x_n]$, $\alpha\,, \beta \in \mathbb{N}^n$; then $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$ if*

i) $\alpha_1 + \cdots + \alpha_n < \beta_1 + \cdots + \beta_n$,

ii) or $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$ *and exists* $i = 1, ..., n$ *such that*

$$\alpha_1 = \beta_1\,,\ \alpha_2 = \beta_2\,, \ldots,\ \alpha_{i-1} = \beta_{i-1}\ \text{ and }\ \alpha_i < \beta_i.$$

Which is clearly a monomial order.

2.1.2 Smith Normal Form

Given a matrix $L \in M_m(\mathbb{Z})$, the cokernel of L, denoted by $coker(L)$, is defined as

$$coker(L) = \mathbb{Z}^m / L\,\mathbb{Z}^m.$$

Since $\mathbb{Z}$ is a Bézout domain, L is equivalent to a unique diagonal matrix

$$D = \operatorname{diag}(d_1, \dots, d_k, 0, \dots, 0) \text{ with } d_i \in \mathbb{Z}_{>0},\ i = 1, \dots, k \text{ and } d_1 | \cdots | d_k.$$

Let $U, V \in GL_n(\mathbb{Z})$ such that $ULV = D$, since V is invertible $U(L\mathbb{Z}^n) = D\mathbb{Z}^n$, and since U is invertible

$$coker(L) = \mathbb{Z}^m / L\mathbb{Z}^m \xrightarrow{\cong} coker(D) = \mathbb{Z}^m / D\mathbb{Z}^m \xrightarrow{\cong} T \oplus \mathbb{Z}^{n-k},$$

where $T = \mathbb{Z}^k / \operatorname{diag}(d_1, \dots, d_k)\,\mathbb{Z}^k$ is a finite group. T and $\mathbb{Z}^{n-r}$ are called the $torsion$ and the $free$ parts of $coker(L)$ respectively. The unique diagonal matrix D is called the *Smith Normal Form* of L.
We understand for r-minor the determinant of an r-square submatrix. Then the Smith Normal Form $\operatorname{diag}(d_1, \dots, d_k, 0, \dots, 0)$ of L is characterized by

$$k = \max\{i | \operatorname{minors}_i(L) \neq 0\}, \text{ and } \ d_i = \gcd(\operatorname{minors}_i(L)) \text{ for each } 1 \leq i \leq k,$$

where $\operatorname{minors}_i(L)$ denotes the set of i-minors of the matrix L.

2.2 Graph Theory

2.2.1 The Laplacian Matrix of a Graph

In this section we define and present some properties of the Laplacian matrix of a graph as well as the critical group of a graph. With that purpose we start defining the concept of a graph that we are going to need for this section.

Definition 2.2.1. *A graph is a pair $G = (V, E)$ where V is a finite set and E is a finite collection of non-ordered pairs of elements of V. We call the elements of V vertices, and the elements of E are called edges. The order of G, denoted by $|G|$, is the number of vertices of G.*

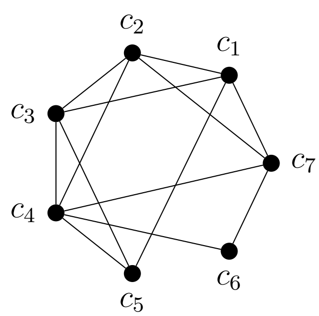
Let $G = (V, E)$ be a graph and $(x, y) \in E$, the vertices $x, y \in V$ are called ends of the edge (x, y); if $x = y$, (x, y) is called a *loop*. We say that two vertices $x, y \in V$ are *adjacent* if $(x, y) \in E$. To the number of edges that has a vertex x as an end is called the degree of x and denoted by $d(x)$ or $d_G(x)$.

Example 2.2.2. *We can use a drawing of the graph for its description, for example if $G = (V, E)$ is the graph where*

$$V = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and

$E = \{ (v_1, v_2), (v_1, v_2), (v_1, v_3), (v_1, v_5), (v_1, v_7), (v_2, v_3), (v_2, v_4), (v_2, v_7), (v_3, v_4), (v_3, v_5), (v_4, v_5), (v_4, v_6), (v_4, v_7), (v_6, v_7)\}$, *Then the following is a drawing of the graph G*



Note that a pair of vertices can have multiple edges, so for $x, y \in V$ is useful to denote the number of edges between them by $m_{x,y}$. We call G a simple graph if it has no loops neither multiple edges, and for now on with graph we refer to simple graphs unless contrary is stated.

Definition 2.2.3. *Let $G = (V, E)$ be a graph, G is called t-regular ($t \in \mathbb{N}$) if every vertex of G has degree equal to t.*

Definition 2.2.4. *Let $G = (V, E)$ be a graph, we define the cone graph of G, denoted by $c(G)$, as the graph with*

$$V(c(G)) = V \cup \{u\} \text{ and } E(c(G)) = E \cup \{(u, v) \mid v \in V\}$$

For more information on general graph theory see [8]. Let us now define the Laplacian matrix of a graph.

Definition 2.2.5. *Let $G = (V, E)$ be a graph with $|G| = n$, we define the Laplacian matrix of G, by*

$$L(G)_{u,v} = \begin{cases} d_G(x) & \text{if } u = v, \\ -m_{u,v} & \text{if } u \neq v. \end{cases}$$

note that $L(G) \in \mathbb{Z}^{n^2}$

The critical group of a graph

Now we define the critical group of a graph

Definition 2.2.6. *Let $G = (V, E)$ be a graph with n vertices, then the critical group, denoted by $K(G)$, of G is torsion part of the cokernel of the Laplacian matrix $L(G)$, that is*

$$\mathbb{Z}^n \Big/ \mathrm{Im} L(G)^t = \mathbb{Z} \oplus K(G)$$

Since $L(G)$ has rank $n-1$ the Smith normal form of $L(G)$ has the form $\mathrm{diag}(f_1, f_2, \ldots, f_{n-1}, 0)$, and

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}}.$$

The integers $f_1, \ldots, f_{n-1}$ are called *invariant factors* of $K(G)$. Since $\mathbb{Z}_1$ is the trivial group, if $f_k = 1$ for some $K = 1, \ldots, n-1$ then, we say that $K(G)$ hast at leats k invariant factors and if $f_{k+1} \neq 1$ then, we say that $K(G)$ has exactly k invariant factors. The following is an important known result.

Theorem 2.2.7. *If G is a connected graph, then $K(G)$ has order number $\tau(G)$ of spanning trees of G.*

2.2.2 Critical Ideals of Graphs

In this section we generalize the concept of Laplacian Matrix and define some new objects called critical ideals.

Definition 2.2.8. *Let $G = (V, E)$ be a graph with $|G| = n$, we define the generalized Laplacian matrix of G, by*

$$L(G, X_G)_{u,v} = \begin{cases} x_u & \text{if } u = v, \\ -m_{u,v} & \text{if } u \neq v. \end{cases}$$

So now we can define the critical ideals of a graph

Definition 2.2.9. *Given a graph G with n vertices, for $1 \leq i \leq n$, let*

$$I_i(G, X_G) = \langle minors_i(L(G, X_G)) \rangle \subseteq P[X_G]$$

be the i-th critical ideal of G.

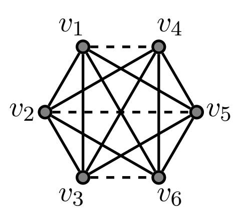
Definition 2.2.10. *Let $G = (V, E)$ be a graph. We define the algebraic co-rank of G as follows*

$$\gamma(G) = \max\{i \mid I_i(G, X_G) = \langle 1 \rangle\}$$

the maximum integer i such that $I_i(G, X_G)$ is trivial.

Now that $\gamma(G) \leq n-1$, since $I_n(G, X_G) = \langle \det(L(G, X_G)) \rangle \neq \langle 1 \rangle$. the algebraic co-rank of a graph is closely related to the combinatorial properties of the graph. For instance, if H is an induced subgraph of G, then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $1 \leq i \leq |V(H)|$, therefore $\gamma(G) \leq \gamma(H)$.
Now, we present an example that illustrates the concept of critical ideal.

Example 2.2.11. *Let H be the complete graph with six vertices minus the perfect matching formed by the edges $M_3 = \{v_1v_4, v_2v_5, v_3v_6\}$ and $\mathcal{P} = \mathbb{Z}$. Then the following is a drawing of H,*



and its generalized Laplacian matrix is

$$L(H) = \begin{pmatrix} x_1 & -1 & -1 & 0 & -1 & -1 \\ -1 & x_2 & -1 & -1 & 0 & -1 \\ -1 & -1 & x_3 & -1 & -1 & 0 \\ 0 & -1 & -1 & x_4 & -1 & -1 \\ -1 & 0 & -1 & -1 & x_5 & -1 \\ -1 & -1 & 0 & -1 & -1 & x_6 \end{pmatrix}$$

Using any algebraic system, it is not difficult to see that $I_i(H,X) = \langle 1\rangle$ *for* $i = 1,2$ *and*

$$I_i(H,X) = \begin{cases} \langle 2, x_1, x_2, x_3, x_4, x_5, x_6 \rangle & \text{if } i = 3,\\ \langle \{x_r x_s \,|\, v_r v_s \in E(H)\} \cup \{2x_r + 2x_s + x_r x_s \,|\, v_r v_s \notin E(H)\} \rangle & \text{if } i = 4,\\ \langle \{x_k x_l(x_r + x_s + x_r x_s) \,|\, (r,s,k,l) \in S(H)\} \cup \{p_{(r,s,k,l)} \,|\, v_r v_s, v_k v_l \notin E(H)\} \rangle & \text{if } i = 5,\\ \langle x_1 x_2 x_3 x_4 x_5 x_6 - \sum_{(r,s,k,l)\in S(H)} x_r x_s x_k x_l - 2\sum_{(r,s,k)\in T(H)} x_r x_s x_k \rangle & \text{if } i = 6, \end{cases}$$

where

$$S(H) = \{(r,s,k,l) \,|\, v_r v_s \notin E(H),\ v_k v_l \in E(H),\ \text{and } \{i,j\} \cap \{k,l\} = \emptyset\},$$

$T(H)$ *are the triangles of* H*, and*

$$p_{(r,s,k,l)} = (x_r + x_s)(x_k + x_l + x_k x_l) + (x_k + x_l)(x_r + x_s + x_r x_s).$$

Note that the expressions of the critical ideals of H *depend heavily on their combinatorics and note that the algebraic co-rank of* H *then is* 2.

Critical Ideals of Graphs with Twin Vertices

In this section we briefly describe the critical ideals of graphs with twin vertices, for this, we define what we mean with *twin vertices*.

Definition 2.2.12. *Given a graph* $G = (V,E)$ *and a vertex* $v \in V$*, the duplication* $\mathbf{d}(G,v)$ *of* v *is the graph obtained from* G *by adding to* V *a new vertex* x *and the arcs* $xu : u \in N(v)$*; and the replication of* v $\mathbf{r}(G,v)$ *is the graph given by* $\mathbf{d}(G,v)$ *with the extra edge* $e = xv$.

So, the vertex x, in the above definition of duplication and replication is called a weak or strong twin vertex of v, respectively.
Now, let $\mathbf{d} \in \mathbb{Z}^{|V|}$, then $G^{\mathbf{d}}$ is the graph obtained from "duplicating" the vertex v, $\mathbf{d}_v$ times, if $\mathbf{d}_v > 0$ and "replicating" it $-\mathbf{d}_v$ times if $\mathbf{d}_v < 0$.
Given a graph G and $\delta \in \{0,1,-1\}^{|V|}$, let

$$\tau_\delta(G) = \{ G^{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{|V|} \text{ such that } \operatorname{supp}(\mathbf{d}) = \delta \}$$

where

$$\operatorname{supp}(\mathbf{d})_v = \begin{cases} -1, & \text{if } \mathbf{d}_v < 0\\ 1, & \text{if } \mathbf{d}_v > 0\\ 0, & \text{otherwise} \end{cases}$$

With what we know about **critical ideals of graphs with twin vertices [1]**, we have then the following corollary:

Corollary 2.2.13. *If G is a multigraph with n vertices, then $\gamma\,(G^{\mathbf{d}}) \leq n$ for all $\mathbf{d} \in \mathbb{Z}^n$. Furthermore $\gamma\,(G^{\mathbf{d}}) = \gamma\,(G^{\mathrm{supp}(\mathbf{d})})$.*

Proof. Let $\gamma = \gamma\,(G^{\delta})$ and $\mathbf{d} \in \mathbb{Z}^n$ such that $\mathrm{supp}(\mathbf{d}) = \delta$. Then

$I_{\gamma+1}\,(G^{\delta}, X) \subseteq \langle\, \{x_{v^0}, x_{v^1} : \mathbf{d}_v = 1\,\}\,, \{x_{v^0+1}, x_{v^1+1} \,: \mathbf{d}_v = -1\,\}\,, I_{\gamma+1}\,(G, X)\,|_{X=\phi\,(\delta)}\,\rangle$, *and*

$$I_{\gamma+1}\,(G, X)\,|_{X=\phi\,(\delta)} \neq \langle\, 1\,\rangle.$$

On the other side, given that $(x_v\,p)\,|_{x_v=0} = 0$ and $((x_v+1)p)\,|_{x_v=-1} = 0$ for every $p \in P[X]$, then,

$$I_{\gamma+1}\,(G^{\delta}, X)\,|_{X=\phi\,(\mathbf{d}-\delta)} \subseteq I_{\gamma+1}\,(G, X)\,|_{X=\phi\,(\mathbf{d})} \neq \langle\, 1\,\rangle$$

Now, applying the theorem to G^{δ} and $\mathbf{d} - \delta$, we have that

$$I_{\gamma+1}\,((G^{\delta})^{d-\delta}, X) = I_{\gamma+1}\,(G^{d}, X) \neq \langle\, 1\,\rangle\,\forall\,\mathbf{d} \text{ with } \mathrm{supp}(\mathbf{d}) = \delta,$$

this is, $\gamma\,(G^{d}) = \gamma\,(G^{\delta})$ with $\mathrm{supp}(\mathbf{d}) = \delta$.
Finally, note that $I_{n+1}\,(G, X) = \langle\, 0\,\rangle$, therefore

$$I_{n+1}\,(G^{d}, X) \subseteq \langle\, \{x_{v^0}, \ldots, x_{v^{d_v}} : d_v \geq 1\}\,, \{x_{v^0+1}, \ldots, x_{v^{d_v}+1} : d_v \leq -1\} \neq \langle\, 1\,\rangle.$$

□

2.2.3 Arithmetical Structures of Graphs

Now we present the concept of *arithmetical structure of a graph*, in which we use the generalize Laplacian matrix of a graph as well.

Definition 2.2.14. *An arithmetical graph is a triplet $(G, \mathbf{d}, \mathbf{r})$, given by a graph G and a pair of vectors $\mathbf{d}\,, \mathbf{r} \in \mathbb{N}_+^V$ such that $\gcd\{\mathbf{r}_v \mid v \in V(G)\} = 1$ and*

$$L(G, \mathbf{d}\,)\,\mathbf{r}^{\,t} = \mathbf{0}^{\,t}$$

Given an arithmetical graph $(G, \mathbf{d}, \mathbf{r})$ we say that the pair $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure of G.

Remark 2.2.15. *Note that any graph G has a canonical arithmetical structure, given by $(\mathbf{d}, \mathbf{r}) = (\mathbf{deg}_G, \mathbf{1})$, where $\mathbf{deg}_G \in \mathbb{N}^{|V(G)|}$ is the vector of degrees of G, i.e., $\mathbf{deg}_{Gv} = d_G(v)$.*

The concept of arithmetical graphs was introduced by Lorenzini as some intersection matrices that arise in the study of degenerating curves in algebraic geometry [11]. One of the most important result given in [11] is that the number of arithmetical structures of a simple connected graph is finite.

Theorem 2.2.16 (Lemma 1.6 [11])**.** *There exist only finitely many arithmetical structures on any simple connected graph.*

Next proposition gives us some basics properties of Laplacian matrices. These properties are essentially equivalent to those given in [11, Proposition 1.1 and Corollary 1.3], for connected simple graphs. Let $M(u, v)$ be a $|V| - 1$ minor of M obtained by deleting the u-th row and the v-th column.

Proposition 2.2.17. *Let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure of a connected multigraph $G = (V, E)$, then:*

i) *M has rank equal to $|V| - 1$ and $\ker_{\mathbb{Q}}(M) = \langle \mathbf{r} \rangle$.*

ii) *Exist a positive integer m such that $\operatorname{adj}(M) = m\, \mathbf{r}\, \mathbf{r}^t$.*
Furthermore, $m = \det(M(u, v))(\mathbf{r}_u \mathbf{r}_v)^{-1}$.

iii) *The cokernel of M is a finite generated abelian group of the form $\mathbb{Z} \oplus \Phi(G)$ where $\Phi(G)$ is a finite group order m.*

Remark 2.2.18. *Note that the condition $\ker_{\mathbb{Q}}(M) = \langle \mathbf{r} \rangle$ is equivalent to $\gcd\{\mathbf{r}_v \mid v \in V\} = 1$. Thus, by proposition 2.2.17 (i), in any arithmetical structure $(\mathbf{d}, \mathbf{r})$ we may assume that $\ker_{\mathbb{Q}}(M) = \langle \mathbf{r} \rangle$.*

In the appendix A.1 of this thesis we show a simple algorithm to compute the arithmetical structures of a given graph and in section 3.3.3 we list the arithmetical structures of graphs with $n \leq 5$ vertices.

M-Matrices

In this part we recall the classical concept of M-matrix and we introduce a new class of M-matrices whose proper principal minors are positive, which are called almost non-singular M-matrices, see [2, Theorem 6.4.16, pag 156]. After that we introduce the following set,

$$\mathcal{A}_\alpha(M) = \{ \mathbf{d} \in \mathbb{N}_+^n \mid A = \text{diag}(\mathbf{d}) - M \text{ is an } M\text{-matrix and } \det(A) = \alpha \}$$

for any $\alpha > 0$ and a non-negative integral $n \times n$ matrix M with all the diagonal entries equal to zero. We prove that $\mathcal{A}_\alpha(M)$ is finite for all $\alpha > 0$, see theorem 2.2.23. In the following, matrix always means square matrix. Recall that a real matrix is called non-negative if all their entries are non-negative real numbers.
We begin by recalling the classical definition of a M-matrix.

Definition 2.2.19. *A real matrix A is said to be an M-matrix if*

$$A = \alpha I - M,$$

for some non-negative matrix M with $\alpha \geq \rho(M)$.

Where $\rho(M)$ is the spectral radius of the square matrix M and is defined by

$$\rho(M) = \max\{ |\lambda| \mid \lambda \in \sigma(M) \},$$

where $\sigma(M)$ is the spectrum of M, that is, the set of complex eigenvalues of M. It turns out that a M-matrix $A = \alpha - M$ is singular if and only if $\alpha = \rho(M)$. The class of M-matrices admit many equivalent definitions, for instance Berman [2] enlists more than 80 ways to characterize M-matrices.

The study of M-matrices is divided in two big parts: non-singular M-matrices (see [2, Section 6.2]) and singular M-matrices (see [2, section 6.4]). Singular M-matrices have been more difficult to study that non-singular M-matrices. M-matrices are very important in a broad range of mathematical disciplines. The book by Berman and Plemmons, [2], studies non-singular and singular M-matrix. Recently M-matrices have been studied in the context of chip-firing games, see [10] and the references contained there.

Since a M-matrix $A = (a_{i,j})$ is equal to $\alpha I - M$ with M non-negative, it turns out that $a_{i,j} \leq 0$ for all $i \neq j$ and $a_{i,i} \geq 0$. A real matrix which satisfies these last conditions is called a L-matrix. In this paper we restricted our attention to the next subclass of singular M-matrices.

Definition 2.2.20. *A real matrix $A = (a_{i,j})$ is called an almost non-singular M-matrix if A is a Z-matrix ($a_{i,j} \leq 0$ for all $i \neq j$) and all the proper principal minors are positive.*

A very important fact of an arithmetical graph is that its associated Laplacian matrix is singular of maximal rank. It is well known that Z-matrix is a non-singular M-matrix if and only if all its proper principal minors are positive and is a singular M-matrix if and only if all its proper principal minors are non-negative. In this sense, the class of almost non-singular M-matrices is between the class of singular M-matrices and non-singular M-matrices. For example, a singular irreducible M-matrix is an almost non-singular M-matrix, see [2, Theorem 6.4.16, pag. 156]. The class of almost non-singular M-matrices admit the next characterization.

Theorem 2.2.21. *If $M = (m_{i,j}) \in M_{n\times n}$ is a real Z-matrix, then the following conditions are equivalent:*

(a) *M is an almost non-singular M-matrix.*

(b) *$M + D$ is a non-singular M-matrix for any diagonal matrix $D \gneq 0$.*

(c) $\det(M + D) \gneq \det(M + D^{'}) \geq 0$ *for all the diagonal matrices* $D \gneq D^{'} \geq 0$.

The proof can be found in [6, Theorem 2.3]. Using any algebraic computational system like Sage, Macaulay, Maple or Mathematica is not difficult to check whether a matrix is indeed an M-matrix or it is not. In particular it is interesting how to check if a specific Z-matrix is or not an M-matrix.

Remark 2.2.22. *Let M be a real Z-matrix and*

$$f_M(\mathbf{x}) = \det(M + \mathrm{diag}(x_1, \ldots, x_n)) \in \mathbb{R}[x_1, \ldots, x_n].$$

Then M is an M-matrix (non-singular M-matrix) if and only if the coefficients of the polynomial f_M are non-negative (positive). In a similar way, M is an almost non-singular M-matrix if and only if all the coefficients except maybe the constant term of *the polynomial f_M are positive. For instance, if*

$$M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

then $f_M(\mathbf{x}) = x_1x_2x_3 + 2x_1x_2 + x_1x_3 + x_2x_3 + x_1 + 2x_2 + x_3$. Thus, M is an almost non-singular matrix M-matrix, but not a non-singular M-matrix.

Given $\alpha \geq 0$ and a non-negative integral $n \times n$ matrix M with all the diagonal entries equal to zero, let

$$\mathcal{A}_{\geq\alpha}(M) = \{\mathbf{d} \in \mathbb{N}_+^n \mid A = \mathrm{diag}(\mathbf{d}) - M \text{ is an } M\text{-matrix and } \det(A) \geq \alpha\}.$$

Therefore, let $\mathcal{A}_\alpha(M) = \{\mathbf{d} \in \mathcal{A}_{\geq\alpha}(M) \mid \det(\mathrm{diag}(\mathbf{d}) - M) = \alpha\}$. This set is closely related to the set of arithmetical structures of a graph. More precisely is related to the case when M is equal to the adjacency matrix of G and $\alpha = 0$. If M is an almost non-singular M-matrix, then by [6, Theorem 2.3] we have that

$$\mathcal{A}_{\geq\alpha}(M) = \mathcal{A}_\alpha(M) + (\mathbb{N}_+ \cup 0)^n.$$

That is, $\mathcal{A}_{\geq\alpha}(M)$ is a monoid and infinite. But now, what about the finiteness of $\mathcal{A}_\alpha(M)$?

Theorem 2.2.23. *If M is a non-negative integral matrix, then $\mathcal{A}_\alpha(M)$ is finite for any $\alpha > 0$.*

Proof. We claim that

$$\mathcal{A}_\alpha(M) \subseteq \min\mathcal{A}_{\geq\alpha}(\mathrm{M}) = \{\mathbf{d} \in \mathcal{A}_{\geq\alpha}(\mathrm{M}) \mid \text{ if } \mathbf{d}' \leq \mathbf{d} \text{ for some } \mathbf{d}' \in \mathbb{N}_+^{\mathrm{n}}, \text{ then } \mathbf{d}' = \mathbf{d}\}.$$

Lets prove this by contradiction, let $\mathbf{d} \in \mathcal{A}_\alpha(M)$ and assume that $\mathbf{d} \notin \min\mathcal{A}_{\geq\alpha}(\mathrm{M})$. This means that exist $\mathbf{e} \in \mathcal{A}_{\geq\alpha}(M)$ such that $\mathbf{e} \gneq \mathbf{d}$. Since

$$\det(\mathrm{diag}(\mathbf{e}) - M) \geq \alpha > 0, \text{ then } \mathrm{diag}(\mathbf{e}) - M \text{ is a non-singular } M\text{-matrix.}$$

By theorem [6, Theorem 2.3], $\det(\mathrm{diag}(\mathbf{d}) - M) > \det(\mathrm{diag}(\mathbf{e}) - M) \geq \alpha$, which is a contradiction since $\det(\mathrm{diag}(\mathbf{d}) - M) = \alpha$.
Now, since $\mathcal{A}_{\geq\alpha}(M) \subseteq \mathbb{N}_+^n$, we have (by Dickson's lemma) that $\min\mathcal{A}_{\geq\alpha}(\mathrm{M})$ is finite and then $\mathcal{A}_\alpha(M)$ is also finite.

□

Example 2.2.24. *If*

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then $\mathcal{A}_6(M) = \{(3,2,2)^t, (2,2,3)^t\}$ *and* $\min\mathcal{A}_{\geq 6} = \{(3,2,2)^{\mathrm{t}}, (2,3,2)^{\mathrm{t}}, (2,2,3)^{\mathrm{t}}\}$.

The special case of $\mathcal{A}_\alpha(M)$ when α is equal to zero is more difficult to treat. For example, if M is reducible, then $\mathcal{A}_0(M)$ can be infinite, as next example shows.

Example 2.2.25. *Let*

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Is not difficult to check that $\{(1,x,1,y)^t | x, y \in \mathbb{N}_+\} \subsetneq \mathcal{A}_0(M)$. *That is* $\mathcal{A}_0(M)$ *is infinite. On the other hand, since*

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we have that M is reducible.

2.3 Threshold Graphs

We start this section with the following

Definition 2.3.1. *We define the set of Threshold Graphs as the family of all graphs that fulfill the following properties:*

i) K_1 *(the trivial graph) is a threshold graph.*

ii) If G *is a threshold graph, then* $G + v$, *with* $v \notin V(G)$, *is a threshold graph.*

iii) If G *is a threshold graph, then* $c(G)$, *the cone graph of* G, *is a threshold graph.*

Remark 2.3.2. *An important result is that* G *is a threshold graph if and only if* G *is* C_4, P_4, *and* $2K_2$*-free. This also explains why threshold graphs are closed under taking complements; because the complement of* C_4 *is* $2K_2$ *and* P_4 *is self-complementary.*

2.3.1 A characterization of Threshold Graphs

From the next theorem is an easy exercise to prove that one can check if a graph is a threshold graph or if it is not in linear time.

Theorem 2.3.3 (Golumbic, [9]). *Let $G = (V, E)$ be a graph with degree partition $V = D_0 + D_1 + \ldots + D_m$. Then the following statements are equivalents:*

(a) G is a threshold graph.

(b) There exist a labeling $c : V \longrightarrow \mathbb{N}$ of V and $t \in \mathbb{R}$ such that for every pair of vertices $x, y \in V, x \neq y$.

$$xy \in E \Longleftrightarrow c(x) + c(y) > t;$$

(c) for every pair of vertices $x \neq y$, with $x \in D_i$, $y \in D_j$,

$$xy \in E \Longleftrightarrow i + j > m;$$

(d) The following recursions are satisfied:

$$\delta_{i+1} = \delta_i + |\, D_{m-i}\,|, (i = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor - 1)$$

$$\delta_i = \delta_{i+1} - |\, D_{m-i}\,|\,, (i = m, m-1, \ldots, \lfloor \frac{m}{2} \rfloor, +1)$$

We already defined a general *threshold graph* (2.3.1), but we just give a fine characterization of threshold graphs, so that we can define a particular family $\{T_n\}_{n\geq 1}$ of threshold graphs that we can see as a base (in a certain way) for any threshold graph.

Definition 2.3.4. *Let T_n be the threshold graph generated by adding an isolated vertex and then another vertex adjacent to all the previous vertices (cone graph), this is,*

$$T_n = \begin{cases} K_2, & \text{if } n = 1, \\ c(T_{n-1} + v), & \text{if } n > 1. \end{cases}$$

Remark 2.3.5. *Note that $|V(T_n)| = 2n$ and $|E(T_n)| = n^2$.*

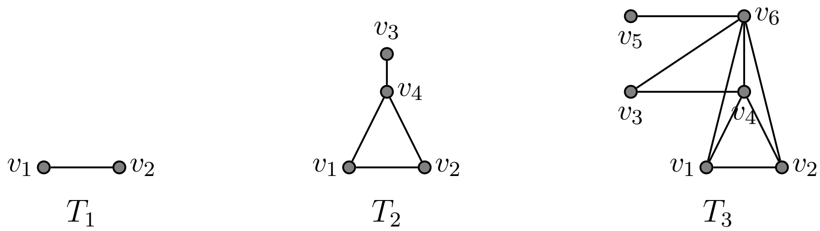


Figure 2.1: The first three graphs in $\{T_n\}_{n\geq 1}$.

If we write the Laplacian matrix of T_n giving the same order to columns and rows that we give to the vertices of T_n in its construction, $L(T_n)$ does not have a nice form, but if we put the vertices (on rows and columns) in increasing order with respect to their degrees we have the following visualization of the Laplacian matrix of T_n,

$$
L(T_n) \equiv
\begin{array}{c}
 \\
v_{2n-1} \\ v_{2n-3} \\ \vdots \\ v_1 \\ v_3 \\ v_2 \\ v_4 \\ \vdots \\ v_{2n-2} \\ v_{2n}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
v_{2n-1} & v_{2n-3} & \cdots & v_3 & v_1 & v_2 & v_4 & \cdots & v_{2n-2} & v_{2n}
\end{array} \\
\left(\begin{array}{ccccc|ccccc}
x_{2n-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & x_{2n-3} & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x_3 & 0 & 0 & -1 & \cdots & -1 & -1 \\
0 & 0 & \cdots & 0 & x_1 & -1 & -1 & \cdots & -1 & -1 \\
\hline
0 & 0 & \cdots & 0 & -1 & x_2 & -1 & \cdots & -1 & -1 \\
0 & 0 & \cdots & -1 & -1 & -1 & x_4 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \cdots & -1 & -1 & -1 & -1 & \cdots & x_{2n-2} & -1 \\
-1 & -1 & \cdots & -1 & -1 & -1 & -1 & \cdots & -1 & x_{2n}
\end{array}\right)
\end{array}
$$

Corollary 2.3.6. *Every connected threshold graph can be represented by a graph of the form $T_i^{\mathbf{d}}$ for some $i \in \mathbb{N}$, and with $\mathbf{d} \in \mathbb{Z}^{|V|}$ such that:*

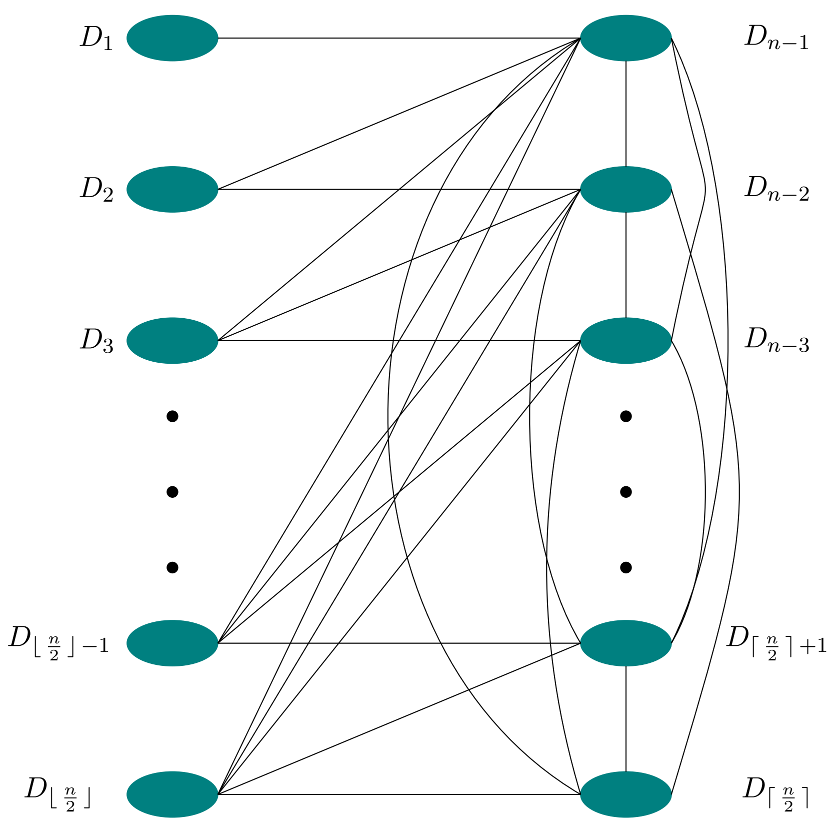
(a) $\mathbf{d}_{v_{2k+1}} \geq 0$, *with* $k \geq 0$ *and,*

(b) $\mathbf{d}_{v_{2k}} \leq 0$, *with* $k \geq 1$.

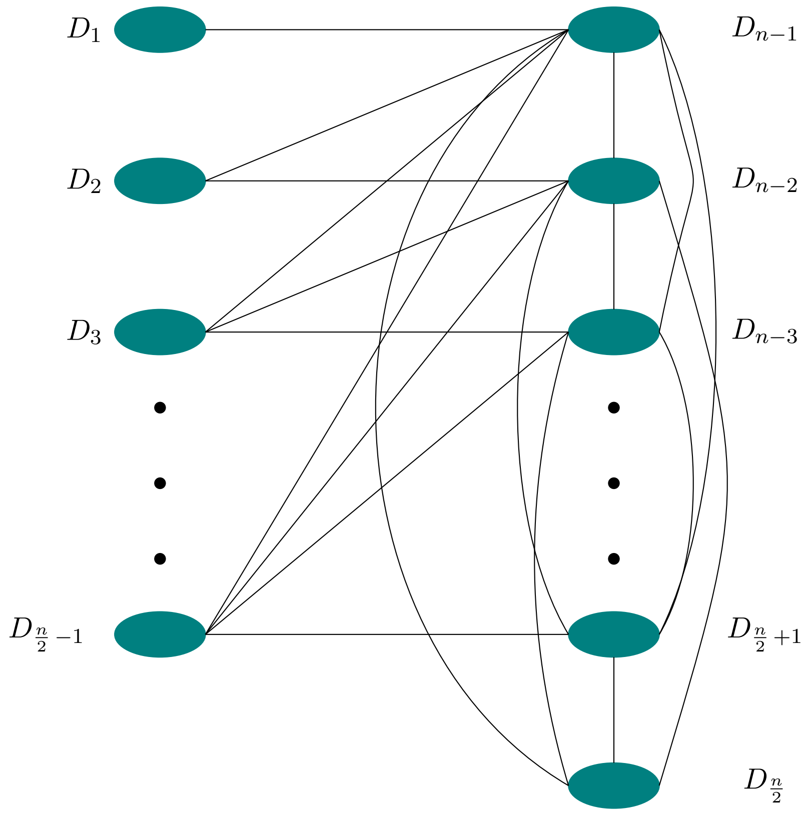
Furthermore, every such graph is indeed a connected threshold graph

Therefore any threshold graph can be obtained from T_n for some $n \in \mathbb{N}$ by duplicate and replicating vertices, i.e., by strong and weak twins (of the odd and even vertices respectively).

The following picture shows the general visualization of a Threshold graph with an odd number of partitions.



And the following figure shows the general visualization of a threshold graph with an even number on his partition of degrees.



Let us think of the vertex-sets on the figure above as single vertices except for the $D_{\frac{n}{2}}$ set, which we replace for two vertices, then we have the T_n graph. From where we can interpret the last corollary.

Chapter 3

Computing Arithmetical Structures of Graphs

3.1 Properties of Arithmetical Structures of Graphs

We know that given any connected graph G, there exist only finitely many arithmetical structures of such graph [Theorem 2.2.14], therefore a natural way to proceed is to try to describe all the possible arithmetical structures that a graph may have. In this section we address this problem and present various different interesting properties of this objects.

Let us write the set of all arithmetical structures of a connected simple graph G as $\mathcal{A}(G)$, and let us remind that the critical group of G, $K(G)$, is defined as the torsion subgroup of the cokernel of $L(G)$, see sections 2.1.2 and 2.2.1. Arithmetical Structures of a graph G and its critical group are closely related because the matrix $L(G, \mathbf{d})$ where $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure of G share many properties with the Laplacian matrix of G.

Now, the definition of critical group can be generalized to any arithmetical graph, more precisely, given $(\mathbf{d}, \mathbf{r})$ an arithmetical structure of a graph G, let

$$K(G, \mathbf{d}, \mathbf{r}) = \ker(\mathbf{r}^t) \Big/ \mathrm{Im} L(G, \mathbf{d})^t \tag{3.1}$$

be the critical group of the arithmetical graph $(G, \mathbf{d}, \mathbf{r})$. In a similar way that for the cokernel of $L(G)$, this new definition is closely related to the critical ideals of the graph (the determinantal ideals associated to the generalized Laplacian matrix, see

2.2.2). Furthermore, by [5, Propositions 3.6 and 3.7] we can recover the critical group of $(G, \mathbf{d}, \mathbf{r})$ as an evaluation of the critical ideals of the graph. Also, given an integer matrix M its critical group $K(M)$ is defined as the torsion part of its cokernel.
Now, we present the theory that we need to generalize the proposition 2.2.17 to any integer irreducible $n \times n$ matrix M such that there exists $\mathbf{r} \in \mathbb{N}_+^n$ with $M\mathbf{r}^t = \mathbf{0}^t$.
Given a connected graph $G = (V, E)$, we define the set of arithmetical structures of a graph, let

$$\mathcal{A}(G) = \{(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^{V(G)} \times \mathbb{N}_+^{V(G)} \mid (\mathbf{d}, \mathbf{r}) \text{ is an arithmetical structure of G}\}.$$

By prop. 2.2.17(i) we have that for any $\mathbf{d} \in \mathbb{N}_+^n$ such that $L(G, \mathbf{d})$ is singular there exist a unique $\mathbf{r} \in \mathbb{N}_+^n$ such that $ker_{\mathbb{Q}}(L(G, \mathbf{d})) = \langle \mathbf{r} \rangle$. In general this is true if and only if M is irreducible. Therefore sometimes will be useful to define the set of the $\mathbf{d}'s$ and the $\mathbf{r}'s$ separately. Given a connected multigraph $G = (V, E)$, let

$$\mathcal{D}(G) = \{\mathbf{d} \in \mathbb{N}_+^n \mid (G, \mathbf{d}, \mathbf{r}) \text{ is an arithmetical graph for some } \mathbf{r} \in \mathbb{N}_+^n\} \text{ and}$$

$$\mathcal{R}(G) = \{\mathbf{r} \in \mathbb{N}_+^n \mid (G, \mathbf{d}, \mathbf{r}) \text{ is an arithmetical graph for some } \mathbf{d} \in \mathbb{N}_+^n\}.$$

This definition can be generalized even further to any non-negative matrix M.

Definition 3.1.1. *Given a non-negative integer $n \times n$ matrix M with all the diagonal entries equal to zero, let*

$$\mathcal{A}(M) = \{(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^n \times \mathbb{N}_+^n \mid [\mathrm{diag}(\mathbf{d}) - M]\mathbf{r}^t = \mathbf{0}^t \text{ and } \gcd\{\mathbf{r}_v \mid v \in V\} = 1\}.$$

Remark 3.1.2. *Let us make a few remarks. (i) It is clear that $\mathcal{A}(G) = \mathcal{A}(\mathcal{A}(G))$; (ii) $\{\mathbf{d} | (\mathbf{d}, \mathbf{r}) \in \mathcal{A}(M)\}$ and; (iii) in general $\mathcal{A}(M) = \mathcal{A}(M - diag(M)) + (diag(M), 0)$ therefore we can assume without loss of generalization that M is a non-negative matrix with zero diagonal.*

The next result is the beginning towards the generalization of theorem 2.2.14 to the general case of a multigraph.

Theorem 3.1.3 (Theorem 3.4, [6])**.** *Let M is a Z-matrix. If there exists $\mathbf{r} > 0$ such that $M\mathbf{r}^t = \mathbf{0}^t$, then M is a M-matrix. Morover, M is almost non-singular M-matrix with $\det(M) = 0$ if and only if M is an irreducible and there exists $\mathbf{r} > 0$ such that $M\mathbf{r}^t = \mathbf{0}^t$.*

An important immediate consequence of this theorem is the following result.

Corollary 3.1.4. *If M is an irreducible Z-matrix, then there exists $\mathbf{r} > 0$ such that $M\mathbf{r}^t = 0$ if and only if there exists $\mathbf{s} > 0$ such that $M^t\mathbf{s}^t = 0$.*

Proof. The result follows from the fact that M is irreducible almost non-singular M-matrix with $\det(M) = 0$ if and only if M^t is irreducible almost non-singular M-matrix with $\det(M^t) = 0$. □

Now let us present a way to compute the adjoint matrix of M in function of $\mathbf{r}$ and $\mathbf{s}$.

Proposition 3.1.5. *Let M be a Z-matrix. Then M is an almost non-singular M-matrix with $\det(M) = 0$ if and only if $\mathbf{r} > 0$ and*

$$\mathrm{Adj}(M) = |K(M)|\, \mathbf{r}\mathbf{s}^t > 0,$$

where $\ker_{\mathbb{Q}}(M) = \langle \mathbf{r} \rangle$ and $\ker_{\mathbb{Q}}(M^t) = \langle \mathbf{s} \rangle$.

Another possible characterization of an almost non-singular M-matrix is that its principal submatrices of maximal size ($M_{i,i} = M[i^c, i^c]$) are non-singular M-matrices. The converse of this is in the next result.

Proposition 3.1.6. *If M is anon-singular M-matrix, then there exists an irreducible almost non-singular M-matrix M' with $\det(M') = 0$ such that $M'_{1,1} = M$.*

Remark 3.1.7. *The last proposition means that, in some sense, any non-singular M-matrix can be extended to a graded M-matrix. Moreover, some of its associated ideals, as its matrix ideal, are graded.*

Corollary 3.1.8. *If $(G, \mathbf{d}, \mathbf{r})$ is a strongly connected arithmetical graph, then $L(G, \mathbf{d})$ is an almost non-singular M-matrix with $\det(L(G, \mathbf{d})) = 0$.*

Proof. Let $M = L(G, \mathbf{d}) = \mathrm{diag}(\mathbf{d}) - A(G)$. Since G is strongly connected if and only if M is irreducible, then the result follows by applying Theorem 3.1.3

□

If G is a multidigraph, its adjacency matrix $A(G)$ is always a non-negative matrix with zeros on the diagonal. On the other hand, if M is a non negative matrix with zeros on the diagonal, then there exists a unique multidigraph G_M such that $M = A(G_M)$. The graph G_M is called the underlying multidigraph of M. The next theorem use this correspondence to establish the necessary and sufficient conditions over a non-negative matrix M in order that $\mathcal{A}(M)$ be finite.

Theorem 3.1.9 ([6], Theorem 3.9)**.** *If M is a non-negative matrix with all the diagonal entries equal to zero, then $A(M) \neq \emptyset$. Even more, $\mathcal{A}(M)$ is finite if and only if M is irreducible.*

Proof. Let G_M be the multigraph such that $A(G_M) = M$. Note that G_M is strongly connected if and only if the underlying graph of M is strongly connected. Because of Corollary 3.1.8 we only need to prove that G_M has at least one arithmetical structure. Let $\mathbf{d} \in \mathbb{N}_+^{V(G_M)}$ be the vector defined by

$$\mathbf{d}_v = \begin{cases} 1, & \text{if } \sum\limits_{\omega \in V(G_M)} M_{v,w} = 0, \\ \\ \sum\limits_{\omega \in V(G_M)} M_{v,w}, & \text{otherwise.} \end{cases}$$

for each $v \in V(G_M)$.
It is obvious that the triplet $(G_M, \mathbf{d}, \mathbf{1})$ is an arithmetical graph, and then $\mathcal{A}(M) \neq \emptyset$. Now we proceed with the second statement of the theorem.
$(\Rightarrow)$ We prove this by contradiction. Assuming that M is reducible we can suppose that

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A, B are square matrices of size $r \times r$ and $s \times s$ respectively and A is irreducible. If $C = 0$, then

$$\mathcal{A}(M) = \{(u\mathbf{r}, v\mathbf{s}) | A\mathbf{r}^t = \mathbf{0}^t, (\mathbf{r}_1, \dots, \mathbf{r}_r) = 1. B\mathbf{s}^t = \mathbf{0}^t, (\mathbf{s}_1, \dots, \mathbf{s}_s) = 1, (u, v) = 1\}$$

is infinite. On the other hand, if $C \neq 0$ and let $\mathbf{d} \in \mathcal{A}_{\geq 1}(A)$ and $L = \operatorname{diag}(\mathbf{d}) - A$. If $(\operatorname{diag}(\mathbf{e}) - B)\mathbf{s} = \mathbf{0}$ for some $(\mathbf{e}, \mathbf{s}) \in \mathbb{N}_+^s$ with $(\mathbf{s}_1, \dots, \mathbf{s}_s) = 1$, then

$$(\operatorname{diag}(, \mathbf{e}) - M)(-vL^{-1}C\mathbf{s}, v\mathbf{s})^t = \mathbf{0} \text{ for all } \mathbf{d} \in \mathcal{A}_{\geq 1}(A) \text{ and } v \in \mathbb{N}_+.$$

Since $\mathbf{d} \in \mathcal{A}_{\geq 1}(A)$, L is an irreducible non-singular M-matrix. We have that $L^{-1} > 0$ (see [2, Theorem 6.2.7, pag 141]) and since $C \leq 0$, then $-L^{-1}C\mathbf{r} > 0$. Now, since L is integer, then there exists $v \in \mathbb{N}_+$ such that

$$-vL^{-1}C\mathbf{r} = -v\frac{1}{det(det(L))}Adj(L)C\mathbf{r}$$

is an integer vector. Furthermore, for every $\mathbf{d} \in \mathcal{A}_{\geq 1}(A)$ (set which is infinite) there exists $v \in \mathbb{N}_+$ such that the entries of $\frac{1}{u}(-vL^{-1}C\mathbf{s}, v\mathbf{s})$ has greatest common divisor equal to one, and then $\mathcal{A}(M)$ is infinite as well.

($\Leftarrow$) We claim that

$$\{\mathbf{d} \mid (\mathbf{d}, \mathbf{r}) \in \mathcal{A}(M)\} \subseteq \min(\mathcal{A}_{\geq 0}(\mathrm{M})).$$

Let $(\mathbf{d}, \mathbf{r}) \in \mathcal{A}(M)$ and suppose that there exists $\mathbf{e} \in \mathcal{A}_{\geq 0}(M)$ such that $< \mathbf{d}$. If we have that $\det(\mathrm{diag}(\mathbf{e}) - M) > 0$ we proceed as in the proof of Theorem 2.2.23.
Now, given that M is irreducible, $\mathrm{diag}(\mathbf{e}) - M$ is a singular and irreducible M-matrix, and then we also have that $\mathrm{diag}() - M$ is an almost non-singular M-matrix, see [2, Theorem 6.4.16, pag 156]. Thus by Theorem 2.2.21, $\det(\mathrm{diag}(\mathbf{d}) - M) > \det(\mathrm{diag}(\mathbf{e}) - M) = 0$; which contradicts the fact that $\mathbf{d} \in \mathcal{A}(M)$.
Therefore, $\mathcal{A}(M) \subseteq \min(\mathcal{A}_{\geq 0}(\mathrm{M}))$ and together with Dickson's Lemma we have proved the result. □

Now we present the result we were aiming for, the generalization of Theorem 2.2.14 to multigraphs.

Corollary 3.1.10. *If G is a multigraph, then $\mathcal{A}(G)$ is finite if and only if G is strongly connected.*

Proof. Since $L(G, \mathbf{d})$ is an almost non-singular M-matrix for each $\mathbf{d} \in \mathcal{D}(G)$, it follows that $\mathcal{D}(G) \subsetneq \mathcal{A}(G)$. The result follows from the theorem above and the fact that $L(G, \mathbf{0})$ is irreducible if and only if G is strongly connected.

□

Remark 3.1.11. *It is clear that $\mathcal{D}(G) \subseteq \mathcal{A}_0(G)$, however we do not always have the equality. For example, if P_5 with vertices $v_1, \ldots, v_5$, then*

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & * & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \mathbf{0}.$$

Therefore $\det(\mathrm{diag}(1,1,*,1,1) - A(P_5)) = 0$ *for all $d \in \mathbb{N}_+$ and $\mathcal{D}(G) \subsetneq \mathcal{A}_0(G)$.*

In the next two propositions we present some arithmetical structures of the cone graph. This represent the first example of how to construct arithmetical structures of graph. Let us recall that given a graph G, the cone graph of G, denoted by $c(G)$, is the graph obtained from adding a vertex v_c to G which is adjacent to every vertex in $V(G)$, that is,

$$V(c(G)) = V(G) \cup \{v_c\} \text{ and } E(c(G)) = E(G) \cup \{(v_c, u) \mid u \in V(G)\}.$$

Proposition 3.1.12. *Let G be a t-regular graph with n vertices. If f is a divisor of n, then $(\mathbf{d}, \mathbf{r})$ given by*

$$\mathbf{d}_u = (\frac{n}{f}, t+f, \dots, t+f) \text{ and } \mathbf{r}_u = (f, 1, \dots, 1)$$

is an arithmetical structure of $c(G)$.

Proof. Just note that $\mathbf{d}, \mathbf{r} \in \mathbb{N}_+^n$ and $\begin{pmatrix} \frac{n}{f} & -\mathbf{1}_n \\ -\mathbf{1}_n^t & L(G, (f+t)\mathbf{1}_n) \end{pmatrix} \begin{pmatrix} f \\ \mathbf{1}_n \end{pmatrix} = \mathbf{0}.$ □

The following result gives a more difficult to find type of arithmetical structures of the cone graph.

Proposition 3.1.13. *Let G be a graph with n vertices and $(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^n \times \mathbb{N}_+^n$ such that $L(G, \mathbf{d})\mathbf{r} = a\mathbf{1}$ and $a | \sum_{i=1}^n \mathbf{r}_i = |\mathbf{r}|$. If $g = \gcd(a, \mathbf{r}_1, \dots, \mathbf{r}_n)$, then $(\tilde{\mathbf{d}}, \tilde{\mathbf{r}})$ given by*

$$\tilde{\mathbf{d}}_u = (\frac{\sum_{i=1}^n \mathbf{r}_i}{a}, \mathbf{d}_1, \dots, \mathbf{d}_n) \text{ and } \tilde{\mathbf{r}}_u = (\frac{a}{g}, \frac{\mathbf{r}_1}{g}, \dots, \frac{\mathbf{r}_n}{g})$$

is an arithmetical structure of $c(G)$.

Proof. The result follows from the fact that $\tilde{\mathbf{d}}, \tilde{\mathbf{r}} \in \mathbb{N}_+^n$ and

$$\begin{pmatrix} \frac{|\mathbf{r}|}{a} & -\mathbf{1}_n \\ -\mathbf{1}_n^t & L(G, \mathbf{d}) \end{pmatrix} \begin{pmatrix} \frac{a}{g} \\ \frac{\mathbf{r}^t}{g} \end{pmatrix} = \mathbf{0}.$$

□

3.2 Arithmetical Graphs and Diophantine equations

The problem of giving a complete characterization of the arithmetical structures of a graph is quite complex. It's equivalent to find some solutions of a Diophantine equation. In the case of the complete graph K_n, one of the most simple and known graphs we have

Proposition 3.2.1. *If* $c = \operatorname{lcm}(\mathbf{d}_1 + 1, \ldots, \mathbf{d}_n + 1)$, *then*

$$\mathcal{A}(K_n) = \left\{ (\mathbf{d}, \mathbf{r}) \in (\mathbb{N}_+^n \times \mathbb{N}_+^n) \;\middle|\; \sum_{i=1}^{n} \frac{1}{\mathbf{d}_i + 1} = 1 \text{ and } \mathbf{r}_i = \frac{c}{\mathbf{d}_i + 1} \text{ for all } i \right\}. \quad (3.2)$$

Therefore, now we are dealing with Egyptian fractions, a known difficult problem in arithmetic. Let us note that even when an arithmetical structure gives a solution of the diophantine equation as in (3.2) above, not every solution of the diophantine equation is an arithmetical structure. We can see the latter, for example, on the path with five vertices P_5 in remark 3.1.11.
Now we present a partial description of the arithmetical structures of a general tree and a description of the arithmetical structures of the star [7].
Let $T = (V, E)$ be any tree, since the blocks of T are its edges, applying Theorem 2.3 [7] we get that the arithmetical structures of T must satisfy the following

Proposition 3.2.2 (Proposition 3.1 [7]). *If T is a tree, let $\overrightarrow{T}$ be the digraph obtained from T by replacing every edge by two arcs, one for each directions, and*

$$a : E(\overrightarrow{T}) \to \mathbb{Q}_+$$

be a weight on the arcs of $\overrightarrow{T}$, then

$$\mathcal{A}(T) = \left\{ (\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^{V(T)} \times \mathbb{N}_+^{V(T)} \;\middle|\; \sum_{uv \in E(T)} \mathbf{a}_{u,v} = \mathbf{d}_v \;\forall\; v \in V(T) \text{ and } \mathbf{a}_{u,v}\mathbf{a}_{v,u} = 1 \;\forall\; uv \in E(T) \right\}.$$

A simpler example of trees are the "star" graphs, trees where all its vertices, except for one of them, are leaves. We denote the star with $n+1$ vertices as S_n. Next result gives a description of the arithmetical structures of a star.

Corollary 3.2.3 (Corollary 3.2, [7]). *If S_n is the star with "center" v and $1, 2, \ldots, n$ as their leaves, then*

$$\mathcal{A}(S_n) = \left\{ (\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^{n+1} \times \mathbb{N}_+^{n+1} \;\middle|\; \mathbf{d}_v = \sum_{i=1}^{n} \frac{1}{\mathbf{d}_i},\; \mathbf{r}_v = \operatorname{lcm}\{\mathbf{d}_i\}_{i=1}^{n} \text{ and } \mathbf{r}_i = \frac{\mathbf{r}_v}{\mathbf{d}_i} \right\}.$$

Proof. The $\supseteq$ part is easy to see. Now, by Proposition 3.2.2, since all the vertices of S_n except its center has degree one, $\det(L(S_n, \mathbf{d})) = 0$ if and only if $\mathbf{d}_v = \sum_{i=1}^{n} \frac{1}{\mathbf{d}_i}$ for some $\mathbf{d}_i \in \mathbb{N}_+$. We know that $L(S_m, \mathbf{d})$ is an almost non-singular M-matrix, so the rest is just to check that $\mathbf{r}$ is in the kernel of $L(S_m, \mathbf{d})$ and then we have the $\subseteq$ part. □

For general trees, we can say something else, is very easy to calculate the order of the critical group associated to any of its arithmetical structures (expresion (3.1)), the next result was proved by Lorenzini [11]

Proposition 3.2.4 (Corollary 2.5 [11]). *Let T be a simple tree. If $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure of T, then*

$$\left|K(T, \mathbf{d}, \mathbf{r})\right| = \prod_{v \in V(T)} \mathbf{r}_v^{d_T(v)-2}$$

3.3 Counting Arithmetical Structures

In this section we treat another interesting aspect about arithmetical graphs. On the past section we give descriptions of the sets of arithmetical structures for some graphs but we know that the sets $\mathcal{A}(G)$ are finite for simple connected graphs G, see Lemma 1.6 [11], therefore a natural way to proceed is trying to count the number of elements in the set $\mathcal{A}(G)$. Actually, in the case of paths we can give the number of arithmetical structures as well as with cycles, see [6] and [4].

3.3.1 Paths

Theorem 3.3.1. *The number of arithmetical structures of the path P_{n+1} is equal to the n-th Catalan number*

$$\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$$

Moreover, the number of arithmetical structures $(\mathbf{d}, \mathbf{r})$ with $\mathbf{r}^{(1)} = \#\{i \mid r_i = 1\} = 2$ is the $(n-1)$-th Catalan number Cat_{n-1}.

With the purpose of giving the prove of this theorem we first present another two other results

Lemma 3.3.2. *Let $\mathbf{r} \in \mathcal{R}(P_n)$ with $n \geq 2$ vertices, then*

$$r_1 = r_n = 1$$

Furthermore, if $r_j = 1$ for some $1 < j < n$, then

$$(r_1, \ldots, r_j) \in \mathcal{R}(P_j) \quad \textit{and} \quad (r_j, \ldots, r_n) \in \mathcal{R}(P_{n-(j-1)})$$

.

Proof. First, note that $(\mathbf{d}.\mathbf{r})$ is an arithmetical structure on P_n if and only if the following equalities hold:

$$r_1 d_1 = r_2;\ r_i d_i = r_{i-1} + r_{i+1} \text{ for } 1 < i < n; \text{ and } r_n d_n = r_{n-1} \tag{3.3}$$

In consequence, we have that:

$$r_1 \mid r_2 \Longrightarrow r_1 \mid r_3 \Longrightarrow \cdots \Longrightarrow r_1 \mid r_n.$$

Since $\mathbf{r}$ is a primitive vector, we conclude that $r_1 = 1$. The same argument starting with the last equation and moving up yields $r_n = 1$.
Now, if $r_j = 1$ for some $1 < j < n$, then we can see that $(r_1, \dots, r_j) \in \mathcal{R}(P_j)$ by changing d_j to $\tilde{d}_j := r_{j-1}$.
A similar argument, using the final $n-(j-1)$ equations defining the pair $(\mathbf{d}, \mathbf{r})$, shows that $(r_j, \dots, r_n) \in \mathcal{R}(P_{n-(j-1)})$. □

Corollary 3.3.3. *Let* $\mathbf{r} = (r_1, \dots, r_n)$ *be a primitive positive integer vector. Then* $\mathbf{r} \in \mathcal{R}(P_n)$ *if and only if*

(a) $r_1 = r_n = 1$*, and*

(b) $r_i | (r_{i-1} + r_{i+1})\ \forall\ i \in [2, n-1]$.

Proof. Condition (a) is part of Lemma 3.3.2, and the necessity of condition (b) follows from equations (3.3). On the other side, if $\mathbf{r}$ satisfies these two conditions then the corresponding $\mathbf{d} \in \mathcal{D}(P_n)$ can be recovered from the equations given in (3.3). □

Proof of Theorem 3.3.1. For the second assertion, the description in Corollary 3.3.3 is a known interpretation of the Catalan numbers, see [13, page 34, Problem 92]. The first assertion then follows from the recurrence $\mathrm{Cat}_n = \sum_{i=0}^{n-1} \mathrm{Cat}_i \mathrm{Cat}_{n-1-i}$, since the same recurrence holds for arithmetical structures by Lemma 3.3.2. □

3.3.2 Cycles

We first need to stablish some notation for multisets. A *multiset* is a list $S = [a_1, \dots, a_l]$, where order does not matter, and repeats are allowed. The number l is the size or *cardinality* of S. We will use square brackets to distinguish multisets from ordinary sets. If $a_i \in T$ for all i then we say that S is a multisubset of T. We write $\mathrm{MSet}_l(T)$ to denote the set of multisubsets of T of size l and let $\left(\!\binom{n}{k}\!\right) = |\mathrm{MSet}_l([n])| = \binom{n+l-1}{l}$.

Similarly, $\mathrm{MSet}_{\leq l}(T)$ denotes the set of multisubsets of T of size at most l. There is a bijection $\mathrm{MSet}_{n-1}(n+1) \to \mathrm{MSet}_{\leq n-1}([n])$ that erases all instances of $n+1$, which implies that $\sum_{l=0}^{n-1} \left(\!\binom{n}{l}\!\right) = \left(\!\binom{n+1}{n-1}\!\right)$.
Now, we can address the case of counting the arithmetical structures of the cycle with n vertices, denoted by C_n.

Theorem 3.3.4 (Theorem 27, [4]). *Let $1 \leq k \leq n$ and $l = n-k$. Then*

$$\#\left\{(\mathbf{d},\mathbf{r}) \in \mathcal{A}(C_n) \;\middle|\; \mathbf{r}^{(1)} = k\right\} = \left(\!\!\binom{n}{n-k}\!\!\right) = \binom{2n-k-1}{n-k}$$

Hence, the total number of arithmetical structures on C_n is

$$\sum_{k=1}^{n} \left(\!\!\binom{n}{n-k}\!\!\right) = \sum_{l=0}^{n-1} \left(\!\!\binom{n}{l}\!\!\right) = \left(\!\!\binom{n+1}{n-1}\!\!\right) = \binom{2n-1}{n-1}.$$

The proof of this theorem requires the developement of more theory, we refer the reader to [4] for an extense reading of it, our intention is to show that in this case we can count the number of arithmetical structues as we did in 3.3.1 with paths. Moreover, from the article [4] we have the next result.

Theorem 3.3.5 (Theorem 26, [4]). *Let $(\mathbf{d},\mathbf{r}) \in \mathcal{A}(C_n)$ be an arithmetical structures of the cycle. Then*

$$\mathbf{r}^{(1)} = 3n - \sum_{j=1}^{n} d_j \tag{3.4}$$

and

$$K(C_n, \mathbf{d}, \mathbf{r}) = \mathbb{Z}_{\mathbf{r}^{(1)}}. \tag{3.5}$$

Thus, the following result comes directly from theorems 3.3.4 and 3.3.5.

Corollary 3.3.6 (Corollary 28, [4]). *The number of arithmetical structures $(\mathbf{d},\mathbf{r}) \in \mathcal{A}(C_n)$ such that $\sum_{i=1}^{n} d_i = k$ is*

$$\left(\!\!\binom{n}{k-2n}\!\!\right) = \binom{k-n-1}{k-2n} \textit{ for every } n \geq 2 \textit{ and } 2n \leq k \leq 3n-1.$$

3.3.3 Small Arithmetical Graphs

In this section we list the arithmetical structures of some simple connected graphs with $2 \leq n \leq 5$ vertices.

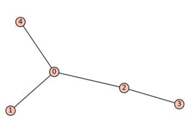
Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$	# of A. S.'s
K_2	$((1,1),(1,1))$	1
P_3	$((2,1,2),(1,2,1))$ $((1,2,1),(1,1,1))$	2
K_3	$((1,5,2),(3,1,2))$ $((1,3,3),(2,1,1))$ $((1,2,5),(3,2,1))$ $((2,5,1),(2,1,3))$ $((2,2,2),(1,1,1))$ $((2,1,5),(2,3,1))$ $((3,3,1),(1,1,2))$ $((3,1,3),(1,2,1))$ $((5,2,1),(1,2,3))$ $((5,1,2),(1,3,2))$	10
P_4	$((2,1,2,1),(1,1,1,1))$ $((2,1,3,1),(1,2,1,1))$ $((3,1,1,2),(1,1,2,1))$ $((2,2,1,3),(1,2,3,1))$ $((3,1,2,2),(1,2,2,1))$	5

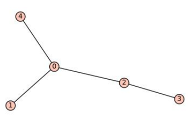
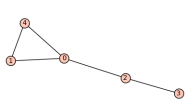
Recall that S_m is the star graph with m leaves, T_2 is defined on definition 2.3.4 and let Q_n be the graph $K_n - \{e\}$ for some $e \in E(V)$, that is, the graph resulting from removing one edge to the complete graph with n vertices.

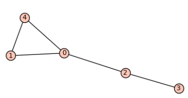
Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
S_3	$((2,1,2,2),(2,2,1,1))$,	$((1,2,6,3),(6,3,1,2))$	14
	$((1,2,4,4),(4,2,1,1))$,	$((1,2,3,6),(6,3,2,1))$	
	$((3,1,1,1),(1,1,1,1))$,	$((2,2,2,1),(2,1,1,2))$	
	$((2,2,1,2),(2,1,2,1))$,	$((1,3,6,2),(6,2,1,3))$	
	$((1,3,3,3),(3,1,1,1))$,	$((1,3,2,6),(6,2,3,1))$	
	$((1,4,4,2),(4,1,1,2))$,	$((1,4,2,4),(4,1,2,1))$	
	$((1,6,3,2),(6,1,2,3))$,	$((1,6,2,3),(6,1,3,2))$	
T_2	$((2,1,6,1),(2,3,1,1))$,	$((1,2,6,1),(3,2,1,1))$	26
	$((3,1,4,1),(1,2,1,1))$,	$((2,2,3,1),(1,1,1,1))$	
	$((1,3,4,1),(2,1,1,1))$,	$((5,1,3,1),(1,3,2,2))$	
	$((3,3,2,1),(1,1,2,2))$,	$((1,5,3,1),(3,1,2,2))$	
	$((6,1,2,5),(2,7,5,1))$,	$((5,2,2,1),(1,2,3,3))$	
	$((2,5,2,1),(2,1,3,3))$,	$((1,6,2,5),(7,2,5,1))$	
	$((7,1,2,3),(1,4,3,1))$,	$((6,2,1,11),(3,7,11,1))$	
	$((4,4,1,3),(1,1,3,1))$,	$((2,6,1,11),(7,3,11,1))$	
	$((1,7,2,3),(4,1,3,1))$,	$((9,1,2,2),(1,5,4,2))$	
	$((8,2,1,5),(1,3,5,1))$,	$((5,5,1,2),(1,1,4,2))$	
	$((2,8,1,5),(3,1,5,1))$,	$((1,9,2,2),(5,1,4,2))$	
	$((11,3,1,2),(1,3,8,4))$,	$((3,11,1,2),(3,1,8,4))$	
	$((14,2,1,3),(1,5,9,3))$,	$((2,14,1,3),(5,1,9,3))$	
C_4	$((2,1,5,3),(3,5,1,2))$,	$((2,1,5,3),(3,5,1,2))$	35
	$((2,1,2,6),(3,4,2,1))$,	$((1,2,6,2),(4,3,1,2))$	
	$((1,2,4,3),(3,2,1,1))$,	$((1,2,3,5),(5,3,2,1))$	
	$((3,1,5,2),(2,5,1,3))$,	$((3,1,2,3),(1,2,1,1))$	
	$((3,1,1,6),(2,3,3,1))$,	$((2,2,6,1),(2,3,1,4))$	
	$((2,2,2,2),(1,1,1,1))$,	$((2,2,1,6),(3,2,4,1))$	
	$((1,3,6,1),(3,2,1,3))$,	$((1,3,3,2),(2,1,1,1))$	
	$((1,3,2,5),(5,2,3,1))$,	$((4,1,3,2),(1,3,1,2))$	
	$((4,1,1,4),(1,2,2,1))$,	$((3,2,4,1),(1,2,1,3))$	

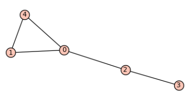
Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$			# of A. S.'s
	((3, 2, 1, 3), (1, 1, 2, 1)),	((2, 3, 3, 1), (1, 1, 1, 2)),	((2, 3, 1, 4), (2, 1, 3, 1))	
C_4	((1, 4, 4, 1), (2, 1, 1, 2)),	((1, 4, 2, 3), (3, 1, 2, 1)),	((6, 1, 2, 2), (1, 4, 2, 3))	35
	((6, 1, 1, 3), (1, 3, 3, 2)),	((5, 2, 3, 1), (1, 3, 2, 5)),	((4, 3, 1, 2), (1, 1, 3, 2))	
	((3, 4, 2, 1), (1, 1, 2, 3)),	((2, 5, 1, 3), (3, 1, 5, 2)),	((1, 6, 3, 1), (3, 1, 2, 3))	
	((1, 6, 2, 2), (4, 1, 3, 2)),	((6, 2, 1, 2), (1, 2, 4, 3)),	((5, 3, 2, 1), (1, 2, 3, 5))	
	((3, 5, 1, 2), (2, 1, 5, 3)),	((2, 6, 2, 1), (2, 1, 3, 4))		
	((2, 1, 21, 3), (6, 11, 1, 4)),	((2, 1, 13, 4), (4, 7, 1, 2)),	((2, 1, 9, 6), (3, 5, 1, 1))	
	((2, 1, 7, 10), (5, 8, 2, 1)),	((2, 1, 6, 18), (9, 14, 4, 1)),	((1, 2, 14, 2), (6, 5, 1, 3))	
	((1, 2, 8, 4), (4, 3, 1, 1)),	((1, 2, 6, 10), (10, 7, 3, 1)),	((3, 1, 21, 2), (4, 11, 1, 6))	
	((3, 1, 9, 3), (2, 5, 1, 2)),	((3, 1, 5, 6), (4, 9, 3, 2)),	((2, 2, 14, 1), (3, 5, 1, 6))	
	((2, 2, 5, 2), (3, 4, 2, 3)),	((1, 3, 11, 1), (4, 3, 1, 4)),	((4, 1, 13, 2), (2, 7, 1, 4))	
	((4, 1, 5, 4), (1, 3, 1, 1)),	((4, 1, 3, 12), (3, 8, 4, 1)),	((3, 2, 2, 6), (2, 3, 3, 1))	
	((2, 3, 3, 2), (1, 1, 1, 1)),	((1, 4, 4, 2), (2, 1, 1, 1)),	((4, 2, 8, 1), (1, 3, 1, 4))	
	((4, 2, 2, 4), (1, 2, 2, 1)),	((2, 4, 4, 1), (1, 1, 1, 2)),	((1, 5, 5, 1), (2, 1, 1, 2))	
	((6, 1, 9, 2), (1, 5, 1, 3)),	((6, 1, 5, 3), (2, 9, 3, 4)),	((6, 1, 3, 6), (1, 4, 2, 1))	
	((6, 1, 2, 30), (5, 18, 12, 1)),	((4, 3, 1, 12), (3, 4, 8, 1)),	((2, 5, 2, 2), (3, 2, 4, 3))	
Q_n	((1, 6, 2, 10), (10, 3, 7, 1)),	((6, 2, 2, 3), (1, 3, 3, 2)),	((6, 2, 1, 30), (5, 12, 18, 1))	63
	((3, 5, 1, 6), (4, 3, 9, 2)),	((2, 6, 1, 18), (9, 4, 14, 1)),	((6, 3, 1, 6), (1, 2, 4, 1))	
	((4, 5, 1, 4), (1, 1, 3, 1)),	((2, 7, 1, 10), (5, 2, 8, 1)),	((1, 8, 2, 4), (4, 1, 3, 1))	
	((10, 1, 7, 2), (1, 8, 2, 5)),	((10, 1, 2, 10), (1, 6, 4, 1)),	((6, 5, 1, 3), (2, 3, 9, 4))	
	((2, 9, 1, 6), (3, 1, 5, 1)),	((10, 2, 6, 1), (1, 7, 3, 10)),	((10, 2, 1, 10), (1, 4, 6, 1))	
	((4, 8, 2, 1), (1, 1, 3, 4)),	((3, 9, 1, 3), (2, 1, 5, 2)),	((1, 11, 3, 1), (4, 1, 3, 4))	
	((12, 1, 3, 4), (1, 8, 4, 3)),	((12, 3, 1, 4), (1, 4, 8, 3)),	((6, 9, 1, 2), (1, 1, 5, 3))	
	((2, 13, 1, 4), (4, 1, 7, 2)),	((1, 14, 2, 2), (6, 1, 5, 3)),	((10, 6, 2, 1), (1, 3, 7, 10))	
	((2, 14, 2, 1), (3, 1, 5, 6)),	((10, 7, 1, 2), (1, 2, 8, 5)),	((4, 13, 1, 2), (2, 1, 7, 4))	
	((18, 1, 6, 2), (1, 14, 4, 9)),	((2, 21, 1, 3), (6, 1, 11, 4)),	((8, 6, 1, 2), (1, 4, 14, 9))	
	((3, 21, 1, 2), (4, 1, 11, 6)),	((30, 1, 2, 6), (1, 18, 12, 5)),	((30, 2, 1, 6), (1, 12, 18, 5))	

The last graph with four vertices is K_4, which have 215 arithmetical structures, we do not include its arithmetical structures in here but in the following table we show some other arithmetical graphs of five vertices, for example, the arithmetical structures of the path and cycle of five vertices.

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
P_5	$((2,2,1,4,2),(2,3,4,1,1))$, $((1,2,2,2,4),(4,3,2,1,1))$, $((3,1,2,3,2),(2,5,3,1,1))$, $((2,2,2,1,1),(1,1,1,1,1))$, $((1,4,1,2,2),(2,1,2,1,1))$, $((1,2,3,1,3),(3,2,1,1,1))$, $((3,1,3,1,1),(1,2,1,1,1))$,	$((1,3,1,3,3),(3,2,3,1,1))$ $((3,2,1,3,1),(1,2,3,1,1))$ $((2,3,1,2,1),(1,1,2,1,1))$ $((2,1,3,2,3),(3,5,2,1,1))$ $((1,3,2,1,2),(2,1,1,1,1))$ $((4,1,2,2,1),(1,3,2,1,1))$ $((2,1,4,1,2),(2,3,1,1,1))$	14
	$((3,1,1,3,2),(2,2,3,1,1))$, $((2,2,1,5,4),(4,2,5,1,1))$, $((2,1,2,2,3),(3,3,2,1,1))$, $((3,2,1,3,1),(2,1,3,1,2))$, $((3,1,2,1,1),(1,1,1,1,1))$, $((2,3,1,4,3),(3,1,4,1,1))$, $((2,2,2,1,2),(2,1,2,2,1))$,	$((2,2,1,7,3),(6,3,7,1,2))$ $((2,2,1,4,6),(6,3,8,2,1))$ $((4,1,1,2,1),(1,1,2,1,1))$ $((3,2,1,2,2),(2,1,4,2,1))$ $((2,3,1,7,2),(6,2,7,1,3))$ $((2,3,1,3,6),(6,2,9,3,1))$ $((2,1,3,1,2),(2,2,1,1,1))$	46

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
	((1, 3, 2, 11, 7), (21, 7, 11, 1, 3)), ((1, 3, 2, 3, 15), (15, 5, 9, 3, 1)), ((1, 2, 3, 3, 8), (8, 4, 3, 1, 1)), ((2, 4, 1, 5, 2), (4, 1, 5, 1, 2)), ((2, 3, 2, 2, 1), (3, 1, 2, 1, 3)), ((1, 4, 2, 2, 12), (12, 3, 8, 4, 1)), ((1, 2, 4, 1, 6), (6, 3, 2, 2, 1)), ((1, 4, 3, 1, 4), (4, 1, 2, 2, 1)), ((1, 2, 5, 1, 4), (4, 2, 1, 1, 1)), ((2, 6, 1, 3, 3), (6, 1, 9, 3, 2)), ((1, 7, 2, 11, 3), (21, 3, 11, 1, 7)), ((1, 4, 5, 1, 2), (4, 1, 1, 1, 2)), ((1, 7, 3, 5, 2), (14, 2, 5, 1, 7)), ((1, 3, 7, 1, 2), (6, 2, 1, 1, 3)), ((1, 8, 3, 3, 2), (8, 1, 3, 1, 4)), ((1, 12, 2, 2, 4), (12, 1, 8, 4, 3)),	((1, 3, 2, 5, 9), (9, 3, 5, 1, 1)) ((1, 2, 3, 5, 7), (14, 7, 5, 1, 2)) ((1, 2, 3, 2, 10), (10, 5, 4, 2, 1)) ((2, 4, 1, 3, 4), (4, 1, 6, 2, 1)) ((2, 2, 3, 1, 1), (2, 1, 1, 1, 2)) ((1, 3, 3, 1, 6), (6, 2, 3, 3, 1)) ((1, 5, 2, 3, 5), (5, 1, 3, 1, 1)) ((1, 3, 4, 1, 3), (3, 1, 1, 1, 1)) ((2, 6, 1, 4, 2), (6, 1, 8, 2, 3)) ((1, 6, 2, 2, 6), (6, 1, 4, 2, 1)) ((1, 6, 3, 1, 3), (6, 1, 3, 3, 2)) ((1, 2, 7, 1, 3), (6, 3, 1, 1, 2)) ((1, 6, 4, 1, 2), (6, 1, 2, 2, 3)) ((1, 9, 2, 5, 3), (9, 1, 5, 1, 3)) ((1, 10, 3, 2, 2), (10, 1, 4, 2, 5)) ((1, 15, 2, 3, 3), (15, 1, 9, 3, 5))	46
	((3, 1, 1, 6, 6), (5, 7, 6, 1, 2)), ((3, 1, 1, 3, 9), (4, 5, 6, 2, 1)), ((2, 2, 1, 6, 8), (5, 3, 6, 1, 1)), ((2, 1, 2, 5, 10), (9, 11, 5, 1, 2)), ((2, 1, 2, 2, 13), (6, 7, 4, 2, 1)), ((3, 2, 1, 2, 5), (3, 2, 6, 3, 1)), ((3, 1, 2, 1, 5), (2, 3, 2, 2, 1)), ((2, 2, 2, 3, 3), (5, 4, 3, 1, 3)), ((2, 1, 3, 1, 9), (4, 5, 2, 2, 1)), ((1, 3, 2, 10, 13), (19, 7, 10, 1, 2)),	((3, 1, 1, 4, 7), (3, 4, 4, 1, 1)) ((2, 2, 1, 12, 6), (11, 7, 12, 1, 3)) ((2, 2, 1, 4, 14), (9, 5, 12, 3, 1)) ((2, 1, 2, 3, 11), (5, 6, 3, 1, 1)) ((4, 1, 1, 2, 5), (2, 3, 4, 2, 1)) ((3, 1, 2, 2, 4), (3, 5, 2, 1, 2)) ((2, 3, 1, 3, 11), (8, 3, 12, 4, 1)) ((2, 2, 2, 1, 5), (3, 2, 3, 3, 1)) ((1, 3, 2, 18, 12), (35, 13, 18, 1, 4)) ((1, 3, 2, 6, 15), (11, 4, 6, 1, 1))	102

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
	((1, 3, 2, 4, 19), (14, 5, 8, 2, 1)), ((1, 2, 3, 10, 15), (29, 16, 10, 1, 3)), ((1, 2, 3, 2, 23), (15, 8, 6, 3, 1)), ((4, 2, 1, 2, 2), (1, 1, 2, 1, 1)), ((3, 3, 1, 2, 3), (2, 1, 4, 2, 1)), ((2, 4, 1, 4, 4), (3, 1, 4, 1, 1)), ((2, 3, 2, 1, 3), (2, 1, 2, 2, 1)), ((1, 4, 2, 14, 7), (27, 8, 14, 1, 5)), ((1, 4, 2, 2, 19), (15, 4, 10, 5, 1)), ((1, 3, 3, 2, 7), (5, 2, 2, 1, 1)), ((1, 2, 4, 5, 10), (19, 11, 5, 1, 3)), ((1, 2, 4, 1, 14), (9, 5, 3, 3, 1)), ((1, 5, 2, 2, 11), (9, 2, 6, 3, 1)), ((7, 1, 1, 2, 2), (1, 3, 2, 1, 2)), ((5, 3, 1, 2, 1), (1, 1, 2, 1, 2)), ((3, 5, 1, 2, 2), (3, 1, 6, 3, 2)), ((2, 6, 1, 12, 2), (11, 3, 12, 1, 7)), ((2, 1, 6, 1, 6), (5, 7, 1, 1, 2)), ((1, 5, 3, 1, 5), (4, 1, 2, 2, 1)), ((1, 2, 6, 1, 8), (5, 3, 1, 1, 1)), ((6, 2, 2, 1, 1), (1, 2, 1, 1, 3)), ((3, 6, 1, 6, 1), (5, 2, 6, 1, 7)), ((1, 7, 2, 14, 4), (27, 5, 14, 1, 8)), ((1, 6, 3, 6, 3), (17, 4, 6, 1, 7)), ((1, 3, 6, 2, 4), (11, 5, 2, 1, 4)), ((3, 7, 1, 4, 1), (3, 1, 4, 1, 4)), ((1, 7, 3, 2, 3), (5, 1, 2, 1, 2)), ((1, 9, 2, 4, 4), (7, 1, 4, 1, 2)), ((2, 7, 4, 1, 1), (3, 1, 1, 1, 4)), ((2, 10, 2, 5, 1), (9, 2, 5, 1, 11)), ((2, 6, 6, 1, 1), (5, 2, 1, 1, 7)),	((1, 3, 2, 3, 27), (20, 7, 12, 4, 1)) ((1, 2, 3, 4, 17), (11, 6, 4, 1, 1)) ((5, 1, 1, 2, 3), (1, 2, 2, 1, 1)) ((4, 1, 2, 1, 3), (1, 2, 1, 1, 1)) ((3, 2, 2, 1, 2), (1, 1, 1, 1, 1)) ((2, 3, 2, 3, 2), (5, 3, 3, 1, 4)) ((2, 1, 4, 1, 7), (3, 4, 1, 1, 1)) ((1, 4, 2, 4, 9), (7, 2, 4, 1, 1)) ((1, 3, 3, 6, 6), (17, 7, 6, 1, 4)) ((1, 3, 3, 1, 11), (8, 3, 4, 4, 1)) ((1, 2, 4, 2, 11), (7, 4, 2, 1, 1)) ((2, 5, 1, 3, 5), (4, 1, 6, 2, 1)) ((1, 3, 4, 2, 5), (7, 3, 2, 1, 2)) ((6, 1, 2, 1, 2), (1, 3, 1, 1, 2)) ((4, 3, 2, 1, 1), (1, 1, 1, 1, 2)) ((3, 4, 2, 2, 1), (3, 2, 2, 1, 5)) ((2, 5, 2, 1, 2), (3, 1, 3, 3, 2)) ((1, 6, 2, 3, 6), (5, 1, 3, 1, 1)) ((1, 4, 4, 1, 4), (3, 1, 1, 1, 1)) ((7, 2, 1, 2, 1), (1, 2, 2, 1, 3)) ((4, 5, 1, 2, 1), (2, 1, 4, 2, 3)) ((3, 5, 2, 1, 1), (2, 1, 2, 2, 3)) ((1, 7, 2, 2, 7), (6, 1, 4, 2, 1)) ((1, 5, 4, 2, 3), (7, 2, 2, 1, 3)) ((1, 2, 7, 2, 7), (13, 8, 2, 1, 3)) ((2, 8, 1, 6, 2), (5, 1, 6, 1, 3)) ((1, 4, 6, 2, 3), (11, 4, 2, 1, 5)) ((3, 9, 1, 3, 1), (4, 1, 6, 2, 5)) ((2, 11, 1, 3, 3), (8, 1, 12, 4, 3)) ((2, 9, 3, 1, 1), (4, 1, 2, 2, 5)) ((1, 11, 2, 2, 5), (9, 1, 6, 3, 2))	102

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
 	((2, 11, 2, 3, 1), (5, 1, 3, 1, 6)), ((1, 11, 3, 1, 3), (8, 1, 4, 4, 3)), ((1, 8, 6, 1, 2), (5, 1, 1, 1, 3)), ((1, 2, 12, 1, 6), (11, 7, 1, 1, 3)), ((1, 11, 4, 2, 2), (7, 1, 2, 1, 4)), ((2, 13, 2, 2, 1), (6, 1, 4, 2, 7)), ((1, 15, 3, 10, 2), (29, 3, 10, 1, 16)), ((1, 6, 12, 1, 2), (11, 3, 1, 1, 7)), ((1, 19, 2, 4, 3), (14, 1, 8, 2, 5)), ((1, 23, 3, 2, 2), (15, 1, 6, 3, 8)),	((1, 12, 2, 18, 3), (35, 4, 18, 1, 13)) ((1, 10, 4, 5, 2), (19, 3, 5, 1, 11)) ((1, 7, 7, 2, 2), (13, 3, 2, 1, 8)) ((1, 13, 2, 10, 3), (19, 2, 10, 1, 7)) ((2, 14, 1, 4, 2), (9, 1, 12, 3, 5)) ((1, 15, 2, 6, 3), (11, 1, 6, 1, 4)) ((1, 14, 4, 1, 2), (9, 1, 3, 3, 5)) ((1, 17, 3, 4, 2), (11, 1, 4, 1, 6)) ((1, 19, 2, 2, 4), (15, 1, 10, 5, 4)) ((1, 27, 2, 3, 3), (20, 1, 12, 4, 7))	102
C_5	((2, 2, 1, 7, 2), (3, 4, 5, 1, 2)), ((2, 2, 1, 4, 5), (3, 5, 7, 2, 1)), ((1, 3, 1, 4, 4), (3, 2, 3, 1, 1)), ((1, 2, 2, 5, 4), (7, 5, 3, 1, 2)), ((1, 2, 2, 2, 7), (5, 4, 3, 2, 1)), ((3, 2, 1, 4, 2), (1, 2, 3, 1, 1)) ((3, 1, 2, 6, 2), (3, 7, 4, 1, 2)) ((3, 1, 2, 3, 5), (3, 8, 5, 2, 1)) ((2, 3, 1, 3, 2), (1, 1, 2, 1, 1)) ((2, 2, 2, 7, 1), (4, 3, 2, 1, 5)) ((2, 2, 2, 1, 7), (2, 3, 4, 5, 1)) ((2, 1, 3, 3, 4), (3, 5, 2, 1, 1)) ((1, 4, 1, 6, 2), (5, 2, 3, 1, 3)) ((1, 4, 1, 2, 6), (3, 2, 5, 3, 1)) ((1, 3, 2, 2, 3), (2, 1, 1, 1, 1)) ((1, 2, 3, 5, 3), (8, 5, 2, 1, 3)) ((1, 2, 3, 1, 7), (4, 3, 2, 3, 1)) ((4, 2, 1, 3, 3), (1, 3, 5, 2, 1)) ((4, 1, 2, 3, 2), (1, 3, 2, 1, 1)) ((3, 3, 1, 4, 1), (1, 1, 2, 1, 2))	((2, 2, 1, 5, 3), (2, 3, 4, 1, 1)) ((1, 3, 1, 6, 3), (5, 3, 4, 1, 2)) ((1, 3, 1, 3, 6), (4, 3, 5, 2, 1)) ((1, 2, 2, 3, 5), (4, 3, 2, 1, 1)) ((3, 2, 1, 7, 1), (2, 3, 4, 1, 3)) ((3, 2, 1, 3, 5), (2, 5, 8, 3, 1)) ((3, 1, 2, 4, 3), (2, 5, 3, 1, 1)) ((2, 3, 1, 7, 1), (3, 2, 3, 1, 4)) ((2, 3, 1, 2, 6), (2, 3, 7, 4, 1)) ((2, 2, 2, 2, 2), (1, 1, 1, 1, 1)) ((2, 1, 3, 5, 3), (5, 8, 3, 1, 2)) ((2, 1, 3, 2, 6), (4, 7, 3, 2, 1)) ((1, 4, 1, 3, 3), (2, 1, 2, 1, 1)) ((1, 3, 2, 6, 2), (7, 3, 2, 1, 4)) ((1, 3, 2, 1, 7), (3, 2, 3, 4, 1)) ((1, 2, 3, 2, 4), (3, 2, 1, 1, 1)) ((4, 2, 1, 5, 1), (1, 2, 3, 1, 2)) ((4, 1, 2, 6, 1), (2, 5, 3, 1, 3)) ((4, 1, 2, 2, 5), (2, 7, 5, 3, 1)) ((3, 3, 1, 2, 4), (1, 2, 5, 3, 1))	126

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
	$((3,2,2,3,1),(1,1,1,1,2))$	$((3,2,2,1,5),(1,2,3,4,1))$	
	$((3,1,3,6,1),(3,5,2,1,4))$	$((3,1,3,2,2),(1,2,1,1,1))$	
	$((3,1,3,1,6),(2,5,3,4,1))$	$((2,4,1,5,1),(2,1,2,1,3))$	
	$((2,4,1,2,3),(1,1,3,2,1))$	$((2,3,2,4,1),(2,1,1,1,3))$	
	$((2,3,2,1,4),(1,1,2,3,1))$	$((2,2,3,5,1),(3,2,1,1,4))$	
	$((2,2,3,1,3),(1,1,1,2,1))$	$((2,1,4,5,2),(5,7,2,1,3))$	
	$((2,1,4,2,3),(2,3,1,1,1))$	$((2,1,4,1,6),(3,5,2,3,1))$	
	$((1,5,1,4,2),(3,1,2,1,2))$	$((1,5,1,2,4),(2,1,3,2,1))$	
	$((1,4,2,3,2),(3,1,1,1,2))$	$((1,4,2,1,5),(2,1,2,3,1))$	
	$((1,3,3,4,2),(5,2,1,1,3))$	$((1,3,3,1,4),(2,1,1,2,1))$	
	$((1,2,4,3,3),(5,3,1,1,2))$	$((1,2,4,1,5),(3,2,1,2,1))$	
	$((5,1,2,4,1),(1,3,2,1,2))$	$((5,1,2,2,3),(1,4,3,2,1))$	
	$((4,1,3,3,1),(1,2,1,1,2))$	$((4,1,3,1,4),(1,3,2,3,1))$	
	$((3,1,4,4,1),(2,3,1,1,3))$	$((3,1,4,1,3),(1,2,1,2,1))$	
C_5	$((2,1,5,3,2),(3,4,1,1,2))$	$((2,1,5,1,4),(2,3,1,2,1))$	126
	$((6,2,1,4,1),(1,3,5,2,3))$	$((6,2,1,3,2),(1,4,7,3,2))$	
	$((5,3,1,2,3),(1,3,8,5,2))$	$((5,2,2,1,4),(1,3,5,7,2))$	
	$((4,4,1,3,1),(1,1,3,2,3))$	$((4,2,3,2,1),(1,1,1,2,3))$	
	$((3,5,1,2,2),(1,1,4,3,2))$	$((3,4,2,1,3),(1,1,3,5,2))$	
	$((3,2,4,1,2),(1,1,1,3,2))$	$((2,6,1,4,1),(3,1,3,2,5))$	
	$((2,4,3,3,1),(3,1,1,2,5))$	$((2,2,5,4,1),(5,3,1,2,7))$	
	$((1,7,1,3,2),(4,1,3,2,3))$	$((1,7,1,2,3),(3,1,4,3,2))$	
	$((1,6,2,1,4),(3,1,3,5,2))$	$((1,5,3,2,2),(4,1,1,2,3))$	
	$((1,4,4,1,3),(3,1,1,3,2))$	$((1,3,5,3,2),(8,3,1,2,5))$	
	$((1,2,6,2,3),(7,4,1,2,3))$	$((1,2,6,1,4),(5,3,1,3,2))$	
	$((7,1,2,3,1),(1,4,3,2,3))$	$((7,1,2,2,2),(1,5,4,3,2))$	
	$((6,3,1,3,1),(1,2,5,3,4))$	$((6,1,3,1,3),(1,4,3,5,2))$	
	$((5,4,1,2,2),(1,2,7,5,3))$	$((5,3,2,2,1),(1,1,2,3,4))$	
	$((5,3,2,1,3),(1,2,5,8,3))$	$((5,1,4,2,1),(1,2,1,2,3))$	
	$((4,3,3,1,2),(1,1,2,5,3))$	$((4,1,5,1,2),(1,2,1,3,2))$	
	$((3,6,1,3,1),(2,1,4,3,5))$	$((3,3,4,2,1),(2,1,1,3,5))$	
	$((3,1,6,3,1),(3,4,1,2,5))$	$((2,7,1,2,2),(2,1,5,4,3))$	
	$((2,6,2,3,1),(4,1,2,3,7))$	$((2,6,2,1,3),(2,1,4,7,3))$	

Graph	Arithmetical Structures $(\mathbf{d}, \mathbf{r})$		# of A. S.'s
C_5	$((2,3,5,3,1),(5,2,1,3,8))$ $((2,1,7,2,2),(4,5,1,2,3))$ $((1,7,2,2,2),(5,1,2,3,4))$ $((1,4,5,2,2),(7,2,1,3,5))$ $((7,2,2,2,1),(1,2,3,4,5))$ $((6,2,3,1,2),(1,2,3,7,4))$ $((4,5,2,2,1),(2,1,3,5,7))$ $((3,5,3,2,1),(3,1,2,5,8))$ $((3,2,6,2,1),(3,2,1,4,7))$ $((2,5,4,1,2),(3,1,2,7,5))$	$((2,3,5,1,2),(2,1,1,4,3))$ $((2,1,7,1,3),(3,4,1,3,2))$ $((1,6,3,1,3),(4,1,2,5,3))$ $((1,3,6,1,3),(5,2,1,4,3))$ $((7,1,3,2,1),(1,3,2,3,4))$ $((6,1,4,1,2),(1,3,2,5,3))$ $((4,1,6,2,1),(2,3,1,3,5))$ $((3,5,3,1,2),(2,1,3,8,5))$ $((3,1,7,1,2),(2,3,1,4,3))$ $((2,2,7,1,2),(3,2,1,5,4))$	126

For the rest of graphs with five vertices the number of arithmetical structures grows; in fact the complete graph with five vertices, K_5, have more than 2300 arithmetical structures. Moreover, we finish this chapter with the following (see [6]).

Conjecture 3.3.7. *Let G be a simple connected graph with n vertices, then*

$$\Big|\mathcal{A}(P_n)\Big| \leq \Big|\mathcal{A}(G)\Big| \leq \Big|\mathcal{A}(K_n)\Big|.$$

Chapter 4

Critical ideals of Threshold graphs

4.1 On the algebraic co-rank of Threshold Graphs

Given that every threshold graph can be obtained from exploding vertices of a graph T_n [1] for some positive integer n, then we may concentrate to the Critical Ideals of the family $\{T_n \mid n \geq 1\}$ and in particular we study the co-rank of this family of threshold graphs. For this, let us take the corollary 2.3.6 and number every vertex on T_n accordingly with the degree class they belong.

Theorem 4.1.1. *For every $n \in \mathbb{N}$, there exists a square sub-matrix U_n of $L(T_n)$ of size $n + \lfloor \frac{n}{3} \rfloor$ and with determinant equal to ± 1, that is, $\gamma(T_n) \geq n + \lfloor \frac{n}{3} \rfloor$*

Proof. Let $k \in \mathbb{N}$, and let $L(T_n).(I, J)$ be the sub-matrix of $L(T_n)$ obtained from taking the rows and the columns in I and J respectively, then we exhibit the matrix U_n.

$$U_n = \begin{cases} L(T_n).([2,4,5,7,8,...,(2n-3),(2n-2),(2n)],[3,4,6,7,9,10,...,(2n-1),(2n)]), & \text{if } n = 3k-1, \\ L(T_n).([1,3,4,6,7,...,(2n-3),(2n-2),(2n)],[2,3,5,6,...,(2n-1),(2n)]), & \text{if n=3k }, \\ L(T_n).([1,3,5,6,8,9,...,(2n-3),(2n-2),(2n)],[2,4,5,7,8,...,(2n-1),(2n)]), & \text{if n=3k + 1 }. \end{cases}$$

where the row (column) i (j) is the row (column) that corresponds to the variable x_i (x_j). □

The reader can see in the appendix A.2 the algorithm which helped to find the submatrices that we just presented for the proof. Moreover, on the above theorem we have a lower bound for the algebraic co-rank of the graph T_n for a given $n \in \mathbb{N}$, yet we think this is an upper bound as well.

Conjecture 4.1.2. $\gamma(T_n) = n + \lfloor \frac{n}{3} \rfloor$ *for every positive integer n*

4.2 The base/stable case

In this case we present the stable case of the family of T_n graphs. Now we define the graph $T_n^{'}$ as the graph obtained from T_n by duplicating every odd numbered vertex, therefore we write the vertices of $T_n^{'}$ as follows:

$$V(T_n^{'}) = V(T_n) \cup \{ v_1^{'}, v_2^{'}, \ldots v_{2n-1}^{'} \}$$

Remark 4.2.1. *Note that, in other words,* $T_n^{'} = T_n^{\mathbf{d}}$ *with* $\mathbf{d} = (1,0,1,0,\ldots,1,0)$,

For this family of graphs we have the next result [see Corollary 2.2.11]

Theorem 4.2.2. $\gamma(T_n^{'}) = 2n$

Before presenting the proof of the theorem above we need to recall some basic properties of the algebraic co-rank of a graph related to the stability and the clique number of the graph.
For this, let us first define the stability and the clique numbers of a graph. A subset S of the vertices of a graph G is called *stable* or *independent* if there is no edge of G with ends in S. A stable set is called *maximal* if it is under the inclusion of sets.
The *stability number* of G, denoted by $\alpha(G)$, is given by

$$\alpha(G) = max\{|S| \mid \text{S is a stable set of G}\}.$$

In a similar way, a subset C of the vertices of a graph G is called a *clique* if all the pairs of vertices in C are joined by an edge of G. A clique set is called *maximal* if it is under the inclusion of sets.
The *clique number* of G, denoted by $\omega(G)$, is given by

$$\omega(G) = max\{|C| \mid \text{C is a clique set of G}\}.$$

Lemma 4.2.3 ([5], Lemma 3.11). *If G is a graph (possibly directed and with multiple edges) and V is a vertex of G, then*

$$\gamma(G) - \gamma(G \setminus v) \leq 2.$$

We refer the reader to [5] for the proof of the Lemma above. Now, is easy to prove that

Theorem 4.2.4 ([5], Theorem 3.13). *If G is a graph (possibly directed graph with multiple edges) with n vertices, then*

$$\gamma(G) \leq 2(n - \omega(G)) + 1 \text{ and } \gamma(G) \leq 2(n - \alpha(G)).$$

Proof. From the last lemma we have the following inequality $\gamma(G) - \gamma(G \setminus v) \leq 2$ for all $v \in V(G)$ and together with facts that the trivial graph (the graph with no edges) has algebraic co-rank zero, and that the complete graph has algebraic co-rank 1, we have the result □

Therefore, we now can prove our theorem.

Proof of Theorem 4.2.2. First, the fact that $\gamma(T_n^{'}) \leq 2n$ comes from the above theorem, because clearly the stability number of $T_n^{'}$, $\alpha(T_n^{'})$, is equal to $2n$, and then we have that $\gamma(T_n^{'}) \leq 2(3n - \alpha(T_n^{'})) = 2(3n - 2n) = 2n$.
Now, to see the other inequality we present the following square sub-matrix of the Laplacian matrix of $T_n^{'}$ with length $2n$ and *unitary* determinant (we re-order rows and columns to see this clear);

$$A(T_n^{'}) = \left(\begin{array}{ccccc|ccccc}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & -1 & -1 \\
0 & 0 & \dots & 0 & 0 & -1 & -1 & \dots & -1 & -1 \\
\hline
0 & 0 & \vdots & 0 & -1 & x_2 & -1 & \dots & -1 & -1 \\
0 & 0 & \dots & -1 & -1 & -1 & x_4 & \dots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \dots & -1 & -1 & -1 & -1 & \dots & x_{2n-2} & -1 \\
-1 & -1 & \dots & -1 & -1 & -1 & -1 & \dots & -1 & x_{2n}
\end{array}\right)$$

In other words,
$A(T_n^{'}) = L(T_n^{'}).[((2n-1)^{'}, \dots 3^{'}, 1^{'}, 2, 4, \dots 2n), ((2n-1), \dots 3, 1, 2, 4, \dots 2n)].$

□

Finally, the following is our conjecture for the structure of the first non-trivial critical ideals of the $T_n^{'}$'s graphs.

Conjecture 4.2.5. *Let* $n \in \mathbb{N}$,

$$I_{2n+1}(T_n^{'}, X, X^{'}) = \Bigg\{ x_1x_2x_1^{'} - x_1 - x_1^{'}, x_3(x_1x_1^{'} + x_1 + x_1^{'}), x_3^{'}(x_1x_1^{'} + x_1 + x_1^{'}),$$
$$\Big\{ (x_{2i}+1)x_{2i-1}x_{2i+1}, (x_{2i}+1)x_{2i-1}x_{2i+1}^{'}, (x_{2i}+1)x_{2i-1}^{'}x_{2i+1}, (x_{2i}+1)x_{2i-1}^{'}x_{2i+1}^{'} \Big\}_{i=1}^{i=n-1},$$
$$\Big\{ x_{2i-1}(x_{2i+1}x_{2i+1}^{'}x_{2i+2} + x_{2i+1}x_{2i+1}^{'} - x_{2i+1} - x_{2i+1}^{'}), x_{2i-1}^{'}(x_{2i+1}x_{2i+1}^{'}x_{2i+2} + x_{2i+1}x_{2i+1}^{'} - x_{2i+1} - x_{2i+1}^{'}) \Big\}_{i=1}^{i=n-1},$$
$$\Big\{ x_{2i+1}x_{2i+1}^{'}x_{2i+2} + x_{2i+1}x_{2i+1}^{'}x_{2i} + 2x_{2i+1}x_{2i+1}^{'} - x_{2i+1} - x_{2i+1}^{'} \Big\}_{i=1}^{i=n-1},$$
$$\Big\{ (x_{2i+1} + x_{2i+1}^{'})x_{2i-1}x_{2i+3}, (x_{2i+1} + x_{2i+1}^{'})x_{2i-1}x_{2i+3}^{'}, (x_{2i+1} + x_{2i+1}^{'})x_{2i-1}^{'}x_{2i+3},$$
$$(x_{2i+1} + x_{2i+1}^{'})x_{2i-1}^{'}x_{2i+3}^{'} \Big\}_{i=1}^{i=n-2} \Bigg\}$$

where $X^{'} = x_1^{'}, x_3^{'}, \ldots, x_{2n-1}^{'}$ *are the variables that correspond to the duplicated odd vertices of* T_n *on* $T_n^{'}$.

Appendix A

Pseudocodes

A.1 Computing Arithmetical Structures of Graphs

Algorithm A.1.1. *Arithmetical Structures Genrator* (M, K)

Input : *A matrix M (adjacency matrix of a graph) of size $n \times n$ and a positive integer K (depht of search)*

Output: *Arithmetical structures of the graph*

$A \leftarrow \emptyset$
$Aaux \leftarrow \emptyset$
$P \leftarrow [0, [1 \text{ } for \text{ } i \text{ } in \text{ } range(0, n)], 0]$
$NP \leftarrow \emptyset$
$k \leftarrow 0$

while $k \leq K$ *and* $len(P) > 0$ **do**

  for $g \in P$ **do**

    $h \leftarrow g[0]$

  $d \leftarrow g[1]$

  $L \leftarrow \text{Laplacian}(M, g[1])$ ▷ *The funtion Laplacian(M, d) gives the matrix M but with the vector d on the diagonal*

  if $\det(L) = 0$ **then**

      $r \leftarrow \text{LeadingMinors}(L)$ ▷ *This function computes the principal lieading minors of size* $n-1$ *of L and it changes the sign so that the leading term is positive*

    if $r \geq \mathbf{0}$ **then**

        $gcdr \leftarrow \text{gcd}(r)$ ▷ *the greatest common divisor from the elemets of r*

      $rker \leftarrow [\sqrt{(\frac{r_i^2}{gcdr})}$ *for* $i = 1, \ldots, n]$ ▷ *where* r_i *is the i-th entry of r*

      append the triplet $[d, rker, gcdr]$ *to A*

      else

        $S \leftarrow DiophantineEquationSolutions(L)$ ▷ *solves the diophantine equation "Axy+Bx+Cy+D" associated to the Laplacian L (with the last two entries as variables).*

      $AddArithmStruct(L, S, A)$ ▷ *This function adds to A the Arith. Structures obtained by solving the dioph. eq.*

      $Addchildren(L, g, NP)$ ▷ *adds the elements obtained from the "descendents" (next level) of g*

      end

    else

    end

    $S \leftarrow DiophantineEquationSolutions(L)$

  $AddArithmStruct(Laux(L), S, A)$

  $Addchildren(L, g, NP)$

  $P = Cut(NP, A)$ ▷ *The function Cut(P, A) reduce the list P eliminating the case where* $d \geq$ *to some arithm. struct. in A*

  $NP \leftarrow \emptyset$

  $k \leftarrow k + 1$

  end

  return A

end

A.2 Finding SubMatrix of Threshold Graphs

Algorithm A.2.1. *First submatrix of size γ with unitary determinant* (L)
Input : *A matrix L (the generalize Laplacian) of size $m \times m$*

Output: *the sets I and J that define the submatrix*

$n \leftarrow (m/2)$
$n \leftarrow n + \lfloor \frac{n}{3} \rfloor$
$C \leftarrow combinations[(1, 2, \ldots, m), n]$ ▷ *Combinations of the vector $(1, \ldots, m)$ of size n*
$S \leftarrow \emptyset$
$i \leftarrow 0$
for $I \in C$ **do**
  $i \leftarrow i + 1$
  for $J \in C[i :]$ **do**
    $p \leftarrow \det(L(I, J))$
    if $|p| = 1$ **then**
      return $(L(I, J), I, J)$
    end
  end
end

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