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Resumen

Los haces de superficies son haces fibrados suaves cuya fibra es una superficie cerrada orientable. Una forma adecuada de clasificarlos es a través de sus clases características, las cuales resultan ser elementos de los grupos de cohomología del espacio clasificante del grupo de difeomorfismos de la superficie que preservan orientación. Para superficies de género mayor o igual a dos, la cohomología de este espacio coincide con la cohomología del grupo modular de la superficie; más aún, existen clases características llamadas clases de Mumford-Morita-Miller, que permitieron en el trabajo desarrollado por Madsen-Weiss [18], dar una descripción de la cohomología racional del grupo modular para géneros muy grandes. Para el caso de género uno la cohomología del grupo modular está relacionada con el espacio de formas automorfas a través del isomorfismo de Eichler-Shimura [12]. Este trabajo aborda la definición y existencia de clases características de haces de superficies.

Abstract

Surface bundle are differentiable fiber bundles with fiber a orientable closed surface. Characteristic classes are a very useful tool in the attempting to classify such bundles, characteristic classes are elements of cohomology groups of classifying space of orientation-preserving diffeomorphism group of a surface. In the case when the genus of surface is higher than one the cohomology of such a space turns out to be the same as the cohomology of mapping class group of surface; in fact, there exist classes called Munford-Morita-Miller classes that were used by Madsen and Weiss to give a characterization of the rational cohomology group of stable mapping class group in terms of these classes [18]. If the genus of surface is equal to one, the cohomology of classifying space of orientation- preserving diffeomorphism group of torus is related to space of automorphic forms using the Eichler-Shimura isomorphism [12]. This work deals with the definition and non-triviality of characteristic classes of surface bundles.

Introduction

Surface bundles are a natural generalization of vector bundles. In this theory as in the theory of vector bundles characteristic classes of surface bundles provide a powerful tool to measure how nontrivial a given surface bundle is. Surface bundles are, roughly speaking, differentiable fiber bundles with fiber a closed orientable surface. Characteristic classes of surface bundles are elements of cohomology group of classifying space of orientation-preserving diffeomorphism group of a surface. In the case when the genus of surface is higher than one the cohomology of such a space turns out to be the same as the cohomology of mapping class group of surface; in fact, there exist classes called Munford-Morita-Miller classes that were used by Madsen and Weiss to give a characterization of the rational cohomology group of stable mapping class group in terms of these classes [18]. If the genus of surface is equal to one, the cohomology of classifying space of orientation-preserving diffeomorphism group of torus is related to space of automorphic forms using the Eichler–Shimura isomorphism [12].

These notes follow the work which was worked out in the late 80's by Shigeyuki Morita [25, 26, 27]. We study the definition, existence of characteristic classes of surface bundles and do some explicit calculations.

In the first chapter we introduce the definition of characteristic classes of a surface bundle and its classification according to the surface used as fiber, namely S^2, T^2 or Σ_g, the 2-dimensional sphere, the torus or a surface of higher genus, respectively. We argue why we only study the case when the fiber is either a torus or a surface of higher genus.

In the second chapter we talk about the torus bundles, we give the dimensions of the rational cohomology groups of the classifying space of orientation-preserving diffeomorphism group of T^2 and provide an example of non-trivial characteristic class of a torus bundle.

In the last chapter we define certain characteristic classes of a Σ_g-bundle, that is, the Miller-Morita-Mumford classes, and prove the non-triviality of the first characteristic class of a Σ_g-bundle.

1

Surface Bundles

In this chapter we define the notion of surface bundle, and we give the motivation to define the characteristic classes of such surface bundles.

1.1 Characteristic classes of differentiable fiber bundles

In this section we give the motivation and the definition of characteristic classes for differentiable fiber bundles.

Definition Let F be a C^∞ manifold. An F-**bundle** is a differentiable fiber bundle whose fibers are diffeomorphic to F.

Here $\mathrm{Diff}\,F$, the diffeomorphism group of F equipped with the C^∞ topology, is the structure group of such bundles. It is interesting in its own right, it also has a relationship with K-theory but here it is related to the fundamental problem of determine the set of all isomorphism classes of F-bundles

$$\pi : E \longrightarrow M$$

over a given manifold M. It is well-known ([21] or [16] chapter 4 sect. 11–13) that if $\mathrm{BDiff}\,F$ denotes its classifying space, then there is a natural bijection:

$$\{\text{isomorphism classes of } F\text{-bundles over } M\} \cong [M, \mathrm{BDiff}\,F]$$

where the right hand side stands for the set of homotopy classes of continuous mappings from M to $\mathrm{BDiff}\,F$.

Example Suppose that M is the n-dimensional sphere S^n, we have canonical identifications ([31], Corollary 18.6)

$$[S^n, \mathrm{BDiff}\ F] \cong \pi_n(\mathrm{BDiff}\ F)/\pi_1(\mathrm{BDiff}\ F)$$

$$\cong \pi_{n-1}(\mathrm{Diff}\ F)/\pi_0(\mathrm{Diff}\ F).$$

Here the quotient is under the action of $\pi_1(\mathrm{BDiff}\ F)$ which is by conjugation. So that we need to know the homotopy groups of BDiff F or Diff F. It becomes almost impossible to compute these groups for a general manifold F, for example in the case when $n = 1$, $[S^1, \mathrm{BDiff}\ F]$ can be identified with the set of all conjugancy classes of $\pi_0(\mathrm{Diff}\ F)$ which is the group of path components of Diff F. Unfortunately, however, it is almost impossible to compute these groups for a general manifold F. The problem to determine the set $[M, \mathrm{BDiff}\ F]$ should be even more difficult.

In such a situation, it is natural to seek methods of determining whether two given F-bundles (over the same manifold) are isomorphic to each other o not. One such method is obtained by applying characteristics classes of F-bundles.

Definition Let A be an abelian group and let k be a nonnegative integer. Suppose that, to any F-bundle $\pi : E \longrightarrow M$, there is associated a certain cohomology class $\alpha(\pi) \in H^k(M; A)$ of the base space in such a way that it is natural with respect to any bundle map. Then we say that $\alpha(\pi)$ is a **characteristic class of F-bundles of degree k with coefficients in A**. Here by natural we mean that for any bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\bar{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

between two F-bundles $\pi_i : E_i \longrightarrow M_i$ with $i = 1,2$, we have the equality

$$\alpha(\pi_1) = f^*(\alpha(\pi_2))$$

In the terminology of the classifying space, we can write $\alpha \in H^k(\mathrm{BDiff}\, F; A)$ and if $f : X \longrightarrow \mathrm{BDiff}\, F$ is the classifying map of the given bundle $\pi : E \longrightarrow M$, then we have $\alpha(\pi) = f^*(\alpha)$. Namely characteristic classes of F-bundles are nothing but elements of cohomology group of $\mathrm{BDiff}\, F$. It follows immediately from the definition that two F-bundles over the same base space which have a different characteristic class are not isomorphic to each other. Thus it is desirable to define as many characteristic classes as possible.

1.2 Surface bundles

In this section we define what a surface bundle is, we also give the concept of tangent bundle along the fiber which is going to be useful to define characteristic classes of a surface bundle. Finally we rewrite the main problem stated before in terms of surface bundles.

A 2-dimensional C^∞ manifold, which is compact, connected and without boundary, will simply be called a closed surface. The classifying of closed surfaces was done already in the beginning of twentieth century. As it is well know, the Euler characteristic together with the property of being orientable or not can serve as a complete set of invariants. In particular, the set of the diffeomorphism classes of closed orientable surfaces can be described by the series:

$$S^2, \quad T^2, \quad \Sigma_g \quad (g = 2,3,\cdots).$$

Here S^2 and T^2 denote the 2-dimensional sphere and torus, respectively, and Σ_g stands for a closed orientable surface of genus g. Of course we have $\Sigma_0 = S^2$, $\Sigma_1 = T^2$. Henceforth we assume that an orientation is fixed on each Σ_g.

Definition A differentiable fiber bundle with fiber Σ_g is called a **surface bundle** or a Σ_g-bundle.

Let $\pi : E \longrightarrow M$ be a Σ_g-bundle. Then the set of all tangent vectors on the total space E which are tangent to the fibers, namely the set:

$$\xi = \{X \in TE \mid \pi_*(X) = 0\}$$

becomes a 2-dimensional vector bundle over E. We call ξ the **tangent bundle along the fiber** of the given Σ_g-bundle. Sometimes the notation $T\pi$ will also be used for ξ. This concept is defined not only for surface bundles but also for general fiber bundles.

Definition A surface bundle $\pi : E \longrightarrow M$ is said to be orientable if its tangent bundle along the fiber $T\pi$ is orientable. If a specific orientation is given on $T\pi$, then it is called an **oriented surface bundle**.

Henceforth in this work, all surface bundles are assumed to be oriented and all bundle maps between them are assumed to preserve the orientation on each fiber.

Definition Two Σ_g-bundle $\pi_i : E_i \longrightarrow M$ with $i = 1{,}2$ over the same manifold M are said to be isomorphic if there exist a diffeomorphism $\widetilde{f} : E_1 \to E_2$ such that the following diagram

$$\begin{tikzcd}
E_1 \arrow[r, "\overline{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

commutes and $\widetilde{f}$ preserves the orientation on each fiber.

Our main problem can now be stated as follows:

Determine the set of isomorphism classes of

Σ_g-bundles over a given manifold.

Let $\mathrm{Diff}_+ \Sigma_g$ denote the group of all the orientation preserving diffeomorphisms of Σ_g equipped with the C^∞ topology. It serves as the structure group of oriented Σ_g-bundles.

1.3 Mapping class group of surfaces

Next we introduce the mapping class group of a surface. It is connected with many areas in mathematics. Here, for instance, is related to the cohomology groups of classifying space of structural group of the surface bundles as it is pointed out in Section 1.4.

Let F be a C^∞ manifold and let $\operatorname{Diff} F$ be its diffeomorphism group. The group of path components of $\operatorname{Diff} F$, namely

$$\pi_0(\operatorname{Diff} F),$$

is called the **diffeotopy group** of F. In this work, we denote this group by $\mathcal{D}(F)$. If we write $\operatorname{Diff}_0 F$ for the identity component of $\operatorname{Diff} F$, then we have

$$\mathcal{D}(F) = \operatorname{Diff} F / \operatorname{Diff}_0 F.$$

We can also define this group as follows:

Definition Two diffeomorphism $\varphi, \psi \in \operatorname{Diff} F$ of a C^∞ manifold F are said to be **isotopic** to each other if there exist a C^∞ mapping

$$\Phi : F \times [0,1] \longrightarrow F$$

such that $\Phi(-,t) : F \times \{t\} \longrightarrow F$ is a diffeormorphism for any $t \in [0,1]$ and $\Phi(-,0) = \varphi, \Phi(-,1) = \psi$.

It can be shown that this notion of isotopy is an equivalence relation. In fact, two diffeomorphisms are isotopic if and only if they belong to the same connected component of $\operatorname{Diff} F$. Hence we can say that $\mathcal{D}(F)$ is the group of all isotopy classes of diffeomorphisms of F. It follows that in the case of bundles over $M = S^1$, we have a natural identification:

$$\{\text{isomorphism classes of } F\text{-bundles over } S^1\}$$

$$\cong \{\text{conjugacy classes of } \mathcal{D}(F)\}.$$

Now we restrict to the case when F is a closed orientable surface Σ_g and consider only orientation preserving diffeomorphisms $f : \Sigma_g \to \Sigma_g$. In the same way as before we can consider the orientation preserving diffeotopy group of Σ_g, $\mathcal{D}_+(\Sigma_g)$, also known as the **mapping class group** $\mathcal{M}_g$ of Σ_g.

As it will be mentioned later in the last chapter, the mapping class group $\mathcal{M}_g$ plays an important role also in the Teichmüller theory regarding complex structures on Σ_g. For this reason, $\mathcal{M}_g$ is also called the **Teichmüller modular group**.

It is clear from the definition that isotopy is a much stronger condition than homotopy. However, in the case of two-dimensional manifolds, it is classically known that they are equivalent. Hence we can say that $\mathcal{M}_g$ is the group of all homotopy classes of orientation preserving diffeomorphism of Σ_g. Moreover it is also known that $\mathcal{M}_g$ is canonically isomorphic to the group of all homotopy classes of orientation preserving homotopy equivalences of Σ_g. In fact, since the time of Nielsen, who flourished in the first half of the twentieth century, it has been known that there exist a natural isomorphism:

$$\mathcal{M}_g \cong \mathrm{Out}_+\pi_1(\Sigma_g) = \mathrm{Aut}_+\,\pi_1(\Sigma_g)/\,\mathrm{Inn}\,\pi_1(\Sigma_g).$$

Here $\mathrm{Aut}_+\pi_1(\Sigma_g)$ denotes the normal subgroup of the automorphism group of $\pi_1(\Sigma_g)$, with index two, consisting of those elements which act on $H_2(\Sigma_g;\mathbb{Z}) = H_2(\pi_1(\Sigma_g);\mathbb{Z})$ trivially, and $\mathrm{Inn}\pi_1(\Sigma_g)$ denotes the normal subgroup of all the inner automorphism.

We refer the reader to the book [4] for basic facts about the mapping class group.

1.4 Classification of surface bundles

In this section we overview what we are going to carry out, in a thorough way, in the next two chapters.

In the case where $g = 0$, namely for the sphere, it was proved by Smale [29] that the natural inclusion

$$SO(3) \subset \mathrm{Diff}_+\,S^2$$

is a homotopy equivalence. It follows from this fact that any S^2-bundle is isomorphic to the sphere bundle of some uniquely defined 3-dimensional oriented vector bundle. Hence the classification of S^2-bundles over a given manifold M is equivalent to that of 3-dimensional oriented vector bundles over M. Since the homotopy type of the classifying space $BSO(3)$ is known [23], we may say that this problem is solved.

♠

Next we consider the case $g = 1$, namely surface bundle whose fibers are diffeomorphic to the torus T^2. If we identify T^2 with $\mathbb{R}^2/\mathbb{Z}^2$, then T^2 acts on itself by diffeomorphism (which just are translations.) Hence T^2 can be naturally considered as a subgroup of $\operatorname{Diff}_0 T^2$ which is the indentity component of $\operatorname{Diff} T^2$. Moreover it is known by Earle-Ells [10] that the inclusion

$$T^2 \subset \operatorname{Diff}_0 T^2$$

is a homotopy equivalence. On the other hand, we have an isomorphism

$$\operatorname{Diff}_+ T^2 / \operatorname{Diff}_0 T^2 = \mathcal{M}_1 \cong SL(2;\mathbb{Z}).$$

Passing to classifying space we obtain a fibration

$$BT^2 \longrightarrow \mathrm{BDiff}_+ T^2 \longrightarrow BSL(2;\mathbb{Z})$$

(see [24], Proposition 8.1 & Theorem 11.4.) Notice that $\mathrm{B}\,SL(2;\mathbb{Z})$ is an Eilenberg-MacLane space $K(SL(2;\mathbb{Z}),1)$, and thus the cohomology of $\mathrm{B}\,SL(2;\mathbb{Z})$ is that of the discrete group $SL(2;\mathbb{Z})$. The structure of the group $SL(2;\mathbb{Z})$ is classically well known, and we have a homotopy equivalence $BT^2 \cong \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ (it is because $ES^1 \times ES^1$ is contractible space with free action $S^1 \times S^1$, the quotient by this action is $BS^1 \times BS^1$ so $B(T^2) = B(S^1 \times S^1) \cong BS^1 \times BS^1$, therefore $BT^2 \cong \mathbb{CP}^\infty \times \mathbb{CP}^\infty$.) Based on these facts we can compute the cohomology of $\mathrm{BDiff}_+ T^2$ which serve as the characteristic classes of T^2-bundles. More about this is going to be developed in the second chapter.

♠

In the case where $g \geq 2$, the situation changes drastically. More precisely, Earle and Eells proved in [11] Theorem 1.c, that $\operatorname{Diff}_0 \Sigma_g$ is contractible so that

$$B \operatorname{Diff}_+ \Sigma_g = K(\mathcal{M}_g,1).$$

It follows immediately from this that

Proposition 1.1 *Let $g \geq 2$. Then for any C^∞ manifold M, we have a natural bijection:*

$$\{\text{isomorphism class of } \Sigma_g\text{-bundle over } M\}$$

$$\cong \{\text{conjugacy class of homomorphism } \pi_1(M) \longrightarrow \mathcal{M}_g\}$$

(see [13] lemma 1.19.) In particular if M is simply connected, then any Σ_g-bundle over it is trivial since $\pi_1(M) = 0$ and the constant homomorphism to the identity would be the only one, therefore just one class of isomorphisms of Σ_g-bundle would be, in fact, it is going to be the class of the trivial bundle. However in general, it is almost impossible to determine the set of all conjugacy classes of homomorphism from a given group to $\mathcal{M}_g$. It may be better to understand the above proposition as starting point for the construction of a classification theory rather than a direct role.

Now let α be a characteristic class of Σ_g-bundle of degree k with coefficients in a abelian group A. Then we can write

$$\alpha \in H^k(\mathrm{BDiff}\, \Sigma_g; A) = H^k(K(\mathcal{M}_g,1); A) =: H^k(\mathcal{M}_g; A)$$

In other words, characteristic classes of surfaces bundles of genus $g \geq 2$ are nothing but cohomology classes of mapping class group $\mathcal{M}_g$.

2

Characteristic Classes of Torus Bundles

In this chapter we study the problem to determine the non-triviality of characteristic classes of differentiable fiber bundles whose fibers are diffeomorphic to the 2-dimensional torus T^2, that is, the problem to compute the cohomology group $H^*(\mathrm{BDiff}_+ T^2)$. More precisely, we determine $H^*(\mathrm{BDiff}_+ T^2; R)$ for $R = \mathbb{Q}$ or $\mathbb{Z}_p$ with $p \neq 2,3$.

2.1 Introduction

Let $\mathrm{Diff}_0 T^2$ be the connected component of the identity of $\mathrm{Diff}_+ T^2$. Then as is well-known the factor group $\mathrm{Diff}_+ T^2 / \mathrm{Diff}_0 T^2$, which is the mapping class group of T^2, can be naturally identified with $SL(2;\mathbb{Z})$. Therefore we have a fibration

$$\mathrm{BDiff}_0 T^2 \longrightarrow \mathrm{BDiff}_+ T^2 \longrightarrow K(SL(2;\mathbb{Z}),1). \tag{2.1}$$

T^2 acts on itself by "translations" (viewed $T^2 = \mathbb{R}^2/\mathbb{Z}^2$) and hence it can be considered as subgroup of $\mathrm{Diff}_0 T^2$. We see that the action by conjugation of $SL(2;\mathbb{Z})$ on this group $T^2 \subset \mathrm{Diff}_0 T^2$ is the same as the standard one. Earle and Eells (see [10], Corollary 7.G) proved that the inclusion $T^2 \subset \mathrm{Diff}_0 T^2$ is a homotopy equivalence so that $\mathrm{BDiff}_0 T^2$ has the homotopy type of $\mathrm{B}\, T^2 \simeq \mathrm{B}\, S^1 \times \mathrm{B}\, S^1 \simeq \mathbb{CP}^\infty \times \mathbb{CP}^\infty$. If we choose suitable elements $x,y \in H^2(\mathrm{BDiff}_0 T^2; \mathbb{Z})$, we can write

$$H^*(\mathrm{BDiff}_0 T^2; \mathbb{Z}) = \mathbb{Z}[x,y]$$

on which $SL(2;\mathbb{Z})$ acts through the automorphism of it given by $\gamma \to {}^t\gamma^{-1}$, where $\gamma \in SL(2;\mathbb{Z})$.

Now let $\{E_r^{s,t},d_r\}$ be the Serre spectral sequence for cohomology (with coefficients in a commutative ring R) of the fibration 2.1. Then by the above argument, the E_2-term is given by

$$\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL(2;\mathbb{Z});R[x,y]) \quad \Rightarrow \quad H^*(\mathrm{BDiff}_+\,T^2).$$

Since $SL(2;\mathbb{Z}) = \langle \alpha,\beta \mid \alpha^4 = \alpha^2\beta^{-3} = 1\rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (see [28], I.4) where

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

therefore the abelianization $H_1(SL(2;\mathbb{Z}))$ of $SL(2;\mathbb{Z})$ is a cyclic group of order 12 and the kernel of the natural surjection $SL(2;\mathbb{Z}) \to H_1(SL(2;\mathbb{Z}))$ is the commutator subgroup of $SL(2,\mathbb{Z})$, which in turn is isomorphic to a free group of rank 2 (see [5], Exercise 8 in II.4). Moreover the composition of the restriction map with the transfer map

$$\begin{tikzcd} H^*(SL(2;\mathbb{Z})) \arrow[r, "\mathrm{res}"] \arrow[rr, bend left, "\times 12"] & H^*(F_2) \arrow[r, "\mathrm{tr}"] & H^*(SL(2;\mathbb{Z})) \end{tikzcd}$$

is multiplication by 12 (see [1], Chapter II.5.) Hence applying an argument of group cohomology (see [6], Proposition III.10.1), we obtain

Proposition 2.1 *If $s \geq 2$, then $\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL(2;\mathbb{Z});R[x,y])$ is annihilated by 12. In particular if $R = \mathbb{Q}$ or $\mathbb{Z}_n$ with $(n,12) = 1$, then*

$$\bigoplus_{t=0}^{\infty} E_2^{s,t} = H^s(SL(2;\mathbb{Z});R[x,y]) = 0 \qquad \text{for } s \geq 2.$$

∎

Corollary 2.2 *Let $k = \mathbb{Q}$ or $\mathbb{Z}_p$ (p is a prime different from 2 and 3). Then*

$$H^n(\mathrm{BDiff}_+\,T^2;k) \cong E_2^{0,n} \oplus E_2^{1,n-1}.$$

∎

2.2 The action of $SL(2;\mathbb{Z})$ on cohomology of the fiber

As is well-known $SL(2;\mathbb{Z})$ has the following presentation (see [28])

$$SL(2;\mathbb{Z}) = \langle \alpha,\beta \mid \alpha^4 = \alpha^2\beta^{-3} = 1\rangle.$$

Here, for the convenience of later computations, we choose two generators $\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. The action of $SL(2;\mathbb{Z})$ on $H^*(\mathrm{BDiff}_0\, T^2;\mathbb{Z}) = \mathbb{Z}[x,y]$ is given by

$$\alpha(x) = -y, \quad \alpha(y) = x$$
$$\beta(x) = x - y, \quad \beta(y) = x$$

because ${}^t\alpha^{-1} = \alpha$ and ${}^t\beta^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Now for each $q \in \mathbb{N}$, let L_q be the submodule of $\mathbb{Z}[x,y]$ consisting of homogeneous elements of degree $2q$. Equivalently, L_q is the $2q$-th cohomology group $H^{2q}(\mathrm{BDiff}_0\, T^2;\mathbb{Z}) = H^{2q}(\mathrm{B}\, T^2;\mathbb{Z})$. We choose a basis $\{x^q, x^{q-1}y, \cdots, xy^{q-1}, y^q\}$ for L_q and let

$$A_q, B_q \in SL(q+1;\mathbb{Z})$$

be the matrix representations of the actions of α and β on L_q with respect to the above basis. Observe that

$$A_q = (a^{(q)}_{i,j}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^q \\ 0 & 0 & \cdots & (-1)^{q-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$B_q = (b^{(q)}_{i,j}) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ -q & -(q-1) & \cdots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^{q+1}q & (-1)^{q+1} & \cdots & 0 & 0 \\ (-1)^{q+2} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

are given by

$$a^{(q)}_{i,j} = \delta_{ij} := \begin{cases} (-1)^{q+1-i} & j = q+2-i, \\ 0 & \text{otherwise} \end{cases} \qquad (i,j = 1,\cdots,q+1).$$

$$b^{(q)}_{i,j} = (-1)^{i+1}\binom{q-j+1}{i-1} \text{ where } \binom{s}{t} = 0 \text{ if } t > s \qquad (i,j = 1,\cdots,q+1).$$

In the following section we study some basic properties of the matrices A_q and B_q, their mod p reduction and rationalization. In particular we begin computing their minimal polynomials. We will use these facts in Section 2.4 to compute $H^0(SL(2;\mathbb{Z});L_q(p))$ and $H^1(SL(2;\mathbb{Z});L_q(p))$ with rational and mod p coefficients. Then we will assemble all this information to get $H^*(\mathrm{BDiff}_+ T^2;\mathbb{Q})$ and $H^*(\mathrm{BDiff}_+ T^2;\mathbb{Z}_p)$ for $p \neq 2,3$.

2.3 Some technical lemmas

Let p denotes either a prime or 0. We write $A_q(p)$ and $B_q(p)$ for the corresponding elements of $SL(q+1,\mathbb{Z}_p)$ if p is a prime or of $SL(q+1;\mathbb{Q})$ if $p=0$. It is easy to prove

Lemma 2.3 *1. If q is odd, then $A_q^2 = B_q^3 = -I$. Moreover the minimal polynomials of A_q and B_q are t^2+1 and t^3+1 respectively.*

2. If q is even, then $A_q^2 = B_q^3 = I$ and the minimal polynomials of A_q and B_q are t^2-1 and t^3-1 respectively.

∎

Corollary 2.4 *If q is odd, then both of $A_q(p)+I$ and $B_q(p)-I$ are invertible provided $p \neq 2$. In fact we have*

$$\begin{aligned}(A_q(p)+I)^{-1} &= -\frac{1}{2}(A_q(p)-I) \quad \text{and} \\ (B_q(p)-I)^{-1} &= -\frac{1}{2}(B_q^2(p)+B_q(p)+I).\end{aligned}$$

∎

Now let $L_q(p)$ be either $L_q \otimes \mathbb{Z}_p$ if p is a prime or $L_q \otimes \mathbb{Q}$ if $p=0$. $A_q(p)$ and $B_q(p)$ act on $L_q(p)$. We assume q is even and define

$$\begin{aligned}L_q^-(p) &= \{u \in L_q(p) \mid A_q(p)u = -u\}) \\ L_q'(p) &= \{u \in L_q(p) \mid (B_q^2(p)+B_q(p)+I)u = 0\}.\end{aligned}$$

Lemma 2.5 *If $p \neq 2$ and $q = 2r$, then*

$$\dim L_q^-(p) = \begin{cases} r+1 & r : \text{odd} \\ r & r : \text{even} \end{cases}$$

Proof. It is easy to see that

$$\{x^q - y^q, x^{q-1}y + xy^{q-1}, x^{q-2}y^2 - x^2y^{q-2}, \cdots, x^{r+1}y^{r-1} - x^{r-1}y^{r+1}, x^ry^r\} \quad r : \text{odd}$$

or

$$\{x^q - y^q, x^{q-1}y + xy^{q-1}, x^{q-2}y^2 - x^2y^{q-2}, \cdots, x^{r+1}y^{r-1} + x^{r-1}y^{r+1}\} \quad r : \text{even}$$

forms a basis of $L_q^-(p)$. ∎

Next we determine $\dim L_q'$. We first consider the case $p=0$.

Lemma 2.6 $\operatorname{Trace} B_q = 1, 1, 0, -1, -1, 0$ *according as* $q \equiv 0, 1, 2, 3, 4, 5 \mod 6$.

Proof. Observe that $B_q = (b_{ij}^{(q)})$, where

$$b_{i,j}^{(q)} = (-1)^{i+1}\binom{q-j+1}{i-1} \quad (i,j = 1, \cdots, q+1).$$

(Here we understand that $\binom{s}{t} = 0$ if $t > s$). In other words the j-th column of B_q consist of coefficients of the polynomial $(1-t)^{q-j+1}$.

$$\begin{array}{cccccccl}
 & (1-t)^5 & (1-t)^4 & (1-t)^3 & (1-t)^2 & (1-t) & (1-t)^0 & \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & = \mathrm{Trace}\, B_1 \\
\cdots & 10 & 6 & 3 & 1 & 0 & 0 & = \mathrm{Trace}\, B_2 \\
\cdots & -10 & -4 & -1 & 0 & 0 & 0 & = \mathrm{Trace}\, B_3 \\
\cdots & 5 & 1 & 0 & 0 & 0 & 0 & = \mathrm{Trace}\, B_4 \\
\cdots & -1 & 0 & 0 & 0 & 0 & 0 & = \mathrm{Trace}\, B_5 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

B_q is naturally a minor matrix of B_{q+1} and if we multiply $(1-t)^{q-j+1}$ by t^{q-j+1}, we find out that

$$\mathrm{Trace}\, B_q = \text{the coefficient of } t^q \text{ in the power series}$$

$$1 + t(1-t) + t^2(1-t)^2 + \cdots$$

But we have

$$\begin{aligned}
\sum_{n=0}^{\infty}(t(1-t))^n &= \frac{1}{1-t+t^2} \\
&= \frac{1}{(t-\omega)(t-\overline{\omega})}
\end{aligned}$$

where $\omega = \exp(2\pi i/6) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

Note that

$$\frac{1}{\omega - \overline{\omega}} = -\frac{\sqrt{3}}{3}i$$

so

$$\frac{1}{(t-\omega)(t-\overline{\omega})} = -\frac{\sqrt{3}}{3}i \cdot \frac{1}{t-\omega} + \frac{\sqrt{3}}{3}i \cdot \frac{1}{t-\overline{\omega}}$$

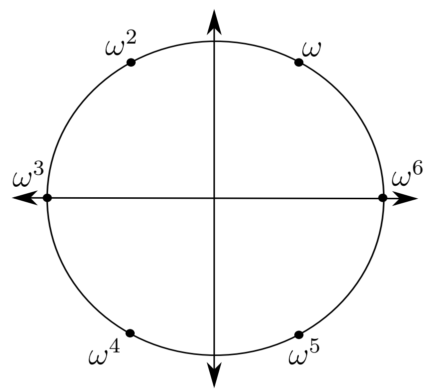

Figure 2.1

but

$$\frac{1}{t-\omega} = -\sum_{n=0}^{\infty} \omega^{-(1+n)} t^n, \qquad \frac{1}{t-\overline{\omega}} = -\sum_{n=0}^{\infty} \overline{\omega}^{-(1+n)} t^n \qquad \& \qquad \sqrt{3}\, i = 2\omega - 1.$$

Thus

$$\frac{1}{(t-\omega)(t-\overline{\omega})} = \frac{1}{3}(2\omega-1)\Big(\sum_{n=0}^{\infty} \omega^{-(1+n)} - \overline{\omega}^{-(1+n)}\Big) t^n$$

therefore

$$\begin{aligned}
\text{Trace}\, B_q &= \frac{1}{3}(2\omega-1)(\omega^{-(1+q)} - \overline{\omega}^{-(1+q)}) \\
&= \frac{1}{3}(2\omega^{5q} - 2\omega^{q+2} - \omega^{5(q+1)} + \omega^{q+1}) && (\text{because } \omega^{-1} = \overline{\omega} = \omega^5) \\
&= \frac{1}{3}(\omega^{5q} - \omega^{q+2} + \omega^{q+1} - \omega^{q+2} + \omega^{5q} - \omega^{5(q+1)}) \\
&= \frac{1}{3}(\omega^{5q} - \omega^{q+2} + \omega^{q} + \omega^{5q} - \omega^{5(q+1)}) && (\text{because } \omega^q(\omega-\omega^2) = \omega^q \\
& && \text{since } \omega - \omega^2 = 1) \\
&= \frac{1}{3}(\omega^{q} - \omega^{q+2} + \omega^{5q} - \omega^{5q+4}) && (\text{because } -\omega^{5q}(\omega^5 - 1) = -\omega^{5q+4} \\
& && \text{since } \omega^5 - 1 = \omega^4.)
\end{aligned}$$

Then the desired result follows from a direct computation. ∎

Lemma 2.7 *If q is even, then*

$$\text{rank}(B_q^2 + B_q + I) = 2k+1 \quad \text{for } q = 6k, 6k+2 \text{ or } 6k+4.$$

Proof. According to Lemma 2.3 (ii), the characteristic polynomial of B_q is

$$(t-1)^a(t^2+t+1)^b$$

for some $a,b\in\mathbb{N}$. Moreover $L_q=\mathrm{Ker}(B_q-I)\oplus\mathrm{Ker}(B_q^2+B_q+I)$, where $\dim\mathrm{Ker}(B_q-I)=a$ and $\dim\mathrm{Ker}(B_q^2+B_q+I)=2b$. Because the degree of characteristic polynomial of B_q is $q+1$ and

$$\begin{aligned}(t-1)^a(t^2+t+1)^b &= \left(t^a-at^{a-1}+\frac{(a-1)a}{2}t^{a-2}+\cdots\right)\left(t^{2b}+bt^{2b-1}+bt^{2b-2}+\cdots\right)\\ &= t^{a+2b}+(b-a)t^{a+2b-1}+\cdots\end{aligned}$$

we obtain

$$a+2b=q+1 \qquad \text{and} \qquad a-b=\mathrm{Trace}\,B_q.$$

but by Lemma 2.6, $\mathrm{Trace}\,B_q=1,0,-1$ as $q=6k,6k+2$ or $6k+4$ respectively.
A simple computation implies the result. ∎

Next we show that the above lemma also holds even if we replace B_q by $B_q(p)$ for $p\neq 3$.

Lemma 2.8 *Let $B_q=(b_{ij}^{(q)})$ and define $C_q=(c_{ij}^{(q)})$ by*

$$c_{ij}^{(q)}=b_{q+2-i,q+2-j}^{(q)}.$$

Then we have $C_q=B_q^{-1}$. In other words, B_q and B_q^{-1} are mutually symmetric with respect to the "center" of them.

Proof. We use induction on q. If $q=1$, then

$$B_1C_1=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}=I.$$

We assume that $B_iC_i=I$ for $i=1,\cdots,q-1$. Now let $b_i^{(q)}$ be the i-th row of B_q and let $c_j^{(q)}$ be the j-th column of C_q. We can write

$$B_q=\begin{pmatrix} * & B_{q-1}\\ (-1)^q & \mathbf{0}\end{pmatrix}, \qquad C_q=\begin{pmatrix}\mathbf{0} & \\ & c_{q+1}^{(q)}\\ C_{q-1} & \end{pmatrix}$$

Hence by the induction assumption, it sufficies to prove

$$b_i^{(q)} c_{q+1}^{(q)} = \delta_{i,q+1}$$

for $i = 1, \cdots, q+1$. Now

$$\begin{aligned}
\sum_{k=1}^{i} b_{kj}^{(q)} &= \sum_{k=1}^{i} (-1)^{k+1} \binom{q-j+1}{k-1} \\
&= \binom{q-j+1}{0} + \sum_{k=2}^{i} (-1)^{k+1} \left[\binom{q-j}{k-2} + \binom{q-j}{k-1} \right] \\
&= (-1)^{i+1} \binom{q-j}{i-1} \\
&= b_{i,j+1}^{(q)} = b_{ij}^{(q-1)}
\end{aligned}$$

for any i,j where $j \leq q$. Hence we have

$$\begin{aligned}
b_1^{(q)} + b_2^{(q)} + \cdots + b_i^{(q)} &= \begin{pmatrix} b_{i1}^{(q-1)} & b_{i2}^{(q-1)} & \cdots & b_{iq}^{(q-1)} & 1 \end{pmatrix} \\
&= \begin{pmatrix} b_i^{(q-1)} & 1 \end{pmatrix} \qquad\qquad (i = 1, \cdots, q) \quad \text{and} \\
b_1^{(q)} + b_2^{(q)} + \cdots + b_{q+1}^{(q)} &= \begin{pmatrix} (-1)^q \binom{q-1}{q} & (-1)^q \binom{q-2}{q} & \cdots & (-1)^q \binom{0}{q} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} & 1 \end{pmatrix}.
\end{aligned}$$

From this we can deduce

$$b_i^{(q)} = \begin{pmatrix} b_i^{(q-1)} & 1 \end{pmatrix} - \begin{pmatrix} b_{i-1}^{(q-1)} & 1 \end{pmatrix} \qquad (i = 2, \cdots, q).$$

Also we have

$$\begin{aligned}
\sum_{k=1}^{i} c_{k,q+1}^{(q)} &= \sum_{k=1}^{i} b_{q+2-k,1}^{(q)} \\
&= \sum_{k=1}^{i} (-1)^{q+3-k} \binom{q}{q+1-k} \\
&= (-1)^{q+2} \binom{q}{q} + \sum_{k=2}^{i} (-1)^{q+3-k} \left[\binom{q-1}{q-k} + \binom{q-1}{q+1-k} \right] \\
&= (-1)^{q+3-1} \binom{q-1}{q-i} \\
&= -b_{q-i+1,2}^{(q)} = -c_{i+1,q}^{(q)} = -c_{iq}^{(q-1)}
\end{aligned}$$

so

$$\begin{aligned}\sum_{k=1}^{i-1} c_{k,q+1}^{(q)} + c_{i,q+1}^{(q)} = & \ -c_{iq}^{(q-1)} \\ -c_{iq}^{(q)} + c_{i,q+1}^{(q)} = & \end{aligned}$$

thus we obtain

$$c_{q+1}^{(q)} = c_q^{(q)} - \begin{pmatrix} c_q^{(q-1)} \\ \\ 0 \end{pmatrix}.$$

Now it is easy to see that

$$\begin{aligned} b_1^{(q)} c_{q+1}^{(q)} = & \sum_{k=1}^{q+1} c_{k,q+1} \\ = & \sum_{k=1}^{q+1} b_{k1}^{(q)} = (-1)^q \binom{q-1}{q} = 0 \end{aligned}$$

and $b_{q+1}^{(q)} c_{q+1}^{(q)} = 1$.

On the other hand if $2 \le i \le q$, then

$$\begin{aligned} b_i^{(q)} c_{q+1}^{(q)} & = b_i^{(q)} \left(c_q^{(q)} - \binom{c_q^{(q-1)}}{0} \right) \\ & = -b_i^{(q)} \binom{c_q^{(q-1)}}{0} \\ & = ((b_{i-1}^{(q-1)} \quad 1) - (b_i^{(q-1)} \quad 1)) \binom{c_q^{(q-1)}}{0} \\ & = 0 \end{aligned}$$

by the induction assumption (the second equality follows from the fact that $b_i^{(q)} c_q^{(q)} = b_i^{(q-1)} c_q^{(q-1)}$). This completes the proof. ∎

Lemma 2.9 *For each q let $B_{q,s}^{(r)}$ where $1 \le r \le q+1$ and $1 \le s \le q+2-r$ be the matrix defined by*

$$B_{q,s}^{(r)} = \begin{pmatrix} b_{1\ s}^{(q)} & b_{1\ s+1}^{(q)} & \cdots & b_{1\ s+r-1}^{(q)} \\ \vdots & & & \vdots \\ b_{r\ s}^{(q)} & b_{r\ s+1}^{(q)} & \cdots & b_{r\ s+r-1}^{(q)} \end{pmatrix}.$$

Then we have $\det B_{q,s}^{(r)} = 1$ *for all r,s.*

Proof. First observe that $B_{q,s}^{(r)} = B_{q-s+1,1}^{(r)}$. Hence we may assume that $s = 1$ and we simply write $B_q^{(r)}$ instead of $B_{q,1}^{(r)}$. If $r = q + 1$, then $\det B_q^{q+1} = \det B_q = 1$. So assume that $r < q + 1$. As in the proof of Lemma 2.8 we have

$$\sum_{k=1}^{i} b_{kj}^{(q)} = b_{ij}^{(q-1)}$$

for any i,j where $j \leq q$. Hence if we define $\overline{B}_q^{(r)}$ to be the matrix obtained from $B_q^{(r)}$ by the following rule:

$$\text{the } i\text{-th row of } \overline{B}_q^{(r)} = \sum_{k=1}^{i} (\text{the } k\text{-th row of } B_q^{(r)}),$$

then we have

$$\overline{B}_q^{(r)} = B_{q-1}^{(r)}$$

and clearly $\det B_q^{(r)} = \det \overline{B}_q^{(r)} = \det B_{q-1}^{(r)}$. Hence inductively we have

$$\det B_q^{(r)} = \det B_{q-1}^{(r)} = \cdots = \det B_{r-1}^{(r)} = \det B_{r-1} = 1.$$

This completes the proof. ∎

Lemma 2.10 *Assume that q is even and $p \neq 3$. Then we have*

$$\operatorname{rank}(B_q^2(p) + B_q(p) + I) = 2k + 1 \qquad \text{if } q = 6k, 6k + 2 \text{ or } 6k + 4.$$

Proof. Clearly we have

$$\operatorname{rank}(B_q^2(p) + B_q(p) + I) \leq \operatorname{rank}(B_q^2 + B_q + I).$$

Hence, in view of Lemma 2.7 we have only to show the existence of a minor of $(B_q^2 + B_q + I)$ of size $(2k + 1) \times (2k + 1)$ (for $q = 6k, 6k + 2$ or $6k + 4$), whose determinant is a power of 3. Now observe that if $i + j > q + 2$, then

$$b_{ij}^{(q)} = 0.$$

We are assuming that q is even so that $B_q^2 = B_q^{-1}$ (see Lemma 2.3 (ii)). Hence by Lemma 2.8, if $i + j < q + 2$, then

$$c_{ij}^{(q)} = 0$$

Therefore the (i,j)-component of $B_q^2+B_q+I$ coincides with that of B_q if (i,j) belongs to the set

$$K=\{(i,j)\mid i+j<q+2 \text{ and } j>i\}.$$

If $q=6k+2$ or $6k+4$, then the minor matrix $B_{q,2k+2}^{(2k+1)}$ of B_q is completely contained in the region of B_q corresponding to K

$$\left(\begin{array}{ccc|cccc|ccc}
b_{11} & b_{12} & \cdots & b_{1,2k+2} & b_{1,2k+3} & \cdots & b_{1,4k+2} & \cdots & b_{1q} & b_{1,q+1} \\
b_{21} & b_{22} & \cdots & b_{2,2k+2} & b_{2,2k+3} & \cdots & b_{2,4k+2} & \cdots & b_{2q} & b_{2,q+1} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
b_{2k,1} & b_{2k,2} & \cdots & b_{2k,2k+2} & b_{2k,2k+3} & \cdots & b_{2k,4k+2} & \cdots & b_{2k,q} & b_{2k,q+1} \\
b_{2k+1,1} & b_{2k+1,2} & \cdots & b_{2k+1,2k+2} & b_{2k+1,2k+3} & \cdots & b_{2k+1,4k+2} & \cdots & b_{2k+1,q} & b_{2k+1,q+1} \\
\vdots & \vdots & & & & & & & \vdots & \vdots \\
b_{q+1,1} & b_{q+1,2} & \cdots & & & & & \cdots & b_{q+1,q} & b_{q+1,q+1}
\end{array}\right)$$

so that $B_{q,2k+2}^{(2k+1)}$ can also be considered to be minor matrix of $B_q^2+B_q+I$. But we have

$$\det B_{q,2k+2}^{(2k+1)}=1$$

by Lemma 2.9. Now if $q=6k$, choose the minor matrix $B_{q,2k+1}^{(2k+1)}$, then the bottom elements of the first and the last columns of $B_{q,2k+1}^{(2k+1)}$ are not contained in the region of B_q corresponding to K.

$$\left(\begin{array}{ccc|cccc|ccc}
b_{11} & b_{12} & \cdots & b_{1,2k+1} & b_{1,2k+2} & \cdots & b_{1,4k+1} & \cdots & b_{1q} & b_{1,q+1} \\
b_{21} & b_{22} & \cdots & b_{2,2k+1} & b_{2,2k+2} & \cdots & b_{2,4k+1} & \cdots & b_{2q} & b_{2,q+1} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
b_{2k,1} & b_{2k,2} & \cdots & b_{2k,2k+1} & b_{2k,2k+2} & \cdots & b_{2k,4k+1} & \cdots & b_{2k,q} & b_{2k,q+1} \\
b_{2k+1,1} & b_{2k+1,2} & \cdots & b_{2k+1,2k+1} & b_{2k+1,2k+2} & \cdots & b_{2k+1,4k+1} & \cdots & b_{2k+1,q} & b_{2k+1,q+1} \\
\vdots & \vdots & & & & & & & \vdots & \vdots \\
b_{q+1,1} & b_{q+1,2} & \cdots & & & & & \cdots & b_{q+1,q} & b_{q+1,q+1}
\end{array}\right)$$

If we denote $D_{q,2k+1}^{(2k+1)}=(d_{ij})$ for the corresponding minor matrix of $B_q^2+B_q+I$, then all the entries of $D_{q,2k+1}^{(2k+1)}$ coincide with those of $B_{q,2k+1}^{(2k+1)}$ except the following two components:

$$d_{2k+1,1}=b_{2k+1,2k+1}^{(q)}+1$$

$$d_{2k+1,2k+1} = b^{(q)}_{2k+1,4k+1} + 1 = 2.$$

Here we have used Lemma 2.8 to deduce the second equality. Then by Lemma 2.9 we conclude that

$$\det D^{(2k+1)}_{q,2k+1} = \det B^{(2k+1)}_{q,2k+1} + \det \begin{pmatrix} b_{1,2k+2} & \cdots & b_{1,4k+1} \\ \vdots & & \vdots \\ b_{2k,2k+2} & \cdots & b_{2k,4k+1} \end{pmatrix} + \det \begin{pmatrix} b_{1,2k+1} & \cdots & b_{1,4k} \\ \vdots & & \vdots \\ b_{2k,2k+1} & \cdots & b_{2k,4k} \end{pmatrix} = 3.$$

This completes the proof. ∎

2.4 Cohomology of $\mathrm{BDiff}_+ T^2$ with twisted coefficients

In this section we compute $H^*(SL(2;\mathbb{Z}); k[x,y])$ for $k = \mathbb{Q}$ or $\mathbb{Z}_p$ for $p \neq 2,3$. Notice that $H^*(\mathrm{BDiff}_+ T^2; k)$ can be deduced immediately from here using Proposition 2.1 and Corollary 2.2.

Recall that we denote $L_q(p)$ for $L_q \otimes \mathbb{Z}_p$ is p is a prime or for $L_q \otimes \mathbb{Q}$ if $p = 0$. Now let $Z^1(SL(2;\mathbb{Z}))$ be the set of all 1-cocycles of $SL(2;\mathbb{Z})$ with values in $L_q(p)$, namely it is the set of all crossed homomorphisms

$$f : SL(2;\mathbb{Z}) \longrightarrow L_q(p).$$

defined by $f(ab) = f(a) + a \cdot f(b)$ where the action is the multiplication of the matrix represented in $SL(q+1;\mathbb{Z}_p)$. Since $SL(2;\mathbb{Z})$ is generated by two elements α and β, crossed homomorphism $f : SL(2;\mathbb{Z}) \to L_q(p)$ is completely determined by two values $f(\alpha)$ and $f(\beta)$. We have the following properties of the crossed homomorphism:

1. $f(1) = 0$,
2. $f(\alpha^4) = f(\alpha) + \alpha \cdot f(\alpha) + \alpha^2 \cdot f(\alpha) + \alpha^3 \cdot f(\alpha)$,
3. $f(\alpha^2) = f(\alpha) + \alpha \cdot f(\alpha)$,
4. $f(\beta^3) = f(\beta) + \beta \cdot f(\beta) + \beta^2 f(\beta)$.

Moreover the two relations $\alpha^4 = 1$ and $\alpha^2 = \beta^3$ imply

$$(A_q^3(p) + A_q^2(p) + A_q(p) + I)f(\alpha) = 0$$

$$(A_q(p) + I)f(\alpha) = (B_q^2(p) + B_q(p) + I)f(\beta).$$

Conversely if two elements $f(\alpha)$ and $f(\beta)$ of $L_q(p)$ satisfy the above two equations, then there is defined the associated crossed homomorphism $f : SL(2;\mathbb{Z}) \to L_q(p)$ with prescribed values at α,β. If we combine the above argument with Lemma 2.3, we can conclude

Lemma 2.11 *1. If q is odd, then*

$$Z^1(SL(2;\mathbb{Z}); L_q(p)) = \{(u,v) \in L_q(p) \times L_q(p) \mid (A_q(p)+I)u = (B_q^2(p)+B_q(p)+I)v\}.$$

2. If q is even, then

$$Z^1(SL(2;\mathbb{Z}); L_q(p)) = \{(u,v) \in L_q(p)\times L_q(p) \mid (A_q(p)+I)u = 0,\ (B_q^2(p)+B_q(p)+I)v = 0\}.$$

Now let

$$\delta : L_q(p) \longrightarrow Z^1(SL(2;\mathbb{Z}); L_q(p))$$

be the homomorphism defined by

$$\delta(u)(\gamma) = (\gamma - 1)u \qquad (u \in L_q(p), \gamma \in SL(2;\mathbb{Z})).$$

Then by the definition of cohomology of groups ([7] Chaper IX.4), we have

$$\begin{aligned} H^0(SL(2;\mathbb{Z}); L_q(p)) &= \ker\delta \\ &= \{u \in L_q(p) \mid A_q(p)u - u = B_q(p)u - u = 0\} \quad \text{and} \\ H^1(SL(2;\mathbb{Z}); L_q(p)) &= \operatorname{Coker}\delta. \end{aligned}$$

Proposition 2.12 $H^0(SL(2;\mathbb{Z}); \mathbb{Q}[x,y]) = \mathbb{Q}$.

Proof. It suffices to prove that the only polynomials in $\mathbb{Q}[x,y]$ which are left invariant under the action of $SL(2;\mathbb{Z})$ are constants. This follows from a direct computation details of which are omitted. ∎

Remark 1 According to a classical result of Dickson [9] (see also Tezuka [32]), the subring of $\mathbb{Z}_p[x,y]$ consisting of those elements which are invariant by the action of $SL(2;\mathbb{Z})$, namely $H^0(SL(2;\mathbb{Z});\mathbb{Z}_p[x,y])$, is the polynomial ring generated by the following two elements

$$x^p y - xy^p \qquad \text{and} \qquad \frac{x^{p^2}y - xy^{p^2}}{x^p y - xy^p} \equiv y^{p(p-1)} + (x^p - xy^{p-1})^{p-1}.$$

Hence if we write $d_q(p)$ for $\dim H^0(SL(2;\mathbb{Z}); L_q(p))$, then we have

$$\sum_{q=0}^{\infty} d_q(p) t^q = \frac{1}{(1-t^{p+1})(1-t^{p(p-1)})}.$$

Proposition 2.13 *If q is odd and $p \neq 2$, then*

$$H^0(SL(2;\mathbb{Z}); L_q(p)) = H^1(SL(2;\mathbb{Z}); L_q(p)) = 0.$$

Proof. According to Corollary 2.4, $B_q(p) - I$ and $A_q(p) - I$ are invertibles and so the homomorphism $\delta : L_q(p) \to Z^1(SL(2;\mathbb{Z}); L_q(p))$ is injective. Hence $H^0(SL(2;\mathbb{Z}); L_q(p)) = 0$. Next let $(u,v) \in Z^1(SL(2;\mathbb{Z}); L_q(p))$ be any element (see Lemma 2.11 (1)) so that

$$(A_q(p) + I)u = (B_q^2(p) + B_q(p) + I)v.$$

By Corollary 2.4, we have

$$u = -\frac{1}{2}(A_q(p) - I)(B_q^2(p) + B_q(p) + I)v.$$

Since $B_q(p) - I$ is invertible, there is an element $w \in L_q(p)$ such that $v = (B_q(p) - I)w$. Then

$$u = (A_q(p) - I)w.$$

Therefore

$$(u,v) = ((A_q(p) - I)w, (B_q(p) - I)w) = \delta w$$

and hence $H^1(SL(2;\mathbb{Z}); L_q(p)) = 0$. ∎

Henceforth we assume that q is even and consider $H^1(SL(2,\mathbb{Z}); L_q(p))$. According to Lemma 2.11 (2), we have an identification

$$Z^1(SL(2;\mathbb{Z}); L_q(p)) = L_q^-(p) \oplus L_q'(p) \qquad (p \neq 2)$$

where $L_q^-(p)$ and $L'_q(p)$ have been defined in Section 2.2.

Proposition 2.14 *If q is even, then*

$$\dim H^1(SL(2;\mathbb{Z});L_q(0)) = \begin{cases} 2m-1 & q=12m \\ 2m+1 & q=12m+2, 12m+4, 12m+6, \\ & \text{or } 12m+8 \\ 2m+3 & q=12m+10. \end{cases}$$

Proof. We know that the homomorphism $\delta : L_q(0) \to Z^1(SL(2;\mathbb{Z});L_q(0))$ is injective (Proposition 2.12). Hence we have

$$\begin{aligned} \dim H^1(SL(2;\mathbb{Z});L_q(0)) &= \dim Z^1(SL(2;\mathbb{Z});L_q(0)) - (q+1) \\ &= \dim L_q^-(0) + \dim L'_q(0) - (q+1). \end{aligned}$$

Then the result follows from Lemma 2.5 and Lemma 2.7. ∎

Proposition 2.15 *Assume q is even and let $d_q(p) = \dim H^0(SL(2;\mathbb{Z});L_q(p))$ (see the Remark 1) Then for p $\neq$ 2,3, we have*

$$\dim H^1(SL(2;\mathbb{Z});L_q(p)) = \dim H^1(SL(2;\mathbb{Z});L_q(0)) + d_q(p).$$

Proof. By a similar argument as in the proof of Proposition 2.14, we have

$$\dim H^1(SL(2;\mathbb{Z});L_q(p)) = \dim L_q^-(p) + \dim L'_q(p) - (q+1) + d_q(p).$$

Then the result follows because we have

$$\dim L_q^-(p) = \dim L_q^-(0) \qquad (p \neq 2)$$

by Lemma 2.5 and also we have

$$\dim L'_q(p) = \dim L'_q(0) \qquad (p \neq 3)$$

by Lemma 2.7 and Lemma 2.10. ∎

Finally, the next two theorems follow from the previous computations

Theorem 2.16

$$\dim \widetilde{H}^n(\mathrm{BDiff}_+ T^2;\mathbb{Q}) = \begin{cases} 0 & n \not\equiv 1 \pmod 4 \\ 2m-1 & n = 24m+1 \\ 2m+1 & n = 24m+5,\ 24m+9,\ 24m+13 \\ & \text{or } 24m+17 \\ 2m+3 & n = 24m+21 \end{cases}$$

Proof. From Corollary 2.2 we have

$$H^n(\mathrm{BDiff}_+ T^2;\mathbb{Q}) \cong E_2^{0,n} \oplus E_2^{1,n-1}$$

but by Proposition 2.12 it turns out

$$H^n(\mathrm{BDiff}_+ T^2;\mathbb{Q}) \cong E_2^{1,n-1}.$$

Thus Theorem 2.16 follows from Proposition 2.13 and Proposition 2.14. ∎

Remark 2 Since $5 \equiv 1 \pmod 4$ and $5 = 24 \cdot 0 + 5$, this implies that the first non-trivial group is $H^5(\mathrm{BDiff}_+ T^2;\mathbb{Q}) \cong \mathbb{Q}$, on the other hand, taking $4k+1 = 24m+13$ implies that $k = 3(2m+1)$, therefore the $\dim H^{4k+1}(\mathrm{BDiff}_+ T^2;\mathbb{Q})$ is approximately $\frac{1}{3}k$. Note that the ring structure on $H^*(\mathrm{BDiff}_+ T^2;\mathbb{Q})$ defined by the cup product is trivial.

We can also obtain information on the torsion of $H^*(\mathrm{BDiff}_+ T^2;\mathbb{Z})$ and use it to obtain:

Theorem 2.17 *Mod 2 and 3 torsion, we have*

$$\widetilde{H}_n(\mathrm{BDiff}_+ T^2;\mathbb{Z}) = \begin{cases} \text{torsion} & n \equiv 0 \pmod 4 \\ \text{free abelian group of rank} & n \equiv 1 \pmod 4 \\ \text{indicated in Theorem 2.16} & \\ 0 & n \equiv 2,3 \pmod 4. \end{cases}$$

Proof. If $p \neq 2, 3$, Corollary 2.2, Proposition 2.13 and Proposition 2.15 imply

$$\dim H^n(\mathrm{BDiff}_+ T^2; \mathbb{Z}_p) = \begin{cases} d_q(p) & n = 2q \quad (q: \text{ even}) \\ \dim H^n(\mathrm{BDiff}_+ T^2; \mathbb{Q}) + d_q(p) & n = 2q+1 \quad (q: \text{ even}) \\ 0 & n \equiv 2{,}3 \pmod 4. \end{cases}$$

where $d_q(p) = \dim H^0(SL(2;\mathbb{Z}); L_q(p))$ and the $d_q(p)$'s are given by the generating function

$$\sum_{q=0}^{\infty} d_q(p) t^q = \frac{1}{(1-t^{p+1})(1-t^{p(p-1)})}$$

Hence if $n \equiv 2, 3 \pmod 4$, then

$$H_n(\mathrm{BDiff}_+ T^2; \mathbb{Z}) = 0 \qquad \bmod 2, 3 \text{ torsions}$$

by the universal coefficient theorem. Similarly it is easy to deduce that $H_n(\mathrm{BDiff}_+ T^2; \mathbb{Z})$ has no p-torsion ($p \neq 2, 3$) if $n \equiv 1 \mod 4$. This completes the proof. ∎

Moreover it turns out that p-torsion appears in $H_{4k}(\mathrm{BDiff}_+ T^2; \mathbb{Z})$ for any prime p.

Remark 3 $H_*(\mathrm{BDiff}_+ T^2; \mathbb{Z})$ has actually 2 and 3 torsion. This follows from the following argument. The projection $\mathrm{BDiff}_+ T^2 \to K(SL(2;\mathbb{Z}),1)$ has a right inverse because $SL(2,\mathbb{Z})$ can be naturally considered as a subgroup of $\mathrm{Diff}_+ T^2$. Hence the homology

$$H_*(SL(2;\mathbb{Z}); \mathbb{Z}) \cong H_*(K(\mathbb{Z}_{12},1); \mathbb{Z})$$

embeds into $H_*(\mathrm{BDiff}_+ T^2; \mathbb{Z})$ as a direct summand. It is easy to check that $H_1(\mathrm{BDiff}_+ T^2; \mathbb{Z}) \cong \mathbb{Z}_{12}$ and $H_2(\mathrm{BDiff}_+ T^2; \mathbb{Z}) = 0$.

Remark 4 By Theorem 2.16 and Theorem 2.17, we have an isomorphism

$$H^{4k}(\mathrm{BDiff}_+ T^2; \mathbb{Z}_p) = \mathrm{Hom}(H_{4k}(\mathrm{BDiff}_+ T^2; \mathbb{Z}),\mathbb{Z}_p) \quad (p \neq 2, 3).$$

On the other hand we have

$$H^{4k}(\mathrm{BDiff}_+ T^2; \mathbb{Z}_p) \cong L_{2k}(p)^{SL(2;\mathbb{Z})}$$

by Corollary 2.2, where the right hand side denotes the subspace of $L_{2k}(p)$ consisting of those elements which are left invariant by the action of $SL(2;\mathbb{Z})$. Then in view of Remark 1, we can conclude that the p-primary part of $H_{4k}(\mathrm{BDiff}_+ T^2;\mathbb{Z})$ is non-trivial provided $2k$ can be expressed as a linear combination of $p+1$ and $p(p-1)$ with coefficients in non-negative integers. Also it can be shown that mod 2 and 3 torsion we have an isomorphism

$$H_{4k}(\mathrm{BDiff}_+ T^2;\mathbb{Z}) \cong L_{2k}/K_{2k}$$

where K_{2k} denotes the submodule of L_{2k} generated by elements $\gamma(u)-u$ where $u \in L_{2k}$, and $\gamma \in SL(2;\mathbb{Z})$.

2.5 Non-triviality of the characteristic classes

In this final section we construct an element of $H_5(\mathrm{BDiff}_+ T^2;\mathbb{Z})$ which has infinite order. First it can be shown by a direct computation that the crossed homomorphism

$$f : SL(2;\mathbb{Z}) \longrightarrow L_2(0)$$

given by $f(\alpha) = x^2 - y^2$ and $f(\beta) = 0$ represents a non-zero element of $H^1(SL(2;\mathbb{Z}); L_2(0)) \cong \mathbb{Q}$ (see Proposition 2.14.) We write $[f] \in H^5(\mathrm{BDiff}_+ T^2;\mathbb{Q})$ for the corresponding element (see Corollary 2.2.) Now let η be the canonical line bundle over $\mathbb{C}P^2$ and let $T^2 \to E(k,l) \to \mathbb{C}P^2$ be the T^2-bundle associated to the complex 2-plane bundle $\eta^k \oplus \eta^l$ on $\mathbb{C}P^2$ with $k,l \in \mathbb{Z}$. Let $T^2 \to E'(k,l) \to \mathbb{C}P^1$ be the restriction of $E(k,l)$ to $\mathbb{C}P^1 \subset \mathbb{C}P^2$. Then we can write

$$E'(k,l) = D^2 \times S^1 \times S^1 \bigcup_{g_{k,l}} D^2 \times S^1 \times S^1$$

where the pasting map $g_{k,l} : \partial D^2 \times S^1 \times S^1 \to \partial D^2 \times S^1 \times S^1$ is given by

$$g_{k,l}(z_1,z_2,z_3) = (z_1^{-1}, z_1^k z_2, z_1^l z_3)$$

where $z_1 \in \partial D^2$, and $z_2,z_3 \in S^1$. Now for an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$, let $h_\gamma : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ be the diffeomorphism defined by

$$h_\gamma(z_1,z_2,z_3) = (z_1, z_2^a z_3^b, z_2^c z_3^d)$$

with $z_1 \in D^2$ and $z_2, z_3 \in S^1$. It is easy to show that if two relations:

$$ak + bl = k \qquad \text{and} \qquad ck + dl = l$$

are satisfied, then h_γ extends to a diffeomorphism $h'_\gamma : E'(k,l) \to E'(k,l)$ which is an automorphism as a T^2-bundle. Then since $\pi_3(\mathrm{Diff}_+ T^2) = 0$, we can extend h'_γ to an automorphism $H_\gamma : E(k,l) \to E(k,l)$. H_γ is nothing but the automorphism of $E(k,l)$ as a *principal* T^2-bundle defined by the automorphism of T^2 given by γ. Let $M_\gamma(k,l)$ be the mapping torus of H_γ. The natural projection

$$M_\gamma(k,l) \longrightarrow S^1 \times \mathbb{C}P^2$$

has the structure of a T^2-bundle. Clearly the classifying map of this T^2-bundle is given by

$$\begin{tikzcd}
\mathbb{C}P^2 \arrow[r] \arrow[d, "i_0"'] & S^1 \times \mathbb{C}P^2 \arrow[r] \arrow[d, "i"'] & S^1 \arrow[d, "\widetilde{i}"'] \\
\mathrm{BDiff}_0\, T^2 \arrow[r] & \mathrm{BDiff}_+\, T^2 \arrow[r] & K(SL(2,\mathbb{Z}),1)
\end{tikzcd}$$

where i_0 is characterized by the induced map $i_0^* : H^2(\mathrm{BDiff}_0\, T^2; \mathbb{Z}) \to H^2(\mathbb{C}P^2; \mathbb{Z})$ which is given by $i_0^*(x) = k\iota$, $i_0^*(y) = l\iota$ where $\iota \in H^2(\mathbb{C}P^2; \mathbb{Z})$ is the first Chern class of η, and the map $\widetilde{i}$ represents $\gamma^{-1} \in \pi_1(K(SL(2;\mathbb{Z}),1)) = SL(2;\mathbb{Z})$. Therefore we conclude that

$$\langle [S^1 \times \mathbb{C}P^2], i^*([f]) \rangle = i_0^*(f(\gamma^{-1})) \in H^4(\mathbb{C}P^2; \mathbb{Q}) \cong \mathbb{Q}.$$

If we choose $\gamma = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and $k = l = 1$, then $\gamma = \beta^{-1}\alpha\beta^{-1}$ so that $f(\gamma^{-1}) = y^2 - 2xy$ and hence $i_0^*(f(\gamma^{-1})) = -\iota^2$. This proves that the corresponding T^2-bundle represents a non-zero element of $H_5(\mathrm{BDiff}_+\, T^2; \mathbb{Q})$. Similarly we can construct non-zero elements of $H_{4k+1}(\mathrm{BDiff}_+\, T^2; \mathbb{Q})$, for $k > 1$, explicitly, but we stop here.

3

Characteristic classes of Σ_g-bundles with $g \geq 2$

This chapter is focused in the non-triviality of the first Miller-Morita-Munford characteristic class e_1 of a surface bundle, with fiber of genus greater than one taking as base space a surface. In order to define these classes and prove the non-triviality of e_1 we introduce some technical tools like the Gysin homomorphism and ramified coverings, as well as some of their properties.

3.1 The Gysin homomorphism

In Section 3.4 we define characteristic classes of surface bundle where we shall make essential use of the Gysin homomorphism. This homomorphism is very important for the study of surface bundles as well as general manifolds. In this section we briefly summarize basic facts concerning it.

Let F be an oriented closed manifold and let

$$\pi : E \longrightarrow M$$

be an F-bundle over M. We assume this bundle is oriented; that is the tangent bundle along the fiber of π, denoted by $\xi = \{X \in TE \mid \pi_* X = 0\}$, is orientable and is given a specific orientation. Although we are only concerned with the case $F = \Sigma_g$, the Gysin homomorphism is defined for general F-bundles. If we denote by $\{E_r^{p,q}\}$ the spectral sequence for the cohomology of the above F-bundle, then its E_2 term is given by

$$E_2^{p.q} \cong H^p(M; \mathcal{H}^q(F))$$

(see [20], Theorem 5.2.) Here $\mathcal{H}^q(F)$ stands for the local coefficient system associated to the q-dimensional cohomology $H^q(\pi^{-1}(p); \mathbb{Z})$ with $p \in M$ of the fibers. If F is n-dimensional, then clearly $\mathcal{H}^q(F) = 0$ for $q > n$ so that

$$E_2^{p,q} = 0 \quad (q > n).$$

By the assumption, $\mathcal{H}^n(F)$ is isomorphic to the constant local system $\mathbb{Z}$. Hence

$$E_2^{p,n} \cong H^p(M; \mathbb{Z})$$

On the other hand, the homomorphism $d_r : E_r^{p-r,n+r-1} \longrightarrow E_r^{p,n}$ is trivial for any p and $r \geq 2$, it is because $E_2^{p-r,n+r-1} = H^{p-r}(M; \mathcal{H}^{n+r-1}(F))$ and $n + r - 1 \geq n + 1$ for $r \geq 2$, since $\mathcal{H}^q = 0$ for $q > n$ so that $E_{r+1}^{p-r,n+r-1} \subset E_2^{p-r,n+r-1} = 0$ for $r \geq 2$.

$$\begin{tikzcd} 0 = E_r^{p-r.n+r-1} \arrow[dr, "d_r=0"] & & \\ & E_r^{p,n} \arrow[dr, "d_r"] & \\ & & E_r^{p+r,n-r+1} \end{tikzcd}$$

where $E_{r+1}^{p,n} = \ker d_r \subset E_r^{p,n}$, so that we obtain a series of monomorphism

$$E_\infty^{p,n} \subset \cdots \subset E_3^{p,n} \subset E_2^{p,n} \cong H^p(M; \mathbb{Z}).$$

Now we denote the homomorphism

$$H^p(E; \mathbb{Z}) \longrightarrow E_\infty^{p-n,n} \subset E_2^{p-n,n} \cong H^{p-n}(M; \mathbb{Z})$$

which is the composition of the natural projection $H^p(E; \mathbb{Z}) \longrightarrow E_\infty^{p-n,n}$ with the above monomorphism (with shifted degree) by

$$\pi_* : H^p(E; \mathbb{Z}) \longrightarrow H^{p-n}(M; \mathbb{Z}) \tag{3.1}$$

and call it the **Gysin homomorphism** of the F-bundle $\pi : E \longrightarrow M$. Sometimes the symbol $\pi_!$ is used instead of π_*. Note that this homomorphism goes in the opposite direction to the usual one which is induced by the projection π and also that it decreases the degree by n, namely the dimension of fiber. Similarly we have the Gysin homomorphism

$$\pi^* : H_p(M;\mathbb{Z}) \longrightarrow H_{p+n}(E;\mathbb{Z}) \tag{3.2}$$

in homology.

The above method of defining the Gysin homomorphism using the spectral sequence is valid over $\mathbb{Z}$, and we may say that it is theoretically the best one. However, it might not be easy to see its geometrical meaning. To cover this point, let us examine the Gysin homomorphism in the context of de Rham cohomology, although the coefficients must be reduce to $\mathbb{R}$. The Gysin homomorphism can be explained by means of an operation

$$\pi_* : A^p(E) \longrightarrow A^{p-n}(M)$$

defined in the de Rham complex called **integration along the fiber**. Any differential p-form on E can be expressed locally as a sum of the forms like

$$\omega = \sum_{i,j} f_{i,j}(x,y) dx_{i_1} \wedge \cdots \wedge dx_{i_s} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_t}.$$

Here $x_1, \cdots, x_m$ with $m = \dim M$ and $y_1, \cdots y_n$ is assumed to coincide with the given orientation of F. Here the summation is taken over the multiindices $i = (i_1, \cdots, i_s)$, $i_1 < \cdots < i_s$, $j = (j_1, \cdots, j_t)$, $j_1 < \cdots < j_t$ with $s + t = p$. We now set

$$\pi_*(\omega) = \sum_{i,j(t=n)} \left(\int_F f_{ij}(x,y) dy_1 \wedge \cdots \wedge dy_n \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{p-n}}.$$

It can be easily shown that π_* is in fact uniquely define by the above. The integration along the fiber commutes with the exterior differential d, namely

$$d \circ \pi_* = \pi_* \circ d.$$

Hence it induces a homomorphism

$$\pi_* : H^p(E;\mathbb{R}) \longrightarrow H^{p-n}(M;\mathbb{R}),$$

and it can be shown that this coincides with the Gysin homomorphism which was defined by using the spectral sequence.

In the cases where the base space M is an oriented closed manifold, there is a simple interpretation of the Gysin homomorphism in terms of Poincaré duality. Namely, given a continuous mapping $f : N \longrightarrow N'$ between two oriented closed manifolds N,N', there is a homomorphism

$$f_* : H^p(N;\mathbb{Z}) \longrightarrow H^{p-d}(N';\mathbb{Z})$$

(also called the Gysin homomorphism) defined by the composition

$$\begin{tikzcd} H^p(N;\mathbb{Z}) \arrow[r, dashed] \arrow[d, "D"', "\cong"] & H^{p-d}(N';\mathbb{Z}) \\ H_{n-p}(N;\mathbb{Z}) \arrow[r, "f_*"] & H_{n-p}(N';\mathbb{Z}) \arrow[u, "\cong", "(D')^{-1}"'] \end{tikzcd}$$

Here $n = \mathrm{dim}N$, $d = \mathrm{dim}N - \mathrm{dim}N'$, and D and D' denote the Poincaré duality maps of N,N' respectively. Similarly, the Gysin homomorphism

$$f^* : H_p(N';\mathbb{Z}) \longrightarrow H_{p+d}(N;\mathbb{Z})$$

in homology is defined by setting $f^* = D^{-1} \circ f^* \circ D'$. Now let us go back to the original situation where we are given an F-bundle $\pi : E \longrightarrow M$ and assume that M is an oriented closed manifold. Then the total space E is also a closed manifold with the induced orientation which is locally equal to the one on the product $M \times F$. In this case, it can be shown that the Gysin homomorphism

$$\pi_* : H^p(E;\mathbb{Z}) \longrightarrow H^{p-n}(M;\mathbb{Z})$$

$$\pi^* : H_p(M;\mathbb{Z}) \longrightarrow H_{p+n}(E;\mathbb{Z})$$

associated to the projection π defined through Poincaré duality coincide with the former definition (3.1), (3.2).

The following proposition concerns basic properties of the Gysin homomorphism.

Proposition 3.1 *1. Let F be an oriented closed manifold and let $\pi : E \longrightarrow M$ be an oriented F-bundle. Then for any $\alpha \in H^p(M;\mathbb{Z})$ and $\beta \in H^q(E;\mathbb{Z})$, the equality*

$$\pi_*(\pi^*(\alpha) \cup \beta) = \alpha \cup \pi_*(\beta)$$

holds.

2. *For any* $u \in H_p(M;\mathbb{Z})$ *and* $\gamma \in H^{p+n}(E;\mathbb{Z})$, *we have*

$$\langle \gamma, \pi^*(u) \rangle = \langle \pi_*(\gamma), u \rangle$$

where $\langle -,- \rangle$ *is the Kronecker pairing.*

In particular, in the situation of (1), *if we further assume that* M *is an oriented closed manifold and* $p + q = \dim E$, *then*

$$\langle \pi^*(\alpha) \cup \beta, [E] \rangle = \langle \alpha \cup \pi_*(\beta), [M] \rangle.$$

3. *The Gysin homomorphism is natural with respect to bundle maps. More precisely, given a map of oriented* F*-bundles*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\widetilde{f}"] & M' \end{tikzcd}$$

the following diagram commutes

$$\begin{tikzcd} H^*(E) \arrow[d, "\pi_*"'] & H^*(E') \arrow[l, "f^*"'] \arrow[d, "\pi'_*"] \\ H^*(M) & H^*(M') \arrow[l, "\widetilde{f}^*"'] \end{tikzcd}$$

∎

The following lemma gives a geometric description of the Gysin homomorphism in the special case for covering maps. We will use it in Section 3.6.

Lemma 3.2 *Let* M *be an* n*-dimensional oriented closed manifold and let* $\pi : \widetilde{M} \longrightarrow M$ *be a finite covering. We give* $\widetilde{M}$ *the orientation induced from* M. *Suppose that the Poincaré dual* $[M] \cap \alpha \in H_{n-k}(M;\mathbb{Z})$ *of a cohomology class* $\alpha \in H^k(M;\mathbb{Z})$ *is represented by an* $(n-k)$*-dimensional oriented submanifold* B *of* M. *Then the Poincaré dual of* $\pi^*(\alpha) \in H^k(\widetilde{M};\mathbb{Z})$ *can be represented by the oriented submanifold* $\widetilde{B} = \pi^{-1}(B)$ *of* $\widetilde{M}$.

Proof. Let $N(B)$ be a closed tubular neighborhood of B in M. If we denote by ν the normal bundle of B, then we can identify $N(B)$ with the disk bundle $D(\nu)$ with respect to a suitable metric. We set $W = M \setminus \mathrm{Int}N(B)$ and consider the following natural homomorphism:

$$H^k(D(\nu),\partial D(\nu);\mathbb{Z}) \cong H^k(N(B),\partial N(B);\mathbb{Z})$$
$$\cong H^k(M,W;\mathbb{Z}) \longrightarrow H^k(M;\mathbb{Z}).$$

If we denote by $U \in H^k(D(\nu),\partial D(\nu);\mathbb{Z})$ the Thom class; then as is well known (see, for example, [3] Proposition 6.24 (a)), the image of U under the above homomorphism is nothing but the Poincaré dual of $[B]$, namely $\alpha \in H^k(M;\mathbb{Z})$. On the other hand, $\pi^{-1}(N(B))$ can serve as a closed tubular neighborhood $N(\widetilde{B})$ of $\widetilde{B}$, and moreover under the homomorphism

$$H^k(N(B),\partial N(B);\mathbb{Z}) \xrightarrow{\pi^*} H^k(N(\widetilde{B}),\partial N(\widetilde{B});\mathbb{Z})$$

induced by the projection, the above Thom class U clearly goes to that of the normal bundle of $\widetilde{B}$. The claim follows from this immediately. ∎

3.2 Ramified coverings

In Section 3.6 we prove the non-triviality of the characteristic classes of surfaces bundles defined in Section 3.4. The proof will be given by explicitly constructing surface bundles with non-zero characteristic classes. In this section, we briefly discuss ramified coverings, which are essential in such construction.

The concept of **ramified covering** (or branched covering) is obtained by generalizing that of covering spaces, and there are various formulations in the framework of algebraic varieties, complex manifolds or differentiable manifolds. Roughly speaking, a submanifold, called the **ramification locus** or branch locus, is given in the base manifold, and away from there it is a usual covering space. Suitable conditions are required on the ramification locus according to each framework mentioned above.

Here we consider only the most simple type of ramified coverings, namely **cyclic ramified coverings**.

Let m be a positive integer. An m-fold cyclic ramified covering is defined by taking the map

$$\mathbb{C} \ni z \mapsto z^m \in \mathbb{C}$$

as a model. This is the identity at the origin, and the usual covering map in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. From a slightly different viewpoint we can also interpret it as follows. The cyclic group $\mathbb{Z}/m$ acts on $\mathbb{C}$ naturally by

$$\mathbb{C} \ni z \mapsto \zeta \cdot z = \exp\left(\frac{2\pi i}{m}\right) z \in \mathbb{C}$$

where ζ denotes the generator of $\mathbb{Z}/m$. This action is free outside of the origin, and the quotient space can be canonically identified with $\mathbb{C}$. Moreover it is easy to see that the projection to the quotient space

$$\mathbb{C} \longrightarrow \mathbb{C}/(\mathbb{Z}/m) \cong \mathbb{C}$$

is equivalent to the above map.

Now a ramified covering is defined, locally, by taking a direct product of this model with other manifolds. More concretely, assume that the cyclic group $\mathbb{Z}/m$ acts on an oriented C^∞ manifold N by orientation preserving diffeomorphism satisfying the following condition: the fixed point set

$$F = \{p \in N \mid \zeta \cdot p = p\}$$

is a submanifold on N of codimension 2, and the action is free outside of F. Then, it can be checked that the quotient space $\overline{N} = N/(\mathbb{Z}/m)$ has a natural structure of an oriented C^∞ manifold by investigating the action of $\mathbb{Z}/m$ on the normal bundle of each connected component of F. If we denote by

$$\pi : N \longrightarrow \overline{N}$$

the natural projection to the quotient space, then $\overline{F} = \pi(F)$ becomes a submanifold of $\overline{N}$ of codimension 2. Moreover the restriction $\pi : F \longrightarrow \overline{F}$ is a diffeomorphism and $\pi : N \setminus F \longrightarrow \overline{N} \setminus \overline{F}$ is a covering map in the usual sense. Finally it is easy to see that the map $\pi : N \longrightarrow \overline{N}$ is equivalent to the above model $F \times \mathbb{C} \ni (p,z) \mapsto (p,z^m) \in \overline{F} \times \mathbb{C}$ near F.

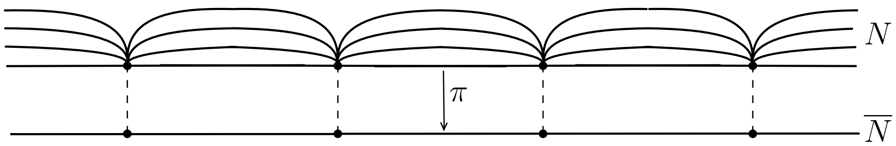

Figure 3.1

In such a situation, we call $\pi : N \longrightarrow \overline{N}$ an m-fold cyclic ramified covering, ramified along F. It is also called simply an m-fold ramified covering.

3.3 Construction of ramified coverings

Let M be an oriented closed C^∞ manifold. Assume that there is given an oriented submanifold $B \subset M$ of codimension 2. Following Atiyah [2] and Hirzebruch [14], let us recall a sufficient condition for the existence of an m-fold ramified covering of M ramified along B.

Let $\alpha \in H^2(M; \mathbb{Z})$ be the Poincaré dual of the fundamental homology class of B, $[B] \in H_{n-2}(M; \mathbb{Z})$, where $n = \dim M$. Recall that there is a canonical bijection

$$\{\text{isomorphism classes of line bundles over } M\} \cong H^2(M; \mathbb{Z}),$$

taking a line bundle L to its first Chern class $c_1(L) \in H^2(M; \mathbb{Z})$ (see [8] Theorem 5.A.) Hence there exist a complex line bundle η over M which corresponds to α. This bundle can be constructed explicitly as follows: let ν be the normal bundle of B in M and denote by $E(\nu)$ its total space. Then ν is a 2-dimensional real vector bundle over B, and it has a natural orientation induced by those of M and B. Hence we can consider ν as a complex line bundle. Let $N(B)$ be a closed tubular neighborhood of B. Then as is well known, by choosing a Hermitian metric on ν, we can construct a diffeomorphism

$$\varphi : N(B) \cong \{v \in E(\nu) \mid ||v|| < \varepsilon\} \quad \text{where } \varepsilon > 0$$

such that $B \subset N(B)$ is sent to the 0-section of ν (cf. Figure 3.2).

Let $\pi : E(\nu) \longrightarrow B$ be the projection and set

$$\pi' = \pi \circ \varphi : N(B) \longrightarrow B.$$

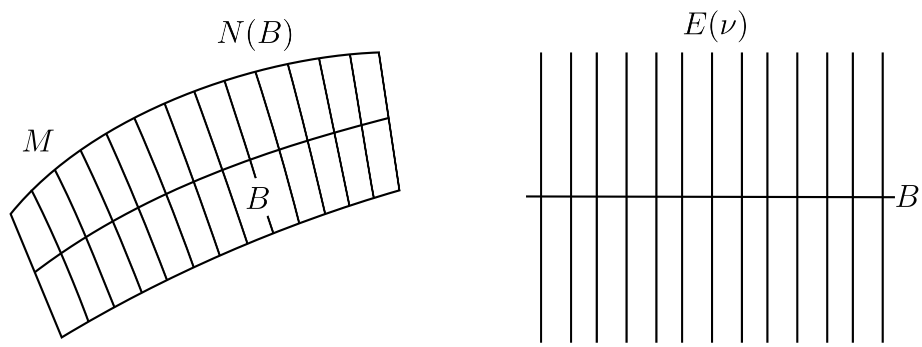

Figure 3.2

The total space of the pullback bundle $\eta_0 = (\pi')^*(\nu)$ of ν by the map π', which is a complex line bundle over $N(B)$, can be described as

$$E(\eta_0) = \{(p,v) \mid p \in N(B), v \in \pi^{-1}(\pi'(p))\}.$$

Then a natural section $s : N(B) \longrightarrow E(\eta_0)$ is defined by

$$N(B) \ni p \longmapsto s(p) = (p,\varphi(p)) \in E(\eta_0).$$

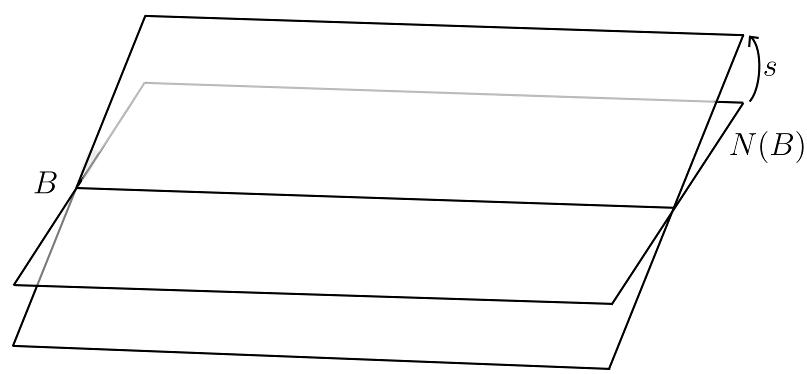

Figure 3.3

This section never vanishes over $N(B) \setminus B$. Hence it induces a trivialization

$$\eta_0|_{N(B)\setminus B} \cong (N(B) \setminus B) \times \mathbb{C}.$$

Now set $W = M \setminus B$ and paste the trivial bundle $W \times \mathbb{C}$ to η_0 by the above trivilization to obtain a complex line bundle η over M. More precisely, we identify $(p,z) \in (N(B)\setminus B) \times \mathbb{C}$ with $(p,zs(p)) \in E(\eta_0)$. We can extend the section s of η_0 to that of η by setting $s = 1$ over W.

It is clear from the construction that $\eta|_B = \nu$. Moreover the zero locus of s is precisely equal to B, and the image $\mathrm{Im} s$ of s intersects the 0-section $M \subset E(\eta)$ transversely (cf. Figure 3.4).

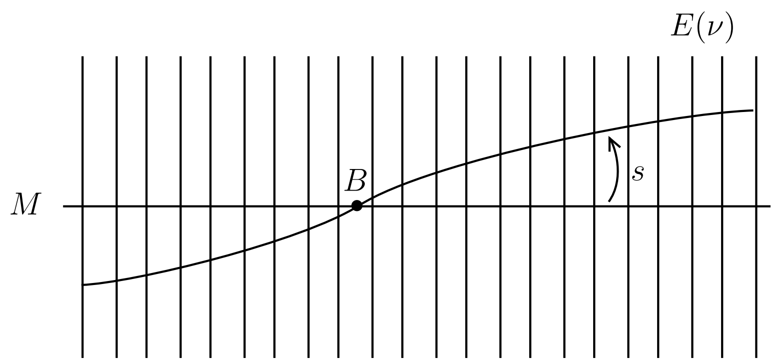

Figure 3.4

In the above argument, we first assumed that M is a closed manifold. However, it can be shown that this assumption is unnecessary for the construction of the complex line bundle η.

In the case M is a complex manifold and B is a complex submanifold of codimension 1, the above construction can be done entirely in the complex analytic category. Namely we can construct η as a holomorphic line bundle over M, and the section s can be chosen to be holomorphic. More concretely, choose a family U_i with $i \geq 1$ of coordinate neighborhoods of M with the property that $B \subset \cup_i U_i$ and there is a coordinate function $f_i : U_i \longrightarrow \mathbb{C}$ such that

$$B \cap U_i = \{p \in B \cap U_i \mid f_i(p) = 0\}.$$

We also set $U_0 = M \setminus B$ and consider the constant function $f_0 = 1$ on U_0. If we put $f_{ij} = f_i \circ f_j^{-1}$, then $\{f_{ij}\}$ becomes a 1-cocycle associated to the open covering $\{U_i\}_{i \geq 0}$ of M with values in $\mathbb{C}^\times$. We can now define η to be the holomorphic line bundle determined by this 1-cocycle. Also the section s which is induced by f_i is holomorphic and clearly satisfies the above condition. In [14] there is a description of a more general construction including the case where B is expressed as the difference of two complex submanifolds.

With the above preparation in mind, we prove the following proposition.

Proposition 3.3 ([14]) *Let M be a closed oriented C^∞ manifold and let $B \subset M$ be an oriented submanifold of codimension 2. Suppose that, for some positive integer m, the homology class $[B] \in H_{n-2}(M;\mathbb{Z})$ determined by B is divisible by m in $H_{n-2}(M;\mathbb{Z})$. Then there exist an m-fold cyclic ramified covering $\widetilde{M} \longrightarrow M$ ramified along B.*

Proof. Let $\alpha \in H^2(M;\mathbb{Z})$ denote the Poincaré dual of $[B] \in H_{n-2}(M;\mathbb{Z})$. By the assumption, there is an element $\beta \in H^2(M;\mathbb{Z})$ such that $m\beta = \alpha$. If we denote by η the complex line bundle corresponding to α, namely $c_1(\eta) = \alpha$, then there exist a section $s : M \longrightarrow E(\eta)$ satisfying the following three conditions:

1. $s = 0$ on B.

2. $s \neq 0$ on $M \setminus B$.

3. $\operatorname{Im} s$ meets with $M \subset E(\eta)$ transversely.

Let η' be the complex line bundle with $c_1(\eta') = \beta$. Then we have $(\eta')^{\otimes m} \cong \eta$. Hence we can define a mapping

$$f : E(\eta') \longrightarrow E(\eta)$$

by setting $f(v) = v \otimes \cdots \otimes v$ with $v \in E(\eta')$. If we set $\widetilde{M} = f^{-1}(\operatorname{Im} s)$, then $f : \widetilde{M} \longrightarrow \operatorname{Im} s = M$ is the desired ramified covering. ∎

3.4 Definition of characteristic classes

Let Σ_g be an oriented closed surface of genus g and let

$$\pi : E \longrightarrow M$$

be an oriented Σ_g-bundle. If we denote by ξ the tangent bundle along the fiber of π, that is, $\xi = \{X \in TE : \pi_*(X) = 0\}$, then by definition of oriented surface bundle ξ has a structure of an oriented 2-dimensional real vector bundle. Hence its Euler class

$$e = \chi(\xi) \in H^2(E;\mathbb{Z})$$

is defined. For each non-negative integer i, we consider the power

$$e^{i+1} \in H^{2(i+1)}(E;\mathbb{Z})$$

of the Euler class e. We then apply the Gysin homomorphism (cf. Section 3.1)

$$\pi_* : H^{2(i+1)}(E;\mathbb{Z}) \longrightarrow H^{2i}(M;\mathbb{Z})$$

to e^{i+1} and obtain a cohomology class of the base space M which we denote by

$$e_i(\pi) = \pi_*(e^{i+1}) \in H^{2i}(M;\mathbb{Z}).$$

Definition The cohomology class $e_i(\pi) \in H^{2i}(M;\mathbb{Z})$ which is defined for any Σ_g-bundle $\pi : E \longrightarrow M$ as above is called the **i-th characteristic class of surface bundle**.

The fact that e_i in fact defines a characteristic class of surface bundles, namely that is natural under tha bundle maps, can be checked as follows. Let $\pi_i : E_i \longrightarrow M_i$, with $i = 1,2$, be two Σ_g-bundles and let

$$\begin{tikzcd} E_1 \arrow[r, "\overline{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

be a bundle map. Then by definition, the restriction of $\overline{f}$ to each fiber is an orientation preserving diffeormorphism. Hence if ξ_i denotes the tangent bundle along the fibers of π_i, then we have

$$\chi(\xi_1) = \overline{f}^*(\chi(\xi_2))$$

The naturality of the Gysin homomorphism (cf. Proposition 3.1(iii)) implies that

$$e_i(\pi_1) = f^*(e_i(\pi_2))$$

which shows that e_i is indeed a characteristic class. Hence from the description of made in Section 1.4, we can write

$$e_i \in H^{2i}(B\,\mathrm{Diff}_+\,\Sigma_g;\mathbb{Z}) = H^{2i}(\mathcal{M}_g;\mathbb{Z})$$

for $g \geq 2$. In other words, e_i can be considered as a cohomology class of $\mathcal{M}_g$ of degree $2i$.

3.5 The first characteristic class e_1 and the signature

The first characteristic class e_1 is closely related to the signature which is an important invariant of closed oriented 4-manifolds. Just to make sure, we recall the definition of the signature, denoted $\operatorname{sign} M$, of a closed oriented $4k$-dimensional manifold M. It is defined as the signature of the symmetric bilinear form

$$H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \longrightarrow H^{4k}(M;\mathbb{Q}) \cong \mathbb{Q}$$

which is induced by the cup product, namely the number of positive eigenvalues minus that of negative eigenvalues.

Now to state the relation between the signature and e_1, let

$$\pi : E \longrightarrow M$$

be an oriented Σ_g-bundle and assume that M is also an oriented closed surface. Then the total space E becomes a closed 4-manifold equipped with a natural orientation so that its signature $\operatorname{sign} E$ is defined.

Proposition 3.4 *Let $\pi : E \longrightarrow M$ be an oriented Σ_g-bundle over closed oriented surface M. Then we have the equality*

$$\langle e_1, [M] \rangle = 3 \operatorname{sign} E.$$

Proof. Let ξ be the tangent bundle along the fiber of the given Σ_g-bundle and let TE and TM be the tangent bundles of E and M respectively. Consider the following diagrams

$$\begin{tikzcd} E(\xi) \arrow[r, hook] \arrow[dr] & TE \arrow[r] \arrow[d] & TM \arrow[d] \\ & E \arrow[r, "\pi"] & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \pi^*(TM) \arrow[r] \arrow[d] & TM \arrow[d] \\ E \arrow[r, "\pi"] & M \end{tikzcd}$$

Then we have

$$TE \cong \xi \oplus \pi^*(TM).$$

If p_1 is the first Pontryagin class and χ is the Euler class, it follows that

$$\begin{aligned} p_1(E) &= p_1(\xi \oplus \pi^*(TM)) \\ &= p_1(\xi) + p_1(\pi^*TM) \quad \text{(by a property of first Pontryagin class)} \\ &= p_1(\xi) + \pi^*(p_1(M)) \qquad \text{(by naturality)} \\ &= \chi(\xi)^2 = e^2 \end{aligned}$$

because $p_1 = \chi^2$ for any 2-dimensional oriented real vector bundle (see [22], Theorem 15.3 and 15.8), and clearly $0 = p_1(M) \in H^4(M;\mathbb{Z})$. We can conclude from this that

$$\begin{aligned} \langle e_1,[M]\rangle &= \langle \pi_*(e^2),[M]\rangle \\ &= \langle e^2,\pi^*[M]\rangle \\ &= \langle e^2,[E]\rangle \\ &= \langle p_1(E),[E]\rangle \\ &= 3 \text{ sign } E \end{aligned}$$

where we have used Proposition 3.1 (ii) of Section 3.1 for the third equality and the Hirzebruch signature theorem (see [15], §4) for the last equation. ∎

In view of the above proposition, to prove the non-triviality of e_1 it is enough to construct a surface bundle over a surface which has non-zero signature. Such a surface bundle was first constructed by Kodaira [17] in the framework of the theory of complex surfaces. Slightly later, but independently, Atiyah [2] gave similar surface bundles.

In the next section, we prove the non-triviality of the first characteristic class e_1 of a surface bundle. The proof is given by slightly modifying Atiyah's argument in [2]. While Atiyah's argument proves the signature is non-zero, we prove directly that the first characteristic class of the constructed surface bundle is non-trivial.

3.6 Non-triviality of the first characteristic class e_1

In this section we construct a surface bundle over a surface which has non-trivial first characteristic class e_1.

First we set $M_1 = \Sigma_{g_1}$ with $g_1 \geq 2$. Choose an m-fold cyclic covering $\rho_1 : M_2 \longrightarrow M_1$ of M_1 and let $\sigma : M_2 \longrightarrow M_2$ be a generator of its covering transformation group. If we denote by g_2 the genus of M_2, then from the equality $2 - 2g_2 = m(2 - 2g_1)$ we obtain $g_2 = mg_1 - m + 1$. Next let $\rho_2 : M_3 \longrightarrow M_2$ be the covering induced by the kernel of surjective homomorphism

$$\begin{tikzcd} \varphi : \pi_1(M_2) \arrow[r] \arrow[rr, bend left] & H_1(M_2;\mathbb{Z}) \arrow[r] & H_1(M_2;\mathbb{Z}/m) \cong (\mathbb{Z}/m)^{2g_2} \end{tikzcd}$$

which is a normal subgroup of $\pi_1(M_2)$. Notice that $[\pi_1(M_2) : \ker\varphi] = \mathrm{o}(\pi_1(M_2)/\ker\varphi) = \mathrm{o}((\mathbb{Z}/m)^{2g_2})$ and thus ρ_2 is a m^{2g_2}-fold covering and so the genus of M_3 is equal to $m^{2g_2}(g_2 - 1) + 1$.

In the product manifold $M_3 \times M_2$, let $\Gamma_{\sigma^i\rho_2}$ denote the graph of the map $\sigma^i\rho_2$ with $i = 1, \cdots, m$

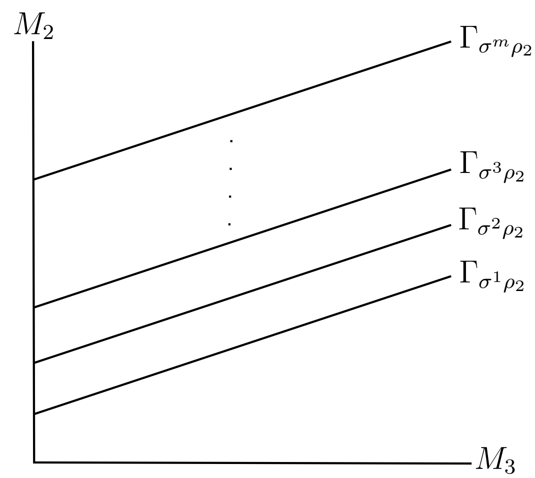

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Figure 3.5

and set

$$D = \Gamma_{\sigma\rho_2} + \cdots + \Gamma_{\sigma^m\rho_2}.$$

Here we have $\Gamma_{\sigma^m\rho_2} = \Gamma_{\rho_2}$ because $\sigma^m = \mathrm{id}$. If we fix a complex structure on M_1, namely a structure of a Riemann surface, then M_2 and M_3 also have the induced structure of

Riemann surface and D becomes a non-singular divisor of the complex surface $M_3 \times M_2$. However, topologically it is enough to understand D as a topological sum of the m submanifolds $\Gamma_{\sigma^i \rho_2}$ with $i = 1, \cdots, m$ of codimension 2.

We will show below the homology class $[D]$ of D is divisible by m in $H_2(M_3 \times M_2; \mathbb{Z})$. In view of Proposition 3.3, it will then follow that there exist an m-fold cyclic covering $f : E \longrightarrow M_3 \times M_2$ ramified along D, and we obtain the following commutative diagram consisting of bundle maps between surface bundles

$$\begin{tikzcd}
\widetilde{D} & D = f_2^{-1}(D_2) & D_2 = f_1^{-1}(D_1) & D_1 \\
\cap & \cap & \cap & \cap \\
E \arrow[r, "f"] \arrow[d, "\pi"'] & M_3 \times M_2 \arrow[r, "f_2"] \arrow[d, "p"'] & M_2 \times M_2 \arrow[r, "f_1"] \arrow[d, "p"'] & M_1 \times M_1 \arrow[d, "p"'] \\
M_3 \arrow[r, equal] & M_3 \arrow[r, "\rho_2"] & M_2 \arrow[r, "\rho_1"] & M_1
\end{tikzcd} \tag{3.3}$$

Here $f_1 = (\rho_1, \rho_1)$, $f_2 = (\rho_2, \mathrm{id}_{M_2})$ and p denotes the projection to the first factor. Also $D_1 \subset M_1 \times M_1$ is the diagonal set and $D_2 = f_1^{-1}(D_1)$. The point here is the fact that $D = f_2^{-1}(D_2)$.

Now to prove that $[D]$ is divisible by m, it is enough to show that the Poincaré dual of $[D]$, denoted by $[D]^* \in H^2(M_3 \times M_2; \mathbb{Z})$, is divisible by m. For that we will show that the mod m reduction of $[D]^*$, which we denote by $[D]^*_m \in H^2(M_3 \times M_2; \mathbb{Z}/m)$, vanishes. Using here Lemma 3.2, we obtain

$$\begin{array}{ccccccc}
H_2(M_1 \times M_1; \mathbb{Z}) & \rightarrow & H^2(M_1 \times M_1; \mathbb{Z}) & \xrightarrow{f_1^*} & H^2(M_2 \times M_2; \mathbb{Z}) & \leftarrow & H_2(M_2 \times M_2; \mathbb{Z}) \\
[D_1] & \mapsto & [D_1]^* & \mapsto & f_1^* [D_1]^* = [D_2]^* & \leftarrow\!\shortmid & [D_2]
\end{array}$$

and because $D = f_2^{-1}(D_2)$, we have

$$\begin{array}{ccccccc}
H_2(M_2 \times M_2; \mathbb{Z}) & \rightarrow & H^2(M_2 \times M_2; \mathbb{Z}) & \xrightarrow{f_2^*} & H^2(M_3 \times M_2; \mathbb{Z}) & \leftarrow & H_2(M_3 \times M_2; \mathbb{Z}) \\
[D_2] & \mapsto & [D_2]^* & \mapsto & f_2^* [D_2]^* = [D]^* & \leftarrow\!\shortmid & [D],
\end{array}$$

then

$$[D]^*_m = f_2^*([D_2]^*_m) = f_2^* f_1^*([D_1]^*_m).$$

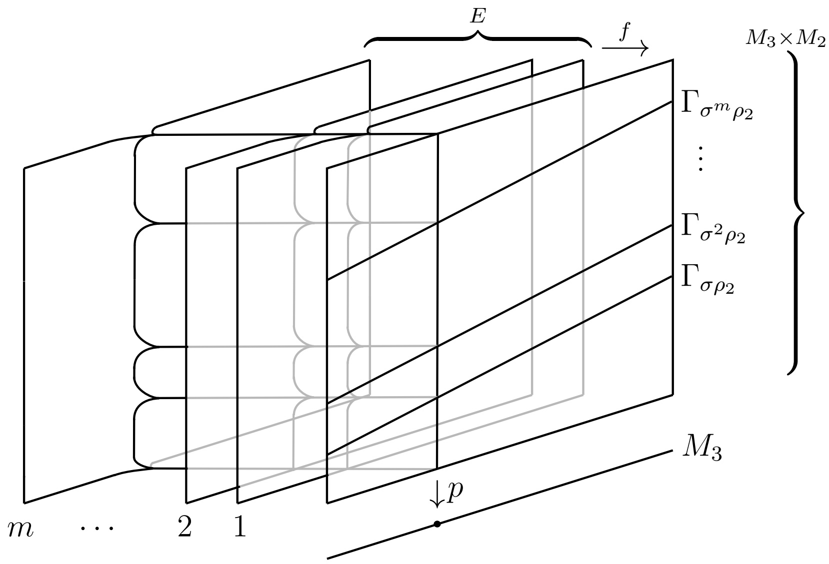

Figure 3.6

On the other hand, the group $H^2(M_1 \times M_1; \mathbb{Z})$ can be expressed as a direct sum

$$H^2(M_1 \times \{pt\}; \mathbb{Z}) \oplus (H^1(M_1; \mathbb{Z}) \otimes H^1(M_1; \mathbb{Z})) \oplus H^2(\{pt\} \times M_1; \mathbb{Z})$$

by the Künneth theorem. As an element of this group, we can write

$$[D_1]^* = [M_1]^* \times 1 \ + \ \sum_i \alpha_i \times \beta_i \ + \ 1 \times [M_1]^*$$

for some α_i,$\beta_i \in H^1(M_1; \mathbb{Z})$, and $[M_1]^*$ is the Poincaré dual of the identity element in $H_0(M_1; \mathbb{Z}) \cong \mathbb{Z}$. Applying f_1^* we obtain

$$\begin{aligned} f_1^*([D_1]_m^*) \ &= \rho_1^*[M_1]_m^* \times \rho_1^*(1) \quad + \textstyle\sum_i \rho_1^*\alpha_1 \times \rho_1^*\beta_i \quad + \rho_1^*(1) \times \rho_1^*[M_1]_m^* \\ &= \textstyle\sum_i \rho_1^*\alpha_1 \times \rho_1^*\beta_i \end{aligned}$$

since $\rho_1^* : H^2(M_1; \mathbb{Z}_m) \to H^2(M_2; \mathbb{Z}_m)$ is trivial. Now applying f_2^* to $f_1^*([D_1]_m^*)$ we get the following

$$\begin{aligned} f_2^* f_1^*([D_1]_m^*) \ &= (\rho_2 \times \rho_2)^*(\textstyle\sum_i \rho_1^*\alpha_1 \times \rho_1^*\beta_i) \\ &= \textstyle\sum_i \rho_2^*(\rho_1^*\alpha_i) \times \rho_2^*(\rho_1^*\alpha) \\ &= 0 \end{aligned}$$

since $\rho_2^* : H^1(M_2; \mathbb{Z}_m) \to H^1(M_3; \mathbb{Z}_m)$ is also trivial. Hence we obtain $[D]_m^* = f_2^* f_1^*([D_1]_m^*)$ $= 0$. Therefore, the mapping

$$\pi : E \longrightarrow M_3$$

in the diagram constructed above (cf. diagram (3.3)) is a surface bundle over M_3. Its fiber is an m-fold ramified covering of M_2.

This construction of surface bundles is obtained by generalizing Atiyah's argument in [2], which treats the case $m = 2$, for arbitrary m. Hirzebruch [14] also develops another method which yields surface bundles with smaller genera. However, our method above is more suitable for generalizations in higher dimensions.

Finally, let us prove that the characteristic class e_1 of this surface bundle is non-zero. For that, we first prove the following general proposition which is used in our argument.

Proposition 3.5 *Let $\pi : E \longrightarrow M$, $\widetilde{\pi} : \widetilde{E} \longrightarrow M$ be two surface bundles over the same base space M. Suppose that there is given a mapping $f : \widetilde{E} \longrightarrow E$, between the total spaces, which is an m-fold cyclic ramified covering ramified along an oriented submanifold $D \subset E$ of codimension 2. Suppose further that D intersects each fiber of π transversely at exactly m points and the following diagram is commutative where $\widetilde{D} = f^{-1}(D)$.*

$$\begin{tikzcd}
\widetilde{D} \arrow[d, phantom, "\cap"] & D \arrow[d, phantom, "\cap"] \\
\widetilde{E} \arrow[r, "f"] \arrow[d, "\widetilde{\pi}"'] & E \arrow[d, "\pi"'] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

Then we have the following two equalities:

1. $f^*(\nu) = m\,\widetilde{\nu}$

2. $\widetilde{e} = f^*(e) - (m-1)\widetilde{\nu} = f^*(e - (1 - \frac{1}{m})\nu)$

where $\nu \in H^2(E; \mathbb{Z})$, $\widetilde{\nu} \in H^2(\widetilde{E}; \mathbb{Z})$ denote the Poincaré duals of D and $\widetilde{D}$, respectively, and $e \in H^2(E; \mathbb{Z})$, $\widetilde{e} \in H^2(\widetilde{E}; \mathbb{Z})$ denote the Euler classes of the tangent bundles along the fiber of π and $\widetilde{\pi}$, respectively.

Proof. 1. Following the proof of Lemma 3.2, the Poincaré dual of $f^*(\nu)$ is $f^*(\nu) \cap [\widetilde{M}] = [\widetilde{D}]$. Applying f_* we obtain

$$\begin{aligned} f_*([\widetilde{D}]) &= f_*(f^*(\nu) \cap [\widetilde{M}]) \\ &= \nu \cap f_*([\widetilde{M}]) && \text{(by Proposition 3.1 (i))} \\ &= \nu \cap m[M] && (f \text{ is an } m\text{-fold}) \\ &= m[D]. && \text{(Poincaré Duality)} \end{aligned}$$

Therefore, dualizing we get $f^*(\nu) = m\,\widetilde{\nu}$.

2. Let $N(D)$, $N(\widetilde{D})$ be closed tubular neighborhoods of D,$\widetilde{D}$, respectively. Then in the commutative diagram

$$\begin{tikzcd} H^2(E) \arrow[r, "f^*"] \arrow[d] & H^2(\widetilde{E}) \arrow[d] \\ H^2(E \setminus \operatorname{Int} N(D)) \arrow[r] & H^2(\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D})) \end{tikzcd}$$

the images of two Euler classes $e \in H^2(E;\mathbb{Z})$, $\widetilde{e} \in H^2(\widetilde{E};\mathbb{Z})$ in $H^2(\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D}))$ clearly coincide. On the other hand, from the exact sequence

$$\cdots \longrightarrow H^2(\widetilde{E},\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D})) \longrightarrow H^2(\widetilde{E}) \longrightarrow H^2(\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D})) \longrightarrow \cdots$$

together with the isomorphism

$$\begin{aligned} H^2(\widetilde{E},\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D})) &\cong H^2(N(\widetilde{D}),\partial N(\widetilde{D})) && \text{(by excision)} \\ &\cong H_{n-2}(\widetilde{D}) && \text{(Thom isomorphism and Poincaré duality)} \end{aligned}$$

we conclude there exist an integer $a \in \mathbb{Z}$ such that

$$\widetilde{e} = f^*(e) + a\widetilde{\nu}.$$

In fact, the commutative diagram above implies that $\widetilde{e} - f^*(e)$ is in the kernel of $H^2(\widetilde{E}) \to H^2(\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D}))$, because of exactness, by the above isomorphism, and the fact that $H^2(\widetilde{E}) \cong H_{n-2}(\widetilde{E})$, we conclude that the homomorphism $H^2(\widetilde{E},\widetilde{E} \setminus \operatorname{Int} N(\widetilde{D})) \longrightarrow H^2(\widetilde{E})$ is the multiplication homomorphism by some integer.

Now if we denote by g, g' the genera of π,$\widetilde{\pi}$, respectively, since D intersects each fiber of π transversely at m points, then the covering along the fiber has m ramified points, and so

$$2 - 2g' = m(2 - 2g - m) + m$$

On the other hand, if we restrict both sides of $\widetilde{e} = f^*(e) + a\widetilde{\nu}$ to the fiber of $\widetilde{\pi}$, and using the part 1. of this proposition, then we obtain

$$2 - 2g' = m(2 - 2g) + am.$$

From the last two equalities we can now conclude

$$a = 1 - m.$$

∎

Let us go back to the commutative diagram (3.3) and compute the first characteristic class e_1 of the surface bundle $\pi : E \longrightarrow M_3$. If we denote by $\widetilde{e}$ the Euler class of π, and since the Euler class e of the trivial bundle $M_3 \times M_2 \to M_3$ is equal to $(2 - 2g_2)[M_2]^*$, then by the above proposition we have

$$\widetilde{e} = f^*((2 - 2g_2)[M_2]^* - (1 - \frac{1}{m})[D]^*).$$

Hence

$$\begin{aligned}
\langle e_1, [M_3]\rangle &= \langle \widetilde{e}^2, [E]\rangle \\
&\quad \text{(cf. the proof of Preposition 3.4)} \\
&= \langle \big(f^*((2 - 2g_2)[M_2]^* - (1 - \tfrac{1}{m})[D]^*)\big)^2, [E]\rangle \\
&\quad \text{(by the above equality)} \\
&= \langle \big((2 - 2g_2)[M_2]^* - (1 - \tfrac{1}{m})[D]^*\big)^2, f_*[E]\rangle \\
&\quad \text{(a property of Kronecker pairing)} \\
&= m\langle \big((2 - 2g_2)[M_2]^* - (1 - \tfrac{1}{m})[D]^*\big)^2, [M_3 \times M_2]\rangle
\end{aligned}$$

(f is a m-fold)

$= m\langle\big((2-2g_2)f_2^*[M_2]^* - (1-\frac{1}{m})f_2^*[D_2]^*\big)^2, [M_3 \times M_2]\rangle$

(since $f_2^*[M_2]^* = [M_2]^*$ and $f_2^*[D_2]^* = [D]^*$)

$= m\langle\big((2-2g_2)[M_2]^* - (1-\frac{1}{m})[D_2]^*\big)^2, f_{2*}[M_3 \times M_2]\rangle$

(a property of Kronecker pairing)

$= m^{2g_2+1}\langle\big((2-2g_2)[M_2]^* - (1-\frac{1}{m})[D_2]^*\big)^2, [M_2 \times M_2]\rangle$

(since f_2 is an m^{2g_2}-fold)

$$\begin{aligned}
= m^{2g_2+1}\langle &-2(1-\tfrac{1}{m})(2-2g_2)[M_2]^* \cup [D_2]^* \\
&+(2-2g_2)^2[M_2]^* \cup [M_2]^* \\
&+(1-\tfrac{1}{m})^2[D_2]^* \cup [D_2]^*, [M_2 \times M_2]\rangle
\end{aligned}$$

$$\begin{aligned}
= m^{2g_2+1}[&\langle -2(1-\tfrac{1}{m})(2-2g_2)[M_2]^* \cup [D_2]^*, [M_2 \times M_2]\rangle \\
&+\langle (2-2g_2)^2[M_2]^* \cup [M_2]^*, [M_2 \times M_2]\rangle \\
&+\langle (1-\tfrac{1}{m})^2 f_1^*([D_1]^* \cup [D_1]^*), [M_2 \times M_2]\rangle]
\end{aligned}$$

(since $f_1^*[D_1]^* = [D_2]^*$)

$$\begin{aligned}
= m^{2g_2+1}[&\langle -2(1-\tfrac{1}{m})(2-2g_2)[M_2]^* \cup [D_2]^*, [M_2 \times M_2]\rangle \\
&+m^2\langle (1-\tfrac{1}{m})^2[D_1]^* \cup [D_1]^*, [M_1 \times M_1]\rangle]
\end{aligned}$$

(since $f_1 = \rho_1 \times \rho_1$)

$$\begin{aligned}
= m^{2g_2+1}[&-2(1-\tfrac{1}{m})(2-2g_2)[M_2]^* m \\
&+m^2(1-\tfrac{1}{m})^2(2-2g_1)]
\end{aligned}$$

(since $\langle [D_1]^* \cup [D_1]^*, [M_1 \times M_1]\rangle = 2-2g_1$)

$= m^{2g_2+1}\big(-2(1-\frac{1}{m})(2-2g_2)m + (1-\frac{1}{m})^2 m(2-2g_2)\big)$

(since $2-2g_2 = m(2-2g_1)$)

$= (2g_2-2)m^{2g_2}(m^2-1).$

Since this number is clearly positive we have finished the proof of non-triviality of e_1. ∎

We summarize this construction simply as follows (cf. diagram (3.3)): we set $M_1 = \Sigma_g$ and consider the trivial Σ_g-bundle $M_1 \times M_1 \longrightarrow M_1$. Then the image of the section $M_1 \ni p \mapsto (p,p) \in M_1 \times M_1$, namely the diagonal set D_1, is a submanifold of the total space of codimension 2 and intersects each fiber transversely. However, for any integer $m > 1$, its homology class $[D_1]$ is no divisible by m in $H_2(M_1 \times M_1; \mathbb{Z})$. We then take a suitable finite covering of the above trivial bundle along the fiber as well as the base so that the homology class of the inverse image of D_1 will be divisible by a given number m. If we consider the associated ramified covering, applying Proposition 3.3, then the resultant surface bundle satisfies the required conditions.

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