**Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional**

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# El principio del máximo para sistemas de control a tiempo discreto en horizonte infinito

Tesis que presenta
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# The maximum principle for discrete-time control systems in infinite horizon

Thesis presented by
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**Master in science**
in the specialty of
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Thesis advisor: **Dr. Onésimo Hernández-Lerma**

# Chapter 1

# Introduction

In this work, we study first order conditions for deterministic discrete-time optimal control problems in infinite horizon. Our main purpose is to establish a discrete maximum principle (MP), in analogy with the continuous case, and a transversality condition (TC) as a necessary and sufficient condition for optimality. We also present an extension of the maximum principle to dynamic games.

We propose a different one from those and show how it can be useful to solve certain problems. The continuous-time maximum principle was established at the end of the 1950s for the finite dimensional case [19]. See [8] for the history of this discovery. Right after this discovery the corresponding theory was developed for discrete-time dynamics in finite horizon, under appropriate convexity assumptions. See, for example, [13] and [14]. An account of the state of the art in the infinite horizon framework, can be found in the recent book by Blot and Hayek [5]; however there is not a single example in that book. In Section 3.2.3 of [5], Blot and Hayek prove a maximum principle for bounded processes without considering control constraints; they prove that there exist a sequence of vectors, usually called *adjoint variables* that satisfy certain conditions; their proof relies in the classical methods for optimization in Banach spaces, so there is no construction of the adjoint variables. We give an explicit form of the adjoint variables. In [17], Michel introduced several types of transversality conditions (some of them known from earlier publications). We propose a new transversality condition from those in [17] and show how it can be useful to solve certain problems. In [3], Aseev et al. developed a necessary condition in the form of a maximum principle for weakly overtaking solutions based on ideas of the continuous-time framework.

*Our approach.* We use Gâteaux differentials to obtain the maximum principle. We calculate a Gâteaux derivative, under a technical assumption, and use it to obtain necessary conditions for optimality. Right after that we consider convexity-concavity assumptions to establish sufficient conditions.

We consider the extension our results to case where the sets are constrained by the state (in addition of time); to that end we consider Markov strategies. Using the maximum principle and the transversality condition, we find the Euler equation and a new transversality condition in the Euler equation approach.. Finally, we consider the maximum principle for dynamic games.

*Organization of this work.* In Chapter two, we introduce the optimal control model that we are interested and study the first order conditions for optimality. Chapter 3 contains variants of the theory developed in Chapter 2. Finally, Chapter 4 considers the related topics to the subject.

# Contents

# Notation

- If $M$ is a matrix, the $M^*$ denotes the transpose of $M$.
- Given a function $g = (g^1, \dots, g^k) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, we denote

$$\frac{\partial g}{\partial x}(x,y) = \begin{bmatrix} \frac{\partial g^1}{\partial x_1}(x,y) & \dots & \frac{\partial g^1}{\partial x_n}(x,y) \\ \vdots & \ddots & \\ \frac{\partial g^k}{\partial x_1}(x,y) & \dots & \frac{\partial g^k}{\partial x_n}(x,y) \end{bmatrix},$$

and

$$\frac{\partial g}{\partial y}(x,y) = \begin{bmatrix} \frac{\partial g^1}{\partial x_{n+1}}(x,y) & \dots & \frac{\partial g^1}{\partial x_{n+m}}(x,y) \\ \vdots & \ddots & \\ \frac{\partial g^k}{\partial x_{n+1}}(x,y) & \dots & \frac{\partial g^k}{\partial x_{n+m}}(x,y) \end{bmatrix},$$

where, for each $i = 0, \dots, n+m$, $\frac{\partial g^j}{\partial x_i}(x,y)$ is the partial derivative of $g^j$ at $(x,y)$.

- As usual, $\mathbb{N}$ denotes the set $\{1, 2, \dots\}$ and $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$.
- Let $A_1, A_2 \dots$ be a sequence of square matrices, we write

$$\prod_{s=\tau}^{t} A_s = \begin{cases} A_\tau A_{\tau+1} \cdots A_t & if \quad \tau \le t \\ \mathbb{I} & if \quad \tau > t, \end{cases}$$

where $\mathbb{I}$ is the identity matrix.

# Chapter 2

# The Maximum Principle

In this chapter, we are concerned with deterministic nonstationary discrete-time optimal control problems in infinite horizon. We show, using Gâteaux differentials, that the so-called Maximum Principle (MP) and a transversality condition (TC) are necessary conditions for optimality. Under additional assumptions, the MP and the TC are also sufficient for optimality.

## 2.1 The Optimal Control Model

In this section, we present the model concerning discrete-time nonstationary (or time-varying) deterministic dynamic optimization problems in infinite horizon. Dynamic optimization problems are also known as optimal control problems. As usual, $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$ and $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$.

Let $X \subset \mathbb{R}^n$ be the state space, and $U \subset \mathbb{R}^m$ be the control set. Consider a sequence $\{X_t \mid t \in \mathbb{N}_0\}$ of non-empty subsets of the state space, and $\{U_t \subset U \mid t \in \mathbb{N}_0\}$ the family of feasible control sets. For each $t \in \mathbb{N}_0$, $x \in X_t$, and $u \in U_t$, we denote by $f_t(x, u)$ the corresponding state in $X_{t+1}$. Thus, given an initial state $x_0$, the state of the system evolves according to the equation

$$x_{t+1} = f_t(x_t, u_t), \tag{2.1}$$

where, for each $t \in \mathbb{N}_0$, $f_t : X_t \times U_t \to X_{t+1}$. We want to optimize the performance index

$$\sum_{t=0}^{\infty} g_t(x_t, u_t), \tag{2.2}$$

where $g_t : X_t \times U_t \to \mathbb{R}$ is a given function for each $t \in \mathbb{N}_0$.

A sequence $\psi = \{u_t\}$ is called an open-loop strategy, or simply a plan, whenever $u_t$ is in $U_t$ for all $t \in \mathbb{N}_0$; we denote the set of plans from $x_0$ as $\Psi(x_0)$. Given a plan $\psi = \{u_t\}$, we denote by $\left\{x_t^\psi\right\}$ the sequence induced by $\psi$ in (2.1), i.e.,

$$\begin{aligned} x_0^\psi &= x_0 \\ x_{t+1}^\psi &= f_t(x_t^\psi, u_t) \quad \text{for } t = 0, 1, \ldots. \end{aligned}$$

The Optimal Control Problem (OCP) is to find a plan $\psi$, also called control policy, that maximizes the performance index (2.2) subject to (2.1).

In a compact form, a nonstationary OCP can be described by the three-tuple

$$(\Psi(x_0), \{f_t\}, \{g_t\}), \tag{2.3}$$

with components as above.

For the OCP to be well-defined, the following assumption is supposed to hold throughout the remainder of this chapter.

**Assumption 2.1.** *The three-tuple in (2.3) satisfies the following for each $x_0 \in X_0$:*

*(a) the set $\Psi(x_0)$ is nonempty;*

*(b) for each $\psi = (u_0, u_1, \ldots) \in \Psi(x_0)$,*

$$\sum_{t=0}^{\infty} g_t(x_t^\psi, u_t) < \infty;$$

*(c) there exists $\psi = (u_0, u_1, \ldots) \in \Psi(x_0)$ such that*

$$\sum_{t=0}^{\infty} g_t(x_t^\psi, u_t) > -\infty;$$

*(d) for each $t \in \mathbb{N}_0$, $f_t$ and $g_t$ are differentiable in the interior of $X_t \times U_t$.*

For $x_0 \in X_0$, define the OCP performance index (also known as objective function) $v : \Psi(x_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\psi) = \sum_{t=0}^{\infty} g_t(x_t^\psi, u_t). \tag{2.4}$$

Assumption 2.1(a)-(b) ensures that $v$ is well defined. For the three-tuple (2.1) and $x_0 \in X_0$, the OCP is to find $\hat{\psi} \in \Psi(x_0)$ such that

$$v(\hat{\psi}) \geq v(\psi)$$

for all $\psi \in \Psi(x_0)$. In such a case, we say that $\hat{\psi}$ is an optimal plan. The optimization problem makes sense by Assumption 2.1(c).

## 2.2 Necessary Conditions

In this section, we introduce the Maximum Principle (MP) (2.5)-(2.6) and the Transversality Condition (TC) (2.7) as necessary conditions for the existence of an optimal plan. We suppose that the initial state $x_0 \in X_0$ is fixed. Recall that Assumption 2.1 holds. We will require the concept of Gâteaux differential.

**Definition 2.2.** *[11, pp.2-4] Let $\mathcal{X}$ be a linear space and $\mathcal{V}$ a subset of $\mathcal{X}$. Let $p \in \mathcal{V}$ and $q \in \mathcal{X}$.*

*(a) We say that $p$ is an internal point in the direction $q$ if there exist a real number $\varepsilon_0 > 0$ such that $p + \varepsilon q$ is in $\mathcal{V}$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.*

*(b) Suppose $p$ is an internal point in the direction $q$. Let $h : \mathcal{V} \to \mathbb{R}$ be a function. If the derivative*

$$\delta_h(p;q) := \frac{dh}{d\varepsilon}(p + \varepsilon q)\bigg|_{\varepsilon=0}$$

*exists, we say that $\delta_h(p;q)$ is the Gâteaux differential of $h$ at $p$ in the direction $q$.*

The next proposition shows an application of Gâteaux differentials.

**Proposition 2.3.** *Let $\mathcal{X}$ be a linear space and $\mathcal{V}$ a subset of $\mathcal{X}$. Let $p \in \mathcal{V}$ be an internal point in the direction $q \in \mathcal{X}$ and $h : \mathcal{V} \to \mathbb{R}$ a given function. If the Gâteaux differential of $h$ at $p$ in the direction $q$ exists and $h$ has a maximum at $p$, then*

$$\delta_h(p;q) = 0.$$

*Proof.* See Theorem 1 in page 178 of [16]. □

The plan to prove the MP (2.5)-(2.6) and the TC (2.7) below is straightforward: we will calculate the Gâteaux differential, in a certain direction, of the performance index (2.4) at the optimal plan.

We will need the following assumption to estimate the Gâteaux differential of the function $v$ in (2.4). As usual, $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ denote the gradients corresponding to the first and the second variables, respectively.

**Assumption 2.4.** *Let $\hat{\psi} = (\hat{u}_0, \hat{u}_1, \dots) \in \Psi(x_0)$. For each $\tau \in \mathbb{N}_0$, define the the sequence of functions $\rho_t^\tau : U_\tau \to \mathbb{R}^n$ as*

$$\rho_t^\tau(u) = \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}^\tau(u)}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}^\tau(u)}, \hat{u}_s),$$

*where $\hat{\psi}^\tau(u) = (\hat{u}_0, \dots, \hat{u}_{\tau-1}, u, \hat{u}_{\tau+1}, \dots)$. We suppose that, for each $\tau \in \mathbb{N}_0$, there exists an open neighborhood $O_\tau \subset U_\tau$ of $\hat{u}_\tau$ such that the series $\displaystyle\sum_{t=\tau+1}^{\infty} \rho_t$ converges uniformly on $O_\tau$.*

If $A_1, A_2 \dots$ is a sequence of square matrices, we write

$$\prod_{s=\tau}^{t} A_s = \begin{cases} A_\tau A_{\tau+1} \cdots A_t & if \quad \tau \le t \\ \mathbb{I} & if \quad \tau > t, \end{cases}$$

where $\mathbb{I}$ is the identity matrix.

Computing the Gâteaux differential of $v$ may be too technical; calculations are in the following lemma. The proof is in the Appendix A. We consider the vector space $\Lambda$ of all sequences in $\mathbb{R}^m$ with the standard addition and scalar multiplication. We consider row vectors $y \in \mathbb{R}^m$, so the transpose $y^*$ is a column vector.

**Lemma 2.5.** *Let $\hat{\psi} = \{\hat{u}_t\} \in \Psi(x_0)$ be a plan for which Assumption 2.4 holds. Let $y \in \mathbb{R}^m$ and $\tau \in \mathbb{N}_0$. Then $\hat{\psi}$ is an internal point in the direction $\psi^{\tau,y} \in \Lambda$, where $\psi^{\tau,y}$ is defined as*

$$\psi_t^{\tau,y} := \begin{cases} y & if \quad t = \tau \\ 0 & if \quad t \neq \tau, \end{cases}$$

*for all $t \in \mathbb{N}_0$. Moreover, the Gâteaux differential of $v$ at $\hat{\psi}$ in the direction $\psi^{\tau,y}$ exists and is given by*

$$\delta_v(\hat{\psi}; \psi^{\tau,y}) = \left( \sum_{t=\tau+1}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right) \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* + \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^*,$$

*where $v$ is the function in (2.4).*

We can now state one of our main results.

**Theorem 2.6.** *Let $\hat{\psi} = \{\hat{u}_t\} \in \Psi(x_0)$ be a plan for which Assumption 2.4 holds. If $\hat{\psi}$ is an optimal plan of the OCP (2.1)-(2.3), then there exists a sequence $\{\lambda_t\}_{t=1}^{\infty}$ in $\mathbb{R}^n$ such that*

*(a) For all $t \in \mathbb{N}$,*

$$\frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) + \lambda_{t+1} \frac{\partial f_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) = \lambda_t, \tag{2.5}$$

*(b) For all $t \in \mathbb{N}_0$,*

$$\frac{\partial g_t}{\partial y}(x_t^{\hat{\psi}}, \hat{u}_t) + \lambda_{t+1} \frac{\partial f_t}{\partial y}(x_t^{\hat{\psi}}, \hat{u}_t) = 0, \tag{2.6}$$

*(c) For every $h \in \mathbb{N}$, we have the transversality condition (TC)*

$$\lim_{t \to \infty} \lambda_t \prod_{s=h}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) = 0. \tag{2.7}$$

*Moreover, each* $\lambda_t$ *is given by*

$$\lambda_t = \sum_{k=t}^{\infty} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=t}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s). \tag{2.8}$$

*Proof.* (a) Pick an arbitrary $\tau \in \mathbb{N}$. Then

$$\begin{aligned}
\lambda_\tau &:= \sum_{t=\tau}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \\
&= \frac{\partial g_\tau}{\partial x}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) + \sum_{t=\tau+1}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \\
&= \frac{\partial g_\tau}{\partial x}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) + \left( \sum_{t=\tau+1}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right) \frac{\partial f_\tau}{\partial x}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) \\
&= \frac{\partial g_\tau}{\partial x}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) + \lambda_{\tau+1} \frac{\partial f_\tau}{\partial x}(x_\tau^{\hat{\psi}}, \hat{u}_\tau).
\end{aligned}$$

(b) Fix $\tau \in \mathbb{N}_0$ and $y \in \mathbb{R}^m$ arbitrary. By Lemma 2.5 and Proposition 2.3,

$$\left( \sum_{t=\tau+1}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right) \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* + \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* = 0,$$

that is

$$\left[ \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* + \lambda_{\tau+1} \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* \right] = 0.$$

Since this holds for any $y \in \mathbb{R}^m$, $(b)$ follows.

(c) By Assumption 2.4, the series

$$\sum_{k=h}^{\infty} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=h}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s),$$

converges for any $h \in \mathbb{N}$ and

$$\begin{aligned}
\lambda_t \prod_{s=h}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) &= \left( \sum_{k=t}^{\infty} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=t}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right) \prod_{s=h}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \\
&= \sum_{k=t}^{\infty} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=h}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s).
\end{aligned}$$

Letting $t$ tend to infinity, $(c)$ follows.

□

Actually, the MP (2.5)-(2.6) is so-named in analogy with the well-known maximum principle in the continuous case (see [19]). See [8] for the history of the MP.

It turns out that the sequence $\{\lambda_t\}$ given in Theorem 2.6 must be unique.

**Proposition 2.7.** *Suppose that a plan $\hat{\psi} \in \Psi(x_0)$ satisfies (2.5) and the TC (2.7). Then*

$$\lambda_t = \sum_{k=t}^{\infty} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=t}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s). \tag{2.9}$$

*Proof.* Let $\{\lambda'_t\}_{t=1}^{\infty}$ be a sequence satisfying (2.5) and (2.7). It can be proved by induction that

$$\lambda'_t = \sum_{s=t}^{h} \frac{\partial g_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \prod_{i=t}^{s-1} \frac{\partial f_i}{\partial x}(x_i^{\hat{\psi}}, \hat{u}_i) + \lambda'_{h+1} \prod_{i=t}^{h} \frac{\partial f_i}{\partial x}(x_i^{\hat{\psi}}, \hat{u}_i),$$

for $h \geq t$. Now, letting h tend to infinite

$$\lambda'_t = \sum_{s=t}^{\infty} \frac{\partial g_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \prod_{i=t}^{s-1} \frac{\partial f_i}{\partial x}(x_i^{\hat{\psi}}, \hat{u}_i).$$

Comparing with (2.9), the result follows. □

**Corollary 2.8.** *The sequence $\{\lambda_t\}_{t=1}^{\infty}$ given in Theorem 2.6 is unique.*

## 2.3 Sufficient Conditions

We have seen that the MP (2.5)-(2.6) and the TC (2.7) are necessary conditions for optimality. Under suitable assumptions, they are also sufficient; see Theorem 2.11 and Assumption 2.10 below.

As in the previous section, we require a proposition concerning Gâteaux differentials.

**Proposition 2.9.** *Let $\mathcal{X}$ be a linear space. Suppose $\mathcal{V}$ is a convex subset of $\mathcal{X}$ and $h : \mathcal{V} \to \mathbb{R}$ a concave function. If $p \in \mathcal{V}$ satisfies $\delta_h(p; q-p) = 0$ for all $q \in \mathcal{V}$, then $p$ maximizes $h$.*

*Proof.* Since $\mathcal{V}$ is convex, $p$ is an internal point in the direction $q - p$ for any $q \in \mathcal{V}$. By concavity of $h$,

$$h(p + \varepsilon(q-p)) \geq h(p) + \varepsilon(h(q) - h(p)),$$

for all $0 \leq \varepsilon \leq 1$. That is

$$\frac{h(p + \varepsilon(q-p)) - h(p)}{\varepsilon} \geq h(q) - h(p).$$

Letting $\varepsilon \downarrow 0$ yields the result. □

The following convexity-concavity assumption ensures the sufficiency of the MP (2.5)-(2.6) and the TC (2.7).

**Assumption 2.10.** *The optimal control model (2.3) satisfies the following:*

*(a) the set of plans* $\Psi(x_0)$ *is convex;*

*(b) the performance index* $v$ *in (2.4) is concave;*

*(c) there exists a sequence of non-positive numbers* $m_t$ *with* $\sum_{t=0}^{\infty} m_t > -\infty$ *such that* $g_t(x_t^{\psi}, u_t) \geq m_t$ *for all* $\psi = (u_0, u_1, \dots) \in \Psi(x_0)$.

**Theorem 2.11.** *Let* $\hat{\psi} \in \Psi(x_0)$ *be a plan for which Assumption 2.4 holds. Suppose that* $\hat{\psi}$ *satisfies the MP (2.5)-(2.6) and the TC (2.7). If Assumption 2.10 holds, then* $\hat{\psi}$ *is an optimal plan for the OCP (2.1)-(2.3).*

*Proof.* For each $k \in \mathbb{N}_0$, consider the function $v_{\hat{\psi}}^k : \Psi(x_0) \to \mathbb{R}$ given by $v_{\hat{\psi}}^k(u_0, u_1, \dots) = v(u_0, \dots, u_k, \hat{u}_{k+1}, \hat{u}_{k+2}, \dots)$, where $v$ is the performance index (2.4). Proceeding as in the proof of Lemma 2.5, we find

$$\delta_{v_{\hat{\psi}}^k}(\hat{\psi}, \psi - \hat{\psi}) =$$
$$\sum_{\tau=0}^{k} \left( \sum_{t=\tau+1}^{\infty} \left[ \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right] \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) + \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) \right) (u_\tau - \hat{u}_\tau)^*.$$

By Proposition 2.7 and (2.6), we have $\delta_{v_{\hat{\psi}}^k}(\hat{\psi}, \psi) = 0$, which by Proposition 2.9 yields that $\hat{\psi}$ is a maximum of $v_{\hat{\psi}}^k$. Let $\psi \in \Psi(x_0)$ be any plan. By Assumption 2.10(c)

$$\begin{aligned} v(\hat{\psi}) &= v_{\hat{\psi}}^k(\hat{\psi}) \\ &\geq v_{\hat{\psi}}^k(\psi) \\ &\geq \sum_{t=0}^{k} g_t(x_t^{\psi}, u_t) + \sum_{t=k+1}^{\infty} m_t. \end{aligned}$$

Letting $k \to \infty$, we obtain $v(\hat{\psi}) \geq v(\psi)$. □

## 2.4 Examples

**Example 2.12** (A consumption-investment problem)**.** *Let* $\beta, \gamma \in (0,1)$ *and* $r > 0$ *such that* $\frac{(r\beta)^{\frac{1}{\gamma}}}{r} < 1$. *Assume that* $x_t$ *is the wealth of certain investor at time* $t \in \mathbb{N}_0$. *At each time* $t = 0, 1, \dots,$ *the investor consumes a fraction* $u_t \in (0,1)$ *of the assets. Suppose that the investor wishes to maximize*

$$\sum_{t=0}^{\infty} \beta^t (x_t u_t)^{1-\gamma},$$

*subject to the dynamics of the assets*

$$x_{t+1} = r(1-u_t)x_t,$$

*where $x_0 > 0$ is given.*

*In the present context, our control model in Section 2.1 has the following components*

- *state space $X_t \equiv X := (0,\infty)$;*
- *control space $U_t \equiv U := (0,1)$;*
- *system functions $f_t : X \times U \to X$ with $f_t(x,u) := r(1-u)x$;*
- *return functions $g_t : X \times U \to \mathbb{R}$ with $g_t(x,u) := \beta^t (xu)^{1-\gamma}$.*

*To use Theorem 2.6, we proceed as follows. From (2.5)-(2.6):*

$$\lambda_t = \beta^t (1-\gamma)(x_t^{\hat{\psi}} \hat{u}_t)^{-\gamma} \hat{u}_t + \lambda_{t+1} r (1-\hat{u}_t) \quad \forall_{t\in\mathbb{N}}, \tag{2.10}$$

$$0 = \beta^t (1-\gamma)(x_t^{\hat{\psi}} \hat{u}_t)^{-\gamma} - \lambda_{t+1} r \quad \forall_{t\in\mathbb{N}_0}. \tag{2.11}$$

*Combining these equations, we obtain $\lambda_t = \beta^t (1-\gamma)(x_t^{\hat{\psi}} \hat{u}_t)^{-\gamma}$ and*

$$1/r = \frac{\lambda_{t+1}}{\lambda_t} = \beta \left( \frac{x_{t+1}^{\hat{\psi}} \hat{u}_{t+1}}{x_t^{\hat{\psi}} \hat{u}_t} \right)^{-\gamma}.$$

*Using the fact that $x_{t+1}^{\hat{\psi}} = r(1-\hat{u}_t) x_t^{\hat{\psi}}$, we obtain*

$$(r\beta)^{\frac{1}{\gamma}} = \frac{x_{t+1}^{\hat{\psi}} \hat{u}_{t+1}}{x_t^{\hat{\psi}} \hat{u}_t} = \frac{(1-\hat{u}_t)}{\hat{u}_t} r \hat{u}_{t+1}.$$

*Solving this difference equation, we find*

$$\hat{u}_t = \frac{1-a}{1-c(1/a)^t},$$

*where $c$ is some constant and $a = \frac{(r\beta)^{\frac{1}{\gamma}}}{r}$. $\{\hat{u}_t\}$ is a bounded sequence, since $\hat{u}_t \in (0,1)$ for all $t \in \mathbb{N}_0$. Hence, it cannot diverge to infinite. Thus, $c$ must be* *zero since* $a < 1$, *otherwise* $\hat{u}_t = \frac{1-a}{1-c(1/a)^t} \to \infty$. *Therefore* $\hat{u}_t = 1 - a$ *for all* $t \in \mathbb{N}_0$.

*We prove that Assumption 2.4 holds. Let* $\tau \in \mathbb{N}_0$ *and consider* $\rho_t^\tau : [0,1] \to \mathbb{R}$ *as in the assumption. Observe that*

$$\begin{aligned} x_t^\psi &= r(1-u_{t-1})x_{t-1}^\psi \\ &= r(1-u_{t-1})r(1-u_{t-2})x_{t-2}^\psi \\ &\vdots \\ &= \prod_{s=\tau+1}^{t-1} r(1-u_s)x_{\tau+1}^\psi \\ &= \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^\psi, u_s)x_{\tau+1}^\psi, \end{aligned}$$

*for any plan* $\psi$. *Since* $u_t \in (0,1)$ *for all* $t \in \mathbb{N}_0$, *we have* $|x_t^\psi| < r^t|x_0|$ *for any plan* $\psi$.

*Now, take* $O_\tau = (\eta', \eta)$ *as a small neighborhood of* $1-a$ *properly contained in* $(0,1)$, *we have*

$$\begin{aligned} |\rho_t^\tau(u)| &= \left| \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}^\tau(u)}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}^\tau(u)}, \hat{u}_s) \right| \\ &= \left| \beta^t r(1-\gamma)(x_t^{\hat{\psi}(u)}\hat{u}_t)^{-\gamma}\hat{u}_t \frac{x_t^{\hat{\psi}(u)}}{x_{\tau+1}^{\hat{\psi}(u)}} \right| \\ &= \left| \beta^t(1-\gamma)(x_t^{\hat{\psi}(u)}\hat{u}_t)^{1-\gamma} \frac{1}{(1-u)x_\tau^{\hat{\psi}}} \right| \\ &< \frac{|x_0|^{1-\gamma}}{(1-\eta)x_\tau^{\hat{\psi}}}(\beta r^{1-\gamma})^t. \end{aligned}$$

*Since* $\dfrac{(r\beta)^{\frac{1}{\gamma}}}{r} < 1$ *implies* $\beta r^{1-\gamma} < 1$, *we have by the Weierstrass M-test that* $\sum_{t=\tau+1}^{\infty} \rho_t^\tau$ *converges uniformly on* $O_\tau$. △

**Example 2.13** (A linear regulator problem)**.** *An OCP with linear system equation and a quadratic cost function is known as a LQ problem (also called a linear regulator problem). LQ problems have been widely studied. See, for instance, Chapter 5 of [15]. We consider a particular deterministic scalar case. The state of the system evolves according to*

$$x_{t+1} = x_t + u_t, \tag{2.12}$$

*for* $t \in \mathbb{N}_0$*. The performance index is*

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}x_t^2 + \frac{1}{2}u_t^2\right], \tag{2.13}$$

*where* $0 < \beta < 1$*. Given* $x_0 \in \mathbb{R}$*, we want to minimize (2.13) subject to (2.12).*

*In the present context, our control model in Section 2.1 has the following components*

- *state space* $X_t \equiv X := \mathbb{R}$*;*
- *control space* $U_t \equiv U := \mathbb{R}$*;*
- *system functions* $f_t : X \times U \to X$ *with* $f_t(x, u) := x + u$*;*
- *cost functions* $g_t : X \times U \to \mathbb{R}$ *with* $g_t(x, u) := \beta^t \left[x^2 + u^2\right]$*.*

*Considering (2.5)-(2.6) of Theorem 2.6:*

$$\lambda_t = \beta^t x_t^{\hat{\psi}} + \lambda_{t+1} \quad \forall_{t \in \mathbb{N}}, \tag{2.14}$$

$$0 = \beta^t \hat{u}_t + \lambda_{t+1} \quad \forall_{t \in \mathbb{N}_0}. \tag{2.15}$$

*From these equations, we obtain*

$$\begin{aligned}
\beta^t x_t^{\hat{\psi}} &= \lambda_t - \lambda_{t+1} \\
&= -\beta^{t-1}\hat{u}_{t-1} + \beta^t \hat{u}_t \\
&= -\beta^{t-1}(x_t^{\hat{\psi}} - x_{t-1}^{\hat{\psi}}) + \beta^t (x_{t+1}^{\hat{\psi}} - x_t^{\hat{\psi}}),
\end{aligned}$$

*which is equivalent to the difference equation*

$$\beta x_{t+1}^{\hat{\psi}} - (1 + 2\beta)x_t^{\hat{\psi}} + x_{t-1}^{\hat{\psi}} = 0. \tag{2.16}$$

*The solution of (2.16) is* $x_t^{\hat{\psi}} = k_1 r_1^t + k_2 r_2^t$ *for some constants* $k_1$ *and* $k_2$*; where* $r_1$ *and* $r_2$ *are the roots of equation* $\beta x^2 - (1 + 2\beta)x + 1 = 0$*. The transversality condition* (2.7) *reduces to*

$$\lambda_t \to 0.$$

*From this fact and* (2.14)*, we conclude that* $\beta^t x_t^{\hat{\psi}} \to 0$*. Now,*

$$\beta^t x_t^{\hat{\psi}} = k_1(\beta r_1)^t + k_1(\beta r_2)^t.$$

*Since* $\beta r_1 > 1$ *and* $\beta r_2 < 1$*, we conclude that* $k_1 = 0$*, and by the initial condition,* $k_2 = x_0$*. So*

$$\hat{u}_t = x_0 r_2^{t+1} - x_0 r_2^t = (r_2 - 1)r_2^t x_0.$$

*To prove that Assumption 2.4 holds, let* $\tau \in \mathbb{N}_0$ *and consider* $\rho_t^\tau : \mathbb{R} \to \mathbb{R}$ *as in the assumption. Take* $O_\tau = (-x_0, x_0)$ *, then we have*

$$\begin{aligned}
|\rho_t^\tau(u)| &= \left| \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}^\tau(u)}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}^\tau(u)}, \hat{u}_s) \right| \\
&= \left| \beta^t x_t^{\hat{\psi}^\tau(u)} \right| \\
&= \beta^t \left| x_\tau^{\hat{\psi}} + u + \sum_{s=\tau+1}^{t-1} (r_2 - 1) x_0 r_2^t \right| \\
&< |x_0| \, |t - \tau + 1| \beta^t .
\end{aligned}$$

*Thus, by the Weierstrass M-test,* $\sum_{t=\tau+1}^{\infty} \rho_t^\tau$ *converges uniformly on* $O_\tau$*.* △

# Chapter 3

# Variants of The Maximum Principle

## 3.1 Finite Horizon

In this section we consider again the non-stationary OCP (2.1)-(2.3), except that the performance index (2.2) is now replaced by the finite-horizon function

$$\sum_{t=0}^{T-1} g_t(x_t, u_t) + g_T(x_T). \tag{3.1}$$

In particular, the dynamic control model is as in (2.1), that is

$$x_{t+1} = f_t(x_t, u_t), \tag{3.2}$$

for $t \in \{0, \ldots, T-1\}$, with a given initial condition $x_0$.

As before, given a plan $\psi = (u_0, \ldots, u_{T-1})$, we denote by $\left\{x_t^\psi\right\}$ the sequence induced by $\psi$ in (3.2), i.e.,

$$x_0^\psi = x_0 \tag{3.3}$$

$$x_{t+1}^\psi = f_t(x_t^\psi, u_t). \tag{3.4}$$

In this optimal control model, we want to find a plan $\psi$, also called a control policy, that maximizes the performance index (3.1) subject to (3.2).

In compact form, the optimal control model can be described by the three-tuple.

$$(\Psi_T(x_0), \{f_t\}, \{g_t\}), \tag{3.5}$$

with components as above.

The following assumption is supposed to hold throughout the remainder of this section.

**Assumption 3.1.** *The three-tuple in (3.5) satisfies the following for each $x_0 \in X_0$:*

*(a) the set $\Psi_T(x_0)$ is nonempty;*

*(b) for each $t \in \{0, \ldots, T-1\}$, $f_t$ and $g_t$ are differentiable in the interior of $X_t \times U_t$ and $g_T$ in the interior of $X_T$.*

Throughout the following we fix the initial state $x_0$. Define $v_T : \Psi_T(x_0) \to \mathbb{R}$ by

$$v_T(\psi) = \sum_{t=0}^{T-1} g_t(x_t^{\psi}, u_t) + g_T(x_T^{\psi}). \tag{3.6}$$

For the three-tuple (3.5) and $x_0 \in X_0$, we want to find $\hat{\psi} \in \Psi_T(x_0)$ such that

$$v_T(\hat{\psi}) \geq v_T(\psi),$$

for all $\psi \in \Psi_T(x_0)$. In this case, we say that $\hat{\psi}$ is an optimal plan.

The following theorem is a consequence of Theorem 2.6 and Corollary 2.8. Observe that the TC (2.7) reduces to the terminal condition (3.9).

**Theorem 3.2.** *Let $\hat{\psi} = (\hat{u}_0, \ldots, \hat{u}_{T-1}) \in \Psi_T(x_0)$ be a plan such that each $\hat{u}_t$ is in the interior of $U_t$. If $\hat{\psi}$ is an optimal plan of the control model (3.1)-(3.5), then there exist unique $\lambda_1, \ldots, \lambda_T$ in $\mathbb{R}^n$ such that*

*(a) For all $t \in \{1, \ldots, T-1\}$,*

$$\frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) + \lambda_{t+1} \frac{\partial f_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) = \lambda_t, \tag{3.7}$$

*(b) For all $t \in \{0, \ldots, T-1\}$,*

$$\frac{\partial g_t}{\partial y}(x_t^{\hat{\psi}}, \hat{u}_t) + \lambda_{t+1} \frac{\partial f_t}{\partial y}(x_t^{\hat{\psi}}, \hat{u}_t) = 0, \tag{3.8}$$

*(c)*

$$\lambda_T = \frac{\partial g_T}{\partial x}(x_T^{\hat{\psi}}) \tag{3.9}$$

*Moreover, each $\lambda_t$ is given by*

$$\lambda_t = \sum_{k=t}^{T} \frac{\partial g_k}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k) \prod_{s=t}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s). \tag{3.10}$$

*Proof.* It suffices to consider the special case of Theorem 2.6 in which $g_T$ only depends of the first variable and $g_t \equiv 0$ for all $t \geq T+1$ and then apply Corollary 2.8. $\square$

To establish sufficient conditions, we need the following assumption.

**Assumption 3.3.** *The control model (3.5) satisfies the following:*

*(a) the set of plans $\Psi_T(x_0)$ is convex;*

*(b) the performance index $v_T$ (3.6) is concave.*

The next theorem is consequence of Theorem 2.11.

**Theorem 3.4.** *Suppose that a plan $\hat{\psi} \in \Psi_T(x_0)$ satisfies (3.7)-(3.9). If Assumption 3.3 holds, then $\hat{\psi}$ is an optimal plan for the control model (3.2)-(3.5).*

*Proof.* Considering the case when $g_T$ only depends of the first variable and $g_t \equiv 0$ for all $t \geq T+1$, Theorem 2.11 yields the result. □

## 3.2 Markov Strategies

In this section, we present a similar model to (2.1)-(2.3), but we consider that the control set and the policies may depend of the state.

As usual, let $X \subset \mathbb{R}^n$ be the state space and $U \subset \mathbb{R}^m$ be the control set. Consider a sequence $\{X_t \mid t \in \mathbb{N}_0\}$ of nonempty subsets of $X$, and $\{U_t(x) \mid x \in X_t, t \in \mathbb{N}_0\}$ the family of feasible control sets. For each $t \in \mathbb{N}_0$, we define

$$\mathbb{K}_t = \{(x,u) \mid x \in X_t, u \in U_t(x)\}.$$

For each $t \in \mathbb{N}_0$, $x \in X_t$, and $u \in U_t(x)$. We denote by $f_t(x,u)$ the corresponding state in $X_{t+1}$, where, for each $t \in \mathbb{N}_0$, $f_t : \mathbb{K}_t \to X_{t+1}$ is a given function.

A sequence $\varphi = \{\varphi_t\}$ of functions $\varphi_t : X_t \to \mathbb{R}^m$ is called a Markovian strategy whenever $\varphi_t(x) \in U_t(x)$ for all $x \in X_t$, $t \in \mathbb{N}_0$. We denote the set of Markovian strategies from $x_0$ as $\Phi(x_0)$. Given a Markovian strategy $\varphi = \{\varphi_t\}$, we denote by $\left\{x_t^{\varphi}\right\}$ the state sequence induced by $\varphi$, i.e.,

$$x_0^{\varphi} = x_0 \tag{3.11}$$
$$x_{t+1}^{\varphi} = f_t(x_t^{\varphi}, \varphi_t(x_t^{\varphi})) \quad \forall_{t \in \mathbb{N}_0}. \tag{3.12}$$

We want to optimize

$$\sum_{t=0}^{\infty} g_t(x_t^{\varphi}, \varphi_t(x_t^{\varphi})), \tag{3.13}$$

where $g_t : \mathbb{K}_t \to \mathbb{R}$ for each $t \in \mathbb{N}_0$. That is, we want to find a Markovian strategy $\varphi \in \Phi(x_0)$ that maximizes the performance index (3.13).

In reduced form, the optimal control model can be described by the three-tuple.

$$(\Phi(x_0), \{f_t\}, \{g_t\}), \tag{3.14}$$

with components as above.

For the OCP to be well defined, the following assumption is supposed to hold throughout the remainder of the section.

**Assumption 3.5.** *The three-tuple in (3.14) satisfies the following for each* $x_0 \in X_0$*:*

*(a) the set* $\Phi(x_0)$ *is nonempty;*

*(b) for each* $\varphi = (\varphi_0, \varphi_1, \dots) \in \Phi(x_0)$*,*

$$\sum_{t=0}^{\infty} g_t(x_t^{\varphi}, \varphi_t(x_t^{\varphi})) < \infty;$$

*(c) there exists* $\varphi = (\varphi_0, \varphi_1, \dots) \in \Phi(x_0)$ *such that*

$$\sum_{t=0}^{\infty} g_t(x_t^{\varphi}, \varphi_t(x_t^{\varphi})) > -\infty;$$

*(d) for each* $t \in \mathbb{N}_0$*,* $f_t$ *and* $g_t$ *are differentiable in the interior of* $\mathbb{K}_t$*.*

For $x_0 \in X_0$, define the performance index $v : \Phi(x_0) \to \mathbb{R} \cup \{-\infty\}$ by

$$v(\varphi) = \sum_{t=0}^{\infty} g_t(x_t^{\varphi}, \varphi_t(x_t^{\varphi})). \tag{3.15}$$

Assumption 3.5(a)-(b) ensures that the function $v$ is well defined. For the three-tuple (3.14) and $x_0 \in X_0$, the optimal control problem is to find $\hat{\varphi} \in \Phi(x_0)$ such that

$$v(\hat{\varphi}) \geq v(\varphi),$$

for all $\varphi \in \Phi(x_0)$. If this holds, we say that $\hat{\varphi}$ is an optimal plan. The optimization problem makes sense by Assumption 3.5(c).

**Remark 3.6.** *For notational convenience, for every* $t \in \mathbb{N}_0$ *and* $\varphi \in \Phi(x_0)$*, we will write*

$$g_t(x, \varphi_t) := g_t(x, \varphi_t(x)) \quad \textit{and} \quad f_t(x, \varphi_t) := f_t(x, \varphi_t(x)). \tag{3.16}$$

△

To proceed as in Chapter 2, we need the following assumption, which is analogous to Assumption 2.4.

**Assumption 3.7.** *Let* $\hat{\varphi} = (\hat{\varphi}_0, \hat{\varphi}_1, \dots) \in \Phi(x_0)$ *be such that each* $\hat{\varphi}_t$ *is differentiable in the interior of* $X_t$*. For each* $\tau \in \mathbb{N}_0$*, define the sequence of functions* $\rho_t^{\tau} : U_\tau(x_\tau^{\hat{\varphi}}) \to \mathbb{R}^n$ *as*

$$\rho_t^{\tau}(u) = \frac{\partial g_t}{\partial x}\left(x_t^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_t\right) \prod_{s=\tau+1}^{t-1} \left[\frac{\partial f_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) + \frac{\partial f_s}{\partial y}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) \frac{\partial \varphi_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}\right)\right] +$$
$$\frac{\partial g_t}{\partial y}\left(x_t^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_t\right) \frac{\partial \varphi_t}{\partial x}\left(x_t^{\hat{\varphi}^{\tau}(u)}\right) \prod_{s=\tau+1}^{t-1} \left[\frac{\partial f_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) + \frac{\partial f_s}{\partial y}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) \frac{\partial \varphi_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}\right)\right],$$

*where* $\hat{\varphi}^{\tau}(u) = (\hat{\varphi}_0, \ldots, \hat{\varphi}_{\tau-1}, \varphi_u, \hat{\varphi}_{\tau+1}, \ldots)$ *and* $\varphi_u(x) = u$ *for all* $x \in X_\tau$. *Given* $\tau \in \mathbb{N}_0$, *we suppose that there exists an open neighborhood* $O_\tau \subset U_\tau(x_\tau^{\hat{\varphi}})$ *of* $\hat{\varphi}_\tau(x_\tau^{\hat{\varphi}})$ *such that* $\displaystyle\sum_{t=\tau+1}^{\infty} \rho_t^\tau$ *converges uniformly on* $O_\tau$.

The following lemma contains the computation of the Gâteaux differential of the performance index $v$. Its proof is analogous to the one of Lemma 2.5, except that now we consider $\Lambda := \{\{\varphi_t\} \mid \varphi_t : X_t \to \mathbb{R}^m, t \in \mathbb{N}_0\}$.

**Lemma 3.8.** *Let* $\hat{\varphi} = \{\hat{\varphi}_t\} \in \Phi(x_0)$ *such that Assumption 3.7 holds. Let* $y \in \mathbb{R}^m$ *and* $\tau \in \mathbb{N}_0$. *Then* $\hat{\varphi}$ *is an internal point in the direction* $\varphi^{\tau,y} \in \Lambda$, *where* $\varphi^{\tau,y}$ *is defined as*

$$\varphi_t^{\tau,y}(x) := \begin{cases} y & \text{if } t = \tau \text{ and } x = x_\tau^{\hat{\varphi}} \\ \\ 0 & \text{otherwise.} \end{cases}$$

*Moreover, the Gâteaux differential of* $v$ *at* $\hat{\varphi}$ *in the direction* $\varphi^{\tau,y}$ *exists and is given by*

$$\delta_v(\hat{\varphi}; \varphi^{\tau,y}) = \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\varphi}}, \hat{\varphi}_\tau)y^* + \lambda_{\tau+1}\frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\varphi}}, \hat{\varphi}_\tau)y^*,$$

*where* $v$ *is the function in (3.15) and*

$$\begin{aligned}\lambda_{\tau+1} := \sum_{k=\tau+1}^{\infty} \Bigg[ & \frac{\partial g_k}{\partial x}\left(x_k^{\hat{\varphi}}, \hat{\varphi}_k\right) \prod_{s=\tau+1}^{k-1} \left( \frac{\partial f_s}{\partial x}\left(x_s^{\hat{\varphi}}, \hat{\varphi}_s\right) + \frac{\partial f_s}{\partial y}\left(x_s^{\hat{\varphi}}, \hat{\varphi}_s\right)\frac{\partial \hat{\varphi}_s}{\partial x}(x_s^{\hat{\varphi}})\right) + \\ & + \frac{\partial g_k}{\partial y}\left(x_k^{\hat{\varphi}}, \hat{\varphi}_k\right)\frac{\partial \hat{\varphi}_k}{\partial x}(x_k^{\hat{\varphi}}) \prod_{s=\tau+1}^{k-1} \left( \frac{\partial f_s}{\partial x}\left(x_s^{\hat{\varphi}}, \hat{\varphi}_s\right) + \frac{\partial f_s}{\partial y}\left(x_s^{\hat{\varphi}}, \hat{\varphi}_s\right)\frac{\partial \hat{\varphi}_s}{\partial x}(x_s^{\hat{\varphi}})\right)\Bigg].\end{aligned}$$

Repeating the same arguments in Theorem 2.6 and with the aid of Lemma 3.8, we can prove the next theorem.

**Theorem 3.9.** *Let* $\hat{\varphi} = \{\hat{\varphi}_t\} \in \Phi(x_0)$ *be such that Assumption 3.7 holds. If* $\hat{\varphi}$ *is an optimal plan for the control model (3.11)-(3.14), then there exists sequence* $\{\lambda_t\}_{t=1}^{\infty}$ *in* $\mathbb{R}^n$ *such that*

*(a) For all* $t \in \mathbb{N}$,

$$\frac{\partial g_t}{\partial x}(x_t^{\hat{\varphi}}, \hat{\varphi}_t) + \lambda_{t+1}\frac{\partial f_t}{\partial x}(x_t^{\hat{\varphi}}, \hat{\varphi}_t) = \lambda_t, \tag{3.17}$$

*(b) For all* $t \in \mathbb{N}_0$,

$$\frac{\partial g_t}{\partial y}(x_t^{\hat{\varphi}}, \hat{\varphi}_t) + \lambda_{t+1}\frac{\partial f_t}{\partial y}(x_t^{\hat{\varphi}}, \hat{\varphi}_t) = 0, \tag{3.18}$$

*(c) For all $h \in \mathbb{N}$,*

$$\lim_{t\to\infty} \lambda_t \prod_{s=h}^{t-1} \left[ \frac{\partial f_s}{\partial x} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) + \frac{\partial f_s}{\partial y} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) \frac{\partial \hat{\varphi}_s}{\partial x} (x_s^{\hat{\varphi}}) \right] = 0. \tag{3.19}$$

*Moreover,*

$$\lambda_t := \sum_{k=t}^{\infty} \left[ \frac{\partial g_k}{\partial x} \left( x_k^{\hat{\varphi}}, \hat{\varphi}_k \right) \prod_{s=t}^{k-1} \left( \frac{\partial f_s}{\partial x} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) + \frac{\partial f_s}{\partial y} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) \frac{\partial \hat{\varphi}_s}{\partial x} (x_s^{\hat{\varphi}}) \right) + \right.$$
$$\left. + \frac{\partial g_k}{\partial y} \left( x_k^{\hat{\varphi}}, \hat{\varphi}_k \right) \frac{\partial \hat{\varphi}_k}{\partial x} (x_k^{\hat{\varphi}}) \prod_{s=t}^{k-1} \left( \frac{\partial f_s}{\partial x} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) + \frac{\partial f_s}{\partial y} \left( x_s^{\hat{\varphi}}, \hat{\varphi}_s \right) \frac{\partial \hat{\varphi}_s}{\partial x} (x_s^{\hat{\varphi}}) \right) \right]. \tag{3.20}$$

**Proposition 3.10.** *Suppose that a plan $\hat{\psi} \in \Psi(x_0)$ satisfies the MP (3.17)-(3.18) and the TC (3.19). Then $\{\lambda_t\}$ is given by (3.20).*

**Corollary 3.11.** *The sequence $\{\lambda_t\}$ given in Theorem 3.9 is unique.*

We can proceed as in Section 2.3 to obtain sufficient conditions.

**Assumption 3.12.** *The control model (3.11)-(3.14) satisfies the following:*

*(a) the set of plans $\Phi(x_0)$ is convex;*

*(b) the performance index $v$ in (3.15) is concave.*

*(c) there exists a sequence of non-positive numbers $m_t$ with $\sum_{t=0}^{\infty} m_t > -\infty$ such that $g_t(x_t^{\varphi}, \varphi_t) \geq m_t$ for all $\varphi = (\varphi_0, \varphi_1, \ldots) \in \Phi(x_0)$.*

**Theorem 3.13.** *Let $\hat{\varphi} \in \Phi(x_0)$ such that Assumption 3.7 holds. Suppose that $\hat{\varphi}$ satisfies (3.17)-(3.19). If Assumption 3.12 holds, then $\hat{\varphi}$ is an optimal plan for the control model (3.11)-(3.14).*

**Example 3.14** (Optimal economic growth)**.** *One of the most studied models in economic growth is the Brock and Mirman model. Capital is represented by $x_t$, and $u_t$ denotes the consumption. The system's dynamics is given by*

$$x_{t+1} = A_t x_t^{\alpha} - u_t,$$

*where $\alpha \in (0, 1)$. The performance index to be maximized is*

$$\sum_{t=0}^{\infty} \beta^t \log u_t.$$

*In the present context, our control model in Section 3.2 has the following components*

- *state space* $X_t \equiv X := (0, \infty)$*;*
- *control space* $U := (0, \infty)$ *and control constraint sets* $U_t(x) := (0, A_t x^{\alpha})$ *for all* $x \in X$*;*
- *system functions* $f_t : \mathbb{K}_t \to X$ *with* $f_t(x, u) := A_t x^{\alpha} - u$*;*
- *cost functions* $g_t : \mathbb{K}_t \to \mathbb{R}$ *with* $g_t(x, u) = \beta^t \log u$*.*

*Theorem 3.9 for an optimal Markov strategy is*

$$0 = \beta^t \frac{1}{\hat{\varphi}_t(x_t^{\hat{\varphi}})} - \lambda_{t+1} \quad \forall_{t \in \mathbb{N}_0}, \tag{3.21}$$

$$\lambda_t = \alpha A_t [x_t^{\hat{\varphi}}]^{\alpha - 1} \lambda_{t+1} \quad \forall_{t \in \mathbb{N}_0}. \tag{3.22}$$

*In page 33 of [9], Chow solved these equations using the guess and verify method; he proposes a solution of the form* $\hat{\varphi}_t(x) = dA_t x^{\alpha}$*. By using this conjecture for* $\hat{\varphi}_t(x_t^{\hat{\varphi}})$ *and combining* (3.21) *and* (3.22)*, one obtains* $\lambda_t = \dfrac{\alpha\beta^t}{dx_t^{\hat{\varphi}}}$*. We can use this to evaluate*

$$\lambda_{t+1} = \frac{\alpha\beta^t}{d([A_t x_t^{\varphi}]^{\alpha} - dA_t [x_t^{\varphi}]^{\alpha})},$$

*on the right hand side of* (3.22) *and equating coefficients on both sides of* (3.22)*, one obtains* $d = 1 - \alpha\beta$*.*

*We verify Assumption 3.7. Let* $\tau \in \mathbb{N}_0$ *and consider* $\rho_t^{\tau} :\to \mathbb{R}$ *as in the assumption. Observe that*

$$\alpha \frac{x_{s+1}^{\hat{\varphi}^{\tau}(u)}}{x_s^{\hat{\varphi}^{\tau}(u)}} = \frac{\partial f_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) + \frac{\partial f_s}{\partial y}\left(x_s^{\hat{\varphi}^{\tau}(u)}, \hat{\varphi}_s\right) \frac{\partial \varphi_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}\right).$$

*Now, take* $O_{\tau} = (\eta', \eta)$ *as a small neighborhood of* $(1 - \alpha\beta) A_t x_{\tau}^{\hat{\varphi}}$ *properly contained in* $[0, A_t [x_{\tau}^{\hat{\varphi}}]^{\alpha}]$*. We have*

$$\begin{aligned} |\rho_{\tau}(u)| &= \left| \frac{\beta^t}{x_t^{\hat{\varphi}^{\tau}(u)}} \prod_{s=\tau+1}^{t-1} \alpha \frac{x_{s+1}^{\hat{\varphi}^{\tau}(u)}}{x_s^{\hat{\varphi}^{\tau}(u)}} \right| \\ &= \left| \frac{\beta^t \alpha^{t-\tau+1}}{A_{\tau}[x_{\tau}^{\hat{\varphi}}]^{\alpha} - u} \right| \\ &< \left| \frac{\beta^t}{A_{\tau}[x_{\tau}^{\hat{\varphi}}]^{\alpha} - \eta} \right|. \end{aligned}$$

*Thus, by the Weierstrass M-test,* $\displaystyle\sum_{t=\tau+1}^{\infty} \rho_t^{\tau}$ *converges uniformly on* $O_{\tau}$*.* $\triangle$

# Chapter 4

# Further Topics

## 4.1 The Euler Equation

Let us now go back to the optimal control model (3.11)-(3.14) in Section 3.2. As usual in the Euler equation approach, we will consider the particular case when the functions $f_t$ in (3.11)-(3.12) satisfy, for each $t \in \mathbb{N}_0$, $f_t(x,u) = u$ for all $(x,u) \in \mathbb{K}_t$. We assume this and the well-posedness Assumption 3.5 to hold during this section.

The particular form of the dynamic functions means that at time $t$, we are directly determining the following state of the system, since, a plan $\varphi \in \Phi(x_0)$ will determine $x_{t+1}^{\varphi}$ identically as a function $\varphi_t$ of $x_t^{\varphi}$. Thus, we want to maximize the performance index

$$v(\varphi) = \sum_{t=0}^{\infty} g_t(x_t^{\varphi}, x_{t+1}^{\varphi}), \tag{4.1}$$

where

$$x_0^{\varphi} = x_0 \tag{4.2}$$

$$x_{t+1}^{\varphi} = \varphi_t(x_t^{\varphi}) \quad \forall_{t\in\mathbb{N}_0}, \tag{4.3}$$

and for each $t \in \mathbb{N}_0$, $\varphi_t(x) \in U_t(x)$ for all $x \in X_t$.

For this problem, Assumption 3.7 reduces to the following one.

**Assumption 4.1.** *Let $\hat{\varphi} = (\hat{\varphi}_0, \hat{\varphi}_1, \dots) \in \Phi(x_0)$ such that each $\hat{\varphi}_t$ is differentiable in the interior of $X_t$. For each $\tau \in \mathbb{N}_0$, define the sequence of functions $\rho_t^{\tau} : U_\tau(x_\tau^{\hat{\varphi}}) \to \mathbb{R}^n$ as*

$$\rho_t^{\tau}(u) = \left[\frac{\partial g_t}{\partial x}\left(x_t^{\hat{\varphi}^{\tau}(u)}, x_{t+1}^{\hat{\varphi}^{\tau}(u)}\right) + \frac{\partial g_t}{\partial y}\left(x_t^{\hat{\varphi}^{\tau}(u)}, x_{t+1}^{\hat{\varphi}^{\tau}(u)}\right)\frac{\partial \varphi_t}{\partial x}\left(x_t^{\hat{\varphi}^{\tau}(u)}\right)\right]\prod_{s=\tau+1}^{t-1}\frac{\partial \varphi_s}{\partial x}\left(x_s^{\hat{\varphi}^{\tau}(u)}\right),$$

*where* $\hat{\varphi}^{\tau}(u) = (\hat{\varphi}_0, \dots, \hat{\varphi}_{\tau-1}, \varphi_u, \hat{\varphi}_{\tau+1}, \dots)$ *and* $\varphi_u(x) = u$ *for all* $x \in X_\tau$. *Given* $\tau \in \mathbb{N}_0$, *we suppose that there exists an open neighborhood* $O_\tau \subset U_\tau(x_\tau^{\hat{\varphi}})$ *of* $x_{\tau+1}^{\hat{\varphi}}$ *such that* $\sum_{t=\tau+1}^{\infty} \rho_t^{\tau}$ *converges uniformly on* $O_\tau$.

The following theorem is a consequence of Theorem 3.9. Equation (4.4) is the so-called Euler Equation (EE).

**Theorem 4.2.** *Let* $\hat{\varphi}$ *be an optimal plan for the control model (4.1)-(4.3). Suppose that* $\hat{\varphi}$ *satisfies Assumption 4.1. Then*

*(a) For each* $t \in \mathbb{N}$,

$$\frac{\partial g_{t-1}}{\partial y}\left(x_{t-1}^{\hat{\varphi}}, x_t^{\hat{\varphi}}\right) + \frac{\partial g_t}{\partial x}\left(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}}\right) = 0; \tag{4.4}$$

*(b) for each* $h \in \mathbb{N}$,

$$\lim_{t\to\infty} \frac{\partial g_{t-1}}{\partial y}\left(x_{t-1}^{\hat{\varphi}}, x_t^{\hat{\varphi}}\right) \prod_{s=h}^{t-1} \frac{\partial \hat{\varphi}_s}{\partial x}(x_s^{\hat{\varphi}}) = 0. \tag{4.5}$$

*Proof.* From Theorem 3.9 (a)-(b), there exist a sequence $\{\lambda\}_{t=1}^{\infty}$ such that, for each $t \in \mathbb{N}$

$$\lambda_t = \frac{\partial g_t}{\partial x}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}}).$$

And, for each $t \in \mathbb{N}_0$

$$0 = \frac{\partial g_t}{\partial y}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}}) + \lambda_{t+1}.$$

These facts yield $(a)$. Part $(b)$, follows from part $(c)$ of Theorem 3.9 and the fact that $\lambda_t = -\frac{\partial g_{t-1}}{\partial y}(x_{t-1}^{\hat{\varphi}}, x_t^{\hat{\varphi}})$. □

To establish sufficient conditions we use the next assumption, which is identical to Assumption 3.12.

**Assumption 4.3.** *The control model (4.1)-(4.3) satisfies the following:*

*(a) the set of plans* $\Phi(x_0)$ *is convex;*

*(b) the performance index is concave;*

*(c) there exists a sequence of non-positive numbers* $m_t$ *with* $\sum_{t=0}^{\infty} m_t > -\infty$ *such that* $g_t(x_t^{\varphi}, x_{t+1}^{\varphi}) \geq m_t$ *for all* $\varphi = (\varphi_0, \varphi_1, \dots) \in \Phi(x_0)$ .

Sufficient conditions follow from Theorem 3.13.

**Theorem 4.4.** *Let $\hat{\varphi} \in \Phi(x_0)$ be such that Assumption 4.1 holds. Suppose that $\hat{\varphi}$ satisfies (4.4)-(4.5). If Assumption 4.3 holds, then $\hat{\varphi}$ is an optimal plan for the control model (4.1)-(4.3).*

*Proof.* Define, for each $t \in \mathbb{N}$, $\lambda_t := \frac{\partial g_t}{\partial x}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}})$. From (4.4), $\lambda_{t+1} = -\frac{\partial g_t}{\partial y}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}})$ for $t \in \mathbb{N}_0$. Thus, for each $t \in \mathbb{N}$,

$$\lambda_t = \frac{\partial g_t}{\partial x}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}}).$$

In addition, for each $t \in \mathbb{N}_0$

$$0 = \frac{\partial g_t}{\partial y}(x_t^{\hat{\varphi}}, x_{t+1}^{\hat{\varphi}}) + \lambda_{t+1}.$$

Therefore, Theorem 3.13 yields the result. □

**Example 4.5** (An economic growth model)**.** *Consider the following problem concerning to an optimal growth model known as de Ak model; see section 2.3.2 of [12]. Let $\beta \in (0,1)$, $\theta < 0$ and $a > 1$ such that $(a\beta)^{\frac{1}{\theta-1}} > 1$. The performance index is*

$$\sum_{t=0}^{\infty} \frac{\beta^t}{\theta}(ax_t - x_{t+1})^{\theta},$$

*subject to $x_{t+1} \in [0, ax_t]$, for all $t \in \mathbb{N}$.*

*Our control model in this section has the following components*

- *state space $X_t \equiv X := (0,\infty)$ with control constraints sets $U_t(x) = [0, ax]$ for all $x \in X$;*
- *return functions $g_t : X_t \times X_{t+1} \to \mathbb{R}$ with $g_t(x,u) = \frac{\beta^t}{\theta}(ax-u)^{\theta}$.*

*Hence, the Euler equation*

$$-(ax_{t-1}^{\hat{\varphi}} - x_t^{\hat{\varphi}})^{\theta-1} + \beta a(ax_t^{\hat{\varphi}} - x_{t+1}^{\hat{\varphi}})^{\theta-1}, \qquad t = 1,2,\ldots,$$

*can be expressed as the difference equation*

$$bx_{t+1}^{\hat{\varphi}} - (1+ab)x_t^{\hat{\varphi}} + ax_{t-1}^{\hat{\varphi}} = 0, \tag{4.6}$$

*with $b := (a\beta)^{\frac{1}{\theta-1}} > 1$. Considering a linear solution $\hat{\varphi}_t(x) = \alpha x$, and substituting in (4.6), $x_{t-1}^{\hat{\varphi}}$ by $\alpha^{-1}x_t^{\hat{\varphi}}$; we obtain $\alpha = b^{-1}$. To prove that Assumption 4.1 holds, let $\tau \in \mathbb{N}_0$ and consider $\rho_t^{\tau} : \mathbb{R} \to \mathbb{R}$ as in the assumption. Take $O_\tau = (\eta, \eta')$ as a small neighborhood of $b^{-1}x_\tau^{\hat{\varphi}}$ properly contained in $[0, ax_\tau^{\hat{\varphi}}]$. Then we have*

$$\begin{aligned}
|\rho_t^{\tau}(u)| &= \left|(a-b^{-1})\beta^t[(a-b^{-1})x_t^{\hat{\varphi}^{\tau}(u)}]^{\theta-1}b^{-t+\tau+1}\right| \\
&< \left|(a-b^{-1})^{\theta}\beta^t[x_t^{\hat{\varphi}^{\tau}(u)}]^{\theta-1}\right| \\
&= \left|(a-b^{-1})^{\theta}\beta^t[b^{\tau-t}u]^{\theta-1}\right| \\
&< \left|(a-b^{-1})^{\theta}\beta^t\eta^{\theta-1}\right|.
\end{aligned}$$

*Thus, by the Weierstrass M-test,* $\sum_{t=\tau+1}^{\infty} \rho_t^{\tau}$ *converges uniformly on* $O_\tau$*.* △

## 4.2 Dynamic Games

In this section, we consider non-cooperative dynamic games with $N$ players and state space $X \subset \mathbb{R}^n$.

Assume that the state dynamics is given by

$$x_{t+1} = f_t(x_t, u_t^1, \dots, u_t^N), \tag{4.7}$$

where, for each $j = 1, \dots, N$, $u_t^j$ is chosen by player $j$ in the control set $U_t^j \subset \mathbb{R}^{m_j}$. We suppose that player $j$ wants to "maximize" a performance index (also known as payoff function) of the form

$$\sum_{t=0}^{\infty} g_t^j(x_t, u_t^1, \dots, u_t^N), \tag{4.8}$$

subject to (4.7) and a given initial state $x_0$.

We denote by $\Psi^j(x_0)$ the set of plans, or strategies, of player $j$, that is, $\psi^j = (u_0^j, u_1^j, \dots)$ with $u_t^j \in U_t^j$ for all $t \in \mathbb{N}_0$. The set of so-called multistrategies $\psi = (\psi^1, \dots, \psi^N)$ is denoted by $\Psi(x_0) := \Psi^1(x_0) \times \cdots \times \Psi^N(x_0)$ .

Given a multistrategy $\psi = (\psi^1, \dots, \psi^N) \in \Psi(x_0)$, we denote by $\left\{x_t^{\psi}\right\}$ the sequence induced by $\psi$ in (4.7), i.e.,

$$\begin{aligned} x_1^{\psi} &= x_0 \\ x_{t+1}^{\psi} &= f_t(x_t^{\psi}, u_t^1, \dots, u_t^N). \end{aligned}$$

We can specify a dynamic game in reduced form as

$$\left(\Psi(x_0), \{f_t\}, \left\{g_t^j \mid j = 1, \dots, N\right\}\right), \tag{4.9}$$

with components as above.

The following assumption is supposed to hold throughout the remainder of the section.

**Assumption 4.6.** *The three-tuple in (4.9) satisfies the following for each* $x_0 \in X_0$ *and each* $j = 1, \dots, N$*:*

*(a) the set* $\Psi^j(x_0)$ *is nonempty,*

*(b) for each* $\psi \in \Psi(x_0)$*,*

$$\sum_{t=0}^{\infty} g_t^j(x_t, u_t^1, \dots, u_t^N) < \infty;$$

(c) *there exist a* $\psi \in \Psi(x_0)$ *such that*

$$\sum_{t=0}^{\infty} g_t^j(x_t^{\psi}, u_t^1, \ldots, u_t^N) > -\infty;$$

(d) *for each* $t \in \mathbb{N}_0$, $f_t$ *and* $g_t^j$ *are differentiable in the interior of* $X \times U_t^1 \times \cdots \times U_t^N$.

For $x_0 \in X_0$ and $j = 1, \ldots N$, define $v^j : \Psi(x_0) \to \mathbb{R}$ by

$$v^j(\psi) = \sum_{t=0}^{\infty} g_t^j(x_t^{\psi}, u_t^1, \ldots, u_t^N). \tag{4.10}$$

Assumption 4.6(a)-(b) ensures that the function $v^j$ is well defined.

We say that $\hat{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^N) \in \Psi(x_0)$ is a Nash equilibrium if, for each player $j = 1, \ldots, N$,

$$v^j(\hat{\psi}) \geq v^j(\hat{\psi}^1, \ldots, \hat{\psi}^{j-1}, \psi^j, \hat{\psi}^{j+1}, \ldots, \hat{\psi}^N) \quad \forall_{\psi^j \in \Psi^j(x_0)}.$$

We want to use the Theorem 2.6 to characterize Nash equilibria (NE). To that end we consider the following assumption.

**Assumption 4.7.** *Let* $\hat{\psi} = (\hat{\psi}^1, \ldots, \hat{\psi}^N) \in \Psi^1(x_0) \times \cdots \times \Psi^N(x_0)$. *For each* $\tau \in \mathbb{N}_0$ *and* $j = 1, \ldots, N$, *define the sequence of functions* $\rho_t^{\tau,j} : U_\tau \to \mathbb{R}^n$ *as*

$$\rho_t^{\tau,j}(u) = \frac{\partial g_t^j}{\partial x}(x_t^{\hat{\psi}^{\tau,j}(u)}, \hat{u}_t^1, \ldots, \hat{u}_t^N) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}^{\tau,j}(u)}, \hat{u}_s^1, \ldots, \hat{u}_s^N),$$

*where* $\hat{\psi}^{\tau,j}(u) = (\hat{\psi}^1, \ldots, \hat{\psi}^{j-1}, \hat{\psi}_j^{\tau}(u), \hat{\psi}^{j+1}, \ldots, \hat{\psi}^N)$ *and* $\hat{\psi}_j^{\tau}(u) = (\hat{u}_0^j, \ldots, \hat{u}_{\tau-1}^j, u, \hat{u}_{\tau+1}^j, \ldots)$. *Given* $\tau \in \mathbb{N}_0$ *and* $j = 1, \ldots, N$, *we suppose that there exists an open neighborhood* $O_\tau^j \subset U_\tau^j$ *of* $\hat{u}_\tau^j$ *such that* $\sum_{t=\tau+1}^{\infty} \rho_t^{\tau,j}$ *converges uniformly on* $O_\tau^j$.

The next theorem follows from Theorem 2.6.

**Theorem 4.8.** *Let* $\hat{\psi} \in \Psi(x_0)$ *for which Assumption 4.7 holds. If* $\hat{\psi}$ *is a Nash equilibrium, then, for each* $j = 1, \ldots, N$, *there exists a sequence* $\left\{\lambda_t^j\right\}_{t=1}^{\infty}$ *in* $\mathbb{R}^n$ *such that*

(a) *For all* $t \in \mathbb{N}$,

$$\frac{\partial g_t^j}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t^1, \ldots, \hat{u}_t^N) + \lambda_{t+1}^j \frac{\partial f_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t^1, \ldots, \hat{u}_t^N) = \lambda_t^j, \tag{4.11}$$

(b) *For all* $t \in \mathbb{N}_0$,

$$\frac{\partial g_t^j}{\partial y_j}(x_t^{\hat{\psi}}, \hat{u}_t^1, \ldots, \hat{u}_t^N) + \lambda_{t+1}^j \frac{\partial f_t}{\partial y_j}(x_t^{\hat{\psi}}, \hat{u}_t^1, \ldots, \hat{u}_t^N) = 0, \tag{4.12}$$

(c) *For all* $h \in \mathbb{N}$,

$$\lim_{t\to\infty} \lambda_t^j \prod_{s=h}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s^1, \ldots, \hat{u}_s^N) = 0. \tag{4.13}$$

*Moreover, each* $\lambda_t^j$ *is given by*

$$\lambda_t^j = \sum_{k=t}^{\infty} \frac{\partial g_k^j}{\partial x}(x_k^{\hat{\psi}}, \hat{u}_k^1, \ldots, \hat{u}_k^N) \prod_{s=t}^{k-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s^1, \ldots, \hat{u}_s^N). \tag{4.14}$$

**Assumption 4.9.** *Let* $\hat{\psi} \in \Psi(x_0)$. *We assume that the game (2.3) satisfies the following for each* $j = 1, \ldots, N$,

(a) $\Psi^j(x_0)$ *is convex;*

(b) *the performance index* $v^j$ *is concave;*

(c) *there exists a sequence of non-positive numbers* $m_t$ *with* $\sum_{t=0}^{\infty} m_t > -\infty$ *such that* $g_t(x_t^{\hat{\psi}}, \hat{u}_t^1, \ldots, \hat{u}_t^{j-1}, u_t^j, \hat{u}_t^{j+1}, \ldots, \hat{u}_t^N) \geq m_t$ *for all* $\psi^j = (u_0^j, u_1^j, \ldots) \in \Psi^j(x_0)$.

Theorem 2.11 yields the next theorem.

**Theorem 4.10.** *Let* $\hat{\psi} \in \Psi(x_0)$ *be such that Assumption 4.7 holds. Suppose that* $\hat{\psi}$ *satisfies (4.11)-(4.13). If Assumption 4.9 holds, then* $\hat{\psi}$ *is a Nash Equilibrium.*

**Example 4.11.** *Consider the following game with linear dynamics*

$$x_{t+1} = x_t + u_t^1 + \cdots + u_t^N,$$

*with* $x_0 \in \mathbb{R}$ *given and performance index*

$$\sum_{t=0}^{\infty} \beta^t \frac{1}{2} \left[ x_t^2 + [u_t^j]^2 \right],$$

*for each player* $j = 1, \ldots, N$.

*From (4.11)-(4.12), for each* $j = 1, \ldots, N$.

$$\lambda_t^j = \beta^t x_t^{\hat{\psi}} + \lambda_{t+1}^j \quad \forall_{t \in \mathbb{N}}, \tag{4.15}$$

$$0 = \beta^t \hat{u}_t^j + \lambda_{t+1}^j \quad \forall_{t \in \mathbb{N}_0}. \tag{4.16}$$

*First, note that by (4.14),* $\lambda_t^j = \lambda_t^1$ *for* $j = 1, \ldots, N$. *And by (4.16),* $u_t^j = u_t^1$ *for* $j = 1, \ldots, N$. *Proceeding as in Example 2.13, we find* $x_t^{\hat{\psi}} = x_0 r^t$, *where*

$r = \min\left\{x | \beta x^2 - [1 + (1+N)\beta]x + 1 = 0\right\}$. *Thus,*

$$\begin{aligned} \hat{u}_t^j &= \frac{x_{t+1}^{\hat{\psi}} - x_t^{\hat{\psi}}}{N} \\ &= \frac{x_0(r-1)r^t}{N} \qquad \forall_{j=1,\ldots,N} \end{aligned}$$

*Assumption 4.7 can be proved exactly as Assumption 2.4 was proved in Example 2.13.* △

# Appendix A

# Proof of Lemma 2.5

For the convenience of the reader, we restate here Lemma 2.5.

**Lemma 2.5.** *Let $\hat{\psi} = \{\hat{u}_t\} \in \Psi(x_0)$ be a plan for which Assumption 2.4 holds. Let $y \in \mathbb{R}^m$ and $\tau \in \mathbb{N}_0$. Then $\hat{\psi}$ is an internal point in the direction $\psi^{\tau,y} \in \Lambda$, where $\psi^{\tau,y}$ is defined as*

$$\psi_t^{\tau,y} := \begin{cases} y & if \quad t = \tau \\ \\ 0 & if \quad t \neq \tau, \end{cases}$$

*for all $t \in \mathbb{N}_0$. Moreover, the Gâteaux differential of $v$ at $\hat{\psi}$ in the direction $\psi^{\tau,y}$ exists and is given by*

$$\delta_v(\hat{\psi};\psi^{\tau,y}) = \left( \sum_{t=\tau+1}^{\infty} \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}}, \hat{u}_t) \prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}}, \hat{u}_s) \right) \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^* + \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}}, \hat{u}_\tau) y^*,$$

*where $v$ is the function in (2.4).*

*Proof.* First, we prove that $\hat{\psi}$ is an internal point in the direction $\psi^{\tau,y}$. If $t = \tau$; there exists $\varepsilon_\tau > 0$ such that $\hat{u}_\tau + \varepsilon y \in U_\tau$ for all $\varepsilon \in (-\varepsilon_\tau, \varepsilon_\tau)$, since, by assumption, $\hat{u}_\tau$ belongs to an open neighborhood $O_\tau \subset U_\tau$. So $\hat{\psi} + \varepsilon\psi^{\tau,y}$ is in $\Psi(x_0)$ for all $\varepsilon \in (-\varepsilon_\tau, \varepsilon_\tau)$. To prove the assertion about the Gâteaux differential, we compute the derivatives of the functions in (a)-(c) below.

(a) For each $t > \tau$, define $h_t : (-\varepsilon_\tau, \varepsilon_\tau) \to \mathbb{R}^n$ as $h_t(\varepsilon) := f_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})$. We prove by induction that

$$h_t'(\varepsilon) = \left[ \prod_{s=\tau+1}^{t} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_s) \right] \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y) y^*.$$

If $t + 1 = \tau$,

$$h_{t+1}'(\varepsilon) = \frac{\partial f_{t+1}}{\partial y}(x_t^{\hat{\psi}}, \hat{u}_{t+1} + \varepsilon y) y^*.$$

Suppose that it holds for $t$. Then, by the chain rule,

$$\begin{aligned}
h'_{t+1}(\varepsilon) &= [f_{t+1}(x_{t+1}^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_{t+1} + \varepsilon\psi_{t+1}^{\tau,y})]' \\
&= \left[f_{t+1}(h_t(\varepsilon), \hat{u}_{t+1} + \varepsilon\psi_{t+1}^{\tau,y})\right]' \\
&= \frac{\partial f_{t+1}}{\partial x}(h_t(\varepsilon), \hat{u}_{t+1} + \varepsilon\psi_{t+1}^{\tau,y})h'_t(\varepsilon) + \frac{\partial f_{t+1}}{\partial y}(h_t(\varepsilon), \hat{u}_{t+1} + \varepsilon\psi_{t+1}^{\tau,y})[\psi_{t+1}^{\tau,y}]^* \\
&= \left[\prod_{s=\tau+1}^{t+1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_s + \varepsilon\psi_s^{\tau,y})\right] \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y)y^*.
\end{aligned}$$

(b) Define, for $t > \tau$, $k_t : (-\varepsilon_\tau, \varepsilon_\tau) \to \mathbb{R}$ as

$$k_t(\varepsilon) := g_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y}).$$

For $t > \tau$, we have

$$\begin{aligned}
k'_t(\varepsilon) &= [g_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})]' \\
&= [g_t(h_{t-1}(\varepsilon), \hat{u}_t + \varepsilon\psi_t^{\tau,y})]' \\
&= \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})h'_{t-1}(\varepsilon) + \frac{\partial g_t}{\partial y}(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})[\psi_t^{\tau,y}]^* \\
&= \frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t)\left[\prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_s)\right] \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y)y^*
\end{aligned}$$

(c) For each $T \in \mathbb{N}_0$, define $l_T : (-\varepsilon_\tau, \varepsilon_\tau) \to \mathbb{R}$ as

$$l_T(\varepsilon) := \sum_{t=0}^{T} g_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})$$

For $T > \tau$, we have

$$\begin{aligned}
l'_T(\varepsilon) &= \left[\sum_{t=0}^{T} g_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t + \varepsilon\psi_t^{\tau,y})\right]' \\
&= \left[\sum_{t=0}^{\tau-1} g_t(x_t^{\hat{\psi}}, \hat{u}_t)\right]' + [g_\tau(x_\tau^{\hat{\psi}}, \hat{u}_\tau + \varepsilon y)]' + \left[\sum_{t=\tau+1}^{T} g_t(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t)\right]' \\
&= \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y)y^* + \sum_{t=\tau+1}^{T} k'_t(\varepsilon) \\
&= \sum_{t=\tau+1}^{T}\left[\frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_t)\left(\prod_{s=\tau+1}^{t-1} \frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_s)\right) \frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y)y^*\right] \\
&\quad + \frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}+\varepsilon\psi^{\tau,y}}, \hat{u}_\tau + \varepsilon y)y^*.
\end{aligned}$$

Finally,

$$\begin{aligned}
\delta_v(\hat{\psi};\psi^{\tau,y}) &= \frac{d}{d\varepsilon}\left[v(\hat{\psi}+\varepsilon\psi^{\tau,y})\right]\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left[\lim_{T\to\infty} l_T(\varepsilon)\right]\Big|_{\varepsilon=0} \\
&= \lim_{T\to\infty}\left[l'_T(\varepsilon)\right]\Big|_{\varepsilon=0} \\
&= \sum_{t=\tau+1}^{\infty}\left[\frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}},\hat{u}_t)\left(\prod_{s=\tau+1}^{t-1}\frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}},\hat{u}_s)\right)\frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}},\hat{u}_\tau)y^*\right]+\frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}},\hat{u}_\tau)y^* \\
&= \left(\sum_{t=\tau+1}^{\infty}\frac{\partial g_t}{\partial x}(x_t^{\hat{\psi}},\hat{u}_t)\prod_{s=\tau+1}^{t-1}\frac{\partial f_s}{\partial x}(x_s^{\hat{\psi}},\hat{u}_s)\right)\frac{\partial f_\tau}{\partial y}(x_\tau^{\hat{\psi}},\hat{u}_\tau)y^*+\frac{\partial g_\tau}{\partial y}(x_\tau^{\hat{\psi}},\hat{u}_\tau)y^*.
\end{aligned}$$

Assumption 2.4 ensures the interchange between the limit and the derivative. □

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