Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional

Unidad Zacatenco
Departamento de Matemáticas

Un estudio de juegos con equilibrios de Nash que tambien son óptimos de Pareto

Tesis que presenta

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Para obtener el grado de Maestro en Ciencias En la Especialidad de Matemáticas

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Ciudad de México Marzo, 2019

CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

Campus Zacatenco
Department of Mathematics

A survey of games with Pareto-optimal Nash equilibria

A DISSERTATION PRESENTED BY

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TO OBTAIN THE DEGREE OF Master in Science In the Specialty of Mathematics

THESIS ADVISOR: Dr. Onésimo Hernández-Lerma

Mexico City March, 2019

Agradecimientos

A mi padre y madre por darme los valores y principios que me han guiado toda mi vida, a mis hermanos Fernando y Ángel por su apoyo incondicional a cada momento, a mis amigos de toda la vida Atziri, Carlos, Salvador y Larisa por estar siempre a mi lado, a Sergio, Agustin y mis tocayos por todas las aventuras que tuvimos juntos durante esta etapa, a mis amigos de la ESFM por siempre estar presentes y a mis excelentes profesores que he tenido durante mi vida. Muchas gracias a todos por formar parte de esto.

Especial dedicación al Dr. Onésimo Hernández-Lerma por orientar mi preparación profesional y a mis sinodales por su orientación.

Finalmente, agradezco al CONACYT por el financiamiento para la realización de esta tesis.

Abstract

This thesis deals with some kinds of games that have Nash equilibria that are also Pareto optimal. Our main objective is to create a collection that brings together results that characterize these kinds of games.

We present results for static games and dynamic games: the first are studied with an approach on *potential games* and *discontinuous games* and the second are studied from an approach of *potential differential games* and *stochastic games*.

The results presented are illustrated with examples to be able to appreciate how they are used and observe that we do not work with the empty case.

Resumen

Esta tesis trata de tipos de juegos que tienen equilibrios de Nash que también son óptimos de Pareto. Nuestro principal objetivo es crear una colección que reúna los resultados que caracterizan este tipo de juegos.

Presentamos resultados para juegos estáticos y juegos dinámicos: los primeros se estudian con un enfoque sobre *juegos potenciales* y *juegos discontinuos*, los segundos se estudian desde un enfoque de *juegos diferenciales porenciales* y *juegos estocsticos*.

Los resultados presentados se ilustran con ejemplos para poder apreciar cómo se utilizan y observar que no trabajamos con el caso vacío.

Contents

1 Introduction

In game theory one of the main problems is knowing how to choose which is the best strategy for each player.

Two of the most studied and known strategies are the so-called Nash equilibria and the Pareto-optimal, the first is a type of best strategy for non-cooperative games, while the seconds is a best strategy for cooperative games. At the outset, these concepts are incompatible [8], [9]. But, on the other hand, there are particular games in which Nash equilibria turn out to be Pareto-optimal, for instance [7], [20].

Cooperative and non-cooperative games are different kinds of games, in problems of economy, natural resources, electronics, etc., we will be able to find Nash equilibria that are also Pareto-optimal.

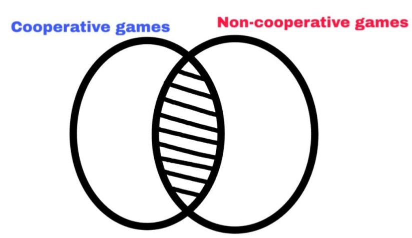

Figure 1: Cooperative and non-cooperative games

The motivation to study this kind of games with Nash equilibria that are also Pareto-optimal, is to obtain criteria to choose the best strategy that turns out to be the solution for the game.

The following examples allow us to observe what is studied in this thesis.

Example 1.1. Players are maximizing. $(1,1)$, $(2,2)$, $(3,3)$ are Nash equilibria (see Definition 2.3). Moreover, $(3,3)$ is the only one which is also Pareto-optimal (see Definition 2.2).

Players	1	2	3
1	1,1	-1,-2	-1,-3
2	-2,-1	2,2	-2,-3
3	-3,-1	-3,-2	3,3

Example 1.2. The battle of the sexes. Players are maximizing. (F, F) and (C, C) are both Nash equilibria (see Definition 2.3), and both Pareto-optimal (see Definition 2.2).

Players	F	C
F	3,2	1,1
C	0,0	2,3

Example 1.3. Matching pennies. Two platers $1, 2$. $A_1 = A_2 = \{a, s\}$.The players wish to maximize their payoffs given by:

Players	a	s
a	-1,1	1,-1
s	1,-1	-1,1

There is no Nash equilibrium (see Definition 2.3) in $A := A_1 \times A_2$. However, all pairs in $A_1 \times A_2$ are Pareto-optimal (see Definition 2.2).

In the first example we have three Nash equilibria and only one is Pareto-optimal. In the second we have that all the Nash equilibria are Pareto-optimal. Finally, in the third example we do not have Nash equilibria but all are Pareto-optimal. This shows us that studying this problem makes sense.

The thesis is structured as follows. In section 2 we study the static games, from two approaches: *potential games* and *discontinuous games*. The first approach was introduced by Monderer and Shapley [21], and the second by Scalzo [29]. Section 3 studies dynamic games. This section is divided in two parts, (deterministic) differential games, and stochastic games. The former part is based on the work of Fonseca-Morales and Hernández-Lerma [10, 11, 12]. On the other hand, the part on stochastic games considers the discrete-time case [13, 14] and stochastic differentials games [10, 11]. Finally, in Section 4, we present the conclusions obtained in this thesis.

2 Static games

In this section we consider games in normal form defined as follows.

Definition 2.1. A game G in normal form can be expressed as a triplet

$$G := (N, \{A_i, i \in N\}, \{r_i, i \in N\}), \tag{2.1}$$

where $N = \{1, 2, \ldots, n\}$ is the set of players and, for every $i \in N$, A_i is the action set for the player i, and $r_i : A \to \mathbb{R}$ is the payoff function for player i, with

$$A := A_1 \times \cdots \times A_n. \tag{2.2}$$

For notational convenience, sometime we write (2.1) as $G = (N, A, r)$, with A as in (2.2), and $r := (r_1, \ldots, r_n)$.

If the action sets A_i are all finite, then G is said to be a *finite game*.

A vector $a = (a_1, \ldots, a_n) \in A$ is called a *strategy profile*.

For every $i \in N$, we write $A_{-i} := \prod_{j \neq i} A_j$. Hence the vectors $a_{-i} \in A_{-i}$ are of the form

$$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n).$$

Given a strategy profile $a = (a_1, \ldots, a_n) \in A$ and given an action $a'_i \in A_i$, for some $i \in N$, we define the strategy profile

$$(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n).$$

Notation. For each $a \in A$, let $r(a) := (r_1(a), \ldots, r_n(a)) \in \mathbb{R}^n$. The *game's payoff set* is $\mathbf{R} := \{ r(a) \,|a \in A\} \subset \mathbb{R}^n$. If $u := (u_1, \ldots, u_n)$ and $v := (v_1, \ldots, v_n)$ are in $\mathbb{R}^n$.

$$\begin{aligned}
&u \geqslant v \text{ means}: \ u_i \geqslant v_i \quad \forall i \in N, \\
&u > v \text{ means}: \ u \geqslant v \quad \text{and} \quad u \neq v, \\
&u \gg v \text{ means}: \ u_i > v_i \quad \text{for all} \quad i = 1, \ldots, n.
\end{aligned}$$

In game theory, two of the most important solution concepts are Nash equilibria and Pareto optimal.

The following definitions characterize the strategies called Pareto optimal and Nash equilibrium.

Definition 2.2. A strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ in A is said to be

- *weakly maximal Pareto efficient strategy* (or *maximal Slater optimal*) for G if, there does not exist $a \in A$ such that

$$r(a) \gg r(a^*)$$

- *Pareto optimal* (or *efficient* or *non inferior* or *non improvable*) for G if there does not exist $a \in A$ such that

$$r(a) > r(a^*).$$

In this case $r(a^*) \in \mathbb{R}^n$ is called a *Pareto optimal value*, and the set of all Pareto optimal values is called the *Pareto frontier*.

Definition 2.3. An strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ is called a *Nash equilibrium* (NE) for the game G if, for each player $i \in N$,

$$r_i(a_{-i}^*, a_i^*) \geq r_i(a_{-i}^*, a_i), \quad \forall a_i \in A_i$$

For convenience we give the following definition.

Definition 2.4. An strategy profile $a^* = (a_1^*, \ldots, a_n^*)$ is said to be a *Pareto-optimal Nash equilibrium* (*Pareto efficient Nash equilibrium*) if it is both a Nash equilibrium and a Pareto optimum.

From Definition 2.3, one can see that the difficulty for obtaining Nash equilibria lies in the fact that one has to solve *n coupled optimization problems*.

We now analyze two approaches: the first using potential games, which has been the most classic approach to studying the Pareto-optimal Nash equilibria, and the second presenting discontinuous games, a different approach to the traditional, which presents us with new techniques for studying the Pareto-optimal Nash equilibria.

2.1 Potential games

In this subsection $G = (N, A, r)$ denotes a game in normal form, as in Definition 2.1. We now start whit formal definitions for different classes of potential games.

Definition 2.5. Let $G = (N, A, r)$ and $P : A \to \mathbb{R}$ be a certain function. The game G is called

a) an *exact potential game* if, for every $i \in N$ and $a \in A$,

$$r_i(a_{-i}, x_i) - r_i(a_{-i}, x_i') = P(a_{-i}, x_i) - P(a_{-i}, x_i') \quad \forall x_i, x_i' \in A_i;$$

b) a *weighted potential game* if there exist positive numbers $w_1, \ldots, w_n$ such that, for every $i \in N$ and $a \in A$,

$$r_i(a_{-i}, x_i) - r_i(a_{-i}, x_i') = w_i \left[P(a_{-i}, x_i) - P(a_{-i}, x_i')\right] \quad \forall x_i, x_i' \in A_i;$$

c) a *best-response potential game* if, for every $i \in N$ and $a \in A$,

$$\arg\max_{x_i \in A_i} r_i(a_{-i}, x_i) = \arg\max_{x_i \in A_i} P(a_{-i}, x_i).$$

The function P is respectively called *exact potential, weighted potential* and *best-response potential*.

Figure 2: Some classes of potential games

The potential games help us to have an optimization approach, which is easier to solve and give conditions to have a Pareto-optimal Nash equilibrium.

In the following subsections we introduce ways to characterize Pareto-optimal Nash equilibria in an environment of best-response potential games, weighted potential games and exact potential games.

2.1.1 Best-response potential game

The next theorem illustrates one of the main advantages of identifying best-response potential games; namely, if we have a potential function P for such a game, then finding a Nash equilibrium can be done by optimizing P.

Theorem 2.6 (See [34]). *Let $G = (N, A, r)$ be a best-response potential game with potential P.*

a) A strategy $a^* \in A$ is a Nash equilibrium for G if and only if a^* is a Nash equilibrium for the coordination game (or team problem)

$$(N, A, (P, \ldots, P)). \tag{2.3}$$

b) If a^* is a global maximizer for P, then a^* is a Nash equilibrium for G.

The converse of Theorem 2.6(b) fails. (See Remark 2.11.)

If the game (2.3) is *finite*, then P attains its maximum value. This yield the following corollary.

Corollary 2.7. *Let $G = (N, A, r)$ be a finite best-response potential game. Then there exist a least one pure Nash equilibrium for G.*

The following definition will allow us to characterize another class of games.

Definition 2.8 (See [31]). Let G be a game in which, for each $i \in N$, A_i is an interval of real numbers, and r_i is of class $\mathcal{C}^1$. A differentiable function $P : A \to \mathbb{R}$ is said to be a *fictitious-objective function* for G if, for every $i \in N$,

$$\frac{\partial P}{\partial a_i}(a) = \frac{\partial r_i}{\partial a_i}(a) \quad \forall\, a \in A.$$

A fictitious-objective function is a best-response potential; see Definition 2.5(c).

The following theorem characterizes the class of fictitious-objective functions.

Theorem 2.9 (See [31]). *Let G be a game as in Definition 2.8. Then the following statements are equivalents:*

(a) A function $P : A \to \mathbb{R}$ is a fictitious-objective function for the game G.

(b) For every $i \in N$, there is a function $f_i : \Pi_{j \neq i} A_j \to \mathbb{R}$ such that

$$r_i(a) = P(a) + f_i(a_{-i}) \quad \forall\, a \in A. \tag{2.4}$$

Proposition 2.10. *Let G be a game with fictitious-objective function P such that, for each $i \in N$, $f_i \equiv c_i$ with $c_i \in \mathbb{R}$. If a strategy profile a^* maximize P, then a^* is a NE (let us remember that NE means Nash equilibrium according to Definition 2.3) that is also Pareto optimal for the game G.*

Proof. Let a^* be a maximizer of P, then, for every $i \in N$ and $a \in A$,

$$P(a^*) = r_i(a^*) - c_i \geq P(a) = r_i(a) - c_i.$$

This implies that

$$r_i(a^*) \geq r_i(a). \tag{2.5}$$

In particular,

$$r_i(a^*) \geq r_i(a^*_{-i}, a_i) \quad \forall\, i \in N, \forall a_i \in A_i. \tag{2.6}$$

Thus, (2.5)-(2.6) implies that a^* is a NE that is also Pareto optimal for the game G. □

Remark 2.11. If G is a best-response potential game with potential P, then not all the Nash equilibria of G are maximizers of P. For example, Mallozzi [18] considers a game with two players, action sets $A_1 = A_2 = [0, 1]$, and payoffs $r_i(a_1, a_2) = a_1 a_2 - 1$, $i = 1, 2$. Then the set of Nash equilibria is $\{(0, 0), (1, 1)\}$, but only $(1, 1)$ maximizes the potential $P(a_1, a_2) = a_1 a_2$.

Lemma 2.12. *Suppose $r_i(a) = g_i(a_i)$ for all $i \in N$. Then $P(a) = g_1(a_1) + \cdots + g_n(a_n)$ satisfies Definition 2.8. Hence, if $a^* \in A$ maximizes P, then a^* is a NE for G. If, in addition, the A_i and g_i are all convex, then a^* is also Pareto optimal.*

2.1.2 Exact and weighted potential games

The following results illustrate how to identify Pareto-optimal Nash equilibria in exact potential and weighted potential games.

Proposition 2.13. *Let G be a game with weighted potential function P. If $a^* \in A$ maximizes P, then a^* is a NE for the game G.*

Proof. Let a^* be a maximizer of P, then, for every $a \in A$,

$$P(a^*) \geq P(a).$$

This implies that

$$P(a^*) \geq P(a^*_{-i}, a_i) \quad \forall\, i \in N, \forall\, a_i \in A_i.$$

Moreover, for every $i \in N$ and $a_i \in A_i$,

$$r_i(a^*) - r_i(a^*_{-i}, a_i) = w_i \left[P(a^*) - P(a^*_{-i}, a_i)\right] \geq 0,$$

then, for every $i \in N$ and $a_i \in A_i$,

$$r_i(a^*) \geq r_i(a^*_{-i}, a_i).$$

Thus, a^* is a NE for the game G. □

Proposition 2.14. *Let G be a weighted potential game, such that*

$$r_i(a) = w_i P(a) \;\; \text{for all} \quad i \in N \quad \text{and} \quad a \in A.$$

If a strategy profile a^* maximizes P, then a^* is a NE that is also Pareto optimal for the game G.

Proof. Let a^* be a maximizer of P. Then, by Proposition 2.13, a^* is a NE for the game G.

Suppose that a^* is not a Pareto-optimal, then, there exist $a \in A$ such that

$$r(a) > r(a^*).$$

This implies that

$$\sum_{i=1}^{n} r_i(a) = P(a) \sum_{i=1}^{n} w_i > \sum_{i=1}^{n} r_i(a^*) = P(a^*) \sum_{i=1}^{n} w_i,$$

then,

$$P(a) > P(a^*).$$

This is a contradiction, since a^* is maximizer of P. Therefore, a^* is also Pareto-optimal for the game G. □

Theorem 2.15 (See [25])**.** *Let $G = (N, A, r)$ be a game as in (2.1) such that A_i is convex in some $\mathbb{R}^{k_i}$ and r_i is bounded for each $i \in N$. Suppose that there is a $\mathcal{C}^1$ exact potential P for G. If P is concave, then the set of maximizers of P equals the set of Nash equilibria of G.*

2.2 Multi-portfolio optimization: a potential game approach

In this subsection, we present a multi-portfolio problem which is studied with exact potential games.

We now consider a problem of multi-portfolio optimization. Yang *et al.* [35] studied this problem as a generalization of Markowitz's [19] mean-variance problem, in which it is justified that the portfolio should be determined based on the trade-off between maximizing the expected return and minimizing the risk.

Let $\mathbf{w} \in \mathbb{R}^N$ be the vector of weights defining the proportion of wealth allocated among a total number of N-assets, and assume that the return of the i-th asset over a single-period investment horizon is modeled as a random variable denoted by r_i. Let $\boldsymbol{\mu} = (\mu_i)_{i=1}^N$ be the vector of expected returns where $\mu_i = \mathbb{E}[r_i]$, and let $\mathbf{R} = (\mathbf{R}_{ij})_{i,j}$ be the positive definite covariance matrix where $\mathbf{R}_{ij} = \mathbb{E}[(r_i - \mu_i)(r_j - \mu_j)]$. In Markowitz's mean-variance portfolio optimization framework, the expected return of the portfolio is $\boldsymbol{\mu}^T\mathbf{w}$ while the risk of the portfolio is $\mathbf{w}^T\mathbf{R}\mathbf{w}$. Then, considering the trade-off between the expected return and the risk, the optimal portfolio is the solution to the following problem:

$$\underset{\mathbf{w}\in\mathcal{W}}{\text{maximize}} \quad \boldsymbol{\mu}^T\mathbf{w} - \frac{1}{2}\rho\mathbf{w}^T\mathbf{R}\mathbf{w}, \tag{2.7}$$

where ρ is a given positive constant specifying the investor's level of risk aversion, and $\mathcal{W}$ is the set of feasible portfolios specified by various trading constraints. This formulation reveals that among the portfolios that have the same risk, we should choose the one with largest expected return.

Let $TC(\cdot)$ be the market impact cost function. The market impact cost associated with rebalancing from the current position $\mathbf{w}^0$ to a new position $\mathbf{w}$ is given by $TC(\mathbf{w} - \mathbf{w}^0)$. Then the optimization problem (2.7) should be revised as

$$\underset{\mathbf{w}\in\mathcal{W}}{\text{maximize}} \quad \boldsymbol{\mu}^T\mathbf{w} - \frac{1}{2}\rho\mathbf{w}^T\mathbf{R}\mathbf{w} - TC(\mathbf{w} - \mathbf{w}^0). \tag{2.8}$$

Mathematically, suppose there are M accounts and denote by $\mathbf{w}_m$ the portfolio vector of the m-th account. Then, the market impact cost of account m is $TC(\sum_{m=1}^M(\mathbf{w}_m - \mathbf{w}_m^0))$ rather than $TC(\mathbf{w} - \mathbf{w}^0)$.

We analyze the multi-portfolio optimization problem under the mean-variance framework (2.7)-(2.8). Specifically, the market impact cost function

$TC(\mathbf{w})$ is modeled as

$$TC(\mathbf{w}) = \langle [\mathbf{w}]^+, \mathbf{c}^+(\mathbf{w})\rangle + \langle [\mathbf{w}]^-, \mathbf{c}^-(\mathbf{w})\rangle \tag{2.9}$$

where $[\mathbf{w}]^+$, $[\mathbf{w}]^-$ ($[\mathbf{w}]^+ = \max(\mathbf{w}, \mathbf{0})$ and $[\mathbf{w}]^- = \max(-\mathbf{w}, \mathbf{0})$) are the buy and sell vector's, and $\mathbf{c}^+(\mathbf{w})$, $\mathbf{c}^-(\mathbf{w})$ are the market impact price functions for buy and sell giving the cost per unit for each asset.

For the market impact price function $\mathbf{c}^+(\mathbf{w})$, we assume that it is separable among assets, i.e., $\mathbf{c}^+(\mathbf{w}) = (c_i^+(w_i))_{i=1}^N$, and $c_i^+(w_i) = \mathbf{\Omega}_{ii}^+([w_i]^+)^p$ with $p \in [0.5, 1]$, where $\mathbf{\Omega}^+$ is a positive diagonal matrix representing market impact coefficients; the modeling is similar for sells. We assume the usual choice $p = 1$.

In the presence of multiple accounts, the market impact price function depends on the aggregate trade from all accounts, i.e.,

$$\mathbf{c}^+(\mathbf{w}_1, \ldots, \mathbf{w}_M) = \mathbf{\Omega}^+ \left(\sum_{m=1}^{M} [\mathbf{w}_m]^+ \right),$$
$$\mathbf{c}^-(\mathbf{w}_1, \ldots, \mathbf{w}_M) = \mathbf{\Omega}^- \left(\sum_{m=1}^{M} [\mathbf{w}_m]^- \right),$$

and the market impact cost for each account is proportional to their individual trade amount. Under this consideration, the utility function for account m is

$$\begin{aligned} u_m(\mathbf{w}_{-m}, \mathbf{w}_m) = \boldsymbol{\mu}^T \mathbf{w}_m - \frac{1}{2}\rho_m \mathbf{w}_m^T \mathbf{R} \mathbf{w}_m \\ - \frac{1}{2}\langle \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^+, \mathbf{\Omega}^+ \sum_{j=1}^{M} \left[\mathbf{w}_j - \mathbf{w}_j^0\right]^+ \rangle \\ - \frac{1}{2}\langle \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^-, \mathbf{\Omega}^- \sum_{j=1}^{M} \left[\mathbf{w}_j - \mathbf{w}_j^0\right]^- \rangle. \end{aligned} \tag{2.10}$$

Since the mean-variance framework focuses on a single-period investment, we assume that $\boldsymbol{\mu}$, $\mathbf{R}$, $\boldsymbol{\rho}$, $\mathbf{\Omega}^{+(-)}$ are fixed.

As in (2.7)-(2.8), the feasible trading strategy $\mathbf{w}$ is in a closed and convex constraint set $\mathcal{W}$. In general, these portfolio constraints may consist of two categories: individual constraints and global constraints.

1. *Individual Constraints*:

- Holding constraint: To reduce risk, a portfolio should not exhibit large concentrations in any specific asset. Minimal and maximal holdings can be controlled by constraints of this form: $\boldsymbol{l}_m \leq \mathbf{w}_m \leq \boldsymbol{u}_m$.
- Long-only constraint (no short-selling constraint): In the process of short-selling, we sell an asset that we borrowed from someone else, and repay our loan after buying the asset back at a later date. Short-selling is profitable if the asset price declines. Because of the risk nature, it is prohibited or purposely avoided sometimes. Mathematically, the long-only constraint corresponds to $\mathbf{w}_m \geq \mathbf{0}$ and it is a special case of the holding constraint where $\boldsymbol{l}_m = \mathbf{0}$ and $\boldsymbol{u}_m = \infty$.
- Budget constraint: $\sum_{i=1}^{N} w_{m,i} \leq b_m$.

2. *Global Constraints*: In some circumstances, there may exist regulations on *all* accounts, and these regulations can be modeled as global (coupling) constraints.

   - Turnover or transaction size constraints over multiple accounts, which are used to limit the average daily trade volume associated with the k-th asset:

$$\sum_{m=1}^{M} |w_{m,i} - w_{m,i}^0| \leq D_i,\ i = 1, \ldots, N \tag{2.11}$$

   - Limitations on the amount invested over groups of assets with related characteristics:

$$\sum_{m=1}^{M} \sum_{i \in \mathcal{J}_l} |w_{m,i} - w_{m,i}^0| \leq U_l,\ l = 1, \ldots, L. \tag{2.12}$$

   Where $\mathcal{J}_l \subset \{1, \ldots, N\}$.

We formulate the optimization as a Nash equilibrium problem (NEP): each account m competes against the others by choosing a strategy that maximizes his own utility function. Stated in mathematical terms, given the strategies of other accounts $\mathbf{w}_{-m}$, account m solves the following optimization problem:

$$\underset{\mathbf{w}_m \in \mathcal{W}_m}{\text{maximize}} \quad u_m(\mathbf{w}_{-m}, \mathbf{w}_m) \quad \forall\, m, \tag{2.13}$$

where u_m is defined in (2.10), and $\mathcal{W}_m$ is a non-empty, closed, and convex set specified by the individual portfolio constraints. Since each accounts strategy set is independent of the rival accounts, the joint strategy set of all accounts has a Cartesian structure, i.e., $\mathcal{W}_1 \times \cdots \times \mathcal{W}_M$.

We will now reinterpret the problem as an exact potential game. As in Subsection 2.1.2 a key rule in the study of potential games is played by the following standard optimization problem, where the objective function is just the exact potential function P:

$$\underset{\mathbf{w}\in\mathcal{W}_1\times\cdots\times\mathcal{W}_M}{\text{maximize}} \quad P(\mathbf{w}). \tag{2.14}$$

The relationship between the NEP (2.13) and optimization problem (2.14) is given in the following lemma.

Lemma 2.16 (See [28])**.** *Suppose that the NEP* (2.13) *is a potential game with a concave potential function P. If $\mathbf{w}^*$ is a Pareto optimal solution of* (2.14)*, then it is a NE of the NEP* (2.13)*. Conversely, if P is continuously differentiable and $\mathbf{w}_{ne}$ is a NE of the NEP* (2.13)*, then $\mathbf{w}_{ne}$ is a Pareto optimal solution of* (2.14)*.*

Notation. $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{J}_n$ is a $n \times n$ matrix with all entries 1 and $\text{diag}(\boldsymbol{\rho})$ is a diagonal matrix with diagonal vector $\boldsymbol{\rho}$.

If $\mathbf{X}$ is an $m \times n$ matrix and $\mathbf{Y}$ is a $p \times q$ matrix, then the Kronecker product $\mathbf{X} \otimes \mathbf{Y}$ is the $mp \times nq$ block matrix:

$$\mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} X_{11}\mathbf{Y} & \dots & X_{1n}\mathbf{Y} \\ \vdots & \ddots & \vdots \\ X_{m1}\mathbf{Y} & \dots & X_{mn}\mathbf{Y} \end{bmatrix}.$$

2.2.1 Reformulation of the problem

Let us rewrite (2.13) in a more convenient form. In fact, the projections in the utility functions $[\cdot]^+$ and $[\cdot]^-$ are generally difficult to handle because of the non convexity and non differentiability they bring about. To cope with these difficulties, we introduce new nonnegative variables $\widetilde{\mathbf{w}}_m = [\widetilde{\mathbf{w}}_m^+; \widetilde{\mathbf{w}}_m^-]$ and make the following variable substitutions:

$$\begin{aligned} \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^+ &= \widetilde{\mathbf{w}}_m^+, \\ \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^- &= \widetilde{\mathbf{w}}_m^-, \\ \mathbf{w}_m - \mathbf{w}_m^0 &= \widetilde{\mathbf{w}}_m^+ - \widetilde{\mathbf{w}}_m^- \quad \forall\, m. \end{aligned}$$

Then the utility function (2.10) in terms of the new variable $\widetilde{\mathbf{w}}$ is:

$$\tilde{u}_m\left(\widetilde{\mathbf{w}}_{-m}, \widetilde{\mathbf{w}}_m\right) = \widetilde{\boldsymbol{\mu}}_m \widetilde{\mathbf{w}}_m - \frac{1}{2}\rho_m \widetilde{\mathbf{w}}_m^T \widetilde{\mathbf{R}} \widetilde{\mathbf{w}}_m - \frac{1}{2}\widetilde{\mathbf{w}}_m^T \widetilde{\boldsymbol{\Omega}} \left(\sum_{j=1}^{M} \widetilde{\mathbf{w}}_j\right), \tag{2.15}$$

where

$$\widetilde{\boldsymbol{\mu}}_m = \left[\boldsymbol{\mu} - \rho_m \mathbf{R}\mathbf{w}_m^0; -\boldsymbol{\mu} + \rho_m \mathbf{R}\mathbf{w}_m^0\right],$$
$$\widetilde{\boldsymbol{\Omega}} = \left[\boldsymbol{\Omega}^+ \quad \boldsymbol{\Omega}^-\right],$$
$$\widetilde{\mathbf{R}} = \begin{bmatrix} \mathbf{R} & -\mathbf{R} & 0 \\ 0 & -\mathbf{R} & \mathbf{R} \end{bmatrix}.$$

With this change of variable, the new constraint set is

$$\widetilde{\mathcal{W}}_m = \{\widetilde{\mathbf{w}}_m : [\mathbf{I} - \mathbf{J}]\, \widetilde{\mathbf{w}} + \mathbf{w}_m^0 \in \mathcal{W}, \widetilde{\mathbf{w}} \geq \mathbf{0}\},$$

which is convex in $\widetilde{\mathbf{w}}_m$.

Note that u_m is not necessarily equivalent to $\widetilde{u}_m$ because $[\mathbf{w}_m - \mathbf{w}_m^0]^+$ is by definition orthogonal to $[\mathbf{w}_m - \mathbf{w}_m^0]^-$. However such an orthogonality is not imposed between $\widetilde{\mathbf{w}}_m^+$ and $\widetilde{\mathbf{w}}_m^-$; instead, $\widetilde{\mathbf{w}}_m^+$ and $\widetilde{\mathbf{w}}_m^-$ are only assumed to be nonnegative. Nevertheless, in the following lemma we prove that this orthogonality property is automatically satisfied at the optimal $\widetilde{\mathbf{w}}_m^+$ and $\widetilde{\mathbf{w}}_m^-$.

Lemma 2.17 (See [35])**.** *In the optimization problem* (2.13) *of account m, given any arbitrary but fixed feasible $(\widetilde{\mathbf{w}}_r)_{r\neq m}$, the optimal buy vector $\widetilde{\mathbf{w}}_m^+$ and optimal sell vector $\widetilde{\mathbf{w}}_m^-$ are orthogonal.*

Now, we consider the more general scenario in which there is also coupling in each accounts strategy set. For example, one accounts trading volume on a particular asset can be limited by other accounts because of the average daily trading volume (ADV) of the assets in the common investment universe.

The coupling in each accounts strategy set can be modeled as global constraints over all accounts. This can be modeled as a NEP, in which account m solves the following problem in terms of the new variable $\widetilde{\mathbf{w}}$:

$$\underset{\widetilde{\mathbf{w}}_m \in \widetilde{\mathcal{W}}_m,\, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) \leq \mathbf{0}}{\text{maximize}} \quad \widetilde{u}_m(\widetilde{\mathbf{w}}_{-m}, \widetilde{\mathbf{w}}_m) \quad \forall\, m, \tag{2.16}$$

where $\widetilde{\mathbf{g}}$ is the global constraint:

$$\widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) = \sum_{m=1}^{M} \widetilde{\mathbf{g}}_m(\widetilde{\mathbf{w}}_m) - \begin{bmatrix} (D_i)_{i=1}^N \\ (U_l)_{l=1}^L \end{bmatrix},$$

with

$$\widetilde{\mathbf{g}}_m(\widetilde{\mathbf{w}}_m) = \begin{bmatrix} \left(\widetilde{w}^+_{m,i} + \widetilde{w}^-_{m,i}\right)_{i=1}^N \\ \left(\sum_{j\in\mathcal{J}_l}\left(\widetilde{w}^+_{m,j} + \widetilde{w}^-_{m,j}\right) - U_l\right)_{l=1}^L \end{bmatrix}.$$

It is easy to see from Definition 2.5 that the definition of potential functions can readily be extended to the NEP and if solution maximizes the potential function, then is also a NE of the NEP. The NE of the NEP (2.16) does not necessarily maximize the potential function over the joint strategy set:

$$\underset{\widetilde{\mathbf{w}}\in\widetilde{\mathcal{W}}_1\times\cdots\times\widetilde{\mathcal{W}}_M,\, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})\leq \mathbf{0}}{\text{maximize}} \quad P_{ne}(\widetilde{\mathbf{w}}) = \widetilde{\boldsymbol{\mu}}^T\widetilde{\mathbf{w}} - \frac{1}{2}\widetilde{\mathbf{w}}^T\mathbf{M}_{ne}\widetilde{\mathbf{w}}, \tag{2.17}$$

where

$$\mathbf{M}_{ne} = \text{diag}(\boldsymbol{\rho}) \otimes \widetilde{\mathbf{R}} + \frac{1}{2}(\mathbf{I}_M + \mathbf{J}_M) \otimes \widetilde{\boldsymbol{\Omega}}.$$

This is because the Cartesian structure in the joint strategy set of all accounts is destroyed by the global constraints.

Yang *et al.*, inspired by [27], use a well-known result in convex analysis to derive a relationship between a NE of the NEP (2.16) and a Pareto optimal solution of (2.17): for a convex optimization problem with strong duality, the pair consisting of a primal optimal solution and a dual optimal solution is a saddle point of the Lagrangian [26]. Specifically, they assume that some constraint qualifications such as Slater's condition are satisfied for (2.16) and (2.17). Then let $\widetilde{\mathbf{w}}_{ne} = (\widetilde{\mathbf{w}}^*_m)_{m=1}^M$ be a NE of the NEP (2.16), there exists $(\boldsymbol{\lambda}^*_m)_{m=1}^M \geq \mathbf{0}$ such that

$$\begin{aligned} &\widetilde{\mathbf{w}}^*_m = \underset{\widetilde{\mathbf{w}}_m\in\widetilde{\mathcal{W}}_m}{\text{argmax}} \quad \widetilde{u}_m(\widetilde{\mathbf{w}}^*_{-m}, \widetilde{\mathbf{w}}_m) - \langle\boldsymbol{\lambda}^*_m, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}^*_{-m}, \widetilde{\mathbf{w}}_m)\rangle, \\ &\mathbf{0} \leq \boldsymbol{\lambda}^*_m, \quad \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}^*_{-m}, \widetilde{\mathbf{w}}_m) \leq \mathbf{0}, \quad \boldsymbol{\lambda}^*_m \perp \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}^*_{-m}, \widetilde{\mathbf{w}}_m) \quad \forall\, m, \end{aligned} \tag{2.18}$$

where $\mathbf{a} \perp \mathbf{b}$ means $\mathbf{a}^T\mathbf{b} = 0$. Similarly, let $\widetilde{\mathbf{w}}^*$ be a Pareto optimal solution of (2.17). Then there exists $\boldsymbol{\xi}^* \geq \mathbf{0}$ such that

$$\begin{aligned} &\widetilde{\mathbf{w}}^* = \underset{\widetilde{\mathbf{w}}\in\widetilde{\mathcal{W}}_1\times\cdots\times\widetilde{\mathcal{W}}_M}{\text{argmax}} \quad P_{ne}(\widetilde{\mathbf{w}}) - \langle\boldsymbol{\xi}^*, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})\rangle, \\ &\mathbf{0} \leq \boldsymbol{\xi}^*, \quad \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) \leq \mathbf{0}, \quad \boldsymbol{\xi}^* \perp \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}^*). \end{aligned} \tag{2.19}$$

A comparison of (2.18) and (2.19) enables us to give a precise connection between the NE of a NEP (2.16) and the Pareto optimal of its potential game formulation (2.17), as summarized in the following proposition.

Proposition 2.18 (See [35]). *Suppose that $\widetilde{\mathbf{w}}^*$ is a Pareto optimal solution of* (2.17) *and satisfies* (2.19). *Then $\widetilde{\mathbf{w}}^*$ is a NE of the NEP* (2.16), *and* (2.18) *holds with $\boldsymbol{\lambda}_1^* = \boldsymbol{\lambda}_2^* = \cdots = \boldsymbol{\lambda}_N^* = \boldsymbol{\xi}^*$. Conversely, suppose that $\widetilde{\mathbf{w}}_{ne}$ is a NE of the NEP* (2.16), *and* (2.18) *holds with $\boldsymbol{\lambda}_1^* = \boldsymbol{\lambda}_2^* = \cdots = \boldsymbol{\lambda}_N^* = \boldsymbol{\xi}^*$, then $\widetilde{\mathbf{w}}_{ne}$ is a Pareto optimal solution of* (2.17) *and satisfies* (2.19).

To summarize, a NE of the NEP (2.16) is generally not a Pareto optimal of (2.17), unless at the NE, the dual variables associated with the global constraints for all accounts are identical. The NE of the NEP (2.16) that is also the Pareto optimal of (2.19) is called a *Variational Equilibrium* (VE). From now on, we mainly focus on the VE of the NEP (2.16), whose (existence and) uniqueness comes readily from the strong convexity of (2.17).

Corollary 2.19 (See [35]). *The NEP* (2.16) *has a unique VE.*

Now, we consider a generic NEP where, for each m, the account m solves the following convex optimization problem

$$\underset{\mathbf{w}_m \in \mathcal{W}_m,\, \mathbf{g}(\mathbf{w}) \leq \mathbf{0}}{\text{maximize}} \quad u_m(\mathbf{w}_{-m}, \mathbf{w}_m), \tag{2.20}$$

where $u_m(\mathbf{w}_{-m}, \cdot)$ is concave on $\mathcal{W}_m$, $\mathbf{g}$ is convex on $\mathcal{W}_1 \times \cdots \times \mathcal{W}_M$, and $\mathcal{W}_m$ is closed and convex. Suppose the NEP (2.20) has a differentiable concave potential function P while some constraint qualifications such as Slater's condition are satisfied. We also introduce a new NEP

$$\underset{\mathbf{w}_m \in \mathcal{W}_m}{\text{maximize}} \quad u(\mathbf{w}_{-m}, \mathbf{w}_m) - \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{w}_{-m}, \mathbf{w}_m) \quad \forall\, m, \tag{2.21}$$

and denote its NE for a given $\boldsymbol{\lambda}$ as $\mathbf{w}_{ne}(\boldsymbol{\lambda})$.

The relationship between (2.20) and the NEP (2.21) is given in the following theorem.

Theorem 2.20 (See [35]). *In the setting above, $\mathbf{w}_{ve}$ is a VE of the NEP* (2.20) *if and only if $\mathbf{w}_{ve} = \mathbf{w}_{ne}(\boldsymbol{\lambda}^*)$, where $(\mathbf{w}_{ne}(\boldsymbol{\lambda}^*), \boldsymbol{\lambda}^*)$ satisfies*

$$\mathbf{0} \leq \boldsymbol{\lambda}^*, \quad \mathbf{g}(\mathbf{w}_{ne}(\boldsymbol{\lambda}^*)) \leq \mathbf{0}, \quad \boldsymbol{\lambda}^* \perp \mathbf{g}(\mathbf{w}_{ne}(\boldsymbol{\lambda}^*)). \tag{2.22}$$

Proof. A variable is a VE of the NEP (2.20) if and only if it solves the following optimization problem:

$$\underset{\mathbf{w} \in \mathcal{W}_\mathbf{1} \times \cdots \times \mathcal{W}_\mathbf{M},\, \mathbf{g}(\mathbf{w}) \leq \mathbf{0}}{\text{maximize}} \quad P(\mathbf{w}). \tag{2.23}$$

Since (2.23) is a convex optimization problem, the Pareto optimal solution of (2.23) can be equally achieved from its dual problem, provided Slater's condition is satisfied [5]:

$$\underset{\boldsymbol{\lambda} \geq \mathbf{0}}{\text{minimize}} \quad Q(\boldsymbol{\lambda}), \tag{2.24}$$

where $Q(\boldsymbol{\lambda}) = \max_{\mathbf{w} \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_N} P(\mathbf{w}) - \boldsymbol{\lambda}^* \mathbf{g}(\mathbf{w})$ and $\boldsymbol{\lambda}$ is the Lagrange multiplier associated with $\mathbf{g}(\mathbf{w}) \leq \mathbf{0}$.

For a fixed $\boldsymbol{\lambda}$, the inner maximization problem in (2.24) is a potential game equivalent to the following NEP:

$$\underset{\mathbf{w}_n \in \mathcal{W}_m}{\text{maximize}} \quad u_m(\mathbf{w}_{-m}, \mathbf{w}_m) - \boldsymbol{\lambda}^* \mathbf{g}(\mathbf{w}) \quad \forall\, m. \tag{2.25}$$

Since $(\mathbf{w}^*, \boldsymbol{\lambda}^*)$ is a saddle point of the minimax problem (2.24), $\mathbf{w}^*$ can be obtained by solving (2.25) with $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ while $(\mathbf{w}^*, \boldsymbol{\lambda}^*)$ are primal feasible, dual feasible and satisfy the complementary slackness condition. □

2.3 Discontinuous games

In this subsection, we give conditions to characterize another class of games with Pareto-optimal Nash equilibria. Our presentation is based on the results of Scalzo and Parvulescu [29] and Nessah [23]. The study of this topic can be traced back to Aumann [2] and his concept of strong Nash equilibrium (SNE). Later on, Ichiishi [16] introduced the notation of social coalition equilibrium and proved its existence. Scalzo [29] provided a theorem for the existence of a weakly Pareto-optimal Nash equilibrium in discontinuous games based on an aggregate function. Scalzo [30] investigates a class of discontinuous games where the Pareto-optimal Nash equilibria are stable with respect to perturbations of the characteristics of players. Nessah *et al.* [22] proved the existence of a Pareto efficient strong Berge equilibrium (strong Berge equilibrium was introduced by Berge [3]) which is also a SNE.

This subsection gives a characterization and existence of pure Pareto-optimal Nash equilibria in discontinuous games, For this purpose, we proceed in two steps. First, we give a characterization of the pure Pareto-optimal Nash equilibria in games satisfying the usual conditions of convexity, compactness and continuity. However, for the sufficiency conditions, the concept of *non-stationary* substitutes the usual assumption of strict (quasi) concavity. Second, we give a set of sufficient conditions for the existence of such equilibria. In this respect, *U-dominance* and *individual dominance* are introduced. Under strict quasiconcavity of the payoff function and its weakened continuity, we prove two sufficient conditions.

We begin with the following definitions.

Definition 2.21. Let E be a vector space and $Z \subset E$ a convex set.

- A function $f : Z \to \mathbb{R}$ is *quasiconcave* on Z iff, any z_1, z_2 in Z and for any $\theta \in [0,1]$,

$$\min\{f(z_1), f(z_2)\} \leq f(\theta z_1 + (1-\theta)z_2).$$

- A function $f : Z \to \mathbb{R}$ is *strictly-quasiconcave* on Z iff, for any z_1, z_2 in Z with $z_1 \neq z_2$ and for any $\theta \in (0,1)$, we have

$$\min\{f(z_1), f(z_2)\} < f(\theta z_1 + (1-\theta)z_2).$$

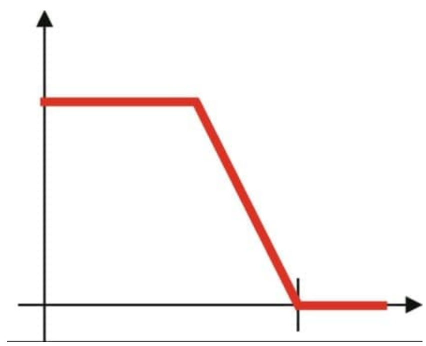

Figure 3: Function that are quasiconcave but not concave

This definition introduces a larger class of admissible payoff functions, which may be discontinuous functions, to have a Pareto-optimal Nash equilibria.

Definition 2.22. The game G is called *non-stationary* if for each $a, a' \in A$, with $r(a) \neq r(a')$, then there exists $\hat{a} \in A$ such that

$$\min\left(r_i(a), r_i(a')\right) < r_i(\hat{a}) \quad \text{for each} \quad i \in N.$$

Now, we prove existence results of pure Pareto-optimal Nash equilibria in discontinuous games. We recall the concepts of transfer continuity and transfer quasiconcavity of a function, which are key elements to or main first result.

Let Δ be the $n-1$-dimensional simplex defined as

$$\Delta = \{\lambda \in \mathbb{R}^n \,\text{such that}\, \lambda_i \geq 0,\ \forall i \in N \,\text{and} \sum_{j \in N} \lambda_j = 1\}.$$

Consider the function $\Gamma : (A \times \Delta) \times (A \times A) \to \mathbb{R}$ defined by

$$\Gamma((a, \lambda), (a', a'')) = \sum_{i \in N} r_i(a_{-i}, a'_i) + \sum_{j \in N} \lambda_j r_j(a'').$$

Ansari *et al.* [1] and Nessah and Tian [24] defined the following transfer of continuity and quasiconcavity.

Definition 2.23. The function Γ is said to be *transfer continuous* in (a, λ) if for each $(a', a'') \in A \times A$ where the following condition is satisfied

$$\Gamma((a, \lambda), (a', a'')) > \Gamma((a, \lambda), (a, a)),$$

then there exist a neighborhood $\mathcal{N}(a, \lambda)$ of (a, λ) and two elements $b, c \in A$ such that

$$\Gamma((\hat{a}, \hat{\lambda}), (b, c)) > \Gamma((\hat{a}, \hat{\lambda}), (\hat{a}, \hat{a})) \quad \text{for each} \quad (\hat{a}, \hat{\lambda}) \in \mathcal{N}(x, \lambda).$$

Definition 2.24 (See [24])**.** The function Γ is said to be *transfer quasiconcave in* (y, z) if, for any subset $\{(y^1, z^1), \ldots, (y^m, z^m)\} \subset (A \times A)$, there exist a corresponding finite subset $\{(x^1, \lambda^1), \ldots, (x^m, \lambda^m)\} \subset A \times \Delta$ such that, for any subset $L \subset \{1, \ldots, m\}$ and any $(x, \lambda) \in co\{(x^h, \lambda^h) | h \in L\}$ (*co* denotes the *convex hull*), we have

$$\min_{h \in L} \Gamma((x, \lambda), (y^h, z^h)) \leq \Gamma((x, \lambda), (x, x)).$$

Definition 2.25. Let X and Y be two topological spaces. A correspondence $C : X \to 2^Y$ be said to be *transfer closed-valued* on X if for every $x \in X$, $y \notin C(x)$ implies that there exist some $x' \in X$ such that $y \notin \overline{C(x')}$ (where $\overline{C(x')}$ is the *topological closure* of $C(x')$).

Definition 2.26. Let X be a topological space and let Z be a non-empty convex subset of E (where E be a Hausdorff topological vector space). A correspondence $C : X \to 2^Z$ is said to be *transfer FS-convex* on X if, for any finite subset $\{x_1, \ldots, x_p\} \subset X$, then

$$\text{co}\{x_1, \ldots, x_p\} \subset \cup_{j=1}^{p} C(x_j).$$

Remark. Note that transfer FS-convexity of C implies that every point $x \in X$ is a fixed point of $C(x)$, i.e., $x \in C(x)$.

The previous definitions provide the necessary conditions for the following theorem.

Theorem 2.27. *Assume that the game G is convex, compact and non-stationary. In addition, if Γ is transfer continuous in (x, λ) and it is transfer quasiconcave in (y, z), then the game G has a Pareto-optimal Nash equilibrium.*

Proof. Let consider the following correspondence $C : A \times A \to A \times \Delta$ defined by

$$C(y, z) = \{(x, \lambda) \in A \times \Delta \quad \text{such that} \quad \Gamma((x, \lambda), (y, z)) \leq \Gamma((x, \lambda), (x, x))\}.$$

Since Γ is transfer continuous in (x, λ) and transfer quasiconcave in (y, z), then the correspondence C is transfer closed-valued and transfer FS-convex on $A \times A$. Then by Lemma 1 of Tian [32], the set $\cap_{(y,z) \in A \times A} C(y, z)$ is nonempty and compact. Let $(\bar{x}, \bar{\lambda}) \in A \times \Delta$ be an element in $\cap_{(y,z) \in A \times A} C(y, z)$. Then for each $(y, z) \in A \times A$, we have

$$\Gamma((\bar{x}, \bar{\lambda}), (y, z)) \leq \Gamma((\bar{x}, \bar{\lambda}), (\bar{x}, \bar{x})). \tag{2.26}$$

First, if we fix $z = \bar{x}$ in (2.26), then we deduce that $\bar{x}$ is a Nash equilibrium of G. Second, if we fix $y = \bar{x}$ in (2.26), then obtain

$$\sum_{j \in N} \bar{\lambda}_j r_j(z) \leq \sum_{j \in N} \bar{\lambda} r_j(\bar{x}) \quad \text{for each} \quad z \in A. \tag{2.27}$$

If $\bar{x}$ is not maximal weakly Pareto efficient of G, there is a strategy $\bar{z} \in A$ such that for each $i \in N$, $r_i(\bar{z}) > r_i(\bar{x})$. Since $\lambda \in \Delta$ then

$$\begin{cases} \forall\, i \in N, & \bar{\lambda}_i \geq 0 \\ \exists\, h \in N, & \bar{\lambda}_h > 0. \end{cases}$$

Therefore,

$$\begin{cases} \forall\, i \in N, & \bar{\lambda}_i r_i(\tilde{z}) \geq \bar{\lambda}_i r_i(\bar{x}) \\ \exists\, h \in N, & \bar{\lambda}_h r_h(\tilde{z}) > \bar{\lambda}_h r_h(\bar{x}). \end{cases} \tag{2.28}$$

System (2.28) implies that $\sum_{j \in N} \bar{\lambda}_j r_j(\tilde{z}) > \sum_{j \in N} \bar{\lambda}_j r_j(\bar{x})$ which is in contradiction with (2.27) and then $\bar{x}$ is maximal weakly Pareto efficient of G. $\square$

Definition 2.28 (See [33]). A function $F : X \to \mathbb{R}$ is said to be *transfer weakly upper continuous* if the condition

$$F(x') > F(x)$$

implies that there exist a neighborhood N_x of x and $x'' \in X$ such that

$$F(x'') \geq F(z)$$

for any $z \in N_x$.

Scalzo [29] using the previous definition introduced by Tian and Zhou [33] showed the following lemma to characterize some Pareto-optimal Nash equilibria.

Lemma 2.29 (See [29]). *Assume that A is a compact and convex subset of a topological vector space and:*

(a) the function $F : x \mapsto F(x) = U(x, x)$ is transfer weakly upper continuous on A, where $U(x, y) = \sum_{i \in N} r_i(x_i, y_{-i})$,

(b) the function $U(\cdot, z)$ is strictly quasiconcave on A for any $z \in A$,

(c) is x is a maximizer of F and if $z \in A \backslash \{x\}$, then there exist at least a $\lambda \in (0, 1)$ such that

$$U(x, x) \geq U(\lambda x + (1 - \lambda)z, x).$$

Then, the game G has at least one Pareto-optimal Nash equilibrium.

Example 2.30. Considerer a concession game with two players and the unit square $A_1 = A_2 = [0, 1]$. For player $i = 1, 2$ and $a = (a_1, a_2) \in A = [0, 1]^2$, let the payoff functions of the players be given by

$$r_i(a_1, a_2) = \begin{cases} a_{-i} - a_i + 1 \quad \text{if} \quad a_i < a_{-i}, \\ 1 \quad \text{if} \quad a_i = a_{-i}, \\ a_{-i} + a_i - 1 \quad \text{if} \quad a_i > a_{-i}. \end{cases}$$

This game does not satisfy condition (iii) of Lemma 2.29 so this lemma cannot be applied. Indeed, the aggregate function is $U(x, y) = r_1(x_1, y_2) + r_2(y_1, x_2)$. Hence, we obtain

$$F(x) = \begin{cases} 2x_1 \quad \text{if} \quad x_1 > x_2, \\ 2 \quad \text{if} \quad x_1 = x_2, \\ 2x_2 \quad \text{if} \quad x_1 < x_2. \end{cases}$$

Let us consider $x = (1, 1)$ and $z = (0, 0)$. Clearly, x maximizes F and $x \neq z$. For each $\lambda \in (0, 1)$, we have $U(\lambda x+(1-\lambda)z, x)) = r_1(\lambda, 1)+r_2(1, \lambda) = 4-2\lambda$. Since $0 < \lambda < 1$ and $U(x, x) < U(\lambda x + (1 - \lambda)z, x)$ for each $\lambda \in (0, 1)$. The considered game is convex, compact and non-stationary. Let us prove that:

a) Γ is transfer continuous in (x, λ). Indeed, for each $x, y, z \in A$ and $\lambda \in \Delta$, if $\Gamma((x, \lambda), (y, z)) > \Gamma((x, \lambda), (x, x))$, then there exist $y', z' \in A$ with $z'_1 = z'_2$, $y'_1 = y'_2 = 0$ and neighborhood $\mathcal{N}(x, \lambda)$ such that $\Gamma((x', \lambda'), (y', z')) > \Gamma((x', \lambda'), (x', x'))$ for each $(x', \lambda') \in \mathcal{N}(x, \lambda)$.

b) Γ is also transfer quasiconcave in (y, z). Indeed, for any finite subset $\{(y^1, z^1), \ldots, (y^m, z^m)\} \subset A \times A$, there exist a corresponding finite subset $\{(x^1, \lambda^1), \ldots, (x^m, \lambda^m)\} \subset A \times \Delta$ defined by $x^1 = \cdots = x^m = (0, 0)$ and $\lambda^1 = \cdots = \lambda^m = (\frac{1}{2}, \frac{1}{2})$ such that, for any subset $L \subset \{1, \ldots, m\}$ and any $(x, \lambda) \in co\{(x^h, \lambda^h)|h \in L\}$, we have $\min_{h\in L} \Gamma((x, \lambda), (y^h, z^h)) \leq \Gamma((x, \lambda), (x, x))$

Then by Theorem 2.27, the game G has a Pareto-optimal Nash equilibrium.

Example 2.31. Let us consider the normal form game of an oligopoly market *á la Cournot* with n firms producing homogeneous goods $\mathcal{O} = (N, \{A_i, i \in N\}, \{\pi_i\}_{i\in N}$ where $N = \{1, \ldots, n\}$ is the set of firms. For each $i \in N$, $A_i = [0, \bar{a}_i]$ is the production set or ith firm i's capacity of production and $A = \Pi_{i\in N} A_i$ is the production set of the entire industry. Denote $C_i(x_i)$ the cost function of the ith firm and $P(\xi(x))$ the inverse demand function, where $\xi(x) = \sum_{i\in N} x_i$ is the total supply. The profit of the i-th firm is given by $\pi_i(x) = x_i P(\xi(x)) - C_i(x_i)$. Let us introduce the following function $\Pi : (A \times \Delta) \times (A \times A) \to \mathbb{R}$ defined by

$$\Pi((x, \lambda), (y, z)) = \sum_{i\in N} \pi_i(x_{-i}, y_i) + \sum_{j\in N} \lambda_j \pi_j(z).$$

Proposition 2.32 (See [23])**.** *If $\mathcal{O}$ in Example 2.31 above is non-stationary, Π is transfer continuous in (x, λ) and transfer quasiconcave in (y, z), then the oligopoly market $\mathcal{O}$ has a Pareto-optimal Nash equilibrium.*

In the following, we give some result of existence of Pareto-optimal Nash equilibria. We introduce new concepts. Let us consider the following aggregate function:

$$U(x, y) = \sum_{i\in N} r_i(x_i, y_{-i}).$$

The game (2.1) is said to be *U-dominated* if for each $x, y \in A$, we have, if

$$U(x, x) > U(y, y),$$

then there exist $x' \neq y$ such that

$$U(x', y) \geq U(y, y).$$

Then U-dominance of the game (2.1) means that for each $x, y \in A$, if x dominates y ($U(x, x) > U(y, y)$), then there is a strategy x' which is weakly preferred to y ($U(x', y) \geq U(y, y)$).

Definition 2.33. The game (2.1) is said *individually dominated* if for each $x, y \in A$, the condition $U(x, x) > U(y, y)$, implies there is a player j and a strategy $x'_j \neq y_j$ such that

$$r_j(x'_j, y_{-j}) \geq r_j(y).$$

The game (2.1) is individuality dominated if for each x, y where x dominates y then there is a player j and a strategy x'_j which is weakly preferred to y.

Finally we have the following definitions and theorems demonstrated by Nessah and Parvulescu [23].

Definition 2.34. A function $F : A \times A \to \mathbb{R}$ is *diagonal transfer continuous* in y if the condition

$$F(x, y) > F(y, y),$$

implies there is a point $x' \in A$ and a neighborhood $\mathcal{N}$ of y such that

$$F(x', z) > F(z, z)$$

for each $z \in \mathcal{N}$.

Theorem 2.35 (See [23])**.** *Let A be convex and compact and $U(x, y)$ be diagonally transfer continuous in y as in the previous definitions. Assume that:*

(a) the game (2.1) *is U-dominated, and*

(b) the function $x \mapsto U(x, y)$ is strictly quasiconcave on A, for each $y \in A$.

Then, the game (2.1) *has at least one Pareto-optimal Nash equilibrium.*

Definition 2.36. A correspondence $\Gamma : A \to B$ is said to be *upper hemicontinuous* at the point $a \in A$ if for any open neighbourhood V of $\Gamma(a)$ there exists a neighbourhood U of a such that for all x in U, $\Gamma(x)$ is subset of V.

Definition 2.37. A game G is said to be *correspondence secure* if whenever $\bar{x} \in A$ is not a Nash equilibrium, there exist a neighborhood $\mathcal{N}$ of $\bar{x}$ and a well-behaved correspondence (ϕ is upper hemicontinuous with nonempty and closed values) $\phi : \mathcal{N} \to A$ ($\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$) so that for each $z \in \mathcal{N}$, there exist a player j for whom $z_j \notin co\{t_j \in A_j \,|\, r_j(t_j, z_{-j}) \geq r_j(y_j, x_{-j})\}$ holds for each $(x, y_j) \in \mathrm{Graph}(\phi_j)$.

Theorem 2.38 (See [23])**.** *Let the game in* (2.1) *be convex, compact and correspondence secure. Assume that:*

(a) the game (2.1) *is individually dominated, and*

(b) the function $x_i \mapsto r_i(x_i, y_{-i})$ is strictly quasiconcave on A_i, for each $y_{-i} \in A_{-i}$, $i \in N$.

Then the game (2.1) *has at least one Pareto-optimal Nash equilibrium.*

Example 2.39. Consider a game between two players on the unit square $A_1 = A_2 = [0, 1]$. For player $i = 1, 2$ and $t = (t_1, t_2) \in A = [0, 1]^2$, let the payoff functions for the players be given by

$$r_1(t_1, t_2) = \begin{cases} -\frac{t_2}{1-t_2} t_1 + \frac{2-t_2}{1-t_2} t_2 & \text{if} \quad t_1 > t_2, \\ 3 \quad \text{if} \quad t_1 = t_2, \\ t_1 + t_2 \quad \text{if} \quad t_1 < t_2, \end{cases} ,$$

$$r_2(t_1, t_2) = t_1 + t_2.$$

It is clear that this game is convex, compact, and the function $x_i \mapsto r_i(x_i, y_{-i})$ is strictly quasiconcave in A_i for each $y_{-i} \in A_{-i}$, $i = 1, 2$.

Let us prove that

$$U(x, y) = r_1(x_1, y_2) + r_2(y_1, x_2)$$

is not strictly quasiconcave in x, for each $y \in A$. Let

$$x^{(1)} = \left(1, \frac{1}{2}\right), \quad x^{(2)} = \left(\frac{1}{2}, 1\right) \quad \text{and} \quad z = (1, 1).$$

We obtain then

$$U(x^{(1)}, z) = 4 + \frac{1}{2} \quad \text{and} \quad U(x^{(2)}, z) = 3 + \frac{1}{2},$$

therefore,

$$\min\big(U(x^{(1)}, z), U(x^{(2)}, z)\big) = 3 + \frac{1}{2}.$$

Let $\lambda \in (0,1)$ and $x^{(\lambda)} = (1-\lambda)x^{(1)} + \lambda x^{(2)} = \big(\frac{2-\lambda}{2}, \frac{1+\lambda}{2}\big)$. This implies that

$$U(x^{(\lambda)}, z) = \frac{3+\lambda}{2} + \frac{4-\lambda}{2} = 3 + \frac{1}{2} = \min\big(U(x^{(1)}, z), U(x^{(2)}, z)\big).$$

So Lemma 2.29 cannot be applied.

Let us prove that Γ is no transfer continuous in (x,λ). Indeed, let $x = (\lambda, \lambda) = (\frac{1}{2}, \frac{1}{2})$. Then for $y = (\frac{1}{2}, \frac{1}{2})$ and $z = (1,1)$, we have

$$\Gamma((x,\lambda),(y,z)) = \frac{13}{2} > 6 = \Gamma((x,\lambda),(x,x)).$$

We have, for any neighborhood $\mathcal{N}(x) \times \mathcal{N}(\lambda)$ of (x,λ) and for all $y', z' \in A$, there exist $\lambda' = \frac{1}{2}$ and $x' \in \mathcal{N}(x)$ with $x'_1 = x'_2 \neq y'_1$ such that

$$\Gamma((x',\lambda'),(y',z')) \leq \frac{9}{2} + x'_1 + x'_2 \leq \frac{9}{2} + \frac{3}{2}(x'_1 + x'_2) = \Gamma((x',\lambda'),(x',x')).$$

So Theorem 2.35 cannot be applied. However, all conditions of Theorem 2.38 are satisfied. Indeed,

a) The game is convex, compact and correspondence secure: let $x = (x_1, x_2)$ be any strategy non equilibrium. If $x_1 \neq x_2$ then there exist a neighborhood $\mathcal{N}(x)$ of x (with $x'_1 \neq x'_2$) and a well-behaved correspondence $\phi_1 : \mathcal{N}(x) \to [0,1]$ defined by $\phi_1(z) = \{z_2\}$ so that for each $z \in \mathcal{N}(x)$, we have $r_1(y_1, x'_2) = 3 > r_1(z)$ for each $(x', y_1) \in \mathrm{Graph}(\phi_1)$. If $x_1 = x_2$ (since x is not an equilibrium then $x_1 < 1$), then there exist a neighborhood $\mathcal{N}_\epsilon(x)$ of x (with $x'_1, x'_2 < 1 - 3\epsilon$, for each $x' \in \mathcal{N}_\epsilon(x)$ and for some $\epsilon > 0$) and a well-behaved correspondence $\phi_2 : \mathcal{N}_\epsilon(x) \to [0,1]$ defined by $\phi_2(z) = \{1\}$ so that for each $z \in \mathcal{N}_\epsilon(x)$, we have $r_2(x'_1, y_2) = 1 + x'_1 > z_1 + z_2 = r_2(z)$ for each $(x', y_1) \in \mathrm{Graph}(\phi_2)$.

b) The game is individually dominated, let $x, y \in A$ be such that $U(y,y) < U(x,x)$. If $y_1 = y_2$, then $U(y,y) = 3 + 2y_1$. Since $U(y,y) < U(x,x)$, hence $x_1 = x_2 > y_1$. Consequently there exist a player $j = 2$ and strategy $x'_2 = x_2 > y_1$ such that $r_2(y_1, x'_2) = y_1 + x'_2 > 2y_1 = r_2(y)$. If $y_1 \neq y_2$, there exist a player $j = 1$ and a strategy $x'_1 = y_2$ such that $r_1(x'_1, y_2) = 3 > r_1(y)$.

c) The function $x_i \mapsto r_i(x_i, y_{-i})$ is strictly quasiconcave on $[0, 1]$, $i = 1, 2$.

Therefore by Theorem 2.38 the game has a Pareto-optimal Nash equilibrium.

3 Dynamic games

In this section we study games that change over time, so called dynamic games.

3.1 Differential games

We introduce the class of open-loop differential games we are concerned with.

Let $\widetilde{N} = \{1, \ldots, N\}$, with $N \geq 2$, be the set of players and $T := [0, h]$, $h \leq \infty$, where h is the game's time *horizon*. For each $i \in \widetilde{N}$, the *set of feasible states* for player i is $X_i \subset \mathbb{R}^{l_i}$, and the *set of feasible controls* is $U_i \subset \mathbb{R}^{m_i}$. Let $X := X_1 \times \cdots \times X_N \in \mathbb{R}^l$, with $l := l_1 + \cdots + l_N$, and $U := U_1 \times \cdots \times U_N \in \mathbb{R}^m$, with $m := m_1 + \cdots + m_N$.

Define, for each $i \in \widetilde{N}$, the *open-loop strategy space* for player i as

$$\mathbf{U}_i := \{\mathbf{u}_i \,:\, T \to U_i | \ \mathbf{u}_i \text{ is Borel} - \text{measurable}\}, \tag{3.1}$$

and let $\mathbf{U} := \mathbf{U}_1 \times \cdots \times \mathbf{U}_N$ be the space of *open-loop multistrategies.*

Given a multistaregy $\mathbf{u} \in \mathbf{U}$, a function $\mathbf{x} : T \to X$ is called the *admissible state path* for the game, corresponding to the multistrategy $\mathbf{u}$, if $\mathbf{x}$ is the unique solution to the system of ordinary differential equations

$$\begin{aligned} \dot{\mathbf{x}}(s) &= f(s, \mathbf{x}(s), \mathbf{u}(s)), \\ \mathbf{x}(0) &= x_0, \end{aligned} \tag{3.2}$$

where f is a given $\mathbb{R}^l$-valued function defined on $T \times X \times U$, and $x_0 := (x_{10}, \ldots, x_{N0}) \in X$ is a given *initial condition.*

For each $i \in \widetilde{N}$, let $L^i : T \times X \times U \to \mathbb{R}$ be an *instantaneous* (or *current*) *payoff function* for player i, and $S^i : X \to \mathbb{R}$ a *terminal* (or *final*) *payoff function*, which is also known as a *salvage* or *bequest* function. The function S^i vanishes when $h = \infty$.

The *payoff function* for player i is defined for each $\mathbf{u} \in \mathbf{U}$ by

$$J_i^h(\mathbf{u}) := \begin{cases} \int_0^h L^i(s, \mathbf{x}(s), \mathbf{u}(s))ds + S^i(\mathbf{x}(h)) & \text{when } h < \infty, \\ \int_0^\infty e^{-\beta s} L^i(s, \mathbf{x}(s), \mathbf{u}(s))ds & \text{when } h = \infty, \end{cases} \tag{3.3}$$

where $\mathbf{x}$ is the admissible state path to the multistrategy $\mathbf{u}$, and $\beta > 0$ is an *intertemporal discount rate*, which is considered to be the same for every player.

Notation. To simplify the notation, we type J_i instead of J_i^∞ from now on.

Remark. Note that we can write f as a vector $(f^1, \ldots, f^N)$ where each coordinate f^i is an $\mathbb{R}^{l_i}$-valued function defined over $T \times X \times U$.

If an index $i \in \widetilde{N}$ is such that $J_i = 0$, then we will understand that player i has no state variable in the game.

In compact form, the class of differential games we are interested in can be expressed as

$$\Gamma_{x_0}^h := \{\widetilde{N}, \{\mathbf{U}_i\}_{i \in \widetilde{N}}, \{J_i^h\}_{i \in \widetilde{N}}, f\},\ h \leq \infty. \tag{3.4}$$

In the infinite horizon case, we write (3.4) as $\Gamma_{x_0}^\infty$.

Definition 3.1. Consider two function $P : T \times X \times U \to \mathbb{R}$ and $S : X \to \mathbb{R}$. These functions define an optimal control problem (OCP) in which a single player (or controller) wants to maximize the payoff function defined, for each $\mathbf{u} \in \mathbf{U}$, by

$$J^h(\mathbf{u}) := \begin{cases} \int_0^h P(s, \mathbf{x}(s), \mathbf{u}(s))ds + S(\mathbf{x}(h)) & \text{when } h < \infty, \\ \int_0^\infty e^{-\beta s} P(s, \mathbf{x}(s), \mathbf{u}(s))ds & \text{when } h = \infty, \end{cases} \tag{3.5}$$

subject to (3.2). A function $\mathbf{u}^* \in \mathbf{U}$ that solves this OCP is called an *open-loop optimal control* or simply an *optimal control*.

When $h = \infty$, we write J instead of J^∞.

In the dynamic case, the Nash equilibrium and Pareto-optimal are defined analogously to the static case. (See Definition 2.2(b) and Definition 2.3.)

We recall the following known facts for cooperative games.

Lemma 3.2. *(a) Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ be such that $\lambda_i > 0$ for all $i \in \widetilde{N}$, and $\lambda_1 + \cdots + \lambda_N = 1$. If $\mathbf{u}^* \in \mathbf{U}$ maximizes the scalar product $\lambda \cdot r(\mathbf{u}) = \sum_{i=1}^N \lambda_i J_i(\mathbf{u})$, that is,*

$$\lambda \cdot r(\mathbf{u}^*) = \max_{\mathbf{u}} \lambda \cdot r(\mathbf{u}),$$

then $\mathbf{u}^*$ is Pareto optimal.

(b) The converse of (a) is true provided that $\mathbf{U}$ is convex and $J_1, \ldots, J_N$ are all concave.

Lemma 3.3. *A multistrategy $\mathbf{u}^* \in \mathbf{U}$ is Pareto optimal if and only if, for every $i \in \widetilde{N}$, $\mathbf{u}^*$ maximizes J_i on the set*

$$\mathcal{U}_i := \{\mathbf{u} \in \mathbf{U} \mid J_j(\mathbf{u}) \geq J_j(\mathbf{u}^*)\, \forall j \neq i\}.$$

3.1.1 Potential differential games

As in the static case, finding an open-loop Nash equilibrium (OLNE) for an N-player differential game is a difficult task due to the fact that a differential game is the coupling of N- OCP's.

Fonseca-Morales and Hernández-Lerma [10, 11, 12] studied this problem, by means of potential differential games obtaining, defined as follows.

Definition 3.4. A differential game $\Gamma^h_{x_0}$, $h \leq \infty$, as in (3.4) is called an *open-loop PDG* (OL-PDG) if there exist an OCP such that an open-loop optimal solution of this OCP is an OLNE for $\Gamma^h_{x_0}$.

The simplest example of a PDG with Pareto-optimal Nash equilibria is a *team game*, defined as follow.

Example 3.5. (See Example 1 in [11].) Team games. The game (3.2)-(3.3) is said to be a team game if the functions L^i in (3.3) are all the same. In other words, there is a function $P(t, \mathbf{x}, \mathbf{u})$ such that

$$L^i(t, \mathbf{x}(t), \mathbf{u}(t)) = P(t, \mathbf{x}(t), \mathbf{u}(t)) \quad \forall\, i \in \widetilde{N}.$$

With this P in (3.5) (with $h = \infty$), it is trivially seen that a team game is a PDG. That is, if $\mathbf{u}^* \in \mathbf{U}$ optimizes (3.5) subject (3.2), then $\mathbf{u}^*$ is a Nash equilibrium for (3.2)-(3.3); see [11]. Moreover, $\mathbf{u}^*$ is Pareto optimal for (3.2)-(3.3). Therefore, a team game lies in the class of games we are interested in.

Theorem 3.6 (See [11]). *Consider a differential game as in* (3.2)-(3.3) *with* $f = (f^1, \ldots, f^N)$. *Suppose that there are functions* $\hat{L}^i, \hat{f}^i$ *such that one of the following conditions holds for every* $i \in \widetilde{N}$:

(a) $L^i(t, \mathbf{x}, \mathbf{u}) = \hat{L}^i(t, \mathbf{u}_i)$.

(b) $L^i(t, \mathbf{x}, \mathbf{u}) = \hat{L}^i(t, \mathbf{x}, \mathbf{u}_i)$, $f^i(t, \mathbf{x}, \mathbf{u}) = \hat{f}^i(t, \mathbf{x})$.

(c) $L^i(t, \mathbf{x}, \mathbf{u}) = \hat{L}^i(t, \mathbf{x}_i, \mathbf{u}_i)$, $f^i(t, \mathbf{x}, \mathbf{u}) = \hat{f}^i(t, \mathbf{x}_i, \mathbf{u}_i)$.

Then the differential game (3.2)-(3.3) *is a PDG. The associated OCP has an objective function* J *as in* (3.5) *with potential function*

$$P = \hat{L}^1 + \cdots + \hat{L}^N \tag{3.6}$$

Hence, if $\mathbf{u}^* = (\mathbf{u}_1, \ldots \mathbf{u}_N) \in \mathbf{U}$ *maximizes* J*, then* $\mathbf{u}^*$ *is an open-loop Nash equilibrium. In addition, if* $\mathbf{U}$ *is convex and* J_i *is concave on* $\mathbf{U}_i$ *for every* $i \in \mathbf{N}$*, then* $\mathbf{u}^*$ *is also Pareto optimal.*

Proof. This result follows from Corollary 1 in [11] and Lemma 3.2 above. □

The following example illustrates the previous theorem.

Example 3.7. Extraction of exhaustible resources under common access. (See [11].)

Replace (3.2) and (3.3), respectively, by

$$\dot{\mathbf{x}}(t) = -\mathbf{q}_i(t) - \sum_{j\neq i} \mathbf{q}_j(t), \quad \mathbf{x}(0) = x_0 > 0, \tag{3.7}$$

$$J_i(\mathbf{u}(\cdot)) = \int_0^\infty e^{-\rho t} \mathbf{q}_j^{a_i}(t) dt \quad \forall i \in \widetilde{N}, \tag{3.8}$$

with $\mathbf{u}(\cdot) = (\mathbf{q}_1(\cdot), \ldots, \mathbf{q}_N(\cdot))$, where $\mathbf{q}_i(t) \geq 0$, $\lim \mathbf{x}(t) \geq 0$ as $t \to \infty$, $0 < a_i < 1$, and $\rho > 0$ is the discount rate. This game is a PDG with potential function

$$P(\mathbf{u}) := \sum_{i=1}^{N} \mathbf{q}_i^{a_i}.$$

The associated OCP has a unique solution, which is an open-loop Nash equilibrium for the game (3.7)-(3.8) (see example 3 in [11]). Furthermore, by the previous Theorem 3.6(a) this Nash equilibrium is also Pareto optimal.

Now, we impose the following hypothesis.

H: Let (3.2)-(3.3) be a PDG where the associated OCP has the objective function (3.5).

Theorem 3.8. *Assume **H**. If* $\mathbf{u}^*$ *is a multistrategy such that, for every* $i \in \widetilde{N}$, $\mathbf{u}^*$ *maximizes* J_i *on* $\mathcal{U}_i$ *and, in addition,* $\mathbf{u}^*$ *maximizes* (3.5), *then* $\mathbf{u}^*$ *is a Pareto-optimal Nash equilibrium.*

Proof. The theorem follows directly from Lemma 3.3 and the definition of a PDG. □

The following corollaries follow from Theorem 3.8 and Lemma 3.3.

Corollary 3.9. *Assume **H**. If a multistrategy* $\mathbf{u}^*$ *is the unique maximizer of* J_k *for some* $k \in \widetilde{N}$, *and if* $\mathbf{u}^*$ *is also maximizer for* (3.5)*: that is, there is an index* $k \in \widetilde{N}$ *such that, for every* $\mathbf{u} \in \mathbf{U}$,

$$J_k(\mathbf{u}^*) \quad \geq \quad J_k(\mathbf{u}) \quad \text{and} \tag{3.9}$$

$$J(\mathbf{u}^*) \quad \geq \quad J(\mathbf{u}), \tag{3.10}$$

then $\mathbf{u}^*$ *is a Pareto-optimal Nash equilibrium. Furthermore, if* $\mathbf{u}^*$ *is not the unique multistrategy satisfying* (3.9)*, then* $\mathbf{u}^*$ *is a Nash equilibrium and it is also weakly Pareto optimal.*

Corollary 3.10. *Consider the game* (3.2)-(3.3)*. If a multistrategy* $\mathbf{u}^*$ *is such that, for each* $i \in \widetilde{N}$*,*

$$J_i(\mathbf{u}^*) \geq J_i(\mathbf{u}) \quad \mathbf{u} \in \mathbf{U}, \tag{3.11}$$

then $\mathbf{u}^*$ *is a Nash equilibrium that is also Pareto optimal.*

In the following example, the Corollary 3.9 is satisfied.

Example 3.11. Consider two players. Let $X = [0, \infty)$ be the state space of the game, and fix an initial state $x_0 \in X$. The feasible control set for player 1 is $U_1 = [0, \infty)$ and for player 2 is $U_2 = [0, 1]$.

The payoff function for player 1 is

$$J_1(\mathbf{u}) = \int_0^\infty e^{-\rho t} \left[\mathbf{u}_2(t) - \mathbf{x}(t) - \frac{\alpha}{2}\mathbf{u}_1^2(t)\right] dt,$$

with $\alpha, \rho > 0$ and for player 2 is

$$J_2(\mathbf{u}) = \int_0^\infty e^{-\rho t} \left[\mathbf{u}_2(t) - \mathbf{x}(t)\right] dt,$$

which are subject to the system equation

$$\dot{\mathbf{x}}(t) = 1 + \mathbf{u}_2(t) - \mathbf{u}_1(t)\sqrt{\mathbf{x}(t)}. \quad \mathbf{x}(0) = x_0. \tag{3.12}$$

This model is PDG with potential function

$$P(t, \mathbf{x}, \mathbf{u}) = \mathbf{u}_2 - \mathbf{x} - \frac{\alpha}{2}\mathbf{u}_1^2.$$

The potential function P and (3.12) define the associated OCP to this game. To obtain an optimal solution for the associated OCP, we have the following Hamiltonian system:

$$\begin{aligned}
H(\cdot) \quad &= \quad \mathbf{u}_2 - \mathbf{x} - \frac{\alpha}{2}\mathbf{u}_1^2 + \lambda\left[1 + \mathbf{u}_2 - \mathbf{u}_1\sqrt{\mathbf{x}}\right], \\
\mathbf{u}_1 \quad &= \quad -\frac{\lambda}{\alpha}\sqrt{\mathbf{x}}, \\
\lambda(t) + 1 \quad &\geq \quad 0, \quad 0 \leq \mathbf{u}_2 \leq 1, \\
\dot{\lambda} \quad &= \quad 1 + \rho\lambda - \frac{1}{2\alpha}\lambda^2, \\
\dot{\mathbf{x}} \quad &= \quad 1 + \mathbf{u}_2 - \mathbf{u}_1\sqrt{\mathbf{x}}, \quad \mathbf{x}(0) = x_0.
\end{aligned}$$

Solving this Hamiltonian system, we have that the optimal solution $\mathbf{u}^* = (\mathbf{u}_1^*, \mathbf{u}_2^*)$ is

$$\begin{aligned} \mathbf{u}_1^*(t) &= -\frac{\lambda^*(t)}{\alpha}\sqrt{\mathbf{x}^*(t)}, \\ \mathbf{u}_2^*(t) &= \begin{cases} 0 & if \quad \lambda^*(t) < -1, \\ \\ 1 & if \quad \lambda^*(t) \geq -1, \end{cases} \end{aligned}$$

where

$$\lambda^*(t) = \frac{\alpha\left[(\rho + C) + k_0 e^{-Ct}(\rho - C)\right]}{1 + k_0 e^{-Ct}},$$

$$C = \sqrt{\rho^2 + \frac{2}{\alpha}},\ \lambda_0 = \alpha\rho + \frac{2\alpha C}{\sqrt{2\alpha} + 2},$$

$$k_0 = \frac{(\frac{\sqrt{2\alpha}}{2} - 1)C - \left[\rho - \frac{\lambda_0}{\alpha}\right]}{(\frac{\sqrt{2\alpha}}{2} + 1)C + \left[\rho - \frac{\lambda_0}{\alpha}\right]}.$$

The corresponding state variable is given by

$$\mathbf{x}^*(t) = \exp\left(\int_0^t \frac{\lambda^*(\tau)}{\alpha} d\tau\right)\left[\int_0^t (1 + \mathbf{u}_2^*(s))\exp\left(\int_0^s \frac{\lambda^*(\tau)}{\alpha} d\tau\right) ds + x_0\right].$$

In addition, (3.9)-(3.10) hold. Therefore, the multistrategy $\mathbf{u}^*$ is a Nash equilibrium and Pareto optimal for the game.

3.1.2 Stochastic differential games

To define a stochastic differential game we replace (3.2) and (3.3) with

$$\begin{aligned} &d\mathbf{x}(s) = f(s, \mathbf{x}(s), \mathbf{u}(s))ds + \sigma(s, \mathbf{x}(s))dW(s), \\ &\mathbf{x}(0) = x_0, \quad s \geq 0, \end{aligned} \tag{3.13}$$

where $\mathbf{x} \in \mathbb{R}^N$, $W(\cdot)$ is a d-dimensional Brownian motion, and

$$J_i(\mathbf{u}) := E\left[\int_0^\infty e^{-\rho t} g_i(t, \mathbf{x}(t), \mathbf{u}(t))dt\right], \tag{3.14}$$

respectively. In (3.13) and (3.14), $\mathbf{u}(\cdot)$ is an open-loop multistrategy in $\mathbf{U}$ for which (3.13) and (3.14) are well defined. For each $i \in \widetilde{N}$, let $\sigma_i := (\sigma_{i1}, \ldots, \sigma_{iN})$ be row i of the $N \times d$ matrix σ in (3.14).

On the other hand, if we consider a stochastic differential game as in (3.13)-(3.14), Theorem 3.6 becomes as follows.

Theorem 3.12. *Consider a stochastic differential game as in* (3.13)-(3.14)*, with* $f := (f_1, \ldots, f_N)$*. Suppose that there are functions* $\hat{g}_i$*,* $\hat{f}_i$*, and* $\hat{\sigma}_i$*, such that one of the following conditions holds for every* $i \in \widetilde{N}$*:*

(a) $g_i(t, \mathbf{x}, \mathbf{u}) = \hat{g}_i(t, \mathbf{u}_i)$.

(b) $g_i(t, \mathbf{x}, \mathbf{u}) = \hat{g}_i(t, \mathbf{x}, \mathbf{u}_i)$, $f_i(t, \mathbf{x}, \mathbf{u}) = \hat{f}_i(t, \mathbf{x})$.

(c) $g_i(t, \mathbf{x}, \mathbf{u}) = \hat{g}_i(t, \mathbf{x}_i, \mathbf{u}_i)$, $f_i(t, \mathbf{x}, \mathbf{u}) = \hat{f}_i(t, \mathbf{x}_i, \mathbf{u}_i)$, $\sigma_i(t, \mathbf{x}) = \hat{\sigma}_i(t, \mathbf{x}_i)$.

Then the game (3.13)-(3.14) *is a stochastic PDG where the associated OCP has objective function* J *as in* (3.5) *and potential function*

$$P = \hat{g}_1 + \cdots + \hat{g}_N. \tag{3.15}$$

Hence, if $\mathbf{u}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_N^*) \in \mathbf{U}$ *maximizes* J*, then* $\mathbf{u}^*$ *is an open-loop Nash equilibrium. If, in addition,* $\mathbf{U}$ *is convex and* J_i *is concave on* $\mathbf{U}_i$ *for every* $i \in \widetilde{N}$*, then the reward vector* $r(\mathbf{u}^*)$ *is a Pareto point.*

Proof. See Theorem 4.1 in [10]. □

The following example illustrates Theorem 3.12.

Example 3.13. *Competition for consumption of a productive asset.* Assume there are N players. The control sets are $U_i := [0, \infty)$ for all $i \in \widetilde{N}$. The players wish to maximize the expected discounted utility of consumption

$$J_i(\mathbf{u}) := E\left[\int_0^\infty e^{\rho t} L^i(\mathbf{u}_i(t)) dt\right]$$

with $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$, subject to the stock dynamics

$$d\mathbf{x}(t) = \left[F(\mathbf{x}(t)) - \sum_{i=1}^N \mathbf{u}_i(t)\right] dt + \sigma(\mathbf{x}(t)) dW(t), \quad \mathbf{x}(0) = x_0,$$

where F and σ are given functions [17]. This game is as in Theorem 3.12 (a). Hence it is a stochastic PDG with potential function

$$P(\mathbf{u}) := \sum_{i=1}^N L^i(\mathbf{u}_i).$$

Assuming that the instantaneous utility functions L^i are strictly concave, the optimal solutions of the OCP are both Nash equilibria and Pareto optimal.

3.2 Discrete-time stochastic games

We consider dynamic stochastic games with N players and state space $X \subset \mathbb{R}^m$. Let $\{\xi_t\}$ be a sequence of independent random variables, and suppose that each ξ_t takes values in a Borel space S_t ($t = 0, 1, \dots$), that is, a Borel subset of a complete and separable metric space. Assume that the state dynamics is given by

$$x_{t+1} = f_t(x_t, u_t^1 \dots, u_t^N, \xi_t), \quad t = 0, 1, \dots, \tag{3.16}$$

where u_t^j is chosen by player j in the control set $U^j \subset \mathbb{R}^{m_j}$ ($j = 1, \dots, N$). In general, the set U^j may depend on time t, the current state x_t, the action u_t^i of each player $i \neq j$, and the value s_t taken by ξ_t, for each $t = 0, 1, \dots,$. We suppose that player j wants to maximize a performance index (also known as reward or payoff function) of the form

$$\mathbb{E} \sum_{t=1}^{\infty} r_t^j(x_t, u_t^1, \dots, u_t^N) \tag{3.17}$$

subject to (3.16) and the given initial pair (x_0, s_0), which is supposed to be *fixed* throughout the following.

Consider a game with dynamics and reward functions given by (3.16) and (3.17), respectively. The state spaces $\{X_t\}$ are subsets of $\mathbb{R}^m$ and each control set $U_j \subset \mathbb{R}^{m_j}$ for $j = 1, \dots, N$. Finally, we consider the sets Ψ^j ($j = 1, \dots, N$) of open loop multistrategies. In reduced form, the game can be expressed as:

$$\left(\{X_t\}, \{\xi_t\}, \{U_j \,| j \in \widetilde{N}\}, \{f_t\}, \{r_t^j \,| j \in \widetilde{N}\}, \{\Psi^j | j \in \widetilde{N}\}\right). \tag{3.18}$$

Now, we consider Pareto solutions to dynamic games. That is, a (Markov or open-loop) multi-strategy ϕ is called a *Pareto solution* for the game (3.18) if ϕ maximizes the convex combination

$$\mathbb{E} \sum_{t=0}^{\infty} \left[\lambda_1 r_t^1(x_t, u_t) + \cdots + \lambda_N r_t^N(x_t, u_t)\right], \tag{3.19}$$

subject to (3.16), for some $\lambda_j > 0$ ($j = 1, \dots, N$) such that $\lambda_1 + \cdots + \lambda_N = 1$.

González-Sánchez and Hernández-Lerma [13, 14] gave the following theorem.

Theorem 3.14 (See [13]). *Suppose that the functions r_t^j in* (3.18) *are of the form*

$$r_t^j(x_t, u_t) = g_t^j(u_t^j) \quad j = 1, \dots, N,\ t = 0, 1, \dots. \tag{3.20}$$

Then the game (3.18) *is an open-loop potential game. Moreover, each open-loop Pareto solution to the game* (3.18) *is also an OLNE (let us remember that OLNE means open-loop Nash equilibrium according with the first paragraph of Section 3.1.1).*

Example 3.15. We consider the following deterministic game. The functions f_t and r_t^j in (3.16)-(3.17) are

$$f_t(x_t, u_u^1, u_u^2) = (x_t - u_t^1 - u_t^2)^\alpha, \quad r_t^j(x_t, u_u^1, u_u^2) = \beta^t \log(u_t^j), \tag{3.21}$$

for $j = 1, 2$, where $\alpha, \beta \in (0, 1)$. In addition, suppose that $x_0 > 0$ is given, the controls u_t^j $(j = 1, 2)$ are positive, and $u_t^1 + u_t^2 < x_t$ for each $t = 0, 1, \dots$. Moreover, for any pair of positive numbers λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$, and $t = 0, 1, \dots,$ the open-loop strategies for this example are given as follow

$$\hat{\psi}^j(t) = \lambda_j(1 - \alpha\beta)\hat{x}_t, \quad j = 1, 2, \tag{3.22}$$

where

$$\log(\hat{x}_t) = \left(\log(x_0) - \frac{\alpha}{1-\alpha}\log(\alpha\beta)\right)\alpha^t + \frac{\alpha}{1-\alpha}\log(\alpha\beta). \tag{3.23}$$

Sufficient conditions for open-loop strategies to be Nash equilibria are also given in [13] and [14]; in particular, it is shown that the strategies (3.22) are open-loop Nash equilibria. Moreover, the strategies (3.22) are also *Pareto solutions*, i.e., they maximize the weighted sum

$$\lambda_1 \sum_{t=0}^{\infty} \beta^t \log(u_t^1) + \lambda_2 \sum_{t=0}^{\infty} \beta^t \log(u_t^2)$$

subject to the dynamics $x_{t+1} = (x_t - u_t^1 - u_t^2)^\alpha$.

4 Conclusions

In this work we analyze games with Pareto-optimal Nash equilibria, in the static and dynamic cases. The main objective is to write a bibliographic reference where one can consult results on some solutions to these games. The static games are introduced in the Section 2, studying them firstly from the point of view of the potential games and subsequently using the approach of discontinuous games. The point of view of potential games yields results for this kind of games . Subsequently we pay attention to a particular example in the Subsection 2.2. Then we introduce the case of dynamic games in Section 3.

The main results and contributions are presented in Section 2. We proposed conditions on the fictitious-objective function to have a Pareto-optimal Nash equilibrium (see Proposition 2.10).

Subsection 2.3 is focused to present a different approach to the potential games. This approach based on discontinuous games, extends the class of admissible payoff functions to have a Pareto-optimal Nash equilibria.

We illustrate with various examples both approaches to the case of the static games. Section 3 presents the results obtained by Fonseca-Morales and Hernández-Lerma [10, 11, 12], and González-Sánchez and Hernández-Lerma [13, 14]. These results are also illustrated with different examples. There are many open problems, like the following ones:

1. Is it possible to weaken the conditions imposed in the discontinuous games in order to obtain similar results?
2. Is it possible to extend the discontinuous games to the dynamical case?
3. In reference to the results presented in Section 3.1, do they hold in the discrete-time case?

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