CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL
INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE

MATEMÁTICAS

Espuma cuántica topológica gravitacional como sistema crítico de organización espontánea

T E S I S

Que presenta

Ramiro Hum-Sah López Vázquez

Para obtener el grado de

MAESTRO EN CIENCIAS

EN LA ESPECIALIDAD DE MATEMÁTICAS

Director de la Tesis

Dr. Ernesto Lupercio Lara

Ciudad de México ENERO, 2019

Dedicatoria:

A la mujer que vuelve brillantes mis días, a quien veo en todo lo que es hermoso y a quien amo más que al todo: A mi hermana Lakshmi Daniela.

A mis padres José Luis López y Leonor Vázquez por hacer de su destino mis cuidado, de sus días mi felicidad y de sus ilusiones las mías. Gracias por su amor.

Con todo el amor del mundo para José Luis López y mis sobrinos María José y Luis Fernando Lopéz.

A los hombres invencibles que me enseñaron a luchar y a llenarlo todo de alegría, Adrián Romero y Enrique Celis. A mi amigo Eric Melchor por lo mejor de la vida y al encanto de Elizabeth Ortiz.

"Yes, he is a great counterexample" for give me reasons and dreams to fight for, to Luboš Motl.

Agradecimientos:

"La labor de un maestro no es enseñar conocimientos, en cuyo caso una biblioteca sería mucho mejor. Un buen maestro enseña una forma de ver el mundo, de lidiar con el inscrutable universo."

Agradezco al profesor Ernesto Lupercio por la increíble aventura intelectual que representó trabajar bajo su supervisión. Fue un verdadero honor conocer a un auténtico mago de las matemáticas.

A ese respecto mi mayor aprendizaje es el tesoro de su frase:

"Las mentes supremas como Borges, Leonardo y Grothendieck lo sabían: La meta es la simplicidad extrema."

Prometo revisitar toda la vida el libro de arena y el libro de las noches para admirar la meta.

Le ofrezco disculpas por haber hecho de este texto un mapa como el de "Silvia y Bruno" de Lewis Carroll, aquel del que Borges se mofó y advirtió en conocido cuento, terrible cuando la topografía de la arena no es complicada y es la marca de una obra maestra. Esos mapas y documentos que llevaron a cierto hombre del futuro y a su utopía a estar cansados.

Aagradezco a un hombre, maestro y científico extraordinario, Hugo García Compean a quien todo debo, quien me formó y quien por años me ha honrado con ejemplo, consejo y enseñanzas. Su confianza en mi es el mayor mérito académico que tengo, gracias mantener vivo mi entusiasmo por seguir aprendiendo.

Agradezco infinitamente al grupo de estudiantes extraordinariamente brillates del profesor Compean. Gracias a Carlos Ramírez, Eduardo López, Erick Chacón y especialmente a Luis Lozano y José de la Cruz quienes convirtieron la escuela de graduados en mi segundo hogar.

A Héctor Zapata por la inmensa, desinteresada e incondicional amabilidad con la que me trató en los tiempos difíciles.

Finalmente doy las gracias a CONACYT y a los contribuyentes mexicanos por financiar mi eduación de graduado.

Special thanks to the right part of the CFT whom with i have entangled the bridge that have transmuted this scrambled modular flow time of my life into an exciting and restriction-free (except for the Papadodimas-Raju ones) pre-wholeographic adventure, a state-dependent acknowledgment from this boundary to my "living in the bulk" french-physics friend: Mmanu.

Resumen:

La motivación principal de este trabajo es revisar la teoría básica de del autómata celular de pilas de arena. La principal inovación ofrecida es la reinterpretación de dicho autómata como espuma cuántica gravitacional para una variedad tórica de tipo Calabi-Yau.

En el capítulo uno se puede encontrar una motivación general para el estudio de el fenómeno de criticalidad organizada espóntanea.

El capítulo dos tiene por objetivo definir las propiedades elementales de "una pila de arena".

El capítulo tres introduce la dinámica de una pila de arena, la teoría de cadenas de Markov es usada para exhibir la propiedad de autoorganización crítica organizada de dicha pila.

El objetivo de el capítulo cuatro es servir como un repaso de los fundamentos de un juego matemático conocido como "el juego del dólar", la dinámica de la pila de arena es mostrada como un caso particular de la dinámica de dicho juego, importantes propiedades de la pila de arena se deducen de este embebimiento.

La meta del capítulo cinco es reinterpretar estados distinguidos de la pila de arena como árboles generadores en una gráfica.

El apartado más importante del presente trabajo es el capítulo seis, reinterpretamos los mencionados estados de pila de arena como cubrimientos dimer en una gráfica canonicamente asociada a la que contiene la pila. Esta clase de gráficas ha aparecido en la literatura como "adoquinados de branas".

En los capítulos siete y ocho revisamos como la existencia de una teoría clásica de campos inducida por la estructura de Kähler en una variedad tórica de tipo Calabi-Yau. La cuantización de dicha teoría de gravedad induce una mecánica cuántica supersimétrica cuya dinámica señalamos equivalente (a nivel de funciones de partición) con la dinámica de la pila de arena y que reproduce teoría de cuerdas topológicas de tipo A en el límite de baja temperatura.

El capítulo nueve concierne a especulaciónes, aplicaciones y trabajo futuro. Como especulación señalamos relaciónes intrigantes entre teorías del campo supersimétricas en cinco dimensiones y pilas de arena, a modo de aplicaciones parametrizamos los microestados de hoyos negros supersimétricos en dimensiones cuatro y seis con estados de pila de arena, relaciónes entre pilas de arena y teoría de representación emergen en esta aplicación.

Concluimos el trabajo en el capítulo diez argumentando brevemente que la reformulación encontrada de la pila de arena no es una operación trivial (en un sentido que es detallado) y amerita investigación.

Contents

1 Motivation and objectives.

"A vast similitude interlocks all,
All spheres, grown, ungrown, small, large, suns, moons, planets,
All distances of place however wide,
All distances of time, all inanimate forms,
All souls, all living bodies though they be ever so different, or in different worlds...."
Walt Whitman.

1.1 Motivation:

This thesis is about a phenomenon called self-organized criticality (SOC) in dynamical systems.

The knowledge of the nature and dynamics of the fundamental constituents of some physical system typically does not allow us to fully predict the possible states that some system can reach, the fundamental obstruction is complexity, there are laws that emerge only under the collective motion of a large number of constituents. An example is that humans know that the earth crust is primarily made of granite, chips of granite obey the fully understood Newton's laws, nevertheless we are incapable to predict how often earthquakes occur,

20th century physics have teach us that even the fundamental laws are statistical in nature, quantum mechanics is an example of this. Quantum objects have intrinsic statistical behaviour and everything in the universe obey the rules of quantum mechanics. Then, statistical and emergent laws are important.

Another important dynamical aspect is the idea of "criticality", we say that a system is in a critical state if the characteristic scale of the correlations among constituents is comparable to the size of the system. An example could be the cracks that appear when a piece of ice is melting, there is no typical size that those fissures typically can have, in particular they can fracture the entire piece of ice into two because the system is experiencing a phase transition and the correlations are as large as the piece of ice.

The very definition of a **self-organized critical system** is no well understood, but it can be phrased as follows:

A self-organized critical system is a dynamical system that have a critical point of its dynamics as an attractor.

In simple words: their dynamics do not require external agents that tune delicately his parameters to reach criticality, examples of critical systems that are *not* critical self-organized are a piece of ice and typical quantum field theories. The piece of ice of water have an internal organization (crystalline structure) but require for its existence that the environment remain with a temperature below of the fusion point of water; quantum field theories typically does not have manifest symmetries, they emerge when energy (temperature) is increased.

Examples of self-organized critical systems are: ecologies, the intensity distribution of earthquakes, free market economies, a beaver and two dimensional conformal field theories (because of the c-theorem).

There is an aspect of SOC that is fascinating with no explanation until now. This aspect is the appearance of power laws relating observables of self organized-critical systems. A power law is an expression of the following type:

$$f(x) = cx^{\alpha},$$

where c and α are constants.

The presence of such power laws in self-organized critical systems is so ubiquitous that it is possible to give an alternative definition SOC:

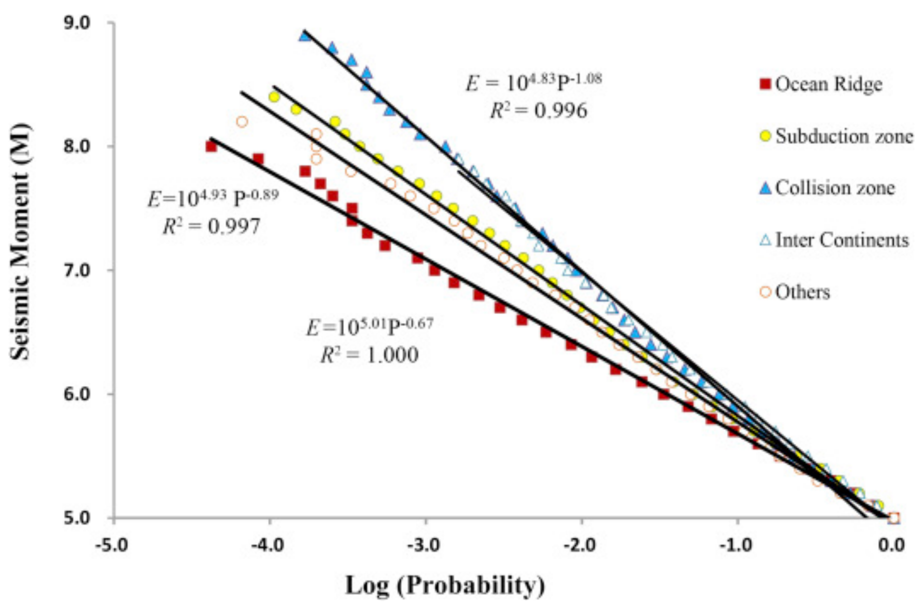
A self-organized critical system is an isolated dynamical system whose correlations between its observables obey power laws over large spatio-temporal windows.

An interesting example is the Gutenberg–Richter law, this law relates the number N of earthquakes of intensity E by the following relation:

$$N = cE^{\alpha},$$

here the constants c and α depend on the zone that is studied; what is intriguing, is the fact that the law appears to be universal in the sense that the functional form is valid independently of the conditions of the studied zone.

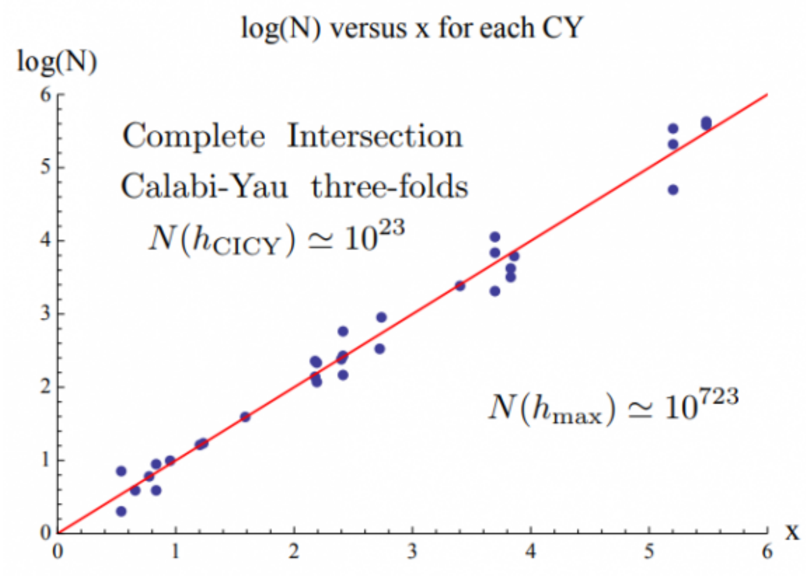
The following figure (extracted from [18]) illustrate this fact:



The figure shows Log-log plots of the energy – probability relations calculated from earthquakes with M $\geq$ 5 that occurred in western Pacific tectonic zones.

It is estimated in the wonderful paper [19] the number of minimally extended supersymmetric standard models (MSSM) that can be realized as compactifications of the $E_8 \times E_8$ heterotic string on Calabi-Yau threefolds. It is find the extraordinary fascinating fact that the logarithm of the number N of MSSMs grow lineally with the Picard number h of the analized threefolds.The amazing relation is:

$$Log[N] \simeq -5 + 1.5h.$$



The creation of a universe is a phase transition (on inflationary scenarios), this beautiful result suggest that the probability within string theory creates universes like ours appears to be critical self-organized!.

This is why self-organized criticality is so interesting, because many fundamental systems exhibit this property, because this type of organization is no evident at all when the system is divided into its smallest parts and fundamentally because it is a new perspective on what *fundamental* and *composite* mean.

The power law that is an example of a completely emergent law, this by itself is a refutation of the ancient greek ideas about reductionism on the fundamental laws. That is ontologically fascinating and invite us to rethink the way we search for the laws of nature.

In summary: Critical self-organization radically change the human conception of what *fundamental* means. Our motivation in this thesis is to understand a platonic object that exhibit self-organized criticality.

1.2 Objectives

The objectives of the present thesis are the following:

-To review the basic mathematical theory of a cellular automata that exhibit the property of begin critical self-organized as a dynamical system. This automata is called: "the abelian sandpile model"

-To explain the origin of long range correlations in sandpiles.

-To explain basic aspects of a mathematical game called "the dollar game" from which the sandpile model has its origin, this is done with the intention to make manifest the relations between sandpiles and algebraic geometry.

-We show strong evidence of the equivalence of the sandpile model with a completely solvable model of random surfaces solved by Percy McMahon in [20] and systematically studied by Andrei Okounkov and Nikolai Reshetikhin in [21] showing mysterious relations with tropical geometry.

The later is original work. To the poor knowledge of the author, there are no previous connections between mathematical sandpiles, topological string theory, black holes and supersymmetric field theories in higher dimensions.

-It is proposed to exploit the above connection to make analytical computations like the probability of appearance of certain states, Lyapunov exponents and the origin of power laws in sandpiles.

-We make observations and conjectures about the physical origin and meaning of fractal patterns on sandpiles. To support our conjectures we propose a mathematical game that allows to recover the identity sandpile state over a convex set via electric magnetic duality in five dimensions.

-As an application of the crystal melting - sandpiles equivalence, we construct the Hilbert space of untwisted RR D0-brane charges of certain BPS black holes in six dimensions realized by branes wrapped over non trivial cycles in the orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ by constructing mathematical sandpiles whose states are in one to one correspondence with the aforementioned charges. It is also indicated how to extend our results to more general black holes.

1.3 Plan of the thesis.

In chapter two is introduced the sandpile model and theory of Markov chains on it, this allow us to concepts of *allowed* and *forbidden* emerge.

In chapter three we present algorithms to characterize the aforementioned type of states, it is explained in detail how the allowed states constitute an abelian group under addition among them that completely characterize the random evolution of a sandpile.

In chapter four we introduce *the dollar game* to suggest a connection between sandpiles and algebraic geometry, it is also given a presentation for the sandpile group.

It is shown in chapter five that the combinatorial problem of counting spanning trees on a graph is equivalent to the knowledge of all the allowed states of a sandpile in the sense of the existence of bijective correspondence between both sets. It is argued that this is the origin of the sandpile self-organization.

Chapter six is devoted to introduce the combinatorial problem of counting dimer covers on a graph, it is shown that this problem is equivalent to the knowledge of all recurrent sandpile states on a canonically related graph.

In chapter seven we briefly review the interpretation of the dimer problem on a graph as "the quantum topological gravitational foam" for a Calabi-Yau threefold a system that reproduces (in certain limit) the closed string sector of Witten´s topological A - Model on the same space.

In chapter eight we show the equivalence of the partitions functions of sandpiles and the quantum gravitational foam. We also make some conjectures in favour of the non triviality of the bijection between allowed sandpile states and the states of the gravitational foam. We speculate about the possible uses of the equivalence.

Chapter nine is dedicated to some basic observations concerning the possible relevance of sandpiles to supersymmetric field theories, BPS black holes and string theory.

We observe curious parallelisms between sandpile dynamics, Hanany-Witten like transitions and monopole condensation. By taking the similarities seriously we make a conjecture towards a possible physical explanation of the meaning of the mysterious patterns found in sandpiles. In this work it is argued that basic physical considerations for five dimensional supersymmetric quantum chromodynamics (with eight supercharges) and Seiberg duality are culprits of the self-similarity found in sandpiles; to support the conjecture we propose an algorithm to recover sandpile patterns (in certain convex sets) by taking subsets of the draw and generating other parts of the draw by repeated use of S-duality and consistency restrictions coming from the hypothesis that particular sandpile draws are $(p,q)-5$ webs that realize universality classes in the Higgs branch of a certain 5d SQCD; the algorithm is used to analyse the sandpile identity on a square.

We also make intriguing observations about sandpile Markov chains, Tong/Lambert kinky D-strings and dyon condensation in five dimensions.

The equivalence between the scaling limit of the sandpile partition function and the Witten´s closed A-topological model and the OSV theorem motivate us to relate certain problems of BPS-state counting in black holes and quiver gauge theories with sandpiles in one dimension. We believe that our results extend to two dimensional sandpiles.

Chapter ten concerns conclusions and possible extensions of our work.

1.4 Main goal.

What is the history that we want to tell?

Our basic goal is to prove a bijection between the set of allowed states of the sandpile on a graph within the set of states of the dimer problem in a canonically related graph and to argue why that the later problem is a non trivial one in topological string theory.

Little is known about sandpiles, the author does not understand what does the found relation possibly mean, but we argue and believe that the reformulation should point out to something interesting.

2 What is the mystery of the sandpile model?

“Complexity matters!
Computation takes time and space.
We also have learned that in some sense computation creates time and space.”
Juan Maldacena - The 50th anniversary of string theory - Strings 2018.

The work of an artist is to transmute ideas or feelings into beauty, the labor of a painter is to do so at one instant of time and as the only known representation of an instant (and the eternity) is an image, then the destiny of a pintor is to get mastered in what nature (the ultimate master of all pintors) has taught them about instants and eternities: colors and symmetries.

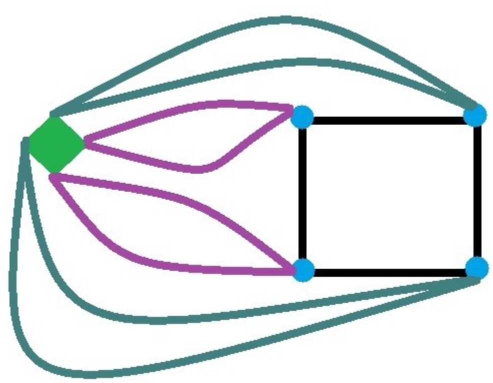
This is the history of a platonic pintor with unknown motivations that produce this type of captive beauty, his nature is that of a cellular automaton, his brush is a unique local and simple law and their colors a number of grains of sand.

As all paint lovers, our hope is to understand what are the eternal and inevitable motivations behind his drawings.

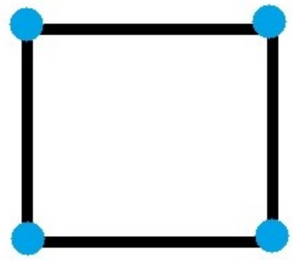
Definition 2.1. By a **multigraph** $G = (V, E)$ we understand a pair of multisets composed of a finite set of *vertices* V and a finite multiset of *edges* $E \subseteq V \times V$; to maintain the notation simple we shall write an edge $\{u, v\}$ simply as uv , furthermore, any sequence of edges of the form $u_1u_2, u_2u_3, ..., u_{n-1}u_n$ will be called a **path** and will be denoted by $u_1u_2...u_{n-1}u_n$.

Observation 2.2. *In the following we demand multigraphs G with the property that there is a unique vertex $\bullet$ such that if we remove it (together with all the edges whose boundaries contain $\bullet$) the resulting multigraph is in fact just a finite graph G (which means that V is finite), connected (there is always a path from u to v for any pair of different vertices u and v) and without loops (there are no edges of the form uu).*

The following draw shows an example:



After removing the vertex marked with a diamond $\diamond$ in green and the loops in purple an blue we obtain a finite graph without loops:



Definition 2.3. *A sandpile state* ϕ is an element of the following set:

$$\mathbb{N}^V = \{\sum_{v \in V} \phi(v)v : \phi(v) \in \mathbb{N}\}.$$

Any multigraph with those properties will be called simply a graph in what follows.

The beautiful allegory is to think on ϕ as a *sand mound* or a *sand pile* over the graph G, more precisely, for every vertex v, $\phi(v)$ represents the number of *chips* or *grains of sand* of the pile ϕ at v. To give a flavor of why the allegory is beautiful, and this work pretend to let us show how real grains of beach sand look like when seen under a microscope:

Definition 2.4. *The degree* of a vertex in a graph is the number of edges incident on it. It will be denoted in the following as *d(v)*.

Definition 2.5. The *sum* of two sandpile states ϕ and ψ can be defined at every vertex v of G as:

$$+ : \mathbb{N}^V \times \mathbb{N}^V \to \mathbb{N}^V$$

$$(\phi(v), \psi(v)) \mapsto \phi(v) + \psi(v)$$

Definition 2.6. Let $G = (V, E)$ be a graph with a distinguished vertex $\bullet$ called *the sink* an a fixed *initial state* $\phi \in \mathbb{N}^V$. ***The abelian sandpile model*** is a cellular automaton defined by the following discrete time evolution rules:

1) If $\phi(v) \leq v$ for some vertex v, we say that v is *unstable* and the system evolves (or *relaxes*) as $\phi \mapsto \phi + \Delta_v$ where

$$\Delta_v(w) := \Delta_{v,w} = \begin{cases} -d(v) & if \ v = w \\ 1 & if \ v \sim w \\ 0 & otherwise \end{cases}$$

Here $v \sim w$ means that v is adyacent to w, or, equivallently: vw is an edge; in that case we say that v *topples.*

2) $\bullet$ never topples.

3) The chips at $\bullet$ are erased from the system at every time during the evolution.

4) If neither of the vertices of G is unstable, then, the evolution stops.

The final state will be writed ϕ° and called *stable* or *relaxed.*

Lemma 2.7. *The order at which the vertices of some unstable sandpile ϕ are toppled during a relaxation process is unimportant (the final state is unique).*

Proof. Let u and v unstable and different vertices at some time t_0. Suppose that u topples first, then the system evolves as $\phi \mapsto \phi + \Delta_u$ at time t_1, now notice that if v was unstable at t_0 then it will be at t_2 because the relaxation process only increases the amount of chips on v if v is not toppled, then system evolves as $\phi + \Delta_u \mapsto \phi^\circ(1) := \phi + \Delta_u + \Delta_v$ at time t_3.

Similarly, if v is toppled at time t_1 and u at t_2, then at time t_3 the system will be $\phi^\circ(2) := \phi+\Delta_v+\Delta_u$. Finally, commutativity of the addition of integers assures $\phi^\circ(1) = \phi^\circ(2)$. ☐

Proposition 2.8. *Given any sandpile ϕ the relaxation process always stops.*

Proof. We say that a vertex adjacent to the sink $\bullet$ is *a boundary vertex* and those which are not will be called *bulk vertices.* Notice that the relaxation process never increase the total amount of sand of ϕ and when a boundary vertex topples a grain of sand is lost, it follows that boundary vertices can not be toppled more that the number of chips in the system.

Now, an infinite number of topplings in a vertex v adjacent to a boundary one w will produce an infinite number of topplings on w and that contradicts the last inference. Suppose that an infinite amount of topplings should not occur at vertices n *sites away from* $\bullet$ but actually happen at a vertex w adjacent to v that is $(n + 1)$ *sites away from* $\bullet$, that necessarily produce an infinite number of topplings on v, contradiction, w can not be toppled an infinite amount of times.

Conclusion: Every vertex topples a finite amount of times in a relaxation process, then the evolutions stops. ☐

Definition 2.9. Let M be a set closed under a binary operation $+ : S \times S \to S$, then the pair $(S, +)$ is a *monoid* if it satisfies the following two properties:

Existence of an identity element: There is an element e in S such that for every ϕ in S the equations $e + a = a + e = a$ hold.

Associativity: For every a, b and c in S, the equation $(a + b) + c = a + (b + c)$ holds.

Examples: 2.10. *The following are examples of monoids:*

I) *The set of natural numbers equipped with the usual addition is a monoid with the zero element as the identity.*

II) *The set of homeomorphism classes of complex compact surfaces with the connected sum. The identity of this monoid is the two sphere S^2.*

III) *(Informal example) Consider the set of three dimensional (non charged an non rotating) BTZ black holes of mass m (recall that for a given mass m there is a unique black hole with this mass), to produce a monoid from this data take two black holes $\bullet_1$ and $\bullet_2$ of mass m_1 and m_2, respectively, and define a binary operation + as "black hole fusion", more precisely, we define + as $\bullet_1 + \bullet_2 := \bullet_{12}$, where $\bullet_{12}$ is the unique black hole of mass $m_{12} = m_1 + m_2$, and we observe that the non existence of gravitational waves in three dimensions (and energy conservation) makes the operation + well defined, on contrary, the result of black hole merging in higher dimensions is "unpredictable" in the sense that an arbitrary fusion can occur as $\bullet_1 + \bullet_2 := \bullet_3$ + (gravitational waves), where the mass of $\bullet_3 = m_1 + m_2 - \epsilon$ with $\epsilon \in \mathbb{R}_{\geq 0}$ the energy dissipated as gravitational waves; notice that the identity under + is the unique black hole whose mass is zero.*

Associativity $(\bullet_1 + \bullet_2) + \bullet_3 = \bullet_1 + (\bullet_2 + \bullet_3)$ also follows from energy conservation and the triviality of three dimensional gravity. But it can also be (physically) proved by using the second law of black hole thermodynamics as follows: Suppose that $(\bullet_1 + \bullet_2) + \bullet_3 \neq \bullet_1 + (\bullet_2 + \bullet_3)$ an take the black hole of greater area between $(\bullet_1 + \bullet_2) + \bullet_3$ and $\bullet_1 + (\bullet_2 + \bullet_3)$, the selected black hole is more probable to appear in a fusion process between $\bullet_1, \bullet_2, \bullet_3$ than the other because of the second law of thermodynamics, this implies the existence of selection rules for black hole merging between $\bullet_1, \bullet_2, \bullet_3$ but such rules are not predicted by general relativity, so, associativity holds.

Observation 2.11. *For a given graph G the set S of all the stable sandpiles states over G equipped with the operation of pointwise addition followed by relaxation (denoted simply as $+$) is a commutative monoid.*

Definition 2.12. Let $(M, +)$ be a commutative monoid, then a non empty subset J of M is an *ideal* of M if $\phi + \psi \in J$ for all $\phi \in J$ an all $\psi \in M$.

Lemma 2.13. *Let M be a finite commutative monoid and $I = \{I \in M; I \text{ is ideal of } M\}$, then:*

$$N := \bigcap_{J \in I} J \qquad \textit{is an abelian group.}$$

Proof. Let $\mid N \mid = n \in \mathbb{N}$ be the number of elements of N.

First notice that N is the minimal ideal of M in the sense that there are no proper ideals of M contained in N.

I) N *is nonempty* since it contains at least the sum of all the elements of M.

II) N *has an identity:*

We say that $\phi \in N$ *has an identity* if there is $e_\phi \in N$ such that $\phi + e_\phi = \phi$. As N is finite we can enumerate all its different elements as $\phi_1, \ldots ,\phi_n$ in such a way that $\Omega_1 := \{\phi_1, ..., \phi_m\}$ are all the elements of N that have at least one identity and $\Omega_2 := \{\phi_{m+1}, ..., \phi_n\}$ are the ones that do not have, then $N = \Omega_1 \sqcup \Omega_2$, indeed $\Omega_1 \neq \emptyset$, because if $\Omega_1 = \emptyset$ and $\Omega_2 = N$ then for any ϕ ,$\sigma \in N$ we have

$\psi + \sigma \neq \phi$, $\psi + \sigma \neq \sigma$ otherwise ψ would be and identity for σ (or vice versa) and $\phi_1 + \cdots + \phi_n$ is a new element of N not on ϕ_1,...,ϕ_n contradiction, then $\Omega_1 \neq \emptyset$.

Our objective is to show that $N = \Omega_1$, in fact for any $\psi \in \Omega_1$ there is e_ψ such that $\psi + e_\psi = \psi$, taking any $\sigma \in M$ we notice that $\psi + \sigma = (\psi + e_\psi) + \sigma = (\psi + \sigma) + e_\psi$ and this implies that $\psi + \sigma \in \Omega_1$ since $\psi + \sigma \in N$ because N is ideal and $\psi \in N$, therefore Ω_1 is an ideal of M that is properly contained in N (the minimal ideal of M), it follows that $N = \Omega_1$.

To finish, we only need to show that there is a common identity to all elements of N, more precisely: we need to show that there is an element $e \in N$ such that $e + \psi = \psi \; \forall \; \psi \in N$. Or, equivalently: for every $\psi \in N$ consider the set $\Upsilon_\psi = \{e_\psi \in N \; ; \; \psi + e_\psi = e_\psi\}$, the objective is to show $\cap_{\psi \in N} \Upsilon_\psi \neq \emptyset$. As N is closed $\sum_{a=1}^{n} \phi_a \in N$ and satisfy $\sum_{a=1}^{n} \phi_a = \phi_1 + \sum_{a=2}^{n} \phi_a = \phi_1$ (up to a relabel of indices),by adding ϕ_i on both sides we obtain: $\phi_i + \phi_1 + \sum_{a=2}^{n} \phi_a = \phi_i + \phi_1$, this imply that $\sum_{a=2}^{n} \phi_a \in \Upsilon_{\phi_1 + \phi_i} \; \forall \; i \in \{1, ..., n\}$.

Now consider $\Omega_1 = \{\psi \in N \;:\; \psi = \phi_1 + \phi_m \text{ for some } n \in \mathbb{N}\}$ and $\Omega_2 = N - \Omega_1$ (the complement of $\Omega_1 \in N$), it is obvious that $N = \Omega_1 \sqcup \Omega_2$, the affirmation is that Ω_1 is an ideal of M that is properly contained in N (what in turn imply that $\Omega_2 = \emptyset$); to verify this notice that $\phi_n \in \Omega_1$ and $\psi \in M$,as there is $m \in \mathbb{N}$ such that $\phi_n = \phi_1 + \phi_m$ it follows that $\psi + \phi_n = \psi + (\phi_1 + \phi_m) = \phi_1 + (\psi + \phi_m)$, but there is $p \in \mathbb{N}$ with $\psi + \phi_m = \phi_p$ because $\psi + \phi_m \in N$, then $\psi + \phi_n \in \Omega_1$ as $\psi + \phi_n = \psi + \phi_p$, this allows us to conclude that Ω is an ideal and $N = \Omega_1$, or equivalently: $\sum_{a=2}^{n} \phi_a \in \cap_{\psi \in N} \Upsilon_\psi$.

It is also obvious that the cardinality of $\cap_{\psi \in N} \Upsilon_\psi$ is one, otherwise we take e and f, different elements of $\cap_{\psi \in N} \Upsilon_\psi$, then $e + f = e = f$ (because both are identities for all the elements of N) which is a contradiction. Denote by e the unique identity element of N.

III) *There is an inverse for every $\phi \in N$.*
Notice that e is its own inverse; now we take $\phi \neq e$, the sequence ϕ,2ϕ,...,$n\phi$,... (where $2\phi = \phi + \phi, 3\phi = \phi + \phi + \phi$ etc.) should have repetitions (otherwise N have infinite elements) take $p \in \mathbb{N}$ the minimum number which the sequence have the first recurrence (i.e. ϕ,2ϕ,...,$p\phi$ are all the different elements in the infinite sequence). If there is $n \in \{1, ..., p\}$ such that $n\phi = e$ then $\phi + (n-1)\phi = e$ indicates that $(n-1)\phi$ is the inverse of ϕ. On the contrary, if $n\phi \neq e$ for every $n \in \{1, ..., p\}$ then $(p+1)\phi$ is an element not on ϕ,2ϕ,...,$p\phi$, to see this notice that if $(p+1)\phi$ is in the list, then there is $q \in \mathbb{N}$, $1 < q < p+1$ such that $q\phi = (p+1)\phi$, now, if $q = p$ it follows that $p\phi = p\phi + \phi$ which means $\phi = e$, contradiction; therefore $q < p$, but this contradicts that p is the minimum number at which the sequence starts repeating. We conclude that $e \in \{\phi$,2ϕ,...,$p\phi\}$ and we can construct the inverse of ϕ (using the above argument) and the proof finishes. □

Corollary: 2.14. *Let $N = (\{\phi_1, \phi_2, ...\phi_n\}, +)$ be the sandpile group of the graph G, if $\sum_{a=1}^{n} \phi_a = \phi_i$ then the following statements are true:*

I)*The identity element of N can be computed as $e = \sum_{a=1, a\neq i}^{n} \phi_a$.*

II) *Denote by $-\phi_j$ inverse of the element $\phi_j \in N$, then $-\phi_j$ can be calculated as $-\phi_j = \sum_{a=1, a\neq i,j} \phi_a$.*

Proof. I) Follows from the proof of the above theorem.

II) Follows from the following equalities and the uniqueness of inverses in a group:

$$\phi_j + (-\phi_j) = \phi_j + \sum_{a=1,\neq i,j}^{n} \phi_a = \sum_{a=1,a\neq j} = e.$$

□

Sandpiles have many remarkable properties, but we try to bias the reader attention to one of their most intriguing epistemological properties: their astonishing ability to produce highly complex behaviour "without doing anything".

Typical (abstract or physical) Turing machines require a lot of non trivial hardware and instructions to display graphics (pre-loaded by humans) or to emulate some laws of physics; physics engines for modern videogames require specialized work of hundreds of developers using dedicated hardware and high level tools to be created, the mystery of cellular automatons is that they do not require more than a few laws (and very generic physical implementation) to produce beautiful patterns or to be *Turing complete*, so clearly the platonic existence of cellular atomatas is interesting and at this point is appropiate to ask: Why the sandpile model deserves special attention in between all the cellular automatons discovered up to date?

The answer is that the sandpile model looks like a completely new computational system far beyond the cellular automata level, it is true that it is defined in the same way as an ordinary one, but the sandpile dynamics is conjectured to be ready to solve problems that are demonstrably beyond the solving capability of any complete Turing machine, a very nice example of this is the problem of decide whether a given number is rational or not, stated in that way the puzzle is a not "well posed problem" because the complication of how to represent the input number, but even at this level of vagueness there is a precise sense in wich sandpile dynamics have something interesting to say about this puzzle [1] that is unattainable by classical computational schemes even in highly weaker forms [2].

On top of that, the output that sandpile dynamics gives highly non trivial outputs (for not well understood reasons), some notable examples are: self-organized behaviour [3],[4], soliton dynamics [5], relations to circle packing [6], proportional pattern formation [7], relations with turbulent hydrodynamics [8], pattern emergence in biology [9], combinatorial realization of logarithmic conformal field theories [10], etc. What could be a feature shared by most of the above examples?, a possible answer would be that most of them are computationally intractable.

Now we hope to make precise the idea that sandpiles generate highly complex behaviour "without doing anything" by studying random secuences of sandpile topplings on a given graph, always keeping in mind that there are no free (adjustable) parameters in the model but what there is one (and only one) simple local rule capable of producing a beautiful global (and highly correlated) pattern.

Mathematics is about all eternal and necessary forms of beauty, the goal of anyone interested in sandpiles is to understand how, why and what kind of beauty are sandpiles creating ... without the necessity of mathematicians.

2.1 Markov chains and sandpiles.

Notation 2.15. *Given a vertex v we denote by $\delta_v \in \mathbb{N}^V$ the sandpile that represents "a single chip at v" defined as follows:*

$$\delta_v(w) := \begin{cases} 1 & if \ v = w \\ 0 & otherwise \end{cases}$$

Definition 2.16. Let $G = (V, E)$ be a graph and M its set of stable sandpile configurations. By a **Markov chain or markovian evolution** in the sandpile model over the graph G with *initial state* $\phi \in M$ we understand any infinite secuence of the form:

$$\phi \mapsto \phi + \delta_{v(1)} \mapsto (\phi + \delta_{v(1)}) + \delta_{v(2)} \mapsto \cdots \mapsto ((\cdots(\phi + \delta_{v(1)}) + \cdots) + \delta_{v(n)} \mapsto \cdots$$

To stay the notation as simple as possible we write the above chain as:

$$\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$$

where $\phi_n = \phi_{n-1} + \delta_{v(n)} = \phi + \sum_{a=1}^{n} \delta_{v(a)}$ and it is understood that the addition of $\delta_{v(i)}$ (one grain of sand at the vertex $v_{v(i)}$) to $\phi_{(i-1)}$ at the $(i+1)-$step of the chain is *random* in the sense that the election of the vertex $v_{(i)}$ is arbitrary, in other words: $\{v(a)\}_{a=1}^{n}$ is an ordered multiset of randomly selected vertices of G. We also want to recall that the notation $\phi + \psi$ for ϕ and ψ sandpile states is an abuse of notation that always represents the stable state produced by the relaxation of the $\phi + \psi$ state.

Notation 2.17. *Let $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ be a Markov chain and we want to create an allegory of time evolution by saying that the state ϕ is the initial state of the markovian evolution and ϕ_n is the state at the time n of the evolution.*

Observation 2.18. *We are interested in to know the probability of appereance of a given sandpile state at "long times" during the evolution, so, we are interested in a time scale at which one should be allowed to make statistics, In the hereinafter we understand by "long times" in the course of a Markov chain $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ bigger indices that the time t defined as the time at which all vertices have suffered at least one topple.*

By "probability of appereance" in a Markov chain $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ with initial state ϕ "after a very long time t have passed" we mean to take $N \in \mathbb{N}$ and $\psi \in M$ and to produce a sequence $p_1, ..., p_m, ...$ where:

$$p_m = \frac{number \ of \ times \ that \ \psi \ appears \ in \ the \ set \ \{\phi_{t+mN}, \phi_{t+mN+1}, ..., \phi_{t+(m+1)N}\}}{N}$$

If $\lim_{m \longrightarrow \infty} p_m$ exist we say that this value is the probability of appereance of ψ in the Markov chain $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ at long times.

We now give a set of important definitions that pretend to formalize certain attributes that sandpile states should acquire under typical markovian evolutions.

Definition 2.19. An stable sandpile state ϕ is **allowed in a Markov chain** if it appears at least one time in the alluded Markov chain at long times.

It is obvious that this set is not empty, otherwise there are no dynamics at all, what it is much more exciting is the possibility of a state common to all possible Markov chains.

Definition 2.20. An stable sandpile state ϕ is **allowed** if it appears at least one time in all Markov chains at long times. The subset of allowed states will be denoted by $\mathcal{A}$.

The dynamics begin to be interesting once is recognized that there are local obstructions for a state to appear at late times, is not obvious at all why a particular state could be "forbidden".

Definition 2.21. An stable sandpile state ϕ is **forbidden in a Markov chain** if it never appears in the alluded Markov chain at long times.

Definition 2.22. An stable sandpile state σ is **forbidden** if never appears at long times in all Markov chains.The subset of forbidden states will be denoted by $\mathcal{F}$.

Although it is simple (as in allowed state definition) that this the following set is not empty, the existence of a state that appears infinite many times in a chain is interesting, because whit it we can potentially define interesting dynamical parameters like the concept of *recurrence time.*

Definition 2.23. An stable sandpile state ϕ is **recurrent in a Markov chain** if it appears with probability one in the alluded Markov chain at long times.

It is intuitively obvious that some states should appear an infinite number of times in a particular Markov chain because the chain is infinite and the number of possible states finite, what could be a little bit surprising is the fact that all Markov chains are eventually the same and in some sense independent of the state used as initial condition.

Definition 2.24. An stable sandpile state ψ is **recurrent** if it has probability one of appearing in any Markov chain at long times.The subset of recurrent states will be denoted by $\mathcal{R}$.

Jorge Luis Borges used to say that because the destiny of human aesthetic search is fundamentally the amazement, all the great tales should start with a mystery worthy of the size of their aspiration. Trying to imitate the teaching of a supreme mind (with immense humility and poor execution) we adopt the goal to state what the immense mystery of sandpiles actually is to produce the amazement of the reader as quicky as possible. For this reason we state (and we use) three important theorems about sandpile dynamics and postpone their respective proofs to the next chapter.

Theorem 2.25. *The following three statements are correct:*

I*)If ϕ is allowed at some Markov chain, then, it is allowed.*

II*)If ϕ is forbidden at some Markov chain, then, it is forbidden.*

III*)If the sandpile ϕ is recurrent at some Markov chain, then, it is recurrent.*

Theorem 2.26. *The set of allowed states is the same as the set of recurrent states. In symbols: $\mathcal{A} = \mathcal{R}$.*

Theorem 2.27. *The set M of stable sandpile states can be written as $M = \mathcal{A} \bigsqcup \mathcal{F}$.*

We want to make two remarks about why is important and appropriate to translate the sandpile dynamics into an algebraic language.

The first one is simply to take a second to appreciate that theorem 2.22 is an equality between the algebra of stable sandpiles (M) and the dynamical characterization ($\mathcal{A}$ and $\mathcal{R}$) that sandpile generates on them; this facts deserves attention because this is typical of highly contrived physical systems like quantum field theories (where physical states always "transform" as irreducible representations of vertex, Lie algebras, etc.) but surely is not a common feature of cellular automatas not to mention that the level of algebra used by sandpiles is as minimum as possible: no algebras, no vector spaces or groups only the additive structure of the natural numbers.

Along the worlds of physics and mathematics we can find very useful *slogans* that even when not precise can serve as principles to develop intuition; beautiful examples are the following:

Art	Slogan
String theory	Every object is a soliton
Algebraic Geometry	Every ring wants to be the ring of coordinates of some variety
Differential topology	Continuous shapes can be approximated by smooth shapes
Two dimensional conformal field theory	Every modular invariant function should be the partition function (or conformal block) of some unitary compact 2d CFT

Given a multigraph G, M its monoid of stable sandpiles and understanding by *ideals* the ideals of M, we can state our slogan to develop intuition for the study of sandpile dynamics as:

"Ideals can not be evaded, because ideals are the *attractors* of sandpile evolution."

This slogan is made precise by the following:

Lemma 2.28. *Let M be the monoid of stable sandpiles of a multigraph G, for arbitrary $\phi \in M$, every ideal J of M and any Markov chain $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$, there is $m \in M$ such that for all $n \in \mathbb{N}$ with $n \geq m$ it follows that $\phi_n \in J$.*

Proof. Notice that if for some markovian evolution $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ there is $m \in \mathbb{N}$ and an ideal J of M such that $\phi_m \in J$ then $\phi_{m+1} = \phi_m + \delta_{v(m)} \in J$ because J is an ideal of M and $\delta_{v(m)} \in M$ no matter what vertex $v(m)$ is. It is immediate to show using induction and the same argument, that $\phi_N \in J$ for all $N \geq m$.

Suppose that there could be a particular Markov chain $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ with the property that for long times $\phi_N \notin J$ for all N greater that the long time parameter of the chain (the time at which all vertices have been toppled at least once), it follows that every state $\sigma \in J$ is a forbidden one because is forbidden for this particular Markov chain; now consider any Markov chain starting at some particular $\sigma \in J$ as we have seen, every state σ_m of $\sigma \mapsto \sigma_1 \mapsto \sigma_2 \mapsto \cdots \mapsto \sigma_n \mapsto \cdots$ verifies that $\sigma_m \in J$ then, by theorem) σ_m allowed in J imply σ_m allowed, contradiction, there are not markovian evolutions that not *reach* J at some time.

□

We are ready to state the theorem that makes platonic the particular mystery of the sandpile that we are interested in.

Theorem 2.29. *The set $\mathcal{R}$ of recurrent sandpiles is the minimal ideal N of the monoid M of all stable sandpiles.*

Proof. To see that $R \subset N$ take $\phi \in R$, and any Markov chain σ_m of $\sigma \mapsto \sigma_1 \mapsto \sigma_2 \mapsto \cdots \mapsto \sigma_n \mapsto \cdots$ using our lemma 2.27 (our slogan) there is $N \in \mathbb{N}$ such that for all $N \leq m$ follows that $\sigma_m \in N$, furthermore, as ϕ is recurrent there is $n \geq N$ such that $\sigma_n = \phi$ but as σ_n is in N we conclude that $\phi \in M$ and $R \subset N$.

To show $N \subset R$ let $\mathcal{A}_N := \{\phi \in N | \ \phi \ is \ recurrent\}$ and $\mathcal{F}_N := \{\psi \in N | \ \psi \ is \ forbidden\}$, clearly $N = \mathcal{A}_N \sqcup \mathcal{F}_N$, we affirm that $\mathcal{A}_N$ is an ideal of M; notice that $\mathcal{A}_N$ is non empty because all the states of all Markov chains are eventually elements of N, if $\mathcal{A}_N$ was empty, all the states of any Markov chain would be forbidden which is a dramatic contradiction, then $\mathcal{A}_N$ is non empty; now take $\phi \in \mathcal{A}_N$ and $\delta_v \in M$ an notice that $\phi + \delta_v \in N$ for all vertex v simply because N is ideal of M, and if $\phi \in N$ and $\delta_v \in M$ then $\phi + \delta_v \in N$; furthermore $\phi + \delta_v \in \mathcal{A}_N$ for all vertices v because ϕ is recurrent and it appears infinitely many times in all Markov chains and at least in one of them the next step is $\phi + \delta_v$ (otherwise we are contradicting the fact that the election of v in δ_v is equiprobable over all the vertices of G and we are considering all possible Markov chains) that means that $\phi + \delta_v$ is allowed and by theorem 2.22 recurrent for any vertex v. By repeated use of the same argument one can prove that $\phi + \delta_{v(1)} + \delta_{v(2)} + \ldots + \delta_{v(n)}$ is an element of $\mathcal{A}_N$ for any sequence of vertices $\{v(1), ..., v(n)\}$, finally, as any $\psi \in M$ can be writed as $\psi = \sum_{all \ vertices} n_v \delta_v$ with $n_v \leq d(v)$, it follows from this and the above conclusion that $\phi + \psi = \phi + \sum_{all \ vertices} n_v \delta_v \in \mathcal{A}_N$ what shows that $\mathcal{A}_N$ is an ideal of M contained in the minimal ideal N of M, it follows that $N = \mathcal{A}_N$ and that every element of N is recurrent, then $N \subset R$.

We conclude that $N = R$ and the proof finishes.

□

2.2 A gallery of sandpile mysteries

The mystery of the identity

We want to note that is intriguing about the way to equip the ideal N with a group structure as was done in lemma 2.12, because is not obvious at all what the group identity should be. Using our slogan for sandpile dynamics we understand that the identity ϵ of N is important because if some recurrent state ϕ appears in a given Markov chain then the expected time that takes the element $\phi + \epsilon$ to be generated by the Markov chain is exactly the Poincaré recurrence time of the sandpile dynamics. So, sandpile have demonstrated to have very interesting dynamics, but if they also have an interesting statistics (that in fact they have) then there should be an interesting thermodynamics and the identity element would be intimately related to the time scale at which we expect the breakdown of the *second law*.

We start the exposure with the following mystery:

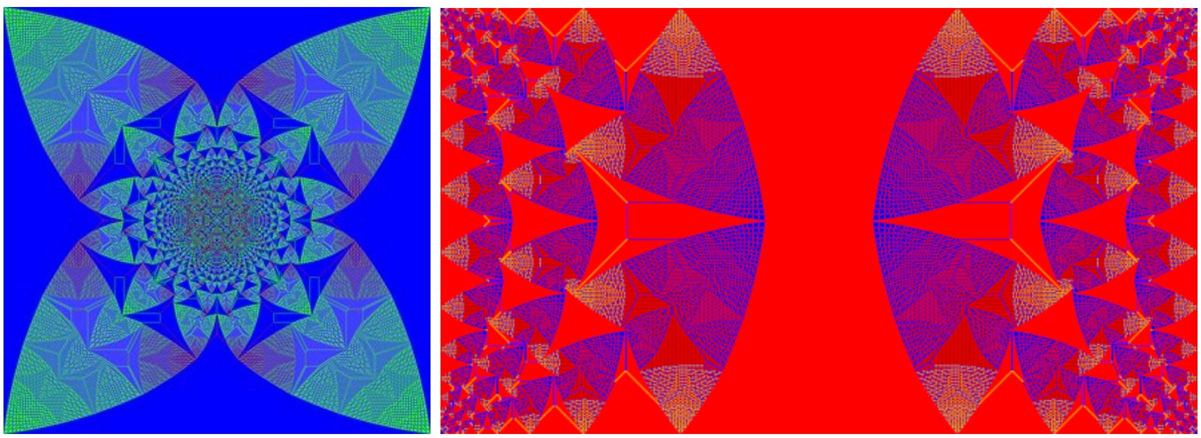
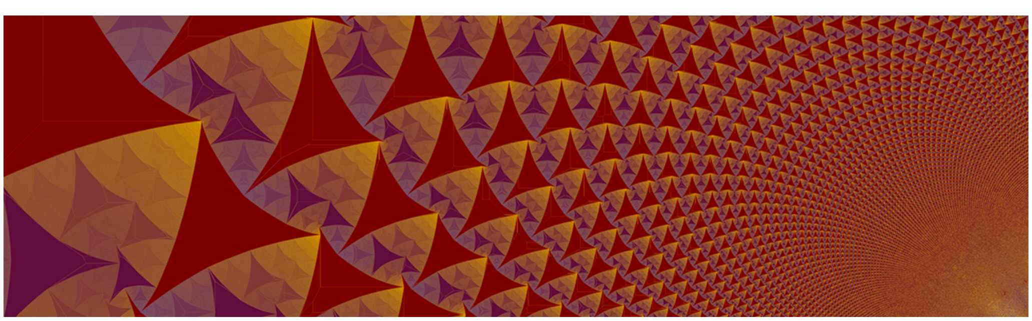


Figure on the left: The identity element of the sandpile group on a square grid of size 521 $\times$ 521, the colors in the draw represent the number of grains of sand that are over a given vertex, the convention here is: orange = 0 grains, red = 1 grain, green = 2 grains and blue = 3 grains. The image was constructed in [11].

Figure on the right: The identity element of the sandpile group on a rectangular lattice of size 780 $\times$ 390. the convention here is: yellow = 0 grains, orange = 1 grain, blue = 2 grains and red = 3 grains. The image was constructed in [12]

The mystery of a typical element:

This mystery can be stated by showing how a generic recurrent elements of the sandpile group look like:



A sandpile recurrent element on a circular grid (in which every vertex has degree four) obtained as the relaxation on the element that have 2^{20} chips at the center and zero in all the remaining sites; the colors in the draw represent the number of grains of sand that are over a given vertex, the convention here is: blue = 0 grains, yellow = 1 grain, purple = 2 grains and brown = 3 grains. The image was constructed in [13].

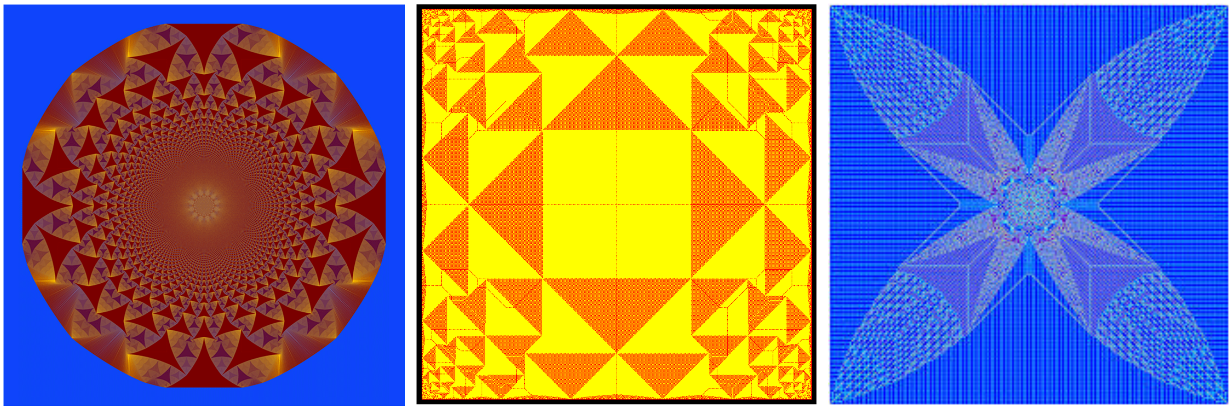
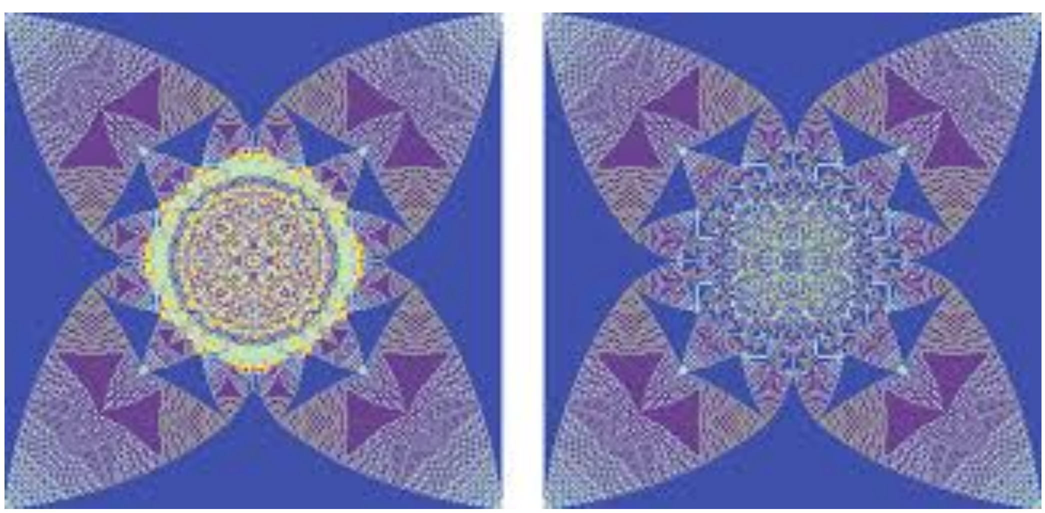


Figure on the left: A complete view of the above figure on a circular grid.
Figure on the center: The identity element of the sandpile group on a variant of the square grid, for details see [14].
Figure at the right: The relaxation of the state with three chips at every vertex plus 15000 chips at the center of a square grid. Later we prove that this state is a recurrent one (an in fact all recurrent states can be obtained in that way). The figure was constructed in [15]

Holography:

This example is extracted from [16]. Sandpiles can be defined for higher dimensional lattices. The intriguing fact is that slices of d-dimensional sandpiles in a d-dimensional grid look like (d-1) sandpiles in a (d-1)-slice on the referred d dimensional lattice. So, it looks like the mystery essentially *lives* in two dimensions.

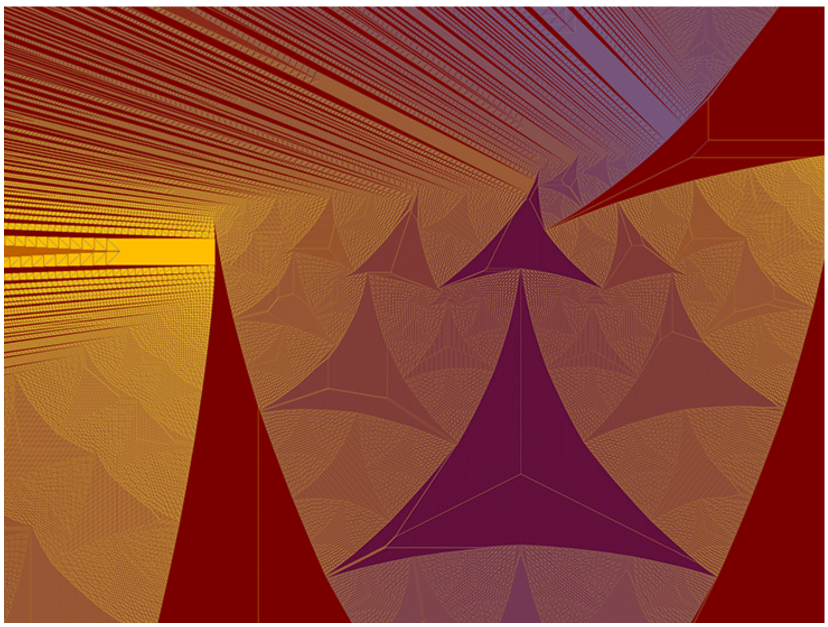


Left: A two-dimensional slice through the origin of the three dimensional sandpile constructed as the relaxation of the configuration that have four chips at every vertex of $\mathbb{Z}^3$ and is perturbed by the addition of $n = 5 \cdot \times 10^6$ grains at the center of the grid. Color scheme on left: sites colored blue have 5 particles, turquoise 4, yellow 3, red 2, gray 1 and white 0.

Right: Sandpile constructed as the relaxation of the configuration that have three chips at every vertex of $\mathbb{Z}^2$ and is perturbed by the addition of $m = 47465$ grains at the center of the grid. Color scheme: blue 3 particles, turquoise 2, yellow 1 and red 0.

What kind of geometry is that?

Classical differential geometry studies topological spaces that locally look like real vector spaces, the vector space structure is vital to represent some geometric operations (scallings or reflexions) as multiplication of vectors by real numbers, a similar story holds for complex geometry, brilliant ideas and 20th century developments made it possible to use rings to define geometry unraveling a deep world of relationships between geometry and arithmetics. We present our final mystery:



It was proven in [17] that recurrent states over the square grid have a *deviation set* from the state with three chips at every vertex (this would be defined later) that is a balanced graph (in the scalling limit). In the figure can be appreciated as white strands.

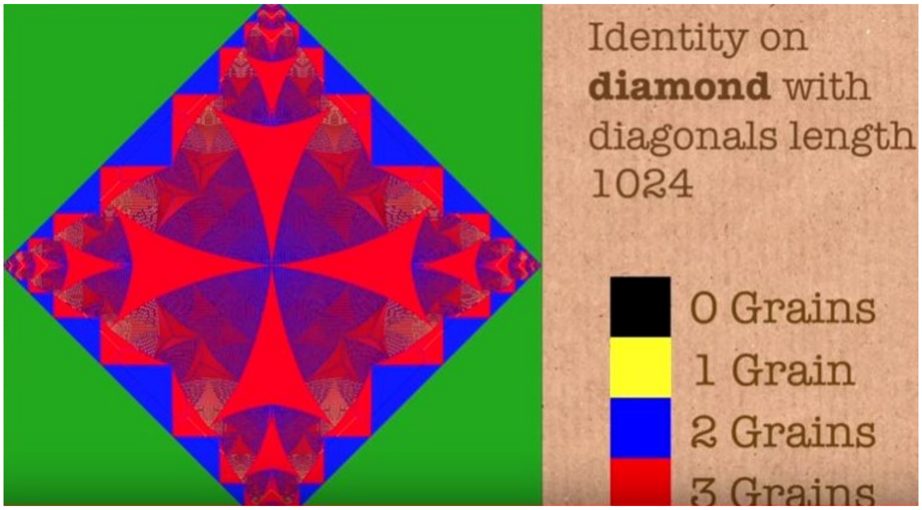
Of course the amazement is because of the impressive level of complexity reached by such a simple system, balanced graphs for a mathematician means tropical curves and for a physicist webs of (p,q) 5-branes from string theory. But the supreme amazement starts when one try to think about the graphs as representing tropical curves inside local two-dimensional proyective spaces, but this is actually not the case, the triangles could not serve as ordinary bases to construct a local projective space because their edges are *curved*, we are forced to think that this is a new type of geometry.

A new type of toric geometry ... whose origin is *just* the additive structure of the natural numbers.

We finish with a very simple and exciting observation as a motivation for the following chapter :

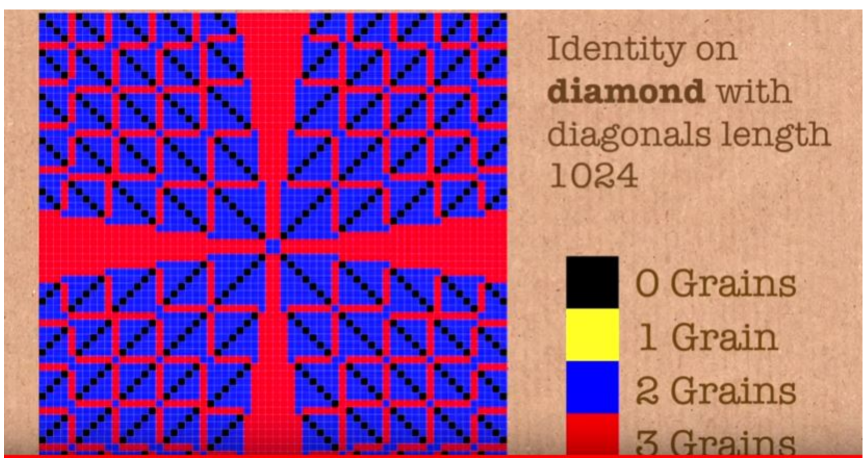
After the shock caused by the beauty of the sandpiles perhaps the first intriguing aspect that is evident from the pictures showed, is that it is clear that there are very few vertices with zero chips on it and apparently in average there are a very few vertices with zero chips over it.

The following picture extracted from [26] pretends to be illustrative in this respect.



Notice how few colored in black sites are on this sandpile, it looks plausible that the sandpile contain in fact no vertices with zero chips.

But a *zoom* into the center of the picture amazingly shows that in fact there are some vertices with no chips on over it.



And something beautiful emerges! the vertices with no chips tend to group into diagonals!, it looks like they prefer to not to be together, to not being adjacent.

This could be very easily explained as follows:

Theorem 2.30. *Two adjacent vertices with zero chips on them are impossible in any allowed configuration.*

Proof. On the contrary, suppose that there are two adjacent vertices v and w with zero chips on some allowed state, by the definition of allowed both vertices have been toppled at least once; let v the one that toppled most recently. When v toppled, it gave a chip to each of its neighbors, in particular one to w. But w has not been toppled since then still has a grain, and can not have zero chips. □

Corollary: 2.31. *Two adjacent vertices with zero chips on them are impossible in any recurrent configuration.*

Proof. Follows immediately from the fact that $\mathcal{R} \subset \mathcal{A}$ and the later theorem. $\square$

The lesson from those extremely simple observation is that it is in fact possible to characterize or discard some clusters in recurrent sandpiles from general principles. Even when the observation looks childish, for the author looks magical and it is likely that if we had the tools to create similar simple theorems about how vertices with some amount of chips group, maybe then would be able to understand the fractal structure of the sandpile.

The next sections explore certain basic aspects of how sand distributes in stable sandpile configurations.

3 Sandpile dynamics

"Esa infinitud condice de admirable manera con los sinuosos números del Azar y con el Arquetipo Celestial de la Lotería, que adoran los platónicos...
Jorge Luis Borges Acevedo - La lotería en Babilonia - Ficciones (1944).

This chapter is dedicated to characterize recurrent (and forbidden) sandpile states by imposing constrains on which sub-configurations (*clusters*) can appear or not on generic Markov chains at large times. As has been done so far, every affirmation presupposes that a graph $G = (V, E)$ has been given and that M denotes the set of stable sandpile states over G.

Proposition 3.1. *In any Markov chain every vertex topples an infinite amount of times.*

Proof. As G is supposed to be finite and the election of the vertex to which we add a chip at some time during the chain $\phi_0 \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ is random, if there would be a vertex v that does not topple during the evolution then v is accumulating an infinite number of chips so $d(v)$ (the degree of v) is infinite which is a contradiction to our hypothesis about G. Then every vertex topples at least once.

In fact, if some v topples only a finite number of times there is some time t at which topples ceases on v, then, after this time v is accumulating and infinite number of chips and this is again a contradiction. $\square$

Definition 3.2. To any graph G we can associate a distinguished sandpile state known as the *maximally stable state* that would be denoted as Ξ_G and defined as:

$$\Xi_G(v) \ = \ d(v), \qquad \forall\, v \in V.$$

In what follows we drop the subindex G on Ξ_G to write simply Ξ.

Definition 3.3. To every sandpile state ϕ we define its *deviation set from* Ξ by

$$D[\phi] := \{v \in V \ : \ \phi(v) \neq \Xi(v)\}$$

In what follows we refer to $D[\phi]$ simply as *the deviation set of* ϕ.

Ξ is in fact our first example of a recurrent sandpile state, although this statement (and its proof) is an "almost tautology", we state it as a theorem because it is the pillar from which we would deduce a lot of important properties (even characterize) the long time evolution of any Markov chain.

Theorem 3.4. Ξ *is recurrent.*

Proof. Let $\mathcal{C} = \phi_0 \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ be any Markov chain, it is clear that the set of recurrent states of $\mathcal{C}$ is non empty because the set of stable sandpile states is finite and the number of steps on the Markov chain infinite, then, there is at least one state that should be repeated an infinite amount of times, if Ξ is one of them the proof finishes, if not take $\psi \neq \Xi$ be one of those states; now notice that necessarily there is $N \in \mathbb{N}$ such that $\phi_N = \psi$ and $\phi_{N+t} = \psi + \sum_{v \in D[\psi]} n_v \delta_v = \Xi$ (here t is the order of $D[\psi]$ and $n_v = d(v) - \psi(v)$) otherwise there is no sequence of addition of exactly one grain to ψ at every vertex of $D[\psi]$ and $\mathcal{C}$ could not be completely random, in fact, this would be happen an infinite amount of times during the markovian evolution (there should be an infinite number of different numbers $m \in \mathbb{N}$ such that $\phi_m = \Xi$) because otherwise, again $\mathcal{C}$ could not be random.The later is a contradiction, then Ξ is recurrent in $\mathcal{C}$, and as $\mathcal{C}$ is arbitrary we conclude that Ξ is recurrent. $\square$

There should be stressed that the way that was argued that Ξ appears an infinite amount of times during a markovian chain suggests that every recurrent state can be obtained from addition of grains of sand to a known recurrent one, in particular to Ξ. This is in fact true and is now what we want to formalize.

Corollary: 3.5. *Let v be a vertex of the graph G, then $\Xi + \delta_v$ is recurrent.*

Proof. By using the last theorem we know that given any Markov chain $\mathcal{C} = \phi_0 \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ there are an infinite number of different subindices t such that $\phi_t = \Xi$ and also there should be an infinite subset of those ones such that $\phi_t = \Xi + \delta_v$, because on the contrary $\mathcal{C}$ would not be random. Then $\Xi + \delta_v$ appear an infinite number of times on $\mathcal{C}$, as $\mathcal{C}$ is arbitrary $\Xi + \delta_v$ is recurrent. □

Corollary: 3.6. *Let W be a set of vertices of G, then $\Xi + \sum_{v\in W} \delta_v$ is recurrent.*

Proof. It is immediate using induction and the last corollary. □

Theorem 3.7. *Every recurrent configuration ψ can be represented as $\Xi + \sum_{v\in W} \delta_v$ for some subset W of vertices of G.*

Proof. If we suppose that the assertion is false, then in any Markov chain $\mathcal{C}$ once Ξ have appeared at some time ψ can not be a state at following times (because all of them are of the form $\Xi + \sum_{v\in X} \delta_v$ for some set of vertices X). It follows that ψ is forbidden, contradiction. □

Corollary: 3.8. *Let ψ be a recurrent state, then $\psi + \phi$ where ϕ is any element of $\mathbb{N}^V$ is recurrent.*

Proof. Immediate. □

We are now able to prove a theorem stated in the last chapter without a proof.

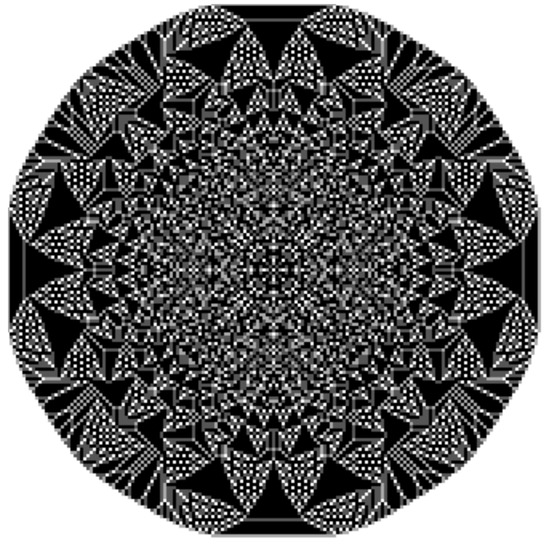
Proof of part I) of theorem 2.25:

Let ψ be a recurrent configuration on $\mathcal{C}_\psi = \phi_0 \mapsto \cdots$; as Ξ is recurrent in every configuration, in particular, it is on $\mathcal{C}_\psi$, which means that there is a large time $t \in \mathbb{N}$ such that $\Xi = \phi_t$; but there is also $s \in \mathbb{N}$ such that $\phi_{t+s} = \psi = \Xi + \sum_{v\in W} \delta_v$ for some multi set W with $|W| = s$ and we finish by using theorem 3.7 to conclude that ψ is recurrent at every Markov chain. □

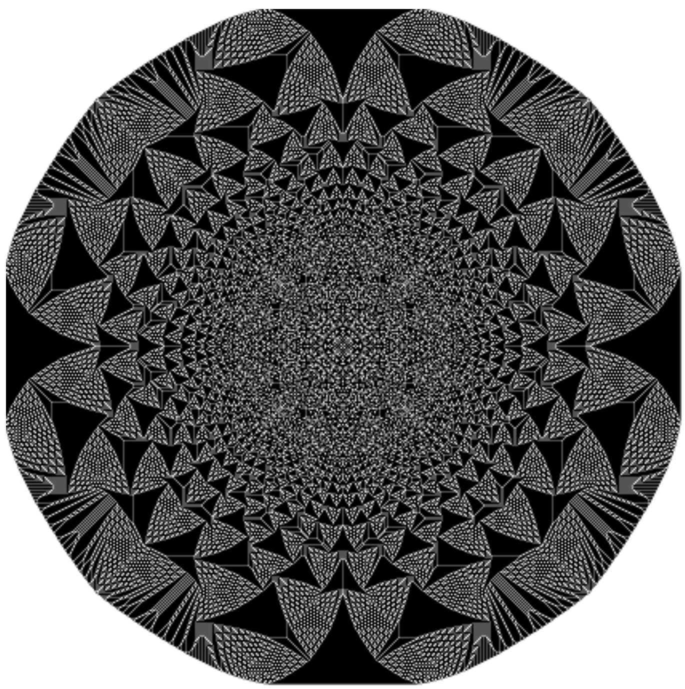
The significance of the above statements is morally a reinforcement of our proposed slogan *"Ideals are attractors of the sandpile evolution"* in the sense that once a Markov chain has reached any configuration on the intersection of all ideals (attractors) it never get out the intersection.

Example: 3.9. *As much of the astonishment within sandpiles is visual, we love to illustrate how is generated a recurrent sandpile putting a grain of sand at the center of the maximally stable state Ξ on a circular table and appreciating how the sandpile relaxes. It is pointed that the degree of every vertex on the table is four.*

Below to the left is shown the relaxation process after 1000 toppled vertices and below to the right the relaxation after 50000 toppled vertices.

After 333333 topples the fractal structure of the sandpile will begin to be evident:



In [34] is discussed how to create algorithms to display sandpile images on a computer after n steps by perturbing arbitrary initial states.

3.1 Forbidden configurations:

Definition 3.10. Let $G = (V, E, \circ)$ be a graph and ϕ a sandpile state on G. *The graph of* the sandpile state ϕ is defined as:

$$\Gamma[\phi] := \{[v, \phi(v)]\ ; v \in G - \{\circ\}, \phi(v) \in \mathbb{N}\}.$$

$\Gamma[\phi]$ can be imagined as a *sand mound* over G that can be thinked as a *table* (if G admits a planar embedding).

Definition 3.11. Let ϕ be a sandpile state, a *sub configuration* $\mathcal{F}[\phi] \subset \Gamma[\phi]$ will be called a *forbidden cluster* if:

$$\phi(v) \;\leq\; deg_{\mathcal{F}[\phi]}(v) \;=\; \sum_{v \neq w, w \in \mathcal{F}[\phi]} \Delta_{v,w}$$

where we denote by $deg_{\mathcal{F}}(v)$ the number of bonds conecting the site v with the other sites of the subset $\mathcal{F}[\phi]$, in what follows we will simply write $d_{\mathcal{F}}(v)$ instead of $deg_{\mathcal{F}}(v)$.

It is important to notice that the order of $\mathcal{F}[\phi]$ is always greater of equal to two.

Examples: 3.12. *The following sandpile states on a rectangular grid (of size 3×4 for the left state and 3×5 for the right one) contain forbidden sub configurations (the sequence **1 1** for the left state and **1 2 1** for the right one):*

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & \boldsymbol{1} & \boldsymbol{1} & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & \boldsymbol{1} & \boldsymbol{2} & \boldsymbol{1} & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

*Another sandpile states with forbidden clusters are shown, below at the left we have a rectangular grid of size 3×6 with a forbidden cluster **0 0** marked on bold numbers. For the below at the left state we have a dramatic example of a grid of size 4×5 for which all its vertices constitute a forbidden configuration.*

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & \boldsymbol{0} & \boldsymbol{0} & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} \end{pmatrix}.$$

Observation 3.13. *The next example is one without forbidden clusters:*

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

It is shown with the intention to avoid the potential confusion of thinking that vertices with zero chips are forbidden. The definition $\phi(v) \;\leq\; deg_{\mathcal{F}[\phi]}(v)$ requires that the order of $\mathcal{F}[\phi]$ is at least two, but neither of the adjacent vertices of to the center of the array (the vertex with zero chips) is forbidden.

It is also easy to construct a recurrent configuration on the 3×3 square grid with exactly one vertex with zero chips on it:

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

the matrix on the right hand of the equality side is recurrent because the left hand side is recurrent as a consequence of corollary 3.6.

Notation 3.14. *To stay the language as simple as possible we will incur in the following abuse of notation: whenever is indicated that "$v \in \mathcal{F}[\phi]$" we should understand that if $\mathcal{F}[\phi] = (W, \phi(W))$ for some sandpile state ϕ an $W \subset V$ we are indicating that $v \in W$.*

Theorem 3.15. *If ϕ is a late time configuration of a Markov chain $\mathcal{C}_\phi$ then $\mathcal{F}[\phi] = \emptyset$.*

Proof. Suppose on the contrary that $\mathcal{F}[\phi] \neq \emptyset$ and notice that as ϕ is a late time configuration, every vertex on G has suffered at least one topple before $\mathcal{C}$ reached the ϕ state, now take $v \in \mathcal{F}[\phi]$ such that all its neighbours in $\mathcal{F}[\phi]$ have been toppled before v, then the number of grains of sand on v satisfy $\phi(v) \geq d_{\mathcal{F}}(v)$, but if v is going to topple after all its neighbours do it then there is at least one extra grain of sand on v (otherwise v is unable to topple) then $\phi(v) \geq d_{\mathcal{F}}(v) + 1$, so $v \notin \mathcal{F}[\phi]$ an this is a contradiction because it was supposed that $v \in \mathcal{F}[\phi]$. □

Corollary: 3.16. *If $\mathcal{F}[\phi] \neq \emptyset$ for some sandpile state ϕ, then, ϕ can not be allowed ($\phi \notin \mathcal{A}$).*

Proof. If we suppose that on the contrary, for some Markov chain $\mathcal{C}$ ϕ is allowed but $\mathcal{F}[\phi] \neq \emptyset$ then we contradict the later theorem. □

Corollary: 3.17. *A sandpile state ϕ is forbidden if and only if $\mathcal{F}[\phi] = \emptyset$.*

Proof. Immediate. □

As was recalled many times, every result reinforces our slogan about ideals, in particular, we are able to prove that at late times once a Markov chain has reached an allowed configuration then every subsequent state is also an allowed configuration. In particular, it will be proved at the end of this subsection that $\mathcal{A} = \mathcal{R}$ (theorem 2.25) that in view of the intuition that we have been gained mean that for a sandpile element the property of *non to being an element of the minimal ideal* of M is the basic dynamical obstruction to be allowed.

The following theorem states that the set of allowed states is closed under a Markov evolution. We warn that the prove of this fact is lengthy and non trivial, for this reason is explained in detail and we offer some examples of how the observations work.

Theorem 3.18. *If ϕ is allowed, then all process of addition of chips to ϕ followed by relaxation finishes in an allowed sandiple state, in particular, any Markov chain starting at ϕ.*

Proof. Let W be an arbitrary set of vertices of V, consider the process of *addition of one chip* to ϕ at every vertex of W, name the relaxed state as $\psi = \phi + \sum_{w \in W} \delta_w$ but suppose that ψ is forbidden, notice that as W is an arbitrary set and ψ is an intermediate state of a Markow chain starting at ϕ.

Because ψ is forbidden, then $\mathcal{F}[\psi] \neq \emptyset$ and we can take $v \in \mathcal{F}[\psi]$ such that there was at least one topple on v during the relaxation process $\phi \mapsto \psi$ (if it were not possible to take such a v then there was a copy of a subset of $\mathcal{F}[\psi]$ contained in $\mathcal{F}[\phi]$ what contradicts that ϕ is allowed); it is affirmed that there is a copy of $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\}$ such that v_i have toppled for all $i \in \{1, ..., n\}$ during $\phi \mapsto \psi$ on $\mathcal{F}[\phi]$ that is forbidden on $\Gamma[\phi]$.

As an aside we illustrate how the affirmation works on a very simple example:

Consider the following avalanche process between two forbidden configurations in which time goes from left to right:

$$\phi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & \mathbf{0} & 2 & 3 & 3 \\ 3 & \mathbf{0} & 3 & 2 & 3 \\ 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \quad \mapsto \quad \phi + \ \delta_{center} = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 2 & 3 & 3 \\ 3 & 0 & 4 & 2 & 3 \\ 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \quad \mapsto \quad \psi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & \mathbf{0} & 3 & 3 & 3 \\ 3 & \mathbf{1} & \mathbf{0} & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

The bold numbers conform forbidden sub configurations when present in stable states. ϕ is explicitly forbidden as a consequence of theorem 2.30.

As can be noticed only the vertex of the array have suffered a topple.

Now let's see the process in reversing time and marking the center of the array with the symbol $\diamond$.

$$\psi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & \mathbf{0} & 3 & 3 & 3 \\ 3 & \mathbf{1} & \diamond & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \qquad \mapsto \qquad \phi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & \mathbf{0} & 2 & 3 & 3 \\ 3 & \mathbf{0} & \diamond & 2 & 3 \\ 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

In fact, the cluster of bold numbers in ψ produce under time reversal a forbidden cluster on ϕ just by eliminating the only vertex that have been toppled. The example also shows that there is a copy of $\mathcal{F}[\psi] - (\diamond, \psi(\diamond))$ on $\Gamma[\phi]$, then $\mathcal{F}[\phi] \neq \emptyset$.

On the contrary, if we mark any other vertex on ψ (forbidden or not) that was not toppled during the relaxation process it does not necessarily produce a forbidden cluster on ϕ as can be showed by the following example:

$$\psi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & \diamond & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix} \qquad \mapsto \qquad \phi = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 2 & 3 & 3 \\ 3 & \diamond & 3 & 2 & 3 \\ 3 & 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

In this example it is seen that no forbidden configurations are manifest even when both of ϕ and ψ are forbidden.

It is worthy emphasized that the marked point is not literally deleted on our graph nor from any configuration, it is only highlighted in order to detect a prohibited sub configuration on $\Gamma\phi]$ given one in $\Gamma[\psi]$.

Now we return to our goal of proving the affirmation $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \subset \mathcal{F}[\phi]$ with $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \neq \emptyset$.

We show that $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\}$ is non empty by induction.

Consider process $\phi \mapsto \phi^1$ (adding grains to ϕ and relaxing the resulting supposed unstable state to ϕ^1) where ϕ is an allowed sandpile state and ϕ^1 is a state with $\mathcal{F}[\phi^1] \neq \emptyset$ and with exactly one forbidden vertex v_1 toppled during the avalanche $\phi \mapsto \phi^1$. It can be noticed the existence of at least one non toppled (during $\phi \mapsto \phi^1$) and forbidden neighbour vertex of v_1, otherwise v_1 can not be forbidden because the very definition of forbidden require one forbidden neighbour and since this neighbour can not be toppled (during $\phi \mapsto \phi^1$) since by hypothesis v_1 is the only toppled one, we can guarantee that $\mathcal{F}[\phi^1] - \{v_1\} \neq \emptyset$.

By induction hypothesis suppose that every process $\phi \mapsto \phi^k$ from an allowed sandpile state ϕ to a state ϕ^k with exactly k forbidden vertices $\{v_1, v_2, ..., v_k\}$ not toppled during the avalanche $\phi \mapsto \phi^k$ verify that $\mathcal{F}[\phi^k] - \{v_1, v_2, ..., v_k\} \neq \emptyset$.

Now consider the process $\phi \mapsto \phi^{k+1}$ from an allowed sandpile state ϕ to a state ϕ^{k+1} with exactly $k+1$ forbidden vertices $\{v_1, v_2, ..., v_{k+1}\}$ but suppose that contrary to our expectations $\mathcal{F}[\phi^{k+1}] - \{v_1, v_2, ..., v_{k+1}\} = \emptyset$ or $\mathcal{F}[\phi^{k+1}] = \{v_1, v_2, ..., v_{k+1}\}$. In that case we can add grains of sand to any vertex on $\mathcal{F}[\phi^{k+1}]$ (without toppling it) to transform it into an allowed one, this always can be done, and such addition of grains never produce more forbidden states, denote this process of adding grains to ϕ^{k+1} as $\phi^{k+1} \mapsto \psi$; then the set of forbidden vertices of ψ is exactly the one of ϕ^{k+1} with exactly one vertex less (the aforementioned vertex with extra grains), it is to say the order of $\mathcal{F}$ is k.

To conclude, notice that the process $\phi \mapsto \phi^{k+1} \mapsto \psi$ is a process form an allowed configuration ϕ to a configuration with precisely k forbidden vertices toppled during the avalanche $\phi \mapsto \phi^{k+1} \mapsto \psi$, in fact, those one produced after the step $\phi \mapsto \phi^{k+1}$ because $\phi^{k+1} \mapsto \psi$ does not have produced topples, but this contradicts the induction hypothesis, then $\mathcal{F}[\phi^{k+1}] - \{v_1, v_2, ..., v_{k+1}\} = \emptyset$ is impossible.

Then $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \neq \emptyset$.

We now prove that $\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \subset \mathcal{F}[\phi]$ by using again an inductive argument.

Suppose that during the avalanche $\phi \to \psi$ only one toppling at the vertex v takes place. As ϕ contain forbidden sub configurations and ψ not it follows that $v \in \mathcal{F}[\psi]$, also we know that $\mathcal{F}[\psi] - \{v\}$ is non empty, then take $w \in \mathcal{F}[\psi] - \{v\}$ and notice that:

$$\psi(w) = \phi(w) + \eta_{vw},$$

where η_{vw} is the number of grains that v gives to w, more precisely:

$$\eta_{vw} = \begin{cases} 1 & if\ v \sim w \\ 0 & otherwise \end{cases}$$

On the other hand, we know that $w \in \mathcal{F}[\phi]$ is equivalent to:

$$\psi(v) \leq deg_{\mathcal{F}[\psi]}(v),$$

therefore

$$\phi(w) \leq deg_{\mathcal{F}[\psi]} - \eta_{vw}.$$

in fact

$$deg_{\mathcal{F}[\psi]} = deg_{\mathcal{F}[\psi]-\{v\}} + \epsilon_{vw},$$

where

$$\epsilon_{vw} = \begin{cases} 1 & if\ v \sim w \\ 0 & otherwise \end{cases}$$

then

$$\phi(w) \leq deg_{\mathcal{F}[\psi]-\{v\}} + \epsilon_{vw} - \eta_{vw},$$

and because

$$\epsilon_{vw} - \eta_{vw} = 0$$

we can deduce that

$$\phi(w) \leq deg_{\mathcal{F}[\psi]-\{v\}},$$

or

$$w \in \mathcal{F}[\phi],$$

since w was an arbitrary element of $\mathcal{F}[\psi] - \{v\}$ we conclude that:

$$\emptyset \neq \mathcal{F}[\psi] - \{v\} \subset \mathcal{F}[\phi]$$

in clear contradiction with the hypothesis that ϕ is allowed.

By induction hypothesis suppose that every avalanche $\phi \to \psi$ from an allowed configuration ϕ to a state ψ that produce n toppled and forbidden vertices $\{v_1, v_2, ..., v_n\}$ verify that $\emptyset \neq \mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \subset \mathcal{F}[\phi]$.

Now consider an avalanche $\phi \to \psi$ from an allowed configuration ϕ to a state ψ that produce $n+1$ toppled and forbidden vertices $\{v_1, v_2, ..., v_n, v_{n+1}\}$, take one of them, say v_{n+1}, we can make v_{n+1} allowed on ψ by putting enough chips on it without toppling it (in fact it suffices to put $deg_{\mathcal{F}[\psi]} + 1 - \psi(v_{n+1})$ chips on it) call the resulting configuration after the adding of chips σ, it is clear that the set of forbidden and toppled vertices (after the avalanche $\phi \to \psi \to \sigma$) of σ is $\{v_1, v_2, ..., v_n\}$, in fact we know that $\emptyset \neq \mathcal{F}[\sigma] - \{v_1, v_2, ..., v_n\}$. Then from the induction hypothesis follows that

$$\emptyset \neq \mathcal{F}[\sigma] - \{v_1, v_2, ..., v_n\} \subset \mathcal{F}[\phi].$$

Notice that this suffices to show that ϕ is was forbidden, but we can do more, we can detect with precision the forbidden subset of ϕ:

By using the same argument of adding chips to ψ at v_1 to make the later an allowed site and using the induction hypothesis we can guarantee that:

$$\emptyset \neq \mathcal{F}[\xi] - \{v_2, ..., v_{n+1}\} \subset \mathcal{F}[\phi],$$

where ξ is the state that results from ψ after adding $(deg_{\mathcal{F}[\psi]} + 1 - \psi(v_1))$ grains at v_1.

Then

$$(\mathcal{F}[\sigma] - \{v_1, v_2, ..., v_n\}) \cup (\mathcal{F}[\xi] - \{v_2, ..., v_{n+1}\}) \subset \mathcal{F}[\phi].$$

We finish by observing that as a consequence of

$$\emptyset \neq \mathcal{F}[\psi] - \{v_1, ..., v_{n+1}\} \subset (\mathcal{F}[\sigma] - \{v_1, v_2, ..., v_n\}) \cup (\mathcal{F}[\xi] - \{v_2, ..., v_{n+1}\})$$

we can conclude that:

$$\mathcal{F}[\psi] - \{v_1, v_2, ..., v_n\} \subset \mathcal{F}[\phi].$$

With this we have demonstrated that is not possible to create forbidden configurations by adding chips to an allowed configuration. □

Corollary: 3.19. *All allowed sandpiles are recurrent, in symbols:* $\mathcal{A} \subset \mathcal{R}$.

Proof. Let $\mathcal{C} = \phi_0 \mapsto \cdots$ a Markov chain on G an let ϕ be an allowed configuration, recall that Ξ is recurrent and allowed. Then at some large time t it happens that $\phi_t = \Xi$, then, using the later theorem and the fact that every recurrent sandpile state can be obtained as addition (followed by relaxation) of sand to Ξ we conclude that the property of being recurrent implies to be being allowed. $\square$

Now we are ready to pay our debt:

Proof of theorem 2.26 It was shown before that $\mathcal{R} \subset \mathcal{A}$ and by corollary 3.19 $\mathcal{A} \subset \mathcal{R}$ then $\mathcal{A} = \mathcal{R}$. $\square$

Proof of theorem 2.25 part ii) Suppose ψ forbidden at $\mathcal{C}$ but allowed at $\mathcal{D}$; theorem 2.26 imply that ψ is recurrent on $\mathcal{D}$, by the part i) of theorem 2.25 ψ is recurrent at every Markov chain, in particular in $\mathcal{C}$, then ψ is allowed on $\mathcal{C}$ and we have a contradiction. Such a ψ can not exist. $\square$

Proof of theorem 2.25 part iii) Take ψ allowed on $\mathcal{C}$, by theorem 2.26 ψ is recurrent at $\mathcal{C}$, theorem 2.25 guarantee that ψ is recurrent on all Markov chains and as $\mathcal{A} = \mathcal{R}$. (theorem 2.26) then ψ is allowed on all Markov chains. $\square$

Proof of theorem 2.27 It is obvious that the monoid M of all stable sandpiles states of G can be written as $M = \mathcal{A} \sqcup \mathcal{F}$, by using theorem 2.26 we also can write it as $M = \mathcal{R} \sqcup \mathcal{F}$. $\square$

To finish this chapter we offer a systematic characterisation of an allowed sandpile state.

3.2 Algorithm to determine whether a sandpile state is allowed or not.

Let ϕ be a stable sandpile on a graph G.

The followed algorithm is called *Algorithm of toppling from the sink* with input ϕ.

Input: ϕ

To know if ϕ is allowed or not follow the next steps:

i) Topple $\bullet$ (or in other words: put a grain at every vertex adjacent to the sink).

ii) Stabilize the resulting configuration.

If all the vertices of G topple during the relaxation process, then return as **output** the label **allowed** otherwise return as **output** the label **forbidden.**

First we prove that the algorithm is well defined in the sense that when decide to assign the label allowed (forbidden) to a state is because this state verify the definition of allowed (forbidden) for sandpile states.

If the reader wants some examples of how the algorithm works before reading the argument that support the well definedness of the algorithm he is invited to follow the examples given in 3.21.

Theorem 3.20. *If all the vertices topple during the relaxation process, then ϕ is allowed, otherwise is forbidden, furthermore, the non toppled vertices are exactly the set of forbidden sub configurations of ϕ.*

Proof. If neither of the vertices of the boundary becomes unstable after the donation of one grain from $\bullet$ then ϕ is forbidden because that means that all vertices on the boundary have less grains than the number of adjacent vertices. Then $\phi(v) \geq d_{\mathcal{F}}(v)$ where v is adjacent to $\bullet$ and $w \in \mathcal{F}$ if w is adjacent to v.

Now, if the avalanche stops without toppling v then at least one neighbour was not toppled, in fact $\phi(v) \geq$ *number of neighbours that were not be toppled* otherwise there would have been a topple on v, even more, for all w adjacent to v that was not toppled by the avalanche verify that $\phi(w) \geq$ *number of neighbours that were not be toppled*, it follows that at least $\{v, w\} \subset \mathcal{F}[\phi]$. Also it follows that the vertices without topples are $\mathcal{F}[\phi]$. $\square$

Examples: 3.21. *We consider the following four sandpile states evolving under the toppling from the sink algorithm. Whatever a state becomes unstable we highlight it using bold numbers an whenever a site topples we mark it using a diamond symbol $\diamond$.*

$$\Xi = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{4} & \mathbf{4} & \mathbf{4} \\ \mathbf{4} & 3 & \mathbf{4} \\ \mathbf{4} & \mathbf{4} & \mathbf{4} \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ \diamond & 7 & \diamond \\ \diamond & \diamond & \diamond \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 2 & \mathbf{4} & \mathbf{4} \\ \mathbf{4} & 2 & \mathbf{4} \\ 2 & \mathbf{4} & 2 \end{pmatrix} \mapsto \begin{pmatrix} 4 & \diamond & \diamond \\ \diamond & 6 & \diamond \\ 4 & \diamond & 4 \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{pmatrix}$$

$$\beta = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 2 & 1 \\ 3 & 3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{4} & \mathbf{4} & \mathbf{4} \\ 1 & 2 & 1 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ 3 & \mathbf{4} & 3 \\ \diamond & \diamond & \diamond \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ \mathbf{4} & \diamond & \mathbf{4} \\ \diamond & \diamond & \diamond \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \\ \diamond & \diamond & \diamond \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{4} & \mathbf{4} & \mathbf{4} \\ 1 & 0 & 1 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} \end{pmatrix} \mapsto \begin{pmatrix} \diamond & \diamond & \diamond \\ 2 & 2 & 2 \\ \diamond & \diamond & \diamond \end{pmatrix}$$

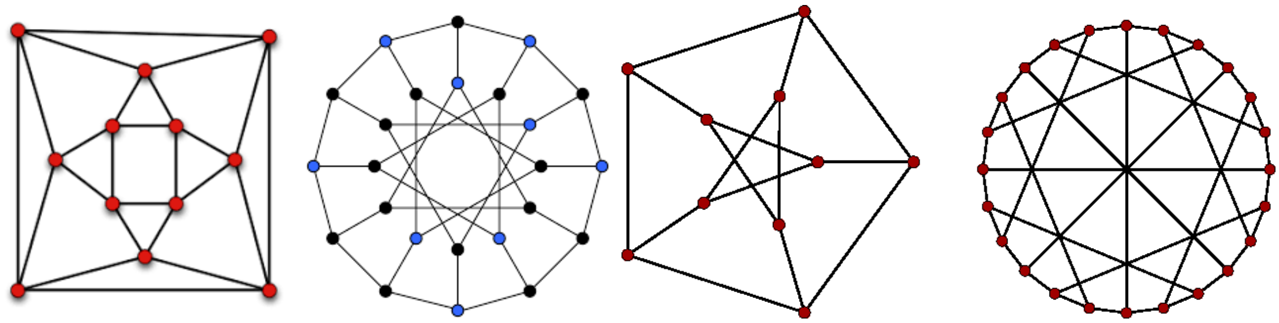
where we conclude that Ξ, α and β are allowed and γ is forbidden.

Definition 3.22. Let d be an integer number grater that two, a $d-$regular graph G is a graph such that every vertex has the same number of neighbours, more precisely, every vertex v of G satisfies:

$$d(v) = d$$

.

Examples: 3.23. *The following are beautiful examples of regular graphs.*



The next theorem is of extreme importance in what follows, although it is a consequence from theorem 3.20 applied to the particular class of $d-$regular graphs, we stress in advance that without this result neither of the material of the chapter five is valid and the correctness of much of those results (and the proof of the following theorem) depends very subtly on the $d-$regularity condition.

Theorem 3.24. *Let G be a connected $d-$regular graph and suppose that ϕ is an allowed configuration over G, then, in the algorithm of toppling from the sink (with input ϕ) every vertex topples exactly once.*

Proof. At time zero the algorithm produces ϕ, at time one $\phi+\sum_{v \text{ adjacent to } \bullet} \delta_v$ and we could guarantee that there is at least one vertex v adjacent to $\bullet$ such that $\phi(v) = d(v)$ (otherwise the algorithm stops and ϕ is forbidden), because the sandpile is abelian we can select arbitrarily the order at which we want to topple the set of vertices that satisfy the later condition, in that case, let's topple v at time two, then $\phi(v) = 0$ and notice that one grain from the system is lost; as ϕ is allowed, every non sink neighbour vertex w of v topples at least once, it follows that there is a time t at which all of the adjacent vertices to v have been toppled i.e. $\phi(v) = d(v) - 1$ at some time t, now it is clear that in order to produce two topples on v one its neighbours have to be toppled two times to donate one grain to v to have $\phi(v) = d(v)$, suppose that this is in fact the case and call w_1 the *two times toppled* neighbour of v.

It can be noticed that in order to produce two topples at w_1 it is necessary the existence of a vertex $w_2 \neq v$ adjacent to w_1 that have been toppled two times before w_1 does, otherwise the maximum number of grains that w_1 could have after its first topple is $d(v)-1$ (here is where the $d(v)-$ regularity is needed) because v has not been toppled yet and a second topple on w_1 could be impossible.

The same reasoning can be inductively used to prove that there is a path without loops $vw_1w_2...w_n$ for every $n \in \mathbb{N}$ with the property that during the evolution w_i topples two times before w_{i-1} does for all $i \in \{2, ..., n\}$; but as our graph is finite for some $m(1) \in \mathbb{N}$ there is a path $vw_1w_2...w_{m(1)}$ such that $w_{m(1)}$ is adjacent to $\bullet$ and the two topples on it makes the system to lose two grains of sand; by the same argument there is another non sink vertex $w_{m(1)+1} \neq w_{m(1)-1}$ adjacent to $w_{m(1)}$ that have been toppled two times before $w_{m(1)}$ does and we can guarantee the existence of a path $vw_1w_2...w_{m(1)}...w_{m(2)}$ such that w_i topples two times before w_{i-1} does for all $i \in \{2, ..., m(2)\}$ such that $w_{m(2)}$ is adjacent to $\bullet$, by repeating the argument enough times and using the finiteness of G we can show that there is some vertex at the boundary that have been toppled two times before all its neighbours were toppled, but this is impossible as we have shown in the first paragraph of this proof, then every vertex v of G have been toppled at most one time but also we know that all vertices topple at least once and it follows that every vertex topples exactly one time. $\square$

4 The Dollar Game.

"La meta es el olvido."
Jorge Luis Borges Acevedo.

4.1 What is a catastrophe?

The abelian sandpile model was serendipitously discovered as an effort to understand the emergence of complex behaviour, deeply linked to the emergence of complexity is the human idea of "catastrophe". As an informal explanation of what a catastrophe is, we can cite the idea that typical "parameter spaces" $\mathcal{M}$ associated with dynamical systems are sometimes studied via his space of smooth functions $C^{\infty}(\mathcal{M}, \mathbb{R})$, a general lesson extracted from this fruitful approach is the fact that critical values and singularities of such functions are of enormous importance because most dynamical systems behave generically outside a set of $\mathcal{M}$ where generic functions have critical points or singularities, those points could be intuitively understood as catastrophes.

The following points encode three supremely beautiful examples of catastrophes: The emergence of classical limits on some quantum systems, darwinian evolution and phase transitions (the born of our universe is a particular case of this).

I) The *mechanics* of a system is grossly studied via its associated phase space $\mathcal{M}$, important to this enterprise are the critical points of its Morse functions (observables) because by Sard's theorem this set is a zero measure set. This is related to the fact that upon quantization, most (large) states are nearly classical and receive little quantum corrections, in particular the probability of tunneling between two saddle points (of the action) is suppressed by inverse powers of the density of states associated to every saddle. If the critical points of the action were a set of finite measure, then the system in question could not admit consistent quantization, this is very far from been sufficient but highlight the fact that smooth manifolds satisfy a minimal requirement to be quantized.

II) *Darwinian evolution* is a mechanism in which live organisms with determined structural skills survive certain environmental pressures and propagate those characteristics to their descendants, evolution is not a "bounded grow" dynamical process and do not have gaussian statistics. Explotions in diversity of species are concentrated after great extinctions that take place every 250 million years in average and endure 30 years on average.

III) *Cosmic inflation* is the masterpiece of science that solve three important cosmological problems and offers a mechanism for creation of universes via phase transitions produced by scalar fields (in simplest scenarios), the inflation of our universe was a catastrophic event (signed by divergences on partition functions for the referred fields) that is characterized as a process in which the universe has expanded its size by a factor of 10^{78} during a period 10^{-33} seconds (it is to say: distances of one nanometre are mapped to distances of 10.6 light years) and then followed by a period of constant expansion of 72 kilometre per second per megaparsec during 3.3 million light years.

Now we want to embed the sandpile dynamics into a much more general dynamical system that is inspired in the way that a closed economy with hierarchical organized competitors self organizes via a simple local rule using the knowledge of its relations (a graph of relations). We hope to explain during the chapter how sandpiles are embedded and reveal the relevance of catastrophe theory in this archetype of self-organization (even when catastrophic events are not a priori happening in the dynamics of this kind of systems).

4.2 What is the dollar game?.

As in the case of the sandpile model, we consider again a graph $G = (V, E)$ without loops, in this context the set V of vertices could be thinked as a graphical representation of individuals in an economy (one $v \in V$ for every individual) and the set E of edges as a relationship between the participants of this economy; $e = uv \in E$ where $u, v \in V$ means that u can "share money" with v (in a way that would be made precise in a moment).

It is useful to have an algebraic representation of G or a list with all the participants in the economy with all the relationships between them.

Definition 4.1. *The adjacency matrix* of G is an element $A(G) \in M_{|V| \times |V|}(\mathbb{Z})$ defined by his entries as:

$$A_{vw} = \text{Number of distinct edges between the vertices } v \text{ and } w.$$

In what follows we write $A(G)$ simply by A.

We assigned to every vertex an integer number that represents "the number of dollars that v has".

Definition 4.2. The set of *divisors* of G is a formal $\mathbb{Z}-$combination of the vertices of G.

$$Div(G) = \{\phi = \sum_{v \in V} n_v v \mid n_v \in \mathbb{Z}, v \in V\}$$

.

It is also important to quantify how many dollars does the economy has.

Definition 4.3. The degree of $\phi = \sum_{v \in V} n_v v$ is the number given by:

$$deg(\phi) = \sum_{v \in V} n_v$$

.

Of course economies are as interesting as the number of dollars they contain.

Definition 4.4. The set of divisors of G of degree $n \in \mathbb{Z}$ can be grouped as:

$$Div^n(G) = \{\phi \in Div(G) \mid deg(\phi) = n\}$$

.

Observation 4.5. *$Div(G)$ is an a abelian group isomorphic to the free abelian group spanned by the vertices of G.*

Observation 4.6. *$Div(G)$ can be settled into a partially ordered set by declaring that two divisors ϕ and ψ obey $\phi \geq \psi$ if $\phi(v) \geq \psi(v)$ for all $v \in V$.*

Notation 4.7. *An economy is a pair (G, ϕ) where G is a graph and ϕ a divisor of G.*

Dynamics in economy is a synonym of the different ways in which any set of participants can share money, in particular, the situation can be (and usually is) studied by analyzing how the local environment of a participant affects it on average. To show how can this be achieved in the dollar game over a graph we motivate the central object of our study.

The laplacian operator of a graph.

Motivation: A *derivative* of on the graph $\mathbb{Z}$ can be defined by his action on functions $f:\mathbb{Z}\to\mathbb{Z}$ at the vertex n as $\partial f(n) := f(n) - f(n-1)$ an this integer number it is said to be the derivative of f at n. Following the smooth case, a *laplacian* on $\mathbb{Z}$ can be defined by $\Delta(f)(n) := \partial\partial f(n) = f(n) + f(n-2) - 2f(n-1)$.

This could be easily generalized to define a laplacian Δ the square grid $\mathbb{Z}^2$ by:

$$\Delta f(i,j) := \partial_i^2 f(i,j) + \partial_j^2 f(i,j) = f(i-1,j) + f(i+1,j) + f(i,j-1) + f(i,j+1) - 4f(i,j).$$

where $\partial_i^2 f(i,j) = f(i+1,j) + f(i-1,j) - 2f(i,j)$ and $\partial_j^2 f(i,j) = f(i,j+1) + f(i,j-1) - 2f(i,j)$, in that context we say that $\Delta f(i,j)$ is the laplacian of the function $f:\mathbb{Z}^2\to\mathbb{Z}^2$ at $(i,j)\in\mathbb{Z}$.

Now consider and arbitrary graph $G=(V,E)$ and $v\in V$, then we define:

Definition 4.8. Let $G=(V,E)$ be a graph, $f:V\to\mathbb{Z}$ and $v\in V$, we denote the laplacian of f at v by $\Delta f(v)$ an it is computed as:

$$\Delta f(v) := \sum_{w\sim v} f(w) - d(v)f(v) = \sum_{w\sim v}(f(w) - f(v)).$$

where it is recalled that $w\sim v$ means that w is adjacent to v.

We can note that the negative value of the laplacian of f at v measures how much $f(v)$ deviate from the average value of f around v by writing:

$$\frac{-1}{d(v)}\Delta f(v) := f(v) - \frac{1}{d(v)}\sum_{w\sim v} f(w).$$

Definition 4.9. Let (G,ϕ) be an economy. **The dollar game** of the economy (G,ϕ) is defined at the cellular automata level by the following evolution rules:

1) Call all sites v such that $\phi(v)\leq 0$ *site in debt.*

2) If $\phi(v)\geq d(v)$ for some vertex v at some time t, we say that v is *lender* or *philanthropic* at time t and the system evolves as $\phi\mapsto\phi+\Delta\sigma_v$ where:

$$\sigma_v(w) := \begin{cases} 1 & if\ v=w \\ 0 & otherwise \end{cases}$$

In such case we colloquially say that there is happening a lending move at v or that v is sharing dollars.

3) If at some time t neither of the sites is at debt the evolution stops and we say that ϕ wins the dollar game, otherwise the economy is said to be *bankrupt.*

Notation 4.10. *Given $\phi\in Div(G)$ we call the integer $\phi(v)$ the number of dollars of v.*

Proposition 4.11. *The order at which the vertices lend money is unimportant.*

Proof. Suppose that u and v are philanthropic at some time t_0. Also suppose that u realizes a lending move first, then the system evolves as $\phi \mapsto \phi + \Delta\sigma_u$ at time t_1, now notice that if v was philanthropic at t_0 then it will be at t_2 because the charitable process only increases the amount of dollars on v if v does not lend dollars, then, system evolves as $\phi + \Delta\sigma_u \mapsto \phi^\circ(1) := \phi + \Delta\sigma_u + \Delta\sigma_v$ at time t_3.

Similarly, if v is lender at time t_1 and u at t_2, then at time t_3 the system will be $\phi^\circ(2) := \phi + \Delta\sigma_v + \Delta\sigma_u$. Finally, commutativity of the addition of integers assures $\phi^\circ(1) = \phi^\circ(2)$. □

A very important aspect of the dollar game is that the amount of dollars on some economy remains constant during the evolution of the system. Therefore is timely to consider two divisor as equivalent if one can be obtained by lending moves from the other. More precisely, we declare that the divisors ϕ and ψ are equivalent if there is $\sigma \in Div(G)$ such that $\phi = \psi + \Delta\sigma$ and in such a case we write $\phi \sim \psi$.

Proposition 4.12. *The relation $\sim$ is an equivalence relation over the set $Div(G)$.*

Proof. I) $\sim$ is reflexive: $\phi = \phi + \Delta\sigma$ with $\sigma(v) = 0 \ \forall \ v \ \in \ V$.

II) $\sim$ is symmetric: because if $\phi = \psi + \Delta\sigma$ then $\psi = \phi + \Delta(-\sigma)$.

III) $\sim$ is transitive: Notice that $\Delta(\sigma + \eta) = \Delta\sigma + \Delta\eta$ for any σ, $\eta \in Div(G)$, this can be proved by direct computation :

$$\begin{aligned}\Delta(\sigma + \eta)(v) &= \sum_{vw \in E} (\sigma(v) + \eta(v) - \sigma(w) - \eta(w)) \\ &= \sum_{vw \in E} (\sigma(v) + \sigma(w)) - \sum_{vw \in E} (\eta(v) - \eta(w)) \\ &= \Delta\sigma(v) + \Delta\eta(v).\end{aligned}$$

It follows that if $\phi = \psi + \Delta\sigma$ and $\psi = \omega + \Delta\eta$, then $\phi = \omega + \Delta\sigma + \Delta\eta = \Delta(\sigma + \eta)$. □

Notation 4.13. *In what follows we use the notation $Pic(G) = \frac{Div(G)}{\sim}$ and this group will be referred as the Picard group of G.*

Notation 4.14. *A class $[\phi] \in Pic(G)$ will be called a closed economy.*

Proposition 4.15. *$Pic(G)$ can be equipped with group structure by defining the operation $+ :$ $Pic(G) \to Pic(G)$ given by $[\phi] + [\psi] = [\phi + \psi]$.*

Proof. Immediate. □

Lemma 4.16. *The identity $deg(\phi + \psi) = deg(\phi) + deg(\psi)$ holds for any two divisors ϕ and ψ of G.*

Proof.
$$deg(\phi + \psi) = \sum_v (\phi(v) + \psi(v)) = \sum_v \phi(v) + \sum_v \psi(v) = deg(\phi) + deg(\psi)$$
□

Lemma 4.17. *$deg(\Delta\sigma) = 0$ for every divisor σ of G.*

Proof. The affirmation follows from a direct computation:

$$deg(\Delta\sigma) = \sum_{v\in V}(d(v)f(v) - \sum_{w\sim v} f(w))$$

$$= \sum_{v\in V}(d(v)f(v)) - \sum_{v\in V}\sum_{w\sim v} f(w))$$

$$= \sum_{v\in V}(d(v)f(v)) - \sum_{w\sim v}(d(w)f(w)) = 0$$

□

Theorem 4.18. *The map $deg : Div(G) \to \mathbb{Z}$ induce a morphism of groups $\tilde{deg} : Pic(G) \to \mathbb{Z}$ defined as:*

$$\tilde{deg}[\phi] = deg(\psi) \text{ where } \psi \in [\psi].$$

Proof. We show that $\tilde{deg}$ is constant on every class of $Pic(G)$, it is to say, if ψ and ω are representatives of the same class $[\phi]$ then $deg(\psi) = deg(\omega)$; to do this simply notice that if $\psi = \omega + \Delta\sigma$ for some $\sigma \in Div(G)$ then:

$$deg(\psi) = deg(\omega + \Delta\sigma) = deg(\omega) + deg(\Delta\sigma) = deg(\omega).$$

By the above lemma.

We conclude that the function $\tilde{deg}$ does not depend on the representative, then $\tilde{deg}$ is well defined morphism because it was observed before that $deg(\phi + \psi) = deg(\phi) + deg(\psi)$. □

Corollary: 4.19. *If $\tilde{deg}(\phi)$ is a negative integer number, then the economy (G, ϕ) is bankrupt.*

Proof. If we suppose that the closed economy $[\phi]$ wins the dollar game then there is a divisor ψ such that $\psi \sim \phi$ and $\psi(v) \leq 0$ for all sites v, it follows that $deg(\psi) = deg(\phi) \leq 0$ in contradiction with the above theorem. □

Observation 4.20. *For any graph G :*

$$Pic(G) = \frac{Div(G)}{\Delta Div(G)}.$$

where:

$$\Delta Div(G) = \{\phi \in Div(G) \mid \exists \psi \in Div(G), \phi = \Delta\psi\}.$$

The following is a very simple statement, but it is also very cute and it will be argued later that there are many beautiful connections of it with the theory of sandpiles, in fact it is the key that allow us to deeply appreciate and get admired by how absolutely incredible the sandpile toppling rule is.

Theorem 4.21. *If the economy (G, ϕ) can win the dollar game, then the number of steps to win is independent of which lending moves can be performed at every step and also independent of the order in which they are realized.*

Proof. Play the dollar game with (G, ϕ), enumerate all the lender sites of (G, ϕ) at the time t_0 of the game as $\{v_1, ..., v_{n_0}\}$ for some $n_0 \in \mathbb{N}$, by proposition 4.11 the order at which those sites share money is unimportant, then, after all those sites lend money (at time t_{n_0}) the state uniquely evolves to $\phi + \sum_{i=1}^{n_0} \sigma_{v_i}$, if every vertex of this state have zero or more dollars the proof finishes; if that is not the case, enumerate again all the lender sites for the state $\phi + \sum_{i=1}^{n_0} \sigma_{v_i}$ at time t_{n_0} as $\{v_{n_0+1}, ..., v_{n_1}\}$ for $n_1 \in \mathbb{N}$ with n_1 n_0, and again by proposition 4.11 the system evolves to $\phi + \sum_{i=1}^{n_1} \sigma_{v_i}$ at time t_{n_1} irrespectively of the order in which the lending moves have been performed, if every site is free of debt at this time the proof finishes, if not this process can be reiterated in principle an arbitrary number of times.

But the later can not be the case because we know that the economy (G, ϕ) wins the dollar game, so, necessarily there is $n_k \in \mathbb{N}$ such that every site of $\phi + \sum_{i=1}^{n_k} \sigma_{v_i}$ have zero or more dollars. □

Corollary: 4.22. *If the economy (G, ϕ) is able to win the dollar game, then the final state is unique.*

Proof. Immediate from the proof of the later theorem. □

4.3 The jacobian group of a graph.

Notation 4.23. *In what follows we write deg instead of $\tilde{deg}$ to allude the morphism $\tilde{deg} : Pic(G) \rightarrow Pic(G)$ and also write classes $[\phi] \in Pic(G)$ simply as ϕ.*

Definition 4.24. We call **the jacobian group of** G to the set of degree zero divisors of G, more precisely:

$$Jac(G) = \{\phi \in Pic(G) \mid deg(\phi) = 0\}$$

Proposition 4.25. *$Jac(G)$ equipped with the addition of divisors is a subgroup of $Pic(G)$.*

Proof. It suffices to notice that if $\phi, \psi \in Pic(G)$ then:

$$deg(\phi + \psi) = deg(\phi) + deg(\psi) = 0$$

that means $\phi + \psi \in Jac(G)$.

□

Notation 4.26. *In what follows we understand by the phrase "solve the dollar game on G" the knowledge of an explicit realization of $Jac(G)$. The meaning of this mysterious phrase will be explained in subsection 4.5.*

At this point we know that $Jac(G)$ is an abelian group, this (as in the case of the sandpile dynamics) is a signal that the dynamics of the dollar game is not as chaotic as it is expected for general mathematical games. In general, it will be desirable to know tools to compute $Jac(G)$ and amazingly the answer is that it is always possible to do that in a systematic way, it is a generic fact of nature and the arts that *integrable* or completely *solvable* models are of extreme importance, even it is possible to say that if we appropriately generalize the word *solvable*, then physics and mathematics study simple *solvable* blocks and then try to "glue" them together, examples of this are physical fields, CW complexes or smooth manifolds which are respectively: a minimally coupled ensemble of harmonic oscillators, continuous attachments of (contractible) n-cells and locally linear spaces, respectively, here the words "harmonic oscillators", "contractible" and "linear" are the appropriate lift of the solvable concept. It is irresistible to mention that the supreme example of this ontological phenomenon is the miracle of the solvability of the dynamics of a quantum relativistic string.

It is a general rule that solvable models are the heart of some science or art. If that is true, then it is wise to figuratively ask: **Of what kind of theory is the dollar game (or the sandpile model) a local trivialization?**, this is a fascinating unanswered question.

What would be important for us at this point is to know how to solve the dollar game, as this is done by the knowledge of $Jac(G)$ then we follow the philosophy that given an unknown finite abelian group the first relevant information to be known about it is its order because the structure theorem for finite abelian groups give us a canonical decomposition of it in terms of its Sylow groups.

4.4 Solving the dollar game.

Observation 4.27. *Given an ordering $\{v_1, .., v_{|V|}\}$ of the vertices of G we can fix a base for the finitely generated abelian group $\mathbb{Z}^V$ using the identification $v_i \mapsto e_i$ where the set $\{e_i\}_{i=1}^{|V|}$ is the canonical basis of $\mathbb{Z}^V$.*

Proposition 4.28. *The linearly extension of the map $\alpha : Div(G) \to \mathbb{Z}^V$ defined by $\alpha(v) = e_v$ is a group isomorphism.*

Proof. From the fact that $\alpha(\sum_{v\in V} n_v v) = \sum_{v\in V} n_v e_v$ for all $n_v \in \mathbb{Z}$ and every $\sum_{v\in V} n_v v \in Div(G)$ follows that α is a group morphism. It is easily checked that the linearly extension of $\alpha^{-1} : \mathbb{Z}^V \to Div(G)$ given by $\alpha^{-1}(e_v) = v$ is the inverse of α. $\square$

Observation 4.29. *We call a representation of the laplacian Δ to the map $L : \mathbb{Z}^V \to \mathbb{Z}^V$ that can be computed as $\alpha^{-1} \circ \Delta \circ \alpha$ and makes the following diagram commutative:*

$$\begin{array}{ccc} Div(G) & \xrightarrow{\Delta} & Div(G) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathbb{Z}^V & \xrightarrow{L} & \mathbb{Z}^V \end{array}$$

L can be computed as $\alpha^{-1} \circ \Delta \circ \alpha$ or explicitly: $L(x) = \sum_{w\sim v}(x(v) - x(v))$ for any $x \in \mathbb{Z}^V$.

Notation 4.30. *In what follows we denote by $\vec{c}$ the element of $\mathbb{Z}^V$ that has the value $c \in \mathbb{Z}$ in all its entries and we say that $\vec{c}$ is a constant element.*

Proposition 4.31. *$Ker(L)$ is isomorphic to the free subgroup of $\mathbb{Z}^V$ generated by $\vec{1}$, in symbols:*

$$Ker(L) = Span\{\vec{1}\}$$

Proof. Let's prove that the only elements on $Ker(L)$ are constant elements. In order to show this take $x \in \mathbb{Z}^V$ such that $L(x) = 0$ and suppose that x has a maximum at the vertex v, that is to say $x(v) \geq x(w)$ for all $w \in V$, summing over all the neighbours of v we arrive at $d(v)x(v) \geq \sum_{w\sim v} x(w)$.

On the other hand, the fact that $L(x)(v) = d(v)x(v) - \sum_{w\sim v} x(w) = 0$ is equivalent to $d(v)x(v) = \sum_{w\sim v} x(w)$ and using the above inequality we conclude that $x(v) = x(w)$ for $v \sim w$; as G is connected we can repeat the argument to show that $x(v) = x(w)$ for any two vertices v and w. $\square$

Definition 4.32. For a given graph G and a vertex s we consider the graph $\tilde{G}$ resulted from deleting s from the set of vertices of G and also all the edges of G that have s as boundary. We define *the configuration space of s on G* as:

$$\mathrm{Conf}(G, s) = \mathrm{Div}(\tilde{G})$$

Proposition 4.33. *For any graph G*

$$Conf(G, s) \simeq Conf(G, p)$$

For any two vertices s and q of G.

Proof. To see this it suffices to linearly extend the map $\chi : \mathrm{Conf}(G, s) \to \mathrm{Conf}(G, p)$ given by

$$\chi(v) := \begin{cases} s & if\ v = p \\ v & otherwise \end{cases}$$

that is clearly invertible. □

Proposition 4.34. *The map $f\colon Conf(G, s) \to Jac(G)$ given by:*

$$\phi \mapsto \phi - deg(\phi)s$$

is an epimorphism.

Proof. The fact that f is a morphism of groups follows from proposition 4.24. To see that it is surjective, take $\psi \in Jac(G)$ given by $\phi = \sum_{v\in V} n_v v$, because $deg(\phi) = 0$ we have:

$$\sum_{v\in V, v\neq s} n_v = -n_s.$$

Then consider $\phi = \sum_{v\in V, v\neq s} n_v \in \mathrm{Conf}(G, s)$ and notice that:

$$f(\phi) = \sum_{v\in V v\neq s} n_v v - deg(\sum_{v\in V, v\neq s} n_v v)s,$$

that gives:

$$f(\phi) = \sum_{v\in V, v\neq s} n_v v - (\sum_{v\in V, v\neq s} n_v)s,$$

and using that $deg(\phi) = 0$:

$$f(\phi) = \sum_{v\in V, v\neq s} n_v v - n_s s = \psi,$$

then f is epimorphism. □

Observation 4.35. *The homomorphism f induces by the first isomorphism theorem an isomorphism $g : \frac{Conf(G,s)}{ker(f)} \to Jac(G)$ such that the following diagram commutes:*

$$\begin{tikzcd} Conf(G,s) \arrow[r, "f"] \arrow[d, "p"'] & Jac(G) \\ \frac{Conf(G,s)}{Ker(f)} \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

where p is the canonical quotient morphism.

The importance of the above observation is that using the fact that $\mathrm{Conf}(G,s) \simeq \mathbb{Z}^{n-1}$ we are able to give a presentation of $Jac(G)$ once we have been explicitly computed $Ker(f)$. This is what we now want to do.

Definition 4.36. Let G be a graph. We say that the laplacian operator on $\tilde{G}$ is *the s-reduced laplacian* of G and we will denote it by $\tilde{L}$, also in the hereinafter we say simply that $\tilde{L}$ is the *reduced laplacian of* G.

Lemma 4.37.

$$Ker(f) \simeq \tilde{L}\mathbb{Z}^{n-1}.$$

Proof. Take $x = \sum_{v,v\neq s} n_v e_v \in \mathbb{Z}^{n-1}$ such that $f(x) = 0$, then:

$$f(x) = deg(x)e_s,$$

or explicitly:

$$\sum_{v,v\neq s} n_v e_v = (\sum_{v,v\neq s} n_v)e_s,$$

which implies that $n_v = 0$ for all v vertex. An easy consequence is that $\sum_{v\in V} n_v = 0$, or, equivalently: $deg(x) = 0$. From what follows that $x = \tilde{L}y$ for some $y \in \mathbb{Z}^{n-1}$. □

Theorem 4.38. *The Jacobian group of a simple, connected and planar graph G can be computed as:*

$$Jac(G) \simeq \frac{\mathbb{Z}^{n-1}}{\tilde{L}\mathbb{Z}^{n-1}}$$

Proof. Follows immediately from the representation $\mathrm{Conf}(G,s) \simeq \mathbb{Z}^{n-1}$ and the later lemma. □

The following theorem is highly non trivial and aims to gives us a presentation of $Jac(G)$.

Theorem 4.39. *Let G be a graph and $\tilde{L}$ its reduced laplacian, then $Ker(\tilde{L})$ is trivial.*

Proof. Let us write $L = [x_1, ..., x_n]$ where $\{x_i\}_{i=1}^n$ are the set of column vectors of L. We know from proposition 4.30 that $Ker(L)$ is a finite generated abelian group of rank one, that means that the number of generators of $im(L)$ is $n-1$ or that the last column of L is a linear combination (with integer coefficients) of the remaining $n-1$ columns, in symbols:

$$x_n = r_1 x_1 + \cdots + r_{n-1}x_{n-1},$$

where $\{r_i\}_{i=1}^n$ are integers.

We can also write $L = [y_1, ..., y_n]$ where $\{y_i\}_{i=1}^n$ are the vector rows of L, using the fact that L is a symmetric matrix we arrive at a similar conclusion for the rows of L:

$$y_n = s_1 y_1 + \cdots + s_{n-1}y_{n-1},$$

Where $\{s_i\}_{i=1}^n$ are integers.

Now suppose that $\tilde{k} \in \mathbb{Z}^{n-1}$ is in the kernel of the reduced laplacian $\tilde{L}$ of G i.e. $\tilde{L}[\tilde{k}] = 0$, then define the vector $k = (\tilde{k}, 0) \in \mathbb{Z}^n$ and notice that k has the property that the dot product of k with x_i is zero for every $i \in \{1, ..., n-1\}$, therefore the dot product of k with x_n is also zero, so $k \in Ker(L)$ and by proposition 4.31 $k = 0$, then $\tilde{k} = 0$.

As $\tilde{k}$ was arbitrary $Ker(\tilde{L})$ is trivial. □

Definition 4.40. Let R be a principal ideal domain and M be a non zero $m \times n$ matrix over R. We say that M has a **Smith normal form** if there exist invertible matrices S and T of respective sizes $m \times m$ and $n \times n$ and $r \in \mathbb{N}$ such that SMT can be written in block diagonal form as follows:

$$SMT = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$$

Here $D_r = diag(\alpha_1, ..., \alpha_r)$ is an $r \times r$ diagonal matrix with the property that $\alpha_i | \alpha_{i+1}$ for all $1 \leq i < r$ and every α_i can be computed (up to multiplication by a unit) as:

$$\alpha_i = \frac{d_i(A)}{d_{i-1}(A)},$$

where $d_i(A) \in R$ (called the $i-$th determinant divisor) equals the greatest common divisor of all $i \times i$ minors of M and $d_0 := 1$.

In that case we say that SMT is the Smith normal form of M.

It is recognized that the Smith normal form is an analogue over a PID of the Jordan canonical form of a matrix with coefficients on a field. The above definition is made over a PID because of the following remarkable fact:

Theorem 4.41. *Any matrix with entries on a principal ideal domain have a Smith normal form.*

The proof is omitted.

It is clear that we are interested in the possibility that some matrix could be represented as block diagonal because the work of computing the cokernel of this kind of matrices is easier than the non diagonal ones. In particular, we want to compute the cokernel of the reduced laplacian of some graph G.

We define over the set of $n \times m$ matrices with integer entries the relation $\sim$ by saying: $M \sim N$ if there are P and Q invertible matrices (with integer coefficients) of sizes $m \times m$ and $n \times n$ such that $N = PMQ$.

Proposition 4.42. *The relation $\sim$ is an equivalence relation.*

Proof. Immediate □

Lemma 4.43. *Let M and N be $n \times m$ matrices with integer entries such that $M \sim N$, then:*

$$Cok(M) = Cok(N).$$

Proof. By hypothesis there are P and Q invertible matrices (with integer coefficients) of sizes $m \times m$ and $n \times n$ such that:

$$N = PMQ,$$

therefore:

$$(P \circ M \circ Q)\mathbb{Z}^n = N\mathbb{Z}^n.$$

As Q is isomorphism on $\mathbb{Z}^n$, $Q\mathbb{Z}^n \simeq \mathbb{Z}^n$ holds and:

$$P(M\mathbb{Z}^n) \simeq N\mathbb{Z}^n.$$

Similarly, because$P$ is isomorphism on $\mathbb{Z}^m$ it follows that $P\mathbb{Z}^r \simeq \mathbb{Z}^r$ for any free subgroup $\mathbb{Z}^r$ of $\mathbb{Z}^m$, in particular $P(M\mathbb{Z}^n) \simeq M\mathbb{Z}^n$ as $M\mathbb{Z}^n$ is subgroup of $\mathbb{Z}^n$ because M is homomorphism of $\mathbb{Z}^n$. Then:

$$P(M\mathbb{Z}^n) \simeq M\mathbb{Z}^n \simeq N\mathbb{Z}^n,$$

from what follows that:

$$Cok(M) \simeq Cok(N).$$

□

Theorem 4.44. *Suppose that a $m \times n$ matrix M is such that $M \sim N$ with $N = diag(m_1, ..., m_n)$ where $\{m_1, ..., m_n\}$ is a set of positive integers. Then:*

$$Cok(M) \simeq \prod_{i=1}^{n} \mathbb{Z}_{m_i}$$

Proof. By the above lemma it suffices to compute $Cok(N)$. Let $\{e_i\}_{i=1}^n$ be the set of canonical generators of $\mathbb{Z}^n$ i.e. $\mathbb{Z}^n = Span\{e_i\}_{i=1}^n$, then $im(N) = Span\{m_i e_i\}_{i=1}^n$.

To calculate the quotient $\frac{\mathbb{Z}^n}{im(N)}$ simply notice that under the quotient e_i gets identified with $e_i + m_i$ for every $i \in \{1, ..., n\}$, it follows that:

$$Cok(N) \simeq \prod_{i=1}^{n} \frac{\mathbb{Z}}{m_i\mathbb{Z}} \simeq \prod_{i=1}^{n} \mathbb{Z}_{m_i}.$$

□

We have reached our objective, to give a presentation of $Jac(G)$.

Corollary: 4.45. *Let G be a graph and $\tilde{L}$ its reduced laplacian, then:*

$$Jac(G) \simeq \prod_{i=1}^{n} \mathbb{Z}_{m_i},$$

where $\{m_i\}_{i=1}^n$ are the determinant divisors of the unique normal Smith form of $\tilde{L}$.

Proof. We notice that because of theorem 4.38 $\tilde{L}$ satisfy the theorem 4.44 hypothesis, we also know that $Jac(G) \simeq Cok(\tilde{L})$ and using theorem 4.44 we obtain the desired conclusion. □

Corollary: 4.46. *Let G be a graph and $\tilde{L}$ his reduced laplacian with respect to any vertex. Then:*

$$|Jac(G)| \simeq |det(\tilde{L})|.$$

Proof. Follows immediately from computing the order of $Jac(G)$ as presented by the above corollary.

□

4.5 The open economy origin of sandpile dynamics.

"The fun is beginning now."
Andrew Strominger - The fun is just beginning.

Real economies never follow the socialist-like scheme that the dollar game presupposes, individuals (vertices) of them are not altruism with those (adjacent) ones whom they have relation (edges), the economy is not closed in the sense that the amount of available money is not constant over time. On contrary, real capitalist schemes allow the entrance and lose of money because they interact with other capitalist-like economies when some individuals do trades with another individuals of other economies.

The dollar game can be modified to be more realistic. The importance of this generalization for us would be that the sandpile dynamics can been shown to be explicitly embedded in this new dynamics, although the generalization (and sandpile subcase) is pretty obvious, it is convenient to do it because it reveals many surprises like an isomorphism between the jacobian group of a graph and the sandpile group of the same graph, a *least action principle* for the sandpile toppling rule, a link between recurrent sandpiles and combinatorial invariants of graphs that will allow us to recognize a probabilistic prove of the *matrix tree theorem* and in the conceptual side the importance of catastrophe theory in mathematical games.

Definition 4.47. We introduce at the cellular automata level *the open economy dollar game* for the triplet $(G, \phi, \bullet)$ where $\bullet$ is a vertex of G, $\phi \in Conf(G, \bullet)$ by the following evolution rules:
1) Call all sites v such that $\phi(v) \leq 0$ *site in debt.*

2) If $\phi(v) \leq v$ for some vertex v at some time t, we say that v is *lender* or *philanthropic* at time t and the system evolves as $\phi \mapsto \phi + \Delta\sigma_v$ where:

$$\sigma_v(w) := \begin{cases} 1 & if\ v = w \\ 0 & otherwise \end{cases}$$

In such case we colloquially say that there is happening a lending move at v or that v is sharing dollars.

3) The dollars shared with $\bullet$ disappear from the system.

4) If at some time t neither of the sites is at debt the evolution stops and we say that the economy wins the open dollar game and it is at equilibrium. Otherwise we say that the economy $(G, \phi, \bullet)$ is bankrupt.

Now we rewrite the basic ideas about Markov chains in sandpiles almost word by word for the open dollar game.

4.6 Markov chains in the open dollar game

Notation 4.48. *Given a vertex v we denote by $\Delta\sigma_v \in Div(G)$ the lending move that represents "v share dollars with its neighbours" where σ_v is defined as follows:*

$$\Delta\sigma_v(w) := \begin{cases} 1 & if\ v = w \\ 0 & otherwise \end{cases}$$

Definition 4.49. Let $(G, \phi, \bullet)$ be an economy. By a **Markov chain or markovian evolution** in the dollar game over the graph G with *initial state* $\phi \in Div(G)$ we understand any infinite sequence of the form:

$$\phi \mapsto \phi + \Delta\sigma_{v(1)} \mapsto (\phi + \Delta\sigma_{v(1)}) + \Delta\sigma_{v(2)} \mapsto \cdots \mapsto ((\cdots(\phi + \Delta\sigma_{v(1)}) + \cdots) + \Delta\sigma_{v(n)} \mapsto \cdots$$

To stay the notation as simple as possible we write the above chain as:

$$\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$$

where $\phi_n = \phi_{n-1} + \Delta\sigma_{v(n)} = \phi + \sum_{a=1}^{n} \Delta\sigma_{v(a)}$ and it is understood that the addition of $\Delta\sigma_{v(i)}$ ($v_{v(i)}$ is philanthropic) to $\phi_{(i-1)}$ at the $(i+1)-$step of the chain is *random* in the sense that the election of the vertex $v_{(i)}$ is arbitrary, in other words: $\{v(a)\}_{a=1}^{n}$ is an ordered multiset of randomly selected vertices of G. We also want to recall that the notation $\phi + \psi$ for ϕ and ψ divisors is an abuse of notation that always represents the stable state produced by lending moves as the $\phi + \psi$ divisor.

Notation 4.50. *Let $\phi \mapsto \phi_1 \mapsto \phi_2 \mapsto \cdots \mapsto \phi_n \mapsto \cdots$ be a Markov chain and we want to create an allegory of time evolution by saying that the state ϕ is the initial state of the markovian evolution and ϕ_n is the state at the time n of the evolution.*

Is not our objective to develop the theory derived from this dynamics in part because there is nothing to be developed, it is easily seen that the late time dynamics of the open economy dollar game is exactly the same as the sandpile one, this is because if some state at some step on a Markov chain wins the dollar game before this step the dollar game dynamics is exactly the same as the theory of Markov chains on sandpiles, also every chain wins the same at some moment.

Then, what is the necessity of being bothered in defining Markov chains on the open economy dollar game? The reason is twofold.

First, the lesson extracted from subsection 4.4. is that the dollar game only precise the knowledge of $\tilde{L}$ (not L!) to compute $Jac(G)$, this motivate us to define an open economy because one of the vertices (and all the edges that merge on it) are redundant in the sense that Markov chains on the dollar game are bad defined because at some point the amount of dollars on the graph vertices should be so large that the configuration is never stable, but, as the computation of $Jac(G)$ suggest this is resolved by *"ignoring"* one vertex, once this is done, a beautiful organized dynamical world emerges, the sandpile dynamics.

Second, on the conceptual side, economies that are recurrent on markovian chains for the open dollar game are "good" in the sense that they amount of dollars on vertices is exactly equilibrated like in a free market economy. In this way, open economy dollar game recurrent states are desirable points of market regularization. They are the configurations that appear once the dollar game is wined in the realistic (open) case.

A beautiful concrete realization of this vague reasoning are the following results:

Definition 4.51. Let G be a graph with sink $\bullet$, $1_{\tilde{G}}$ the constant divisor that assigns one chip to every vertex of $\tilde{G}$ (G without $\bullet$ and all the edges connected to it) and Ξ its maximally stable state (definition 3.2) we define the following two distinguished states:

$$c_{big} := \Xi + 1_{\tilde{G}}$$

$$c_{null} := c_{big} - c_{big}^{\circ}$$

The following lemma help us to understand the above definitions.

Lemma 4.52. c_{null} *satisfy:*

$$c_{null} = 0 \quad mod \ \tilde{L}$$

$$c_{null} \geq 1_{\tilde{G}}$$

Proof. Notice that $c^{\circ}_{big} = c_{big} + \tilde{L}\sigma$ for some σ divisor $\sigma \geq 0$, then:

$$c_{null} = c_{big} - (c_{big} - \Delta\sigma),$$

or

$$c_{null} = \tilde{L}(-\sigma);$$

this proves the first equality. Now suppose that there is a vertex v such that $c_{null}(v) < 1$, or

$$c_{big}(v) - c^{\circ}_{big}(v) < 1$$

$$c_{big}(v) < c^{\circ}_{big}(v) + 1,$$

and using the definition of c_{big} :

$$c_{max}(v) + 1 < c^{\circ}_{big}(v),$$

we arrive at:

$$\Xi(v) < c^{\circ}_{big}(v),$$

which is a contradiction because c°_{big} is stable and $c_{big} \geq \phi$ for all ϕ stable. $\square$

c_{null} is not an special state, what is important of it for us is that the above lemma can be stated as the affirmation that $c_{null} = \Delta\sigma$ with $\sigma \geq 0$, this will be proven to be useful because of the following two reasons:

I) c_{null} can serve as a representative of the "zero class" on $Jac(G)$.

II) Given an unstable sandpile state ϕ we can compute its relaxation as $\phi^{\circ} = \phi + \Delta\sigma$ for some $\sigma \geq 0$ (this is in fact the toppling rule), then we would always write $\phi^{\circ} = \phi + c$ for some $c \sim c_{null}$.

In other words: c_{null} is just a convenient way of write the zero class of $Cok(\tilde{L})$ in a way that can realize the toppling rule by addition of it to an unstable pile.

The proves of the following results are delicate, an issue concerning their proves is made, although technical in nature it can help us to prove perhaps the most beautiful result of this work, a *least action principle* for the sandpile toppling rule.

Observation 4.53. *The notation ϕ° alludes that ϕ° is the relaxation of the sandpile state ϕ, the emphasis is made to notice that only makes sense for states such that $\phi \geq 0$, that is not the case for generic divisors, so, a relaxation process $\phi \mapsto \phi_{\circ}$ only makes sense for sandpile states. We can **not** apply lemma 2.7 to "relax" a generic divisor $(\phi + \psi)^{\circ} \mapsto \phi^{\circ} + \psi^{\circ}$ because the above expression does not make sense until we can guarantee that ϕ and ψ are both sandpile states.*

The following result is remarkable concerning its beauty, the moral is that if we have an unstable sandpile state there is no mechanical way to stabilize it more quickly that the way the sandpile toppling rule does. This is basically we the magic lives, the toppling rule solves an optimization problem.

Suppose that v is unstable in a configuration ϕ, we say that a lending move on v **legal for** ϕ if after the lending move on v the amount of money on v is non- negative. If ψ can be reached from ϕ after a sequence $\sigma = [v_1, ..., v_k]$ of lending moves on the vertices $v_1, ..., v_k$ we write $\phi \mapsto_\sigma \psi$

Theorem 4.54. ***(Least action principle for the sandpile toppling rule)*** *Let $\phi \in$ Conf(G, s), suppose that $\sigma, \tau \geq 0$ are such that σ arising from a legal sequence for ϕ and $\phi \mapsto_\tau \tilde{\phi}$. Then $\tau \geq \sigma$*

Proof. Consider a legal sequence $v_1, ..., v_k$ corresponding to σ, it is to say $\sigma = \sum_i v_i$. We proceed by induction, the $k = 0$ case is obvious. Let us suppose $k > 0$.Because by hypothesis v_1 is unstable in ϕ and stable in $\tilde{\phi}$ it follows that at least one lending move according to τ have been taken place on v_1, this implies that $\tau(v_1) > 0$. By making this lending move on v_1 we get $\tilde{\phi}$ and let $\tilde{\tau} := \tau - v_1$. Then $v_2, ..., v_k$ is a legal firing sequence for $\tilde{\phi}$, and $\tilde{\phi} \mapsto_{\tilde{\tau}} \tilde{\phi}$.

It follows by induction that $\sigma - v_1 \leq \tilde{\tau}$, then:

$$\sigma \leq \tilde{\tau} + v_1 = \tau.$$

□

Lemma 4.55. *Let ϕ be a recurrent sandpile configuration on the graph G, then:*

$$(\phi + c_{null})^\circ = \phi$$

Proof. As ϕ is recurrent we can find a state $\psi + 1_{\tilde{G}}$ such that $\phi = (\psi + 1_{\tilde{G}} + \Xi)^\circ$ (theorem 3.7), or equivalently: $\phi = (\psi + \Xi)^\circ$.

As the order at which an unstable state is relaxed is unimportant (lemma 2.7) we can relax the state $\psi + c_{big} + c_{null}$ in the following way:

$$(\psi + c_{big} + c_{null})^\circ \mapsto (\psi + c_{big})^\circ + c_{null} = \phi + c_{null}$$

$$\mapsto (\phi + c_{null})^\circ.$$

Notice that the relaxation process is well defined because c_{null} is a sandpile state ($c_{null} \geq 0$) by lemma 4.52 and $\psi + c_{big} \geq 0$ because $\psi \geq 0$ and $c_{big} = \Xi + 1_{\tilde{G}} \geq 0$.

By using the definition of c_{null} we can also write the same state as:

$$\psi + c_{big} + c_{null} = \psi + c_{big} + c_{big} - c_{big}^\circ,$$

and notice that another way relax this state is to first topple c_{big} as follows:

$$(\psi + c_{big} + c_{null})^\circ = (\psi + c_{big} + c_{big} - c_{big}^\circ)^\circ$$

$$\mapsto (\psi + c_{big} - c_{big}^\circ) + c_{big}^\circ = \psi + c_{big}$$

$$\mapsto (\psi + c_{big})^\circ,$$

again, this relaxation process is well defined because c_{big} is a sandpile state and $\psi + c_{big} - c^{\circ}_{big} \geq 0$ because $\psi + c_{big} - c^{\circ}_{big} = \psi + c_{null}$ and $\psi, c_{null} \geq 0 = \phi$ by construction and by lemma 4.25, respectively.

As the result of the avalanche process is independent on how the intermediate states have been performed (lemma 2.7) we conclude that:

$$(c_{null} + \phi)^{\circ} = \phi$$

□

Lemma 4.56. *Let ϕ be a recurrent sandpile configuration on the graph G and $k \in \mathbb{N}$, then:*

$$(\phi + kc_{null})^{\circ} = \phi$$

Proof. We proceed by induction. The above lemma is the step $k = 1$ of the induction; now suppose that $(\phi + kc_{null}) = \phi$ for some $k \in \mathbb{N}$, and relax the state $\phi + (k+1)c_{null} = \phi$ as follows:

$$\phi + (k+1)c_{null} = \phi + kc_{null} + c_{null} \mapsto (\phi + kc_{null})^{\circ} + c_{null}$$

$$= \phi + c_{null} \mapsto (\phi + c_{null})^{\circ}.$$

□

Theorem 4.57. *Let G be a graph and denote by $\mathcal{S}(G)$ its sandpile group,then:*

$$\mathcal{S}(G) \simeq Jac(G)$$

Proof. Define the map $f : \mathcal{S}(G) \to Cok(\tilde{L})$ by

$$\phi \mapsto [\phi]$$

It is clear this is a group morphism.

f is epimorphism:

Take $[\phi] \in Cok(\tilde{L})$ and represent this class by $\psi + \tilde{L}\sigma$ with ψ and σ divisors and pick $k \in \mathbb{N}$ large enough to guarantee that $\psi + \tilde{L}\sigma + kc_{null} \geq \Xi$. then $\psi + \tilde{L}\sigma + kc_{null}$ is an unstable sandpile state.

Let's relax it

$$\psi_r = (\psi + \tilde{L}\sigma + kc_{null})^{\circ} = \psi + \tilde{L}\sigma + kc_{null} + \tilde{L}\eta_k,$$

where $\eta_k \geq 0$ and ψ_r is a stable sandpile state.

In fact, ψ_r is a recurrent one because it can be reached by adding grains of sand to Ξ as can be seen from $\psi + \tilde{L}\sigma + kc_{null} \geq \Xi$ and by theorem 3.7 all the recurrent configurations of the sandpile are of this form.

Therefore $\psi_k \in [\phi]$, or:

$$f(\psi_r) = [\phi],$$

this shows that f is epimorphism.

To show that f is monomorphism, take $[\phi] \in Cok(\tilde{L})$ and suppose that there are two different recurrent states ψ and χ on that class, that means that there is a divisor σ such that $\psi = \phi + \tilde{L}\sigma$

If $\sigma = 0$ the proof finishes, otherwise write $\sigma = \sigma_+ + \sigma_-$ in a way that $\sigma_+ \geq 0$ and $\sigma_- < 0$, then the condition $\psi = \phi + \tilde{L}\sigma$ translates into:

$$\psi + \tilde{L}(-\sigma_-) = \phi + \tilde{L}\sigma_+,$$

now take $k \in \mathbb{N}$ large enough to guarantee that:

$$(\psi + \tilde{L}(-\sigma_-) + kc_{null})(v) \geq max_{w \in V}\{|n_w| d(w)\}, \quad \forall\, v \in V,$$

thus, each vertex v can be toppled $|n_w|$ times in $\psi + \tilde{L}(-\sigma_-)$ irrespectively of the order in which this is done.

Then the following stabilization is valid:

$$\psi + \tilde{L}(-\sigma_-) + kc_{null} \mapsto \psi + kc_{null} \mapsto \psi,$$

in the last step we have used lemma 4.55.

By the same reasoning one can show that:

$$(\chi + \tilde{L}(-\sigma_-) + kc_{null})^\circ = (\chi + \tilde{L}(\sigma_+) + kc_{null})^\circ = \chi$$

By the uniqueness of the result of an avalanche process we have $\psi = \chi$. Thus f is monomorphism, then f is isomorphism. □

Corollary: 4.58. *There is exactly one representative recurrent sandpile state under the isomorphism $f : \mathcal{S}(G) \to Jac(G)$ for every equivalence class of $Jac(G)$.*

Proof. Immediate. □

The next observation is made to clarify notational difficulties used in this work and in the literature that could cause important conceptual confusion.

Observation 4.59. *Theorem 3.7 states that every recurrent configuration ψ can be represented as a perturbation of the maximally stable configuration state Ξ of some graph, more precisely $\psi = \Xi + \sum_{v \in W} \delta_v$ for some multiset of vertices W. In simple words: ψ can be obtained from adding grains of sand to Ξ and relaxing the resulting state, it is **false** that this can be written as $\psi = \Xi + \Delta\sigma$ for some divisor $\sigma \geq 0$, then looks that apparently $\psi \sim \Xi$ is in contradiction with the above corollary, however, there is not such contradiction. The formula $\xi = \phi + \Delta\sigma$ can only be used if ϕ is unstable and there is an avalanche of the form $\phi \to \xi$ with ξ stable and certainly this is not our case because Ξ is stable.*

*It is important to remember that the notation $\psi = \Xi + \sum_{v \in W} \delta_v$ is an abuse of notation for the correct expression $\psi = (\Xi + \sum_{v \in W} \delta_v)^\circ$ where $(\Xi + \sum_{v \in W} \delta_v)^\circ$ is the relaxation of $\Xi + \sum_{v \in W} \delta_v$, using this correct expression we see that $\psi = (\Xi + \sum_{v \in W} \delta_v)^\circ = \Xi + \sum_{v \in W} \delta_v + \Delta\sigma$ for $\sigma \geq 0$, then the moral is that $\psi \sim \Xi + \sum_{v \in W} \delta_v$ but **not** $\psi \sim \Xi$, in total agreement with the above corollary.*

The following is a straightforward affirmation that nevertheless provide an efficient way to compute the sandpile group identity, it is in fact the way in which the identity is computed in popular sandpile explanations like the fantastic one of [26].

Corollary: 4.60. *The identity i of the sandpile group can be calculated as:*

$$i = (2\Xi - (2\Xi)^\circ)^\circ.$$

Proof. Call $j = (2\Xi - (2\Xi)^\circ)^\circ$ and notice that j is recurrent because:

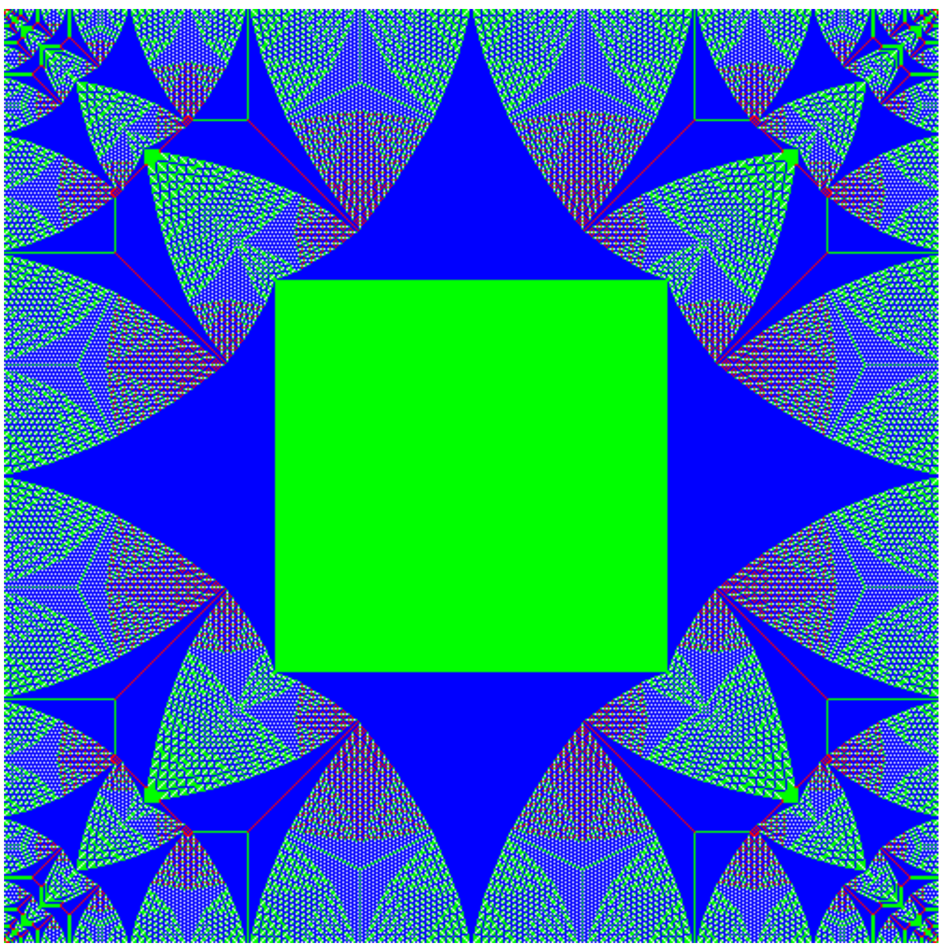
$$(2\Xi - (2\Xi)^\circ)^\circ = (\Xi + (\Xi - 2\Xi)^\circ)^\circ,$$

and

$$\Xi - (2\Xi)^\circ \geq 0.$$

Because j is on the same class as c_{null} which by lemma 4.25 represents the zero class on $\frac{\mathbb{Z}}{\tilde{L}\mathbb{Z}}$ it follows that the recurrent configurations j and i are on the same class, but by the above corollary there is only one recurrent configuration in each equivalence class of $Jac(G)$, then $i = j$. □

Example: 4.61. *To notice how non trivial the above result is, it is very stimulating to visit [33] to learn how to compute the identity in a $n \times n$ square grid from a purely algorithmic approach, to take a beautiful example we show the identity on a 500×500 square grid. The number of grains on every vertex is indicated by the color of the vertex according to the following rule: blue = 3, green = 2, red = 1, white = 0.*



5 Why are sandpiles self-organized?

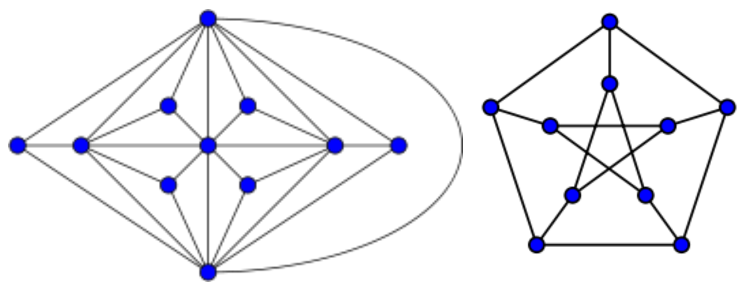
"The loss of locality suggested by string perturbation theory breakdown is precisely sufficient to resolve some versions of the information paradox."
Suvrat Raju -The Breakdown of String Perturbation Theory and Implications for Locality. - Strings 2017

Let us begin with some technical clarifications:

In the course of this work we have used repetidelly the allegory that some graph G can be used as a "planar table" where we can put sand and study how avalanches occur. But this is not strictly true, all the past results are valid independently of whether the graph is planar or not. But now the requirement of planarity is essential.

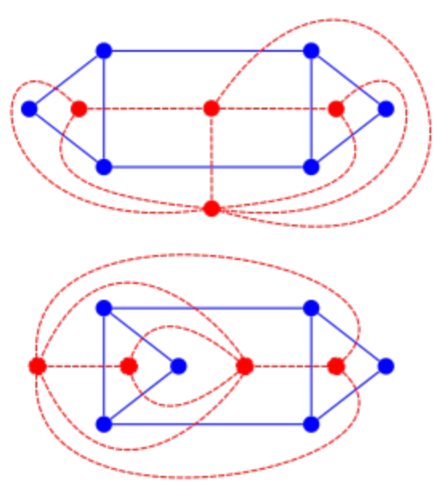
Definition 5.1. A graph is *planar* if it can be drawn in a plane without graph edges crossing.

Examples: 5.2. *At the left (below) we show the Goldner-Harary graph as example of a planar graph and at the right the Peterson graph that is not planar.*



Notation 5.3. *In the rest of the work we always presuppose that G is simple, connected, planar and $d-$regular. Also in this chapter G is always finite.*

Observation 5.4. *Every time we refer some planar graph G we are implicitly assuming that a particular planar embedding was chosen. This is important because there are ambiguities on the algorithms that we want to discuss that can be avoided once an embedding is given. Here there are an example of two isomorphic planar graphs with non isomorphic embeddings.*



There are several ways to be amazed by the simplicity of the unique toppling rule that governs sandpile dynamics, but as we have tried to emphasize, the fundamental amazement is the obvious to the human view (but elusive to the human understanding) beautiful global pattern that appears when a large number of grains of sand are introduced on to graph. What could the incredible complex global pattern emerging from the simple local rule is trying to indicate us?.

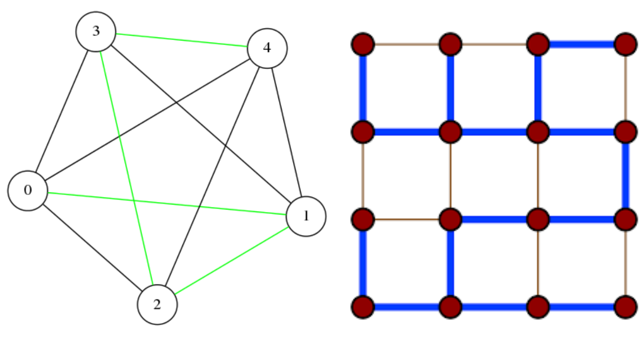
In this chapter we want to partially answer that question by showing a beautiful interplay between sand dynamics and the space in which the dynamics is taken place. Stated simply the answer is that recurrent sandpiles are in one to one correspondence with "long range correlations" between graph vertices, by "correlation" between to vertices on a graph we understand any path that passes throw that points, paths with cycles could be thinked as "redundancies" on correlations and if necessary we can always build any cycle on a graph as union of paths (remember that we are supposing graphs without loops).

Definition 5.5. A *tree* in a graph G is any connected path T without cycles.

What is more, this *long range* correlations are *maximal* in the sense that they actually relate any two vertices of G ; this is extremely non trivial because this kind of behaviour is typical of physical systems at the critical point, and as we have showed in the last chapter, once the sandpile dynamics have reached a recurrent (critical) state it never becomes forbidden (non critical) and this strongly suggest that the behaviour of the sandpile model is the proper of a conformal field theory in two dimensions. By "maximal correlation" on a graph we understand a tree that passes through all the vertices of G. This is how we formalize what we will understand by "maximal correlation" without redundancies.

Definition 5.6. A *spanning tree* of a graph G is a tree that passes through all the vertices of G.

Examples: 5.7. *The image (below) on the left shows an spanning tree (remarked with green) over the (non planar) complete graph K_5 and the image at right contains an example an spanning tree (remarked with blue) on the 4×4 square grid.*



The following definition is a technicality, but it is also a class of graphs for which the results of this section are valid, in fact, most of the interesting cases are examples of this class.

Definition 5.8. An *almost $d-$regular graph* is a graph with a distinguished vertex s such that removing s and all the edges of the form $e = vs$ from G the resulting graph is $d-$regular.

Examples: 5.9. *I) Any regular polygon P of n sides is an almost $2-$regular graph if we connect every vertex of P with a point (not on the polygon) using exactly one edge.*

II) More generally: Let $m \in \mathbb{N}$ and let Ω be the convex hull of m distinct points contained on a plane is an almost $2-$regular graph if we connect every vertex of Omega with a point (not on Ω) using exactly one edge.

III) Let $N, M \in \mathbb{N}$, the square grid of size $N \times M$ is an almost $4-$regular graph if we connect every vertex at the boundary of the square grid with a point (not on the grid) using exactly one edge.

Observation 5.10. *All the results of this chapter are valid for an almost $d-$regular graphs under the condition that the distinguished vertex can be selected as the sink.*

The reason is that the following theorem can be proved by using word by word the proof of theorem 3.24.

Theorem 5.11. *Let (G,s) be a connected $d-$regular graph and suppose that ϕ is an allowed configuration over G and let s be the sink, then, in the toppling from the sink algorithm (with input ϕ) every vertex topples exactly once*

Proof. See the proof of theorem 3.24. □

We state our new slogan:

> "Spanning trees are the graph for which they are sub graphs without redundancies."

And we discover that long range correlations are present in the dynamics of sandpiles by the following algorithm:

5.1 Burning Algorithm

Let ϕ be a stable sandpile on a graph G embedded on the plane with a distinguished vertex $\bullet$ as sink.

We associate a sub graph $T[\phi]$ of G to ϕ inductively by the next steps:

Input. $(G, \bullet)$ & ϕ.

1) **At time t=0** topple $\bullet$ (or in other words: put a grain of sand at every vertex adjacent to the sink).

2) **At time t=1** color the edge $\bullet v$ if the vertex v becomes unstable after the toppling of $\bullet$.

3) Topple *simultaneously* all the sites that are unstable **at time t=n**.

4) For a given site v that became unstable **at time t = n +1**, consider the set of all sites that have shared a grain with v at time $t = n$, then color an edge vw where w is chosen from this set according to the sense of the clock hands, it is to say: If some w is at "the north of v" (twelve o'clock) select w, if not, color the first edge that appears following the clock sense.

5) Denote by $T[\phi]$ the sub graph of G generated by the union of all colored edges by the algorithm.

Output. A sub graph $T[\phi]$ of G.

The algorithm is well defined because it stops as the number of sites is finite, the algorithm does not act two times on the same vertex because any of them topple at most once (theorem 3.24), finally, the way in which the algorithm color an edge is non ambiguous for every vertex because the prescription IV avoids any potential ambiguity in the coloring process.

The sink of G is important to make the dynamics of the sandpile model well defined (in the sense that the dynamics stops), but it is irrelevant for correlations in the same way that, in a finite state thermal machine coupled to a reservoir, we are not interested in correlations "machine to reservoir", the reservoir is only relevant to make the system reach thermal equilibrium acting as a "sink" (or a "source") of heat; in this analogy, the sandpile state behave as a thermodynamic system and $\bullet$ as a reservoir.

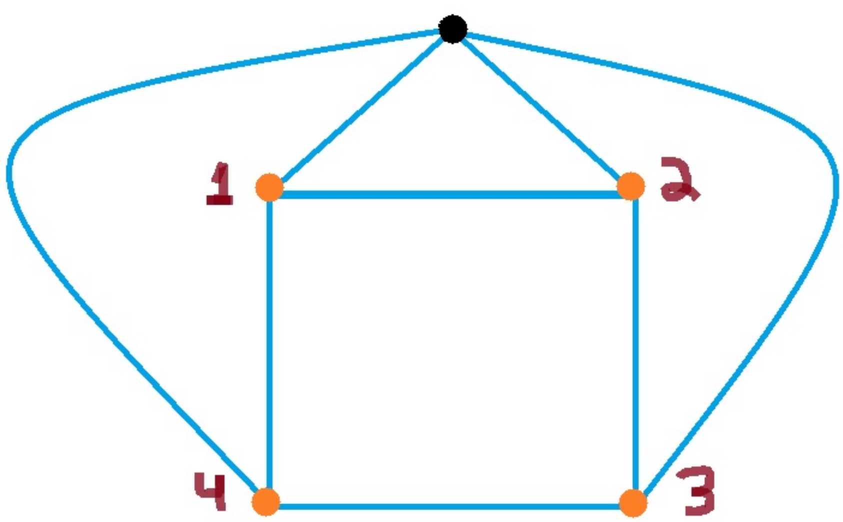
Theorem 5.12. *If ϕ is a recurrent configuration, $T[\phi]$ is a spanning tree.*

Proof. As ϕ is recurrent, theorem 3.20 guarantee that all the vertices of G topple if we apply the burning algorithm to it, then $T[\phi]$ passes through all vertices.

It can also be noticed that $T[\phi]$ does not have cycles, because suppose that there is in fact a path on $T[\phi]$ ordered as $v_1 v_2 ... v_n v_1$ where v_i has toppled at some time t_i contributing with one chip to producing the toppling of v_{i+1} at time t_{i+1}, then v_1 is toppled at times t_1 and t_{n+1} in contradiction with theorem 3.24, therefore $T[\phi]$ has no cycles.

By the above we conclude that $T[\phi]$ is a spanning tree. $\square$

Example: 5.13. *Let us consider the following almost $3-$regular graph G:*



We declare that the uppermost (black) vertex will be the sink of the game.

It is easily verified that the following configuration is recurrent: (either from the toppling from the sink algorithm or by noticing that is obtained by adding one grain of sand at the first vertex to the maximally stable configuration):

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Either from the toppling from the sink algorithm or by noticing that is obtained by adding one grain of sand at the first vertex to the maximally stable configuration:

The objective is to show how to construct a spanning tree for G from this recurrent sandpile state by using the burning algorithm.

Burning algorithm:

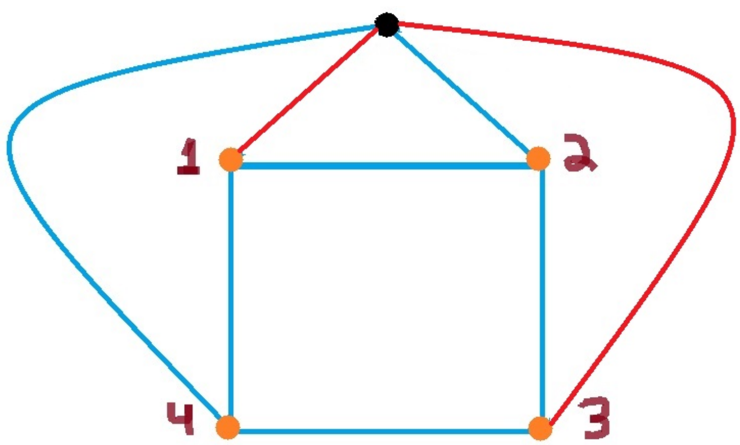
1) Applying the "toppling from the sink algorithm" we have:

$$\beta = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \ \textit{Toppling the sink} \ \mapsto \ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \ \textit{Sandpile toppling rules} \ \mapsto \ \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \ \mapsto \ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

2) Because the vertices 1 and 3 became unstable after the sink has toppled, we color with red the edges connecting them with the sink.

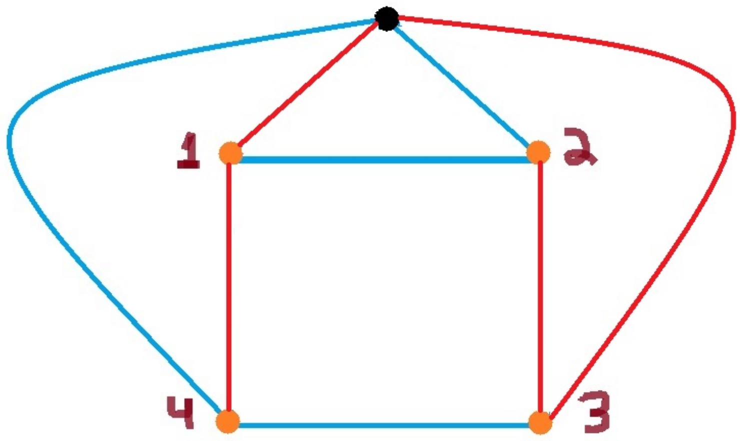
3) Topple vertices 1 and 3 simultaneously and notice that vertices 2 and 4 become unstable.

4) Fix your attention on vertex 2; because vertices 1 and 3 caused the instability of 2 at step 3), we need to decide if we color with red the edge 12 or the edge 23, this is done using the prescription of the burning algorithm, sweeping a vector with tail at vertex 2 starting at the north from 2 in the sense of the clock, we find that the vector intersects first the edge 23, then we color with red the juncture 23.



Now we focus or attention on vertex 4, because vertices 1 and 3 caused the instability of 4 at step 3), we need to decide if we color with red the edge 41 or the edge 43, this is done using the prescription of the burning algorithm, sweeping a vector with tail with center at vertex 4 starting at the north from 4 in the sense of the clock, we find that the vector intersects immediately the edge 41, then we color with red the juncture 41.

Before the topplings of vertices 2 and 4 have been performed the configuration is stable, the algorithm stops and the set of edges of G marked with red are a spanning tree of G.



Graphs are artefacts that abstractly encode relationships between vertices, its spanning trees are all the possible ways to fully correlate them without redundancies and recurrent sandpile states are precisely those correlations.

Finally we show how to produce a recurrent sandpile state from a spanning tree.

5.2 Dhar's tree bijection algorithm.

Is the objective of this subsection to show how to produce a recurrent sandpile state from a spanning tree on a graph G with a chosen sink, this is done basically by *"reverting"* the steps of the burning algorithm. We warn that the algorithm is more subtle that the preceding one.

Let T be a spanning tree on a graph G embedded on the plane with a distinguished vertex $\bullet$ as sink.

Let us associate a sandpile state ϕ_T on G to T recursively by the following steps:

Input $(G, \bullet)$ & T.

At time t=0. If there is path $\bullet u \in T$ where u is a vertex of G, put $d(u) - 1$ grains of sand at u, it is to say $\phi_T(u) = d(u) - 1$. After that happen we say that $\bullet$ and the vertices u with grains are *burnt*, the vertices without grains on it will be called *unburnt*.

At time t=n. For every unburnt vertex u with at least one burnt adjacent vertex define the following number:

$$\xi(u) := \text{Number of edges of } G \text{ connecting } u \text{ with an unburnt vertex},$$

and the following sets:

$$\mathscr{B}_n(u) := \{v \text{ burned neighbour of } u : uv \in T\},$$

$$\mathscr{N}\mathscr{B}_n(u) := \{v \text{ burned } \mathbf{at\ t = n\ \text{-}\ 1} \text{ neighbour of } u : uv \in T\},$$

$$\mathscr{F}(u)_n := \{e \text{ edge of } G\text{: } e \text{ connect } u \text{ with a vertex on } \mathscr{N}\mathscr{B}_n(u)\},$$

where we can enumerate the elements of the last set as:

$$\mathscr{F}(u)_n = \{e_1, ..., e_m\}.$$

Following the prescription of the step 4) of the burning algorithm, that is, put a vector with tail at u pointing towards the north, sweep the vector in the sense of the clock hands and call e_1 the first edge of $\mathscr{F}(u)_n$ that the sweeping vector intersects, call e_2 the second and so one.

Then put $\phi_T(u) = \xi(u)+(l-1)$ grains of sand at u where l is unique sub index such that $e_l \in \mathscr{F}(u)_n \cap T$.

Output: A stable sandpile state ϕ_T.

The resulting sandpile state ϕ_T is defined via its values at the vertices of G that the algorithm assign.

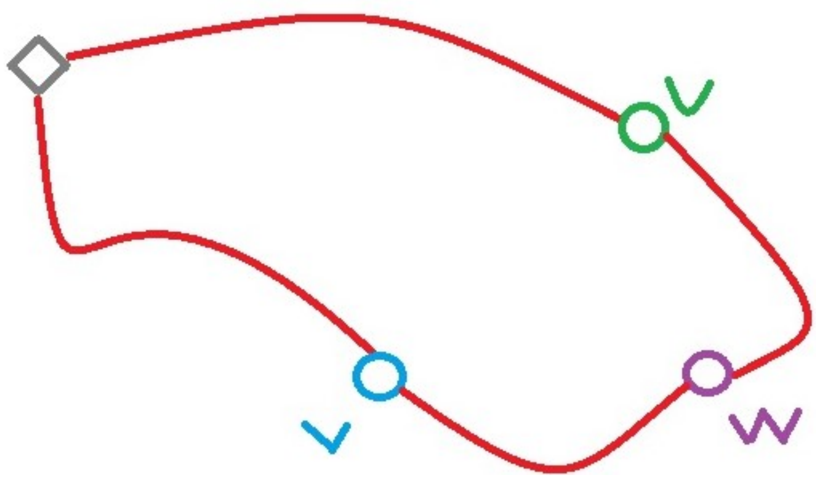
Now we prove that the algorithm is well defined in the sense that there is no ambiguity on in, the output is really an stable sandpile state and that the algorithm always stops.

Theorem 5.14. *The Dhar's tree bijection algorithm is well defined.*

Proof. The algorithm stops simply because T pass trough all the vertices of G. What is really interesting is to show that when the algorithm is applied to an unburnt vertex at time $t = n$ there is exactly one element e_l on $\mathscr{F}(u)_n \cap T$, this is the only potential ambiguity in the algorithm. We prove it by induction.

Let u and v vertices that were burnt at time $t = 0$, in particular that means that the edges $\bullet u$ and $\bullet v$ are elements of T, suppose that at time $t = 1$ there is a vertex w such that $\bullet u, \bullet v \in \mathscr{F}(u)_n \cap T$, in particular $\bullet u$ and $\bullet v$ are in T, this statement is a contradiction because it imply that the loop $\bullet uwv\bullet$ is in T something that it is impossible as T is a spanning tree. Then at time $t = 1$ there is just one element in $\mathscr{F}(u)_n \cap T$ and no ambiguities on the algorithm so far.

The following picture illustrate how is impossible for a vertex w (in lilac) at time t=1 to have burned neighbours (in blue and green) connected by edges on T(in red) which in turn are connected with the sink by edges on T without contradicting that T is acyclic.



By induction hypothesis suppose that there is just one element on $\mathscr{F}(u)_n \cap T$, or in other words, the algorithm is ambiguity free at time $t = n$, now it is easy to see that if there were two elements on $\mathscr{F}(u)_n \cap T$ for some unburnt vertex u at the time $t = n + 1$ then we can guarantee the existence of a loop in T along the same line of reasoning as in the base of the induction.

The algorithm is ambiguity free.

As T covers all the vertices of G then ϕ_T is well defined at every vertex, so is a sandpile state.

The only think that remains to be proven is that ϕ_T is a stable sandpile state. to show this it suffices to prove that $d(u) > \phi_T(u) = \xi(u) + (l-1)$ for all u. But this is in fact very easy to see because $\xi(u) + l < d(u)$. Then ϕ_T is stable. □

It is interesting to notice that the algorithm is subtle because the burning algorithm is well defined when applied to every sandpile state, but it is only when is applied to a recurrent one that the output of the algorithm produces a sandpile state. On contrary, Dhar's tree bijection algorithm is not well defined if his input is not a spanning tree.

Theorem 5.15. *Let G be a graph ,when Dhar's tree bijection algorithm is applied to a spanning tree T of G the algorithm produces a recurrent sandpile state ϕ_T.*

The idea and the structure of the proof is very beautiful.The strategy is to apply the toppling from the sink algorithm to ϕ_T to show that every very vertex topples once to guarantee that ϕ_T is recurrent, the magic is that in the proof makes manifest why the burning and Dhar's tree bijection algorithm are fundamentally the same process applied to the same object expressed in different variables.

Proof. Notice that if we apply the burning algorithm to ϕ_T **at time t=0** we have that $\bullet$ is toppled and all the vertices u such that $\bullet u \in T$ are unstable, denote the later set by $\mathscr{U}_0$ (unstable at $t = 0$); denote by $\mathscr{T}_1$ the set of toppled vertices **at time t=1**, it is clear that $\mathscr{T}_1 = \{\bullet\} \cup \mathscr{U}_0$.

On the other hand, if we apply the Dhar's tree bijection algorithm to T we have that at **at time t=0** $\bullet$ is burnt and all the vertices u such that $\bullet u \in T$ are burnable, this set is exactly $\mathscr{U}_0$ from the above paragraph, in fact the burned vertices **at time t=1** are $\mathscr{B}_1 = \{\bullet\} \cup \mathscr{U}_0$.

By induction hypothesis suppose that $\mathscr{T}_k = \mathscr{B}_k$ **at time t=k**, were $\mathscr{T}_k$ is the set of toppled vertices at time t=k when the burning algorithm is applied to ϕ_T and $\mathscr{B}_k$ is the set of burned vertices at time t=k when Dhar's algorithm is applied to T.

Now take $v \in \mathscr{B}_{k+1}$ in Dhar's algorithm and let us analyze what happen with v at time t=k+1 when the burning algorithm is applied to ϕ_T. By induction hypothesis at t=k v have been received one grain of sand from all his burned neighbours, in fact, v have received a total of $d(v) - \xi(v)$ grains, then at time t=k $v \in \mathscr{N}\mathscr{B}_k$ and it has $\phi_T(v) + d(v) - \xi(v)$ grains or:

$$\phi_T(v) + d(v) - \xi(v) = \xi(u) + (l_v - 1) + d(v) - \xi(v) = d(v) + (l_v - 1),$$

but from the fact that $l_v \leq 1$ it follows that v is unstable at t=k when the burning algorithm is applied to ϕ_T, then $v \in \mathscr{T}_{k+1}$.

Then

$$\mathscr{T}_{k+1} \subseteq \mathscr{B}_{k+1}.$$

Similarly, take $w \in \mathscr{T}_{k+1}$ in the burning algorithm and let us analyze what happen with w at time t=k+1 when Dhar's algorithm is applied to T. The observation is that by induction hypothesis w has burned neighbours at time t=k and as T passes trough all the vertices, in particular it passes trough w; from this follows that Dhar's algorithm burn w at time t=k+1.

Then

$$\mathscr{B}_{k+1} \subseteq \mathscr{T}_{k+1}.$$

Therefore:

$$\mathscr{B}_{k+1} = \mathscr{T}_{k+1}.$$

The conclusion is that the set of burned vertices when Dhar's algorithm is applied to T and the set of vertices that the burning algorithm topple when applied to ϕ_T is exactly the same set. Because Dhar's algorithm burn all the vertices of G it follows that all the vertices of G topple if the burning algorithm is applied to ϕ_T, then by theorem 3.18 ϕ_T is recurrent.

□

Theorem 5.16. ***(Matrix tree)*** *Let G be a planar, connected and d-regular (or almost d-regular) graph, then the number of spanning trees on G is equal to $|det(\tilde{L})|$.*

Proof. We know from corollary 4.46 that $|Jac(G)| = |det(\tilde{L})|$, theorem 4.57 state that $Jac(G) = \mathcal{S}(G)$ and we now know that $|\mathcal{S}(G)|$ is the number of spanning trees on G. □

It is worthy emphasized that the matrix tree theorem is a result of deep importance and the result is much stronger, it holds in any connected graph G. We have only proved it under the hypothesis of the later theorem as an easy consequence of the study of the dollar game and sandpile dynamics.

5.3 Conclusion.

"Mathematics is about all platonic forms of beauty.
To know that something is true is not enough,
proving is not guarantee of understanding.
The goal is to reduce proofs to obvious facts,
it is the only way that humans have to reach the beauty of understanding."
Ernesto Lupercio.

We have stated in the introduction that the deep meaning of critical self-organization is still lacking, but we have offered a pragmatic definition:

"A self-organized critical system is a dynamical system whose attractors are its critical points".

Now we have showed that the attractors of the sandpile dynamics are its recurrent states, this is not a tautology because we have proved that once the critical state was reached the sandpile never fall intro a transient configuration under a random evolution. We have also characterized recurrent states as long range correlations in the sense that they are spanning trees on graphs, maximality of spanning trees is related to emergence of global patterns from local rules, the place where the magic remains is the (proved) fact that the paths on the graph that sandpile states represent are acyclic, this fact hides the mystery, the delicate *local to global interplay.*

Then we argue that a pragmatic (but not epistemologically satisfying!) answer is that the sandpile is self-organized because its attractors under a random evolution are states that are by itself long range correlations and because the appearance of transient states is obstructed by the dynamics.

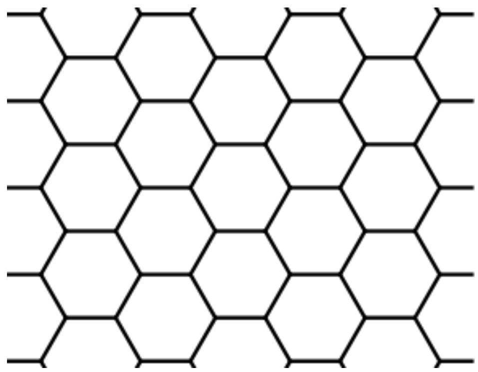
6 The dimer problem.

> "One often hear that people say that we know a lot about two dimensional conformal field theories. I am not sure to share that feeling."
> Xi Yin - Super Conformal Bootstrap in 2D - Strings 2016.

6.1 A graphene story.

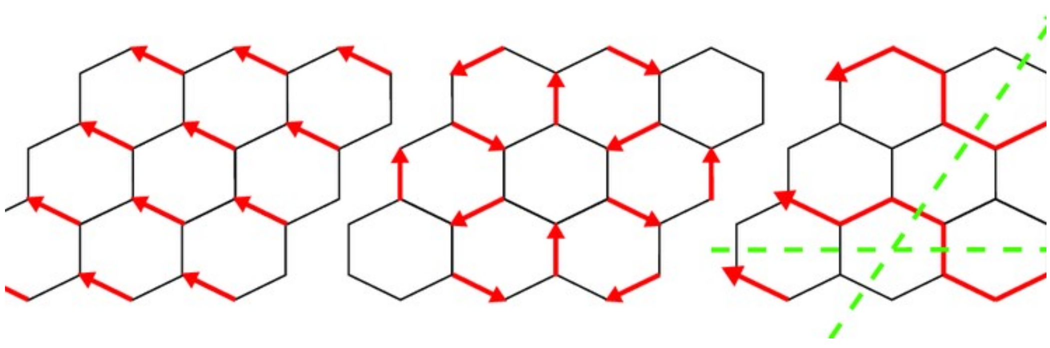
As we have mentioned, integrable models are of enormous importance in mathematics and physics. Mathematicians have encountered one very rich solvable system by studying the following physicist story:

Consider a crystalline one layer surface, one layer of graphene could be a particularly exciting example, a layer of this material can be model with excellent approximation by the honeycomb lattice, here is a picture of this lattice:



Any vertex of this lattice represent a carbon molecule.

Any diatomic molecule of any other compound would be called a *dimer*, a *dimer cover* on a layer of graphene is an arrangement of diatomic molecules such that every every carbon molecule is paired with exactly one molecule of one dimer.**The dimer problem** can be stated as follows: **How many dimer covers are there for a layer of graphene?**



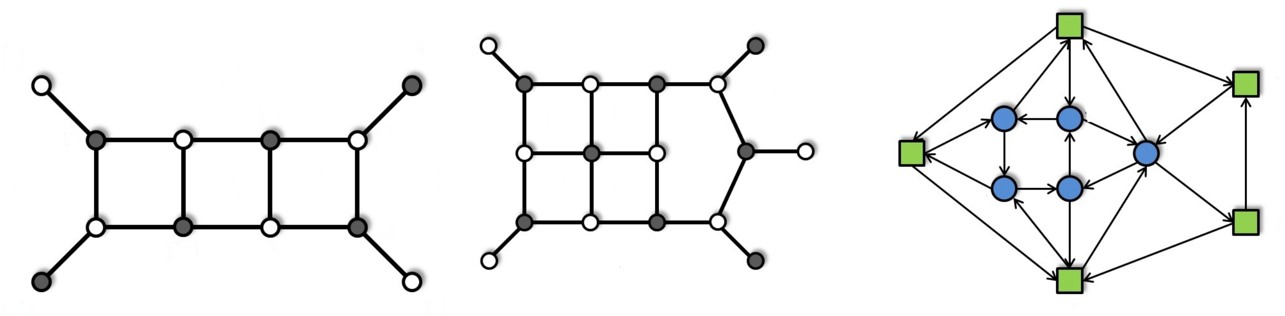
The leftmost and center figures show different dimer covers on the honeycomb lattice, the edges on black represent the graphene and the arrows in red correspond to dimers, the key point is that every (not boundary) vertex is at most one time the head or the tail of an arrow. The rightmost figure show an arrangement of dimers that is not a dimer cover, the dashed lines in green intersect at the center of an hexagon whose all (non boundary) vertices are covered by two dimers.

Is our objective to make precise this problem and show how Fisher, Kasteleyn and Temperley have solved it for general crystalline layer.

6.2 The dimer problem.

Definition 6.1. A *bipartite graph* is a graph such that using two different colors all vertices can be colored in such a way that all the adjacent vertices of a vertex v are of the same color and this color is different from the color of v.

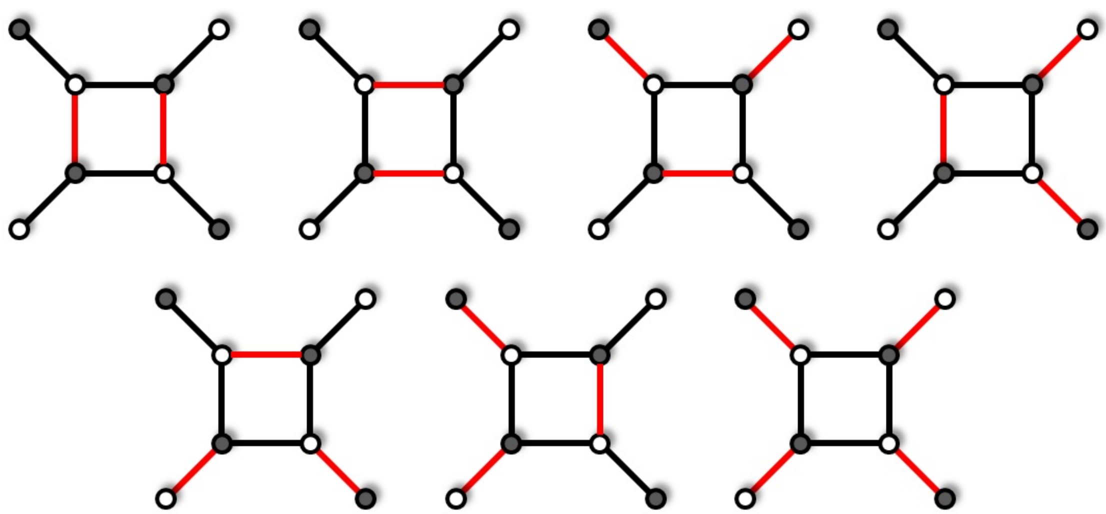
Examples: 6.2. *Below at the left and below at the center we show examples of a bipartite graphs and below at right an example of one that is not:*



Notation 6.3. *In the rest of the chapter we always assume that whenever we say "Let G be a graph" we are demanding that G is a planar with an specified embedding, oriented, connected graph.*

Definition 6.4. Let G be a graph, a *dimer cover* or *perfect matching* for G is a subset PM of edges of G such that every vertex of G belongs to exactly one edge of PM. The element of PM will be called *dimers.*

Example: 6.5. *The following are examples of dimer configurations, dimers are marked in red and edges that are not dimers are marked on black, it is notices that all the dimer configurations of the first row and the first two of the second row are not dimer covers, on contrary, the last one that it is in fact a perfect matching:*



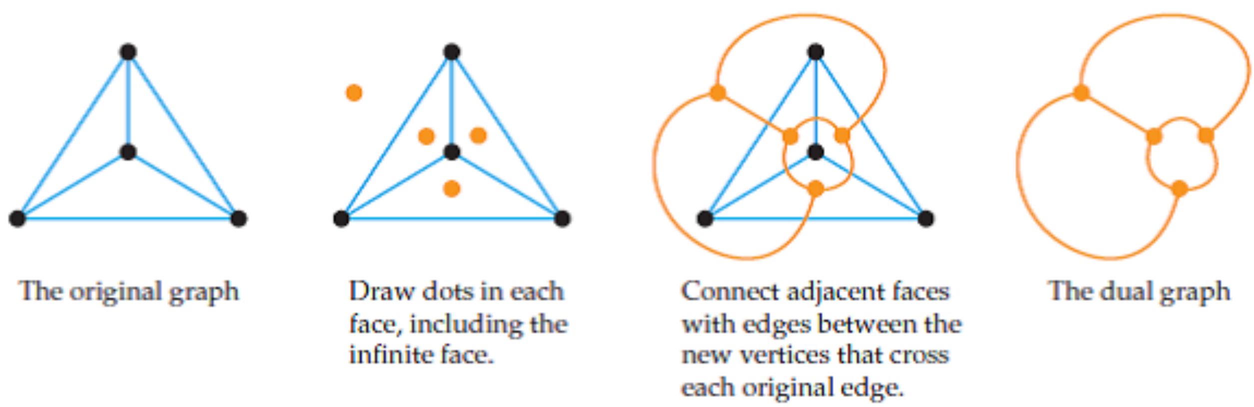
The dimer problem is stated as the challenge of computing how many perfect matching exist for a given graph G.

6.3 The Kasteleyn-Fisher-Temperley solution to the dimer problem

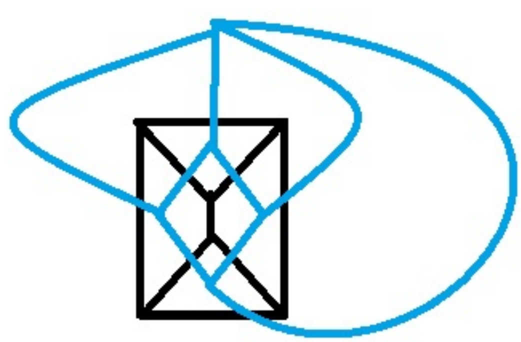
6.4 Temperley's Bijection

Definition 6.6. Let G be a graph, The dual graph G^* of G has graph vertices each of which corresponds to a face of G and each of whose faces corresponds to a graph vertex of G. Two vertices in G^* are connected by an graph edge if the corresponding faces in G have a boundary graph edge in common. As a result, each edge of a graph G has a corresponding dual edge e^* in G^* corresponding to the edge that connects the two faces on either side of e, meaning the edge counts are the same.

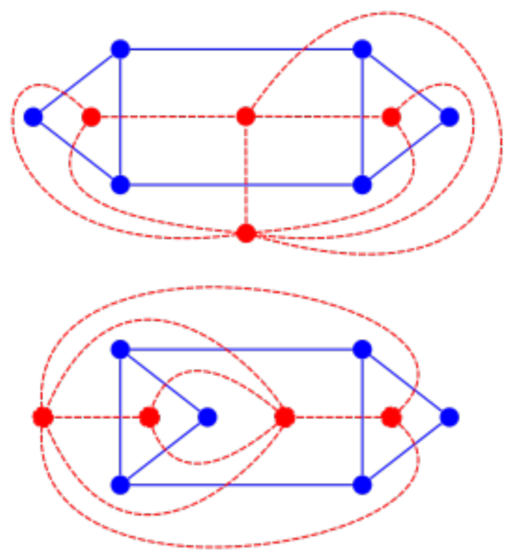
Example: 6.7. *The following figure illustrates how to generate step by step the dual graph of another 3-regular graph.*



Example: 6.8. *Now we show the described construction of the dual graph G^* in blue to the graph 3-regular G shown in black. It is noticed that the dual graph is 3-regular except at one point where the edges associated to the boundary edges of G cross.*



Observation 6.9. *It is remarked the importance that is not enough to say that G is planar, it is necessary to prescribe an embedding of it into the plane because the notion of dual graph depends on that embedding. The following example shows two different embeddings of the same graph with non isomorphic duals.*



Definition 6.10. The doubled bipartite graph G^d of a graph G is the graph constructed by following algorithm:

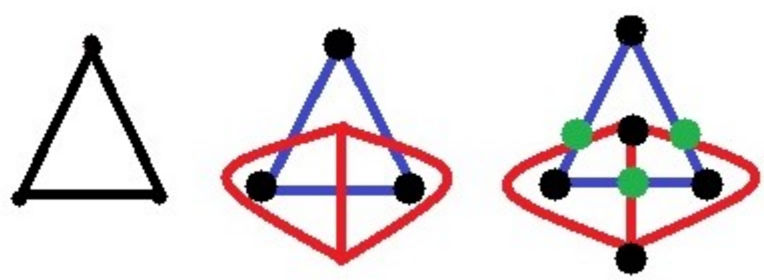
I) Embed G and G^* simultaneously in the plane in such a way that every edge e of G intersect the corresponding dual edge e^* of G^* exactly once and it does not cross other edge of G.

II) Color the vertices of G and G^* in black.

III) Take the intersections of the edges of G and G^* as new vertices and color them on white.

G^d is the resulting graph.

Example: 6.11. *It is shown how to construct the double graph of a triangle (in black at the leftmost), below at the center is drawn how to superpose the triangle (in blue) with its dual (in red), finally below at the rightmost it shown how to color the intersection of edges between the triangle and its dual (dots in green), this is precisely the double graph of the triangle and is evident from the figure that the double graph is bipartite.*



Observation 6.12. *Given a graph G (bipartite or not) the dual graph constructed in definition 6.2 is by construction bipartite.*

The above construction is a little bit contrived, we want to emphasize that the idea of the construction is very simple, is just a matter of how to canonically assign to a given graph G a bipartite graph.

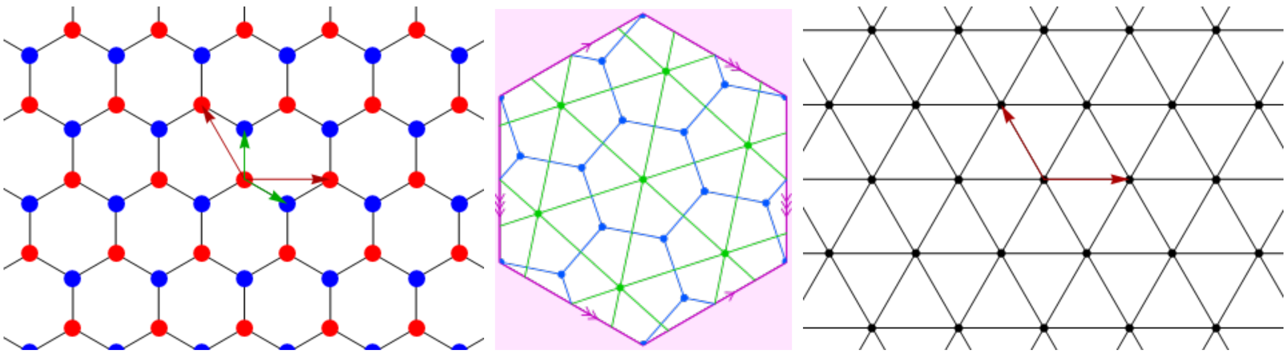
To make the double graph construction more intuitive, it is extremely illuminating to consider how natural the construction is on tillings of the plane which can be understood as "very large graphs".

Definition 6.13. A tilling is a collection of disjoint open sets of the plane such that its closures cover the plane.

Example: 6.14. *The honeycomb lattice is a tilling, the interior of the hexagons are the open sets that define it as a tilling.*

The definition of the double of a lattice mimics the one given for graphs.

Example: 6.15. *Below to the left it is shown the honeycomb lattice, below at the center is illustrated how to superpose the honeycomb lattice (in blue) with its dual (in green) to obtain the doubled lattice of the honeycomb lattice that is shown below to the right.*



6.5 The Euler formula

Definition 6.16. Let G be a graph, a *forest* on G is a sub graph that is a disjoint union of trees.

Now is presented one of the greatest formulas ever discovered by humans:

Definition 6.17. The Euler characteristic of a graph with F faces, E edges and V vertices is the following number: $\chi = F - E + V$

Lemma 6.18. *The Euler characteristic of a connected planar graph is 2.*

The reference [23] collect twenty (!) simple and beautiful proofs of this fact. Next we use the proof number one of the above list because the key observation of the proof would be very useful for us latter.

Proof. Draw G^d in the plane. The wonderful observation is the following:

Any cycle in G disconnects G^* (Jordan curve theorem) and any sub graph of G without cycles is a forest that does not disconnects G^*.

It is noticed that every tree T verifies that $E = V - 1$ and $F = 1$ and take any spanning tree T of G. The dual edges of its complement form a connected sub graph of G^* that is also a spanning tree T^*. The Euler formula applied to $G = T \cup T^*$ gives $E = (V - 1) + (F - 1)$, then the Euler characteristic of G is 2. □

Proposition 6.19. *The number of vertices of G^d is even.*

Proof. By construction the number of vertices V_{G^d} of G^d satisfy:

$$V_{G^d} = V_G + V_{G^*} + E_G,$$

where V_G, V_{G^*} and E_G are the number of vertices of G, the number of vertices of G^* and the number of edges of G, respectively.

We notice that if F_{G^*} is the number of faces of G^*, by definition we have $V_G = F_{G^*}$, therefore:

$$V_{G^d} = F_{G^*} + V_{G^*} + E_G,$$

Applying the above lemma to G^* we obtain:

$$2 = F_{G^*} + V_{G^*} + E_{G^*},$$

where F_{G^*} and E_{G^*}, are the number of faces and edges of G^* respectively.

Therefore:

$$V_{G^d} = (F_{G^*} + V_{G^*}) + E_G = 2 + E_{G^*} + E_G.$$

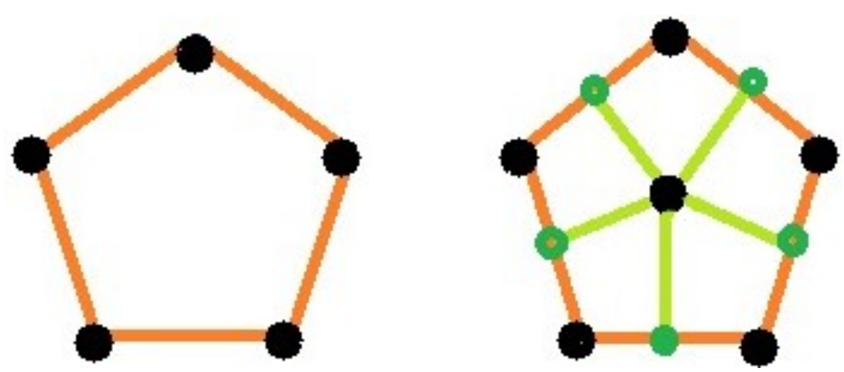
But by construction of G^*, we know that $E_{G^*} = E_G$, then V_{G^d} is divisible by two. □

Notation 6.20. *In what follows we will be saying that a face f of G bounds n vertices of G^d to abbreviate the following fact: there is a copy of G on G^d(with extra vertices and edges of G^d) the boundary of f in this copy of G is a closed simple curve that divide the plane into two components (Jordan curve theorem) one that could be assumed compact and the other open and unbounded; when we use the refereed phrase we mean: the number of vertices that the bounded part produced by applying the Jordan curve theorem to the boundary of f. on G^d*

Lemma 6.21. *Let f be a face of G, when G^d is drawn on the plane the region that f encloses on G^d contain an odd number of vertices of G^d.*

Proof. Every face of G is (topologically) a polygon of n sides or an n-agon, it suffices to prove the affirmation for n-agons.

The figure in below to the left shows a pentagon as a possible face of a graph, below at the right is shown how that face looks like when G is doubled, as is possible to see, the original five vertices are doubled to ten by the vertices that come from intersections of edges of G (in orange) with vertices of G^*(in green) plus the black vertex at the intersection of apothems of the pentagon that comes as the vertex dual to G in G^*. Eleven vertices in total, an odd number.



The same story holds for an n-agon, the number of vertices of G^d that should bound on G^d is $2n+1$. □

Lemma 6.22. *If a face of G is topologically an n-agon, then the number of faces that this face enclose on G^d is $2n+1$.*

Proof. Immediate from the proof of the above theorem. □

Notation 6.23. *In what follows we will be considering loops on graphs. We use the word loops as an abbreviation for a loop without self intersections and with winding number one, a simple loop. Those loops divide the plane into two components (Jordan curve theorem) one that we can assume compact and the other open and unbounded; when we use the phrase "the number of vertices that a loop enclose" we mean the number of vertices in the compact component of the plane associated to the loop.*

The next lemma is a very important part of the proof of the result that we want to show.

Lemma 6.24. *Let G be a graph, let $\mathscr{C}$ be a cycle on G, then the number of vertices of G^d that $\mathscr{C}$ encloses on G^d is an odd number, furthermore, the number of vertices of G^d on the unbounded component associated to G^d is also an odd number.*

Proof. We proceed by induction on the number of faces that $\mathscr{C}$ could enclose. At every step of the induction we denote by $\mathcal{B}$ the compact region of the plane associated to and by $\mathcal{U}$ the unbounded one.

Suppose that $\mathscr{C}$ encloses exactly one face of G, by lemma 6.21 the number of vertices of G^d on $\mathcal{B}$ is odd and by proposition 6.20 the number of vertices of G^d is even, noticing that the number of vertices on G^d is the sum of those on $\mathcal{B}$ plus the ones on $\mathcal{U}$ follows that the number of vertices on $\mathcal{U}$ is also odd.

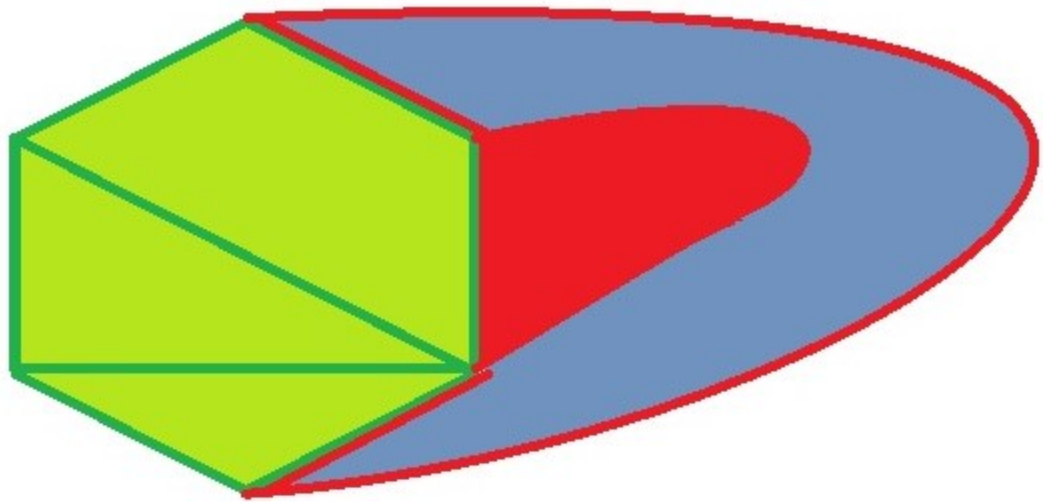
By induction hypothesis suppose that whenever $\mathscr{C}$ encloses n faces of G the number of vertices on $\mathcal{B}$ and $\mathcal{U}$ are both odd numbers.

Now suppose that $\mathscr{C}$ encloses $n+1$ faces of G and let us compute the number of vertices contained on $\mathcal{B}$. Select n faces of G enclosed on by and call the remaining face f, if we count the vertices of G^d on those n faces of G contained on $\mathcal{B}$ we obtain an odd number (induction hypothesis) say $2a+1$ with $a \in \mathbb{N}$, it remains to count the vertices on f and subtract those ones on the edges that this face share with the first n ones. The key point to do this count is to notice that the number of vertices of G^d contained on an edge of G on G^d is three because every edge •—• of G looks like •—○—• on G^d.

By lemma 6.22 the number of vertices on f is $2b+1$ for some $b \in \mathbb{N}$ (f is an $b-$gon), suppose that f share just one edge with the first n faces, then the number of edges inside $\mathcal{B}$ is $(2a+1)+(2b+1)-3 = 2(a+b)-1$ and odd number, if f share two adjacent edges of G with the first n faces, then the number of vertices over $\mathcal{B}$ is $(2a+1)+(2b+1)-5 = 2(a+b)-3$ because two adjacent edges of G of the form •—•—• look like •—○—•—○—• on G^d; in general, if f share k faces with the another n ones then the number of vertices on $\mathcal{B}$ is $2(a+b)-2k-1$ an odd number. Notice that $\mathcal{U}$ also have an odd number of vertices of G^d because the later conclusion and the fact that number of vertices on G^d is even.

Finally notice the later case by case computation exhausted all the possibilities, because the situation in which f share two non adjacent edges of G with the chosen n faces is impossible, because in that case $\mathscr{C}$ encloses an extra face of G that is not f not one of the chosen n ones contradicting the hypothesis that $\mathcal{C}$ encloses exactly $n+1$ faces of G on G^d; contradiction.

We offer a picture to illustrate the described situation. The region shaded in green represent the union of n faces of G, the region shaded in blue with contour in blue is a face that share two edges with the green region and finally the red region represents the new face of G in G^d predicted by the argument of the later paragraph.

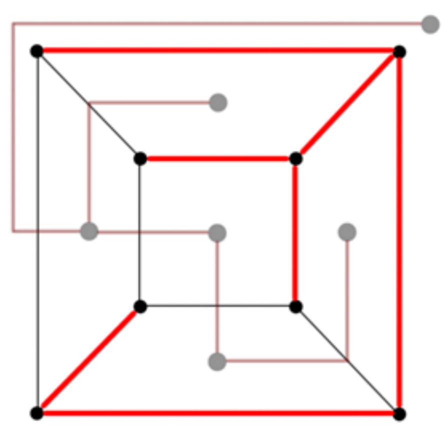


We conclude that $\mathscr{C}$ encloses an odd number of vertices, that in turn imply that the number of vertices on $\mathcal{U}$ is also odd. $\square$

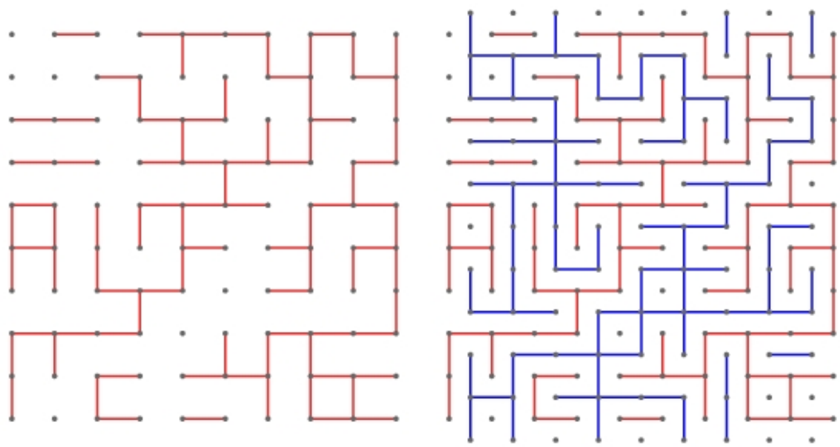
The construction of the double graph is a little contrived but it equip us with some tricks, the simplicity of the proof of the following theorem is an example of this.

Theorem 6.25. *Let G be a graph, there is a bijective correspondence between the set of spanning trees of G and those ones of G^*.*

The proof of this affirmation is visual and constructive in nature, for that matters it is convenient to have an image of what will be happen. The following picture have solid vertices and edges in black and red and black, respectively, they constitute a 3-regular graph G, the union of the red lines is a spanning tree T of G, there are also diffusely marked vertices and lines in gray and lilac, they constitute the associated spanning tree in G^* that will be constructed in the course of the proof.



Below at the left we show a spanning tree on the square lattice and below at the right its dual spanning tree (images taken from [40])



Proof. Let T be a spanning tree of G, we use the double graph G^d of G to define the following subset of edges on T^*.

Construct G^d and for every vertex v^* of G^* consider its set of adjacent vertices on G^*, if there is an edge $e^* = v^*w^*$ on G^* that not intersects T in G^d color it with green and denote by T^* the resulting set green edges of G^*. The affirmation is that T^* is a spanning tree of G^*.

In fact, if some vertex v^* of G^* is not a boundary of some edge e^* in T^* then this vertex in G^d is surrounded by a loop of T on G, but that its impossible as T is spanning tree. Then all the vertices of G^* are boundary of some edge of T^*.

To see that T^* does not have loops suppose that on the contrary, there is one, this loop is also a loop on G^* and in fact bound on G^d at least one face of G^* , it follows that the vertices of G associated to those faces are not boundary of edges of T because otherwise T and T^* intersect on G^d that it is impossible by how T^* was constructed but this contradicts the fact that T is spanning tree, contradiction; T^* has no loops.

To see that T^* is connected suppose that it is not, then there is a pair of vertices v^* and w^* of G^* that can not be connected by a path contained in T^* this statement is equivalent to the existence of a loop on G^d contained in T bounding the face of G associated to v^* on G^d, this contradicts that T does not have loops. It follows that T^* is connected.

It is easily shown that with the same construction starting with T^* allow us to reconstruct construct T simply by replacing the symbols G by G^* and T by T^* and vice versa in the above argument. Then the association $T \mapsto T^*$ is bijective. $\square$

Observation 6.26. *The orientation of G induces an orientation on T in the obvious way, that is, if an edge $e = uv$ is oriented from u to v and if $e \in T$ we orient $e = uv$ in T from u to v. A spanning tree with a chosen orientation will be called* ***an oriented spanning tree.***

Lemma 6.27. *Let G be an oriented graph, there is a bijective correspondence between the set of oriented spanning trees of G and the set of oriented spanning trees of G^*.*

Proof. Immediate. □

Notation 6.28. *Every vertex of G has one and only one outgoing edge in T if T is a spanning tree of G except at exactly one vertex v, If such a vertex did not exist T would have loops. The special vertex of T will be called the **root** of T and we say that T is **rooted at** v.*

Definition 6.29. Given the double graph G^d of a graph G, we associate to the pair (G^d,u) where u is a vertex of G^d a graph $G^d(u)$ obtained by deleting v from G^d together with all the incident edges of G^d in u.

Observation 6.30. *The graph $G^d(u)$ above defined is oriented and bipartite.*

We arrive at the central part of this work, the result is called the *Temperley bijection*, the result states that the number of spanning trees on G is the exactly the number of dimer covers of its reduced double graph, we use this result as the main ingredient for what we want to say. We warn that the proof that we present is lengthy, but done with the intention of being as explicative as possible and with every step ilustrated in detail. The decition was done because we believe that there a deep mystery not yet understood about it. In any case, the standard and original proof can be found on [23], we follow the approach of [24] which actually is our guide in what we want to say about sandpiles.

It is appropriate to remark that although the result presented here just concerns finite, connected and planar graphs, the Temperley bijection is a deep and vast phenomenon that has analogous statements that hold in much larger classes of graphs (that could be weighted, non planar, non finite etc.), originally the correspondence was discovered for square grids in [22] and later impressively generalized in [23] to more general classes of graphs; [23] is also a source for detailed and independent explanations of the correspondence in specific graphs and lattices.

Theorem 6.31. ***(Temperley bijection)*** *Let G be a graph and u any vertex of it, there is a bijection between the set of spanning trees of G and the set of dimer covers of $G^d(u)$.*

As in the case of lemma 6.15, the proof of this statement is visual and constructive, because of this we try to be as explicit (and visual) as possible when proving it.

Proof. **Tree → dimer cover.**

Let T be an oriented spanning tree on G, we want to associate a perfect matching PM to T, in order to achieve this we use the following algorithm:

1) Color T on G^d.

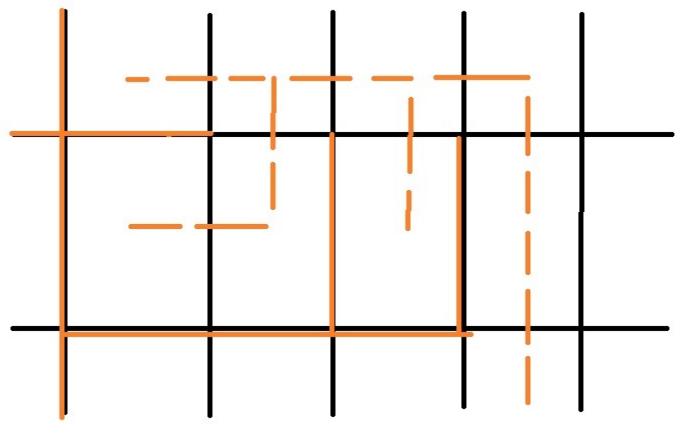
2) Using lemma 6.16 paint with dashed lines of the same color of T in G^d the unique oriented spanning tree T^* on G^* that do not intersect T at any point.

3) Take any vertex in black $\bullet$ of G^d and notice that $\bullet$ is boundary of an edge on T or on T^*, if $\bullet$ is on T consider the set of G^d-neighbours (in white) $\circ$ of $\bullet$ such that the edge $\bullet\!-\!\circ$ is oriented (according to T) from $\bullet$ to $\circ$, if there are no such neighbours do nothing, if there is one, say $\circ$, ask if there is a G^d-neighbour $\blacklozenge$ of $\circ$ such that the path $\bullet\!-\!\circ\!-\!\blacklozenge$ is contained in T and oriented with it; if that is the case then use a new color to color $\bullet\!-\!\circ$ on the contrary do not color $\bullet\!-\!\circ$.

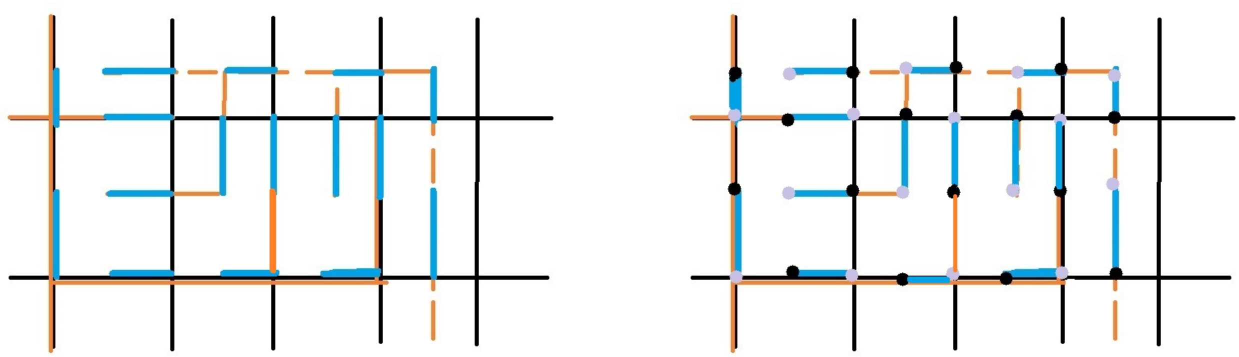
The same procedure can be applied to all the vertices of G^d, if some vertex is at T^* then replace in the above set of instructions T by T^* an apply the algorithm to it.

The affirmation is that the set PM of edges colored by the set of instructions 3) is a dimer cover of G^d.

The following picture pretends to illustrate the step 2) of the algorithm. A spanning tree colored in orange is constructed for a graph (in black) together with his dual painted with dashed lines in orange.



The images in below do not try to be explicative, they just try be to give a visual idea of how a dimer cover emerges from the above picture once the step 3) of the algorithm was performed, the resulting dimers are colored in blue.



To show that PM is a dimer cover of G^d just notice that by construction any vertex v is in the boundary of some colored edge in PM, in fact this edge is unique for v because if that were not the case, v is in T and also in T^*, this contradicts the fact that T and T^* does not have points in common. Thus PM is a perfect matching.

Dimer cover $\rightarrow$ spanning tree.

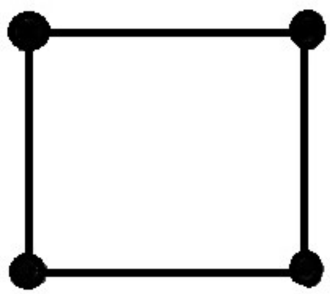
Let PM be a dimer cover of G^d, is now our objective to construct a spanning tree T on G from PM in such a way that T produces PM if the above algorithm is applied to it.

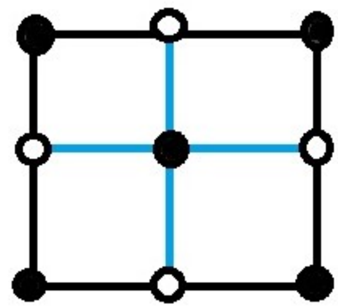
The set of vertices of G^d could be partitioned into three classes, the first class are vertices of G (in black), the second contain vertices of G^* (also in black) and the third one are white vertices ,or equivalently, those ones that arise in G^d as intersection of one edge of G with one from G^*. Notice that by construction of G^d every edge e on G has a unique white vertex $\bar{e}$ of G^d drawn over it.

Define T as the set of edges e of G such that his associated vertex $\bar{e}$ is paired in G^d by a dimer with a black vertex of that is a vertex of G.

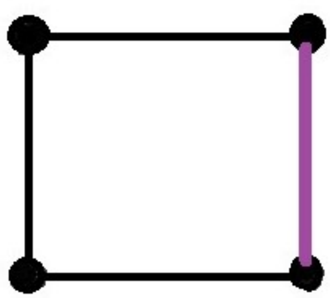
We explain how this works.

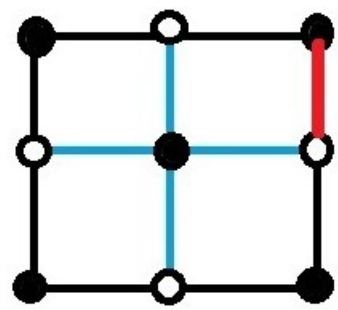
To avoid the difficulties inherent to the construction a dual graph, let us consider as G a large square grid, a face of G is of course a square shown above to the left in black, after the construction of the double graph of G the face now looks like the figure above to the right shows; the blue lines represent edges of G^* and the vertex in black at the center in which they intersect is a vertex of G^*, the novelty are the white vertices that arise as intersections of edges of G and G^*.



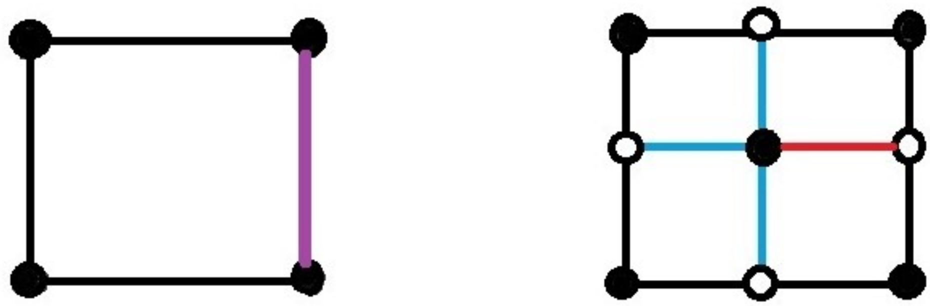


Suppose that a dimer cover was given on G^d and we want to know if the edge marked in purple of the square below on the left is part of T or not; if the white vertex associated to this edge is paired by a dimer (in red) with a vertex on G^d that it is also a vertex of G as the image in below to the left show, then the purple edge in G is an element of T.





Now suppose that other dimer cover was given and again we want to know if the same edge in purple of G is in T, in that case the white vertex associated to the purple edge is paired with a black vertex that is *not* a vertex of G (is a vertex of G^*), then for this case the purple edge is *not* an element of T.



The affirmation is that T is a spanning tree on G.

T is a sub graph of G by construction, in fact it passes trough all the vertices of G, to see this take any vertex $\bullet$ of G and notice that it is paired with a white vertex $\circ$ of G^d by a dimer, therefore the edge of G associated to this white vertex (the unique edge of G that contain $\bullet$—$\circ$ on G^d) is an element of T, as $\bullet$ is contained in this edge it follows that T passes trough $\bullet$.

Now we show that T does not have cycles on G.

On the contrary, suppose that T does have one cycle $\mathscr{C}$ on G, this cycle divides the plane into two regions, one compact $\mathcal{B}$ and other open and unbounded $\mathcal{U}$; by lemma 6.24 the number of vertices of G^d on those components are both odd numbers.

The first observation is that as the number of vertices on $\mathcal{B}$ is odd and the number of vertices of G^d on $\mathcal{C}$ (the boundary of $\mathcal{B}$) is even (n white vertices and n black vertices), then the number of vertices of G^d in the interior of $\mathcal{B}$ is odd. The second observation is that PM pair every vertex $\bullet$ on the interior of $\mathcal{B}$ in a unique way by a dimer as $\bullet$—$\circ$ with another vertex $\circ$ on $\mathcal{B}$ possibly on $\mathcal{C}$.

Combining those observations we infer that the number of vertices on $\mathcal{B}$ is even but we have proved that according to lemma 6.24 this number is odd. Contradiction.

Thus, $\mathcal{T}$ does not have cycles. $\square$

6.6 What does the dimer problem can teach us about sandpiles and self-organized criticality?

The big picture that we wished to show is summarized by the following points:

1) Counting recurrent sandpiles on a graph is equivalent to count spanning trees over the same graph.

2) The problem of counting spanning trees on a graph is equivalent to counting dimer covers on the double graph.

The conclusion is that for every recurrent sandpile state on a graph G there is exactly one perfect matching on the doubled graph of G. In physics language:

$$Z_{sand}[G] = Z_{dimer}[G^d],$$

where $Z_{sand}[G]$ is the partition function of recurrent sandpiles on the graph G and $Z_{dimer}[G^d]$ is the partition function for perfect matchings on G^d.

Just to be sure: $Z_{sand}[G]$ is the number of recurrent sandpile states on G and similarly $Z_{dimer}[G^d]$ is the number of dimer cover of G^d

Why this should be interesting?. Is not just a reformulation of the same problem?. After all physics and mathematics are kingdoms of analogies.

Take the following example: The differential equation for a charging RC electric circuit is formally the same as the equation describing a particle moving in a viscous media, this curiosity is just that, a curiosity, but it hides something important: A kind of universality of Newton´s second law on classical systems.

The reformulation of the sandpile dynamics as a the combinatorial problem of counting dimers on graphs is just a curiosity?. If it is, what is the potentially universal lesson that the equality is pointing out?.

What if this is not simply an analogy?, we believe that it is interesting to explore this question. The following chapter pretends to clarify why we believe that the reformulation of the computation of the sandpile model as a dimer problem is a non trivial statement in view of the non triviality of the dimer problem

7 The Vafa-Okounkov-Reshetikin missing corner.

"Wholeography is a theory of the whole bulk theory.
We would like to understand if there is something similar to a quantum equivalence principle that governs this theory. "
Juan Maldacena - Towards Wholeography - Strings 2017.

It is remarked that we do not pretend in any way to be explicative or rigorous in this section, our goal is very humble, we just want to explain with precision what the dimer problem means in the context of topological string theory and to give a very brief sketch of the main properties of this beautiful interpretation.

7.1 The missing corner in our understanding of quantum gravity.

Notation 7.1. *In the AdS/CFT correspondence context, the topological interior of the AdS spacetime times a five sphere* $\mathcal{S}^5$ *is called by physicist "the bulk", this is jargon adopted with the intention to distinguish this spacetime from the one in which the CFT is defined. Also notice that the topological boundary of AdS(5)* $\times\mathcal{S}^5$ *is not the spacetime of the CFT. In this brief motivation we adopt this jargon.*

One of the many revolutionary insights that we have learned from, for example, the BFSS model or the AdS/CFT correspondence, is the non-perturbative consistency and completeness of quantum gravity, or equivalently, string theory. The best definition we have of string theory is as the large N limit of a gauge theory and this is absolutely revolutionary. Nevertheless it is pointed out by many eminent physicist like Juan Maldacena [28] , Edward Witten [29] or Cumrun Vafa [30] that this is not satisfactory because the knowledge of the equivalence between two systems do not imply that we understand both sides of the correspondence, in this case we don´t know what the bulk is (beyond its tree level description).

In [28] Cumrun Vafa propose to call this gap *the missing corner* and to be more precise, the missing gap is to find the variables that describe the physics between the Planck and the string scales. The emphasis is the fact that this description can not be just a description of the quantum fluctuations of the metric or a sum over topologies and geometries because topologies and geometries can not be observables (or properties of states) in quantum gravity for many reasons like the majestically beautiful Papadodimas-Raju realization of black hole complementarity on the context of AdS/CFT.

This is the missing corner, a conjectured quantum theory of the bulk.

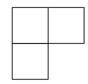
Andrei Okounkov, Nikolai Reshetikhin and Cumrun Vafa have demostrated in [27] that in topological strings is possible to describe the physics at the Planck scale and unexpectedly ...

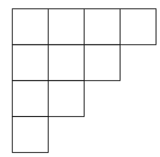
The missing corner of the topological sector is a dimer problem!.

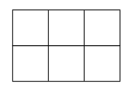
7.2 Melting crystals.

Definition 7.2. Let $\lambda = [\lambda_1, \lambda_2, ..., \lambda_k]$ with $(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0)$ be a partition of the natural number $|\lambda| = \lambda_1 + \lambda_2 + \ldots \lambda_k$, a **Young tableau** is a left-justified arrangement of boxes of length $\lambda_1, \lambda_2, ..., \lambda_k$.

Examples: 7.3. *The following are examples of partitions and their associated Young tableau.*

$\mu = [2, 1] \quad \rightarrow$ 

$\nu = [4, 3, 2, 1] \quad \rightarrow$ 

$\nu = [3, 3] \quad \rightarrow$ 

Definition 7.4. A **semistandard Young tableau** of shape λ is a filling with natural numbers of the Young tableau associated to λ in such a way that the entries weakly increase along the rows and strictly increase along columns, that is to say, the numbers on the rows increase or repeat when read from left to the right and the numbers in columns increase when they are read from up to down.

Examples: 7.5. *The following are examples of partitions and associated semistandard Young tableau.*

$$\mu = [3, 2, 1] \quad \rightarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 6 & 7 \\ \cline{1-2}\end{array}$$

$$\nu = [4, 2] \quad \rightarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

Definition 7.6. A **standard Young tableau** of shape λ is a filling with natural numbers of the Young tableau associated to λ that it is a bijective assignment of the set $\{1, 2, ..., |\lambda|\}$.

Examples: 7.7. *The following are examples of partitions and associated standard Young tableaus.*

$$\mu = [3, 2, 1] \quad \rightarrow \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

$$\nu = [4, 2] \quad \rightarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$$

The following is the fundamental object for the missing corner.

Definition 7.8. A plane partition Π is a two-dimensional array of nonnegative integers $\pi_{i,j}$ (with positive integer indices i and j) such that:

$$\phi_{i,j} \geq \phi_{i,j+1} \text{ and } \phi_{i,j} \geq \phi_{i+1,j}, \text{ for all } i \text{ and } j.$$

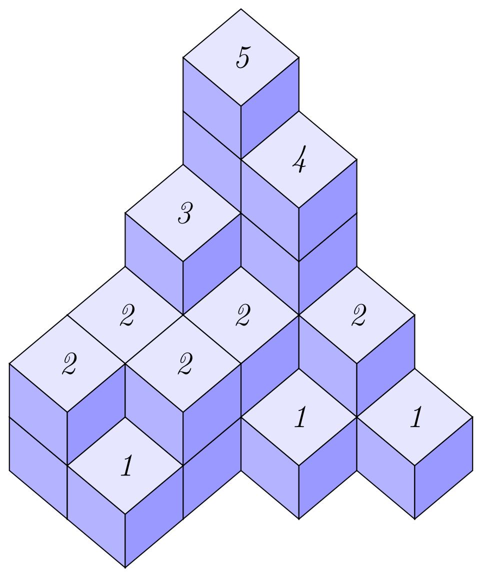
The size of the partition is defined as $|\Pi| = \sum_{i,j} \pi_{i,j}$.

Plane partitions are in some sense three dimensional generalization of Young tableaus. We clarify this with an example.

Example: 7.9. *Consider the following plane partition:*

$$\Pi = \begin{array}{|c|c|c|c|} \hline 5 & 4 & 2 & 1 \\ \hline 3 & 2 & 1 & \\ \cline{1-3} 2 & 2 & & \\ \cline{1-2} 2 & 1 & & \\ \cline{1-2} \end{array}$$

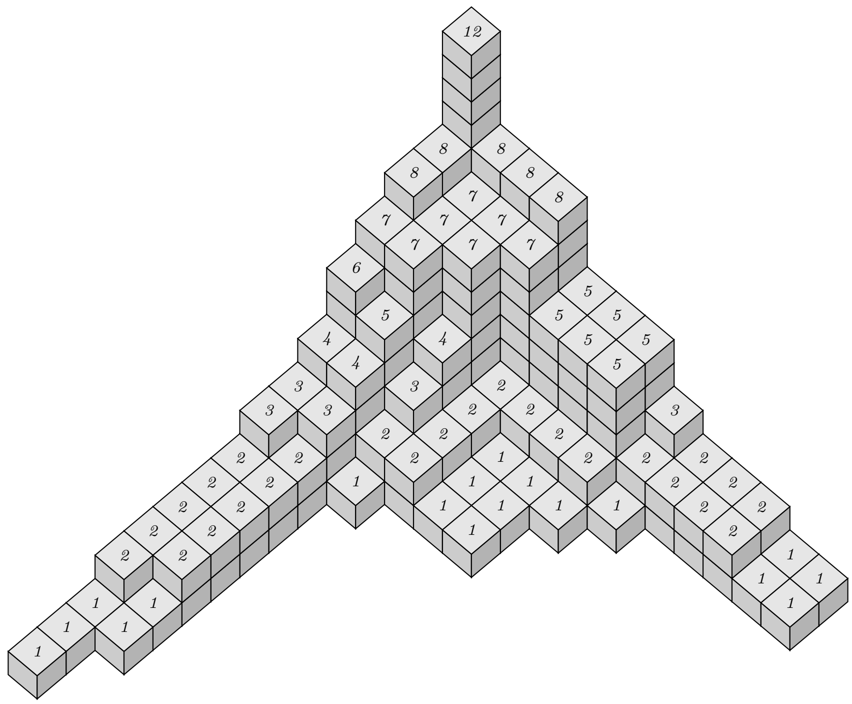
It is usual to represent Π *as a three dimensional array of boxes by imagining that over every box of the plane partition there is a pile of cubes such a pile has as many cubes as the number in the box of* Π *indicates.*



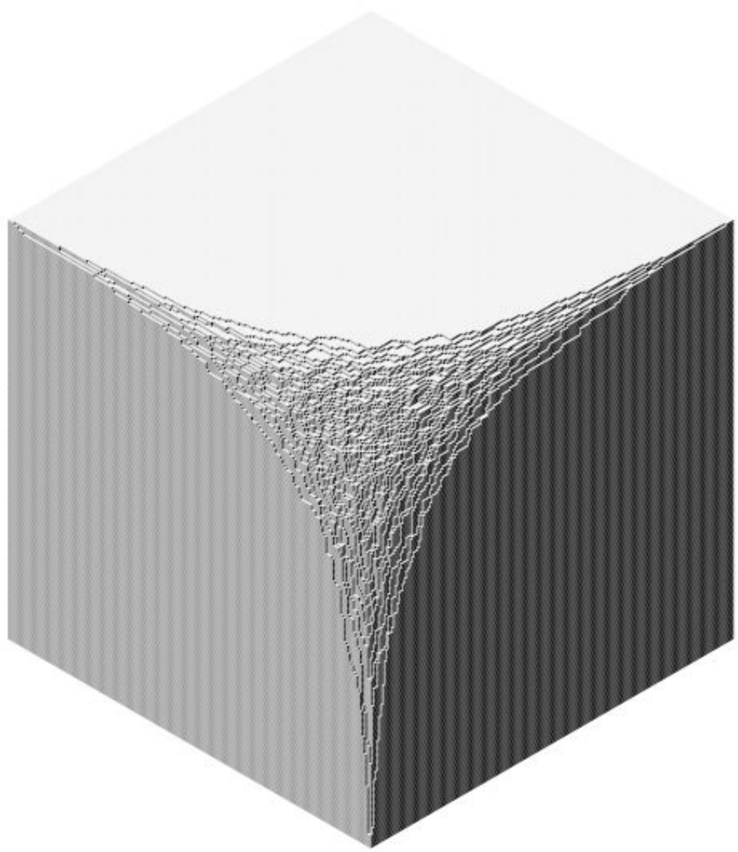
Notation 7.10. *We call* ***melting crystal*** *to the complement of some plane partition sited at the origin of the positive octant of* $\mathbb{R}^3_+$

Physicst call to the complement of plane partition a ***melting crystal*** *because if we inscribe a sphere into every cube of a given plane partition the resulting "sphere packing" correspond to a FCC crystalline structure and because it looks like a piece of ice melting.*

The following is a representation of how a plane partition can be sited at the positive octant.

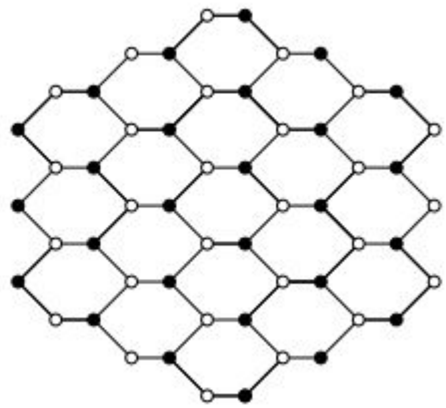


The complement of a large plane partition in the positive three dimensional octant looks like a melting ice cube like the following image taken from [31] shows:

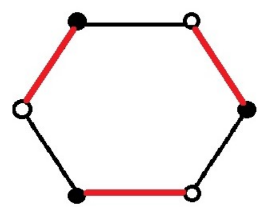


7.3 Dimer cover to plane partition (melting crystal) bijection.

To begin with we want to motivate the construction of a plane partition from a perfect matching on a bipartite graph, we do not develop the theory in detail, we will be focusing only on the honeycomb lattice and latter we explain how to extend the presented results to the square grid.



Observation 7.11. *It is possible to show that any dimer cover on the honeycomb lattice contain at least one face covered by dimers in the following way:*



The edges in red are dimers.

Definition 7.12. Let G be a bipartite planar graph that is locally the honeycomb lattice and PM a perfect matching on it, for PM there is at least one face on G that is covered by dimers as in the observation 7.11, call this face F_0 and define the **height function of PM** by:

$$h(F_0) = 0,$$

and for any faces A and B that share an edge e as:

$$h_{PM}(A) - h_{PM}(B) = \begin{cases} 1 & if\ e \in PM \\ 2 & if\ e \notin PM \end{cases}$$

Algorithm dimer cover to plane partition.

Input: A dimer cover PM of the honeycomb lattice.

Recipe:

1) Select a face that verifies observation 7.11 and call it F_0.

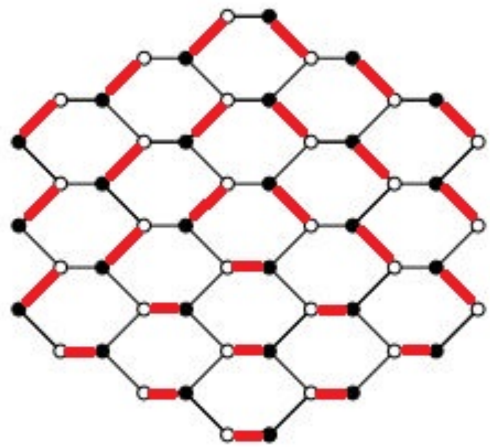
2) Construct h_{PM} and draw the value of $h_{PM}(F)$ at every face F of the honeycomb lattice.

3) Consider an arbitrary face F, suppose that is marked with the integer n; draw a straight line from the center of this face towards the center of an adjacent face $F_{adjacent}$ and passing trough the common edge if and only if $F_{adjacent}$ is marked with the integer $n-1$.

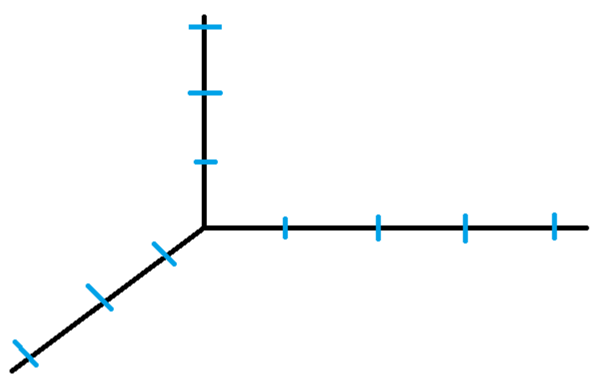
4) The set of drown lines gives the profile a plane partition.

Output: A plane partition Π.

Example: 7.13. *The following dimer cover is know as the* ***The empty room configuration***

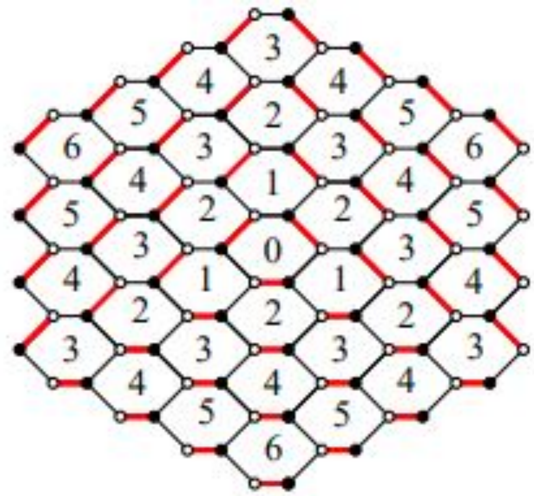


The reason is that the associated plane partition is the zero partition i.e. the positive octant with no cubes on it or equivalently: the unique plane partition Π *such that* $|\Pi| = 0$.

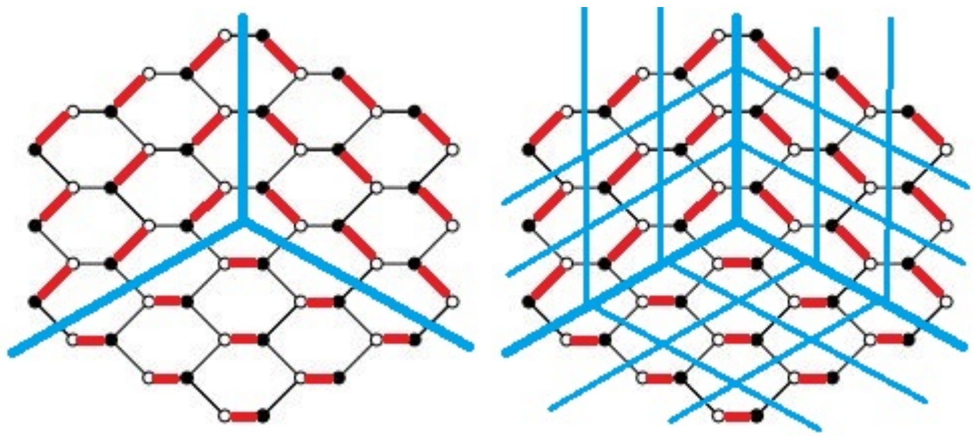


In the physics jargon this is the frozen crystal because there are no atoms removed or melted at all.

Let us construct the associated height function for the empty room configuration noticing that the face marked with a zero is F_0*:*



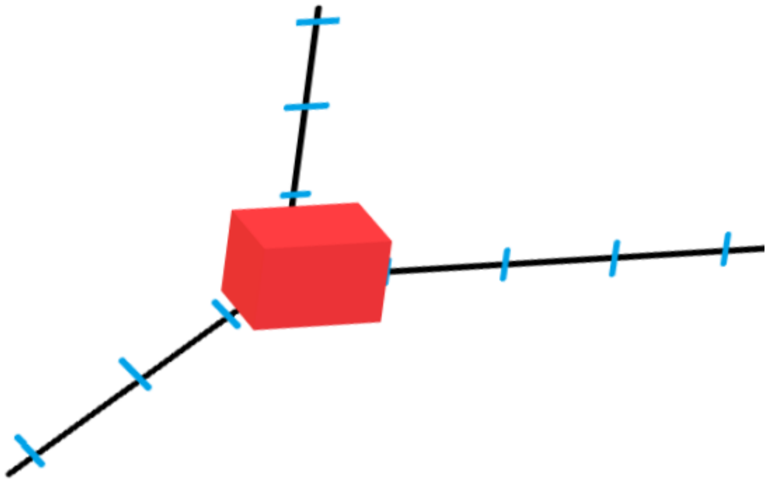
The following picture shows the result of applying the algorithm dimer to plane partition to the empty room configuration, the figure shows the coordinate axes of the positive octant in gross blue lines and the thin blue lines join points with integer coordinates:



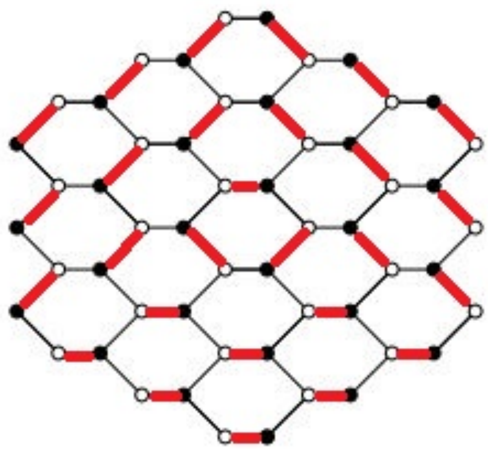
the crucial detail to be noticed is the fact that that every blue rhombus bounded by blue lines contain one and just one dimer in its interior; a simple consequence of this is that no blue line intersects a dimer.

Another way to obtain a the zero plane partition from the empty room dimer cover is to send blue lines starting from F_0 and crossing its edges to the center of the adjacent vertices only if the edge in question is not a dimer.

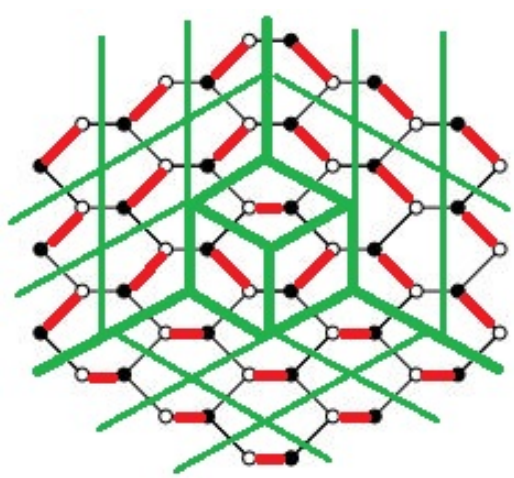
Example: 7.14. *Now we want to obtain the positive octant with one box on it.*



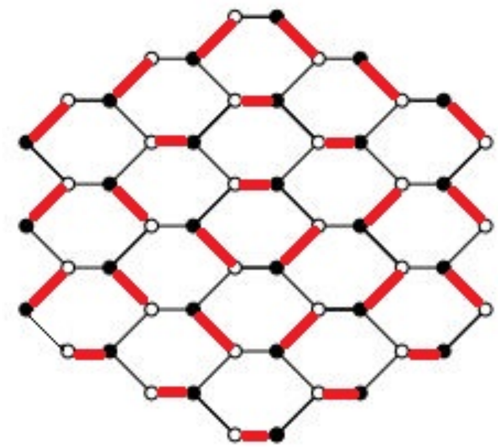
The following image is the ***corner with one box*** *perfect matching:*



In this example we obtain the corresponding plane partition without requiring the height function, just by drawing one rhombus around every dimer.

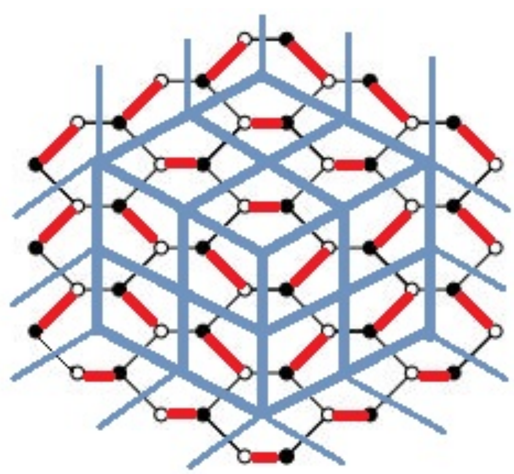


Example: 7.15. *Now we take a less obvious example, the following dimer cover:*



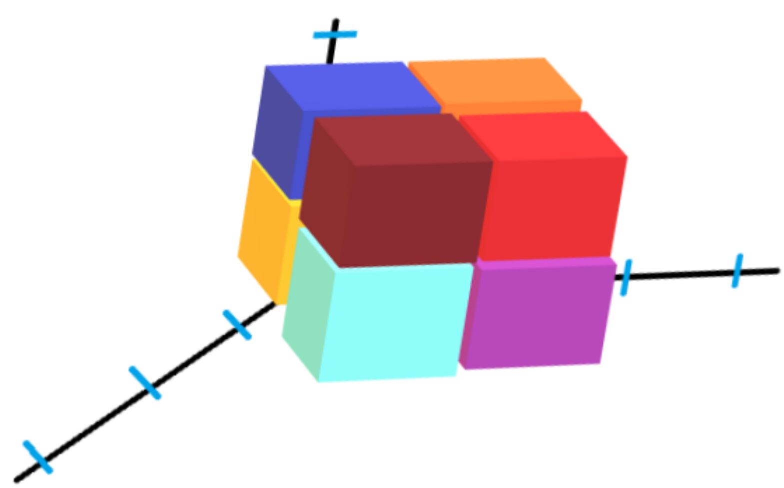
It is asked: what is the plane partition associated to this cover?

Applying our algorithm dimer cover to plane partition we obtain:

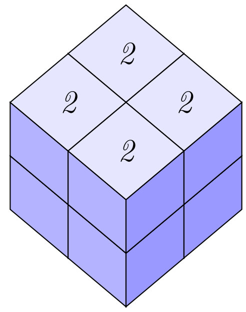


The gross lines in blue determine the profile of the plane partition that solves our problem.

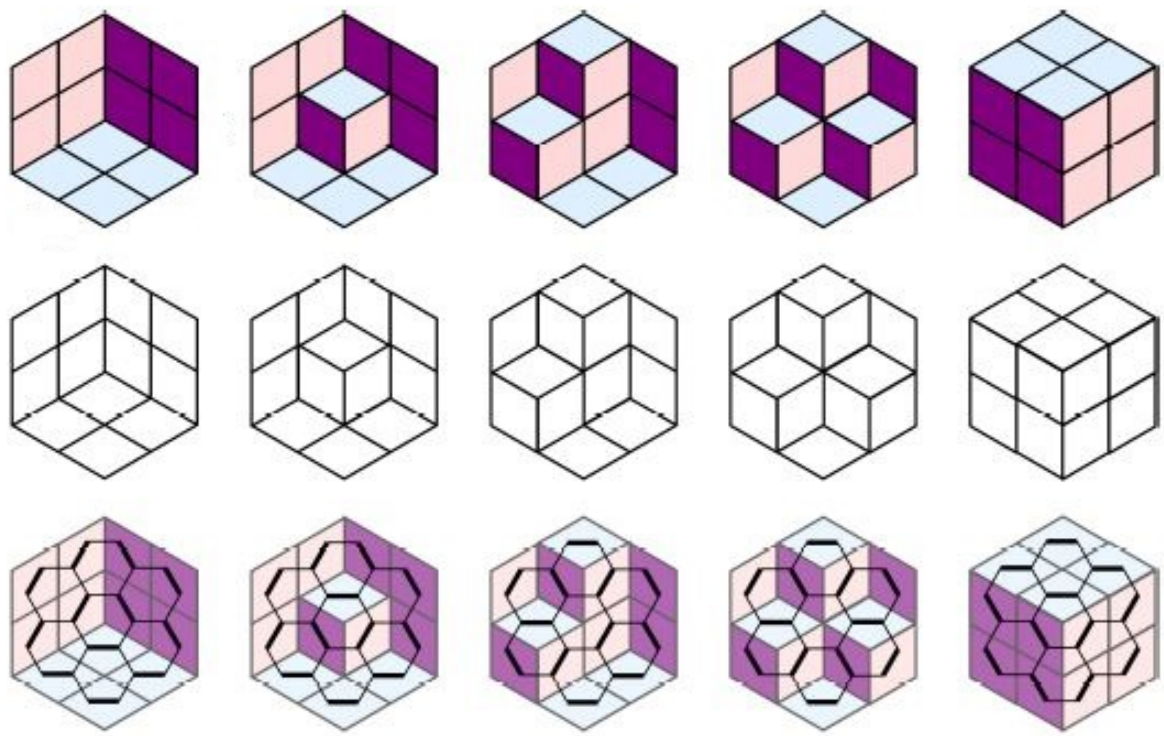
We also offer a picture of how this partition looks on the positive octant:



The answer written explicitly as a plane partition is:



Examples: 7.16. *The following figure shows in the first row five plane partitions, the second one contain five melting crystals and the final row contain five dimer covers. Every column shows a plane partition a melting crystal and a dimer cover (respectively from top to down) that are equivalent in the sense that from every one is possible to algorithmically obtain the other two. The paper [32] is an excellent reference to understand the equivalence.*



The general principle that gives the converse should be obvious at this point (although is non trivial).

Algorithm plane partition to dimer.

Input.: A plane partition Π.

1) Draw Π in isometric perspective on the plane.

2) For every vertex on picture draw an hexagon with center the mentioned vertex. This can be done in exactly one way if we demand that whenever the coordinate and Π profile lines in the draw of Π cross with the edges of the hexagons they do it at the mild point of the hexagon edges.

3) If an hexagon edge do not intersect the lines of the draw of Π then color it in red.

4) Denote by PM the set of red edges of the honeycomb lattice.

Output. A dimer cover PM on a honeycomb lattice.

7.3.1 McMahon solution to the plane partition enumeration problem.

In the paper [20] Percy McMahon proposes (and solves) the following interesting combinatorial problem:

Plane partition enumeration problem: Compute the partition function for all the plane partitions embedded on the positive octant of $\mathbb{R}^3$:

$$Z_{McMahon}[R^3_{\geq 0}] = \sum_{n=0}^{\infty} PL(n) q^n,$$

where it is assumed that every cube on every plane partition has unit volume and $PL(n)$ is the number of plane partitions volume of volume $n \in \mathbb{N}$.

Solution(McMahon):

$$Z_{McMahon}[R^3_{\geq 0}] = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

It was observed by as a curiosity Robert Dijkgraaf in private communication to Cumrun Vafa that this is in fact equal to the topological string A model on $\mathbb{C}^3$ (source: the introduction of [27]) in symbols:

If

$$Z_A[\mathbb{C}^3](g_s) = exp\{\sum_g g_s^{2g-2} \int_{\mathcal{M}(\Sigma_g)} c^3_{g-1}\}$$

Where $\mathcal{M}_g$ is the moduli space of Riemann surfaces with genus g, and c_{g-1} is the $(g-1)^{th}$ Chern class of the Hodge bundle $\mathcal{H} \to \mathcal{M}_g$.

Is this just a curiosity?.

8 Topological string interpretation

It was noticed by Vafa and Gopakumar in [32] that large N dualities in topological strings can be implemented only if the integral of the Kähler form over any homology cycle of a Calabi-Yau threefold is a integer multiple of the string coupling constant, for example; Topological strings on resolved conifold are equivalent to the large N limit of U(N) Chern-Simons on S^3 provided that the Kähler class of the blown up $\mathbb{P}^1$ gives:

$$\int_{\mathbb{P}^1} K = g_s N,$$

here g_s is the topological string coupling constant and K the Kähler form of the resolved conifold.

This is suspicious ...

It looks like consistency with holography demands that the volume of the target space is quantized in units of the coupling constant!.

Is it possible at all to derive a theory of Kähler gravity in a threefold?

Vafa, Okunkov, Iqbal, Reshetikin and Nekrasov answered positively those questions in a series of beautiful papers [28] [31], their solution also explain why the Dijkgraaf observation $Z_A[\mathbb{C}^3] = Z_{McMahon}[\mathbb{R}^3_{\geq 0}]$ was not a curiosity. We briefly explain their supremely beautiful reasoning.

8.1 The missing corner for $\mathbb{C}^3$

$\mathbb{C}^3$ with complex coordinates (z_1, z_2, z_3) can be described as a special lagrangian $T^2 \times \mathbb{R}$ fibration (T^2 is a 2-torus).

We start by writing:

$$z_i = |z_i| e^{i\theta_i},$$

for $i = 1, 2, 3$. With Kähler form k given by:

$$k = dz_i \wedge d\bar{z}_i = d|z_i|^2 \wedge d\theta_i,$$

where the Einstein summation convention is used to facilitate the notation.

The geometry of $\mathbb{C}^3$ can be encoded on a graph Γ embedded on $\mathbb{R}^3_{\geq 0}$ given by the positive part of the coordinate edges of $\mathbb{R}^3$, the edges of Γ are tagged by elements of $H^1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, such labels encode where the fibration degenerates, it is to say, if an edge e of Γ is labelled by (p, q) then the cycle $(p, q) \in H^1(T^2)$ degenerates on the fiber $T^2 \times \mathbb{R}$ over every point on e. We precise that:

$$e_1 + e_2 + e_3 = 0,$$

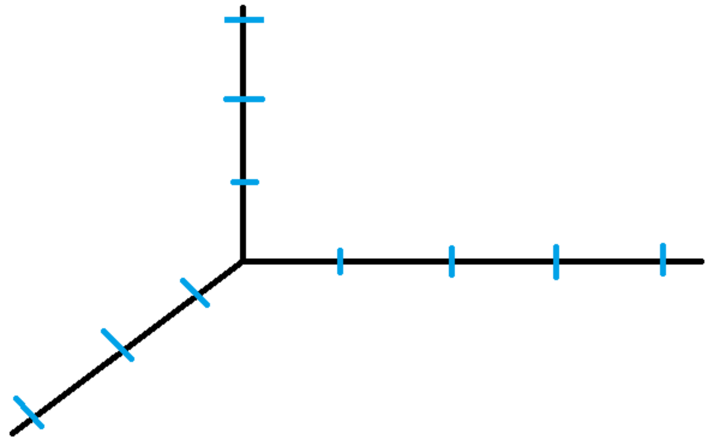
where all edges are take to be incoming at the vertex of Γ.

To guarantee smoothness we require:

$$|e_i \wedge e_j| = 1.$$

All the above is done to naturally relate $\mathbb{R}^3_{\geq 0}$ (the geometry where the plane partitions are defined) and $\mathbb{C}^3$ (the geometry where topological strings propagate).

The important reasoning is as follows: as was noticed, the integral of the Kähler form over any cycle of homology is quantized in units of the string coupling, so to make this manifest we label every edge of the graph Γ in integer multiples of g_s as follows:



Two important facts arise:

The first one is that as the description of the topological string is valid at $g_s \sim 0$, the appearance of discreteness on the target space require $g_s > 0$, is not only that the discrete description of the target require strong coupling, the amazing fact is that this description dominates all the parameter space of the A-topological string, in fact the string scale is a dynamically generated scale in a degenerate limit g_s of this description, later we will make some comments on how this happen.

$\mathbb{R}^3_{\geq 0}$ is the image of a moment map for $\mathbb{C}^3$, therefore $\mathbb{C}^3$ is know "discrete" in the sense that is locally divided into compact sets (the inverse image under the moment map of a cube on $\mathbb{R}^3_{\geq 0}$) of integer quantized volume (in units of g_s^6).

The authors of [31] very naturally discover that the action for a compact subset $X \subset \mathbb{C}^3$ is simply its volume:

$$I[X] := Vol(X) = \frac{1}{g_s^2} \int_X k \wedge k \wedge k.$$

Observation 8.1. *In this very elementary situation that we are describing it is absolutely important to consider that the volume of X is very large, this comes from the fact that gravitational instantons are suppressed in inverse powers of $Vol(X)$.*

$I[X]$ define a quantum theory for the quantum Kähler gravitational foam, to be mathematically precise: fixed a Kähler form k_0 on X what $I[X]$ define is a moduli space of solutions $F \in H^{(1,1)}[X, \mathbb{Z}]$ for the Euler-Lagrange equations associated to $I[X]$ such that:

$$\int_X F \wedge F \wedge F, \int_X F \wedge F \wedge k_0 < \infty$$

that also obey the condition:

$$\int_\alpha F = 0,$$

for every $\alpha \in H_2[X, \mathbb{Z}]$.

It is proved in [31] that if $X = \mathbb{C}^3$ those solutions are in one to one correspondence with the set of all the plane partitions on $\mathbb{R}^3_{\geq 0}$ and beautifully the following: statement:

$$Z_{A-Top}[\mathbb{C}^3] = Z_{McMahon}[\mathbb{R}^3_{\geq 0}].$$

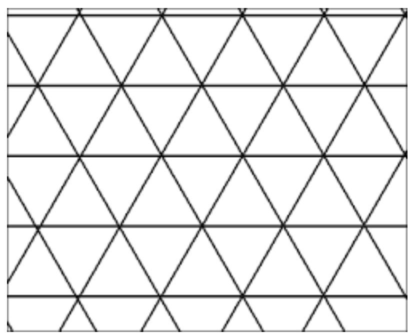
We also have explained why

$$Z_{Dimer}[Honeycomb\ lattice] = Z_{McMahon}[\mathbb{R}^3_{\geq 0}],$$

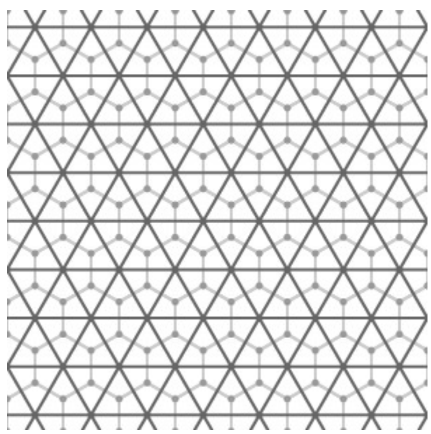
It is recalled that $Z_{Dimer}[Honeycomb\ lattice]$ is defined as:

$$Z_{Dimer}[Honeycomb\ lattice] := \sum_m d(m) q^m,$$

where $d(m)$ is the number of dimer covers on a honeycomb graph of $m \times m$ hexagon faces. Now consider as G the triangular lattice:

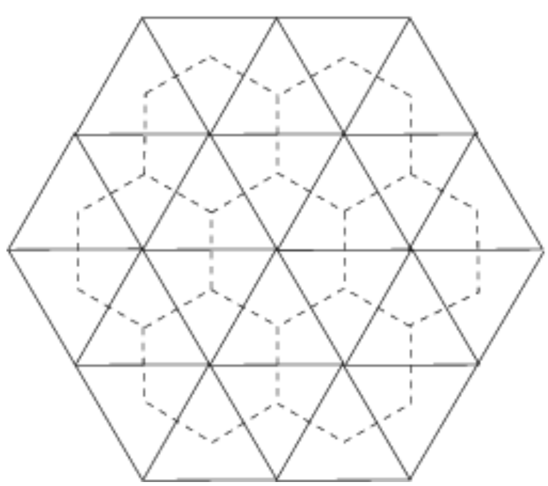


It is a know fact that his doubled graph G^d is the honeycomb lattice:



The later image was taken from [41].

We offer a image of the explicit construction of the honeycomb from the triangular lattice:



8.2 The scaling limit of the sandpile is a topological string theory!

The result that we offer in this thesis is the following sequence of equivalence of partition functions in the case of the honeycomb lattice:
Vafa, Iqbal and Reshetikin have proved that:

$$Z_{A-Top}[\mathbb{C}^3] = Z_{McMahon}[\mathbb{R}^3_{\geq 0}],$$

It is a classical result that

$$Z_{McMahon}[\mathbb{R}^3_{\geq 0}] = Z_{Dimer}[Honeycomb\ lattice]$$

We have dedicated chapter six to use Temperley bijection to argue why

$$Z_{Trees}[Triangular\ lattice] = Z_{Dimers}[Honeycomb\ lattice]$$

where:

$$Z_{Dimer}[Honeycomb\ lattice] := \sum_m d(m) q^m$$

$$Z_{Dimer}[Triangular\ lattice] := \sum_m t(m) q^m$$

and$d(m)$ is the number of dimer covers on a honeycomb graph of $m \times m$ hexagon faces and $t(m)$ is the number of spanning trees on the unique triangular graph whose double graph is the honeycomb graph of $m \times m$ hexagon faces. Temperley bijection guarantee that $d(m) = t(m)$.

It was in fact the objective of chapters three, four and five to prove that if

$$Z_{Dimer}[Triangular\ lattice] = \sum_m t(m) q^m,$$

$$Z_{Sandpile}[Triangular\ lattice] := \sum_m s(m) q^m,$$

where $s(m)$ is the number of recurrent sandpile states on the unique triangular graph whose double graph is the honeycomb graph of $m \times m$ hexagon faces, then:

$$Z_{Dimer}[Triangular\ lattice] = Z_{Sandpile}[Triangular\ \ lattice]$$

where $s(m)$ is the number of recurrent sandpile states on the unique triangular graph whose double graph is the honeycomb graph of $m \times m$ hexagon faces.

Our conclusion is that:

$$Z_{sandpile}[G] = Z_{McMahon}[\mathbb{R}^3_{\geq 0}] = Z_{A-Top}[\mathbb{C}^3]$$

The sandpile on the triangular lattice is equivalent to the crystal melting problem on $\mathbb{C}^3$!

The scaling limit of the sandpile on the triangular lattice is a topological string $\mathbb{C}^3$!.

8.2.1 Our conjecture for toroidal graphs.

Let G be a non finite graph and G^d his doubled graph, then:

$$Z_{Sandpile}[G] = Z_{Dimer}[G^d] = Z_{A-top}[X],$$

where $Z_{A-top}[X]$ is the A topological string partition function for the toric Calabi-Yau threefold X whose toric base is defined by the dimer problem on G^d.

In the case of toroidal graphs our conjecture can be formulated as follows.

Let G be a tilling for the 2-torus and G^d its doubled graph, then:

$$Z_{Sandpile}[G] = Z_{Dimer}[G^d] = Z_{N-D3}[X],$$

where $Z_{N-D3}[X]$ is the partition function for the quiver gauge theory of N D3 branes proving a conical singularity on the singular toric Calabi-Yau threefold X whose toric base is defined by the dimer problem on G^d.

The last conjecture is interesting for us and we now have purely field theoretical arguments to support it [40]. In this respect the author is very grateful with professor Hugo Compean for allowing him to present his results at his seminar and for useful suggestions, also many thanks to Eduardo López and José de la Cruz for the immense care and kindness with which they studied the arguments.

8.3 The dimer problem on topological strings.

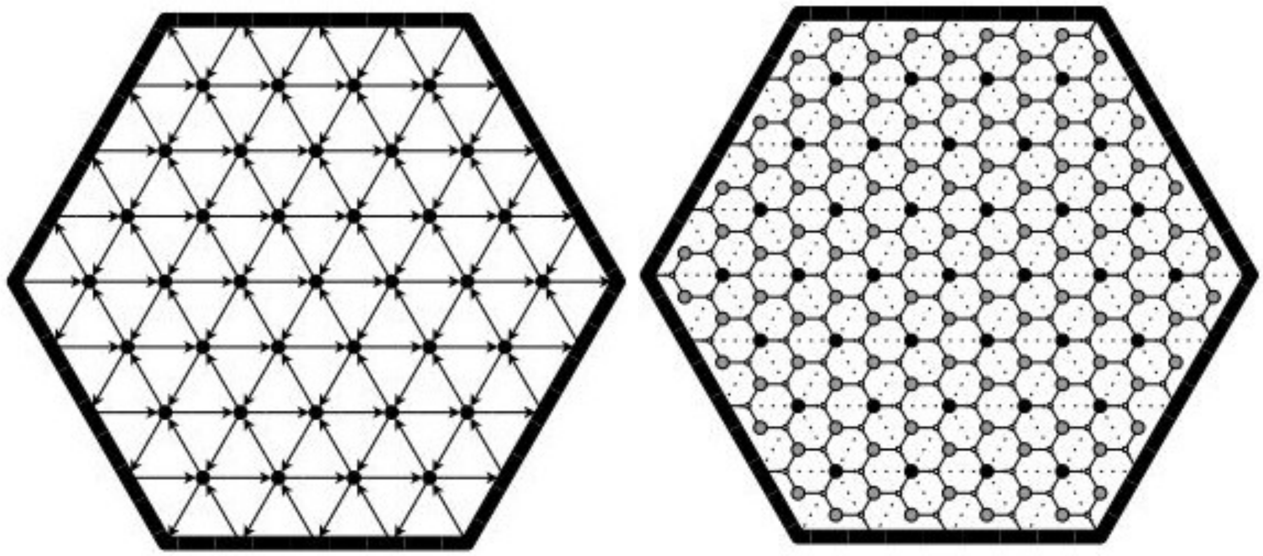
It is conjectured in [27] that the partition function for the dimer problem on an arbitrary planar biparite graph coincide with the topological vertex on a toric variety fibered over a toric base defined by the mentioned graph.

The above conjecture is proved in the references [35] and [36], also the dimer problem is formulated in a mathematically precise language and a clear physical picture of the meaning of the bipartite graph is offered.

8.3.1 Our result for finite graphs.

During the first six sections of this works we have worked with finite graphs. What is our conclusion for this class of graphs?

As an example consider a finite triangular graph G and its finite hexagonal doble graph G^d (images taken from [23]):



We have proved that:

$$Z_{Sandpile}[G] = Z_{Dimer}[G^d],$$

what would be the meaning of $Z_{Dimer}[G^d]$ in topological strings?.

8.4 A hint for the physical missing corner?

Notice that when it is taken the near horizon limit of a stack of N D_3 branes the metric adopts the form:

$$ds^2 = H^{\frac{-1}{2}}(-dt^2 + d\vec{x}^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2,)$$

$$H = 1 + \frac{L^4}{r^4} \quad , \quad L^4 = 4\pi g_s l_s^4 N,$$

where $d\Omega_5$ is the volume element for the transverse directions to the D_3 branes, $(t, \vec{x}, r)$ the Poincaré coordinates of AdS(5), L is the Poincaré length and l_s is the string scale.

It is noticed that the same history that runs for the topological string works here!. In the topological setting the Kähler form is the analogue of the Ramond-Ramond three-form C_{abc} of the full theory, the lagrangian A-branes are the sources for K_{ab} in the topological version and the D_3 are the sources of the three form in the full version. The absolutely exciting observation is that the volume of (a spacelike sice of) the AdS(5) space is quantized in units of $4\pi g_s l_s^4$!

It is clear that to make the locality of the bulk manifest we need to understand the strong coupling behaviour of N=4 d=4 SU(N) SYM. What is not obvious at all (and this observation suggest) is that perhaps there is a U(1) theory that describes the bulk physics below the string scale, we close this observation with a mysterious cite in this respect:

> "The thermofield double looks like an energy superconductor.
> If we take this observation seriously, it is worthy to ask:
> Where is the BCS theory of spacetime?"
> Juan Maldacena - Towards Wholeography - Strings 2017.

A ese respecto mi mayor aprendizaje es el tesoro de su frase:

9 Sandpiles, supersymmetric field theories, superconductors and black holes.

9.1 Supersymmetric field theories and sandpiles

Of course the big question concerning sandpiles is the meaning of the beautiful patterns that emerge in the scaling limit associated to recurrent states.

We make some very intriguing observations from the physical point view about those draws.

We start with observations concerning the Lupercio et. al. paper [9] and Kalinin et.al. one [17]

Definition 9.1. A convex closed subset $\Omega \subset \mathbb{R}^2$ is said to be *not admissible* if at least one of the following conditions is satisfied:

- Ω has empty topological interior $\Omega^\circ = \emptyset$ or equivalently: Ω is a piece of a line,
- Ω is $\mathbb{R}^2$,
- Ω is a half-plane with the boundary of irrational slope,
- Ω is a band between two parallel lines of irrational slope.

If neither of the above takes place we say that Ω is **admissible**

We want to consider the sandpile model on a 4-regular grid inside a convex admissible domain $\Omega \subset \mathbb{Z}^2$.

Notation 9.2. *We identify by an abuse of notation the convex domain Ω with the 4-regular grid inside of it.*

Definition 9.3. Let ϕ be a sandpile state on Ω and ϕ° his relaxation, we define the **odometer** function $H : \mathbb{N}^V \to \mathbb{N}$ (where V is the set of vertices of Ω) as follows:

$$H(i,j) := \text{number of times that there was a toppling at the vertex } (i,j) \text{ in the process } \phi \to \phi^\circ,$$

for any $(i,j) \in V$.

The importance of the odometer function can be remarked by the following observation that follows immediately from the sandpile toppling rules:

Observation 9.4. *Let ϕ be a sandpile state on Ω and ϕ° his relaxation, then for every $(i,j) \in V$:*

$$\phi^\circ(i,j) = \phi(i,j) + \Delta H(i,j).$$

Definition 9.5. Given a fixed vertex $p = (i_0, j_0)$, we define the **toppling operator** $T_p : \mathbb{N}^V \to \mathbb{N}^V$ by the operator that topples the site p if $\phi(p) \geq 4$.

Definition 9.6. Given a fixed vertex $p = (i_0, j_0)$, we define the **wave operator** $W_p : \mathbb{N}^V \to \mathbb{N}^V$ by:

$$W_p(\phi) := (T_p(\phi + \delta_p) - \delta_p)^\circ$$

Definition 9.7. An Ω**-tropical series** is a function $f : \Omega \to \mathbb{R}_{\geq}$, such that

$$f|_{\partial\Omega} = 0,$$

and

$$f(x,y) = \inf_{(i,j)\in\mathcal{A}} (ix + jy + a_{ij}), \ a_{ij} \in \mathbb{R},$$

with $\mathcal{A} \subset \mathbb{Z}^2$.

Definition 9.8. A **Ω-tropical analytic curve** $C(f)$ on Ω° is the corner locus of an Ω-tropical series f i.e. the set of points of Ω° where the polynomial is not locally linear on

Theorem 9.9. ***(Kalinin-Shkolnikov)*** *[17] Consider the avalanche $\phi \to \phi^\circ$, if $\phi^\circ = \phi + \Delta F$ is stable and $F \geq 0$ then*

$$F(i,j) \geq H(i,j)$$

for all $(i,j) \in \mathbb{Z}^2$.

The critical observation of [17] is that the odometer function is a picewise linear function (in the scaling limit).

Theorem 9.10. *Let P be a finite subset of points on Ω, consider the family $\mathcal{F}_p$ of functions with domain Ω with following properties:*

- *Piecewise linear with integral slopes.*
- *Non-negative over Ω and zero on $\partial\Omega$*
- *Concave.*
- *Non smooth at every point of P.*

Then $H = F_P$ where F_P is the pointwise minimum of all the elements of $\mathcal{F}_P$.

Combining the facts that the odometer function H is piecewise linear and

$$\phi^\circ(i,j) = \phi(i,j) + \Delta H(i,j)$$

we conclude that ϕ° is equal to 3 everywhere except at the loci where $\Delta H \neq 0$.

Definition 9.11. Let Ω be an admissible polygon and P a finite non-empty subset of Ω°. Consider the set $\Gamma_N := N\Omega \cap \mathbb{Z}^2$. Given a sandpile state $\phi_N = (3 + \sum_{v \in P} \delta_v)^\circ$ on Γ_N we define **the deviation set of ϕ_N** by

$$C_N := \frac{1}{N}\{w \in \Gamma_N | \phi_N(w) < 3\}.$$

Theorem 9.12. ***(Kalinin-Shkolnikov)*** *[17] The sequence $\{C_N\}_{N \in \mathbb{N}}$ Hausdorff converges to a planar graph $\tilde{C}_{\Omega,P}$ that pass through the points of P, furthermore, each edge of $\tilde{C}_{\Omega,P}$ is a straight segment with rational slope.*

Definition 9.13. A finite planar weighted graph $C \subset \Omega$ is called an Ω-tropical curve if

- The intersection of C with $\partial\Omega$ is the set of all vertices of Ω,
- Each edge e of C has a rational slope with integer weight p_e,
- If $v \in \Omega^\circ$ is a vertex in C, then

$$\sum_{e \in E_v} p_e l_e = 0,$$

where E_v is the set of all edges of $C_{\Omega,P}$ incident to v and l_e is the primitive vector of $e \in E_v$ oriented out of v.

- there exist a unique labelling $d_s \in \mathbb{Z}_{>0}$ for each edge e of Ω such that for each side s of Ω

$$\sum_{e \in E_v} p_e l_e = d_{s_1} l_{s_1} + d_{s_2} l_{s_2},$$

where s_1 and s_2 are the two sides of Ω incident to v and l_e is the primitive vector of $s_i \in E_v$ oriented out of v.

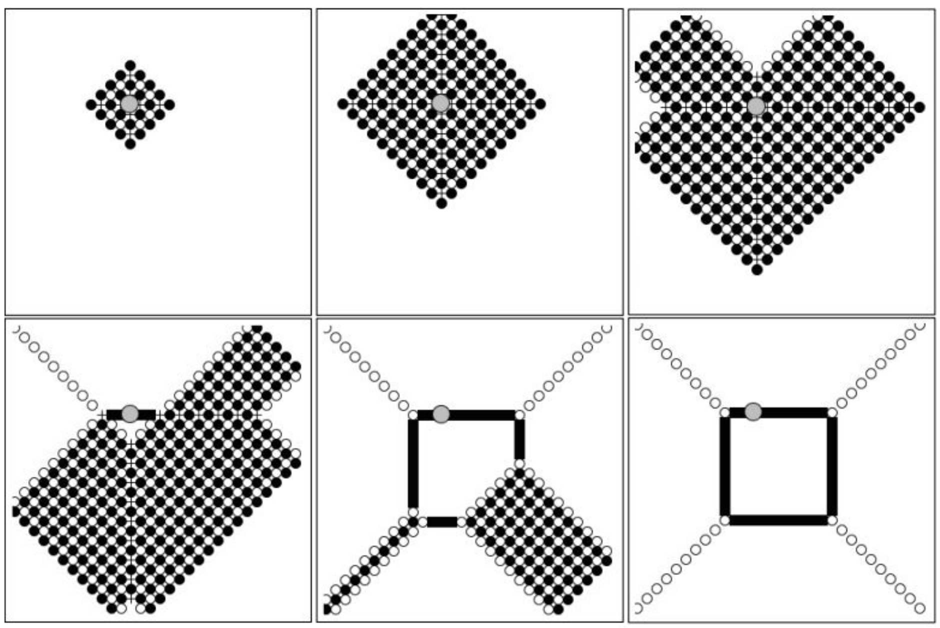
Theorem 9.14. ***Kalinin and Shkolnikov*** *[17] The graph $C_{\Omega,P} = \tilde{C}_{\Omega,P} \cap \Omega^{\circ}$ is a $\Omega-$tropical curve. Furthermore, $C_{\Omega,P}$ has the minimal tropical symplectic area among all $\Omega-$tropical curves passing through the points of P.*

Theorem 9.15. ***Kalinin and Shkolnikov*** *[17] Consider a 4-regular graph inside an admissible convex subset of the plane Ω and a finite subset of vertices of the graph P. Let H be the odometer function of the avalanche $3 + \sum_{v \in P} \delta_v \to (3 + \sum_{v \in P} \delta_v)^{\circ}$*

Then the corner locus of the odometer function H is $C_{\Omega,P}$

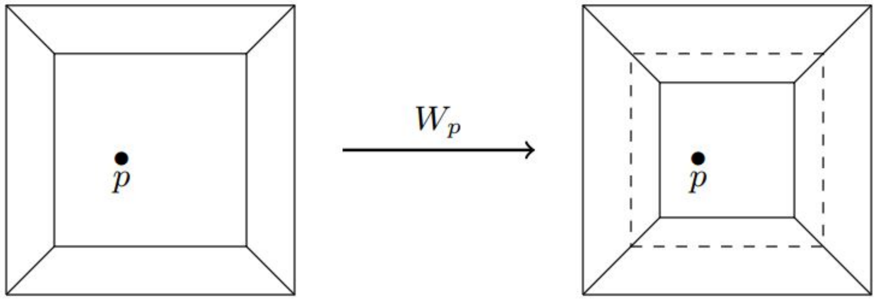
Theorems 9.14 and 9.15 are enough to guarantee that corner locus of a odometer function associated to a perturbation of the maximally stable state verify the definition of a $(p, q) - 5$ web.

The following picture (extracted from [17]) shows the relaxation process of the maximally stable state on a 4-regular square grid inside a square when a single grain of sand is added at some point on the interior:



We see that the result looks like a generic (p, q) web associated to the Coulomb branch of a pure (global) $SU(2)$ SQCD theory with eight supercharges in five dimensions. Is this just a coincidence?

The (p, q)-web is the deviation locus of a tropical series on the square, say f, it is interesting to consider a point p on the face of the web and to apply the wave operator W_p to the corner locus of f (the next is an image taken from [9]):



We observe that the result is a local deformation of the web!.

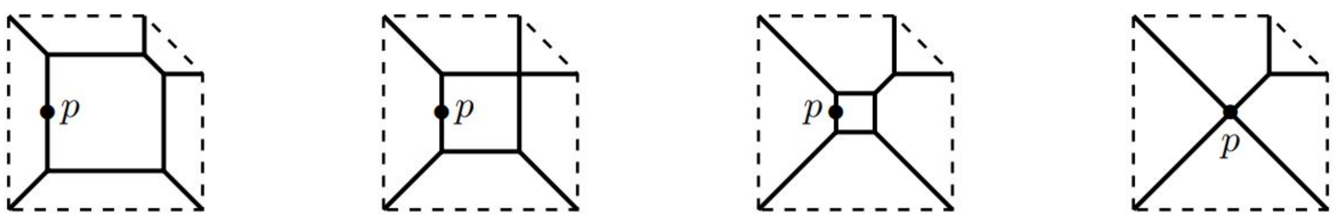
What is interesting is that in principle there is no reason of why the wave operator does not produce a global deformation. But, on contrary, from the IIB superstring point of view, wave operators look like a finite mass deformation of the web (a finite modification in the VeVs of the scalars in the vector multiplets).

Conjecture: The addition of a single grain of sand to the maximally stable state on the square produce a generic state on the Coulomb branch of the pure $SU(2)$ gauge theory, subsequent addition of grains produce via Hanany-Witten transitions webs that are universality classes of Higgs phases of the aforementioned $SU(2)$ gauge theory with possible extra local gauge symmetries but in such a way that the global symmetry group is preserved.

We precise our idea; consider two points on the Higgs branch of the $SU(2)$ theory as equivalent if it is possible to deform the web associated to one point into the web associated to the other point by a sequence of local deformations, in the language of sandpiles this reads as: two tropical series are equivalent if one can be deformed to another by a sequence of applications of wave operators.

After adding a grain of sand to Ξ we obtain a tropical series f on the square gird, subsequent additions of grains produce "higgsings" of $\mathcal{C}(f)$ it is to say: webs on the Higgs branch of the gauge theory of associated to the web.

The following picture (extracted again from [17]) shows four extremely intriguing figures, any of them is the result of adding a single grain of sand at different points of the maximally stable state on a pentagon, the result is revealing:



Notice that in all cases the tropical curves have the same number of external legs that vertices of the polygon. In the first picture (from left to right) appears a $SU(2)$ gauge theory with one flavor charge ($N_f = 1$); in the second picture there is again a web associated to a $SU(2)$ with lighter monopoles that is a mesonic root for the Higgs branch; the third one show again lighter monopoles that the second picture and it is identified as the root for the barionic branch and finally the rightmost picture show a web such that the monopole ($D3-$brane wrapping the face of the web) tension is zero because such a tension is equal to the area of the face ... this web is a point of gauge symmetry enhancement!.

We learn from this that apparently the position of the first grain added to the maximally stable state determine where on the Coulomb branch the gauge theory associated to the web sits on. Subsequent addition of grains only develop sub-webs via Hanany-Witten transitions that do not modify the global gauge groups, but could develop new faces on the web, then the number of colours changes.

What is intriguing about the later observation is that there is no a priori reason of why the sandpile does neither generate theories with different global gauge groups nor global deformations of the webs on it, are sandpile Markov chains classifying in some sense five dimensional theories in some sense?.

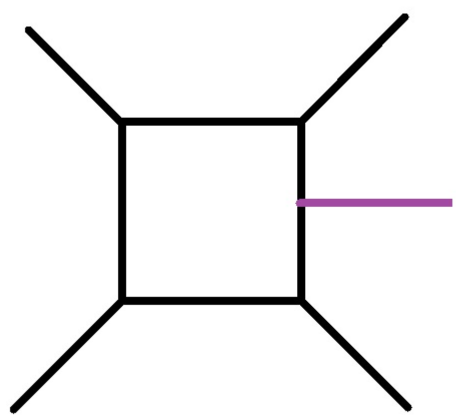
It looks suspicious.

Now suppose that someone have perturbed the maximally stable state on the polygon by the addition of a single grain of sand (in field theoretic language: a SQCD is sited at some generic point on his Coulomb branch), we have talked about that the addition of a few grains on some examples forces the web to enter into to the Higgs branch of the SQCD, this is very nice but in our examples the way the SQCD *higgses* is always by preserving the global symmetries of the gauge theory, in physics language: the deformations always give a VEV to a meson but it is well know that this is not the only way to enter to the Higgs branch, in fact it is also possible to enter the Higgs branch by giving a VEV to a baryon in general field theories, surprisingly this is in fact not possible in five dimensions ([45]), we analyse the case of baryons in the following subsection.

9.1.1 Electric-magnetic duality in supersymmetric systems and self-similarity in sandpiles

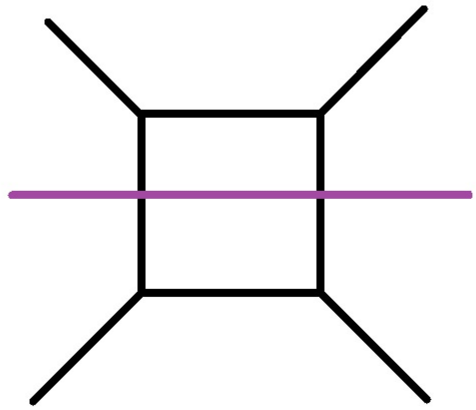
We believe that the hypothesis that the sandpile "explore" the Higgs branch of generic webs is very relevant for the fractal structure of the draws that recurrent states exhibit.

Let us analyse a pure $SU(2)$ web with one flavour of quark added:

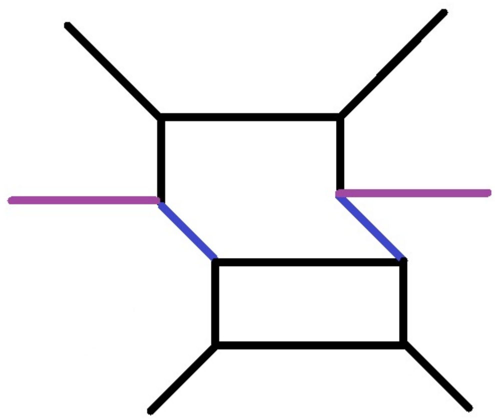


The semi-infinite line in purple geometrize the added quark flavor. As is evident this figure is *not* left-right symmetric, this is good because (from the field-theoretical perspective) the sandpile is not adding flavors in this way, is our objective to suppose that what is taking place is that the sandpile "travels" on the Higgs branch of generic webs to derive some very basic observations.

We want to ask the question: How do we give a VEV to a meson in a $SU(2)$ theory with $N_f = 1$? The following web shows the root of the mesonic branch:

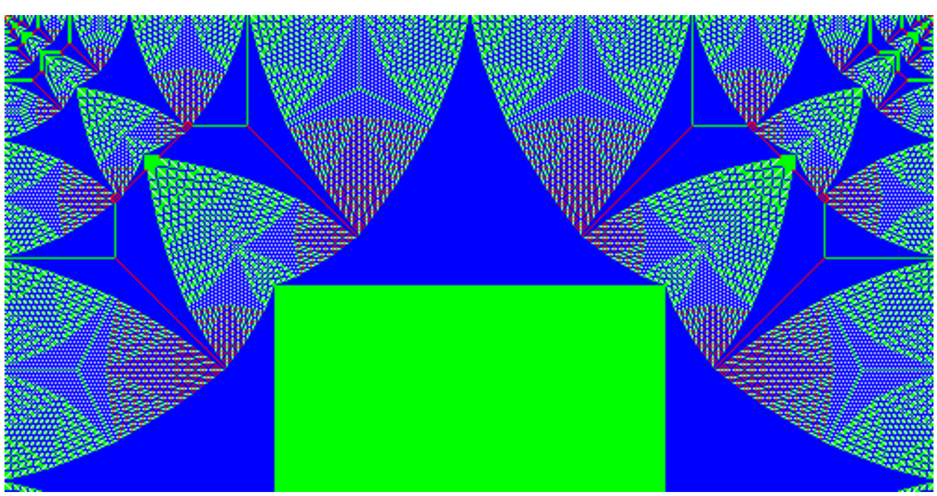


The next figure is a generic configuration on the Higgs branch:

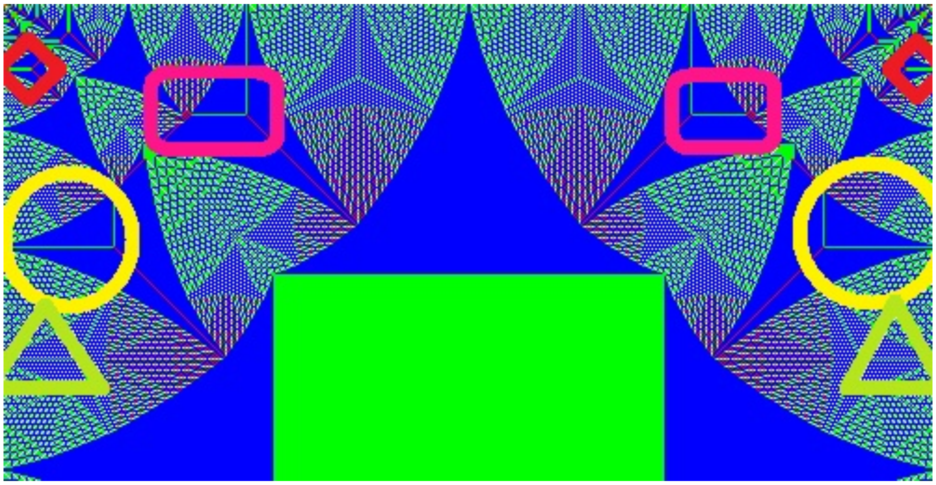


The observation is that the signal of VEVs for mesons in the web are external horizontal legs one at the left and one at the right at the same height, in our picture the ones in purple. In general webs the signal of VEVs for gauge invariant mesonic operators are external legs that if prolonged intersect the edges of the admissible polygon that supports the sandpile at 90 degrees.

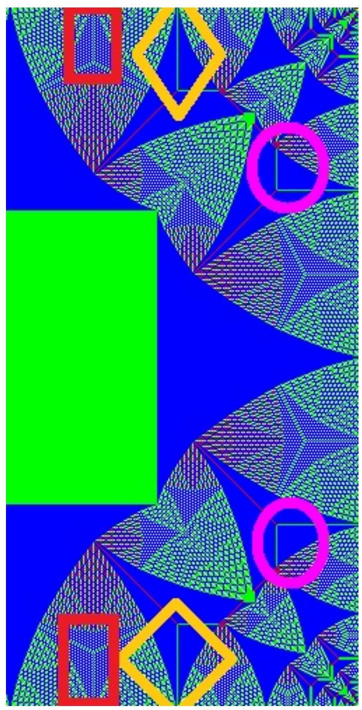
In the particular case of the square we expect that a generic web on Higgs branch posses many horizontal (vertical) pairs of symmetrically positioned legs. Let us look at the upper part of the identity sandpile on a 500×500 square grid.



In the following picture we use figures to highlight what appear to be pairs of quarks that develop mesons with finite VEVs.

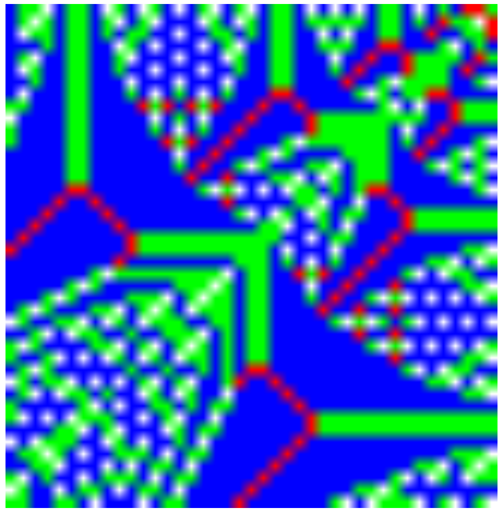


Now we show candidates to S-dual mesons (confined 't Hooft–Polyakov monopoles)

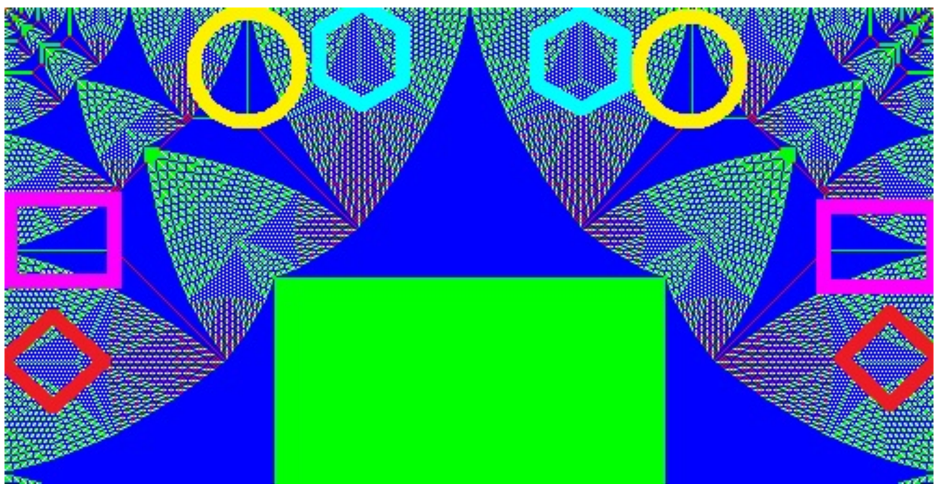


Notice that is not only that the segments of lines are collinear, they have also the same length (mass)!

Let us make a zoom at the upper right the most of the identity to show the intriguing insistence of the sandpile on develop (green) legs of the web that intersect at 90 degrees the edges of the boundary square.



It appear to be a rule that the legs of the web that intersect the boundary (not at the vertices of the) square always do it at 90 degrees as expected from our physical hypothesis.



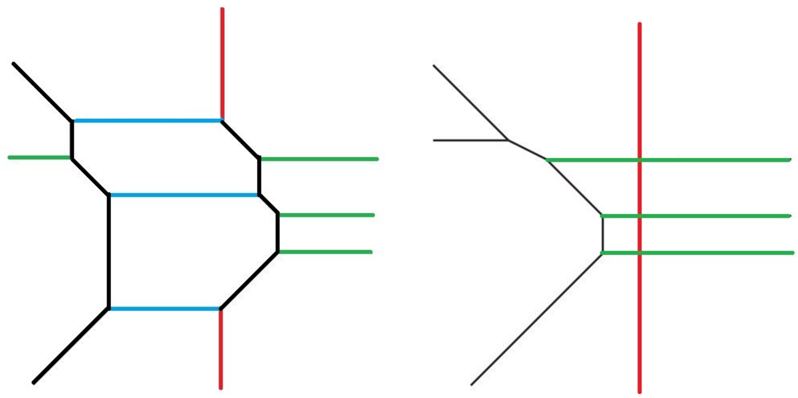
Observation 9.16. *It was highly emphasized by us that the sandpile have a lot of care in not to develop extra global gauge symmetries on the web; a new global symmetry can be developed if we reach a web with a leg that intersect the boundary of the square, then it looks that the above pictures with lines intersecting edges of the square are counterexamples of our affirmation; except that they are not.*

The idea is that the lines that we have highlighted in our examples ***do not*** *add new global gauge interactions because they intersect the boundary edges of the square at 90 degrees angle and always come in pairs (as was noticed), because of this they are associated to mesons (if the lines are horizontal) such that its valence quarks transform into the fundamental representation of some local gauge group associated to vertical edges of some boxes of the faces of the web.*

The problem would be if a single line appear intersecting the boundary at an angle different from 90 degrees, this in fact also change the global group and destroy the self-similarity of the web (tropical sandpile curve) but fortunately it appears not to be the case; even more the sandpile draws do not develop lines intersecting the boundary edges at 90 degrees without a partner, this suggest that every time a quark is added by the sandpile to the web this is accompanied by another quark in such a way that in the low energy theory associated to the web a meson adquieres a non zero VEV.

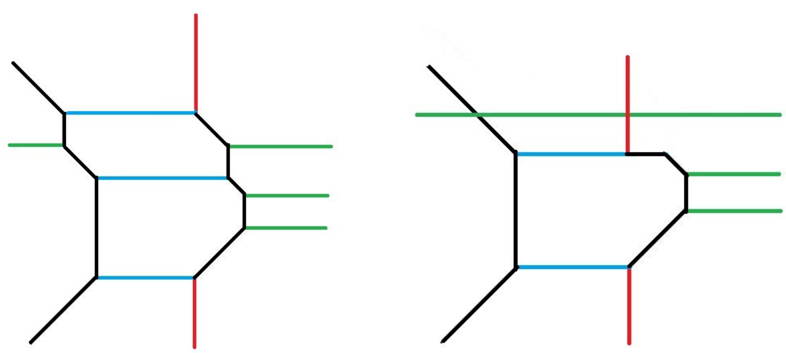
The sandpile do not create isolated fermions and do not destroy or add new global gauge groups

A basic consistency check would be to not observe baryons with finite VEV on the webs that sandpiles draw because they do not exist in five dimensions. To those matters let us analyse the following $N_c = 3, N_f = 4$ webs:

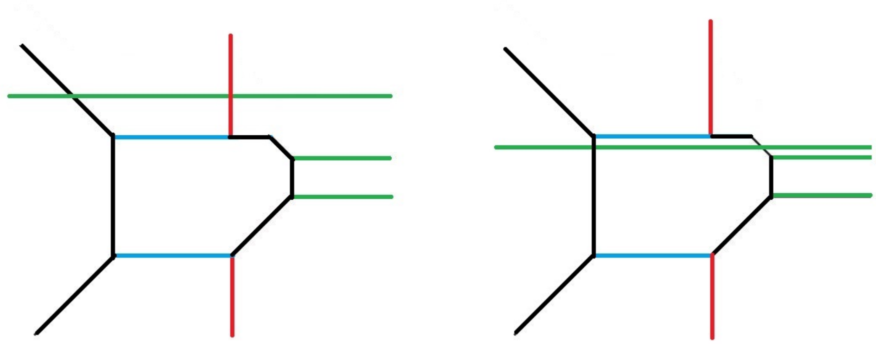


In the figure above at the left show a generic configuration, color branes are in blue, quarks in green and the 't Hooft–Polyakov monopole in red. The figure in the right is the root for a baryonic branch. The observation is that in order to deform the figure on the left to the figure on the right we need to align the D5-branes in blue with the quarks in green lines.

We want to exemplify what the problem is by trying to deform the generic web into the root of the baryonic branch. Consider the following deformation starting from left to right:

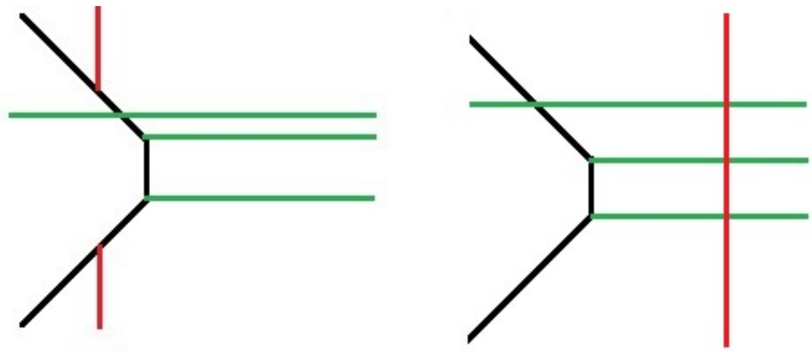


In the deformation from the leftmost web to the web at the center a sitting of the theory at the root of a mesonic branch takes place.



The process from the left to the right the most web traverse a point of gauge symmetry enhancement from $U(1)^3$ to $SU(2) \times U(1)$. No problems up to this point.

The problem emerges when trying to move the vertical red line from right to left, when this is done the color gauge group is broken because the color branes does not have more a vertical support.



That is the problem and is generic, is not possible to give finite VEVs to bayons in five dimensional gauge theories without destroying the gauge group.

The amazement is that sandpiles appear to know it, there are no webs associated to baryonic branches on the identity.

Our point now is the observation that if the sandpile know about the moduli space of SQCDs in five dimensions, then it is very plausible that the symmetries of such moduli spaces became manifest by the sandpile in some way. In particular we are talking about S-duality.

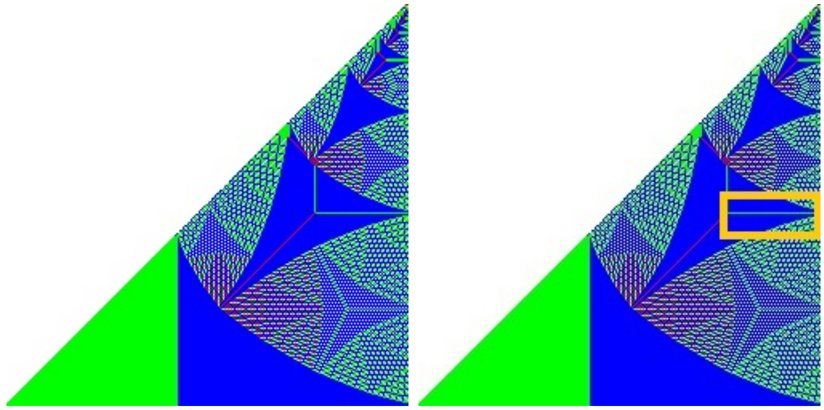
9.1.2 Jigsaw puzzles and sandpile recurrent states via electric-magnetic duality.

We propose a funny experiment:

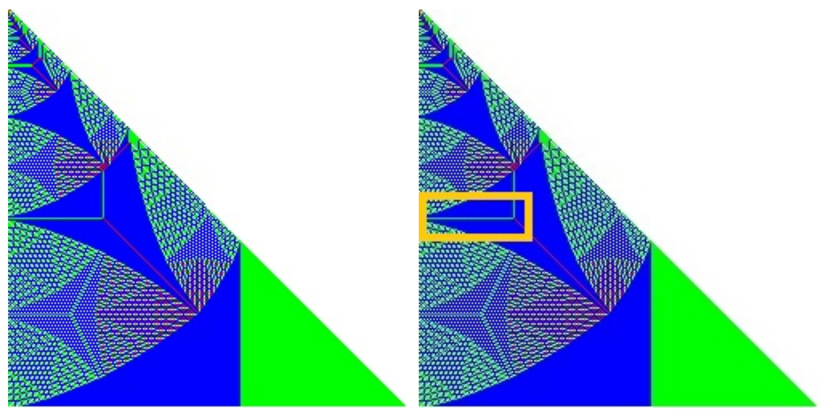
To show some evidence we select (judiciously) a domain on the identity and reconstruct the global pattern of the identity by using physical considerations as if it were a puzzle, crucially S-duality.

Our fundamental hypothesis is that the sandpile create only (p, q)5-webs duals to confining phases and our work tool is the electric magnetic duality.

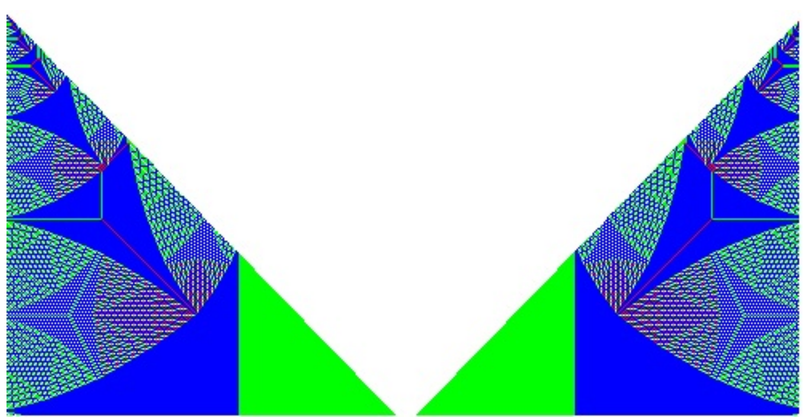
Let us consider a domain and we take one quark on it (but the analysis can be done with any quark on the figure):



We are supposing that the web is a Higgs phase of a SQCD with global $SU(2)$ symmetry, because of this we expect (by Higgs phase-confinement phase) that the quark in question is confined into a meson, then the picture below at the left is part of the draw because of the necessity of introducing the quark highlighted on the picture below at the right:

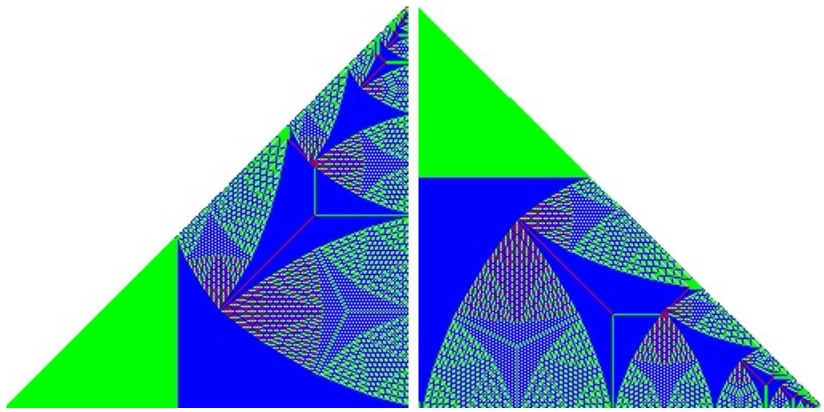


We obtain

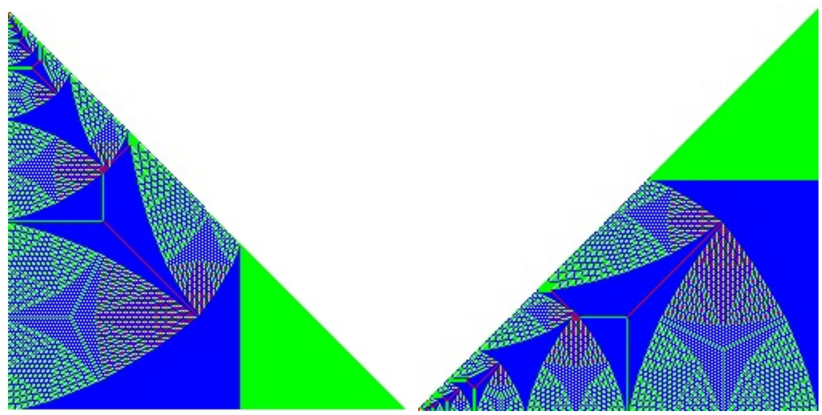


Know we use S-duality to argue that for any one of the quarks in question there should be 't Hooft–Polyakov monopoles, to include those objects it suffices to include rotate 90 degrees the triangles that we have so far.

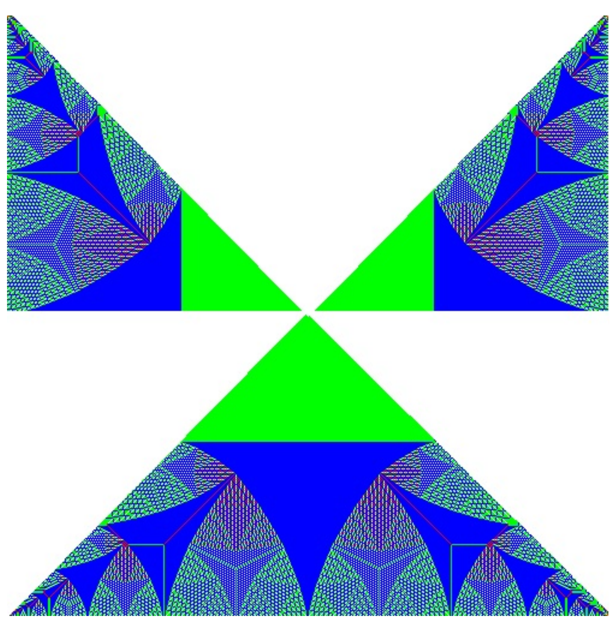
Below at the left is the triangle we have started with and below at the right there is a 90 degrees rotation of it.



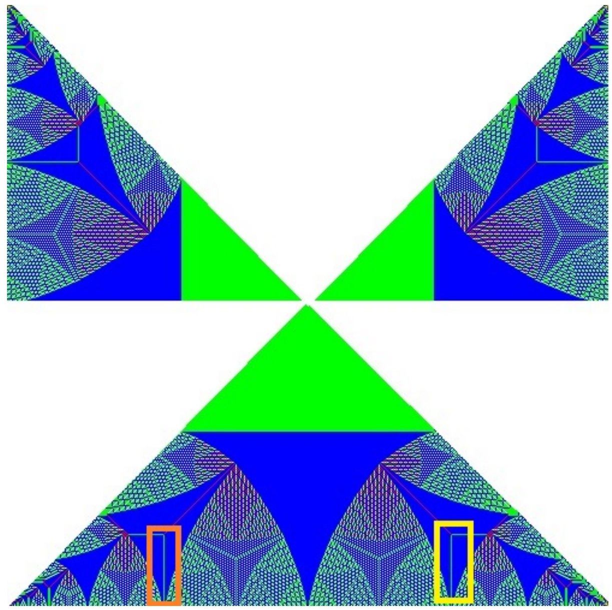
below at the left is the triangle we have encountered and below at the right a 90 degrees rotation of it.



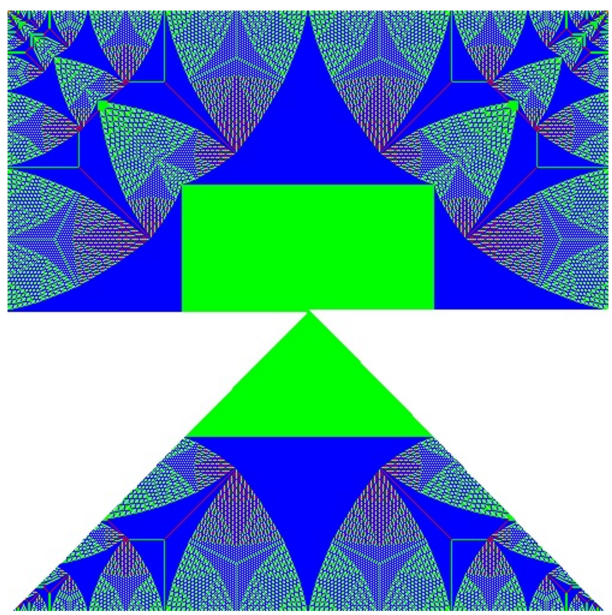
We arrive at the following pattern



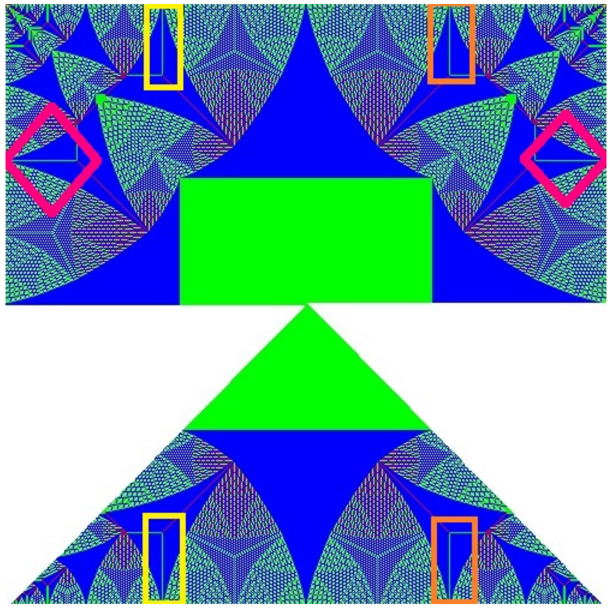
But this new pattern is not on dual to a confined phase because the monopoles inside the rectangles in orange and lemon are not confined:



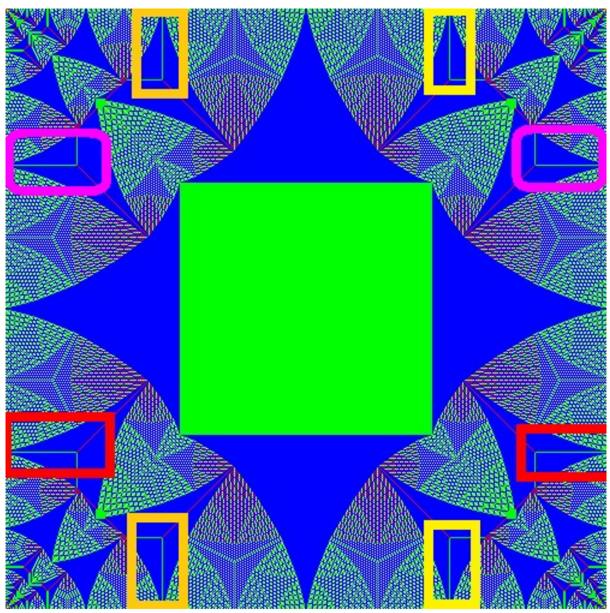
To solve this we extend our picture in the following way:



Another problem arise, the new pattern is not consistent with S-duality in the sense that we have two confined monopoles (the pairs in orange and lemon rectangles) and just one meson (the one defined by the pair of segments inside the pink diamonds):



We can use S-duality again to show that the only consistent way to complete the picture is the following:

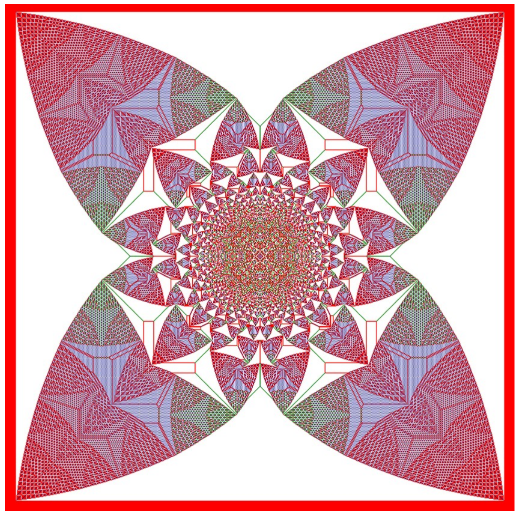


The pairs of vertical lines into the (orange and lemon) rectangles are confined monopoles and the pairs of horizontal lines into the rectangles with bisel are mesons. Those two mesons and confined monopoles are states on S-dual theories that are exchanged by S-Duality.

We have recovered the identity sandpile.

9.1.3 Do sandpiles know the six dimensional origin of five dimensional gauge theories?

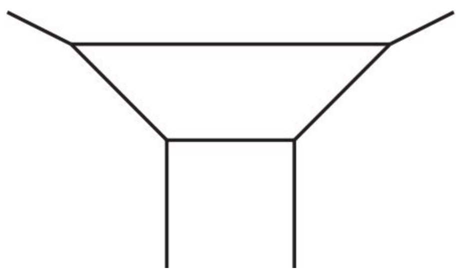
We begin this subsection by showing a beautiful recurrent state on the square grid inside a square, the color code is as follows: white = 3 chips, red = 2 chips, green = 1 chip and blue = 0 chips. The draw is extracted from [42]



The first amazement is that despite the complexity of the web all the external legs of it are exactly four!, even when the number of faces (the genus of the tropical curve) developed by Hanany-Witten transitions is enormous the sandpile strives to preserve the global symmetries.

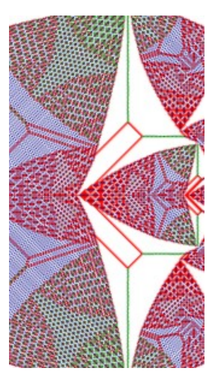
This is impressive but the focus of our attention is another mystery.

Consider the following web:

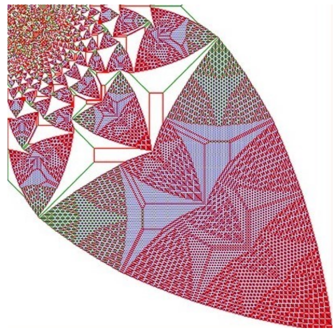


It is no known (to the humble knowledge of the author) if there is a consistent $SU(2)$ theory associated to this web on the same Coulomb branch as the pure $SU(2)$ theory, the problem is caused by the parallel vertical legs that represent two NS5 branes, we can not guarantee a well-defined IR fixed point because is not clear that we can decouple states on the spectrum of the six dimensional "M-strings" between those NS5 branes; if we suppose that the aforementioned web is not part of a sub web those six dimensional states are charged under the global symmetries of the theory.

The mystery that we want to highlight is that this phenomenon of parallel external legs takes place many times on sub-webs of our recurrent state:



Remarkably the sandpile force this kind of legs to reach the vertices of the square apparently because the sandpile want that the six dimensional particles to became charged under the global $SU(2)$ symmetry, it is to say, the sandpile apparently prohibit the parallel lines to reach the edges of the square. The following picture exhibits this:



The first fact is that there is really a (p, q)-web in the scaling limit of this recurrent state but is not clear at all that there is a well defined gauge theory associated to this web; the second fact is that the sandpile in the square generates (by the addition of a single chip to the maximally stable state not at his center) the web whose low energy theory is a pure $SU(2)$ theory; the second fact is that webs such that the one exhibited by this recurrent state are connected by addition of grains to the pure $SU(2)$ web, this is perturbing, if the sandpile in an admissible polygon is sort of classifying 5d QCDs with with fixed global rank (the number of vertices of the polygon minus three), how one could physically explain that this two theories are "connected" in some way?.

9.2 Black holes and sandpiles.

"As usual all roads lead to black holes"
Andrew Strominger - Lectures on the Infrared Structure of Gravity and Gauge Theory.

In view that now we know that the scaling limit of the sandpile model on some lattice G reproduces the closed string sector of the A-model on a Calabi-Yau threefold canonically associated to G^d it is natural to think on some applications of this fact.

One of the most impressive applications of the topological string is the fact that it allow us to compute the exact entropy of some supersymmetric black holes via the Vafa-Ooguri-Strominger relation:

$$Z_{BH} = |Z_{Top}|^2,$$

where Z_{BH} is the black hole partition function and $|Z_{Top}|^2 = Z_{Top}\bar{Z}_{Top}$ is the square of the topological string partition function.

It is then natural to expect that sandpiles are related to black holes.

As an application we show how to parametrize the BPS states that produce the entropy of a four dimensional black hole in type IIA superstring theory, we also make some intriguing observations about a possible important relation between sandpiles and black holes that could benefit both.

9.3 Sandpiles and six dimensional supersymmetric black holes.

The Vafa-Strominger black hole entropy computation and the string theoretic microscopic understanding of it are in between the greatest intellectual achievements of all times. The computations are holographic in nature, morally they replace highly curved backgrounds (with for example D-branes or other solitons) with a weakly coupled field theoretical description using open-closed duality where enummerative problems are easy and then certain countings are extrapolated by non-renormalization theorems.

The later is absolutely wonderful and revolutionary but there is a missing corner: **Is it possible to compute the black hole entropy by a statistical sum of ensembles of atoms of spacetime?**.The beauty of the melting crystal picture is that it is capable to fill this missing gap [39] [48].

We consider type IIA superstrings on the supersymmetric orbifold $\mathbb{C}^2/\mathbb{Z}_n$ in the large n limit. As $\mathbb{C}^2/\mathbb{Z}_n$ is non compact gravity decouples in the effective field theory.

The strategy: following the techniques of [46], [47] and [48] we construct a base for fractional branes given by blown down D2 branes wrapping the 2-cycles of the geometry at the orbifold point (where D2 are blown down), this base would serve us as a base for the Hilbert space of the effective quiver quantum mechanics, before this is discussed we show how to stablish a one to one correspondence between the aforementioned basis and all the possible two dimensional quiver tableaus of class A (two dimensional melting crystals) and finally we indicate how to construct a one dimensional sandpile model whose allowed states are precisely those A-type quiver tableaus and how to generalize our construction to more general orbifolds.

Computation:

The quiver mechanics in question is given by a infinite one dimensional graph, every node a has attached a $U(N_a)$ gauge group to it and three chiral superfields $X_{a,a+1}, Y_{a+1,a}$ and Z_a transforming in the representations $(N_a, \tilde{N}_{a+1}), (N_{a+1}, \tilde{N}_a)$ and $(N_a, \tilde{N}_a)$ respectively.

A generic state on the A_∞^∞ quiver is given as:

$$\vec{Q} = \sum_{a \in \mathbb{Z}} \alpha_a$$

where $\{\alpha_a\}_{a\in\mathbb{Z}}$ is the set of simple roots of the type A Lie algebra.

The idea is to generate the BPS spectrum produced by the fractional branes proving the orbifold singularities by using flopping one by one 2-cycles of the geometry. Intuitively this corresponds to count how many deformations of the geometry that respect the asymptotic symmetries of $\mathbb{C}^2/\mathbb{Z}_n$ (at large n) exist, a flop at some cycle modifies the geometry but there is always an open set that contains the 2-cycle such that in his complement the geometry is unchanged.

From the effective field theory description point of view a flop over a 2-cycle corresponds to a Seiberg duality (an inversion of the square of the coupling constant) on the quiver node associated to that cycle.

Result: (Cachazo-Fiol-Intriligator-Katz-Vafa [46]) The S-duality grop of the quiver theory is isomorphic to the Weyl group of the infinite A-series algebra.

Let us perform a Weyl reflection on the k node at an arbitrary state $\vec{Q}$:

$$\sigma_{\alpha_k}[\vec{Q}] := \vec{Q} - (\alpha_k, \vec{Q})\alpha_k,$$

where the inner product $(\alpha_k, \vec{Q})$ is given by the Cartan matrix of the A-infinity algebra defined as:

$$(\alpha_k, \alpha_{k\pm1}) = -1 \ , \ (\alpha_k, \alpha_k) = 2 \ , \ (\alpha_k, \alpha_l) = 0 \text{ otherwise.}$$

we obtain

$$\sigma_{\alpha_k}[\vec{Q}] = \vec{Q} - [N_k(\alpha_k, \alpha_k) + N_{k-1}(\alpha_k, \alpha_{k-1}) + N_{k+1}(\alpha_k, \alpha_{k+1})],$$

or

$$\sigma_{\alpha_k}[\vec{Q}] = \vec{Q} - [2N_k - N_{k-1} - N_{k+1}]\alpha_k.$$

Now consider the positive cuadrant on $\mathbb{R}^2$

Definition 9.17. Consider the positive cuadrant on $\mathbb{R}^2_{\geq 0}$, **the heigh function of a A-series quiver Young Tableau** is a function $h : \mathbb{Z} \to \mathbb{Z}$ such that for all n

$$h_n - h_{n+1} = \pm 1,$$

and

$$h_n \geq n \text{ for all } n \in \mathbb{N} \text{ and } h_n = n \text{ except for a finite set of } \mathbb{N}.$$

Definition 9.18. An **A-series quiver Young Tableau** is a Young tableau constructed from the following algorithm:

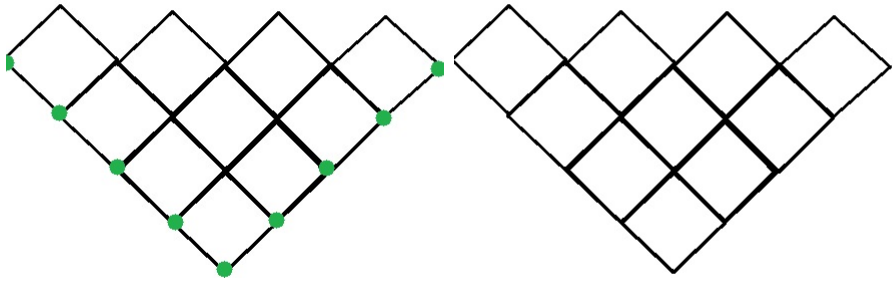
Input: An A-series quiver Young tableau height function h.
Algorithm:
1) For every $n \in \mathbb{Z}$ join the point (n, h_n) with the points $(n \pm 1, h_{n\pm1})$ on $\mathbb{R}^2$ by straight lines call this picewise linear set Γ.
2) Draw the graph of the absolute value function $|\bullet| : \mathbb{R} \to \mathbb{R}$ (the one that sends $x \mapsto |x|$ for $x \in \mathbb{R}$) on $\mathbb{R}^2$.
3) The union to the graph of the absolute value function and Γ define (by Jordan´s curve theorem) a compact set on $\mathbb{R}^2$ that we denote by $\mathcal{Y}$.
4) Join every point on $\mathcal{Y} \cap \mathbb{Z}^2$ with nearest neighbours on $\mathcal{Y} \cap \mathbb{Z}^2$.
The resulting set of boxes is the desired A-series quiver Young tableau associated to h.

Definition 9.19. A **two dimensional melting crystal** is the complement on the positive cuadrant of $\mathbb{R}^2$ of an A-series quiver Young tableau.

The vacuum state of the melting crystal problem for $\mathbb{C}^2$ is given by the empty corner configuration:

$$|0>= \sum_{n\in\mathbb{Z}} |n|\alpha_n,$$

the quiver tableau associated to the empty corner configuration is the zero tableau; in terms of melting crystals is the "completely frozen" (not melted at all) crystal that we draw at the right, at the left we mark in green dots the values of his associated heigh function.



It is shown in [48] that the Hilbert space of the crystal melting problem is given as:

$$\mathcal{H}_{\mathbb{C}^2} = \text{Span } \{\sigma_{\alpha_k}\}|0>$$

Where $\{\sigma_{\alpha k}\}$ is the group of Weyl reflections (or the S-duality group) of the A_∞^∞ series algebra.

Let us make a flop at the k^{th} cycle of the geometry (a Seiberg duality at the k^{th} node of the quiver).

$$\sigma_{\alpha_k}|0>= |0> -[2|k| - |k-1| - |k+1|]\alpha_k,$$

if $k = 0$

$$\sigma_{\alpha_k}|0>= |0> +2\alpha_0,$$

but if $k < 0$ or $k > 0$ then

$$\sigma_{\alpha_k}|0>= |0>,$$

in summary

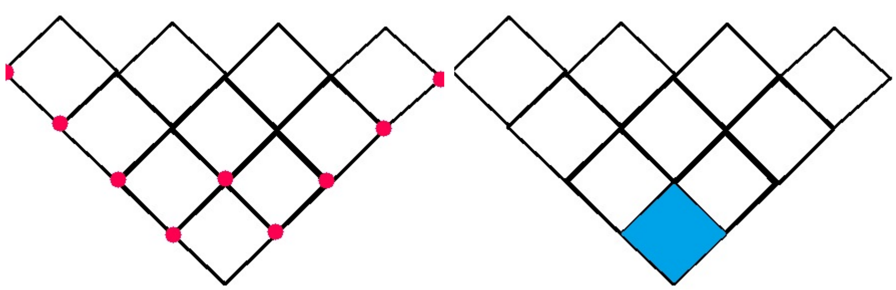
$$\sigma_{\alpha_k}|0>= |0> -[2|k| - |k-1| - |k+1|]\delta_{k,0}\alpha_k.$$

The lesson is that it is not possible make a flop at an arbitrary cycle in the geometry starting from the empty corner configuration, only a flop at α_0 is allowed:

$$\sigma_{\alpha_0}|0>= |0> +2\alpha_0,$$

that operation increases in two units the value of the height function of the empty corner configuration.

Below at the right it is shown in pink dots the values of the heigh function for $|0> +2\alpha_0$, below at the right the crystal melting configuration (the tableau quiver is in blue):



Now starting from the empty corner plus a flop at zero configuration $|0> +2\alpha_0$ we make a flop at α_l:

$$\sigma_l(|0> +2\alpha_0) = |0> +2\alpha_0 - (\alpha_l, |0> +2\alpha_0)\alpha_l$$

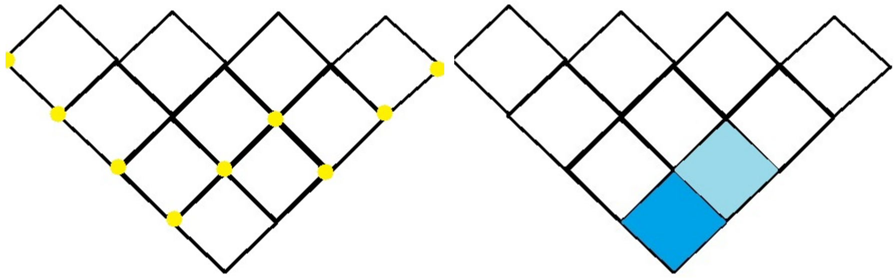
from where we obtain:

$$\sigma_l(|0>+2\alpha_0) = |0>+2\alpha_0 - [2|l| - |l-1| - |l+1|]\delta_{l,0}\alpha_0 - 2(\alpha_l, \alpha_0)\alpha_l,$$

and again, is not possible to produce a flop at an arbitrary node of the quiver the only values of l that produce non trivial configurations are $l = 1, -1$.

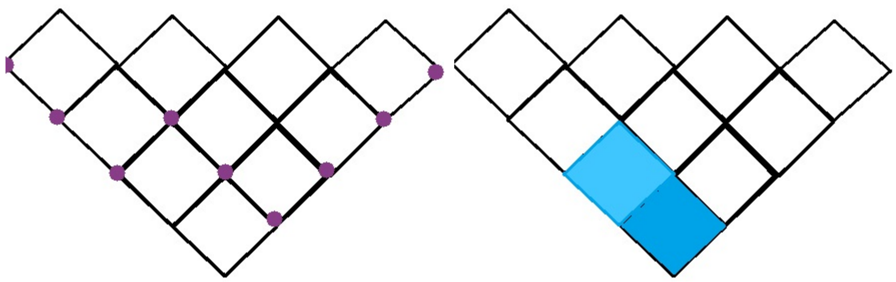
For $l = 1$ we obtain:

$$\sigma_1(|0>+2\alpha_0) = |0>+2\alpha_0 + 2\alpha_1$$

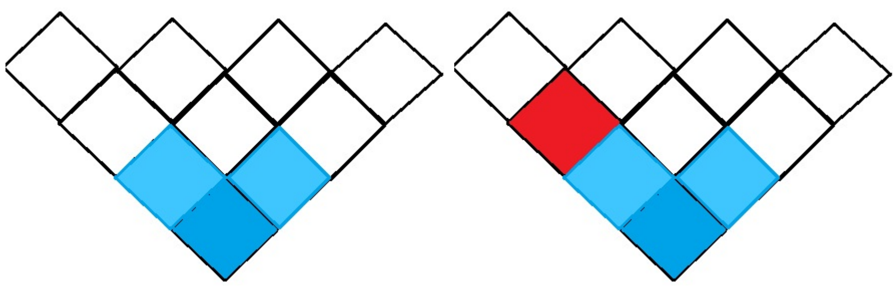


In the $l = -1$ case:

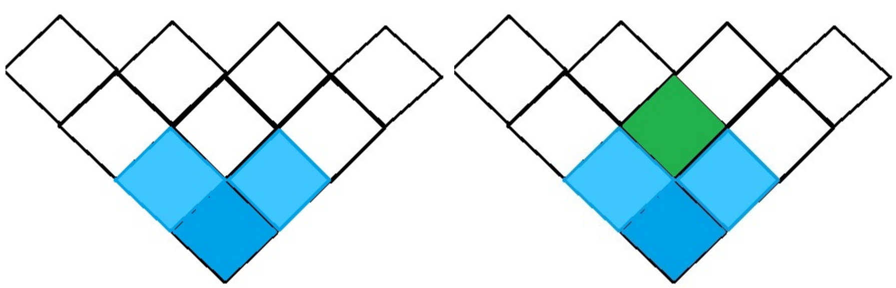
$$\sigma_1(|0>+2\alpha_0) = |0>+2\alpha_0 + 2\alpha_{-1}$$



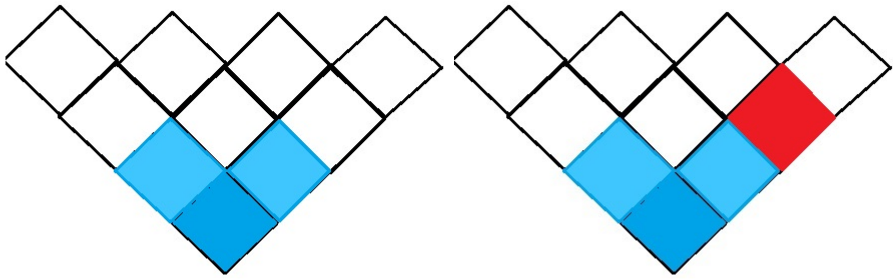
Starting with the configuration $\sigma_{-1}\sigma_0\sigma_1|0>=|0>2\alpha_0 + 2\alpha_{-1} + 2\alpha_1$ we can not produce a flop at -2^{th} node.



Or we can produce a flop at the 0^{th} node starting from $\sigma_{-1}\sigma_0\sigma_1|0>$



A flop at the 2^{th} node on $\sigma_{-1}\sigma_0\sigma_1|0>$ exhaust all the possibilities.



The key observation that is evident form the examples and connect two dimensional melting crystals with sandpiles is the following:

A flop at the k^{th} node of the quiver obtained from the empty corner configuration increases the value of his height function on two units.

9.3.1 Sandpile states for A-series quiver tableaus.

Is our objective now to define a sandpiles over the Dynkin diagram of the A_∞^∞ Lie algebra.

Definition 9.20. A one dimensional sandpile state is a formal positive integer linear combination of the roots of the A_∞^∞.

Notice that this definition coincides with what is called a "charge vector" in the physics literature.

Definition 9.21. A one dimensional sandpile is defined at the cellular automata level as follows:

Input: A sandpile state $\phi = \sum_{n\in\mathbb{Z}} \phi_n \alpha_n \ , \ \phi_n \in \mathbb{N}$.

Cellular automata rule: Define the local slopes of ϕ as $z_i = \phi_i - \phi_{i+1}$

1) If there is a site i such that $z_i > 2$ the relaxes as:

$$z_i \mapsto z_i - 2$$

$$z_{i\pm1} \mapsto z_{i\pm1} + 1$$

Rule for generate A-series Young tableaus from one dimensional sandpiles.

We simply state our results, our goal at this point is simply to report the bijection between two dimensional melting crystals and the states of sandpile in one dimension starting from a specific state, we also exhibit the exact agreement of states at every energy level and the agreement between unphysical (ghost) degrees of freedom in the melting crystal picture and "bad evolutions" in the sandpile.The picture we offer is interesting because it shows very graphically and intuitively what is the difference between BPS states and ghost states.

A derivation from first (cominatorial) principles and a detailed physical analysis will appear soon [41].

Start with the double ramp one dimensional sandpile state:

$$R = \sum_{a\in\mathbb{Z}} |n|\alpha_n = (..., 4, 3, 2, 1, 0, 1, 2, 3, 4, ...),$$

Intuitively R is the state puts n grains of sand at $\alpha_{\pm n}$.
Let us show how this works:

Starting from the double ramp sandpile state $R = (..., 4, 3, 2, 1, 0, 1, 2, 3, 4, ...)$ (the ground state), we highlight "the middle" (the symmetry axis) as $R = (..., 4, 3, 2, 1, \mathbf{0}, 1, 2, 3, 4, ...)$ to facilitate the lecture for the reader. We note that the only site at which we can add grains of sand without producing topples on R is the zero node, in fact, the addition of one grain at every other position produces a chain of topples without a finite time relaxation, for example let us add a grain to R at α_{-1}, the result produce the following relaxation process:

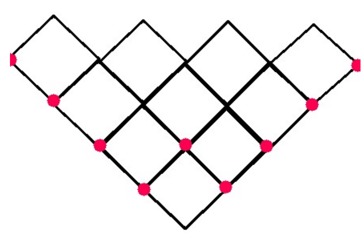
$$R+\alpha_{-1} = (..., 4, 3, 2, 2, \mathbf{0}, 1, 2, 3, 4, ...) \mapsto (..., 4, 3, 3, 0, \mathbf{1}, 1, 2, 3, 4, ...) \mapsto (..., 4, 4, 1, 1, \mathbf{1}, 1, 2, 3, 4, ...) \mapsto \dots$$

As can be noticed this sequence of toppling does not stop at finite time, the amazement is that $R + \alpha_{-1}$ is in fact a spurious state from the crystal melting picture, more generally, any state $R + \alpha_k$ with $|k| \geq 1$ is spurious and as can easily be verified is a state without finite time relaxation in the sandpile picture.

The rule is to select a sate α_k that admit the addition of chips without producing topples and to put on it the highest number of chips n in such a way that $R + n\alpha_k$ remains stable but $R + (n + 1)\alpha_k$ produce topples.

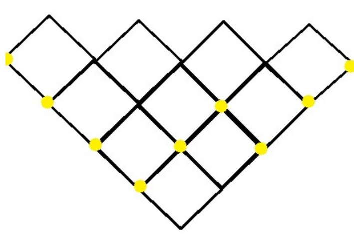
First sandpile excited state:

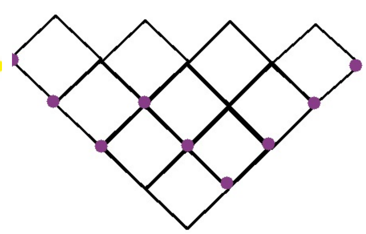
We have mentioned that the only site at which we can add chips in order to produce the first excited sandpile state is α_0, then the first excited state is $R+2\alpha_0$; we notice immediately that if we see $R+2\alpha_0$ as a function from the Dynkin diagram to the natural numbers it is exactly the heigh function of the empty corner with one atom removed crystal:



Sandpile stable states at the second energy level.

Now start with $R + 2\alpha_0$, the only sites at which we can add grains of sand without destabilizing the configuration are $\alpha_{\pm 1}$, select for example α_{-1}, the maximum number of chips that we can add to α_{-1} in such a way that an extra grain destabilize the resulting configuration is 2, then $R + 2\alpha_0 + 2\alpha_{-1}$ is an admissible second excited sandpile state. The other one is $R + 2\alpha_0 + 2\alpha_1$.



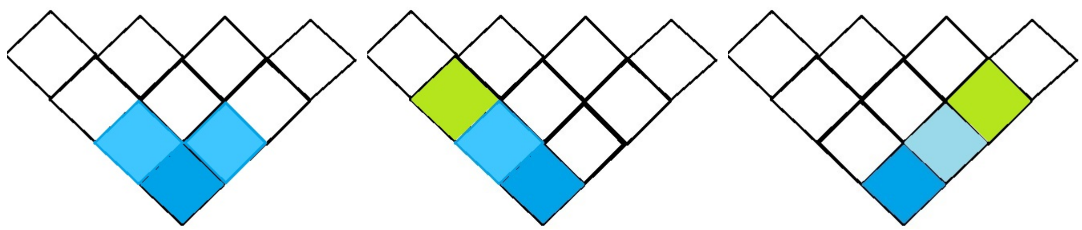


Notice that the pass from one state at one energy level to the next energy level is made only by adding grains at a one and just one site as if the addition of grains were the action of a creation operator in field theoretical language, in this sense although $R + 2\alpha_0 + 2\alpha_{-1} + 2\alpha_1$ is stale **is not** a sandpile state at the second energy level (it is in fact a state at the third energy level).

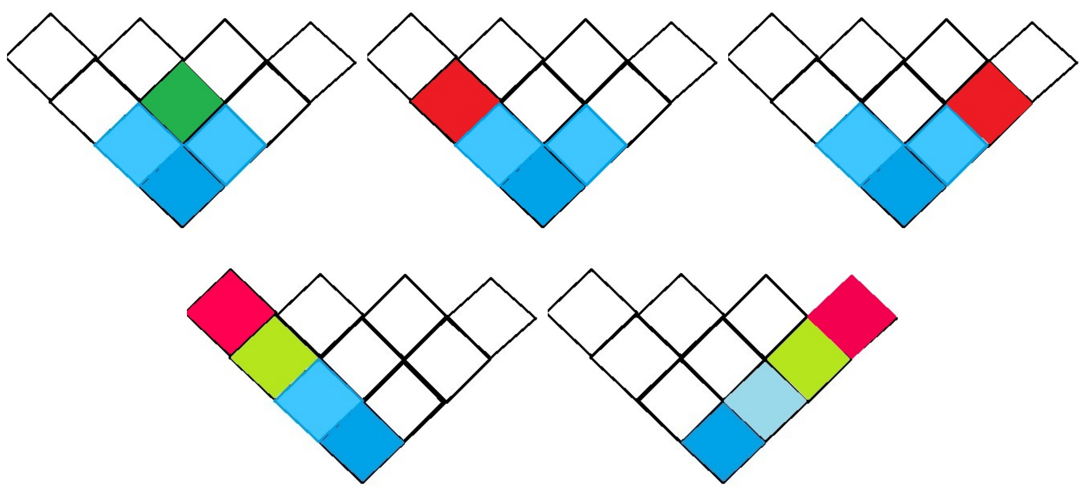
Sandpile states at the third energy level:

At this point it should start to be evident that the levels of sandpile states will be filled by quiver tableaus or two dimensional melting crystals, in fact is expected that the number of quiver tableaus obtained from the empty corner configuration by the action of n flops is exactly the set of one dimensional sandpile states at level n.

The possible states at the third level are the heigh functions for the following three crystals:



Sandpile states at the fourth energy level:



And so one and so forth.

The conclusion is that we have produced a one to one correspondence between sandpile states in one dimension starting from the double ramp state and the melting crystal configurations that parametrize the untwisted RR D0 charge on the the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ (in the large n limit), we also have produced a one to one correspondence between unstable sandpile states in our model with ghost states on the same background.

9.4 Four dimensional BPS black holes and two dimensional sandpiles.

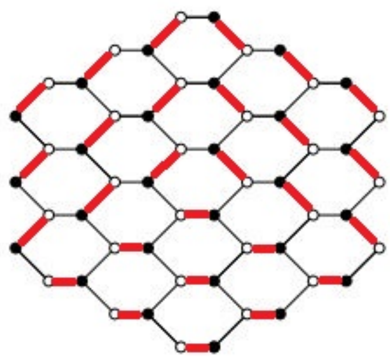
The analysis that follows is very delicate, as indicated we only present the results that show a one to one correspondence between the crystal melting configuration that contribute to the entropy of

$D3-$branes proving the singularities of the orbifold $X = \mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$ in the large n-limit and sandpile states on a related lattice. We also observe that unstable sandpile states on this lattice are in bijective correspondence with melting crystal states that posses bifundamental scalars with negative kinect energy therm.

The strategy: As in the two dimensional case the idea is that as the volume of X is infinite gravity decouples and we can restrict our attention to the BPS spectrum of the quiver theory that arises from D0, D2 and D4 bound states on X. It is proposed in [49] to deal with this problem by passing to the mirror manifold of X denoted by $\tilde{X}$ and constructing the base of $D3$ fractional branes that wrap homology 3-spheres $\Delta_i \in H_3(\tilde{X}, \mathbb{Z})$ for $i = 1, ..., n^2$.

Geometrically we want to characterize all the flop transitions over non trivial 3-cycles of $\tilde{X}$ that preserve the asymptotic form of the metric of $\tilde{X}$ outside an open set that contain the flopped 3-cycle, this is an absolutely beautiful way to generate the BPS spectrum because it computes candidates for black hole microstates by counting certain geometries, this is the spirit of missing corner. But our objective is to exhibit new relations between sandpiles and BPS states, for this reason we review the techniques developed on [49] to relate dimer covers on the honeycomb lattice with the quiver gauge theory generated by D3 branes proving orbifold singularities.

Computation: Consider the empty room perfect matching on the honeycomb lattice:

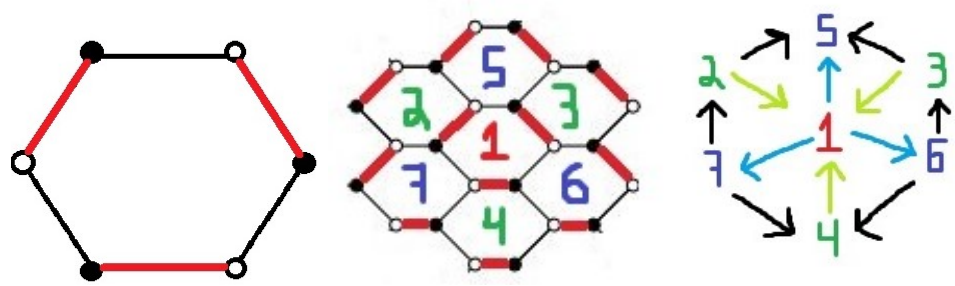


Crystal melting configurations are realized by local rearrangement of dimers over the empty room matching in such a way that the resulting configuration of dimers is still a perfect matching.

The profile of the crystal is constructed via the height function h defined as $h(F_0) = 0$ in the unique face F_0 of the honeycomb lattice with three edges on the perfect matching and for any faces A and B that share an edge e as:

$$h_{PM}(A) - h_{PM}(B) = \begin{cases} \pm 1 & if\ e \in PM \\ \mp 2 & if\ e \notin PM \end{cases}$$

The idea that we pursuit is to dualize F_0 to create a unit cell on the two dimensional quiver diagram that define the holographic dual gauge theory.



Above at the left we show F_0, above at the center are shown the nearest neighbours of F_0 with labels that help to identify it as nodes on the quiver diagram shown above at the left. The cycles Δ_i represented as faces at the center obey that: Δ_2, Δ_3 and Δ_4 are incoming branes and Δ_2, Δ_3 and Δ_4 are outgoing branes, notice that the dimers in the honeycomb graph indicate which arrows are incoming into the 1^{th} node.

We want to produce Seiberg dualities at some node on the quiver diagram in such a way that the superpotential and adjacency of bifundamentals in the theory are preserved, there are the two possible ways to achieve this, one is by fractionalizing a D3-brane wrapping Δ_1 into the outgoing branes and the other is by doing the same over the ingoing branes. The former case is realized by the action of the operator L_R defined as follows:

$$L_R(\Delta_j) = \Delta_j + (\Delta_1 \cap \Delta_j)\Delta_1$$

We remember the adjacency relations between the 3-spheres of $\tilde{X}$ Δ_i as

$$\Delta_i \cap \Delta_i = 0, \Delta_1 \cap \Delta_{out} = 1, \Delta_1 \cap \Delta_{in} = -1,$$

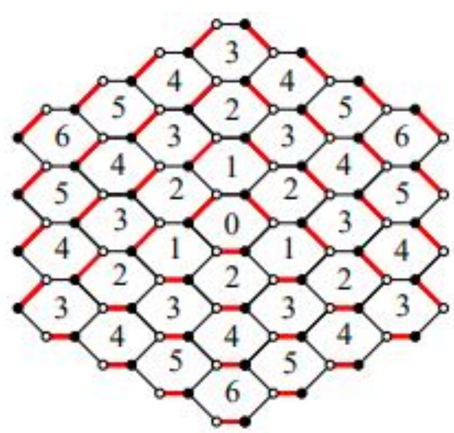
it is easy to verify that the intersection form is preserved:

$$L_R(\Delta_i) \cap L_R(\Delta_j) = [\Delta_i + (\Delta_1 \cap \Delta_i)\Delta_1] \cap [\Delta_j + (\Delta_1 \cap \Delta_j)\Delta_1]$$

$$= \Delta_i \cap \Delta_j + (\Delta_1 \cap \Delta_j)(\Delta_1 \cap \Delta_1) + (\Delta_1 \cap \Delta_i)(\Delta_1 \cap \Delta_j) + (\Delta_1 \cap \Delta_i)(\Delta_1 \cap \Delta_j)(\Delta_1 \cap \Delta_1)$$

$$= \Delta_i \cap \Delta_j$$

Now consider the empty room height function: $h = \sum_F N_F \Delta_F$ where the sum is over the faces of the honeycomb lattice and the set of natural numbers N_F are given as follows



And begin acting with the operator L_R on h:

$$L_R(h) = N_1 L_R(\Delta_1) + \sum_{a=2}^{4} N_a L_R(\Delta_a) + \sum_{b=5}^{7} N_b L_R(\Delta_b) + \sum_{c=8}^{\infty} N_c \Delta_c,$$

where $L_R(\Delta_i)$ is an abuse of notation that represents a change in the rank on the quiver node Δ_i produced by the action of L_R on the 3-cycle Δ_i

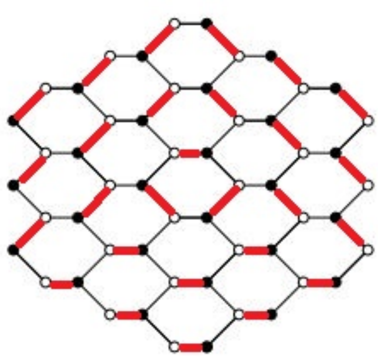
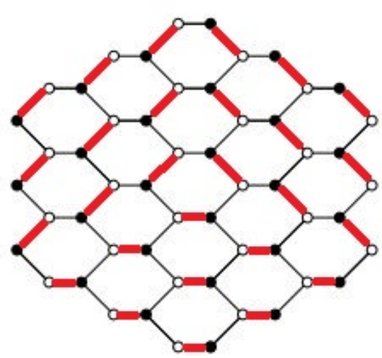
$$L_R(h) = N_1 \Delta_1 + \sum_{a=2}^{4} N_a(\Delta_a - \Delta_1) + \sum_{b=5}^{7} N_b(\Delta_b + \Delta_1) + \sum_{c=8}^{\infty} N_c \Delta_c$$

$$= h + \sum_{a=2}^{4} N_a(-\Delta_1) + \sum_{b=5}^{7} N_b \Delta_1$$

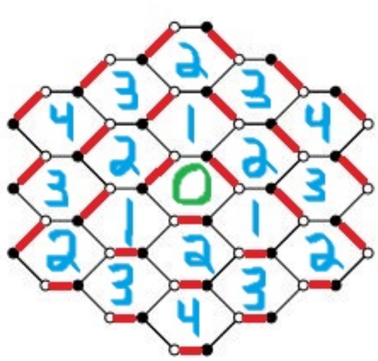
Using that $N_{2,3,4} = 1$ and $N_{5,6,7} = 2$ we arrive at

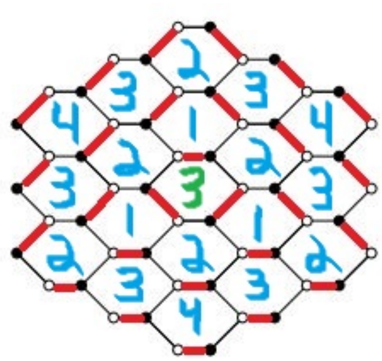
$$L_R(h) = h + (-1 - 1 - 1 + 2 + 2 + 2)\Delta_1 = h + 3\Delta_1,$$

we see that $L_R(h)$ is equal to h except at F_0 where L_R raises the value from zero to three.The perfect matching associated to the action of L_R at the empty room configuration is the crystal with one crystal atom removed shown below at the right, notice the local "rotation" of the dimers on F_0 with respect to the empty room configuration shown at the left:

We notice that the result of the computation agrees with the change in heigh of the crystal when a single atom is melted from the completely frozen one.





Observation 9.22. *The computations above can be redone by applying the operator L_L (the operator that rotate faces of the honeycomb lattice against the clock hands) to the empty room (or another crystal) configuration. The result is extraordinarily interesting:*

$$L_L(h) = h + \sum_{a=2}^{4} N_a(+\Delta_1) + \sum_{b=5}^{7} N_b(-\Delta_1),$$

or

$$L_L(h) = h + (+1+1+1-2-2-2)\Delta_1 = h - 3\Delta_1,$$

the resulting change in heigh (or the rank at the zero node of the associated quiver) of the crystal is negative!

This strongly suggest that the operator L_h is able to produce ghost branes at the cycles of the orbifold geometry that we are proving, perhaps there is an extension of our set up that includes ghost branes whose associated low energy theory is given by a superquiver $U(N|M)$ quantum mechanics. We leave this as an interesting speculation.

In fact we can apply the operator L_R n-times at F_0 at the empty room configuration to obtain:

$$L_h^n[h] = h + 3n\Delta_1,$$

the issue is that it is argued in [48] that those extra configurations $L_h^n[h]$ exist but they are unstable even at the supergravity level, we briefly explain how this works and what our algorithm to parametrize the entropy of BPS black holes in four dimensions is.

9.4.1 Algorithm four dimensional BPS black hole to sandpile state.

The key is to notice how curious is the fact that the system becomes unstable for the first time exactly when the rank of the gauge group at the zero node of the quiver diagram associated to the low energy theory of the worldvolume of $D3$-proving $\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}$ (that is a triangular 6-regular graph) acquire rank six; or, viewed from the geometric point of view, exactly when we put six units of D0-brane charge at some non-trivial cycle of the geometry, we also stress the point that even when the system $L_R[h]^n$ with $n \geq 2$ (bound states of more than six $D0$-branes) is unstable the charge created by the operator L_R^n can not possibly disappear from the system without violating RR-charge conservation, in fact it can disappear by the action of $L_L[h]$ but we are not using at all this operator, what we are doing is putting n units of RR-charges at some cycle (with six nearest neighbours) of the geometry and noticing that when we add six units of charge the system destabilize and because of charge conservation the cycle Δ_1 pass exactly one unit of charge to every one of its neighbours Δ_i for $i = 2, ..., 7$, of course this analysis can be repeated over any node of the triangular quiver diagram and the rule should be the same, this observation produce an astonishing result:

The counting of black hole microstates have produced a sandpile cellular automata over the quiver triangular lattice!.

It is our objective now to create a precise bijection between black hole microstates and sandpile recurrent states over the triangular lattice.

Definition 9.23. Let X be a non-compact Calabi-Yau threefold, an **exceptional collection** of sheaves $\mathcal{E}^V = (E_1, E_2, ..., E_n)$ supported on a complex surface V embedded on X is an ordered set of sheaves which satisfy the following special properties:

- Each E_i verify: $\mathrm{Ext}^q(E_i, E_i) = 0$ for $q > 0$ and $\mathrm{Ext}^0(E_i, E_i) = Hom(E_i, E_i) = \mathbb{C}$.
- $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i > j$ and all q. If in addition $\mathcal{E}$ satisfy:
- $\mathrm{Ext}^q(E_i, E_j) = 0$ for $j > i$ for $q > 0$. The collection is said to be **strongly exceptional**.

It was proved in [47] that the physical (ghost-free) base of fractional brane states that spans the Hilbert space of the quiver theory is given by strongly exceptional collections of sheaves supported on a complex surface V of $\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}$ obtained by resolving some of the non-trivial 3-cycles of the geometry; a subsequent development in [50] have proved that there is a bijection (up to tensoring all sheaves by a common line bundle) between this set of strongly exceptional sheaves and perfect matchings on the honeycomb lattice. The conclusion is that the spectrum of BPS black hole configurations are generated by crystal melting configurations, or equivalently, by dimer covers of the honeycomb lattice, this affirmation was explicitly stated for the first time in the beautiful Heckman´s paper [48].

A consequence of the way in which we have used Temperley´s bijection in the course of this work is the fact that recurrent sandpile states on the triangular lattice also parametrize those exceptional collections because recurrent sandpile states on the triangular lattice and dimer covers on the honeycomb lattice are the same set, but what we want to exhibit what exactly is the sandpile doing at the field theoretical point of view.

It is interesting to ask where does the instabilities of the bound states of states with more than six units of RR-charge come from, after all the geometry is supersymmetric, the charges are BPS, the GSO projection removes the open string states with negative square mass $M^2 = \frac{-1}{2}$ from the spectrum and the no-ghost theorem guarantee that no tachyonic vector bosons (Pauli-Villiars ghost) exist. The answer is that generic configurations contain uncanceled ghost; more precisely, the problematic states are exceptional collections that are *not* strongly exceptional, if there is $\mathcal{E}^V = (E_1, E_2, ..., E_n)$ exceptional collection with V a complex surface obtained by resolving some cycles of the geometry such that $\mathrm{Ext}^q(E_i, E_j) = 0$ for some $i \neq j$.

For example, if E_i and E_j were locally free

$$\mathrm{Ext}^q(E_i, E_j) = \mathrm{H}^q(X, \, \mathrm{Hom}(E_i, E_j))$$

where X is a geometry obtained from $\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}$ by resolving some singularities to obtain V and the number q is the ghost number. The application of L_R more that two times at the zero face of the empty room configuration produce exceptional collections $\mathcal{E} = (B_1, ..., B_i, ..., \bar{B}_j, ..., \bar{B}_n)$ over the geometry that contain brane B_i anti-brane $\tilde{B}_j$ bound states on V, clearly this configuration is unstable due to tachyon condensation, more precisely, the non trivial elements of $\mathrm{H}^q(X, \, \mathrm{Hom}(E_i, E_j))$ force the bound state to decay.

The perfect matchings of the honeycomb lattice parametrize precisely the exceptional collections such that $\mathrm{Ext}^q(E_i, E_j) = 0$. What is the picture that we want to offer with sandpiles?.

The observation is the that a non trivial element of $\mathrm{H}^q(X, \mathrm{Hom}(E_i, E_j))$ induce non trivial morphisms in between , we have proved that to a perfect matching on the honeycomb (an exceptional collection) corresponds an oriented spanning tree on the quiver triangular lattice, that is the definition of a **Bellinson quiver** an oriented sub-quiver that passes trough all the nodes and with not oriented loops (no bifundamental vectors), the recurrent sandpile states over the triangular quiver diagram parametrize all the possible Bellinson quivers that can be obtained from the triangular lattice by eliminating a minimal number of bifundamentals in such a way that not oriented loops remain.

That is the prediction of the dimer cover - recurrent sandpile equivalence. The physical base for black hole microstates over the geometry $\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}$ (at large n) is given by Bellinson quivers over the triangular quiver diagram, or equivalently: by reccurent sandpile states over the triangular lattice.

Algorithm to parametrize four dimensional BPS black hole microstates via recurrent sandpiles.

Suppose that we want to compute the black hole microstates of black hole of type IIB superstring theory over the geometry $\mathbb{M} \times CY_3$ where $\mathbb{M}$ is Minkowski space and CY_3 is a toric Calabi-Yau threefold.

Input: The toric diagram of CY_3 denoted by $\mathcal{T}$.

Steps:

- Compute the fivebrane diagram of $\mathcal{T}$ ([52])
- Compute the brane tilling (bipartite graph) associated to the fivebrane diagram of $\mathcal{T}$ ([52])
- The perfect matchings over the above tilling parametrize strongly exceptional collections of sheaves (a basis of physical fractional brane states) over CY_3
- Compute the sandpile group over the quiver diagram (the dual graph of the brane tilling)

Output: The recurrent sandpile states over the quiver diagram are in one to one correspondence with all the possible Bellinson sub-quivers of the quiver quantum mechanics, this provide us a basis for fractional branes over CY_3.

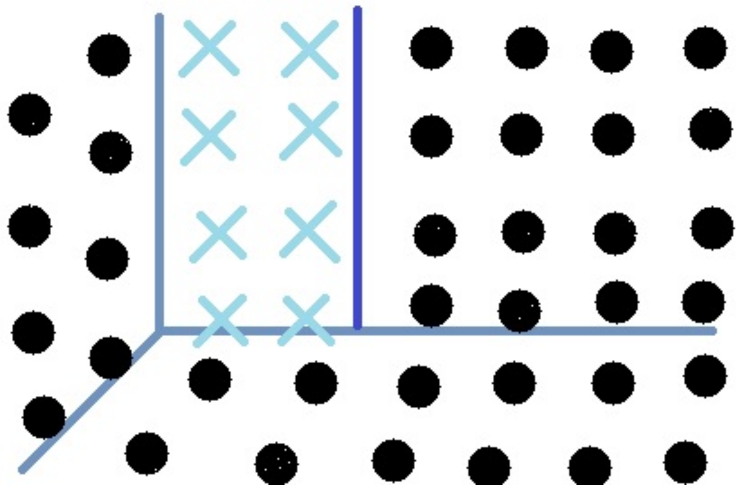
9.4.2 Defects on crystals

It is in fact possible possible to obtain a "crystalographic" illustration of the open string modes that induce the tachyon condensation that forces the states $L_h^n[h]$ with $n \geq 2$ to decay.

The melting rule for the crystal asserts that we are able to remove a crystal atom with coordinates (x_0, y_0, z_0) from the crystal if and only if the following sites are vacant:

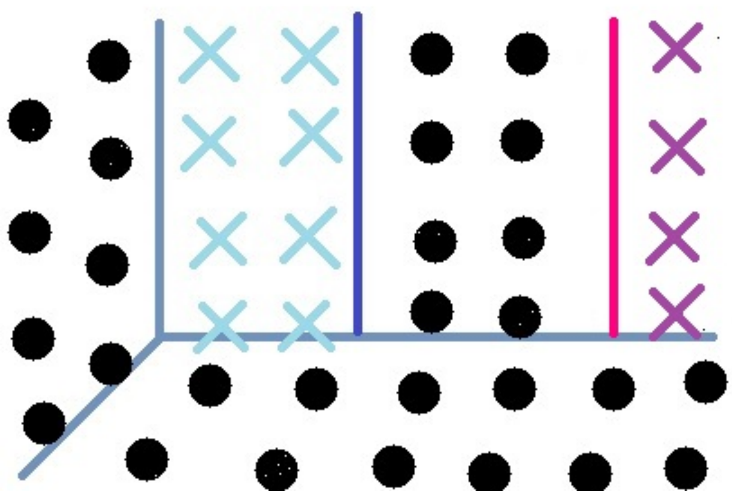
$$(x, y_0, z_0)\ x < x_0, \quad (x_0, y, z_0)\ y < y_0, \quad (x_0, y_0, z)\ z < z_0$$

It is possible to create branes in the crystal by inserting a local defect at the line at $x = a$ over the positive octant on $\mathbb{R}^3$, if we view the positive octant as the toric base over which a toric variety is fibered (by strates and with fiber T^3) the line at $x = a$ is a direction over which a 1-cycle of T^3, if this defect is inserted the atoms in between $0 < x < a$ melt:

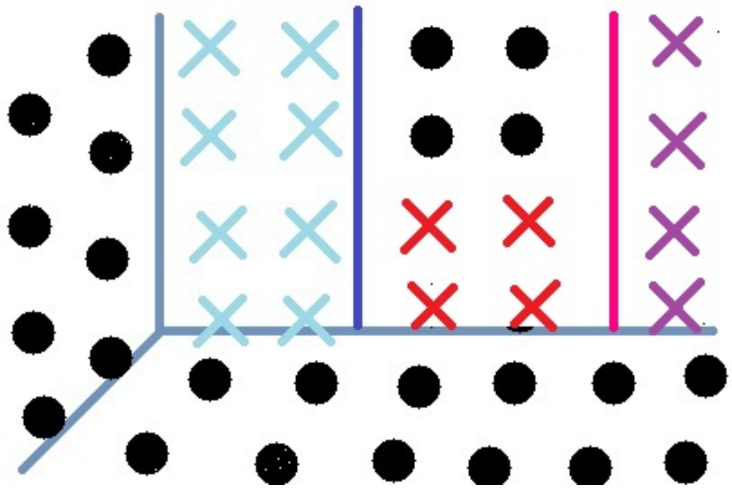


The lines in blue are the base of the fibration (the corners of the room) and the the defect introduced is shown in strong blue, a ball in black indicates a non-melted atom of the crystal and the crosses indicate a melted-atom.

It is equally possible to add an anti D-brane defect for example at $x = a + 3$ that melts the atoms at $a + 3 < 0$, we offer a picture where the anti D-brane defect is shown in pink and the melted atoms it produces in purple.



Acting with the operator L_R arbitrarily on a dimer cover produce melted crystals with this kind of local geometries. This deserves several problems, one of them is the fact that a D-brane / anti D-brane pair is an unstable system if the branes are nearly because of the open string modes that they share. In the following picture we show how those open strings can be created by melting atoms (crosses in red) in between the defects:



9.5 Sandpiles and physical string theory.

This subsection concerns to a very simple comment:

To the humble knowledge of the author there is no relation between up to date between dynamical systems and physical string theory,

The closest relation could be traced out to the worldsheet-spacetime-configuration space duality proposal of Luboš Motl [51] inspired on Kutatsov and Martinec observations about $\mathcal{N} = (2, 1)$ heterotic strings.

The main operational point of the crystal melting-string duality is an analogy target space (Calabi-Yau) – Phase space, RR-fields (Kahler form in the topological case) – symplectic form, string coupling – Planck constant. Viewed in that way "quantizing spacetime" is totally analogous to a geometric quantization of phase space. The deconstruction of spacetime can be steed equivalent to the enumeration of brane tillings on Riemann surfaces (topological worldsheet "deconstruction").

As we have speculated, the sandpile behaves as the moduli space of some quantum field theories, theories that in some cases are dual to sub sectors of string theory, The link we have exhibited is between topology changing transitions at the Planck scale in a toric variety and a dynamical system apparently over a moduli space, perhaps there is some link with Motl´s ideas about the relation between "collisions" in configuration space and topology change in spacetime, but it is worthy to say that the relation (if any) is not evident.

It would be very interesting to investigate the possible connection between dynamic systems and physical string theory.

10 Evidence in favour of the non triviality of the reformulation of the sandpile partition function as a dimer problem.

The strongest argument we have is the first one, for this reason we describe it in some detail.

1) The scaling limit of the abelian sandpile model (on the square grid) can be proved to be equivalent to a two dimensional logarithmic conformal field theory of simplectic fermions with central charge $c = -2$ (see for example [32]). On contrary, the dimer problem in the scaling limit is a two dimensional conformal field theory (in the square grid) with central charge $c = 1$.

This strongly suggests that even when the partition functions are the same the underlying models are behave significantly different as conformal field theories on the same graph (notice that the square grid is self-dual).

-There is no known example (to the knowledge of the author) of two conformal field theories with central charges of distinct sign on the same target space that can be related via "a trivial change of variables", by this we mean that this can not be achieved by deforming the vacua by using relevant or marginal deformations on the respective partition functions.

-It remains to be seen if irrelevant deformations can done the work, even if true, it would be a very interesting exercise to show explicitly how this can be achieved. Our optimistic conjecture is that if those deformations exist they interpolate between two vacua of two different theories on different target spaces.

-The dimer problem on the square grid describes the quantum gravitational foam of the deformed conifold in this case there are irrelevant operators that change the background, it would be possible that this example is the one that captures all the possible complexity of the relation that we propose.

2) The plane partitions associated to the dimer problem on a bipartite graph define a toric base for a blown up Calabi-Yau threefold, it is natural to ask if there is a canonical way to associate a geometry of the above type associated to the sandpile on a graph. The fact that a graph and his double are not isomorphic makes plausible that those geometries are not trivially related.

10.1 Final Conclusions.

1) We have reviewed the basic aspects of sandpiles and his relation with the dollar game.

2) We offered a review of the Temperley bijection and use it to connect sandpiles with melting crystals.

3) We briefly explained the basic idea of the Vafa´s missing corner.

4) We conjectured that sandpiles have a fascinating and not understood relationship with the fluctuations of the quantum topological foam induced by target space Kähler gravity on a toric Calabi-Yau threefold.

5) It was briefly argued that we believe that this relationship is not just the same problem expressed on different variables.

6) We have exhibited some intriguing relations between sandpiles and the moduli spaces of five dimensional quantum field theories.
7) We have offered parametrizations of black hole microstates in six and four dimensions as an application of the relation we have found.

8) Even if the equality of partition functions between sandpiles and melting crystals is just a coincidence, we believe that this relation could potentially serve to make claims from one side of the correspondence were something is clear to the other when the same thing is not so clear, the following sentences are examples of this:

-The quantum gravitational foam is a self-organized critical system.

-Critical exponents, correlation functions, free energies etc. are computable in melting crystals, on contrary, in sandpiles there are no analytic ways to produce such a results, the combination of Temperley bijection and the topological vertex formalism could serve as a via to realize those computations for sandpiles in simple cases.

-Pherhaps the most impressive use of melted crystals are that they are a powerful and beautiful way of understand wall crossing phenomena [30], nevertheless it is difficult to use the formalism for toric varieties that are not conifolds or with few cohomology; sandpile statistics could be very promising to make computations on more general varieties and with a lot of cohomology.

It is worthy emphasized that even though we have algorithms that count the number of recurrent states in polynomial time (computing the determinant of the laplacian) the sandpile mysteries remain untouched, it is to said: Why recurrent sandpiles look fractal and why the sandpile statistics is non-gaussian?. We do not address those deep mathematical mysteries in any sense, but we want to make a series of conjectures about the nature of the sandpile.

10.2 Final comments.

”Cualquier destino, por largo y complicado que sea, consta en realidad de un solo momento: el momento en que el hombre sabe para siempre quién es". - Jorge Luis Borges.

What is the unique moment that resumes all what this thesis work what to say?. Perhaps the following line is an appropriate one:

The quantum gravitational foam is a self-organized critical system!

At the end this is probably not so surprising at all, a physical law is as predictive as less free parameters it has, the laws of string theory are maximally predictive in this sense, there is no parameter on string theory that require fine-tuning, his supreme self-consistency determine dynamically all the values of all his parameters, the strength of the interactions depend on the value of the dilaton and the value of the dilaton is in principle determined by the interactions between stringy states.

Spacetime certainly has a stringy origin, the existence of gravity is a consistency requirement imposed by a theory without free parameters, thus, free of fine-tuning. Is in that sense that it is very unlikely to believe that the quantum fluctuations of a consistency condition (gravity) induced by a free-parameter theory were subject to fine-tunings.

To end it is important to remember that everything said concerns just to the topological sector, but it is recalled that many of the phenomena in this sector have counterparts in the full physical theory, large N, S/T, open-closed dualities are examples of this.

Are sandpiles telling us something important about the nature of spacetime?.

References

[1] Ernesto Lupercio.
Talk: Quantum Toric Geometry, Complex Systems, and Mirror Symmetry.
https://www.youtube.com/watch?v=2q0DZLXZcTo

[2] Existence of algorithm for determining if a given number is rational or not.
Math Stack Exchange
https://math.stackexchange.com/questions/691669/existence-of-algorithm-for-determining-if-

[3] S. Caracciolo, G. Paoletti, and A. Sportiello.
Conservation laws for strings in the abelian sandpile model. EPL (Europhysics Letters), 90(6):60003, 2010.

[4] Per Bak, Chao Tang, and Kurt Wiesenfeld.
Self-organized criticality: An explanation of the 1/f noise. Phys. Rev. Lett. , 59(4):381–384, 1987.

[5] N. Kalinin and M. Shkolnikov.
Sandpile solitons via smoothing of superharmonic functions.
https://arxiv.org/abs/1711.04285

[6] L. Levine, W. Pegden, and C. K. Smart. Apollonian structure in the Abelian sandpile. Geom. Funct. Anal., 26(1):306–336, 2016.

[7] T. Sadhu and D. Dhar. Pattern formation in fast-growing sandpiles. Phys Rev.E, 85(2):021107, 2012

[8] A. Sportiello.
Hydrodynamic behaviour of an abelian sandpile model with laplacian rules.
arXiv:1207.5769, 2012 https://arxiv.org/abs/1207.5769

[9] Nikita Kalinin, Aldo Guzmán-Sáenz, Yulieth Prieto, Mikhail Shkolnikov, Vera Kalinina, Ernesto Lupercio.
Self-Organized Criticality and Pattern Emergence through the lens of Tropical Geometry.
https://arxiv.org/abs/1806.09153

[10] S. Moghimi-Araghi, M. A. Rajabpour, S. Rouhani.
Abelian Sandpile Model: a Conformal Field Theory Point of View, Nucl. Phys. B718 (2005) 362-370.
https://arxiv.org/abs/cond-mat/0410434

[11] Alexander E. Holroyd, Lionel Levine, Karola Meszaros, Yuval Peres, James Propp, David B. Wilson.
Chip-Firing and Rotor-Routing on Directed Graphs,
https://arxiv.org/abs/0801.3306

[12] Sandpiles for Android.
https://androidappsapk.co/detail-sandpiles/

[13] Jordan Ellenberg,
The Amazing, Autotuning Sandpile. Nautilus, April 2, 2015
http://nautil.us/issue/23/dominoes/the-amazing-autotuning-sandpile

[14] Tridib Sadhu and Deepak Dhar.
Identity configurations in sandpile models http://www.theory.tifr.res.in/ tridib/work.html

[15] Anne Fey, Lionel Levine, Yuval Peres.
Growth rates and explosions in sandpiles,
https://arxiv.org/abs/0901.3805

[16] Lionel Levine and James Propp.
What is ... a sandpile?
https://www.ams.org/notices/201008/rtx100800976p.pdf

[17] Nikita Kalinin, Mikhail Shkolnikov.
Tropical curves in sandpiles.
https://arxiv.org/abs/1509.02303

[18] Qui-Ming Cheng, Hong-Yun Sun.
Variation of singularity of earthquake-size distribution with respect to tectonic regime. Geoscience Frontiers, Volume 9, Issue 2, March 2018, Pages 453-458.

[19] Andrei Constantin, Yang-Hui He, Andre Lukas.
"Counting String Theory Standard Models,"
arXiv:1810.00444 [hep-th].

[20] MacMahon, Percy A. (1896).
"XVI. Memoir on the theory of the partition of numbers.-Part I". Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. 187: Article 52.

[21] Andrei Okounkov, Nikolai Reshetikhin.
Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram.
https://arxiv.org/abs/math/0107056

[22] H. N. V. Temperley. In Combinatorics: Being the Proceedings of the Conference on Combinatorial Mathematics held at the Mathematical Institute, Oxford, pages 356–357, 1972.

[23] Richard W. Kenyon, James G. Propp, David B. Wilson.
Trees and Matchings. Electronic Journal of Combinatorics, 7(1):R25, 2000.
https://arxiv.org/abs/math/9903025

[24] David Eppstein.
Twenty Proofs of Euler's Formula: V-E+F=2.
https://www.ics.uci.edu/ eppstein/junkyard/euler/

[25] Wangru Sun.
Toroidal Dimer Model and Temperley's Bijection.
https://arxiv.org/abs/1603.00690

[26] Sandpiles - Numberphile.
https://www.youtube.com/watch?v=1MtEUErz7Gg

[27] Andrei Okounkov, Nikolai Reshetikhin, Cumrun Vafa.
Quantum Calabi-Yau and Classical Crystals. https://arxiv.org/abs/hep-th/0309208

[28] Juan Maldacena.
Towards Wholeography. www.youtube.com/watch?v=L7BXSgSFXuo

[29] Edward Witten.
A few remarks on AdS/CFT. https://www.youtube.com/watch?v=JbxXc2lDpyo

[30] Cumrun Vafa.
The Missing corner. https://www.youtube.com/watch?v=tv1f2KSGkd4

[31] A. Iqbal, N. Nekrasov, A. Okounkov, C. Vafa.
Quantum Foam and Topological Strings.
https://arxiv.org/abs/hep-th/0312022

[32] Thiago Milanetto Schlittler.
PhD. Thesis
Study of quantum dimer and partition models on honeycomb lattices.

[33] R. Gopakumar, C. Vafa,
"On the Gauge Theory/Geometry Correspondence".
Adv.Theor. Math. Phys. 3 (1999) 1415-1443, hep-th/9811131.

[34] Find the identity sandpile.
Programming Puzzles & Code Golf Stack Exchange.
https://codegolf.stackexchange.com/questions/106963/find-the-identity-sandpile

[35] Build a sandpile.
Programming Puzzles & Code Golf Stack Exchange.
https://codegolf.stackexchange.com/questions/92251/build-a-sandpile

[36] Amihay Hanany, Kristian D. Kennaway.
Dimer models and toric diagrams. https://arxiv.org/abs/hep-th/0503149.

[37] Amihay Hanany, Christopher P. Herzog, David Vegh.
Brane Tilings and Exceptional Collections. https://arxiv.org/abs/hep-th/0602041.

[38] Bo Feng,Yang-Hui He, Kristian D. Kennaway and Cumrun Vafa.
Dimer Models from Mirror Symmetry and Quivering Amoebæ.
https://arxiv.org/pdf/hep-th/0511287.pdf

[39] Masahito Yamazaki.
Crystal Melting and Wall Crossing Phenomena.
https://arxiv.org/abs/1002.1709

[40] Ernesto Lupercio, Ramiro López. To appear.

[41] Ernesto Lupercio, Ramiro López. To appear.

[42] Percolation: Slipping through the Cracks.
Feature column. American Mathematical Society.
http://www.ams.org/publicoutreach/feature-column/fcarc-percolation

[43] Tribid Sadhu and Deepak Dhar.
Physical Review E 85, February 2012. Pattern formation in fast-growing sandpiles.

[44] Anne Fey, Lionel Levine, Yuval Peres.
Growth Rates and Explosions in Sandpiles.
https://arxiv.org/abs/0901.3805

[45] Amihay Hanany and Edward Witten.
Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics.
https://arxiv.org/abs/hep-th/9611230

[46] Michael R. Douglas, Gregory Moore.
D-branes, Quivers, and ALE Instantons.
https://arxiv.org/abs/hep-th/9603167

[47] F. Cachazo, B. Fiol, K. Intriligator, S. Katz, C. Vafa.
A Geometric Unification of Dualities.
https://arxiv.org/abs/hep-th/0110028

[48] Jonathan J. Heckman, Cumrun Vafa.
Crystal melting and black holes.
https://arxiv.org/abs/hep-th/0110028

[49] Sebastian Franco, Amihay Hanany, Dario Martelli, James Sparks, David Vegh, Brian Wecht.
Gauge Theories from Toric Geometry and Brane Tilings.
https://arxiv.org/abs/hep-th/0505211

[50] Amihay Hanany, Christopher P. Herzog, David Vegh.
Brane Tilings and Exceptional Collections. https://arxiv.org/abs/hep-th/0602041

[51] Luboš Motl.
Worldsheets and spacetimes: kinship and cross-pollination
https://motls.blogspot.com/2014/05/worldsheets-and-spacetimes-kinship-and.html
Quantum gravity from self-collisions of the configuration space
https://motls.blogspot.com/2019/01/quantum-gravity-from-self-collisions-of.html

[52] Masahito Yamazaki.
Brane Tilings and Their Applications. https://arxiv.org/abs/0803.4474