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Hodge decomposition for absolutely q-convex manifolds

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Descomposición de Hodge para variedades absolutamente q-convexas

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Resumen

Se presenta una exposición esencialmente autocontenida de la teoría clásica de Hodge sobre variedades Riemanianas y Kählerianas compactas sin frontera; para después extender la descomposición de Hodge a variedades q-convexas no compactas. En el primer capítulo se estudian las bases de la geometría compleja. En el segundo se demuestra la descomposición de Hodge y otros teoremas concernientes a la cohomología de variedades Kählerianas. Por último se presenta la noción de espacios q-convexos y se estudian ciertas propiedades de estos.

Abstract

We present an almost-self-contained exposition of the classic Hodge theory for compact Riemannian and Kählerian manifolds with no boundaries. Thereafter, we study Hodge decomposition over non-compact q-convex manifolds. In Chapter 1 we study the foundations of complex geometry. In Chapter 2 we prove the Hodge decomposition and other important theorems concerning the cohomology of kählerian manifolds. Finally, in the last chapter we expose the notions of q-convex spaces and certain properties of them.

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Introduction

The goal of this work is to give an extensive almost self contained and detailed exposition of the principal cohomology groups in Kähler manifolds. All the manifolds are supposed to be compact in the first part of the thesis; and in the second part we ask for additional properties in order to obtain similar results without the compactness assumption.

Consider a Riemannian manifold X and a Euclidean or Hermitian bundle E over X. We assume that E is equipped with a connection D compatible with the metric. In particular, recall that a connection is a differential operator analogous to the exterior differentiation acting on forms of arbitrary degree and with values in E. The most important point is that any connections satisfies and which satisfies the Leibniz rule for the exterior product. The *Laplace-Beltrami operator* is then defined as the self-adjoint differential operator of second order $\Delta_E = D_E D_E^* + D_E^* D_E$, where D_E^* is the Hilbert space adjoint of D_E. One easily shows that Δ_E is an *elliptic operator*. The finiteness theorem for elliptic operators then shows that the space $\mathscr{H}^q(X, E)$ of harmonic q-forms with values in E is finite dimensional, if X is compact (we say that a form u is harmonic if $\Delta u = 0$). If we assume in addition that the connection satisfies $D_E^2 = 0$, the operator D_E acting on forms of all degrees defines a complex called the *de Rham complex* with values in the local system of coefficients defined by E. The corresponding cohomology groups will be denoted by $H^q_{DR}(X, E)$. The fundamental observation of Hodge theory is that any cohomology class contains a unique harmonic representative element, because X is compact. It then leads to an isomorphism, called the *Hodge isomorphism*

$$H^q_{DR}(X, E) \cong \mathscr{H}^q_{DR}(X, E) \tag{1}$$

When the manifold X and the bundle E are holomorphic, there exists a canonical connection D_E called the *Chern connection*, which is compatible with the Hermitian metric on E and has the following properties: D_E splits into a sum $D_E = D'_E + D''_E$ of a connection D'_E of type $(1, 0)$ and a connection D''_E of type $(0, 1)$, such that $D'^2_E = D''^2_E = 0$ and $D'_E D''_E + D''_E D'_E = \Theta(E)$ (the Chern curvature tensor of the bundle). The operator D''_E acting on the forms of bidegree (p, q) then defines for fixed p, a complex called the *Dolbeault complex*. When X is compact, the Dolbeault cohomology groups $H^{p,q}(X, E)$

satisfy a Hodge isomorphism analogous to (1), namely

$$H^{p,q}(X,E) \cong \mathscr{H}^{p,q}(X,E), \tag{2}$$

where $\mathscr{H}^{p,q}(X,E)$ denotes the space of harmonic (p,q)-forms with values in E, relative to the anti-holomorphic Laplacian $\Delta''_E = D''_E D''^*_E + D''^*_E D''_E$. By using this latter result, one easily proves the *Serre duality theorem*

$$H^{p,q}(X,E)^* = H^{n-p,n-q}(X,E^*), \quad n = \dim_{\mathbb{C}} X, \tag{3}$$

which is the complex version of the Poincaré duality theorem. The central theorem of Hodge theory concerns compact Kähler manifolds: A Hermitian manifold (X,ω) is called *Kählerian* if the Hermitian $(1,1)$-form $\omega = i\sum_{j,k}\omega_{jk}\,\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ satisfies $\mathrm{d}\omega = 0$. A fundamental example of a compact Kählerian manifold is given by the projective algebraic manifolds. If X is compact Kählerian and if E is a local system of coefficients on X, the *Hodge decomposition theorem* asserts that

$$H^k_{DR}(X,E) = \bigoplus_{p+q=k} H^{p,q}(X,E) \qquad \text{(Hodge decomposition)} \tag{4}$$

$$\overline{H^{p,q}(X,E)} \cong H^{q,p}(X,E^*) \qquad \text{(Hodge symmetry)} \tag{5}$$

The intrinsic character of these decompositions will be shown in this work, via the the Bott-Chern cohomology groups (also called $\partial\overline{\partial}$-cohomology groups). Different cohomological properties of compact Kähler manifolds are obtained by means of the primitive decomposition and the Hard Lefschetz theorems (which is in turn the result of the existence of an $\mathfrak{sl}(2)$ action on harmonic forms)

In the second part of the thesis, we give a brief exposition of plurisubharmonic functions. We firstly present and study the harmonic and subharmonic functions and its properties. We also introduce the concept of a domain of holomorphy and give a list of equivalent characterizations. The main objective is to define the the domains of holomorphy D as those which has a plurisubharmonic exhausting function.

Then, we introduce the concepts of a ringed space and of a complex model space, in order to define a complex space. We discuss certain properties of these spaces.

Next, we introduce the concept of a strongly q-complete subvariety of a complex analytic space and we show that they have a fundamental system of strongly q-complete neighborhoods. As a consequence, a Demailly's proof of Ohsawa's result is presented: every non compact irreducible n-dimensional analytic space is strongly n-complete. Finally, it is shown that L^2-cohomology theory readily implies both, Ohsawa's Hodge decomposition and the Lefschetz isomorphism theorems for absolutely q-convex manifolds.

Certain methods exposed here are widely used nowadays in research in the field of complex geometry, such as the modifications of Kähler metrics and use of the geometry of the manifolds to obtain properties in the cohomology. One is in particular interested in deducing vanishing theorems. So we intent here to give a background to the use of these techniques for further research in the area of complex geometry.

Chapter 1

Preliminaries

In this chapter, we introduce and study the notion of a complex structure on a differentiable manifold. A complex manifold X of (complex) dimension n is a differentiable manifold locally equipped with a complex-valued coordinates (called holomorphic coordinates) $z_1, \ldots, z_n$, such that the diffeomorphisms from an open set of $\mathbb{C}^n$ to an open set of $\mathbb{C}^n$ given by coordinate changes are biholomorphic.

1.1 Manifolds and vector bundles

A topological manifold is a topological space X equipped with a covering by open sets U_i, which are homeomorphic, via maps ϕ_i called local charts, to open sets of $\mathbb{R}^n$. One can show [Mil65] that such an n is necessarily independent of the index i when X is connected; n is then called the dimension of X.

Definition 1.1.1. *A $\mathscr{C}^k$ differentiable manifold is a topological manifold equipped with a system of local charts $\phi_i : U_i \to \mathbb{R}^n$ such that the open sets U_i cover X,each ϕ_i is bijective onto its image, and the change of chart morphisms*

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are all differentiable bijections of class $\mathscr{C}^k$.

Definition 1.1.2. *A $\mathscr{C}^k$ differentiable function on such a manifold, or on a open set, is any function f such that for each U_i, the composition $f \circ \phi_i^{-1}$ is differentiable of class $\mathscr{C}^k$.*

A real, respectively complex, topological vector bundle of rank m over a topological space X is a topological space E equipped with a map $\pi : E \to X$ such that there exists an open cover $\{U_i\}$ of X, where we have "local trivialisation" homeomorphisms

$$\tau_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{R}^m (\text{ resp. } U_i \times \mathbb{C}^m)$$

such that $\{\pi^{-1}(U_i)\}$ is an open cover of E and:

1. We have
$$\mathrm{pr}_1 \circ \tau_i = \pi$$
on $\pi^{-1}(U_i)$, where $\mathrm{pr}_1(x, y) = x$ is the projection on the first coordinate.

2. The transition functions
$$\tau_j \circ \tau_i^{-1} : \tau_i(\pi^{-1}(U_i \cap U_j)) \to \tau_j(\pi^{-1}(U_i \cap U_j))$$
are continuous on the first variable and $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) on the second variable, i.e. on each fibre $u \times \mathbb{R}^m$, (res. $u \times \mathbb{C}^m$). Such a transformation
$$U_i \cap U_j \times \mathbb{R}^m \to U_i \cap U_j \times \mathbb{R}^m$$
must respect the first projection, by condition 1 above, and thus the second entry is described by a real matrix of type (m, m), whose coefficients, by continuity, are continuous functions of the first variable $u \in U_i \cap U_j$. (In the complex case, we must consider complex matrices.) These matrices are called transition matrices.

Definition 1.1.3. *Given a $\mathscr{C}^k$ differentiable manifold X, a vector bundle E over X is equipped with a $\mathscr{C}^k$ differentiable structure if and only if we are can give local trivialisations whose transition matrices are $\mathscr{C}^k$.*

Remark 1.1.4. *The bundle E is then equipped with the structure of a $\mathscr{C}^k$ manifold for which π is $\mathscr{C}^k$, as well as the local trivialisations τ_i.*

A section of a vector bundle $E \xrightarrow{\pi} X$ is a map $\sigma : X \to E$ such that $\pi \circ \sigma = \mathrm{Id}_x$. This section is said to be continuous, resp. differentiable, or $\mathscr{C}^k$ differentiable, if σ is so. If $\pi : E \to X$ is a vector bundle and $x \in X$, we write $E_x := \pi^{-1}(x)$ and it is called the fibre over x. It is canonically a vector space with a structure given by any of the trivialisations of E in the neighbourhood of x.

A vector bundle $\pi : E \to X$ is said to be trivial if it admits a global trivialisation $\phi : E \cong X \times \mathbb{R}^n$. Equivalently, E must admit n global sections which provide a basis of the fibre E_x at each point $x \in X$. These sections are given by $\sigma = \phi^{-1} \circ \hat{e}_i$, where $\hat{e}_i : X \to X \times \mathbb{R}^n$ is given by $\hat{e}_i(x) = (x, e_i)$, where the e_i form the standard basis of $\mathbb{R}^n$. Let U be an open subset of X. A frame for E over U is a set of n sections $\{s_1, \ldots, s_n\}$, such that $\{s_1(x), \ldots, s_r(x)\}$ is a basis for E_x for any $x \in U$. Any vector bundle E admits a frame in some neighbourhood of any given point in the base space, constructed just as before for the global sections but in a given trivialisation.

Definition 1.1.5. *Let $E \to M$ be a vector bundle. An Euclidean metric (resp. Hermitian) of class $\mathscr{C}^\infty$ over E is an inner product (resp. hermitian inner product) on each fiber E_x of E, varying smoothly with $x \in M$. i.e. such that if $e = (e_1, \dots, e_k)$ is a frame for E, then the functions*

$$h_{ij}(x) = (e_i(x), e_j(x))$$

are $\mathscr{C}^\infty$. A frame e for E is called unitary if $e_1(x), \dots, e_k(x)$ is an orthonormal basis for E_x for each x; unitary frames always exist locally, since we can take any frame and apply the Gram-Schmidt process.

Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be vector bundles over X. A morphism $\phi : E \to F$ of vector bundles is a continuous map such that $\pi_F \circ \phi = \pi_E$, and ϕ is linear on each fibre. This means that in local trivialisations, ϕ becomes linear ($\mathbb{C}$-linear in the case of complex bundles) on the fibres $u \times \mathbb{R}^m$; this definition is independent of the choice of the open set containing $u \in X$, since the transition functions are also linear on the fibres. We have an analogous definition for differentiable bundles, but in this case ϕ is also $\mathscr{C}^k$.

Given a vector bundle E, we can define its dual E^* and its exterior powers $\bigwedge^k E$, which are differentiable of the same class as E. The points of E^* are the linear forms acting on the fibres of $\pi_E : E \to X$. The vector bundle E^* admits a natural trivialisation when E is trivialised, this is

$$\tau_i^* : \pi_{E^*}^{-1}(U_i) \cong U_i \times (\mathbb{R}^m)^* \text{ (resp. } U_i \times (\mathbb{C}^m)^*)$$

with the same open cover $\{U_i\}$. The transition matrices of E^* are the inverses of the transposes of the transitions matrices of E. Similarly, the points of $\bigwedge^k E$ can be identified with the alternating k-linear forms acting on the fibres of $\pi_{E^*} : E^* \to X$.

1.2 The tangent bundle

If X is a $\mathscr{C}^k$ differentiable manifold, the tangent bundle T_X of X is a $\mathscr{C}^{k-1}$ differentiable bundle of rank $n = \dim X$ which we can define as follows. If X is covered by open sets U_i equipped with $\mathscr{C}^k$ diffeomorphisms ϕ_i to open sets of $\mathbb{R}^n$, then T_X is covered by open sets $U_i \times \mathbb{R}^n$, where the identifications (or transition morphisms) between $U_i \cap U_j \times \mathbb{R}^n \subset U_i \times \mathbb{R}^n$ and $U_i \cap U_j \times \mathbb{R}^n \subset U_j \times \mathbb{R}^n$ are given by

$$(u, v) \mapsto (u, \phi_{ij_*}(v)).$$

Here $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is the transition diffeomorphism between the open sets $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ of $\mathbb{R}^n$, and ϕ_{ij_*} is its Jacobian matrix at the point u. A section of the tangent bundle of a differentiable manifold is called a vector field.

There exist two intrinsic ways of describing the elements of the tangent bundle. The points of the tangent bundle can be identified with equivalence classes of differentiable maps $\gamma : [-\epsilon, \epsilon] \to X$ (for an $\epsilon \in \mathbb{R}, \epsilon > 0$ varying with γ) for the equivalence relation

$$\gamma_1 \equiv \gamma_2 \iff \gamma_1(0) = \gamma_2(0), \frac{d}{dt}\gamma_1|_{t=0} = \frac{d}{dt}\gamma_2|_{t=0}.$$

The second equality in this definitions makes sense in any local chart for the neighbourhood of $\gamma(0)$. We call these equivalence classes "jets of order 1". To check that the set defined in this way has the structure of the vector bundle introduced earlier, it suffices to note that the jets of order 1 of an open set U of $\mathbb{R}^n$ can be identified, via the map $\gamma \mapsto (\gamma(0), \dot{\gamma}(0))$, with $U \times \mathbb{R}^n$, and that a diffeomorphism $\psi : U \cong V$ between two open sets of $\mathbb{R}^n$ induces the isomorphism (ψ, ψ_*) between the spaces of jets of order 1 of U and V.

Another definition of the tangent vectors, i.e. of the elements of the tangent bundle, consists in identifying them with the derivations of the algebra of the real differentiable functions on X with values in $\mathbb{R}$ supported at a point $x \in X$. This means that we consider the linear maps

$$\psi : \mathscr{C}^1(X) \to \mathbb{R}$$

satisfying Leibniz rule

$$\psi(fg) = f(x)\psi(g) + g(x)\psi(f)$$

for a point $x \in X$. The equivalence between the two definitions is realised by the map which to a jet γ associates the derivation $\psi_\gamma(f) = \frac{d(f\circ\gamma)}{dt}|_{t=0}$.

Definition 1.2.1. *A differential form of degree k is a section of $\bigwedge^k(T_X)^*$, the k-antisymmetric linear functions defined on the cotangent bundle.*

In general, we write $\bigwedge^1 T^*_{X,\mathbb{R}}$ for the bundle of real differential 1-forms, and $\bigwedge^1 T^*_{X,\mathbb{C}} = \mathrm{Hom}(T_X, \mathbb{C})$ for its complexification, this is its tensor product with $\mathbb{C}$. Similarly, the bundle of real (resp. complex) k-forms is written $\bigwedge^k T^*_{X,\mathbb{R}}$ (resp. $\bigwedge^k T^*_{X,\mathbb{C}}$). We see immediately that if f is a real $\mathscr{C}^k$ differentiable function on X, then df is a $\mathscr{C}^{k-1}$ section of $\bigwedge T^*_{X,\mathbb{R}}$. We also see that if $x_1, \ldots, x_n$ are local coordinates defined on an open set $U \subset X$, we have for any $x \in U$, the n derivations

$$\frac{\partial}{\partial x_1}\Big|_x, \ldots, \frac{\partial}{\partial x_n}\Big|_x \text{ defined by } \frac{\partial}{\partial x_i}\Big|_x f = \frac{\partial f}{\partial x_i}(x)$$

these form a basis for the fibre T_X at x, then the $dx_I = dx_{x_1} \wedge \ldots \wedge dx_{i_k}, 1 \le i_1 < \ldots < i_k \le n$ provide a basis of the fibre of $\bigwedge^1 T^*_{X,\mathbb{R}}$, at each point of the open set U. Indeed by the definition of T_X, the coordinates x_i provide a local trivialisation of T_X, where the corresponding local basis is given at each point $x \in U$ by the derivations $\frac{\partial}{\partial x_i}|_x$. The dx_i simply form the dual basis of $\bigwedge^1 T^*_{X,\mathbb{R}}$ at each point of U.

1.3 Complex manifolds

Definition 1.3.1. *Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be a continuously differentiable function. Then f is said to be holomorphic if*

$$\frac{\partial f}{\partial \bar{z}_i} = 0 \text{ for } i = 1, \dots, n.$$

Let X be a differentiable manifold of dimension $2n$.

Definition 1.3.2. *We say that X is equipped with a complex structure if X is covered by open sets U_i which are diffeomorphic, via maps called ϕ_i, to open sets of $\mathbb{C}^n$, in such a way that the transition diffeomorphisms*

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \circ U_j)$$

are holomorphic functions.

The (complex) dimension of X is by definition equal to n. On a complex manifold, a map f with values in $\mathbb{C}$ defined on an open set U is said to be holomorphic if $f \circ \phi_i^{-1}$ is holomorphic on $\phi_i(U \cap U_i)$. Once again, this definition does not depend on the choice of chart, since the change of chart morphisms is holomorphic and compositions of holomorphic functions are also holomorphic.

We will give some examples

Example 1.3.3. *The complex projective space $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ is the most important compact complex manifold. By definition, $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$ or equivalently*

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}^{n+1}$. The points of $\mathbb{P}^n$ are written as $(z_o : z_1 : \dots : z_n)$. Here, the notation intends to indicate that for a $\lambda \in \mathbb{C}^*$ the two points $(\lambda z_0 : \lambda z_1 : \dots : \lambda z_n)$ and $(z_0 : z_1 : \dots : z_n)$ define the same point in $\mathbb{P}^n$. Only the origin $(0, \dots, 0)$ does not define a point in $\mathbb{P}^n$.

The standard open covering of $\mathbb{P}^n$ is given by the $n+1$ open subsets

$$U_i := \{(z_0 : \dots : z_n) | z_i \neq 0\} \subset \mathbb{P}^n.$$

If $\mathbb{P}^n$ is endowed with the quotient topology via

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^n,$$

then the U_i's are indeed open.

Consider the bijective maps

$$\phi_i : U_i \longrightarrow \mathbb{C}^n, (z_0 : \ldots : z_n) \mapsto (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i})$$

which is well defined. For the transition maps $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_I \cap U_j) \to \phi_i(U_i \cap U_j)$ *one has*

$$\phi_{ij}(w_1, \ldots, w_n) = \left(\frac{w_1}{w_i} \ldots, \frac{w_{i-1}}{w_i} . \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \ldots, \frac{w_n}{w_i} \right).$$

Note that $\phi_j(U_i \cap U_j) = \mathbb{C}^n \setminus Z(w_i)$*, where* $Z(w_i)$ *is the set where* w_i *equals zero. These maps are obviously bijective and holomorphic.*

There is a more elegant way to describe the transition functions. Namely, we may identify $\phi_i(U_i)$ *with the affine subspace* $\{(z_0, \ldots, z_n) | z_i = 1\} \subset \mathbb{C}^{n+1}$*. Then* $\phi_j(U_i \cap U_j) = \{(z_0, \ldots, z_n) | z_j = 1, z_i \neq 0\}$ *and* $\phi_{ij}(z_0, \ldots, z_n) = z_i^{-1} \cdot (z_0, \ldots, z_n)$*.*

Example 1.3.4 (Complex tori)**.** *Let* X *be the quotient* $\mathbb{C}^n/\mathbb{Z}^{2n}$*, where* $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ *is the natural inclusion. Then* X *can be endowed with the quotient topology of* $\pi : \mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}^{2n} = X$*. If* $U \subset \mathbb{C}^n$ *is a small open subset such that* $(U + (a_1 + ib_1, \ldots, a_n + ib_n)) \cap U = \varnothing$ *for all* $0 \neq (a_1, b_1, \ldots, a_n, b_n) \in \mathbb{Z}^{2n}$*, then* $U \to \pi(U)$ *is bijective. Covering* X *by those provides a holomorphic atlas of* X*. The transition functions are just translations by vectors in* $\mathbb{Z}^{2n}$*. Explicitly, if* $z \in \mathbb{C}^n$*, then the polydis* $U = B_\varepsilon(z)$ *with* $\varepsilon = (1/2, \ldots, 1/2)$ *has the above property.*

We can also define the notion of a holomorphic vector bundle.

Definition 1.3.5. *A differentiable complex vector bundle, this is that its fibres are* $\mathbb{C}$*-vector spaces,* $\pi_E : E \to X$ *over a complex manifold* X *is said to be equipped with a holomorphic structure if we have trivialisations*

$$\tau_i : \pi_i^{-1}(U_i) \cong U_i \times \mathbb{C}^n$$

such that the transition matrices $\tau_{ij} = \tau_j \circ \tau_i^{-1}$ *have holomorphic coefficients.*

The above trivialisations will be called "holomorphic trivialisations". If E is a holomorphic vector bundle, E is in particular a complex manifold such that the projection π_E is holomorphic. Indeed we can assume, in the definition above, that the sets U_i are charts, i.e. identified via ϕ_i with open sets of $\mathbb{C}^n$; then the $(\phi_i \times \mathrm{Id}_{\mathbb{C}^n}) \circ \tau_i$ give charts for E whose transition functions are clearly holomorphic.

A holomorphic section of a holomorphic vector bundle $\pi_E : E \to X$ over an open set U of X is a section $s : X \to E$ of π_E which is a holomorphic map. For example, a holomorphic local trivialisation τ_i of E as above is given by the choice of a family of holomorphic sections of E, whose values at each point u of U_i form a basis of the fibre E_u over $\mathbb{C}$.

Example 1.3.6 (The holomorphic tangent bundle)**.** *This bundle is defined exactly like the real tangent bundle of a differentiable manifold. Given a system of charts $\phi : U_i \cong V_i \subset \mathbb{C}^n$, we define T_X as the union of the $U_i \times \mathbb{C}^n$, glued by identifying $U_i \cap U_j \times \mathbb{C}^n \subset U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n \subset U_j \times \mathbb{C}^n$ via*

$$(u, v) \mapsto (u, \phi_{ij_*}(v)).$$

Here the holomorphic Jacobian matrix ϕ_{ij_*} is the matrix with holomorphic coefficients $\frac{\partial \phi^k_{ij}}{\partial z_l}(u)$, where $\phi_{ij} = \phi_j \circ \phi_i^{-1}$, and the operator $\frac{\partial}{\partial z_l}$ is defined as

$$\frac{\partial}{\partial z_l} = \frac{1}{2}(\frac{\partial}{\partial x_l} - i\frac{\partial}{\partial y_l}).$$

We can also, as in section 1.2, define the holomorphic tangent bundle as the set of complex-valued derivations of the $\mathbb{C}$-algebra of holomorphic functions, or as the set of jets of order 1 of holomorphic maps from the complex disk to X.

1.4 Integrability of almost complex structures

In the following, V shall denote a finite-even-dimensional real vector space.

Definition 1.4.1. *An endomorphism $I : V \to V$ with $I^2 = -\operatorname{Id}$ is called an almost complex structure on V.*

Clearly, if I is an almost complex structure then $I \in \mathrm{Gl}(V)$, the general linear group, this is I is invertible. If V is the real vector space underlying a complex vector space then $v \mapsto i \cdot v$ defines an almost complex structure I on V in the following way, if $z_j = x_j + iy_j$ then

$$i \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -y_1 \\ x_1 \\ \vdots \\ -y_n \\ x_n \end{pmatrix}$$

. The converse holds true as well:

Lemma 1.4.2. *If I is an almost complex structure on a real vector space V, then V admits in a natural way the structure of a complex vector space.*

Proof. The $\mathbb{C}$-module structure on V is defined by $(a + ib) \cdot v = a \cdot v + b \cdot I(v)$, where $a, b \in \mathbb{R}$. The $\mathbb{R}$-linearity of I and the assumption $I^2 = -\operatorname{Id}$ yield $((a + ib)(c + id)) \cdot v = (a + ib)((c + id) \cdot v)$ and in particular $i(i \cdot v) = -v$. □

Thus, almost complex structures and complex structures are equivalent notions for vector spaces. In particular, an almost complex structure can only exist on an even dimensional real vector space.

Corollary 1.4.3. *Any almost complex structure on V induces a natural orientation on V.*

Proof. Using the lemma, the assertion reduces to the statement that the real vector space $\mathbb{C}^n$ admits a natural orientation. We may assume $n = 1$ and use the orientation given by the basis $(1, i)$. The orientation is well-defined, as it does not change under $\mathbb{C}$-linear automorphisms. □

For a real vector space V the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is denoted by $V_{\mathbb{C}}$ and it is called the complexification of V. Thus, the real vector space V is naturally contained in the complex vector space $V_{\mathbb{C}}$ via the map $v \mapsto v \otimes 1$. V is then called the real part of the complexification $V_{\mathbb{C}}$. Moreover, $V \subset V_{\mathbb{C}}$ is the part that is left invariant under complex conjugation on $V_{\mathbb{C}}$, which is defined by $(v \bar{\otimes} \lambda) := v \otimes \overline{\lambda}$ for all $v \in V$ and $\lambda \in \mathbb{C}$.

Suppose that V is endowed with an almost complex structure I, then we will also denote by I its $\mathbb{C}$-linear extension to an endomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$. Clearly the only eigenvalues of I on $V_{\mathbb{C}}$ are $\pm i$, because the identity $I^2 = -\operatorname{Id}$ still holds in $V_{\mathbb{C}}$.

Definition 1.4.4. *Let I be an almost complex structure on a real vector space V and let $I : V_{\mathbb{C}} \to \mathbb{C}$ be its $\mathbb{C}$-linear extension. Then the $\pm i$ eigenspaces are denoted $V^{1,0}$ and $V^{0,1}$, respectively; i.e.,*

$$V^{1,0} = \{v \in V_{\mathbb{C}} | I(v) = i \cdot v\} \text{ and } V^{0,1} = \{v \in V_C | I(v) = -i \cdot v\}$$

Lemma 1.4.5. *Let V be a real vector space endowed with an almost complex structure I. Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

Complex conjugation on $V_{\mathbb{C}}$ induces an $\mathbb{R}$-linear isomorphism between $V^{1,0}$ and $V^{0,1}$.

Proof. Notice that $V^{1,0} \cap V^{0,1} = 0$, because $v = 0$ is the only vector in $V_{\mathbb{C}}$ which satisfies $iv = -iv$. Hence, the canonical map

$$\begin{aligned} V^{1,0} \oplus V^{0,1} &\to V_{\mathbb{C}} \\ (v, w) &\mapsto v + w \end{aligned}$$

is injective. The first assertion follows from the existence of the inverse map

$$v \mapsto \frac{1}{2}(v - iI(v)) \oplus \frac{1}{2}(v + iI(v)).$$

Because of $I(v \mp iI(v)) = I(v) \pm iv = \pm i(v \mp iI(v))$.

For the second assertion we write $v \in V_{\mathbb{C}}$ as $v = x + iy$ with $x, y \in V$. Then $\overline{(v - iI(v))} = (x - iy + iI(x) + I(y)) = (\overline{v} + iI(\overline{v}))$. Hence, complex conjugation interchanges the two factors. □

One should be aware of the existence of two almost complex structures of $V_{\mathbb{C}}$. One is given by I and the other one by i. They coincide on the subspace $V^{1,0}$ but differ by a sign on $V^{0,1}$. In $V^{1,0}$, we have $I(v - iI(v)) = i(v - iI(v))$, as shown in the last proof. This is, it is equivalent to multiply for i. But as in $V^{0,1}, I$ has eigenvalue $-i$ it will differ for a sign.

Obviously, $V^{1,0}$ and $V^{0,1}$ are complex subspaces of $V_{\mathbb{C}}$ with respect to both almost complex structures. In the sequel, we will always regard $V_{\mathbb{C}}$ as the complex vector space with respect to i. The $\mathbb{C}$-linear extension of I is the additional structure that gives rise to the above decomposition.

Lemma 1.4.6. *Let V be a real vector space endowed with an almost complex structure I. Then the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $I(f)(v) = f(I(v))$. The induced decomposition on $(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$ is given by*

$$(V^*)^{1,0} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) | f(I(v)) = if(v)\} = (V^{1,0})^*$$

$$(V^*)^{0,1} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) | f(I(v)) = -if(v)\} = (V^{0,1})^*$$

Also note that $(V^*)^{1,0} = \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$

The proof is natural. Now we endow this structure to the tangent bundle of a complex manifold.

Lex X be a complex manifold, and let $\phi_i : U_i \to \mathbb{C}^n$ be holomorphic local charts. Then the real tangent bundle $T_{U_i,\mathbb{R}}$ can be identified, via the differential ϕ_{i*}, with $U_i \times \mathbb{C}^n$. Moreover, the change of chart morphisms $\phi_j \circ \phi_i^{-1}$ are holomorphic by hypothesis, i.e. have $\mathbb{C}$-linear differentials, for the natural identifications:

$$T_{\mathbb{C}^n,x} \cong \mathbb{C}^n, \ \forall x \in \mathbb{C}^n$$

It follows that the $\mathbb{R}$-linear operators

$$I_i : T_{U_i.\mathbb{R}} \to T_{U_i,\mathbb{R}},$$

identified with $\mathrm{Id} \times i$ acting on $U_i \times \mathbb{C}^n$, glue together on $U_i \cap U_j$ and define a global endomorphism, written I, of the bundle $T_{X,\mathbb{R}}$. Obviously I satisfies the identity $I^2 = \mathrm{Id} \times (-\mathrm{Id})$; thus I defines an almost complex structure on each fibre $T_{X,x}$, for every fixed point $x \in X$. The differentiability of I even shows that $T_{X,\mathbb{R}}$ is thus equipped with the structure of a differentiable complex vector bundle. This leads us to introduce the following definition.

Definition 1.4.7. *An almost complex structure on a differentiable manifold is an endomorphism I of $T_{X,\mathbb{R}}$ such that $I^2 = -\operatorname{Id}; I^2 = \operatorname{Id}\times(-\operatorname{Id})$; i.e. $I^2(x.v) = (x, -v)$ equivalently, it is the structure of a complex vector bundle on $T_{X,\mathbb{R}}$.*

We saw that a complex structure on X naturally induces an almost complex structure.

Definition 1.4.8. *An almost complex structure I on a manifold X is said to be integrable if there exists a complex structure on X which induces I.*

In the case of a complex manifold, the relation between $T_{X,\mathbb{R}}$, seen as a complex vector bundle, and the holomorphic tangent bundle T_X of X is as follows: the bundle T_X is generated, in the charts U_i, by the elements

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

which are naturally elements of $T_{U_i} \otimes \mathbb{C}$. Thus, in fact, we have and inclusion of complex vector bundles

$$T_X \subset T_{X,\mathbb{R}} \otimes \mathbb{C}.$$

Moreover, for an almost complex manifold (X, I), the complexified tangent bundle $T_{X,\mathbb{R}} \otimes \mathbb{C}$ contains a complex vector subbundle, denoted by $T_X^{1,0}$ and defined as the bundle of eigenvectors of the complex structure I associated to the eigenvalue i. As a real vector bundle, $T_X^{1,0}$ is naturally isomorphic to $T_{X,\mathbb{R}}$ via the application $\Re$ (real part), which to a complex field $u + iv$ associates its real part u. Moreover, this identification relates the operators i on $T_X^{1,0}$ and I on $T_{X,\mathbb{R}}$. Clearly $T_X^{1,0}$ is generated by the $u - iIu$, for all $u \in T_{X,\mathbb{R}}$.

In conclusion, we have shown the following.

Proposition 1.4.9. *If X is a complex manifold, then X admits an almost complex structure I, and the subbundle $T_X^{1,0} \subset T_{X,\mathbb{R}} \otimes \mathbb{C}$ defined by I is equal, as a complex vector subbundle of $T_{X,\mathbb{R}} \otimes \mathbb{C}$, to the holomorphic tangent bundle T_X.*

Complex conjugation acts naturally on the complexified tangent bundle $T_{X,\mathbb{C}}$ of a differentiable manifold X. If I is an almost complex structure on X, we have the subbundle $T_X^{0,1}$ of $T_{X,\mathbb{C}}$, defined as the complex conjugate of $T_X^{1,0}$. We can also define it as the set of the complexified tangent vectors which are the eigenvectors of I associated to the eigenvalue $-i$. Thus, it is clear that we have a direct sum decomposition

$$X, \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}. \tag{1.1}$$

Remark 1.4.10. *When X is an almost complex manifold, the vector bundle $T_X^{1,0}$ does not have a priori the structure of a holomorphic bundle. In what follows, if X is a complex manifold, a section of T_X will be taken to mean a holomorphic section of T_X, while a section of $T_X^{1,0}$ will be a differentiable section.*

A $\mathscr{C}^l$ vector field χ over a manifold X naturally defines a derivation

$$\chi : \mathscr{C}^k(X) \to \mathscr{C}^{k-1}(X),\ k \leq l+1,\ \chi(f) = df(\chi),$$

i.e. a linear map satisfying Leibniz rule: $\chi(fg) = f\chi(g) + g\chi(f)$. As $df \in \bigwedge^1 T^*_{X,\mathbb{R}}$, it acts naturally on χ. Conversely, as explained previously, such a derivation gives a tangent vector at each point of X, and thus a vector field which is easily shown to be $\mathscr{C}^{k-1}$. This enable us to define the Lie bracket of two $\mathscr{C}^l$ fields, thanks to the following elementary lemma.

Lemma 1.4.11. *Let χ, ψ be two derivations*

$$\chi, \psi : \mathscr{C}^{l+1}(X) \to \mathscr{C}^l(X),\ l \geq 1$$

Then the commutator

$$\chi \circ \psi - \psi \circ \chi : \mathscr{C}^2(X) \to \mathscr{C}^0(X)$$

is again a derivation.

Proof. For arbitrary $f, g \in \mathscr{C}^2$, we compute

$$\begin{aligned}(\chi \circ \psi - \psi \circ \chi)(fg) =& \chi\big(\psi(fg)\big) - \psi\big(\chi(fg)\big)\\ =& \chi\big(f\psi(g) + g\psi(f)\big) - \psi\big(f\psi(g) + g\psi(f)\big)\\ =& \chi(f)\psi(g) + f\chi\psi(g) + \chi(g)\psi(f) + g\chi\psi(f)\\ & - \psi(f)\chi(g) - f\psi\chi(g) - \psi(g)\chi(f) - g\psi\chi(f)\\ =& f\chi\psi(g) + g\chi\psi(f) - f\psi\chi(g) - g\psi\chi(f)\\ =& f\big(\chi \circ \psi - \psi \circ \chi\big)(g) + g\big(\chi \circ \psi - \psi \circ \chi\big)(f)\end{aligned}$$

□

Thus we can give the following definition.

Definition 1.4.12. *The bracket $[\chi, \psi]$ of the vector fields χ, ψ is the vector field corresponding to the derivation $\chi \circ \psi - \psi \circ \chi$.*

In local coordinates x_i on X, the vector field χ can be written uniquely as $\chi = \sum_i \chi_i \frac{\partial}{\partial x_i}$, and we have a similar expression for the vector field $\psi = \sum_i \psi_i \frac{\partial}{\partial x_i}$.

Lemma 1.4.13. *We have the formula*

$$[\chi, \psi] = \sum_i \big(\chi(\psi_i) - \psi(\chi_i)\big) \frac{\partial}{\partial x_i}$$

Proof. We must check that for a $\mathscr{C}^2$ function f, we have

$$[\chi, \psi](f) = \sum_i \big(\chi(\psi_i) - \psi(\chi_i)\big)\frac{\partial f}{\partial x_i}.$$

But,

$$\psi(f) = df(\psi) = \big(\sum_i \frac{\partial f}{\partial x_i} dx_i\big)\big(\sum_j \psi_j \frac{\partial}{\partial x_j}\big) = \sum_{i,j} \psi_j \frac{\partial f}{\partial x_i} dx_i\big(\frac{\partial}{\partial x_j}\big) = \sum_i \psi_i \frac{\partial f}{\partial x_i}$$

and

$$\chi(f) = \sum_i \chi_i \frac{\partial f}{\partial x_i}$$

so we obtain

$$\chi \circ \psi(f) = \sum_{j,i} \chi_i \frac{\partial}{\partial x_i}\big(\psi_j \frac{\partial f}{\partial x_j}\big) = \sum_{j,i} \chi_i \big(\frac{\partial \psi_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \psi_j \frac{\partial^2 f}{\partial x_i \partial x_j}\big).$$

and a similar expression for $\psi \circ \chi(f)$. The symmetry of the second derivatives then gives

$$[\chi, \psi](f) = \sum_{i,j} \big(\chi_i \frac{\partial \psi_j}{\partial x_i} - \psi_i \frac{\partial \chi_j}{\partial x_i}\big)\frac{\partial f}{\partial x_j}.$$

□

The following is an immediate consequence of lemma 1.4.13.

Corollary 1.4.14. *If χ, ψ are two $\mathscr{C}^1$ vector fields and f is a $\mathscr{C}^1$ function, all of which are differentiable, then*

$$[\chi, f\psi] = f[\chi, \psi] + \chi(f)\psi.$$

Proof. By 1.4.13 we have

$$\begin{aligned}[\chi, f\psi] &= \sum_i \big(\chi(f\psi_i) - f\psi(\chi_i)\big)\frac{\partial}{\partial x_i} \\ &= \sum_i \big(f\chi(\psi_i) - f\psi(\chi_i) + \psi_i\chi(f)\big)\frac{\partial}{\partial x_i} \\ &= f[\chi, \psi] + \chi(f)\psi\end{aligned}$$

□

In the following we consider X as a differentiable manifold, unless additional information is explicitly stated, therefore T_X will refer to the tangent bundle of this manifold.

Definition 1.4.15. *If X and Y are differentiable manifolds and $\phi : X \to Y$ is a differentiable map, for each $x \in X$ we define a map*

$$d\phi : T_{X,x}X \to T_{Y,\phi(x)}Y$$

called the differential of ϕ at x, as follows: given $v \in T_{X,x}X$, we let $d\phi_x(v)$ be the derivation at $\phi(x)$, in the sense of tangent vectors, that acts on $f \in \mathscr{C}^k(Y)$ by the rule

$$d\phi_x(v)(f) = v(f \circ \phi)$$

Definition 1.4.16. *If $\phi : X \to Y$ is a differentiable map and $v(y) = \sum v_J(y)dy_J$ is a differential p-form on Y, the pull-back ϕ^*v is the differential p-form on X obtained after making the substitution $y = \phi(x)$ in v, i.e.*

$$\phi^*v(x) = \sum v_I(\phi(x))d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_p}$$

If we have a second map $\psi : Y \to Y'$ and if w is a differentiable form on Y', then $\phi^*(\psi^*w)$ is obtained by means of the substitutions $z = \psi(y)$, $y = \phi(x)$, thus

$$\phi^*(\psi^*w) = (\phi \circ \psi)^*w$$

Moreover, we always have $d(\phi^*v) = \phi^*(dv)$

Definition 1.4.17. *Let X be an n-dimensional manifold, and let $E \subset T_X$ be a $\mathscr{C}^1$ vector subbundle of rank k, its rank as vector bundle. Such and E is called a distribution on X. We say that the distribution E is integrable if X is covered by open sets U such that there exists a $\mathscr{C}^1$ map*

$$\phi_U : U \to \mathbb{R}^{n-k}$$

such that for every $x \in U$, the vector subspace $E_x \subset T_{X,x}$ is equal to $\operatorname{Ker} d\phi_x$

As the differential is linear, therefore the image of l.i vectores are l.i., hence ϕ is a submersion, and each fibre $\phi^{-1}(v)$ is a closed submanifold of U having the property that its tangent space at each point is equal to the fibre of E at that point. The following theorem characterises the integrable distributions. As the proof would lead as astray, we omit it, and refer to [Voi02].

Theorem 1.4.18 (Frobenius)**.** *A distribution E is integrable if and only if for all $\mathscr{C}^1$ vector fields χ, ψ contained in E, the bracket $[\chi, \psi]$ is also contained in E.*

Note firstly that the bracket of vector fields over a differentiable manifold X extends by $\mathbb{C}$- linearity to the complexified vector fields, i.e. to the differentiable seccions of $T_{X,\mathbb{C}}$. Now, let (X, I) be an almost complex manifold. As mentioned before, the almost complex structure operator I splits the bundle $T_{X,\mathbb{C}}$ into elements of type $(1,0)$ and elements of type $(0,1)$. The bundle $T_X^{1,0}$ is the complex conjugate of the bundle $T_X^{0,1}$. The following theorem gives an exact description of the integrable almost complex structures.

Theorem 1.4.19 (Newlander-Nirenberg)**.** *The almost complex structure I is integrable if and only if we have*

$$[T_X^{0,1}, T_X^{0,1}] \subset T_X^{0,1}$$

Remark 1.4.20. *By passing to the conjugate, this is equivalent to the condition that the bracket of two vector fields of type* $(1,0)$ *is of type* $(1,0)$.

This theorem is a difficult one to prove in analysis, for it implies, in particular, that the manifold X whenever it is assumed to be only differentiable actually admits the structure of a real analytic manifold. Following Weil (1957), we will show that when (X, I) are assumed to be real analytic, The Newlander-Nirenberg Theorem follows easily from the following analytic version of the Frobenius theorem 1.4.18.

Theorem 1.4.21. *Let X be a complex manifold of dimension n, and let E be an holomorphic distribution of rank k over X, i.e. a holomorphic vector subbundle of rank k of the holomorphic tangent bundle T_X. Then E is integrable in the holomophic sense if and only if we have the integrability condition*

$$[E, E] \subset E$$

Here, the integrability in the holomorphic sense means that X is covered by open sets U such that there exists a holomorphic submersive map

$$\phi_U : U \to \mathbb{C}^{n-k}$$

satisfying

$$E_u = \ker(\phi_* : T_{U,u} \to T_{\mathbb{C}^{n-k},\phi(u)})$$

for every $u \in U$.

Proof. We first reduce the problem to use the real Frobenius theorem, by noting that the conditions that E is holomorphic and that $[E, E] \subset E$ automatically imply that the real distribution $\Re E \subset T_{X,\mathbb{R}}$ also satisfies the Frobenius integrability condition, so it is integrable. X is then covered by open sets U such that there exists a submersion

$$\phi_U : U \to V$$

where V is open in $\mathbb{R}^{2(n-k)}$, satisfying

$$(\Re E)_u = \operatorname{Ker}(\phi_{*,u} : T_{U,u,\mathbb{R}} \to T_{\mathbb{R}^{2(n-k)},\phi(u)}),\ \forall u \in U$$

Next, we show that there exists a complex structure on the image of ϕ, for which ϕ is holomorphic. To do so, we first note that if $v = \phi(u), T_{V,v} = T_{U,u}/(\Re E)_u$, as quotient of vector spaces as one result of linear algebra implies, and $\Re E$ is stable under the endomorphism I corresponding to the almost complex structure on T_U, there is then an induced complex structure on $T_{V,v}$ via the differential of ϕ_U. As v can be the image of different u_i, this structure would depend of the u_i chosen, but as E is of constant rank, all of $(\Re E)_{u_i}$ are isomorphic, this is we have the same complex structure on $T_{V,v}$ regardless the point u_i. Thus, there exists an almost complex structure on T_V for which the differential of ϕ is $\mathbb{C}$-linear at every point, and if we show that we can give a V one complex structure ϕ would be holomorphic.

Finally, to see that this almost complex structure is integrable, we take a complex submanifold of U transverse to the fibres of ϕ_U, which exists up to restricting U. Via ϕ_U, this submanifold becomes locally isomorphic to V, by the Theorem of the Inverse Function and the fact that ϕ_U is a submersion, and this isomorphism is compatible with the almost complex structures. Thus, the almost complex structure on T_V is integrable, and it makes ϕ_U into a holomorphic map. □

Proof of the theorem 1.4.19 in the real analytic case. Theorem 1.4.21 implies the Newlander-Nirenberg theorem in the real analytic case as follows. Since everything is local, we may assume that X is an open set U of $\mathbb{R}^{2n}$ and that I is a real analytic map with values in $\operatorname{End}\mathbb{R}^{2n}$, satisfying $I \circ I = -\operatorname{Id}$. Up to restricting U, we may assume that I is given by a convergent power series. If we consider $\mathbb{R}^{2n}$ as a subspace of $\mathbb{C}^{2n}$ this power series extends to the whole complex domain and gives a holomorphic map I from an open set $U_{\mathbb{C}}$ of $\mathbb{C}^{2n}$ (a neighbourhood of U) to $\operatorname{End}\mathbb{C}^{2n}$. This map of course satisfies the condition $I \circ I = -1$. Now, this map I gives a holomorphic distribution $E_{\mathbb{C}}$ of rank n on $U_{\mathbb{C}}$, where we define

$$E_{\mathbb{C},u} \subset T^{1,0}_{U_{\mathbb{C}},u} \cong \mathbb{C}^{2n}$$

to be the eigenspace associated to the eigenvalue $-i$ of I. Note that by definition, along U, we have $E_{\mathbb{C},u} = T^{0,1}_{U,u} \subset T_{U,u} \otimes \mathbb{C} = \mathbb{C}^{2n}$

By definition, the sections of $T^{0,1}_X$ on U are generated over $\mathbb{C}$ by the $\chi + iI\chi$, where χ is a real vector field over U. Similarly, the sections of $E_{\mathbb{C}}$ on $U_{\mathbb{C}}$ are generated by the $\chi + iI\chi$, where χ is a real or complex vector field on $U_{\mathbb{C}}$. It follows immediately that if I satisfies the integrability condition of theorem 1.4.19, then the holomorphic distribution $E_{\mathbb{C}}$ is thus integrable, which gives (at least locally) a holomorphic submersion

$$\phi : U_{\mathbb{C}} \to \mathbb{C}^n$$

whose fibres are integral holomorphic submanifolds of the distribution $E_{\mathbb{C}}$. □

1.5 The operators ∂ and $\overline{\partial}$

Let (X, I) be an almost complex manifold; the decomposition (1.1) $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$ induces a dual decomposition

$$T_X^* = (T_X^*)^{1,0} \oplus (T_X^*)^{0,1} \tag{1.2}$$

When X is a complex manifold, the bundle $(T_X^*)^{1,0}$ of complex differential forms of type $(1,0)$, i.e. $\mathbb{C}$-linear forms, is generated in holomorphic local coordinates $z_1, \ldots, z_n$ by the dz_i, i.e. a form α of type $(1,0)$ can be written locally as $\alpha = \sum_i \alpha_i dz_i$, where the α_i are $\mathscr{C}^k$ functions if α is $\mathscr{C}^k$. Since $d(dz_i) = 0$, it follows that

$$d\alpha = \sum_i d\alpha_i \wedge dz_i \tag{1.3}$$

Furthermore, the decomposition (1.2) also induces the decomposition of the complex k-forms into forms of type (p,q), for $p + q = k$:

$$\bigwedge^k T_{X,\mathbb{C}}^* = \bigoplus_{p+q=k} \bigwedge^{p,q} T_X^*, \tag{1.4}$$

where the bundle $\bigwedge^{p,q} T_X^*$ is equal to

$$\bigwedge^p (T_X^*)^{1,0} \otimes \bigwedge^q (T_X^*)^{0,1}$$

More generally, the bundle $\bigwedge^{p,q} T_X^*$ admits as generators in holomorphic local coordinates $z_1, \ldots, z_n$ the differential forms

$$dz_I \wedge d\overline{z} = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q},$$

where I, J are sets of ordered indices $1 \leq i_i < \ldots < i_p \leq n$ and $1 \leq j_i < \ldots < j_q \leq n$. Note that these forms are closed, i.e. annihilated by the exterior differential operator d, again because of $d(dz_i) = 0$. A form α of type (p,q) can thus be written locally as $\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\overline{z}_J$. It follows that

$$d\alpha = \sum_{I,J} d\alpha_{I,J} \wedge dz_I \wedge d\overline{z}_J$$

is the sum of a form of type $(p, q+1)$ and a form of type $(p+1, q)$.

Definition 1.5.1. *For a $\mathscr{C}^1$ differential form α of type (p,q) on a complex manifold X, we define $\overline{\partial}\alpha$ to be the component of type $(p,q+1)$ of $d\alpha$. Similarly, we define $\partial\alpha$ to be the component of type $(p+1,q)$ of $d\alpha$.*

For $(p,q) = (0,0)$, a form of type (p,q) is a function f. Therefore $\overline{\partial} f$ is then the $\mathbb{C}$-antilinear part of df, and this it vanishes if and only if f is holomorphic.
By definition, we have

$$df = \sum_i \frac{\partial f}{\partial z_i} dz_i + \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i,$$

and thus

$$\overline{\partial} f = \sum_i \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i.$$

As mentioned above, a k-differential form α decomposes uniquely into components $\alpha^{p,q}$ of type (p,q), $p+q=k$. We then set

$$\overline{\partial}\alpha = \sum_{p,q} \overline{\partial}\alpha^{p,q}, \quad \partial\alpha = \sum_{p,q} \partial\alpha^{p,q}.$$

The following lemmas describe the essential properties of the operators $\partial, \overline{\partial}$.

Lemma 1.5.2. *The operator $\overline{\partial}$ satisfies Leibniz' rule*

$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \overline{\partial}\beta,$$

where k is the degree of the form α. Similarly, the operator ∂ satisfies Leibniz' rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge +(-1)^k \alpha \wedge \partial\beta.$$

Proof. The second assertion follows from the first, since by definition of the operators ∂ and $\overline{\partial}$, we have the relation

$$\partial\alpha = \overline{\overline{\partial}\overline{\alpha}}.$$

As for the first relations, it suffices to prove it for α of type (p,q) and β of type (p',q'). We then obtain it immediately in this case, by taking the component of type $(p+p', q+q'+1)$ of $d(\alpha \wedge \beta)$. □

Lemma 1.5.3. *We have the following relations between the operators ∂ and $\overline{\partial}$.*

$$\overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0, \partial^2 = 0.$$

Proof. This follows from the formulas

$$d \circ d = 0, \quad d = \partial + \overline{\partial}.$$

Indeed, these relations imply that

$$\partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = d^2 = 0.$$

Now, if α is a form of type (p, q), then $\partial^2\alpha$ is of type $(p + 2.q), (\partial\overline{\partial} + \overline{\partial}\partial)\alpha$ is of type $(p+1, q+1)$ and $\overline{\partial}^2\alpha$ is of type $(p, q+2)$. Thus, $d^2\alpha = 0$ implies that $\partial^2\alpha = (\partial\overline{\partial}+\overline{\partial}\partial)\alpha = \overline{\partial}^2\alpha = 0$. □

The Poincaré lemma shows the local exactness of the operator d:

Lemma 1.5.4 (See [Lee03])**.** *Let α be a closed differential form of strictly positive degree on a differentiable manifold. Then, locally there exists a differential form β such that $\alpha = d\beta$.*

We say that α is locally exact.

Now consider a complex manifold X. Let $\alpha = \overline{\partial}\beta$ be a form of type (p, q) which is $\overline{\partial}$-exact. Then we have $\overline{\partial}\alpha = 0$ by Lemma 1.5.3. With the following propositions we show a partial converse, which is the analogue of the Poincaré lemma for the operator $\overline{\partial}$.

Theorem 1.5.5. *Let f be a $\mathscr{C}^k$ function (for $k \geq 1$) on an open set of $\mathbb{C}$. Then, locally on this open set, there exists a $\mathscr{C}^k$ function g (for $k \geq 1$), such that*

$$\frac{\partial g}{\partial \overline{z}} = f. \tag{1.5}$$

Remark 1.5.6. *Such a function g is defined up to the addition of a holomorphic function.*

Proof. We set

$$g(z) := \frac{1}{2i\pi} \int_{B_\varepsilon} \frac{f(w)}{w - z} dw \wedge d\overline{w}.$$

Note first, that for $w = x+iy$ one has $dw\wedge d\overline{w} = (dx+idy)\wedge(dx-idy) = -2idx\wedge dy$. The existence of g as well as the assertion that $\overline{\partial}g = f$ will be shown by splitting g into two parts. This splitting will depend on a chosen point $z_o \in B_\varepsilon$ or rather on a neighbourhood of such a point.

Let $z_0 \in B := B_\varepsilon$ and let $\psi : B \to \mathbb{R}$ be a differentiable function with compact $\text{supp}(\psi) \subset B$ and such that $\psi|_V \equiv 1$ for some open neighbourhood of $z_0 \in V \subset B$. If

$f_1 := \psi \cdot f$ and $f_2 := (1-\psi)\cdot f$, then $f = f_1 + f_2$. In order to see that the above integral is well-defined we consider first the following integrals

$$g_i(z) := \frac{1}{2\pi i}\int_B \frac{f_i(w)}{w-z} dw \wedge d\overline{w},\ \ i = 1, 2.$$

Since $f_2|_V \equiv 0$, the second one is obviously well defined for $z \in V$. The first integral can be rewritten as

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi i}\int_B \frac{f_1(w)}{w-z} dw \wedge d\overline{w} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(2)}{w-z},\ \text{since}\ \operatorname{supp}(f_1) \subset B \text{ is compact} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(u+z)}{u} du \wedge d\overline{u},\ \text{for}\ u := w - z \\ &= \frac{1}{\pi}\int_{\mathbb{C}} f_1(z + re^{i\varphi})e^{-i\varphi} d\varphi \wedge dr,\ \text{for}\ u = re^{i\varphi} \text{ and } du \wedge d\overline{u} = 2ird\varphi \wedge dr. \end{aligned}$$

The last integral is clearly well-defined. Since the integral defining g splits into the two integrals just considered, we see that the function g in the assertion is well-defined on V and thus everywhere on B.

In order to compute $\bar{\partial} g$, we use the same splitting of $g = g_1 + g_2$ as before. Let us first consider $\bar{\partial} g_2$. Since $(w-z)^{-1}$ is holomorphic as a function of z for w in the complement of V, one finds

$$\frac{\partial g_2}{\partial \bar{z}}(z) = \frac{1}{2\pi i}\int_B f_2(w)\frac{\partial (w-z)^{-1}}{\partial \bar{z}} dw \wedge d\bar{w} = 0$$

for all $z \in V$

Using the above expression for g_1 we get

$$\begin{aligned} \frac{\partial g_1}{\partial \bar{z}}(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f_1(z+re^{i\varphi})}{\partial \bar{z}} e^{-i\varphi} d\varphi \wedge dr \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{\partial f_1}{\partial \bar{w}}\frac{\partial(\bar{z} + re^{-i\varphi})}{\partial \bar{z}} + \frac{\partial f_1}{\partial w}\frac{\partial (z + re^{i\varphi})}{\partial \bar{z}} \right) e^{-i\varphi} d\varphi \wedge dr \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar{w}}(z + re^{i\varphi}) e^{-i\varphi} d\varphi \wedge dr \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f_1}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}. \end{aligned}$$

Thus, for $z \in V$ one has

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial g_1}{\partial \bar{z}} + \frac{\partial g_2}{\partial \bar{z}} = \frac{\partial g_1}{\partial \bar{z}} = \frac{1}{2\pi i}\int_B \frac{\partial f_1}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z} \overset{(*)}{=} f_1(z) = f(z).$$

Here, $(*)$ is a consequence of Stokes' theorem:

$$\begin{aligned}
\frac{1}{2\pi i}\int_B \frac{\partial f_1}{\partial \bar w}(w)\frac{dw\wedge d\bar w}{w-z} &= \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{B\setminus B_\delta(z)}\frac{\partial f_1}{\partial \bar w}(w)\frac{dw\wedge \bar w}{w-z}\\
&= \frac{-1}{2\pi i}\lim_{\delta\to 0}\int_{B\setminus B_\delta(z)} d\left(\frac{f_1(w)}{w-z}dw\right), \text{since } (w-z)^{-1} \text{ is holomorphic on } B\setminus B_\delta(z)\\
&= \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\partial B_\delta(z)}\frac{f_1(w)}{w-z}dw. \text{ since } \operatorname{supp}(f_1)\subset B\\
&= \frac{1}{2\pi}\mathit{lim}_{\delta\to 0}\int_0^{2\pi} f_1(z+de^{i\varphi})d\varphi = f_1(z)
\end{aligned}$$

□

The following proposition is known as the Grothendieck-Poincaré lemma. The first proof of it is due to Grothendieck and was presented by Serre in the Séminaire Cartan in 1958.

Proposition 1.5.7 ($\overline{\partial}$ -Poincaré lemma in several variables)**.** *Let α be a $\mathscr{C}^1$ form of type (p,q) with $q>0$. If $\overline{\partial}\alpha=0$, then there locally exists on X a $\mathscr{C}^1$ form β of type $(p,q-1)$ such that $\alpha=\overline{\partial}\beta$.*

Proof. We first reduce to the case where $p=0$ by the following argument. Locally, we can write in holomorphic coordinates $z_1,\dots,z_n$:

$$\alpha=\sum_{I,J}\alpha_{I,J}dz_I\wedge d\bar z_J,$$

where the sets of indices I are of cardinal p and the sets of indices J are of cardinal q. Then

$$\overline{\partial}\alpha=\sum_{I,J}\overline{\partial}\alpha_{I,J}\wedge dz_I\wedge d\bar z_J$$

by lemma 1.5.2. It follows that if $\overline{\partial}\alpha=0$, for every I of cardinal p the form α_I of type $(0,q)$ defined by

$$\alpha_I=\sum_J\alpha_{I,J}d\bar z_J$$

is $\overline{\partial}$-closed. If the proposition is proved for forms of type $(0,q)$, then locally we have $\alpha_I=\overline{\partial}\beta_I$, and

$$\alpha=(-1)^p\overline{\partial}\Big(\sum_I dz_I\wedge\beta_I\Big).$$

It remains to show the proposition for forms of type $(0, q)$. Such a form can be written $\alpha = \sum_J \alpha_J d\bar{z}_J$. Choose k minimal such that no $d\bar{z}_i$ occurs in this sum for $i > k$. Thus, we can write $\alpha = \alpha_1 \wedge d\bar{z}_k + \alpha_2$, with α_2 free of $d\bar{z}_i$ for $i \geq k$. By assumption, $0 = \overline{\partial}\alpha = (\overline{\partial}\alpha_1) \wedge d\bar{z}_k + \overline{\partial}\alpha_2$. If we set $\overline{\partial}_i := (\partial/\partial\bar{z}_i)d\bar{z}_i$, then this implies $\overline{\partial}_i\alpha_1 = \overline{\partial}_i\alpha_2 = 0$ for $i > k$. Therefore, the functions α_J are holomorphic in $z_{k+1}, \ldots, z_n$.

By the one-dimensional Poincaré lemma 1.5 the function

$$g_J(z) = \frac{1}{2\pi i}\int_{B_{\varepsilon_k}} \frac{\alpha_J(z_1.\ldots, z_{k-1}, w, z_{k+1}, \ldots, z_n)}{w - z_k} dw \wedge d\bar{w}$$

satisfies $\frac{\partial g_J}{\partial \bar{z}_k} = \alpha_J$ on $B_{\varepsilon_k} \subset \mathbb{C}$. Moreover, the function g_J is holomorphic in $z_{k+1}, \ldots, z_n$ and differentiable in the other variables.

Set $\gamma := (-1)^q \sum_{k\in J} g_J d\bar{z}_{J\setminus\{k\}}$, where the sum runs over all the indices J such that $k \in J$. Then $\overline{\partial}_i\gamma(z) = 0$ for $i > k$ and $\overline{\partial}\gamma(z) = -\alpha_1 \wedge d\bar{z}_k$. Hence, $\alpha + \overline{\partial}\gamma = \alpha_2$ is still $\overline{\partial}$-closed, but it does not involve any $d\bar{z}_i$ for $i \geq k$ anymore. Then one concludes by induction. □

In the following we will use results from sheaf theory, we refer to Chapter A, and B in [GR77].

Let $\mathscr{A}^{p,q}$ be the sheaf of germs of differential forms of bidegree (p, q) with complex valued $\mathscr{C}^\infty$ coefficients. The Grothendieck-Poincaré Lemma asserts that all $\overline{\partial}$-closed forms of type (p, q) with $q > 0$ are locally $\overline{\partial}$-exact. In other words, the complex of sheaves $(\mathscr{A}^{p,\bullet}, \overline{\partial})$ is exact in degree $q > 0$: and in degree $q = 0$, $\operatorname{Ker}\overline{\partial}$ is the sheaf Ω^p_X of germs of holomorphic forms of degree p on X.

More generally, if E is a holomorphic vector bundle of rank r over X, there exists a natural operator $\overline{\partial}$ acting on the space $\mathscr{C}^\infty(X, \bigwedge^{p,q} T^*_X \otimes E)$ of $\mathscr{C}^\infty$ (p, q)-forms with values in E. In a holomorphic trivialisation of E, $\tau_U : E|_U \cong U \times \mathbb{C}^k$, such a section can be written $(\alpha_1, \ldots, \alpha_k)$, where the α_i are $\mathscr{C}^\infty$ forms of type (p, q) on U. We then set

$$\overline{\partial}_U\alpha = (\overline{\partial}\alpha_1, \ldots, \overline{\partial}\alpha_k);$$

it is a section of $\bigwedge^{p,q} T^*_X \otimes_{\mathbb{C}} E$. We will show that this local definition in fact gives a form $\overline{\partial}\alpha \in \mathscr{A}^{p,q+1}(E)$.

Lemma 1.5.8. *Let V be and open subset of X and $\tau_V : E|_V \cong V \times \mathbb{C}^k$ a holomorphic trivialisation of E over V. Then for $\alpha \in \mathscr{A}^{0,q}(E)$, we have*

$$\overline{\partial}_U\alpha|_{U\cap V} = \overline{\partial}_V\alpha|_{U\cap V}$$

Proof. Let M_{UV} be the transition matrix, with holomorphic coefficients, which enables us to pass from the trivialisation τ_U to the trivialisation τ_V. Then, by definition, if α_U is a section of E over U, $\alpha_U = (\alpha_{1,U}, \dots, \alpha_{k,U})$ in the trivialisation τ_U, and α_V is a section of E over V, $\alpha_V = (\alpha_{1,V}, \dots, \alpha_{k,V})$ in the trivialisation τ_V, the sections α_U and α_V coincide on $U \cap V$ if and only if

$$(\alpha_{1,V}, \dots, \alpha_{k,V})^t = M_{UV}(\alpha_{1,U}, \dots, \alpha_{k,U}).$$

We can of course replace the functions α_i by differential forms. The form α can be written $(\alpha_{1,U}, \dots, \alpha_{k,U})$ in the trivialisation τ_U and $(\alpha_{1,V}, \dots, \alpha_{k,V})$ in the trivialisation τ_V, and we have, as above,

$$(\alpha_{1,V}, \dots, \alpha_{k,V})^t = M_{UV}(\alpha_{1,U}, \dots, \alpha_{k,U}).$$

To see that $\overline{\partial}_U \alpha|_{U\cap V} = \overline{\partial}_V|_{U \cap V}$, by the above and the definition of $\overline{\partial}_U, \overline{\partial}_V$, it suffices to show that

$$(\overline{\partial}\alpha_{1,V}, \dots, \overline{\partial}\alpha_{k,V})^t = M_{UV}(\overline{\partial}\alpha_{1,U}, \dots, \overline{\partial}\alpha_{k,U}).$$

But this follows immediately from the Leibniz formula lemma 1.5.2 and the fact that the matrix M_{UV} has holomorphic coefficients. □

It then follows that the Grothendieck-Poincaré Lemma still holds for forms with values in E. For every integer $p = 0, 1, \dots, n$, the Dolbeault cohomology groups $H^{p,q}(X, E)$ are defined as being the cohomology of the complex of global forms of type (p, q) (indexed by q):

$$H^{p,q}(X, E) = H^q(\mathscr{C}^\infty(X, \bigwedge^{p,\bullet} T_X^* \otimes E)). \tag{1.6}$$

There is the following fundamental result of sheaf theory (de Rham-Weil Isomorphism Theorem): Let $(\mathscr{L}^\bullet, \delta)$ be a resolution of a sheaf $\mathscr{F}$ by acyclic sheaves, i.e. a complex $(\mathscr{L}^\bullet, \delta)$ given by an exact sequence of sheaves

$$0 \to \mathscr{F} \xrightarrow{j} \mathscr{L}^0 \xrightarrow{\delta^0} \mathscr{L}^1 \to \dots \to \mathscr{L}^q \xrightarrow{\delta^q} \mathscr{L}^{q+1} \to \dots,$$

where $H^s(X, \mathscr{L}^q) = 0$ for all $q \geq 0$ and $s \geq 1$. (To arrive at this latter condition of acyclicity, it is enough for example that the $\mathscr{L}^q$ are flasque or soft, for example a sheaf of modules over the sheaf of ring $\mathscr{C}^\infty$).)Then there is a functorial isomorphism

$$H^q(\Gamma(X, \mathscr{L}^\bullet)) \to H^q(X, \mathscr{F}). \tag{1.7}$$

We apply this in the following situation. Let $\mathscr{A}^{p,q}(E)$ be the sheaf of germs of $\mathscr{C}^\infty$ sections of $\bigwedge^{p,q} T_X^* \otimes E$. Then $(\mathscr{A}^{p,\bullet}(E), \overline{\partial})$ is a resolution of the locally free $\mathscr{O}_X$-module $\Omega_X^p \otimes \mathscr{O}(E)$ (Grothendieck-Poincaré Lemma), and the sheaves $\mathscr{A}^{p,q}(E)$ are acyclic as $\mathscr{C}^\infty$-modules. According to (1.7), we obtain

Theorem 1.5.9 (Dolbeault Isomorphism Theorem(1953)). *For all holomorphic vector bundles E on X, there exists a canonical isomorphism*

$$H^{p,q}(X,E) \simeq H^p(X, \Omega_X^p \otimes E).$$

If X is projective algebraic and if E is an algebraic vector bundle, the theorem of Serre (GAGA) [Ser56], shows that the algebraic cohomology groups $H^q(X, \Omega_X^p \otimes \mathscr{O}(E))$ computed via the corresponding algebraic sheaf in the Zariski topology are isomorphic to the corresponding analytic cohomology groups. Since our point of view here is exclusively analytic, we will no longer need to refer to this comparison theorem.

1.6 Connections

Definition 1.6.1. *Assume given a real or complex $\mathscr{C}^\infty$ vector bundle E of rank r on a differentiable manifold M of class $\mathscr{C}^\infty$. A connection D on E is a linear differential operator of order* 1

$$D : \mathscr{C}^\infty(M, \bigwedge^q T_M^* \otimes E) \to \mathscr{C}^\infty(M, \bigwedge^{q+1} T_M^* \otimes E)$$

such that D satisfies Leibnitz rule:

$$D(f \wedge u) = \mathrm{d}f \wedge u + (-1)^{\deg f} f \wedge Du \tag{1.8}$$

for all forms $f \in \mathscr{C}^\infty(M, \bigwedge^p T_M^*), u \in \mathscr{C}^\infty(X, \bigwedge^q T_M^* \otimes E)$.

On an open set $\Omega \subset M$ where E admits a trivialization $\tau : E|_\Omega \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, if we let θ be a frame over Ω, then we define the connection matrix $\Gamma(D,\theta)$ associated with the connection D and the frame θ by setting first

$$De_\sigma = \sum_{\rho=1}^r \Gamma_{\rho\sigma}(D,\theta) \cdot e_\rho$$

and then

$$\Gamma(D,\theta) = [\Gamma_{\rho\sigma}(D,\theta)],\ \Gamma_{\rho\sigma}(D,\theta) \in \bigwedge^1 T_M^*.$$

We shall denote the matrix $\Gamma(D,\theta)$ simply by Γ.

We can use the connection matrix to explicitly represent the action of D on a section of E. Namely, if u is a section of E over Ω, then for a given frame θ,

$$\begin{aligned} Du &= D\big(\sum_\lambda u_\lambda(\theta)e_\lambda\big) \\ &= \sum_\sigma \mathrm{d}u_\sigma(\theta)\cdot e_\sigma + \sum_\lambda u_\lambda(\theta) De_\lambda \\ &= \sum_\sigma [\mathrm{d}u_\sigma(\theta) + \sum_\rho u_\rho(\theta)\Gamma_{\sigma\rho}(\theta)]\cdot e_\sigma \\ Du &= \sum_\sigma [\mathrm{d}u(\theta) + \Gamma u]\cdot e_\sigma. \end{aligned} \tag{1.9}$$

Where we have set $\mathrm{d}u \simeq_\tau (\mathrm{d}u_\lambda)_{1\le\lambda\le r}$ and the wedge product inside the brackets in (1.9) is ordinary matrix multiplication of matrices with differential form coefficients. Thus a connection D can be written

$$Du \simeq_\tau \mathrm{d}u + \Gamma \wedge u.$$

Suppose that $E \to M$ is a vector bundle equipped with a connection D. Let $\mathrm{Hom}(E, E)$ be the vector bundle whose fibres are $\mathrm{Hom}(E_x, E_x)$. We want to show that the connection D on E induces in a natural manner an element

$$\Theta_E(D) \in \mathscr{C}^\infty(M, \bigwedge^2 T_M^* \otimes \mathrm{Hom}(E, E)).$$

to be called the curvature tensor.

First we want to give a local description of an arbitrary element $\chi \in \mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes \mathrm{Hom}(E, E))$ Let θ be a frame for E over U in M. Then $\theta = (e_1, \ldots, e_r)$ becomes a basis for the free $\mathscr{C}^\infty(U, \bigwedge^p T_M^*)$-module

$$\mathscr{C}^\infty(U, \bigwedge^p T_M^* \otimes \mathrm{Hom}(E, E)) \cong \mathscr{C}^\infty(U, \bigwedge^p T_M^*) \otimes_{C^\infty(U, \bigwedge^0 T_M^*)} \mathscr{C}^\infty(U, \bigwedge^0 T_M^* \otimes \mathrm{Hom}(E, E)),$$

Since $E|_U \cong U \times \mathbb{C}^r$, by using θ to effect a trivialization, we see that

$$\mathscr{C}^\infty(U, \bigwedge^0 T_M^* \otimes \mathrm{Hom}(E, E)) \cong \mathfrak{M}_r(U) = \mathfrak{M}_r \otimes \mathscr{C}^\infty(U, \bigwedge^0 T_M^*)$$

where $\mathfrak{M}_r$ is the vector space of $r \times r$ matrices and thus $\mathfrak{M}_r(U)$ is the $\mathscr{C}^\infty(U, \bigwedge^0 T_M^*)$-module of $r \times r$ matrices with coefficients $\mathscr{C}^\infty(U, \bigwedge^0 T_M^*)$. Therefore there is associated with χ under the above isomorphisms, an $r \times r$ matrix

$$\chi(\theta) = [\chi(\theta)_{\rho\sigma}],\ \chi(\theta)_{\rho\sigma} \in \mathscr{C}^\infty(U, \bigwedge^p T_M^*).$$

Returning to the problem of defining the curvature, let $E \to M$ be a vector bundle with a connection D and let $\Gamma(\theta) = \Gamma(D, \theta)$ be the associated connection matrix. We define

$$\Theta(D, \theta) := \mathrm{d}\Gamma(\theta) + \Gamma(\theta) \wedge \Gamma(\theta), \tag{1.10}$$

which is an $r \times r$ matrix of 2-forms;i.e.,

$$\Theta_{\rho\sigma} = \mathrm{d}\Gamma_{\rho\sigma} + \sum \Gamma_{\rho\kappa} \wedge \Gamma_{\kappa\sigma}.$$

We call $\Theta(D, \theta)$ the curvature matrix associated with the connection matrix $\Gamma(\theta)$. We have the following two simple propositions, the first showing how $\Gamma(\theta)$ and $\Theta(\theta)$ transform, and the second relating $\Theta(\theta)$ to the operator d+$\Gamma(\theta)$.

Lemma 1.6.2. *Let η be a change of frame and define $\Gamma(\theta)$ and $\Theta(\theta)$ as above. Then*

(a) $\mathrm{d}\eta + \Gamma(\theta)\eta = \eta\Gamma(\theta\eta)$,

(b) $\Theta(\theta\eta) = \eta^{-1}\Theta(\theta)\eta$.

Proof. (a) As η is a change of frame we have that

$$\theta\eta = \big(\sum \eta_{\rho 1} e_\rho, \ldots, \sum \eta_{\rho r} e_\rho\big) = (e'_1, \ldots, e'_r).$$

then

$$\begin{aligned} D(e'_\sigma) &= \sum_{\upsilon} \Gamma_{\upsilon\sigma}(\theta\eta) e'_\upsilon \\ &= \sum_{\upsilon,\rho} \Gamma_{\upsilon\sigma}(\theta\eta)\eta_{\rho\upsilon} e_\rho, \end{aligned}$$

and, on the other hand,

$$D\big(\sum_\rho \eta_{\rho\sigma} e_\rho\big) = \sum_\rho \mathrm{d}\eta_{\rho\sigma} e_\rho + \sum_{\rho,\tau} \eta_{\rho\sigma}\Gamma_{\tau\rho} e_\tau.$$

By comparing coefficients, we obtain

$$\eta\Gamma(\theta\eta) = \mathrm{d}\eta + \Gamma(\theta)\eta. \tag{1.11}$$

(b) Take the exterior derivative of the matrix equation (1.11), obtaining

$$\mathrm{d}\Gamma(\theta) \cdot \eta - \Gamma(\theta) \cdot \mathrm{d}\eta = \mathrm{d}\eta \cdot \Gamma(\theta\eta) + \eta \cdot \mathrm{d}\Gamma(\theta\eta). \tag{1.12}$$

Where the sign minus in the first side of the equations comes from the fact that Γ is a matrix of 1-forms and as η are 0-forms, this is, complex numbers, no sign minus appears.

Also from (1.11):

$$\Gamma(\theta\eta) = \eta^{-1}\,\mathrm{d}\eta + \eta^{-1}\Gamma(\theta)\eta, \tag{1.13}$$

and thus we obtain by substituting (1.13) into (1.12) an algebraic expression for $\eta\,\mathrm{d}\Gamma(\theta\eta)$ in terms of the quantities $\mathrm{d}\Gamma(\theta), \Gamma(\theta), \mathrm{d}\eta, \eta$ and η^{-1}. Then we can write

$$\eta[\mathrm{d}\Gamma(\theta\eta) + \Gamma(\theta\eta)\wedge\Gamma(\theta\eta)] \tag{1.14}$$

in terms of these same quantities. Writing this out and simplifying, we find that (1.14) is the same as

$$[\mathrm{d}\Gamma(\theta) + \Gamma(\theta)\wedge\Gamma(\theta)]\eta$$

which proves part (b).

□

Lemma 1.6.3. $[\mathrm{d}+\Gamma(\theta)][\mathrm{d}+\Gamma(\theta)]\xi(\theta) = \Theta(\theta)\xi(\theta)$.

Proof. By straightforward computation we have (deleting the notational dependence on θ)

$$\begin{aligned}
(\mathrm{d}+\Gamma)(\mathrm{d}+\Gamma)\xi &= \mathrm{d}^2\xi + \Gamma\cdot\mathrm{d}\xi + \mathrm{d}(\Gamma\cdot\xi) + \Gamma\wedge\Gamma\cdot\xi \\
&= \Gamma\cdot\mathrm{d}\xi + \mathrm{d}\Gamma\cdot\xi - \Gamma\cdot\mathrm{d}\xi + \Gamma\wedge\Gamma\cdot\xi \\
&= \mathrm{d}\Gamma\cdot\xi + \Gamma\wedge\Gamma\cdot\xi \\
&= \Theta\cdot\xi.
\end{aligned}$$

□

Definition 1.6.4. *Let D be a connection in a vector bundle $E \to M$. Then the curvature $\Theta_E(D)$ is defined to be that element $\Theta \in \mathscr{C}^\infty(M, \bigwedge^2 T_M^* \otimes \mathrm{Hom}(E,E))$ such that the $\mathbb{C}$-linear mapping*

$$\Theta : \mathscr{C}^\infty(M, \bigwedge^0 T_M^* \otimes E) \longrightarrow \mathscr{C}^\infty(M, \bigwedge^2 T_M^* \otimes E)$$

has the representation with respect to a frame

$$\Theta(\theta) = \Theta(D,\theta) = \mathrm{d}\Gamma(\theta) + \Gamma(\theta)\wedge\Gamma(\theta).$$

We see by Lemma 1.6.2(b) that $\Theta_E(D)$ is well defined, since $\Theta(D,\theta)$ satisfies the transformation property which ensures that $\Theta(D,\theta)$ determines a global element in $\mathscr{C}^\infty(M, \bigwedge^2 T_M^* \otimes$ $\mathrm{Hom}(E,E))$

Proposition 1.6.5. *$D^2 = \Theta$, as an operator mapping*

$$\mathscr{C}^\infty(M, \bigwedge^{p} T_M^* \otimes E) \longrightarrow \mathscr{C}^\infty(M, \bigwedge^{p+2} T_M^* \otimes E), \text{ where } D^2 = D \circ D.$$

The only unproved part is for $p > 0$, but we observe that Lemma 1.6.3 is still valid in this case. Then the curvature is the obstruction to $D^2 = 0$ and therefore the obstruction that the sequence

$$\mathscr{C}^\infty(M, \bigwedge^{0} T_M^* \otimes E) \xrightarrow{D} \mathscr{C}^\infty(M, \bigwedge^{1} T_M^* \otimes E) \xrightarrow{D} \mathscr{C}^\infty(M, \bigwedge^{2} T_M^* \otimes E) \longrightarrow \cdots \longrightarrow$$

be a complex.

In particular we can discuss the operator

$$D^2 : \mathscr{C}^\infty(M, \bigwedge^{0} T_M^* \otimes E) \longrightarrow \mathscr{C}^\infty(M, \bigwedge^{2} T_M^* \otimes E).$$

and the fact that D^2 is linear over $\bigwedge^0 T_M^*$, i.e., for u a section of E and f a $\mathscr{C}^\infty$ function

$$\begin{aligned} D^2(f \cdot u) &= D(\mathrm{d}f \otimes u + f \cdot Du) \\ &= -\mathrm{d}f \wedge Du + df \wedge Du + f \cdot D^2 u \\ &= f \cdot D^2 u. \end{aligned}$$

Now suppose that E is equipped with a Euclidean metric (resp. Hermitian) of class $\mathscr{C}^\infty$ and that the isomorphism $E|_\Omega \cong \Omega \times \mathbb{C}^n$ is given by a $\mathscr{C}^\infty$ frame (e_λ). We then have a canonical bilinear pairing, (resp. sesquilinear)

$$\begin{aligned} \mathscr{C}^\infty(M, \bigwedge^{p} T_M^* \otimes E) \times \mathscr{C}^\infty(M, \bigwedge^{q} T_M^* \otimes E) &\to \mathscr{C}^\infty(M, \bigwedge^{p+q} T_M^* \otimes \mathbb{C}) \\ (u, v) &\mapsto \{u, v\} \end{aligned} \tag{1.15}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \bar{v}_\mu \langle e_\lambda, e_\mu \rangle, \; u = \sum u_\lambda \otimes e_\lambda, \; v = \sum v_\mu \otimes e_\mu.$$

The connection D is called Hermitian if it satisfies the additional property

$$\mathrm{d}\{u, v\} = \{Du, v\} + (-1)^{\deg u}\{u, Dv\}.$$

By assuming that (e_λ) is orthonormal, we have that D is Hermitian if and only if $\Gamma^* = -\Gamma$ where $*$ denotes the conjugate transpose of the matrix. This is because, if we suppose D hermitian

$$\begin{aligned}0 = \mathrm{d}\{e_\lambda, e_\mu\} &= \{De_\lambda, e_\mu\} + \{e_\lambda, De_\mu\}\\ &= \Big\{\sum_{\rho=1}^{n}\Gamma_{\rho\lambda}e_\rho, e_\mu\Big\} + \Big\{e_\lambda, \sum_{\rho}^{n}\Gamma_{\rho\mu}e_\rho\Big\}\\ &= \Gamma_{\mu\lambda} + \overline{\Gamma}_{\lambda\mu},\end{aligned}$$

which is also valid if $\Gamma^* = -\Gamma$. This means that $i\Gamma$ is a 1-form with values in the space $\mathrm{Herm}\big(\mathbb{C}^n, \mathbb{C}^n\big)$ of hermitian matrices. The identity $\mathrm{d}^2 = 0$ implies

$$\begin{aligned}0 = \mathrm{d}^2\{u, v\} &= \mathrm{d}(\{Du, v\} + (-1)^p\{u, Dv\})\\ &= \mathrm{d}\{Du, v\} + (-1)^p\mathrm{d}\{u, Dv\}\\ &= \{D^2u, v\} + (-1)^{p+1}\{Du, Dv\} + (-1)^p\{Du, Dv\} + \{u, D^2v\}\\ &= \{D^2u, v\} + \{u, D^2v\},\end{aligned}$$

i.e. $\{\Theta(D)\wedge u, v\} + \{u, \Theta(D)\wedge v\} = 0$. Therefore $\Theta(D)^* = -\Theta(D)$ and the curvature $\Theta(D)$ is such that

$$i\Theta(D) \in \mathscr{C}^\infty(M, \bigwedge^2 T_M^* \otimes \mathrm{Herm}(E, E)).$$

Now we study those properties of connections governed by the existence of a complex structure on the base manifold. Let $M = X$ be a complex manifold, $\dim_{\mathbb{C}} X = n$ and E a $\mathscr{C}^\infty$ vector bundle of rank r over X, here, E is not assumed to be holomorphic. We denote by $\mathscr{C}^\infty_{p,q}(X, E)$ the space of $\mathscr{C}^\infty$ sections of the bundle $\bigwedge^{p,q} T^*X \otimes E$. We have therefore a direct sum decomposition

$$\mathscr{C}^\infty_l(X, E) = \bigoplus_{p+q=l} \mathscr{C}^\infty_{p,q}(X, E).$$

Connections of type $(1, 0)$ or $(0, 1)$ are operators acting on vector valued forms, which imitate de usual operators $\partial, \overline{\partial}$ acting on $\mathscr{C}^\infty_{p,q}(X, \mathbb{C})$. More precisely, a connection of type $(1, 0)$ on E is a differential operator D' of order 1 acting on $\mathscr{C}^\infty_{\bullet,\bullet}(X, E)$ and satisfying the two following properties

$$D' : \mathscr{C}^\infty_{p,q}(X, E) \longrightarrow \mathscr{C}^\infty_{p+1,q}(X, E), \tag{1.16}$$

$$D'(f \wedge s) = \partial f \wedge s + (-1)^{\deg f} f \wedge D's \tag{1.17}$$

for any $f \in \mathscr{C}^\infty_{p_1,q_1}(X, \mathbb{C}), s \in \mathscr{C}^\infty_{p_2,q_2}(X, E)$. The definition of a connection D'' of type $(0,1)$ is similar. If $\theta : E|_\Omega \to \Omega \times \mathbb{C}^r$ is a $\mathscr{C}^\infty$ trivialisation of $E|_\Omega$ and if $\sigma = (\sigma_\lambda) = \theta(s)$, then all such connections D' and D'' can be written

$$D's \simeq_\theta \partial\sigma + \Gamma' \wedge \sigma. \tag{1.18}$$

$$D''s \simeq_\theta \overline{\partial}\sigma + \Gamma'' \wedge \sigma \tag{1.19}$$

where $\Gamma' \in \mathscr{C}^\infty_{1,0}(\Omega, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$,$\Gamma'' \in \mathscr{C}^\infty_{0,1}(\Omega, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ are forms with matrix coefficients.

We have then that $D := D' + D''$ is a connection in the sense of Definition 1.6.1; conversely any connection D admits a unique decomposition $D = D' + D''$ in terms of a $(1,0)$-connection and a $(0,1)$-connection.

Theorem 1.6.6. *If H is a Hermitian metric on a holomorphic vector bundle $E \to X$, then H induces canonically a connection, $D(H)$, on E which satisfies, for W and open set in X,*

1. *For $\xi, \eta \in \mathscr{C}^\infty(W, \bigwedge^0 T^*_X \otimes E)$*

$$\mathrm{d}\langle \xi, \eta\rangle = \langle D\xi, \eta\rangle + \langle \xi, D\eta\rangle;$$

   i.e., D is compatible with the metric H.

2. *If $\xi \in \mathscr{O}(W, E)$, i.e., is a holomorphic section of E, then $D''\xi = 0$.*

Proof. First, we point out that 2 is equivalent to the fact that the connection matrix $\Gamma(\theta)$ is of type $(1,0)$ for a holomorphic frame θ. This follows, since for $\xi \in \mathscr{O}(W, E)$ and θ a holomorphic frame, we have

$$\begin{aligned} D\xi(\theta) &= (\mathrm{d}{+}\Gamma(\theta))\xi(\theta) \\ &= (\partial + \Gamma^{(1,0)}(\theta))\xi(\theta) + (\overline{\partial} + \Gamma^{(0,1)}(\theta))\xi(\theta), \end{aligned}$$

where $\Gamma = \Gamma^{(1,0)} + \Gamma^{(0,1)}$ is the natural decomposition. Therefore

$$D'\xi(\theta) = (\partial + \Gamma^{(1,0)}(\theta))\xi(\theta)$$

and

$$D''\xi(\theta) = (\overline{\partial} + \Gamma^{(0,1)}(\theta))\xi(\theta).$$

But $\overline{\partial}\xi(\theta) = 0$ since ξ and θ are holomorphic. Thus

$$D''\xi(\theta) = \Gamma^{(0,1)}(\theta)\xi(\theta).$$

Suppose now that we have a connection D satisfying 1 and 2. Then let $\theta = (e_1, \dots, e_r)$ be a holomorphic frame over $U \subset X$ and Γ the associated connection matrix. Since D is compatible with the metric H, we have

$$\mathrm{d}H = H\Gamma + \overline{\Gamma}^t H.$$

Since, in addition, D satisfies 2, we have seen that Γ is of type $(1,0)$. Thus, by examining types we see that

$$\partial H = H\Gamma$$

and

$$\overline{\partial} H = \overline{\Gamma}^t H,$$

from which it follows that

$$\Gamma = H^{-1}\partial H. \tag{1.20}$$

We can define then Γ by (1.20). Such a connection matrix clearly satisfies 1 and 2. □

This theorem gives a simple formula for the canonical connection, called the Chern connection, in terms of the metric H, namely

$$\Gamma(\theta) = H(\theta)^{-1}\partial H(\theta)$$

for a holomorphic frame θ. Moreover, $D = D' + D''$ has the following representation with respect to a holomorphic frame θ,

$$\begin{aligned} D' &= \partial + \Gamma(\theta), \\ D'' &= \overline{\partial}. \end{aligned} \tag{1.21}$$

Proposition 1.6.7. *Let D be the Chern connection of a Hermitian holomorphic vector bundle $E \to X$, with Hermitian metric H. Let $\Gamma(\theta) := \overline{H}^{-1}\partial\overline{H}$ and $\Theta(E)$ be the connection and curvature matrices defined by D with respect to a holomorphic frame θ. Then*

(a) $\Gamma(\theta)$ *is of type* $(1,0)$*, and* $\partial\Gamma(\theta) = -\Gamma(\theta) \wedge \Gamma(\theta)$.

(b) $\Theta(E) = \overline{\partial}\Gamma(\theta)$*, and* $\Theta(E)$ *is of type* $(1,1)$.

(c) $\overline{\partial}\Theta(E) = 0$.

Proof. Let $H = H(\theta), \Gamma = \Gamma(\theta)$, and $\Theta = \Theta(E)$. Then we first note that Γ is of type $(1,0)$ by Theorem 1.6.6. Then by using

$$\begin{aligned}\partial(\overline{H}^{-1}\overline{H}) &= \partial(\mathrm{Id}) = 0\\ \partial\overline{H}^{-1} &= -\overline{H}^{-1}\cdot\partial\overline{H}\cdot\overline{H}^{-1}\\ \partial^2 &= 0,\end{aligned}$$

we see that

$$\begin{aligned}\partial\Gamma &= \partial(\overline{H}^{-1}\partial\overline{H}) = -\overline{H}^{-1}\cdot\partial\overline{H}\cdot\overline{H}^{-1}\wedge\partial\overline{H}\\ &= -(\overline{H}^{-1}\partial\overline{H})\wedge(\overline{H}^{-1}\partial\overline{H}) = -\Gamma\wedge\Gamma,\end{aligned}$$

just using that the wedge product is matrix multiplication with wedge product of 1-forms, therefore associative, which gives us part (a). Part (b) is simple computation, namely

$$\begin{aligned}\Theta &= \mathrm{d}\Gamma + \Gamma\wedge\Gamma = \partial\Gamma + \Gamma\wedge\Gamma + \overline{\partial}\Gamma\\ &= \overline{\partial}\Gamma\end{aligned}$$

by using part (a). Part (c) then follows from

$$\overline{\partial}\Theta = \overline{\partial}^2\Gamma = 0.$$

□

Chapter 2

Hodge Decomposition

2.1 Differential operators on vector bundles

We first describe some basic concepts concerning differential operators (symbol, composition, adjunction, ellipticity), in the general setting of vector bundles. Let M be a $\mathscr{C}^\infty$ differentiable manifold, $\dim_{\mathbb{R}} M = m$, and let E, F be $\mathbb{K}$-vector bundles over M, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} F = r'$.

Definition 2.1.1. *A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$, $u \mapsto Pu$ of the form*

$$Pu(x) = \sum_{|\alpha| \leq \delta} a_\alpha(x) D^\alpha u(x),$$

where $E|_\Omega \simeq \Omega \times \mathbb{K}^r, F|_\Omega \simeq \Omega \times \mathbb{K}^{r'}$ are trivialized locally on some open chart $\Omega \subset M$ equipped with local coordinates $x_1, \ldots, x_m$, and where $a_\alpha(x) = (a_{\alpha\lambda\mu}(x))_{1 \leq \lambda \leq r', 1 \leq \mu \leq r}$ are $r' \times r$- matrices with $\mathscr{C}^\infty$ coefficients on Ω. Here $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_m})^{\alpha_m}$ as usual, and $u = (u_\mu)_{1 \leq \mu \leq r}, D^\alpha u = (D^\alpha u_\mu)_{1 \leq \mu \leq r}$ are viewed as column matrices.

This expression for P changes quite a bit if we choose a different set of coordinates. However it turns out we can make invariant the top order part of P: if $t \in \mathbb{K}$ is a parameter, we will consider the following for insight purposes $u(x) = u(x_1, x_2)$ and $\alpha = (1, 1)$, then:

$$\begin{aligned} D^\alpha(e^{tu(x)}) &= \frac{\partial^2}{\partial x_1 \partial x_2}(e^{tu(x)}) \\ &= \frac{\partial}{\partial x_1}\big(t\frac{\partial u}{\partial x_2}(x)e^{tu(x)}\big) \\ &= t\frac{\partial^2 u}{\partial x_1 \partial x_2}(x)e^{tu(x)} + t^2\frac{\partial u}{\partial x_1}(x)\frac{\partial u}{\partial x_2}(x)e^{tu(x)}. \end{aligned}$$

This is, by calculating in a similar fashion, we obtain that $e^{-tu(x)}P(e^{tu(x)})$ is a polynomial of degree δ in t. For each $\xi \in T^*_{M,x}$, choose a function u such that $\mathrm{d}u(x) = \xi$ and then set

$$\sigma_P(x, \xi) = \lim_{t\to\infty} t^{-\delta}\left(e^{-tu(x)}P(e^{tu(x)})\right)(x) \in \mathrm{Hom}(E_x, F_x), \tag{2.1}$$

and the top order term at x is of the form

$$t^\delta \sum_{|\alpha|=\delta}(x)\big(\partial^{\alpha_1}_{y_1}u\big)(x)\cdots\big(\partial^{\alpha_m}_{y_m}\big)(x) = t^\delta \sum_{|\alpha|=\delta} a_\alpha(x)\xi_1^{\alpha_1}\ldots\xi_m^{x_m},$$

which shows that 2.1 is independent of the choice of u and hence well-defined. We say that σ_P is the principal symbol of P. Notice that this is not just a smooth function on T^*_M, but in fact a homogeneous polynomial of order δ on each fiber of T^*_M. Now, if E, F, G are vector bundles and

$$P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F), \quad Q : \mathscr{C}^\infty(M, F) \to (M, G)$$

are differential operators of respective degrees δ_P, δ_Q, we have that for $Q \circ P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, G)$ is a differential operator of degree $\delta_P + \delta_Q$ and that

$$\sigma_{Q\circ P}(x, \xi) = \sigma_Q(x, \xi)\sigma_P(x, \xi). \tag{2.2}$$

Here the product of symbols is computed as a product of matrices. This follows directly from (2.1) as

$$\begin{aligned} \sigma_{Q\circ P} &= \lim_{t\to\infty} t^{-(\delta_P+\delta_Q)}\left(e^{-tu}\circ P \circ Q \circ e^{tu}\right) \\ &= \lim_{t\to\infty} t^{-\delta_P}\left(e^{-tu}\circ P(e^{tu})\right) \cdot \lim_{tl\to\infty} t^{-\delta_Q}\left(e^{-tu}\circ Q(e^{tu})\right)(x) \\ &= \sigma_P \cdot \sigma_Q. \end{aligned}$$

Definition 2.1.2. *Let M be a compact differentiable manifold of $\dim_{\mathbb{R}} M = n$. A volume form is a nowhere-vanishing top dimensional form, this is, a section in $\bigwedge^n T^*_M$. M is said to be orientable if there exists a volume form over M.*

Assume that M is oriented and is equipped with a smooth volume form $\mathrm{d}V(x) = \gamma(x)\,\mathrm{d}x_1 \ldots \wedge \mathrm{d}x_m$, where $\gamma(x) > 0$ is a smooth density. This is, the coefficients of $\gamma(x)$ transform by

$$\gamma\,\mathrm{d}x = \gamma\,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m = \tilde{\gamma}(y(x)) \left| \det \frac{\partial y(x)}{\partial x} \right| \mathrm{d}x$$

where $\tilde{\gamma}(y)dy$ is the representation with respect to the coordinates $y = (y_1, \ldots, y_n)$, where $x \to y(x)$ and $\frac{\partial y}{\partial x}$ is the corresponding Jacobian matrix of the change of coordinates.

If E is a euclidean or hermitian vector bundle, we have a Hilbert space $L^2(M, E)$ of global sections u of E with measurable coefficients, satisfying the L^2 estimate

$$\|u\|^2 = \int_M |u(x)|^2\,\mathrm{d}V(x) < +\infty. \tag{2.3}$$

We denote by

$$(u, v) = \int_M \langle u(x), v(x)\rangle\,\mathrm{d}V, \quad u, v \in L^2(M, E) \tag{2.4}$$

the corresponding L^2 inner product.

Definition 2.1.3. *Let M be a differentiable manifold and E, F vector bundles over M. Let $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ be a $\mathbb{C}$-linear map. Then a $\mathbb{C}$-linear map $P^* : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, E)$ is called a formal adjoint of P if*

$$(Pu, v) = (u, P^*v) \textit{ whenever } \operatorname{supp} u \cap \operatorname{supp} v \subset\subset M \tag{2.5}$$

for all $u \in \mathscr{C}^\infty(M, E), v \in \mathscr{C}^\infty(M, F)$.

Theorem 2.1.4. *If $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ is a differential operator and both E, F are euclidean or hermitian, there exists a unique formal adjoint which extends to $L^2(M, E)$ in a unique way.*

Proof. To prove uniqueness suppose there exist $P^*, S : \mathscr{C}^\infty(M, F) \to \mathscr{C}^\infty(M, E)$ such that

$$(Pf, g) = (f, P^*g) = (f, Sg).$$

This is

$$0 = (f, (P^* - S)g) = \int_M \langle f, (P^* - S)g\rangle dV,$$

and by using a partition of unity and sections with compact support we have that locally $P^* = S$ which extends to all M, and by using the density of the set of elements $u \in \mathscr{C}^\infty(M, E)$ with compact support in $L^2(M, E)$ it follows that P^* is unique as well as its extension. Since uniqueness is clear, it is enough, by a partition of unity argument, to show the existence of P^* locally. Now, let $Pu(x) = \sum_{|\alpha|\leq\delta} a_\alpha(x) D^\alpha u(x)$ be the expansion of P with respect to trivializations of E, F given by orthonormal frames over some coordinate open set $\Omega \subset M$ in order to avoid another terms in the sum for $h_{ij} = \langle e_i, e_j\rangle$. Assuming $\operatorname{supp} u \cap \operatorname{supp} v \subset\subset \Omega$, an integration by parts, and using that $D^\alpha(\bar{u}(x)) = \overline{D^\alpha(u(x))}$ yields

$$\begin{aligned}(Pu, v) &= \int_\Omega \sum_{\substack{\lambda,\mu\\|\alpha|\leq\delta}} a_{\alpha\lambda\mu} D^\alpha u_\mu(x) \bar{v}_\lambda(x)\gamma(x)\,\mathrm{d}x \\ &= \int_\Omega \sum_{\substack{\lambda,\mu\\|\alpha|\leq\delta}} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\bar{a}_{\alpha\lambda\mu} v_\lambda(x))}\,\mathrm{d}x \\ &= \int_\Omega \langle u, \sum_{|\alpha|\leq\delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha(\gamma(x)\bar{a}^t_\alpha v(x))\rangle\,\mathrm{d}V(x).\end{aligned}$$

Hence we see that P^* exists and is uniquely defined by

$$P^* v(x) = \sum_{|\alpha|\leq\delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha(\gamma(x)\bar{a}^t_\alpha v(x)). \tag{2.6}$$

□

It follows immediately from 2.6 that the principal symbol of P^* is

$$\sigma_{P^*}(x,\xi) = (-1)^\delta \sum_{|\alpha|=\delta} \bar{a}^t_\alpha \xi^\alpha = (-1)^\delta \sigma_P(x,\xi)^*. \tag{2.7}$$

Definition 2.1.5. *A differential operator P is said to be elliptic if $\sigma_P(x,\xi) \in \operatorname{Hom}(E_x, F_x)$ is injective for every $x \in M$ and $\xi \in T^*_M \setminus \{0\}$.*

On the following we will assume that M is a compact oriented $\mathscr{C}^\infty$ manifold of dimension m, with volume form $\mathrm{d}V$. Let $E \to M$ be a $\mathscr{C}^\infty$ hermitian vector bundle of rank r on M.

Definition 2.1.6. *For any real number s, we define the Sobolev space $W^s(\mathbb{R}^m)$ to be the Hilbert space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^m)$ such that the Fourier transform $\hat{u}$ is a L^2_{loc} function satisfying the estimate*

$$\|u\|^2_s = \int_{\mathbb{R}^m} (1 + |\xi|^2)^s\, |\hat{u}(\xi)|^2\,\mathrm{d}\lambda(\xi) < +\infty. \tag{2.8}$$

Note that $\|\cdot\|_s$ is defined for all $s \in \mathbb{R}$, but we shall deal only with integral values in our applications. Intuitively, $\|\xi\|_s < +\infty$, for s a positive integer, means that ξ has all derivatives $D^\alpha u$ of order $|\alpha| \leq s$ in L^2. This follows from the fact that in $\mathbb{R}^m$ the norm $\|\cdot\|_s$ is equivalent to the norm

$$\left[\sum_{|\alpha|\leq s} \int_{\mathbb{R}^m} |D^\alpha u(x)|^2 \, \mathrm{d}\lambda(x)\right]^{\frac{1}{2}} \tag{2.9}$$

(see [Hö63], Chap. 1). This follows essentially from the basic facts about Fourier transforms that

$$\widehat{D^\alpha} u(y) = y^\alpha \hat{u}(y),$$

where $y^\alpha = y_1^{\alpha_1} \ldots y_n^{\alpha_n}$, $D^\alpha = (-i)^{|\alpha|} D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $D_j = \frac{\partial}{\partial x_j}$, and $\|u\|_0 = \|\hat{u}\|_0$.

More generally, we denote by $W^s(M, E)$ the Sobolev space of sections $u : M \to E$ whose components are locally in $W^s(\mathbb{R}^m)$ on all open charts. More precisely, choose a finite subcovering (Ω_j) of M by open coordinate charts $\Omega \simeq \mathbb{R}^m$ on which E is trivial. Consider an orthonormal frame $(e_{j,\lambda})_{1\leq\lambda\leq r}$ of $E|_{\Omega_j}$ and write u in terms of its components, i.e. $u = \sum u_{j,\lambda} e_{j,\lambda}$. We then set

$$\|u\|_s^2 = \sum_{j,\lambda} \|\psi_j u_{j,\lambda}\|_s^2,$$

where (ψ_j) is a partition of unity subordinate to (Ω_j), such that $\sum \psi_j^2 = 1$.

The norm $\|\|_s$ defined on E depends on the choice of partition of unity and the local trivialization. It is a fact, which we shall not verify here, that the topology on $W^s(M, E)$ is independent of the choices made; i.e., any two such norms are equivalent.

We have a sequence of inclusions of the Hilbert spaces $W^s(X, E)$

$$\cdots \supset W^s \supset W^{s+1} \supset \cdots \supset W^{s+j} \supset \cdots.$$

We will need the following two important results concerning this sequence of Hilbert spaces, see [Hö63].

Lemma 2.1.7 (Sobolev Lemma). *For an integer $k \in \mathbb{N}$ and any real numbers $s \geq k + \frac{m}{2}$, we have $W^s(M, E) \subset \mathscr{C}^k(M, E)$ and the inclusion is continuous.*

It follows immediately from the Sobolev lemma that

$$\bigcap_{s\geq 0} W^s(M, E) = \mathscr{C}^\infty(M, E).$$

Lemma 2.1.8 (Rellich Lemma)**.** *For all $t > s$, the inclusion*

$$W^t(M, E) \hookrightarrow W^s(M, E)$$

is a compact linear operator.

In Lemma 2.1.8 the compactness of X is strongly used, whereas it is inessential for Lemma 2.1.7.

To give some appreciation of these propositions, we shall give proofs of them in special cases to show what is involved. The general results for vector bundles and distributions are essentially formalism and the piecing together of these special cases.

Lemma 2.1.9 (Sobolev)**.** *Let f be a measurable L^2 function in $\mathbb{R}^n$ with $\|f\|_s < \infty$, for $s > [n/2] + k + 1$, a nonnegative integer. Then $f \in \mathscr{C}^k(\mathbb{R}^n)$ (after a possible change on a set of measure zero).*

Proof. Our assumption $\|f\|_s < \infty$ means that

$$\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 (1 + |\xi|^2)^s \, \mathrm{d}\xi < \infty,$$

Let

$$\tilde{f}(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi) \, \mathrm{d}\xi$$

be the inverse Fourier transform, if it exists. We know that if the inverse Fourier transform exists, then $\tilde{f}(x)$ agrees with $f(x)$ almost everywhere, and we agree to say that $f \in \mathscr{C}^0(\mathbb{R}^n)$ if this integral exists, making the appropriate change on a set of measure zero. Similarly, for some constant c

$$D^\alpha f(x) = c \int e^{i\langle x,\xi\rangle} \xi^\alpha \hat{f}(\xi) \, \mathrm{d}\xi$$

will be continuous derivatives of f if the integral converges. Therefore we need to show that for $|\alpha| \leq k$, the integrals

$$\int e^{i\langle x,\xi\rangle} \xi^\alpha \hat{f}(\xi) \, \mathrm{d}\xi$$

converge, and it will follow that $f \in \mathscr{C}^k(\mathbb{R}^n)$. But, indeed, we have

$$\begin{aligned} \int \left|\hat{f}(\xi)\right| |\xi|^{|\alpha|} \, \mathrm{d}\xi &= \int \left|\hat{f}(\xi)\right| (1 + |\xi|^2)^{\frac{s}{2}} \frac{|\xi|^{|\alpha|}}{(1 + |\xi|^2)^{\frac{s}{2}}} \, \mathrm{d}\xi \\ &\leq \|f\|_s \left(\int \frac{|\xi|^{2|\alpha|}}{(1 + |\xi|^2)^s} \, \mathrm{d}\xi \right)^{\frac{1}{2}}. \end{aligned}$$

by the inequality of Cauchy–Schwarz. Let

$$g = \frac{|\xi|^{2|\alpha|}}{(1+|\xi|^2)^s}$$

and S^{n-1} be the surface of the n-dimensional unit ball with respect to Euclidean norm. Defining $\tilde{g}(r,\omega) : \mathbb{R}^+ \times S^{n-1} \to \mathbb{R}$ by

$$\tilde{g}(r,\omega) = g(r\omega)$$

we have

$$\begin{aligned}\int_{\mathbb{R}^n} g(\xi)\,\mathrm{d}\xi &= \int_0^\infty \left(\int_{S^{n-1}} \tilde{g}(r,\omega)\,\mathrm{d}\omega\right) r^{n-1}\,\mathrm{d}r \\ &= \int_0^\infty \left(\int_{S^{n-1}} \frac{r^{2|\alpha|}}{(1+r^2)^s}\,\mathrm{d}w\right) r^{n-1}\,\mathrm{d}r \\ &= c\int_0^\infty \frac{r^{2\left(|\alpha|+\frac{n-1}{2}\right)}}{(1+|r|^2)^s}\,\mathrm{d}r,\end{aligned}$$

where $c = \int_{S^{n-1}} \mathrm{d}\omega$. Now s has been chosen so that this last integral exists, and so we have

$$\int \left|\hat{f}(\xi)\right| |\xi|^{|\alpha|}\,\mathrm{d}\xi < \infty,$$

and the proposition is proved □

Similarly, we can prove a simple version of Rellich's lemma.

Lemma 2.1.10 (Rellich)**.** *Suppose that $f_\nu \in W^s(\mathbb{R}^n)$ and that all f_ν have compact support in $K \subset\subset \mathbb{R}^n$. Assume that $\|f_\nu\|_s \leq 1$. Then for any $t < s$ there exists a subsequence $f_{\nu\iota}$ which converges in $\|\cdot\|_t$.*

Proof. We observe first that for $\xi, \eta \in \mathbb{R}^n, s \in \mathbb{Z}^+$,

$$(1+|\xi|^2)^{s/2} \leq 2^{s/2}(1+|\xi-\eta|^2)^{s/2}(1+|\eta|^2)^{s/2}. \tag{2.10}$$

To see this we write, using the triangle inequality, the fact that $(|\xi|-|\eta|)^2 \geq 0$, and adding positive terms

$$\begin{aligned}1+|\zeta+\eta|^2 &\leq 1+(|\zeta|+|\eta|)^2 \leq 1+2(|\zeta|^2+|\eta|^2) \\ &\leq 2(1+|\zeta|^2)(1+|\eta|^2).\end{aligned}$$

Now let $\xi = \zeta + \eta$, and we obtain (2.10) easily.

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ a $\mathscr{C}^\infty$ function with compact support be chosen so that $\phi \equiv 1$ near K. Then from a standard relation between the Fourier transform and convolution we have that $f_\nu = \phi f_\nu$ implies

$$\hat{f}(\xi) = \int \hat{\phi}(\xi - \eta)\hat{f}_\nu(\eta)\,\mathrm{d}\eta \tag{2.11}$$

Therefore we obtain from 2.10 and 2.11, and using again Cauchy–Schwartz that

$$\begin{aligned}(1+|\xi|^2)^{s/2}\left|\hat{f}_\nu(\xi)\right| &\le 2^{s/2}\int (1+|\xi-\eta|^2)^{s/2}\left|\hat{\phi}(\xi-\eta)\right|(1+|\eta|^2)^{s/2}\left|\hat{f}_\nu(\eta)\right|\mathrm{d}\eta \\ &\le K_{s,\phi}\left\|f_\nu\right\|_s \le K_{s,\phi},\end{aligned}$$

where $K_{s,\phi}$ is a constant depending on s and ϕ. Therefore $\left|\hat{f}_\nu(\xi)\right|$ is uniformly bounded on compact subsets of $\mathbb{R}^n$. Similarly, by differentiating (2.11) we obtain that all derivatives of $\hat{f}_\nu$ are uniformly bounded on compact subsets in the same manner. Therefore, there is, by Arzelà–Ascoli's theorem, a subsequence f_{ν_ι} such that $\hat{f}_{\nu_\iota}$ converges in the $\mathscr{C}^\infty$ topology to a $\mathscr{C}^\infty$ function on $\mathbb{R}^n$. Let us call $\{f_\nu\}$ this new sequence.

Let $\varepsilon > 0$ be given. Suppose that $t < s$. Then there is a ball B_ε such that

$$\frac{1}{(1+|\xi|^2)^{s-t}} < \varepsilon$$

for ξ outside the ball B_ε. Then consider

$$\begin{aligned}\left\|f_\nu - f_\mu\right\|_t^2 &= \int_{\mathbb{R}^n}\frac{\left|(\hat{f}_\nu - \hat{f}_\mu)(\xi)\right|^2}{(1+|\xi|^2)^{s-t}}(1+|\xi|^2)^s\,\mathrm{d}\xi \\ &\le \int_\varepsilon \left|(\hat{f}_\nu - \hat{f}_\mu)(\xi)\right|^2(1+|\xi|^2)^t\,\mathrm{d}\xi \\ &+ \varepsilon\int_{\mathbb{R}^n\setminus B_\varepsilon}\left|(\hat{f}_\nu - \hat{f}_\mu)(\xi)\right|^2(1+|\xi|^2)^s\,\mathrm{d}\xi \\ &= \int_\varepsilon \left|(\hat{f}_\nu - \hat{f}_\mu)(\xi)\right|^2(1+|\xi|^2)^t\,\mathrm{d}\xi + 2\varepsilon,\end{aligned}$$

where we have used the fact that $\left\|f_\nu\right\|_s \le 1$. Since we know that $\hat{f}_\nu$ converges on compact sets, we can choose ν, μ large enough so that the first integral is $< \varepsilon$, and thus f_ν is a Cauchy sequence in the $\left\|\cdot\right\|_t$ norm. □

Definition 2.1.11. *A section $\xi \in \mathscr{C}^\infty(M,E)$ has compact support on a (not necessarily compact) manifold M if $\{x \in X; \xi(x) \neq 0\}$ is relatively compact in M. We shall denote the compactly supported sections by $\mathscr{D}(M,E) \subset \mathscr{C}^\infty(M,E)$.*

If $P = \sum_{|\alpha|\le\delta} a_\alpha(x)D^\alpha$ is a differential operator on $\mathbb{R}^m$, the Fourier inversion formula gives

$$Pu(x) = \int_{\mathbb{R}^m} \sum_{|\alpha|\le\delta} a_\alpha(x)\xi^\alpha \hat{u}(\xi)e^{i\langle x,\xi\rangle}\,\mathrm{d}\lambda(\xi),\ \forall u \in \mathscr{D}(\mathbb{R}^m),$$

where $\langle x, \xi\rangle = \sum_{j=1}^n x_j\xi_j$ is the usual Euclidean inner product, and

$$\hat{u}(\xi) = (2\pi)^{-m}\int u(x)e^{-i\langle x,\xi\rangle}\,\mathrm{d}x$$

is the Fourier transform of u.

A pseudodifferential operator is an operator Op_σ defined by a formula of the type

$$\mathrm{Op}_\sigma(u)(x) = \int_{\mathbb{R}^m} \sigma(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\,\mathrm{d}\lambda(\xi),\ u \in \mathscr{D}(\mathbb{R}^m), \tag{2.12}$$

where σ belongs to a suitable class of functions on $T^*_{\mathbb{R}^m}$. The standard class of symbols $S^\delta(\mathbb{R}^m)$ is defined as follows: Assume given $\delta \in \mathbb{R}$, $S^\delta(\mathbb{R}^m)$ is the class of $\mathscr{C}^\infty$ functions $\sigma(x,\xi)$ on $T^*_{\mathbb{R}^m}$ such that for any $\alpha, \beta \in \mathbb{N}^m$ and any compact subset $K \subset \mathbb{R}^m$ one has an estimate

$$\left|D_x^\alpha D_\xi^\beta \sigma(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{\delta-|\beta|},\ \forall(x,\xi) \in K\times\mathbb{R}^m, \tag{2.13}$$

where $\delta \in \mathbb{R}$ is regarded as the "degree" of σ. Then $\mathrm{Op}_\sigma(u)$ is a well defined $\mathscr{C}^\infty$ function on $\mathbb{R}^m$, since $\hat{u}$ belongs to the class $\mathcal{S}(\mathbb{R}^m)$ of functions having rapid decay.

In the more general situation of operators acting on a bundle E and having values in a bundle F over a compact manifold M, we introduce the analogous space of symbols $S^\delta(M; E, F)$. The elements of $S^\delta(M; E, F)$ are the functions

$$T^*_M \ni (x,\xi) \mapsto \sigma(x,\xi) \in \mathrm{Hom}(E_x, F_x)$$

satisfying condition (2.13) in all coordinate systems. Finally, we take a finite trivializing cover (Ω_j) of M and a "partition of unity" (ψ_j) subordinate to Ω_j such that $\sum \psi^2 = 1$, and we define

$$\mathrm{Op}_\sigma(u) = \sum \psi_j\,\mathrm{Op}_\sigma(\psi_j u),\ u \in \mathscr{C}^\infty(M, E),$$

in a way which reduces the calculations to the situation of $\mathbb{R}^m$. The basic results pertaining to the theory of pseudodifferential operators are summarized below.

If $\sigma \in S^\delta(M; E, F)$, then Op_σ extends uniquely to a continuous linear

$$\mathrm{Op}_\sigma : W^s(M, E) \to W^{s-\delta}(M, F). \tag{2.14}$$

In particular if $\sigma \in S^{-\infty}(M; E, F) := \cap S^{\delta}(M; E, F)$, then Op_σ is a continuous operator sending an arbitrary distributional section of $\mathscr{D}'(M, E)$ into $\mathscr{C}^\infty(M, F)$. Such an operator is called a *regular operator*. As we did before, we will prove a weaker version of this.

Lemma 2.1.12. *Let* Op_σ *defined as in 2.12, then* Op_σ *is a linear operator mapping* $\mathscr{D}(\mathbb{R}^m)$ *into* $\mathscr{C}^\infty(\mathbb{R}^m)$.

Proof. Since $u \in \mathscr{D}(\mathbb{R}^m)$, we have, for any multiindex α,

$$\xi^\alpha \hat{u}(\xi) = (2\pi)^{-m} \int D^\alpha u(x) e^{-i\langle x, \xi\rangle}\, \mathrm{d}x,$$

and hence, since u has compact support, $|\xi^\alpha|\,|\hat{u}(\xi)|$ is bounded for any α, which implies that for any large N,

$$|\hat{u}(\xi)| \leq C(1 + |\xi|^{-N}),$$

i.e., $\hat{u}(\xi)$ goes to zero at ∞ faster than any polynomial. Then we have the estimate for any derivatives of the integrand in (2.12),

$$\left| D_x^\beta \sigma(x, \xi) \hat{u}(\xi) \right| \leq C(1 + |\xi|)^m (1 + |\xi|)^{-N},$$

which implies that the integral in (2.12) converge nicely enough to differentiate under the integral sign as much as we please, and hence $\mathrm{Op}_\sigma(u) \in \mathscr{C}^\infty(\mathbb{R}^m)$. The same estimates give that Op_σ is indeed a continuous linear mapping from $D(\mathbb{R}^m) \to \mathscr{C}^\infty(\mathbb{R}^m)$. □

It is a standard result in the theory of distributions that the class $\mathcal{R}$ of regular operators coincides with the class of operators defined by means of a $\mathscr{C}^\infty$ kernel $K(x, y) \in \mathrm{Hom}(E_y, F_x)$. That is the operators of the form

$$R : \mathscr{D}'(M, E) \to \mathscr{C}^\infty(M, F),\ u \mapsto Ru,\ Ru(x) = \int_M K(x, y) \cdot u(y)\, \mathrm{d}V(y).$$

Conversely, if $\mathrm{d}V(y) = \gamma(y)\, \mathrm{d}y_1 \wedge \ldots \mathrm{d}y_m$ on Ω_j and if we write $Ru = \sum R(\theta_j u)$, where (θ_j) is a partition of unity, the operator $R(\theta \bullet)$ is the pseudodifferential operator associated to the symbol σ defined by the partial Fourier transform

$$\sigma(x, \xi) = \left(\gamma(y)\widehat{\theta_j(t) K}(x, y)\right)_y(x, \xi),\ \sigma \in S^{-\infty}(M; E, F).$$

When one works with pseudodifferential operators, it is customary to work modulo the regular operators and to allow operators more generally of the form $\mathrm{Op}_\sigma + R$ where $R \in \mathcal{R}$ is an arbitrary regular operator.

Composition 2.1.13. *If $\sigma \in S^{\delta}(M;E,F)$ and $\sigma' \in S^{\delta'}(M;F,G)$. δ, $\delta' \in \mathbb{R}$, there exists a symbol $\gamma' \diamondsuit \gamma \in S^{\delta+\delta'}(M;E,G)$ such that $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_{\sigma} = \mathrm{Op}_{\sigma' \diamondsuit \sigma}$ mod $\mathcal{R}$. Moreover*

$$\sigma' \diamondsuit \sigma - \sigma' \cdot \sigma \in S^{\delta+\delta'-1}(M;E,G).$$

Definition 2.1.14. *A pseudodifferential operator Op_{σ} of degree δ is called elliptic if it can be defined by a symbol $\sigma \in S^{\delta}(M;E,F)$ such that*

$$|\sigma(x,\xi) \cdot u| \geq c\,|\xi|^{\delta}\,|u|\,,\ \forall (x,\xi) \in T^*_M,\ \forall u \in E_x$$

for $|\xi|$ large enough, the estimate being uniform for $x \in M$

If E and F have the same rank, the ellipticity condition implies that $\sigma(x,\xi)$ is invertible for large ξ. By taking a suitable truncating function $\theta(\xi)$ equal to 1 for large ξ, one sees that the function $\sigma'(x,\xi) = \theta(\xi)\sigma(x,\xi)^{-1}$ defines a symbol in the space $S^{-\delta}(M;F,E)$, and according to 2.1.14 we have $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_{\sigma} = \mathrm{Id} + \mathrm{Op}_{\rho}$, $\rho \in S^{-1}(M;E,E)$. Choose a symbol τ asymptotically equivalent at infinity to the expansion $\mathrm{Id} - \rho + \rho^{\diamondsuit 2} + \ldots + (-1)^j \rho^{\diamondsuit j} + \ldots$. It is clear then that one obtains an inverse $\mathrm{Op}_{\tau \diamondsuit \sigma'}$ of Op_{σ} modulo $\mathcal{R}$. An easy consequence of this observations is the following:

Proposition 2.1.15 (Gårding Inequality)**.** *Assume given $P : \mathscr{C}^{\infty}(M,E) \to \mathscr{C}^{\infty}(M,F)$ an elliptic differential operator of degree δ, where $\operatorname{rank} E = \operatorname{rank} F = r$, and let $\tilde{P}$ be an extension of P with distributional coefficient sections. For all $u \in W^0(M,E)$ such that $\tilde{P}u \in W^s(M,F)$ one then has $u \in W^{s+\delta}$ and*

$$\|u\|_{s+\delta} \leq C_s \left(\left\| \tilde{P}u \right\|_s + \|u\|_0 \right),$$

where C_s is a positive constant depending only on s.

Proof. Since P is elliptic, there exists a symbol $\sigma \in S^{-\delta}(M;F,E)$ such that

$$\mathrm{Op}_{\sigma} \circ \tilde{P} = \mathrm{Id} + R, \quad R \in \mathcal{R}.$$

Then $\|\mathrm{Op}_{\sigma}(v)\|_{s+\delta} \leq C\,\|v\|_s$ by applying (2.14). Consequently, in setting $v = \tilde{P}u$, we see that $u = \mathrm{Op}_{\sigma}(\tilde{P}u) - Ru$ satisfies the desired estimate, using:

$$\|u\|_{s+\delta} - \|Ru\|_{s+\delta} \leq \|u + Ru\|_{s+\delta} \leq c \left\| \tilde{P}u \right\|_s$$

and finally

$$\begin{aligned} \|u\|_{s+\delta} &\leq c \left\| \tilde{P}u \right\|_s + \|Ru\|_{s+\delta} \\ &\leq c \left\| \tilde{P}u \right\|_s + c' \|u\|_0 \leq C_s \left(\left\| \tilde{P}u \right\|_s + \|u\|_0 \right). \end{aligned}$$

□

We conclude this section with the proof of the following fundamental finiteness theorem, which is the starting point of L^2 Hodge theory.

Theorem 2.1.16 (Finiteness Theorem)**.** *Assume given E, F hermitian vector bundles on a compact manifold M, such that* $\operatorname{rank} E = \operatorname{rank} F = r$*; and given $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ an elliptic differential operator of degree δ. Then:*

1. $\operatorname{Ker} P$ *is finite dimensional.*

2. *$P(\mathscr{C}^\infty(M, E))$ is closed and of finite codimension in $\mathscr{C}^\infty(M, F)$; moreover, if P^* is the formal adjoint of P, there exists a decomposition:*

$$\mathscr{C}^\infty(M, F) = P(\mathscr{C}^\infty(M, E)) \oplus \operatorname{Ker} P^*$$

as an orthogonal direct sum in $W^0(M, F) = L^2(M, F)$.

Proof. 1. Take $u \in \operatorname{Ker} P$ then $Pu = 0$ and $\tilde{P}u = 0$. The Gårding inequality show shows that $u \in W^{s+\delta}$ for all s and that

$$\|u\|_{s+\delta} \leq C_s \|u\|_0 . \tag{2.15}$$

We will show that $\operatorname{Ker} P$ is closed in $W^0(M, E)$. Take $u \in \mathscr{C}^\infty(M, E)$ and $(u_n)_n$ a sequence in $\operatorname{Ker} P$ such that $u_n \to u$ in $W^0(M, E)$. For every k there exists $s > 0$ such that $s + \delta > k + \frac{m}{2}$, and by the Sobolev Lemma

$$W^{s+\delta} \hookrightarrow \mathscr{C}^k(M, E)$$

in a continuous ways, therefore $u_n \to u$ in $\mathscr{C}^\infty(M, E)$, where P is continuous, hence $Pu_n \to Pu$ and finally $Pu = 0$. Therefore $\operatorname{Ker} P$ is closed in $W^0(M, E)$.

By the Rellich Lemma we have that the inclusion $i_0 : W^\delta(M, E) \hookrightarrow W^0(M, E)$ is compact. By (2.15) we obtain

$$\overline{B}_{\|\cdot\|_0}(0, 1) \cap \operatorname{Ker} P \subset i_0(\overline{B}_{\|\cdot\|_\delta}(0, C_0) \cap \operatorname{Ker} P).$$

As $\overline{B}_{\|\cdot\|_\delta}(0, C_0) \cap \operatorname{Ker} P$ is bounded in $W^\delta(M, E)$ we have $i_0(\overline{B}_{\|\cdot\|_\delta}(0, C_0) \cap \operatorname{Ker} P)$ is relatively compact which implies that $\overline{B}_{\|\cdot\|_0}(0, 1) \cap \operatorname{Ker} P$ is a compact in $\operatorname{Ker} P \subset W^0(M, E)$. By the Riesz' Lemma we have that $\dim \operatorname{Ker} P < \infty$.

2. We first show that the extension

$$\tilde{P} : W^{s+\delta}(M, E) \to W^s(M, F)$$

has closed image for all s. For any $\varepsilon > 0$, there exists a finite number of elements $v_1 \dots, v_N \in W^{s+\delta}(M, F), N = N(\varepsilon)$, such that

$$\|u\|_0 \leq \varepsilon \|u\|_{s+\delta} + \sum_{j=1}^{N} |\langle u, v_j \rangle_0| . \tag{2.16}$$

Indeed the set

$$K_{(v_j)} = \left\{ u \in W^{s+\delta}(M, F); \varepsilon \|u\|_{s+\delta} + \sum_{j=1}^{N} |\langle u, v_j \rangle_0| \leq 1 \right\},$$

is relatively compact in $W^0(M, F)$ and $\cap_{(v_j)} \overline{K}_{(v_j)} = \{0\}$. It follows that there are elements (v_j) such that $\overline{K}_{(v_j)}$ are contained in the unit ball of $W^0(M, E)$, as required. Substituting the main term $\|u\|_0$ given by (2.16) in the Gårding inequality; we obtain

$$(1 - C_s \varepsilon) \|u\|_{s+\delta} \leq C_s \left(\left\| \tilde{P} u \right\|_s + \sum_{j=1}^{N} |\langle u, v_j \rangle_0| \right).$$

Define $T = \{u \in W^{s+\delta}(M, E); u \perp v_j, 1, \leq j \leq N\}$ and put $\varepsilon = 1/2C_s$. It follows that

$$\|u\|_{s+\delta} \leq C_s \left\| \tilde{P} u \right\|_s, \ \forall u \in T.$$

This implies that $\tilde{P}(T)$ is closed. As a consequence

$$\tilde{P}(W^{s+\delta}(M, E)) = \tilde{P}(T) + \operatorname{Vect}\left(\tilde{P}(v_1), \dots, \tilde{P}(v_N) \right)$$

is closed in $W^s(M, E)$. Consider now the case $s = 0$. Since $\mathscr{C}^\infty(M, E)$ is dense in $W^\delta(M, E)$, we see that in $W^0(M, E) = L^2(M, E)$, one has

$$\left(\tilde{P}(W^\delta(M, E) \right)^\perp = (P(\mathscr{C}^\infty(M, E)))^\perp = \operatorname{Ker} \tilde{P}^*.$$

We have thus proven that

$$W^0(M, E) = \tilde{P}(W^\delta(M, E)) \oplus \operatorname{Ker} \tilde{P}^*. \tag{2.17}$$

Since P^* is also elliptic, it follows that $\operatorname{Ker} \tilde{P}^*$ is finite dimensional and that $\operatorname{Ker} \tilde{P}^* = \operatorname{Ker} P^*$ is contained in $\mathscr{C}^\infty$. By applying the Gårding inequality, the decomposition formula (2.17) gives

$$W^s(M, E) = \tilde{P}(W^{s+\delta}(M, E)) \oplus \operatorname{Ker} P^*. \tag{2.18}$$

$$\mathscr{C}^\infty(M, E) = P(\mathscr{C}^\infty(M, E)) \oplus \operatorname{Ker} P^*. \tag{2.19}$$

□

2.2 Hodge theory of compact Riemannian manifolds

Definition 2.2.1. *Let V and W be $\mathbb{K}$ finite dimensional vector spaces. A pairing of V and W is a bilinear map $(\ ,\ ): V \times W \to \mathbb{K}$. A pairing is called non-singular if whenever $w \neq 0$ in W, there exists an element $v \in V$ such that $(v, w) \neq 0$, and whenever $v \neq 0$ in V, there exists an element $w \in W$ such that $(v, w) \neq 0$.*

Let V and W be non-singularly paired by $(\ ,\ )$, and define

$$\phi : V \to W^* \text{ by } \phi(v)(w) = (v, w) \quad \text{ for } v \in V, w \in W$$

We have that ϕ is injective. Suppose $\phi(v) = \phi(v')$ this is $(v, w) = (v', w)$ for all $w \in W$ then $(v - v', w) = 0$ for all $w \in W$ as the pairing is non-singular it follows that $v = v'$. Similarly there is an injective map $W \to V^*$. Therefore V and W have the same dimension, and hence ϕ is an isomorphism of V with W^*. Thus a non-singular pairing of V and W in a canonical way yields an isomorphism $\phi : V \to W^*$ and similarly an isomorphism $W \to V^*$.

Definition 2.2.2. *A Riemannian manifold (M, g) consists of a $\mathscr{C}^\infty$-manifold M and an Euclidean inner product g_x on each of the tangent spaces $T_{M,x}$ of M, such that $x \mapsto g_x$ varies smoothly. This means that for any two smooth vector fields X, Y the inner product $g_x(X|x, Y|x)$ is a smooth function of x.*

Let (M, g) be an oriented Riemannian $\mathscr{C}^\infty$ manifold of dimension m, and let $E \to M$ be a hermitian vector bundle of rank r on M. We denote respectively by $(\xi_1, \dots, \xi_m)$ and $(e_1, \dots, e_r)$ orthonormal frames of T_M and of E on a coordinate chart $\Omega \subset M$, and let $(\xi_1^*, \dots, \xi_,^*)$, $(e_1^*, \dots, e_m^*)$ be the corresponding dual coframes of T_M^*, E^* respectively. Further, let $\mathrm{d}V$ be the Riemannian volume element on M. The exterior algebra $\bigwedge^\bullet T_M^*$ is endowed with a natural inner product $\langle \bullet, \bullet \rangle$, given by

$$\langle u_1 \wedge \dots \wedge u_p, v_1 \wedge \dots \wedge v_p \rangle = \det(\langle u_j, v_k \rangle)_{1 \le j,k \le p}, \quad u_j, v_k \in T_M^* \tag{2.20}$$

for all p, with $\bigwedge^\bullet T_M^* = \bigoplus \bigwedge^p T_M^*$ an orthogonal direct sum. Thus the family of covectors $\xi_I^* = \xi_{i_1}^* \wedge \dots \wedge \xi_{i_p}^*$, $i_1 < i_2 < \dots < i_p$, defines an orthonormal basis of $\bigwedge^\bullet T_M^*$. One denotes by $\langle \bullet, \bullet \rangle$ the corresponding inner product on $\bigwedge^\bullet T_M^* \otimes E$, this is if $\alpha, \beta \in \bigwedge T_M^* \otimes E$, $\alpha = \sum \alpha_i \otimes e_i$, $\beta = \sum \beta_j \otimes e_j$ for the frame $e = (e_k)$, then $\langle \alpha, \beta \rangle = \langle \alpha_i, \beta_j \rangle \cdot \langle e_i, e_j \rangle = \langle \langle \alpha_i, \beta_j \rangle e_i, e_j \rangle$.

Definition 2.2.3. *The Hodge operator $\star$ is the endomorphism of $\bigwedge^\bullet T_M^*$ defined by a collection of linear maps such that*

$$\star : \bigwedge^p T_M^* \to \bigwedge^{m-p} T_M^*, \quad u \wedge \star v = \langle u, v \rangle \, \mathrm{d}V, \quad \forall u, v \in \bigwedge^p T_M^*.$$

The existence and uniqueness of this operator follows from the duality pairing

$$\begin{aligned}
&\bigwedge^{p} T_M^* \times \bigwedge^{m-p} T_M^* \to \mathbb{R} \\
&(u, w) \mapsto u \wedge w / \,\mathrm{d}V := \sum \varepsilon(I, \complement I) u_I w_{\complement I},
\end{aligned} \tag{2.21}$$

where $u = \sum_{|I|=p} u_I \xi_I^*$, $w = \sum_{|J|=m-p} w_J \xi_J^*$, and where $\varepsilon(I, \complement I)$ is the sign of the permutation $(1, 2, \ldots, m) \mapsto (I, \complement I)$ defined by I followed by the complementary (ordered) multi-indices $\complement I$. From this, we will deduce that

$$\star v = \sum_{|I|=p} \varepsilon(I, \complement I) v_I \xi_{\complement I}^* =: v'. \tag{2.22}$$

First, it is clear that $u \wedge v' = \langle u, v \rangle \,\mathrm{d}V$. In order to get one expression for $\langle u, v \rangle$ consider

$$\langle u, v \rangle = \sum_{|J|=|I|=p} u_I v_J \langle \xi_I^*, \xi_J^* \rangle$$

now, taking $(\xi) = (\xi_1 \ldots, \xi_m)$ an orthonormal frame of T_M on a coordinate chart $\Omega \subset M$ we have that $\langle \xi_I^*, \xi_J^* \rangle = 0$ if and only if $I \neq J$. Suppose $I \neq J$ then there exists $i \in I$ such that $i \notin J$, therefore $\langle \xi_i^*, \xi_j^* \rangle = 0$ for all $j \in J$, this is $\langle \xi_I^*, \xi_J^* \rangle = 0$ as the determinant will have a row of zeros, hence

$$\langle u, v \rangle = \sum_{|I|=p} u_I v_I \tag{2.23}$$

(2.22) follows from this.

More generally, the sesquilinear pairing $\{\bullet, \bullet\}$ defined by (1.15) induces an operator $\star$ on the vector-valued forms, such that

$$\star : \bigwedge^{p} T_M^* \otimes E \to \bigwedge^{m-p} T_M^* \otimes E, \quad \{s, \star t\} = \langle s, t \rangle \,\mathrm{d}V, \tag{2.24}$$

$$\star t = \sum_{|I|=p,\lambda} \varepsilon(I, \complement I) t_{I,\lambda} \xi_{\complement I}^* \otimes e_\lambda, \quad \forall s, t \in \bigwedge^{p} T_M^* \otimes E, \tag{2.25}$$

for $t = \sum t_{I,\lambda} \xi_I^* \otimes e_\lambda$. Since $\varepsilon(I, \complement I)\varepsilon(\complement I, I) = (-1)^{p(m-p)} = (-1)^{p(m-1)}$, as p and p^2 are both even or odd, we immediately obtain

$$\star \star t = (-1)^{p(m-1)} t \text{ on } \bigwedge^{p} T_M^* \otimes E. \tag{2.26}$$

We have that $\star$ is an isometry of $\bigwedge^\bullet T^*_M \otimes E$, this is for $u, v \in T^*_M \otimes E$ we have $\langle u, v\rangle = \langle \star u, \star v\rangle$, because taking orthonormal frames (ξ) and (e) for T_M and E respectively, and using (2.23) we have

$$\langle u, v\rangle = \sum \varepsilon(I, \complement I)\varepsilon(I, \complement I) u_I v_I = \langle \star u, \star v\rangle.$$

We will need also a variant of the $\star$ operator, namely the antilinear operator

$$\# : \bigwedge^p T^*_M \otimes E \to \bigwedge^{m-p} T^*_M \otimes E^*,$$

defined by $s \wedge \# t = \langle s, t\rangle \,\mathrm{d}V$, where the exterior product $\wedge$ is combined with the canonical pairing $E \times E^* \to \mathbb{C}$. We have

$$\# t = \sum_{|I|=p.\lambda} \varepsilon(I, \complement I)\bar{t}_{I,\lambda} \xi^*_{\complement I} \otimes e^*_\lambda. \tag{2.27}$$

Definition 2.2.4. *Assume given a tangent vector $\theta \in T_M$ and a form $u \in \bigwedge^p T^*_M$. The contraction $\theta \lrcorner u \in \bigwedge^{p-1} T^*_M$ is defined by*

$$\theta \lrcorner u(\eta_1, \ldots, \eta_{p-1}) = u(\theta, \eta_1, \ldots, \eta_{p-1}), \quad \eta_j \in T_M.$$

In terms of the basis (ξ_j), $\bullet \lrcorner \bullet$ is the bilinear operator characterized by

$$\xi_l \lrcorner (\xi^*_{i_1} \wedge \ldots \wedge \xi^*_{i_p}) = \begin{cases} 0 & \text{if } l \notin \{i_1, \ldots, i_p\}, \\ (-1)^{k-1} \xi^*_{i_1} \wedge \ldots \wedge \hat{\xi^*_{i_k}} \wedge \ldots \wedge \xi^*_{i_p} & \text{if } l = i_k. \end{cases}$$

This same formula is also valid when (ξ_j) is not orthonormal, as we are taking derivations.

Proposition 2.2.5. *Let u, v be forms of degree k and l respectively and $\theta \in T_M$ a tangent vector. Then*

$$\theta \lrcorner (u \wedge v) = (\theta \lrcorner u) \wedge v + (-1)^k u \wedge (\theta \lrcorner v),$$

this is $\theta \lrcorner \bullet$ is a derivation of the exterior algebra.

Proof. By linearity it is sufficient to consider the case when θ, u, v are basic vector and forms, i.e.

$$\theta = \xi_j,\ u = \xi^*_{i_1} \wedge \ldots \wedge \xi^*_{i_k},\ v = \xi^*_{j_1} \wedge \ldots \wedge \xi^*_{j_l}$$

as seen before $\theta \lrcorner u \neq 0$ iff $j \in A := \{i_1, \ldots, i_k\}$ analogously $\theta \lrcorner v \neq 0$ iff $j \in B := \{j_1, \ldots, j_k\}$. Now we proceed by cases:

If $j \in A$ but $j \notin B$

$$\theta \lrcorner (u \wedge v) = (-1)^{m-1} \xi^*_{i_1} \wedge \ldots \wedge \hat{\xi^*_{i_m}} \wedge \ldots \wedge \xi^*_{i_k} \wedge \ldots \wedge \xi^*_{j_l} = (\theta \lrcorner u) \wedge v$$

while $u \wedge (\theta \lrcorner v) = 0$.

Similarly, of $j \in B$ but $j \notin A$ then:

$$\theta(u \wedge v) = (-1)^{k+n-1} \xi^*_{i_1} \wedge \ldots \wedge \xi^*_{i_k} \wedge \xi^*_{j_1} \wedge \ldots \wedge \hat{\xi^*_{j_n}} \wedge \ldots \wedge \xi^*_{j_l} = (-1)^k u \wedge (\theta \lrcorner v)$$

while $(\theta \lrcorner u) \wedge v = 0$. Hence in both cases the formula holds.

If $\xi_j \in A \cap B$ then $u \wedge v = 0$, and hence $\theta \lrcorner (u \wedge v) = 0$. On the other hand

$$\begin{aligned}(\theta \lrcorner u) \wedge v + (-1)^k u \wedge (\theta \lrcorner v) =& (-1)^{m-1} \xi^*_{i_1} \wedge \ldots \wedge \hat{\xi^*_{i_m}} \wedge \ldots \xi^*_{i_k} \wedge \ldots \wedge \xi^*_{j_l} + \\ & (-1)^{k+n-1} \xi^*_{i_1} \wedge \ldots \wedge \xi^*_{i_k} \wedge \ldots \wedge \hat{\xi^*_{j_n}} \wedge \ldots \wedge \xi^*_{j_l} = 0\end{aligned}$$

because $\xi^*_{i_1} \wedge \ldots \wedge \hat{\xi^*_{i_m}} \wedge \ldots \xi^*_{i_k} \wedge \ldots \wedge \xi^*_{j_l}$ and $\xi^*_{i_1} \wedge \ldots \wedge \xi^*_{i_k} \wedge \ldots \wedge \hat{\xi^*_{j_n}} \wedge \ldots \wedge \xi^*_{j_l}$ differ only in the position of ξ^*_j. In the first product it is at the $k+n$ position, and in the second at the m position. Hence, the difference in sign is $(-1)^{(n-1)+(k+1-m)}$ which leads to the required cancellation. □

Moreover, if $\tilde{\theta} = \langle \bullet, \theta \rangle \in T^*_M$, the operator $\theta \lrcorner \bullet$ is the adjoint of $\tilde{\theta} \wedge \bullet$, i.e.,

$$\langle \theta \lrcorner u, v \rangle = \langle u, \tilde{\theta} \wedge v \rangle, \quad \forall u, v \in \bigwedge^{\bullet} T^*_M. \tag{2.28}$$

Indeed, using a similar reasoning that in the previous proposition, by linearity, using $\theta = \xi_l$, $u = \xi^*_I$, $v = \xi^*_J$ and using the fact that $\tilde{\theta}$ acts as ξ^*_l this property is immediate.

Let E be a Hermitian vector bundle on M, and let D_E be a Hermitian connection on E. We consider the Hilbert space $L^2(M, \bigwedge^p T^*_M \otimes E)$ of p-forms on M with values in E, with the given L^2 scalar product

$$(s,t) = \int_M \langle s, t \rangle \, \mathrm{d}V$$

already introduced in (2.4). Here $\langle s, t \rangle$ is the specific scalar product on $\bigwedge^p T^*_M \otimes E$ associated to the Riemannian scalar product on $\bigwedge^p T^*_M$ and the Hermitian pairing on E

Theorem 2.2.6. *The formal adjoint of D_E acting on $\mathscr{C}^\infty(M, \bigwedge^p T^*_M \otimes E)$ is given by*

$$D^*_E = (-1)^{mp+1} \star D_E \star .$$

Proof. If $s \in \mathscr{C}^\infty(M, \bigwedge^p T^*_M \otimes E)$ and $t \in \mathscr{C}^\infty(M, \bigwedge^{p+1} T^*_M \otimes E)$ have compact support, we have

$$\begin{aligned}(D_E s, t) &= \int_M \langle D_E s, t \rangle \, \mathrm{d}V = \int_M \{D_E s, \star t\} \\ &= \int_M \mathrm{d}\{s, \star t\} - (-1)^p \{s, D_E \star t\} = (-1)^{p+1} \int_M \{s, D_E \star t\}\end{aligned}$$

by an application of Stokes' theorem and the fact that we have been working with manifolds without boundary. As a consequence (2.24) and (2.26) imply

$$(D_E s, t) = (-1)^{p+1}(-1)^{p(m-1)}\int_M \{s, \star\star D_E \star t\} = (-1)^{mp+1}(s, \star D_E \star t).$$

The desired formula follows. □

Remark 2.2.7. *In the case of the trivial connection* d *on* $E = M\times\mathbb{C}$*, the formula becomes* $\mathrm{d}^* = (-1)^{mp+1}\star\mathrm{d}\star$*. If* m *is even, these formulas reduce to*

$$\mathrm{d}^* = -\star\mathrm{d}\star, \quad D_E^* = -\star D_E \star .$$

Definition 2.2.8. *The Laplace–Beltrami operator is the second order differential operator acting on the bundle* $\bigwedge^p T_M^*\otimes E$*, such that*

$$\Delta_E = D_E D_E^* + D_E^* D_E.$$

In particular, the Laplace-Beltrami operator acting on $\bigwedge^p T_M^*$ is $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

Since D_E^* is the adjoint of D, the Laplacian Δ is formally self-adjoint i.e. $(\Delta_E s, t) = (s, \Delta_E t)$ whenever the forms s, t are $\mathscr{C}^\infty$ and that one of them has compact support.

Example 2.2.9. *Let* M *be an open subset of* $\mathbb{R}^m$ *and* $g = \sum_{i=1}^m \mathrm{d}x_i^2$*. In that case we get*

$$u = \sum_{|I|=p} u_I\,\mathrm{d}x_I, \quad \mathrm{d}u = \sum_{|I|=p,j}\frac{\partial u_I}{\partial x_j}\mathrm{d}x_j\wedge\mathrm{d}x_I,$$

$$(u,v) = \int_M \langle u, v\rangle\,\mathrm{d}V = \int_M \sum_I u_I v_I\,\mathrm{d}V.$$

One can write $\mathrm{d}v = \sum \mathrm{d}x_j\wedge(\partial v/\partial x_j)$ *where* $\partial v/\partial x_j$ *denotes the form* v *in which all coefficients* v_I *are differentiated as* $\partial v_I/\partial x_j$*. An integration by parts combined with contraction and its adjoint gives*

$$\begin{aligned}(\mathrm{d}^*u, v) = (u, \mathrm{d}v) &= \int_M \langle u, \sum_j \mathrm{d}x_j\wedge\frac{\partial v}{\partial x_j}\rangle\,\mathrm{d}V\\ &= \int_M \sum_j \langle \frac{\partial}{\partial x_j}\lrcorner u, \frac{\partial v}{\partial x_j}\rangle\,\mathrm{d}V = -\int_M\langle\sum_j \frac{\partial}{\partial x_j}\lrcorner\frac{\partial u}{\partial x_j}, v\rangle\,\mathrm{d}V,\\ \mathrm{d}^*u = -\sum_j \frac{\partial}{\partial x_j}\lrcorner\frac{\partial u}{\partial x_j} &= -\sum_{I,j}\frac{\partial u_I}{\partial x_j}\frac{\partial}{\partial x_j}\lrcorner\,\mathrm{d}x_I.\end{aligned}$$

Using the same notation for the $\mathrm{d}u$ *as used for* $\mathrm{d}v$. *We get therefore*

$$\mathrm{d}\mathrm{d}^{*}u = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} \mathrm{d}x_k \wedge \left(\frac{\partial}{\partial x_j} \lrcorner \, \mathrm{d}x_I \right),$$
$$\mathrm{d}^{*}\mathrm{d}u = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} \frac{\partial}{\partial x_j} \lrcorner (\mathrm{d}x_k \wedge \mathrm{d}x_I).$$

Since

$$\frac{\partial}{\partial x_j} \lrcorner (\mathrm{d}x_k \wedge \mathrm{d}x_I) = \left(\frac{\partial}{\partial x_j} \lrcorner \, \mathrm{d}x_k \right) \mathrm{d}x_I - \mathrm{d}x_k \wedge \left(\frac{\partial}{\partial x_j} \lrcorner \, \mathrm{d}x_I \right),$$

using the property of derivation. We get that the second term will cancel with $\mathrm{d}\mathrm{d}^{*}u$, *and* $(\partial/\partial x_j) \lrcorner \, \mathrm{d}x_k = \delta_{jk}$. *We obtain*

$$\Delta u = -\sum_{I} \left(\sum_{j} \frac{\partial^2 u_I}{\partial x_j^2} \right) \mathrm{d}x_I.$$

Consequently Δ *has the same expression as the elementary Laplacian operator, up to a minus sign.*

Now we proceed to calculate the symbol of the Laplacian.

For every $\mathscr{C}^\infty$ function f, Leibnitz rule gives $e^{-tf} D_E(e^{tf} s) = t \, \mathrm{d}f \wedge s + D_E s$. By definition of the symbol, we therefore find

$$\sigma_{D_E}(x, \xi) \cdot s = \xi \wedge s, \quad \forall \xi \in T^{*}_{M,x}, \quad \forall s \in \bigwedge^{p} T^{*}_{M} \otimes E.$$

From formula (2.7), we obtain $\sigma_{D_E^*} = -(\sigma_{D_E})^*$, therefore

$$\sigma_{D_E^*}(x, \xi) \cdot s = -\tilde{\xi} \lrcorner s,$$

where $\tilde{\xi} \in T_M$ is the adjoint tangent vector of ξ. The equality $\sigma_{\Delta_E} = \sigma_{D_E} \sigma_{D_E^*} + \sigma_{D_E^*} \sigma_{D_E}$, and $\tilde{\xi} \lrcorner (\xi \wedge s) = (\tilde{\xi} \lrcorner \xi) s - \xi \wedge (\tilde{\xi} \lrcorner s)$ implies that

$$\sigma_{\Delta_E}(x, \xi) \cdot s = -\xi \wedge (\tilde{\xi} \lrcorner s) - \tilde{\xi} \lrcorner (\xi \wedge s) = -(\tilde{\xi} \lrcorner \xi) s,$$
$$\sigma_{\Delta_E}(x, \xi) \cdot s = -\left| \xi \right|^2 s.$$

In particular, Δ_E is always an elliptic operator.

Definition 2.2.10. *Let* E *be a Hermitian vector bundle on a compact Riemannian manifold. A Hermitian connection* D_E *is said to be flat if* $\Theta(D_E) = D_E^2 = 0$.

A standard example is the trivial bundle $E = M \times \mathbb{C}$ with the connection $D_E = \mathrm{d}$
If we assume that D_E is flat it implies that D_E defines a generalized de Rham complex

$$\mathscr{C}^\infty(M, E) \xrightarrow{D_E} \mathscr{C}^\infty(M, \bigwedge^1 T_M^* \otimes E) \to \cdots \to \mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E) \xrightarrow{D_E} \cdots .$$

The cohomology groups of this complex are denoted by $H^p_{DR}(M, E)$.

Definition 2.2.11. *The space of harmonic forms of degree p relative to the Laplace–Beltrami operator $\Delta_E = D_E D_E^* + D_E^* D_E$ is defined by*

$$\mathscr{H}^p(M, E) = \{s \in \mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E); \Delta_E s = 0\}. \tag{2.29}$$

Using the linearity and the adjoint we have that $(\Delta_E s, s) = \|D_E s\|^2 + \|D_E^* s\|^2$, we see that $s \in \mathscr{H}^p(M, E)$ if and only if $D_E s = D_E^* s = 0$.

Theorem 2.2.12. *For any p, there exists an orthogonal decomposition*

$$\mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E) = \mathscr{H}^p(M, E) \oplus \operatorname{Im} D_E \oplus \operatorname{Im} D_E^*,$$
$$\operatorname{Im} D_E = D_E\big(\mathscr{C}^\infty(M, \bigwedge^{p-1} T_M^* \otimes E)\big),$$
$$\operatorname{Im} D_E^* = D_E^*\big(\mathscr{C}^\infty(M, \bigwedge^{p+1} T_M^* \otimes E)\big).$$

Proof. By taking adjoint and using the remark before the Theorem, it is immediate that $\mathscr{H}^p(M, E)$ is orthogonal to both subspaces $\operatorname{Im} D_E$ and $\operatorname{Im} D_E^*$. The orthogonality of these two subspaces is also clear, thanks to the assumption $D_E^2 = 0$:

$$(D_E s, D_E^* t) = (D_E^2 s, t) = 0.$$

We apply now the Finiteness theorem 2.1.16 to the elliptic operator $\Delta_E = \Delta_E^*$ acting on p-forms, i.e. on the bundle $F = \bigwedge^p T_M^* \otimes E$. We get

$$\mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E) = \mathscr{H}^p(M, E) \oplus \Delta_E\big(\mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E)\big),$$
$$\operatorname{Im} \Delta_E = \operatorname{Im}(D_E D_E^* + D_E^* D_E) \subset \operatorname{Im} D_E + \operatorname{Im} D_E^*,$$

where $\operatorname{Im} D_E$ and $\operatorname{Im} D_E^*$ are as defined in the statement of the Theorem. Further, since $\operatorname{Im} D_E$ and $\operatorname{Im} D_E^*$ are orthogonal to $\mathscr{H}^p(M, E)$, these spaces are contained in $\operatorname{Im} \Delta_E$. □

Theorem 2.2.13 (Hodge Isomorphism Theorem)**.** *The de Rham cohomology groups $H^p_{DR}(M,E)$ are finite dimensional; moreover $H^p_{DR}(M,E) \simeq \mathscr{H}(M,E)$.*

Proof. From the decomposition in Theorem 2.2.12, we obtain

$$\begin{aligned} B^p_{DR}(M,E) &= D_E(\mathscr{C}^\infty(M, \bigwedge^{p-1} T^*_M \otimes E)) = \operatorname{Im} D_E, \\ Z^p_{DR}(M,E) &= \operatorname{Ker} D_E = (\operatorname{Im} D^*_E)^\perp = \mathscr{H}^p(M,E) \oplus \operatorname{Im} D_E. \end{aligned}$$

The first equation is by definition. For the second take $u \in \operatorname{Im} D^*_E$ such that $u \in Z^p_{DR}$, we have that $u = D^*_E v$ for certain $v \in \mathscr{C}^\infty(M, \bigwedge^{p+1} T^*_M \otimes E)$ but then

$$0 = (D_E D^*_E v, v) = (D^*_E v, D^*_E v),$$

which implies $D^*_E v = u = 0$. This shows that any de Rham cohomology class contains a unique harmonic representative □

Theorem 2.2.14 (Poincaré duality)**.** *The bilinear pairing*

$$H^p_{DR}(M,E) \times H^{m-p}_{DR}(M,E^*) \to \mathbb{C}, \quad (s,t) \mapsto \int_M s \wedge t$$

is a non-singular pairing.

Proof. We can write

$$s = \sum_{|I|=p,\lambda} \alpha_{I,\lambda} \xi^*_I \otimes e_\lambda, \quad t = \sum_{|J|=m-p,\mu} \beta_{J,\beta} \xi^*_J \otimes e^*_\mu$$

therefore as in 2.27

$$s \wedge t = \sum_{\substack{\lambda,\mu \\ |I|=p \\ |J|=m-p}} e^*_\mu(e_\lambda) \alpha_{I,\lambda} \beta_{J,\mu} \xi^*_I \wedge \xi^*_J$$

A dual connection D_{E^*} is defined in the following way

$$\langle D_{E^*}\phi, s\rangle = \mathrm{d}\langle \phi, s\rangle - \langle \phi, D_E s\rangle, \quad \forall \phi \in E^*, s \in E$$

where $\langle -,- \rangle : E^* \times E \to \mathscr{C}^\infty(M)$ is the pairing. Obviously D_{E^*} is also a flat connection, and for any $s_1 \in \mathscr{C}^\infty_\bullet(M,E), s_2 \in \mathscr{C}^\infty_\bullet(M,E^*)$ we have

$$\mathrm{d}(s \wedge t) = (D_E s) \wedge t + (-1)^{\deg s} s \wedge D_{E^*} t. \tag{2.30}$$

As we are working with forms s, t representing the cohomology classes, we have that $D_E(s) = D_E(t) = 0$. We will prove that the pairing in cohomology is well defined. Given s_1 another representative of the de Rham class s, then $s_1 = s + D_E u$ for some form u, and by Stokes' formula

$$\begin{aligned}\int_M s_1 \wedge t &= \int_M s \wedge t + \int_M D_E s_1 \wedge t \\ &= \int_M s \wedge t + \int_M \mathrm{d}(s_1 \wedge t) = \int_M s \wedge t.\end{aligned}$$

Observe also from its definition that the bilinear function depends on the orientation on M. For $s \in \mathscr{C}^\infty(M, \bigwedge^p T_M^* \otimes E), t \in \mathscr{C}^\infty(M, \bigwedge^{p+1} T_M^* \otimes E)$, we will prove using (2.30) that

$$D_{E^*} = (-1)^{p+1} \# D_E^* s. \tag{2.31}$$

We proceed in a similar way as in the proof of Theorem 2.2.6

$$\begin{aligned}(D_E s, t) &= \int_M \langle D_E s, t\rangle \,\mathrm{d}V = \int_M D_E s \wedge \# t \\ &= \int_M \mathrm{d}(s \wedge \# t) - (-1)^p s \wedge D_{E^*} \# t \\ &= (-1)^{p+1}(-1)^{p(m+1)} \int_M \langle s, \# D_{E^*} \# t\rangle \,\mathrm{d}V \\ &= (-1)^{p+1}(-1)^{p(m+1)} (s, \# D_{E^*} \# t)\end{aligned}$$

by applying # to the left side we obtain (2.31.) Proceeding in a similar way we obtain

$$(D_{E^*})^*(\# s) = (-1)^{p+1} \# D_E s \tag{2.32}$$

and from (2.31) and (2.32) we obtain

$$\Delta_{E^*}(\# s) = \# \Delta_E s.$$

Consequently $s \in \mathscr{H}^{m-p}(M, E^*)$ if and only if $s \in \mathscr{H}(M, E)$. Since

$$\int_M s \wedge \# s = \int_M |s|^2 \,\mathrm{d}V = \|s\|^2 .$$

It follows that the pairing is non-singular. □

2.3 Hermitian and Kähler Manifolds

Let V be a complex vector space, which we can also consider as a real vector space equipped with an endomorphism I of complex structure. Let W : $\mathrm{Hom}(V, \mathbb{R})$. Then $V_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ admits the decomposition

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

into $\mathbb{C}$- linear and $\mathbb{C}$-antilinear forms. Let

$$W^{1,1} = W^{1,0} \otimes W^{0,1} \subset \bigwedge^2 W_{\mathbb{C}}$$

and $W_{\mathbb{R}}^{1,1} := W^{1,1} \cap \bigwedge^2 W_{\mathbb{R}}$. We then have

Lemma 2.3.1. *There is a natural identification between the Hermitian forms on $V \times V$ and the elements of $W_{\mathbb{R}}^{1,1}$ given by*

$$h \mapsto \omega = -\Im h.$$

Here, $\Im$ is the function sending a complex number to its imaginary part, h is a complex-valued bilinear form which is $\mathbb{C}$-linear on the left and $\mathbb{C}$- antilinear on the right, and satisfies $h(u, v) = \overline{h(v, u)}$

Proof. Firstly, we define a bilinear form ω on V by $\omega(u, v) = -\Im h(u, v)$, if h is Hermitian, then

$$\omega(v, u) = -\Im h(v, u) = -\Im \overline{h(u, v)} = \Im h(u, v) = -\omega(u, v)$$

and thus ω is alternating. Thus it is an element of $\bigwedge^2 W_{\mathbb{R}}$, and we need to check that it is also an element of $W^{1,1}$. But by definition, ω is in $W^{1,1}$ if and only if the natural extension of ω (by $\mathbb{C}$-bilinearity) to a 2-form on $V_{\mathbb{C}}$ vanishes on the bivectors $(u, v),\ u,\ v \in V^{1,0}$, this is, it has no element of $W^{1,0} \otimes W^{1,0}$ besides 0, and on the bivectors $(u, v),\ u,\ v \in V^{0,1}$, the second property follows from the first by using complex conjugation. Now, $V^{1,0}$ is generated by the $\tilde{v} = v - iIv,\ v \in V$. Let $v, u \in V$, we have:

$$\omega(\tilde{u}, \tilde{v}) = \omega(u, v) - \omega(Iu, Iv) - i(\omega(u, Iv) + \omega(Iu, v)).$$

As h is $\mathbb{C}$-linear on the left and $\mathbb{C}$-antilinear on the right, we have $h(Iu, Iv) = h(u, v)$ and thus $\omega(u, v) = \omega(Iu, Iv)$. Similarly, the condition

$$h(u, Iv) = -h(Iu, v)$$

implies that $\omega(u, Iv) = -\omega(Iu, v)$. Thus $\omega(\tilde{u}, \tilde{v}) = 0$.

Conversely, let us start from $\omega \in W_{\mathbb{R}}^{1,1}$, and set

$$g(u,v) = \omega(u, Iv), \quad h(u,v) = g(u,v) - i\omega(u,v).$$

We have

$$h(u, Iv) = g(u, Iv) - i\omega(u, Iv) = -\omega(u,v) - ig(u,v) = -ih(u,v).$$

As ω is alternating, we have $\Im h(u,v) = -\Im h(v,u)$. Moreover, as $\omega(u, Iv) = -\omega(Iu, v)$, we have $g(u,v) = g(v,u)$ and thus $h(u,v) = \overline{h(v,u)}$. Thus h is Hermitian. For the definition it is clear that these two constructions are inverses of each other. □

Definition 2.3.2. *We say that a real alternating form ω of type $(1,1)$ on V is positive if the corresponding Hermitian form h is positive definite.*

Take $\mathbb{C}$-linear coordinates $z_1, \ldots, z_n$ on V. Then for $z = (t_1, \ldots, t_n)$, $z' = (t_1, \ldots, t_n)$, we have $h(z,z') = \sum_{i,j} h_{ij} t_i \bar{t}'_j$, with $h_{ij} = \bar{h}_{ji} = h(e_i, e_j)$, where the e_i form the basis of V dual (z_i). Using the elementary $\Im z = (z - \bar{z})/(2i)$, we have

$$w(z,z') = \frac{i}{2} \sum_{i,j} h_{ij} \left(t_i \bar{t}'_j - t'_i \bar{t}_j \right).$$

In other words, we have the equality of bilinear forms on V

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij} z_i \wedge \bar{z}_j \in W^{1,1}$$

The proof of Lemma 2.3.1 show that we can also identify such Hermitian forms h with the symmetric bilinear forms associated to them by the relation $g(u,v) = \Re h(u,v)$. The forms g obtained in this way are exactly those satisfying the condition $g(Iu, Iv) = g(u,v)$.

Definition 2.3.3. *1. A Hermitian manifold is a pair (X, ω) where ω is a positive definite $\mathscr{C}^\infty$ $(1,1)$-form on X.*

2. The metric ω is said to be Kähler if $\mathrm{d}\omega = 0$.

3. X is called a Kähler manifold if X has at least one Kähler metric.

Since ω is real, the condition $\mathrm{d}\omega = 0$, $\partial\omega = 0$, $\overline{\partial}\omega = 0$ are all equivalent. It is clear that the first implies the others by degree reasons, the others imply the first by using $\mathrm{d} = \overline{\mathrm{d}}$. In local coordinates, we see that $\mathrm{d}\omega = 0$ if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}, \quad 1 \le j, k, l \le n.$$

This follows immediately from the definition of $\partial\omega$

$$\partial\omega = \sum_{l,j,k} \frac{\partial h_{jk}}{\partial z_l} \, \mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \bar{z}_k.$$

For $x_0 \in X$ it is possible to choose coordinates $z_1 \ldots, z_n$ such that $h_{jk} = \delta_{jk}$. These are called euclidean coordinates. We calculate the $2n$-form in these coordinates at x_0.

We have $\omega = \frac{i}{2} \sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$, and so

$$\begin{aligned}
\omega^n &= \left(\frac{i}{2} \sum_{j=1}^{n} (\mathrm{d}x_i + i \, \mathrm{d}y_i) \wedge (\mathrm{d}x_i - i \, \mathrm{d}y_i) \right)^n \\
&= \left(\frac{i}{2} \sum_{j=1}^{n} (-2i) \, \mathrm{d}x_i \wedge \mathrm{d}y_i \right)^n \\
&= \left(\sum_{j=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_j \right)^n .
\end{aligned}$$

Using induction we have that

$$\left(\sum_{j=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \right)^k = k! \sum_{1 \le j_1 < \cdots < j_k \le n} \mathrm{d}x_{j_1} \wedge \mathrm{d}y_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k} \wedge \mathrm{d}y_{j_k}.$$

It follows that

$$\omega^n = n! \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

In general coordinates one would get

$$\omega^n = n! \det(h_{jk}) \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Consequently the (n, n) form

$$\mathrm{d}V = \frac{1}{n!} \omega^n \tag{2.33}$$

is positive and coincides with the Hermitian volume element of X. If X is compact then $\int_X \omega^n = n! \operatorname{Vol}_\omega(X) > 0$. This simple observation already implies that a compact Kähler manifold must satisfy certain restrictive topological conditions:

Proposition 2.3.4. *1. If (X, ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X, \mathbb{R})$, then $\{\omega\} \neq 0$.*

2. *If X is compact Kähler, then $H^{2k}(X,\mathbb{R}) \neq 0$ or $0 \leq k \leq n$. Indeed, $\{\omega\}$ is a non-zero class of $H^{2k}(X,\mathbb{R})$.*

Proof. Using Leibnitz rule and induction we get that $\mathrm{d}(\omega^k) = 0$. Suppose that there is some $(2k-1)$-form ϕ with $\mathrm{d}\phi = \omega^k$. Then $\mathrm{d}(\phi \wedge \omega^{n-k}) = \omega^n$, and by Stokes' theorem

$$\int_X \omega^n = \int_X \mathrm{d}(\phi \wedge \omega^{n-k}) = 0.$$

□

Example 2.3.5. *The Fubini-Study metric is a canonical Kähler metric on the projective space $\mathbb{P}^n$. Let $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ be the standard open covering and*

$$\phi_i : U_i \cong \mathbb{C}^n, (z_0 : \ldots : z_n) \mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right).$$

Then one defines

$$\omega_i := \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{l=0}^n \left|\frac{z_l}{z_i}\right|^2\right) \in \mathcal{A}^{1,1}(U_i),$$

which under ϕ_i corresponds to

$$\frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{k=1}^n |w_k|^2 + 1\right).$$

Let us first show that $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$, i.e. that the ω_i glue to a global form $\omega_{FS} \in \mathcal{A}^{1,1}(\mathbb{P}^n)$. Indeed

$$\log\left(\sum_{l=0}^n \left|\frac{z_l}{z_i}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2 \sum_{l=0}^n \left|\frac{z_l}{z_j}\right|^2\right) = \log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \log\left(\sum_{l=0}^n \left|\frac{z_l}{z_j}\right|^2\right).$$

Thus, it suffices to show that $\partial\overline{\partial}\log\left(\left|\frac{z_j}{z_i}\right|^2\right) = 0$ on $U_i \cap U_j$. Since $\frac{z_j}{z_i}$ is the j-th coordinate function on U_i, this follows from

$$\partial\overline{\partial}\log|z|^2 = \partial\left(\frac{1}{z\bar{z}}\overline{\partial}(z\bar{z})\right) = \partial\left(\frac{z\,\mathrm{d}\bar{z}}{z\bar{z}}\right) = \partial\left(\frac{\mathrm{d}\bar{z}}{\bar{z}}\right) = 0.$$

Next, we observe that ω_{FS} is a real $(1,1)$-form. Indeed, $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$, which came from $\mathrm{d}^2 = 0$, yields $\omega_i = \overline{\omega_i}$. Moreover, ω_{FS} is closed, as $\partial\omega_i = \frac{1}{2\pi}\partial^2\overline{\partial}\log() = 0$.

It remains to show that ω_{FS} is positive definite, i.e. that ω_{FS} really is the Kähler form associated to a metric. This can be verified on each U_i separately. A straightforward computation yields

$$\begin{aligned}\partial\overline{\partial}\log\left(1+\sum_{i=1}^{n}|w_i|^2\right) &= \frac{\left(1+\sum|w_i|^2\right)(\mathrm{d}w_i\wedge\mathrm{d}\overline{w}\ :_i)-(\overline{w}_i\,\mathrm{d}w_i)\left(\sum w_i\,\mathrm{d}\overline{w}_i\right)}{\left(1+\sum|w_i|^2\right)^2}\\ &= \frac{1}{\left(1+\sum|w_i|^2\right)^2}\sum h_{ij}\,\mathrm{d}w_i\wedge\mathrm{d}\overline{w}_j,\end{aligned}$$

with $h_{ij}=(1+\sum|w_i|^2)\delta_{ij}-\overline{w}_iw_j$. The matrix (h_{ij}) is positive definite, since for $u\neq 0$ the Cauchy-Schwarz inequality for the standard hermitian product $(\ ,\ )$ on $\mathbb{C}^n$ yields

$$\begin{aligned}u^t(h_{ij})\overline{u} &= (u,u)+(w,w)(u,u)-u^t\overline{w}w^t\overline{u}\\ &= (u,u)+(w,w)(u,u)-(u,w)(w,u)\\ &= (u,u)+(w,w)(u,u)-\overline{(w,u)}(w,u)\\ &= (u,u)+(w,w)(u,u)-|(w,u)|^2>0.\end{aligned}$$

As the Fubini-Study metric is a very prominent example of a Kähler metric, we will dwell on it a bit longer.

Let us consider the natural projection $\pi:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n$. Then

$$\pi^*\omega_{FS}=\frac{i}{2\pi}\partial\overline{\partial}\log(\|z\|^2).$$

Indeed, over $\pi^{-1}U_i=\{(z_0,\ldots,z_n)|z_i\neq 0\}$ one has

$$\begin{aligned}\pi^*\omega_{FS} &= \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_i}\right|^2\right)\\ &= \frac{i}{2\pi}\partial\overline{\partial}(\log(\|z\|^2)-\log(|z_i|^2)),\end{aligned}$$

but $\partial\overline{\partial}\log(|z_i|^2)=0$, as has been shown above.

We conclude this example by proving the equation

$$\int_{\mathbb{P}^1}\omega_{FS}=1,$$

which serve as normalization in the definition of Chern classes. Moreover, since $\mathbb{P}^1\cong S^2$ and thus $H^2(\mathbb{P}^1,\mathbb{Z})=H^2(S^2,\mathbb{Z})\cong\mathbb{Z}$, it shows that $\{\omega_{FS}\}\in H^2(\mathbb{P}^1,\mathbb{Z})$ is a generator.

The integral is explicitly computed as follows

$$\begin{aligned}\int_{\mathbb{P}^1} \omega_{FS} &= \int_{\mathbb{C}} \frac{i}{2\pi} \frac{1}{(1+|w|^2)^2} \, \mathrm{d}w \wedge \mathrm{d}\overline{w} \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{1}{(1+\|(x,y)\|^2)^2} \, \mathrm{d}x \wedge \mathrm{d}y \\ &= 2\int_0^\infty \frac{r}{(1+r^2)^2} \, \mathrm{d}r = 1.\end{aligned}$$

Example 2.3.6. *Any positive definite Hermitian form $\omega = i \sum h_{jk} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ with constant coefficients on $\mathbb{C}^n$ defines a Kähler metric on X.*

Definition 2.3.7. *A complex submanifold M of a complex manifold X is a differentiable submanifold whose tangent space at each point is stable under the almost complex structure operator I of X.*

We note that the induced almost complex structure on M is integrable, since (M, I) also satisfies the Newlander-Nirenberg integrability criterion.

Proposition 2.3.8. *Let X be a Kähler manifold with Kähler metric h and let M be a complex submanifold of X. Then h induces a Kähler metric on M, and with this metric M becomes, therefore, a Kähler manifold.*

Proof. Let $j : M \to X$ be the injection mapping. Then $h_M = j^* h$ defines a metric on M, and $j^*\omega = \omega_M$ is the associated fundamental form to h_M on M. Since $\mathrm{d}\omega_M = j^* \, \mathrm{d}\omega = 0$, we have that ω_M is also a Kähler fundamental form. □

Theorem 2.3.9. *Let ω be a positive definite $\mathscr{C}^\infty$ $(1,1)$-form on X. For ω to be Kähler, it is necessary and sufficient to show that at any point $x_0 \in X$, there exists a holomorphic coordinate system $(z_1 \ldots, z_n)$ centered at x_0 such that*

$$\omega = i \sum_{1 \le l,m \le n} \omega_{lm} \, \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_m, \quad \omega_{lm} = \delta_{lm} + O\left(|z|^2\right). \tag{2.34}$$

If ω is Kähler, the coordinates $(z_j)_{1 \le j \le n}$ can be chosen such that

$$\omega_{lm} = \left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle = \delta_{lm} - \sum_{1 \le j,k \le n} c_{jklm} z_j \overline{z}_k + O\left(|z|^3\right), \tag{2.35}$$

where (c_{jklm}) are the coefficients of the Chern curvature

$$\Theta(T_X)_{x_0} = \sum_{j,k,l,m} c_{jklm} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k \otimes \left(\frac{\partial}{\partial z_l}\right)^* \otimes \frac{\partial}{\partial z_m} \tag{2.36}$$

associated to (T_X, ω) at x_0. Such a system (z_j) will be called a geodesic coordinate system at x_0.

Proof. As 2.34 implies that terms of the form $a_{jlm} z_k + a_{jlm} \overline{z}_k$ do not occur, it implies $\mathrm{d}_{x_0} \omega = 0$, so the condition is sufficient. Assume now that ω is Kähler. Then one can choose local coordinates $(x_1, \ldots, x_n)$ such that $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ is an ω-orthonormal basis of $T^*_{x_0} X$. Therefore

$$\omega = i \sum_{1 \leq l,m \leq n} \tilde{\omega}_{lm} \, \mathrm{d}x_l \wedge \mathrm{d}\overline{x}_m, \quad \text{where}$$
$$\tilde{\omega}_{lm} = \delta_{lm} + O(|x|) = \delta_{lm} + \sum_{1 \leq j \leq n} (a_{jlm} x_j + a'_{jlm} \overline{x}_j) + O\left(|x|^2\right). \tag{2.37}$$

Since ω is real, $\tilde{\omega}_{lm} = \overline{\tilde{\omega}}_{ml}$, and we have $a'_{jlm} = \overline{a}_{jml}$. Furthermore, the Kähler condition $\partial \omega_{lm} / \partial x_j = \partial \omega_{jm} / \partial x_l$ at x_0 o, implies $a_{jlm} = a_{ljm}$. Set now

$$z_m = x_m + \frac{1}{2} \sum_{j,l} a_{jlm} x_j x_l, \quad 1 \leq m \leq n.$$

Then (z_m) is a coordinate system at x_0, and

$$\mathrm{d}z_m = \mathrm{d}x_m + \frac{1}{2} \sum_{j,l} a_{jlm} (\mathrm{d}x_j) x_l + \frac{1}{2} \sum_{j,l} a_{jlm} x_j (\mathrm{d}x_l) = \mathrm{d}x_m + \sum_{j,l} a_{jlm} x_j \, \mathrm{d}x_l$$

and similarly

$$\mathrm{d}\overline{z}_m = \mathrm{d}\overline{x}_m + \sum_{j,l} a'_{jlm} \overline{x}_j \, \mathrm{d}\overline{x}_l.$$

Therefore

$$\begin{aligned} i \sum_m \mathrm{d}z_m \wedge \mathrm{d}\overline{z}_m &= i \sum_m \mathrm{d}x_m \wedge \mathrm{d}\overline{x}_m + i \sum_{j,l,m} a_{jlm} x_j \, \mathrm{d}x_l \wedge \mathrm{d}\overline{x}_m \\ &\quad + i \sum_{j,l,m} a'_{jlm} \overline{x}_j \, \mathrm{d}x_m \wedge \mathrm{d}x_l + O\left(|x|^2\right) \\ &= i \sum_{l,m} \tilde{\omega}_{lm} \, \mathrm{d}x_l \wedge \mathrm{d}\overline{x}_m + O\left(|x|^2\right) = \omega + O\left(|z|^2\right). \end{aligned}$$

Condition (2.34) is proved. Suppose the coordinates (x_m) chosen from the beginning so that (2.34) holds with respect to (x_m). Then the Taylor expansion (2.37) can be refined into

$$\begin{aligned} \tilde{\omega}_{lm} &= \delta_{lm} + O\left(|x|^2\right) \\ &= \delta_{lm} + \sum_{j,k} (a_{jklm} x_j \overline{x}_k + a'_{jklm} x_j x_k + a''_{jklm} \overline{x}_j \overline{x}_k). \end{aligned} \tag{2.38}$$

These new coefficients satisfy the relations

$$a'_{jklm} = a_{kjlm}, \quad a''_{jklm} = \overline{a}'_{jkml}, \quad \overline{a}_{jklm} = a_{kjml}.$$

The Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at $x = x_0$ gives the equality $a'_{jklm} = a'_{lkjm}$; in particular a'_{jklm} is invariant under all permutations of j, k, l. If we set

$$z_m = x_m + \frac{1}{3}\sum_{j,k,l} a'_{jklm} x_j x_k x_l, \quad 1 \leq m \leq n,$$

then by (2.38) we find

$$\begin{aligned} \mathrm{d}z_m &= \mathrm{d}x_m + \sum_{j,k,l} a'_{jklm} x_j x_k \, \mathrm{d}x_l, \quad 1 \leq m \leq n, \\ \omega &= i \sum_{1\leq m\leq n} \mathrm{d}x_m \wedge \mathrm{d}\overline{z}_m + i \sum_{j,k,l,m} a_{jklm} x_j \overline{x_k} \, \mathrm{d}x_l \wedge \mathrm{d}\overline{x}_m + O\left(|x|^3\right), \\ \omega &= i \sum_{1\leq m\leq n} \mathrm{d}z_m \wedge \mathrm{d}\overline{z}_m + i \sum_{j,k,l,m} a_{jklm} z_j \overline{z}_k \, \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_m + O\left(|z|^3\right). \end{aligned} \tag{2.39}$$

It is now easy to compute the Chern curvature $\Theta(T_X)_{x_0}$ in terms of the coefficients a_{jklm}. Indeed

$$\begin{aligned} \left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle &= \delta_{lm} + \sum_{j,k} a_{jklm} z_j \overline{z}_k + O\left(|z|^3\right), \\ \partial \left\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle &= \left\{ D' \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\} = \sum_{j,k} a_{jklm} \overline{z}_k \, \mathrm{d}z_j + O\left(|z|^2\right), \\ \Theta\left(T_X\right) \cdot \frac{\partial}{\partial z_l} &= D'' D' \left(\frac{\partial}{\partial z_l} \right) = -\sum_{j,k,m} a_{jklm} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k \otimes \frac{\partial}{\partial z_m} + O\left(|z|\right), \end{aligned}$$

therefore $c_{jklm} = -a_{jklm}$ and the expansion (2.35) follows from (2.39) □

Remark 2.3.10. *As a by-product of our computations, we find that on a Kähler manifold the coefficients of* $\Theta(T_X)$ *satisfy the symmetry relations*

$$\overline{c}_{jklm} = c_{kjml}, \quad c_{jklm} = c_{lkjm} = c_{jmlk} = c_{lmjk}.$$

Let (X, ω) be a hermitian manifold and let $z_j = x_j + iy_j$, $1 \leq j \leq n$, be analytic coordinates at a point $x \in X$ such that $\omega(x) = i \sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ is diagonalized at this point. The associated hermitian form is the $h(x) = 2\sum \mathrm{d}z_j \otimes \mathrm{d}\overline{z}_j$ and its real part is the euclidean metric $2\sum(\mathrm{d}x_j)^2 + (\mathrm{d}y_j)^2$. It follows from this that $|\mathrm{d}x_j| = |\mathrm{d}y_j| = 1/\sqrt{2}$, $|\mathrm{d}z_j| = |\mathrm{d}\overline{z}_j| = 1$, and that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $(T_x^* X, \omega)$. Formula 2.20 with u_j, v_k in the orthogonal sum $(\mathbb{C} \otimes T_X)^* = T_X^* \oplus \overline{T_X^*}$ defines a natural inner product on the exterior algebra $\bigwedge^\bullet(\mathbb{C} \otimes T_X)^*$. The norm of a form

$$u = \sum_{I,J} u_{I,J}\, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_j \in \bigwedge(\mathbb{C} \otimes T_X)^*$$

at the given point x is then equal to

$$|u(x)|^2 = \sum_{I,J} |u_{I,J}(x)|^2 . \tag{2.40}$$

The Hodge $\star$ operator 2.2.3 can be extended to $\mathbb{C}$-valued forms by the formula

$$u \wedge \star\overline{v} = \langle u, v\rangle\, \mathrm{d}V. \tag{2.41}$$

It follows that $\star$ is a $\mathbb{C}$-linear isometry

$$\star : \bigwedge^{p,q} T_X^* \longrightarrow \bigwedge^{n-q,n-p} T_X^*.$$

this follows from the definition of $\star$ and the fact that for $\gamma_i \in \bigwedge^{p_i,q_i} T_X^*$ with $p_1 + p_2 + q_1 + q_2 = 2n$ but $(p_1 + p_2, q_1 + q_2) \neq (n, n)$ implies $\gamma_1 \wedge \gamma_2 = 0$.

The usual operators of hermitian geometry are the operators $\mathrm{d}, \delta = -\star\mathrm{d}\star, \Delta = \mathrm{d}\delta + \delta\,\mathrm{d}$ already defined, and their complex counterparts

$$\begin{cases} \mathrm{d} = \partial + \overline{\partial} \\ \delta = \partial^* + \overline{\partial}^*, \qquad \partial^* = (\partial)^* = -\star\overline{\partial}\star, \quad \overline{\partial}^* = (\overline{\partial})^* = -\star\partial\star, \\ \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \end{cases}$$

Another important operator, the so called Lefschetz operator, is the operator L of type $(1,1)$ defined by

$$Lu = \omega \wedge u \tag{2.42}$$

and its adjoint Λ:

$$\langle u, \Lambda v\rangle = \langle Lu, v\rangle. \tag{2.43}$$

by simple calculations we obtain

$$\langle Lu, v\rangle\, \mathrm{d}V = Lu \wedge \star\overline{v} = \omega \wedge u \wedge \star\overline{v} = u \wedge (\omega \wedge \star\overline{u}) = \langle u, \star^{-1}(L(\star v))\rangle\, \mathrm{d}V$$

therefore $\Lambda = \star^{-1} \circ L \circ \star$.

Definition 2.3.11. *Let $H : \bigwedge^\bullet T_X^* \to \bigwedge^\bullet T_X^*$ be the counting operator defined by $H|_{\bigwedge^k T_X^*} = (k-n)\cdot \mathrm{Id}$, where $\dim_{\mathbb{R}} T_X^* = 2n$. Equivalently*

$$H = \sum_{k=0}^{2n} (k-n)\cdot \Pi^k.$$

where Π^k is the projection in the subspace of differential forms of degree k.

With H, L, Λ, Π, etc., we dispose of a large number of linear operator on $\bigwedge^\bullet T_X^*$ and one might wonder whether they commute. In fact, they do not, but their commutators can be computed. This is done in the next proposition.

Definition 2.3.12. *If A, B are endomorphism (of pure degree) of the graded module $\mathscr{C}^\infty(X, \bigwedge^{\bullet,\bullet} T_X^*)$, their graded commutator (or graded Lie bracket is defined by)*

$$[A,B] = AB - (-1)^{ab} BA \tag{2.44}$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, one has the following formal Jacobi identity

$$(-1)^{ca}\,[A,[B,C]] + (-1)^{ab}\,[B,[C,A]] + (-1)^{bc}\,[C,[A,B]] = 0. \tag{2.45}$$

Proposition 2.3.13. *Let (X, ω) be a Hermitian manifold and consider the following linear operators on $\bigwedge^\bullet T_X^*$: The associated Lefschetz operator L, its dual Λ, and the counting operator H. They satisfy:*

1. $[H, L] = 2L$
2. $[H, \Lambda] = -2\Lambda$
3. $[L, \Lambda] = H$.

Proof. Let $\alpha \in \bigwedge^k T_X^*$. Then $[H,L](\alpha) = (k+2-n)(\omega\wedge\alpha) - \omega^((k-n)\alpha) = 2\omega\wedge\alpha$. Analogously, $[H,\Lambda](\alpha) = (k-2-n)(\Lambda\alpha) - \Lambda((k-n)\alpha) = -2\Lambda\alpha$.

The third assertion is the most difficult one. We will prove it by induction on the dimension of T_X. Assume we have a decomposition $T_X = W_1 \oplus W_2$ which is compatible with the hermitian product and the almost complex structure, i.e. $(T_X, \langle\ ,\ \rangle, I) = (W_1, \langle\ ,\ \rangle_1, I_1) \oplus (W_2, \langle\ ,\ \rangle_2, I_2)$. Then $\bigwedge^\bullet T_X^* = \bigwedge^\bullet W_1^* \otimes \bigwedge^\bullet W_2^*$ and in particular $\bigwedge^2 T_X^* = \bigwedge^2 W_1^* \oplus \bigwedge^2 W_2^* \oplus W_1^* \otimes W_2^*$. Since $W_1 \oplus W_2$ is orthogonal, the fundamental form ω on T_X decomposes as $\omega_1 \oplus \omega_2$, where ω_i is the fundamental form on W_i (no component in $W_1^* \otimes W_2^*$ for the orthogonality). Hence the Lefschetz operator L on $\bigwedge^\bullet T_X^*$

is the direct sum of the Lefschetz operators L_1 and L_2 acting on $\bigwedge^\bullet W_1^*$ and $\bigwedge W_2^*$, respectively, i.e. $L = L_1 + L_2$ with L_1 and L_2 acting as $L_1 \otimes 1$ respectively $1 \otimes L_2$ on $\bigwedge^\bullet W_1^* \otimes \bigwedge^\bullet W_2^*$.

Let $\alpha, \beta \in \bigwedge^\bullet T_X^*$ and suppose that both are split, i.e. $\alpha = \alpha_1 \otimes \alpha_2$, $\beta = \beta_1 \otimes \beta_2$, with $\alpha_i, \beta_i \in \bigwedge^\bullet W_i^*$. Then $\langle \alpha, \beta \rangle = \langle \alpha_1, \beta_2 \rangle \cdot \langle \alpha_2, \beta_2 \rangle$. Therefore,

$$\begin{aligned}\langle \alpha, L\beta \rangle &= \langle \alpha, L_1(\beta_1) \otimes \beta_2 \rangle + \langle \alpha, \beta_1 \otimes L_2(\beta_2) \rangle \\ &= \langle \alpha_1, L_1\beta_1 \rangle \langle \alpha_2, \beta_2 \rangle + \langle \alpha_1, \beta_1 \rangle \langle \alpha_2, L_2\beta_2 \rangle \\ &= \langle \Lambda_1\alpha_1, \beta_1 \rangle \langle \alpha_2, \beta_2 \rangle + \langle \alpha_1, \beta_1 \rangle \langle \Lambda_2\alpha_2, \beta_2 \rangle \\ &= \langle \Lambda_1(\alpha_1) \otimes \alpha_2, \beta_1 \otimes \beta_2 \rangle + \langle \alpha_1 \otimes \Lambda_2(\alpha_2), \beta_2 \rangle.\end{aligned}$$

Hence $\Lambda_1 + \Lambda_2$, where Λ_i is the dual Lefschetz operator on $\bigwedge^\bullet W_i^*$. This yields

$$\begin{aligned}[L, \Lambda](\alpha_1 \otimes \alpha_2) &= (L_1 + L_2)(\Lambda_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes \Lambda_2(\alpha_2)) \\ &\quad - (\Lambda_1 + \Lambda_2)(L_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes L_2(\alpha_2)) \\ &= [L_1, \Lambda_1](\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes [L_2, \Lambda_2](\alpha_2).\end{aligned}$$

By induction hypothesis $[L_i, \Lambda_i] = H_i$ and therefore.

$$\begin{aligned}[L, \alpha](\alpha_1 \otimes \alpha_2) &= H_1(\alpha_1) \otimes \alpha_2 + \alpha_1 \otimes H_2(\alpha_2) \\ &= (k_1 - n_1)(\alpha_1 \otimes \alpha_2) + (k_2 - n_2)(\alpha_1 \otimes \alpha_2) \\ &= (k_1 + k_2 - n_1 - n_2)(\alpha_1 \otimes \alpha_2),\end{aligned}$$

for $\alpha_i \in \bigwedge^{k_i} W_i^*$ and $n_i = \dim_{\mathbb{C}}(W_i, I_i)$.

It remains to prove the case $\dim_{\mathbb{C}}(T_X, I) = 1$. With respect to a basis x_1, y_1 of T_X one has

$$\begin{aligned}\bigwedge^\bullet T_X^* &= \bigwedge^0 T_X^* \oplus \bigwedge^1 T_X^* \oplus \bigwedge^2 T_X^* \\ &= \mathbb{R} \oplus (x_1^*\mathbb{R} \oplus y_1^*\mathbb{R}) \oplus \omega\mathbb{R}\end{aligned}$$

Moreover, $L : \bigwedge^0 T_X^* \to \bigwedge^2 T_X^*$ and $\Lambda : \bigwedge^2 T_X^* \to \bigwedge^0 T_X^*$ are given by $1 \mapsto \omega$ and $\omega \mapsto 1$, respectively. Hence. $[L, \Lambda]|_{\bigwedge^0 T_X^*} = -\Lambda L|_{\bigwedge^0 T_X^*} = -1$, $[L, \Lambda]|_{\bigwedge^1 T_X^*} = 0$, and $[L, \Lambda]|_{\bigwedge^2 T_X^*} = 1$. □

Corollary 2.3.14. *$[L^r, \Lambda](\alpha) = r(k - n + r - 1)L^{r-1}(\alpha)$ for all $\alpha \in \bigwedge^k T_X^*$.*

Proof. This is easily seen by induction on r as follows:

$$\begin{aligned}
[L^r, \Lambda](\alpha) &= L^r \Lambda\alpha - \Lambda L^r \alpha \\
&= L(L^{r-1}\Lambda\alpha - \Lambda L^{r-1}\alpha) + L\Lambda L^{r-1}\alpha - \Lambda L L^{r-1}\alpha \\
&= L[L^{r-1}, \Lambda](\alpha) + [L, \Lambda](L^{r-1}\alpha) \\
&= (r-1)(k-n+(r-1)-1)L^{r-1}(\alpha) + (2r-2+k-n)L^{r-1}(\alpha) \\
&= r(k-n+r-1)L^{r-1}(\alpha).
\end{aligned}$$

□

A consequence of these identities is the Lefschetz decomposition of the cohomology of a compact Kähler manifold. To put this in proper perspective, we must first digress for a moment and discuss representations of $\mathfrak{sl}(2)$.

We have that $\mathfrak{sl}(2)$ is the Lie algebra of the group $SL(2)$; it is realized as the vector space of 2×2 complex matrices with trace 0, and with the bracket

$$[A, B] = AB - BA.$$

We take as standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the relations

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

Now, let V be a finite-dimensional complex vector space $\mathrm{End}(V)$ its algebra of endomorphisms. We want to study Lie algebra maps

$$\rho : \mathfrak{sl}(2) \to \mathrm{End}(V),$$

i.e., linear maps ρ such that

$$\rho([A, B]) = \rho(A)\rho(B) - \rho(B)\rho(A).$$

Such a map is called a representation of $\mathfrak{sl}(2)$ in V; V is called an $\mathfrak{sl}(2)$-module. A subspace of V fixed under $\rho(\mathfrak{sl}(2))$ is called a submodule , V (or ρ) is called irreducible if V has no trivial submodules. By a fundamental result, which we won't prove here, every submodule W of an $\mathfrak{sl}(2)$-module V has a complementary submodule $W^{\perp}$; thus every $\mathfrak{sl}(2)$-module is the direct sum of irreducible $\mathfrak{sl}(2)$ modules, and to study representations of $\mathfrak{sl}(2)$ we need only look at irreducibles ones.

Suppose then that V is an irreducible $\mathfrak{sl}(2)$-module. The key to analyzing the structure of V is to look at the eigenspaces for $\rho(H)$ (from now on, we will omit the ρ's). These are called weight spaces. First of all, note that if $v \in V$ is an eigen vector of H with eigenvalue λ, then Xv and Yv are also eigenvectors of H, with eigenvalues $\lambda + 2$ and $\lambda - 2$, respectively: this follows from

$$\begin{aligned} H(Xv) &= XHv + [H, X]v \\ &= X\lambda v + 2Xv \\ &= (\lambda + 2)Xv, \end{aligned}$$

and similarly for Yv. Since H can have only a finite number of eigenvalues, we see from this that X and Y are nilpotent. We say that $v \in V$ is primitive if v is an eigenvector for H and $Xv = 0$; primitive elements always exists, indeed, let v_0 be an eigenvector of H, and consider the sequence of eigenvectors or H

$$v_0, Xv_0, X^2v_0, \ldots, X^nv_0, \ldots$$

The nonzero terms in this sequence are linearly independent, since they are eigenvectors with differing eigenvalues, so the sequence must terminate, and hence for some fixed k, $X^kv_0 = 0$, $X^{k-1}v_0 \neq 0$, and $v = X^{k-1}v_0$ is a primitive vector.

Proposition 2.3.15. *If $v \in V$ is primitive, then V is generated as a vector space by*

$$v, Yv, Y^2v, \ldots$$

Proof. Since V is irreducible, we need only show that the linear span V' of $\{Y^iv\}$ is fixed under $\mathfrak{sl}(2)$. For the remarks before the proposition $HV' \subset V'$ and $YV' \subset V'$ by definition of linear span. We show $XV' \subset V'$ by induction: $Xv = 0$ trivially lies in V', and in general

$$XY^nb = YXY^{n-1}v + HY^{n-1}v;$$

so

$$XY^{n-1}v \in V' \Rightarrow XY^nv \in V'.$$

□

As before, the elements $\{Y^nv\}_n$ that are nonzero are linearly independent. Thus we have the picture of $V : V = \oplus V_\lambda$, where V_λ is one-dimensional,

$$H(V_\lambda) = V_\lambda, \quad X(V_\lambda) = V_{\lambda+2}, \quad Y(V_\lambda) = V_{\lambda-2}.$$

Proposition 2.3.16. *All eigenvalues for H are integers, and we can write*

$$V = V_n \oplus V_{n-1} \oplus \ldots \oplus V_{-n+2} \oplus V_{-n}.$$

Proof. Let v be primitive, and suppose $Y^n v \neq 0, \quad Y^{n+1}v = 0$, and $Hv = \lambda v$. Then

$$\begin{aligned} Xv &= 0, \\ XYv &= YXv + Hv = \lambda v, \\ XY^2 v &= YXYv + HYv \\ &= Y\lambda v + (\lambda - 2)Yv = (\lambda + (\lambda - 2))Yv, \end{aligned}$$

and in general $XY^m v = YXY^{m-1}v + HY^{m-1}v$ so we have

$$\begin{aligned} XY^m v &= (\lambda + (\lambda - 2) + (\lambda - 4) + \cdots + (\lambda - 2(m-1)))Y^{m-1}v \\ &= (m\lambda - m^2 + m)Y^{m-1}v, \end{aligned}$$

and since $Y^n \neq 0$, $Y^{n+1}v = 0$,

$$(n+1)\lambda - (n+1)^2 + n + 1 = 0 \Rightarrow \lambda = n.$$

□

In summary, the irreducible $\mathfrak{sl}(2)$ modules are indexed by nonnegative integers n; for each such n the corresponding $\mathfrak{sl}(2)$-module $V(n)$ has dimension $n+1$. The eigenvalues of H acting on $V(n)$ are $-n, -n+2, \ldots, n-2, n$ each appearing with multiplicity 1.

For any $\mathfrak{sl}(2)$-module V, not necessarily irreducible, we define the Lefschetz decomposition of V as follows: Let $PV = \operatorname{Ker}\rho(X)$; then

$$V = PV \oplus YPV \oplus Y^2PV \oplus \cdots,$$

and this decomposition is compatible with the decomposition of V into eigenspaces V_m for H. We also see that the maps

$$V_m \underset{X_m}{\overset{Y^m}{\rightleftarrows}} V_{-m}$$

are isomorphism. Finally, in general,

$$(\operatorname{Ker} X) \cap V_k = \operatorname{Ker}(Y^{k+1} : V_k \to V_{-k-2}).$$

We return to our hermitian manifold (X, ω)

Corollary 2.3.17. *There is a natural action of the Lie algebra* $\mathfrak{sl}(2)$ *on the vector space* $\bigwedge^{\bullet,\bullet} T_X^*$, *i.e. a morphism of Lie algebras* $\rho : \mathfrak{sl}(2) \to \operatorname{End}(\bigwedge^{\bullet,\bullet} T_X^*)$, *given by* $\rho(X) = L$, $\rho(Y) = \Lambda$, *and* $\rho(H) = H$.

Definition 2.3.18. *Let (X, ω) and the induced operators L, Λ, and H be as before. An element $\alpha \in \bigwedge^k T_X^*$ is called primitive if $\Lambda\alpha = 0$. The linear subspace of all primitive elements $\alpha \in \bigwedge^k T_X^*$ is denoted by* $\mathrm{Prim}^k \subset \bigwedge^k T_X^*$

Proposition 2.3.19. *Let (X, ω) be an hermitian manifold, and let L and Λ be the associated Lefschetz operators*

1. *There exists a direct sum decomposition of the form:*

$$\bigwedge^k T_X^* = \bigoplus_{i \geq 0} L^i(\mathrm{Prim}^{k-2i}). \tag{2.46}$$

   This is the Lefschetz decomposition. Moreover, (2.46) is orthogonal with respect to $\langle\ ,\ \rangle$.

2. *If $k > n$, then* $\mathrm{Prim}^k = 0$.

3. *The map $L^{n-k} : \mathrm{Prim}^k \to \bigwedge^{2n-k} T_X^*$ is injective for $k \leq n$.*

4. *The map $L^{n-k} : \bigwedge^k T_X^* \to \bigwedge^{2n-k} T_X^*$ is bijective for $k \leq n$.*

5. *If $k \leq n$, then* $\mathrm{Prim}^k = \{\alpha \in \bigwedge^k T_X^* | L^{n-k+1}\alpha = 0\}$

Proof. 1. Since $\bigwedge^\bullet T_X^*$ is a finite dimensional $\mathfrak{sl}(2)$ representation, it is a direct sum of irreducible ones. We have seen that any finite-dimensional $\mathfrak{sl}(2)$ admits a primitive vector v, i.e. $\Lambda v = 0$. Using Corollary 2.3.14 one finds that for any primitive v the subspace $v, Lv, L^2v, \ldots$ defines a subrepresentation. Thus the irreducible $\mathfrak{sl}(2)$-representations are of this form. Altogether this proof the existence of the direct sum decomposition (2.46).

2. If $\alpha \in \mathrm{Prim}^k$, $k > n$ and $0 < i$ minimal with $L^i\alpha = 0$, then by Corollary 2.3.14 one has $0 = [L^r, \Lambda](\alpha) = r(k - n + r - 1)L^{r-1}\alpha$. This yields $r = 0$, i.e. $\alpha = 0$.

3. Let $0 \neq \alpha \in \mathrm{Prim}$, $k \leq n$ and $0 < i$ minimal with $L^i\alpha = 0$. Then again by Corollary 2.3.14 one finds $0 = [L^r, \Lambda](\alpha) = r(k-n+r-1)L^{r-1}$ and, therefore $K-n+r-1 = 0$. In particular $L^{n-k}(\alpha) \neq 0$. Moreover, $L^{n-k+1}\alpha = 0$, which will be used in the proof of 5.

4. Follows from $1, 2$ and 3.

5. We have seen already that $\mathrm{Prim}^k \subset \mathrm{Ker}(L^{n-k+1})$. Conversely, let $\alpha \in \bigwedge^l T_X^*$ with $L^{n-k+1}\alpha = 0$. Then $L^{n-k+2}\Lambda\alpha = L^{n-k+2}\Lambda\alpha - \Lambda L^{n-k+2}\alpha = (n-k+2)L^{n-k+1}\alpha = 0$. But by 4 the map L^{n-k+2} is injective on $\bigwedge^{k-2} T_X^*$. Hence, $\Lambda\alpha = 0$.

□

Let us consider a few special cases. Obviously, $\bigwedge^0 T_X^* = \mathrm{Prim}^0 = \mathbb{C}$ and $\bigwedge^1 T_X^* = \mathrm{Prim}^1$. In degree two and four one has $\bigwedge^2 T_X^* = \omega\mathbb{C} \oplus \mathrm{Prim}^2$ and $\bigwedge^4 T_X^* = \omega^2\mathbb{C} \oplus L(\mathrm{Prim}^2) \oplus \mathrm{Prim}^4$.

Now we proceed to calculate some famous identities.

Assume first that $X = \Omega \subset \mathbb{C}^n$ is an open subset and that ω is the standard Kähler metric

$$\omega = i \sum_{1\le j\le n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j.$$

For any form $u \in \mathscr{C}^\infty(\Omega, \bigwedge^{p,q} T_X^*)$ we have

$$\partial u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \mathrm{d}z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J, \tag{2.47}$$

$$\overline{\partial} u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} \mathrm{d}\overline{z}_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J. \tag{2.48}$$

Since the global L^2 inner product is given by

$$(u,v) = \int_\Omega \sum_{I,J} u_{I,J}\overline{v}_{I,J}\,\mathrm{d}V,$$

making similar computations as in Example 2.2.9 we show that

$$\partial^* u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} \frac{\partial}{\partial z_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J), \tag{2.49}$$

$$\overline{\partial}^* u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner (\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J). \tag{2.50}$$

We first prove a lemma due to Akizuki and Nakano.

Lemma 2.3.20. *In $\mathbb{C}^n$, we have $[\overline{\partial}^*, L] = i\partial$.*

Proof. Using the same convention as in Example 2.2.9 for the notation, formula (2.50) can be written more briefly as

$$\overline{\partial}^* u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \left(\frac{\partial u}{\partial z_k}\right).$$

Then we get

$$[\partial^*, L]u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \left(\frac{\partial}{\partial z_k}(\omega \wedge u)\right) + \omega \wedge \sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \left(\frac{\partial u}{\partial z_k}\right).$$

Since ω has constant coefficients $\frac{\partial}{\partial z_k}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_k}$ and therefore using Leibnitz' rule

$$\begin{aligned}[\overline{\partial}^*, L]u &= -\sum_k \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \left(\omega \wedge \frac{\partial u}{\partial z_k} \right) - \omega \wedge \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial u}{\partial z_k} \right) \right) \\ &= -\sum_k \left(\frac{\partial}{\partial \overline{z}_k} \lrcorner \omega \right) \frac{\partial u}{\partial z_k}.\end{aligned}$$

By Leibnitz $\frac{\partial}{\partial \overline{z}_k} \lrcorner \omega = -i \, \mathrm{d}z_k$, so

$$[\partial^*, L]u = i \sum_k \mathrm{d}z_k \wedge \frac{\partial u}{\partial z_k} = i\partial u.$$

□

We are now ready to derive the basic commutation relations in the case of an arbitrary Kähler manifold (X, ω).

Theorem 2.3.21. *If (X, ω) is Kähler, then*

$$\begin{aligned}[\overline{\partial}^*, L] &= \quad i\partial, \quad [\partial^*, L] = - \quad i\overline{\partial} \\ \left[\Lambda, \overline{\partial}\right] &= - \quad i\partial^*, \quad [\Lambda, \partial] = \quad i\overline{\partial}^*.\end{aligned}$$

Proof. It is sufficient to verify the first relation, because the second one is the conjugate of the first, and the relations of the second line are the adjoin of those of the first line. If (z_j) is a geodesic coordinate system at a point $x_0 \in X$,we know $\omega = \omega_0 + O(|z|^2)$ where ω_0 is the standard Kähler form on $\mathbb{C}^n$. Since the quantity $[\overline{\partial}^*, L]$ involves only the first derivative, the calculations for $\mathbb{C}^n$ with the standard Kähler form holds also on X, proving the identity. □

Corollary 2.3.22. *If (X, ω) is Kähler, the complex Laplace-Beltrami operators satisfy*

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta.$$

Proof. It will be first shown that $\Delta_{\overline{\partial}} = \Delta_\partial$. We have

$$\Delta_{\overline{\partial}} = [\overline{\partial}, \overline{\partial}^*] = -i \left[\overline{\partial}, [\Lambda, \partial]\right]$$

Since $[\partial, \overline{\partial}] = 0$, in the graded commutator, Jacobi identity 2.45 implies that

$$-\left[\overline{\partial}, [\Lambda, \partial]\right] + \left[\partial, [\overline{\partial}, \Lambda]\right] = 0$$

hence $\Delta_{\overline{\partial}} = \left[\partial, -i[\overline{\partial}, \Lambda]\right] = [\partial, \partial^*] = \Delta_\partial$. On the other hand

$$\Delta = [\partial + \overline{\partial}, \partial^* + \overline{\partial}^*] = \Delta_\partial + \Delta_{\overline{\partial}} + [\partial, \overline{\partial}^*] + [\overline{\partial}, \partial^*].$$

Thus, it is enough to prove:

Lemma 2.3.23. $[\partial, \overline{\partial}^*] = [\overline{\partial}, \partial^*] = 0.$

Proof. We have $[\partial, \overline{\partial}^*] = -i\,[\partial, [\Lambda, \partial]]$ and 2.45 implies

$$-\,[\partial, [\Lambda, \partial]] + [\Lambda, [\partial, \partial]] + [\partial, [\partial, \Lambda]] = 0,$$

hence $-2\,[\partial, [\Lambda, \partial]] = 0$ and $[\partial, \overline{\partial}^*] = 0$. The second relation $[\overline{\partial}, \partial^*] = 0$ is the adjoint of the first. □

□

Theorem 2.3.24. *The operator Δ commutes with all operators $\star, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, L, \Lambda$.*

Proof. The identities $[\partial, \Delta_\partial] = [\partial^*, \Delta_\partial] = 0$, $[\overline{\partial}, \Delta_{\overline{\partial}}] = [\overline{\partial}^*, \Delta_{\overline{\partial}}] = 0$ and $[\Delta, \star] = 0$ are immediate. Furthermore, the equality $[\partial, L] = \partial\omega = 0$ together with the Jacobi identity implies

$$[L, \Delta_\partial] = [L, [\partial, \partial^*]] = -\,[\partial, [\partial^*, L]] = i[\partial, \overline{\partial}] = 0.$$

By adjunction, we also get $[\Delta_\partial, \Lambda] = 0$. □

Let (X, ω) be a Kähler manifold and E a holomorphic hermitian vector bundle of rank r over X. We denote by D_E the Chern connection of E, by $D_E^* = -\star D_E \star$ the formal adjoint of D_E, and by $D_E'^*$, $D_E''^*$ the components of D_E^* of type $(-1, 0)$ and $(0, -1)$.

Corollary 2.3.22 implies that the principal symbol of the operator $\Delta_E'' = D'' D_E''^* + D_E''^* D''$ is one half that of Δ_E. The operator Δ_E'' acting on each space $\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^* \otimes E)$ is thus a self-adjoint elliptic operator. Since $D''^2 = 0$, the following results can be obtained in a way similar to those of 2.2.12.

Theorem 2.3.25. *For every bidegree (p, q), there exists an orthogonal decomposition*

$$\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^* \otimes E) = \mathscr{H}^{p,q} \oplus \operatorname{Im} D_E'' \oplus \operatorname{Im} D_E''^*$$

where $\mathscr{H}^{p,q}(X, E)$ is the space of Δ_E''-harmonic forms in $\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X \otimes E)$.

The above decomposition shows that the subspace of $\overline{\partial}$-cocycles in $\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^* \otimes E)$ is $\mathscr{H}^{p,q}(X, E) \oplus \operatorname{Im} D_E''$. From this, we infer

Theorem 2.3.26 (Hodge isomorphism theorem)**.** *The Dolbeault cohomology group $H^{p,q}(X, E)$ is finite dimensional, and there is an isomorphism*

$$H^{p,q} \simeq \mathscr{H}^{p,q}(X, E).$$

Theorem 2.3.27 (Serre duality theorem)**.** *The pairing*

$$H^{p,q} \times H^{n-p,n-q}(X, E^*) \longrightarrow \mathbb{C}, \quad (s,t) \longmapsto \int_X s \wedge t$$

is non-singular.

Proof. Let $s_1 \in \mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^* \otimes E)$, $s_2 \in \mathscr{C}^\infty(X, \bigwedge^{n-p,n-1-1} T_X^* \otimes E)$. Since $s_1 \wedge s_2$ is of bidegree $(n, n-1)$, we have

$$\mathrm{d}(s_1 \wedge s_2) = \overline{\partial}(s_1 \wedge s_2) = \overline{\partial} s_1 \wedge s_2 + (-1)^{p+q} s_1 \wedge \overline{\partial} s_2. \tag{2.51}$$

Stokes' formula implies that the above bilinear pairing can be factorized through Dolbeault cohomology groups. The # operator defined as in (2.27) satisfies

$$\#\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^* \otimes E) \longrightarrow \mathscr{C}^\infty(X, \bigwedge^{n-p,n-q} T_X^* \otimes E^*).$$

Furthermore, 2.31 and 2.32 imply

$$\overline{\partial}(\#s) = (-1)^{\deg s} \# D_E''^* s, \quad D_{E^*}''^*(\#s) = (-1)^{\deg s+1} \# D_E''^* s,$$

$$\Delta_{E^*}''(\#s) = \#\Delta_E'' s,$$

where D_{E^*} is the Chern connection of E^*. Consequently, $s \in \mathscr{H}^{p,q}(X, E)$ if and only if $\#s \in \mathscr{H}^{n-p,n-q}(X, E^*)$. Theorem 2.3.27 is then a consequence of the fact that the integral $\|s\|^2 = \int_X s \wedge \#s$ does not vanish unless $s = 0$. □

2.4 Cohomology of Compact Kähler Manifolds

Let X be for the moment an arbitrary complex manifold. The following "cohomology" groups are helpful to describe Hodge theory on compact complex manifolds which are not necessarily Kähler.

Definition 2.4.1. *We define the Bott-Chern cohomology groups of X to be*

$$H_{BC}^{p,q}(X, \mathbb{C}) = \left(\mathscr{C}^\infty(X, \bigwedge^{p,q} T_X^*) \cap \operatorname{Ker} \mathrm{d} \right) / \partial\overline{\partial}\mathscr{C}^\infty(X, \bigwedge^{p-1,q-1} T_X^*).$$

Then $H_{BC}^{\bullet,\bullet}(X, \mathbb{C})$ has the structure of a bigraded algebra, which we call the Bott-Chern cohomology algebra of X.

As the group $\partial\overline{\partial}\mathscr{C}^\infty(X, \bigwedge^{p-1,q-1} T_X^*)$ is contained in both coboundary groups $\overline{\partial}\mathscr{C}^\infty(X, \bigwedge^{p,q-1} T_X^*)$ or $\mathrm{d}\mathscr{C}^\infty(X, \bigwedge^{p+q-1}(\mathbb{C} \otimes T_X)^*)$, the are canonical morphisms

$$H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H^{p,q}(X, \mathbb{C}), \tag{2.52}$$
$$H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H_{DR}^{p+q}(X, \mathbb{C}), \tag{2.53}$$

of the Bott-Chern cohomology to the Dolbeault or De Rham cohomology. These morphisms are homomorphisms of $\mathbb{C}$-algebras. From the definition, conjugating and interchanging partial it follows that $H_{BC}^{q,p}(X, \mathbb{C}) = \overline{H_{BC}^{p,q}(X, \mathbb{C})}$. It can be shown from the Hodge-Frölicher spectral sequence that $H_{BC}^{p,q}(X, \mathbb{C})$ is always finite dimensional if X is compact.

We suppose from now on that (X, ω) is a compact Kähler manifold. The equality $\Delta = 2\Delta_{\overline{\partial}}$ shows that Δ is homogeneous with respect to bidegree and that there is an orthogonal decomposition, this u is harmonic for Δ if and only if u is harmonic for $\Delta_{\overline{\partial}}$.

$$\mathscr{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}(X, \mathbb{C}). \tag{2.54}$$

As $\overline{\Delta_{\overline{\partial}}} = \Delta_\partial = \Delta_{\overline{\partial}}$, we also have $\mathscr{H}^{q,p}(X, \mathbb{C}) = \overline{\mathscr{H}^{p,q}(X, \mathbb{C})}$. Using the Hodge isomorphism theorems for the De Rham and Dolbeault cohomology, we get:

Theorem 2.4.2 (Hodge decomposition theorem)**.** *On a compact Kähler manifold, there are canonical isomorphisms*

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}), \qquad \textit{Hodge decomposition}$$
$$H^{q,p}(X, \mathbb{C}) \simeq \overline{H^{p,q}(X, \mathbb{C})}. \qquad \textit{Hodge symmetry.}$$

The only point which is not a priori completely clear is that this decomposition is independent of the Kähler metric. In order to show that this is the case, one can use the following Lemma, which allows us to compare all three types of cohomology groups considered in 2.4.1.

Lemma 2.4.3. *Let u be a* d*-closed (p, q)-form. The following properties are equivalent:*

1. *u is* d*-exact;*
2. *u is ∂-exact;*
3. *u is $\overline{\partial}$-exact;*
4. *u is $\partial\overline{\partial}$-exact, i.e. u can be written $u = \partial\overline{\partial}v$.*

5. u is orthogonal to $\mathscr{H}^{p,q}(X,\mathbb{C})$.

Proof. It is obvious that 4 implies 1, 2, 3, just using $\partial\overline{\partial} = -\overline{\partial}\partial$ for 3 and $\overline{\partial}v$ as the element from the u comes. It is clear as well that 1 or 2 or 3 implies 5 using the Hodge decomposition for harmonic forms. It is thus sufficient to prove that 5 implies 4. As $\mathrm{d}u = 0$, we have $\partial u = \overline{\partial}u = 0$, and as u is supposed to be orthogonal to $H^{p,q}(X,\mathbb{C})$, Theorem 2.3.25 implies $u = \overline{\partial}s$, $s \in \mathscr{C}^\infty(X, \bigwedge^{p,q-1} T_X^*)$. By the analogue of Theorem 2.3.25 for ∂, we have $s = h + \partial v + \partial^*\omega$, with $h \in \mathscr{H}^{p,q-1}(X,\mathbb{C})$, $v \in \mathscr{C}^\infty(X, \bigwedge^{p-1,q-1} T_X^*)$ and $\omega \in \mathscr{C}^\infty(X, \bigwedge^{p+1,q-1} T_X^*)$. Therefore

$$u = \overline{\partial}\partial v + \overline{\partial}\partial^*\omega = -\partial\overline{\partial}v - \partial^*\overline{\partial}w$$

using lemma 2.3.23. As $\partial u = 0$, hence $\partial\partial^*\overline{\partial}w = 0$. Since $(\partial\partial^*\overline{\partial}w, \overline{\partial}\omega) = \left\|\partial^*\overline{\partial}w\right\|^2 = 0$ we conclude $u = -\partial\overline{\partial}v$. □

Corollary 2.4.4. *Let X be a compact Kähler manifold. Then the natural morphisms*

$$H^{p,q}_{BC}(X,\mathbb{C}) \longrightarrow H^{p,}(X.\mathbb{C}), \qquad \bigoplus_{p+q=k} H^{p,q}_{BC}(X,\mathbb{C}) \longrightarrow H^k_{DR}(X,\mathbb{C})$$

are isomorphisms.

Proof. The surjectivity of $H^{p,q}_{BC}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C})$ comes from the fact that every class in $H^{p,q}(X,\mathbb{C})$ can be represented by a harmonic (p,q)-form, thus by a d-closed (p,q)-form; the injectivity means nothing more than the equivalence 2.4.3 3 $\Leftrightarrow$ 2.4.3 4. Hence $H^{p,q}_{BC}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C}) \simeq \mathscr{H}^{p,q}(X,\mathbb{C})$, and the isomorphism $\bigoplus_{p+q=k} H^{p,q}_{BC}(X,\mathbb{C}) \mapsto H^k_{DR}(X,\mathbb{C})$ follows from (2.54). □

Let us quote now two simple applications of Hodge theory. The first of these is a computation of the Dolbeault cohomology groups of $\mathbb{P}^n$. As $H^{2p}_{DR}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C}$ and $H^{p,p}(\mathbb{P}^n,\mathbb{C}) \ni \{\omega^p\} \neq 0$, the Hodge decomposition formula implies

Application 2.4.5. *The Dolbeault cohomology groups of* $\mathbb{P}^n$ *are*

$$H^{p,p}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C} \text{ for } 0 \leq p \leq n, \quad H^{p,q}(\mathbb{P}^n,\mathbb{C}) = 0 \text{ for } p \neq q.$$

Proposition 2.4.6. *Every holomorphic p-form on a compact Kähler manifold X is d-closed*

Proof. If u is a holomorphic form of type $(p,0)$ then $\overline{\partial}u = 0$. Furthermore $\overline{\partial}^*u$ is of type $(p,-1)$, hence $\overline{\partial}^*u = 0$. Therefore $\Delta u = 2\Delta_{\overline{\partial}}u = 0$, from where $d^*du = 0$ which implies $\mathrm{d}u = 0$. □

Example 2.4.7. *Consider the Heisenberg group $G \subset \mathrm{Gl}_3\,\mathbb{C}$, defined as the subgroup of matrices*

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad (x, y, z) \in \mathbb{C}^3.$$

Let Γ be the discrete subgroup of matrices with entries $x, y, z \in \mathbb{Z}[i]$ (or more generally in the ring of integers of an imaginary quadratic field). Then $X = G/\Gamma$ is a compact complex 3-fold, known as the Iwasawa manifold. The equality

$$M^{-1}\,\mathrm{d}M = \begin{pmatrix} 0 & \mathrm{d}x & \mathrm{d}z - x\,\mathrm{d}y \\ 0 & 0 & \mathrm{d}y \\ 0 & 0 & 0 \end{pmatrix}$$

shows that $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z - x\,\mathrm{d}y$ are left invariant holomorphic 1-forms on G. These forms induce holomorphic 1-form on the quotient $X = G/\Gamma$. Since $\mathrm{d}z - x\,\mathrm{d}y$ is not d-closed, we see that X cannot be Kähler.

Remark 2.4.8. *For simplicity of notation we work here with constant coeficients, but analogous results hold for cohomology with values in a local system of coefficients (flat Hermitian bundle), as in 2.2.12. It is enough to replace everywhere in the proof the operator $\mathrm{d} = \partial + \overline{\partial}$ by $D_E = D'_E + D''_E$, and to observe that one still has $\Delta'_E = \Delta''_E = \frac{1}{2}\Delta_E$ (proof identical to that of Corollary 2.3.22). One can then deduce the existence of isomorphisms*

$$H^{p,q}_{BC}(X, E) \to H^{p,q}(X, E), \qquad \bigoplus_{p+q=k} H^{p,q}_{BC}(X, E) \to H^k_{DR}(X, E)$$

and a canonical decomposition

$$H^k_{DR}(X, E) = \bigoplus_{p+q=k} H^{p,q}(X, E).$$

In this context, the symmetry property of Hodge becomes

$$\overline{H^{p,q}(X, E)} \simeq H^{q,p}(X, E^*)$$

via the antilinear operator #. These observations are useful for the study of variations of Hodge structures.

Definition 2.4.9. *The Betti numbers and Hodge number of X are by definition*

$$b_k = \dim_{\mathbb{C}} H^k(X, \mathbb{C}), \quad h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X, \mathbb{C}). \tag{2.55}$$

Thanks to Hodge decomposition, these numbers satisfy the relations

$$b_k = \sum_{p+q=k} h^{p,q}, \quad h^{q,p} = h^{p,q}. \tag{2.56}$$

As a consequence, the Betti numbers b_{2k+1} of a compact Kähler manifold are even. Note that the Serre duality theorem gives the additional relation $h^{p,q} = h^{n-p.n-q}$, which holds as soon as X is compact.

Lemma 2.4.10. *If $u = \sum_{i\geq 0} L^i u_{k-2i}$ is the primitive decomposition of a harmonic k-form u, then all components u_{k-2i} are harmonic.*

Proof. Since $[\Delta, L] = 0$, we get $0 = \Delta u = \sum_i L^i \Delta u_{k-2i}$, hence $\Delta u_{k-2i} = 0$ by uniqueness. □

Definition 2.4.11. *Let (X, ω) be a compact Kähler manifold, then the Primitive cohomology is defined by*

$$H^k(X, \mathbb{C})_p := \mathrm{Ker}(\Lambda : H^k(X, \mathbb{C}) \to H^{k-2}(X, \mathbb{C}))$$

and

$$H^{p,q}(X)_p := \mathrm{Ker}(\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)).$$

Another important result of Hodge theory (which is in fact a direct consequence of 2.3.19.

Proposition 2.4.12 (Hard Lefschetz Theorem)**.** *Let (X, ω) be a compact Kähler manifold of dimension n. Then for $k \leq n$*

$$L^{n-k} : H^k(X, \mathbb{C}) \simeq H^{2n-k}(X, \mathbb{C})$$

and for any k

$$H^k(X, \mathbb{C}) = \bigoplus_{i\geq 0} L^i H^{k-2i}(X, \mathbb{C})_p.$$

Chapter 3

Hodge decomposition for absolutely q-convex manifolds

3.1 Plurisubharmonic Functions

Recall that the elementary Laplace operator Δ in $\mathbb{C}$ is defined by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

where $z = x+iy$. A $\mathscr{C}^2$-function u on a region $D \subset \mathbb{C}$ is called harmonic, as in the previous chapter, if $\Delta u = 0$ (as we have seen that the Laplacians only differ for a minus sign). We state some of the well-known elementary properties of harmonic functions (see [Ahl79]).

Proposition 3.1.1. *A real valued function u is harmonic on $D \subset \mathbb{C}$ if and only if u is locally the real part of a holomorphic function. In particular, harmonic functions are $\mathscr{C}^\infty$, and even real analytic.*

Proposition 3.1.2 (The Mean Value Property)**.** *If u is harmonic on $D \subset \mathbb{C}$, then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta$$

whenever $\{z : |z-a| \leq r\} \subset D$.

Proposition 3.1.3 (The Maximum Principle)**.** *If u is real valued and harmonic on $D \subset \mathbb{C}$, then:*
(Strong version) *If u has a local maximum at the point $a \in D$, then u is constant in a neighborhood of a (and hence on the connected component of D which contains a).*
(Weak version) *If $D \subset\subset \mathbb{C}$ and u extends continuously to $\overline{D}$, then $u(z) \leq \max_{\mathrm{b}D} u$ for $z \in D$.*

Notice that the strong version of the maximum principle implies the weak version.

Theorem 3.1.4 (The Dirichlet Problem)**.** *If $\Omega := \{z| \, |z-a| < r\}$ and $g \in \mathscr{C}(b\Omega)$, then there is a unique continuous function u on Ω, such that $u(z) = g(z)$ for $z \in \mathrm{b}\Omega$. This harmonic extension u is given explicitly by the Poisson integral of g, i.e.,*

$$u(a+\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |\zeta|^2}{|re^{i\theta} - \zeta|^2} g(a + re^{i\theta}) \, \mathrm{d}\theta \quad \text{for } |\zeta| < r.$$

The the solutions of the Laplace equation in one real variable are the linear functions $l(x) = ax + b$. A function $y = u(x)$ is said to be convex if on any interval $[\alpha, \beta]$ in its domain $u(x)$ is less than or equal to the unique linear function l with $u(\alpha) = l(\alpha)$ and $u(\beta) = l(\beta)$. Substituting harmonic functions for linear functions in the definition above leads to the idea of subharmonic functions: A continuous function u is subharmonic on $D \subset \mathbb{C}$ if on every disc $\Omega \subset\subset D$ one has $u \leq h$, where $h \in \mathscr{C}(\bar{\Omega})$ is the unique function harmonic on Ω with $h = u$ on $\mathrm{b}\Omega$. (The function h exists by the solution of the Dirichlet problem for discs.)

For technical reasons it is convenient to include upper semicontinuous functions and to admit the value $-\infty$ in the definition of subharmonic functions. Moreover, one usually replaces discs by more general sets (although this does not really matter, as we will see that it is a local property). As the Dirichlet problem cannot generally be solved in this setting, one is led to the following formulation.

Definition 3.1.5. *A function $u : D \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if u is upper semicontinuous and if for every compact set $K \subset D$ and every function $h \in \mathscr{C}(K)$ which is harmonic on the interior of K and satisfies $u \leq h$ on $\mathrm{b}K$ it follows that $u \leq h$ on K.*

Recall that u is said to be upper semicontinuous on D in

$$\limsup_{z \to a} u(z) \leq u(a) \quad \text{for } a \in D, \tag{3.1}$$

or, equivalently,

$$\{z \in D | u(z) < c\} \text{ is open in } D \text{ for every } c \in \mathbb{R}. \tag{3.2}$$

An upper semicontinuous function takes on a maximum on every compact set (though not necessarily a minimum). A function $u : D \to \mathbb{R}$ is continuous if and only if u and $-u$ are upper semicontinuous.

From the (weak) maximum principle one sees immediately that harmonic functions are subharmonic. We will give another examples after we have discussed some characterizations of subharmonic functions.

Lemma 3.1.6. *Let $D \subset \mathbb{C}$ be open.*

1. *If u is subharmonic on D, so is cu for $c > 0$.*

2. *If $\{u_\alpha | \alpha \in A\}$ is a family of subharmonic functions on D such that $u = \sup u_\alpha$ is finite and upper semicontinuous, then u is subharmonic.*

3. *If $\{u_j, j = 1, 2, \ldots\}$ is a decreasing sequence of subharmonic functions on D, then $u = \lim_{j\to\infty} u_j$ is subharmonic.*

Proof. 1. It follows immediately from the definition.

2. We have that u is upper semicontinuous as hypothesis. Hence given $K \subset D$ a compact and $h \in \mathscr{C}(K)$ which is harmonic on the interior of K and satisfies $u \leq h$ on bK, for definition of u, $u_\alpha \leq h$ also on bK, as they are subharmonic it follow that $u_\alpha \leq h$ on K, and taking the supremum we obtain that $u \leq h$ on K.

3. Suppose $K \subset D$ is compact and $h \in \mathscr{C}(K)$ is harmonic on $\text{int}(K)$ with $h \geq u = \lim u_j$ on bK. Given $\varepsilon > 0$, $E_j = \{z \in \text{b}K | u_j(z) \geq h(z) + \varepsilon\}$ is a closed subset of bK for $j = 1, 2, \ldots$, for this take any Cauchy sequence $(z_l) \in E_j$, as $\limsup_{l\to\infty} u_j(z_l) \leq u_j(\lim z_l)$, our assertion follows easily. Moreover $E_{j+1} \subset E_j$ and it is a decreasing sequence, so that $\cap_{j=1}^{\infty} E_j = \varnothing$. By compactness of b$K$, more specifically, by the finite intersection property, there is $l \in \mathbb{N}$ with $E_l = \varnothing$. Hence $u_l \leq h + \varepsilon$ on bK, and so on K as well, because u_l is subharmonic. This implies that $u \leq h + \varepsilon$ on K for all $\varepsilon > 0$, i.e., $u \leq h$ on K.

□

As an application we present a curious property of arbitrary domains in $\mathbb{C}$.

Given a domain D, $\delta_D(z) = \sup\{r | B_r(z) \subset D\}$ denotes the (Euclidean distance) from the point $z \in D$ to the boundary of D. If $D \neq \mathbb{C}^n$, then $0 < \delta_D(z) < \infty$ for all $z \in D$, and δ_D extends to a continuous function on $\overline{D}$ by setting $\delta_D(z) = 0$ for $z \in \text{b}D$. One has $\delta_D(z) = \inf\{|z - \zeta| : \zeta \in \text{b}D$. The distance between two sets A and B is given by $\text{dist}(A, B) := \inf\{|a - b| : a \in A, b \in B\}$.

Corollary 3.1.7. *For every open set D in $\mathbb{C}$ the function $u(z) = -\log \delta_D(z)$ is subharmonic on D.*

Proof. If $D = \mathbb{C}$, then $u \equiv -\infty$, and there is nothing to prove. If $D \neq \mathbb{C}$, then $u(z)$ is continuous, and for $z \in D$ one has $u(z) = \sup\{-\log |z - \zeta| : \zeta \in \text{b}D\}$. Since $-\log |z - \zeta|$is harmonic, and hence subharmonic on D (it is, locally, the real part of a holomorphic branch of $-\log(z - \zeta)$), the conclusion follows by Lemma 3.1.6. □

We shall now discuss some other characterizations of subharmonic functions which are useful in various situations. In particular, it will follow that subharmonicity is a local property.

Recall from integration theory that for a Borel measure μ on a compact set K and an upper semicontinuous function $u : K \to \mathbb{R} \cup \{-\infty\}$, the integral $\int u \, \mathrm{d}\mu$ is well defined (possibly $= -\infty$). Moreover

$$\int_K u \, \mathrm{d}\mu = \inf \left\{ \int_K \varphi \, \mathrm{d}\mu \mid \varphi \in \mathscr{C}(K) \text{ and } \varphi \geq u \right\}, \tag{3.3}$$

and $u \in L^1(K, \mu)$ if and only if $\int u \, \mathrm{d}\mu > -\infty$.

Theorem 3.1.8. *Let D be open in $\mathbb{C}$. The following statements are equivalent for an upper semicontinuous function $u : D \to \mathbb{R} \cup \{-\infty\}$:*

1. *u is subharmonic.*
2. *For every disc $\Omega \subset\subset D$ and holomorphic polynomial f with $u \leq \Re f$ on $\mathrm{b}\Omega$, one has $u \leq \Re f$ on Ω.*
3. *For every $a \in D$ there exists a positive number $r_a < \delta_D(a)$ such that*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, \mathrm{d}\theta \quad \text{for all } 0 < r \leq r_a.$$

Remark 3.1.9. *The result presented in the point (3) above is called the submean value property . It is clearly a local property and it clearly holds if we change u by the sum $u_1 + u_2$. Therefore, we have:*

Corollary 3.1.10. *If u_1 and u_2 are subharmonic on D, so is $u_1 + u_2$.*

Before proving the Theorem we single out an important ingredient of the proof.

Lemma 3.1.11. *An upper semicontinuous function u which satisfies the submean value property satisfies the strong maximun principle 3.1.3.*

Proof. The argument is identical to the one which is often used to prove the maximum principle for harmonic functions. Suppose u satisfies the submean value property and u has a local maximum at $a \in D$, i.e., there is $\rho > 0$ such that $u(z) \leq u(a)$ for all z with $|z - a| \leq \rho$. We must assume that $\rho \leq r_a$. If there were a point z_0 with $r = |z_0 - a| \leq \rho$ and $u(z_0) < u(a)$, then

$$\{\theta \in [0, 2\pi] | u(a + re^{i\theta} < u(a))$$

would have nonempty interior, by the upper semicontinuity of u; thus

$$\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta < \int_0^{2\pi} u(a)\,\mathrm{d}\theta = 2\pi u(a),$$

in contradiction to the hypothesis. So u must be constant in a neighborhood of a. □

Proof of Theorem 3.1.8. As f is a holomorphic polynomial, hence harmonic we have that 1 implies 2. In order to show 2 $\Rightarrow$ 3,we suppose $\Omega = \{z : |z - a| < r\} \subset\subset D$ and let $\varphi \in \mathscr{C}(\mathrm{b}\Omega)$ with $\varphi \geq u$ on $\mathrm{b}\Omega$. After replacing φ by its Poisson integral, we may assume that φ is continuous on $\overline{\Omega}$ and harmonic on Ω. For $\tau < 1$, the function

$$\varphi_\tau(z) = \varphi(a + \tau(z - a))$$

is harmonic in a neighborhood of $\overline{\Omega}$, and $\varphi_\tau \to \varphi$ on $\overline{\Omega}$ as $\tau \to 1$. Now $\varphi_\tau = \Re f_\tau$, where f_τ is holomorphic on $\overline{\Omega}$, and by considering the partial sums of the Taylor series of f_τ, it follows that for $\varepsilon > 0$, there is a holomorphic polynomial f with $u \leq \varphi \leq \Re f \leq \varphi + \varepsilon$ on $\mathrm{b}\Omega$. By 2 and the mean value property for the harmonic function $\Re f$, one obtains

$$u(a) \leq \Re f(a) = \frac{1}{2\pi}\int_0^{2\pi} \Re f(a + re^{i\theta}\,\mathrm{d}\theta) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})\,\mathrm{d}\theta + \varepsilon.$$

Since ε is arbitrary, we have shown that

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})\,\mathrm{d}\theta$$

for every continuous function $\varphi \leq u$ on $\mathrm{b}\Omega$, and thus 3 follows from 3.3.

Finally, to show 3 $\Rightarrow$ 1, we take $K \subset D$ be compact and suppose $h \in \mathscr{C}(K)$ is harmonic on $\operatorname{int} K$ and $u \leq h$ on $\mathrm{b}K$; we must show $u \leq h$ on K. Notice that 3 and the mean value property for h imply the submean value property for $u - h$ on $\operatorname{int} K$. Therefore, by Lemma 3.1.11, $(u - h)(z) \leq \max_{\mathrm{b}K}(u - h) \leq 0$ for $z \in K$, i.e., $u \leq h$ on K. □

Proposition 3.1.12. *If f is holomorphic on D, then $|f|^\alpha$ for $\alpha > 0$ and $\log|f|$ are subharmonic on D.*

Proof. For the first assertion we take $a \in D$, as D is open, there exists a ball Ω of radius r such that $r < \delta_D(a)$, using the Cauchy integral formula, and an inequality for the integral, we obtain that $|f(a)|^\alpha \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{it})|^\alpha\,\mathrm{d}t$ if $f(a) \neq 0$. If $f(a) = 0$, it is obvious that $|f(a)|^\alpha \leq |f(z)|^\alpha$, this implies that $|f|^\alpha$ is harmonic.

For the second assertion we use the maximum principle and the fact that log is an increasing function, this is, let $c := \max|f|_\Omega$ and we take as the holomorphic polynomial the constant $\log c$ and by 2, $\log|f|$ is subharmonic. □

The following property of the mean values of subharmonic functions is very useful.

Lemma 3.1.13. *If u is subharmonic on the disc $\{|z-a|<\rho\}$, then*

$$A(u;r)=\frac{1}{2\pi}\int_0^{2\pi}u(a+re^{i\theta})\,\mathrm{d}\theta$$

is a nondecreasing function for $0<r<\rho$.

Proof. Let $\Omega(r)=\{|z-a|<r\}$ and suppose $0<r_1<r_2<\rho$. Let $\varphi\in\mathscr{C}(\mathrm{b}\Omega(r_2))$ satisfy $\varphi\geq u$ on $\mathrm{b}\Omega(r_2)$. By taking the Poisson integral of φ, we may assume that $\varphi\in\mathscr{C}(\overline{\Omega(r_2)})$ and φ is harmonic on $\Omega(r_2)$. By the mean value property, $A(\varphi;r)=\varphi(a)$ for $r\leq r_2$, and the subharmonicity of u implies $u\leq\varphi$ on $\Omega(r_2)$. Hence $A(u;r_2)\leq A(\varphi;r_1)=A(\varphi,r_2)$ for all such φ, and it follows that $A(u;r_1)\leq\inf\{A(\varphi;r_2)\mid\varphi$ continuous and $\varphi\geq u$ on $\mathrm{b}\Omega(r_2)\}=A(u;r_2)$. □

It is well known that a $\mathscr{C}^2$ function $u(x)$ on an interval $I\subset\mathbb{R}$ is convex if and only if $u''(x)\geq 0$ on I. An analogous characterization holds for smooth subharmonic functions, giving a simple computational test for subharmonicity.

Proposition 3.1.14. *A real valued function $u\in\mathscr{C}^2(D)$ is subharmonic on D if and only if $\Delta u\geq 0$ on D.*

Proof. We first show that $\Delta u>0$ implies that u is subharmonic. Let $K\subset D$ be compact, $h\in\mathscr{C}(K)$ harmonic on $\operatorname{int}K$, and suppose $v=u-h\leq 0$ on $\mathrm{b}K$. If $v(z)>0$ for some $z\in K$, then v would take on its maximum at a point $a\in\operatorname{int}K$, and it would follow that $\Delta v(a)\leq 0$. Since $\Delta h=0$, this contradicts $\Delta u(a)>0$, so we must have $v\leq 0$, i.e., $u\leq h$, on K. Next, if $\Delta u\geq 0$, the preceding argument applied to $u_j=u+(1/j)\,|z|^2$ for $j=1,2,\ldots$ shows that u_j is subharmonic. As $u_j(z)$ decreases to $u(z)$ as $j\to\infty$, Lemma 3.1.6 implies that u is subharmonic as well.

To prove the converse, let u be subharmonic and suppose there is a $a\in D$ such that $\Delta u(a)<0$. By continuity, $\Delta u<0$ on a neighborhood U of a, and hence, by the first part of the proof, $-u$ is subharmonic on U. Thus u and $-u$ are subharmonic on U, and by the submean value property u is harmonic on U, but this would imply $\Delta u=0$ on U, contradicting $\Delta u(a)<0$. So we must have $\Delta u\geq 0$ on D. □

Definition 3.1.15. *Let D be open in $\mathbb{C}^n$. A function $u:D\to\mathbb{R}\cup\{-\infty\}$ is said to be plurisubharmonic on D if u is upper semicontinuous, and if for every $a\in D$ and $w\in\mathbb{C}^n$ the function $\lambda\mapsto u(a+\lambda w)$ is subharmonic on the region $\{\lambda\in\mathbb{C}:a+\lambda w\in D\}$. The class of plurisubharmonic functions on D is denoted by $PSH(D)$.*

Remark 3.1.16. *Certain properties of subharmonic functions are inherited by plurisubharmonic functions. For example, Lemma 3.1.6 holds for plurisubharmonic functions, $PSH(D)$ is closed under addition, and $u \in PSH(D)$ if and only if u is plurisubharmonic in some neighborhood of every point $a \in D$. If $f \in \mathscr{O}(D)$, then $|f|^\alpha$, $\alpha > 0$, and $\log|f|$ are plurisubharmonic on D. (This follows from Proposition 3.1.12— notice that the restriction of f to a complex line is holomorphic where defined.)*

On the other hand, Corollary 3.1.7 does not extend to higher dimensions. For example, if $D = \mathbb{C}^2 - \{0\}$, let $u = -\log\delta_D(z)$. For $a = (1,0)$ and $w = (0,1)$, $u(a+\lambda w) = -\log\delta_D(1,\lambda) = -\log\sqrt{1+|\lambda|^2}$, and this function has a strict maximum at $\lambda = 0$, so it cannot be subharmonic (Lemma 3.1.11). So u is not plurisubharmonic. We shall see that the regions $D \subset \mathbb{C}^n$ for which $-\log\delta_D$ is plurisubharmonic are precisely the pseudoconvex ones.

For plurisubharmonic functions of class $\mathscr{C}^2$ there is a differential characterization analogous to the one given in Proposition 3.1.14 for subharmonic functions.

Proposition 3.1.17. *Let $D \subset \mathbb{C}^n$ and suppose $u \in \mathscr{C}^2(D)$ is real valued. Then $u \in PSH(D)$ if and only if the complex Hessian of u,*

$$L_z(u;w) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k,$$

is positive semidefinite on $\mathbb{C}^n$ at every point $z \in D$.

Proof. A straightforward computation gives

$$\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}} u(a+\lambda w) = L_{a+\lambda w}(u,w) \tag{3.4}$$

for $w \in \mathbb{C}^n$ and $a + \lambda w \in D$. By Proposition 3.1.14, $u(a+\lambda w)$ is subharmonic in λ if and only if the left side in (3.4) is nonnegative. □

Corollary 3.1.18. *Suppose $\Omega \subset \mathbb{C}^n$ and $D \subset \mathbb{C}^m$ are open, and $F : D \to \Omega$ is holomorphic. Then $u \circ F \in PSH(D)$ if $u \in PSH(\Omega) \cap \mathscr{C}^2(\Omega)$.*

Proof. A computation gives $L_a(u \circ F; w) = L_{F(a)}(u; F'(a)w)$. Now use the Proposition 3.1.17. □

In order to extend Corollary 3.1.18 to arbitrary $u \in PSH(D)$, one needs to locally approximate u by smooth plurisubharmonic functions. In order to get started we need to know that plurisubharmonic functions are in L^1, at least locally, with respect to $2n$-dimensional Lebesgue measure.

Lemma 3.1.19. *Let $D \subset \mathbb{C}^n$ be connected. If $u \in PSH(D)$ and $u \not\equiv -\infty$ on D, then $u \in L^1_{loc}(D)$. In particular, $\{z \in D | u(z) = -\infty\}$ has Lebesgue measure 0.*

Proof. We first show that if $u(a) > -\infty$ at some point $a \in D$, then $u \in L^1(P(a,r))$ for every polydisc, this is the product of n open discs in $\mathbb{C}$, $P(a,r) \subset\subset D$. Since u is bounded from above on such a polydisc, it is enough to show $\int_{P(a,r)} u \, \mathrm{d}V > -\infty$. By applying the submean value property in each coordinate separately, one obtains

$$u(a) \leq (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} u(a + \rho e^{i\theta}) \, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n$$

for all $\rho = (\rho_1, \ldots, \rho_n)$ with $0 \leq \rho \leq r$, where $\rho e^{i\theta} = (\rho_1 e^{i\theta_1}, \ldots, \rho_n e^{i\theta_n})$. After multiplying by $\rho_1 \ldots \rho_n \, \mathrm{d}\rho_1 \ldots \mathrm{d}\rho_n$ and integrating in ρ_j from 0 to r_j, $1 \leq j \leq n$, it follows that

$$-\infty < u(a) \leq [\mathrm{Vol}\, P(a,r)]^{-1} \int_{P(a,r)} u \, \mathrm{d}V.$$

The application of the Fubini-Tonelli theorem is legitimate as u is bounded from above.

Now consider the set $E = \{a \in D | u \text{ is integrable in a neighborhood of } a\}$. E is clearly open, and we just saw that $E \neq \varnothing$. The statement proved above also implies that if $a \in D \setminus E$, then $u(z) = -\infty$ for all z in some neighborhood of a, so $D - E$ is open as well. Since D is connected, $E = D$. □

Theorem 3.1.20. *Let $D \subset \mathbb{C}^n$ and set $D_j = \{z \in D | \, |z| < j \text{ and } \delta_D(z) > 1/j\}$. Suppose $u \in PSH(D)$ is not identically $-\infty$ on any component of D. Then there is a sequence $\{u_j\} \subset \mathscr{C}^\infty(D)$ with the following properties*

1. *u_j is strictly plurisubharmonic on D_j.*
2. *$u_j(z) \geq u_{j+1}(z)$ for $z \in D_j$, and $\lim_{j\to\infty} u_j(z) = u(z)$ for $z \in D$.*
3. *If u is also continuous, the convergence in 2 is compact on D.*

Proof. Let $\varphi \in \mathscr{C}^\infty_c(B(0,1))$, this is with compact support on $B(0,1)$ (the ball centered in 0 with radius 1), such that $\varphi \geq 0$, φ is radial (i.e., $\varphi(z) = \varphi(z')$) if $|z| = |z'|$), and $\int \varphi \, \mathrm{d}V = 1$. Since $D_j \subset\subset D$, by Lemma 3.1.19 one has $u \in L^1(D_j)$ for each $j = 1, 2, \ldots$. The integral $v_j(z) = \int_{D_j} u(\zeta)\varphi(j(z-\zeta)) j^{2n} \, \mathrm{d}V(\zeta)$ is thus defined for each $z \in \mathbb{C}^n$, and standard results give $v_j \in \mathscr{C}^\infty$. We set $u_j(z) = v_j(z) + (1/j)\,|z|^2$. For $z \in D_j$, a linear change of variables gives

$$v_j(z) = \int_{|\zeta|<1} u(z - \zeta/j)\varphi(\zeta) \, \mathrm{d}V(\zeta). \tag{3.5}$$

In order to prove 1 it is enough to show that $v_j \in PSH(D_j)$, i.e., $v_j(a+\lambda w)$ satisfies the submean value property at $\lambda = 0$ for $a \in D_j$ and $w \in C^n$, since then

$$L_a(u_j; w) = L_a(v_j, w) + (1/j)\,|w|^2 \geq (1/j)\,|w|^2 .$$

But this follows easily from the corresponding property of u, as follows: for sufficiently small r one has

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} v_j(a+re^{i\theta}w)\,\mathrm{d}\theta &= \int_{|\zeta|<1}\left[\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta}w-\zeta/j)\,\mathrm{d}\theta\right]\varphi(\zeta)\,\mathrm{d}V(\zeta)\\ &\geq \int_{|\zeta|<1} u(a-\zeta/j)\varphi(\zeta)\,\mathrm{d}V(\zeta)\\ &= v_j(a).\end{aligned}$$

Next, observe that the integral (3.5) is invariant under substitution of ζ by $e^{it}\zeta, t \in \mathbb{R}$. Thus

$$v_j(z) = \int_{|\zeta|<1}\left[\frac{1}{2\pi}\int_0^{2\pi} u(z-e^{it}\zeta/j)\,\mathrm{d}t\right]\varphi(\zeta)\,\mathrm{d}V(\zeta). \tag{3.6}$$

By Lemma 3.1.13 applied to the subharmonic function $\lambda \mapsto u(z+\lambda(-\zeta))$, the inner integral in 3.6 is nondecreasing in $r = 1/j$; thus $v_j(z) \geq v_{j+1}(z)$. Also, 3.6 and the submean value property show $v_j(z) \geq u(z)\int \varphi\,\mathrm{d}V = u(z)$. If $\varepsilon > 0$ is given, by the upper semicontinuity of u there is a ball $B(z,\delta) \subset \{\zeta \in D | u(\zeta) < u(z)+\varepsilon\}$; thus, for $j > 1/\delta$, one obtains from (3.5) and the above that $u(z) \leq v_j(z) < u(z)+\varepsilon$. This completes the proof of 2 for $\{v_j\}$; 3 follows by a similar argument. The corresponding statements for $\{u_j\}$ are then obvious. □

We can now show that plurisubharmonic functions are invariant under holomorphic maps.

Theorem 3.1.21. *If $\Omega \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and $F : D \to \Omega$ is holomorphic, then $u \circ F \in PSH(D)$ for every $u \in PSH(\Omega)$.*

Proof. Without loss of generality we may assume that Ω is connected and that $u \in PSH(\Omega)$ is $\not\equiv -\infty$. Choose a decreasing sequence $\{u_j\}$ with $\lim u_j = u$ as in Theorem 3.1.20. If $D' \subset\subset D$, then $u_j \circ F$ is plurisubharmonic on D' for j sufficiently large, by the plurisubharmonicty of u_j and Corollary 3.1.18. Since $\{u_j \circ F\}$ decreases to $u \circ F$ as $j \to \infty$, the conclusion follows from Lemma 3.1.6. □

In 1906 F. Hartogs discovered the first example exhibiting the remarkable extension properties of holomorphic functions in more than one variable. It is this phenomenon, more that anything else, which distinguishes function theory in several variables from the classical one-variable theory.

The phenomenon of simultaneous extension of all holomorphic functions from one domain to a strictly larger one raises the question of characterizing those domains for which this phenomenon does not occur: these are the so-called domains of holomorphy.

Definition 3.1.22. *A holomorphic function f on D is completely singular at $p \in \mathrm{b}D$ if for every connected neighborhood U of p there is no $h \in \mathscr{O}(U)$ which agrees with f on some connected component of $U \cap D$. D is called a weak domain of holomorphy if for every $p \in \mathrm{b}D$ there is $f_p \in \mathscr{O}(D)$ which is completely singular at p. D is called a domain of holomorphy if there exists $f \in \mathscr{O}(D)$ which is completely singular at every boundary point $p \in \mathrm{b}D$.*

The concept of weak domain of holomorphy is not standard; it is, in fact, equivalent to the concept of domain of holomorphy, but we will not prove that here.

We will characterized these sets and give some equivalences. We set:

$$\begin{aligned} \Gamma = \{ & z \in \mathbb{C}^n | z_j = 0 \text{ for } j < n, |z_n| \leq 1\} \\ & \cup \{z \in \mathbb{C}^n | z_j = 0 \text{ for } j < n-1, |z_{n-1}| \leq 1, |z_n| = 1\}, \end{aligned}$$

and

$$\widehat{\Gamma} = \{z \in \mathbb{C}^n | z_j = 0 \text{ for } j < n-1, |z_{n-1} \leq 1, |z_n| \leq 1| \,;$$

we call the pair $(\Gamma, \widehat{\Gamma})$ the (standard) Hartogs frame in $\mathbb{C}^n$. Note that $\Gamma = \widehat{\Gamma}$ for $n = 1$. A pair $(\Gamma^*, \widehat{\Gamma}^*)$ of compact sets in $\mathbb{C}^n$ is called a Hartogs figure if there is a biholomorphic map $F : \widehat{\Gamma} \to \widehat{\Gamma}^*$, such that $F(\Gamma) = \Gamma^*$.

We will state some results, for the proof we refer to [Ran86].

Lemma 3.1.23. *Let $(\Gamma^*, \widehat{\Gamma}^*)$ be a Hartogs figure. Then every $f \in \mathscr{O}(\Gamma^*)$ has a holomorphic extension $\hat{f} \in \mathscr{O}(\widehat{\Gamma}^*)$.*

Definition 3.1.24. *A domain $D \subset \mathbb{C}^n$ is called Hartogs pseudoconvex if foer every Hartogs figure $(\Gamma^*, \widehat{\Gamma}^*)$ with $\Gamma^* \subset D$ one has $\widehat{\Gamma}^* \subset D$ as well.*

Theorem 3.1.25. *A weak domain of holomorphy is Hartogs pseudoconvex.*

A function $\varphi : D \to \mathbb{R}$ on the open set D is said to be an exhaustion function for D if for every $c \in \mathbb{R}$ the set $D_c = \{z \in D | \varphi(z) < c\}$ is relatively compact in D. An exhaustion function φ satisfies $\varphi(z) \to \infty$ as $z \to \mathrm{b}D$; this is also sufficient if D is bounded.

Definition 3.1.26. *An open set D in $\mathbb{C}^n$ is called pseudoconvex if there is $u \in PSH(D) \cap \mathscr{C}^2(D)$ such that u is an exhaustion function for D, which satisfies $L_z(u, w) > 0$ for every $z \in D$ and $w \in \mathbb{C}^n$.*

Definition 3.1.27. *A region $D \subset \mathbb{C}^n$ is called plurisubharmonic convex if for every compact set $K \subset D$, its plurisubharmonic convex hull*

$$\widehat{K}_{PSH(D)} = \{z \in D : u(z) \leq \sup_K u \text{ for all } u \in PSH(D)\}$$

is relatively compact in D.

Next we introduce a version of the classical "continuity principle" which describes a geometrically very intuitive analogue of linear convexity. If $\Omega \subset\subset \mathbb{C}$ is an open disc and $\varphi : \overline{\Omega} \to D$ is a continuous map which is holomorphic on Ω, we shall say that $\varphi(\overline{\Omega})$ is an analytic disc S in Ω and call the set $\varphi(\mathrm{b}\Omega)$ the "boundary" ∂S of S.

Definition 3.1.28. *A region D in $\mathbb{C}^n$ is said to satisfy the continuity principle if for every family $\{S_\alpha | \alpha \in I\}$ of analytic discs in D with*

$$\bigcup_{\alpha \in I} \partial S_\alpha \subset\subset D,$$

it follows that

$$\bigcup_{\alpha \in I} S_\alpha \subset\subset D.$$

The following Theorem gives the equivalence of all the definitions given before. See [Ran86].

Theorem 3.1.29. *The following properties are equivalent for an open set D in $\mathbb{C}^n$*

1. *There is a $\mathscr{C}^2$ strictly plurisubharmonic exhaustion function for D (i.e., D is pseudoconvex according to the definition in 3.1.26).*

2. *There is a plurisubharmonic exhaustion function for D.*

3. *D is plurisubharmonic convex*

4. *For every analytic disc S in D one has $\mathrm{dist}(S, \mathrm{b}D) = \mathrm{dist}(\partial S, \mathrm{b}D)$.*

5. *D satisfies the continuity principle.*

6. *D is Hartogs pseudoconvex.*

7. *$-\log \delta_D$ is plurisubharmonic on D.*

8. *D is a domain of holomorphy.*

3.2 q-Convex Spaces

In this section we will assume the knowledge of sheaf theory, for which we refer to [Ten75] for a clear and general exposition, and [GR77] for a more complex point of view. We will follow mainly [GR84] and [Dem12].

Definition 3.2.1. *A topological space X together with a sheaf of ring $\mathscr{A}$ on X is called a ringed space.*

We recall briefly that sheaf of rings here means the following: for each point $x \in X$ we have a commutative ring $\mathscr{A}_x$ of "germs at x" with a unit 1_x, and the union $\mathscr{A}$ of all rings $\mathscr{A}_x$ is provided with a topology in such a way that:

1. the map which assigns to every $a \in \mathscr{A}$ the unique $x \in X$ with $a \in \mathscr{A}_x$ is locally a homeomorphism.

2. for any open set U in X the set $\mathscr{A}(U)$ of "sections in $\mathscr{A}$ over U" is a ring with unit (this means the map $x \mapsto 1_x$ is continuous and addition and multiplication are continuous operations). Then, for each U and $x \in U$, we have a natural homomorphism $\mathscr{A}(U) \to \mathscr{A}_x$ of rings attaching to every section $s \in \mathscr{A}(U)$ its germ s_x at x. The knowledge of the rings $\mathscr{A}(U)$ for all open sets U together with the "restriction" homomorphisms $\mathscr{A}(U) \to \mathscr{A}(V)$ for $V \subset U$ completely determines the sheaf $\mathscr{A}$.

Ringed space are usually denoted $(X, \mathscr{A})$, the sheaf $\mathscr{A}$ is called the structure sheaf. Often we simply write X instead of $(X, \mathscr{A})$ if it is clear from the context what the structure sheaf is.

Any domain D in $\mathbb{C}^n$ gives rise, by restriction, to the open ringed subspace $(D, \mathscr{O}_D)$ of $(\mathbb{C}^n, \mathscr{O})$.

If $f_1, \ldots, f_k$ are finitely many continuous functions in a space X the set

$$N(f_1, \ldots, f_k) := \{x \in X | f_1(x) = \ldots = f_k(x) = 0\}$$

is called the set of common zeros of $f_1, \ldots, f_k$ in X. As we can see it as the finite intersection of closed subsets, it is closed in X, for $N(f_1, \ldots, f_k) = N(f_1) \cap N(f_2) \cap \ldots \cap N(f_k)$. We are only interested in zero sets of holomorphic functions.

We next introduce the notion of a complex model space. Let D be a domain in $\mathbb{C}^n$ and let $\mathscr{I}$ be an ideal sheaf in $\mathscr{O}_D$, which is of "finite type" on D, i.e. to every point $z \in D$ there exists an open neighborhood $U \subset D$ of z and functions $f_1, \ldots, f_k \in \mathscr{O}(U)$ such that the sheaf $\mathscr{I}$ is generated over U by $f_1, \ldots, f_k$, i.e.

$$\mathscr{I}_U = \mathscr{O}_U f_1 + \ldots + \mathscr{O}_U f_k$$

The quotient sheaf $\mathscr{O}/\mathscr{I}$ is a sheaf of rings on D. We consider its support $Y := \mathrm{Supp}(O_D/\mathscr{I})$, that is the set of all points $z \in D$, where $(\mathscr{O}_D/\mathscr{I})_z \neq 0$, i.e. where $\mathscr{I}_z \neq \mathscr{O}_z$. Clearly in a neighborhood U of z as above we have $Y \cap U = N(f_1, \ldots, f_k)$, so locally Y is the zero set of finitely many holomorphic functions. The restriction

$$\mathscr{O}_Y := (\mathscr{O}_D/\mathscr{I})|Y$$

of $\mathscr{O}_D/\mathscr{I}$ to Y is a sheaf of rings on Y. The ringed space $(Y, \mathscr{O}_Y)$ is called a *complex model space* (in D), more precisely: the complex model space defined by the ideal $\mathscr{I} \subset \mathscr{O}_D$ of finite type. Clearly D itself and the empty space are complex model spaces (defined by the zero ideal sheaf resp. by $\mathscr{I} := \mathscr{O}_D$).

Any finite set $f_1, \ldots, f_k$ of holomorphic functions defines a complex model space $(Y, \mathscr{O}_Y)$ in D with $Y = N(f_1, \ldots, f_k)$, just perform the above construction for the ideal $\mathscr{I} := \mathscr{O}_D f_1 + \ldots + \mathscr{O}_D f_k$. We give four simple examples:

1. NEIL's *parabola*. This is the complex model space defined in $\mathbb{C}^2$ with complex coordinates w, z by the polynomial $w^2 - z^3$.

2. *The space of coordinate lines in* $\mathbb{C}^2$. This space is defined by the polynomial wz. Its underlying topological space $N(wz)$ consists of two complex lines in $\mathbb{C}^2$ meeting in the origin.

3. *The n-fold point.* This is the complex model space defined in $\mathbb{C}$ with complex coordinate z by the monomial z^n , $n \geq 1$. Here $N(z^n)$ is a single point p (origin) and $\mathscr{O}_p = (\mathscr{O}_\mathbb{C}/\mathscr{O}_\mathbb{C} z^n)|_p$ is a local $\mathscr{C}$-algebra with n generators $1, \varepsilon, \ldots, \varepsilon^{n-1}$ and $\varepsilon^n = 0$(Artin-algebra). We see that in case $n > 1$ there live *nilpotent* germs $\neq 0$ on the n-fold point-. If $n = 1$, we have $(p, \mathscr{O}_0) = (p, \mathbb{C})$, if $n = 2$, the space $(p, \mathscr{O}_p)$ is called *double* point.

4. *The cone in* $\mathscr{C}^3$. This space is given by the polynomial $w^2 - z_1 z_2$. The topological space $N(w^2 - z_1 z_2)\backslash\{0\}$ is a topological manifold of dimension 4, the origin 0 however is a "singular" point of the cone: there is no neighborhood of 0 in $N(w^2 - z_1 z_2)$ which is homeomorphic to a ball in $\mathbb{R}^4$.

Let us denote by $\mathscr{K} := X \times \mathbb{C}$ the *constant* sheaf of field $\mathbb{C}$ over X with projection $(x, c) \mapsto x$. A sheaf of ring $\mathscr{A}$ on X is called a *sheaf of* $\mathbb{C}$-*algebras*, if $\mathscr{A}$ is a sheaf of modules over $\mathscr{K}$ with $\mathrm{Supp}\,\mathscr{A} = X$ such that $c(fg) = (cf)g$ for all $c \in \mathscr{K}_x, f, g \in \mathscr{A}_x, x \in X$. In particular the identity section $1 \in \mathscr{A}$ is nowhere zero and $\iota : \mathscr{K} \to \mathscr{A}, (x, c) \mapsto c1_x$ is a sheaf monomorphism (of rings). We identify $\mathscr{K}$ with $\iota(\mathscr{K}) \subset \mathscr{A}$ and $\mathbb{C}$ with $\mathbb{C}1_x \subset \mathscr{A}_x$.

A sheaf of $\mathbb{C}$-algebras $\mathscr{A}$ is called a sheaf of *local* $\mathbb{C}$-algebras if every stalk $\mathscr{A}_x$ is a *local* ring with (unique) *maximal* ideal $\mathfrak{m}(\mathscr{A}_x)$ so that the quotient epimorphism $\mathscr{A}_x \to$

$\mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x)$ always induces an isomorphism $\mathbb{C} \xrightarrow{\sim} \mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x)$. One identifies $\mathscr{A}_x/\mathfrak{m}(\mathscr{A}_x)$ with $\mathbb{C}$ and thus has a canonical direct sum $\mathscr{A}_x = \mathbb{C} \oplus \mathfrak{m}(\mathscr{A}_x)$ as $\mathbb{C}$-vector space. A sheaf mapping $\alpha : \mathscr{A} \to \mathscr{A}'$ between sheaves of $\mathbb{C}$-algebras is called a $\mathbb{C}$-homomorphism if every stalk map $\alpha_x : \mathscr{A}_x \to \mathscr{A}'_x$ is a $\mathbb{C}$-algebra homomorphism. If $\mathscr{A}, \mathscr{A}'$ are sheaves of local $\mathbb{C}$-algebras such homomorphism are automatically stalk-wise local, i.e. α_x maps $\mathfrak{m}(\mathscr{A}_x)$ into $\mathfrak{m}(\mathscr{A}'_x)$.

A ringed space $(X, \mathscr{A})$ is called a $\mathbb{C}$-*ringed* space if $\mathscr{A}$ is a sheaf of local $\mathbb{C}$-algebras. A *morphism* $(X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ *of* $\mathbb{C}$-*ringed spaces* consists of a continuous map $f : X \to Y$ together with a $\mathbb{C}$-algebra homomorphism $\tilde{f} : \mathscr{A}_Y \to f_*(\mathscr{A}_X)$. Note that the image sheaf $f_*(\mathscr{A}_X)$ always is, in a canonical way, a sheaf of ring (not necessarily of $\mathbb{C}$-algebras) on Y so that maps $\tilde{f}$ can be considered.

A morphism $(f, \tilde{f}) : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ canonically determines stalk maps

$$\tilde{f}_x : \mathscr{A}_{Y,f(x)} \to \mathscr{A}_{X,x}, \quad x \in X,$$

by composing the map $\mathscr{A}_{Y,f(x)} \to f_*(\mathscr{A}_X)_{f(x)}$ induced by f with the natural germ map $\hat{f}_x : f_*(\mathscr{A}_X)_{f(x)} \to \mathscr{A}_{X,x}$. These stalk maps determine $\tilde{f}$ and are automatically *local* $\mathbb{C}$-algebra homomorphism.

If U is open in X clearly $(U, \mathscr{A}_U)$ with $\mathscr{A}_U := \mathscr{A}_X|U$ is a $\mathbb{C}$-ringed space. We call $(U, \mathscr{A}_U)$ an *open* $\mathbb{C}$-*ringed subspace* of $(X, \mathscr{A}_X)$, the injection $\iota : U \to X$ canonically induces a morphism $(U, \mathscr{A}_U) \to (X, \mathscr{A}_X)$.

We are now in position to introduce the notion of a complex space.

Definition 3.2.2. *Let $(X, \mathscr{O}_X)$ be a $\mathbb{C}$-ringed space such that X is a Hausdorff space: we call $(X, \mathscr{O}_X)$ a complex space if every point of X has an open neighborhood U such that the open $\mathbb{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space.*

In other words a complex space is a ringed Hausdorff space which can be locally realized (as a $\mathbb{C}$-ringed space) by the zero set of finitely many holomorphic functions in some domain of a complex number space. All complex model spaces, especially all spaces $(D, \mathscr{O}_D)$ and the double point $(p, \mathscr{O}_p)$, are complex spaces.

Morphisms between complex spaces are called *holomorphic maps*; isomorphisms are called *biholomorphic maps*. We now discruss several possibilities of how good resp. how bad a given complex space $X = (X, \mathscr{O})$ may behave at a point $x \in X$. The situation is optimal if x is a *smooth* point of X, i.e. if there exists an open neighborhood of x which is isomorphic to a domain $(D, \mathscr{O}_D)$. If all points of X are smooth the complex space X is called a *complex manifold*. ([Ten75] prove the equivalence between the two definitions that we have now.) Smooth points are also called *simple* or *regular*. A non-regular point of X is called a *singular* point, e.g. the origin is a singular point of NEIL's parabola $w^2 - z^3 = 0$ and of the cone $w^2 - z_1 z_2 = 0$. The space X is called *irreducible at x* if the stalk $\mathscr{O}_x$ is an integral domain, otherwise X is called *reducible atx*. All smooth points are irreducible points, since for such a point x the stalk $\mathscr{O}_x$ is isomorphic to the ring of convergent power series at $0 \in \mathbb{C}^n$. The origin is an irreducible point of NEIL's parabola and of the cone in $\mathbb{C}^3$, however a reducible point of the space of coordinate lines $wz = 0$ in $\mathbb{C}^2$. The space X is called *locally irreducible* if all point of X are irreducible. Complex manifolds are locally irreducible.

The complex space X is called *reduced at x* if the stalk $\mathscr{O}_x$ is a reduced ring, i.e. does not contain nilpotent elements $\neq 0$. All irreducible point are reduced points of X, the origin of $wz = 0$ also is a reduced point. We call X a *reduced* complex space if X is reduce at all its points. The double point $(p, \mathscr{O}_p)$ is the typical example of a *non-reduced* complex space.

A reduced point $x \in X$ is called a *normal* point of X, if the stalk $\mathscr{O}_x$ is *integrally closed* in its quotient ring. Smooth points are normal, X is irreducible at every normal point. The origin of the cone $w^2 - z_1 z_2 = 0$ is a normal point, while the origin of NEIL's parabola is not. A complex space with normal points only is called a *normal space*.

Now, we will introduce the concept of q-convexity. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$. A function $v \in \mathscr{C}^2(M, \mathbb{R})$ is said to be strongly (resp. weakly) q-convex at a point $x \in M$ if $i\partial\overline{\partial}v(x)$ has at least $(n-q+1)$ strictly positive (resp. non-negative) eigenvalues, or equivalently if there exists a $(n-q+1)$-dimensional subspace $F \subset T_x M$ on which the complex Hessian $H_x v$ is positive definite (resp. semi-positive). Weak 1-convexity is thus equivalent to plurisubharmonicity. Some authors use different conventions for the number of positive eigenvalues in q-convexity. The reason why we introduce the number $n - q + 1$ instead of q is mainly due to the following result:

Proposition 3.2.3. *If $v \in \mathscr{C}^2(M, \mathbb{R})$ is strongly (weakly) q-convex and if Y is a submanifold of M, then $v|_Y$ is strongly (weakly) q-convex.*

Proof. Let $d = \dim Y$. For every $x \in Y$, there exists $F \subset T_x M$ with $\dim F = n-q+1$ such that Hv is (semi-)positive on F. Then $G = F \cap T_x Y$ has dimension $\geq (n-q+1)-(n-d) = d - q + 1$, and $H(v|_Y)$ is (semi)- positive on $G \subset T_x Y$. Hence $v|_Y$ is strongly (weakly) q-convex at x. □

Proposition 3.2.4. *Let $v_j \in \mathscr{C}^2(M, \mathbb{R})$ be a weakly (strongly) q_j-convex function, $1 \leq j \leq s$, and $\chi \in \mathscr{C}^2(\mathbb{R}^s, \mathbb{R})$ a convex function that is increasing (strictly increasing) in all variables. Then $v = \chi(v_1, \ldots, v_s)$ is weakly (strongly) q-convex with $q - 1 = \sum(q_j - 1)$. In particular $v_1 + \cdots + v_j$ is weakly (strongly) q-convex.*

Proof. Using that

$$(D^2(J\circ h)(x))(a,a)=\sum_{i,j,k}\partial_i J(h(x))\partial_j\partial_k h_i(x)a_ja_k+\sum_{i,j,k,l}\partial_i h_j(x)\partial_j\partial_k J(h(x))\partial_k h_l(x)a_ia_l.$$

for a composition of real functions we have that

$$Hv=\sum_j\frac{\partial\chi}{\partial t_j}(v_1,\dots,v_s)Hv_j+\sum_{j,k}\frac{\partial^2\chi}{\partial t_j\partial t_k}(v_1,\dots,v_s)\partial v_j\otimes\overline{\partial v_k}, \tag{3.7}$$

and the second sum defines a semi-positive hermitian form. In every tangent space T_xM there exists a subspace F_j of codimension q_j-1 on which Hv_j is semi-positive (positive definite). Then $F=\cap F_j$ has codimension $\le q-1$ and Hv is semi-positive (positve definite) on F. □

The above result cannot be improved, as shown by the trival example

$$v_1(z)=-2|z_1|^2+|z_2|^2+|z_3|^2,\quad v_2(z)=|z_1|-2|z_2|^2+|z_3|^2\qquad\text{on }\mathbb{C}^3,$$

in which case $q_1=q_2=2$ as they have the 2-dimensional subspace where the complex hessian H_xv_i is positive definite, $\mathscr{L}(z_2,z_3),\mathscr{L}(z_1,z_3)$ but v_1+v_2 is only 3-convex, this is, it is positive definite on $\mathscr{L}(z_3)$. However, formula (3.7) implies the following result:

Proposition 3.2.5. *Let $v_j\in\mathscr{C}^2(M,\mathbb{R})$, $1\le j\le s$, be such that every convex linear combination $\sum\alpha_jv_j,\alpha_j\ge0,\sum\alpha_j=1$, is weakly (strongly) q-convex. If $\chi\in\mathscr{C}^2(\mathbb{R}^s,\mathbb{R})$ is a convex function that is increasing (strictly increasing) in all variables, then $\chi(v_1,\dots,v_s)$ is weakly (strongly) q-convex.*

The invariance property of Proposition 3.2.3 immediately suggests the definition of q-convexity on complex spaces.

Definition 3.2.6. *Let $(X,\mathscr{O}_X)$ be a complex space. A function v on X is said to be strongly (resp. weakly) q-convex of class $\mathscr{C}^k$ on X if X can be covered by patches $G:U\overset{\simeq}{\to}A,A\subset\Omega\subset\mathbb{C}^n$ such that for each patch there exists a function $\tilde v$ on Ω with $\tilde v|_A\circ G=v|_U$ which is strongly (resp. weakly) q-convex of class $\mathscr{C}^k$.*

The notion of q-convexity on a patch U does not depend on the way U is embedded in $\mathbb{C}^N$, as shown by the following lemma.

Lemma 3.2.7. *Let $G:U\to A\subset\Omega\subset\mathbb{C}^N$ and $G':U'\to A'\subset\Omega'\subset\mathbb{C}^{N'}$ be two patches of X. Let $\tilde v$ be a strongly (weakly) q-convex function on Ω and $v=\tilde v|_A\circ G$. For every $x\in U\cap U'$ there exists a strongly (weakly) q-convex function $\tilde v'$ on a neighborhood $W'\subset\Omega'$ of $G'(x)$ such that $\tilde v'_{A'\cap W'}\circ G'$ coincides with v on $G'^{-1}(W')$.*

Proof. The isomorphisms

$$G' \circ G^{-1} : A \supset G(U \cap U') \to G'(U \cap U') \subset A',$$
$$G \circ G'^{-1} : A' \supset G'(U \cap U') \to G(U \cap U') \subset A$$

are restrictions of holomorphic maps $H : W \to \Omega', H' : W' \to \Omega$ defined on neighborhood $W \ni G(x), W' \ni G'(x)$; we can shrink W' so that $H'(W') \subset W$. If we compose the automorphism $(z, z') \mapsto (z, z' - H(z))$ of $W \times \mathbb{C}^{N'}$ with the function $v(z) + |z'|^2$ we see that the function $\varphi(z, z') = \tilde{v}(z) + |z' - H(z)|^2$ is strongly (weakly) q-convex on $W \times \Omega'$. Now, W' can be embedded in $W \times \Omega'$ via the map $z' \mapsto (H'(z'), z')$, so that the composite function

$$\tilde{v}(z') = \varphi(H'(z'), z') = \tilde{v}(H'(z')) + |z' - H \circ H'(z')|^2$$

is strongly (weakly) q-convex on W' by Proposition 3.2.3. Since $H \circ G = G'$ and $H' \circ G' = G$ on $G'^{-1}(W')$, we have $\tilde{v}' \circ G' = \tilde{v} \circ G = v$ on $G'^{-1}(W')$ and the lemma follows. □

A consequence of this lemma is that Proposition 3.2.4 is still valid for a complex space X (all the extension $\tilde{v}_j$ near a given point $x \in X$ can be obtained with respect to the same local embedding).

Definition 3.2.8. *A complex space $(X, \mathscr{O}_X)$ is said to be strongly (resp. weakly) q-convex if X has a $\mathscr{C}^\infty$ exhaustion function ψ which is strongly (resp. weakly) q-convex outside an exceptional compact set $K \subset X$. We say that X is strongly q-complete if ψ can be chosen so that $K = \varnothing$. By convention, a compact complex space X is said to be strongly 0-complete, with exceptional compact set $K = X$.*

We consider the sublevel sets

$$X_c = \{x \in X | \psi(x) < c\} \quad c \in \mathbb{R}. \tag{3.8}$$

If $K \subset X_c$, we may select a convex increasing function χ such that $\chi = 0$ on $]-\infty, c]$ and $\chi' > 0$ on $]c, +\infty[$. Then $\chi \circ \psi = 0$ on X_c, so that $\chi \circ \psi$ is weakly q-convex everywhere in virtue of 3.7. In the weakly q-convex case, we may therefore always assume $K = \varnothing$. The following properties are almost immediate consequences of the definition:

Theorem 3.2.9. *1. If X is strongly (weakly) q-convex , every sublevel set X_c containing the exceptional compact set K is strongly (weakly) q-convex.*

2. If U_j is a weakly q_j-convex open subset of X, $1 \leq j \leq s$, the intersection $U = U_1 \cap \ldots \cap U_s$ is weakly q-convex with $q - 1 = \sum(q_j - 1)$; U is strongly q-convex (resp. q-complete) as soon as one of the sets U_j is strongly q_j-convex (resp. q_j-complete).

Proof. 1. Let ψ be an exhaustion of the required type on X. Then $1/(c-\psi)$ is an exhaustion on X_c. Moreover, this function is strongly (weakly) q-convex on $X_c \setminus K$, thanks to Proposition 3.2.3 and 3.2.4.

2. Note that a sum $\psi = \psi_1 + \ldots + \psi_s$ of exhaustion functions on the sets U_j is an exhaustion on U, choose the ψ_j's weakly q_j-convex everywhere and apply Proposition 3.2.4.

□

Corollary 3.2.10. *Any finite intersection $U = U_1 \cap \ldots \cap U_s$ of weakly 1-convex open subsets is weakly 1-convex. The set U is strongly 1-convex (resp. 1-complete) as soon as one of the sets U_j is strongly 1-convex (resp. 1-complete).*

We prove now a rather useful result asserting the existence of q-complete neighborhoods for q-complete subvarieties. The first step is an approximation-extension theorem for strongly q-convex functions.

Proposition 3.2.11. *Let Y be an analytic set in a complex space X and ψ a strongly q-convex $\mathscr{C}^\infty$ function on Y. For every continuous function $\delta > 0$ on Y, there exists a strongly q-convex $\mathscr{C}^\infty$ function φ on a neighborhood V of Y such that $\psi \leq \varphi|_Y < \psi + \delta$.*

Proof. Let Z_k be a stratification of Y, i.e. Z_k is an increasing sequence of analytic subsets of Y such that $Y = \cup Z_k$ and $Z_k \setminus Z_{k-1}$ is a smooth k-dimensional manifold (possibly empty for some k's). We shall prove by induction on k the following statement:

There exists a $\mathscr{C}^\infty$ function φ_k on X which is strongly q-convex along Y and on a closed neighborhood $\overline{V}_k$ of Z_k in X, such that $\psi \leq \varphi_k|_Y < \psi + \delta$.

We first observe that any smooth extension φ_{-1} of ψ to X satisfies the requirements with $Z_{-1} = V_{-1} = \varnothing$. Assume that V_{k-1} and φ_{k-1} have been constructed. Then $Z_k \setminus V_{k-1} \subset Z_k \setminus Z_{k-1}$ is contained in $Z_{k,\mathrm{reg}}$. The closed set $Z_k \setminus V_{k-1}$ has a locally finite covering (A_λ) in X by open coordinate patches $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda}$ in which Z_k is given by equation $z'_\lambda = (z_{\lambda,k+1}, \ldots, z_{\lambda,N_\lambda}) = 0$. Let θ_λ be $\mathscr{C}^\infty$ functions with compact support in A_λ such that $0 \leq \theta \leq 1$ and $\sum \theta_\lambda = 1$ on $Z_k \setminus V_{k-1}$. We set

$$\varphi_k(x) = \varphi_{k-1}(x) + \sum \theta_\lambda(x)\varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4}\,|z'_\lambda|^2) \quad \text{on } X.$$

For $\varepsilon_\lambda > 0$ small enough, we will have $\psi \leq \varphi_{k-1}|_Y \leq \varphi_k|_Y < \psi + \delta$. Now, we check that φ_k is still strongly q-convex along Y and near any $x_0 \in \overline{V}_{k-1}$, and that ϕ_k becomes strongly q-convex near any $x_0 \in Z_k \setminus V_{k-1}$. We may assume that $x_0 \in \operatorname{Supp}\theta_\mu$ for some μ, otherwise φ_k coincides with φ_{k-1} in a neighborhood of x_0. Select μ and a small neighborhood $W \subset\subset \Omega_\mu$ of x_0 such that

1. if $x_0 \in Z_k \setminus V_{k-1}$, then $\theta_\mu(x_0) > 0$ and $A_\mu \cap W \subset\subset \{x \in A_\mu | \theta_\mu > 0\}$;
2. if $x_0 \in A_\lambda$ for some λ (there is only a finite set I of such λ's), then $A_\mu \cap W \subset\subset A_\lambda$ and $z_\lambda|_{A_\mu \cap W}$ has a holomorphic extension $\tilde{z}_\lambda$ to $\overline{W}$;
3. if $x_0 \in \overline{V}_{k-1}$, then $\varphi_{k-1}|_{A_\mu \cap W}$ has a strongly q-convex extension $\tilde{\varphi}_{k-1}$ to $\overline{W}$.
4. if $x_0 \in Y \setminus \overline{V}_{k-1}$, then $\varphi_{k-1}|_{Y \cap W}$ has a strongly q-convex extension $\tilde{\varphi}_{k-1}$ to $\overline{W}$.

Take an arbitrary smooth extension $\tilde{\varphi}_{k-1}$ of $\varphi_{k-1}|_{A_\mu \cap W}$ to $\overline{W}$ and let $\tilde{\theta}_\lambda$ be an extension of $\theta_\lambda|_{A_\mu \cap W}$ to $\overline{W}$. Then

$$\tilde{\varphi}_k = \tilde{\varphi}_{k-1} + \sum \tilde{\theta}_\lambda \varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4} |\tilde{z}'_\lambda|^2)$$

is an extension of $\varphi_k|_{A_\mu \cap W}$ to $\overline{W}$, resp. of $\varphi_k|_{Y \cap W}$ to $\overline{W}$ in case 4. As the function $\log(1 + \varepsilon_\lambda^{-4} |\tilde{z}'_\lambda|^2)$ is plurisubharmonic and as its first derivative $\langle \tilde{z}'_\lambda, \mathrm{d}\tilde{z}'_\lambda \rangle (\varepsilon_\lambda^4 + |\tilde{z}'_\lambda|^2)^{-1}$ is bounded by $O(\varepsilon_\lambda^{-2})$, we see that

$$i\partial\overline{\partial}\tilde{\varphi}_k \geq i\partial\overline{\partial}\tilde{\varphi}_{k-1} - O\left(\sum \varepsilon_\lambda\right).$$

Therefore, for ε_λ small enough, $\tilde{\varphi}_k$ remains q-convex on $\overline{W}$ in cases 3 and 4. Since all functions $\tilde{z}'_\lambda$ vanish along $Z_k \cap W$, we have

$$i\partial\overline{\partial}\tilde{\varphi}_k \geq i\partial\overline{\partial}\tilde{\varphi}_{k-1} + \sum_{\lambda \in I} \theta_\lambda \varepsilon_\lambda^{-1} i\partial\overline{\partial} |\tilde{z}'_\lambda|^2 \geq i\partial\overline{\partial}\tilde{\varphi}_{k-1} + \theta_\mu \varepsilon_\mu^{-1} i\partial\overline{\partial} \left|z'_\mu\right|^2$$

at every point of $Z_k \cap W$. Moreover $i\partial\overline{\partial}\tilde{\varphi}_{k-1}$ has at most $(q-1)$-negative eigenvalues on TZ_k since $Z_k \subset Y$, whereas $i\partial\overline{\partial}\left|z'_\mu\right|^2$ is positive definite in the normal directions to Z_k in Ω_μ. In case 1, we thus find that $\tilde{\varphi}_k$ is strongly q-convex on $\overline{W}$ for φ_μ small enough; we also observe that only finitely many conditions are required on each φ_λ if we choose a locally finite covering of $\bigcup \operatorname{Supp} \theta_\lambda$ by neighborhoods W as above. Therefore, for ε_λ small enough, φ_k is strongly q-convex on a neighborhood $\overline{V}'_k$ of $Z_k \setminus V_{k-1}$. The function φ_k and the set $V_k = V'_K \cup V_{k-1}$ satisfy the requirements at order k. It is clear that we can choose the sequence φ_k stationary on every compact subset of X; the limit φ and the open set $V = \bigcup V_k$ fulfill the proposition. □

The second step is the existence of almost plurisubharmonic functions having poles along a prescribed analytic set. By an almost plurisubharmonic function on a manifold we mean a function that is locally equal to the sum of a plurisubharmonic function and a smooth function, or equivalently, a function whose complex Hessian has bounded negative part. On a complex space, we require that our function can be locally extended as an almost plurisubharmonic function in the ambient space of an embedding.

Lemma 3.2.12. *Let Y be an analytic subvariety in a complex space X. There is an almost plurisubharmonic function v on X such that $v = -\infty$ on Y with logarithmic poles and $v \in \mathscr{C}^\infty(X \setminus Y)$.*

Proof. Since $\mathscr{I}_Y \subset \mathscr{O}_X$ is a coherent subsheaf, there is a locally finite covering of X by patches A_λ isomorphic to analytic sets in balls $B(0, r_\lambda) \subset \mathbb{C}^{N_\lambda}$, such that $\mathscr{I}_Y$ admits a system of generators $g_\lambda = (g_{\lambda,j})$ on a neighborhood of each set $\overline{A}_\lambda$. We set

$$v_\lambda(z) = \log |g_\lambda(z)|^2 - \frac{1}{r_\lambda^2 - |z - z_\lambda|^2} \quad \text{on } A_\lambda,$$
$$v(z) = m(\ldots, v_\lambda(z), \ldots) \quad \text{for } \lambda \text{ such that } A_\lambda \ni z,$$

where m is a regularized max function defined as follows: select a smooth function ρ on $\mathbb{R}$ with support in $[-1/2, 1/2]$, such that $\rho \geq 0$, $\int_{\mathbb{R}} \rho(u)\,\mathrm{d}u = 1$, $\int_{\mathbb{R}} u\rho(u)\,\mathrm{d}u = 0$, and set

$$m(t_1, \ldots, t_p) = \int_{\mathbb{R}^p} \max\{t_1 + u_1, \ldots, t_p + u_p\} \prod_{1 \leq j \leq p} \rho(u_j)\,\mathrm{d}u_j.$$

It is clear that m is increasing in all variables and convex, thus m preserves plurisubharmonicity. Moreover, we have

$$m(t_1, \ldots, t_j, \ldots, t_p) = m(t_1, \ldots, \widehat{t_j}, \ldots, t_p)$$

as soon as $t_j < \max\{t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_p\} - 1$. As the generators $(g_{\lambda,j})$ can be expressed in terms of one another on a neighborhood of $\overline{A}_\lambda \cap \overline{A}_\mu$, we see that the quotient $|g_\lambda| / |g_\mu|$ remains bounded on this set. Therefore none of the values $v_\lambda(z)$ for $A_\lambda \ni z$ and z near ∂A_λ contributes to the value of $v(z)$, since $1/(r^2 - |z - z_\lambda|^2)$ tends to $+\infty$ on ∂A_λ. It follows that v is smooth on $X \setminus Y$: as each v_λ is almost plurisubharmonic on A_λ, we also see that v is almost plurisubharmonic on X. □

Theorem 3.2.13. *Let X be a complex space and Y a strongly q-complete analytic subset. Then Y has a fundamental family of strongly q-complete neighborhoods V in X.*

Proof. By Proposition 3.2.11 applied to a strongly q-convex exhaustion of Y and $\delta = 1$, there exists a strongly q-convex function φ on a neighborhood W_0 of Y such that $\varphi|_Y$ is an exhaustion. Let W_1 be a neighborhood of Y such that $\overline{W}_1 \subset W_0$ and such that $\varphi|_{\overline{W}_1}$ is an exhaustion. We are going to show that every neighborhood $W \subset W_1$ of Y contains a strongly q-complete neighborhood V. If v is the function given by Lemma 3.2.12, we set

$$\tilde{v} = v + \chi \circ \varphi \quad \text{on } \overline{W},$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function. If χ grows fast enough, we get $\tilde{v} > 0$ on ∂W and the $(q-1)$-codimensional subspace on which $i\partial\overline{\partial}\varphi$ is positive definite (in some ambient space) is also positive definite for $i\partial\overline{\partial}\tilde{v}$ provided that χ' be large enough to compensate the bounded negative part of $i\partial\overline{\partial}v$. Then $\tilde{v}$ is strongly q-convex. Let θ be a smooth convex increasing function on $]-\infty, 0[$ such that $\theta(t) = 0$ for $t < -3$ and $\theta(t) = -1/t$ on $]-1, 0[$. The open set $V = \{z \in W | \tilde{v}(z) < 0\}$ is a neighborhood of Y and $\tilde{\psi} = \varphi + \theta \circ \tilde{v}$ is a strongly q-convex exhaustion of V. □

It it obvious by definition that a n-dimensional complex manifold M is strongly q-complete for $q \geq n+1$. If M is connected an non compact, this property also holds for $q = n$, i.e. there is a smooth exhaustion ψ on M such that $i\partial\overline{\partial}\psi$ has at least one positive eigenvalue everywhere. In fact, one can even show that M has strongly subharmonic exhaustion functions. Let ω be an arbitrary hermitian metric on M. We consider the Laplace operator Δ_ω defined by

$$\Delta_\omega v = \mathrm{Trace}_\omega(i\partial\overline{\partial}v) = \sum_{i \leq j,k \leq n} \omega^{jk}(z) \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k},$$

where (ω^{jk}) is the conjugate of the inverse matrix of (ω_{jk}). Note that Δ_ω may differ from the usual Laplace-Beltrami operator if ω is not Kähler. We say that v is *strongly ω-subharmonic* if $\Delta_\omega v > 0$. Clearly, this property implies that $i\partial\overline{\partial}v$ has at least one positive eigenvalue at each point, i.e. that v is strongly n-convex. Moreover, since

$$\Delta_\omega \chi(v_1, \ldots, v_s) = \sum_j \frac{\partial \chi}{\partial t_j}(v_1, \ldots, v_s)\Delta_\omega v_j + \sum_{j,k} \frac{\partial^2 \chi}{\partial t_j \partial t_k}(v_1, \ldots, v_s)\langle \partial v_j, \partial v_k \rangle_\omega,$$

subharmonicity has the advantage of being preserved by all convex increasing combinations, whereas a sum of strongly n-convex functions is not necessarily n-convex. We shall need the following partial converse.

Lemma 3.2.14. *If ψ is strongly n-convex on M, there is a hermitian metric ω such that ψ is strongly subharmonic with respect to ω.*

Proof. Let $U_\lambda \subset\subset U'_\lambda$, $\lambda \in \mathbb{N}$, be locally finite coverings of M by open balls equipped with coordinates such that $\partial^2\psi/\partial z_1 \partial \overline{z}_1 > 0$ on $\overline{U}'_\lambda$. By induction on λ, we construct a hermitian metric ω_λ on M such that ψ is strongly ω_λ-subharmonic on $U_0 \cup \ldots \cup U_{\lambda-1}$. Starting from an arbitrary ω_0, we obtain w_λ from $\omega_{\lambda-1}$ by increasing the coefficient $\omega^{11}_{\lambda-1}$ in $(\omega^{jk}_{\lambda-1}) = (\omega_{\lambda-1,kj})^{-1}$ on a neighborhood of $\overline{U}_\lambda$. Then $\omega = \lim \omega_\lambda$ is the required metric. □

Lemma 3.2.15. *Let $U, W \subset M$ be open sets such that for every connected component U_s of U there is a connected component $W_{t(s)}$ of W such that $W_{t(s)} \cap U_(s) \neq \varnothing$ and $W_t(s) \setminus \overline{U}_s \neq \varnothing$. Then there exists a function $v \in \mathscr{C}^\infty(M, \mathbb{R}), v \geq 0$, with support contained in $\overline{U} \cup \overline{W}$, such that v is strongly ω-subharmonic and > 0 on U.*

Proof. We first prove that the result is true when U, W are small cylinders with the same radius and axis. Let $a_0 \in M$ be a given point and $z_1, \dots, z_n$ holomorphic coordinates centered at a_0. We set $\Re z_j = x_{2j-1}$, $\Im z_j = x_{2j}$, $x' = (x_2, \dots, x_{2n})$ and $\omega = \sum \tilde{\omega}_{jk}(x)\, \mathrm{d}x_j \otimes \mathrm{d}x_k$. Let U be the cylinder $|x_1| < r$, $|x'| < r$, and W the cylinder $r-\varepsilon < x_1 < r+\varepsilon$, $|x'| < r$. There are constants $c, C > 0$ such that

$$\sum \tilde{\omega}^{jk}(x)\xi_j\xi_k \geq c\,|\xi|^2 \text{ and } \sum \left|\tilde{\omega}^{jk}(x)\right| \leq C \text{ on } \overline{U}.$$

Let $\chi \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ be a non-negative function equal to 0 on $]-\infty, -r] \cup [r+\varepsilon, +\infty[$ and strictly convex on $]-r, r]$. We take explicitly $\chi(x_1) = (x_1+r)\exp(-1/(x_1+r)^2)$ on $[-r, r]$ and

$$v(x) = \chi(x_1)\exp(1/(|x'|^2 - r^2)) \text{ on } U \cup W,\ v = 0 \text{ on } M \setminus (U \cup W).$$

We have $v \in \mathscr{C}^\infty(M, \mathbb{R}), v > 0$ on U, and a simple computation gives

$$\begin{aligned}\frac{\Delta_\omega v(x)}{v(x)} &= \omega^{11}(x)(4(x_1+r)^{-5} - 2(x_1+r)^{-3}) \\ &+ \sum_{j>1} \tilde{\omega}^{1j}(x)(1 + 2(x_1+r)^{-2})(-2x_j)(r^2 - |x'|^2)^{-2} \\ &+ \sum_{j,k>1} \tilde{\omega}^{jk}(x)\left(x_jx_k(4 - 8(r^2 - |x'|^2)) - 2(r^2 - |x'|^2\,\delta_{jk})\right)(r^2 - |x'|^2)^{-4},\end{aligned}$$

for r small, we get

$$\begin{aligned}\frac{\Delta_\omega v(x)}{v(x)} \geq 2c(x_1+r)^{-5} &- C_1(x_1+r)^{-2}\,|x'|\,(r^2 - |x'|^2)^{-2} \\ &+ (2c\,|x'|^2 - C_2r^4)(r^2 - |x'|^2)^{-4}\end{aligned}$$

with constants C_1, C_2 independent of r. The negative term is bounded by $C_3(x_1+r)^{-4} + c\,|x'|^2\,(r^2 - |x'|^2)^{-4}$, hence

$$\Delta_\omega v/v(x) \geq c(x_1+r)^{-5} + (c\,|x'|^2 - C_2r^4)(r^2 - |x'|^2)^{-4}.$$

The last term is negative only when $|x'| < C_4r^2$, in which case it is bounded by $C_5r^{-4} < c(x_1+r)^{-5}$. Hence v is strongly ω-subharmonic on U.

Next, assume that U and W are connected. Then $U \cup W$ is connected. Fix a point $a \in W \setminus \overline{U}$. If $z_0 \in U$ is given, we choose a path $\Gamma \subset U \cup W$ from z_0 to a a which is piecewise linear with respect to holomorphic coordinate patches. Then we can find a finite sequence of cylinders (U_j, W_j) of the type described above, $1 \leq j \leq N$, whose axes are segments contained in Γ, such that

$$U_j \cup W_j \subset U \cup W, \quad \overline{W}_j \subset U_{j+1} \text{ and } z_0 \in U_0,\ a \in W_N \subset W \setminus \overline{U}.$$

For each such pair, we have a function $v_j \in \mathscr{C}^\infty(M)$ with support in $\overline{U}_j \cup \overline{W}_j,\ v_j \geq 0$, strongly ω-subharmonic and > 0 on U_j. By induction, we can find constants $C_j > 0$ such that $v_0 + C_1 v_1 + \ldots + C_j v_j$ is strongly ω-subharmonic on $U_0 \cup \ldots U_j$ and ω-subharmonic on $M \setminus \overline{W}_j$. Then

$$w_{z_0} = v_0 + C_1 v_1 + \ldots + \mathscr{C}_N v_n \geq 0$$

is ω-subharmonic on U and strongly ω-subharmonic > 0 on a neighborhood Ω_0 of the given point z_0. Select a countable covering of U by such neighborhoods Ω_p and set $v(z) = \sum \varepsilon_p w_{z_p}(z)$ where ε_p is a sequence converging sufficiently fast to 0 so that $v \in \mathscr{C}^\infty(M, \mathbb{R})$. Then v has the required properties.

In the general case, we find for each pair $(U_s, W_{t(s)})$ a function v_s with support in $\overline{U}_s \cup \overline{W}_{t(s)}$, strongly ω-subharmonic and > 0 on U_s. Any convergent series $v = \sum \varepsilon_s v_s$ yields a function with the desired properties. □

Lemma 3.2.16. *Let X be a connected, locally connected and locally compact topological space. If U is a relatively compact open subset of X, we let $\tilde{U}$ be the union of U with all compact connected components of $X \setminus U$. Then $\tilde{U}$ is open and relatively compact in X, and $X \setminus \tilde{U}$ has only finitely many connected components, all non compact.*

Theorem 3.2.17 (Greene-Wu). *Every n-dimensional connected non compact complex manifold M has a strongly subharmonic exhaustion function with respect to any hermitian metric ω. In particular, M is strongly n-complete.*

Proof. Let $\varphi \in \mathscr{C}^\infty(M, \mathbb{R})$ be an arbitrary exhaustion function. There exists a sequence of connected smoothly bounded open sets $\Omega'_\nu \subset\subset M$ such that $\overline{\Omega}'_\nu \subset \Omega'_{\nu+1}$ and $M = \bigcup \Omega'_\nu$. Let $\Omega_\nu = \tilde{\Omega}'_\nu$ be the relatively compact open set given by Lemma 3.2.16. Then $\overline{\Omega}_\nu \subset \Omega_{\nu+1}$, $M = \bigcup \Omega_\nu$ and $M \setminus \Omega_\nu$ has no compact connected component. We set

$$U_1 = \Omega_2, \quad U_\nu = \Omega_{\nu+1} \setminus \overline{\Omega}_{\nu-2} \text{ for } \nu \geq 2.$$

Then $\partial U_\nu = \partial\Omega_{\nu+1} \cup \partial\Omega_{\nu-2}$; any connected component $U_{\nu,s}$ of U_ν has its boundary $\partial U_{\nu,s} \not\subset \partial\Omega_{\nu-2}$, otherwise $\overline{U}_{\nu,s}$ would be open and closed in $M \setminus \Omega_{\nu-2}$, hence $\overline{U}_{\nu,s}$ would be a compact component of $M \setminus \Omega_{\nu-2}$. Therefore $\partial U_{\nu,s}$ intersects $\partial\Omega_{\nu+1} \subset U_{\nu+1}$. If $\partial U_{\nu+1,t(s)}$

is a connected component of U_{v+1} containing a point of $\partial U_{\nu,s}$, then $U_{\nu+1,t(s)} \cap U_{\nu,s} \neq \varnothing$ and $U_{\nu+1,t(s)} \setminus \overline{U}_{\nu,s} \neq \varnothing$. Lemma 3.2.15 implies that there is a non-negative function $v_\nu \in \mathscr{C}^\infty(M, \mathbb{R})$ with support in $U_\nu \cup U_{\nu+1}$, which is strongly ω-subharmonic on U_ν. An induction yields constants C_ν such that

$$\psi_\nu = \varphi + C_1 v_1 + \ldots + C_\nu v_\nu$$

is strongly ω-subharmonic on $\overline{\Omega}_\nu \subset U_0 \cup \ldots \cup U_\nu$, thus $\psi = \varphi + \sum C_\nu v_\nu$ is a strongly ω-subharmonic exhaustion function on M. □

By an induction on the dimension, the above result can be generalized to an arbitrary complex space, as was first shown by T. Ohsawa.

Theorem 3.2.18 (Ohsawa)**.** *Let X be a complex space such that all irreducible components have dimension $\leq n$.*

1. *X is always strongly $(n+1)$-complete.*
2. *If X has no compact irreducible component of dimension n, then X is strongly n-complete*
3. *If X has only finitely many irreducible components of dimension n, then X is strongly n-convex.*

Proof. We prove 1 and 2 by induction on $n = \dim X$. For $n = 0$, property 2 is void and 1 is obvious (any function can then be considered as strongly 1-convex). Assume that 1 has been proved in dimension $\leq n-1$. Let X' be the union of X_{sing}, the singular points of X, and of the irreducible components of X of dimension at most $n-1$, and $M = X \setminus X'$ the n-dimensional part of X_{reg}, the regular points of X. As $\dim X' \leq n-1$, the induction hypothesis shows that X' is strongly n-complete. By Theorem 3.2.13, there exists a strongly n-convex exhaustion function φ' on a neighborhood V' of X'. Take a closed neighborhood $\overline{V} \subset V'$ and an arbitrary exhaustion φ on X that extends $\varphi'|_{\overline{V}}$. Since every function on a n-dimensional manifold is strongly $(n+1)$-convex, we conclude that X is at worst $(n+1)$-complete, as stated in 1

In case 2, the hypothesis means that the connected components M_j of $M = X \setminus X'$ have non compact closure $\overline{M}_j$ in X. On the other hand, Lemma 3.2.14 shows that there exists a hermitian metric ω on M such that $\varphi|_{M\cap V}$ is strongly ω-subharmonic. Consider the open sets $U_{j,\nu}$ provided by Lemma 3.2.19 below. By the arguments already used in Theorem 3.2.17, we can find a strongly ω-subharmonic exhaustion $\varphi = \phi + \sum_{j,\nu} C_{j,\nu} v_{j,\nu}$ on X, with $v_{j,\nu}$ strongly ω-subharmonic on $U_{j,v}$, $\operatorname{Supp} v_{j,v} \subset U_{j,\nu} \cup U_{j,\nu+1}$ and $C_{j,\nu}$ large. Then ψ is strongly n-convex on X.

Lemma 3.2.19. *For each j, there exists a sequence of open sets $U_{j,\nu} \subset\subset M_j$, $\nu \in N$, such that*

1. *$M_j \setminus V' \subset \bigcup_\nu U_{j,\nu}$ and $(U_{j,\nu})$ is locally finite in $\overline{M}_j$;*
2. *For every connected component $U_{j,\nu,s}$ of $U_{j,\nu}$ there is a connected component $U_{j,\nu+1,t(s)}$ of $U_{j,\nu+1}$ such that $U_{j,\nu+1,t(s)} \cap U_{j,\nu,s} \neq \varnothing$ and $U_{j,\nu+1,t(s)} \setminus \overline{U}_{j,\nu,s} \neq \varnothing$.*

Proof. By Lemma 3.2.16 applied to the space $\overline{M}_j$, there exists a sequence of relatively compact connected open sets $\Omega_{j,\nu}$ in $\overline{M}_j$ such that $\overline{M}_j \setminus \Omega_{j,\nu}$ has no compact connected component, $\overline{\Omega}_{j,\nu} \subset \Omega_{j,\nu+1}$ and $\overline{M}_j = \bigcup \Omega_{j,\nu}$. We define a compact set $K_{j,\nu} \subset M_j$ and an open set $W_{j,\nu} \subset \overline{M}_j$ containing $K_{j,\nu}$ by

$$K_{j,\nu} = (\overline{\Omega}_{j,\nu} \setminus \Omega_{j,\nu-1}) \setminus V', \quad W_{j,\nu} = \Omega_{j,\nu+1} \setminus \overline{\Omega}_{j,\nu-2}.$$

By induction on ν, we construct an open set $U_{j,\nu} \subset\subset W_{j,\nu} \setminus X' \subset M_j$ and a finite set $F_{j,k} \subset \partial U_{j,\nu} \setminus \overline{\Omega}_{j,\nu}$. We let $F_{j,-1} = \varnothing$. If these sets are already constructed for $\nu - 1$, the compact set $K_{j_\nu} \cup F_{j,\nu-1}$ is contained in the open set $W_{j,\nu}$, thus contained in a finite union of connected components $W_{j,\nu,s}$. We can write $K_{j,\nu} \cup F_{j,\nu-1} = \bigcup L_{j,\nu,s}$ where $L_{j,\nu,s}$ is contained in $W_{j,\nu,s} \setminus X' \subset M_j$. The open set $W_{j,\nu,s} \setminus X'$ is connected and non contained in $\Omega_{j,\nu} \cup L_{j,\nu,s}$, otherwise its closure $\overline{W}_{j,\nu,s}$ would have no boundary point in $\partial\Omega_{j,\nu+1}$, thus would be open and compact $\overline{M}_j \setminus \Omega_{j,\nu-2}$, contradiction. We select a point $a_s \in (W_{j,\nu,s} \setminus X') \setminus (\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s})$ and a smoothly bounded connected open set $U_{j,\nu,s} \subset\subset W_{j,\nu,s} \setminus X'$ containing $L_{j,\nu,s}$ with $a_s \in \partial U_{j,\nu,s}$. Finally, we set $U_{j,\nu} = \bigcup_s U_{j,\nu,s}$ and let $F_{j,\nu}$ be the set of all points a_s. By construction, we have $U_{j,\nu} \supset K_{j,\nu} \cup F_{j,\nu-1}$, thus $\bigcup U_{j,\nu} \supset \bigcup K_{j,\nu} = M_j \setminus V'$, and $\partial U_{j,\nu,s} \ni a_s$ with $a_s \in F_{j,\nu} \subset U_{j,\nu+1}$. Property 2 follows. □

We now prove 3. Let $Y \subset X$ be the union of X_{sing} with all irreducible components of X that are non compact of dimension $< n$. Then $\dim Y \leq n-1$, so Y is n-convex and Theorem 3.2.13 implies that there is an exhaustion function $\psi \in \mathscr{C}^\infty(X, \mathbb{R})$ such that ψ is strongly n-convex on a neighborhood V of Y. Then the complement $K = X \setminus V$ is compact and ψ is strongly n-convex on $X \setminus K$. □

We now follow [Dem90] to give a simple proof of Ohsawa's Hodge decomposition theorem.

Let M be a complex n-dimensional manifold admitting a Kähler metric ω and a strongly q-convex plurisubharmonic exhaustion function ψ. For any convex increasing function $\chi \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$, we consider the new Kähler metric

$$\omega_\chi = \omega + i\partial\overline{\partial}(\chi \circ \psi) = \omega + \chi'(\psi) i\partial\overline{\partial}\psi + \chi''(\psi) i\partial\psi \wedge \overline{\partial}\psi$$

and the associated geodesic distance δ_χ. Then the norm of $\chi''(\psi)^{1/2}\,\mathrm{d}\psi$ with respect to ω_χ is less that 1, thus if ρ is a primitive of $(\chi'')^{1/2}$ we have

$$|\rho(\psi(x)) - \rho(\psi(y))| \leq \delta_\chi(x, y).$$

Hence ω_χ is complete as soon as $\lim_{t\to\infty}\rho(t) = +\infty$, that is $\int_0^\infty \chi''(t)^{1/2}\,\mathrm{d}t = +\infty$. In the sequel, we always assume that χ grows sufficiently fast at infinity so that this condition is fulfilled. We denote by $L^{2,(k)}_\chi(M) = \oplus_{r+s=k} L^{2,(r,s)}_\chi(M)$ the space of L^2 forms of degree k with respect to the metric ω_χ, by $\mathscr{H}^k_\chi(M)$ the subspace of L^2 harmonic forms of degree k with respect to the associated Laplace-Beltrami operator $\Delta_\chi = \mathrm{d}\mathrm{d}^*_\chi + \mathrm{d}^*_\chi\mathrm{d}$ and by $\mathscr{H}^{r,s}_\chi(M)$ the space of L^2-harmonic forms of bidegree (r, s) with respect to $\Delta_{\overline{\partial}_\chi} = \overline{\partial}\overline{\partial}^*_\chi + \overline{\partial}^*_\chi\overline{\partial}$. As ω_χ is Kähler, we have $\Delta_{\partial_\chi} = \Delta_{\overline{\partial}_\chi} = \frac{1}{2}\Delta_\chi$, hence

$$\mathscr{H}^k_\chi(M) = \bigoplus_{r+s=k} \mathscr{H}^{r,s}_\chi(M), \quad \mathscr{H}^{s,r}_\chi(M) = \overline{\mathscr{H}^{r,s}_\chi(M)}, \tag{3.9}$$

for each $k = 0, 1, \ldots, 2n$. Since ω_χ is complete, we have orthogonal decompositions

$$\begin{aligned} L^{2,(r,s)}_\chi(M) &= \mathscr{H}^{r,s}_\chi(M) \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi} \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}^*_\chi} \\ \mathrm{Ker}^{r,s}\,\overline{\partial}_\chi &= \mathscr{H}^{r,s}_\chi(M) \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi}, \end{aligned} \tag{3.10}$$

where $\overline{\partial}_\chi$ is the unbounded $\overline{\partial}$ operator acting on L^2 forms with respect to ω_χ and where $\overline{\mathrm{Im}^{r,s}}$ means closure of the range (in the specified bidegree). In particular $\mathscr{H}^{r,s}_\chi(M)$ is isomorphic to the quotient $\mathrm{Ker}^{r,s}\,\overline{\partial}_\chi/\overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi}$. Of course, similar results also hold for Δ_χ-harmonic forms.

Lemma 3.2.20. *Let u be a form of type (r, s) with L^2_{loc} coefficients on M. If $r+s \geq n+q$, then $u \in L^{2,(r,s)}_\chi(M)$ as soon as χ grows sufficiently fast at infinity.*

Proof. At each point $x \in M$, there is an orthogonal basis $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ of T_xX in which

$$\omega = i \sum_{1\leq j\leq n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j, \quad \omega_\chi = i \sum_{1\leq j\leq n} \lambda_j\,\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j,$$

where $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of ω_χ with respect to ω. Then the volume elements $\mathrm{d}V = \omega^n/2^n n!$ are related by

$$\mathrm{d}V_\chi = \lambda_1 \ldots \lambda_n\,\mathrm{d}V$$

and for a (r, s)-form $u = \sum_{I,J} u_{I,J}\,\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$ we find

$$|u|^2_\chi = \sum_{|I|=r,|J|=s} \left(\prod_{k\in I}\lambda_k \prod_{k\in J}\lambda_k\right)^{-1} |u_{I,J}|^2.$$

In particular

$$|u|_\chi^2 \,\mathrm{d}V_\chi \leq \frac{\lambda_1 \ldots \lambda_n}{\lambda_1 \ldots \lambda_r \lambda_1 \ldots \lambda_s} |u|^2 \,\mathrm{d}V = \frac{\lambda_{r+1} \ldots \lambda_n}{\lambda_1 \ldots \lambda_s} |u|^2 \,\mathrm{d}V.$$

On the other hand, we have upper bounds

$$\lambda_j \leq 1 + C_1 \chi'(\psi), \quad 1 \leq j \leq n-1, \quad \lambda_n \leq 1 + C_1 \chi'(\psi) + C_2 \chi''(\psi),$$

where $C_1(x)$ is the largest eigenvalue of $i\partial\overline{\partial}\psi(x)$ and $C_2(x) = |\partial\psi|^2$; to find the $n-1$ first inequalities, we need only apply the minimax principle on the kernel of $\partial\psi$. As $i\partial\overline{\partial}\psi$ has at most $q-1$ zero eigenvalues on $X \setminus K$, the minimax principle also gives lower bound

$$\lambda_j \geq 1, \quad 1 \leq j \leq q-1, \quad \lambda_j \geq 1 + c\chi'(\psi), \quad q \leq j \leq n,$$

where $c(x) \geq 0$ is the q-th eigenvalue of $i\partial\overline{\partial}\psi(x)$ and $c(x) > 0$ on $X \setminus K$. Assuming $\chi' \geq 1$, we infer easily

$$\begin{aligned} \frac{|u|_\chi^2 \,\mathrm{d}V_\chi}{|u|^2 \,\mathrm{d}V} &\leq \frac{(1 + C_1\chi'(\psi))^{n-r-1}(1 + C_1\chi(\psi) + C_2\chi''(\psi))}{(1 + c\chi'(\psi))^{s-q+1}} \\ &\leq C_3(\chi'(\psi)^{n+q-r-s-1} + \chi''(\psi)\chi'(\psi)^{n+q-r-s-2}) \quad \text{on } X \setminus K, \end{aligned}$$

for $r + s \geq n + q$, this is less than

$$C_3 \left(\chi'(\psi)^{-1} + \chi''(\psi)\chi'(\psi)^{-2} \right),$$

and it is easy to show that this quantity can be made arbitrarily small when χ grows sufficiently fast at infinity on M. □

It is a well-known result of Andreotti-Grauert [AG62] that the natural topology on the cohomology groups $H^k(M, \mathscr{F})$ of a coherent sheaf $\mathscr{F}$ over a strongly q-convex manifold is Hausdorff for $k \geq q$. If $\mathscr{F} = \mathscr{O}(E)$ is the sheaf of sections of a holomorphic vector bundle, this topology is given by the Fréchet topology on the Dolbeault complex of L^2_{loc} forms with L^2_{loc} $\overline{\partial}$-differential. In particular, the morphism

$$\mathrm{Ker}^{r,s}\, \overline{\partial}_\chi \to H^s(M, \Omega^r)$$

is continuous and has a closed kernel, and therefore this kernel contains $\overline{\mathrm{Im}^{r,s}\, \overline{\partial}_\chi}$. We thus obtain a factorization

$$\mathscr{H}_\chi^{r,s}(M) \simeq \mathrm{Ker}^{r,s}\, \overline{\partial}_\chi / \overline{\mathrm{Im}^{r,s}\, \overline{\partial}_\chi} \to H^s(M, \Omega^r).$$

Consider the direct limit

$$\varinjlim_{\chi} \mathscr{H}^{r,s}_{\chi}(M) \to H^s(M,\Omega^r) \tag{3.11}$$

over the set of smooth convex increasing functions χ with the ordering

$$\chi_1 \prec \chi_2 \iff \chi_1 \leq \chi_2 \text{ and } L^{2,(k)}_{\chi_1}(M) \subset L^{2,(k)}_{\chi_2}(M) \text{ for } k = r+s;$$

this ordering is filtering by the proof of Lemma 3.2.22. It is well known that the De Rham cohomology groups are always Hausdorff, hence there is a similar morphism

$$\varinjlim_{\chi} \mathscr{H}^{k}_{\chi}(M) \to H^k(M,\mathbb{C}). \tag{3.12}$$

Theorem 3.2.21 (Ohsawa)**.** *Let (M,ω) be a Kähler n-dimensional manifold. Suppose that M is absolutely q-convex, i.e. admits a smooth plurisubharmonic exhaustion function that is strongly q-convex on $M \setminus K$ for some compact set K. Set $\Omega^r = \mathscr{O}(\bigwedge^r T^*M)$. Then the De Rham Cohomology groups with arbitrary (resp. compact) supports have decompositions*

$$H^k(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H^s(M,\Omega^r), \qquad H^r(M,\Omega^s) \simeq \overline{H^s(M,\Omega^r)}, \qquad k \geq n+q,$$
$$H^k_c(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H^s_c(M,\Omega^r), \qquad H^r_c(M,\Omega^s) \simeq \overline{H^s_c(M,\Omega^r)}, \qquad k \leq n-q,$$

and these groups are finite dimensional. Moreover, there is a Lefschetz isomorphism

$$\omega^{n-r-s} \wedge \bullet : H^s_c(M,\Omega^r) \to H^{n-r}(M,\Omega^{n-s}), \quad r+s \leq n-q.$$

The first decomposition in Theorem 3.2.21 follows now from (3.9) and the following simple lemma

Lemma 3.2.22. *The morphisms (3.11),(3.12) are one-to-one for $k = r+s \geq n+q$.*

Proof. Let us treat for example the case of 3.11, Let u be a smooth $\overline{\partial}$-closed form of bidegree (r,s), $r+s \geq n+q$. Then there is a choice of χ for which $u \in L^{2,(r,s)}_{\chi}(M)$, so $u \in \operatorname{Ker}^{r,s}\overline{\partial}_{\chi}$ and (3.11) is surjective. If a class $\{u\} \in \mathscr{H}^{r,s}_{\chi_0}(M)$ is mapped to zero in $H^s(M,\Omega^r)$, we can write $u = \overline{\partial}v$ for some smooth form v of bidegree $(r,s-1)$. In the case $r+s > n+q$, we have $v \in L^{2,(r,s-1)}_{\chi}(M)$ for some $\chi \succ \chi_0$. Hence the class of $u = \overline{\partial}_{\chi}v$ in $\mathscr{H}^{r,s}_{\chi}(M)$ is zero and (3.11) is injective. When $r+s = n+q$, the form v need not lie in any space $L^{2,(r,s-1)}_{\chi}(M)$, but it suffices to show that $u = \overline{\partial}v$ is in the closure of $\operatorname{Im}^{r,s}\overline{\partial}_{\chi}$ for some χ. Let $\theta \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function such that $\theta(t)=1$ for $t \le 1/2$, $\theta(t)=0$ for $t \ge 1$ and $|\theta'| \le 3$. Then

$$\overline{\partial}\left(\theta(\varepsilon\psi)v\right) = \theta(\varepsilon\psi)\overline{\partial}v + \varepsilon\theta'(\varepsilon\psi)\overline{\partial}\psi \wedge v.$$

By the proof of lemma 3.2.20, there is a continuous function $C(x)>0$ such that $|v|_\chi^2 \,\mathrm{d}V_\chi \le C(1+\chi''(\psi)/\chi'(\psi))\,|v|^2\,\mathrm{d}V$, whereas $\left|\overline{\partial}\psi\right|_\chi^2 \le 1/\chi''(\psi)$ by the definition of ω_χ. Hence we see that

$$\int_M \left|\theta'(\varepsilon\psi)\overline{\partial}\psi \wedge v\right|_\chi^2 \mathrm{d}V_\chi \le 9\int_M C(1/\chi''(\psi)+1/\chi'(\psi))\,|v|^2\,\mathrm{d}V$$

is finite for χ large enough, and $\overline{\partial}(\theta(\varepsilon\psi)v)$ converges to $\overline{\partial}v = u$ in $L_\chi^{2,(r,s)}(M)$.

□

By Poincaré-Serre duality, the groups $H_c^k(M,\mathbb{C})$ and $H_c^s(M,\Omega^r)$ with compact supports are dual to $H^{2n-k}(M,\mathbb{C})$ and $H^{n-s}(M,\Omega^{n-r})$ as soon as the latter groups are Hausdorff and finite dimensional. This is certainly true for $k=r+s\le n-q$, thus we obtain a Hodge decomposition

$$H_c^k(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H_c^s(M,\Omega^r), \quad H_c^r(M,\Omega^s) \simeq \overline{H_c^s(M,\Omega^r)}, \quad k \le n-q \tag{3.13}$$

As in Ohsawa [Ohs81], it is easy to prove that the Lefschetz isomorphism

$$\omega^{n-r-s}\wedge\bullet : \mathscr{H}_\chi^{r,s}(M) \to \mathscr{H}_\chi^{n-s,n-r}(M) \tag{3.14}$$

yields in the limit an isomorphism from the cohomology with compact support onto the cohomology without supports. Indeed, the natural morphism

$$H_c^s(M,\Omega^r) \to \mathrm{Ker}^{r,s}\,\overline{\partial}_\chi/\overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi} \simeq \mathscr{H}_\chi^{r,s}(M), \quad r+s \le n-q \tag{3.15}$$

is dual to $\mathscr{H}_\chi^{n-r,n-s}(M) \to H^{n-s}(M,\Omega^{n-r})$, which is surjective for χ large by Lemma 3.2.20 and the finite dimensionality of the target space. Hence 3.15 is injective for χ large and after a composition with (3.14) we get an injection

$$H_c^s(M,\Omega^r) \to \mathscr{H}_\chi^{n-s,n-r}(M).$$

If we take the direct limit over all χ, combine with the isomorphism (3.11) and observe that ω_χ has the same cohomology class as ω, we obtain an injective map

$$\omega^{n-r-s}\wedge\bullet : H_c^s(M,\Omega^r) \to H^{n-r}(M,\Omega^{n-s}), \quad r+s \le n-q. \tag{3.16}$$

As both sides have the same dimension by Serre duality and Hodge symmetry, this map must be an isomorphism. Since 3.16 can be factorized through $H^s(M, \Omega^r)$ or through $H_c^{n-r}(M, \Omega^{n-s})$, we infer that the natural morphism

$$H_c^s(M, \Omega^r) \to H^s(M, \Omega^r) \tag{3.17}$$

is injective for $r + s \leq n - q$ and surjective for $r + s \geq n + q$. Of course, similar properties hold for the De Rham cohomology groups.

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Glossary of Symbols and Notations

$(\Gamma, \widehat{\Gamma})$ Hartogs frame, 86
$(\Gamma^*.\widehat{\Gamma}^*)$ Hartogs figure, 86
$H_{BC}^{p,q}$ Bott-Chern cohomology, 73
L_z complex Hessian, 83
L_{loc}^1 Locally L^1 functions, 84
$N(f_1, \ldots, f_k)$ set of common zeros of $f_1, \ldots, f_k$, 88
$P(a, r)$ polydisc centered in a of polyradious r, 84
$PS(D)$ plurisubharmonic functions on D, 82
P^* formal adjoint , 35
$[A, B]$ commutator of A, B, 63
$\#$ antilinear operator, 47
$\mathscr{C}_c(D)$ Continuous function with compact support on D, 84
$\mathscr{H}^p(M, E)$ harmonic forms of degree p, 51
Op_σ pseudodifferential operator, 41
$\triangle$ Laplace–Beltrami operator, 49
$\lrcorner$ contraction by a tangent vector, 47
$\mathfrak{sl}(2)$ 2×2 matrices with trace 0, 65
∂S bundary of an analytic disc, 87
σ_P Principal symbol, 34
$\star$ Hodge Operator, 46