Resaerch Center and Advanced Studies
of the
National Polytechnic Institute

Department of Mathematics

# Semiorthogonal Decompostion in Algebraic Geometry

Thesis that presents

**Oscar García Hernández**

to obtain the degree of

**Master of Science**

in the speciality of

**Mathematics**

Advisor

**Dr. Ernesto Lupercio Lara**

Ciudad de México, July 2018

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL
INSTITUTO POLITÉCNICO NACIONAL
DEPARTAMENTO DE MATEMÁTICAS

# Descomposición Semiortogonal en Geometría Algebraica

TESIS QUE PRESENTA

**Oscar García Hernández**

PARA OBTENER EL GRADO DE

**Maestro en Ciencias**

EN LA ESPECIALIDAD DE

**Matemáticas**

Director:

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Ciudad de México, Julio 2018

# Acknowledgements

Agradezco a el CONACyT el apoyo para obtener el título de maestro en ciencias.

Agradezco a todas las personas que me ayudarón mientras escribía este trabajo principalmente a la profesora **Leticia Brambila Paz** de CIMAT. Gto. que me sugirío asistir a *Advanced School and Workshop on Moduli Spaces, Mirror Symmetry and Enumerative Geometry* en ICTP en Trieste del 1 al 19 Agosto 2016 donde conocí el asombroso tema de categorías derivadas en geometría algebraica. También quiero agradecer al profesor **Ernesto Lupercio Lara** por todos los consejos y sujerencias, por la invitación a la conferencia Algebraic Geometry in Mexico, en Los Cabos, Baja California 2016, y en Puerto Escondido, Oaxaca donde descubrí nuevos temas interesantes.

Me gustaría agradecer a Juan Bosco Frías por su ayuda en Oaxaca. Me gustaría agradecer al CINVESTAV por la increíble hospitalidad para trabajar.

*Especialmente agredezco a mi madre por todos sus consejos y apoyo que me ha brindado a lo largo de mi vida, por ser mi amiga, por su incondicional ayuda y por brindarme más que su vida.*

I thank to CONACyT for the support to obtain my master of Science degree.

There are many persons which helped me while I was writing this work mainly profesor **Leticia Brambila Paz** from CIMAT. Gto. who suggested me to attend the *Advanced School and Workshop on Moduli Spaces, Mirror Symmetry and Enumerative Geometry* at ICTP in Trieste from 1 to 19 August 2016 where I found out the amazing topic of derived categories view on algebraic geometry. Also I want to thank profesor **Ernesto Lupercio Lara** for all his advices and suggestions, for the invitations to the conference Algebraic Geometry in Mexico at Los Cabos, Baja California 2016, and at Puerto Escondido, Oaxaca where I found out new amazing topics.

I would like to thank to my friends from CIMAT and CINVESTAV, to Juan Bosco Frías for useful discutions in Oaxaca. I would like to thank CINVESTAV for the wonderful atmosphere.

*Spatially, I thank my mother for all her advice and support that has brought me along my life, for being my friend, for her unconditional help and for bringing me more that her life.*

# Resumen

Este trabajo es una introducción a la descomposición semiortogonal en geometría algebraica y sus aplicaciones a problemas de racionalidad. La descomposición semiortogonal en geometría algebraica esta motivada por la célebre conjetura de Maxim Kontsevich en simetría espejo homológica (homological mirror symetry) el cual es un tema interesante y maravilloso difícil de evitar en este trabajo, pero no lo mensionaremos aquí. Particularmente esta conjetura conecta geometría algebraica y simpléctica el cual es un problema muy interesante en matemáticas y física.

En geometría birracional uno de los más importantes conceptos es la explosión (desingularización) de una variedad y nosotros damos una formula para la descomposición semiortogonal de la categoría derivada de gavillas coherentes en la explosión de una variedad y en particular describimos la categoría derivada de la explosión de $\mathbb{P}^2$ en a lo más 8 puntos en posició general.

# Preface

This work provides an introduction to semiorthogonal decomposition in algebraic geometry as well as some applications to rationality problems. Semiorthogonal decomposition is motivated by the Maxim Kontsevich's celebrated conjecture in homological mirror symmetry which in turn is an interesting and wonderful topic that is complicated to avoid in this work, but we do not talk about it here. Particularly this conjecture connects algebraic and symplectic geometry which is an amazing problem in mathematics and physics.

In birational geometry one of the most important concept is the blow up of a variety and we give a formula for the semiorthogonal decomposition of the derived category of coherent sheaves on the blow up and in particular we describe the derived category of coherent sheaves on the blow up of $\mathbb{P}^2$ at the most on 8 points in general position.

# Contents

# Introduction

Investigation of derived categories of coherent sheaves on algebraic varieties became one of the most important topics in the modern algebraic geometry. Besides other reasons this is mainly caused by the homological mirror symmetry conjecture of Maxim Kontsevich.

Mirror Symmetry was discovered several years ago in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds, more precisely, complex algebraic manifolds possessing holomorphic volume elements without zeroes. The name comes from the symmetry among Hodge numbers. For dual Calabi-Yau manifolds $X$, $Y$ of dimension $n$ (not necessarily equal to 3) one has

$$\mathsf{dim}\ H^p(X, \Omega^q) = \mathsf{dim}\ H^{n-p}(X, \Omega^q).$$

Physicists conjectured that conformal field theories associated with mirror varieties are equivalent. Mathematically, MS is considered now as a relation between numbers of rational curves on such a manifold and Taylor coefficients of periods of Hodge structures considered as functions on the moduli space of complex structures on a mirror manifold. Recently it has been realized that one can make predictions for numbers of curves of positive genera and also on Calabi-Yau manifolds of arbitrary dimensions.

In 1994 Kontsevich at ICM [Kon94] changed the perspective of mirror symmetry and introduced a new approach to it, the homological mirror symmetry and his celebrated conjecture. Briefly and vaguely this conjecture connects algebraic and symplectic geometry, two amazing branches of mathematics that apparently have no anything in common and predicts that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi-Yau variety and the derived Fukai category (Fukaya category of Lagrangian submanifolds) of its mirror. In this sense, the Fukaya category is the category of A-branes and the derived category of coherent sheaves is the category of B-branes.

Whereas the derived category of coherent sheaves is a familiar object, even the definition of the Fukaya category is far more complicated. Moreover, the Fukaya category is not a category in the usual sense, but an $A_\infty$-category.

Of course, even the formulation of this conjecture is vague, as it is difficult to grasp mathematically the real meaning of mirror. The conjecture has been verified for elliptic curves by Polishchuk and Zaslov in [PZ03]. Seidel has undertaken a detailed investigation of a special quartic K3 surface in [Sei03].

In chapter 1 we will give the basic theory of derived and triangulated categories introduced by Verdier in his thesis [Ver65] and following our geometric approach we introduce the Fourier-Mukai functors and Serre functors. Finally, we introduce Hochschild homology and cohomology of a geometric derived category which are important invariants of the derived category of coherent sheaves of a given algebraic variety.

One year later of Kontsevich's Homological Mirror Symmetry conjecture at ICM [Kon94] Boldal and Orlov [BO95] introduce semiorthogonal decomposition for algebraic varieties which nowdays takes part of the modern algebraic geometry [BO02] and [Orl03]. In chapter 2 we introduce semiorthogonal decomposition for a given triangulated category and prove the Beilinson's Theorem 2.1, a start point for this theory. Basically in this work we present only one method for construct a semiorthogonal decomposition which to be honest was discovered by Beilinson [Bei78]. Semiorthogonal decomposition can be thought as "linear algebra on triangulated categories" and what is been trying to do, it is to find a birational invariant in terms of the decomposition of the derived category of an algebraic variety. To be more precise, if we have a semiorthogonal decomposition of the bounded derived category of a cubic fourfold $X$ in terms of exceptional objects $E_1, \dots, E_n$

$$\mathbf{D}(X) = \langle \mathcal{A}_X, E_1, \dots, E_n \rangle,$$

with $\mathcal{A}_X$ is the orthogonal complement of the objects $E_1, \dots, E_n$, there exist evidence and examples [Kuz15b] where the following conjecture holds.

*Conjecture: $X$ is rational if and only if there is a smooth projective $K3$ surface $Y$ such that $\mathcal{A}_X \cong \mathbf{D}(Y)$.*

So following this approach, from the birational point of view Fano varieties are important objects and provide interesting examples of semiorthogonal decompositions, in particular hypersurfaces are Fano varieties and the semiorhogonal decomposition for cubic 3-folds and cubic 4-folds will give us conditions for rationality.

Del Pezzo surfaces are so important in algebraic surfaces as Fano varieties are in the higher dimensional case. In chapter 3 we prove the main results of this work, the Orlov's formula for the semiorhogonal decomposition of the projectivization of a vector bundle and for the blow up of an algebraic variety, Proposition 3.1.1 (Orlov I) and Theorem 3.1 (Orlov II) respectively, which will provide a formula for the semiorhogonal decomposition for del Pezzo surfaces which are blow ups of $\mathbb{P}^2$ at the most on 8 points in general position.

Through this work $X$ will be an algebraic variety in the sence of [Sha74] or [Har77], sometimes is convenient think of $X$ as an algebraic variety in the sence of [GH78].

# Introducción

La categorí derivada de gavillas coherentes en variedades algebraícas se ha convertido en uno de los principales objetos de estudio en gemetría algebraica. Una de las principales razones es la conjetura de Maxim Kontsevich, homological mirror symmetry (simetría especular homológica).

Simetrí espejo o Simetrí de espejo se descubrió a atrás en teoría de cuerdas como una dualidad entre familias de variedades Calabi-Yau de dimensión 3, más precisamente, variedades complejas algebraicas que tienen una forma de volumen nunca cero. El nombre viene de la simetría entre los números de Hodge. Para variedades Calabi-Yau $X$, $Y$ de dimensión $n$ (no necesariamente 3) uno

$$\mathsf{dim}\ H^p(X, \Omega^q) = \mathsf{dim}\ H^{n-p}(X, \Omega^q).$$

Físicos conjeturan que teorias conformes de campo asociadas con variedades espejo son equivalentes. Matemáticamente, SE es considerada ahora una relación entre números de curvas racionales en una variedad y coeficientes de Taylor de períodos de estructuras de Hodge consideradas como funciones en el espacio moduli de estructuras complejas en una variedad espejo. Recientemente se ha encontrado que uno puede hacer predicciones para los números de curvas de género positivo y también en variedades Calabi-Yau de dimensión arbitraria.

En 1994 Kontsevich en ICM [Kon94] cambió la perspectiva de simetría espejo e introdujo un nuevo enfoque de esta, simetría especular homológica (homological mirror symmetry) y su celebrada conjetura. Brevemente y vagamente esta conjetura conecta geometría algebraica y simpléctica, dos ramas asombrosas de las matemáticas que aparentemente no tienen algo en común y predice que hay una equivalencia entre la categoría derivada de gavillas coherentes en una variedad Calabi-Yau y la categoría derivada de Fukaya (categoría de subvariedades Lagrangianas) de su espejo. En este sentido, la categoría derivada de Fukaya es la categoría de A-branas y la categoría derivada de gavillas coherentes es la categoría de B-branas.

Mientras que la categoría de gavillas coherentes es un objeto conocido y bien entendido, incluso la definición de la categoría de Fukaya es más complicada. Más aun, la categoría de Fukaya no es una catagoría en el sentido usual, si no una $A_\infty$-categoría.

Claro, incluso la formilación de esta conjetura es vaga así como difícil de formular, matemáticamente, el concepto de espejo. La conjetura se ha verificado para el caso de curvas elípticas por Polishchuk y Zaslov en [PZ03]. Seidel ha investigado un caso de una cuadrica K3 especial en [Sei03].

En el capítulo 1 vamos a dar la teria básica de categorías trianguladas introducida por Verdier en su tesis [Ver65] y siguiendo nuestro enfoque geométrico introducimos los funtores de Fourier-Mukai y Serre. Finalmente, introducimos la homología y cohomología de Hochschild de una categoría derivada geométrica las cuales son invariantes importantes de la categoría derivada de gavillas coherentes en una variedad algebraica dada.

Un adespués de la conjetura de Kontsevich, simetría especular homológica en el ICM [Kon94] Boldal y Orlov [BO95] introducen la descomposición semiortogonal para variedades algebraicas la cual forma parte de la geometría algebraica moderna actual, citeBO02 y [Orl03]. En el capítulo 2 introducimos la descomposición semiortogonal para categorías trianguladas y provamos el teorema de Beilinson 2.1, el cual es un punto de partida para esta teoría. Básicamente en este trabajo presentamos sólo un metodo para construir una descomposición semiortogonal, el cual para ser honesto fue descubierto por Beilinson [Bei78]. Descomposición semiortogonal se puede pensar como "álgebra lineal en categorías trianguladas" y lo que se esta tratando de hacer, es encontrar un invariante biracional en terminos de la descomposición semiortogonal de la categoría derivada de una variedades algebraica. Para ser más preciso, si tenemos una descomposición semiortogonal de la categoría derivada acotada de una cúbica $X \subset \mathbb{P}^4$ en terminos de objetos excepcionales $E_1, \ldots, E_n$,

$$\mathbf{D}(X) = \langle \mathcal{A}_X, E_1, \ldots, E_n \rangle,$$

con $\mathcal{A}_X$ el complemento ortogonal de los objetos $E_1, \ldots, E_n$, existe evidencia y ejemplos [Kuz15b] donde la siguiente conjetura se comple.

*Conjetura: $X$ es racional si y sólo si existe una superficie $K3$ suave y projectiva $Y$, tal que $\mathcal{A}_X \cong \mathbf{D}(Y)$.*

Siguiendo esta idea, desde el punto de vista birracional las variedades de Fano son objetos importantes y dan ejemplos interesantes de descomposiciones semiortogonales, en particular hipersuperficies son variedades de Fano y la descomposición semiortogonal para una cúbica en $\mathbb{P}^4$ y en $\mathbb{P}^5$ va a darnos una condición de racionalidad.

Superficies de Del Pezzo son tan importantes en superficies algebraicas como lo son las variedades de Fano en el caso de mayor dimensión. En el capítulo 3 probamos los principales recultados, la formula de Orlov para la descomposición semiortogonal de la projectivización de un haz vectorial y para la desingularización o explosión (blow up) de una variedad algebraica, Proposición 3.1.1 (Orlov I) y Teorema 3.1 (Orlov II) respectivamente, las cuales nos darán una descomposición semiortogonal para superficies del Pezzo las cuales se pueden ver como explosiones de $\mathbb{P}^2$ en a lo más 8 puntos en posición general.

En este trabajo $X$ va ha ser una variedad algebraica en el sentido de [Sha74] o [Har77], algunas veces es conveniente pensar en $X$ como una variedad algebraica en el sentido de [GH78].

# 1

# Triangulated Categories

## 1.1 Derived categories

Derived categories were defined by Verdier in his thesis [Ver65] back in 60's. When appeared they were used as an abstract notion to formulate general results, like Grothendieck-Riemann-Roch theorem, for which actually they were devised by Grothendieck. Later on, they were actively used by Hartshorne as a technical tool in [Har66]. The situation changed with appearance of Beilinson's brilliant paper [Bei78], when they attracted attention as the objects of investigation. Finally, results of Bondal and Orlov [BO95], [BO02] put them on their present place in the center of algebraic geometry.

We refer to [GM02] for a classical treatment of derived categories, and to [Huy06] for a more geometrically point of view. Here we restrict ourselves to give an introduction into the subject.

From now on $\mathsf{k}$ will be a field.

**Definition 1.1.** A complex over a $\mathsf{k}$-linear abelian category $\mathcal{A}$ is a pair $(F^\bullet, d_F^\bullet)$, with $F^i \in \mathcal{A}$ is a collection of objects and $d_F^i : F^i \to F^{i+1}$ are morphisms, such that $d_F^{i+1} \circ d_F^i = 0$ for all $i \in \mathbb{Z}$. A complex is bounded if $F_i = 0$ for all $|i| \gg 0$. A morphism of complexes

$$(F^\bullet, d_F^\bullet) \xrightarrow{\varphi} (G^\bullet, d_G^\bullet)$$

is a collection of morphisms $\varphi^i : F^i \to G^i$ in $\mathcal{A}$ commuting with the differentials: $\varphi^{i+1} \circ d_F^i = d_G^i \circ \varphi^i$ for all $i \in \mathbb{Z}$. The category of bounded complexes in $\mathcal{A}$ is denoted by $\mathsf{Com}^b(\mathcal{A})$.

One of the most important functors is the following

**Definition 1.2.** Let $\mathcal{A}$ be a $\mathsf{k}$-linear abelian category and let $(F^\bullet, d_F^\bullet)$ be a complex in $\mathcal{A}$. The $i-$th cohomology of the complex $(F^\bullet, d_F^\bullet)$ is an object of $\mathcal{A}$ defined by

$$\mathcal{H}^i(F^\bullet) = \frac{\mathsf{Ker}(d_F^i : F^i \to F^{i+1})}{\mathsf{Im}(d_F^{i-1} : F^{i-1} \to F^i)}.$$

Clearly a morphism of complexes $\varphi : F^\bullet \to G^\bullet$ induces a morphism in cohomology $\mathcal{H}^i(\varphi) : \mathcal{H}^i(F^\bullet) \to \mathcal{H}^i(G^\bullet)$. Therefore the cohomology is a functor

$$\mathcal{H}^i : \mathsf{Com}^b(\mathcal{A}) \to \mathcal{A}.$$

A morphism of complexes $\varphi : F^\bullet \to G^\bullet$ is called a quasi-isomorphism if for all $i \in \mathbb{Z}$ the morphism $\mathcal{H}^i(\varphi)$ is an isomorphism. The class of all quasi-isomorphism in $\mathsf{Com}^b(\mathcal{A})$ is denoted by $\mathsf{Qiso}$.

**Definition 1.3.** The bounded derived category of an abelian category $\mathcal{A}$ is the localization of $\mathsf{Com}^b(\mathcal{A})$ with respect to the class of quasi-isomorphisms,

$$\mathbf{D}^b(\mathcal{A}) = \mathsf{Com}^b(\mathcal{A})[\mathsf{Qiso}^{-1}].$$

Of course, this definition itself requires an explanation, which we will skip and we refer to [GM02] for describing this in detail instead. Here we just restrict ourselves by saying that a localization of a category $\mathcal{C}$ in a localizing class of morphisms $S$ is a category $\mathcal{C}[S^{-1}]$ with a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $Q(S)$ are isomorphisms and the functor has a universal property, namely any functor $K$ : $\mathsf{Com}^b(\mathcal{A}) \to \mathcal{B}$ transforming the morphisms in $S$ into isomorphisms can be uniquely factorized trough $\mathcal{C}[S^{-1}]$, i.e. there exist a unique functor $G : \mathcal{C}[S^{-1}] \to \mathcal{B}$ such that $K = G \circ Q$.

Although sometimes it is more convenient to consider the unbounded, left bounded or right bounded version of the derived category which are denoted by $\mathbf{D}^*(\mathcal{A})$, $* = \emptyset$[1], $+, -$ respectively, we will only consider the bounded case.

The cohomology functors descend to the derived category, so $\mathcal{H}^i : \mathbf{D}^b(\mathcal{A}) \to \mathcal{A}$ is an additive functor for each $i \in \mathbb{Z}$. Further, there is a full and faithful embedding functor

$$\mathcal{A} \to \mathbf{D}^b(\mathcal{A}),$$

taking an object $F \in \mathcal{A}$ to the complex $\cdots \to 0 \to F \to 0 \to \cdots$ with zeroes everywhere outside of degree zero, in which the object F sits. This complex has only one nontrivial cohomology which lives in degree zero and equals $F$.

## 1.2 Triangulated categories

Triangulated categories are the kind of categories we will be interested in throughout, were introduced independently and around the same time by Puppe and in Verdier's thesis [Ver65] under the supervision of Grothendieck. The most important structure on the derived category is the triangulated structure.

**Definition 1.4.** A triangulated category is an additive category $\mathcal{T}$ equipped with

- an automorphism of $\mathcal{T}$ called the shift functor and denoted by $[1] : \mathcal{T} \to \mathcal{T}$, the powers of the shift functor are denoted by $[k] : \mathcal{T} \to \mathcal{T}$ for all $k \in \mathbb{Z}$;
- a class of chains of morphisms in $\mathcal{T}$ of the form

$$F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \tag{1.1}$$

  called distinguished triangles,

which satisfy a number of axioms see [GM02].

[1] The unbounded case.

Instead of listing all them we will discuss only the most important axioms and properties.

From now on by triangulated category $\mathcal{T}$ we will mean in a $\mathsf{k}$-linear triangulated category.

First, each morphism $F_1 \xrightarrow{\varphi_1} F_2$ can be extended to a distinguished triangle (1.1). The extension is unique up to a noncanonical isomorphism, the third vertex of such a triangle is called a cone of the morphism $\varphi_1$ and is denoted by $\mathsf{Cone}(\varphi_1)$.

Further, a triangle (1.1) is distinguished if and only if the triangle

$$F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \xrightarrow{\varphi_1[1]} F_2[1] \tag{1.2}$$

is distinguished. Such triangle is referred to as the rotation of the original triangle. Clearly, rotating a distinguished triangle in both directions, one obtains an infinite chain of morphisms

$$\cdots \to F_3[-1] \xrightarrow{\varphi_3[-1]} F_1 \xrightarrow{\varphi_1} F_2 \xrightarrow{\varphi_2} F_3 \xrightarrow{\varphi_3} F_1[1] \xrightarrow{\varphi_1[1]} F_2[1] \xrightarrow{\varphi_2[1]} F_3[1] \xrightarrow{\varphi_3[1]} F_1[2] \to \cdots,$$

called a helix. Any consecutive triple of morphisms in a helix is thus a distinguished triangle.

The sequence of $\mathsf{k}$-vector spaces

$$\cdots \to \mathsf{Hom}(G, F_3[-1]) \xrightarrow{\varphi_3[-1]\circ(-)} \mathsf{Hom}(G, F_1) \xrightarrow{\varphi_1\circ(-)} \mathsf{Hom}(G, F_2) \xrightarrow{\varphi_2\circ(-)} \mathsf{Hom}(G, F_3) \xrightarrow{\varphi_3\circ(-)} \mathsf{Hom}(G, F_1[1]) \to \cdots,$$

is obtained by applying the funtor $\mathsf{Hom}(G, -)$ to a helix, is a long exact sequence. Analogously, the sequence of $\mathsf{k}$-vector spaces

$$\cdots \to \mathsf{Hom}(F_1[1], G) \xrightarrow{(-)\circ\varphi_3} \mathsf{Hom}(F_3, G) \xrightarrow{(-)\circ\varphi_2} \mathsf{Hom}(F_2, G) \xrightarrow{(-)\circ\varphi_1} \mathsf{Hom}(G, F_1) \xrightarrow{(-)\circ\varphi_3[-1]} \mathsf{Hom}(F_3[-1], G) \to \cdots,$$

obtained by applying the functor $\mathsf{Hom}(-, G)$ to a helix, is a long exact sequence.

In a triangulated category the isomorphisms are determined by a cone, i.e. a morphism $F_1 \xrightarrow{\varphi_1} F_2$ is and isomorphism if and only if $\mathsf{Cone}(\varphi_1) = 0$. This follows from the following fact, if in the triangle (1.1) one has $\varphi_3 = 0$ the $F_2 \cong F_1 \oplus F_3$.

**Definition 1.5.** Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S} \subset \mathcal{T}$ is called a triangulated subcategory if $\mathcal{S}[1] = \mathcal{S}$ (it is closed under the shift functor) and is closed under isomorphisms and cones, i.e. if we assume that in a distinguished triangle (1.1) we have $F_1, F_2 \in \mathcal{S}$ then $\mathsf{Cone}(\varphi_1) = F_3 \in \mathcal{S}$.

The most important example of a triangulated category is the derived category of an abelian category.

Derived category $\mathbf{D}^b(\mathcal{A})$ carries a natural triangulated structure.

**Definition 1.6.** Let $(F^\bullet, d_F^\bullet)$ a complex and $\varphi : F^\bullet \to G^\bullet$ a morphism of complexes over $\mathcal{A}$. The shift functor $[1] : \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$ is defined by

$$(F[1])^i = F^{i+1}, \qquad d^i_{F[1]} = -d^{i+1}_F, \qquad (\varphi[1])^i = -\varphi^{i+1}, \tag{1.3}$$

and we write $(F[1]^\bullet, d^\bullet_{F[1]})$. The cone of $\varphi$ is defined by

$$\mathsf{Cone}(\varphi)^i = G^i \oplus F^{i+1}, \qquad d_{\mathsf{Cone}(\varphi)} = (d_G + \varphi, -d_F), \tag{1.4}$$

and is denoted by $(\mathsf{Cone}(\varphi)^\bullet, d^\bullet_{\mathsf{Cone}(\varphi)})$.

A morphism of complexes $F^\bullet \xrightarrow{\varphi} G^\bullet$ extends to a distinguished triangle by morphisms

$$\kappa : G^\bullet \to \mathsf{Cone}(\varphi)^\bullet, \quad \kappa(g^i) = (g^i, 0); \qquad \rho : \mathsf{Cone}(\varphi)^\bullet \to F[1]^\bullet \quad \rho(g^i, f^{i+1}) = f^{i+1}.$$

**Definition 1.7.** A distinguished triangle in $\mathbf{D}^b(\mathcal{A})$ is a triangle isomorphic to the triangle

$$F^\bullet \xrightarrow{\varphi} G^\bullet \xrightarrow{\kappa} \mathsf{Cone}(\varphi)^\bullet \xrightarrow{\rho} F[1]^\bullet \tag{1.5}$$

defined above.

The following theorem was proved by Verdier in his thesis [Ver65].

**Theorem 1.1** (Verdier)**.** *The shift functor* $[1] : \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})$ *and the class of distinguished triangles* (1.5) *provide* $\mathbf{D}^b(\mathcal{A})$ *with a structure of a triangulated category*

Note that the previous theorem is also valid for $\mathbf{D}^*(\mathcal{A})$, $* = \emptyset, +, -$ respectively.

For $F, G \in \mathcal{A}$ the spaces of morphisms in the derived category between $F$ and shifts of $G$ are identified with the $\mathsf{Ext}$-groups in the original abelian category

$$\mathsf{Hom}(F, G[i]) = \mathsf{Ext}^i(F, G).$$

For arbitrary objects $F, G \in \mathbf{D}^b(\mathcal{A})$ we will use this as definition.

**Definition 1.8.** Let $F, G \in \mathbf{D}^b(\mathcal{A})$,

$$\mathsf{Ext}^i(F, G) := \mathsf{Hom}(F, G[i]).$$

With this definition the long exact sequences obtained by applying $\mathsf{Hom}$ functors to a helix can be rewritten as

$$\cdots \to \mathsf{Ext}^{i-1}(G, F_3) \to \mathsf{Ext}^i(G, F_1) \to \mathsf{Ext}^i(G, F_2) \to \mathsf{Ext}^i(G, F_3) \to \mathsf{Ext}^{i+1}(G, F_1) \to \cdots,$$

$$\cdots \to \mathsf{Ext}^{i-1}(F_1, G) \to \mathsf{Ext}^i(F_3, G) \to \mathsf{Ext}^i(F_2, G) \to \mathsf{Ext}^i(G, F_1) \to \mathsf{Ext}^{i+1}(F_3, G) \to \cdots.$$

**Example 1.2.1.** Let $\mathcal{A} := \mathsf{Vect}(\mathsf{k})$ be the category of finite dimensional vector spaces over $\mathsf{k}$, $\mathcal{A}$ is and abelian category. Moreover it is a semisimple, i.e. any exact sequence in $\mathcal{A}$ splits, or equivalently is isomorphic to a sequence of the form $0 \to A \xrightarrow{(1_A, 0)} A \oplus B \to B \to 0$. For instance, the category of abelian groups is not semisimple: the sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$ does not split. An object in $\mathsf{Com}(\mathcal{A})$ is cyclic if all its differentials are zero, the subcategory of cyclic objects $\mathsf{Com}_0(\mathcal{A}) \subset \mathsf{Com}(\mathcal{A})$ is isomorphic to the category $\prod_{n=-\infty}^{\infty} \mathcal{A}[n]$, where $\mathcal{A}[n]$ is the "$n$-th copy of $\mathcal{A}$". Since $\mathsf{Vect}(\mathsf{k})$ is semisimple then there is an equivalence of categories $\mathbf{D}(\mathsf{Vect}(\mathsf{k})) \to \prod_{n \in \mathbb{Z}} \mathsf{Vect}(\mathsf{k})[n] = \mathsf{Com}_0(\mathsf{Vect}(\mathsf{k}))$. In fact, any complex $A^\bullet \in \mathbf{D}(\mathsf{Vect}(\mathsf{k}))$ is isomorphic to its cohomology complex $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}^n(A^\bullet)[-n]$ (with trivial differentials). Note that all of this works for any semisimple abelian category $\mathcal{A}$.

**Example 1.2.2.** Let $X$ be an algebraic variety over $\mathsf{k}$. It is well know that the category of quasi-coherent and coherent sheafs on $X$, $\mathsf{Qcoh}(X)$ and $\mathsf{coh}(X)$ respectively, are abelian categories see [Huy06], [GM02] or [Har77] thus we can consider the derived categories of them $\mathbf{D}^*(\mathsf{Qcoh}(X))$ and $\mathbf{D}^*(\mathsf{coh}(X))$, $* =, +, -$ respectively.

In algebraic geometry the most important triangulated category for an algebraic variety $X$ is the bounded derived category of coherent sheaves on $X$. To make notation more simple we will use the shorthand

$$\mathbf{D}(X) := \mathbf{D}^b(\mathsf{coh}(X)).$$

Although most of the results we will discuss are valid in much more generality, we restrict to the case of smooth projective varieties. Sometimes one also may need some assumptions on the base field, so let us assume for simplicity that $\mathsf{k} = \mathbb{C}$, from now on.

## 1.3 Triangulated functors

**Definition 1.9.** A triangulated functor between triangulated categories $\mathcal{T}_1 \to \mathcal{T}_2$ is a pair $(\Phi, \phi)$, where $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a $\mathsf{k}$-linear functor $\mathcal{T}_1 \to \mathcal{T}_2$, which takes distinguished triangles of $\mathcal{T}_1$ to distinguished triangles of $\mathcal{T}_2$, and $\phi : \Phi \circ [1]_{\mathcal{T}_1} \to [1]_{\mathcal{T}_2} \circ \Phi$ is an isomorphism of functors. Usually the isomorphism $\phi$ will be left implicit.

**Definition 1.10.** Let $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor. The kernel of $\Phi$ is defined to be the full subcategory

$$\mathsf{Ker}(\Phi) = \{F \in \mathcal{T}_1 | \Phi(F) = 0\}.$$

As one can expect

**Lemma 1.1.** *Let $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ triangulated functor. Then $\mathsf{Ker}(\Phi)$ is a triangulated category.*

*Proof.* By definition $\Phi(F) = 0$ if and only if $\Phi(F)[1] = 0$. Thus $F[1] \in \mathsf{Ker}(\Phi)$ if and only if $F \in \mathsf{Ker}(\Phi)$. Also if $F_1, F_2 \in \mathsf{Ker}(\Phi)$ such that $\Phi(F_1) \to \Phi(F_2) \to \Phi(F_2) \to \Phi(F_1)[1]$ is a triangle in $\mathcal{T}_2$ then it is isomorphic to $0 \to 0 \to 0 \to 0[1]$, and in particular $\Phi(F_3) = 0$. This proves the lemma. □

There is a powerful theory (see [GM02] for the classical or [Kel06] for the modern approach) which allows to extend an additive functor between abelian categories to a triangulated functor between their derived categories. If an initial functor $\Phi$ is right exact, one extends it as the left derived functor $L\Phi$ by applying the original functor $\Phi$ to a projective resolution of the object, and if the initial functor $\Phi$ is left exact, one extends it as the right derived functor $R\Phi$ by applying $\Phi$ to an injective resolution. We will not develop this theory here. Instead, we list the most important, from the geometric point of view, triangulated functors between derived categories of coherent sheaves, see [Har66], or [Huy06] for more details.

### 1.3.1 Pullbacks and pushforwards, twisted pullbacks, tensor products and local $\mathscr{H}om$'s.

*Pullbacks and pushforwards.*

Let $f : X \to Y$ be a morphism of smooth projective algebraic varieties. It gives an adjoint pair of functors $(f^*, f_*)$, where

- $f^* : \mathsf{coh}(Y) \to \mathsf{coh}(X)$ is the pullback functor,
- $f_* : \mathsf{coh}(X) \to \mathsf{coh}(Y)$ is the pushforward functor..

Since $f^*$ is right exact and $f_*$ is left exact, this adjoint pair induces and adjoint pair of derived functors $(Lf^*, Rf_*)$ on the derived categories, where

- $Lf^* : \mathbf{D}(Y) \to \mathbf{D}(X)$ is the left derived pullback functor,
- $Rf_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ is the right derived pushforward functor.

The cohomology sheaves of the derived pullback and pushforward applied to a coherent sheaf $\mathcal{F}$ are well known as the classical higher pullbacks and pushforwards.

$$L_i f^*(\mathcal{F}) = \mathcal{H}^i(Lf^*(\mathcal{F})), \qquad R^i f_*(\mathcal{F}) = \mathcal{H}^i(Rf_*(\mathcal{F})).$$

When $f : X \to \mathsf{Spec}(\mathsf{k})$ is the structure morphism of a $\mathsf{k}$-scheme, the pushforward functor is identified with the global sections functor $\Gamma(X, -)$, so that $Rf_* = R\Gamma(X, -)$ and $R^i f_* = H^i(X, -)$. On the other hand the functors $\Gamma(X, -)$ and $\mathsf{Hom}(\mathcal{O}_X, -)$ are equal, so their derived functors are isomorphic, thus

$$\mathsf{Ext}^i(\mathcal{O}_X, -) \cong R^i\Gamma(X, -) \cong H^i(X, -).$$

For instance

**Proposition 1.3.1.** *Let $X = \mathbb{P}^n$, then for $0 \leq j, i \leq n$ and $l > 0$*

$$\mathsf{Ext}^l(\mathcal{O}(-i), \mathcal{O}(-j)) = \mathsf{Ext}^l(\omega^i(i), \omega^j(j)) = 0.$$

*And*

$$\mathsf{Ext}^\bullet(\omega^i(i), \omega^i(i)) = \mathsf{Ext}^\bullet(\omega^i(i), \omega^i(i)) = \mathsf{k}.$$

*The last assertion means,* $\mathsf{Ext}^0(\mathcal{O}(-i), \mathcal{O}(-i)) = \mathsf{Ext}^0(\omega^i(i), \omega^i(i)) = \mathsf{k}$ *and* $\mathsf{Ext}^l(\mathcal{O}(-i), \mathcal{O}(-i)) = \mathsf{Ext}^l(\omega^i(i), \omega^i(i)) = 0$ *for $l > 0$.*

*Proof.* It is just a simple cohomology computation, indeed for example

$$\mathsf{Ext}^l(\mathcal{O}(-i), \mathcal{O}(-j)) \cong \mathsf{Ext}^\bullet(\mathcal{O}, \mathcal{O}(i-j)) \cong H^l(X, \mathcal{O}(i-j)).$$

The result follows since it is well know the cohomology in projective spaces, [Har77]. ☐

*Tensor products and local $\mathscr{H}om$'s.*

Another important adjoint pair of functors on the category $\mathsf{coh}(X)$, of coherent sheaves, is $(\otimes, \mathscr{H}om)$. In fact these are bifunctors, and the adjunction is a functorial isomorphism

$$\mathsf{Hom}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) \cong \mathsf{Hom}(\mathcal{F}_1, \mathscr{H}om(\mathcal{F}_2, \mathcal{F}_3)).$$

This adjoint pair induces an adjoint pair of derived functors $(\overset{\mathbb{L}}{\otimes}, \mathsf{R}\mathscr{H}om)$ on the derived categories, where

- $\overset{\mathbb{L}}{\otimes} : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X)$ is the left derived tensor product functor,
- $\mathsf{R}\mathscr{H}om : \mathbf{D}(X)^{\mathsf{opp}} \times \mathbf{D}(X) \to \mathbf{D}(X)$ s the right derived local $\mathscr{H}om$ functor.

One special case of the $\mathsf{R}\mathscr{H}om$ functor is very useful.

**Definition 1.11.** Let $F \in \mathbf{D}(X)$. The object

$$F^{\vee} := \mathsf{R}\mathscr{H}om(F, \mathcal{O}_X)$$

is called the derived dual object.

As in the case of coherent sheaves with the smoothness assumption on $X$ there is a canonical isomorphism

$$\mathsf{R}\mathscr{H}om(F, G) \cong F^{\vee} \overset{\mathbb{L}}{\otimes} G$$

for all $F, G \in \mathbf{D}(X)$.

*Twisted pullbacks.*

The derived pushforward functor $Rf_*$ also has a right adjoint functor $f! : \mathbf{D}(Y) \to \mathbf{D}(X)$, which is called sometimes the twisted pullback functor. The pair $(Rf_*, f^!)$ is an adjoint pair and is known as the Grothendieck duality. The twisted pullback $f^!$ has a very simple relation with the derived pullback functor, under our assumption of smoothness and projectivity

$$f^!(F) \cong Lf_*(F) \overset{\mathbb{L}}{\otimes} \omega_{X/Y}[\mathsf{dim}\ X - \mathsf{dim}\ Y],$$

where $\omega_{X/Y} = \omega_X \otimes f^* \omega_Y^{-1}$, is the relative dualizing sheaf.

The functors we introduced have many interesting properties and obey a long list of relations. Here we mention the most important of them. More details and proofs can be found in [GM02], [Har66] or [Huy06].

*Functoriality.* Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of morphisms. Then

$$R(g \circ f)_* \cong Rg_* \circ Rf_*,$$

$$L(g \circ f)^* \cong Lf^* \circ Lg*,$$

$$(g \circ f)^! \cong f^! \circ g^!.$$

In particular, if $Z = \mathsf{Spec}(\mathsf{k})$ then the first formula gives an isomorphism $R\Gamma \circ Rf_* \cong R\Gamma$.

*Local adjunctions.* There are isomorphisms

$$Rf_* \mathsf{R}\mathscr{H}om(Lf_*(F), G) \cong \mathsf{R}\mathscr{H}om(F, Rf_*(G)),$$

$$Rf_* \mathsf{R}\mathscr{H}om(G, f^!(F)) \cong \mathsf{R}\mathscr{H}om(Rf_*(G), F).$$

If one applies the functor $R\Gamma$ to these formulas, the usual adjunctions are recovered. Another local adjunction is the following isomorphism

$$Rf_* \mathsf{R}\mathscr{H}om(F \overset{\mathbb{L}}{\otimes} G, H) \cong \mathsf{R}\mathscr{H}om(F, \mathsf{R}\mathscr{H}om(G, H)).$$

*Tensor products and pullbacks.* Derived tensor product is associative and commutative, the pullback functor is a tensor functor, i.e.

$$Lf^*(F \overset{\mathbb{L}}{\otimes} G) \cong Lf^*(F) \overset{\mathbb{L}}{\otimes} Lf^*(G)$$

$$Lf^*\mathsf{R}\mathscr{H}om(F,G) \cong \mathsf{R}\mathscr{H}om(Lf^*(F), Lf^*(G)).$$

*The projection formula.* In a contrast with the pullback, the pushforward is not a tensor functor. It has, however, a weaker property

$$Rf_*(Lf^*F \overset{\mathbb{L}}{\otimes} G) \cong F \overset{\mathbb{L}}{\otimes} Rf_*(G), \tag{1.6}$$

which is called projection formula and is very useful. A particular case is the following isomorphism

$$Rf_*(Lf^*F) \cong F \overset{\mathbb{L}}{\otimes} Rf_*(\mathcal{O}_X)$$

*Base change.* Let $f : X \to S$ and $u : T \to S$ be morphisms of schemes consider the fiber product $X_T := X \times_S T$ and the fiber square

$$\begin{array}{ccc} X_T & \xrightarrow{u_X} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{u} & S. \end{array}$$

Figure 1.1: Base change

Using adjunctions and functoriality of pullbacks and pushforwards, one can construct a canonical morphism of functors $Lu^* \circ Lf_* \to Rf_{T*} \circ Lu_X^*$. The base change theorem says that it is an isomorphism under appropriate conditions. To formulate these we need the following

**Definition 1.12.** A pair of morphisms $f : X \to S$ and $u : T \to S$ is called Tor-independent if for all points $x \in X$, $t \in T$ such that $f(x) = s = u(t)$ one has

$$\mathsf{Tor}_i(\mathcal{O}_{X,x}, \mathcal{O}_{T,t}) = 0 \qquad \text{for} \quad i > 0.$$

Recall, for $\mathcal{F}, \mathcal{G} \in \mathsf{coh}(X)$ the classical Tor is $\mathsf{Tor}^i(\mathcal{F}, \mathcal{G}) = \mathcal{H}^{-i}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G})$

**Remark 1.1.** If either $f$ or $u$ is flat then the square is Tor-independent. Furthermore, when $X$, $S$, and $T$ are all smooth there is a simple sufficient condition for a pair $(f, u)$ to be Tor-independent:

$$\mathsf{dim}(X_T) = \mathsf{dim}(X) + \mathsf{dim}(Y) - \mathsf{dim}(S),$$

i.e. the equality of the dimension of $X_T$ and of its expected dimension, see [Kuz05b].

The following theorem is proved in [Kuz05b].

**Theorem 1.2 (Base Change).** *The base change morphism $Lu^* \circ Lf_* \to Rf_{T*} \circ Lu_X^*$ is an isomorphism if and only if the pair of morphisms $f : X \to S$ and $u : T \to S$ is* Tor*-independent.*

If $u$ is flat and $f$ proper, then there exists of a functorial isomorphism, the flat base change

$$u^* \circ Lf_* \to Rf_{T*} \circ u_X^*, \tag{1.7}$$

for any $F \in \mathbf{D}(X)$. Note that $u$ and, therefore, $u_X$ are flat, $u^*$, $u_X^*$ are exact and need not be derived.

### 1.3.2 Fourier-Mukay functors and Serre functor

Fourier-Mukai transform between derived categories is the derived version of the notion of a correspondence, which has been studied for all kinds of cohomology theories, e.g. Chow groups, singular cohomology, etc., for many decades. Functors that are of Fourier-Mukai type behave well in many respects. They are exact, admit left and right adjoints, can be composed, etc. In fact, Orlov's celebrated result [Orl03], says that any equivalence between derived categories of smooth projective varieties is of geometric origin, i.e. of Fourier-Mukai type.

The analogy to the classical Fourier transform is most striking in the case of abelian varieties. Roughly, $L^2$-functions are replaced by complexes of coherent sheaves and, in particular, the usual integral kernel by an object in the derived category of the product of the variety [Muk94].

An intrinsic property of a derived category is the Serre functor. The notion of a Serre functor is a categorical interpretation of the Serre duality.

**Definition 1.13.** A Serre functor in a triangulated category $\mathcal{T}$ is an autoequivalence $\mathcal{S}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ with a bifunctorial isomorphism

$$\mathsf{Hom}(F, G)^{\vee} \cong \mathsf{Hom}(G, \mathcal{S}_{\mathcal{T}}(F)),$$

for any $F, G \in \mathcal{T}$.

If a Serre functor exists, it is unique up to a canonical isomorphism and when $\mathcal{T} = \mathbf{D}(X)$ with $X$ smooth projective algebraic variety, the Serre functor is given by a simple formula [BK89]

$$\mathcal{S}_{\mathbf{D}(X)}(F) = F \overset{\mathbb{L}}{\otimes} \omega_X[\mathsf{dim}\ X]. \tag{1.8}$$

The bifunctorial isomorphism in its definition is the Serre duality for X.

If $X$ is a Calabi-Yau variety i.e. $\omega_X \cong \mathcal{O}_X$ then the corresponding Serre functor $\mathcal{S}_{\mathbf{D}(X)} \cong [\mathsf{dim}\ X]$ is just a shift. This motivates the following

**Definition 1.14.** A triangulated category $\mathcal{T}$ is a Calabi $-$ Yau category of dimension $n \in \mathbb{Z}$ if $\mathcal{S}_{\mathcal{T}} \cong [n]$. A triangulated category $\mathcal{T}$ is a fractional Calabi $-$ Yau category of dimension $\frac{p}{q} \in \mathbb{Q}$ if $\mathcal{S}_{\mathcal{T}}^{q} \cong [p]$.

Of course, $\mathbf{D}(X)$ cannot be a fractional Calabi-Yau category with a non-integer Calabi-Yau dimension, see (1.8).

The crucial properties of a Serre functor are summarized in the theorem below proved in [BK89].

**Theorem 1.3.** *Let $\mathcal{T}$ be a triangulated category. Then a Serre functor $\mathcal{S}_{\mathcal{T}}$, if it exists, is unique up to a unique isomorphism of functors. It is exact i.e., takes exact triangles to exact triangles, commutes with the shift and if $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a triangulated equivalence, then there is a canonical isomorphism*

$$\Phi \circ \mathcal{S}_{\mathcal{T}_1} \cong \mathcal{S}_{\mathcal{T}_2} \circ \Phi.$$

The wonderful thing about Serre functor is that it allows one to convert from a left adjoint functor to a right adjoint functor and vice versa. Specifically

**Theorem 1.4.** *Let $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ be a functor between triangulated categories that admit Serre functors $\mathcal{S}_{\mathcal{T}_1}, \mathcal{S}_{\mathcal{T}_2}$. Assume that $\Phi$ has a left adjoint $\Phi^* : \mathcal{T}_2 \to \mathcal{T}_1$. Then*

$$\Phi^! := \mathcal{S}_{\mathcal{T}_1} \circ \Phi^* \circ \mathcal{S}_{\mathcal{T}_2}^{-1} : \mathcal{T}_2 \to \mathcal{T}_1$$

*is a right adjoint to $F$.*

*Proof.* Lef $F \in \mathcal{T}_1$ and $G \in \mathcal{T}_2$, then

$$\mathsf{Hom}(\Phi(F), G) \cong \mathsf{Hom}(\mathcal{S}_{\mathcal{T}_2}^{-1}(G), \Phi(F))^\vee \cong \mathsf{Hom}(\Phi^* \circ \mathcal{S}_{\mathcal{T}_2}^{-1}(G), F)^\vee \cong \mathsf{Hom}(F, \mathcal{S}_{\mathcal{T}_1} \circ \Phi^* \circ \mathcal{S}_{\mathcal{T}_2}^{-1}(G)),$$

and the result follows. □

Let $X$ and $Y$ be smooth projective algebraic varieties.

**Definition 1.15.** Let $K \in \mathbf{D}(X \times Y)$ an object of the derived category of the product. The $\mathsf{Fourier-Mukay\ funtor}$ with kernel $K$ is defined by

$$\begin{aligned} \Phi_K : &\mathbf{D}(X) \to \mathbf{D}(X) \\ &F \mapsto Rp_{Y*}(K \overset{\mathbb{L}}{\otimes} Lp_X^*(F)) \qquad \forall\ F, \end{aligned}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections, some times we denote them by $p_1, p_2$ respectively.

$\Phi_K$ is also called the kernel functor or the integral functor with kernel K. Fourier-Mukai functors form a nice class of functors, which includes most of the functors we considered before. This class is closed under compositions and adjunctions.

**Example 1.3.1.** The identity functor $1_{\mathbf{D}(X)} : \mathbf{D}(X) \to \mathbf{D}(X)$ is naturally isomorphic to the Fourier-Mukay functor $\Phi_{\mathcal{O}_\Delta}$ with kernel the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$. Indeed, with $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ the diagonal embedding one has

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(F) = Rp_{2*}(\mathcal{O}_\Delta \overset{\mathbb{L}}{\otimes} Lp_1^*(F)) &= Rp_{2*}(\iota_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} Lp_1^*(F)) \\ &\cong Rp_{2*}(R\iota_*(L\iota^*Lp_1^*(F) \overset{\mathbb{L}}{\otimes} \mathcal{O}_X)) \qquad \text{Projection formula (1.6)} \\ &\cong R(p_2 \circ \iota)_* L(p_1 \circ \iota)^*(F) \qquad p_2 \circ \iota = 1 = p_1 \circ \iota \\ &\cong F. \end{aligned}$$

It is a remarkable fact that, when $X = \mathbb{P}^n$ the structure sheaf $\mathcal{O}_\Delta$ has a locally free resolution [Bei78]. Define for $E \in \mathbf{D}(X)$ and $F \in \mathbf{D}(Y)$ the shorthand

$$E \boxtimes F = Lp_X^*(E) \overset{\mathbb{L}}{\otimes} Lp_Y^*(F) \in \mathbf{D}(X \times Y).$$

**Lemma 1.2** (Beilinson). *The following is a locally free resolution of $\mathcal{O}_\Delta$ on $\mathbf{D}(\mathbb{P}^n \times \mathbb{P}^n)$:*

$$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \mathcal{O}(-n+1) \boxtimes \Omega^{n-1}(n-1) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0 \quad (1.9)$$

*Proof.* According to [FL85] we need to construct a section $s$ of $\mathcal{O}(1) \boxtimes T(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ such that the zeros of $s$, $Z(s) = \Delta$, with $T$ the tangent bundle of $\mathbb{P}^n$.

Fix a basis $y_0, \ldots, y_n$ of $H^0(\mathbb{P}^n, \mathcal{O}(1))$. Consider the Euler exact sequence of vector bundles on $\mathbb{P}^n$,

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to T(-1) \to 0.$$

Taking global sections we get an isomorphism $H^0(\mathbb{P}^n, \mathcal{O}^{n+1}) \cong H^0(\mathbb{P}^n, T(-1))$. A basis of $H(\mathbb{P}^n, \mathcal{O}^{n+1})$ is given by a dual basis $y_0^\vee, \ldots, y_n^\vee$ (dual to the basis $y_0, \ldots, y_n$), and denote by $\partial/\partial y_i$ the image of $y_i^\vee$ in $H^0(\mathbb{P}^n, T(-1))$. So there is a global section $s$ of $\mathcal{O}(1) \boxtimes T(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$ given by

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i},$$

where the $x_i$'s and $y_i$'s are coordinates of the first and second factor on $\mathbb{P}^n \times \mathbb{P}^n$, respectively. The claim is that the zeros of $s$, are precisely along the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$, $Z(s) = \Delta$.

Without lost of generality, suppose that $x_0, y_0 \neq 0$. In the chart where $y_0 \neq 0$ we have the affine coordinates $Y_i = y_i/y_0$ for $1 \leq i \leq n$. Then $\partial/\partial Y_i$ for $1 \leq i \leq n$ is a basis for $T$ at this chart. From $y_i = Y_i y_0$ it follows that

$$dY_i = \frac{y_0 dy_i + y_i dy_0}{y_o^2}.$$

If we write

$$\frac{\partial}{\partial y_i} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial Y_i}$$

with $f_i = dY_i(\frac{\partial}{\partial y_i})$, it follows that

$$\frac{\partial}{\partial y_i} = \frac{1}{y_0} \frac{\partial}{\partial Y_i}$$

if $i \neq 0$, and

$$\frac{\partial}{\partial y_0} = -\sum_{i=1}^{n} \frac{y_i}{\partial y_0^2} \frac{\partial}{\partial Y_i}.$$

Therefore

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i} = \sum_{i=1}^{n} \frac{x_i}{y_0} \frac{\partial}{\partial Y_i} - \sum_{i=1}^{n} \frac{x_o y_i}{y_0^2} \frac{\partial}{\partial Y_i} = \sum_{i=1}^{n} \frac{x_i y_0 - y_i x_0}{y_0^2} \frac{\partial}{\partial Y_i}.$$

Thus $s = 0$ precisely when

$$\frac{x_i}{x_0} = \frac{y_i}{y_0}$$

for all $i$, i.e. in the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. Taking the Kuszul resolution of the section $s$ gives the result. □

**Example 1.3.2.** Let $\mathscr{L}$ be a line bundle on $X$. The functor $F \mapsto F \overset{\mathbb{L}}{\otimes} \mathscr{L}$ is an autoequivalence $\mathbf{D}(X) \to \mathbf{D}(X)$ which is isomorphic to the kernel $\iota_*(\mathscr{L})$, with $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ the diagonal embedding again, as in the previous example.

**Example 1.3.3.** Consider once more the diagonal embedding $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$. Then

$$\Phi_{\iota_*(\omega_X^k)} \cong \mathcal{S}^k[-nk],$$

with $\mathcal{S}_X$ the Serre functor (1.8) and $n = \mathsf{dim}\ X$.

It is important know when a Fourier-Mukay functor is fully faithful, i.e. an embedding. In general one has the following proposition which is proved in [Huy06].

**Proposition 1.3.2.** *Let $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ be a functor between triangulated categories such that $\Phi$ has a right adjoint $\Phi^! : \mathcal{T}_2 \to \mathcal{T}_1$. Then*

$$\Phi \quad \textit{is fully faithfull} \iff \exists\ g : 1_{\mathcal{T}_1} \xrightarrow{\sim} \Phi^! \circ \Phi.$$

*Similarly, If $\Phi^*$ is left adjoint to $\Phi$ then*

$$\Phi \ \textit{is fully faithfull} \quad \iff \exists\ h : \Phi^* \circ \Phi \xrightarrow{\sim} 1_{\mathcal{T}_1}.$$

Adjoint functors of Fourier-Mukai functors are also Fourier-Mukai functors.

**Lemma 1.3.** *The right adjoint functor of $\Phi_K$ is the Fourier-Mukai functor*

$$\Phi_K^! \cong \Phi_{K^\vee \overset{\mathbb{L}}{\otimes} \omega_X[\mathsf{dim}\ X]} : \mathbf{D}(Y) \to \mathbf{D}(X).$$

*The left adjoint functor of $\Phi_K$ is the Fourier-Mukai functor*

$$\Phi_K^* \cong \Phi_{K^\vee \overset{\mathbb{L}}{\otimes} \omega_Y[\mathsf{dim}\ Y]} : \mathbf{D}(Y) \to \mathbf{D}(X).$$

*Proof.* The functor $\Phi_K$ is the composition of the derived pullback $Lp^{X*}$ , the derived tensor product with $K$ and the derived pushforward $Rp_{Y*}$ functors. Therefore its right adjoint functor is the composition of their right adjoint functors, i.e. of the twisted pullback functor $p_Y^!$ , the derived tensor product with $K^\vee$, and the derived pushforward functor $Rp_{X*}$. By Grothendieck duality we have $p_Y^!(G) \cong Lp_{Y*}(G) \overset{\mathbb{L}}{\otimes} \omega_X[\mathsf{dim}\ X]$. All of this gives the first formula . For the second formula note that if $K' = K^\vee \overset{\mathbb{L}}{\otimes} \omega_Y[\mathsf{dim}\ Y]$ then the functor $\Phi_{K'} : \mathbf{D}(Y) \to \mathbf{D}(X)$ has a right adjoint, which by the first part of the Lemma coincides with $\Phi_K$. Hence the left adjoint of $\Phi_K$ is $\Phi_{K'}$. □

An important kind of triangulated categories are those that can be embedded in other ones, for instance.

**Definition 1.16.** Let $\mathcal{T}$ be atriangulated category. A full triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ is called right admissible if for the inclusion functor $i : \mathcal{A} \hookrightarrow \mathcal{T}$ there is a right adjoint $i^! : \mathcal{T} \to \mathcal{A}$, and left admissible if there is a left adjoint $i^* : \mathcal{T} \to \mathcal{A}$. Subcategory $\mathcal{A}$ is called admissible if it is both right and left admissible.

### 1.3.3 Spectral sequence

The spectral sequences which we are interested on, are these that appears when two derived functors are composed.

One is advised to imagine a stack of square-lined sheets of paper with each square numbered by a pair of integers $(p, q) \in \mathbb{Z}^2$. An object $E_r^{p,q}$ is assumed to sit in the $(p, q)$-th square at the $r$-th sheet. Objects $E^n$ sit at the last, "transfinite" sheet, and occupy the whole diagonal $p+q = n$. More precisely

**Definition 1.17.** A spectral sequence in an abelian category $\mathcal{A}$ is a collection of objects

$$E = (E_r^{p,q}, E^n),$$

$n, p, q, r \in \mathbb{Z}$, $r > 0$, and morphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

such that

i) $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all $p, q, r$.

ii) For $(E_r^{p,q}, d_r^{p,q})$ we can construct the cohomology

$$H^{p,q}(E_r) = \mathsf{Ker}\ d_r^{p,q} / \mathsf{Im}\ d_r^{p+r,q-r+1},$$

and there are isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \to E_{r+1}^{p,q}$.

iii) For any pair $(p, q)$ there exist $r_0$ such that $d_r^{p,q} = 0$, $d_r^{p+r,q-r+1} = 0$ for $r \geq r_0$. In this case $\alpha_r^{p,q}$ identify all $E_r^{p,q}$ for $r \geq r_o$ and we will denote this object by $E_\infty^{p,q}$.

iv) A decreasing regular filtration $\cdots \supset F^p E^n \supset F^{p+1} E^n \supset \cdots$ on each $E^n$ and isomorphisms $\beta^{p,q} : E_\infty^{p,q} \to F^p E^{p+q} / F^{p+1} E^{p+q}$ are given. Recall that a decreasing filtration is regular if $\bigcap F^p E^n = \{0\}$, $\bigcup_p F^p E^n = E^n$.

If these conditions are satisfied we say that the spectral sequence $(E_r^{p,q})$ converges to $(E^n)$ or that $(E^n)$ is the limit of $(E_r^{p,q})$ and sometimes one writes $E_r^{p,q} \Rightarrow E^n$.

The important fact for us is the following Proposition which is proved in [Huy06].

**Proposition 1.3.3.** *Let $R\Phi : \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ the right derived functor of a left exact functor $\Phi : \mathcal{A} \to \mathcal{B}$. Then there exist a spectral sequence which converges*

$$E_1^{p,q} = R^q\Phi(A^p) \Rightarrow R^{p+q}\Phi(A^\bullet),$$

*for any $A^\bullet \in \mathsf{Com}^b(\mathcal{A})$.*

## 1.4 Hochschild homology and cohomology

Hochschild homology and cohomology of algebras are well known and important invariants. In a geometric situation we have the following, let $X$ be a smooth projective algebraic variety, $\mathsf{dim}\ X = n$ and $\iota : X \xrightarrow{\sim} X \times X$ the diagonal embedding:

**Definition 1.18.** Let $X$ be a smooth projective algebraic variety. Define the Hochschild cohomology of $X$ by

$$\mathsf{HH}^\bullet(X) = \mathsf{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

and the Hochschild homology of $X$ by

$$\mathsf{HH}_\bullet(X) = \mathsf{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta, \iota_*\omega_X[n]).$$

One can define Hochschild homology and cohomology for triangulated categories as well under some technical assumptions [Kuz09].

When $\mathcal{T}$ is geometric, i.e. $\mathcal{T} = \mathbf{D}(X)$ for some variety $X$, Hochschild homology and cohomology have an interpretation in terms of standard geometrical invariants.

**Theorem 1.5** (Hochschild-Kostant-Rosenberg)**.** *Let $X$ be a smooth projective algebraic variety. If $\mathcal{T} = \mathbf{D}(X)$ then*

$$\mathsf{HH}^k(\mathcal{T}) = \bigoplus_{q+p=k} H^q(X, \wedge^p T_X), \qquad \mathsf{HH}_k(\mathcal{T}) = \bigoplus_{q-p=k} H^q(X, \Omega^p_X) = \bigoplus_{q-p=k} H^{p,q}(X).$$

Thus the Hochschild cohomology is the cohomology of polyvector fields, while the Hochschild homology is the cohomology of differential forms, or equivalently, the Hodge cohomology of X with one grading lost. Since $\mathbf{D}(X)$ is a Calabi-Yau category whenever $X$ is a Calabi-You variety, we have the following

$$\mathsf{HH}_\bullet(X) \cong \mathsf{HH}^\bullet(X)[n] \tag{1.10}$$

with $n = \mathsf{dim}\ X$. In fact this result holds for any Calabi-Yau triangulated category see [Kuz09].

**Example 1.4.1.** If $\mathcal{T} = \mathbf{D}(\mathsf{Spec}(\mathsf{k}))$ then $\mathsf{HH}^\bullet(\mathcal{T}) = \mathsf{HH}_\bullet(\mathcal{T}) = \mathsf{k}$

Sometimes, if $X$ is a smooth projective algebraic variety, we will write $\mathsf{HH}_\bullet(\mathbf{D}(X))$ and $\mathsf{HH}^\bullet(\mathbf{D}(X))$ instead of $\mathsf{HH}_\bullet(X)$ and $\mathsf{HH}^\bullet(X)$.

# 2

# Semiorthogonal decomposition

Investigation of derived categories of coherent sheaves on algebraic varieties became one of the most important topics in the modern algebraic geometry. Besides other reasons this is caused by the Homological Mirror Symmetry conjecture of Maxim Kontsevich [Kon94] predicting that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi-Yau variety and the derived Fukai category of its mirror. Thus from the point of view of mirror symmetry it is important to investigate when the derived category of coherent sheaves on a variety admits a semiorthogonal decomposition. In recent years an extensive investigation of semiorthogonal decompositions of derived categories of coherent sheaves on algebraic varieties has been done, and now we know quite a lot of examples and some general constructions. With time it is becoming more and more clear that semiorthogonal components of derived categories can be thought of as the main objects in noncommutative algebraic geometry.

The goal of this chapter is define what it will be called an exceptional collection and give the definition of a semiorthogonal decomposition (s.o.d. for short) of a triangulated category. Basically we will give only one method to construct s.o.d. for a given triangulated category, by means of a exceptional object or an admissible subcategory. Finally, Hochschild homology and cohomolgy are very interesting topics related to s.o.d. of a triangulated category which give us some interesting geometric consequences.

Through this chapter $X$ will always be a smooth projective algebraic variety.

## 2.1 Semiorthogonal decomposition

We begin with two-step semiorthogonal decomposition and then we will generalize s.o.d.

**Definition 2.1.** A (two-step) semiorthogonal decomposition of a triangulated category $\mathcal{T}$ is a pair of full triangulated subcategories $\mathcal{A}$, $\mathcal{B} \subset \mathcal{T}$ such that

1. $\mathsf{Hom}(\mathcal{B}, \mathcal{A}) = 0$,
2. for any $T \in \mathcal{T}$ there is a distinguished triangle

$$T_{\mathcal{B}} \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1], \tag{2.1}$$

with $T_{\mathcal{B}} \in \mathcal{B}$ and $T_{\mathcal{A}} \in \mathcal{A}$.

The semiorthogonal decomposition for $\mathcal{T}$ is denoted by $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$.

The condition definition 2.1, 1., can be thought as a condition of semiorthogonality (as in linear algebra) between the subcategory $\mathcal{B}$ and $\mathcal{A}$ and the condition definition 2.1, 2., tells us that $\mathcal{T}$ is "generated" by $\mathcal{B}$ and $\mathcal{A}$, it is said (2.1) is decomposition triangle for $T$, we will explain what "generated" means later.

Before giving an example let us prove the following lemma.

**Lemma 2.1.** *Assume that is given a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then for any $T \in \mathcal{T}$ the triangle (2.1) is unique and functorial. In particular*

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathcal{A} \\ T &\longmapsto T_{\mathcal{A}}, \end{aligned}$$

*is a functor, left adjoint to the embedding functor $\mathcal{A} \hookrightarrow \mathcal{T}$,i.e. $\mathcal{A}$ is an admissible subcategory. Similarly*

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathcal{B} \\ T &\longmapsto T_{\mathcal{B}}, \end{aligned}$$

*is right adjoint to the embedding functor $\mathcal{B} \hookrightarrow \mathcal{T}$ and $\mathcal{B}$ is an admissible subcategory.*

*Proof.* Firstly, let us check that if $T, T' \in \mathcal{T}$ and $\varphi \in \mathsf{Hom}(T, T')$ then there exist a morphism from the triangle $T_{\mathcal{B}} \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1]$ to the triangle $T'_{\mathcal{B}} \to T' \to T'_{\mathcal{A}} \to T'_{\mathcal{B}}[1]$. Consider the long exact sequence obtained by applying the functor $\mathsf{Hom}(-, T'_{\mathcal{A}})$ to (2.1):

$$\cdots \to \mathsf{Hom}(T_{\mathcal{B}}[1], T'_{\mathcal{A}}) \to \mathsf{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}}) \to \mathsf{Hom}(T, T'_{\mathcal{A}}) \to \mathsf{Hom}(T_{\mathcal{B}}, T'_{\mathcal{A}}) \to \cdots .$$

By definition 2.1, 1., we have that $\mathsf{Hom}(T_{\mathcal{B}}[1], T'_{\mathcal{A}}) = \mathsf{Hom}(T_{\mathcal{B}}, T'_{\mathcal{A}}) = 0$ and then $\mathsf{Hom}(T_{\mathcal{A}}, T'_{\mathcal{A}}) \cong \mathsf{Hom}(T, T'_{\mathcal{A}})$. This means that the composition $T \xrightarrow{\varphi} T' \to T'_{\mathcal{A}}$ factors in an unique way as a composition $T \to T_{\mathcal{A}} \to T'_{\mathcal{A}}$. Denoting the obtained morphism by $T_{\mathcal{A}} \xrightarrow{\varphi_{\mathcal{A}}} T'_{\mathcal{A}}$, the uniqueness implies the functoriality of $T \mapsto T_{\mathcal{A}}$. Furthermore, let $A \in \mathcal{A}$ if we apply the functor $\mathrm{Hom}(-, A)$ to (2.1) and using again definition 2.1, 1., the semiorthogonality, we deduce and isomorphism $\mathsf{Hom}(T_{\mathcal{A}}, A) \cong \mathsf{Hom}(T, A)$, which means that the functor $T \mapsto T_{\mathcal{A}}$ is left adjoint to the embedding $\mathcal{A} \hookrightarrow \mathcal{T}$. The functoriality of $T \mapsto T_{\mathcal{B}}$ and its adjunction property are proved analogously. □

In particular Lemma (2.1) says that the mapping cone is functorial. Note that the composition of the embedding $\mathcal{A} \hookrightarrow \mathcal{T}$ with the projection $\mathcal{T} \to \mathcal{A}$ is the identity. In fact, the construction of lemma 2.1 can be reversed.

**Lemma 2.2.** *Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{T}$ be a left admissible subcategory with $\mathcal{T} \xrightarrow{\alpha^*} \mathcal{A}$ left adjoint to $\alpha$ and suppose $\alpha^* \circ \alpha \cong id_{\mathcal{A}}$. Then $\alpha$ is a full and faithful and there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \alpha(\mathcal{A}), \mathsf{ker}\ \alpha^* \rangle. \tag{2.2}$$

*Analogously, if $\mathcal{B} \xrightarrow{\beta} \mathcal{T}$ is a right admissible subcategory with $\mathcal{T} \xrightarrow{\beta^!} \mathcal{B}$ right adjoint to $\beta$ and $\beta^! \circ \beta \cong id_{\mathcal{B}}$, then $\beta$ is full and faithful and there is a s.o.d.*

$$\mathcal{T} = \langle \mathsf{Ker}\ \beta^!, \beta(\mathcal{B}) \rangle. \tag{2.3}$$

*Proof.* Let $B \in \mathsf{Ker}\ \alpha^*$ then by adjunction $\mathsf{Hom}(B, \alpha(A)) = \mathsf{Hom}(\alpha^*(A), B) = 0$, this proves definition 2.1, 1.

Let $T \in \mathcal{T}$ and consider the morphism $T \xrightarrow{u_T} \alpha\alpha^*(T)$, see Proposition 1.3.2, and extend it to a distinguished triangle

$$T' \to T \xrightarrow{u_T} \alpha\alpha^*(T) \to T'[1]. \tag{2.4}$$

Applying $\alpha^*$ we get a distinguished triangle

$$\alpha^*T' \to \alpha^*T \xrightarrow{\alpha^* u_T} \alpha^*\alpha\alpha^*(T) \to \alpha^*T'[1],$$

since $\alpha^* u_T$ is an isomorphism it follows that $\alpha^*T' = 0$, hence $T' \in \mathrm{Ker}\ \alpha^*$ , so (2.4) is a decomposition triangle for $T$. To see that $\alpha^* u_T$ is an isomorphism, consider the composition

$$\alpha^*T \xrightarrow{\alpha^* u_T} \alpha^*\alpha\alpha^*(T) \to \alpha^*T,$$

where the last morphism is induced by the counit adjunction, see Proposition 1.3.2. The composition of this maps is an isomorphism by Propositon 1.3.2 again, the second morphism is an isomorphism by $\alpha^* \circ \alpha \cong id_{\mathcal{A}}$. Hence the first morphism is also an isomorphism. This proves the first s.o.d.

The second statement is proved analogously. □

Lemmas 2.1 and 2.2 show that a two-step s.o.d is given by admissible subcategories and conversely, a left or right admissible subcategory gives a s.o.d. Later in the chapter we will see that indeed this is one of the most important (probably the unique) way to construct s.o.d.

The main example of a triangulated category is the derived category of an abelian category as we saw in, chapter 1, so it is in our interest to construct a s.o.d for the bounded derived category of an algebraic variety $X$.

**Example 2.1.1.** Let $X$ be a $\mathsf{k}$-scheme with structure morphism $\pi_X : X \to \mathsf{Spec}(\mathsf{k})$. Suppose that $H^\bullet(X, \mathcal{O}_X) = \mathsf{k}$ then there is a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle \mathsf{Ker}\ R\pi_{X*}, L\pi_X^*\mathbf{D}(\mathsf{k})\rangle.$$

Remember $\mathbf{D}(\mathsf{k}) = \mathbf{D}(\mathsf{Spec}(\mathsf{k}))$.

Indeed, the functor $R\pi_{X*}$ is right adjoint to $L\pi_X^*$, subsection 1.3.1 and by the projection formula (1.6)

$$R\pi_{X*}(L\pi_X^*(F)) \cong F \overset{\mathbb{L}}{\otimes} R\pi_{X*}(\mathcal{O}_X) = F \overset{\mathbb{L}}{\otimes} H^\bullet(X, \mathcal{O}_X) = F \overset{\mathbb{L}}{\otimes} \mathsf{k} = F,$$

for any $F \in \mathbf{D}(\mathsf{k})$. Thus, the functor $L\pi_X^*$ is fully faithful and right admissible. The result follows by the second part of lemma 2.2.

**Example 2.1.2.** With the notation of the previous example 2.1.1. There is a semiorthogonal decomposition

$$\mathbf{D}(X) = \langle L\pi_X^*\mathbf{D}(\mathsf{k}), \mathsf{Ker}\ (R\pi_{X*} \circ \mathsf{S}_X)\rangle,$$

$\mathsf{S}_X$ is the Serre functor of $\mathbf{D}(X)$.

By Theorem 1.4 $R\pi_{X*} \circ \mathsf{S}_X$ is left adjoint to $L\pi_X^*$ and for any $E, F \in \mathbf{D}(\mathsf{k})$,

$$\begin{aligned}\mathsf{Hom}(R\pi_{X*} \circ \mathsf{S}_X \circ L\pi_X^*(E), F) &= \mathsf{Hom}(L\pi_X^*(E), L\pi_X^*(F)) \\ &= \mathsf{Hom}(E, R\pi_{X*} \circ L\pi_X^*(F)),\end{aligned}$$

so $R\pi_{X*} \circ \mathsf{S}_X \circ L\pi_X^* \cong id_{\mathbf{D}(\mathsf{k})}$. Then the first semiorthogonal decomposition of the lemma 2.2 applies.

We know that $\mathbf{D}(\mathsf{k})$ is the derived category of $\mathsf{k}$-vector spaces and that the functor $L\pi_X^*$ applied to an object $V^\bullet \in \mathbf{D}(\mathsf{k})$ is just $L\pi_X^*(V^\bullet) = V^\bullet \otimes \mathcal{O}_X$, a complex of trivial vector bundles with zero differentials, Example 1.2.1.

On the other hand $R\pi_{X*}(-) \cong \mathsf{Ext}^\bullet(\mathcal{O}_X, -)$, see subsection 1.3.1 so we have

**Definition 2.2.** The kernel of $\mathsf{Ext}^\bullet(\mathcal{O}_X, -)$ is denoted by

$$\mathcal{O}_X^\perp = \{F \in \mathbf{D}(X) | \mathsf{Ext}^\bullet(\mathcal{O}_X, F) = 0\},$$

and is called the (right) orthogonal subcategory of $\mathcal{O}_X$. Analogously, the (left) orthogonal subcategory of $\mathcal{O}_X$ is

$${}^\perp\mathcal{O}_X = \{F \in \mathbf{D}(X) | \mathsf{Ext}^\bullet(F, \mathcal{O}_X) = 0\}.$$

In general, for any object $E \in \mathbf{D}(X)$ it is defined the (right and left) orthogonal subcategory $E^\perp$ and ${}^\perp E$, respectively.

With this notation, the s.o.d. for $\mathbf{D}(X)$ in examples 2.1.1 and 2.1.2 can be rewritten as

$$\mathbf{D}(X) = \langle \mathcal{O}_X^\perp, \mathcal{O}_X \rangle \tag{2.5}$$

and

$$\mathbf{D}(X) = \langle \mathcal{O}_X, {}^\perp \mathcal{O}_X \rangle,$$

where we write $\mathcal{O}_X$ instead of $L\pi_X^*(\mathbf{D}(\mathsf{k})) = \mathbf{D}(\mathsf{k}) \otimes \mathcal{O}_X$.

Note that if $H^\bullet(X, \mathcal{O}_X) = \mathsf{k}$ then $\mathsf{Ext}^\bullet(\mathcal{O}_X, \mathcal{O}_X) = \mathsf{k}$. In general

**Definition 2.3.** Let $E \in \mathbf{D}(X)$, $E$ is called an exceptional object if $\mathsf{Ext}^\bullet(E, E) = \mathsf{k}$.

Therefore, given an exceptional object $E \in \mathbf{D}(X)$, there are two semiorthogonal decompositions

$$\mathbf{D}(X) = \langle E^\perp, E \rangle \text{ and } \mathbf{D}(X) = \langle E, {}^\perp E \rangle,$$

where, again we write $E$ instead of $\mathbf{D}(\mathsf{k}) \otimes E$.

Let us see some examples.

**Example 2.1.3.** 1) Suppose $H^\bullet(X, \mathcal{O}_X) = \mathsf{Hom}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}$. (For instance, $H^\bullet(X, \mathcal{O}_X) = \mathbb{C}$ if and only if $X$ is connected.) Then by definition

(I) $\mathcal{O}_X$ is exceptional if and only if $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^{>0}(X, \mathcal{O}_X) = 0$.

2) If (I) holds then any line bundle is exceptional. Indeed, it is enough to note

$$\mathsf{Ext}^i(\mathscr{L}, \mathscr{L}) \cong H^i(X, \mathscr{L}^\vee \otimes \mathscr{L}) = H^i(X, \mathcal{O}_X).$$

More generally, if $E \in \mathbf{D}(X)$ is an exceptional object and $\mathscr{L}$ is a line bundle then $\mathscr{L} \overset{\mathbb{L}}{\otimes} E$ is an exceptional object.

Exceptionality is an intrinsic property of an object in a given triangulated category, since clearly $\Phi_{\mathscr{L}}(F) = F \overset{\mathbb{L}}{\otimes} \mathscr{L}$ is an autoequivalence of a derived category.

**Example 2.1.4** ($\mathsf{Spec}(\mathbb{C})$). Suppose $X = \mathsf{Spec}(\mathbb{C}) = \mathsf{pt}$ then $\mathbb{C}$ is an exceptional object in $\mathbf{D}(\mathbb{C})$, more over $\mathbb{C}[i]$ is an exceptional object for any $i \in \mathbb{Z}$.

In fact, any exceptional object is of the form $\mathbb{C}[i]$ for some $i$. If $E$ is an exceptional object and $E \neq \mathbb{C}[i]$ for some $i$ then $E = \mathbb{C}[i] \oplus \mathbb{C}[j]$ [1] and this means that $\mathsf{Hom}(E, E) \neq \mathsf{k}$ because there are at

[1] Since the category $\mathsf{Vect}(\mathbb{C})$ is semisimple each object is a direct sum of irreducible objects.

least two linearly independent homomorphisms of $E$ (namely the identity of $\mathbb{C}[i]$ and $\mathbb{C}[j]$ ), hence $E$ can not be an exceptional object.

**Example 2.1.5** (Curves)**.** Let $C$ be a smooth projective curve of genus $g$. If there is an exceptional object $E \in \mathbf{D}(C)$ then $C \cong \mathbb{P}^1$, see [Oka11], and $E \cong \mathcal{O}_{\mathbb{P}^1}(k)[i]$ for some integers $i$ and $k$. To see the last part, any coherent sheaf in $\mathbb{P}^1$ can be represented as a direct sum of torsion sheafs or as a sum of line bundles; torsion sheafs are not exceptional because their $\mathsf{Ext}^1$'s groups are not equal to zero and sums of line bundles are not exceptional because they have more than one linearly independent morphisms. So $E$ is a line bundle and thus $E \cong \mathcal{O}_{\mathbb{P}^1}(k)[i]$.

### 2.1.1 General semiorthogonal decomposition

The construction of Lemma 2.2 can be iterated to produce a longer (multi-step) semiorthogonal decomposition.

**Definition 2.4.** Let $\mathcal{T}$ a triangulated category. A semiorthogonal decomposition of $\mathcal{T}$:

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle, \tag{2.6}$$

is a sequence $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of full triangulated subcategories of $\mathcal{T}$ such that

(i) $\mathsf{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$;

(ii) for any $T \in \mathcal{T}$ there is a chain of morphisms

$$\begin{array}{ccccccccccc} 0 = T_m & \longrightarrow & & & T_{m-1} & \longrightarrow & & & T_{m-2} \longrightarrow \cdots \longrightarrow T_1 & \longrightarrow & & & T_0 = T \\ & \nwarrow^{[1]} & & \swarrow & & \nwarrow^{[1]} & & \swarrow & & \nwarrow^{[1]} & & \swarrow & \\ & & c_m & & & & c_{m-1} & & & & c_1 & & \end{array}$$

with $c_i = \mathsf{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$, for $1 \leq i \leq m$.

The subcategories $\mathcal{A}_i$ are called components of $\mathcal{T}$ with respect to (2.6). A sequence $\mathcal{A}_1, \ldots, \mathcal{A}_m$ satisfying the fist condition will be called semiorthogonal.

**Remark 2.1.** The first condition implies the objects $T_i \in \mathcal{T}$ and $c_i \in \mathcal{A}_i$ are uniquely determined by and functorial on $T$, see Lemma 2.1. The functor $\mathcal{T} \to \mathcal{A}_i$, $T \mapsto c_i$ are called projection functors, and $c_i$ is called the component of $T$ in $\mathcal{A}_i$ with respect to the decomposition (2.6).

Also the construction in (2.5) can be iterated to produce a semiorthogonal decomposition.

**Definition 2.5.** A sequence of exceptional objects $E_1, \ldots, E_n \in \mathbf{D}(X)$ is called an exceptional collection if $\mathsf{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$.

**Lemma 2.3.** *Assume $E_1, \ldots, E_n \in \mathbf{D}(X)$ is an exceptional collection. Then*

$$\mathbf{D}(X) = \langle E_1^\perp \cap \cdots \cap E_n^\perp, E_1, \ldots, E_n \rangle$$

*is a semiorthogonal decomposition.*

*Proof.* By (2.5) for $E_n \in \mathbf{D}(X)$ we have a s.o.d. $\mathbf{D}(X) = \langle E_n^{\perp}, E_n\rangle$, since $E_1, \ldots, E_{n-1} \in E_n^{\perp}$ there is a s.o.d. $E_n^{\perp} = \langle E_{n-1}^{\perp}, E_{n-1}\rangle$ by (2.5), again. So $\mathbf{D}(X) = \langle E_{n-1}^{\perp} \cap E_n^{\perp}, E_{n-1}, E_n\rangle$ and we repeat this process until the required s.o.d. is obtained. □

The semiorthogonal decomposition of Lemma 2.3 will give us many interesting examples, one of them is the following.

**Definition 2.6.** An exceptional collection $E_1, \ldots, E_n \in \mathbf{D}(X)$ is called a full exceptional collection (for short f.e.c.) if $E_1^{\perp} \cap \cdots \cap E_n^{\perp} = 0$.

**Corollary 2.1.** *If $E_1, \ldots, E_n \in \mathbf{D}(X)$ is a full exceptional collection then $\mathbf{D}(X) = \langle E_1, \ldots, E_n\rangle$.*

**Remark 2.2.** The definitions of orthogonal subcategory, exceptional object, exceptional collection and full exceptional collection can be defined in an arbitrary triangulated category, namely by definition $\mathsf{Ext}^i(E, F) = \mathsf{Hom}(E, F[i])$ for any objects $E$ and $F$, Definition 1.8 so, for example if $E$ is now an object in a $\mathsf{k}$-linear triangulated category then $E$ is exceptional if $\mathsf{Hom}(E, E[i]) = \mathsf{k}$ if $i = 0$ and $\mathsf{Hom}(E, E[i]) = 0$ if $i \neq 0$. Analogously the orthogonal subcategory, exceptional collection and full exceptional collection are defined, and the results showed above about exceptional objects, orthogonal subcategories, etc. are valid for $\mathsf{k}$-linear triangulated categories. For instance see [Huy06].

The following proposition shows that indeed exceptionality is an intrinsic property of objects in a given triangulated category.

**Proposition 2.1.1.** *Let $\mathcal{T}$ be a $\mathsf{k}-$linear triangulated category such that $\mathcal{T} \xrightarrow{\Phi} \mathcal{T}$ is an autoequivalence. If $E_1, \ldots, E_n \in \mathcal{T}$ is a f.e.c then $\Phi(E_1), \ldots, \Phi(E_n)$ is a f.e.c.*

*Proof.* In order to prove that $\Phi(E_1), \ldots, \Phi(E_n)$ are exceptional it is enough to note

$$\mathsf{Hom}(\Phi(E_i), \Phi(E_i)[j]) \cong \mathsf{Hom}(E_i, E_i[j]), \quad \forall i, j$$

To see that the collection is full, it is clear that

$$0 = \Phi(E_1^{\perp} \cap \cdots \cap E_n^{\perp}) \overset{\Phi}{\cong} \Phi(E_1)^{\perp} \cap \cdots \cap \Phi(E_n)^{\perp},$$

and the result follows. □

The functor $L\pi_X^*$ in Example 2.1.1 can be defined in an arbitrary $\mathsf{k}-$linear triangulated category.

**Definition 2.7.** Let $\mathcal{T}$ be a triangulated category and let $E \in \mathcal{T}$. We define the functor

$$\mathbf{D}(\mathsf{k}) \xrightarrow{\Phi_E} \mathcal{T}$$
$$V^{\bullet} \mapsto V^{\bullet} \otimes_{\mathsf{k}} E \qquad \forall\, V^{\bullet}$$

The operation $V^{\bullet} \otimes_{\mathsf{k}} E$ in $\mathcal{T}$ is very simple, $\Phi_E$ takes $\mathsf{k}$ to $E$. For instance, if $V^{\bullet}$ is a graded vector space with a three dimensional space in degree zero and a two dimensional space in degree five then $V^{\bullet} \otimes_{\mathsf{k}} E = (E[0] \oplus E[0] \oplus E[0]) \bigoplus (E[5] \oplus E[5])$.

By Example 2.1.1 we have that the functor $L\pi_X^*$ is fully faithful if $\mathcal{O}_X$ is exceptional and the converse is also true, if $L\pi_X^*$ is fully and faithful then $\mathcal{O}_X$ is exceptional.

**Lemma 2.4.** *Let $\mathcal{T}$ be a* $\mathsf{k}$*−linear triangulated category. Then $E \in \mathcal{T}$ is an exceptional object if and only if* $\mathbf{D}(\mathsf{k}) \xrightarrow{\Phi_E} \mathcal{T}$ *is fully and faithful, i.e an embedding.*

*Proof.* By Proposition 1.3.2, it is enough to prove $\Phi_E$ has a right adjoint functor $\Phi_E^!$ with $\Phi_E^! \circ \Phi_E \cong 1_{\mathbf{D}(\mathsf{k})}$. Since the tensor functor $(-)\otimes_{\mathsf{k}}$ has to $R\mathsf{Hom}$ as right adjoint,

$$\mathsf{Hom}(\Phi_E(V^\bullet), F) = \mathsf{Hom}(V^\bullet \otimes_{\mathsf{k}} E, F) = \mathsf{Hom}((V^\bullet, R\mathsf{Hom}(E, F)),$$

so we define $\Phi_E^!(-) = R\mathsf{Hom}(E, -)$ as the right adjoint to $\Phi_E$.

Now, consider the composition $\Phi_E^! \circ \Phi_E(V^\bullet) = R\mathsf{Hom}(E, V^\bullet \otimes_{\mathsf{k}} E) \cong R\mathsf{Hom}(E, E) \otimes_{\mathsf{k}} V^\bullet$. Then we see that $\Phi_E^! \circ \Phi_E \cong 1_{\mathbf{D}(\mathsf{k})}$ if and only if $R\mathsf{Hom}(E, E) = \mathsf{k}$ if and only if $E$ is exceptional. □

Lemma 2.4 gives us another form to write the s.o.d. for $\mathbf{D}(X)$ in examples 2.1.1 and 2.1.2

$$\mathbf{D}(X) = \langle \mathbf{D}(\mathsf{pt})^\perp, \mathbf{D}(\mathsf{pt})\rangle \text{ and } \mathbf{D}(X) = \langle \mathbf{D}(\mathsf{pt}), {}^\perp\mathbf{D}(\mathsf{pt})\rangle,$$

with $\mathbf{D}(\mathsf{k}) = \mathbf{D}(\mathsf{pt})$, and in general if we have an exceptional object $E \in \mathcal{T}$ then

$$\mathcal{T} = \langle \mathbf{D}(\mathsf{pt})^\perp, \mathbf{D}(\mathsf{pt})\rangle \text{ and } \mathcal{T} = \langle \mathbf{D}(\mathsf{pt}), {}^\perp\mathbf{D}(\mathsf{pt})\rangle.$$

An important problem in the theory of s.o.d. is giving good necessary conditions for a exceptional collection to be full, actually it is so complicated to construct a full exceptional collection. But if we have good luck, we will find out a exceptional collection and so get a s.o.d as in Lemma 2.3. It will be showed when $X = \mathbb{P}^n$ there is full exceptional collection. In fact there is a

**Conjecture 2.1.** *Let $\mathcal{T} = \langle E_1, \dots, E_n\rangle$ be a s.o.d. given by an exceptional collection for triangulated category. Then any exceptional collection of length $n$ in $\mathcal{T}$ is full.*

Recall that for $\mathcal{T} = \mathbf{D}(X)$ we have $\mathsf{HH}^0(\mathcal{T}) = H^0(X, \mathcal{O}_X)$ by the HKR isomorphism, Theorem 1.5. This and Lemma 2.4 motivate the following

**Definition 2.8.** A triangulated category $\mathcal{T}$ is called connected, if $\mathsf{HH}^0(\mathcal{T}) = \mathsf{k}$.

Later, we will give some examples of connected triangulated categories. In retrospective, at this point, it must be naive that if a triangulated category is connected then it has not s.o.d. Nevertheless, there is a stronger version of s.o.d. that will be useful to prove it.

**Definition 2.9.** Let $\mathcal{T}$ be a triangulated category. We say that $\mathcal{T} = \langle \mathcal{A}, \mathcal{B}\rangle$ is a completely orthogonal decomposition if it is a s.o.d. and $\mathsf{Hom}(\mathcal{A}, \mathcal{B}) = 0$. A triangulated category that has a completely orthogonal decomposition is called decomposable otherwise it is called indecomposable.

Later, we will see that the derived category of an integral scheme is indecomposable, Proposition 2.3.1, and as one expects

**Definition 2.10.** A semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m\rangle$ is maximal, if each component $\mathcal{A}_i$ is an indecomposable category, i.e., does not admit a nontrivial semiorthogonal decomposition.

### 2.1.2 Mutations

If a triangulated category $\mathcal{T}$ has a semiorthogonal decomposition then usually it has quite a lot of them. More precisely there are two groups acting on the set of semiorthogonal decompositions (the group of autoeqivalences of $\mathcal{T}$, and certain braid group); the action of the braid group is given by the so called mutations.

Roughly speaking, the mutation drops one of the component of the s.o.d. and extends the obtained semiorthogonal collection by inserting a new component at some other place. More precisely, the basic two operations are defined as follows.

**Definition 2.11.** Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{T}$ be and admissible subcategory. We define the left and right mutation functors by

$$\begin{aligned} \mathbb{L}_{\mathcal{A}} : \mathcal{T} &\longrightarrow \mathcal{T} \\ T &\longmapsto \mathsf{Cone}(\alpha\alpha^{!}T \to T) \qquad \forall T \in \mathcal{T}, \end{aligned}$$

$$\begin{aligned} \mathbb{R}_{\mathcal{A}} : \mathcal{T} &\longrightarrow \mathcal{T} \\ T &\longmapsto \mathsf{Cone}(T \to \alpha\alpha^{*}T)[-1] \qquad \forall F \in \mathcal{T}, \end{aligned}$$

respectively.

The following lemma proved in [Kuz08] describes the above mutation functors.

**Lemma 2.5.** *Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{T}$ be and admissible subcategory so that we have two s.o.d. $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ and $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$. Then the left and right mutation functors vanish on $\mathcal{A}$ and induce mutually inverse equivalences ${}^{\perp}\mathcal{A} \xrightarrow{\mathbb{L}_{\mathcal{A}}} \mathcal{A}^{\perp}$ and $\mathcal{A}^{\perp} \xrightarrow{\mathbb{R}_{\mathcal{A}}} {}^{\perp}\mathcal{A}$.*

Now we will explain what generated means in the paragraph after Definition 2.1.

**Definition 2.12.** Let $\mathcal{S} \subset \mathcal{T}$ be a collection of objects of a triangulated category. The collection $\mathcal{S}$ generated $\mathcal{T}$ if the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$ is equivalent to $\mathcal{T}$, via the inclusion.

Although the smallest triangulated subcategory from Definition 2.12 must be closed under direct summands and cones, in order to know how this subcategory looks like we need to assume the condition of semiorthogonality.

**Proposition 2.1.2.** *Let $\mathcal{T}$ be a triangulated category and suppose that $\mathcal{A}_1, \dots, \mathcal{A}_m$ are full triangulated subcategories of $\mathcal{T}$. If $\mathsf{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $i > j$, then the following conditions*

*(i) $\mathcal{A}_1, \dots, \mathcal{A}_m$ are admissible and $\mathcal{T}$ is generated by $\mathcal{A}_1, \dots, \mathcal{A}_m$,*

*(ii) $\mathcal{T}$ has a s.o.d., $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$*

*are equivalent.*

*Proof.* $(i) \Rightarrow (ii)$ Let $F \in \mathcal{T}$ and defined inductively

$$F_0 := F \text{ and } F_i := \mathbb{R}_{\mathcal{A}_i}(F_{i-1}), \quad i = 1, \dots, n.$$

Note that $F_i \in {}^{\perp}\mathcal{A}_1 \cap \cdots \cap {}^{\perp}\mathcal{A}_i$ for $i = 1, \dots, n$ and $\mathbb{R}_{\mathcal{A}_i}(F_{i-1}$ is on the left of the triangle $F_{i-1} \to \alpha\alpha^*(F_{i-1}) \to \mathsf{Cone}(F_{i-1} \to \alpha\alpha^*(F_{i-1})) \to F_{i-1}[1]$, so we have a sequence of morphism

$$F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 = F. \tag{2.7}$$

Finally $F_m \in {}^{\perp}\mathcal{A}_1 \cap \cdots \cap {}^{\perp}\mathcal{A}_m = 0$, the last equality follows from the fact that $\mathcal{A}_1, \cdots, \mathcal{A}_m$ generate $\mathcal{T}$, so all the properties of Definition 2.4 are satisfied.

$(ii) \Rightarrow (i)$ If $\mathcal{A}_1, \cdots, \mathcal{A}_m$ is a s.o.d. for $\mathcal{T}$ we have a chain of morphisms as (2.7) with the property $\mathsf{Cone}(F_i \to F_{i-1}) \in \mathcal{A}_i$ for all $i$, then the statement that $\mathcal{A}_1, \cdots, \mathcal{A}_m$ generates follows. The only thing that we have to check is that $\mathcal{A}_i$ admissible for any $i$. Let us show that $\mathcal{A}_m$ is admissible, the functor

$$\begin{aligned} \mathcal{T} &\xrightarrow{\alpha^*} \mathcal{A}_m \\ F &\mapsto F_{m-1}, \end{aligned}$$

where $F_{m-1}$ appears in (2.7), is right adjoint of the embedding functor $\alpha A_m \hookrightarrow \mathcal{T}$. Let $A \in \mathcal{A}_m$, applying the functor $\mathsf{Hom}(A, -)$ to (2.7) we obtain $\mathsf{Hom}(A, F_{m-1}) = \mathsf{Hom}(A, F_0)$ which means

$$\mathsf{Hom}(A, \alpha^*(F)) = \mathsf{Hom}(\alpha(A), F),$$

i.e. $\alpha^*$ is right adjoint to $\alpha$ and clearly $\alpha^*\alpha \cong id_{\mathcal{A}_m}$, so $\mathcal{A}_m$ is admissible. On the other hand, since $\mathcal{A}_m$ is admissible then ${\mathcal{A}_m}^{\perp}$ is admissible too, and by the s.o.d. of $\mathcal{T}$ we have

$${\mathcal{A}_m}^{\perp} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{m-1} \rangle, \tag{2.8}$$

applying the same argument to the subcategory $\mathcal{A}_{m-1}$ with the s.o.d. (2.8) we deduce $\mathcal{A}_m$ is admissible and so on. □

## 2.2 Semiorthogonal decomposition for Fano varieties

In this section we will show that for a Fano variety $X$ there exist a s.o.d. given by line bundles, the fist example is the projective space. In fact, $\mathbf{D}(\mathbb{P}^n)$ has a full exceptional collection, Corollary 2.2, which is a result proved by Beilinson. After proving Beilinson's theorem 2.1 we will give a nice application to moduli spaces, namely we will give an alternative construction of the moduli space of rank 2 vector bundles in $\mathbb{P}^2$, and thus we will give examples of s.o.d. for some hypersurfaces, which are Fano varieties. Finally, Hochschild homology is well behaved if one has a s.o.d. for a given triangulated category and this fact has many interesting geometric consequences.

### 2.2.1 Beilinson's Spectral sequence

Beilinson in his brilliant paper [Bei78] proved $\mathbf{D}(\mathbb{P}^n)$ has an exceptional collection, more over duo to the existence of a very special resolution of $\mathcal{O}_{\Delta}$, the structure sheaf of the diagonal $\Delta$ in $\mathbb{P}^n \times \mathbb{P}^n$, there is nevertheless a highly intriguing structure that emerges, the Beilinson's spectral sequence. After giving the proof of the convergence of this spectral sequence we will give some applications. We will also describe Beilinson's original proof, which is beautifully geometric and actually proves more – giving an explicit presentation (2.11) of any $F \in \mathbf{D}(\mathbb{P}^n)$ in terms of the exceptional collection.

Lemma 1.2 says that $\mathcal{O}_\Delta$ has a locally free resolution which we denote by $L^\bullet \to \mathcal{O}_\Delta$, so in $\mathbf{D}(\mathbb{P}^n \times \mathbb{P}^n)$ one has $L^\bullet \simeq \mathcal{O}_\Delta$.

**Theorem 2.1** (Beilinson's Spectral sequence)**.** *For any $\mathcal{F} \in \mathsf{Coh}(\mathbb{P}^n)$ there exist two spectral sequences*

$$E_1^{p,q} := H^q(\mathbb{P}^n, \mathcal{F}(p)) \otimes_{\mathsf{k}} \Omega^{-p}(-p) \Rightarrow E^{p,q} = \begin{cases} \mathcal{F} & , p+q=0, \\ 0 & , p+q \neq 0, \end{cases} \tag{2.9}$$

*and*

$$E_1^{p,q} := H^q(\mathbb{P}^n, \mathcal{F} \otimes \Omega^{-p}(-p)) \otimes_{\mathsf{k}} \mathcal{O}(p) \Rightarrow E^{p,q} = \begin{cases} \mathcal{F} & , p+q=0, \\ 0 & , p+q \neq 0. \end{cases} \tag{2.10}$$

*Proof.* The theorem is a consequence of the spectral sequence given by the right and left derived functor, Proposition 1.3.3.

Let $\mathcal{F} \in \mathsf{Coh}(\mathbb{P}^n)$ and let $A^\bullet := p_1^*(\mathcal{F}) \otimes L^\bullet$, note that in $A^\bullet$ the tensor product need not be derived, as $L^\bullet$ is a complex of locally free sheaves, (here $p_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n, i = 1, 2$ are the projections). Thus

$$A^p := \mathcal{F}(p)) \boxtimes \Omega^{-p}(-p)$$

and

$$\begin{aligned} \Phi_{\mathcal{O}(-p))\boxtimes\Omega^p(p)}(\mathcal{F}) = Rp_{2*}(A^p) &= p_{2*}(\mathcal{F}(p)) \boxtimes \Omega^{-p}(-p)) && \text{projection formula (1.6)} \\ &= \Omega^{-p}(-p) \otimes p_{2*}p_1^*(\mathcal{F}(p)) && \text{flat base change (1.7)} \\ &= \Omega^{-p}(-p) \otimes g^* f_*(\mathcal{F}(p)) \\ &= \Omega^{-p}(-p) \otimes \Gamma(\mathcal{F}(p)). \end{aligned}$$

Hence

$$R^q p_{2*}(A^p) = R^q\Gamma((\mathcal{F}(p))) \otimes_{\mathsf{k}} \Omega^{-p}(-p) = H^q(\mathbb{P}^n, \mathcal{F}(p)) \otimes_{\mathsf{k}} \Omega^{-p}(-p).$$

On the other hand, $L^\bullet \simeq \mathcal{O}_\Delta$ in $\mathbf{D}(\mathbb{P}^n \times \mathbb{P}^n)$, so

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(\mathcal{F}) = \Phi_{\iota_*\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}) &= p_{2*}(p_1^*(\mathcal{F}) \otimes \mathcal{O}_\Delta) \\ &= p_{2*}(p_1^*(\mathcal{F}) \otimes \iota_*\mathcal{O}_{\mathbb{P}^n}) \\ &= p_{2*}\iota_*(\iota^* p_1^*(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n}) && \text{projection formula (1.6)} \\ &= (p_2 \circ \iota)_*(p_1 \circ \iota)^*(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}) && \text{see Example 1.3.1} \\ &= \mathcal{F}. \end{aligned}$$

This proves the spectral sequence (2.9). The proof of the second spectral sequence (2.10) is analogous just interchanging $p_1$ and $p_2$. ☐

**Remark 2.3.** The spectral sequences (2.9) and (2.10) hold for any object $F \in \mathbf{D}(\mathbb{P}^n)$. Indeed, similar computations show that

$$\Phi_{\mathcal{O}_\Delta}(F) = Rp_{2*}(Lp_{1*}F \overset{\mathbb{L}}{\otimes} \mathcal{O}_\Delta) = R(p_2 \circ \iota)_*(L(p_1 \circ \iota)^* F) = F,$$

and

$$\Phi_{\Omega^p(p)\boxtimes\mathcal{O}(-p))}(F) = R\Gamma(\mathbb{P}^n, \Omega^p(p) \overset{\mathbb{L}}{\otimes} F) \otimes_{\mathsf{k}} \mathcal{O}(-p)$$

for $F \in \mathbf{D}(\mathbb{P}^n)$ and we see that $F$ is quasi-isomorphic to the complex

$$F_n \otimes_{\mathsf{k}} \mathcal{O}_{\mathbb{P}^n}(-n) \to F_{n-1} \otimes_{\mathsf{k}} \mathcal{O}_{\mathbb{P}^n}(-(n-1)) \to \cdots F_1 \otimes_{\mathsf{k}} \mathcal{O}_{\mathbb{P}^n}(-1) \to F_0 \otimes_{\mathsf{k}} \mathcal{O}_{\mathbb{P}^n}, \tag{2.11}$$

where $F_i := R\Gamma(\mathbb{P}^n, \Omega^p(p) \overset{\mathbb{L}}{\otimes} F)$. Therefore this can be rewritten to get analogous spectral sequences to those in Theorem 2.1 and they converge to the object $F$ too.

Note that $\Omega^{-p}(-p)$ is non-trivial for $-n \leq r \leq 0$, so $E_1^{p,q}$ in (2.9) and (2.10) are trivial for $p < -n$ or $p > 0$ independently of $\mathcal{F}$. Also $E_1^{p,q} = 0$ for $q < 0$ and $q > n$. Thus both spectral sequences are concentrated in the second quadrant.

**Corollary 2.2.** *The sequence of line bundles*

$$\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \in \mathbf{D}(\mathbb{P}^n)$$

*is a full exceptional collection. Hence, there is a s.o.d.*

$$\mathbf{D}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle.$$

*In general $\mathcal{O}(m), \dots, \mathcal{O}(n+m)$ is a f.e.c. for any $m \in \mathbb{Z}$.*

*Proof.* By Proposition 1.3.1 , $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O})$ is an exceptional collection. In order to show it is full let us prove $\mathcal{O}(-n)^\perp \cap \cdots \cap \mathcal{O}(-1)^\perp \cap \mathcal{O}^\perp = 0$.

Let $\mathcal{F} \in \mathsf{Coh}(\mathbb{P}^n)$ such that $\mathcal{F} \in \mathcal{O}(-n)^\perp \cap \cdots \cap \mathcal{O}(-1)^\perp \cap \mathcal{O}^\perp$, applying (2.9) to $\mathcal{F}$ yields a spectral sequence with

$$E_1^{p,q} = H^q(\mathbb{P}^n, \mathcal{F}(p)) \otimes \Omega^{-p}(-p) \cong \mathsf{Ext}^q(\mathcal{O}, \mathcal{F}(p)) \otimes \Omega^{-p}(-p),$$

$E_1^{p,q}$ is non-trivial for $-n \leq p \leq 0$ but

$$\mathsf{Ext}^q(\mathcal{O}, \mathcal{F}(p)) \cong \mathsf{Ext}^q(\mathcal{O}(-p), \mathcal{F}) = 0,$$

for $n \geq -p \geq 0$. Then $E_1^{p,q} = 0$ and $F \simeq 0$ by (2.9).

For the general case, the resolution $L^\bullet \simeq \mathcal{O}_\Delta$ can be split into short exact sequences,

$$0 \to \mathcal{O}(-n)) \boxtimes \Omega^n(n) \to \mathcal{O}(-(n-1)) \boxtimes \Omega^{n-1}(n-1) \to \mathcal{E}_{n-1} \to 0$$
$$0 \to \mathcal{E}_{n-1} \to \mathcal{O}(-(n-2)) \boxtimes \Omega^{n-2}(n-2) \to \mathcal{E}_{n-2} \to 0$$
$$\vdots$$
$$0 \to \mathcal{E}_1 \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Each short exact sequence can be regarded as a distinguished triangle in $\mathbf{D}(\mathbb{P}^n \times \mathbb{P}^n)$. Let $F \in \mathbf{D}(\mathbb{P}^n)$, tensor product with $Rp_2^*(F)$ and taking derived direct image under $p_1$ yields distinguished triangles

$$\Phi_{\mathcal{E}_{i+1}}(F) \to \Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(F) \to \Phi_{\mathcal{E}_i}(F) \to \Phi_{\mathcal{E}_{i+1}}(F)[1].$$

(Morally we are using (2.10) this time). Clearly, $\Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(F) \cong H^\bullet(\mathbb{P}^n, F \overset{\mathbb{L}}{\otimes} \Omega^i(i)) \otimes_{\mathsf{k}} \mathcal{O}(-i))$ is contained in $\langle \mathcal{O}(-i) \rangle$ for $0 \leq i \leq n$ and

$$\mathsf{Cone}(\Phi_{\mathcal{O}(-n)\boxtimes\Omega^n(n)}(F) \to \Phi_{\mathcal{O}(-(n-1))\boxtimes\Omega^{n-1}(n-1)}(F)) \cong \Phi_{\mathcal{E}_{n-1}}(F) \in \langle \mathcal{O}(-n), \mathcal{O}(-(n-1)) \rangle.$$

By induction $\Phi_{\mathcal{E}_i}(F) \in \langle \mathcal{O}(-n), \dots, \mathcal{O}(-i) \rangle$ for all $i$ and eventually $F \simeq \Phi_{\mathcal{O}_\Delta}(F) \in \langle \mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$. Thus $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$ is a f.e.c. by Proposition 2.1.2.

Since $\mathcal{O}(m+n) \overset{\mathbb{L}}{\otimes} (-) : \mathbf{D}(\mathbb{P}^n) \to \mathbf{D}(\mathbb{P}^n)$, tensor with a line bundle is an autoequivalence, the general result follows by Proposition 2.1.1 □

The spectral sequence (2.9) gives us the

**Corollary 2.3.**

$$\mathcal{O}, \Omega^1(1), \dots, \Omega^n(n) \in \mathbf{D}(\mathbb{P}^n)$$

*is a full exceptional collection. Hence, there is a s.o.d.*

$$\begin{aligned}\mathbf{D}(\mathbb{P}^n) &= \langle \mathcal{O}, \Omega^1(1), \dots, \Omega^n(n) \rangle \\ &= \langle \mathbf{D}(\mathsf{pt}), \mathbf{D}(\mathsf{pt}), \dots, \mathbf{D}(\mathsf{pt}) \rangle.\end{aligned}$$

The last equality follows from Lemma 2.4, since the functors $\mathbf{D}(\mathsf{pt}) \to \mathbf{D}(\mathbb{P}^n)$, $\mathsf{k} \mapsto \mathcal{O}(i)$ as well as $\mathbf{D}(\mathsf{pt}) \to \mathbf{D}(\mathbb{P}^n)$, $\mathsf{k} \mapsto \Omega^i(i)$ are embeddings for all $i$.

### 2.2.2 Moduli space of rank 2 vector bundles in $\mathbb{P}^2$

**The moduli space of rank 2 vector bundles on $\mathbb{P}^2$ .**

Let us consider $\mathbb{P}^2 = \mathbb{P}(V)$ with $V$ a $\mathbb{C}$-vector space of dimension 3. Let $\mathfrak{M}(2;0,2)_{\mathbb{P}^2}$ the moduli space of stable rank 2 vector bundles on $\mathbb{P}^2$ with first and second Chern class $c_1 = 0$ and $C_2 = 2$, respectibely.

In general we know that in $\mathbb{P}^n$, $\Omega^n(n) \cong \mathcal{O}_{\mathbb{P}^n}(-1)$.

Let $F$ be a stable vector bundle of rank 2 with $c_1 = 0$ and $c_2 = 2$, the spectral sequence (2.9) in the first sheet has the form

$$\begin{array}{ccccc}
H^2(\mathbb{P}^2, F(-2)) \otimes_{\mathbb{C}} \mathcal{O}(-1) & \longrightarrow & H^2(\mathbb{P}^2, F(-1)) \otimes_{\mathbb{C}} \Omega^1(1) & \longrightarrow & H^2(\mathbb{P}^2, F) \otimes_{\mathbb{C}} \mathcal{O} \\
H^1(\mathbb{P}^2, F(-2)) \otimes_{\mathbb{C}} \mathcal{O}(-1) & \longrightarrow & H^1(\mathbb{P}^2, F(-1)) \otimes_{\mathbb{C}} \Omega^1(1) & \longrightarrow & H^1(\mathbb{P}^2, F) \otimes_{\mathbb{C}} \mathcal{O} \\
H^0(\mathbb{P}^2, F(-2)) \otimes_{\mathbb{C}} \mathcal{O}(-1) & \longrightarrow & H^0(\mathbb{P}^2, F(-1)) \otimes_{\mathbb{C}} \Omega^1(1) & \longrightarrow & H^0(\mathbb{P}^2, F) \otimes_{\mathbb{C}} \mathcal{O},
\end{array}$$

Figure 2.1: Spectral Sequence of a stable vector bundle $F$.

and in the second sheet the morphisms are given by the dashed arrows. Since $F$ is stable

$$H^0(\mathbb{P}^2, F(-2)) = H^0(\mathbb{P}^2, F(-1)) = H^0(\mathbb{P}^2, F) = 0.$$

Therefore the first line of the Figure 2.1 is zero. On the other hand, by Serre duality

$$H^2(\mathbb{P}^2, F(-k)) = H^0(\mathbb{P}^2, F^\vee(k-3))^\vee,$$

for $0 \leq k \leq 2$ and again by stability

$$H^0(\mathbb{P}^2, F^\vee(-1)) = H^0(\mathbb{P}^2, F^\vee(-2)) = H^0(\mathbb{P}^2, F^\vee(-3)) = 0,$$

thus the third line of the spectral sequence 2.1 is zero. Hence the spectral sequence degenerate to the second line. By Riemann-Roch computations one gets

$$\mathsf{dim} H^1(\mathbb{P}^2, F) = 0, \text{ and } \mathsf{dim} H^1(\mathbb{P}^2, F(-1)) = \mathsf{dim} H^1(\mathbb{P}^2, F(-2)) = 2,$$

at the end we have a exact sequence

$$0 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \Omega^1(1) \oplus \Omega^1(1) \to F \to 0 \tag{2.12}$$

where $H^1(\mathbb{P}^2, F(-2)) \otimes \mathcal{O}(-1) \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $H^1(\mathbb{P}^2, F(-1)) \otimes \Omega^1(1) \cong \Omega^1(1) \oplus \Omega^1(1)$, whenever we have chosen a base of $H^1(\mathbb{P}^2, F(-1))$ and $H^1(\mathbb{P}^2, F(-2))$.

Conversely, we can also do this construction in the oppositive direction, i.e. we can start with a morphism of vector bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \Omega^1(1) \oplus \Omega^1(1)$ and consider cokernels of $\varphi$ and see that they are stable vector bundles with the same parameters that we have fixed.

Thus we have a description of the moduli space $\mathfrak{M}(2; 0, 2)_{\mathbb{P}^2}$ as a parameter space of these morphisms. For short, let us write $A \otimes \mathcal{O}(-1) \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $B \otimes \Omega^1(1) \cong \Omega^1(1) \oplus \Omega^1(1)$ with $A$ and $B$ two dimensional vector spaces, the exact sequence (2.12) can be rewritten as

$$0 \to A \otimes \mathcal{O}(-1) \to B \otimes \Omega^1(1) \to F \to 0. \tag{2.13}$$

Since $\mathsf{Hom}(A \otimes \mathcal{O}(-1), B \otimes \Omega^1(1)) \cong A^\vee \otimes B \otimes \mathsf{Hom}(\mathcal{O}(-1), \Omega^1(1))$ and canonically $V \cong \mathsf{Hom}(\mathcal{O}(-1), \Omega^1(1))$, then the vector space $A^\vee \otimes B \otimes V$ parametrizes the maps $\varphi$, of course there is and action of the group $\mathsf{GL}(A) \times \mathsf{GL}(B)$ in this vector space (changing the base of $A \times B$), and clearly the action of any elemenet of the group does not change the isomorphic class of the cokernel in (2.13).

All together means that the moduli space $\mathfrak{M}(2; 0, 2)_{\mathbb{P}^2}$ is the $(GIT)$ quotient of the form

$$\mathfrak{M}(2; 0, 2)_{\mathbb{P}^2} = A^\vee \otimes B \otimes V / \mathsf{GL}(A) \times \mathsf{GL}(B).$$

See [OSS88] for details.

### 2.2.3 Semiorthogonal decomposition for Fano varieties

Most interesting examples of s.o.d. come from Fano varieties.

**Definition 2.13.** A Fano variety is a smooth projective connected variety $X$ with ample anticanonical class $-K_X$. A Fano variety is prime if $\mathsf{Pic}(X) \cong \mathbb{Z}$. The index of a Fano variety $X$ is the maximal integer $r$, such that $-K_X = rH$ for some $H \in \mathsf{Pic}(X)$.

The main example is a hypersurface.

**Example 2.2.1.** Let $V$ a vector space of dimension $n+1$ and let $Y \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a hypersurface of degree $d$. By Lefschetz hyperplane section theorem $\mathsf{Pic}(Y) = \mathbb{Z}$ and is generated by $H$, the restriction of the class of a hyperplane in $\mathbb{P}(V)$. On the other hand, the canonical sheaf of $Y$ equals

$$\omega_Y = \mathcal{O}_Y(d - n - 1)$$

by adjunction formula and so the canonical class of $Y$

$$K_Y = (d - n - 1)H. \tag{2.14}$$

$Y$ is a Fano variety since $\mathbb{P}(V)$ is so by (2.14) we have that the index of a hypersurface of degree $d$ is $d - n - 1$. See[Har77, II, 8, Example 8.20.2] for details.

**Proposition 2.2.1.** *Let $X$ be a Fano variety of index $r$, $-K_X = rH$. Then the collection of line bundles*

$$\mathcal{O}_X((1-r)H), \ldots, \mathcal{O}_X(-H), \mathcal{O}_X \in \mathsf{Pic}(X),$$

*is an exceptional collection.*

*Proof.* Indeed, for $i > -r$ we have

$$H^{>0}(X, \mathcal{O}_X(iH)) = H^{>0}(X, \mathcal{O}_X(K_X((i+r)H))) = 0,$$

by Kodaira vanishing theorem, [Har77, III, 7]. Moreover,

$$H^0(X, \mathcal{O}_X(iH)) = 0,$$

for $i < 0$ by ampleness of $H$ and $H^0(X, \mathcal{O}_X) = \mathsf{k}$ by connectedness of $X$. Thus the collection is exceptional. □

Note also that $\mathcal{O}_X, \mathcal{O}_X((1)H), \ldots, \mathcal{O}_X((r-1)H)$ is an exceptional collection.

Applying Lemma (2.3) we have a s.o.d.

$$\mathbf{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X((1-r)H), \ldots, \mathcal{O}_X(-H), \mathcal{O}_X \rangle \tag{2.15}$$

where

$$\mathcal{A}_X = \mathcal{O}_X((1-r)H)^{\perp} \cap \cdots \cap \mathcal{O}_X(-H)^{\perp} \cap \mathcal{O}_X^{\perp}.$$

In some cases, the orthogonal complement $\mathcal{A}_X$ in (2.15) vanishes or can be explicitly described.

**Example 2.2.2.** One of the most important example is the projective space, $\mathbb{P}^n$ is a Fano variety of index $n+1$. The orthogonal complement of the maximal exceptional collection vanishes Corollary 2.2, and we have

$$\mathbf{D}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle.$$

**Example 2.2.3.** Let $V$ be a five dimensional vector space. Let $Y \subset \mathbb{P}(V)$ be a hypersurface of degree 3, i.e. a smooth cubic 3-fold, $Y$ is Fano variety of index 2 by (2.14), and so

$$\mathbf{D}(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

**Example 2.2.4.** Let $V$ be a six dimensional vector space. Let $Y \subset \mathbb{P}(V) \cong \mathbb{P}^5$ be a hypersurface of degree 3, i.e. a smooth cubic 4-fold, $Y$ is Fano variety of index 3 by (2.14), and so

$$\mathbf{D}(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle.$$

The following example is the most important for our purpose.

**Example 2.2.5.** Remember a Fano variety $X$ of dimension 2 is a called a del Pezzo surface, if the base field is $\mathbb{C}$, then $\mathbf{D}(X)$ has a full exceptional collection. In order to prove it we need to describe the derived category of the blow up of $\mathbb{P}^2$ in several points, ingeneral this result is Orlov's blow up formula and we will prove it in Chapter 3.

## 2.3 Hochschild Homology and semiorthogonal decomposition

One of the reasons Hochschild Homology is very useful is its additivity with respect to semiorthogonal decompositions. A proof of the following theorem can be found in [Kuz09].

**Theorem 2.2.** *Let $\mathcal{T}$ a triangulated category. If we have a s.o.d. $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ then*

$$\mathsf{HH}_\bullet(\mathcal{T}) = \bigoplus_{i=1}^{m} \mathsf{HH}_\bullet(\mathcal{A}_i).$$

One of the nice consequences of this theorem is the following necessary condition for a category to have a full exceptional collection.

**Corollary 2.4.** *If a triangulated category $\mathcal{T}$ has a full exceptional collection then $\mathsf{HH}_k(\mathcal{T}) = 0$ for $k \neq 0$ and $\dim \mathsf{HH}_0(\mathcal{T}) < \infty$. Moreover, the length of any full exceptional collection in $\mathcal{T}$ equals $\dim \mathsf{HH}_0(\mathcal{T})$.*

*In particular, if $X$ is a smooth projective variety and $\mathbf{D}(X)$ has a full exceptional collection, then $H^{p,q}(X) = 0$ for $p \neq q$, and the length of the exceptional collection equals to $\sum_p \dim H^{p,p}(X)$.*

*Proof.* Suppose $\mathcal{T} = \langle T_1, \dots, T_m \rangle$ is a s.o.d. given by a f.e.c, for $k > 0$ one has

$$\mathsf{HH}_k(\mathcal{T}) = \bigoplus_{i=1}^{m} \mathsf{HH}_k(T_i) = \bigoplus_{i=1}^{m} \mathsf{HH}_k(\mathbf{D}(\mathsf{pt})) = 0$$

by Lemma 2.4 and Example 1.4.1, and if $k = 0$

$$\mathsf{HH}_0(\mathcal{T}) = \bigoplus_{i=1}^{m} \mathsf{HH}_0(\mathbf{D}(\mathsf{pt})) = \bigoplus_{i=1}^{m} \mathsf{k},$$

and so $\dim \mathsf{HH}_0(\mathcal{T}) < \infty$.

For the second part, suppose $\mathbf{D}(X)$ has a f.e.c. of length $m$ then if $k > 0$ we get $p \neq q$, thus

$$0 = \mathsf{HH}_k(\mathbf{D}(X)) = \bigoplus_{p-q=k} H^{p,q}(X),$$

by the first part and Example 1.4.1.

Therefore

$$m = \sum_{i=1}^{m} \dim \mathsf{HH}_0(\mathbf{D}(\mathsf{pt})) = \dim \mathsf{HH}_0(\mathbf{D}(X)) = \sum_p \dim H^{p,p}(X),$$

and the result follows. □

In a contrast, the Hochschild cohomology is not additive, it depends not only on the components of a s.o.d., but also on the way that these are glued together. However, if there is a completely orthogonal decomposition then the Hochschild cohomology is additive, [Kuz09].

**Lemma 2.6.** *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a completely orthogonal decomposition then*

$$\mathsf{HH}^\bullet(\mathcal{T}) = \mathsf{HH}^\bullet(\mathcal{A}) \oplus \mathsf{HH}^\bullet(\mathcal{B}). \tag{2.16}$$

This lemma also has a nice consequence. Remember the definition of a connected triangulated category, Definition 2.8 and indecomposable triangulated category, Definition 2.9.

**Corollary 2.5.** *If $\mathcal{T}$ is a connected triangulated category then $\mathcal{T}$ has no completely orthogonal decompositions.*

*Proof.* Assume $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is completely orthogonal decomposition, then by Lemma 2.6

$$\mathsf{k} = \mathsf{HH}^0(\mathcal{T}) = \mathsf{HH}^0(\mathcal{A}) \oplus \mathsf{HH}^0(\mathcal{B}),$$

hence one of the summands vanishes. But a nontrivial category always has nontrivial zero Hochschild cohomology (since the identity functor always has the identity endomorphism). □

The first examples of triangulated categories which have no nontrivial s.o.d. was found by Bridgeland [Bri05].

**Proposition 2.3.1.** *If $X$ is a smooth projective algebraic variety then $\mathbf{D}(X)$ is indecomposable if and only if $X$ is connected.*

*Proof.* In particular, $\mathsf{HH}^0(\mathbf{D}(X)) = H^0(X, \mathcal{O}_X)$ and $\mathsf{HH}^0(\mathbf{D}(X)) = \mathsf{k}$ if and only if $X$ is connected and the result follows. □

Moreover

**Proposition 2.3.2.** *If $\mathcal{T}$ is a connected Calabi-Yau category then $\mathcal{T}$ has no semiorthogonal decompositions.*

*Proof.* Assume $\mathcal{T}$ is Calabi-Yau of dimension $n$ and $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a s.o.d. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$\mathsf{Hom}(A, B)^\vee = \mathsf{Hom}(B, \mathsf{S}_{\mathcal{T}}(A)) = \mathsf{Hom}(B, A[n]) = 0,$$

since $A[n] \in \mathcal{A}$. Hence the decomposition is completely orthogonal and Corollary 2.5 applies. □

**Corollary 2.6.** *The category $\mathbf{D}(\mathsf{k})$ is indecomposable.*

*Proof.* Indeed, $\mathsf{S}_{\mathbf{D}(\mathsf{k})} \cong 1_{\mathbf{D}(\mathsf{k})}$ so it is Calabi-Yau of dimension 0. □

Besides, one can check that derived categories of curves of positive genus are indecomposable. The following proposition is a consequence of Beilinson's theorem 2.1 and [Oka11], see Example 2.1.5.

**Proposition 2.3.3.** *Let $C$ be a smooth projective curve of genus $g$. If $g > 0$ then $\mathbf{D}(C)$ is indecomposable and if $C \cong \mathbb{P}^1$ then any semiorthogonal decomposition of $\mathbf{D}(C)$ is given by an exceptional pair.*

# 3

# Semiorthogonal decomposition in Birational geometry

It is expected (and has been partially confirmed) that we can understand the minimal model program (MMP) in terms of the semiorthogonal decompositions of the derived category of coherent sheaves. To be precise one expects that it can be defined a suitable triangulated category for each projective variety with mild singularities, which equals to the bounded derived category of coherent sheaves when the variety is smooth, and that each step of MMP yields a non trivial SOD of the category. In particular we expect that a variety whose derived category admits no nontrivial s.o.d is minimal in the sense of MMP (see [BO95]).

On the other hand, Fano varieties are important in the theory of higher dimensional varieties . The interest in Fano varieties increased recently since Mori's program predicts that every uniruled variety is birational to a fiberspace whose general fiber is a Fano variety (with terminal singularities). Thus del Pezzo surfaces are so important in the two dimensional theory as Fano varieties are in the higher dimensional theory. In fact, it has been fount out a mysterious duality between M-Theory (in physics) and del Pezzo surfaces which may provide a hint about the understanding of the symmetries of this theory [Nei03]. Therefore studying s.o.d. for del Pezzo surfaces will be very rewarding. We will prove that there is a s.o.d. for the derived categories of the projectivization of a vector bundle, Proposition 3.1.1 (Orlov I) and of the blow up on a smooth projective algebraic variety, Theorem 3.1 (Orlov I) and hence we will compute a s.o.d. for del Pezzo surfaces.

## 3.1 Orlov I

**Definition 3.1.** Let $f : X \to Y$ be a morphism of smooth projective varieties. An object $E \in \mathbf{D}(X)$ is called a relative exceptional object if

$$Rf_* \mathsf{R}\mathscr{H}om(E, E) \cong \mathcal{O}_Y. \tag{3.1}$$

Note that when $Y = \mathsf{Spec}(\mathsf{k})$ the above condition is just the definition of an exceptional object; since $Rf_*(-) = \mathsf{Ext}^\bullet(\mathcal{O}_X, -)$ and $(\overset{\mathbb{L}}{\otimes}, \mathsf{R}\mathscr{H}om)$ is an adjoint pair, see subsection 1.3.1, we have

$$Rf_* \mathsf{R}\mathscr{H}om(E, E) \cong \mathsf{Ext}^\bullet(\mathcal{O}_X, \mathsf{R}\mathscr{H}om(E, E)) \cong \mathsf{Ext}^\bullet(\mathcal{O}_X \overset{\mathbb{L}}{\otimes} E, E) \cong \mathsf{Ext}^\bullet(E, E) \cong \mathsf{k}.$$

In the same way that an exceptional object gives a s.o.d. Lemma 2.3, also a relative exceptional object gives a s.o.d.

**Lemma 3.1.** *If $E \in \mathbf{D}(X)$ is a relative exceptional object then the functor*

$$\mathbf{D}(Y) \longrightarrow \mathbf{D}(X)$$
$$F \mapsto E \overset{\mathbb{L}}{\otimes} Lf^*(F) \qquad \forall\ F,$$

*is fully faithful and gives a s.o.d.*

$$\mathbf{D}(X) = \langle \mathsf{Ker}\ Rf_* \circ \mathsf{R}\mathscr{H}om(E, -), E \overset{\mathbb{L}}{\otimes} Lf^*(\mathbf{D}(Y)) \rangle.$$

*Proof.* We know that $Rf_*\mathsf{R}\mathscr{H}om(E, -)$ is the left adjoint of $E \overset{\mathbb{L}}{\otimes} Lf^*(-)$ see subsection 1.3.1, and since for any $F \in \mathbf{D}(Y)$ one has

$$Rf_* \circ \mathsf{R}\mathscr{H}om(E, E \overset{\mathbb{L}}{\otimes} Lf^*(F) \cong Rf_*(\mathsf{R}\mathscr{H}om(E, E) \overset{\mathbb{L}}{\otimes} Lf^*(F)) \cong Rf_*(\mathsf{R}\mathscr{H}om(E, E)) \overset{\mathbb{L}}{\otimes} F,$$

the condition that $E$ is a relative exceptional object implies $Rf_* \circ \mathsf{R}\mathscr{H}om(E, E \overset{\mathbb{L}}{\otimes} Lf^*(-) \cong 1_{\mathbf{D}(Y)}$, i.e. the functor $E \overset{\mathbb{L}}{\otimes} Lf^*(F)$ is fully faithful, thus Lemma 2.2 proves the result. □

Let $X = \mathbb{P}_Y(\mathscr{V}) \xrightarrow{f} Y$ be the projectivization of a vector bundles $\mathscr{V}$ of rank $r$ on $Y$, [Har77]. Then $Rf_*\mathcal{O}_X \cong \mathcal{O}_Y$, hence any line bundle $\mathscr{L}$ on $X$ is a relative exceptional object since $\mathsf{R}\mathscr{H}om(\mathscr{L}, \mathscr{L}) \cong \mathcal{O}_X$, in particular we have the tautological line bundle $\mathcal{O}_X(-1) \hookrightarrow f^*(\mathscr{V})$ in $X$ and thus for any $n \in \mathbb{Z}$, the functor $\mathcal{O}_X(n) \overset{\mathbb{L}}{\otimes} Lf^*(-) : \mathbf{D}(Y) \to \mathbf{D}(X)$ is fully faithful by Lemma 3.1 and we denote

$$\mathbf{D}(Y)(n) = \mathcal{O}_X(n) \overset{\mathbb{L}}{\otimes} Lf^*(\mathbf{D}(Y)).$$

So, iterating the construction of Lemma 3.1 we get Orlov's s.o.d. for the projectivization of a vector bundle.

**Proposition 3.1.1** (**Orlov I**). *Let $\mathscr{V}$ be a vector bundle of rank $r$ on $Y$ and let $X = \mathbb{P}_Y(\mathscr{V}) \xrightarrow{f} Y$ be its projectivization. Then there is a s.o.d.*

$$\mathbf{D}(X) = \langle \mathbf{D}(Y)(1-r), \ldots, \mathbf{D}(Y)(-1), \mathbf{D}(Y) \rangle.$$

In order to prove Proposition 3.1.1 we have to rewrite all the statements in the subsection 2.2.1 in such way we get an analogous corollary as that Corollary 2.2. It can be done without any difficulty because there is not any new idea in the proof of Proposition 3.1.1 as Beilinson cleverly observes [Bei78], nevertheless we make some remarks.

**Remarks 3.1.1.** 1. In Corollary 2.2 the $i$ th copy of $\mathbf{D}(\mathsf{pt})$ is embedded in $\mathbf{D}(\mathbb{P}^n)$ via the functor $\mathsf{pt} \to \mathcal{O}(i)$. Here the $i$ th copy of $\mathbf{D}(Y)$ is embedded in $\mathbf{D}(X)$ via de functor $Lf^*(-) \overset{\mathbb{L}}{\otimes} \mathcal{O}(i)$.

2. As in Lemma 1.2 there is a locally free resolution for $\mathcal{O}_\Delta$. Let $X = \mathbb{P}_Y(\mathscr{V}) \xrightarrow{f} Y$ be the projectivization of a vector bundle and let $p, q : \mathbb{P}_Y(\mathscr{V}) \times_Y \mathbb{P}_Y(\mathscr{V}) \to \mathbb{P}_Y(\mathscr{V})$ the projections, then there is a resolution

$$0 \to \mathcal{O}_X(-n) \boxtimes \Omega_X^n(n) \to \mathcal{O}_X(-n+1) \boxtimes \Omega_X^{n-1}(n-1) \to \cdots \to \mathcal{O}_X(-1) \boxtimes \Omega_X^1(1) \to \mathcal{O}_{X \times_Y X} \to \mathcal{O}_\Delta \to 0 \tag{3.2}$$

3. In the same way that we found out the spectral sequences in Theorem 2.1 we can find analogous spectral sequences for the projectivization of a vector bundle, namely for any object $F \in \mathbf{D}(X)$ there is a spectral sequence which converges

$$E_1^{p,q} := Lf^* R^q f_*(E \overset{\mathbb{L}}{\otimes} \mathcal{O}_X(p)) \otimes \Omega_X^{-p}(-p) \Rightarrow E^{p,q} = \begin{cases} F & , p+q = 0, \\ 0 & , p+q \neq 0, \end{cases}$$

and there is another analogous to (2.10).

## 3.2 Orlov II: Semiorthogonal decomposition of a blow up

Remember $X$ and $Y$ are smooth projective algebraic varieties.

In birational geometry, the most important semiorthogonal decomposition is that given by a blow up. Let $X = \mathsf{Bl}_Z(Y)$ be the blow up of $Y$ in a smooth subvariety $Z$ of codimention $c$. Then we have the following blow up diagram

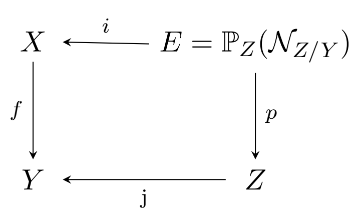


Figure 3.1: Blow up

where the exceptional divisor $E = f^{-1}(Z)$ is isomorphic to the projectivization of the normal bundle, and its natural map to $Z$ is the standard projection of the projectivization. Under this identification the normal bundle $\mathcal{O}_E(E)$ of the exceptional divisor is isomorphic to the Grothendieck line bundle $\mathcal{O}_E(-1)$

$$\mathcal{O}_E(E) \cong \mathcal{O}_E(-1),$$

on the projectivization $E \cong \mathbb{P}_Z(\mathcal{N}_{Z/Y})$.

For each $k \in \mathbb{Z}$, we consider the Fourier-Mukai functor with kernel $\mathcal{O}_E(k)$, i.e.

$$\begin{aligned} \Phi_{\mathcal{O}_E(k)}(-) : \mathbf{D}(Z) &\longrightarrow \mathbf{D}(X) \\ F &\mapsto Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F)) \qquad \forall\ F. \end{aligned}$$

**Proposition 3.2.1.** *Suppose $Z \subset X$ is of codimension $c \geq 2$. Then the functor $\Phi_{\mathcal{O}_E(k)}$ is fully faithful for any $k$. Moreover, $\Phi_{\mathcal{O}_E(k)}$ admits a right adjoint.*

*Proof.* To prove the proposition let us see that $\Phi_{\mathcal{O}_E(k)}$ has a right adjoint $\Phi^!_{\mathcal{O}_E(k)}$ and $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(k)} \cong 1_{\mathbf{D}(Z)}$, Proposition (1.3.2).

Simple computations of $\mathsf{Hom}(\Phi_{\mathcal{O}_E(k)}(F), G)$, with $F \in \mathbf{D}(Z)$ and $G \in \mathbf{D}(X)$ show that the right adjoint of $\Phi_{\mathcal{O}_E(k)}$ is defined by

$$\begin{aligned} \Phi^!_{\mathcal{O}_E(k)}(-) : \mathbf{D}(X) &\longrightarrow \mathbf{D}(Z) \\ G &\mapsto Rp_*(\mathcal{O}_E(-k) \overset{\mathbb{L}}{\otimes} Li^!(F)) \qquad \forall\, G. \end{aligned}$$

On the other hand, the composition $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(k)}$ is given by

$$\begin{aligned} \Phi^!_{\mathcal{O}_E(k)}(\Phi_{\mathcal{O}_E(k)}(F)) &= \Phi^!_{\mathcal{O}_E(k)}(Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F))) \\ &\cong Rp_*(\mathcal{O}_E(-k-1) \overset{\mathbb{L}}{\otimes} Li^*(Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F)))[-1]) \\ &\cong Rp_*(\mathcal{O}_E(-1) \overset{\mathbb{L}}{\otimes} Li^*(Ri_*(Lp^*(F))))[-1]. \end{aligned} \tag{3.3}$$

**Fact**: $i$ is an embedding, hence the composition $Li^* \circ Ri_*$ comes with a distinguished triangle

$$G \overset{\mathbb{L}}{\otimes} \mathcal{O}_E(1)[1] \to Li^* \circ Ri_*(G) \to G \to G \overset{\mathbb{L}}{\otimes} \mathcal{O}_E(1)[2],$$

see [Huy06, Cor. 11.4 (ii)].

If $G = Lp^*(F)$ we get

$$Lp^*(F) \overset{\mathbb{L}}{\otimes} \mathcal{O}_E(1)[1] \to Li^* \circ Ri_*(Lp^*(F)) \to Lp^*(F) \to Lp^*(F) \overset{\mathbb{L}}{\otimes} \mathcal{O}_E(1)[2], \tag{3.4}$$

applying $[-1]$, $(-) \overset{\mathbb{L}}{\otimes} \mathcal{O}_E(-1)$ and $Rp_*(-)$ to (3.4) results

$$Rp_*(Lp^*(F)) \to \Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(k)}(F) \to Rp_*(\mathcal{O}_E(-1) \overset{\mathbb{L}}{\otimes} Lp^*(F))[-1] \to Rp_*(Lp^*(F))[1].$$

Using the projection formula (1.6) and the fact that $Rp_*(\mathcal{O}_E) \cong \mathcal{O}_Z$ and $Rp_*(\mathcal{O}_E(-1)) = 0$, we conclude that $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(k)}(F) \cong F$, hence $\Phi_{\mathcal{O}_E(k)}$ is fully faithful. □

The following proposition is very useful.

**Proposition 3.2.2.** *Let $f : X \to Y$ be a projective morphism of smooth projective varieties such that $Rf_*(-) : \mathbf{D}(X) \to \mathbf{D}(Y)$ sends $\mathcal{O}_X$ to $\mathcal{O}_Y$. Then $Lf^*(-) : \mathbf{D}(Y) \to \mathbf{D}(X)$ is fully faithful and thus $\mathbf{D}(Y)$ is an admissible subcategory of $\mathbf{D}(X)$.*

*Proof.* The first assertion is an immediate consequence of the projection formula (1.6). Indeed, the adjunction morphism $1_{\mathbf{D}(Y)} \to Rf_* \circ Lf^*$ yields isomorphism

$$F \to Rf_* \circ Lf^*(F) \cong F \overset{\mathbb{L}}{\otimes} Rf_*(\mathcal{O}_X) \cong F,$$

for any $F \in \mathbf{D}(Y)$. Hence $1_{\mathbf{D}(Y)} \cong Rf_* \circ Lf^*$ and therefore $Lf^*$ is fully faithful which is equivalent to the fact that $Lf^*$ admits a right adjoint, Proposition (1.3.2). □

There are two particular cases where this Proposition 3.2.2 can be applied: the projectivization of a vector bundle Proposition 3.1.1 (Orlov I) and the blow up of a variety. So if we consider the blow up $X = \mathsf{Bl}_Z(Y)$, Figure 3.1, then by the previous proposition the functor

$$Lf^* : \mathbf{D}(Y) \to \mathbf{D}(X)$$

is fully faithful.

If we denote $\mathbf{D}(Z)(k) = \Phi_{\mathcal{O}_E(k)}(\mathbf{D}(Z))$ and $\mathbf{D}(Y) = Lf^*(\mathbf{D}(Y))$, which makes sense because Proposition 3.2.1 and 3.2.2 tell us these functors are embeddings

**Theorem 3.1** (**Orlov II, Orlov's blow up formula**). *Let $X = \mathsf{Bl}_Z(Y)$ the blow up in Figure 3.1 . Then the Fourier-Mukai functors $\Phi_{\mathcal{O}_E(k)}$ and $Lf^*$ give the following s.o.d.*

$$\mathbf{D}(X) = \langle \mathbf{D}(Z)(1-c), \ldots, \mathbf{D}(Z)(-1), \mathbf{D}(Y) \rangle. \tag{3.5}$$

*Proof.* To show $\mathbf{D}(X)$ has the s.o.d. (3.5) we have to check that the categories $\mathbf{D}(Z)(k)$ and $\mathbf{D}(Y)$, $1-c \le k \le -1$, are admissible, semiorthogonal and generate $\mathbf{D}(X)$. By Proposition 3.2.1 and 3.2.2 these categories are admissible.

Let us prove that they are semiorthogonal, let $F \in \mathbf{D}(Z)$ and $k > l$ integers. Analogous computations as in (3.3) give us for the composition $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(l)}$

$$Rp_*(\mathcal{O}_E(l-k) \overset{\mathbb{L}}{\otimes} Lp^*(F)) \to \Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(l)}(F) \to Rp_*(\mathcal{O}_E(l-k-1) \overset{\mathbb{L}}{\otimes} Lp^*(F))[-1] \to \ldots$$

Recall that $Rp_*(\mathcal{O}_E(-t)) = 0$ for $1 \le t \le c-1$. As for $1-c \le l < k \le -1$ we have $1-c \le l-k, l-k-1 \le -1$ hence $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(l)}(F) \cong 0$, so $\Phi^!_{\mathcal{O}_E(k)} \circ \Phi_{\mathcal{O}_E(l)} = 0$ and this shows that the first $c-1$ components of (3.5) are semiorthogonal.

For the composition $Rf_* \circ \Phi_{\mathcal{O}_E(k)}$ we have

$$Rf_* \circ \Phi_{\mathcal{O}_E(k)}(F) = Rf_*(Ri_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F))) = Rj_*(Rp_*(\mathcal{O}_E(k) \overset{\mathbb{L}}{\otimes} Lp^*(F))) = Rj_*(Rp_*(\mathcal{O}_E(k)) \overset{\mathbb{L}}{\otimes} F) = 0,$$

since $Rp_*(\mathcal{O}_E(k)) = 0$ for $1-c \le k \le -1$, it follows that $\mathbf{D}(Z)(k)$ and $\mathbf{D}(Y)$ are semiorthogonal for $1-c \le k \le -1$.

It remains to show that the components we just described generated the whole category $\mathbf{D}(X)$, for this see [Huy06]. □

Roughly speaking, we can interpret Theorem 3.1 by saying that the "difference" between the derived categories of $X$ and $Y$ is given by a number of derived categories of subvarieties of codimension $\le n-2$, where $n = \mathsf{dim} X = \mathsf{dim} Y$.

As an application of Theorem 3.1 (Orlov II) we will see that if $X$ is a del Pezzo surface then, $\mathbf{D}(X)$ has a full exceptional collection, Example 2.2.5.

The theorem below classifies del Pezzo surfaces, [Kol99] or [Dol12].

**Theorem 3.2.** *Given a del Pezzo surface $X$ then:*

1. *If $\rho(X)$*[1] *$= 1$ then $X \cong \mathbb{P}^2$,*
2. *$X \cong \mathbb{P}^1 \times \mathbb{P}^1$, $X$ is the first minimal rational ruled surface $\mathsf{F}_0$ ,*
3. *$X$ is the blow up of $\mathbb{P}^2$ in at most 8 points in general position.*

If $X \cong \mathbb{P}^2$ then Corollary 2.2 gives us the desired s.o.d. In the other cases

**Corollary 3.1.** *Let $X$ be a del Pezzo surface such that $X \ncong \mathbb{P}^2$ then:*

1. *If $X$ is the blow up of $\mathbb{P}^2$ in at most 8 points then there is a s.o.d.*

$$\mathbf{D}(X) = \langle \mathbf{D}(p_1)(-1), \ldots, \mathbf{D}(p_N)(-1), \mathbf{D}(\mathbb{P}^2) \rangle$$

*where $Z = \{p_1, \ldots, p_N\} \subset \mathbb{P}^2$, $N \leq 8$ is the number of points such that $X = \mathsf{Bl}_Z(\mathbb{P}^2)$.*

2. *If $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ there is a s.o.d.*

$$\mathbf{D}(X) = \langle \mathbf{D}(\mathbb{P}^1)(-1), \mathbf{D}(\mathbb{P}^1) \rangle.$$

*Proof.* 1. Let $N$ the number of points where we are blowing up in $\mathbb{P}^2$, clearly $\mathbf{D}(Z) = \langle \mathbf{D}(p_1), \ldots \mathbf{D}(p_N) \rangle$ by Proposition 2.3.1. If $\mathbf{D}(Z)(-1) = \langle \mathbf{D}(p_1)(-1), \ldots, \mathbf{D}(p_N)(-1) \rangle$ then the first s.o.d. follows from Theorem 3.1 (Orlov II).

2. For the case $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. It is known that the minimal rational ruled surfaces are the projectivization of some vector bundles of rank 2 in $\mathbb{P}^1$, namely $\mathsf{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$, for $n \geq 0$ [Dol12]. Then the second s.o.d. follows immediately from Proposition 3.1.1.

□

The above results suggest the following definition.

**Definition 3.2.** Let $\mathcal{A}$ be a triangulated category. We define the geometric dimension of $\mathcal{A}$ as

$$\mathsf{gdim}(\mathcal{A}) = \min_{\substack{\mathcal{A} \hookrightarrow \mathbf{D}(X) \\ \mathsf{dim}\ X = k}} k$$

with $\mathcal{A} \hookrightarrow \mathbf{D}(X)$ is admissible and $X$ smooth projective and connected variety.

**Example 3.2.1.** If $\mathcal{A}$ is a triangulated category of geometric dimension 0 then $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$. Indeed, be definition $\mathcal{A}$ should be an admissible subcategory of the derived category of a smooth projective connected variety of dimension 0, i.e. of $\mathbf{D}(\mathsf{k}) = \mathbf{D}(\mathsf{Spec}(\mathsf{k}))$. But this category is indecomposable Corollary 2.6, hence $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$.

[1] The rank of $Pic(X)$, see [Kol99].

**Example 3.2.2.** If $\mathcal{A}$ is a indecomposable triangulated category of geometric dimension 1 then $\mathcal{A} \cong \mathbf{D}(C)$, where $C$ is a curve of genus $g \geq 1$. Indeed, by definition $\mathcal{A}$ should be an admissible subcategory of the derived category of a smooth projective connected variety of dimension 1, i.e. of $\mathbf{D}(C)$. If $g \geq 1$ then $\mathbf{D}(C)$ is indecomposable by Proposition 2.3.3 and so $\mathcal{A} = \mathbf{D}(C)$. If $g = 0$, i.e. $C \cong \mathbb{P}^1$, then again by Proposition 2.3.3 any nontrivial decomposition of $\mathbf{D}(C)$ consists of two exceptional objects, so if $\mathcal{A} \subset \mathbf{D}(C)$ is its indecomposable admissible subcategory then $\mathcal{A} \cong \mathbf{D}(\mathsf{k})$, but then its geometric dimension is 0.

A classification of triangulated categories of higher geometric dimension should be much more complicated. For instance, it was found out recently that some surfaces of general type with $p_g = q = 0$ (geometric genus and irregularity equal to zero) contain admissible subcategories with zero Hochschild homology (so called quasiphantom categories). This categories are highly nontrivial examples of categories of geometric dimension 2, [Kuz12].

## 3.3 Higher dimensional varieties

**A noncommutative $K3$ surface associated with a cubic fourfold.**

Interesting 4-dimensional varieties are cubic 4-folds. There are examples of cubic 4-folds which are known to be rational, but general cubic 4-folds are expected to be nonrational. In this section we will mention vaguely how does rationality of cubic 4-folds correlate with the structure of their derived categories.

For cubic 3-folds we have the following, see Clemens-Griffiths [CG72].

**Proposition 3.3.1.** *Let $Y \subset \mathbb{P}(V)$ be a smooth cubic 3-fold, by Example 2.2.3 we have a s.o.d.*

$$\mathbf{D}(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle.$$

*Then the category $\mathcal{A}_Y$ is highly non-trivial and cannot be the derived category of a smooth projective variety.*

*Proof.* Indeed, the Serre functor $\mathcal{S}_{\mathcal{A}_Y}$ is such that $\mathcal{S}_{\mathcal{A}_Y}^3 \cong [5]$ [Kuz15a], so $\mathcal{A}_Y$ is a fractional Calabi-Yau of dimension $\frac{5}{3}$ and cannot be the derived category of a smooth projective variety by (1.8). □

On the other hand, for a smooth cubic 4-fold it is not the case

**Proposition 3.3.2.** *Let $Y \subset \mathbb{P}^5$ be a smooth cubic 4-fold, by Example 2.2.4 one has*

$$\mathbf{D}(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y(-2), \mathcal{O}_Y(-1), \mathcal{O}_Y \rangle. \tag{3.6}$$

*The category $\mathcal{A}_Y$ is a connected Calabi-Yau category of dimension 2 with Hochschild homology isomorphic to that of $K3$ surfaces. In particular, $\mathcal{A}_Y$ is indecomposable and (3.6) is a maximal semiorthogonal decomposition.*

*Proof.* The proof of the fact that $\mathcal{S}_{\mathcal{A}_Y} \cong [2]$ can be found in [Kuz15a]. The Hochschild homology computation, is quite simple. One can compute the Hodge diamond of $Y$, see [GH78], and looks as

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Thus by HKR isomorphism, Theorem 1.5 we have $\mathsf{HH}_\bullet(\mathbf{D}(Y)) = \mathsf{k}[2] \oplus \mathsf{k}^{25} \oplus \mathsf{k}[-2]$. Since $\mathcal{A}_Y$ is the orthogonal complement of an exceptional collection of three objects in the category $\mathbf{D}(Y)$, by additivity of Hochschild homology, Theorem 2.2 it follows that

$$\mathsf{HH}_\bullet(\mathbf{D}(Y)) = \mathsf{k}[2] \oplus \mathsf{k}^{22} \oplus \mathsf{k}[-2].$$

By Theorem 2.2, again this coincides with the dimensions of Hochschild homology of $K3$ surfaces since the Hodge diamond of them looks as

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

Since $\mathcal{A}_Y$ is a 2-Calabi-Yau category there is an isomorphism $\mathsf{HH}_i(\mathcal{A}_Y) \cong \mathsf{HH}^{i+2}(\mathcal{A}_Y)$ see (1.10) and [Kuz15a]. Therefore, from the above description of Hochschild homology it follows that $\mathsf{HH}^0(\mathcal{A}_Y)$ is one-dimensional, i.e., the category $\mathcal{A}_Y$ is connected. Indecomposability of $\mathcal{A}_Y$ then follows from Propositionl 2.3.2 and the components generated by exceptional objects are indecomposable by Corollary 2.6.

□

Being Calabi-Yau category of dimension 2, the nontrivial component $\mathcal{A}_Y$ of $\mathbf{D}(Y)$ can be considered as a noncommutative $K3$ surface.

In retrospective, it is a classic result that smooth cubic 3-folds are not rational, by Clemens-Griffiths [CG72] and the Proposition 3.3.1 tell us that $\mathcal{A}_Y$ cannot be the derived category of a smooth projective variety, which does not happen for cubic 4-folds. This strange fact motivates the

**Conjecture 3.1** ([Kuz08])**.** *A cubic fourfold $Y$ is rational if and only if there is a smooth projective $K3$ surface $S$ and an equivalence $\mathcal{A}_Y \cong \mathbf{D}(S)$.*

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