CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE MATEMÁTICAS

Aplicación del método de perturbación a problemas de valuación de opciones europeas

T E S I S

QUE PRESENTA

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PARA OBTENER EL GRADO DE

MAESTRO EN CIENCIAS

EN LA ESPECIALIDAD DE

MATEMÁTICAS

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CIUDAD DE MÉXICO. MAYO, 2017

CENTER FOR RESEARCH AND ADVANCED STUDIES
OF THE NATIONAL POLYTECHNIC INSTITUTE

ZACATENCO CAMPUS
DEPARTMENT OF MATHEMATICS

Application of the perturbation method to valuation problems of European options

T H E S I S

SUBMITTED BY

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TO OBTAIN THE DEGREE OF

MASTER OF SCIENCE

IN THE SPECIALTY OF

MATHEMATICS

THESIS ADVISOR:

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MEXICO CITY. MAY, 2017

Dedicated to my parents.
For their endless love and support.

Resumen

En esta tesis se estudia el método de perturbación para la valuación de una opción europea cuando los precios del activo no siguen el modelo gaussiano de parámetros constantes; esto debido a observaciones empíricas que no estan presentes en el modelo clásico: la curva de la volatilidad implícita, y que la función de distribución tiene un mayor máximo, colas que decaen menos rápido y que son asimétricas. Se consideran dos modelos, el primero sigue siendo un modelo gaussiano, pero sus parámetros serán funciones continuas que dependen del tiempo; el segundo será un modelo de difusión con saltos y distribución de saltos doble exponencial. Para el primer modelo, las valuaciones son 5-8 veces mas rápidas que la valuación exacta y el error en el intervalo $[K - 15\%, K + 15\%]$ es menor a 0.1ε, donde K es el precio de ejercicio y ε es la magnitud de la perturbación. Para el segundo modelo, las aproximaciones son 50-60 veces mas rápidas que la valuación exacta y el error es menor a 0.5λ en el mismo intervalo, donde λ es la magnitud de la perturbación.

Abstract

This thesis will study the perturbation method for the valuation of an European call option when the asset prices are not governed by the Gaussian model with constant parameters; this will be due to empirical phenomena not well fitted by the classic model: the volatility smile, the higher peak and the assymetric fat tails of the return distribution of assets. Two models are considered, the first one remains a Gaussian model but the parameters will be assumed continuous time-dependent functions, the second one will be a double-exponential jump-diffusion. For the first model, the valuations are 5-8 times faster than the exact valuation and the error in the interval $[K - 15\%, K + 15\%]$ is less than 0.1ε, where K stands for the strike price and ε for the perturbation magnitude. For the second model, the approximations are 50-60 times faster than the exact valuation and the error is less than 0.5λ on the same interval, where λ stands for the perturbation magnitude.

Acknowledgements

I would like to thank all the people who directly or indirectly contributed to the realization of this thesis. First of all, I thank my advisor, Professor Sergei Grudskiy, for the guidance, the patience, and for encourage me to continue the pursuit of knowledge. To him, my most genuine admiration. I also want to thank the rest of my thesis committee: Prof. Michael Porter and Prof. Héctor Jasso, for their helpful observations, thoughtful advice and great classes.

My sincere appreciation goes to all the faculty and staff of the Mathematics Department at CINVESTAV, they have always provided a peaceful working environment. My recognition to all the researchers, they have been an example of hard work and perseverance.

I thank CONACyT for the financial support, and for keeping the state of science moving forward.

To professors Maribel Loaiza, Nikolai Vasilevski, Ruy Fabila and Onésimo Hernández-Lerma. Thank you for interesting courses and for sharing your time and work. I would also like to mention my gratitude to Prof. Carlos Pacheco, thanks for keeping your office door open when I needed it, your suggestions and encouragement were really valuable.

I want to thank Alberto, Ángel, Antonio, German, Rodolfo and Vladimir for the joyful conversations and fun times. Special thanks to Isaac for the sincere advice, the profound discussions and for becoming a great friend.

Last but not least, I would like to state my endless gratitude to my family: my parents Adrián and Virginia, who have always supported me and to whom I owe my entire life; to my brother Gustavo and my sister Linda, for being always there for me.

Contents

Chapter 1

Introduction

Is it well known that despite the success of the Black-Scholes model (which uses the geometric Brownian motion as model for the underlying asset price behavior), it fails into fitting two empirical phenomena observed in real life data. First, the tails of the return distribution are heavier than those of the normal distribution, also, the distribution has asymmetric decay. Second, the implied volatility is not constant, this fact is known as the volatility smile.

As a result, in the last years Lévy processes have become an indispensable tool when modelling market fluctuations, nevertheless, the first time a Lévy processes was used for this purpose seems far away from now. In 1900, Bachelier used the Brownian motion to evaluate stock prices in his PhD thesis *The Theory of Speculation*, this was indeed the first paper in mathematical finance. From a physicist's point of view, Osborne used the exponential of a Brownian motion, $\exp(W_t)$, as a stock price model in 1959, based on the Weber-Fetcher law. In 1965, Samuelson noticed that the formula obtained by Bachelier applied to a perpetual option leaded to an unbounded price, additionally, for an absolute Brownian motion, negative values for the stock prices were possible. In order to ensure non-negative valuations, Samuelson introduced the geometric Brownian motion.

The exponential Gaussian processes could give accurate results in short periods of time, however, as Mandelbrot observed (1963), the logarithm of some financial instruments have a heavy-tailed distribution, his solution was the introduction of a symmetric

α-stable Lévy motion with index $\alpha < 2$. This established the first pure-jump model. Press introduced a non-stable model combining a Brownian motion and an independent compound Poisson process with normally distributed jumps for the log price in 1967. Madan and Seneta used the *variance gamma process* as model for the log prices in 1987. Eberlein and Keller (1995) studied the exponential *hyperbolic Lévy motion* as a stock price model. In 1995, Bandorff-Nielsen, who introduced the *generalized hyperbolic distributions* in 1977, proposed an exponential *normal inverse Gaussian* Lévy process. Later, the whole family of generalized hyperbolic distributions was analyzed by Eberlein and Prause (1998).

The specialized needs of investors and the availability of preferential information to some of them gave rise to over the counter contracts in which the parts were committed to buy/sell stock at a fixed price. Options (not only of stocks) have been negotiated even before the existence of financial markets. It was until 1973 that the Chicago Board of Trade (CBOT) opened the Chicago Board Options Exchange (CBOE), the first regulated options exchange. At present, the derivatives markets trade not only options, and the number of options traded are billions. The regulation of the option trading brought with it a big question: *which is the fair price of a certain option?* Black and Scholes found this theoretical value for European options the same year the CBOT was open (1973), also in this year, Merton gave the option pricing formula for the jump model. Given that trading of American options started only one month before of the Black-Scholes model publication, the price of American options was not considered in the theory. In 1979, Cox, Ross and Rubestein introduced the Binomial Options Pricing Model, a discrete-time numerical method, slower than the Black-Scholes formula, but more accurate for longer-dated options and for not classical European options (American options, options with dividend payments, etc).

As it was mentioned before, the unsuitability of the Gaussian model with constant parameters in the actual financial market has developed the necessity of finding more complex models to fit stock prices; and the attempt to find the theoretical value of an option under non-Gaussian models has resulted in a generalization of the Black-Scholes equation: from a differential to a pseudo-differential problem. The analytic solution of such problem is not always in closed formulas, and most of the time the numerical valuation is not time-efficient, so precise and fast approximation methods are appreciated.

In this work, the perturbation method is considered as an approximation for the valuation problem of an European call option, and two specific models will be considered to see that under appropriate circumstances this method works well. The first case will be a Gaussian model whose parameters are continuous time-dependent functions. The second model for the log prices will be a Lévy process with characteristic exponent a rational function, namely, a *double-exponential jump-diffusion*, known as Kou model. The latter model fits the empirical phenomena stated before, and the parameters contained in it can be easily interpreted.

Chapter 2 will be dedicated to all the necessary financial and stochastic background.

In Chapter 3, the Gaussian model with continuous time-dependent parameters will be analyzed; for this scenario an explicit formula is known, but as mentioned early, the exact valuation is not time-efficient. As an example of the perturbation method, only the volatility will be considered as a continuous time-dependent function of the form $\sigma_0 (1 + \varepsilon\varphi(t))$, and several functions φ will be considered. In this case, two accurate and fast asymptotic approximations were found. The first one is linear respect to ε, so once the main terms are calculated, the valuation for different values of ε could be computed really fast. The second approximation is more precise and with computation times similar to the first one for a single value of ε, but as it is not linear, when a set of values of ε is fixed, the computation time for the new valuation is not negligible and is multiplied for the number of values of ε considered.

In Chapter 4, the Kou model will be used, as stated before, the price of the underlying assets will be modeled as exponentials of Lévy process; specifically, a Brownian motion plus a compound Poisson process with rate of jumps λ and an asymmetrical double exponential distribution. Again, an explicit formula for the European call option exists, expressed as an integral of an exponential function of the characteristic exponent; but it lacks a good numerical implementation as the integrand is highly oscillatory. A numerical valuation is attainable, thanks to the saddle point method, also known as the steepest descent method, given that the oscillation on the new contour is manageable. This valuation is very time consuming, so, a reliable and fast approximation for small values of λ will be developed using the perturbation method.

Finally, Chapter 5 contains a summary of the results obtained in this work.

Chapter 2

Preliminaries

2.1 The financial market

Financial markets are the mechanisms that allows people and organizations to trade negotiable instruments called *financial securities*. Some of the most common contracts are described below.

2.1.1 Bonds

Security debts in which the issuer must pay to the holder the deposit of the bond and some fixed or variable interest (or coupon) at a future date (called maturity). Bonds could be emitted by governments, regional public authorities, credit institutions, companies, or supranational institutions (World Bank, International Monetary Fund, Inter-American Development Bank...). The price of a fixed-rate bond at the time $t \in [0, T]$ is $B_t = B_0 e^{\int_0^t r(h)dh}$, $r(t) \geq 0$; where B_0 is the deposit or initial amount and $r(t)$ is the interest rate.

2.1.2 Commodities

Physical assets such as oil, gold, silver, natural gas, coffee beans, etc. Some commodities like silver, gold or copper have universal prices and are determined daily based on supply and demand. Oil, natural gas and electronic parts, for example, have levels of quality that affects their daily fluctuations.

2.1.3 Shares

A share of stock gives the holder a share of ownership in a corporation. Almost all shares can be traded freely and could give to the owner economic and political rights, such as dividends or voting rights on certain issues related to the company. The stock exchange value of a corporation is the price of all the shares of it. The price of a share is volatile and depends on many factors.

Before defining derivatives, let's mention two practices to trade financial securities:

- *Going long.* A long position is taken if the investor (holder) owns (buy) the security and will profit it if the price of the security goes up.
- *Going short.* It is the selling of a financial security that the seller does not own at the time of the sell. The seller hopes to buy the security at a lower price. The seller expects the price of the financial instrument to decline and to make a profit of this situation.

2.1.4 Derivatives

Derivatives are contracts whose values are derived from other financial instruments (bonds, stocks, commodities, loans, indexes...). The main types of derivatives are:

Swaps

Contracts where two counterparties are committed to exchange amounts of money at different periods of time, these amounts of money are called the legs of the swap.

Options

Options are contracts that give the holder the right (but not the obligation) to buy or sell some particular asset (the *underlying asset*) with spot price S, for an agreed price (*strike price, K*) in a later date (*exercise date*), but before the expiration date. The writer has the obligation to trade the asset if the holder wants to. If the holder has the right to buy the underlying, the options is named a *call option*; in the other hand, if the holder has the right to sell the asset, the option is a *put option*. The gain of money resulting of exercising the option is called the *payoff*.

Options are usually divided in different classes as follow.

Plain vanilla options Options whose payoff is given by

$\max\{0, S - K\}$ for a call option,

$\max\{0, K - S\}$ for a put option.

Plain vanilla options could be differentiated by their exercise date:

- *European options.* The holder has the right to exercise the option only at the expiry.
- *American options.* The holder has the right to exercise the option any time up to the expiration date.

Options with non-vanilla exercise rights Options with the same payoff of the plain vanilla options but different exercise dates, as examples we have:

- *Bermudan options.* The holder has the right to exercise at different fixed times before the expiry.
- *Canary options.* The holder has the right to exercise various times but not before a set time period.

Exotic options with vanilla exercise right Options with the same exercise date of the vanilla ones but with different payoffs. Examples:

- *Basket options.* The payoff depends on the weighted average of several underlyings.
- *Exchange options.* The holder has the right to trade an underlying for another one at the exercise date.

Exotic "path-dependent" options Options whose exercise date and/or payoff depend on the behavior of the underlying in a period of time. Examples:

- *Asian options.* The payoff is not given by the price of the underlying at the exercise date, but by the average of the price over some prefixed period of time.
- *Look-back options.* The payoff is determined by the lowest or highest price of the underlying over some period of time.
- *Binary options.* The payoff is either a fixed amount or nothing at all, depending on the underlying price at the expiry.
- *Barrier options.* The option becomes or vanishes if the underlying price reaches a fixed value.

Futures

Standardized contracts to buy or sell some financial instruments at a certain date in the future at a market determined price.

Forwards

Agreement between two counterparties to buy or sell some financial instruments in a fixed date in the future at a prefixed price. Forwards are different from futures because they are over-the-counter contracts (not regularized) and their price is an agreement, in contrast to the futures, whose price is determined by the market.

Futures and forwards may not be confused with options. Options give the right to buy or sell, in futures and forwards the trade must be done.

2.2 Lévy processes

In order to understand why Lévy processes provide useful models for market fluctuations we will study some basic definitions and properties. The demonstrations of the results in this section can be found on [7].

2.2.1 Probability measures

Let Ω be a set and $\mathfrak{F}$ a $\sigma-$*algebra* of Ω (a collection of subsets of Ω containing $\emptyset, \Omega$; and closed under complement, and numerable unions and intersections), the pair $(\Omega, \mathfrak{F})$ is called a *measurable space.*

A $\sigma-$*additive measure* is a map $\mu : \mathfrak{F} \longrightarrow [0, +\infty)$ that satisfy the conditions

- $\mu[\emptyset] = 0$
- $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ then $\mu[\cup_{n=1}^{\infty} A_n] = \sum_{n=1}^{\infty} \mu[A_n]$.

Additionally,

- if $\mu[\Omega] < +\infty$, the measure μ is called *finite*;
- if there is $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}$ such that $\mu[A_n] < +\infty$ for all n, and $\cup_{n=1}^{\infty} A_n = \Omega$, then the measure μ is called $\sigma-$*finite*.

The triple $(\Omega, \mathfrak{F}, \mu)$ is a *measure space*. Let $(\Omega, \mathfrak{F}, \mu)$, $(\Omega, \mathfrak{F}', \mu)$ be measure spaces, a function $f : \Omega \longrightarrow \Omega$ is said to be $(\mathfrak{F}, \mathfrak{F}')-$*measurable*, or shortly, a *measurable function* if $f^{-1}(A') \in \mathfrak{F}$ for all $A' \in \mathfrak{F}'$.

Let $\mathbb{P}$ be the measure corresponding to the triple $(\Omega, \mathfrak{F}, \mathbb{P})$; if $\mathbb{P}[\Omega] = 1$, $\mathbb{P}$ is called a *probability measure*, and $(\Omega, \mathfrak{F}, \mathbb{P})$ is said to be a *probability space*. The sets in $\mathfrak{F}$ are called *events*, and for any event $A \in \mathfrak{F}$, $\mathbb{P}[A]$ is the probability of the event A.

Let μ and ν be two $\sigma-$finite measures on $(\Omega, \mathfrak{F})$. μ is called *absolutely continuous* w.r.t. ν if for $A \in \mathfrak{F}$, $\nu[A] = 0$ implies $\mu[A] = 0$; in such case we denote $\mu << \nu$. In the case where $\mu << \nu$ and $\nu << \mu$, μ and ν are called *equivalent measures*.

Theorem 2.2.1 (*Radon – Nikodým*)

Let μ, ν be $\sigma-$finite measures on $(\Omega, \mathfrak{F})$. The following statements are equivalent

i. $\mu << \nu$.

ii. There is a non-negative measurable function f, such that $\mu = f\nu$. The function f is called the Radon-Nikodým derivative of μ relative to ν, and it is denoted by $f = \frac{d\mu}{d\nu}$.

The *Borel $\sigma-$algebra* of $\mathbf{R}^n$, denoted by $\mathcal{B}(\mathbf{R}^n)$, is the smallest $\sigma-$algebra that contains all the open sets in $\mathbf{R}^n$. A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is called *measurable* if it is $(\mathcal{B}(\mathbf{R}^n), \mathcal{B}(\mathbf{R}))-$measurable. If $\mathbb{P} << \lambda$, where λ is the Lebesgue measure, $p = \frac{d\mathbb{P}}{d\lambda}$ is called the *density* of the measure $\mathbb{P}$.

2.2.2 Random variables

A $\mathbf{R}^n-$valued *random variable* is a $(\mathfrak{F}, \mathcal{B}(\mathbf{R}^n))-$measurable mapping $X : \Omega \longrightarrow \mathbf{R}^n$. It will be denoted $\mathbb{P}[X \in B]$ for $\mathbb{P}[\omega \in \Omega : X(\omega) \in B]$. Thus, we define a probability measure on $\mathcal{B}(\mathbf{R}^n)$ by the map $\mathcal{B}(\mathbf{R}^n) \ni B \longmapsto \mathbb{P}[X \in B]$, such probability is called the *distribution* of X and it is denoted by $\mathbb{P}_X$.

Let X be a measurable function on a measure space, if $X(\omega)$ satisfies a property $\mathfrak{B}$ outside a 0 measure set, X is said to satisfy the property $\mathfrak{B}$ *almost everywhere* (a.e.). If the measure space is a probability space, instead of a.e., we will say that X satisfies the property $\mathfrak{B}$ *almost surely* (a.s.).

Les $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. If the set $\{\mathfrak{F}_k\}_{k=1}^{\infty}$ of $\sigma-$sub-algebras of $\mathfrak{F}$, satisfies $\mathbb{P}\left[\bigcap_{k=1}^{n} A_{j_k}\right] = \prod_{k=1}^{n} \mathbb{P}[A_{j_k}]$ whenever $A_{j_k} \in \mathfrak{F}_k$, and all j_k are distinct; the sub$-\sigma-$algebras $\mathfrak{F}_1, \mathfrak{F}_2, ...$ are said to be *independent*. Let X_j be a $\mathbf{R}^{n_j}$-valued random variable for $j = 1, ...$ the collection $\{X_j\}$ is called *independent* if the $\sigma-$algebras $\sigma(X_1), ...$ are *independent*.

Let X be a real-valued random variable, if the integral $\int_{\Omega} X(\omega) \mathbb{P}(d\omega)$ exists, it is called the *expectation* of X and it is denoted by $E[X]$. Given X and a measurable function g, the expectation of $g(X)$ (if exists) is given by
$E[g(X)] = \int_{\mathbf{R}^n} g(x) \mathbb{P}_X(d\omega)$. If $U \in \mathfrak{F}$ it will be written $E[X; U] := E[X\chi_U] := \int_U X(\omega) \mathbb{P}(d\omega)$.

Let $\mathfrak{F}'$ be a $\sigma-$sub-algebra of $\mathfrak{F}$, and X be a random variable such that
$E[|X|] < +\infty$. There is a random variable Y, called a *version* of the conditional expectation $E[X \mid \mathfrak{F}']$, that satisfies the following conditions:

i. Y is $\mathfrak{F}'-$measurable;

ii. $E[|Y|] < +\infty$;

iii. $E[Y;U] = E[X;U]$ for any $U \in \mathfrak{F}'$.

We denote $Y = E[X \mid \mathfrak{F}']$ for such random variable.

Example 2.2.1 *A random variable X follows a normal distribution with drift μ and variance σ^2, if its density is of the form*

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Its distribution is given by $\mathbb{P}_X(x) = \mathbb{P}(X \leq x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz.$

Example 2.2.2 *A positive random variable Y is said to follow an exponential distribution with parameter $\lambda > 0$ if it has a probability density function of the form $\lambda e^{-\lambda y}\chi_{y\geq 0}$. The distribution function of Y is given by*

$$\mathbb{P}_Y(y) = \mathbb{P}(Y \leq y) = 1 - e^{-\lambda y}.$$

2.2.3 Lévy processes

A stochastic process is a family $X = (X(t))_{t\geq 0}$ of $\mathbf{R}^n-$valued random variables, all of them defined on $(\Omega, \mathfrak{F}, \mathbb{P})$. Given $\omega \in \Omega$, a trajectory of the process X is a map $[0, +\infty) \ni t \longmapsto X(t, \omega) \in \mathbf{R}^n$.

The stochastic process Y is said to be a *modification* of the stochastic process X if $\mathbb{P}[X(t) = Y(t)] = 1$ for $t \in [0, +\infty)$.

Let Ω be a non-empty set, suppose that $\mathfrak{F}(t)$ is a $\sigma-$algebra for any $t \geq 0$ such that: if $s \leq t$, every set in $\mathfrak{F}(s)$ is also in $\mathfrak{F}(t)$. The collection of such $\sigma-$algebras, $\{\mathfrak{F}(t)\}_{t\geq 0}$, is called a *filtration*. An *adapted stochastic process* is a stochastic process $(X(t))_{t\geq 0}$ such that $X(t)$ is $\mathfrak{F}_t-$measurable, where $\mathfrak{F}(t)$ belongs to the filtration $\{\mathfrak{F}(t)\}_{t\geq 0}$.

Definition 2.2.1 *A stochastic process $(X(t))_{t\geq 0}$ is a Lévy process if the following properties are satisfied*

1. $X(0) = 0$ *a.s.;*

2. *given a trajectory of the process, almost surely, it is right-continuous on* $[0, +\infty)$ *and for any* $t > 0$ *the left limit exists;*

3. *it has independent increments, that is, for* $0 \leq t_0 < t_1 < ... < t_m$*, the random variables* $X(t_0), X(t_1) - X(t_0), ..., X(t_m) - X(t_{m-1})$ *are independent;*

4. *the distribution of* $X(t+s) - X(t)$ *does not depend on s;*

5. *it is stochastically continuous, that is,* $\lim_{s \longrightarrow t} \mathbb{P}[|X(s) - X(t)| > \varepsilon] = 0$ *for all* $t \geq 0$ *and* $\varepsilon > 0$.

If a stochastic process satisfy all the above conditions but (2), it is called a Lévy process in law, and it can be constructed a version of that process such that (2) is satisfied.

Example 2.2.3 *The classical example of Lévy process is the Brownian motion. Given a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, suppose that for each* $\omega \in \Omega$ *there is a continuous function* $W(t)$ *for* $t \geq 0$ *such that* $W(0) = 0$ *and that depends on* ω*;* $W = (W(t))_{t \geq 0}$ *is called a Brownian motion if for all* $0 = t_0 < t_1 < ... < t_m$ *the increments* $W(t_1) = W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, ... , $W(t_m) - W(t_{m-1})$ *are independent and each of these increments is normally distributed with* $E[W(t_{i+1}) - W(t_i)] = 0$ *and* $Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i$.

Example 2.2.4 *Let* $z_i, i \geq 1$ *be a sequence of independent exponential random variables with parameter* λ *and* $Z_n = \sum_{i=1}^{n} z_i$*. The process*$(N(t))_{t \geq 0}$ *defined by*

$$N(t) = \sum_{n \geq 1} \chi_{t \geq Z_n}$$

is called a Poisson process with intensity λ.

Example 2.2.5 *A compound Poisson process with intensity* $\lambda > 0$ *and jump size distribution* f *is a stochastic process* Z *defined as*

$$Z = \sum_{i=1}^{N_t} Y_i,$$

where the jump sizes Y_i *are i.i.d. with distribution* f*, and* $(N(t))_{t\geq 0}$ *is a Poisson process with intensity* λ*, independent from* $(Y_i)_{i\geq 1}$.

Another useful definition will be martingales. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, T a fixed positive number, and $\{\mathfrak{F}(t)\}_{t\in[0,T]}$ a filtration of sub$-\sigma-$algebras of $\mathfrak{F}$, an adapted stochastic process $(M(t))_{t\in[0,T]}$ is said to be a *martingale* if $E[M(t)|\mathfrak{F}(s)] = M(s)$ for all $0 \leq s \leq t \leq T$.

2.2.4 Laplace and Fourier transforms

Definition 2.2.2 *Let the function* $f(\tau)$ *be measurable in* $\tau \in (0, +\infty)$*, and such that* $|f(\tau)| \leq Me^{\delta\tau}$ *for some positive constants* M *and* δ*. The Laplace transformation of the function* $f(\tau)$ *is defined by*

$$[\mathcal{L}f](\omega) = \int_0^{+\infty} e^{-\omega\tau} f(\tau)\, d\tau \qquad \omega \in \mathbf{C}, \Re\omega > \delta.$$

The inverse formula is given by

$$f(\tau) = \frac{1}{2\pi i}\int_{\Gamma_b} [\mathcal{L}f](\omega)\, e^{\omega\tau} d\omega,$$

where $\Gamma_b = \{\xi \in \mathbf{C} : \Re\xi = b\}$ *for a fixed* $b > \delta$.

Definition 2.2.3 *The Fourier transform of a function* f*, denoted by* $\widehat{f} = \mathcal{F}f$*, is defined as*

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbf{R}^n} e^{-i\langle\xi,x\rangle} f(x)\, dx.$$

The inverse Fourier transform $\mathcal{F}^{-1}$ *is defined by*

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{i\langle\xi,x\rangle} \widehat{f}(\xi)\, dx.$$

The existence of the Fourier transform is assured for all the functions in $L_1(\mathbf{R}^n)$. If

μ is a measure on $\mathbf{R}^n$, its Fourier transform is:

$$\widehat{\mu}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle \xi, x\rangle} \mu(dx),$$

and exists for all finite measures. For the particular case of a probability measure $\mathbb{P}_Y$ with density p and absolutely continuous w.r.t. to the Lebesgue measure, the Fourier transform is given by $\widehat{\mathbb{P}}_Y(\xi) = \widehat{p}(\xi)$. The following one is a very similar definition used in probability theory.

2.2.5 Generating triplet and characteristic exponent

Definition 2.2.4 *The characteristic function of a measure μ is defined as*

$$\breve{\mu}(\xi) = \widehat{\mu}(-\xi);$$

and for a distribution of a random variable Y, as

$$E\left[e^{i\langle \xi, Y\rangle}\right] = \int_{\mathbf{R}^n} e^{i\langle \xi, Y\rangle} \mathbb{P}_Y(dx);$$

or $\breve{\mathbb{P}}_Y(\xi) = \widehat{\mathbb{P}}_Y(-\xi)$, if it is gd by $\breve{\mathbb{P}}_Y$.

Let the m-fold self-convolution of a probability measure μ be denoted by

$$\mu^m = \mu * \cdots * \mu \quad (m \text{ times}).$$

Definition 2.2.5 *The probability measure μ (on $\mathbf{R}^n$) is indefinitely divisible if, there is a probability measure μ_m (on $\mathbf{R}^n$) for any $m \in \mathbf{N}$ such that $\mu = \mu_m^m$.*

The characterization mentioned exists since all the Lévy processes are in a one-to-one correspondence to all the indefinitely divisible probability measures.

Lemma 2.2.1 *There exists a unique continuous function $\phi : \mathbf{R}^n \longrightarrow \mathbf{C}$ for each infinitely divisible μ, such that $\phi(0) = 0$ and* $\exp[\phi(\xi)] = \widehat{\mu}(\xi)$ $\forall t \geq 0$.

$\mu^t = \mathcal{F}^{-1} \exp[t\phi]$ *is well-defined and infinitely divisible, also there is a Lévy process in law,* X*, such that* $\mathbb{P}_{X_t} = \mu^t$.

Lemma 2.2.2 *Let* $(X_t)_{t\geq 0}$ *be a Lévy process in law on* $\mathbf{R}^n$*, then for any* $t \geq 0$*,* $\mathbb{P}_{X_t}$ *is infinitely divisible and* $\mathbb{P}_{X_t} = \mu^t$*, where* $\mu = \mathbb{P}_{X_1}$.

Theorem 2.2.2 *If* X *and* Y *are Lévy processes in law on* $\mathbf{R}^n$ *such that* $\mathbb{P}_{X_1} = \mathbb{P}_{Y_1}$*,* X *and* Y *are identical in law.*

Theorem 2.2.3 *Any* X *Lévy process in law on* $\mathbf{R}^n$ *has a modification that is a Lévy process.*

The last results give the following representation for the characteristic function of the distribution of the $n-$dimensional Lévy process $(X_t)_{t\geq 0}$, called the *Lévy-Khintchine formula*:

$$E\left[e^{i\langle \xi, X_t\rangle}\right] = e^{-t\psi(\xi)}, \qquad \xi \in \mathbf{R}^n, t \geq 0, \tag{2.1}$$

where $\psi(\xi)$ is called the *characteristic exponent* of X, and two Lévy processes have the same characteristic exponent if they have the same law.

Theorem 2.2.4 .

i. Let X *be a Lévy process on* $\mathbf{R}^n$*. Then its characteristic exponent admits the representation*

$$\psi(\xi) = \frac{1}{2}\langle B\xi, \xi\rangle - i\langle a, \xi\rangle - \int_{\mathbf{R}^n} \left(e^{i\langle x, \xi\rangle} - 1 - i\langle x, \xi\rangle \chi_{|x|\leq 1}(x)\right) \Pi(dx), \tag{2.2}$$

where B *is a symmetric non-negative-definite* $n \times n$ *matrix,* $a \in \mathbf{R}^n$*, and* Π *is a measure on* $\mathbf{R}^n$ *satisfying*

$$\Pi(\{0\}) = 0, \qquad \textstyle\int_{\mathbf{R}^n} \min\left\{|x|^2, 1\right\} \Pi(dx) < \infty. \tag{2.3}$$

ii. The representation (2.2) *is unique.*

iii. *Conversely, if B is a symmetric non-negative-definite $n \times n$ matrix, $a \in \mathbf{R}^n$, and Π is a measure on $\mathbf{R}^n$ satisfying* (2.3)*; then, there exists a Lévy process X defined by Equations* (2.1) *and* (2.2)*.*

The triple (a, B, Π) is called the *generating triplet* of X, and Π is called the *Lévy measure* of X.

Definition 2.2.6 *Let X be a Lévy process on $\mathbf{R}^n$ with generating triplet (a, B, Π), then, the infinitesimal generator of X is the operator*

$$\begin{aligned}[Lf](x) &= \frac{1}{2}\sum_{j,k=1}^{n} b_{j,k}\frac{\partial^2 f}{\partial x_j \partial x_k} + \langle a, f'(x)\rangle \\ &\quad + \int_{\mathbf{R}^n}\left(f(x+y) - f(x) - \chi_{|y|\le 1}(y)\langle y, f'(x)\rangle\right)\Pi(dy),\end{aligned}$$

where f is in the space of infinitely differentiable functions vanishing faster than any rational function at infinity: $S(\mathbf{R}^n)$.

For the one-dimensional case with generating triplet (μ, σ^2, Π):

$$\begin{aligned}Lf(x) &= \frac{\sigma^2}{2}f''(x) + \mu f'(x) \\ &\quad + \int_{\mathbf{R}}\left(f(x+y) - f(x) - \chi_{|y|\le 1}(y)\, y f'(x)\right)\Pi(dy) \\ \psi(\xi) &= \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbf{R}}\left(1 - e^{iy\xi} + iy\xi\chi_{|y|\le 1}(y)\right)\Pi(dy),\end{aligned}$$

so, using $f(x) = e^{ix\xi}$ $(f'(x) = i\xi e^{ix\xi},\ f''(x) = -\xi^2 e^{ix\xi})$

$$\begin{aligned}Le^{ix\xi} &= -\frac{\sigma^2}{2}\xi^2 e^{ix\xi} + i\mu\xi e^{ix\xi} \\ &\quad + \int_{\mathbf{R}}\left(e^{ix\xi}e^{iy\xi} - e^{ix\xi} - \chi_{|y|\le 1}(y)\, i\xi y e^{iy\xi}\right)\Pi(dy) \\ &= -e^{ix\xi}\left(\frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbf{R}}\left(1 - e^{iy\xi} + i\xi y\chi_{|y|\le 1}(y)\right)\Pi(dy)\right) \\ &= -e^{ix\xi}\psi(\xi) \qquad (2.4)\end{aligned}$$

2.2.6 Regular Lévy processes of exponential type

Definition 2.2.7 *Let $\delta_- < 0 < \delta_+$, and $\nu \in (0,2]$. A Lévy process is called a Regular Lévy process of Exponential type $[\delta_-, \delta_+]$ and order $\nu > 0$ (RLPE) if the following two conditions are satisfied:*

- *the characteristic exponent admits a representation*

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{2.5}$$

  where ϕ is holomorphic in the strip $\Im\xi \in (\delta_-, \delta_+)$, it is continuous up to the boundary of the strip, and it admits a representation

$$\phi(\xi) = c|\xi|^\nu + O(|\xi|^{\nu_1}), \tag{2.6}$$

  as $\xi \to \infty$ in the strip $\Im\xi \in [\delta_-, \delta_+]$, where $\nu_1 < \nu$;

- *there exist $\nu_2 < \nu$ and C such that the derivative of ϕ in Equation (2.5) admits a bound*

$$|\phi'(\xi)| \leq C(1+|\xi|)^{\nu_2}, \qquad \Im\xi \in [\delta_-, \delta_+]. \tag{2.7}$$

2.2.7 Equivalent martingale measures

Equivalent changes of measure take an important role in arbitrage pricing theory. The results in this section can be found in [2]. Let X be a Lévy process such that the stock price is modeled as $S_t = S_0 e^{X_t}$, for an initial known price S_0, and for all $t \geq 0$. Let S_t^* denote the discounted stock price, *i.e*, $S_t^* = e^{-\int_0^t r(h)dh} S_t$.

Proposition 2.2.1 *(Fundamental theorem of asset pricing) The market model defined by $(\Omega, \mathfrak{F}, \mathbb{P})$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free, if and only if, there exists an equivalent probability measure $\mathbb{Q}$ such that the discounted assets $(S_t^*)_{t\in[0,T]}$ are martingales with respect to $\mathbb{Q}$.*

Proposition 2.2.2 *(Second fundamental theorem of asset pricing) A market defined by* *the assets* $\left(S_t^0, S_t^1, ..., S_t^d\right)_{t\in[0,T]}$ *described as stochastic processes on* $(\Omega, \mathfrak{F}, \mathbb{P})$ *is complete, if and only if, there is a unique martingale measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$.

Definition 2.2.8 *An equivalent martingale measure (EMM) is a probability measure* $\mathbb{Q}$ *which is equivalent (in other words, it has the same null sets) to the given (historical) probability measure* $\mathbb{P}$ *and under which the process* S_t^* *is a martingale.*

Given an equivalent measure, the characteristic exponent under such measure must satisfies:

$$\begin{aligned} S_0 &= S_0^* = E^{\mathbb{Q}}\left[S_t^*\right] = S_0 E^{\mathbb{Q}}\left[e^{-rt}e^{X_t}\right] \\ &= S_0 e^{-rt} E^{\mathbb{Q}}\left[e^{X_t}\right] = S_0 e^{-rt} E^{\mathbb{Q}}\left[e^{i(-i)X_t}\right] \\ &= S_0 e^{-rt} e^{-t\psi^{\mathbb{Q}}(-i)} = S_0 e^{-t\left(\psi^{\mathbb{Q}}(-i)+r\right)} \end{aligned}$$

$$\Rightarrow$$

$$\psi^{\mathbb{Q}}\left(-i\right) + r = 0$$

which is known as the *EMM condition*.

2.2.8 Esscher transform

The equivalent measure most used in the finance field is the so-called Esscher transform, it was introduced in actuarial science since 1932 by F. Esscher. It is defined for a parameter $\theta \in \mathbf{R}$ by

$$\begin{aligned} \frac{d\mathbb{Q}|_{\mathfrak{F}_t}}{d\mathbb{P}|_{\mathfrak{F}_t}} &= \frac{e^{\theta X_t}}{E^{\mathbb{P}}\left[e^{\theta X_t}\right]} = \frac{e^{\theta X_t}}{E^{\mathbb{P}}\left[e^{i(-i\theta)X_t}\right]} \\ &= \frac{e^{\theta X_t}}{e^{-t\psi^{\mathbb{P}}(-i\theta)}} = \exp\left\{\theta X_t + t\psi^{\mathbb{P}}\left(-i\theta\right)\right\}. \end{aligned}$$

The Esscher transform is considered because the minimal entropy martingale (MEM) could be characterized as an Esscher transform, where the MEM is the solution of

$$\inf\left\{\mathcal{E}\left(\mathbb{Q},\mathbb{P}\right) \mid \mathbb{Q} \sim \mathbb{P} \text{ and } \mathbb{Q} \text{ is a martingale}\right\},$$

with $\mathcal{E}\left(\mathbb{Q},\mathbb{P}\right) = E^{\mathbb{Q}}\left[\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$ being the Kullback-Leibler distance (or relative entropy). As $f(x) = x \ln x$ is strictly convex, the relative entropy is a convex functional of $\mathbb{Q}$, and by the Jensen's inequality $\mathcal{E}(\mathbb{Q}, \mathbb{P}) \geq 0$ and $\mathcal{E}(\mathbb{Q}, \mathbb{P}) = 0$ if and only if $\frac{d\mathbb{Q}}{d\mathbb{P}} = 1$ $a.s.$

Given this EMM, the characteristic exponential of the model under the new measure is related to the one under the original measure as follow

$$\begin{aligned} e^{-t\psi^{\mathbb{Q}}(\xi)} &= E^{\mathbb{Q}}\left[e^{i\xi X_t}\right] = E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} e^{i\xi X_t}\right] = E^{\mathbb{P}}\left[e^{\theta X_t + t\psi^{\mathbb{P}}(-i\theta) + i\xi X_t}\right] \\ &= e^{t\psi^{\mathbb{P}}(-i\theta)} E^{\mathbb{P}}\left[e^{i(\xi - i\theta)X_t}\right] = e^{-t\left[\psi^{\mathbb{P}}(\xi - i\theta) - \psi^{\mathbb{P}}(-i\theta)\right]} \end{aligned}$$

$$\Leftrightarrow$$

$$\psi^{\mathbb{Q}}(\xi) = \psi^{\mathbb{P}}(\xi - i\theta) - \psi^{\mathbb{P}}(-i\theta). \tag{2.8}$$

The last equation give us a particular EMM condition for the Esscher transform

$$\psi^{\mathbb{Q}}(-i) = \psi^{\mathbb{P}}(-i(1+\theta)) - \psi^{\mathbb{P}}(-i\theta)$$

$$\psi^{\mathbb{P}}(-i(1+\theta)) - \psi^{\mathbb{P}}(-i\theta) + r = 0 \tag{2.9}$$

Lemma 2.2.3 *Let X be a RLPE of type $[\delta_-, \delta_+]$, with $\delta_+ - \delta_- > 1$, under the historic measure $\mathbb{P}$. Then,*

i. the function $f(\theta) = -\psi^{\mathbb{P}}(-i(1+\theta)) + \psi^{\mathbb{P}}(-i\theta)$ is strictly increasing on $[-\delta_+, -\delta_- - 1]$;

ii. the Equation (2.9) has at most one root on $[-\delta_+, -\delta_- - 1]$;

iii. the Equation (2.9) has a root on $(-\delta_+, -\delta_- - 1)$ if and only if

$$\lim_{\theta \to -\delta_+ + 0} f(\theta) < r < \lim_{\theta \to -(\delta_- - 1) - 0} f(\theta). \tag{2.10}$$

2.3 Stochastic calculus

Itô calculus has a lead role in the valuation of assets in continuous time, in the present section, the Itô integral and its properties are introduced. The demonstrations of the results in this section could be found in [8].

Given a contract that occurs in the interval of time $[0,T]$, for some positive real number T, we want to know the set of feasible events for each time $t \in [0,T]$, these sets must be $\sigma-$algebras, in fact, the stochastic process used to model the price fluctuations must be an adapted stochastic process to some filtration $\{\mathfrak{F}(t)\}_{t\in[0,T]}$.

It is desirable to obtain a meaning for $\int_0^T Y(t)\,dW(t)$, where $(W(t))_{t\geq 0}$ is a Brownian motion with a filtration $\{\mathfrak{F}(t)\}_{t\geq 0}$, and $(Y(t))_{t\geq 0}$ is an adapted stochastic process to the filtration $\{\mathfrak{F}(t)\}_{t\geq 0}$. The reason of such desire is that $Y(t)$ will be the position we take in an asset at time t, and this position depends on the path of the price path of the asset up to time t. The problem in finding that meaning is that Brownian motion paths cannot be differentiated with respect to time.

The typical construction of the Itô integral is the following. Let's start defining the Itô integral $I(t) = \int_0^t Y(u)\,dW(u)$ for simple processes $(Y(t))_{t\geq 0}$, those simple processes are defined with paths to be constant in the intervals $[t_j, t_{j+1})$ for a given partition $\Pi = \{t_0 = 0, t_1, t_2, ..., t_{n-1}, t_n = T\}$, $t_0 < t_1 < ... < t_n$. The Itô integral at time $t_k \leq t \leq t_{k+1}$ is defined by

$$I(t) = Y(t_k)\left[W(t) - W(t_k)\right] + \sum_{j=0}^{k-1} Y(t_j)\left[W(t_{j+1}) - W(t_j)\right]. \tag{2.11}$$

Theorem 2.3.1 *The Itô integral defined by* (2.11) *is a martingale.*

Theorem 2.3.2 *(Itô Isometry) The Itô integral defined by* (2.11) *satisfies*

$$E\left[I^2(t)\right] = E\left[\int_0^t Y^2(h)\,dh\right].$$

Definition 2.3.1 *Given a function $f(t)$ defined for $0 \leq t \leq T$, its quadratic variation up to time T is*

$$[f,f](T) = \lim_{\|\Pi\|\longrightarrow 0} \sum_{j=0}^{n-1} \left[f(t_{j+1}) - f(t_j)\right]^2.$$

Theorem 2.3.3 *The quadratic variation up to time T of a Brownian motion W is T almost surely for all $T \geq 0$.*

The last results could be interpreted as the statement that the Brownian motion accumulates quadratic variation at rate one per unit of time, and the two forms of writing this are

$$\begin{aligned} [W, W](T) &= T, \\ dW(t)\, dW(t) &= dt, \end{aligned}$$

Theorem 2.3.4 *The quadratic variation accumulated up to time t by the Itô integral* (2.11) *is*

$$[I, I](t) = \int_0^t Y^2(h)\, dh.$$

Using the differential expression of the Itô integral, $dI(t) = Y(t)\, dW(t)$, the last result can be written as

$$dI(t)\, dI(t) = Y^2(t)\, dW(t)\, dW(t) = Y^2(t)\, dt,$$

which means that the Itô integral accumulates quadratic variation at rate $Y^2(t)$ per unit of time.

Now, it is time to define the Itô integral for integrands that are allowed to vary continuously with time, and also to jump. Let $(Y(t))_{t\geq 0}$ be an adapted stochastic process to the filtration $\{\mathfrak{F}(t)\}_{t\geq 0}$, and assume that

$$E\left[\int_0^T Y^2(t)\, dt\right] < \infty. \tag{2.12}$$

Aiming to define the Itô integral, the process $(Y(t))_{t\geq 0}$ must be approximated by simple processes in the following way: first choose an arbitrary partition of the interval $[0, T]$, $0 = t_0 < t_1 < ... < t_m = T$, and use as approximation of $(Y(t))_{t\geq 0}$ the simple process with value $Y(t_j)$ in the interval $[t_j, t_{j+1})$ for each $j = 1, ..., m$. As the norm of the partition approaches to zero, the approximation will be more accurate. It is possible to choose a sequence of simple processes $(Y_n(t))_{t\geq 0}$ such that

$$\lim_{n\longrightarrow\infty} E\left[\int_0^T |Y_n(t) - Y(t)|^2\, dt\right] = 0.$$

The Itô integral of the process $(Y(t))_{t\geq 0}$ is defined by

$$I(t) = \int_0^t Y(h)\,dW(h) = \lim_{n\longrightarrow\infty}\int_0^t Y_n(h)\,dW(h)\,, \quad 0\leq t\leq T. \tag{2.13}$$

This definition has the same properties of the Itô integral for simple processes, this can be seen in the next result.

Theorem 2.3.5 *Let T be an arbitrary positive number, and let $(Y(t))_{t\geq 0}$ be an adapted stochastic process to the filtration $\{\mathfrak{F}(t)\}_{t\geq 0}$ that satisfies (2.12); the Itô integral defined in (2.13) has the following properties.*

i. *(Continuity) As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

ii. *(Adaptivity) For each t, $I(t)$ is $\mathfrak{F}-$measurable.*

iii. *(Linearity) If $I(t) = \int_0^t Y(u)\,dW(h)$ and $J(t) = \int_0^t Z(h)\,dW(h)$, then*

$$I(t) \pm J(t) = \int_0^t [Y(u) \pm Z(t)]\,dW(u)\,;$$

furthermore, for any constant c, $cI(t) = \int_0^t cY(u)\,dW(u)$.

iv. *(Martingale) $I(t)$ is a martingale.*

v. *(Itô isometry) $E[I^2(t)] = E\left[\int_0^t Y^2(h)\,dh\right]$.*

vi. *(Quadratic Variation) $[I,I](t) = \int_0^t Y^2(h)\,dh$.*

As we want to model the prices of assets with functions of stochastic processes, it is needed a rule to differentiate expressions like $f(W(t))$, where $f(x)$ is a differentiable function and $W(t)$ is a Brownian motion.

Theorem 2.3.6 *(Itô formula for Brownian motion) Let $f(x,t)$ be a function for which the partial derivatives $f_x(x,t)$, $f_t(x,t)$ and $f_{xx}(x,t)$ are defined and continuous, and* *let $W(t)$ be a Brownian motion, then, for every $t \geq 0$*

$$df(W(t), t) = f_t(W(t), t)\, dt + \frac{1}{2} f_{xx}(W(t), t)\, dt + f_x(W(t), t)\, dW(t).$$

The last result is very important, but in the practice we need a similar rule for more general processes, for example the processes covered by the next definition.

Definition 2.3.2 *Let $(W(t))_{t \geq 0}$ be a Brownian motion, an let $\{\mathfrak{F}(t)\}_{t \geq 0}$ be an associated filtration. An Itô process is a stochastic process, $(X(t))_{t \geq 0}$, of the form*

$$X(t) = X(0) + \int_0^t \sigma(h)\, dW(h) + \int_0^t \mu(h)\, dh, \tag{2.14}$$

where, $X(0)$ is non-random, and $\sigma(t)$ and $\mu(t)$ are adapted stochastic processes.

Usually Itô processes are also known as diffusion Itô processes. In finance, $\sigma(t)$ and $\mu(t)$ are called volatility and drift respectively.

Lemma 2.3.1 *The quadratic variation of the Itô process $(X(t))_{t \geq 0}$ is*

$$[X, X](t) = \int_0^t \sigma^2(h)\, dh.$$

The previous lemma gives us an equivalent expression for the Itô process $(X(t))_{t \geq 0}$ given by

$$dX(t) = \sigma(t)\, dW(t) + \mu(t)\, dt,$$

which is more used in the financial field. The next definition lets us integrate adapted processes respect to this wider class of stochastic processes.

Definition 2.3.3 *Let $(X(t))_{t \geq 0}$ be an Itô process represented as above, and let $(Y(t))_{t \geq 0}$ be an adapted process. The integral of $Y(t)$ respect to an Itô process is defined by*

$$\int_0^t Y(h)\, dX(h) = \int_0^t Y(h)\, \sigma(h)\, dW(h) + \int_0^t \mu(h)\, Y(h)\, dh.$$

And for this new integral a new differentiation rule arises.

Theorem 2.3.7 *(Itô formula for an Itô process) Let $(X(t))_{t\geq 0}$ be an Itô process represented as in (2.14), and let $f(x,t)$ be a function for which the partial derivatives $f_x(x,t)$, $f_t(x,t)$ and $f_{xx}(x,t)$ are defined and continuous, then, for every $t \geq 0$*

$$\begin{aligned} df(X(t),t) &= f_t(X(t),t)\,dt + f_x(X(t),t)\,dX(t) \\ &\quad + \frac{1}{2} f_{xx}(X(t),t)\,dX(t)\,dX(t). \end{aligned}$$

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, and let $\{\mathfrak{F}_t\}_{t\geq 0}$ be a filtration on this space. All processes will be adapted to this filtration. We want to define the stochastic integral

$$\int_0^t f(h)\,dX(h),$$

were the process X can have jumps. Let's consider in particular that X is right-continuous and of the form $X(t) = X(0) + I(t) + R(t) + J(t)$, where $X(0)$ is a non-random initial condition; the process

$$I(t) = \int_0^t \Gamma(h)\,dW(h)$$

is an Itô integral of an adapted process $\Gamma(h)$ with respect to a Brownian motion relative to the filtration. We will call $I(t)$ the Itô integral part of X. The process $R(t)$ is a Lebesgue integral

$$R(t) = \int_0^t \Theta(h)\,dh$$

for some adapted process $\Theta(t)$, and its called the *Lebesgue integral part* of X. The continuous part of $X(t)$ is defined to be

$$X^c(t) = X(0) + I(t) + R(t) = X(0) + \int_0^t \Gamma(h)\,dW(h) + \int_0^t \Theta(h)\,dh. \tag{2.15}$$

The quadratic variation of this process is

$$\begin{aligned} [X^c, X^c](t) &= \int_0^t \Gamma^2(h)\,dh \\ dX^c(t)\,dX^c(t) &= \Gamma^2(t). \end{aligned}$$

Finally, $J(t)$ is an adapted process, pure jump process such that $J(0) = 0$, and

$$J(t) = \lim_{h \downarrow t} J(h) \qquad \forall t \geq 0,$$

and is called the *pure jump part* of X. The left-continuous version of this process will be denoted by $J(t-)$. We assume that J does not jump at time zero, it has only finitely many jumps on each finite time interval $(0, T]$, and it is constant between jumps. Both the Poisson and the compound Poisson process have these properties.

Definition 2.3.4 *Let $(X(t))_{t \geq 0}$ be a jump process of the form (2.15), and with its parts described as above, such process will be called a jump process. Its continuous part is $X^c(t) = X(0) + I(t) + R(t)$, and for an adapted process $(Y(t))_{t \geq 0}$, the stochastic integral of $Y(t)$ respect to the jump process is defined by*

$$\int_0^t Y(h)\, dX(h) = \int_0^t Y(h)\, \Gamma(h)\, dW(h) + \int_0^t Y(h)\, \Theta(h)\, dh + \sum_{0 \leq h \leq t} Y(h)\, \Delta J(h).$$

Theorem 2.3.8 *(Itô formula for jump process) Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined continuous. Then*

$$\begin{aligned} f(X(t)) \quad &= \quad f(X(0)) + \int_0^t f'(X(h))\, dX^c(h) + \frac{1}{2}\int_0^t f''(X(h))\, dX^c(h)\, dX^c(h) \\ &\quad + \sum_{0 \leq h \leq t} \left[f(X(h)) - f(X(h-)) \right], \end{aligned}$$

where X^c denotes the continuous part of X, and $X(t-)$ is the left continuous version of $X(t)$.

Chapter 3

The Gaussian model

Let's use the sub-index notation to express the dependency of time. Let $V(S_t, t)$ be the value of an European call option with payoff $V(S_T, T) = \varphi(S_T)$, where T is the expiry date. The follow assumptions about the market are made:

- The trading takes place continuously in time.
- Unrestricted borrowing and lending of funds are possible at the same interest rate at a specified time.
- The market is frictionless, that is, there are no transaction costs or taxes.
- There is no discrimination against the short sale.

3.1 The Black-Scholes model

The reasoning and the results in the following two sections can be found in [11]. Let's consider a market with two assets, the first one a risk-less bond of a bank account given by

$$dB_t = r(t) B_t dt, \;\; B_0 = 1$$
$$B_t = \exp\left\{\int_0^t r(h)\, dh\right\}.$$

The second, a stock with the stochastic differential equation

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t, \tag{3.1}$$

for a fixed S_0; where $(W_t)_{t\geq 0}$ is a Brownian motion, and S_t is on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let's assume that r, μ and σ are measurable functions in t. The last equation can also be written as

$$S_t = S_0 + \int_0^t \mu(h) S_h dh + \int_0^t \sigma(h) S_h dW_h \qquad \forall t \in [0, T], \tag{3.2}$$

where the first integral is a Lebesgue integral, and the second one is understood in the Itô sense, and both are assumed well-defined.

Lemma 3.1.1 *The process given by the formula*

$$S_t = S_0 \exp\left\{ \int_0^t \sigma(h) dW_h + \int_0^t \left(\mu(h) - \frac{\sigma^2(h)}{2} \right) dh \right\} \qquad \forall t \in [0, T], \tag{3.3}$$

is the unique solution of the stochastic differential equation (3.1)*, or equivalently, the Itô integral equation* (3.2)*.*

Let $\phi(t) = (\phi_1(t), \phi_2(t)) \in \mathbf{R}^2$ represent the portfolio of an investor with a short position in one call option, with $\phi_1(t)$ being the amount of shares of stocks held at time t, and $\phi_2(t)$ be the amount of money deposited on a bank account or borrowed from a bank. Let $M(\phi(t))$ denote the wealth of the portfolio at time t, therefore

$$M(\phi(t)) = \phi_1(t) S_t + \phi_2(t) B_t.$$

Definition 3.1.1 *A trading strategy $\phi(t)$ is called self-financing if its wealth process $M(\phi(t))$ satisfies*

$$dM(\phi(t)) = \phi_1(t) dS_t + \phi_2(h) dB_h \qquad \forall t \in [0, T],$$

or in integral form

$$M(\phi(t)) = M(\phi(0)) + \int_0^t \phi_1(h) dS_h + \int_0^t \phi_2(h) dB_h \qquad \forall t \in [0, T],$$

where again, the first integral is understood in the Itô sense, and the second one is a

Lebesgue integral, and both are assumed well-defined.

Since $M(\phi(t)) - \phi_1(t) S_t = \phi_2(t) B_t$,

$$dM(\phi(t)) = (r(t) M(\phi(t)) + (\mu(t) - r(t)) \phi_1(t) S_t) dt + \phi_1(t) \sigma(t) S_t dW_t.$$

Let's introduce $\gamma_t = B_t^{-1} = \exp\left\{-\int_0^t r(h) dh\right\}$, then the discounted stock price is

$$S_t^* = \gamma_t S_t = S_0^* \exp\left\{\int_0^t \sigma(h) W_h + \int_0^t \left(\mu(h) - r(h) - \frac{\sigma^2(h)}{2}\right) dh\right\},$$

and

$$d(\gamma_t M(\phi(t))) = (\mu(t) - r(t)) \phi_1(t) \gamma_t S_t dt + \phi_1(t) \sigma(t) \gamma_t S_t dW_t,$$

or in integral form

$$\gamma_t M(\phi(t)) = M(\phi(0)) + \int_0^t (\mu(h) - r(h)) \phi_1(h) \gamma_h S_h dh + \int_0^t \phi_1(h) \sigma(h) \gamma_h S_h dW_h.$$

$M^*(\phi(t)) = \gamma_t M(\phi(t))$ is the discounted wealth process and $\overline{M}^*(\phi(t)) = \gamma_t M(\phi(t)) - M(\phi(0))$ is called the discounted gain process.

Definition 3.1.2 *A trading strategy $\phi(t)$ is called an arbitrage opportunity if*

$$\mathbb{P}[M^*(\phi(t)) \geq 0] = 1$$

and

$$\mathbb{P}[M^*(\phi(t)) > 0] > 0.$$

Additionally, we call a market arbitrage-free if no such portfolio exists in it.

Since the fundamental theorem of asset pricing give us a sufficient condition to check for a market to be arbitrage-free, we will find such EMM. Let's introduce the function

$$\alpha(t) = \frac{r(t) - \mu(t)}{\sigma(t)},$$

and suppose that μ, r and σ are such that

$$\int_0^T |\alpha(h)|^2 \, dh < \infty. \tag{3.4}$$

Let's consider the exponential process

$$z_t = \exp\left\{ \int_0^t \alpha(h) \, dW_t - \frac{1}{2} \int_0^t \alpha^2(h) \, dh \right\}.$$

Lemma 3.1.2 *The unique martingale measure $\mathbb{Q}$ for the discounted stock price process S^* is given by the Radon-Nikodým derivative*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = z_T.$$

Under the martingale measure $\mathbb{Q}$, the discounted stock price S^* satisfies

$$dS_t^* = \sigma(t) \, S_t^* dW_t^*,$$

and the process $W_t^* = W_t - \int_0^t \alpha(h)$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

Hence, the market is both arbitrage-free and complete if the condition (3.4) holds.

Definition 3.1.3 *A trading strategy $\phi(t)$ is called $\mathbb{Q}$-admissible if the discounted wealth process $M^*(\phi(t))$ follows a martingale under $\mathbb{Q}$. The class of all $\mathbb{Q}$-admissible trading strategies will be denoted by $\Phi(\mathbb{Q})$.*

The triple $[S, B, \Phi(\mathbb{Q})]$ is called the arbitrage-free Black-Scholes model of a financial market, or briefly, the Black-Scholes model.

3.2 The Black-Scholes formula

For the particular case when μ, r and σ are continuous functions of time, (3.4) holds, then the considered market $[S, B, \Phi(\mathbb{Q})]$ is a Black-Scholes model.

Definition 3.2.1 *An European contingent claim Y that settles at time T, is attainable in the Black-Scholes model if it can be replicated by the means of an admissible strategy, i.e, there exists ϕ such that $M(\phi(T)) = Y$.*

Given an attainable European contingent claim Y that settles at time T, we denote its arbitrage price as $\pi_t(Y) = M(\phi(t))$.

Proposition 3.2.1 *Let Y be an attainable European contingent claim that settles at time T. Then the arbitrage price $\pi_t(Y)$ at time $t \in [0,T]$ in the Black-Scholes model $[S, B, \Phi(\mathbb{Q})]$ is given by the risk-neutral valuation formula*

$$\pi_t(Y) = B_t E^{\mathbb{Q}}[\gamma_T Y \mid \mathfrak{F}_t] \qquad \forall t \in [0,T].$$

Let's consider an European call option written on a stock S with expiry date T, strike price K, and payoff $\varphi(S_T) = (S_T - K)_+$. If $\widehat{\gamma}_t = \exp\left\{-\int_t^T r(h)\,dh\right\}$, then the option price can be calculated as

$$V(S_t, t) = E^{\mathbb{Q}}[\widehat{\gamma}_t \varphi(S_T) \mid \mathfrak{F}_t]. \tag{3.5}$$

Let's note that

$$dW_t = dW_t^* + \alpha(t)\,dt,$$

then

$$S_T = S_t \exp\left\{\int_t^T \sigma(h)\,dW_h^* + \int_t^T \left(r(h) - \frac{\sigma^2(h)}{2}\right)dh\right\}.$$

Let $\Sigma = \int_t^T \sigma^2(h)\,dh$, and $l = \int_t^T \sigma(h)\,dW_h^*$; then its probability density under the measure $\mathbb{Q}$ is a normal distribution with volatility $\sqrt{\Sigma}$. Then (3.5) can be rewritten as

$$V(S_t, t) = \frac{\widehat{\gamma}_t}{\sqrt{2\pi\Sigma}} \int_{\mathbf{R}} \varphi\left(S_t \exp\left\{l + \int_t^T \left(r(h) - \frac{\sigma^2(h)}{2}\right)dh\right\}\right) \exp\left\{-\frac{l^2}{2\Sigma}\right\} dl.$$

Since

$$S_T \geq K \Leftrightarrow -D_2(t) = -\ln\frac{S_t}{K} - \int_t^T \left(r(h) - \frac{\sigma^2(h)}{2}\right)dh \;\leq\; \int_t^T \sigma(h)\,dW_h^*,$$

$$V(S_t,t)=\frac{\widehat{\gamma}_t}{\sqrt{2\pi\Sigma}}\int_{-D_2(t)}^{+\infty}\left(S_t\exp\left\{\begin{array}{c}\int_t^T\left(r(h)-\frac{\sigma^2(h)}{2}\right)dh\\+l\end{array}\right\}-K\right)\exp\left\{-\frac{l^2}{2\Sigma}\right\}dl.$$

Making the change of variable $L=\frac{l}{\sqrt{\Sigma}}$ $\left(-d_2(t)=\frac{-D_2(t)}{\sqrt{\Sigma}}\right)$,

$$V(S_t,t)=\left[\begin{array}{c}S_t\int_{-d_2(t)}^{+\infty}\exp\left\{-\frac{\left(L-\sqrt{\Sigma}\right)^2}{2}\right\}dL\\-\widehat{\gamma}_tK\Phi(d_2(t))\end{array}\right].$$

With the translation $\widehat{L}=L-\sqrt{\Sigma}$,

$$V(S_t,t)=S_t\Phi(d_1(t))-\exp\left\{-\int_t^T r(h)\,dh\right\}K\Phi(d_2(t)) \tag{3.6}$$

$$\begin{aligned}d_1(t)&=\frac{\ln\frac{S_t}{K}+\int_t^T r(h)\,dh}{\sqrt{\int_t^T\sigma^2(h)\,dh}}+\frac{1}{2}\sqrt{\int_t^T\sigma^2(h)\,dh}\\d_2(t)&=\frac{\ln\frac{S_t}{K}+\int_t^T r(h)\,dh}{\sqrt{\int_t^T\sigma^2(h)\,dh}}-\frac{1}{2}\sqrt{\int_t^T\sigma^2(h)\,dh}\end{aligned}$$

which is the Black-Scholes pricing formula for an European call option.

3.3 Constant parameters

The classical Black-Scholes scenario is when μ, r and σ are constants. In particular, the valuation of an European call option is

$$\begin{aligned}u(S,t)&=S\Phi(d_1)-e^{-r(T-t)}K\Phi(d_2)\\d_1&=\frac{\ln\frac{S}{K}+r(T-t)}{\sigma\sqrt{(T-t)}}+\frac{1}{2}\sigma\sqrt{(T-t)}\\d_2&=\frac{\ln\frac{S}{K}+r(T-t)}{\sigma\sqrt{(T-t)}}-\frac{1}{2}\sigma\sqrt{(T-t)}\end{aligned}$$

An example is given in Figure 3.1.

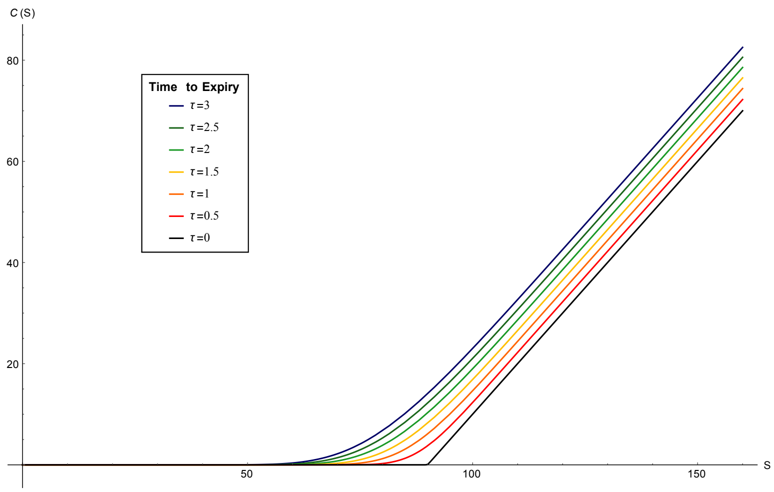


Figure 3.1: The European call value $C(S,t)$ as a function of several values of time to expiry and constant parameters: $\sigma = 0.1$, $r = 0.05$, $K = 0.9$, $0 \leq t \leq 3 = T$.

3.4 Proportional dividends

In this case a new supposition is considered: in a time dt the underlying asset pays out a dividend $D(S,t) = \widehat{D}(t)\,S$. Let

$$\widehat{S} = Se^{-\int_t^T \widehat{D}(\theta)d\theta}, \tag{3.7}$$

the Itô formula gives a representation of this new process:

$$d\widehat{S} = \mu(t)\,\widehat{S}dt + \sigma(t)\,\widehat{S}dW. \tag{3.8}$$

(3.1) and (3.8) are very similar, with $\widehat{S}$ instead of S, so the effect of the dividends on the asset price is that it is discounted by $e^{-\int_t^T \overline{D}(\theta)d\theta}$, moreover, with the change of variable (3.7), at the maturity $S = \widehat{S}$, so the payoff for a call option is the same, hence, the Equation (3.6) holds and the solution is given by

$$\begin{aligned} u(S,t) &= \widehat{S}\Phi\left(\widehat{d}_1(t)\right) - \exp\left\{-\int_t^T r(h)\,dh\right\} K\Phi\left(\widehat{d}_2(t)\right) \\ &= Se^{-\int_t^T \widehat{D}(\theta)d\theta}\Phi\left(\widehat{d}_1(t)\right) - Ke^{-\int_t^T r(h)dh}\Phi\left(\widehat{d}_2(t)\right), \end{aligned}$$

where

$$\widehat{d}_1(t) = \frac{\ln\left(\frac{S}{K}\right) - \int_t^T \widehat{D}(\theta)\, d\theta + \int_t^T r(h)\, dh}{\sqrt{\int_t^T \sigma^2(h)\, dh}} + \frac{1}{2}\sqrt{\int_t^T \sigma^2(h)\, dh},$$

$$\widehat{d}_2(t) = \frac{\ln\left(\frac{S}{K}\right) - \int_t^T \widehat{D}(\theta)\, d\theta + \int_t^T r(h)\, dh}{\sqrt{\int_t^T \sigma^2(h)\, dh}} - \frac{1}{2}\sqrt{\int_t^T \sigma^2(h)\, dh}.$$

Examples are given in Figures 3.2 - 3.5.

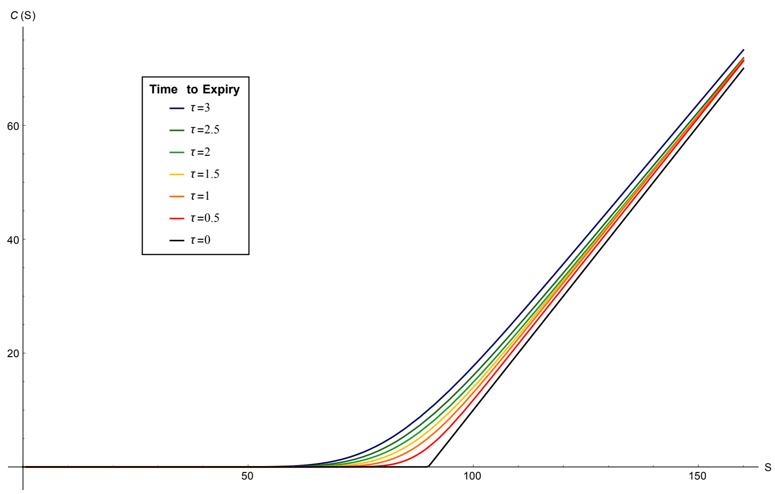


Figure 3.2: The European call value with parameters $\sigma = 0.1$, $r = 0.05$, $K = 90$, $0 \leq t \leq 3 = T$ and dividends $D = D_0 \sin t$, $D_0 = 0.03$.

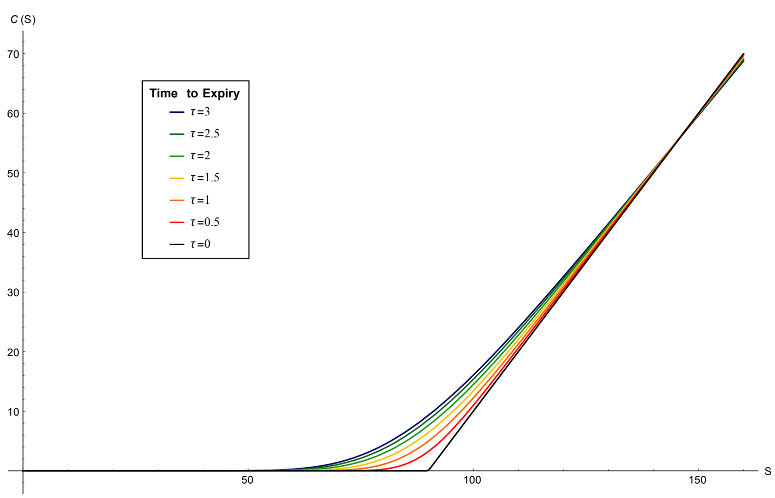


Figure 3.3: The European call value with parameters, $r = 0.05$, $K = 90$, $D(t) = D_0 = 0.03$, $\sigma(t) = 0.1(1 + 0.3 \sin t)$, $0 \leq t \leq 3 = T$.

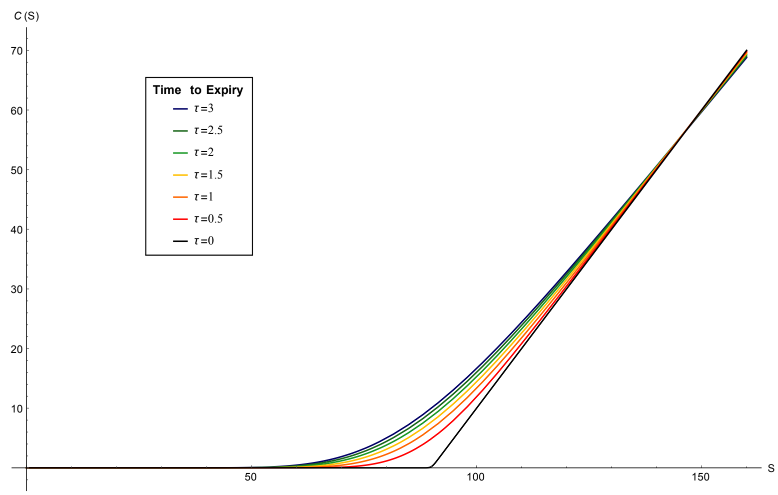


Figure 3.4: The European call value with parameters, $r = 0.05$, $K = 90$, $D(t) = D_0 = 0.03$, $\sigma(t) = 0.1(1 + 0.05e^t)$, $0 \leq t \leq 3 = T$.

3.5 Perturbation method

As an example of the perturbation method, the asymptotic expansion for an European call option without dividends and constant parameters but $\sigma(t)$ will be considered, it will be supposed that $\sigma(t)$ is of the form $\sigma_0(1 + \varepsilon\varphi(t))$.

Initially, the term containing $\sigma(t)$ will be expanded:

$$\int_t^T \sigma^2(\theta)\, d\theta = \sigma_0^2(T-t) + 2\varepsilon\sigma_0^2 \int_t^T \varphi(\theta) + O\left(\varepsilon^2\right)$$

Now, the volatility terms included in d_1 and d_2 will be approximated using the Taylor's series of $\sqrt{1+\delta}$ and $\frac{1}{1+\delta}$:

$$\begin{aligned}
\left(\int_t^T \sigma^2(\theta)\, d\theta\right)^{1/2} &= \sigma_0\sqrt{T-t}\left(1 + \frac{\varepsilon}{T-t}\int_t^T \varphi(\theta)\, d\theta\right) + O\left(\varepsilon^2\right) \\
\left(\int_t^T \sigma^2(\theta)\, d\theta\right)^{-1/2} &= \frac{1}{\sigma_0\sqrt{T-t}}\left(1 - \frac{\varepsilon}{T-t}\int_t^T \varphi(\theta)\, d\theta\right) + O\left(\varepsilon^2\right)
\end{aligned}$$

It is important to verify the values of ε where the conditions for the approximations are valid, *i.e.* the ε for which the second derivatives are bounded:

$$\left|\frac{2\varepsilon}{T-t}\int_t^T \varphi(\theta)\, d\theta\right| < 1. \tag{3.9}$$

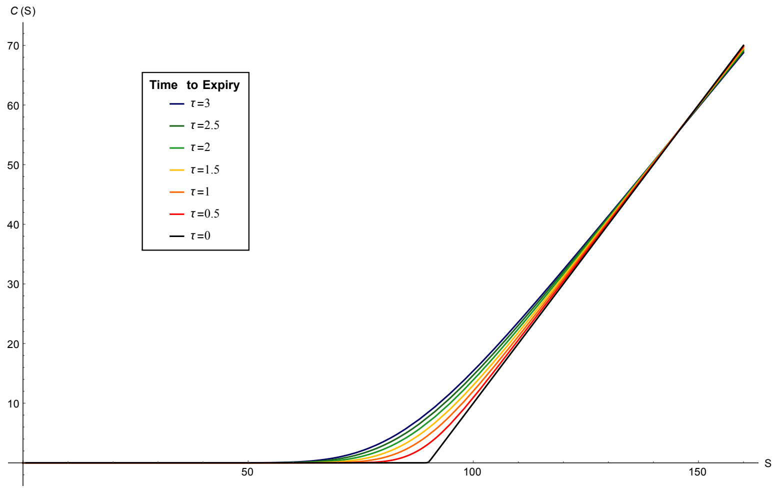


Figure 3.5: The European call value with parameters, $r = 0.05$, $K = 90$, $D(t) = D_0 = 0.03$, $\sigma(t) = 0.1(1 + 0.03t)$, $0 \leq t \leq 3 = T$.

Using these approximations in $d_{1,2}$ we have

$$\begin{aligned}
d_{1,2}(\varepsilon) &= \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma_0\sqrt{T-t}} \pm \frac{1}{2}\sigma_0\sqrt{T-t} \\
&\quad + \frac{\varepsilon}{T-t}\int_t^T \varphi(\theta)\,d\theta\left(-\frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma_0\sqrt{T-t}} \pm \frac{1}{2}\sigma_0\sqrt{T-t}\right) + O\left(\varepsilon^2\right) \\
&= d_{1,2}(0) + \varepsilon\frac{-d_{2,1}(0)}{T-t}\int_t^T \varphi(\theta)\,d\theta + O\left(\varepsilon^2\right).
\end{aligned}$$

Now let's work the approximation one step further, that is, taking the approximation of the cumulative function in terms of the one obtained in the Constant parameters Section,

$$\Phi\left(d_{1,2}(\varepsilon)\right) = \Phi\left(d_{1,2}(0)\right) + \varepsilon\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(d_{1,2}(0)\right)^2}{2}\right)\frac{\partial d_{1,2}}{\partial\varepsilon}(0) + O\left(\varepsilon^2\right)$$

where

$$\frac{\partial d_{1,2}}{\partial\varepsilon} = -\frac{d_{2,1}(0)}{T-t}\int_t^T \varphi(\theta)\,d\theta.$$

With

$$I_{1,2} = -\frac{1}{\sqrt{2\pi}}\frac{d_{2,1}(0)}{T-t}\exp\left(-\frac{\left(d_{1,2}(0)\right)^2}{2}\right)\int_t^T \varphi(\theta)\,d\theta,$$

then

$$\Phi\left(d_{1,2}\left(\varepsilon\right)\right)=\Phi\left(d_{1,2}\left(0\right)\right)+\varepsilon I_{1,2}+O\left(\varepsilon^{2}\right),$$

and the solution is approximated by

$$\begin{aligned} u\left(S,t\right) &= S\Phi\left(d_{1}\left(\varepsilon\right)\right)-Ke^{-r(T-t)}\Phi\left(d_{2}\left(\varepsilon\right)\right)+O\left(\varepsilon^{2}\right) \\ &= u_{0}\left(S,t\right)+\varepsilon\left[SI_{1}-Ke^{-r(T-t)}I_{2}\right]+O\left(\varepsilon^{2}\right) \end{aligned} \tag{3.10}$$

$$u_{asy}=u_{0}+\varepsilon\left[SI_{1}-Ke^{-r(T-t)}I_{2}\right],$$

where u_0 is the one given in Section 3.3. Denoting

$$u_{1}\left(S,t\right)=SI_{1}\left(S,t\right)-Ke^{-r(T-t)}I_{2}\left(S,t\right),$$

the expansion is given by

$$u_{asy}=u_{0}+\varepsilon u_{1}.$$

And it relates to the theoretical valuation as

$$u=u_{asy}+O\left(\varepsilon^{2}\right)$$

As the corresponding figures will show, this linear approximation is much better than to just consider the volatility as a constant parameter.

From (3.10) an alternative approximation is considered:

$$u_{alt}\left(S,t\right)=S\Phi\left(d_{1}\left(\varepsilon\right)\right)-K\exp\left(-r\left(T-t\right)\right)\Phi\left(d_{2}\left(\varepsilon\right)\right),$$

where,

$$u=u_{alt}+O\left(\varepsilon^{2}\right).$$

In all the examples, the option will be considered with a maturity of 3 years, strike price 90, interest rate 0.05, and no dividend returns. The volatility will be $\sigma_0 = 0.1$ affected by a different perturbation in each case. The option will be valuated

- at time $t = 1$ (two years before expiry);
- at discrete increments of size 0.25 in the interval $S \in [80, 110]$, that is, 120 points;
- considered as a constant parameter case (Classic), using the exact formulas for the case with time-dependent parameters, and using the two approximations considered (Asymptotic, Alternative);
- for 10 perturbation magnitudes, these magnitudes will be according to the appropriate ε given by (3.9) and specified in each case.

Tables comparing the computation time taken to do the calculations, and plots showing the percent error (against the exact valuation) are shown. The calculations were performed using *Mathematica* [15] on a PC with Intel®Core™ 2 Duo P8600 (2.40GHz, 3MB L2 Cache, 1066MHz FSB), under an Arch Linux 64-bit operating system with 8GB of RAM.

3.5.1 Sine Case: $\sigma(t) = \sigma_0 (1 + \varepsilon \sin(t))$

For this case $\varepsilon \leq 0.65$.

Table 3.1: Computation times (Seconds)

		Perturbation:	$\mathfrak{T} = \mathfrak{T}(C_0) + \mathfrak{T}(C_1) + \sum \mathfrak{T}(\varepsilon)$
		Classic term:	$\mathfrak{T}(C_0) = 0.013147$
		Linear term:	$\mathfrak{T}(C_1) = 2.04989$
ε	**Theoretical**	$\mathfrak{T}(\varepsilon)$ (ms)	**Alternative**
0.50	16.2249	+0.945	2.06025
0.45	15.7925	+0.986	2.06688
0.40	16.6018	+0.997	2.26601
0.35	16.7844	+1.060	2.15497
0.30	16.3418	+1.096	2.13372
0.25	16.1353	+1.131	2.07219
0.20	15.8643	+1.513	2.03498
0.15	15.7780	+1.237	2.06195
0.10	15.6929	+1.163	2.07520
0.05	15.9888	+1.175	2.05208

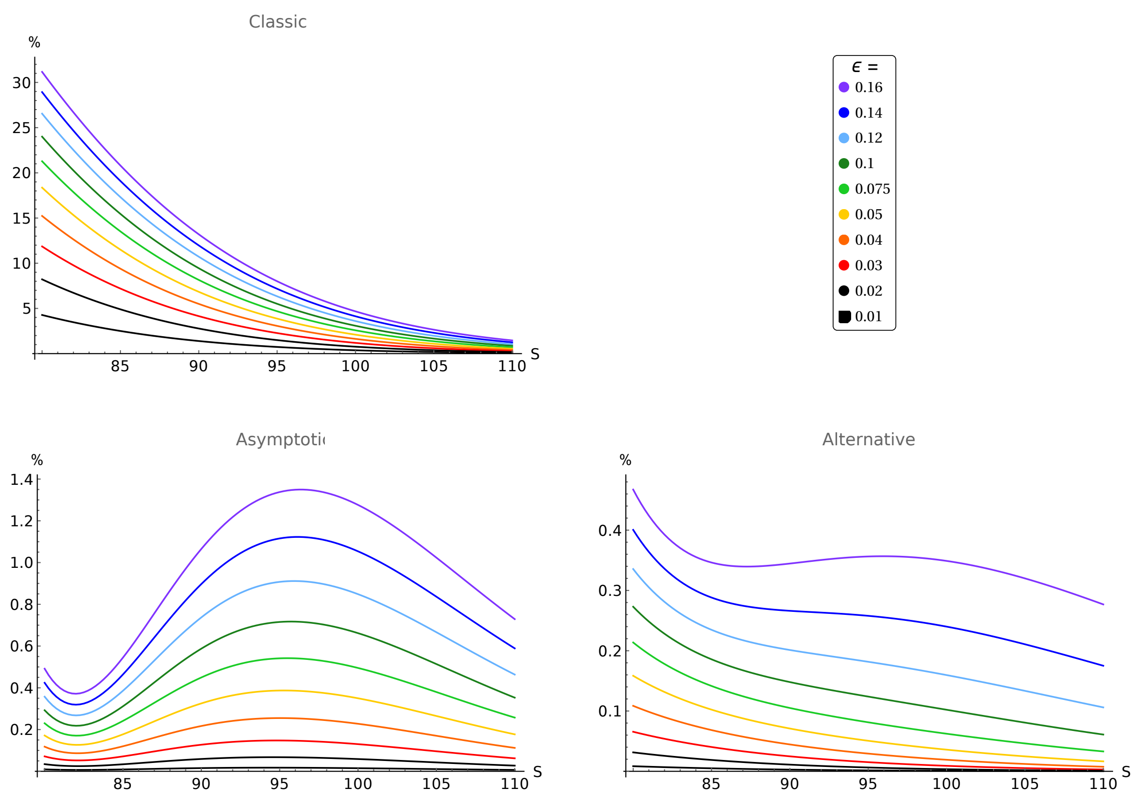


Figure 3.6: Percent error of valuation approximations.

3.5.2 Exponential Case: $\sigma(t) = \sigma_0 \left(1 + \varepsilon e^{\frac{t}{4}}\right)$

For this case $\varepsilon \leq 0.24$.

Table 3.2: Computation times (Seconds)

		Perturbation:	$\mathfrak{T} = \mathfrak{T}(C_0) + \mathfrak{T}(C_1) + \sum \mathfrak{T}(\varepsilon)$
		Classic term:	$\mathfrak{T}(C_0) = 0.01282$
		Linear term:	$\mathfrak{T}(C_1) = 1.59957$
ε	**Theoretical**	$\mathfrak{T}(\varepsilon)$ (ms)	**Alternative**
0.240	7.58800	+0.922	1.54766
0.225	7.78794	+0.951	1.48927
0.200	7.69392	+1.054	1.56674
0.175	7.50699	+1.095	1.54849
0.150	7.51256	+1.223	1.49520
0.125	7.50834	+1.102	1.50432
0.100	7.61747	+1.076	1.54081
0.075	7.50119	+1.147	1.52674
0.050	7.48821	+1.207	1.50973
0.025	7.50762	+1.166	1.58159

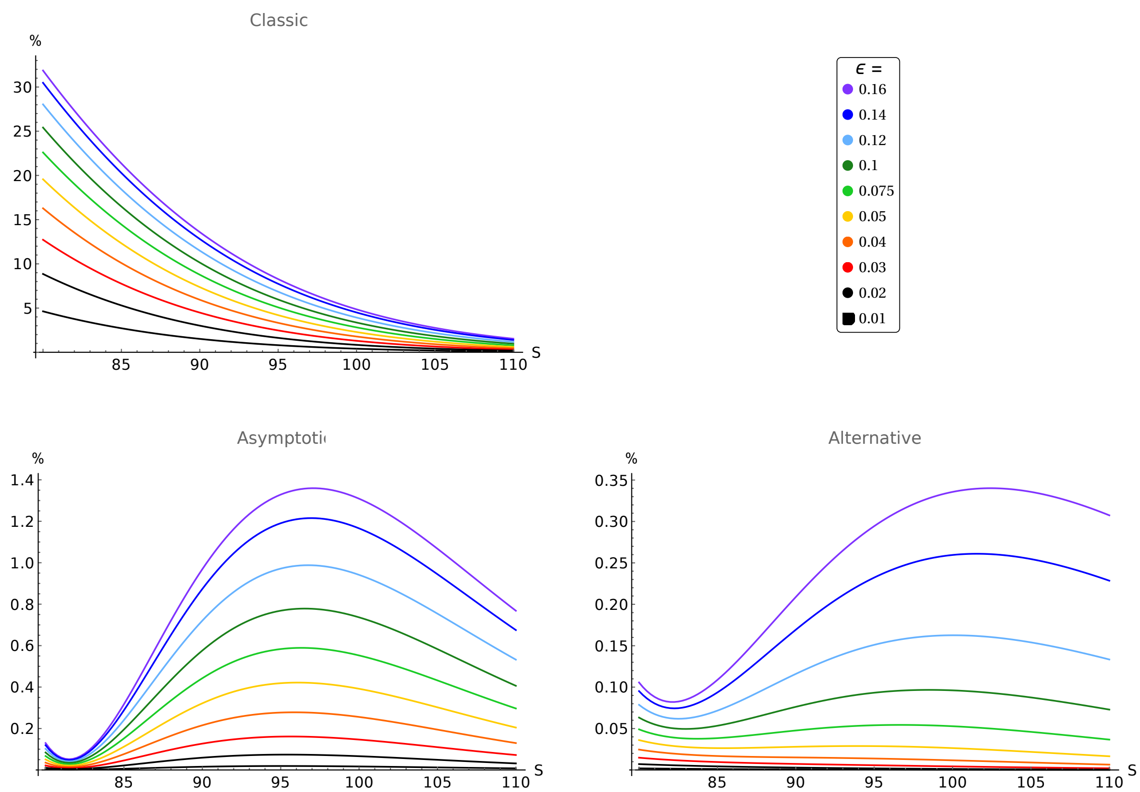


Figure 3.7: Percent error of valuation approximations.

3.5.3 Negative Exponential Case: $\sigma(t) = \sigma_0 \left(1 + \varepsilon e^{-\frac{t}{4}}\right)$

For this case $\varepsilon \leq 0.71$.

Table 3.3: Computation times (Seconds)

		Perturbation:	$\mathfrak{T} = \mathfrak{T}(C_0) + \mathfrak{T}(C_1) + \sum \mathfrak{T}(\varepsilon)$
		Classic term:	$\mathfrak{T}(C_0) = 0.013959$
		Linear term:	$\mathfrak{T}(C_1) = 1.55556$
ε	**Theoretical**	$\mathfrak{T}(\varepsilon)$ (ms)	**Alternative**
0.70	9.59455	+0.956	1.51937
0.60	9.17233	+1.022	1.49523
0.50	9.41991	+1.055	1.65938
0.40	9.20302	+1.027	1.49084
0.30	9.21084	+1.045	1.51117
0.25	9.27951	+1.038	1.54642
0.20	9.23386	+1.175	1.51708
0.15	9.26316	+1.102	1.52667
0.10	9.27299	+1.179	1.54929
0.05	9.22038	+1.225	1.51071

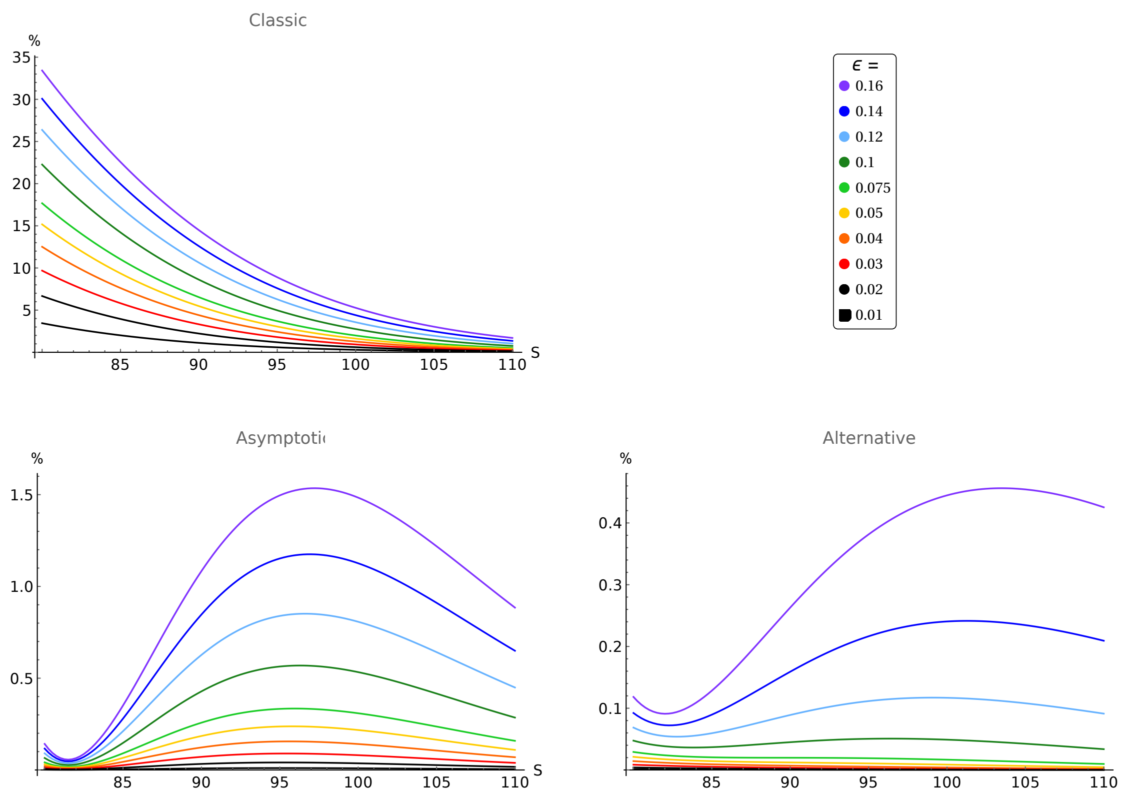


Figure 3.8: Percent error of valuation approximations.

3.5.4 Linear Case: $\sigma(t) = \sigma_0 (1 + \varepsilon t)$

For this case $\varepsilon \leq 0.173$.

Table 3.4: Computation times (Seconds)

		Perturbation:	$\mathfrak{T} = \mathfrak{T}(C_0) + \mathfrak{T}(C_1) + \sum \mathfrak{T}(\varepsilon)$
		Classic term:	$\mathfrak{T}(C_0) = 0.013742$
		Linear term:	$\mathfrak{T}(C_1) = 0.69997$
ε	**Theoretical**	$\mathfrak{T}(\varepsilon)$ (ms)	**Alternative**
0.160	3.89090	+0.935	0.703176
0.140	3.96905	+1.025	0.678662
0.120	3.86180	+0.968	0.721801
0.100	3.84995	+0.997	0.714328
0.075	3.88358	+1.044	0.671299
0.050	3.92113	+1.082	0.668144
0.040	3.82112	+1.174	0.664001
0.030	3.87661	+1.111	0.653976
0.020	3.85492	+1.169	0.666319
0.010	3.78734	+1.607	0.681194

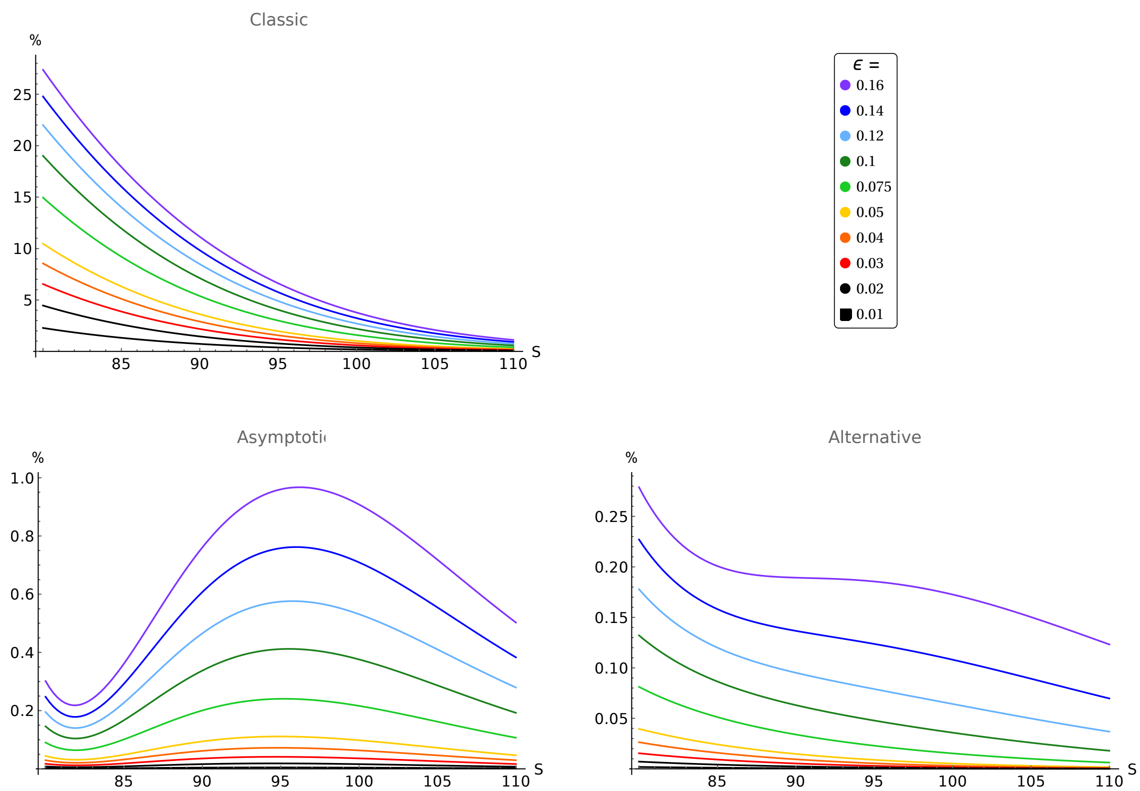


Figure 3.9: Percent error of valuation approximations.

3.6 Conclusions

It seems that the perturbation method works well when the volatility is allowed to be a continuous function of time of the form $\sigma_0 (1 + \varepsilon\varphi(t))$, and the value of ε is small enough that the correspondent linear approximation is valid. For the particular set of parameters considered, the perturbation method shows errors of less than 0.1ε w.r.t. the analytic valuation on the interval $[K - 15\%, K + 15\%]$. Additionally, another approximation algorithm was proposed. The numerical valuation of the given examples shows that the alternative method is approximately 3-4 times more precise than the perturbation method. For a particular value of ε, both approximations are roughly 5-8 times faster than the theoretical valuation; however, the perturbation method has the advantage that once u_0 and u_1 are calculated, the re-evaluation for a different value of ε takes less than a millisecond; where for the alternative method the re-evaluation takes the full amount of time.

Chapter 4

Kou model

4.1 The model

The Kou model assumes that the logarithm of the asset price follows a Brownian motion plus a compound Poisson process whose jumps sizes are distributed double exponentially:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} Y_i - 1\right),$$

where W_t is a Brownian motion, $N_t \sim Poi\,(\lambda)$, $\{Y_i\}$ is a collection of independent identically distributed non-negative random variables with density given by

$$f\,(dz) = \left[p\eta_+ e^{-\eta_+ z}\chi_{z>0}\,(z) + (1-p)\,\eta_- e^{-\eta_- z}\chi_{z<0}\,(z)\right] dz,$$

with p being the probability of an upward jump, this means

$$Y_i \overset{d}{=} \left\{ \begin{array}{ll} Z^+, & \text{with probability } p \\ -Z^- & \text{with probability } 1-p \end{array} \right\},$$

where Z^+ and $-Z^-$ are exponential random variables with means η_+^{-1} and η_-^{-1} respectively. In other words, $\eta_+, \eta_- > 0$ govern the decay of the tails for the distribution of positive and negative jump sizes. It will be assumed that every source of randomness is independent.

This control over the weight of the tails will cover one of the missing empirical properties of the Black-Scholes model, the asymmetric distribution. The jump part also contributes to a higher peak, another missing empirical property. This absences on the traditional model, correspond to the overreaction and under-reaction to outside news in the market.

If $S_t = S_0 e^{X_t}$, the model is represented by

$$X_t = at + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where $a = \mu - \frac{\sigma^2}{2}$.

Let's notice that in order to get bounded expectation for the stock price and the valuation, the average upward jump size must not exceed 100% , i.e., $\eta_+ > 1$.

The other missing property of the Black-Scholes framework is the volatility smile. In the Kou model, if the implied volatility regarding the strike price is calculated, the solution is a strict convex function (see [12], [14]). As if the same is done on the classic Black-Scholes model, the solution is a constant volatility function.

The purpose of this Chapter is to show that the perturbation method is a good approximation for the valuation problem when using the Kou model. First of all, the theoretical valuation will be calculated under a risk-free market, using the Esscher transform as the EMM. For the analytic valuation, it is desirable to know the probability density of the model, unfortunately, for this model it is not available in closed form, so the characteristic exponent will be a useful tool. In the Section 4.2, it will be shown that the Esscher transform preserves the structure of the model, that is, the characteristic exponent under the risk-neutral measure is also a rational function of the the same form.

The numeric computation of the solution is not straightforward, the integrand is highly oscillatory on the integration line ($\mathbf{R}_\rho$), this makes the numeric integration both slow and imprecise. To find a reliable computation to compare the perturbation method to, in the Section 4.5, we will use the *saddle point method*, also known as the *steepest descent method*; in it, the integration line will be transformed into a more suitable contour for the numerical integration; for this, it will be necessary to find the saddle point ξ_0 for each pair (S,t), and to use the steepest descent contour as the integration line. As the characteristic exponent under the EMM is also a rational function, for some points (S,t), the saddle point will be near (or even on) simple poles or essential singularities, the method to address this caveat will be to use different representations and use the correspondent contours as the integration contour.

Given the complexity of the analytic valuation for the double-exponential jump-diffusion model, the necessity of a good and fast approximation is evident. As an example, the perturbation method for relative small values of λ will be studied. Another approximation could be done using the Fast Fourier Transform, but the accuracy of the method is not well controlled. Additionally, the Integration-along-a-cut method developed in [7] is not applicable on this model.

The next result will be useful to find a characterization of the Kou model with the Lévy-Khintchine formula, the proof can be found in [2].

Proposition 4.1.1 *Let Z be a compound Poisson process on $\mathbf{R}^n$, with jump intensity λ, and jump size distribution f. Its characteristic exponent is given by*

$$E\left[e^{i\langle\xi,Z\rangle}\right] = \exp\left\{t\lambda\int_{\mathbf{R}^n}\left(e^{i\langle\xi,Z\rangle}-1\right)f(dh)\right\}.$$

Since, for the jump part in the model

$$\begin{aligned}&\int_{\mathbf{R}}\left(e^{i\xi z}-1\right)\left[p\eta_+e^{-\eta_+z}\chi_{z>0}(z)+(1-p)\,\eta_-e^{-\eta_-|z|}\chi_{z<0}(z)\right]dz\\ = \;& i\xi\left[p\left(\frac{1}{\eta_+-i\xi}\right)+(1-p)\left(\frac{1}{\eta_-+i\xi}\right)\right],\end{aligned}$$

then,

$$E\left[e^{i\xi Z}\right] = \exp\left\{it\lambda\xi\left[p\left(\frac{1}{\eta_+-i\xi}\right)+(1-p)\left(\frac{1}{\eta_-+i\xi}\right)\right]\right\}.$$

Using the independence of the variables, the exponent characteristic of the model can be found:

$$\begin{aligned} e^{-t\psi(\xi)} &= E\left[e^{i\xi(G+Z)}\right] = E\left[e^{i\xi G}\right]E\left[e^{i\xi Z}\right] \\ &= \exp\left\{t\left(-\frac{1}{2}\sigma^2\xi^2 + ia\xi + i\xi\lambda\left[p\left(\frac{1}{\eta_+ - i\xi}\right) + (1-p)\left(\frac{1}{\eta_- + i\xi}\right)\right]\right)\right\}, \end{aligned}$$

and finally

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ia\xi - i\xi\lambda\left(\frac{p}{\eta_+ - i\xi} - \frac{1-p}{\eta_- + i\xi}\right). \tag{4.1}$$

The last expression will be denoted by $\psi^{\mathbb{P}}(\xi)$ to specify that it is taken under the historical probability measure $\mathbb{P}$.

4.2 Risk-neutral universe

4.2.1 Geometric Brownian case

As $\lambda \to 0$, the jumps vanishes; and at $\lambda = 0$, the process becomes a geometric Brownian motion; so $S = S_0 e^G$, $G = at + \sigma W_t$ with $a = \mu - \frac{\sigma^2}{2}$. It is well known that such variable G has the density function

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left\{-\frac{(x-at)^2}{2\sigma^2 t}\right\},$$

and its characteristic exponent is given by

$$E\left[e^{i\xi G}\right] = \exp\left\{iat\xi - \frac{1}{2}\sigma^2 t\xi^2\right\}.$$

Let's find out if the structure of a geometric Brownian motion part is preserved under a EMM; we know that the characteristic exponent is

$$\psi^{\mathbb{P}}(\xi) = \frac{\sigma_{\mathbb{P}}^2}{2}\xi^2 - ia\xi,$$

then, if under the EMM the process structure is preserved, the characteristic exponent must be

$$\psi_0^{\mathbb{Q}}(\xi) = \frac{\sigma_{\mathbb{Q}}^2}{2}\xi^2 - i\mu_{\mathbb{Q}}\xi.$$

The Proposition 9.8 in [2] tells us that $\sigma_{\mathbb{P}} = \sigma_{\mathbb{Q}} = \sigma$, and the EMM condition lets us know the new drift:

$$\mu_{\mathbb{Q}} = r - \frac{\sigma^2}{2}.$$

As (3.4) holds for constant parameters, the EMM is unique and must coincide with the one given by the Esscher transform; the correspondent parameter θ_0 could be found using Equation (2.9),

$$\theta_0 = \frac{r-\mu}{\sigma^2}.$$

Then, the characteristic exponent under the EMM is

$$\psi_0^{\mathbb{Q}}(\xi) = \frac{\sigma^2}{2}\xi^2 - i\left(r - \frac{\sigma^2}{2}\right)\xi.$$

4.2.2 Kou model

For the Kou model it is easy to see that for

$$\phi^{\mathbb{P}}(\xi) = \frac{\sigma^2}{2}\xi^2 - i\xi\lambda\left(\frac{p}{\eta_+ - i\xi} - \frac{1-p}{\eta_- + i\xi}\right),$$

$\phi^{\mathbb{P}}(\xi)$ is holomorphic in $\Im\xi \in (-\eta_+, \eta_-)$, it is also continuous up to the boundary of the strip; and as $\xi \to \infty$ in $\Im\xi \in (-\eta_+, \eta_-)$, (2.5) – (2.6) holds for: $c = \frac{\sigma^2}{2}$, $0 = \nu_1 < \nu = 2$, $1 = \nu_2 < \nu$ and $C = \sigma^2$. So, the model considered is a $RLPE$ of order $\nu = 2$, intensity $c = \frac{\sigma^2}{2}$ and steepness parameters $\{-\eta_+, \eta_-\}$.

For the double-exponential jump-diffusion model, let's first find the parameter θ

that defines the Esscher transform; the EMM condition (2.9) is

$$\frac{\sigma^2}{2}+a-r+\sigma^2\theta+\lambda\left[\begin{array}{c}\frac{p(1+\theta)}{\eta_+-(1+\theta)}-\frac{p\theta}{\eta_+-\theta}\\ -\left(\frac{(1-p)(1+\theta)}{\eta_-+(1+\theta)}-\frac{(1-p)\theta}{\eta_-+\theta}\right)\end{array}\right]=0,$$

that after some calculations could be expressed as

$$\mu-r+\sigma^2\theta+\lambda\left[\frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))}-\frac{(1-p)\,\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))}\right]=0. \qquad (4.2)$$

One might wonder if finding θ will be reasonable, and the Lemma 2.2.3 assure the existence of such θ. In our case the condition in Equation (2.10) turns into

$$\lim_{\theta\to-\eta_-+0} f(\theta)<r<\lim_{\theta\to(\eta_+-1)-0} f(\theta),$$

where

$$f(\theta)=\mu+\sigma^2\theta+\lambda\left[\frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))}-\frac{(1-p)\,\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))}\right].$$

Since the condition holds, then (4.2) has one root on $(-\eta_-,\eta_+-1)$.

Now, from Equation (2.9), the characteristic exponent under the new measure is

$$\begin{aligned}\psi^{\mathbb{Q}}(\xi) &= \frac{\sigma^2}{2}\xi^2-i\left(\sigma^2\theta+a\right)\xi-i\lambda\xi\left\{\frac{p}{\eta_+-i\xi-\theta}-\frac{1-p}{\eta_-+i\xi+\theta}\right\}\\ &\quad -\lambda\theta\left\{\frac{p}{\eta_+-i\xi-\theta}-\frac{1-p}{\eta_-+i\xi+\theta}-\left(\frac{p}{\eta_+-\theta}-\frac{1-p}{\eta_-+\theta}\right)\right\}.\end{aligned}$$

As with the geometric Brownian motion, we would like to know if the structure of the model is preserved once the Esscher transform was applied, fortunately the structure is preserved once new parameters are specified (see [13]):

$$\begin{aligned}
b &= \sigma^2\theta + a = \sigma^2\left(\theta - \frac{1}{2}\right) + \mu \\
\widehat{\eta}_+ &= \eta_+ - \theta \\
\widehat{\eta}_- &= \eta_- + \theta \\
\zeta &= \frac{p\eta_+}{\eta_+ - \theta} + \frac{(1-p)\,\eta_-}{\eta_- + \theta} \\
\widehat{p} &= \frac{p\eta_+}{\zeta\,(\eta_+ - \theta)} = \frac{p\eta_+}{\zeta\widehat{\eta}_+} \\
\widehat{\lambda} &= \zeta\lambda.
\end{aligned}$$

Then

$$\psi^{\mathbb{Q}}(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi - i\widehat{\lambda}\xi\left\{\frac{\widehat{p}}{\widehat{\eta}_+ - i\xi} - \frac{1-\widehat{p}}{\widehat{\eta}_- + i\xi}\right\}.$$

Since the double-exponential jump-diffusion model will be seen as a perturbation to the geometric Brownian model, the parameter λ acquires a lead role, then it must be noted that Equation (4.1) depends on λ, so θ and $\psi^{\mathbb{Q}}$ are functions of λ too, and this will be remarked with the following notation: $\psi^{\mathbb{Q}}(\xi, \lambda)$.

4.3 Pseudo-differential problem

Given that at time T the value of the option, the payoff $\varphi(S_T)$, is known, the option value is given by the martingale property of the EMM:

$$V(S_t, t) = e^{-r(T-t)}E^{\mathbb{Q}}\left[\varphi(S_T) \mid S_t\right].$$

Considering

$$\begin{aligned}
u(X_t, t) &= V\left(S_0 e^{X_t}, t\right) = V(S_t, t), \\
g(X_T) &= \varphi\left(S_0 e^{X_T}\right) = \varphi(S_T),
\end{aligned}$$

we have

$$u(x, t) = e^{-r(T-t)}E^{\mathbb{Q}}\left[g(X_T) \mid X_t = x\right].$$

Using $\tau = T-t$, along with $X_T = X_\tau + X_t$ from the independent increment property of a Lévy process:

$$\begin{aligned} u(x,t) &= e^{-r\tau}\int_{\mathbf{R}} g(x+y)\, p_\tau(y)\, dy \\ &= e^{-r\tau}\int_{\mathbf{R}} g(y)\, p_\tau(y-x)\, dy \qquad y \mapsto y-x \end{aligned}$$

where p_τ is the density of X_τ under $\mathbb{Q}$ (and associated to its generating triplet), and can be expressed using the Lévy-Khintchine formula

$$\begin{aligned} e^{-\tau\psi^{\mathbb{Q}}(-\xi)} &= E^{\mathbb{Q}}\left[e^{-i\xi X_\tau}\right] \\ &= \int_{\mathbf{R}} e^{-i\xi x} p_\tau(x)\, dx \\ &= \widehat{p}_\tau(\xi) \end{aligned}$$

and the inverse Fourier Transform

$$\begin{aligned} p_\tau(x) &= \frac{1}{2\pi}\int_{\mathbf{R}} e^{i\xi x}\widehat{p}_\tau(\xi)\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbf{R}} e^{-i\xi x} e^{-\tau\psi^{\mathbb{Q}}(\xi)} d\xi. \end{aligned}$$

As $\psi^{\mathbb{Q}}(\xi)$ oscillates, we want to shift the line of integration to a more suitable one, the following definition will help.

Definition 4.3.1 *Let $e^{\rho x} f \in L^1(\mathbf{R})$ for some $\rho \in \mathbf{R}$. The generalized Fourier transform of f is given by*

$$\widehat{f}(\xi) = (\mathcal{F}_\rho f)(\xi) = \int_{\mathbf{R}} e^{-i\xi x} f(x)\, dx, \quad \text{for } \xi \in \mathbf{R}_\rho,$$

where $\mathbf{R}_\rho = \{v + i\rho : v \in \mathbf{R}\}$. If f is piece-wise-continuously differentiable, then the inverse generalized Fourier transformation is given by

$$f(x) = \left(\mathcal{F}_\rho^{-1}\widehat{f}\right)(x) = \frac{1}{2\pi}\int_{\mathbf{R}_\rho} e^{i\xi x}\widehat{f}(\xi)\, d\xi, \quad \text{for } x \in \mathbf{R}.$$

Given the form of $\psi^{\mathbb{Q}}$, $e^{-i\xi x-\tau\psi^{\mathbb{Q}}(\xi)}$ is analytic in a strip on the complex plane, so, using the Cauchy Integral Theorem

$$p_\tau(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-i\xi x-\tau\psi^{\mathbb{Q}}(\xi)}d\xi = \frac{1}{2\pi}\int_{\mathbf{R}_\rho} e^{-i\xi x-\tau\psi^{\mathbb{Q}}(\xi)}d\xi.$$

Then,

$$\begin{aligned} u(x,t) &= e^{-r\tau}\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}_\rho} e^{-i\xi(y-x)-\tau\psi^{\mathbb{Q}}(\xi)}g(y)\,d\xi dy \\ &= \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}_\rho} e^{-i\xi(y-x)-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}g(y)\,d\xi dy, \end{aligned}$$

as the integral converges absolutely, the order of integration can be changed, so

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbf{R}_\rho} e^{i\xi x-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}\int_{\mathbf{R}} e^{-i\xi y}g(y)\,dyd\xi \\ &= \frac{1}{2\pi}\int_{\mathbf{R}_\rho} e^{i\xi x}e^{-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}\left(\mathcal{F}_\rho g\right)(\xi)\,d\xi \\ &= \left(\mathcal{F}_\rho^{-1}e^{-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}\mathcal{F}_\rho g\right)(x), \end{aligned} \tag{4.3}$$

where ρ is such that $e^{\rho x}g \in L^1(\mathbf{R})$.

Using the inverse generalized Fourier transformation and (4.3):

$$Lf(x) = -\left[\mathcal{F}_\rho^{-1}\left(\psi^{\mathbb{Q}}\right)\mathcal{F}_\rho f\right](x).$$

We will use the following notation to specify the measure under which the risk-neutral valuation will be calculated, the shifted line of integration, and the variable on which the function is defined

$$\left[L^{\mathbb{Q}}_{\rho,x}f\right](x) = -\left[\mathcal{F}_\rho^{-1}\left(\psi^{\mathbb{Q}}\right)\mathcal{F}_\rho f\right](x),$$

and then, $L^{\mathbb{Q}}_{\rho,x}$ is a pseudo-differential operator with symbol $-\psi^{\mathbb{Q}}$. In particular

$$\left[L^{\mathbb{Q}}_{\rho,x}u\right](x) = -\left[\mathcal{F}_\rho^{-1}e^{-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}\mathcal{F}_\rho g\right](x)$$

Given that the pseudo-differential problem from which $u(x,\tau)$ is the solution will give us the price behavior through time, it is reasonable to differentiate in τ the expression (4.3):

$$\frac{\partial u}{\partial \tau} = -ru + L_{\rho,x}^{\mathbb{Q}} u = \left(-r + L_{\rho,x}^{\mathbb{Q}}\right) u$$

which can be represented as the differential equation

$$\frac{\partial u}{\partial \tau} + \left(r - L_{\rho,x}^{\mathbb{Q}}\right) u = 0,$$

called the *generalized Black-Scholes equation*; adding the boundary condition, the complete problem is obtained

$$\begin{cases} \frac{\partial u}{\partial \tau}(x,\tau) + \left(r - L_{\rho,x}^{\mathbb{Q}}\right) u(x,\tau) = 0 & x \in \mathbf{R}, \tau \in (0,T) \\ u(x,0) = g(x) & x \in \mathbf{R} \end{cases}.$$

4.4 Generalized Black-Scholes equation

First, let's apply the generalized Fourier transform to the problem, for the partial derivative it results

$$\begin{aligned} \left[\mathcal{F}_\rho \frac{\partial u}{\partial \tau}\right](\xi,\tau) &= \int_{\mathbf{R}} e^{-i\xi x} \frac{\partial u}{\partial \tau}(x,\tau)\, dx \\ &= \frac{d}{d\tau} \int_{\mathbf{R}} e^{-i\xi x} u(x,\tau)\, dx = \frac{d}{d\tau} \left[\mathcal{F}_\rho u\right](\xi,\tau); \end{aligned}$$

also, for $\xi \in \mathbf{R}_\rho$,

$$\begin{aligned} \mathcal{F}_\rho \left[\left(r - L_{\rho,x}^{\mathbb{Q}}\right) u\right](\xi,\tau) &= \int_{\mathbf{R}} e^{-i\xi x} \left(r - L_{\rho,x}^{\mathbb{Q}}\right) u(x,\tau)\, dx \\ &= \left(r + \psi^{\mathbb{Q}}(\xi)\right) \left[\mathcal{F}_\rho u\right](\xi,\tau); \end{aligned}$$

and for the boundary, as $e^{\rho x}(e^x - K)_+$ must be in $L^1(\mathbf{R})$, so $\rho < -1$. Then, in particular for an European call option, for $\xi \in \mathbf{R}_\rho$,

$$\begin{aligned}
[\mathcal{F}_\rho u](\xi, 0) &= [\mathcal{F}_\rho g](\xi) \\
&= \int_{\mathbf{R}} e^{-i\xi x}(e^x - K)_+ \, dx = \int_{\ln K}^{+\infty} e^{-i\xi x}(e^x - K)\, dx \\
&= -\frac{Ke^{-i\xi \ln K}}{(\xi + i)\,\xi}.
\end{aligned} \tag{4.4}$$

Then, the problem becomes

$$\begin{cases} \left(\frac{d}{d\tau} + r + \psi^{\mathbb{Q}}(\xi)\right)[\mathcal{F}_\rho u](\xi, \tau) = 0, & \xi \in \mathbf{R}_\rho,\ \tau \in (0, +\infty) \\ [\mathcal{F}_\rho u](\xi, 0) = -\frac{Ke^{-i\xi \ln K}}{(\xi+i)\xi}, & \xi \in \mathbf{R}_\rho. \end{cases}.$$

Now, the problem can be written to include the boundary condition into the differential equation.

So, let's apply the Laplace transformation to the problem, for $\xi \in \mathbf{R}_\rho$, $\rho < -1$, $\omega \in \mathbf{C}$, δ,

$$\begin{aligned}
&\mathcal{L}\left[\left(\frac{d}{d\tau} + r + \psi^{\mathbb{Q}}(\xi)\right)(\mathcal{F}_\rho u)\right](\xi, \omega) \\
= \;& \left(r + \psi^{\mathbb{Q}}(\xi)\right)[\mathcal{L}\mathcal{F}_\rho u](\xi, \omega) + \int_0^{+\infty} e^{-\omega\tau} d(\mathcal{F}_\rho u)(\xi, \tau).
\end{aligned}$$

Integrating by parts

$$\int_0^{+\infty} e^{-\omega\tau} d(\mathcal{F}_\rho u)(\xi, \tau) \;=\; \left(r + \psi^{\mathbb{Q}}(\xi) + \omega\right)[\mathcal{L}\mathcal{F}_\rho u](\xi, \omega) + \frac{Ke^{-i\xi \ln K}}{(\xi + i)\,\xi}.$$

This can be summarized as

$$\left(r + \omega + \psi^{\mathbb{Q}}(\xi)\right)[\mathcal{L}\mathcal{F}_\rho u](\xi, \omega) = -\frac{Ke^{-i\xi \ln K}}{(\xi+i)\xi} \qquad \xi \in \mathbf{R}_\rho, \omega \in \mathbf{C}, \Re\omega > \delta.$$

The last expression can be divided by $\left(r + \omega + \psi^{\mathbb{Q}}(\xi)\right)$ to obtain an equation for $[\mathcal{L}\mathcal{F}_\rho u](\xi, \omega)$, and then apply the Laplace and Fourier inverse transforms to obtain a solution in a integral form:

$$\begin{aligned}&[\mathcal{F}_\rho u](\xi,\tau) = -\tfrac{1}{2\pi i}\tfrac{Ke^{-i\xi\ln K}}{(\xi+i)\xi}\int_{\Gamma_b}\tfrac{e^{\omega x}}{\omega+r+\psi^{\mathbb{Q}}(\xi)}d\omega,\\ &\xi\in\mathbf{R}_\rho,\, \Gamma_b = \{\xi\in\mathbf{C}:\Re\xi=b\}, b>\delta\ .\end{aligned}$$

Considering the Residue Theorem

$$\int_{\Gamma_b}\frac{e^{\omega x}}{\omega+r+\psi^{Q}(\xi)}d\omega = 2\pi i e^{-\left(r+\psi^{\mathbb{Q}}(\xi)\right)},$$

then

$$\begin{aligned}[\mathcal{F}_\rho u](\xi,\tau) &= -K\frac{e^{-i\xi\ln K-\tau\left(r+\psi^{\mathbb{Q}}(\xi)\right)}}{(\xi+i)\xi},\\ u(x,\tau) &= -\frac{Ke^{-r\tau}}{2\pi}\int_{\mathbf{R}_\rho}\frac{e^{i\xi(x-\ln K)-\tau\psi^{\mathbb{Q}}(\xi)}}{(\xi+i)\xi}d\xi,\end{aligned} \tag{4.5}$$

$\rho < -1$. Which is the generalized Black-Scholes equation for the valuation of an European call option.

4.5 Numerical valuation

As with other non-Gaussian models, the integral does not converge fast enough once the Fourier transform has been performed, this is a result of the oscillatory nature of the characteristic exponent; the easiest way to calculate the value is to use a numerical approximation by deforming the integration contour into one in which $\Im\xi$ is constant, this happens in a steepest ascent or descent path, and it passes through a saddle point.

4.5.1 Saddle Point Method

We must express the integral in the form

$$\int_{\mathbf{R}_\rho} f(\xi)\, e^{g(\xi)}d\xi.$$

In our case

$$u(S,\tau) = -\frac{Ke^{-\tau r}}{2\pi}\int_{\mathbf{R}_\rho} f(\xi)\, e^{g(\xi)} d\xi$$

$$\begin{aligned} f(\xi) &= \frac{\exp\left\{i\tau\widehat{\lambda}\xi\left\{\frac{\widehat{p}}{\widehat{\eta}_+ - i\xi} - \frac{1-\widehat{p}}{\widehat{\eta}_- + i\xi}\right\}\right\}}{(\xi+i)\,\xi} \\ g(\xi) &= i\left[\ln\frac{S}{K} + b\tau\right]\xi - \frac{\sigma^2}{2}\tau\xi^2. \end{aligned}$$

Let's call $H = H(S,\tau) = \ln\frac{S}{K} + b\tau$, then

$$\begin{aligned} g(\xi) &= iH\xi - \frac{\sigma^2\tau}{2}\xi^2, \\ g'(\xi) &= iH - \sigma^2\tau\xi \\ g''(\xi) &= -\sigma^2\tau = \sigma^2\tau e^{i\pi}. \end{aligned} \tag{4.6}$$

The stationary point ξ_0 must satisfy

$$\xi_0 = i\frac{H}{\sigma^2\tau}.,$$

Given (4.6), not only the directions of steepest descent are given by $\Theta = 0, \pi$; but the single paths of steepest descent are

$$\begin{aligned} \Gamma_+ &= \{u + \xi_0 : u > 0\} \\ \Gamma_- &= \{-u + \xi_0 : u > 0\}. \end{aligned}$$

We will use the following representation given by the Taylor's series

$$g(\xi) - g(\xi_0) = -\frac{\sigma^2\tau}{2}(\xi - \xi_0)^2 \in \mathbf{R} \text{ when } \xi \in \Gamma,$$

where

$$g(\xi_0) = -\frac{1}{2}\frac{H^2}{\sigma^2\tau}.$$

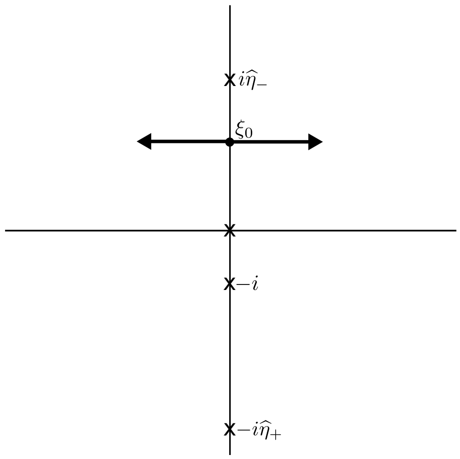


Figure 4.1: Local contour.

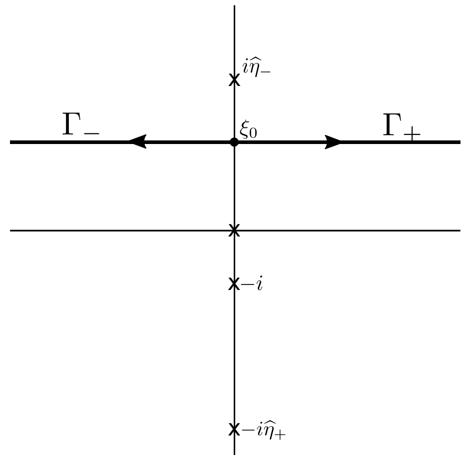


Figure 4.2: Steepest descent.

Taking the change of variable $-w^2 = g(\xi) - g(\xi_0)$,

$$\begin{aligned} w_+ &= \sqrt{\frac{\sigma^2\tau}{2}}(\xi - \xi_0) \to d\xi = \sqrt{\frac{2}{\sigma^2\tau}} dw_+ \\ w_- &= -\sqrt{\frac{\sigma^2\tau}{2}}(\xi - \xi_0) \to d\xi = -\sqrt{\frac{2}{\sigma^2\tau}} dw_-. \end{aligned}$$

Noting that for the deformed contour $\Gamma(\xi_0)$ given by the steepest descent path, $\int_{\Gamma(\xi_0)} d\xi = \int_{\Gamma_+} d\xi - \int_{\Gamma_-} d\xi$; and taking $w = w_+ = -w_- : \int_{\Gamma(\xi_0)} d\xi = \int_{-\infty}^{+\infty} dw$. So, we have

$$\begin{aligned} & e^{g(\xi_0)} \int_{\Gamma(\xi_0)} f(\xi(w)) e^{g(\xi) - g(\xi_0)} d\xi \\ = & \sqrt{\frac{2}{\sigma^2\tau}} e^{-\frac{1}{2}\frac{H^2}{\sigma^2\tau}} \int_{-\infty}^{+\infty} f(\xi(w)) e^{-w^2} dw, \end{aligned}$$

where

$$\xi(w) = \xi(w, S, \tau) = \sqrt{\frac{2}{\sigma^2\tau}} w + i\frac{H}{\sigma^2\tau}.$$

The contour will not be appropriate when ξ_0 is near the essential singularities $-i\widehat{\eta}_+, i\widehat{\eta}_-$, and the poles. So we need to divide the imaginary axis and to choose an appropriate approximation in each subset; as the value of ξ_0 is determined by S and τ, it will be more convenient to specify each case based on those parameters.

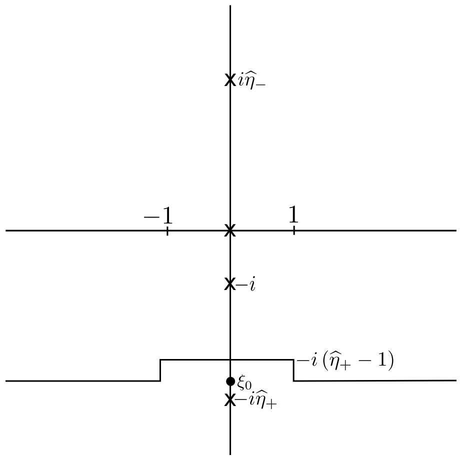


Figure 4.3

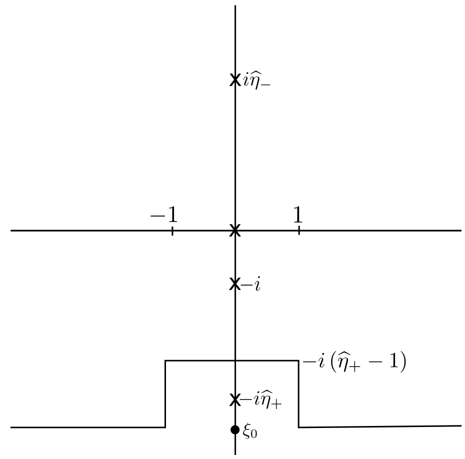


Figure 4.4

1. $\Im\xi_0 \in \left(-\infty, -\left(\widehat{\eta}_+ - 1\right)\right)$

   If we use the Laurent series of f and the Taylor series of e^g, the integrand will take the form of a infinite product that will be problematic to compute. So in that case, the contours will be taken as the example in Figure 4.3.

   For the particular cases when $\Im\xi_0 \leq -\widehat{\eta}_+$. The contours will be considered as in Figure 4.4, since the residue could only be obtained from the series expansion.

   We will take $\Gamma\left(\xi_0\right) = \Gamma_1 + \Gamma_2 + \Delta_1 + \Delta_2$, where

$$\begin{aligned} \Gamma_1 &= \left\{v + \xi_0 : v \in \mathbf{R} \backslash \left(-1, 1\right)\right\}, \\ \Gamma_2 &= \left\{v - i\left(\widehat{\eta}_+ - 1\right) : v \in \left(-1, 1\right)\right\}, \\ \Delta_1 &= \left\{-1 + i\delta : \delta \in \left[\Im\xi_0, -\left(\widehat{\eta}_+ - 1\right)\right]\right\}, \\ \Delta_2 &= \left\{1 + i\delta : \delta \in \left[-\left(\widehat{\eta}_+ - 1\right), \Im\xi_0\right]\right\}. \end{aligned}$$

   And for each part

$$\begin{aligned} A_1 &= \sqrt{\frac{2}{\sigma^2\tau}} e^{\frac{1}{2}\frac{H^2}{\sigma^2\tau}} \int_{\mathbf{R}\backslash\left(-\sqrt{\frac{\sigma^2\tau}{2}}, \sqrt{\frac{\sigma^2\tau}{2}}\right)} f\left(\xi_1\left(w\right)\right) e^{-w^2} dw, \\ \xi_1\left(w\right) &= \sqrt{\frac{2}{\sigma^2\tau}} w + i\frac{H}{\sigma^2\tau}; \end{aligned}$$

$$
\begin{aligned}
A_2 &= \sqrt{\frac{2}{\sigma^2\tau}} e^{g(-i(\widehat{\eta}_+-1))} \int_{-\sqrt{\frac{\sigma^2\tau}{2}}}^{\sqrt{\frac{\sigma^2\tau}{2}}} f\left(\xi_2\left(w\right)\right) e^{-w^2} dw, \\
g\left(-i\left(\widehat{\eta}_+ - 1\right)\right) &= H\left(\widehat{\eta}_+ - 1\right) + \frac{\sigma^2\tau}{2}\left(\widehat{\eta}_+ - 1\right)^2, \\
\xi_2\left(w\right) &= \sqrt{\frac{2}{\sigma^2\tau}} w - i\left(\widehat{\eta}_+ - 1\right); \\
A_3 &= \int_{\Im\xi_0}^{-(\widehat{\eta}_+-1)} f\left(-1+i\delta\right) e^{g(-1+i\delta)} d\delta; \\
A_4 &= \int_{-(\widehat{\eta}_+-1)}^{\Im\xi_0} f\left(1+i\delta\right) e^{g(1+i\delta)} d\delta.
\end{aligned}
$$

To get

$$
u\left(S,\tau\right) = -\frac{Ke^{-\tau r}}{2\pi}\left(A_1 + A_2 + A_3 + A_4\right).
$$

2. $\Im\xi_0 \in \left[-\left(\widehat{\eta}_+ - 1\right), -1\right)$

   We will use

$$
\begin{aligned}
u\left(S,\tau\right) &= -\frac{Ke^{-\tau r}}{2\pi}\sqrt{\frac{2}{\sigma^2\tau}} e^{-\frac{1}{2}\frac{H^2}{\sigma^2\tau}} \int_{-\infty}^{+\infty} f\left(\xi\left(w\right)\right) e^{-w^2} dw, \\
\xi\left(w\right) &= \sqrt{\frac{2}{\sigma^2\tau}} w + i\frac{H}{\sigma^2\tau},
\end{aligned}
\tag{4.7}
$$

   as explained in the presentation of the method.

3. $\xi_0 = -i$ The integrand could be represented as

$$
f\left(\xi\right) e^{g(\xi)} = \frac{f_0\left(\xi\right) e^{g(\xi)}}{\xi + i},
$$

   where f_0 does not have a pole in $-i$. Then

$$
\int_{\mathbf{R}-i} f\left(\xi\right) e^{g(\xi)} d\xi = \sqrt{\frac{2}{\sigma^2\tau}} e^{g(-i)} \left[\begin{array}{c} \int_{-\infty}^{+\infty} \frac{\left[f_0(\xi(w)) - f_0(-i)\right]}{\xi(w)+i} e^{-w^2} dw \\ +f_0\left(-i\right) \int_{-\infty}^{+\infty} \frac{e^{-w^2}}{\xi(w)+i} dw \end{array}\right]
$$

with

$$\begin{aligned} f_0(-i) &= -i\exp\left\{\tau\widehat{\lambda}\left\{\frac{\widehat{p}}{\widehat{\eta}_+ - 1} - \frac{1-\widehat{p}}{\widehat{\eta}_- + 1}\right\}\right\} \\ g(-i) &= H + \frac{\sigma^2}{2}\tau \\ \xi(w) &= \sqrt{\frac{2}{\sigma^2\tau}}w - i, \end{aligned}$$

then

$$\int_{\mathbf{R}-i} f(\xi)\, e^{g(\xi)} d\xi = e^{g(-i)} \left[\begin{array}{c} \int_{-\infty}^{+\infty} \frac{[f_0(\xi(w)) - f_0(-i)]}{w} e^{-w^2} dw \\ +f_0(-i) \int_{-\infty}^{+\infty} \frac{e^{-w^2}}{w} dw \end{array} \right],$$

where the first integral is well defined thanks to L'Hopital's rule, and for the second integral we will use the formula

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{-w^2}}{w - i\varepsilon} dw &= i\pi e^{\varepsilon^2}\left[1 - \phi(\varepsilon)\right], \\ \phi(\varepsilon) &= \frac{2}{\sqrt{\pi}} \int_0^{\varepsilon} e^{-v^2} dv. \end{aligned} \tag{4.8}$$

Then

$$u(S,\tau) = -\frac{Ke^{-\tau r}}{2\pi} e^{g(-i)} \left[\begin{array}{c} \int_{-\infty}^{+\infty} \frac{[f_0(\xi(w)) - f_0(-i)]}{w} e^{-w^2} dw \\ +i\pi f_0(-i) \end{array} \right].$$

4. $\Im\xi_0 \in (-1, 0)$ We will use Equation (4.7) but, we must add

$$2\pi i Res[fe^g, -i] = -2\pi \exp\left\{H + \frac{\sigma^2\tau}{2} + \widehat{\lambda}\tau\left\{\frac{\widehat{p}}{\widehat{\eta}_+ - 1} - \frac{1-\widehat{p}}{\widehat{\eta}_- + 1}\right\}\right\}$$

since the new contour is above the pole $-i$. So,

$$\begin{aligned} u(S,\tau) &= -\frac{Ke^{-\tau r}}{2\pi} \left(\begin{array}{c} \sqrt{\frac{2}{\sigma^2\tau}} e^{-\frac{1}{2}\frac{H^2}{\sigma^2\tau}} \left(\int_{-\infty}^{+\infty} f(\xi(w))\, e^{-w^2} dw \right) \\ +2\pi i Res[fe^g, -i] \end{array} \right), \\ \xi(w) &= \sqrt{\frac{2}{\sigma^2\tau}} w + i\frac{H}{\sigma^2\tau}. \end{aligned}$$

5. $\xi_0 = 0$

The procedure is similar to the previous pole. The integrand is written as

$$f(\xi)\, e^{g(\xi)} = \frac{f_0(\xi)\, e^{g(\xi)}}{\xi},$$

then

$$\int_{\mathbf{R}} f(\xi)\, e^{g(\xi)} d\xi = \sqrt{\frac{2}{\sigma^2 \tau}} e^{g(0)} \left[\begin{array}{c} \int_{-\infty}^{+\infty} \frac{[f_0(\xi(w)) - f_0(0)]}{\xi(w)} e^{-w^2} dw \\ + f_0(0) \int_{-\infty}^{+\infty} \frac{e^{-w^2}}{\xi(w)} dw \end{array} \right]$$

with

$$\begin{aligned} f_0(0) &= -i, \\ g(0) &= 0, \\ \xi(w) &= \sqrt{\frac{2}{\sigma^2 \tau}} w. \end{aligned}$$

Again, the first integral is well defined thanks to L'Hopital's rule, and we know the value of the second integral (Equation (4.8)), so

$$\int_{\mathbf{R}} f(\xi)\, e^{g(\xi)} d\xi = \begin{array}{c} \int_{-\infty}^{+\infty} \frac{[f_0(\xi(w)) + i]}{w} e^{-w^2} dw \\ + \pi. \end{array}$$

We must add $2\pi i Res[fe^g, -i]$ again since the new contour is above the pole $-i$, then

$$u(S, \tau) = -\frac{K e^{-\tau r}}{2\pi} \left[\begin{array}{c} \int_{-\infty}^{+\infty} \frac{[f_0(\xi(w)) + i]}{w} e^{-w^2} dw \\ + \pi + 2\pi i Res[fe^g, -i] \end{array} \right].$$

6. $\Im \xi_0 \in \left(0, \frac{\widehat{\lambda}(1-\widehat{p})}{\sigma^2 \widehat{\eta}_-}\right]$

We will use Equation (4.7) again, but now we must add $2\pi i Res[fe^g, -i]$ and

$2\pi i Res[fe^g, 0] = 2\pi$, so

$$
\begin{aligned}
u(S,\tau) &= -\frac{Ke^{-\tau r}}{2\pi}\left(\begin{array}{c}\sqrt{\frac{2}{\sigma^2\tau}}e^{-\frac{1}{2}\frac{H^2}{\sigma^2\tau}}\left(\int_{-\infty}^{+\infty} f(\xi(w))\, e^{-w^2}dw\right)\\ +2\pi i Res[fe^g,-i]+2\pi i Res[fe^g,0]\end{array}\right),\\
\xi(w) &= \sqrt{\frac{2}{\sigma^2\tau}}w + i\frac{H}{\sigma^2\tau}.
\end{aligned}
$$

7. $\Im\xi_0 \in \left(\frac{\widehat{\lambda}(1-\widehat{p})}{\sigma^2\widehat{\eta}_-}, +\infty\right)$

At first thought, one may attempt to apply an analog method to the first case, but the equivalent contours to be chosen in order to avoid the essential singularity lead to an approximation error that it is not so easy to estimate. In order to avoid such scenario, the saddle point method will be taken with the following representation

$$
\begin{aligned}
u(S,\tau) &= -\frac{Ke^{-\tau r}}{2\pi}\int_{\mathbf{R}_\rho} f(\xi)\, e^{g(\xi)}d\xi\\
f(\xi) &= \frac{\exp\left\{-\frac{\sigma^2}{2}\tau\xi^2 + \frac{i\tau\widehat{\lambda}\widehat{p}\xi}{\widehat{\eta}_+ - i\xi}\right\}}{(\xi+i)\,\xi},\\
g(\xi) &= iH\xi - i\tau\widehat{\lambda}(1-\widehat{p})\frac{\xi}{\widehat{\eta}_- + i\xi},\\
g'(\xi) &= i\left[H - \frac{\tau\widehat{\lambda}(1-\widehat{p})\,\widehat{\eta}_-}{(\widehat{\eta}_- + i\xi)^2}\right].
\end{aligned}
$$

If $g'\left(\overline{\xi}_0\right) = 0$, $\overline{\xi}_0 = i\left(\widehat{\eta}_- \mp \sqrt{\frac{\tau\widehat{\lambda}(1-\widehat{p})\widehat{\eta}_-}{H}}\right)$. Writing $C = \sqrt{\frac{\tau\widehat{\lambda}(1-\widehat{p})\widehat{\eta}_-}{H}} > 0$ we have the pair of saddle points $\overline{\xi}_0^+ = i(\widehat{\eta}_- + C)$ and $\overline{\xi}_0^- = i(\widehat{\eta}_- - C)$, as $\overline{\xi}_0^+$ is always above the singularity, we will work with $\overline{\xi}_0^-$, and it will be denoted simply by $\overline{\xi}_0$.

$$
g\left(\overline{\xi}_0\right) = (\widehat{\eta}_- - C)\left[\frac{\tau\widehat{\lambda}(1-\widehat{p})}{C} - H\right] \in \mathbf{R}
$$

Given that on the steepest descent path $\Im g\left(\overline{\xi}\right) = \Im g\left(\overline{\xi}_0\right) = 0$, and being $\overline{\xi}_0$ a saddle point, we can parametrize the curve as $g\left(\overline{\xi}\right) - g\left(\overline{\xi}_0\right) = -w^2$.

Now, to find the path in order to use it as the contour of integration let's denote

$$B = \tau\widehat{\lambda}\left(1-\widehat{p}\right) - H\widehat{\eta}_{-} + g\left(\overline{\xi}_0\right),$$

then

$$\overline{\xi}^{\pm} = i\left(\frac{-\left(B-w^2\right) \pm \sqrt{\left(B-w^2\right)^2 + 4H\widehat{\eta}_{-}\left(g\left(\overline{\xi}_0\right) - w^2\right)}}{2H}\right),$$

in particular for $w = 0$,

$$\widehat{\eta}_{-} - C \quad = \quad \frac{-B \pm \sqrt{B^2 + 4H\widehat{\eta}_{-}g\left(\overline{\xi}_0\right)}}{2H},$$

that after some substitutions leads to

$$-B^2 = 4H\widehat{\eta}_{-}g\left(\overline{\xi}_0\right),$$

changing the equation of the contour to

$$\overline{\xi} = i\left(\widehat{\eta}_{-} - C + \frac{w}{2H}\left(w + \sqrt{w^2 - 4HC}\right)\right).$$

For $w \in \left[-\sqrt{4HC}, \sqrt{4HC}\right]$, $\left(\Re\overline{\xi}\right)^2 + \left(\Im\overline{\xi} - \widehat{\eta}_{-}\right)^2 = C$; and for $w \notin \left[-\sqrt{4HC}, \sqrt{4HC}\right]$, $\overline{\xi}$ is purely imaginary.

This contours are shown in Figure 4.5, given the orientation of the curves, we will take $-\int_{-\infty}^{+\infty} f\left(\overline{\xi}\left(w\right)\right) e^{-w^2} d\xi\left(w\right)$.

A problem arises as we transform the original contour of integration $(\mathbf{R}_\rho)$ to this new one, the integrand is not well defined on the vertical lines, so a new suitable contour must be taken, our election is shown in Figure 4.6, where the integral over the segments of line could be computed directly. As displayed, we need to know the value of w that corresponds to the semicircle, this value is $w_0 = \sqrt{2HC}$.

Now we need $d\overline{\xi}$ over the semicircle in order to calculate the integral on the

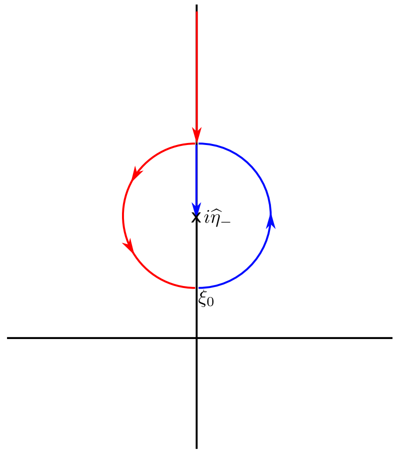


Figure 4.5

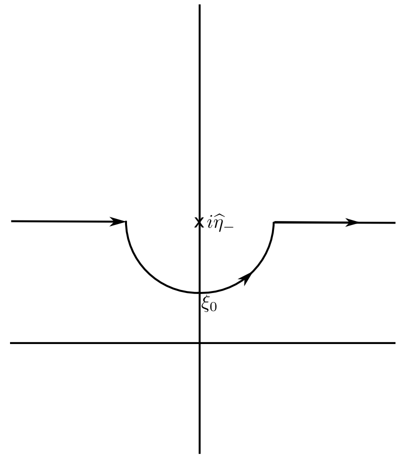


Figure 4.6

parametrized contour correspondent for the circle:

$$d\overline{\xi} = \frac{i}{2H}\left(2w + \sqrt{w^2 - 4HC} + \frac{w^2}{\sqrt{w^2 - 4HC}}\right) dw.$$

And now we can calculate the valuation as

$$u\left(S,\tau\right) \quad = \quad -\frac{Ke^{-\tau r}}{2\pi}\begin{bmatrix} - & e^{g\left(\overline{\xi}_0\right)}\int_{-w_0}^{+w_0} f\left(\overline{\xi}\left(w\right)\right)e^{-w^2}d\overline{\xi}\left(w\right) \\ + & \int_{-\infty}^{-C} f\left(z + i\widehat{\eta}_-\right)e^{g\left(z+i\widehat{\eta}_-\right)}dz \\ + & \int_{+C}^{+\infty} f\left(z + i\widehat{\eta}_-\right)e^{g\left(z+i\widehat{\eta}_-\right)}dz \\ + & 2\pi iRes[fe^g, -i] + 2\pi iRes[fe^g, 0] \end{bmatrix},$$

where the residues match the ones given before.

One might wonder if a similar procedure could be performed for the lower essential singularity with the analogous representation, unfortunately, the correspondent contour consists of vertical segments over the imaginary axis.

Given that we have covered all the possible values of the saddle point, the valuation problem has been completely approximated. As an example, the Figure 4.7 shows the value of an option for several times.

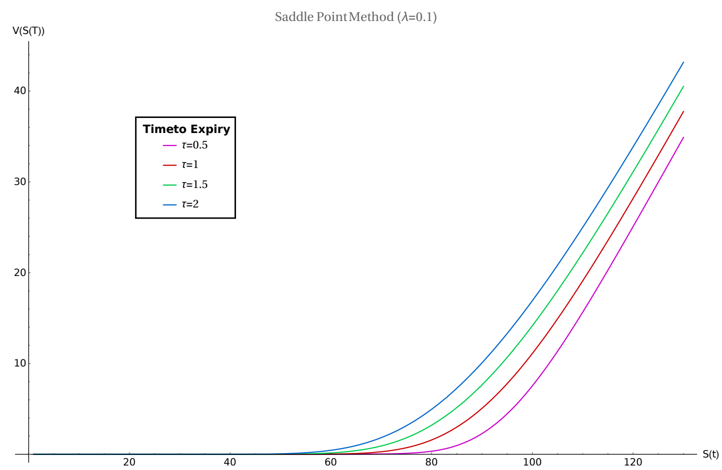


Figure 4.7: Saddle-point approximations for several values of time to expiry and parameters: $\mu = 0.12$, $\sigma = 0.16$, $r = 0.05$, $K = 98$, $\lambda = 0.1$, $p = 0.4$, $\eta_+ = 10$, $\eta_- = 5$.

4.6 Perturbation method

It is the purpose of this work to find an easy way of get numerical valuations, as mentioned before, the jumps in the stock price will be interpreted as a perturbation to the geometric Brownian model; then, to compute numerical data, a linear approximation of the option value as a function of the intensity of the jumps (λ) will be used; from (4.5):

$$\begin{aligned} u(x,\tau) &= u(x,\tau,\lambda) = -\frac{Ke^{r\tau}}{2\pi}\int_{\mathbf{R}_\rho}\frac{e^{i\xi(x-\ln K)-\tau\psi(\xi)}}{(\xi+i)\,\xi}d\xi \\ &= u_0(x,\tau) + \lambda u'_\lambda(x,\tau,0) + O\left(\lambda^2\right) \\ &= u_0(x,\tau) + \lambda u_1(x,\tau) + O\left(\lambda^2\right), \end{aligned}$$

where $u_0(x,\tau)$ can be interpreted as the solution for constant parameters found in Section 3.3. On the other hand

$$\begin{aligned} u_1(x,\tau) &= u'_\lambda(x,\tau,0) \\ &= \frac{\tau Ke^{-r\tau}}{2\pi}\int_{\mathbf{R}_\rho}\frac{\exp\{i\xi(x-\ln K)\}\exp\left\{-\tau\psi_0^{\mathbb{Q}}(\xi)\right\}\left(\frac{\partial\psi^{\mathbb{Q}}}{\partial\lambda}(\xi,0)\right)}{(\xi+i)\,\xi}d\xi. \end{aligned}$$

To find the valuation, first we need to calculate

$$\begin{aligned}
\frac{\partial \psi^{\mathbb{Q}}}{\partial \lambda}(\xi,\lambda) &= -i\sigma^2\xi\frac{\partial \theta}{\partial \lambda} - i\xi\left\{\frac{p}{\eta_+ - i\xi - \theta} - \frac{1-p}{\eta_- + i\xi + \theta}\right\} \\
&\quad -\theta\left\{\frac{p}{\eta_+ - i\xi - \theta} - \frac{1-p}{\eta_- + i\xi + \theta} - \frac{p}{\eta_+ - \theta} + \frac{1-p}{\eta_- + \theta}\right\} \\
&\quad +\lambda D(\xi).
\end{aligned}$$

For some function D, particularly,

$$\begin{aligned}
\frac{\partial \psi^{\mathbb{Q}}}{\partial \lambda}(\xi,0) &= -i\sigma^2\xi\theta'(0) - i\xi\left\{\frac{p}{\eta_+ - i\xi - \theta_0} - \frac{1-p}{\eta_- + i\xi + \theta_0}\right\} \\
&\quad -\theta_0\left\{\frac{p}{\eta_+ - i\xi - \theta_0} - \frac{1-p}{\eta_- + i\xi + \theta_0} - \frac{p}{\eta_+ - \theta_0} + \frac{1-p}{\eta_- + \theta_0}\right\}.
\end{aligned}$$

With the changes

$$\begin{aligned}
\bar{\eta}_+ &= \eta_+ - \theta_0, \\
\bar{\eta}_- &= \eta_- + \theta_0, \\
\bar{p} &= \frac{\eta_+}{\bar{\eta}_+}p, \\
\bar{q} &= \frac{\eta_-}{\bar{\eta}_-}(1-p),
\end{aligned}$$

the equation can be reformulated as

$$\frac{\partial \psi^{\mathbb{Q}}}{\partial \lambda}(\xi,0) = -i\xi\left\{\sigma^2\theta'(0) + \frac{\bar{p}}{\bar{\eta}_+ - i\xi} - \frac{\bar{q}}{\bar{\eta}_- + i\xi}\right\}.$$

To find $\theta'(0)$ let's use the Equation (4.2) and define

$$\Psi(\theta,\lambda) = \mu - r + \sigma^2\theta + \lambda\left[\begin{array}{c} \frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))} \\ -\frac{(1-p)\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))} \end{array}\right],$$

then the EMM condition can be written as

$$\begin{aligned} 0 &= \Psi(\theta,\lambda) \\ \Rightarrow 0 &= \Psi'_\theta(\theta,\lambda)\,\theta'(\lambda) + \Psi'_\lambda(\theta,\lambda) \\ \Rightarrow 0 &= \Psi'_\theta(\theta,0)\,\theta'(0) + \Psi'_\lambda(\theta,0) \end{aligned}$$

now,

$$\begin{aligned} \Psi'_\lambda(\theta,\lambda) &= \sigma^2\theta'(\lambda) + \lambda\frac{d}{d\lambda}\left[\begin{array}{c} \frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))} \\ -\frac{(1-p)\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))} \end{array}\right] \\ &\quad + \left[\begin{array}{c} \frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))} \\ -\frac{(1-p)\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))} \end{array}\right], \end{aligned}$$

in particular

$$\Psi'_\lambda(\theta,0) = \sigma^2\theta'(0)\left[\frac{\bar{p}}{\bar{\eta}_+-1} - \frac{\bar{q}}{\bar{\eta}_-+1}\right],$$

and

$$\Psi'_\theta(\theta,\lambda) = \sigma^2 + \lambda\frac{d}{d\theta}\left[\begin{array}{c} \frac{p\eta_+}{(\eta_+-\theta)(\eta_+-(1+\theta))} \\ -\frac{(1-p)\eta_-}{(\eta_-+\theta)(\eta_-+(1+\theta))} \end{array}\right],$$

then

$$\Psi'_\theta(\theta,0) = \sigma^2;$$

and finally,

$$\theta'(0) = -\frac{1}{2\sigma^2}\left[\frac{\bar{p}}{\bar{\eta}_+-1} - \frac{\bar{q}}{\bar{\eta}_-+1}\right].$$

So,

$$u_1(x,\tau) = \tau K e^{-r\tau}\mathcal{F}_\rho^{-1}\left(\frac{\partial\psi^{\mathbb{Q}}}{\partial\lambda}(\xi,0)\,\frac{\exp\{-i\xi\ln K\}}{(\xi+i)\,\xi}\exp\left\{-\tau\psi_0^{\mathbb{Q}}(\xi)\right\}\right)(x).$$

Using again the convolution property of the Fourier transform:

$$\begin{aligned} u_1(x,\tau) &= \tau K e^{-r\tau}\int_{-\infty}^{\infty}\mathcal{F}_\rho^{-1}\left(\frac{\partial\psi^{\mathbb{Q}}}{\partial\lambda}(\xi,0)\,\frac{\exp\{-i\xi\ln K\}}{(\xi+i)\,\xi}\right)(v) \\ &\quad \times\mathcal{F}_\rho^{-1}\left(\exp\left\{-\tau\psi_0^{\mathbb{Q}}(\xi)\right\}\right)(x-v)\,dv. \end{aligned}$$

Let's find $\mathcal{F}_{\rho}^{-1}\left(\exp\left\{-\tau\psi_{0}^{\mathbb{Q}}\left(\xi\right)\right\}\right)\left(x-v\right)$, denoting by $\Lambda=\frac{\sigma^{2}\tau}{2}$, and $\beta=\frac{x-v+d}{\sigma^{2}\tau}$ with $d=\tau\left(r-\frac{\sigma^{2}}{2}\right)$, this expression could be simplified:

$$\begin{aligned}
& \mathcal{F}_{\rho}^{-1}\left(\exp\left\{-\tau\psi_{0}^{\mathbb{Q}}\left(\xi\right)\right\}\right)\left(x-v\right) \\
= \; & \frac{1}{2\pi}e^{-\Lambda\beta^{2}}\int_{\mathbf{R}_{\rho}}\exp\left\{-\Lambda\left(\xi-i\beta\right)^{2}\right\}d\xi.
\end{aligned}$$

Additionally, the change of variable $\phi=\xi-i\beta$ leads to an integral easier to handle,

$$\frac{1}{2\pi}e^{-\Lambda\beta^{2}}\int_{\mathbf{R}_{\rho-\beta}}\exp\left\{-\Lambda\phi^{2}\right\}d\phi.$$

Considering the parametrization of the line $\mathbf{R}_{\rho-\beta}=\left[-\infty+iw,+\infty+iw\right]$ given by $\gamma\left(y\right)=y+iw$ where $w=\rho-\beta$:

$$\int_{\mathbf{R}_{\rho-\beta}}\exp\left\{-\Lambda\phi^{2}\right\}d\phi=\sqrt{\frac{\pi}{\Lambda}}.$$

Therefore,

$$\begin{aligned}
& \mathcal{F}_{\rho}^{-1}\left(\exp\left\{-\tau\psi_{0}^{\mathbb{Q}}\left(\xi\right)\right\}\right)\left(x-v\right) \\
= \; & \frac{1}{\sqrt{2\pi\sigma^{2}\tau}}\exp\left\{-\frac{\left(x-v+\tau\left(r-\frac{\sigma^{2}}{2}\right)\right)^{2}}{2\sigma^{2}\tau}\right\}.
\end{aligned}$$

For the other part,

$$\begin{aligned}
& \mathcal{F}_{\rho}^{-1}\left(\frac{\partial\psi^{\mathbb{Q}}}{\partial\lambda}\left(\xi,0\right)\frac{\exp\left\{-i\xi\ln K\right\}}{\left(\xi+i\right)\xi}\right)\left(v\right) \\
= \; & -\frac{i}{2\pi}\sigma^{2}\theta'\left(0\right)\int_{\mathbf{R}_{\rho}}\frac{\exp\left\{i\xi\left(v-\ln K\right)\right\}}{\xi+i}d\xi && (4.9)\\
& -\frac{i}{2\pi}\bar{p}\int_{\mathbf{R}_{\rho}}\frac{\exp\left\{i\xi\left(v-\ln K\right)\right\}}{\left(\xi+i\right)\left(\bar{\eta}_{+}-i\xi\right)}d\xi && (4.10)\\
& +\frac{i}{2\pi}\bar{q}\int_{\mathbf{R}_{\rho}}\frac{\exp\left\{i\xi\left(v-\ln K\right)\right\}}{\left(\xi+i\right)\left(\bar{\eta}_{-}+i\xi\right)}d\xi. && (4.11)
\end{aligned}$$

The integrand in (4.9) has only one simple pole at $-i$, since the integration line is below the pole, we could consider it as

$$\int_{\mathbf{R}_\rho} \frac{\exp\{i\xi(v-\ln K)\}}{\xi+i} d\xi$$
$$= \begin{cases} 2\pi i e^{v-\ln K} & v-\ln K > 0 \\ 0 & v-\ln K \leq 0. \end{cases}$$

For the integral in (4.11) the poles at $-i$ and $i\bar{\eta}_-$are both above the line of integration,

$$\int_{\mathbf{R}_\rho} \frac{\exp\{i\xi(v-\ln K)\}}{(\xi+i)(\bar{\eta}_-+i\xi)} d\xi$$
$$= \begin{cases} 2\pi i \left[\frac{e^{v-\ln K}}{\bar{\eta}_-+1} - \frac{e^{-\bar{\eta}_-(v-\ln K)}}{\bar{\eta}_-+1}\right], & v-\ln K > 0 \\ 0, & v-\ln K \leq 0. \end{cases}$$

In the case of the integral (4.10), we have a pole at $-i\bar{\eta}_+$below the line of integration, and a pole at $-i$ above the line,

$$\int_{\mathbf{R}_\rho} \frac{\exp\{i\xi(v-\ln K)\}}{(\xi+i)(\bar{\eta}_+-i\xi)} d\xi$$
$$= \begin{cases} 2\pi i \frac{e^{v-\ln K}}{\bar{\eta}_+-1}, & v-\ln K > 0 \\ 2\pi i \frac{e^{\bar{\eta}_+(v-\ln K)}}{\bar{\eta}_+-1}, & v-\ln K \leq 0. \end{cases}$$

Putting all together

$$\mathcal{F}_\rho^{-1}\left(\frac{\partial\psi^{\mathbb{Q}}}{\partial\lambda}(\xi,0)\frac{\exp\{-i\xi\ln K\}}{(\xi+i)\xi}\right)(v)$$
$$= \begin{cases} e^{v-\ln K}\frac{1}{2K}\left[\frac{\bar{p}}{\bar{\eta}_+-1} - \frac{\bar{q}}{\bar{\eta}_-+1}\right] + \frac{\bar{q}}{\bar{\eta}_-+1}e^{-\bar{\eta}_-(v-\ln K)} & v-\ln K > 0 \\ \frac{\bar{p}}{\bar{\eta}_+-1}e^{\bar{\eta}_+(v-\ln K)} & v-\ln K \leq 0. \end{cases}$$

Denoting

$$\begin{aligned}
\hat{C} &= \frac{1}{2K}\left[\frac{\bar{p}}{\bar{\eta}_+ - 1} - \frac{\bar{q}}{\bar{\eta}_- + 1}\right], \\
\hat{C}_- &= \frac{\bar{q}}{\bar{\eta}_- + 1} K^{\bar{\eta}_-}, \\
\hat{C}_+ &= \frac{\bar{p}}{\bar{\eta}_+ - 1}\frac{1}{K^{\bar{\eta}_+}};
\end{aligned}$$

$$u_1(x,\tau) = \tau K e^{-r\tau}\hat{C}_+ \int_{-\infty}^{\ln K} \frac{\exp\left\{-\frac{\left(x-v+\tau\left(r-\frac{\sigma^2}{2}\right)\right)^2 - 2\sigma^2\tau\bar{\eta}_+ v}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv \tag{4.12}$$

$$+\tau K e^{-r\tau}\hat{C}_- \int_{\ln K}^{+\infty} \frac{\exp\left\{-\frac{\left(x-v+\tau\left(r-\frac{\sigma^2}{2}\right)\right)^2 + 2\sigma^2\tau\bar{\eta}_- v}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv \tag{4.13}$$

$$+\tau K e^{-r\tau}\hat{C} \int_{\ln K}^{+\infty} \frac{\exp\left\{-\frac{\left(x-v+\tau\left(r-\frac{\sigma^2}{2}\right)\right)^2 - 2\sigma^2\tau v}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv \tag{4.14}$$

Let's work with the numerator of the exponents, let

$$d = \tau\left(r - \frac{\sigma^2}{2}\right).$$

For (4.12), with $D_+ = d + \sigma^2\tau\bar{\eta}_+$,

$$(x - v + d)^2 - 2\sigma^2\tau\bar{\eta}_+ v = (v - (x + D_+))^2 + (x + d)^2 - (x + D_+)^2.$$

For (4.13), with $D_- = d - \sigma^2\tau\bar{\eta}_-$,

$$(x - v + d)^2 + 2\sigma^2\tau\bar{\eta}_- v = (v - (x + D_-))^2 + (x + d)^2 - (x + D_-)^2.$$

For (4.14), with $D = d + \sigma^2\tau$,

$$(x - v + d)^2 + 2\sigma^2\tau v = (v - (x + D))^2 + (x + d)^2 - (x + D)^2.$$

$$
\begin{aligned}
u_1(x,\tau) &= \tau K e^{-r\tau}\hat{C}_+ \exp\left\{-\frac{(x+d)^2-(x+D_+)^2}{2\sigma^2\tau}\right\} \\
&\quad \times \int_{-\infty}^{\ln K} \frac{\exp\left\{-\frac{(v-(x+D_+))^2}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv \\
&\quad + \tau K e^{-r\tau}\hat{C}_- \exp\left\{-\frac{(x+d)^2-(x+D_-)^2}{2\sigma^2\tau}\right\} \\
&\quad \times \int_{\ln K}^{+\infty} \frac{\exp\left\{-\frac{(v-(x+D_-))^2}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv \\
&\quad + \tau K e^{-r\tau}\hat{C} \exp\left\{-\frac{(x+d)^2-(x+D)^2}{2\sigma^2\tau}\right\} \\
&\quad \times \int_{\ln K}^{+\infty} \frac{\exp\left\{-\frac{(v-(x+D))^2}{2\sigma^2\tau}\right\}}{\sqrt{2\pi\sigma^2\tau}} dv.
\end{aligned}
$$

With

$$
\begin{aligned}
C &= \hat{C} \exp\left\{-\frac{(x+d)^2 (x+D)^2}{2\sigma^2\tau}\right\}, \\
C_- &= \hat{C}_- \exp\left\{-\frac{(x+d)^2 (x+D_-)^2}{2\sigma^2\tau}\right\}, \\
C_+ &= \hat{C}_+ \exp\left\{-\frac{(x+d)^2 (x+D_+)^2}{2\sigma^2\tau}\right\};
\end{aligned}
$$

finally,

$$
u_1(x,\tau) = \tau K e^{-r\tau} \begin{pmatrix} C_+(x,\tau)\left[1-\Phi\left[\frac{x-\ln K+D_+}{\sigma\sqrt{\tau}}\right]\right] \\ +C_-(x,\tau)\,\Phi\left[\frac{x-\ln K+D_-}{\sigma\sqrt{\tau}}\right] \\ +C(x,\tau)\,\Phi\left[\frac{x-\ln K+D}{\sigma\sqrt{\tau}}\right] \end{pmatrix}.
$$

4.6.1 Behavior of the approximation for a particular set of parameters

It is know that the linear approximation used is precise only in a small interval around the reference point, in this case $\lambda = 0$, so it is desirable to know appropriate conditions in which the perturbation method is a good alternative.

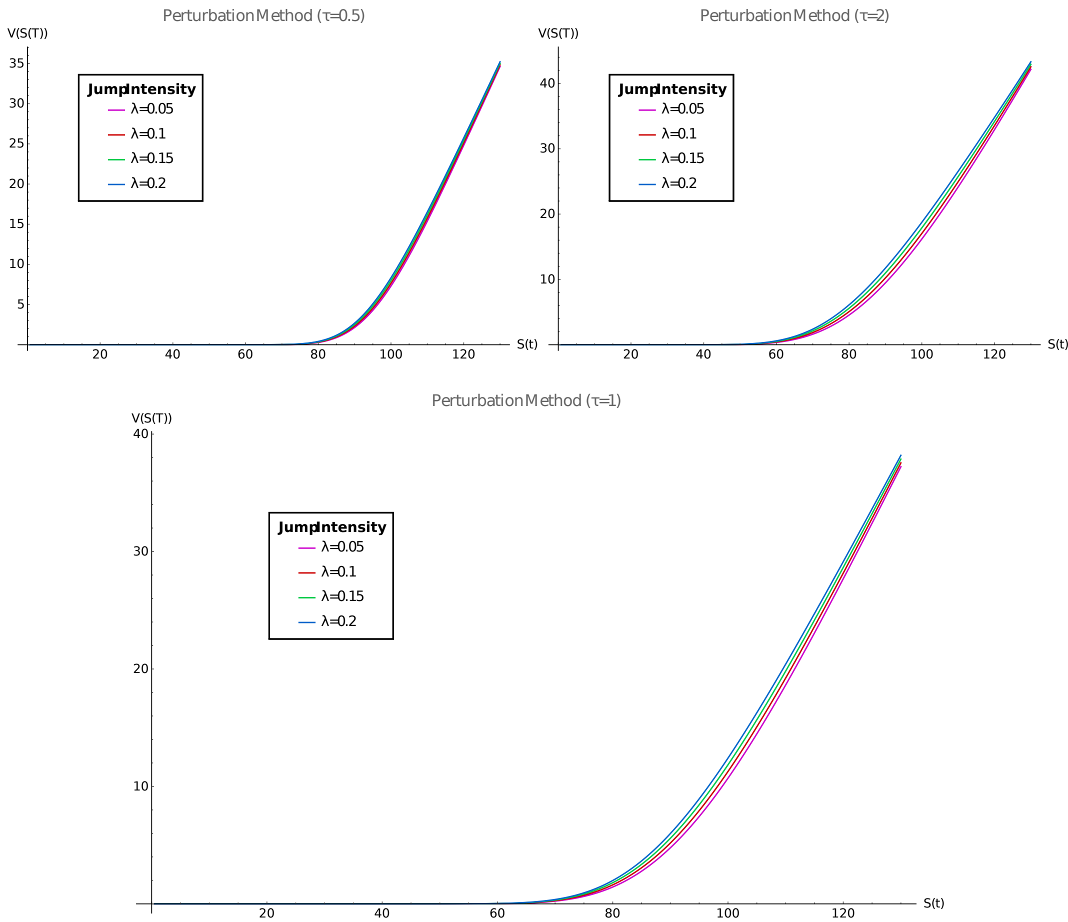


Figure 4.8: Impact of λ

It is easy to study the behavior of the approximation as λ and τ vary. To continue with the previous example, the valuation approximations will be taken for the following parameters: $K = 98$, $\mu = 0.12$, $\sigma = 0.16$, $r = 0.05$, $p = 0.4$, $\eta_+ = 10$, $\eta_- = 5$. The Figure 4.8 shows valuations for several values of λ using the perturbation method at different times.

A comparison between the different valuations for a particular value of λ are displayed in Figure 4.9.

So far, it seems that the perturbation method is good for times to expiry not so far away, and small values of λ; but to give a more concise analysis of the viability of the approximation, let's not only perform an analysis of the error compared to consider just a classic Black-Scholes framework, but taking in consideration the resources (computation time) necessary to valuate the option to different levels of precision, by the

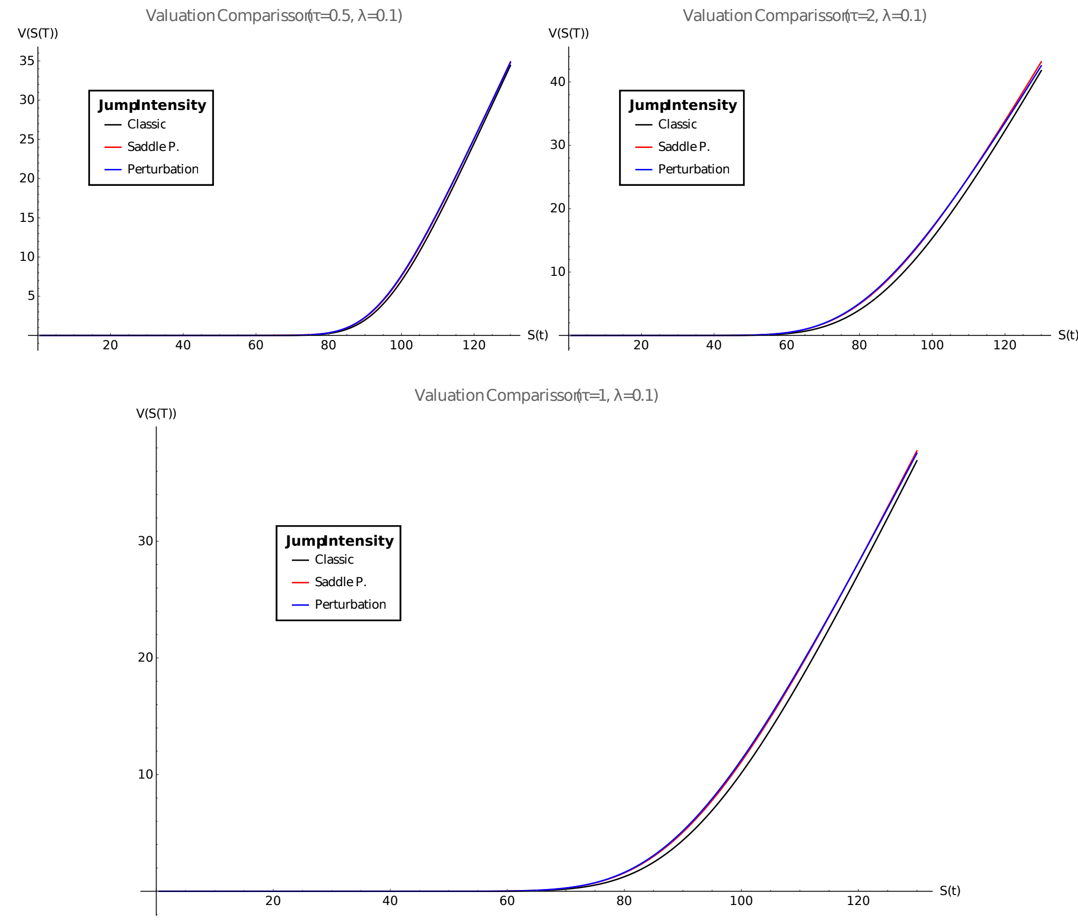


Figure 4.9: Comparison of Methods

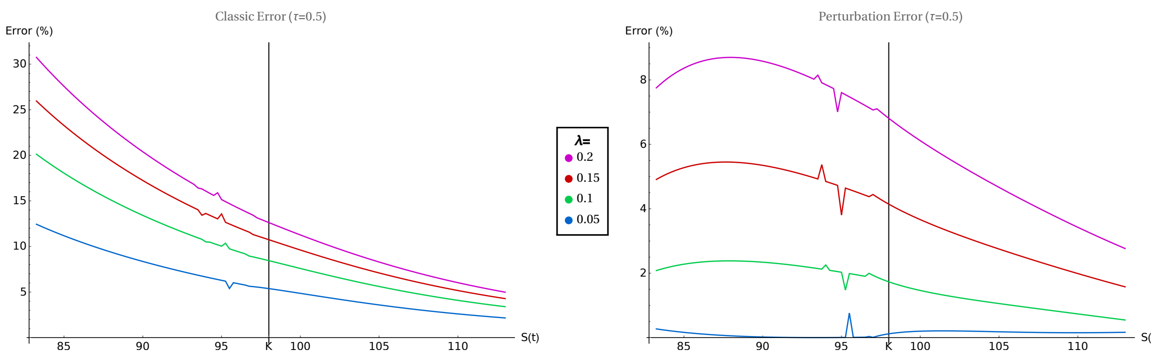
means of using the developed methods: the explicit value (saddle-point method), the classic method, and the perturbation one.

The computations presented in the following tables were performed on the same machine as the one in the Chapter 3. 120 points were calculated in each case, at fixed intervals of lenght 0.25 across the interval $[83.25, 113]$, the interval corresponds to $K \pm 15\%$.

Case 1: $\tau = 0.5$

Table 4.1: Computation Times (Milliseconds)

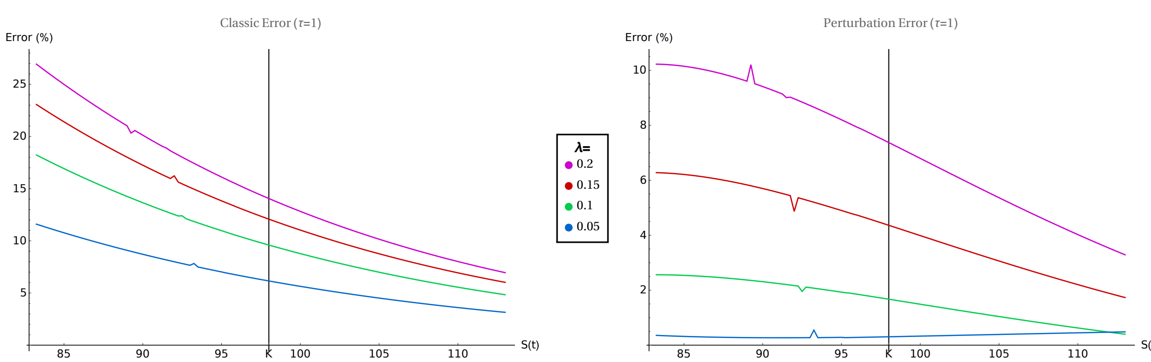
	Perturbation:	$\mathfrak{T} = \mathfrak{T}(u_0) + \mathfrak{T}(u_1) + \sum \mathfrak{T}(\lambda)$
	Classic term:	$\mathfrak{T}(u_0) = 8.190$
	Linear term:	$\mathfrak{T}(u_0) = 20.575$
λ	$\mathfrak{T}(\lambda)$	**Saddle P.**
0.20	+0.810	1543.68
0.15	+0.793	1581.12
0.10	+0.813	1671.01
0.05	+0.878	1906.81



Case 2: $\tau = 1$

Table 4.2: Computation Times (Milliseconds)

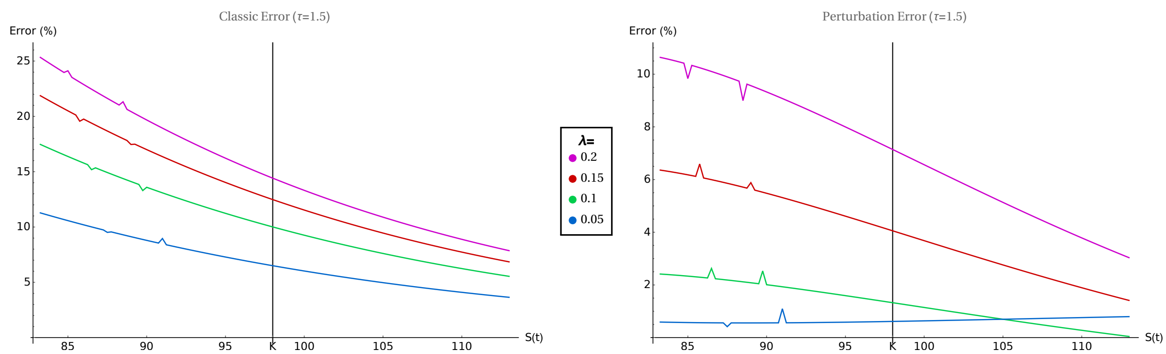
	Perturbation:	$\mathfrak{T} = \mathfrak{T}(u_0) + \mathfrak{T}(u_1) + \sum \mathfrak{T}(\lambda)$
	Classic term:	$\mathfrak{T}(u_0) = 8.028$
	Linear term:	$\mathfrak{T}(u_0) = 19.684$
λ	$\mathfrak{T}(\lambda)$	**Saddle P.**
0.20	+0.759	1613.99
0.15	+0.749	1663.86
0.10	+1.089	1663.49
0.05	+0.810	1786.21



Case 3: $\tau = 1.5$

Table 4.3: Computation Times (Milliseconds)

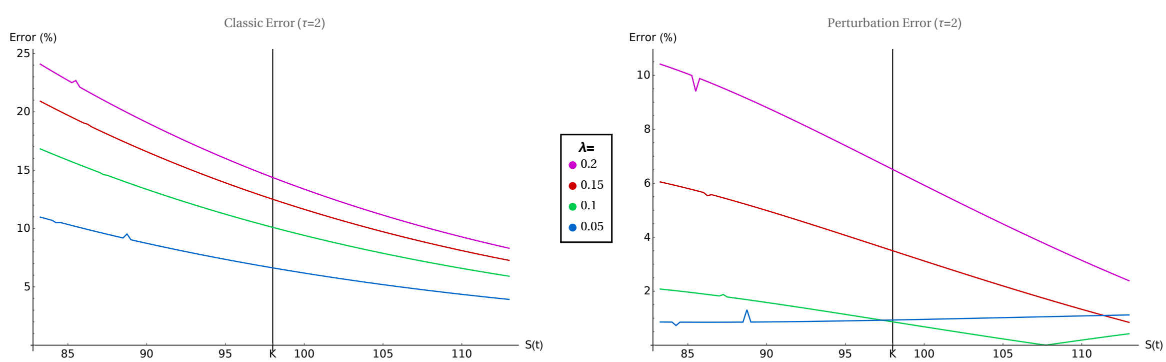
	Perturbation:	$\mathfrak{T} = \mathfrak{T}(u_0) + \mathfrak{T}(u_1) + \sum \mathfrak{T}(\lambda)$
	Classic term:	$\mathfrak{T}(u_0) = 7.693$
	Linear term:	$\mathfrak{T}(u_0) = 19.666$
λ	$\mathfrak{T}(\lambda)$	**Saddle P.**
0.20	+0.767	1569.26
0.15	+0.770	1545.55
0.10	+0.742	1769.03
0.05	+0.858	1776.42



Case 4: $\tau = 2$

Table 4.4: Computation Times (Milliseconds)

	Perturbation:	$\mathfrak{T} = \mathfrak{T}(u_0) + \mathfrak{T}(u_1) + \sum \mathfrak{T}(\lambda)$
	Classic term:	$\mathfrak{T}(u_0) = 9.566$
	Linear term:	$\mathfrak{T}(u_0) = 21.686$
λ	$\mathfrak{T}(\lambda)$	**Saddle P.**
0.20	+0.788	1597.14
0.15	+1.044	1524.60
0.10	+1.068	1690.53
0.05	+0.747	1649.26



4.6.2 Conclusions

The saddle point method turned to be a great tool to find the analytic value, the only error is present on the lower end of the spectrum for the asset price, due to the essential singularity in the lower half plane. Unlike the essential singularity in the upper half plane, no workaround was feasible. An exact numerical solution is always desired, but the computation times for this method are too high, nonetheless, this theoretical value worked as the reference to measure the precision of the perturbation method.

The approximation obtained, aside from being easy and fast to compute, has the advantage (being linear) of allowing to get almost instant valuations for different values of the perturbation magnitude (λ), once the classic valuation and the linear coefficient have been computed. This will be useful for model fitting.

Additionally, the perturbation method can lead to concise observations for a defined sets of parameters. For the parameters considered in the numerical example, a clear conclusion could be made: the approximation is no more than 10% off the theoretical value for $S \in [K - 15\%, K + 15\%]$, $\tau \leq 2$ and $\lambda \leq 0.2$. For the particular values studied, it can be concluded that the error w.r.t. the saddle-point method on the mentioned interval is bounded by 0.5λ. Additionally the computations are 50-60 times faster than the analytic valuation.

Chapter 5

Summary

- The numeric computation of the valuation using a Gaussian model with non-constant parameters is not time-efficient, leading to the necessity of fast and accurate approximations, two algorithms were found.

- For Gaussian models with parameters and dividends of the form $1 + \varepsilon\varphi(t)$ with adequate values of ε, the perturbation method gives a precise and time-efficient algorithm for the valuation of European call options. For the numerical examples studied, the computation times are 5-8 times faster than the theoretical valuation, and the error is less than 0.1ε on the interval $[K - 15\%, K + 15\%]$. Also, as the perturbation method is based on a linear approximation w.r.t. ε, once the main terms u_0 and u_1 were calculated, the re-evaluation for different values of ε takes only an extra millisecond each.

- For Gaussian models as mentioned in the last point, the alternative algotithm found resulted 4 times more precise than the perturbation method, and equally fast for a fixed value of ε, but once the value is changed, the re-evaluation takes the full time to compute.

- The numerical implementation of the generalized Black-Scholes formula is not straightforward for the Kou model, the saddle-point method was used to address this problem, however, it is not fast to compute.

- The perturbation method is a fast and accurate valuation approximation for European call options under the Kou model. In particular, times to expiry $\tau < 2$ and jump intensities $\lambda < 0.2$ were studied. The perturbation method is 50-60 times faster than the analytic valuation given by the saddle-point method, and the error is less than 0.5λ on the interval $[K - 15\%, K + 15\%]$. Additionally, being linear w.r.t. λ, the perturbation method allows fast re-evaluations for different values of λ at an additional millisecond each.

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