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El anillo de cohomología con 2 invertido de espacios de configuraciones en espacios proyectivos reales

Tesis que presenta

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A Karla,
mis papás,
y mis hermanos.

Abstract

In this work we compute the cohomology ring of the space $\mathrm{Conf}(\mathbb{RP}^m, k)$, the space of k ordered pairwise different points in the m-dimensional real projective space $\mathbb{RP}^m$, with coefficients in a commutative ring with unit where 2 is invertible.

The computation is based on the observation that the configuration space of k ordered pairwise different orbits in the m-dimensional sphere (with respect to the antipodal action) is a 2^k-fold covering of $\mathrm{Conf}(\mathbb{RP}^m, k)$. This observation also helps us in the computation of the cohomology ring of the configuration space of k ordered points in the punctured real projective space $\mathbb{RP}^m - \star$ with the same coefficients. Finally we compute the Lusternik-Schnirelmann category and topological complexity of some of the auxiliary orbit configuration spaces.

Resumen

En este trabajo calculamos el anillo de cohomología del espacio $\mathrm{Conf}(\mathbb{RP}^m, k)$, el espacio de k puntos distintos en el espacio proyectivo real m-dimensional $\mathbb{RP}^m$, con coeficientes en un anillo conmutativo con unidad donde 2 es invertible.

El cálculo está basado en la observación de que el espacio de configuraciones de k órbitas disjuntas ordenadas es un 2^k-recubrimiento de $\mathrm{Conf}(\mathbb{RP}^m, k)$. Esta observación nos ayuda en el cálculo del anillo de cohomología del espacio de configuraciones del espacio proyectivo agujereado $\mathbb{RP}^m - \star$ con los mismos coeficientes. Por último calculamos la categoría de Lusternik-Schnirelmann y la complejidad topológica de algunos espacios de configuraciones de órbitas auxiliares.

Contents

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Chapter 1

Introduction

This chapter contains a very informal –and concrete– motivation for this work, as well as a brief exposition of the main results. We begin with the motivations for the computation of the cohomology ring away from 2 of $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)$, the configuration space of k points in $\mathbb{R}\mathrm{P}^n$.

1.1 Topological complexity

In recent years, there have been many developments regarding applications of algebraic topology to problems of a slightly less abstract nature. One of such applications is topological complexity which, speaking in very broad strokes, tells us the smallest number of plans required to move around a space. So, for instance, we can ask about a certain collection of points in a space and to whether it is possible to arrive at other different collection of points without too much ambiguity in our choice (these points can be positions of some airplanes in the sky, for instance, and we could try to determine if they can all arrive at their destinations with some specified number of turns without colliding—this analogy is fairly vague, but gives the feeling of what topological complexity sees. See Figure 1.1).

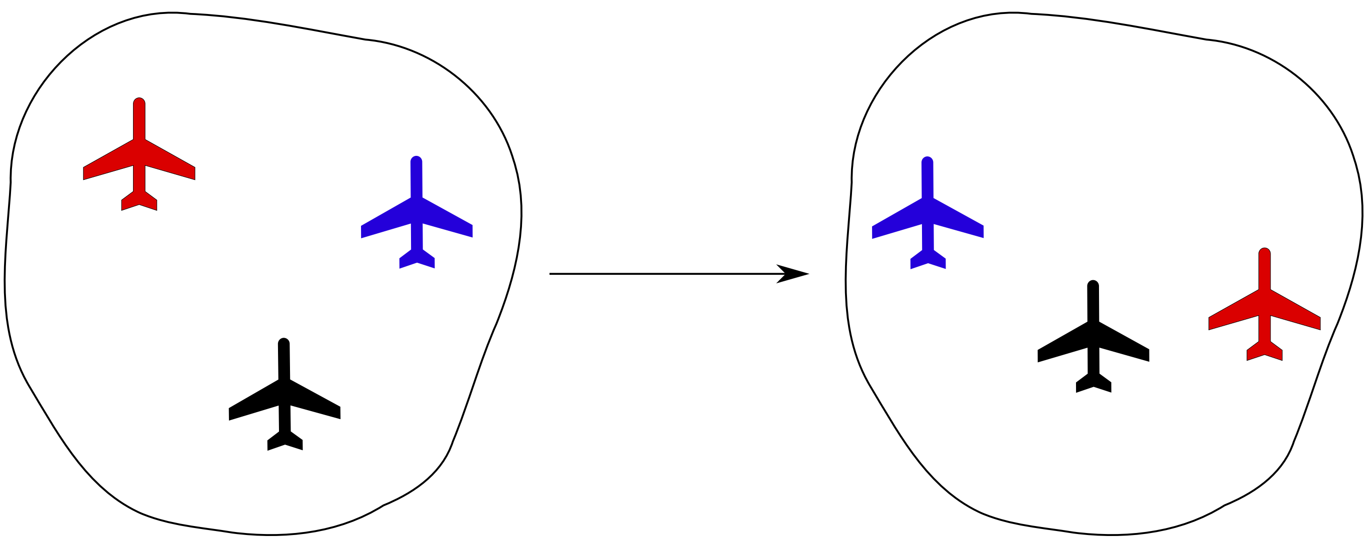

Figure 1.1: Choose a way to go from one configuration of airplanes to another (avoiding collisions).

A little more specifically, given a topological space X, with PX its path space, its topological complexity is defined as the sectional category of the fibration $PX \longrightarrow X \times X$ given by evaluation of endpoints of a curve. So, in essence, we are asking whether it is possible to chose, for each pair of endpoints in X, some path of X connecting them in such a way that the choosing is continuous in some finite number of chunks of $X \times X$. The points of discontinuities among these local sections correspond to instabilities for the choice of the path connecting two points x and y (see Figure 1.2). A very good introductory reference for this subject is [7].

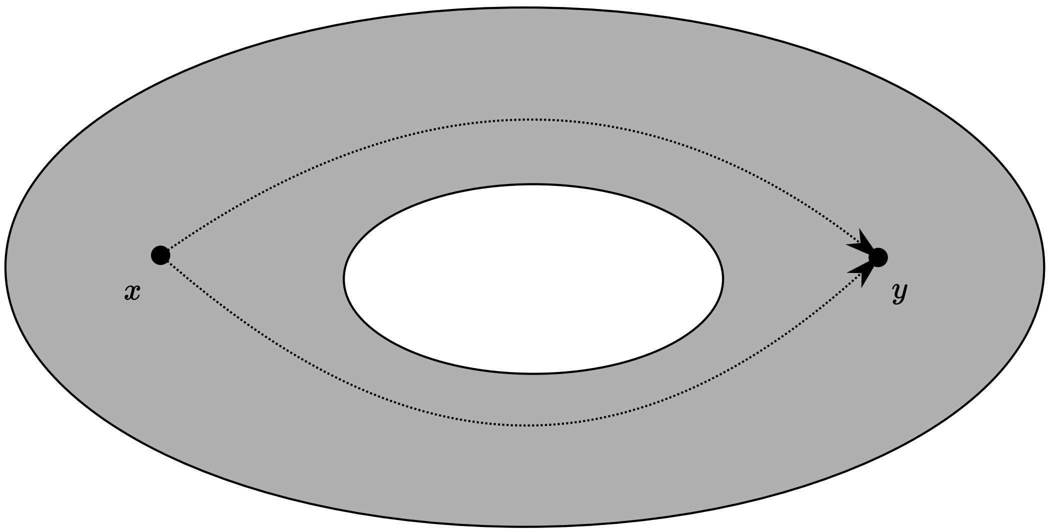

Figure 1.2: Which path is from x to y the right one?

Topological complexity has the desirable property, from an algebraic topology standpoint, that it is homotopy invariant. So if the space that is currently being studied proves too complicated or cumbersome to work with, we can replace it with an easier one. Two of the main tools to compute the topological complexity of a space, which are used in this work, are a lower bound given by the so-called zero-cup-length of the cohomology ring of the space in question, and an upper bound related to the homotopical dimension of the space.

The importance of the computation of $H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k))$ is now apparent: it gives information pertaining a possible lower bound for the topological complexity of $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k)$ and, if we are lucky enough, perhaps even enough information to completely determine $\mathrm{TC}(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k))$!

1.2 Configuration spaces

The configuration space of n points on a space X is simply the space of all ordered n-tuples of elements in X with pairwise different entries. This by itself may sound a little bit artificial, but a very simple, yet powerful way to think about configuration spaces, is to think about n bodies in 3-dimensional space. Since the positions of the bodies are all different between them, this can be modeled using the space $\mathrm{Conf}(\mathbb{R}^3, n)$. We can then study this space to try to get some information regarding, say, the n body problem.

There has been work about understanding the algebraic topology of configuration spaces, both ordered and unordered versions of them. Perhaps the first cases studied in this sense were $\mathrm{Conf}(\mathbb{R}^2, n)$ and $B(\mathbb{R}^2, n)$ (B denotes unordered configurations), and it turns out that their fundamental groups are the pure braid group and the Artin braid group on n strings.

Much more is known about configuration spaces, however. A couple of examples of results giving better understanding about them are the work of Fadell and Neuwirth [5], where in particular we can find the very useful Fadell-Neuwirth fibrations, and the work of Bödigheimer, Cohen and Taylor [2], where they give an additive description of the homology of $B(M, k)$, where B is a manifold.

As a contribution to the body of knowledge regarding algebraic topology of configuration spaces, in this work we aim to give a multiplicative structure for $H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k); R)$, where R is a commutative ring with unit where 2 is invertible. Perhaps more important that the given description itself are the techniques used to get it. A part of said techniques is the use of orbit configuration spaces, a generalization of configuration spaces for spaces which admit a group action, and an isomorphism between invariants in the cohomology of orbit configuration spaces and the cohomology of usual configurations. Here it is important that 2 is invertible, otherwise we get a rather complicated relationship between said cohomologies.

1.3 Main results

The main results are stated next where k and n stand for integers greater than 1.

Theorem 3.3.8. *Suppose R is a commutative ring with unit where 2 is invertible. For n odd, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k); R) \cong \Lambda(\iota_n) \otimes R[\mathcal{C}^+]/\mathcal{K},$$

where

- *ι_n has degree n and is the image of the generator in $\mathbb{R}\mathrm{P}^n$ under the projection on the first coordinate $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k) \xrightarrow{\pi_1} \mathbb{R}\mathrm{P}^n$,*
- *the generators in $\mathcal{C}^+$ have degree $n-1$ and are detailed at the beginning of Section 4, and*
- *$\mathcal{K}$ is the ideal generated by the relations specified in Theorem 3.3.6.*

Theorem 3.3.15. *Let R be a commutative ring with unit where 2 is invertible. For n even, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k); R) \cong \Lambda(\omega) \otimes R[\mathcal{E}]/\mathcal{J},$$

where

- *ω is a generator of degree $2n-1$ specified in Theorem 3.1.9,*

- *the set of generators $\mathcal{E}$ have degree $2n-2$ and are defined just above Lemma 3.3.12, and*
- *$\mathcal{J}$ is the ideal generated by the relations in Lemma 3.3.12.*

Remark 1.3.1. Theorem 3.3.8 and the known description of the cohomology ring of configuration spaces on spheres ([6, 9]) imply that there is a ring isomorphism

$$H^*(\mathrm{Conf}(S^n, k); R) \cong H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k); R)$$

provided n is odd. Compare with Remark 3.3. But there is no such an isomorphism if n is even, in view of Theorem 3.3.15.

The approach used in this work for Theorems 3.3.8 and 3.3.15 allows us to get information on the cohomology ring of configuration spaces on punctured real projective spaces.

Theorem 3.4.1. *Let R be a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k); R) \cong R[\mathcal{C}^+]/\mathcal{K}.$$

Theorem 3.4.4. *Let R be a commutative ring with unit where 2 is invertible. For $n \geq 2$ even, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k); R) \cong R[\mathcal{E}']/\mathcal{J}',$$

where the generators $\mathcal{E}'$ and the relations $\mathcal{J}'$ are detailed in Section 3.4.

Since the cohomology groups described here are R-free of rank independent of the actual ring R, we deduce:

Corollary 1.3.2. *There is no odd torsion in the integral cohomology of $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)$ and $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k)$.*

The general strategy for proving these results is finding covering projections and studying their cohomological properties. The viewpoint used in this work corrects and extends the method used in [16]. In addition, these results fix a couple of errors in the descriptions given in [10] for some of these cohomology rings (see Remarks 3.1 and 3.2).

1.4 Applications

Although it is not mentioned in the main results, a key ingredient to get them was a computation of $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ as a ring, which corrects previous computations found in [16]. Now we state some aplications regarding higher topological complexity and Lusternik-Schnirelmann category of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. These results are found in full detail in Chapter 4.

The description of $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ gives lower bounds for topological complexity and Lusternik-Schnirelmann category of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. These lower bounds, combined with results regarding upper bounds for topological complexity and Lusternik-Schnirelmann category found in [4],[1] and [11] yield the following Corollaries:

Corollary 1.4.1. *For $n > 2$, $\mathrm{cat}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = k$.*

Corollary 1.4.2. *Let $n > 2$. Then $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = sk$ if n is odd, whereas, if n is even, $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \in \{sk - 1, sk\}$.*

Corollary 1.4.3. *Let $n > 2$ (any parity). Then $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ agrees with the s-th zero-divisors cup-length of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$.*

Chapter 2

Preliminaries

In this Chapter we will define and state some of the results used often during the remainder of this work. These results are in nearly all cases given with references, so we will just provide statements here and we will point the reader to the respective sources of the results for their proofs.

2.1 Some algebraic lemmas

A lemma that aids our computations in Section 3.3 is Lemma 2.1.2. Here we state first Nakayama's Lemma and then, as a corollary, we obtain the lemma adapting an argument found in [14].

Theorem 2.1.1 (Nakayama's Lemma)**.** *Let R be a commutative ring with identity. Let M be a finitely generated module over R. Then if there is an ideal $I \subseteq R$ such that $IM = M$, then there exists an element $r \in R$ with $r \equiv 1 \mod I$ such that $rM = 0$.*

Lemma 2.1.2. *Let R, M as above. If M is a free R-module of rank n, then any set $\mathcal{A}$ of n elements generating M is a basis.*

Proof (Following [14, Proposition 1.2]).

Consider a basis $\mathcal{B}$ of M and a bijection $\bar{f}$ from $\mathcal{B}$ to $\mathcal{A}$. This extends to a surjective endomorphism $f : M \longrightarrow M$. Let $S = R[x]$. We consider M as an S-module with the action of x given by $xm = f(m)$. Note that we have $\langle x \rangle M = M$, where $\langle x \rangle$ denotes the ideal generated by x in S. By Nakayama's Lemma, there is $s \in S$ such that $sM = 0$ with $s - 1 \in \langle x \rangle$. Therefore $s = rx + 1$ for some $r \in S$ and we have $-rxm = m$ for all $m \in M$. Thus f is an isomorphism and $\mathcal{A}$ is a basis. □

2.2 Some topological lemmas

In this section we give an account of results used repeatedly in section 3.1 to compute degrees of certain compositions of maps.

Lemma 2.2.1. *Let $f : S^n \longrightarrow S^n$ be a map. If f is not surjective, then it is nulhomotopic.*

Corollary 2.2.2. *Let $f : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ be a map such that $f(x) = kx + c$ where $k > 0$ and $c \in \mathbb{R}^n$ is constant. If $\|c\| > k$, then $\bar{f}(x) = \dfrac{f(x)}{\|f(x)\|}$ is nulhomotopic.*

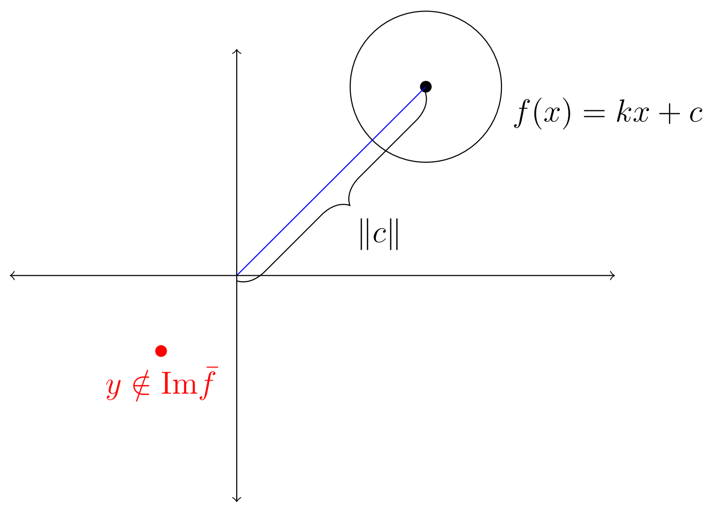

Figure 2.1: The normalization of f misses y, therefore is nulhomotopic by Lemma 2.2.1.

The argument for proving this corollary is illustrated in Figure 2.1. Note that when we look at the image of $\bar{f}$, we see that it misses at least one point in the sphere that corresponds to some point antipodal to the image of f. Instead of proving this corollary, we will prove the following generalization:

Corollary 2.2.3. *Let $f : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ be a map such that*

$$f(x) = g(x) + c$$

where $g : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ is a map and $c \in \mathbb{R}^n - \{0\}$ is constant. If $\|c\| > \|g(x)\|$ for all $x \in S^n$, then $\bar{f}(x) = \dfrac{f(x)}{\|f(x)\|}$ is nulhomotopic.

Proof. Assume that the conditions of the corollary hold for some g, c, and let $y_0 = -\frac{c}{\|c\|} \in S^n$. We claim that $y_0 \notin \mathrm{Im}\bar{f}$. Indeed, suppose on the contrary that $y_0 \in \mathrm{Im}\bar{f}$, so there exists an element $x_0 \in S^n$ such that $f(x_0) = \lambda c$, with $\lambda \in \mathbb{R}_-$. We have

$$\lambda c = f(x_0) = g(x_0) + c,$$

which implies

$$(\lambda - 1)c = g(x_0)$$

and, since $\lambda < 0$,

$$(1-\lambda)\|c\| = \|(\lambda - 1)c\| = \|g(x_0)\|.$$

Recall that, by hypothesis, $0 < \|c\| - \|g(x_0)\|$. Therefore

$$0 < \|c\| - (1-\lambda)\|c\| = \|c\| - \|(\lambda-1)c\| = \|c\| - \|g(x_0)\| \leq \|g(x_0) + c\|,$$

$$0 < (1 - (1-\lambda))\|c\|,$$

$$0 < \lambda,$$

which is a a contradiction. The result follows from Lemma 2.2.1. □

The previous Corollaries in essence are saying that if we have a map to $\mathbb{R}^n - \{0\}$ of the form $g(x) + c$ such that c is not enclosed in the image of $g(x)$, then the map in question is nulhomotopic.

Lemma 2.2.4. *Let $f : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ be a map such that $f(x) = kx + c$ where $k > 0$ and $c \in \mathbb{R}^n$. If $\|c\| < k$, then $\bar{f}(x) = \dfrac{f(x)}{\|f(x)\|}$ is homotopic to the identity map.*

Proof. We have

$$\bar{f}(x) = \frac{f(x)}{\|f(x)\|} = \frac{kx + c}{\|kx + c\|} = \frac{x + \frac{c}{k}}{\|x + \frac{c}{k}\|}.$$

Consider the map

$$H(x,t) = \frac{x + t\frac{c}{k}}{\|x + t\frac{c}{k}\|}.$$

This map satisfies $H(x,0) = x$ and $H(x,1) = \bar{f}(x)$. We claim that this is a well-defined homotopy $H : S^n \times I \longrightarrow S^n$. Indeed, suppose $x + t\frac{c}{k} = 0$ for some $x \in S^n$ and $t \in [0,1]$. Then

$$kx + tc = 0,$$

therefore

$$k = \|kx\| = \|tc\| \leq \|c\| < k$$

which is a contradiction. □

2.3 Generalities on configuration spaces

Definition 2.3.1. *Let X be a topological space. We define the configuration space of k points in X as*

$$\mathrm{Conf}(X,k) = \{(x_1, \ldots, x_k) \quad | \quad x_i \neq x_j \text{ if } i \neq j\} \subseteq X^k,$$

equipped with the subspace topology.

In this section and the following one, all manifolds will be connected and without boundary.

Theorem 2.3.2 ([5, Theorem 1]). *Let M be a manifold of dimension greater than 1. Then*

$$\mathrm{Conf}(M - \{x_0\}, k-1) \longrightarrow \mathrm{Conf}(M, k) \xrightarrow{\pi} M,$$

where π is the projection on the first coordinate and $x_0 \in M$, is a locally trivial fiber bundle.

This theorem generalizes nicely as follows:

Theorem 2.3.3 (Theorem 1.1 [6]). *Let M be a manifold of dimension greater than 1. Then*

$$\mathrm{Conf}(M - \{x_0, \dots, x_{l-1}\}, k-l) \longrightarrow \mathrm{Conf}(M, k) \xrightarrow{\pi} \mathrm{Conf}(M, l),$$

is a locally trivial fiber bundle, where π is the projection on the first l coordinates and $x_0, \dots, x_{l-1} \in M$.

Corollary 2.3.4. *Let M be a manifold of dimension greater than 1, π a projection on l different coordinates of $\mathrm{Conf}(M, k)$ and $x_0, \dots, x_{l-1} \in M$. Then*

$$\mathrm{Conf}(M - \{x_0, \dots, x_{l-1}\}, k-l) \longrightarrow \mathrm{Conf}(M, k) \xrightarrow{\pi} \mathrm{Conf}(M, l),$$

is a locally trivial fiber bundle.

2.4 Generalities on orbit configuration spaces

Definition 2.4.1. *Let X be a topological space and G be a group acting on X. We define the orbit configuration space of k G-orbits in X as*

$$\mathrm{Conf}_G(X, k) = \{(x_1, \dots, x_k) \quad | \quad Gx_i \neq Gx_j \text{ if } i \neq j\} \subseteq X^k,$$

equipped with the subspace topology. We will usually call this space just the orbit configuration space of k orbits in X.

Note 2.4.2. *In Definition 2.4.1 we can put the condition $Gx_i \cap Gx_j = \emptyset$ if $i \neq j$. Both definitions are equivalent.*

Note 2.4.3. *While the previous definition and Definition 2.3.1 make sense for $k = 1$, we will always assume $k > 1$ for both situations throughout this work.*

Note 2.4.4. *The notation $\mathrm{Conf_G}(X, k)$ is overloaded. This is because G can act in several different ways on X, yielding possibly distinct orbit configuration spaces. Thus the action of G on X has to be kept in mind while working with these objects.*

We will just say configuration space of k orbits when the group G and its action on X are clear from the context. In [15], we can find the following generalization of Theorem 2.3.3 for orbit configuration spaces:

Theorem 2.4.5 ([15, Theorem 2.2.2]). *Let G be a finite group acting freely on a manifold M of dimension greater than 1. Then*

$$\mathrm{Conf_G}(M - Q_l^G, k-l) \longrightarrow \mathrm{Conf_G}(M,k) \xrightarrow{\pi} \mathrm{Conf_G}(M,l),$$

is a locally trivial fiber bundle, where π is the projection on the first l coordinates and Q_l^G denotes the union of l disjoint G-orbits in M.

Corollary 2.4.6. *Let G be a finite group acting freely on a manifold M of dimension greater than 1, π a projection on l different coordinates of $\mathrm{Conf_G}(M,k)$ and Q_l^G the union of l disjoint G-orbits in M. Then*

$$\mathrm{Conf_G}(M - Q_l^G, k-l) \longrightarrow \mathrm{Conf_G}(M,k) \xrightarrow{\pi} \mathrm{Conf_G}(M,l),$$

is a locally trivial fiber bundle.

Example 2.4.7. *Take $G = \mathbb{Z}_2$ acting antipodally on $M = S^n$. The antipodal action is free and therefore, by Corollary 2.4.6, we have a fibration*

$$\mathrm{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k-1) \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(S^n,k) \xrightarrow{\pi_1} S^n.$$

Example 2.4.8. *Take $G = \mathbb{Z}_2$ acting on $M = \mathbb{R}^n - \{0\}$ by*

$$x \longmapsto -\frac{x}{\|x\|^2}.$$

This action is obviously free, so we have a fibration

$$(\mathbb{R}^n - \{0\}) - Q_{k-1}^{\mathbb{Z}_2} \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \xrightarrow{\pi} \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1).$$

The preceding examples will be of great importance for the computations in Section 3.1.

Lemma 2.4.9. *Let G be a group acting on spaces X, Y. Let $f : X \longrightarrow Y$ be an equivariant injective map. Then the image of*

$$f^{\times k} : \mathrm{Conf_G}(X,k) \longrightarrow Y^k$$

is contained in $\mathrm{Conf_G}(Y,k)$, and we have

$$f^{\times k} : \mathrm{Conf_G}(X,k) \longrightarrow \mathrm{Conf_G}(Y,k).$$

Proof. Suppose $f^{\times k}(x_1, \dots, x_k) \notin \mathrm{Conf_G}(Y,k)$ for some $(x_1, \dots, x_k) \in \mathrm{Conf_G}(X,k)$. Then there exist $l, l' \in \{1, \dots, k\}$ with two possibilities:

1. $f(x_l) = f(x_{l'})$. This contradicts the fact that f is injective.
2. $f(x_l) = gf(x_{l'})$ for some $g \in G$. In this case we have $f(x_l) = f(gx_{l'})$, and by injectivity $x_l = gx_{l'}$, which contradicts the fact that $(x_1, \dots, x_k) \in \mathrm{Conf_G}(X,k)$. □

The following fibration is useful to do computations regarding the cohomology of configuration spaces in terms of cohomology of orbit configuration spaces

Corollary 2.4.10 (of [15, Proposition 2.2.1])**.** *Let G be a finite group acting freely on a manifold M. Then there is a fibration*

$$\mathrm{Conf}_{\mathrm{G}}(M,k) \longrightarrow \mathrm{Conf}(M/G,k) \longrightarrow BG^k. \tag{2.4.1}$$

The Serre Spectral Sequence of the fibration (2.4.1) and the finiteness of $|G|$ yield

Theorem 2.4.11. *Let G be a finite group acting freely on a connected manifold X and let R be a commutative ring with 1 such that $|G|$ is a unit. Then there is an algebra isomorphism:*

$$H^*(\mathrm{Conf}(X/G,k);R) \cong H^*(\mathrm{Conf}_{\mathrm{G}}(X,k);R)^{G^k}.$$

Proof. Consider the Serre spectral sequence associated to

$$\mathrm{Conf}_{\mathrm{G}}(M,k) \longrightarrow \mathrm{Conf}(M/G,k) \longrightarrow BG^k.$$

The spectral sequence has E_2 term given by $E_2^{p,q} = H^p(BG^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k)))$, where $G^k = \pi_1(BG^k)$ acts on the cohomology of the fiber. Thus we have

$$E_2^{p,q} \cong H^p(BG^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k))) \cong H^p(G^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k))).$$

Now, since $|G|$ is invertible, this implies that the composition

$$H^p(G^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k))) \xrightarrow{res} H^p(\{e\}; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k))) \xrightarrow{tr} H^p(G^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k)))$$

is an isomorphism, therefore $H^p(G^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k)))$ vanishes for all $p > 0$.

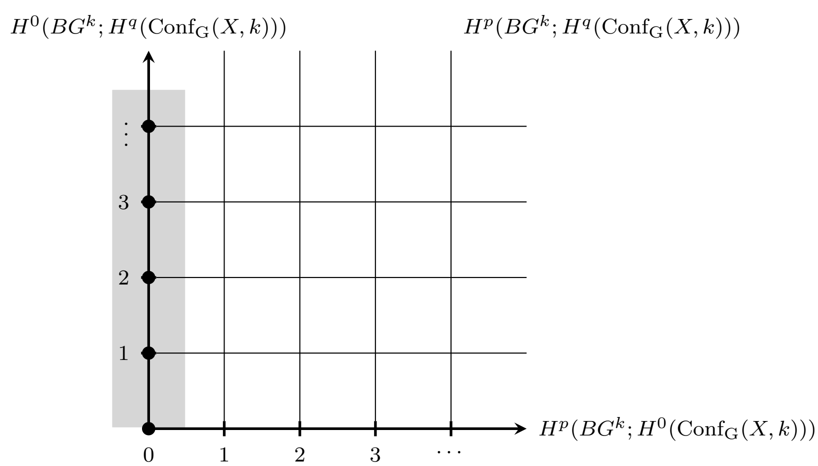

Figure 2.2: $E_2 = E_\infty$ term, all cohomology concentrated in horizontal degree 0.

Since the zeroth cohomology of a group has the form

$$H^0(BG^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k);R)) \cong H^0(G^k; H^q(\mathrm{Conf}_{\mathrm{G}}(X,k))) \cong H^q(\mathrm{Conf}_{\mathrm{G}}(X,k);R)^{G^k},$$

we get the desired isomorphism. ∎

2.5 Homotopy types of $\mathrm{Conf}(-, k)$

A natural question that arises when studying configuration spaces is whether it is a homotopy functor. In this section we give some remarks and results regarding this question.

Remark 2.5.1. $\mathrm{Conf}(-, k)$ is not necessarily homotopy invariant. Indeed, it is enough to consider the open unit interval $I = (0, 1)$, which is contractible, while $\mathrm{Conf}(I, k)$ is non-connected.

Perhaps one can ask for more interesting examples. One of such examples is given in an article by P. Salvatore and R. Longoni:

Theorem 2.5.2 ([13, Theorem 2])**.** *The spaces* $\mathrm{Conf}(L_{7,1}, k)$ *and* $\mathrm{Conf}(L_{7,2}, k)$ *are not homotopy equivalent (whereas* $L_{7,1} \simeq L_{7,2}$*).*

One of the main features of this theorem is that $L_{7,1}$ and $L_{7,2}$ are closed manifolds of the same dimension. In somewhat different spirit, as a consequence of the main results of this work, we have the following family of examples:

Theorem 2.5.3. *For n odd and $k \geq 3$,* $\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)$ *and* $\mathrm{Conf}(\mathbb{R}\mathrm{P}^{n+1} - \star, k)$ *are not homotopy equivalent, even though* $\mathbb{R}\mathrm{P}^n \simeq \mathbb{R}\mathrm{P}^{n+1} - \star$*. Moreover, these configuration spaces do not have isomorphic cohomology groups, therefore they are not stably homotopy equivalent.*

Proof. Theorem 3.3.8 implies that

$$H^{n-1}(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)) \neq 0$$

while, Theorem 3.4.4 implies that

$$H^{n-1}(\mathrm{Conf}(\mathbb{R}\mathrm{P}^{n+1} - \star, k)) = 0.$$

Therefore

$$H^{n-1}(\mathrm{Conf}(\mathbb{R}\mathrm{P}^{n+1} - \star, k)) \ncong H^{n-1}(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)). \qquad \square$$

These examples are not in the same spirit as the ones given by Salvatore and Longoni, since we are working with manifolds of different dimension and one of them is open.

2.6 The cohomology of $\mathrm{Conf}(\mathbb{R}^n, k)$

The cohomology of $\mathrm{Conf}(\mathbb{R}^n, k)$ was determined in [3].

Theorem 2.6.1. *The cohomology of* $\mathrm{Conf}(\mathbb{R}^n, k)$ *is isomorphic to the R-algebra generated by elements* $A'_{i,j}$*, for* $1 \leq j < i \leq k$*, subject to the relations*

$$A'_{r,j}A'_{r,i} = A'_{i,j}(A'_{r,i} - A'_{r,j}). \tag{2.6.1}$$

The generators $A'_{i,j}$ *correspond to the classes* $(p'_{i,j})^*(\iota_{n-1})$*, where the maps* $p'_{i,j} : \mathrm{Conf}(\mathbb{R}^n, k) \longrightarrow S^{n-1}$ *are given by*

$$p'_{i,j}(x_1, \ldots, x_k) = \frac{x_i - x_j}{\|x_i - x_j\|}$$

and ι_{n-1} *denotes the cohomology fundamental class of* S^{n-1}.

Chapter 3

The cohomology away from 2 of ordered configurations on (punctured) projective spaces

For our purposes we consider the antipodal action of the group $\mathbb{Z}_2$ on the sphere S^n. The cohomology algebra of $\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$ was determined in [16] for $n > 2$, albeit with some minor corrections required. We start by addressing the needed corrections, and extending the argument to $n \geq 2$.

3.1 The cohomology of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$

Xicotencatl's approach is to look at the Serre spectral sequence associated to the fibration (cf. Example 2.4.7)

$$\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \approx \mathrm{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k-1) \to \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \to S^n, \tag{3.1.1}$$

where the arrow on the right is the projection onto the first coordinate, and the homeomorphism on the left is induced by the stereographic projection

$$S^n - Q_1^{\mathbb{Z}_2} \xrightarrow{\approx} \mathbb{R}^n - \{0\}, \tag{3.1.2}$$

where $Q_1^{\mathbb{Z}_2}$ is the orbit of $(1, 0, \ldots, 0)$.

Lemma 3.1.1. *The action of $\mathbb{Z}_2$ on $\mathbb{R}^n - \{0\}$ in fibration (3.1.1) is given by*

$$\tau(x) = -\frac{x}{\|x\|^2}, \tag{3.1.3}$$

and it makes (3.1.2) an equivariant homeomorphism.

Proof. We consider the stereographic projection $p : S^n - Q_1^{\mathbb{Z}_2} \longrightarrow \mathbb{R}^n - \{0\}$ given by

$$(x_0, \dots, x_n) \overset{p}{\longmapsto} \frac{1}{1-x_0}(x_1, \dots, x_n),$$

with inverse

$$x = (x_1, \dots, x_n) \overset{p^{-1}}{\longmapsto} \frac{1}{\|x\|^2+1}\left(\|x\|^2-1, x_1, \dots, x_n\right).$$

Then, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n - \{0\}$ with $\|x\| < 1$,

$$\begin{aligned}
p^{-1}\tau(x) &= p^{-1}(-\frac{x}{\|x\|^2}) \\
&= \left(\frac{\|\frac{x}{\|x\|^2}\|^2 - 1}{\|\frac{x}{\|x\|^2}\|^2+1}, -\frac{\frac{x_1}{\|x\|^2}}{\|\frac{x}{\|x\|^2}\|^2+1}, \dots, -\frac{\frac{x_n}{\|x\|^2}}{\|\frac{x}{\|x\|^2}\|^2+1} \right) \\
&= \left(\frac{\frac{1}{\|x\|^2} - 1}{\frac{1}{\|x\|^2}+1}, -\frac{\frac{x_1}{\|x\|^2}}{\frac{1}{\|x\|^2}+1}, \dots, -\frac{\frac{x_n}{\|x\|^2}}{\frac{1}{\|x\|^2}+1} \right) \\
&= \left(\frac{1-\|x\|^2}{1+\|x\|^2}, -\frac{x_1}{1+\|x\|^2}, \dots, -\frac{x_n}{1+\|x\|^2} \right),
\end{aligned}$$

and

$$\begin{aligned}
-p^{-1}(x) &= -\left(\frac{\|x\|^2-1}{\|x\|^2+1}, \frac{x_1}{\|x\|^2+1}, \dots, \frac{x_n}{\|x\|^2+1} \right) \\
&= \left(\frac{1-\|x\|^2}{1+\|x\|^2}, -\frac{x_1}{1+\|x\|^2}, \dots, -\frac{x_n}{1+\|x\|^2} \right).
\end{aligned}$$

□

Remark 3.1.2. In this work, while $\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$ will denote orbit configurations on S^n with respect to the antipodal action on S^n, the notation $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ will mean orbit configurations on $\mathbb{R}^n - \{0\}$ ***with respect to the action of*** τ ***on*** $\mathbb{R}^n - \{0\}$, which is different to the antipodal action on $\mathbb{R}^n - \{0\}$ (τ in fact acts as an inversion with respect to the unit sphere followed by the antipodal map, as is apparent from Equation 3.1.3. See Figure 3.1).

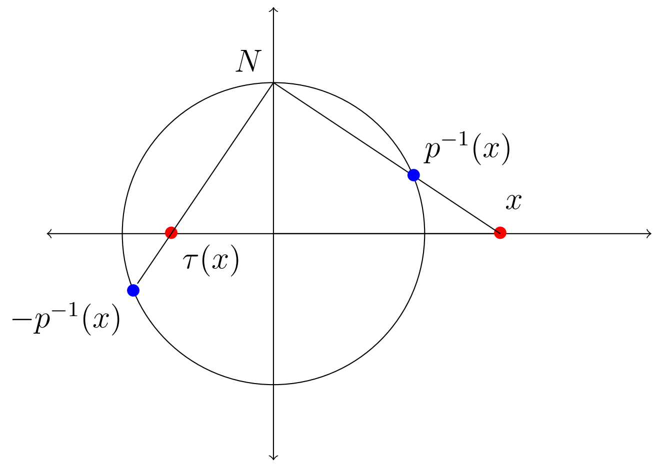

Figure 3.1: τ in action.

Xicoténcatl then computes the cohomology of the fiber in (3.1.1) using, in an inductive way, the Serre spectral sequence associated to the fibration (cf. Example 2.4.8)

$$(\mathbb{R}^n - \{0\}) - Q_{k-2}^{\mathbb{Z}_2} \to \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \to \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-2). \tag{3.1.4}$$

Proposition 3.1.3 ([10, Proposition 1, Remark 10], [16, Theorem 2.5])**.** *The system of coefficients in (3.1.4) is trivial and the Serre Spectral Sequence associated to it collapses. Therefore we have an R-module isomorphism*

$$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)) \cong M_1 \otimes M_2 \otimes \cdots \otimes M_{k-1}, \tag{3.1.5}$$

where the tensor product corresponds to the cohomology ring structure, and M_i denotes an R-free module generated by a zero dimensional class 1 and by $(n-1)$-dimensional spherical classes $\{A_{i,0}\} \cup \{A_{i,j}, A_{i,-j}\}_{1 \le j < i}$.

We next describe these generators and determine their multiplicative relations while correcting a small typographical error found in [16]. Namely, we do *not* have

$$A_{r,j}A_{r,-i} = (-1)^n(A_{j,0} + A_{i,0} - A_{i,j})(A_{r,-i} - A_{r,j}),$$

as it is claimed there, but rather the second expression in (c) of Proposition 3.1.5 below. First we define the maps which will define our generators, as well as their duals.

For $0 \le |j| < i < k$, define maps $p_{i,j} : \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \to S^{n-1}$ given by:

$$\begin{aligned} p_{i,0}(x_1, \ldots, x_{k-1}) &= \frac{x_i}{\|x_i\|} \\ p_{i,j}(x_1, \ldots, x_{k-1}) &= \frac{x_i - x_j}{\|x_i - x_j\|} \\ p_{i,-j}(x_1, \ldots, x_{k-1}) &= \frac{x_i - \tau x_j}{\|x_i - \tau x_j\|}, \end{aligned} \tag{3.1.6}$$

where the last two formulas hold for $j > 0$. As before, let ι_{n-1} be the cohomology fundamental class of S^{n-1}. Define for $j > 0$

$$\begin{aligned} A_{i,0} &= p_{i,0}^*(\iota_{n-1}) \\ A_{i,j} &= p_{i,j}^*(\iota_{n-1}) \\ A_{i,-j} &= p_{i,-j}^*(\iota_{n-1}). \end{aligned}$$

Define maps $f_{i,j} : S^{n-1} \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3)$ by

$$\begin{array}{lll} f_{1,0}(x) = (x, 2e, 3e), & f_{2,0}(x) = (e, \frac{x}{2}, 3e), & f_{2,1}(x) = (e, e + \frac{x}{2}, 3e), \\ f_{2,-1}(x) = (e, -e + \frac{x}{2}, 3e), & f_{3,0}(x) = (e, \frac{3}{2}e, \frac{x}{2}), & f_{3,1}(x) = (e, \frac{3}{2}e, e + \frac{x}{3}), \\ f_{3,2}(x) = (e, \frac{3}{2}e, \frac{3}{2}e + \frac{x}{4}), & f_{3,-1}(x) = (e, \frac{3}{2}e, -e + \frac{x}{4}), & f_{3,-2}(x) = (e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}), \end{array} \tag{3.1.7}$$

where $e = (1, 0 \ldots, 0) \in \mathbb{R}^n$.

Lemma 3.1.4. *With the notations above and with $0 \leq |s| < r < 4$ and $0 \leq |j| < i < 4$ then we have the following homotopies*

$$p_{r,s} f_{i,j} \simeq \begin{cases} \text{identity}, & \textit{if } r = i \textit{ and } s = j; \\ \text{constant}, & \textit{otherwise}, \end{cases} \tag{3.1.8}$$

Proof. We will explicitly show the result for two cases, the remaining cases are proved in a similar way.

1. $p_{r,s} f_{3,-2}$

   (a) $p_{1,0} f_{3,-2}(x) = p_{1,0}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = e = \text{constant}$.

   (b) $p_{2,0} f_{3,-2}(x) = p_{2,0}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = e = \text{constant}$.

   (c) $p_{2,1} f_{3,-2}(x) = p_{2,1}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = e = \text{constant}$.

   (d) $p_{2,-1} f_{3,-2}(x) = p_{2,-1}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = e = \text{constant}$.

   (e) $p_{3,0} f_{3,-2}(x) = p_{3,0}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = \dfrac{-\frac{2}{3}e + \frac{x}{4}}{\|-\frac{2}{3}e + \frac{x}{4}\|} \simeq$ constant, by Corollary 2.2.2.

   (f) $p_{3,1} f_{3,-2}(x) = p_{3,1}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = \dfrac{-\frac{5}{3}e + \frac{x}{4}}{\|-\frac{5}{3}e + \frac{x}{4}\|} \simeq$ constant, by Corollary 2.2.2.

   (g) $p_{3,-1} f_{3,-2}(x) = p_{3,-1}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = \dfrac{\frac{1}{3}e + \frac{x}{4}}{\|\frac{1}{3}e + \frac{x}{4}\|} \simeq$ constant, by Corollary 2.2.2.

(h) $p_{3,2}f_{3,-2}(x) = p_{3,2}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = \dfrac{-\frac{13}{6}e + \frac{x}{4}}{\|-\frac{13}{6}e + \frac{x}{4}\|} \simeq$ constant, by Corollary 2.2.2.

(i) $p_{3,-2}f_{3,-2}(x) = p_{3,-2}(e, \frac{3}{2}e, -\frac{2}{3}e + \frac{x}{4}) = \dfrac{\frac{x}{4}}{\|\frac{x}{4}\|} = x.$

2. $p_{r,s}f_{2,1}$

(a) $p_{1,0}f_{2,1}(x) = p_{1,0}(e, e + \frac{x}{2}, 3e) = e =$ constant.

(b) $p_{2,0}f_{2,1}(x) = p_{2,0}(e, e + \frac{x}{2}, 3e) = \dfrac{e + \frac{x}{2}}{\|e + \frac{x}{2}\|} \simeq$ constant, by Corollary 2.2.2.

(c) $p_{2,1}f_{2,1}(x) = p_{2,1}(e, e + \frac{x}{2}, 3e) = \dfrac{\frac{x}{2}}{\|\frac{x}{2}\|} = x.$

(d) $p_{2,-1}f_{2,1}(x) = p_{2,-1}(e, e + \frac{x}{2}, 3e) = \dfrac{2e + \frac{x}{2}}{\|2e + \frac{x}{2}\|} \simeq$ constant, by Corollary 2.2.2.

(e) $p_{3,0}f_{2,1}(x) = p_{3,0}(e, e + \frac{x}{2}, 3e) = e =$ constant.

(f) $p_{3,1}f_{2,1}(x) = p_{3,1}(e, e + \frac{x}{2}, 3e) = e =$ constant.

(g) $p_{3,-1}f_{2,1}(x) = p_{3,-1}(e, e + \frac{x}{2}, 3e) = e =$ constant.

(h) $p_{3,2}f_{2,1}(x) = p_{3,2}(e, e + \frac{x}{2}, 3e) = \dfrac{2e - \frac{x}{2}}{\|2e - \frac{x}{2}\|} \simeq$ constant, by Corollary 2.2.2.

(i) $p_{3,-2}f_{2,1}(x) = p_{3,-2}(e, e + \frac{x}{2}, 3e) = \frac{3e - \tau(e + \frac{x}{2})}{\|3e - \tau(e + \frac{x}{2})\|}$.
Note that, since $\frac{1}{2} \leq \|e + \frac{x}{2}\|$ for all $x \in S^n$, $\|-\tau(e + \frac{x}{2})\| \leq 2 < \|3e\|$ for all $x \in S^n$. Therefore, by Corollary 2.2.3,

$$p_{3,-2}f_{2,1}(x) \simeq \text{constant}.$$

□

Now put

$$\mathcal{A} = \{A_{i,j}, \,|\, 1 \leq j < i < k\} \cup \{A_{i,-j} \,|\, 1 \leq j < i < k\} \cup \{A_{i,0} \,|\, 1 \leq i < k\}. \tag{3.1.9}$$

Proposition 3.1.5. *$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is the graded commutative R-algebra generated by the set $\mathcal{A}$ subject to the relations*

(a) For $0 \leq j < i < k$,

$$A_{i,j}^2 = A_{i,-j}^2 = 0.$$

(b) For $1 \leq i < r < k$

$$\begin{aligned} A_{r,0}A_{r,i} &= A_{i,0}(A_{r,i} - A_{r,0}), \\ A_{r,0}A_{r,-i} &= (-1)^n A_{i,0}(A_{r,-i} - A_{r,0}), \\ A_{r,i}A_{r,-i} &= (-1)^n A_{i,0}(A_{r,-i} - A_{r,i}). \end{aligned}$$

(c) *For* $1 \leq j < i < r < k$

$$\begin{aligned} A_{r,j}A_{r,i} &= A_{i,j}(A_{r,i} - A_{r,j}), \\ A_{r,j}A_{r,-i} &= (-1)^n(A_{j,0} + A_{i,0} - A_{i,-j})(A_{r,-i} - A_{r,j}), \\ A_{r,i}A_{r,-j} &= (-1)^n A_{i,-j}(A_{r,-j} - A_{r,i}), \\ A_{r,-j}A_{r,-i} &= (-1)^n(A_{i,0} - A_{i,j} + (-1)^n A_{j,0})(A_{r,-i} - A_{r,-j}). \end{aligned}$$

Since [16] offers little detail on the actual derivation of the multiplicative relations above, for the sake of completeness, we next give the full argument giving the correct relation for $A_{r,j}A_{r,-i}$. Our method differs slightly from the one sketched in [16].

Proof of the second relation in (c) of Proposition 3.1.5. In order to obtain the relation

$$A_{r,j}A_{r,-i} = (-1)^n(A_{j,0} + A_{i,0} - A_{i,-j})(A_{r,-i} - A_{r,j}),$$

we start by considering the map $\alpha : \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \longrightarrow \mathrm{Conf}(\mathbb{R}^n, 3)$ given by

$$\alpha(x, y, z) = (x, \tau y, z).$$

We will show that by applying α^* to the known relation

$$A'_{3,1}A'_{3,2} = A'_{2,1}(A'_{3,2} - A'_{3,1})$$

we obtain the desired equality. We clearly have

$$\begin{aligned} \alpha^*(A'_{3,2}) &= A_{3,-2} \\ \alpha^*(A'_{3,1}) &= A_{3,1}, \end{aligned}$$

and we next compute $\alpha^*(A'_{2,1})$. This will be achieved by computing the degrees of the compositions $p'_{2,1}\alpha f_{i,j} \colon S^{n-1} \to S^{n-1}$, as those degrees give the coefficients $\alpha^*(A'_{2,1})$ in terms of the basis $\mathcal{A}$. Let $N : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ be the normalization map. We have

$$p'_{2,1}\alpha f_{1,0}(x) = N(\frac{-e}{2} - x) = -N(\frac{e}{2} + x), \tag{3.1.10}$$

$$p'_{2,1}\alpha f_{2,0}(x) = N(-2x - e) = -N(2x + e), \tag{3.1.11}$$

$$p'_{2,1}\alpha f_{2,1}(x) = N(\tau(e + \frac{x}{2}) - e), \tag{3.1.12}$$

$$p'_{2,1}\alpha f_{2,-1}(x) = N(\tau(-e + \frac{x}{2}) - e) = -N(-e + \frac{x}{2} + \|-e + \frac{x}{2}\|^2 e), \tag{3.1.13}$$

$$\begin{aligned} p'_{2,1}\alpha f_{3,0}(x) &= p'_{2,1}\alpha f_{3,1}(x) = p'_{2,1}\alpha f_{3,2}(x) \\ &= p'_{2,1}\alpha f_{3,-1}(x) = p'_{2,1}\alpha f_{3,-2}(x). \end{aligned} \tag{3.1.14}$$

We need to compute the degrees of all these maps, and for most of them it is easily computed:

- The maps in (3.1.10) and (3.1.11) are obviously homotopic to the antipodal map.
- By Corollary 2.2.3, the map in (3.1.12) is homotopic to the constant map since e is not enclosed by the image of $\tau(e + \frac{x}{2})$.
- The maps in (3.1.14) are all obviously constant maps.

Identifying the degree of $p'_{2,1}\alpha f_{2,-1}$, however, requires some work: Let $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the map given by

$$F(t, x) = F(t, (t_1, t_2, \ldots, t_n)) = (tt_1, t_2, \ldots, t_n).$$

Note that $F(1, x) = x$ and $F(-1, x)$ is x reflected across the hyperplane $t_1 = 0$. As maps $S^{n-1} \longrightarrow S^{n-1}$, we have

$$N(-e + \frac{x}{2} + \|-e + \frac{x}{2}\|^2 e) = N(F(1, \frac{x}{2}) + (-1 + \|-e + \frac{x}{2}\|^2)e) \simeq N(F(-1, \frac{x}{2}) + (-1 + \|-e + \frac{x}{2}\|^2)e).$$

The homotopy on the right is given by

$$N(F(t, \frac{x}{2}) + (-1 + \|-e + \frac{x}{2}\|^2)e) \qquad \text{for } t \in [-1, 1],$$

and it is well defined: suppose there exist $t \in [-1, 1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that

$$F(t, \frac{x}{2}) + (-1 + \|-e + \frac{x}{2}\|^2)e = 0.$$

Then $F(t, \frac{x}{2}) = (1 - \|-e + \frac{x}{2}\|^2)e$ and so we have $\frac{tt_1}{2} = 1 - \|-e + \frac{x}{2}\|^2$ and $t_i = 0$ for $i > 1$. The latter condition, in turn, implies $t_1 = \pm 1$.

- Suppose $t_1 = 1$. Then $\frac{t}{2} = 1 - \|-e + \frac{e}{2}\|^2 = 1 - \|-\frac{e}{2}\|^2 = \frac{3}{4}$, so $t = \frac{3}{2} > 1$.
- Suppose $t_1 = -1$. Then $-\frac{t}{2} = 1 - \|-e - \frac{e}{2}\|^2 = 1 - \|-\frac{3e}{2}\|^2 = -\frac{5}{4}$, so $t = \frac{5}{2} > 1$.

Both assumptions lead to a contradiction, so the homotopy is well defined. Now we will prove that, as maps $S^{n-1} \longrightarrow S^{n-1}$,

$$N(F(-1, \frac{x}{2}) + (-1 + \|-e + \frac{x}{2}\|^2)e) \simeq N(F(-1, \frac{x}{2})).$$

Consider the homotopy

$$N(F(-1, \frac{x}{2}) + t(-1 + \|-e + \frac{x}{2}\|^2)e) \qquad \text{for } t \in [0, 1].$$

This homotopy is well defined: suppose there exist $t \in [0, 1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that

$$F(-1, \frac{x}{2}) + t(-1 + \|-e + \frac{x}{2}\|^2)e = 0.$$

Then $F(-1, \frac{x}{2}) = t(1 - \|-e + \frac{x}{2}\|^2)e$ and so we have $-\frac{t_1}{2} = t(1 - \|-e + \frac{x}{2}\|^2)$ and $t_i = 0$ for $i > 1$. The latter condition, in turn, implies $t_1 = \pm 1$.

- Suppose $t_1 = 1$. Then $-\frac{1}{2} = t(1 - \|-e + \frac{e}{2}\|^2) = t(1 - \|-\frac{e}{2}\|^2) = t\frac{3}{4}$, so $t = -\frac{2}{3} < 0$
- Suppose $t_1 = -1$. Then $\frac{1}{2} = t(1 - \|-e - \frac{e}{2}\|^2) = t(1 - \|-\frac{3e}{2}\|^2) = -t\frac{5}{4}$, so $t = -\frac{2}{5} < 0$.

Both assumptions lead to a contradiction, so the homotopy is well defined. Therefore

$$p'_{2,1}\alpha f_{2,-1}(x) \simeq -N(F(-1, \frac{x}{2})) = -N(F(-1, x))$$

which is clearly a map of degree $(-1)^{n+1}$.

Having understood the maps $p'_{2,1}\alpha f_{i,j}(x)$, we can read off the expression for $\alpha^*(A'_{2,1})$, namely, $\alpha^*(A'_{2,1}) = (-1)^n(A_{1,0} + A_{2,0} - A_{2,-1})$. Then, by applying α^* to the relation

$$A'_{3,1}A'_{3,2} = A'_{2,1}(A'_{3,2} - A'_{3,1}),$$

we get

$$A_{3,1}A_{3,-2} = (-1)^n(A_{1,0} + A_{2,0} - A_{2,-1})(A_{3,-2} - A_{3,1}),$$

which is the second relation asserted in ((c)) above in the case $(r, i, j) = (3, 2, 1)$. The general case follows by applying the maps

$$\pi_{r,i,j} : F_{\mathbb{Z}_2}(\mathbb{R}^n - 0, k - 1) \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - 0, 3)$$

given by $\pi_{r,i,j}(x_1, \ldots, x_{k-1}) = (x_j, x_i, x_r)$, and which evidently satisfy

$$\begin{array}{rclrclrcl}
\pi^*_{r,i,j}(A_{1,0}) & = & A_{j,0}, & \pi^*_{r,i,j}(A_{2,0}) & = & A_{i,0}, & \pi^*_{r,i,j}(A_{2,1}) & = & A_{i,j}, \\
\pi^*_{r,i,j}(A_{2,-1}) & = & A_{i,-j}, & \pi^*_{r,i,j}(A_{3,0}) & = & A_{r,0}, & \pi^*_{r,i,j}(A_{3,1}) & = & A_{r,j}, \\
\pi^*_{r,i,j}(A_{3,2}) & = & A_{r,i}, & \pi^*_{r,i,j}(A_{3,-1}) & = & A_{r,-j}, & \pi^*_{r,i,j}(A_{3,-2}) & = & A_{r,-i}.
\end{array}$$

□

The product relations in (a) and (b) can be obtained by considering the maps $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) \longrightarrow \mathrm{Conf}(\mathbb{R}^n, 3)$ given by:

$$\begin{array}{rcl}
(x, y) & \longmapsto & (0, x, y) \\
(x, y) & \longmapsto & (0, \tau x, y)
\end{array}$$

and computing the images of their cohomological counterparts applying arguments similar to the one given in our previous proof.

Similarly, the first, third, and fourth product relations in (c) can be obtained, respectively, by considering the maps $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \longrightarrow \mathrm{Conf}(\mathbb{R}^n, 3)$ given by:

$$\begin{array}{rcl}
(x, y, z) & \longmapsto & (x, y, z) \\
(x, y, z) & \longmapsto & (\tau x, y, z) \\
(x, y, z) & \longmapsto & (\tau x, \tau y, z)
\end{array}$$

and computing their images in cohomology.

These relations imply that the cohomology of the fiber in (3.1.1) is additively generated by products of the form $A_{i_1,j_1} \cdots A_{i_r,j_r}$ where $i_l < i_{l'}$ if $l < l'$. Furthermore, such products are in fact an additive basis in view of (3.1.5). In summary, we have the following theorem.

Theorem 3.1.6 (Corrected form of [16, Theorem 1.1]). *For $n \geq 2$, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)) \cong R[\mathcal{A}]/I, \tag{3.1.15}$$

where I denotes the ideal generated by the relations (a), (b), (c) *above, and $\mathcal{A}$ is defined in (3.1.9).*

Now we determine the cohomology ring of the total space of (3.1.1), following Section 4 of [16]. Since $n \geq 2$, S^n is simply connected and we have trivial coefficients on the Serre spectral sequence associated to (3.1.1). Also, S^n has torsion free cohomology, therefore

$$E_2^{p,q} \cong H^p(S^n; H^q(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))) \cong H^p(S^n) \otimes H^q(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)).$$

Proposition 3.1.7. *For n odd, the Serre spectral sequence associated to*

$$\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \approx \mathrm{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k-1) \to \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \xrightarrow{\pi_1} S^n,$$

collapses.

Proof. Consider a nowhere vanishing vector field on S^n. This gives a section

$$\sigma : S^n \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$$

of 3.1.1, by choosing $k-1$ vectors along the direction of the vector field and applying the exponential map. Consequently we have a monomorphism

$$\pi_1^* : H^*(S^n) \longrightarrow H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)),$$

so the differentials with image in the 0-th row cannot be nonzero. Extending this using the description of E_2 we already have computed, we get the result. □

Theorem 3.1.8 ([10, Proposition 14],[16, Proposition 5.2(a)]). *For $n > 2$ odd, there is an R-algebra isomorphism $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong H^*(S^n) \otimes H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$.*

Note that the multiplicative structure in $E_\infty = E_2$, which is just the tensor product of the multiplicative structures for the base and fiber, already gives the multiplicative structure of $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k))$, by dimensional considerations —recall n is an odd integer greater than 1.

For n even, Xicoténcatl shows that the differential $d_n^{0,n-1} : E_n^{0,n-1} \longrightarrow E_n^{n,0}$ is determined by $d_n(A_{i,j}) = 2\iota_n$ for all $A_{i,j} \in \mathcal{A}$ (see also [10, Proposition 13]). In particular, if the characteristic of R is 2, the conclusion of (and argument for) Theorem 3.1.8 holds also for any even n (the case $n = 2$ requires an additional argument based on Brown representability, see the proof of Theorem 3.1.9 below). We close the section with a description of the R-cohomology algebra of $\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$ for n even under the additional hypothesis—in force throughout the rest of this section—that the characteristic of R is either zero (e.g. $R = \mathbb{Z}$ or $R = \mathbb{Q}$) or an odd integer (e.g. $R = \mathbb{Z}_t$, odd t), so that the map $2\colon R \to R$ given by multiplication by 2 is injective.

It will be convenient to make a change of basis by defining $B_{i,j} = A_{i,j} - A_{1,0}$, for $|j| < i < k$, and $\mathcal{B} = \{B_{i,j} \,|\, |j| < i < k \text{ and } 1 < i\}$. A straightforward computation shows that a product of two given elements in $\mathcal{B}$ satisfies the exact same relation holding for the product of the corresponding two elements in $\mathcal{A}$ (keeping in mind that, by definition, $B_{1,0} = 0$). Let us denote by J the resulting set of relations among the $B_{i,j}$'s. It is also clear that a new basis for $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is obtained from the basis described just before Theorem 3.1.6 by replacing each factor $A_{i,j}$ with $i > 1$ by the corresponding $B_{i,j}$. In these conditions, the hypothesis on the characteristic of R, and the fact that the differential sends every $A_{i,j}$ to $2\iota_n$ imply that $\mathcal{B}$ is a basis for the kernel of $d_n^{0,n-1}$. More generally, let $\mathbb{K} = \ker\, d_n^{0,n-1}$ denote the (free) R-module generated by $\mathcal{B}$, and let $\mathbb{K}^j$ denote the R-module generated by products of j factors in $\mathbb{K}$, where $\mathbb{K}^0$ and $\mathbb{K}^{-1}$ are set to be R and 0 respectively. Then a basis for $\mathbb{K}^j$ is given by the degree $j(n-1)$ elements in the above modified basis for $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ which do not contain the factor $A_{1,0}$ (e.g. $\mathbb{K}^{k-1} = 0$). It is then clear that the only non-trivial terms in the $(n+1)$-stage of the spectral sequence are given by

$$\begin{aligned} E_{n+1}^{0,j(n-1)} &= \mathbb{K}^j, && \text{for } 0 \le j \le k-2; \\ E_{n+1}^{n,j(n-1)} &= \iota_n A_{1,0}\mathbb{K}^{j-1} \oplus (\iota_n \mathbb{K}^j)_2, && \text{for } 0 \le j \le k-1; \end{aligned}$$

where $(-)_2$ denotes the mod 2 reduction of the given module (that is, tensoring with $\mathbb{Z}_2$). There are no extension problems in the spectral sequence since its $p = 0$ column is R-free. Further, just as with Theorem 3.1.8, if $n > 2$, the multiplicative structure of the cohomology of the total space follows by dimensional considerations from that for the E_∞-term of the spectral sequence. In fact:

Theorem 3.1.9. *Assume that the characteristic of R is either zero or an odd integer. For even $n \geq 2$ there is an isomorphism of graded R-algebras*

$$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong R[\mathcal{B}]/J \otimes \Lambda(\lambda, \omega)/(2\lambda, \lambda\omega)$$

where λ and ω are represented in the spectral sequence by ι_n and $\iota_n A_{1,0}$, respectively.

Proof. It only remains to argue the assertion about the multiplicative structure when $n = 2$. (The issue is mentioned without explanation by Feichtner and Ziegler on the first half of page 100 in [10].) The point is that, for any even n, $2B_{i,j}^2 = 0$ by anticommutativity. But for $n = 2$ we need to rule out the possibility that, as an element in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k))$, the square of a 1-dimensional class $B_{i,j}$ agrees with the 2-dimensional 2-torsion class λ. This follows from Brown representability when the coefficients are $\mathbb{Z}$. For other coefficients R the assertion holds since the definition of the classes $B_{i,j}$ is natural with respect to the canonical ring morphism $\mathbb{Z} \to R$. □

Note $R[\mathcal{B}]/J = \bigoplus_{0 \le j \le k-2} \mathbb{K}^j$, a basis of which has already been described. In the E_∞ term of the spectral sequence, this R-subalgebra corresponds to the left hand side tower supported by 1. Besides, two additional "copies" of this tower show up: one copy (tensored with $\mathbb{Z}_2$) is supported by λ; another copy (shifted one level up) is supported by ω:

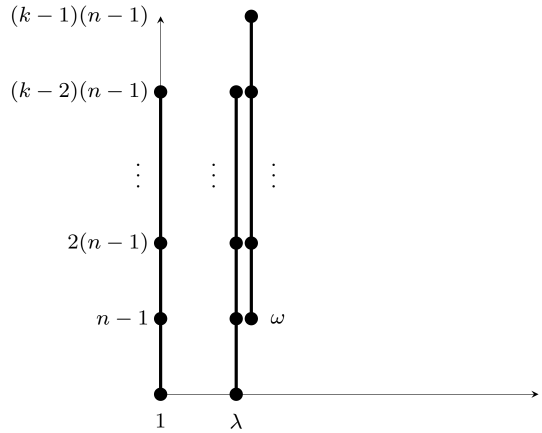

Figure 3.2: The E_∞ term.

Remark 3.1.10. The additive version of Theorem 3.1.9 is obtained in [16, Theorem 5.2, items (b) and (c)] assuming implicitly $n > 2$. On the other hand, for $n > 2$, the multiplicative relations among generators in Theorems 3.1.6 and 3.1.9 correct those found in [10]. In fact, the multiplicative relations described by Feichtner and Ziegler in [10, Proposition 11] for their generators in the cohomology of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ lead to inconsistencies. We illustrate the problem using Feichtner-Ziegler's notation, which the reader is assumed to be familiar with. (In particular, the notation for the fiber in (3.1.1) will momentarily change to $F_{\langle\varphi\rangle}(\mathbb{R}^k \setminus \{0\}, n)$). Take $1 \leq i < j \leq n$, and let k be odd (so that the generators $c_i, c_{i,j}^+, c_{i,j}^-$ are even dimensional and, therefore, commute without introducing signs). Then Lemma 7 and Proposition 11 in [10] imply

$$c_{i,j}^- c_{i,j}^+ + c_i(c_{i,j}^+ + c_{i,j}^-) = 0 = A_i(0) = A_i\left(c_{i,j}^- c_{i,j}^+ + c_i(c_{i,j}^+ + c_{i,j}^-)\right) = c_{i,j}^- c_{i,j}^+ - c_i(c_{i,j}^+ + c_{i,j}^-).$$

This yields $c_i c_{i,j}^+ + c_i c_{i,j}^- = 0$, if we work with integral coefficients. However the latter relation contradicts Proposition 8(2) in [10].

3.2 $(\mathbb{Z}_2)^k$-Action

In this section, R will denote a commutative ring with unit where 2 is (still) not necessarily invertible. In [16], the action of the group $(\mathbb{Z}_2)^k$ on $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k))$ induced via antipodal maps on each coordinate was determined for $k \leq 3$, with most details omitted; here we generalize Xicoténcatl's result for all k, providing full details in typical cases, and correcting the description for $k = 3$.

Let us denote by $\epsilon_i : \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$ the antipodal map on the i-th coordinate and, by abuse of notation, its induced map in cohomology. We will work with the Serre spectral sequence of (3.1.1), and determine the action of $(\mathbb{Z}_2)^k = \langle \epsilon_1, \epsilon_2, \ldots, \epsilon_k \rangle$ on the cohomology of the total space by understanding the action of each ϵ_i on the cohomology of the base and the fiber. We first state the main results of this section (dealing with the action on the fiber), and then we recall (from [16]) the details on how $(\mathbb{Z}_2)^k$ can be thought of as acting on the fiber of (3.1.1).

Theorem 3.2.1. *For $n \geq 2$, the action of $(\mathbb{Z}_2)^k$ on $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is given by*

$$\epsilon_l A_{i,j} = \begin{cases} (-1)^{n-1}A_{j,0} - A_{i,0} + A_{i,j} & \text{if } l = 1,\ j > 0; \\ -A_{|j|,0} - A_{i,0} + A_{i,j} & \text{if } l = 1,\ j < 0; \\ -A_{i,0} & \text{if } l = 1,\ j = 0,\ i \geq 1; \\ A_{i,-j} & \text{if } l > 1,\ |j| = l-1; \\ (-1)^n A_{i,0} & \text{if } l > 1,\ i = l-1,\ j = 0; \\ (-1)^n A_{j,0} + (-1)^n A_{i,0} + (-1)^{n-1} A_{i,-j} & \text{if } l > 2,\ i = l-1,\ j > 0; \\ A_{|j|,0} + (-1)^n A_{i,0} + (-1)^{n-1} A_{i,|j|} & \text{if } l > 2,\ i = l-1,\ j < 0; \\ A_{i,j} & \text{otherwise.} \end{cases} \tag{3.2.1}$$

Theorem 3.2.2. *For $n \geq 2$ even, the action of $(\mathbb{Z}_2)^k$ on the permanent cycles $\mathbb{K}^* \subseteq H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is given by*

$$\epsilon_l B_{i,j} = \begin{cases} -B_{|j|,0} - B_{i,0} + B_{i,j} & \text{if } l = 1,\ |j| > 0; \\ -B_{i,0} & \text{if } l = 1,\ j = 0,\ i > 1; \\ B_{i,-j} & \text{if } l > 1,\ |j| = l-1; \\ B_{|j|,0} + B_{i,0} - B_{i,-j} & \text{if } l > 2,\ i = l-1,\ |j| > 0; \\ B_{i,j} & \text{otherwise.} \end{cases} \tag{3.2.2}$$

Note that $B_{1,0} = 0$ in (3.2.2), and that the formulas in (3.2.2) are the same ones as those in (3.2.1) for n even and replacing each A with B.

Unlike the maps ϵ_l for $l > 1$, ϵ_1 does not preserve the fiber in (3.1.1). Indeed, ϵ_1 covers the antipodal map. This issue is dealt with in [16] by using the rotation

$$T = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -I_2 \end{pmatrix} \in SO(n+1)$$

that interchanges the north and south poles $N = (0, \ldots, 0, 1), S = (0, \ldots, 0, -1) \in S^n$. In detail, the restriction of T to S^n is $\mathbb{Z}_2$-equivariant and it is $\mathbb{Z}_2$-equivariantly isotopic to the identity, therefore it induces a map $T^{\times k} : \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)$ homotopic to the identity such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) & \xrightarrow{T^{\times k} \circ \epsilon_1} & \mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ S^n & \xrightarrow{-T} & S^n. \end{array}$$

Since $-T$ fixes the north pole, $T^{\times k} \circ \epsilon_1$—which is homotopic to ϵ_1—restricts to a map on the corresponding fiber. This allows us to understand the effect of $T^{\times k} \circ \epsilon_1$ (and, consequently, of ϵ_1) on the spectral sequence. With this in mind note that, after removing the poles and taking into account the stereographic projection, the map T induces a map $\tilde{T} : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$, given by

$$\tilde{T}(x) = \frac{\bar{x}}{\|x\|^2},$$

where $\bar{x} = (t_1, \ldots, t_{n-1}, -t_n)$ for $x = (t_1, \ldots, t_n)$. Thus, the action of $T^{\times k} \circ \epsilon_1$ restricted to the fiber is given by $\tilde{T}^{\times(k-1)}$, so the action of ϵ_1 on the cohomology of the fiber is the same as the action of the map $\epsilon_1'(x_1, \ldots, x_{k-1}) = (\tilde{T}(x_1), \ldots, \tilde{T}(x_{k-1}))$, which from now on we will also denote by ϵ_1. The remaining actions restricted to the fiber are given by $\epsilon_l(x_1, \ldots, x_{k-1}) = (x_1, \ldots, \tau x_{l-1}, \ldots, x_{k-1})$ for $1 < l \leq k$. The maps $\epsilon_1, \epsilon_2 \cdots \epsilon_k\colon \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \to \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)$ are related as follows:

Lemma 3.2.3. *For n odd, $\epsilon_1 \simeq \epsilon_2 \cdots \epsilon_k$. For n even, $\epsilon_1 \simeq h^{\times(k-1)}\epsilon_2 \cdots \epsilon_k$, with*

$$h : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$$

given by $h(x) = \bar{x}$.

Proof. Let $g, f : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$ be the maps $f(x) = \dfrac{x}{\|x\|^2}$ and $g(x) = -x$. We have that any $Q \in O(n)$ is $\mathbb{Z}_2$-equivariant (with $\mathbb{Z}_2 = \langle \tau \rangle$):

$$Q(\tau x) = Q\left(\frac{-x}{\|x\|^2}\right) = \frac{-Q(x)}{\|x\|^2} = \frac{-Q(x)}{\|Q(x)\|^2} = \tau Q(x).$$

This, coupled with the injectivity of Q, implies that $Q^{\times(k-1)}$ sends orbit configurations to orbit configurations. Therefore $h^{\times(k-1)}, g^{\times(k-1)}$ are maps of orbit configurations spaces. We also have that f is injective, and it is also $\mathbb{Z}_2$-equivariant:

$$f(\tau x) = f(-\frac{x}{\|x\|^2}) = \frac{-\dfrac{x}{\|x\|^2}}{\|\dfrac{-x}{\|x\|^2}\|^2} = \frac{-\left(\dfrac{x}{\|x\|^2}\right)}{\|\dfrac{x}{\|x\|^2}\|^2} = \tau\left(\frac{x}{\|x\|^2}\right) = \tau f(x),$$

therefore $f^{\times(k-1)}$ is a map between orbit configurations spaces. Note that $\tau = gf$ and $\tilde{R} = hf$. Therefore we have $\epsilon_1 = \tilde{R}^{\times(k-1)} = (hf)^{\times(k-1)} = h^{\times(k-1)} f^{\times(k-1)}$. For n odd, it is known that there is a homotopy through $O(n)$ between g and h, so we have $g^{\times(k-1)} \simeq h^{\times(k-1)}$ as maps of orbit configurations spaces, therefore

$$\epsilon_1 = h^{\times(k-1)} f^{\times(k-1)} \simeq g^{\times(k-1)} f^{\times(k-1)} = \tau^{\times(k-1)} = \epsilon_2 \cdots \epsilon_k$$

as maps of orbit configuration spaces. For n even, there is a homotopy through $O(n)$ between g and the identity. Therefore

$$\epsilon_1 = h^{\times(k-1)} f^{\times(k-1)} \simeq h^{\times(k-1)} g^{\times(k-1)} f^{\times(k-1)} = h^{\times(k-1)} \tau^{\times(k-1)} = h^{\times(k-1)} \epsilon_2 \cdots \epsilon_k$$

as maps of orbit configurations spaces. □

Of course, Theorem 3.2.1 can be used to give a description of the effect in cohomology of the map $h^{\times(k-1)}\colon \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \to \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)$ that arises in Lemma 3.2.3 for n even. We omit the details as we will not have occasion of using such information. Yet, in the next section we will need to describe the behavior of the map $h^{\times(k-1)}$ on the permanent cycles $\mathbb{K}^*$ of the previous section.

Note that ϵ_1 acts as multiplication by $(-1)^{n+1}$ on the generator of the cohomology of the base space of (3.1.1), and that ϵ_l acts trivially on said generator for $l > 1$. Thus we have the following description of the action of $(\mathbb{Z}_2)^k$ on the total space of (3.1.1).

Corollary 3.2.4. *For $n > 1$ odd, the action of $(\mathbb{Z}_2)^k$ on*

$$H^*(\operatorname{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong H^*(S^n) \otimes H^*(\operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)) = \Lambda(\iota_n) \otimes R[\mathcal{A}]/I$$

is the tensor product of the corresponding actions on each factor of the tensor product.

Corollary 3.2.5. *Assume that the characteristic of R is either zero or an odd integer. For $n \geq 2$ even, the action of $(\mathbb{Z}_2)^k$ on*

$$H^*(\operatorname{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong R[\mathcal{B}]/J \otimes \Lambda(\lambda, \omega)/(2\lambda, \lambda\omega)$$

satisfies

$$\epsilon_l(\lambda) = \begin{cases} -\lambda, & \text{if } l = 1; \\ \lambda, & \text{if } l > 1, \end{cases}$$
$$\epsilon_l(\omega) = \omega, \qquad \forall\ l \geq 1,$$

and restricts to the action of $(\mathbb{Z}_2)^k$ on $\mathcal{B}$ stated in Theorem 3.2.2.

Theorem 3.2.2 is a straightforward consequence of the definitions and Theorem 3.2.1. In turn, it suffices to prove the latter result in the special case $k = 3$. Indeed, on the one hand, Theorem 3.2.1 is elementary for $k = 2$. On the other, for $k \geq 3$ and $0 < j < i < k$, the map

$$\pi_{i,j}\colon \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \to \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2)$$

given by

$$\pi_{i,j}(x_1, \ldots, x_{k-1}) = (x_j, x_i)$$

sends $A_{1,0}$, $A_{2,0}$, $A_{2,1}$, and $A_{2,-1}$ respectively to $A_{j,0}$, $A_{i,0}$, $A_{i,j}$, and $A_{i,-j}$, whereas, for $1 \leq l \leq k$, $\pi_{i,j}$ fits in the commutative diagram

$$\begin{array}{ccc} \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) & \xrightarrow{\epsilon_l} & \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) \\ \downarrow{\scriptstyle \pi_{i,j}} & & \downarrow{\scriptstyle \pi_{i,j}} \\ \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) & \xrightarrow{\bar{\epsilon}} & \operatorname{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2), \end{array}$$

where

$$\bar{\epsilon}(x,y)=\begin{cases}\epsilon_3(x,y), & \text{if } i=l-1;\\ \epsilon_2(x,y), & \text{if } j=l-1;\\ \epsilon_1(x,y), & \text{if } l=1;\\ (x,y), & \text{otherwise.}\end{cases}$$

The rest of this section is devoted to proving Theorem 3.2.1 in the case $k=3$, i.e. to the proof of the following set of equalities—which corrects the action reported in Table 2 of [16]:

$$\begin{aligned}
\epsilon_1 A_{1,0} &= -A_{1,0},\\
\epsilon_1 A_{2,0} &= -A_{2,0},\\
\epsilon_1 A_{2,1} &= (-1)^{n-1}A_{1,0}-A_{2,0}+A_{2,1},\\
\epsilon_1 A_{2,-1} &= -A_{1,0}-A_{2,0}+A_{2,-1},\\
\epsilon_2 A_{1,0} &= (-1)^{n}A_{1,0},\\
\epsilon_2 A_{2,0} &= A_{2,0},\\
\epsilon_2 A_{2,1} &= A_{2,-1},\\
\epsilon_2 A_{2,-1} &= A_{2,1},\\
\epsilon_3 A_{1,0} &= A_{1,0},\\
\epsilon_3 A_{2,0} &= (-1)^{n}A_{2,0},\\
\epsilon_3 A_{2,1} &= (-1)^{n}A_{1,0}+(-1)^{n}A_{2,0}+(-1)^{n-1}A_{2,-1},\\
\epsilon_3 A_{2,-1} &= A_{1,0}+(-1)^{n}A_{2,0}+(-1)^{n-1}A_{2,1}.
\end{aligned}$$

Recall the maps $p_{i,j}$ and $f_{r,s}$ introduced in (3.1.6) and (3.1.7). By abuse of notation, for $|j|<i\le 2$, we will denote by $f_{i,j}$ the composition $\pi_{2,1}f_{i,j}: S^{n-1}\longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2)$. These maps, together with the corresponding maps $p_{r,s}$, detect the generators for $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2)$ in the sense of (3.1.8). To prove the above set of relations, we will compute the degree of the compositions

$$S^{n-1}\xrightarrow{f_{ij}}\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2)\xrightarrow{\epsilon_l}\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2)\xrightarrow{p_{r,s}}S^{n-1}$$

for $0<l\le 3$, $|j|<i\le 2$, and $|s|<r\le 2$. We start by computing the action of ϵ_1.

1. $\epsilon_1 A_{1,0}$: We have

   $$p_{1,0}\epsilon_1 f_{1,0}(x) = \bar{x},\quad p_{1,0}\epsilon_1 f_{2,0}(x) = e,\quad p_{1,0}\epsilon_1 f_{2,1}(x) = e,\quad p_{1,0}\epsilon_1 f_{2,-1}(x) = e.$$

   The first map is a reflection and the rest are constant maps, therefore

   $$\deg(p_{1,0}\epsilon_1 f_{1,0})=-1,\quad \deg(p_{1,0}\epsilon_1 f_{2,0})=0,\quad \deg(p_{1,0}\epsilon_1 f_{2,1})=0,\quad \deg(p_{1,0}\epsilon_1 f_{2,-1})=0.$$

   Thus, $\epsilon_1 A_{1,0}=-A_{1,0}$.

2. $\epsilon_1 A_{2,0}$: Clearly, $p_{2,0}\epsilon_1 f_{1,0}(x) = N(\frac{\bar{e}}{2})$, which implies $\deg(p_{2,0}\epsilon_1 f_{1,0}) = 0$. Note that $N(\tilde{R}(y)) = N(\bar{y})$ for all $y \in \mathbb{R}^n - \{0\}$, therefore

$$\begin{array}{rcccl} p_{2,0}\epsilon_1 f_{2,0}(x) & = & N(\bar{x}) & = & \bar{x}, \\ p_{2,0}\epsilon_1 f_{2,1}(x) & = & N(\tilde{R}(e + \frac{x}{2})) & = & N(e + \frac{\bar{x}}{2}), \\ p_{2,0}\epsilon_1 f_{2,-1}(x) & = & N(\tilde{R}(-e + \frac{x}{2})) & = & N(-e + \frac{\bar{x}}{2}); \end{array}$$

The second and third maps are not surjective, therefore we have

$$\deg(p_{2,0}\epsilon_1 f_{2,0}) = -1, \quad \deg(p_{2,0}\epsilon_1 f_{2,1}) = 0, \quad \deg(p_{2,0}\epsilon_1 f_{2,-1}) = 0.$$

Thus $\epsilon_1 A_{2,0} = -A_{2,0}$.

3. $\epsilon_1 A_{2,1}$: We have

$$p_{2,1}\epsilon_1 f_{1,0}(x) \;=\; N(\tfrac{e}{2} - \bar{x}) \;=\; -N(\bar{x} - \tfrac{e}{2}) \;\simeq\; -\bar{x},$$

therefore

$$\deg(p_{2,1}\epsilon_1 f_{1,0}) = (-1)^{n-1}.$$

We also have

$$p_{2,1}\epsilon_1 f_{2,0}(x) \;=\; N(2\bar{x} - e) \;=\; N(\bar{x} - \tfrac{e}{2}),$$

so

$$\deg(p_{2,1}\epsilon_1 f_{2,0}) = -1.$$

Recall the map F defined earlier, given by $(t, (t_1, \ldots, t_n)) \longmapsto (tt_1, t_2, \ldots, t_n)$. We have

$$\begin{array}{rclcl} p_{2,1}\epsilon_1 f_{2,1}(x) & = & N(\tilde{R}(e + \frac{x}{2}) - e) & = & N(e + \frac{F(1,\bar{x})}{2} - \|e + \frac{\bar{x}}{2}\|^2 e) \\ & \simeq & N(e + \frac{F(-1,\bar{x})}{2} - \|e + \frac{\bar{x}}{2}\|^2 e) & \simeq & N(\frac{F(-1,\bar{x})}{2}). \end{array}$$

The first homotopy is given by

$$N(e + \frac{F(t,\bar{x})}{2} - \|e + \frac{\bar{x}}{2}\|^2 e) \qquad \text{with } t \in [-1, 1].$$

As before, we have to check that this homotopy is well defined: suppose there exist $t \in [-1, 1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that $F(t, \frac{\bar{x}}{2}) + (1 - \|e + \frac{\bar{x}}{2}\|^2)e = 0$. Then

$$F(t, \frac{\bar{x}}{2}) = (-1 + \|e + \frac{\bar{x}}{2}\|^2)e$$

and so we have $\frac{tt_1}{2} = -1 + \|e + \frac{\bar{x}}{2}\|^2$ and $t_i = 0$ for $i > 1$. This, in turn, implies $t_1 = \pm 1$.

- Suppose $t_1 = 1$. Then $\frac{t}{2} = -1 + \|e + \frac{e}{2}\|^2 = -1 + \|\frac{3e}{2}\|^2 = \frac{5}{4}$, so $t = \frac{5}{2} > 1$.
- Suppose $t_1 = -1$. Then $-\frac{t}{2} = -1 + \|e - \frac{e}{2}\|^2 = -1 + \|\frac{e}{2}\|^2 = -\frac{3}{4}$, so $t = \frac{3}{2} > 1$.

Both assumptions lead to a contradiction, so the homotopy is well defined. The second homotopy is

$$N(\frac{F(-1,\bar{x})}{2} + t(1 - \|e + \frac{\bar{x}}{2}\|^2)e), \qquad \text{with } t \in [0,1].$$

Let us verify that this homotopy is well defined: suppose there exist $t \in [0,1]$ and $x = (t_1, \ldots, t_n) \in S^{n-1}$ such that $\frac{F(-1,\bar{x})}{2} + t(1 - \|e + \frac{\bar{x}}{2}\|^2)e = 0$. Then

$$\frac{F(-1,\bar{x})}{2} = t(-1 + \|e + \frac{\bar{x}}{2}\|^2)e$$

and so we have $-\frac{t_1}{2} = t(-1 + \|e + \frac{\bar{x}}{2}\|^2)$ and $t_i = 0$ for $i > 1$. This, in turn, implies $t_1 = \pm 1$.

- Suppose $t_1 = 1$. Then $-\frac{1}{2} = t(-1 + \|e + \frac{e}{2}\|^2) = t(-1 + \|\frac{3e}{2}\|^2) = t\frac{5}{4}$, so $t = -\frac{2}{5} < 0$.
- Suppose $t_1 = -1$. Then $\frac{1}{2} = t(-1 + \|e - \frac{e}{2}\|^2) = t(-1 + \|\frac{e}{2}\|^2) = -t\frac{3}{4}$, so $t = -\frac{2}{3} < 0$.

Both assumptions lead to a contradiction, consequently the homotopy is well defined. Therefore $p_{2,1}\epsilon_1 f_{2,1}$ is homotopic to a composition of two reflections. Thus

$$\deg(p_{2,1}\epsilon_1 f_{2,1}) = 1.$$

Finally, since e is not enclosed by the image of $\tilde{R}(-e + \frac{x}{2})$, we have that the map

$$p_{2,1}\epsilon_1 f_{2,-1}(x) = N(\tilde{R}(-e + \frac{x}{2}) - e)$$

is not surjective. Therefore

$$\deg(p_{2,1}\epsilon_1 f_{2,-1}) = 0,$$

and we conclude that $\epsilon_1 A_{2,1} = (-1)^{n-1} A_{1,0} - A_{2,0} + A_{2,1}$.

4. $\epsilon_1 A_{2,-1}$: We clearly have

$$p_{2,-1}\epsilon_1 f_{1,0}(x) \;=\; N(\tfrac{e}{2} - \tau(\bar{x})) \;=\; N(\bar{x} + \tfrac{e}{2}) \;\simeq\; \bar{x}$$

and

$$p_{2,-1}\epsilon_1 f_{2,0}(x) \;=\; N(2\bar{x} + e) \;=\; N(\bar{x} + \tfrac{e}{2}) \;\simeq\; \bar{x}.$$

Therefore

$$\deg(p_{2,-1}\epsilon_1 f_{2,0}) = \deg(p_{2,-1}\epsilon_1 f_{1,0}) = -1.$$

On the other hand, since $-e$ is not enclosed by the image of $\tilde{R}(e + \frac{x}{2})$, the map

$$p_{2,-1}\epsilon_1 f_{2,1}(x) = N(\tilde{R}(e + \frac{x}{2}) + e)$$

is not surjective. Therefore

$$\deg(p_{2,-1}\epsilon_1 f_{2,1}) = 0.$$

Finally,

$$\begin{array}{rclcl} p_{2,-1}\epsilon_1 f_{2,-1}(x) & = & N(\tilde{R}(-e+\frac{x}{2})+e) & \simeq & N(-e+\frac{F(-1,\bar{x})}{2}+\|-e+\frac{\bar{x}}{2}\|^2 e) \\ & \simeq & N(\frac{F(-1,\bar{x})}{2}) & \simeq & x, \end{array}$$

where the first homotopy is given by

$$N(-e+\frac{F(t,\bar{x})}{2}+\|-e+\frac{\bar{x}}{2}\|^2 e), \qquad \text{with } t\in[-1,1],$$

and the second one by

$$N(\frac{F(-1,\bar{x})}{2}+t(-1+\|-e+\frac{\bar{x}}{2}\|^2)e), \qquad \text{with } t\in[0,1].$$

We can show that these homotopies are well defined in a similar fashion to the previous case. Therefore,

$$\deg(p_{2,-1}\epsilon_1 f_{2,-1})=1,$$

And we conclude that $\epsilon_1 A_{2,-1} = -A_{1,0} - A_{2,0} + A_{2,-1}$.

From now on we will just record the results of the computations, without writing out the details, for these computations are entirely analogous to the computation of the action of ϵ_1. Next we consider ϵ_2.

1. $\epsilon_2 A_{1,0}$: We have

   $$p_{1,0}\epsilon_2 f_{1,0}(x) = -x, \quad p_{1,0}\epsilon_2 f_{2,0}(x) = -e, \quad p_{1,0}\epsilon_2 f_{2,1}(x) = -e, \quad p_{1,0}\epsilon_2 f_{2,-1}(x) = -e,$$

   therefore

   $$\deg(p_{1,0}\epsilon_2 f_{1,0})=(-1)^n, \quad \deg(p_{1,0}\epsilon_2 f_{2,0})=0, \quad \deg(p_{1,0}\epsilon_2 f_{2,1})=0, \quad \deg(p_{1,0}\epsilon_2 f_{2,-1})=0.$$

   Thus, $\epsilon_2 A_{1,0} = (-1)^n A_{1,0}$.

2. $\epsilon_2 A_{2,0}$: We have

   $$\begin{array}{rclcl} p_{2,0}\epsilon_2 f_{1,0}(x) & = & e, & & \\ p_{2,0}\epsilon_2 f_{2,0}(x) & = & N(\frac{x}{2}) & = & x, \\ p_{2,0}\epsilon_2 f_{2,1}(x) & = & N(e+\frac{x}{2}) & \simeq & 0, \\ p_{2,0}\epsilon_2 f_{2,-1}(x) & = & N(-e+\frac{x}{2}) & \simeq & 0; \end{array}$$

   therefore

   $$\deg(p_{2,0}\epsilon_2 f_{1,0})=0, \quad \deg(p_{2,0}\epsilon_2 f_{2,0})=1, \quad \deg(p_{2,0}\epsilon_2 f_{2,1})=0, \quad \deg(p_{2,0}\epsilon_2 f_{2,-1})=0.$$

   Thus, $\epsilon_2 A_{2,0} = A_{2,0}$.

3. $\epsilon_2 A_{2,1}$: We have

$$\begin{array}{rclcl} p_{2,1}\epsilon_2 f_{1,0}(x) &=& N(2e+x) &\simeq& 0, \\ p_{2,1}\epsilon_2 f_{2,0}(x) &=& N(\frac{x}{2}+e) &\simeq& 0, \\ p_{2,1}\epsilon_2 f_{2,1}(x) &=& N(e+\frac{x}{2}+e) &\simeq& 0, \\ p_{2,1}\epsilon_2 f_{2,-1}(x) &=& N(-e+\frac{x}{2}+e) &=& N(\frac{x}{2}) \;=\; x; \end{array}$$

therefore

$$\deg(p_{2,1}\epsilon_2 f_{1,0}) = 0, \quad \deg(p_{2,1}\epsilon_2 f_{2,0}) = 0, \quad \deg(p_{2,1}\epsilon_2 f_{2,1}) = 0, \quad \deg(p_{2,1}\epsilon_2 f_{2,-1}) = 1.$$

Thus $\epsilon_2 A_{2,1} = A_{2,-1}$.

4. $\epsilon_2 A_{2,-1}$: Note that $\epsilon_2^2 =$ identity. Application of the previous case yields $\epsilon_2 A_{2,-1} = A_{2,1}$.

Next, ϵ_3.

1. $\epsilon_3 A_{1,0}$: We have

$$p_{1,0}\epsilon_3 f_{1,0}(x) = x, \quad p_{1,0}\epsilon_3 f_{2,0}(x) = e, \quad p_{1,0}\epsilon_3 f_{2,1}(x) = e, \quad p_{1,0}\epsilon_3 f_{2,-1}(x) = e;$$

therefore

$$\deg(p_{1,0}\epsilon_3 f_{1,0}) = 1, \quad \deg(p_{1,0}\epsilon_3 f_{2,0}) = 0, \quad \deg(p_{1,0}\epsilon_3 f_{2,1}) = 0, \quad \deg(p_{1,0}\epsilon_3 f_{2,-1}) = 0.$$

Thus, $\epsilon_3 A_{1,0} = A_{1,0}$.

2. $\epsilon_3 A_{2,0}$: We have

$$\begin{array}{rclcl} p_{2,0}\epsilon_3 f_{1,0}(x) &=& N(-\frac{e}{2}), && \\ p_{2,0}\epsilon_3 f_{2,0}(x) &=& N(\tau(\frac{x}{2})) &=& -x, \\ p_{2,0}\epsilon_3 f_{2,1}(x) &=& N(\tau(e+\frac{x}{2})) &\simeq& 0, \\ p_{2,0}\epsilon_3 f_{2,-1}(x) &=& N(\tau(-e+\frac{x}{2})) &\simeq& 0; \end{array}$$

therefore

$$\deg(p_{2,0}\epsilon_3 f_{1,0}) = 0, \quad \deg(p_{2,0}\epsilon_3 f_{2,0}) = (-1)^n, \quad \deg(p_{2,0}\epsilon_3 f_{2,1}) = 0, \quad \deg(p_{2,0}\epsilon_3 f_{2,-1}) = 0.$$

Thus, $\epsilon_3 A_{2,0} = (-1)^n A_{2,0}$.

3. $\epsilon_3 A_{2,1}$: We have

$$\begin{array}{rclclcl} p_{2,1}\epsilon_3 f_{1,0}(x) &=& N(-\frac{e}{2}-x) &=& -N(\frac{e}{2}+x) &\simeq& -x, \\ p_{2,1}\epsilon_3 f_{2,0}(x) &=& N(-2x-e) &=& -N(2x+e) &\simeq& -x, \\ p_{2,1}\epsilon_3 f_{2,1}(x) &=& N(\tau(e+\frac{x}{2})-e) &\simeq& 0, && \end{array}$$

therefore

$$\deg(p_{2,1}\epsilon_3 f_{1,0}) = (-1)^n, \quad \deg(p_{2,1}\epsilon_3 f_{2,0}) = (-1)^n, \quad \deg(p_{2,1}\epsilon_3 f_{2,1}) = 0.$$

Lastly,

$$\begin{array}{rcl} p_{2,1}\epsilon_3 f_{2,-1}(x) & = & N(\tau(-e+\frac{x}{2})-e) \;\simeq\; N(e+\frac{F(-1,-x)}{2} - \|-e+\frac{x}{2}\|^2 e) \\ & \simeq & N(\frac{F(-1,-x)}{2}) \qquad\qquad\quad = \; -F(-1,x), \end{array}$$

where the first homotopy is given by

$$N(e+\frac{F(t,-x)}{2} - \|-e+\frac{x}{2}\|^2 e), \qquad \text{with } t\in[-1,1],$$

and the second one is given by

$$N(\frac{F(-1,-x)}{2} + t(1-\|-e+\frac{x}{2}\|^2)e), \qquad \text{with } t\in[0,1].$$

Therefore

$$\deg(p_{2,1}\epsilon_3 f_{2,-1}) = (-1)^{n-1}.$$

And we conclude $\epsilon_3 A_{2,1} = (-1)^n A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,-1}$.

4. $\epsilon_3 A_{2,-1}$: Note that $\epsilon_3^2 =$ identity. By our previous computations,

$$\begin{array}{rcl} A_{2,1} = \epsilon_3(\epsilon_3 A_{2,1}) & = & \epsilon_3((-1)^n A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,-1}) \\ & = & (-1)^n A_{1,0} + A_{2,0} + (-1)^{n-1}\epsilon_3 A_{2,-1}. \end{array}$$

Therefore $\epsilon_3 A_{2,-1} = A_{1,0} + (-1)^n A_{2,0} + (-1)^{n-1} A_{2,1}$.

Remark 3.2.6. Theorems 3.2.1 and 3.2.2 correct results in [10]. The situation is closely related to our discussion, in Remark 3.1, of the existence of inconsistencies with the determination in [10] of a presentation for the cohomology ring of the fiber and base spaces in (3.1.1). As described next, the problem can be traced back to the description in [10, Lemma 7] of the action of the various ϵ_i on cohomology rings. To simplify the explanation, once again we adopt momentarily Feichtner-Ziegler's notation in [10]—which the reader is assumed to be familiar with. The proof of Lemma 7(iv) in [10] is based on the asserted equality $(A_2\circ A_1)^*(c^+_{1,2}) = (-1)^k c^+_{1,2}$ whose proof, in turn, is reduced to showing that the obvious map

$$A_2\circ A_1\colon \mathcal{M}(\{U_1,U_2,U^+_{1,2}\}) \to \mathcal{M}(\{U_1,U_2,U^+_{1,2}\}) \tag{3.2.3}$$

satisfies

$$(A_2\circ A_1)^*(\widetilde{c}_{1,2}) = (-1)^k \widetilde{c}_{1,2}. \tag{3.2.4}$$

(Note that (3.2.3) is not to be understood as a composition of maps from $\mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ to itself.) Feichtner-Ziegler's argument for (3.2.4) then proceeds by considering the central sphere S (of radius $\sqrt{2}$) in ${}^{\perp}U_{1,2}^+ \setminus \{0\}$ which retracts from $\mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ (with retraction p). It is observed that

$$\text{(3.2.3) } \textit{restricts on } S \textit{ as the antipodal map} \tag{3.2.5}$$

and, from this, (3.2.4) is concluded. But such a conclusion is clearly flawed: The assertion in (3.2.5) is right, and gives the (strict) commutativity of the diagram

$$\begin{array}{ccc} S & \hookrightarrow & \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \\ \downarrow{\scriptstyle \text{antipodal}} & & \downarrow{\scriptstyle A_2 \circ A_1} \\ S & \hookrightarrow & \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}). \end{array}$$

But (3.2.4) cannot be drawn from this, since the map induced in cohomology by the inclusion $S \hookrightarrow \mathcal{M}(\{U_1, U_2, U_{1,2}^+\})$ has a nontrivial kernel. Indeed, $H^{k-1}(\mathcal{M}(\{U_1, U_2, U_{1,2}^+\}))$ is free of rank 3, while $H^{k-1}(S)$ is free of rank 1. Instead, what would certainly give (3.2.4) is the existence of a commutative diagram (at least up to homotopy)

$$\begin{array}{ccc} \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) & \xrightarrow{p} & S \\ \downarrow{\scriptstyle A_2 \circ A_1} & & \downarrow{\scriptstyle \text{antipodal}} \\ \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) & \xrightarrow{p} & S. \end{array}$$

But (3.2.4) is false according to Theorem 3.2.1, so that such a diagram is impossible.

3.3 $(\mathbb{Z}_2)^k$-Invariants

In this section R will denote a commutative ring with unit where 2 is invertible, and n will be an integer greater than or equal to 2. First we will compute the $(\mathbb{Z}_2)^k$-invariants in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ starting with the case n odd, assumption that will be in force until Theorem 3.3.8.

For $0 < i < k$ we let $C_{i,0}$ stand for $A_{i,0}$, and for $0 < j < i < k$ we define

$$\begin{aligned} C_{i,j}^+ &= A_{i,j} + A_{i,-j} - A_{i,0}, \\ C_{i,j}^- &= -A_{i,j} + A_{i,-j} - A_{j,0}. \end{aligned}$$

For ease of notation, for a positive j we will also use the notation $C_{i,j}$ and $C_{i,-j}$ to stand respectively for $C_{i,j}^+$ and $C_{i,j}^-$. Put

$$\mathcal{C}^+ = \{C_{i,j}^+ \,|\, 1 \le j < i < k\},$$
$$\mathcal{C}^- = \{C_{i,j}^- \,|\, 1 \le j < i < k\},$$

$$\mathcal{C}_0 = \{C_{i,0} \,|\, 1 \leq i < k\}$$

and $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \cup \mathcal{C}_0$. Clearly, $\mathcal{C}$ is a basis for $H^{n-1}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ with inverse change of basis map given by

$$\begin{aligned} A_{i,j} &= \frac{C^+_{i,j} - C^-_{i,j} + C_{i,0} - C_{j,0}}{2}, \\ A_{i,-j} &= \frac{C^+_{i,j} + C^-_{i,j} + C_{i,0} + C_{j,0}}{2}, \\ A_{i,0} &= C_{i,0}, \end{aligned}$$

for $0 < j < i < k$. These formulas make it clear that:

Proposition 3.3.1. *$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ is additively generated by the products*

$$C_{i_1,j_1} \cdots C_{i_r,j_r} \tag{3.3.1}$$

with $|j_l| < i_l < k$ for $l = 1, \ldots, r$.

Our first goal is to show that, in fact, an additive basis is formed by such products that satisfy in addition

$$i_l < i_{l'} \text{ if } l < l'. \tag{3.3.2}$$

Example 3.3.2. *For $n \geq 2$ odd, the multiplicative relations among the $A_{i,j}$'s yield*

$$\begin{aligned} C^-_{3,2} C_{3,0} &= -A_{2,0}A_{3,0} + A_{3,-2}A_{3,0} - A_{3,0}A_{3,2} \\ &= -A_{2,0}A_{3,-2} + A_{2,0}A_{3,0} - A_{2,0}A_{3,2} \\ &= -C^+_{3,2}C_{2,0}. \end{aligned}$$

Example 3.3.3. *For odd n, the multiplicative relations among the $A_{i,j}$'s yield*

$$\begin{aligned} C^+_{4,3}C^-_{4,2} &= -A_{2,0}A_{4,-3} + A_{4,-3}A_{4,-2} + A_{2,0}A_{4,0} - A_{4,-2}A_{4,0} - A_{4,-3}A_{4,2} + A_{4,0}A_{4,2} \\ &\quad -A_{2,0}A_{4,3} + A_{4,-2}A_{4,3} - A_{4,2}A_{4,3} \\ &= A_{2,0}A_{4,-3} - A_{3,-2}A_{4,-3} + A_{3,2}A_{4,-3} - A_{3,-2}A_{4,-2} + A_{3,0}A_{4,-2} - A_{3,2}A_{4,-2} \\ &\quad -A_{2,0}A_{4,0} + A_{3,-2}A_{4,2} - A_{3,0}A_{4,2} + A_{3,2}A_{4,2} - A_{2,0}A_{4,3} + A_{3,-2}A_{4,3} - A_{3,2}A_{4,3} \\ &= -C^-_{3,2}C^-_{4,3} - C^-_{4,2}C^+_{3,2} - C^+_{3,2}C_{2,0} - C^-_{3,2}C_{3,0} - C_{2,0}C_{4,0}. \end{aligned}$$

Therefore, by the previous example,

$$C^+_{4,3}C^-_{4,2} = -C^-_{3,2}C^-_{4,3} - C^-_{4,2}C^+_{3,2} - C_{2,0}C_{4,0}.$$

Note that the set of products in (3.3.1) satisfying (3.3.2) is in bijective correspondence with the basis described just before Theorem 3.1.6. By Lemma 2.1.2 we see that the former set will be in fact an additive basis of $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}), k-1)$ as long as it additively generates. In turn, the latter condition follows directly from the fact that the products $A_{i_1,j_1}\cdots A_{i_r,j_r}$ satisfying the condition (3.3.2) form an additive basis, from the explicit form of the relations expressing the $A_{i,j}$'s in terms of the $C_{i,j}$'s, and from the relations in item 2 of Lemma 3.3.4 below—which generalizes the calculation illustrated in Example 3.3.2. The proof of the lemma is straightforward and left to the reader.

Lemma 3.3.4. *For $n \geq 2$ odd, the elements of $\mathcal{C}$ satisfy the following multiplicative relations:*

1. *For $0 < j < i < r < k$,*

$$\begin{aligned} C^+_{r,i}C^+_{r,j} &= -C^+_{i,j}C^+_{r,j} + C^+_{i,j}C^+_{r,i}, \\ C^+_{r,i}C^-_{r,j} &= -C^-_{i,j}C^-_{r,i} - C^+_{i,j}C^-_{r,j} - C_{j,0}C_{r,0}, \\ C^-_{r,i}C^+_{r,j} &= C^-_{i,j}C^-_{r,j} + C^+_{i,j}C^-_{r,i} - C_{i,0}C_{r,0}, \\ C^-_{r,i}C^-_{r,j} &= C^-_{i,j}C^+_{r,j} - C^-_{i,j}C^+_{r,i} + C_{j,0}C_{i,0}. \end{aligned}$$

2. *For $0 < i < r < k$,*

$$\begin{aligned} C^+_{r,i}C_{r,0} &= -C_{i,0}C^-_{r,i}, \\ C^-_{r,i}C_{r,0} &= -C_{i,0}C^+_{r,i}. \end{aligned}$$

3. *For $0 \leq j < i < k$,*

$$\begin{aligned} (C^+_{i,j})^2 &= 0, \\ (C^-_{i,j})^2 &= 0, \\ C^+_{i,j}C^-_{i,j} &= -C_{j,0}C_{i,0}. \end{aligned}$$

The advantage of using $\mathcal{C}$ over $\mathcal{A}$ to compute invariants becomes apparent when describing the action of $(\mathbb{Z}_2)^k$ as a straightforward verification yields

$$\epsilon_l C^+_{ij} = C^+_{ij} \text{ for all } 0 < j < i \text{ and all } l,$$

$$\epsilon_l C^-_{ij} = \begin{cases} -C^-_{ij}, & \text{if } i = l-1 \text{ or } j = l-1; \\ C^-_{ij}, & \text{otherwise,} \end{cases}$$

$$\epsilon_l C_{i,0} = \begin{cases} -C_{i,0}, & \text{if } i = l-1 \text{ or } l = 1; \\ C_{i,0}, & \text{otherwise.} \end{cases}$$

Theorem 3.3.5. *Suppose R is a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, the $(\mathbb{Z}_2)^k$-invariants in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ are multiplicatively generated by the set $\mathcal{C}^+$. In fact, an additive basis of the invariants is formed by the products (3.3.1) satisfying (3.3.2) and $j_l > 0$ for $l = 1, \ldots, r$.*

Proof. Let $x \in H^{m(n-1)}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ be an invariant. We will show that each of the basis elements appearing with a nontrivial coefficient in the expression of x as a linear combination of the basis of products (3.3.1) satisfying (3.3.2) has no factors belonging to $\mathcal{C}^-$ or $\mathcal{C}_0$. Write

$$x = \sum a_I C_{i_1,j_1} \cdots C_{i_m,j_m}, \tag{3.3.3}$$

where each coefficient a_I is non-zero and the summation runs over some multi-indices

$$I = ((i_1, j_1), \ldots, (i_m, j_m))$$

such that $|j_l| < i_l$ and $i_l < i_{l'}$ if $l < l'$. Note that given our description of the $(\mathbb{Z}_2)^k$-action on $\mathcal{C}$, each monomial $C_{i_1,j_1} \cdots C_{i_m,j_m}$ is sent to a multiple of itself under the action of any element in $(\mathbb{Z}_2)^k$. Since 2 is invertible, this means that each term $C_{i_1,j_1} \cdots C_{i_m,j_m}$ appearing in (3.3.3) is invariant. Fix I, and consider the corresponding invariant monomial $z = C_{i_1,j_1} \cdots C_{i_m,j_m}$. Suppose that the set of integers i such that we have a factor of the form $C^-_{i,j}$ in z is non-empty, and let i_0 be the greatest element of this set. By applying ϵ_{i_0+1} to z we get that $-z = \epsilon_{i_0+1} z = z$, which is a contradiction, so z has no factors belonging to $\mathcal{C}^-$. An entirely analogous argument shows that there are no factors belonging to $\mathcal{C}_0$ in z either. □

Theorem 3.3.6. *Suppose R is a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))^{(\mathbb{Z}_2)^k} \cong R[\mathcal{C}^+]/\mathcal{K},$$

where $\mathcal{K}$ is the ideal generated by the elements ${C^+_{i,j}}^2$ and $C^+_{r,i}C^+_{r,j} - C^+_{i,j}(C^+_{r,i} - C^+_{r,j})$ for $0 < j < i < r < k$.

Proof. Lemma 3.3.4 gives an obvious ring map (with domain in $R[\mathcal{C}^+]/\mathcal{K}$). This is an isomorphism since it sets a bijective correspondence between the basis described in Theorem 3.3.5 and the usual basis in the domain. □

Remark 3.3.7. Note that the second relation in the preceding Theorem is identical to the known relation (2.6.1). In particular, the cohomology ring described in Theorem 3.3.6 is isomorphic to the cohomology ring of the standard configuration space of $k-1$ ordered points in $\mathbb{R}^n$.

Since the canonical projection $S^n \to \mathbb{R}\mathrm{P}^n$ induces a $(\mathbb{Z}_2)^k$ covering space $\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k) \to \mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)$, Theorem 3.1.8, the fact that the $(\mathbb{Z}_2)^k$-action on $H^*(S^n)$ is trivial for odd n, and the preceding theorem imply the following result:

Theorem 3.3.8. *Suppose R is a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n, k)) \cong H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k))^{(\mathbb{Z}_2)^k} \cong \Lambda(\iota_n) \otimes R[\mathcal{C}^+]/\mathcal{K}.$$

Next, we describe $(\mathbb{Z}_2)^k$-invariant permanent cycles in $\mathbb{K}^* \subseteq H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ for n even, assumption that will be in force throughout the rest of the section. For $0 < j < i < k$ define

$$\begin{aligned} D^+_{i,j} &= B_{i,j} + B_{i,-j} - B_{i,0} - B_{j,0}, \\ D^-_{i,j} &= B_{i,j} - B_{i,-j}, \\ D_{i,0} &= B_{i,0}, \end{aligned}$$

and, for $|j| < i$ with $i \geq 2$, let

$$D_{i,j} = \begin{cases} D^+_{i,j} & \text{if } j > 0; \\ D^-_{i,|j|}, & \text{if } j < 0; \\ D_{i,0}, & \text{if } j = 0. \end{cases}$$

(Recall $B_{1,0} = 0$.) Put $\mathcal{D} = \{D^+_{i,j} \,|\, 0 < j < i < k\} \cup \{D^-_{i,j} \,|\, 0 < j < i < k\} \cup \{D_{i,0} \,|\, 1 < i < k\}$. With this notation, the action described in (3.2.2) takes the form

$$\epsilon_l D^+_{i,j} = \begin{cases} -D^+_{i,j}, & \text{if } i = l-1; \\ D^+_{i,j}, & \text{otherwise,} \end{cases}$$

$$\epsilon_l D^-_{i,j} = \begin{cases} -D^-_{i,j}, & \text{if } j = l-1; \\ D^-_{i,j}, & \text{otherwise,} \end{cases}$$

and

$$\epsilon_l D_{i,0} = \begin{cases} -D_{i,0}, & \text{if } l = 1; \\ D_{i,0}, & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{D}$ forms a basis for $\mathbb{K}$ with inverse change of basis given by

$$\begin{aligned} B_{i,j} &= \frac{D^+_{i,j} + D^-_{i,j} + D_{i,0} + D_{j,0}}{2}, \\ B_{i,-j} &= \frac{D^+_{i,j} - D^-_{i,j} + D_{i,0} + D_{j,0}}{2}, \\ B_{i,0} &= D_{i,0}, \end{aligned}$$

for $0 < j < i < k$. We leave to the reader the verification of the following multiplicative relations among the elements of $\mathcal{D}$:

Lemma 3.3.9. *Let R be a commutative ring with unit where 2 is invertible. Suppose $n \geq 2$ even. The elements of $\mathcal{D}$ satisfy the following multiplicative relations:*

1. *For $0 < j < i < r < k$,*

$$\begin{aligned} D^+_{r,i} D^+_{r,j} &= D^-_{i,j} D^-_{r,j} - D^+_{i,j} D^-_{r,i} - D_{j,0} D_{i,0} + D_{j,0} D_{r,0} - D_{i,0} D_{r,0}, \\ D^+_{r,i} D^-_{r,j} &= D^-_{i,j}(D^+_{r,j} - D^+_{r,i}), \\ D^-_{r,i} D^+_{r,j} &= D^+_{i,j}(D^+_{r,j} - D^+_{r,i}), \\ D^-_{r,i} D^-_{r,j} &= -D^-_{i,j} D^-_{r,i} + D^+_{i,j} D^-_{r,j}. \end{aligned}$$

2. *For $0 < i < r < k$,*

$$\begin{aligned} D^+_{r,i} D_{r,0} &= -D_{i,0} D^+_{r,i}, \\ D^-_{r,i} D_{r,0} &= -D_{i,0} D^-_{r,i}. \end{aligned}$$

3. For $0 < j < i < k$,

$$\begin{aligned} (D^+_{i,j})^2 &= 0, \\ (D^-_{i,j})^2 &= 0, \\ (D_{i,0})^2 &= 0, \\ D^+_{i,j}D^-_{i,j} &= 0. \end{aligned}$$

By repeated applications of Lemma 3.3.9 we see that

Proposition 3.3.10. $\mathbb{K}^*$ *is additively generated by products of the form*

$$D_{i_1,j_1} \cdots D_{i_r,j_r}, \tag{3.3.4}$$

where

$$i_l < i_{l'} \text{ if } l < l'. \tag{3.3.5}$$

The set of these products is in bijective correspondence with the basis consisting of products $B_{i_1,j_1} \cdots B_{i_r,j_r}$ satisfying condition (3.3.5), and so by Lemma 2.1.2 the set of products of the form (3.3.4) satisfying (3.3.5) is an additive basis of the permanent cycles in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$.

Remark 3.3.11. The previous discussion and our description of $\epsilon_\ell(D^{\pm}_{i,j})$ easily yield that the map $h^{\times(k-1)}$ in Lemma 3.3 acts on the permanent cycles in $\mathbb{K}^m$ as multiplication by $(-1)^m$.

Next we define elements which are clearly $(\mathbb{Z}_2)^k$-invariants; in fact we will show in Theorem 3.3.14 below that they are multiplicative generators for all $(\mathbb{Z}_2)^k$-invariants. For $0 < j < i < r < k$, put

$$\begin{aligned} I^+_{r,i,j} &= D^+_{i,j}D^-_{r,i}, \\ I^-_{r,i,j} &= D^-_{i,j}D^-_{r,j}, \end{aligned}$$

and for $1 < j < i < k$ put

$$I_{i,j,0} = D_{j,0}D_{i,0}.$$

For $j > 0$, we will sometimes write $I_{r,i,j}$ and $I_{r,i,-j}$ instead of $I^+_{r,i,j}$ and $I^-_{r,i,j}$ respectively. Accordingly, we will sometimes write $I^+_{i,j,0}$ or even $I^-_{i,j,0}$ as a substitute for $I_{i,j,0}$. Let

$$\mathcal{E}^+ = \{I^+_{r,i,j} \,|\, 0 < j < i < r < k\},$$

$$\mathcal{E}^- = \{I^-_{r,i,j} \,|\, 0 < j < i < r < k\},$$

$$\mathcal{E}_0 = \{I_{i,j,0} \,|\, 1 < j < i < k\}$$

and

$$\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \cup \mathcal{E}_0.$$

The corroboration of the existence of the following relations is straightforward, although it can take quite some time to write down. As such, it is left to the reader.

Lemma 3.3.12. *Let R be a commutative ring with unit where 2 is invertible. Suppose $n \geq 2$ even. The elements of $\mathcal{E}$ satisfy the relations listed below. Relations* (a) *through* (c) *express a product $I^{\pm}_{r,i,j} I^{\pm}_{s,a,b}$ with*

$$0 \leq j < i < r < k, \;\; 0 \leq b < a < s < k, \;\; 2 \leq i, \;\; 2 \leq a, \;\; \text{and} \;\; r \leq s \tag{3.3.6}$$

as a linear combination of such products satisfying in addition

$$r < s \;\; \text{and} \;\; a \notin \{r, i\}. \tag{3.3.7}$$

Those relations are listed according to the several possibilities for the indices r, i, j, s, a, and b when they satisfy (3.3.6) but not (3.3.7).

(a) $r = s$

(a.a) $j \neq 0 \neq b$

(a.a.a) $|\{i, j, a, b\}| = 4$ *(can assume $a < i$)*

(a.a.a.a) $b < a < j$

$$\begin{aligned}
I^+_{r,i,j} I^+_{r,a,b} &= I^+_{j,a,b}(I^-_{r,i,a} - I^+_{r,i,a} + I^+_{r,i,j}) + (I_{i,b,0} - I^+_{i,j,a} - I_{i,j,0} - I_{j,b,0}) I^+_{r,a,b}, \\
I^-_{r,i,j} I^-_{r,a,b} &= I^-_{j,a,b} I^-_{r,i,j} - I^+_{i,j,b} I^-_{r,a,b}, \\
I^+_{r,i,j} I^-_{r,a,b} &= I^-_{j,a,b}(I^+_{r,i,j} - I^+_{r,i,b}) + (I^-_{i,j,b} - I^+_{i,j,b} - I_{j,b,0} + I_{i,b,0} - I_{i,j,0}) I^-_{r,a,b}, \\
I^-_{r,i,j} I^+_{r,a,b} &= I^+_{j,a,b} I^-_{r,i,j} - I^+_{i,j,a} I^+_{r,a,b}.
\end{aligned}$$

(a.a.a.b) $b < j < a$

$$\begin{aligned}
I^+_{r,i,j} I^+_{r,a,b} &= (I^-_{a,j,b} - I^+_{a,j,b} + I_{a,b,0} - I_{a,j,0} - I_{j,b,0})(I^+_{r,i,a} - I^-_{r,i,a} - I^+_{r,i,j}) \\
&\quad + I^+_{i,j,b}(I^+_{r,a,j} - I^+_{r,a,b}) + (I_{i,b,0} - I_{i,j,0} - I_{j,b,0}) I^+_{r,a,b}, \\
I^-_{r,i,j} I^-_{r,a,b} &= -I^-_{a,j,b} I^-_{r,i,j} - I^+_{i,j,b} I^-_{r,a,b}, \\
I^+_{r,i,j} I^-_{r,a,b} &= I^-_{a,j,b}(I^+_{r,i,b} - I^+_{r,i,j} - I^-_{r,i,b}) + (I_{i,b,0} - I^+_{i,j,b} - I_{j,b,0} - I_{i,j,0}) I^-_{r,a,b}, \\
I^-_{r,i,j} I^+_{r,a,b} &= I^+_{i,j,b}(I^+_{r,a,j} - I^+_{r,a,b}) + (I_{j,b,0} - I_{a,b,0} + I_{a,j,0} - I^-_{a,j,b} + I^+_{a,j,b}) I^-_{r,i,j}.
\end{aligned}$$

(a.a.a.c) $j < b < a$

$$\begin{aligned}
I^+_{r,i,j} I^+_{r,a,b} &= (I^+_{a,b,j} - I^-_{a,b,j} - I_{a,j,0} + I_{a,b,0} + I_{b,j,0})(I^+_{r,i,a} - I^-_{r,i,a} - I^+_{r,i,j}) \\
&\quad + I^-_{i,b,j}(I^+_{r,a,j} - I^+_{r,a,b}) + (I_{i,b,0} - I_{i,j,0} + I_{b,j,0}) I^+_{r,a,b}, \\
I^-_{r,i,j} I^-_{r,a,b} &= -I^-_{i,b,j} I^-_{r,a,b} - I^+_{a,b,j} I^-_{r,i,j}, \\
I^+_{r,i,j} I^-_{r,a,b} &= I^+_{a,b,j}(I^+_{r,i,b} - I^-_{r,i,b} - I^+_{r,i,j}) + (I_{b,j,0} - I_{i,j,0} + I_{i,b,0} - I^-_{i,b,j}) I^-_{r,a,b}, \\
I^-_{r,i,j} I^+_{r,a,b} &= I^-_{i,b,j}(I^+_{r,a,j} - I^+_{r,a,b}) + (I^-_{a,b,j} - I^+_{a,b,j} - I_{b,j,0} + I_{a,j,0} - I_{a,b,0}) I^-_{r,i,j}.
\end{aligned}$$

(a.a.b) $|\{i, j, a, b\}| = 3$ *(can assume $a \leq i$)*

(a.a.b.a) $a = i$ *(can assume $b < j$)*

$$I^{\pm}_{r,i,j} I^{\pm}_{r,i,b} \quad = \quad 0.$$

(a.a.b.b) $a = j$

$$I^{\pm}_{r,i,j} I^{\pm}_{r,j,b} \quad = \quad 0.$$

(a.a.b.c) $b = i$ *is impossible.*

(a.a.b.d) $b = j$

$$I^{\pm}_{r,i,j} I^{\pm}_{r,a,j} \quad = \quad 0.$$

(a.a.c) $|\{i, j, a, b\}| = 2$

$$I^{\pm}_{r,i,j} I^{\pm}_{r,i,j} \quad = \quad 0.$$

(a.b) $j = 0 \neq b$ *(the case $j \neq 0 = b$ is symmetric)*

(a.b.a) $|\{i, a, b\}| = 3$

(a.b.a.a) $i < b < a$

$$\begin{aligned} I_{r,i,0} I^{+}_{r,a,b} &= I_{b,i,0} I^{+}_{r,a,b}, \\ I_{r,i,0} I^{-}_{r,a,b} &= I_{b,i,0} I^{-}_{r,a,b}. \end{aligned}$$

(a.b.a.b) $b < i < a$ *or* $b < a < i$

$$\begin{aligned} I_{r,i,0} I^{+}_{r,a,b} &= -I_{i,b,0} I^{+}_{r,a,b}, \\ I_{r,i,0} I^{-}_{r,a,b} &= -I_{i,b,0} I^{-}_{r,a,b}. \end{aligned}$$

(a.b.b) $|\{i, a, b\}| = 2$

(a.b.b.a) $a = i$

$$I_{r,i,0} I^{\pm}_{r,i,b} \quad = \quad 0.$$

(a.b.b.b) $b = i$

$$I_{r,i,0} I^{\pm}_{r,a,i} \quad = \quad 0.$$

(a.c) $j = 0 = b$ *(can assume $a \leq i$)*

$$I_{r,i,0} I_{r,a,0} \quad = \quad 0.$$

(b) $a = r < s$

(b.a) $j \neq 0 \neq b$

(b.a.a) $|\{i,j,b\}| = 3$

(b.a.a.a) $j < i < b$

$$\begin{aligned}
I^+_{r,i,j} I^+_{s,r,b} &= I^+_{b,i,j}(I^+_{s,r,b} - I^+_{s,r,i}),\\
I^-_{r,i,j} I^-_{s,r,b} &= I^-_{b,i,j} I^-_{s,r,b} + I^-_{r,i,j} I^+_{s,b,j},\\
I^+_{r,i,j} I^-_{s,r,b} &= I^+_{b,i,j} I^-_{s,r,b} + I^+_{r,i,j} I^+_{s,b,i},\\
I^-_{r,i,j} I^+_{s,r,b} &= I^-_{b,i,j}(I^+_{s,r,b} - I^+_{s,r,j}).
\end{aligned}$$

(b.a.a.b) $j < b < i$

$$\begin{aligned}
I^+_{r,i,j} I^+_{s,r,b} &= (I^-_{i,b,j} - I^+_{i,b,j} - I_{b,j,0} + I_{i,j,0} - I_{i,b,0})(I^+_{s,r,i} - I^+_{s,r,b}),\\
I^-_{r,i,j} I^-_{s,r,b} &= I^-_{r,i,j} I^+_{s,b,j} - I^-_{i,b,j} I^-_{s,r,b},\\
I^+_{r,i,j} I^-_{s,r,b} &= (I^+_{r,i,j} - I^+_{r,i,b}) I^+_{s,b,j} + (-I^-_{i,b,j} + I^+_{i,b,j} + I_{b,j,0} - I_{i,j,0} + I_{i,b,0}) I^-_{s,r,b},\\
I^-_{r,i,j} I^+_{s,r,b} &= I^-_{i,b,j}(I^+_{s,r,j} - I^+_{s,r,b}).
\end{aligned}$$

(b.a.a.c) $b < j < i$

$$\begin{aligned}
I^+_{r,i,j} I^+_{s,r,b} &= (I^-_{i,j,b} - I^+_{i,j,b} - I_{j,b,0} + I_{i,b,0} - I_{i,j,0})(I^+_{s,r,b} - I^+_{s,r,i}),\\
I^-_{r,i,j} I^-_{s,r,b} &= I^-_{r,i,j} I^-_{s,j,b} - I^+_{i,j,b} I^-_{s,r,b},\\
I^+_{r,i,j} I^-_{s,r,b} &= (I^+_{r,i,j} - I^+_{r,i,b}) I^-_{s,j,b} + (I^-_{i,j,b} - I^+_{i,j,b} - I_{j,b,0} + I_{i,b,0} - I_{i,j,0}) I^-_{s,r,b},\\
I^-_{r,i,j} I^+_{s,r,b} &= I^+_{i,j,b}(I^+_{s,r,j} - I^+_{s,r,b}).
\end{aligned}$$

(b.a.b) $|\{i,j,b\}| = 2$

(b.a.b.a) $b = i$

$$I^\pm_{r,i,j} I^\pm_{s,r,i} = 0.$$

(b.a.b.b) $b = j$

$$I^\pm_{r,i,j} I^\pm_{s,r,j} = 0.$$

(b.b) $j = 0 \neq b$

(b.b.a) $|\{i,b\}| = 2$

(b.b.a.a) $i < b$

$$\begin{aligned}
I_{r,i,0} I^+_{s,r,b} &= I_{b,i,0} I^+_{s,r,b},\\
I_{r,i,0} I^-_{s,r,b} &= I_{b,i,0} I^-_{s,r,b}.
\end{aligned}$$

(b.b.a.b) $b < i$

$$\begin{aligned} I_{r,i,0}I^+_{s,r,b} &= -I_{i,b,0}I^+_{s,r,b}, \\ I_{r,i,0}I^-_{s,r,b} &= -I_{i,b,0}I^-_{s,r,b}. \end{aligned}$$

(b.b.b) $|\{i,b\}| = 1$

$$I_{r,i,0}I^\pm_{s,r,i} = 0.$$

(b.c) $j \neq 0 = b$

$$\begin{aligned} I^+_{r,i,j}I_{s,r,0} &= I^+_{r,i,j}I_{s,j,0}, \\ I^-_{r,i,j}I_{s,r,0} &= I^-_{r,i,j}I_{s,j,0}. \end{aligned}$$

(b.d) $j = 0 = b$

$$I_{r,i,0}I_{s,r,0} = 0.$$

(c) $a = i < r < s$

(c.a) $j \neq 0 \neq b$

(c.a.a) $|\{j,b\}| = 2$

(c.a.a.a) $j < b$

$$\begin{aligned} I^+_{r,i,j}I^+_{s,i,b} &= (I^-_{i,b,j} - I_{b,j,0} + I_{i,j,0} - I_{i,b,0} - I^+_{i,b,j})I^-_{s,r,i}, \\ I^-_{r,i,j}I^-_{s,i,b} &= I^-_{r,b,j}I^-_{s,i,b} + I^-_{r,i,j}I^+_{s,b,j}, \\ I^+_{r,i,j}I^-_{s,i,b} &= (I^+_{r,i,j} - I^+_{r,i,b})I^+_{s,b,j}, \\ I^-_{r,i,j}I^+_{s,i,b} &= I^-_{r,b,j}(I^+_{s,i,b} - I^+_{s,i,j}). \end{aligned}$$

(c.a.a.b) $b < j$

$$\begin{aligned} I^+_{r,i,j}I^+_{s,i,b} &= (-I^-_{i,j,b} + I^+_{i,j,b} + I_{j,b,0} - I_{i,b,0} + I_{i,j,0})I^-_{s,r,i}, \\ I^-_{r,i,j}I^-_{s,i,b} &= I^-_{r,i,j}I^-_{s,j,b} + I^+_{r,j,b}I^-_{s,i,b}, \\ I^+_{r,i,j}I^-_{s,i,b} &= (I^+_{r,i,j} - I^+_{r,i,b})I^-_{s,j,b}, \\ I^-_{r,i,j}I^+_{s,i,b} &= I^+_{r,j,b}(I^+_{s,i,b} - I^+_{s,i,j}). \end{aligned}$$

(c.a.b) $|\{j,b\}| = 1$

$$I^\pm_{r,i,j}I^\pm_{s,i,j} = 0.$$

(c.b) $j = 0 \neq b$

$$\begin{aligned} I_{r,i,0}I^+_{s,i,b} &= I_{r,b,0}I^+_{s,i,b}, \\ I_{r,i,0}I^-_{s,i,b} &= I_{r,b,0}I^-_{s,i,b}. \end{aligned}$$

(c.c) $j \neq 0 = b$

$$\begin{aligned} I^+_{r,i,j} I_{s,i,0} &= I^+_{r,i,j} I_{s,j,0}, \\ I^-_{r,i,j} I_{s,i,0} &= I^-_{r,i,j} I_{s,j,0}. \end{aligned}$$

(c.d) $j = 0 = b$

$$I_{r,i,0} I_{s,i,0} = 0.$$

(d) *For* $0 \leq j < t < s < i < r < k$,

$$\begin{aligned} I^-_{i,s,j} I^-_{r,t,j} &= I^-_{s,t,j} I^-_{r,i,j}, \\ I^-_{i,t,j} I^-_{r,s,j} &= -I^-_{s,t,j} I^-_{r,i,j}. \end{aligned}$$

(e) *For* $0 < j < i < t < s < r < k$,

$$\begin{aligned} I^-_{s,t,i} I^+_{r,i,j} &= I^+_{t,i,j} I^-_{r,s,i}, \\ I^+_{s,i,j} I^-_{r,t,i} &= -I^+_{t,i,j} I^-_{r,s,i}. \end{aligned}$$

Remark 3.3.13. Relations (d) and (e) in the previous lemma are not a consequence of the multiplicative relations among the elements in $\mathcal{D}$, but rather a consequence of the fact that, in some cases, there are different alternatives for associating four D's to form a product of two I's.

The relations in Lemma 3.3.12 imply that every product

$$I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m} \quad \text{with } |j_l| < i_l < r_l < k \text{ and } 1 < i_l \text{ for } l = 1, \ldots, m \tag{3.3.8}$$

can be written as a linear combination of products of the form (3.3.8) satisfying in addition

$$r_l < r_{l'} \text{ if } l < l', \tag{3.3.9}$$
$$\text{the sets } \{i_l, r_l\}, \text{ with } 1 \leq l \leq m, \text{ are pairwise disjoint,} \tag{3.3.10}$$
$$\text{if } j_a = j_b \leq 0, \text{ say with } r_a < r_b, \text{ then in fact } r_a < i_b, \tag{3.3.11}$$
$$\text{if } j_a > 0 \text{ and } i_a = -j_b, \text{ then in fact } r_a < i_b. \tag{3.3.12}$$

Theorem 3.3.14. *Suppose R is a commutative ring with unit where 2 is invertible. For $n \geq 2$ even, the $(\mathbb{Z}_2)^k$-invariants in $\mathbb{K}^* \subseteq H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1))$ are multiplicatively generated by the set $\mathcal{E}$. In fact, an additive basis for the invariants is given by all products of the form (3.3.8) satisfying (3.3.9)–(3.3.12).*

Proof. Suppose m odd and let $x \in \mathbb{K}^m$ be an invariant. By Remark 3.3, we have that $x = \epsilon_1 x = -\epsilon_2 \cdots \epsilon_k x = -x$, so $x = 0$.

Suppose now m is even and, as above, let $x \in \mathbb{K}^*$ be an invariant. Write

$$x = \sum a_I D_{i_1,j_1} \cdots D_{i_m,j_m},$$

where each a_I is non-zero and the summation runs over all multi-indices

$$I = ((i_1, j_1), \ldots, (i_m, j_m))$$

such that $|j_l| < i_l$ for $l = 1, \ldots, m$ and $i_l < i_{l'}$ if $l < l'$. Recall that an ϵ_l sends each monomial $D_{i_1,j_1} \cdots D_{i_m,j_m}$ to a multiple of itself, therefore, since 2 is invertible, each $D_{i_1,j_1} \cdots D_{i_m,j_m}$ is invariant. Fix I, and consider the corresponding monomial $z = D_{i_1,j_1} \cdots D_{i_m,j_m}$. Note that the action of ϵ_1 on z implies that an even number of factors in z are of the form $D_{i,0}$. Further, such factors can be matched in pairs to yield a product of the form

$$I_{i_1,j_1,0} I_{i_2,j_2,0} \cdots \text{ where } j_1 < i_1 < j_2 < i_2 < \cdots . \tag{3.3.13}$$

Likewise, for each l between 2 and k, we have two possibilities:

1. There is no factor $D^+_{l-1,*}$ in z (e.g. if $l = 2$). In this case, there is an even number of factors of the form $D^-_{*,l-1}$, because otherwise we would have $z = \epsilon_l z = -z$.

2. There is (exactly) one factor $D^+_{l-1,*}$ in z. In this case, there is an odd number of factors of the form $D^-_{*,l-1}$ in z.

The first case allows us to associate products of the form $D^-_{i,j} D^-_{r,j}$, and the second allows us to associate a product of the form $D^+_{i,j} D^-_{r,i}$ and products of the form $D^-_{i,j} D^-_{r,j}$. Further, just as with (3.3.13), the new matchings can be done so to yield, together with (3.3.13), a unique expression of $D_{i_1,j_1} \cdots D_{i_m,j_m}$ as a product of the form (3.3.8) satisfying in addition (3.3.9)–(3.3.12).

The above analysis shows that the $(\mathbb{Z}_2)^k$-invariants in $\mathbb{K}^*$ are generated by the products of the form (3.3.8) satisfying in addition (3.3.9)–(3.3.12). In fact, this is a basis, since such generators are a subset of the additive basis of $\mathbb{K}^*$ given by the products (3.3.1) satisfying (3.3.5). □

We arrive at the complete description of the invariants for the case n even.

Theorem 3.3.15. *Let R be a commutative ring with unit where 2 is invertible. For n even, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{RP}^n, k)) = H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k))^{(\mathbb{Z}_2)^k} \cong \Lambda(\omega) \otimes R[\mathcal{E}]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by the relations in Lemma 3.3.12.

Proof. Since we are assuming that 2 is a unit in R, the isomorphism of Theorem 3.1.9 reduces to

$$H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n, k)) \cong \Lambda(\omega) \otimes R[\mathcal{B}]/J.$$

Recall from Corollary 3.2.5 that ω is fixed by the action of $(\mathbb{Z}_2)^k$. The result follows from Theorem 3.3.14, which implies that the subring of $\mathbb{Z}_2$-invariants in the tensor factor $R[\mathcal{B}]/J$ has the presentation $R[\mathcal{E}]/\mathcal{J}$. □

3.4 Punctured real projective spaces

In this section we maintain the assumption that n is an integer greater than or equal to 2, and that R will denote a commutative ring with unit where 2 is invertible. Note that the canonical projection $S^n \to \mathbb{R}\mathrm{P}^n$ induces a $(\mathbb{Z}_2)^k$-covering space

$$\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \approx \mathrm{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k) \longrightarrow \mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k)$$

which, given that 2 is invertible in R, induces an isomorphism

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k)) \cong H^*(\mathrm{Conf}_{\mathbb{Z}_2}(S^n - Q_1^{\mathbb{Z}_2}, k))^{(\mathbb{Z}_2)^k} \cong H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)^k}.$$

In order to reuse the notation of the previous section when computing invariants, we will consider that the group action in this section is that of the subgroup $(\mathbb{Z}_2)^k = \langle \epsilon_2, \ldots, \epsilon_{k+1} \rangle < (\mathbb{Z}_2)^{k+1}$ on $\mathbb{R}^n - \{0\} \approx S^n - Q_1^{\mathbb{Z}_2}$. We can do this because in this case each ϵ_l preserves fibers, so there is no need for the correcting rotation T used at the beginning of Section 3.2. In practice this means that, when computing invariants, we just have to ignore the action of ϵ_1. Thus Lemma 3.2.3 and Theorem 3.3.6 yield:

Theorem 3.4.1. *Let R be a commutative ring with unit where 2 is invertible. For $n \geq 2$ odd, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{R}\mathrm{P}^n - \star, k)) \cong H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)^k} \cong R[\mathcal{C}^+]/\mathcal{K}.$$

Note that the role of the parameter k in Theorem 4.5 changes here to $k+1$. For instance, the generators $C^+_{i,j}$ of $\mathcal{C}^+$ are now defined for $0 < j < i \leq k$.

For n even, we have only computed invariants in the permanent cycles $\mathbb{K}^*$, but we now have to account for all the invariants in the cohomology of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. Start by noticing that the considerations following Theorem 3.1.8 and Lemma 3.3.9 show that $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ is multiplicatively generated by $A_{1,0}$ and the elements $D_{i,j}$ with $|j| < i \leq k$ subject only to the relations in Lemma 3.3.9 together with $A_{1,0}^2 = 0$. Further, an additive basis is given by all products of the form (3.3.1) and products of the form

$$A_{1,0} D_{i_1,j_1} \cdots D_{i_r,j_r} \tag{3.4.1}$$

satisfying (3.3.5).

It is natural to expect now more invariants than those found in the previous section. In fact, all the elements in the set

$$\mathcal{E}' = \{I^+_{r,i,j} \,|\, 0 < j < i < r \leq k\} \cup \{I^-_{r,i,j} \,|\, 0 < j < i < r \leq k\} \cup \{D_{i,0} \,|\, 1 < i \leq k\} \cup \{A_{1,0}\}$$

are clearly $(\mathbb{Z}_2)^k$-invariant. Before showing these generate all other invariants, we describe their multiplicative relations. First of all, while all relations in Lemma 3.3.12 are clearly inherited (albeit with upper bound $k+1$ instead of k for indices r, i, j), the relations involving terms of the form $I_{i,j,0}$ are evidently not in the most primitive form. Instead, we have the following easy-to-check relations:

Lemma 3.4.2. *Let R be a commutative ring with unit where 2 is invertible. Suppose $n \geq 2$ even. For $0 < j < i < r \leq k$ we have:*

$$\begin{aligned} I^+_{r,i,j} D_{i,0} &= I^+_{r,i,j} D_{j,0}, \\ I^+_{r,i,j} D_{r,0} &= I^+_{r,i,j} D_{j,0}, \\ I^-_{r,i,j} D_{i,0} &= I^-_{r,i,j} D_{j,0}, \\ I^-_{r,i,j} D_{r,0} &= I^-_{r,i,j} D_{j,0}. \end{aligned}$$

Therefore, any product of the form

$$D_{s_1,0} \cdots D_{s_{m'},0} I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m} \quad \begin{array}{l} \text{with } 0 < |j_l| < i_l < r_l \leq k \text{ for } l = 1, \ldots, m \\ \text{and } 1 < s_l \leq k \text{ for } l = 1, \ldots, m' \end{array} \tag{3.4.2}$$

or

$$A_{1,0} D_{s_1,0} \cdots D_{s_{m'},0} I_{r_1,i_1,j_1} \cdots I_{r_m,i_m,j_m} \quad \begin{array}{l} \text{with } 0 < |j_l| < i_l < r_l \leq k \text{ for } l = 1, \ldots, m \\ \text{and } 1 < s_l \leq k \text{ for } l = 1, \ldots, m' \end{array} \tag{3.4.3}$$

can be written as a linear combination of products of the form (3.4.2) or (3.4.3) satisfying

$$s_l < s_{l'} \text{ if } l < l', \tag{3.4.4}$$

$$s_l \notin \{r_1, \ldots, r_m\} \cup \{i_1, \ldots, i_m\} \text{ for } l = 1, \ldots, m', \tag{3.4.5}$$

as well as conditions (3.3.9)–(3.3.12).

Theorem 3.4.3. *Let R be a commutative ring with unit where 2 is invertible. For $n \geq 2$ even, the $(\mathbb{Z}_2)^k$-invariants in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ are multiplicatively generated by the set $\mathcal{E}'$. Moreover, an additive basis is given by products of the form (3.4.2) together with products of the form (3.4.3) all of which satisfy (3.4.4), (3.4.5), and (3.3.9)–(3.3.12).*

Proof. The proof is almost the same as the proof of Theorem 3.3.14, except for two differences:

1. We ignore the action of ϵ_1. This, however, only removes the condition of having an even number of terms $D_{i,0}$, and now we can just associate all terms $D_{i,0}$.

2. We add $A_{1,0}$ as a potential factor to all monomials. This does not affect our previous proof, because $A_{1,0}$ is already an invariant. □

We thus get:

Theorem 3.4.4. *Let R be a commutative ring with unit where 2 is invertible. For $n \geq 2$ even, there is an R-algebra isomorphism*

$$H^*(\mathrm{Conf}(\mathbb{RP}^n - \star, k)) \cong H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))^{(\mathbb{Z}_2)^k} \cong R[\mathcal{E}']/\mathcal{J}',$$

where $\mathcal{J}'$ is the ideal generated by the relations in Lemma 3.3.12 not involving a term $I_{i,j,0}$ together with the relations of Lemma 3.4.2 and the relation $A^2_{1,0} = 0$.

Chapter 4

Applications to the Lusternik-Schnirelmann category and topological complexity

4.1 Preliminaries

In this section we give the definitions of higher topological complexity, as in found in [1], and Lusternik-Schnirelmann category, as in [4]. The remainder of the results from outside sources used in this chapter are given with detailed references of their location in their respective sources.

Definition 4.1.1. *The sectional category of a map $p : E \to B$ is the least number k such that there is an open covering $U_0, U_1, \ldots, U_k$ of B such that there exist maps $s_i : U_i \to E$ with ps_i homotopic to the inclusion $U_i \hookrightarrow B$.*

Definition 4.1.2. *Let X be a path-connected space. The Lusternik-Schnirelmann category of X, $cat(X)$, is the sectional category of the fibration*

$$P_*X \xrightarrow{p} X,$$

where P_*X denotes the based path space of X and p is evaluation at 1.

Definition 4.1.3. *Let X be a path-connected space. The n-th topological complexity of X, $TC_n(X)$, is the sectional category of the fibration*

$$e_n^X = e_n : X^{J_n} \to X^n, \qquad e_n(\gamma) = (\gamma(1_1), \ldots, \gamma(1_n))$$

where J_n is the wedge of n closed intervals $[0,1]$ (each with 0 as basepoint) and 1_i stands for 1 in the i^{th} interval.

4.2 Results

The results in the previous chapters are now used to study the category and all the higher topological complexities[1] of the auxiliary orbit configuration space $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ for $n > 2$ (hypothesis that will be in force throughout this section, unless explicitly noted otherwise). In what follows all references to cohomology use integer coefficients—omitting the coefficients from the notation.

The homotopy exact sequences associated to the fibrations in (3.1.4) inductively yield that $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is $(n-2)$-connected. Further, from (3.1.5), the cohomology of this space is torsion-free, and vanishes above dimension $k(n-1)$. Therefore $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ has the homotopy type of a cell complex X which is $(n-2)$-connected and $k(n-1)$-dimensional (see [12, Section 4.C]). Then the upper bounds in [4, Theorem 1.50] for the Lusternik-Schnirelmann category, and in [1, Theorem 3.9] for the higher topological complexities immediately yield

$$\mathrm{cat}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \leq k \quad \text{and} \quad \mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \leq sk. \tag{4.2.1}$$

The former inequality is in fact sharp, as the product $A_{1,0} \cdots A_{k,0} \in H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ is non-zero. We thus get:

Corollary 4.2.1. *For $n > 2$, $\mathrm{cat}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = k$.*

Alternatively, we could use the observation that the rule $A'_{i,j} \mapsto A_{i-1,j-1}$, $1 \leq j < i \leq k+1$, determines a ring monomorphism

$$H^*(\mathrm{Conf}(\mathbb{R}^n, k+1)) \hookrightarrow H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)).$$

so that the non-triviality of the product $A'_{2,1} \cdots A'_{k+1,1}$ yields the existence of a non-trivial product with k factors in $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$. Further, since these rings are torsion-free, we also get a ring monomorphism

$$H^*(\mathrm{Conf}(\mathbb{R}^n, k+1)) \otimes H^*(\mathrm{Conf}(\mathbb{R}^n, k+1)) \hookrightarrow H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \otimes H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)).$$

Consequently, the s-th zero-divisors cup-length of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is bounded from below by the s-th zero-divisors cup-length of $\mathrm{Conf}(\mathbb{R}^n, k+1)$. Thus, the second inequality in (4.2.1) and [11, Proposition 4.2] yield:

Corollary 4.2.2. *Let $n > 2$. Then $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) = sk$ if n is odd, whereas, if n is even, $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)) \in \{sk-1, sk\}$.*

Note that item (d) of Theorem 2.2 in [8] implies that the indetermination by one unit in the case with an even n in Corollary 4.2.2 is resolved in terms of the s-th zero-divisors cup-length of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$):

[1]We use the reduced versions of these homotopy invariants, so that the category and all the higher topological complexities of a contractible space are 0.

Corollary 4.2.3. *Let $n > 2$ (any parity). Then* $\mathrm{TC_s}(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ *agrees with the s-th zero-divisors cup-length of* $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$.

Since $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ and $H^*(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^{n+2} - \{0\}, k))$ differ only by degree scaling, Corollary 4.2.3 implies that, for fixed s and k, $\mathrm{TC}_s(\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k))$ depends only on the parity of n. In particular, the indeterminacy by one in Corollary 4.2.2 could be settled by considering the situation for a single value of n. In our setting, $n = 4$ would be the most reasonable instance to explore. However, for the analogous situation in [8] and [11], $n = 2$ is the right choice, in view of the standard splitting

$$\mathrm{Conf}(\mathbb{R}^2, k) \simeq X \times S^1 \tag{4.2.2}$$

with X a CW complex of dimension $k - 2$. Indeed, standard cohomology considerations give $\mathrm{TC}_s(\mathrm{Conf}(\mathbb{R}^{2m}, k)) \in \{s(k-1) - 1, s(k-1)\}$, and then (4.2.2) implies that the actual answer is given by the lower value.

Note that, just as above, the indeterminacy by one in Corollary 4.2.2 would be resolved with the smallest value (and the restriction $n > 2$ in this section would be waived) by answering affirmatively the following analogue of (4.2.2) in the case of $\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{R}^2 - \{0\}, k)$. So the question is: Is it true that the latter space has the homotopy type of a product $S^1 \times X$ for some CW complex X of dimension $k - 1$?

Evidence towards an affirmative answer for the previous question comes from the fact that the result is true for $k = 2$. Indeed, note that $\mathbb{R}^2 - \{0\} \approx \mathbb{C}^*$, and consider the fibration

$$S^1 \longrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{C}^*, 2) \longrightarrow M, \tag{4.2.3}$$

where S^1 acts on $\mathbb{C}^*$ by multiplication and on the total space diagonally, and where M denotes the corresponding orbit space. First we show that M has the homotopy type of a wedge of three circles. Note that there is a well-defined map

$$M \xrightarrow{h} \{(r, w) \in \mathbb{R}^+ \times \mathbb{C}^* \,|\, r > 0,\, w \neq -\frac{1}{r},\, w \neq r\} =: S$$

given by

$$[z_1, z_2] \longmapsto (\sqrt{z_1\overline{z_1}}, \frac{\sqrt{z_1\overline{z_1}}}{z_1} z_2),$$

where the space on the right has the subspace topology. This map has continuous inverse

$$(r, w) \longmapsto [r, w]$$

and therefore is a homeomorphism. An illustration is given in Figure 4.1.

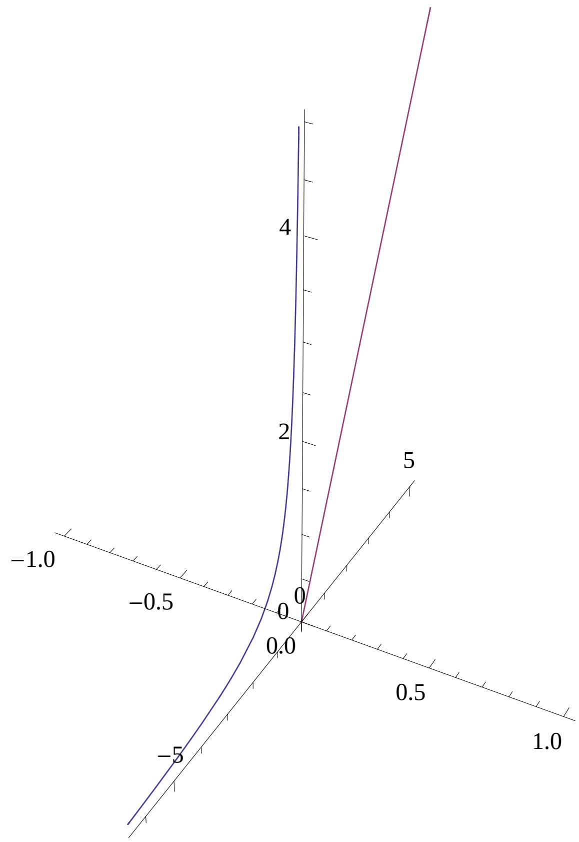

Figure 4.1: S is homeomorphic to the complement of the hyperbola, the diagonal and the vertical line in $\mathbb{R}^+ \times \mathbb{C}^*$.

Denote now by L_3 the union of the lines $(z, 0, 0)$, $(z, -1, 0)$, $(z, 1, 0)$ in $\mathbb{R}^3$. We consider now $f : S \longrightarrow \mathbb{R}^+ \times \mathbb{R}^2 - L_3$ and $g : \mathbb{R}^+ \times \mathbb{R}^2 - L_3 \longrightarrow S$ given by

$$f(r, w) = f(r, x, y) = \begin{cases} (r, \dfrac{x}{r}, y) & \text{if } x \geq 0, \\ (r, xr, y) & \text{otherwise;} \end{cases}$$

and

$$g(r, x, y) = \begin{cases} (r, xr, y) & \text{if } x \geq 0, \\ (r, \dfrac{x}{r}, y) & \text{otherwise.} \end{cases}$$

These are well-defined maps and are inverses of each other, therefore $S \approx \mathbb{R}^+ \times \mathbb{R}^2 - L_3$, and so $S \simeq S^1 \vee S^1 \vee S^1$. Also note that 4.2.3 admits a section: the composition

$$M \xrightarrow{h} S \subseteq \mathbb{R}^+ \times \mathbb{C}^* \hookrightarrow \mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{C}^*, 2)$$

and, since 4.2.3 is a principal fibration, we have

$$\mathrm{Conf}_{\mathbb{Z}_2}(\mathbb{C}^*, 2) \simeq S^1 \times M \simeq S^1 \times (S^1 \vee S^1 \vee S^1).$$

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