Center for Research and Advanced Studies
of the National Polythechnic Institute

Campus Zacatenco
Department of Mathematics

Holes and Integer Grids.

A DISSERTATION PRESENTED BY

Frank Rodrigo Duque Patiño

TO OBTAIN THE DEGREE OF

Doctor in Science

IN THE SPECIALITY OF

Mathematics

Thesis Advisor: **Dr. Ruy Fabila Monroy**

Mexico City September 2016

Centro de Investigación y de Estudios
Avanzados del Instituto Politécnico Nacional

Unidad Zacatenco
Departamento de Matemáticas

Hoyos y Mallas Enteras.

TESIS QUE PRESENTA

Frank Rodrigo Duque Patiño

PARA OBTENER EL GRADO DE

Doctor en Ciencias

EN LA ESPECIALIDAD DE

Matemáticas

Director de tesis: **Dr. Ruy Fabila Monroy**

Ciudad de México Septiembre 2016

Abstract

In 1935, Erdős and Szekeres proved that every set of n points in general position in the plane contains the vertices of a convex polygon of $\frac{1}{2}\log_2(n)$ vertices. In 1961, they constructed for every positive integer t, a set of $n := 2^{t-2}$ points in general position in the plane, such that every convex polygon with vertices in this set has at most $\log_2(n) + 1$ vertices. The set obtained from that construction is now called *the Erdős-Szekeres set.*

In 1978, Erdős asked if every sufficiently large set of points in general position in the plane contains the vertices of a convex polygon of k vertices, with the additional property that no other point of the set lies in its interior. Shortly after, Horton provided a construction with no such convex polygon of 7 vertices. The set obtained from that construction is now called *the Horton set.*

In 2001, Károlyi, Pach and Tóth introduced a family of point sets to solve an Erdős-Szekeres type problem; this set has been used to solve several other Edős-Szekeres type problems. In this thesis we refer to these sets as *nested almost convex sets.* A nested almost convex set $\mathcal{X}$ has the property that the interior of every triangle determined by three points in the same convex layer of $\mathcal{X}$, contains exactly one point of $\mathcal{X}$.

In this thesis we study the Erdős-Szekeres set, the Horton set, the nested almost convex sets, and their representations in the plane with integer coordinates of small absolute values.

For the Erdős-Szekeres set of n points, we show how to realize its construction with integer coordinates of absolute values at most $O(n^2 \log_2(n)^3)$.

For the Horton set set of n points, we show how to realize its construction with integer coordinates of absolute values at most $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$. We also prove that any set of points with integer coordinates combinatorially equivalent (with the same order type) to the Horton set, contains a point with a coordinate of absolute value at least $c \cdot n^{\frac{1}{24}\log(n/2)}$, where c is a positive constant.

For the nested almost convex sets, we obtain a characterization. Our characterization implies that there exists at most one (up to order type) nested almost convex set of n points. We use our characterization to obtain a linear time algorithm to construct nested almost convex sets of n points, with integer coordinates of absolute values at most $O(n^{\log_2 5})$. Finally, we use our characterization to obtain an $O(n \log n)$-time algorithm to determine whether a set of points is a nested almost convex set.

Resumen

En 1935, Erdős y Szekeres demostraron que todo conjunto de n puntos en el plano en posición general, contiene los vértices de un polígono convexo de $\frac{1}{2}\log_2(n)$ vértices. En 1961, ellos construyeron, para todo entero positivo t, un conjunto de $n := 2^{t-2}$ puntos en el plano en posición general, tal que todo polígono convexo con vértices en este conjunto contiene a los más $\log_2(n)+1$ vértices. El conjunto obtenido por esta construcción es ahora conocido como *el conjunto de Erdős-Szekeres.*

En 1978, Erdős preguntó si todo conjunto en posición general lo suficientemente grande contiene los vértices de un polígono convexo de k vértices, con la propiedad adicional de que no hay otro punto del conjunto en su interior. Poco tiempo después, Horton proporcionó una construcción de un conjunto sin tales polígonos convexos de 7 vértices. El conjunto obtenido por esta construcción es ahora conocido como *el conjunto de Horton.*

En 2001 Károlyi, Pach y Tóth, introdujeron una familia de conjuntos de puntos para solucionar un problema tipo Erdős-Szekeres; este ha sido utilizado para solucionar varios otros problemas tipo Erdős-Szekeres. En esta tesis nos referimos a estos conjuntos como *conjuntos casi convexos anidados.* Un conjunto casi convexo anidado $\mathcal{X}$ tiene la propiedad de que el interior de todo triángulo determinado por tres puntos de la misma capa convexa de $\mathcal{X}$, contiene exactamente un punto de $\mathcal{X}$.

En esta tesis nosotros estudiamos el conjunto de Erdős-Szekeres, el conjunto de Horton, los conjuntos casi convexos anidados, y sus representaciones en el plano con coordenadas enteras de valor absoluto pequeño.

Para el conjunto de Erdős-Szekeres de n puntos, nosotros mostramos como realizar su construcción con coordenadas enteras de valor absoluto a lo más $O(n^2 \log_2(n)^3)$.

Para el conjunto de Horton de n puntos, nosotros mostramos como realizar su construcción con coordenadas enteras de valor absoluto a lo más $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$; nosotros también demostramos que todo conjunto de puntos combinatoriamente equivalente (con el mismo tipo de orden) al conjunto de Horton, contiene un punto con coordenadas con valor absoluto al menos $c \cdot n^{\frac{1}{24}\log(n/2)}$, donde c es una constante positiva.

Para los conjuntos casi convexos anidados, nosotros obtenemos una caracterización. Como consecuencia de nuestra caracterización existe a lo más (bajo tipos de orden) un conjunto casi convexo anidado de n puntos. Nosotros usamos nuestra caracterización para obtener un algoritmo de tiempo lineal para construir conjuntos casi convexos anidados de n puntos, con coordenadas enteras de valor absoluto a lo más $O(n^{\log_2 5})$. Finalmente, nosotros usamos nuestra caracterización para obtener un algoritmo de tiempo $O(n \log n)$ para determinar si un conjunto de puntos es un conjunto casi convexo anidado.

Acknowledgements

Agradezco al Consejo Nacional de Ciencia y Tecnología, CONACyT, por la beca que me otorgó para realizar mis estudios de doctorado. Agradezco al Dr. Ruy Fabila Monrroy por la ayuda que me brindó durante estos años, y por permitirme participar en el Proyecto 253261 ”Conjuntos de Puntos en Mallas Enteras” de Ciencia Básica, CONACyT.

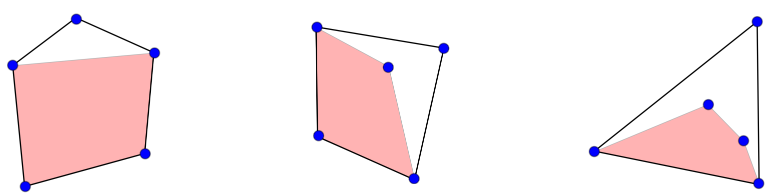
Contents

List of Figures

Chapter 1

Introduction

Let $\mathcal{X}$ be a set of five points in the plane in general position, *i.e.* such no three of them are collinear. Consider the points in the convex hull of $\mathcal{X}$. If there are three, four or five points in the convex hull of $\mathcal{X}$, it looks like in Figure 1.1a, Figure 1.1b or Figure 1.1c, respectively. Note that $\mathcal{X}$ always defines a convex quadrilateral, *i.e.* four points in convex position. This was observed by Esther Klein in 1935 [18]. According to this, she proposed the following problem.



(a) 5 points in the convex hull

(b) 4 points in the convex hull

(c) 3 points in the convex hull

Figure 1.1: Illustration of a convex quadrilateral in five points sets, according to the number of points in their convex hull.

Problem 1 (The happy ending problem). *Is it true that, for every s there exist an $ES(s)$, so that if there are given $ES(s)$ points in the plane in general position, one can always find s of them which determine the vertices of a convex s-gon?*

One of the most prolific mathematicians of the 20th century was Paul Erdős. He wrote around 1525 mathematical articles, solved and proposed numerous problems in different areas of mathematics. One of them solved problems is “The happy ending problem”, it was solved by Erdős and Szekeres in 1935 [18]. This problem let to the marriage of George Szekeres and Esther Klein; for this reason Paul Paul Erdős name it “The happy ending problem”.

In [18] Erdős and Szekeres proved that

$$ES(s) \leq \binom{2s-4}{s-2} + 1$$

and conjectured that $ES(s) = 1 + 2^{s-2}$ for $s \geq 3$. More than 80 years later the value of $ES(s)$ is still unknown for $s > 6$. For $s \leq 6$ it is known that: $ES(3) = 3$, it follows from that every three points in general position are in convex position; $ES(4) = 5$, it was observed by Klein in 1935 [18]; $ES(5) = 9$, according to Erdős and Szekeres in [18] it was first proved by Makai; $ES(6) = 17$, it was proved by Szekeres and Peters in 2006 in [43].

On the lower bound of the happy ending problem, Erdős and Szekeres prove in 1960 [19] that $ES(s) \geq 2^{s-2} + 1$. For this Erdős and Szekeres provided a set of $n := 2^{s-2}$ points in general position in the plane such that every convex $0k$-gon of this set has at most $s-1$ vertices. This set is known as the **Erdős-Szekeres set**. On the upper bound of the happy ending problem, Suk prove in 2016 [42] that

$$ES(s) \leq 2^{s+2s^{4/5}} + 1.$$

In [20], Erdős introduced the following variation of The happy ending problem.

Problem 2 (Erdős problem)**.** *For any positive integers $s \geq 3$ and $l \geq 1$ determine the smallest positive integer $E(s,l)$, if it exist, that satisfies the following. Every set of at least $E(s,l)$ points, contains s points in convex position with at most l points in their interior.*

In [20], Erdős also says that he did not get satisfactory bounds for $E(s,l)$, the reason of that would be given by Horton few years later. In 1983, Horton surprised the community with a simple proof that $E(s, 0)$ does not exist for $s \geq 7$ [29]. Horton construct an arbitrarily large point set with no empty convex heptagons; this set is known as the **Horton Set**.

On the values of $E(s, 0)$ for $s \leq 6$ it is known that: $E(3, 0) = 3$, it follows from the fact that every three points in general position are in convex position; $E(4, 0) = 5$, it follows from the observation of Klein on convex quadrilaterals in five points set; the value of $E(5, 0)$ is between 30 and 463, this bounds were obtained by Overmars and Koshelev [39, 34].

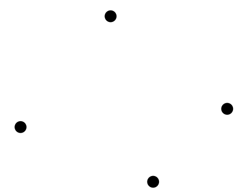
In 2001 [32] Károlyi, Pach and Tóth prove a modular version of the Erdős problem. They prove that every set of at least $B(s, l)$ points in general position, contains s points in convex position such that the number of points in the interior of their convex hull is 0 mod(l). For this Károlyi, Pach and Tóth introduce a some sets that, although they had no name, they have been used in other works related to the problem of Erdős. We refer to this sets as the **nested almost convex sets**.

In this thesis we are interested good representations of three points sets: the the Horton set, the Erdős-Szekeres set and the nested almost convex set. In the way, we introduce structural properties and some result related to this sets.

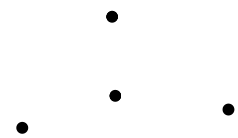
In the remaining of this chapter we introduce some concepts, and afterwards we specify the results obtained in this thesis.

Order types

Although the number of distinct sets of n points in the plane (in general position) is infinite, for most problems in Combinatorial Geometry only a finite number of them can be considered as essentially distinct. For the case $n = 3$, any three points in the plane in general position look like the vertices of a triangle. For the case $n = 4$, any four points in general position look like the vertices of a convex quadrilateral, or look like the vertices of a triangle with a point in its interior (See Figure 1.2). In this thesis we use a equivalence relation on point sets, known as the order type [24] to decide if two sets are essentially distinct.



(a) Convex position.

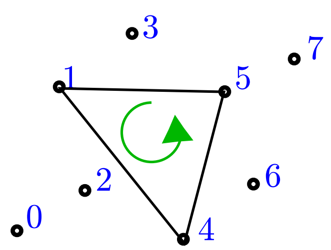


(b) No convex position.

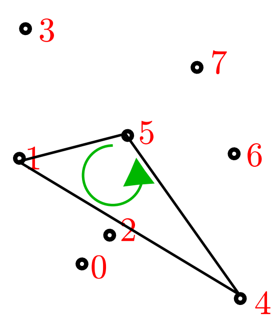
Figure 1.2: The two possible order types for a four points set.

The order type is one of the various equivalence relations on point sets proposed by Goodman and Pollack [23, 25, 26, 24]. It has been widely used in Combinatorial Geometry to classify point sets; two sets of points are consider essentially the same if they have the same order type.

The *order type* of a point set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ is a mapping that assigns to each ordered triplet (x_i, x_j, x_k) an orientation. If x_k is to the left of the directed line from x_i to x_j, the orientation of (x_i, x_j, x_k) is counterclockwise. If x_k is to the right of the directed line from x_i to x_j, the orientation of (x_i, x_j, x_k) is clockwise. We say that two set of points have the same order type, if there exist a bijection between these sets that preserves the orientation of all triplets. In Figure 1.3 there is an example of two sets with the same order type.



(a) A points set, $\mathcal{X}$.



(b) A drawing of $\mathcal{X}$.

Figure 1.3: An example of two different points sets with the same order type.

Drawings

Since their inception, order types were defined with computational applications in mind (see [24] for example). The orientation of a triple is determined by the sign of a determinant; many algorithms use precisely this determinant as their geometric primitive. Given that the determinant of an integer valued matrix is an integer, for numerical computations it is best if a point set has integer coordinates. Two main reasons are that integer arithmetic is much faster than floating point arithmetic, and that floating point arithmetic is prone to rounding errors. The latter is easily taken care of with an integer representation that can handle arbitrarily large numbers.

If a set of n points has already integer coordinates, it is best if these coordinates have as small absolute value as possible—again, for computational reasons. Even though rounding errors can be avoided using arbitrarily large integers, the cost of computation increases as the numbers get larger. Also, if we wish to store the point set, the number of bits needed depend on the size of the coordinates.

Definition 1. *Let S be a set of n points in general position in the plane. A* drawing *of S is a set of points with integer coordinates and with the same order type as S. The* size *of a drawing is the maximum of the absolute values of its coordinates.*

For the reasons mentioned above, it is of interest to find the drawing of S of minimum size. In [27] Goodman, Pollack and Sturmfels presented sets of n points in general position whose smallest drawings have size $2^{2^{c_1 n}}$, and proved that every point set has a drawing of size at most $2^{2^{c_2 n}}$ (where c_1 and c_2 are positive constants).

Aichholzer, Aurenhammer and Krasser [1] have assembled a database of drawings. For $n = 3, \ldots, 11$, the database contains a drawing of every possible set of n points in general position in the plane. The main advantage of having these drawings is that one can use them to compute certain combinatorial parameters of all point sets up to eleven points. The order type data base stops at eleven because the size of the database grows prohibitively fast. Thus, we cannot hope to store drawings for all point sets beyond small values of n; it is convenient however, to have programs that generate small drawings of infinite families of point sets which are of known interest in Combinatorial Geometry. In this direction, some sets have been drawn with small size.

Convex sets: In 1926 Jarník [30] showed how to found a draw of a set of points in convex position with size $O(n^3/2)$. He also proved that his construction is optimal.

Double circle: The Double Circle of $2n$ points is constructed as follows. Start with a convex n-gon; arbitrarily close to the midpoint of each edge, place a point in the interior of this polygon; finally place a point at each vertex of the polygon. In 2013, Bereg et al. [10] provided a linear time algorithm to generate a drawing of the Double Circle. Their drawing has size $O(n^{3/2})$; they also proved a lower bound of $\Omega(n^{3/2})$ on the size of every drawing of the Double Circle.

ComPoSe: ComPoSe is a multinational Collaborative Research Project (CRP) within the EUROCORES (EUROpean COllaborative RESearch) program EuroGIGA (Graphs in Geometry and Algorithms) of the European Science Foundation (ESF). In [21] the Authors provide drawings for up to 1000 points for: Convex sets, Horton set, Double chain, Double circle, Double ZigZag chain and Anomalous sets. The sets Double chain, Double ZigZag chain and Anomalous sets were constructed similar to Double circle and Convex sets. The Horton set was constructed using our method (Chapter 2) from [8].

We are mainly interested in having an algorithm that generates small drawings of the Horton set, the Erdős-Szekeres set and the nested almost convex sets. However, the problem of finding small drawings also raises interesting theoretical questions; we provide some of these later.

1.1 The Horton Set

A *k-hole* of a point set S, is a subset of k points of S, that are the vertices of a convex polygon with no other point of S in its interior. Besides being the Horton set, is the first construction of an arbitrarily large set without a 7-hole, every construction that we know of without 7-holes, contains the Horton set as a subset. Additionally, the Horton set has been the better extremal example in other problem related to holes. Some of them are the following.

Problem 3. *What is the minimum number of k-holes in every set of n points in the plane?*

The case of empty triangles of Problem 3 was first considered by Katchalski and Meir [33]. They constructed a set of n points with $200n^2$ empty triangles and showed that every set of n points contains $\Omega(n^2)$ of them. This bound was later improved by Bárány and Füredi [6], who showed that the Horton set has $2n^2$ empty triangles.

The Horton set was then used in a series of papers as a building block to construct sets with fewer and fewer k-holes. The first of these constructions was given by Valtr [46]; it was later improved by Dumitrescu [17] and the final improvement was given by Bárány and Valtr [7].

Problem 4. *What is the minimum number of empty monochromatic triangles in every two-colored set of n points in the plane?*

Since every set of 10 points contains a 5-hole, every two-colored set of at least 10 points contains an empty monochromatic triangle. The first non trivial lower bound of $\Omega(n^{5/4})$, on the number of empty monochromatic triangles in every two-colored set of n points, was given by Aichholzer et al [2]. This was later improved by Pach and Tóth [40] to $\Omega(n^{4/3})$. The known set with the least number of empty monochromatic triangles is given in [2]; it is based on the known set with the fewest number of empty triangles, which in turn is based on the Horton set. Devillers et al. [16] considered other chromatic variants of these problems. In particular, they described a three-coloring of the points of the Horton set with no empty monochromatic triangles.

In Chapter 2, we study the problem of drawing the Horton set in an integer grid with small size. We provide a drawing of size $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$ of the Horton set of n points; our drawing can be easily constructed in linear time. We also show a lower bound of $c \cdot n^{\frac{1}{24}\log(n/2)}$ (for some constant $c > 0$) on the minimum size of any drawing of the Horton set.

1.2 The Erdős-Szekeres Set

The first construction of the Erdős-Szekeres set was introduced in 1960 in [19]. This construction had some inaccuracies, which were corrected by Kalbfleisch and Stanton in 1995 in [31]. The construction described in [31] uses integer-valued coordinates; where the size of these coordinates grows quickly with respect to n. This has led some researchers to conjecture that the Erdős-Szekeres construction cannot be carried out with small integer coordinates.

As an example here are some excerpts from the book "Research Problems in Discrete Geometry" [12] by Brass, Moser and Pach regarding the Erdős-Szekeres construction.

> "The complexity of this construction is reflected by the fact that none of the numerous papers on the Erdős-Szekeres convex polygon problem includes a picture of the 16-point set without a convex hexagon."

> "Kalbfleisch and Stanton [31] gave explicit coordinates for the 2^{t-2} points in the Erdős-Szekeres construction. However, even in the case of $t = 6$ the coordinates are so large that they cannot be used for a reasonable illustration."

> "The exponential blowup of the coordinates in the above lower bound constructions may be necessary. It is possible that all extremal configurations belong to the class of order types that have no small realizations."

Also, in the survey [37] on the Erdős-Szekeres problem by Morris and Soltan we find the following.

> "The size of the coordinates of the points in the configurations given by Kalbfleisch and Stanton [31] that meet the conjectured upper bound on $N(n)$ grows very quickly. A step toward showing that this is unavoidable was taken by Alon et al. [4]."

In Chapter 3 we prove that the Erdős-Szekeres construction can be realized in a rather small integer grid of size $O(n^2 \log_2(n)^3)$. This solves an open problem of [12], which we discuss, together with other problems, at the end of Chapter 3.

1.3 The Nested Almost Convex Sets

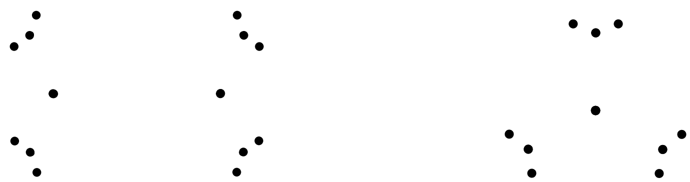
We define formally the nested almost convex sets as follows.

Definition 2. *Let $\mathcal{X}$ be a point set; let k be the number of convex layers of $\mathcal{X}$; and for $1 \leq j \leq k$, let R_j be the set of points in the j-th convex layer of $\mathcal{X}$. We say that $\mathcal{X}$ is a nested almost convex set if:*

1. *$\mathcal{X}_j := R_1 \cup R_2 \cup \cdots \cup R_j$ is in general position,*
2. *the vertices in the convex hull of $\mathcal{X}_j$ are the elements of R_j, and*
3. *any triangle determined by three points of R_j contains precisely one point of $\mathcal{X}_{j-1}$ in its interior.*

In previous papers, two constructions of nested almost convex sets have been presented. The first construction was introduced by Károlyi, Pach and Tóth in [32]. The second construction was introduced by Valtr, Lippner and Károlyi in [48] six years later.

Construction 1: Let X_1 be a set of two points. Assume that $j > 0$ and that X_j has been constructed. Let $z_1, \ldots z_r$ denote the vertices of R_j in clockwise order. Let P_j be the polygon with vertices in R_j. Let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let ℓ_i denote the line through z_i orthogonal to the



(a) Construction 1 (b) Construction 2

Figure 1.4: Examples of nested almost convex sets

bisector of the angle of P_j at z_i. Let z_i' and z_i'' be two points in ℓ_i at distance ε_j of z_i. Finally, move z_i' and z_i'' away from P_j at distance δ_j, in the direction orthogonal to ℓ_i, and denote the resulting points by u_i' and u_i'', respectively. Let $R_{j+1} = \{u_i', u_i'' : i = 1 \ldots r\}$ and $X_{j+1} = X_j \cup R_{j+1}$. It is easy to see that if ε_j and $\frac{\varepsilon_j}{\delta_j}$ are sufficiently small, then X_{j+1} is an almost convex set. See Figure 1.4a.

Construction 2: Let X_1 be a set of one point. Let R_2 be a set of three points such that, the point in X_1 is in the interior of the triangle determined by R_2. Let $X_2 = X_1 \cup R_2$. Now recursively, suppose that X_j and R_j have been constructed and construct the next convex layer R_{j+1} as in Construction 1. See Figure 1.4b.

This sets have been used in the following problems.

A modular version of the Erdős problem. In 2001 [32] Károlyi, Pach and Tóth use the nested almost convex sets to prove that, for any $s \geq 5l/6 + O(1)$, there is an integer $B(s, l)$ with the following property. Every set of at least $B(s, l)$ points in general position contains s points in convex position such that the number of points in the interior of their convex hull is 0, modulo (l). This "modular" version of the Erdős problem was proposed by Bialostocki, Dierker, and Voxman [11]. This was proved for $s \geq l + 2$ by Bialostocki et al. The original upper bound on $B(s, l)$ was later improved by Caro in [14].

A version of the Erdős problem in almost convex sets. We say that $\mathcal{X}$ is an *almost convex* set if every triangle with vertices in $\mathcal{X}$ contains at most one point of $\mathcal{X}$ in its interior. Let $N(s)$ be the smallest integer such that every almost convex set of at least $N(s)$ points contains an s-hole. In 2007 [48] Valtr Lippner and Károlyi use the nested almost convex sets to prove that:

$$N(s) = \begin{cases} 2^{(s+1)/2} - 1 & \text{if } s \geq 3 \text{ is odd} \\ \frac{3}{2}2^{s/2} - 1 & \text{if } s \geq 4 \text{ is even.} \end{cases} \tag{1.1}$$

The authors use the nested almost convex sets to attain the equality in (1.1). The existence of $N(s)$ was first proved by Károlyi, Pach and Tóth in [32]. The upper bound for $N(s)$ was improved by Kun and Lippner in [35], and it was improved again by Valtr in [47].

Maximizing the number of non-convex 4-holes. In 2014 [3] Aichholzer, Fabila-Monroy, González-Aguilar, Hackl, Heredia, Huemer, Urrutia and Vogtenhuber prove that the maximum number of non-convex 4-holes in a set of n points is at most $n^3/2 - \Theta(n^2)$. The authors use the nested almost convex sets to prove that some sets have $n^3/2 - \Theta(n^2 \log(n))$ non-convex 4-holes.

Blocking 5-holes. A set of points B blocks the convex k-holes in $\mathcal{X}$, if any k-hole of $\mathcal{X}$ contains at least one element of B in the interior of its convex hull. In 2015 [13] Cano, Garcia, Hurtado, Sakai, Tejel and Urritia use the nested almost convex sets to prove that: $n/2 - 2$ points are always necessary and sometimes sufficient to block the 5-holes of a point set with n elements in convex position and $n = 4k$. The authors use the nested almost convex sets as an example of a set for which $n/2 - 2$ points are sufficient to block its 5-holes.

In Chapter 4, we introduce a characterization of nested almost convex sets. Our characterization implies that there exists at most one (up to order type) nested almost convex set of n points. We use our characterization to obtain a linear time algorithm to construct nested almost convex sets of n points, with integer coordinates of absolute values at most $O(n^{\log_2 5})$.

Finally, we use our characterization to obtain an $O(n \log n)$-time algorithm to determine whether a set of points is a nested almost convex set.

Chapter 2

The Horton Set

In this chapter we are mainly interested in having small drawings of the Horton set. First, in Section 2.1, we define the Horton set and exhibit two of its constructions. Afterwards, in Section 2.2, we provide a drawing of size $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$ of the Horton set of n points. Finally, in Section 2.3, we introduce a lower bound of $c \cdot n^{\frac{1}{24}\log(n/2)}$ (for some $c > 0$) on the minimum size of any drawing of the Horton set.

2.1 Background

2.1.1 Definition of the Horton Set

Let $\mathcal{X}$ be a set of n points in the plane with no two points having the same x-coordinate. Let $p_0, p_1, \ldots, p_{n-1}$ be the points in $\mathcal{X}$ sorted from left to right by their x-coordinate. In this chapter we denote by $\mathcal{X}_{\text{even}}$ the subset of the even-indexed points, and we denote by $\mathcal{X}_{\text{odd}}$ the subset of the odd-indexed points. Thus

$$\mathcal{X}_{\text{odd}} = \{p_i : i \text{ is odd}\} \quad \mathcal{X}_{\text{even}} = \{p_i : i \text{ is even}\}.$$

Definition 3. *Let X and Y be two sets of points in the plane. We say that X is* **high above** *Y if:*

- *every line determined by two points in X is above every point in Y,* *and*

- *every line determined by two points in Y is below every point in X.*

In Figure 2.1 we introduce an example of a set of points $\mathcal{X}$ where $\mathcal{X}_{\text{odd}}$ is high above $\mathcal{X}_{\text{even}}$.

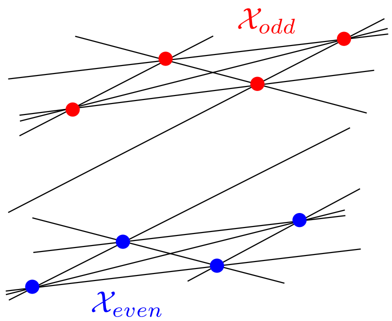


Figure 2.1: The eight points Horton set, as example of a set of points $\mathcal{X}$ where $\mathcal{X}_{\text{odd}}$ is high above $\mathcal{X}_{\text{even}}$.

Definition 4. *A* ***Horton set*** *is a set H^k of 2^k points, with no two points having the same x-coordinate, that satisfies the following properties. See Figure 2.1.*

1. *H^0 is a Horton set;*
2. *both H^k_{even} and H^k_{odd} are Horton sets ($k \geq 1$);*
3. *H^k_{odd} is high above H^k_{even} ($k \geq 1$).*

2.1.2 Previous construction

In this subsection we provide two ways to obtain the Horton set. First we introduce the construction given by Horton in [29]; it was the first construction of the Horton set. Afterwards we introduce a more intuitive way to define the Horton set given in [6] by Bárány and Füredi.

Horton's Construction: Let k be a positive integer. Let $a_1, a_2, \ldots, a_k$ be the binary expansion of the integer i, $0 \leq i \leq 2^k$. Note that leading 0's are not omitted. Let $c = 2^k + 1$ and define

$$d_i = \sum_{j=1}^{k} a_j c^{j-1}$$

let p_i be the point (i, d_i) and define H^k to be the set of points $\{p_i : i = 0 \dots 2^{k-1}\}$.

Bárány and Füredi's Construction: Let

$$H^0 := \{(1,1)\} \quad \text{and} \quad H^1 := \{(1,1),(2,2)\}.$$

For $k \geq 2$ obtain H^k using recursively:

$$\begin{aligned} H^k := &\{(2x-1, y) : (x,y) \in H^{k-1}\} \\ &\cup \{(2x, y + 3^{2^{k-1}}) : (x,y) \in H^{k-1}\}. \end{aligned}$$

The previous two constructions have exponential size; we have not seen in the literature a drawing of subexponential size.

2.2 Construction of the Horton set with small integer coordinates.

Let

$$f(i) := \begin{cases} 0 & \text{if } i = 1. \\ 2^{\frac{i(i-1)}{2}-1} & \text{if } i \geq 2. \end{cases}$$

$$g(i) := \begin{cases} 0 & \text{if } i = 1. \\ f(i) - f(i-1) & \text{if } i \geq 2. \end{cases}$$

Let $P^0 := \{(0,0)\}$. For $1 \leq i \leq k$, let

$$P^i_{\text{even}} := \{(2x, y) : (x, y) \in P^{i-1}\}$$
$$P^i_{\text{odd}} := \{(2x+1, y + g(i)) : (x, y) \in P^{i-1}\}$$
$$P^i := P^i_{\text{even}} \cup P^i_{\text{odd}}$$

In this section we prove that P_k is a drawing of the Horton set of $n = 2^k$ points; with size $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$. Note that P_k can be easily constructed in linear time.

Theorem 1. *There exist a drawing of the Horton set of n points of size $\frac{1}{2}n^{\frac{1}{2}\log(n/2)}$ for $n \geq 16$.*

Proof. We prove by induction on k that P^k is the desired drawing. It can be verified by hand that P^4 has size equal to $32 = \frac{1}{2}16^{\frac{1}{2}\log(16/2)}$; assume that $k \geq 5$. By induction P^k_{even} and P^k_{odd} are Horton sets; it only remains to show that P^k_{odd} is high above P^k_{even}. We only prove that every point of P^k_{odd} is above every line through two points of P^k_{even}; the proof that every point of P^k_{even} is below every line through two points of P^k_{odd} is analogous.

Let $p_0, p_1, \ldots, p_{n-1}$ be the points of P^k sorted by their x-coordinate. Let $0 \leq i < j \leq n-1$ be two even integers, and let ℓ be the directed line from p_i to p_j. By definition P^k_{odd} is above the horizontal line passing through p_1; in particular, since the smallest y-coordinate of P^k_{odd} is equal to $g(k)$, P^k_{odd} is above the line segment joining the points $(1, g(k))$ and $(n-3, g(k))$. Therefore, it suffices to show that $(1, g(k))$, $(n-3, g(k))$ and p_{n-1} are above ℓ.

We will define a line ℓ' with the property that if $(1, g(k))$ and $(n-3, g(k))$ are above ℓ', then $(1, g(k))$, $(n-3, g(k))$ and p_{n-1} are above ℓ. Afterwards we will show that indeed $(1, g(k))$ and $(n-3, g(k))$ are above ℓ'.

If the slope of ℓ is non-positive, define ℓ' to be the line passing through the points $(n-6, f(k-1))$ and $(n-4, 0)$; if the slope of ℓ is positive, define ℓ' to be the line passing through the points $(0,0)$ and $(2, f(k-1))$. Note that the largest y-coordinate of P^k_{even} is equal to $\sum_{i=1}^{k-1} g(i) = f(k-1)$. Therefore the slope of ℓ is at least $-f(k-1)/2$ and at most $f(k-1)/2$; in particular

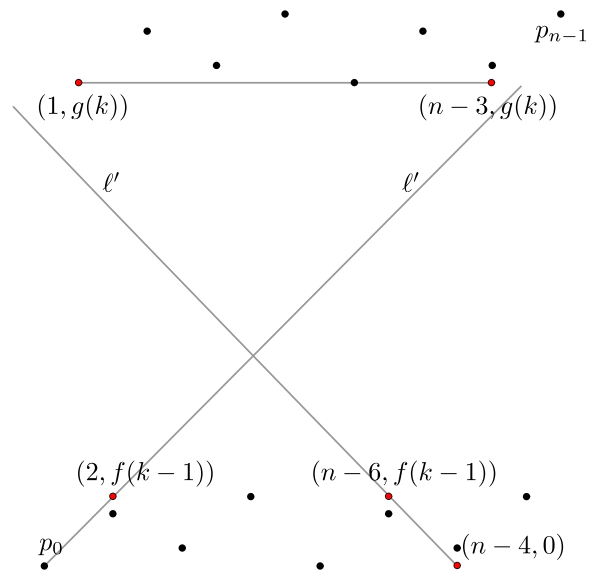


Figure 2.2: The two possible definitions of ℓ' in the proof of Theorem 1.

the absolute value of the slope of ℓ' is larger or equal to the absolute value of the slope of ℓ. The farthest point of P^k_{even} to the right that ℓ can contain while having non-positive slope is p_{n-4} (which has x-coordinate equal to $n-4$); the farthest point of P^k_{even} to the left that ℓ can contain while having positive slope is p_0. Therefore in both cases if $(1, g(k))$ and $(n-3, g(k))$ are above ℓ', then they are also above ℓ; see Figure 2.2. (As $(2, f(k-1))$, $(n-6, f(k-1))$, $(n-4, 0)$, $(n-3, g(k))$ are not in P^k, they may have the same x coordinate that a point in P^k.)

If ℓ has non-positive slope and $(1, g(k))$ is above ℓ, then p_{n-1} is also above ℓ' since p_{n-1} has larger x-coordinate. If ℓ has positive slope and $(n-3, g(k))$ is above ℓ', then p_{n-1} must also be above ℓ. Otherwise ℓ intersects the line segment joining $(n-3, g(k))$ and p_{n-1}; this line segment has slope equal to $f(k-1)/2$, since the y-coordinate of p_{n-1} is equal to $\sum_{i=1}^{k} g(i) = f(k)$. This in turn would imply that ℓ has slope larger than $f(k-1)/2$—a contradiction.

Suppose ℓ has non-positive slope. Then it suffices to show that $(1, g(k))$ is above ℓ'. This is the case since:

$$\begin{aligned}
&\begin{vmatrix} n-6 & f(k-1) & 1 \\ n-4 & 0 & 1 \\ 1 & g(k) & 1 \end{vmatrix} \\
= \;& 2g(k) - (n-5)f(k-1) \\
= \;& 2f(k) - (n-3)f(k-1) \\
= \;& 2f(k) - 2^k f(k-1) + 3f(k-1) \\
= \;& 3f(k-1) \\
> \;& 0.
\end{aligned}$$

Suppose ℓ has positive slope. Then it suffices to show that $(n-3, g(k))$ is above ℓ'. This is the case since:

$$\begin{aligned}
&\begin{vmatrix} 0 & 0 & 1 \\ 2 & f(k-1) & 1 \\ n-3 & g(k) & 1 \end{vmatrix} \\
= \;& 2g(k) - (n-3)f(k-1) \\
= \;& 2f(k) - (n-1)f(k-1) \\
= \;& 2f(k) - 2^k f(k-1) + f(k-1) \\
= \;& f(k-1) \\
> \;& 0.
\end{aligned}$$

Finally the largest x-coordinate of P^k is equal to $n-1$, and the largest y-coordinate of P^k is equal to

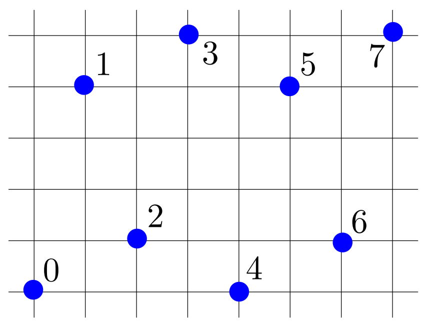
$$\sum_{i=1}^{k} g(i) = f(k) = 2^{\frac{k(k-1)}{2}-1} = \frac{1}{2} n^{\frac{1}{2}\log(n/2)},$$

since $k = \log n$. Therefore, P^k is a drawing of the Horton set of n points of size $\frac{1}{2} n^{\frac{1}{2}\log(n/2)}$. $\square$

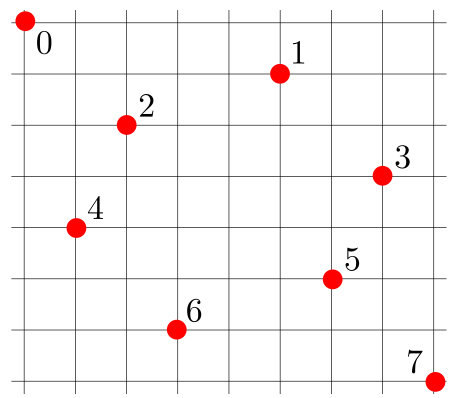
2.3 Lower Bound for the Size of the Horton Sets Drawings

Consider the set in the Figure 2.3b; it is a drawing of a Horton Set since it has the same order type than the set in Figure 2.3a. However, the set in the Figure 2.3b does not satisfy Definition 4.

Definition 5. *We call a drawing that satisfies Definition 4, an* isothetic *drawing of the Horton set.*



(a) An isothetic drawing of the Horton set.



(b) A no isothetic drawing of the Horton set.

Figure 2.3: An example of a drawing of the Horton set, and a other drawing of the same Horton set that does not satisfy the definition of Horton set.

In this section we prove the following lower bounds on the size of drawings of the Horton set.

Theorem 2. *For a sufficiently large value of k, every isothetic drawing of the Horton set of $n = 2^k$ points has size at least $n^{\frac{1}{8}\log n}$.*

Theorem 3. *Every drawing of the Horton set of $n = 2^k$ points has size at least $c \cdot n^{\frac{1}{24}\log(n/2)}$, for a sufficiently large value of n and some positive constant c.*

This section is divided in three parts. First, we introduce some information on the structure of the isothetic drawings of the Horton set.

Afterwards, we obtain a recursive expression for the lower bounds for the size of isothetic drawings of the Horton set. Finally, we prove Theorem 2 and Theorem 3.

In the following two subsections P is an isothetic drawing of the Horton set of $n := 2^k$ points, and $p_0, p_1, \ldots, p_{n-1}$ are the points of P sorted by their x-coordinate.

2.3.1 Isothetic drawings

In this subsection, first we introduce an auxiliary structure for the isothetic drawings of the Horton set; afterward we introduce a measure for the width of these sets.

We recursively define a complete rooted binary tree T, as follows. P is the root of T; and if $Q \subset P$ is a vertex of T, of at least two points, then Q_{even} and Q_{odd} are its left and right children, respectively. Furthermore, for each vertex in T, label the edge incident to its left child with a "0" and the edge incident with its right child with a"1"; the labels encountered in a path from a leaf $\{p_i\}$ to the root are precisely the bits in the binary expansion of i; see Figure 2.4.

By construction, the vertices of T are sets of 2^i points of P (for some $0 \leq i \leq k$). Let T_i be the set of vertices of T that consist of exactly 2^i points of P: we call it the i-th[1] level of T. The first level, T_1, are the vertices of T that consist of a pair of points of P. For each such pair, we consider the line defined by them.

Let R be the closed vertical slab bounded by the vertical lines through $p_{n/4}$ and $p_{3n/4-1}$. Let Q be a vertex at the first level of T and let p_i and p_j be its leftmost and rightmost points respectively. Suppose that Q is a left child. Then the two most significant bits in the binary expansion of i are "00", and the two most significant bits in the binary expansion of j are "10". This implies that $i \leq n/4$ and $j - i = n/2$; in particular p_j is contained in R, while p_i is to the left of R. In this case, we say that Q is *left-to-right* crossing. By similar arguments if Q is a right child, then p_i

[1]In the literature the i-th level of a binary tree are those vertices at distance i from the root; we have precisely the opposite order.

is contained in R, while p_j is to the right of R. In this case we say that Q is *right-to-left* crossing. Note that the vertices in the first level of T, in their left to right order in T, are alternatively *left-to-right* and *right-to-left* crossing (see Figure 2.4). The following lemma relates the left to right order of these vertices in T, to the bottom-up order of their corresponding pairs of points of P.

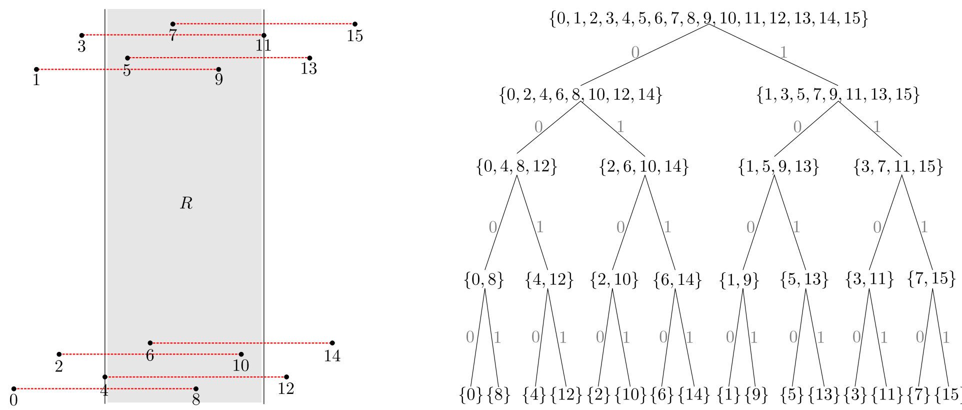


Figure 2.4: The Horton set and its associated tree T.

Lemma 4. *The lines defined by the vertices of the first level of T do not intersect inside R. In particular, the bottom-up order of convex hull of these vertices corresponds to their left to right order in T.*

Proof. Let Q_1 and Q_2 be two vertices in the first level of T such that Q_1 is a left child and Q_2 is a right child. Without loss of generality assume that in the left to right order in T, Q_1 is before Q_2. Then, Q_1 is left-to-right crossing and Q_2 is right-to-left crossing. Let γ_1 and γ_2 be the lines defined by Q_1 and Q_2, respectively. If γ_1 and γ_2 intersect inside R, then the leftmost point of Q_1 is above γ_2 or the rightmost point of Q_2 is below γ_1—a contradiction to property 3 of Definition 4. Since between every two left children there is a right child, and between every two right children there is a left child, the result follows. $\square$

By construction every vertex of T is an isothetic drawing of the Horton set. The main idea behind the proof of the lower bound on the size of isothetic drawings of the Horton set is to lower bound the size of these drawings in terms of the size of their children. We define some parameters on the vertices of T, that make this idea more precise.

Let $2 \leq t \leq k$ be an integer. Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four vertical lines sorted from left to right, such that:

- All of them are contained in the interior of R.
- There are exactly 2^{k-t} points of P between both pairs (ℓ_1, ℓ_2) and (ℓ_3, ℓ_4).

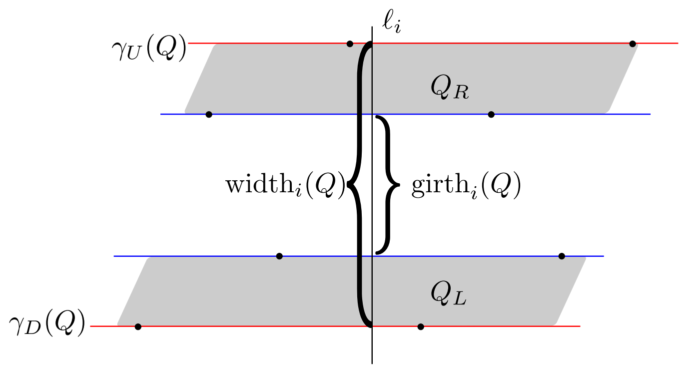


Figure 2.5: The bounding lines of Q, together with its width and girth.

Let Q be a vertex of T with more than two points. For each of the ℓ_i, we define two parameters of Q. Let $\gamma_D(Q)$ be the line defined by the leftmost descendant of Q in T_1. Let $\gamma_U(Q)$ be the line defined by the rightmost descendant of Q in T_1. Note that Q is bounded from below by $\gamma_D(Q)$ and from above by $\gamma_U(Q)$ (Lemma 4). Let Q_L and Q_R be the left and right children of Q, respectively. Define width$_i(Q)$ as the distance between the points $\gamma_D(Q) \cap \ell_i$ and $\gamma_U(Q) \cap \ell_i$, and girth$_i(Q)$ as the distance between the points $\gamma_U(Q_L) \cap \ell_i$ and $\gamma_D(Q_R) \cap \ell_i$; see Figure 2.5.

2.3.2 Pushing girths and widths

In this subsection we give a lower bound the girth of a vertex of T, in terms of the girth of one of its children. This bound is expressed in Lemma 5.

Before proceeding we need one more definition. Let Q be a vertex of T with more than two points and let $P(Q)$ be its parent. If Q is the left child of $P(Q)$, let $S(Q)$ be the right child of Q; otherwise let $S(Q)$ be the left child of Q.

Lemma 5. *Let Q be a vertex at the l-th level of T, for some $1 \leq t < l < k$. If the distance between ℓ_1 and ℓ_2 is d_1, and the distance between ℓ_3 and ℓ_4 is d_2, then:*

(1) $\operatorname{girth}_1(P(Q)) \geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{l-t-1} \operatorname{girth}_4(Q) - \operatorname{width}_1(S(Q))$ *and,*

(2) $\operatorname{girth}_4(P(Q)) \geq \left(\frac{(d_2)^2}{(d_1+d_2)d_1}\right) 2^{l-t-1} \operatorname{girth}_1(Q) - \operatorname{width}_4(S(Q))$.

Proof. We will prove inequality (1); the proof of (2) is analogous. Assume that Q is the left child of $P(Q)$ and let Q' be the right child of $P(Q)$; the case when Q is the right child of $P(Q)$ can be proven with similar arguments. Note that is $S(Q)$ is the right child, Q_R, of Q.

Let p'_1 and p'_2 be two consecutive points in Q_L lying between ℓ_3 and ℓ_4 at a horizontal distance of at most $\Delta_x := d_2/2^{l-t-1}$ from each other; such a pair exists as there are 2^{l-t-1} points of Q_L between ℓ_3 and ℓ_4. Let p'' be the point in Q_R that lies between p'_1 and p'_2 (in the x-coordinate order). Let φ be the line through p'_2 and p''. Let $\Delta_y := \min\{\operatorname{girth}_3(Q), \operatorname{girth}_4(Q)\}$; note that the slope of φ is at most $-\Delta_y/\Delta_x$. Recall that by Lemma 4, $\gamma_D(Q_R)$ and $\gamma_U(Q_L)$ do not intersect between ℓ_1 and ℓ_4; this implies that $\operatorname{girth}_3(Q) \geq \frac{d_1}{d_1+d_2} \operatorname{girth}_4(Q)$, in particular $\Delta_y \geq \frac{d_1}{d_1+d_2} \operatorname{girth}_4(Q)$. Therefore, the slope of φ is at most

$$-\Delta_y/\Delta_x = -\left(\frac{d_1}{d_1+d_2} \operatorname{girth}_4(Q)\right)/\Delta_x = \frac{d_1}{(d_1+d_2)d_2} 2^{l-t-1} \operatorname{girth}_4(Q).$$

Define the following points $q_1 := \gamma_D(Q_R) \cap \ell_1$, $q_2 := \varphi \cap \ell_1$ and $q_3 := \gamma_D(Q') \cap \ell_1$. See Figure 2.6 (in it the straight lines φ and φ' were represented by curved lines). Note that the leftmost point of $\gamma_D(Q') \cap Q'$ is to the left of ℓ_1; since this point is above φ, q_2 cannot be above q_3. Therefore, the distance from q_1 to q_2 is at most the distance from q_1 to q_3; the distance from q_1 to q_3 is precisely $\operatorname{girth}_1(P(Q)) + \operatorname{width}_1(S(Q))$. We now show that the distance

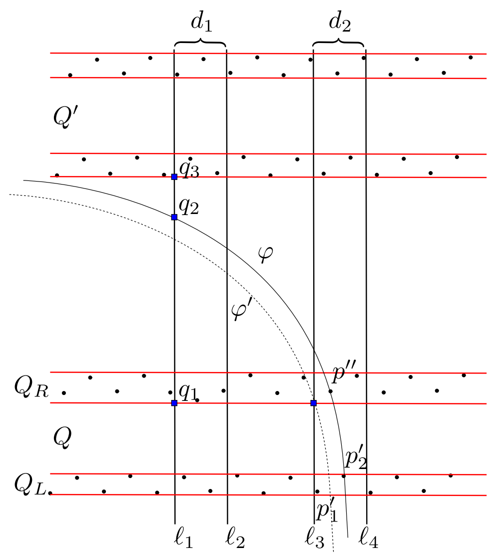


Figure 2.6: Schematic depiction of the proof of Lemma 5.

from q_1 to q_2 is at least $\frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\mathrm{girth}_4(Q)$—this completes the proof of (1).

Let φ' be the line parallel to φ and passing through the intersection point of ℓ_3 and $\gamma_D(Q_R)$. Note that φ' is below φ. Therefore, the distance from q_1 to q_2 is at least the distance of q_1 to the intersection point of φ': this is at least $d_1(\Delta_y/\Delta_x) = \frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\mathrm{girth}_4(Q)$. □

Two obstacles prevent us from directly applying Lemma 5. One is that the difference between d_1 and d_2 may be too big and in consequence $\frac{(d_1)^2}{(d_1+d_2)d_2}$ or $\frac{(d_2)^2}{(d_1+d_2)d_1}$ too small. This situation can be fixed with following Lemma.

Lemma 6. *For $t := \lceil 2\log k\rceil$ and $k \geq 16$, P has size at least $n^{\frac{1}{2}\log n}$ or $\ell_1, \ell_2, \ell_3, \ell_4$ can be chosen so that the ratio between d_1 and d_2 is at least $1/2$ and at most 2.*

Proof. Let $\varphi_1, \ldots, \varphi_{2^{t-1}}$ be consecutive vertical lines such that:

- all of them lie in the interior of R and,

- between every pair of two consecutive lines $(\varphi_i, \varphi_{i+1})$ there are exactly 2^{k-t} points of P.

For $1 \leq i < 2^{t-1}$, let Δ_i be the distance between φ_i and φ_{i+1}; let $\Delta'_1 \leq \Delta'_2 \leq \cdots \leq \Delta'_{2^{t-1}-1}$ be these distances sorted by size. We look for a pair $\Delta'_i \leq \Delta'_j$, such that one is at most two times the other. Suppose there is no such pair; then $\Delta'_{i+1} \geq 2\Delta'_i$. Since between the two lines defining Δ'_1 there are exactly 2^{k-t} points of P, and no three of them have the same integer x-coordinate, $\Delta'_1 \geq 2^{k-t-1}$. Therefore,

$$\Delta'_{2^{t-1}-1} \geq 2^{k-t-1} \cdot 2^{2^{t-1}-2} \geq 2^{\frac{1}{2}k^2+k-t-3} \geq 2^{\frac{1}{2}k^2+k-2\log(k)-4} \geq n^{\frac{1}{2}\log n}.$$

The latter part of the inequality follows from our assumption that $k \geq 16$. Therefore, if there is no such pair, P has size at least $n^{\frac{1}{2}\log n}$.

□

The second obstacle is that the second term in the right hand sides of inequalities (1) and (2) of Lemma 5 may be too large. In this case, we prune T to get rid of vertices of large width; this is done by choosing an integer $l \leq k-1$ and then removing from P all the points contained in either: all the vertices of T_l that are a left child to their parent, or all the the vertices of T_l that are a right child to their parent (see Figure 2.7). We call this operation *pruning* the l-th level of T. The resulting set is a drawing of the Horton set, as shown by the following lemma.

Lemma 7. *Let P' be the subset of P that results from pruning the l-th level of T. Then:*

(1) P' is an isothetic drawing of the Horton set of $n/2$ points.

(2) Suppose that $l \leq k-3$. Let T' be the tree associated to P', and Q' be any vertex at the l-th level of T' (for some $l' > l$). Then there exist a vertex Q at the $(l'+1)$-level of T that contains Q'. Moreover, $S(Q') \subset S(Q)$.

Proof. Assume without loss of generality that the left children are removed when pruning T. If $l = k-1$, (2) holds trivially, and (1) holds because in

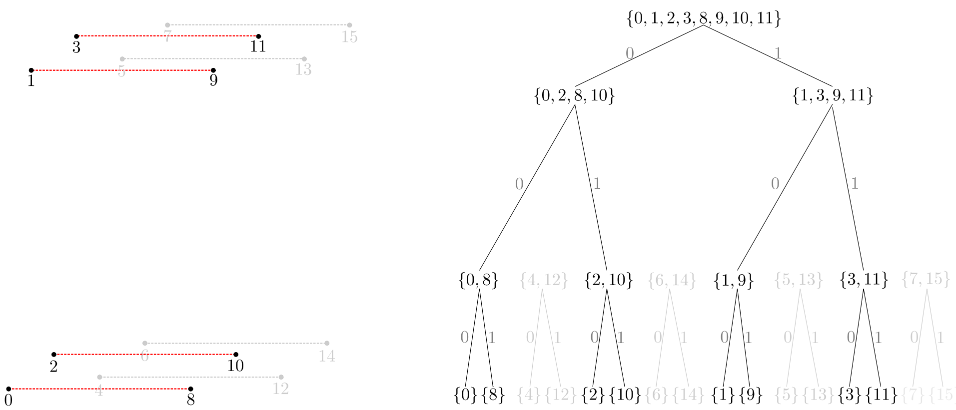


Figure 2.7: P and T after the removal of the vertices of T_1 that are a right child to their parent.

that case $P' = P_{\text{odd}}$. Assume that $l \leq k-2$, and let $s := k - l$; we proceed by induction on s.

Note that P_{even} and P_{odd} are each an isothetic drawing of the Horton set of $n/2$ points. Moreover, their corresponding trees, T_{even} and T_{odd}, are the subtrees of T rooted at P_{even} and P_{odd}, respectively. Therefore, when we prune the l-th level of T, we also prune the l-th level of T_{even} and T_{odd}. By induction and (1), this produces two isothetic drawings of the Horton set of $n/4$ points; let $P_0'' \subset P_{\text{even}}$ and $P_1'' \subset P_{\text{odd}}$ be these drawings, respectively.

We first prove $(*)$: *P' can be constructed from P by, starting at p_0, alternatively removing and keeping intervals of 2^{k-l-1} consecutive points of P.*

For $s = 1$, this is trivial since $2^{k-l-1} = 1$ and $P' = P_{\text{odd}}$. Thus, by induction, P_0'' and P_1'' are constructed from P_{even} and P_{odd} by, starting at their leftmost point, alternatively removing and keeping intervals of 2^{k-l-2} consecutive points of P_{even} and P_{odd}, respectively. Let $I_1', \dots, I_{2^{l+1}}' \subset P_{\text{even}}$ and $J_1', \dots, J_{2^{l+1}}' \subset P_{\text{odd}}$ be these intervals (in order). Finally, $(*)$ follows by letting $I_i := I_i' \cup J_i'$.

We now prove *1* and *2*.

(1) Note that $(*)$ implies that $P_{\text{even}}' = P_0''$ and $P_{\text{odd}}' = P_1''$. Thus

$P'_{\text{even}} \subset P_{\text{even}}$ and $P'_{\text{odd}} \subset P_{\text{odd}}$; in particular P'_{odd} is high above P'_{even}. Therefore, P' is an isothetic drawing of the Horton set of $n/2$ points.

(2) Consider the following algorithm. Remove from T the subtrees rooted at the vertices in the l-th level of T that are a left child to their parent; afterwards, remove from each vertex of T the points in $P \setminus P'$. After this last step, each vertex at the l-th level of T that was not removed is equal to its parent—producing a loop; remove these loops. We claim that this algorithm produces T'. For $s = 1$, this follows from $(*)$. Let T'_{even} and T'_{odd} be the left and right subtrees of the root of T', respectively. By induction T'_{even} and T'_{odd} can constructed from T_{even} and T_{odd} with the above algorithm, respectively. Since the root of T' is precisely the root of T minus the points in $P \setminus P'$, the algorithm produces T'.

Now, suppose that $l \leq k - 3$ and let Q' be a vertex at the l'-th level of T', for some $l' > l$. By the algorithm, there is a vertex Q such that $Q' = Q \setminus (P \setminus P')$; this vertex is in the $(l' + 1)$-level of T. Finally, also by the algorithm we have that $S(Q') \subset S(Q)$.

□

2.3.3 Lower bound

We are now ready to prove our lower bound on the size of isothetic drawings of the Horton set.

Proof of Theorem 2. Set $t := \lceil 2 \log k \rceil$ and assume that $k \geq 16$. By Lemma 6 ℓ_1, ℓ_2, ℓ_3 and ℓ_4 can be chosen so that, the ratio of the distance between d_1 and d_2 is at least $1/2$ and at most 2. Without loss of generality assume that $d_1 \leq d_2$. Let D be the distance between ℓ_1 and ℓ_4. We may assume that

$$D < n^{\frac{1}{8} \log n};$$

as otherwise we are done.

Let Q be a vertex in the $(t + 1)$-th level of T. Note that between two consecutive points in every vertex at the l-th level of T there are exactly

$2^{k-l}-1$ points of P. This trivially holds for $l=k$; it holds for smaller values of l, by induction on $k-l$. In particular, there are $(2^{t+1}-1)(2^{k-t-1}-1)+2^{t+1}-2=2^k-2^{k-t-1}-1$ points of P between the leftmost and rightmost point of Q.

As we will see, there are exactly two points of Q between ℓ_1 and ℓ_2, and exactly two points of Q between ℓ_3 and ℓ_4.

Suppose that there were less than two points of Q between ℓ_1 and ℓ_2, then the number of points of P would be at least the sum of the following.

- The number of points of P between ℓ_1 and ℓ_2; recall that this is equal to 2^{k-t}.

- The number of points of P between the leftmost and rightmost point of Q that are not between ℓ_1 and ℓ_2; since there are exactly $2^{k-t-1}-1$ points of P between two consecutive points of Q, and at most one point of Q between ℓ_1 and ℓ_2, this is at least $(2^k-2^{k-t-1}-1)-2^{k-t-1}=2^k-2^{k-t}$.

- Two, for the leftmost and rightmost point of Q.

In total this is at least $2^k+1=n+1$—a contradiction; similar arguments hold for ℓ_3 and ℓ_4.

Suppose that there are more than two points of Q between ℓ_1 and ℓ_2, then the number of points of P between ℓ_1 and ℓ_2 is at least $2(2^{k-t-1})+3=2^{k-t}+3$; this is a contradiction to the assumption that there are exactly 2^{k-t} points of P between ℓ_1 and ℓ_2. The same argument holds for ℓ_3 and ℓ_4.

The two points of Q between ℓ_1 and ℓ_2, and the two points of Q between ℓ_3 and ℓ_4, have integer coordinates. Therefore, by Pick's theorem [41] the area of their convex hull is at least one. Since these points are contained in trapezoid bounded by $\gamma_D(Q)$, $\gamma_U(Q)$, ℓ_1 and ℓ_4, the area of this trapezoid is also at least one. But this area is at most $D(\text{width}_1(Q)+\text{width}_4(Q))/2$. Therefore

$$\max\{\text{width}_1(Q),\text{width}_4(Q)\}\geq 1/D. \tag{2.1}$$

This bound also holds for every vertex at a level higher than $t+1$ (since all of these vertices contain vertices at the $t+1$-level as subsets).

Let $t < l \leq k$ be the largest positive integer such that there exists a vertex R in the l-th level of T that satisfies:

$$\max\{\mathrm{width}_1(S(R)), \mathrm{width}_4(S(R))\} \geq \frac{2^{(l-t-6)(l-t-7)/2}}{D}. \tag{2.2}$$

Such an l and R exist since (2.2) holds for every vertex at the $(t+6)$-th level of T. Indeed if Q is a vertex at the $(t+6)$-th level of T, then $S(Q)$ is in the $(t+5)$ level of T and by (2.1):

$$\max\{\mathrm{width}_1(S(Q)), \mathrm{width}_4(S(Q))\} \geq 1/D = \frac{2^{((t+6)-t-6)((t+6)-t-7)/2}}{D}.$$

Without loss of generality assume that

$$\mathrm{width}_1(S(R)) \geq (2^{(l-t-6)(l-t-7)/2})/D$$

and that R is a left child. We may assume that $l < k$, otherwise (2.2) implies that P has size at least $n^{\frac{1}{8}\log n}$ (for a sufficiently large value of k).

We now apply Lemma 7 to prune T of all the vertices of large width (that, is that satisfy (2.2)). Prune the l-th level of T by removing all the vertices that are a left child to their parent. Let P' be the resulting point set and T' its corresponding tree. No vertex of T' in a level higher than l satisfies (2.2); otherwise, by part (2) of Lemma 7 there would be a vertex at level of T higher than l that satisfies (2.2).

Let $(P(R)' = Q'_l, Q'_{l+1}, \ldots, Q'_{k-1} = P')$ be the path from $P(R)'$ to the root of T'. We prove inductively for $l \leq m \leq k-1$, that:

$$\mathrm{girth}_1(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \equiv l \mod 2, \tag{2.3}$$

$$\mathrm{girth}_4(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \not\equiv l \mod 2 \tag{2.4}$$

(2.3) holds for $m = l$ since $\mathrm{girth}_1(Q'_{l+1}) = \mathrm{girth}_1(P(R)') \geq \mathrm{width}_1(S(R)) \geq$

$(2^{(l-t-6)(l-t-7)/2})/D$. Assume that $m > l$ and that both (2.3) and (2.4) hold for smaller values of m. Suppose that m has the same parity as l. Then by inequality (1) of Lemma 5 and inequalities (2.2) and (2.3):

$$\begin{aligned}
\operatorname{girth}_1(Q'_m) &\geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{m-t-2} \operatorname{girth}_4(Q'_{m-1}) - \operatorname{width}_1(S(Q'_{m-1})) \\
&\geq 2^{m-t-5} \operatorname{girth}_4(Q'_{m-1}) - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-5} \frac{2^{(m-t-7)(m-t-8)/2}}{D} - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-6} \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&= \frac{2^{(m-t-6)(m-t-7)/2}}{D}
\end{aligned}$$

Therefore P' has size at least $\frac{2^{(k-t-7)(k-t-8)/2}}{D}$. This at least $n^{\frac{1}{8}\log n}$, for a sufficiently large value of k. Since $P' \subset P$, the result follows. The proof when m has different parity as l is similar, but uses inequality (2) of Lemma 5 instead. □

To prove the general lower bound we do the following. Take a drawing of the Horton set; find a subset of half of its points, for which we know that there exists a linear transformation that maps it into an isothetic drawing; afterwards, apply Lemma 2 to the image and use the obtained lower bound to lower bound the size of original drawing.

Proof of Theorem 3. Let P' be a (not necessarily isothetic) drawing of the Horton set of n points. As P and P' have the same order type we can label P' with the same labels as P, such that corresponding triples of points in P and P' have the same orientation. Let $\{p'_0, \ldots, p'_{n-1}\}$ be P' with these labels.

Note that the clockwise order by angle of P'_{odd} around p'_0 is $(p'_1, p'_3, \ldots)$, and that p'_0 lies in an unbounded cell of the line arrangement of the lines defined by every pair of points of P'_{odd}; thus, point p'_0 can be moved towards infinity without changing this radial order around p'_0. Therefore, there is a direction $\vec{d}$ in which if P'_{odd} is projected orthogonally the order of the projection is precisely $(p'_1, p'_3, \dots)$. We may rotate $\vec{d}$ as long as it does not coincide with a direction defined by a pair of points of P' and the order of P'_{odd} in this projection does not change. Let v' and v'' be the first vectors, defined by pairs of points of P', encountered when rotating $\vec{d}$ to the left and to the right, respectively; let $v = v' + v'' = (a, b)$.

We may assume that $||v|| = \sqrt{a^2 + b^2} \leq 4^{1/3}(n/2)^{\frac{1}{24}\log(n/2)}$; otherwise one of v' and v'' has length at least $(1/2)4^{1/3}(n/2)^{\frac{1}{24}\log(n/2)}$, and therefore a coordinate of value at least $(1/4)^{2/3}(n/2)^{\frac{1}{24}\log(n/2)}$. Let $v^{\perp} = (b, -a)$. Consider a change of basis from the standard basis to $\{v, v^{\perp}\}$. Note that under this transformation (x, y) is mapped to $\left(\frac{ax+by}{a^2+b^2}, \frac{ay-bx}{a^2+b^2}\right)$. We multiply the image of P' under this mapping by a^2+b^2, to obtain an isothetic drawing of the Horton set on $n/2$ points. By Theorem 2, this drawing has size at least $(n/2)^{\frac{1}{8}\log(n/2)}$. Therefore, P' has size at least $((n/2)^{\frac{1}{8}\log(n/2)})/(a^2 + b^2) \geq (1/4)^{2/3}(n/2)^{\frac{1}{24}\log(n/2)}$. □

2.4 Observations and Comments

- In this chapter, we are mainly interested in having small drawings of the Horton set. However, the problem of finding small drawings also raises interesting theoretical questions. For example, after learning of our lower bound, Alfredo Hubard posed the following problem.

  Problem 5. *Does every sufficiently large set of points, for which there exist a drawing of polynomial size, contains an empty 7-hole?*

  In particular our lower bound implies that any set of points that has a drawing of polynomial size, cannot have large copies of the Horton set. On this way, the problem of Erdős (Problem 2) is still open for points sets on integer grids of polynomial size.

- In Matoušek's book [36] (page 36), there is definition of the Horton set very similar to the one given in Definition 4. The only difference is that in that definition either H^k_{even} is high above H^k_{odd} or H^k_{odd} is high above H^k_{even}; i.e. this relationship is allowed to change at each step of the recursion. As a result, for a fixed value of k, one gets a family of "Horton sets" (with different order types), rather than a single Horton set. Normally, this does not affect the properties that make Horton sets notable. For example, none of the them have empty heptagons. In some circumstances it does; as is the case of the constructions with few k-holes [46, 17, 7].

We fixed one of these two options in order to make the proof of our lower bound more readable, but our results should hold for the general setting. Note that this choice fixes the order type of the Horton set. However, an arbitrary drawing of the Horton set need not satisfy Definition 4.

- We also believe that the machinery developed to prove Theorem 3 will be useful for analyzing Horton sets in other settings.

- We point out that the constants in the exponent of the lower bounds of Theorems 2 and 3 can be improved. We simplified the exposition at the expense of these worse bounds.

Chapter 3

The Erdős-Szekeres Set

In this chapter we prove that the Erdős-Szekeres construction can be realized in a rather small integer grid of size $O(n^2 \log_2(n)^3)$. First, in Section 3.1, we define the Erdős-Szekeres set. Afterwards, in Section 3.2, we provide a drawing of size $O(n^2 \log_2(n)^3)$ of the Erdős-Szekeres set of n points. Finally, in Section 3.3, we discuss some Erdős-Szekeres type problems.

3.1 Background

3.1.1 Large Sets with Short Caps and Cups.

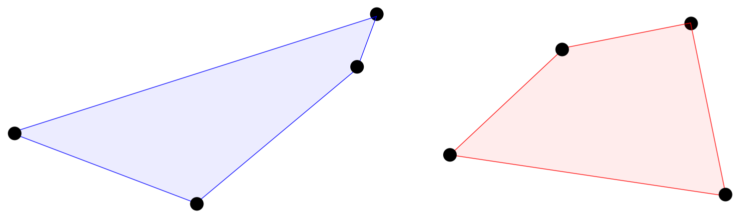
The Erdős-Szekeres set is made from smaller point sets, $S_{k,l}$, which we now describe.

Let $\mathcal{X}$ be a set of r points in general position. We call $\mathcal{X}$ an r-**cup** if $\mathcal{X}$ is in convex position and its convex hull is bounded from above by a single edge. Similarly, we call $\mathcal{X}$ an r-**cap** if $\mathcal{X}$ is in convex position and its convex hull is bounded from below by a single edge. See Figure 3.1.

Lemma 8. *Let $k \geq 2$ and $l \geq$ be integers. Then there is a set $S_{k,l}$ with $\binom{k+l-4}{k-2}$ points that it does not contains a k-cup or an l-cap.*

Proof. For $k \leq 2$ or $l \leq 2$, let $S_{k,l} := \{(0,0)\}$. In other case, let

$$S_{k,l} := L_{k,l} \cup R_{k,l};$$



(a) A 4-cup (b) A 4-cup

Figure 3.1: An example of a 4-cup and a 4-cap

where:

$$L_{k,l} := S_{k-1,l};$$
$$R_{k,l} := \{(x + \delta_{k,l}, y + \delta'_{k,l}) : (x, y) \in S_{k,l-1}\}.$$

$\delta_{k,l}$ is chosen large enough so that $R_{k,l}$ is to the right of $L_{k,l}$; and $\delta'_{k,l}$ is chosen large enough with respect to $\delta_{k,l}$ so that $R_{k,l}$ is high above $L_{k,l}$.

It can be shown by induction on $k + l$ that $S_{k,l}$ has $\binom{k+l-4}{k-2}$ points, and that $S_{k,l}$ does not contain a k-cup nor a l-cap. □

3.1.2 Definition of the Erdős-Szekeres Set.

Definition 6. *Let $\mathcal{X}$ be a set of 2^{k-2} points. We say that $\mathcal{X}$ is an Erdős-Szekeres set if $\mathcal{X}$ contains disjoint sets T_i for $0 \leq i \leq k-2$ such that:*

1. *For $0 \leq i \leq k-2$, $|T_i| = \binom{k-2}{i}$.*

2. *For $0 \leq i \leq k-2$, T_i does not contains a $(i+2)$-cap or a $(k-i)$-cup.*

3. *For $0 \leq i \leq k-2$, every two points in T_i are connected by a line having positive slope.*

4. *For $0 \leq i, j \leq k-2$, every two points in T_i and T_j, respectively, they are connected by a line having negative slope.*

5. *Let $i(1)$, $i(2)$, $i(3)$ be integers such that $0 \leq i(1) \leq i(2) \leq i(3) \leq k-2$. Let $p_{i(1)}$, $p_{i(2)}$, $p_{i(3)}$ be points in $T_{i(1)}$, $T_{i(2)}$ and $T_{i(3)}$ respectively. Then*

the sequence of points $(p_{i(1)}, p_{i(2)}, p_{i(3)})$ *is in clockwise orientation.*

Theorem 9 (Erdős-Szekeres)**.** *The Erdős-Szekeres set of* 2^{k-2} *points does not contains* k *points in convex position.*

Proof. Let $\mathcal{X}$ be the Erdős-Szekeres set of 2^{k-2} points, and for $0 \leq i \leq k-2$, let T_i be as in Definition 6. Let P be a convex polygon of $\mathcal{X}$. We prove that P has at most $k-1$ points.

Let p and q be the leftmost and rightmost point in P, respectively. Note that p and q splits the convex hull of P in two polygonal chains. Let U and L be the upper and lower polygonal chains respectively (See Figure 3.2). Let r be the index such that p is in T_r, and let s be the index such that q is in T_s.

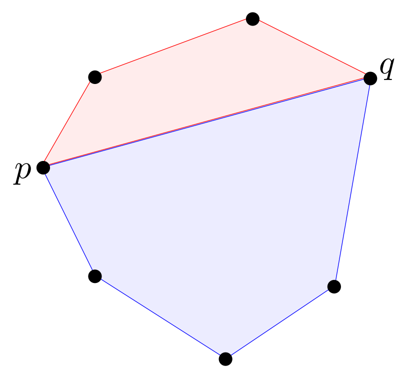


Figure 3.2: Example of the cap and the cup from a convex

In case that $r = s$, by Definition 2, U has at most $r+1$ points and L has at most $k-r-1$ points. As p and q are in both U and L, in this case P has at most $(r+1)+(k-r-1)-2 = k-2$ points. Assume that $r < s$.

Let p' be the highest point in U. Let U_1 be the polygon chain from p to p', and let U_2 be the polygonal chain from p' to q. Note that, the edges in U_1 have positive slope and the edges in U_2 have negative slope. Thus, the points of U_1 are in T_r and there are not two points of U_2 in T_i for $r \leq i \leq s$. Then,

$$|U| \leq (r+1) + (s-r+1) - 1 \leq s+1$$

Let q' be the lowest point in L. Let L_1 be the polygon chain from p to

q', and let L_2 be the polygonal chain from q' to q. Note that, the edges in L_1 have negative slope and the edges in L_2 have positive slope. Thus, the points of L_2 are in T_s and there are at most two points in L_1. Then,

$$|L| \leq (k - s - 1) + (2) - 1 \leq k - s$$

As p and q are in both U and L, then

$$|P| \leq (s + 1) + (k - s) - 2 \leq k - 1$$

□

3.2 Construction of the Erdős-Szekeres set with Small Integer Coordinates

3.2.1 A Superset of $S_{k,l}$

Let $r \geq 1$ be an integer. In this section we construct a point set P_r in general position in the plane with integer coordinates such that P_r has 2^r points, and for $r = k + l - 1$, P_r contains $S_{k,l}$ as a subset.[1]

We define P_r recursively as follows.

- $P_0 := \{(0, 0)\}$;
- $P_r := L_r \cup R_r$; where:

$$\begin{aligned} L_r :=& P_{r-1} \\ R_r :=& \{(x + \delta_r, y + \delta'_r) : (x, y) \in L_r\} \\ \delta_r :=& 3 \cdot 4^{r-1}; \qquad (3.1) \\ \delta'_r :=& (3r + 1) \cdot 4^{r-1}. \qquad (3.2) \end{aligned}$$

Let X_r be the value of the largest x-coordinate of P_r; note that for $r \geq 1$

$$X_r = X_{r-1} + \delta_r.$$

[1]More accurately, P_r contains a subset with the same order type as $S_{k,l}$.

Since $X_0 = 0$, by induction we have that

$$X_r = 4^r - 1. \tag{3.3}$$

Let Y_r be the value of the largest y-coordinate of P_r; note that for $r \geq 1$

$$Y_r = Y_{r-1} + \delta'_r.$$

Since $Y_1 = 0$, by induction we have that

$$Y_r = r \cdot 4^r. \tag{3.4}$$

Since

$$\delta_r > X_{r-1},$$

every point of L_r is to the left of every point of R_r.

For $r \geq 2$, let p_r be the rightmost point of L_r and let q_r be the leftmost point of R_r; let ℓ_r be the straight line passing through p_r and q_r; see Figure 3.3. By construction of P_r, the point p_r is the point of L_r of largest y-coordinate, and the point q_r is the point of R_r of smallest y-coordinate; therefore, the slope m_r of ℓ_r is given by

$$\begin{aligned} m_r =& \frac{Y_r - 2Y_{r-1}}{X_r - 2X_{r-1}} \\ =& \frac{r4^r - 2(r-1)4^{r-1}}{4^r - 1 - 2(4^{r-1} - 1)} \\ =& \frac{2r \cdot 4^{r-1} + 2 \cdot 4^{r-1}}{2 \cdot 4^{r-1} + 1} \\ =& r + 1 - \frac{r+1}{2 \cdot 4^{r-1} + 1}. \end{aligned} \tag{3.5}$$

The m_r are increasing, since

$$m_r - m_{r-1} = 1 + \frac{r}{2 \cdot 4^{r-2} + 1} - \frac{r+1}{2 \cdot 4^{r-1} + 1} > 0;$$

the last inequality follows from the fact that $\frac{r+1}{2 \cdot 4^{r-1} + 1} \leq \frac{1}{3}$ for $r \geq 2$.

Let L'_{r-1} and R'_{r-1} be the translations of L_{r-1} and R_{r-1} in R_r,

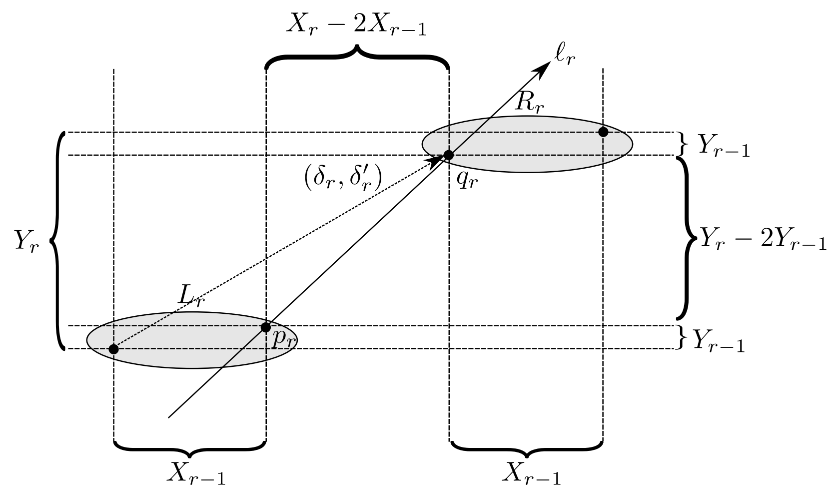


Figure 3.3: Illustration of P_r

respectively. Let ℓ'_{r-1} be the line defined by the rightmost point of L'_{r-1} and the leftmost point of R'_{r-1}. Thus, ℓ'_{r-1} is the translation of ℓ_{r-1} in R_r. We now prove some properties of P_r.

Lemma 10. *Among the lines passing through two points of P_r, ℓ_r is the line with the largest slope.*

Proof. We proceed by induction on r. For $r = 0$ and $r = 1$, the lemma holds trivially. So assume that $r > 1$ and that the lemma holds for smaller values of r. Let ℓ be a line passing through two points of P_r.

Suppose that ℓ passes through two points of L_r or through two points of R_r. By induction the slope of ℓ is at most m_{r-1}. Since $m_r > m_{r-1}$, the slope of ℓ_r is larger than the slope of ℓ.

Suppose that ℓ passes through a point p of L_r and a point q of R_r. Consider the polygonal chain $C := (p, p_r, q_r, q)$. Since p is to the left of p_r and q_r is to the left of q, the slope of ℓ is at most the maximum of the slopes of the edges of C. By induction each of these edges has slope at most m_r. Therefore, the slope of ℓ is at most the slope of ℓ_r. □

Lemma 11. *The rightmost point of P_r is above ℓ_{r-1} and the leftmost point of P_r is below ℓ'_{r-1}.*

Proof. The result holds trivially for $r = 0$ and $r = 1$; assume that $r \geq 2$.

First we prove that the rightmost point p of P_r is above ℓ_{r-1}. Note that $p = (X_r, Y_r)$. Let q be the point in ℓ_{r-1} with x-coordinate equal to X_r; note that since ℓ_{r-1} contains the point (X_{r-2}, Y_{r-2}), the y-coordinate of q is equal to $Y_{r-2} + m_{r-1}(X_r - X_{r-2})$. Therefore, it is sufficient to show that:

$$Y_r > Y_{r-2} + m_{r-1}(X_r - X_{r-2}).$$

Equivalently that

$$\frac{Y_r - Y_{r-2}}{X_r - X_{r-2}} > m_{r-1}.$$

This follows from

$$\begin{aligned}\frac{Y_r - Y_{r-2}}{X_r - X_{r-2}} =& \frac{r4^r - (r-2)4^{r-2}}{4^r - 1 - (4^{r-2} - 1)} \\ =& \frac{15r \cdot 4^{r-2} + 2 \cdot 4^{r-2}}{15 \cdot 4^{r-2}} \\ =& r + \frac{2}{15},\end{aligned}$$

and that by (3.5)

$$m_{r-1} = r - \frac{r}{2 \cdot 4^{r-2} + 1}.$$

Now we prove that the leftmost point of P_r is below ℓ'_{r-1}. Note that $(0, 0)$ is the leftmost point of P_r. Let q' be the point in ℓ'_{r-1} with x-coordinate equal to 0; note that since ℓ'_{r-1} contains the point $(X_{r-2} + \delta_r, Y_{r-2} + \delta'_r)$, the y-coordinate of q' is equal to $Y_{r-2} + \delta'_r - m_{r-1}(X_{r-2} + \delta_r)$. Therefore, it is sufficient to show that:

$$Y_{r-2} + \delta'_r - m_{r-1}(X_{r-2} + \delta_r) > 0.$$

Equivalently that

$$\frac{Y_{r-2} + \delta'_r}{X_{r-2} + \delta_r} > m_{r-1}.$$

This follows from

$$\begin{aligned}\frac{Y_{r-2}+\delta'_r}{X_{r-2}+\delta_r} =& \frac{(r-2)4^{r-2}+(3r+1)4^{r-1}}{4^{r-2}-1+3\cdot 4^{r-1}} \\ =& \frac{13r\cdot 4^{r-2}+2\cdot 4^{r-2}}{13\cdot 4^{r-2}-1} \\ =& r+\frac{2\cdot 4^{r-2}+r}{13\cdot 4^{r-2}-1},\end{aligned}$$

and that by (3.5)

$$m_{r-1}=r-\frac{r}{2\cdot 4^{r-2}+1}.$$

□

Lemma 12. *The following properties hold.*

(a) *R_r is above ℓ_{r-1};*

(b) *L_r is below ℓ'_{r-1};*

(c) *no point of L_{r-1} is below ℓ_{r-1};*

(d) *no point of L'_{r-1} is below ℓ'_{r-1};*

(e) *no point of R_{r-1} is above ℓ_{r-1}; and*

(f) *no point of R'_{r-1} is above ℓ'_{r-1}.*

Proof. For $r=0,1,2$ the lemma can be verified directly or holds trivially; assume that $r>2$.

(a) By Lemma 11, the rightmost point of R_r is above ℓ_{r-1}. If a point p of R_r is below ℓ_{r-1}, then the line defined by p and the rightmost point of R_r has slope larger than m_{r-1}— a contradiction to Lemma 10 and the fact that R_r is a translation of P_{r-1}.

(b) By Lemma 11, the leftmost point of L_r is below ℓ'_{r-1}. If a point p of L_r is above ℓ'_{r-1}, then the line defined by p and the leftmost point of L_r has slope larger than the slope of ℓ'_{r-1}— a contradiction to Lemma 10 and the fact that ℓ'_{r-1} is parallel to ℓ_{r-1}.

(c) If a point p of L_{r-1} is below ℓ_{r-1}, then the line defined by p and the rightmost point of L_{r-1} has slope larger than m_{r-1}— a contradiction to Lemma 10.

(d) Follows from *(c)* and the fact that R_r is a translation of L_r.

(e) If a point p of R_{r-1} is above ℓ_{r-1}, then the line defined by p and the leftmost point of R_{r-1} has slope larger than m_{r-1}— a contradiction to Lemma 10.

(f) Follows from *(e)* and the fact that R_r is a translation of L_r.

□

Lemma 13. *R_r is high above L_r.*

Proof. For $r = 0, 1, 2$ the lemma holds trivially or can be verified directly; assume that $r > 2$ and that lemma holds for smaller values of r. We proceed by induction on r.

We first prove that R_r is above every line ℓ defined by two points of L_r. By *(a)* of Lemma 12, R_r is above ℓ_{r-1}. By Lemma 10, the slope of ℓ is at most the slope of ℓ_{r-1}. Thus we may assume that ℓ does not contain the rightmost point of L_{r-1} nor the leftmost point of R_{r-1}. Suppose that ℓ passes through a point of R_{r-1}; then, by *(e)* of Lemma 12, this point is below ℓ_{r-1}. Since the slope of ℓ is at most the slope of ℓ_{r-1}, R_r is above ℓ in this case. Suppose that ℓ passes through two points of L_{r-1}. Then, by induction the leftmost point of R_{r-1} is above ℓ. Since the slope of ℓ is at most the slope of ℓ_{r-1}, R_r is above ℓ in this case.

We now prove that L_r is below every line ℓ defined by two points of R_r. By *(b)* of Lemma 12, L_r is below ℓ'_{r-1}. Since R_r is a translation of P_{r-1}, by Lemma 10, we have that the slope of ℓ is at most the slope of ℓ'_{r-1}. Thus we may assume that ℓ does not contain the rightmost point of L'_{r-1} nor the leftmost point of R'_{r-1}. Suppose that ℓ passes through a point of L'_{r-1}; then, by *(d)* of Lemma 12, this point is above ℓ'_{r-1}. Since the slope of ℓ is at most the slope of ℓ_{r-1}, L_r is below ℓ in this case. Suppose that ℓ passes through two points of R'_{r-1}. Since R_r is translation of P_{r-1}, by induction we have that the rightmost point of L'_{r-1} is below ℓ. Since the slope of ℓ is at most the slope of ℓ'_{r-1}, R_r is above ℓ in this case. The result follows. □

Proposition 14. *P_r can be realized with non-negative integer coordinates of size at most $r4^r$.*

Proof. This follows from $X_r = 4^r - 1$ and $Y_r = r4^r$. □

Proposition 15. *Let k, l be positive integers. If $r := k + l - 1$ then $S_{k,l}$ is a subset of P_r.*

Proof. The result holds for $k \leq 2$ or $l \leq 2$, since in these cases $S_{k,l} = \{(0, 0)\}$. Therefore, the result holds for $r \leq 4$. Assume that $k, l \geq 2$, $r \geq 5$ and that the result holds for smaller values of r. By induction $S_{k-1,l}$ and $S_{k,l-1}$ are subsets of P_{r-1}. The result follows from Lemma 13 and by setting $\delta_{k,l} := \delta_r$ and $\delta'_{k,l} := \delta'_r$. □

3.2.2 An small drawing of the Erdős-Szekeres set.

In this section we use the set of points described in Section 3.2.1 to realize, with small integer coordinates, the construction given by Erdős and Szekeres in [19]. The Erdős-Szekeres construction is made from a small number of translations of $S_{k,l}$ (for some values of k and l). We first describe these translations.

Let $t > 0$ be an integer and let $n := 2^{t-2}$. For every integer $1 \leq i \leq t-2$ we define the vector

$$v_i := (3(t - i), -3i).$$

Using these vectors, we define a set of $t - 1$ points $w_0, \dots, w_{t-2}$ recursively as follows.

- $w_0 := (0, 0)$;
- $w_{i+1} := w_i + v_i$ for $i = 0, \dots, t - 3$.

For $i = 0, \dots, t-2$, let C_i be the unit square whose lower left corner is equal to w_i.

Lemma 16. *The union, $\bigcup C_i$, of the squares C_i lies in a $3t^2 \times 3t^2$ integer grid.*

Proof. Note that the largest absolute value of the x-coordinates of the w_i's is equal to

$$\sum_{i=0}^{t-3}(3t - i) < 3t^2;$$

and the largest absolute value of the y-coordinates of the w_i's is equal to

$$\sum_{i=0}^{t-3} 3i < 3t^2.$$

Therefore, $\bigcup C_i$ lies in a $3t^2 \times 3t^2$ integer grid. □

Let D_i be the square C_i scaled by factor of $(t+1)4^{t+1}$, that is

$$D_i := \{\big((t+1)4^{t+1}x, (t+1)4^{t+1}y\big) : (x, y) \in C_i\}.$$

Lemma 17. *Let $0 \le i < j < k \le t-2$ be three integers; let p_i, p_j, and p_k be points in D_i, D_j and D_k, respectively. Then (p_i, p_j, p_k) is a right turn.*

Proof. For $i = 0, \ldots, t-3$, let W_i be the set of vectors of the form $u := q - q'$ where q is a point of C_{i+1} and q' is a point of C_i. Note that the endpoints of these vectors lie in a 2×2 square centered at v_i; let γ_i be the smallest cone with apex at the origin and that contains C_i. By the previous observation the γ_i only intersect at the origin; see Figure 3.4.

Let $m_{i-1,i}$ be the slope of a line passing through a point of C_{i-1} and a point of C_i and let $m_{i,i+1}$ be the slope of a line passing through a point of C_i and a point of C_{i+1}. The vector defining $m_{i-1,i}$ lies in γ_{i-1} and the vector defining $m_{i,i+1}$ lies in γ_i. This implies that $m_{i-1,i} > m_{i,i+1}$. Let $0 \le i < j < k \le t-2$ be three integers. Let $m_{i,j}$ be the slope of a line passing through a point of C_i and a point of C_j and let $m_{j,k}$ be the slope of a line passing through a point of C_j and a point of C_k. Thus, we have that

$$m_{i,j} > m_{j,k}. \tag{3.6}$$

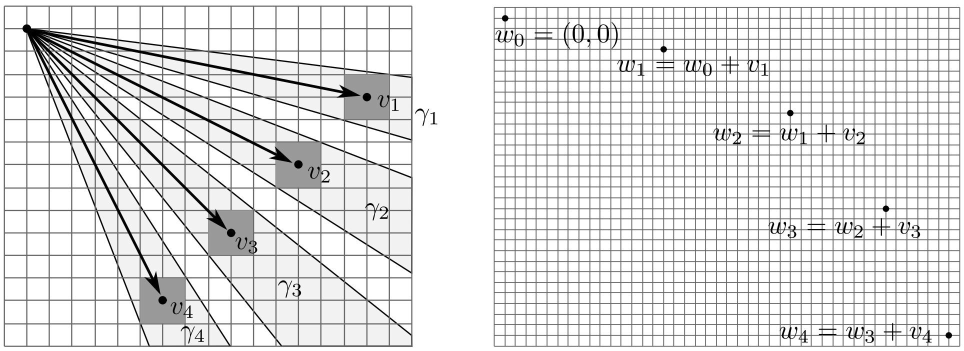


Figure 3.4: The vectors v_i, the cones γ_i and the points w_i, for $t = 6$

Note that (3.6) also holds for the lines defined by pairs points in the D_i's. Therefore, (p_i, p_j, p_k) is a right turn. □

Let q_i be the lower left corner of D_i, and let $S'_{t-i,i+2}$ be the translation of $S_{t-i,i+2}$ by q_i. That is

$$S'_{t-i,i+2} := \{p + q_i : p \in S_{t-i,i+2}\}.$$

The Erdős-Szekeres construction is given by

$$S_t = \bigcup_{i=0}^{t-2} S'_{t-i,i+2}.$$

Note that

$$|S_t| = \sum_{i=0}^{t-2} |S_{t-i,i+2}| = \sum_{j=0}^{t-2} \binom{t-2}{j} = 2^{t-2} = n.$$

Proposition 18. *S_t lies in an integer grid of size $O(n^2 \log_2(n)^3)$.*

Proof. Recall that D_i is a scaling of C_i by a factor of $(t+1)4^{t+1}$. Therefore, by Lemma 16, S_t lies in an integer grid of size

$$3t^2(t+1)4^{t+1} = 192n^2 \log_2(4n)^2 \log_2(8n) = O(n^2 \log_2(n)^3).$$

□

Proposition 19. *S_t is in general position.*

Proof. Let p_1, p_2, p_3 be three points of S_t. If these three points are contained in a same D_i, then they are not collinear since $S_{k,l}$ is in general position. If the three of them are in different D_i, then by Lemma 17 they are not collinear. If two of them lie on a same D_i and one of them in some D_j, then they are not collinear since the slope of a line joining a point in D_i and a point in D_j is negative, while the slope of a line defined by two points in $S_{k,l}$ is greater or equal to zero. Therefore, S_t does not contain three collinear points. □

Theorem 20. *Every convex k-gon of S_t has at most $t-1$ vertices.*

Proof. Let P be a convex k-gon of S_t. Let U and L be the upper and lower polygonal chains of P, respectively. Let s be the index such that the leftmost point of U (and L) is in D_s, and let r be the index such that the rightmost point of U (and L) is in D_r.

Note that for all $0 \leq i < j \leq t-2$, the slopes of an edge joining a point of D_i with a point of D_j are negative; since the slope of an edge defined by a pair of points in $S'_{t-i,i+2}$ is greater or equal to zero, neither U nor L contain two consecutive vertices in D_i for $s < i < r$. By Lemma 17 such a D_i cannot contain a vertex of L. Therefore, P contains at most $r-s-1$ vertices not in $D_s \cup D_r$.

The vertices of P contained in S_s must form a cap and thus consists of at most $s+1$ vertices. Similarly, the vertices of P contained in S_r must form a cup and therefore consists of at most $t-r-1$ vertices. Therefore, P has at most $(r-s-1)+(s+1)+(t-r-1) = t-1$ vertices; the result follows. □

3.3 Observations and Comments

3.3.1 Implementation

A direct implementation of Section 3.2 gives way to an efficient algorithm to compute the Erdős-Szekeres construction. By Proposition 18, the size of the grid needed by this algorithm is asymptotically small; however, there are large constants hidden in such an implementation. In this section we mention some optimizations we have done to further reduce the size of the integer grid needed for the Erdős-Szekeres construction.

- **Decrease the horizontal distance between left and right parts of $S_{k,l}$.**

  In Section 3.2.1, to construct P_r, we gave explicit values to δ_r and δ'_r. This allowed us to show that L_r is to the left of R_r and that R_r is high above L_r. However, it is enough to show that L_r is to the left of R_r and that the rightmost point of R_r is above ℓ_r. That is that

  $$Y_r > \frac{Y_r - Y_{r-2}}{X_r - X_{r-2}} (X_r - X_{r-2}) + Y_{r-2}. \tag{3.7}$$

  Let $c > 0$ be a constant and set $X_r = (2+c)^r$. It can be shown that if we replace inequality (3.7) by an equality and solve for Y_r, then Y_r is of order $O\left(\left(2+\frac{3}{c}\right)^r\right)$. Therefore, if we set $c = \sqrt{3}$, then both X_r and Y_r are of order $O\left(\left(2+\sqrt{3}\right)^r\right)$. In the actual implementation we choose δ_r so that

  $$X_r = \left\lceil \left(2+\sqrt{3}\right)^r \right\rceil.$$

  Then we choose δ'_r so that

  $$Y_r = \left\lceil \frac{Y_r - Y_{r-2}}{X_r - X_{r-2}} (X_r - X_{r-2}) + Y_{r-2} + 1 \right\rceil.$$

  The addition of the ceiling functions has prevented us from proving that Y_r is of order $O\left(\left(2+\frac{3}{c}\right)^r\right)$. If this is the case then P_r can be realized in an integer grid of size $O\left(n^{\log_2(2+\sqrt{3})}\right) = O\left(n^{1.8999\ldots}\right)$. In Section 3.2.1, we opted to avoid using ceiling functions at the expense of being able to show a slightly worse upper bound.

  Inspired by this, we do likewise when constructing $S_{k,l}$. First we construct $S_{k-1,l}$ and $S_{k,l-1}$. Let $X_{k,l}$ be the horizontal length of $S_{k,l}$.

We choose

$$\delta_{k,l} := \left\lceil \left(1+\sqrt{3}\right)\left(\frac{X_{k-1,l}+X_{k,l-1}}{2}\right)\right\rceil .$$

For any two positive integers k and l, let $\ell_{k,l}$ be the straight line passing through the rightmost point of $S_{k-1,l}$ and the leftmost point of the copy of $S_{k,l-1}$ in $S_{k,l}$. (This definition is similar to the definition of ℓ_r for P_r.) We choose $\delta'_{k,l}$ so that; the rightmost point in the translation of $S_{k,l-1}$ is above $\ell_{k-1,l}$; and the leftmost point of $S_{k-1,l}$ is below the corresponding translation of $\ell_{k,l-1}$ in $R_{k,l}$.

- **Separate the left and right parts of $S_{k,l}$ by one in the last step of the recursion.**

  The reason for choosing a relatively large horizontal separation between the left and right parts of $S_{k,l}$ is so that the slope of $\ell_{k,}$ does not increase too quickly. We do not need to do this in the last step of the construction. At each step in the construction of $S_{i,j}$ for $2 \leq i \leq k$, $2 \leq j \leq k$ and $i+j < k$, we separate the corresponding left and right parts as before. In the last step, when constructing $S_{k,l}$, we separate $S_{k-1,l}$ from the copy of $S_{k,l-1}$ by one.

- **Decrease the size of the squares (rectangles) D_i.**

  In Section 3.2, for $i = 0, \ldots, t-2$ we defined a square D_i, inside which we placed a copy of $S_{t-i,i+2}$. D_i was chosen large enough so that P_r fits inside D_i for $r = t+1$. Since we only need to fit $S_{t-i,i+2}$ we replace D_i by a rectangle of length $X_{t-1,i+2}$ and height $Y_{t-i,i+2}$. The definitions of the v_i's and w_i's are changed accordingly.

In Figure 3.5 we show $S_{5,5}$ in 55×109 integer grid; in Figure 3.6 we show S_6 in a 58×62 integer grid; and in Figure 3.7 we show S_7 in a 230×310 integer grid. For a comparison, we note that Kalbfleisch and Stanton [31] realize S_6 in a 6970×1828 integer grid.

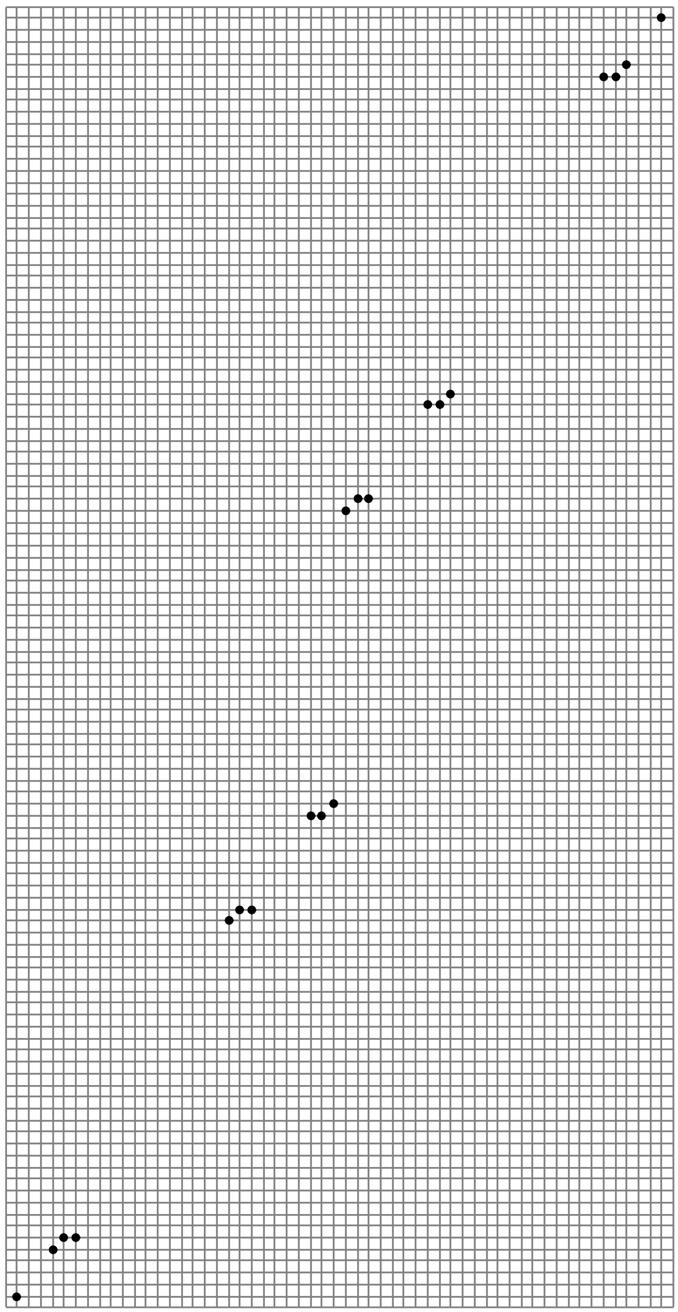


Figure 3.5: $S_{5,5}$ in 55×109 integer grid.

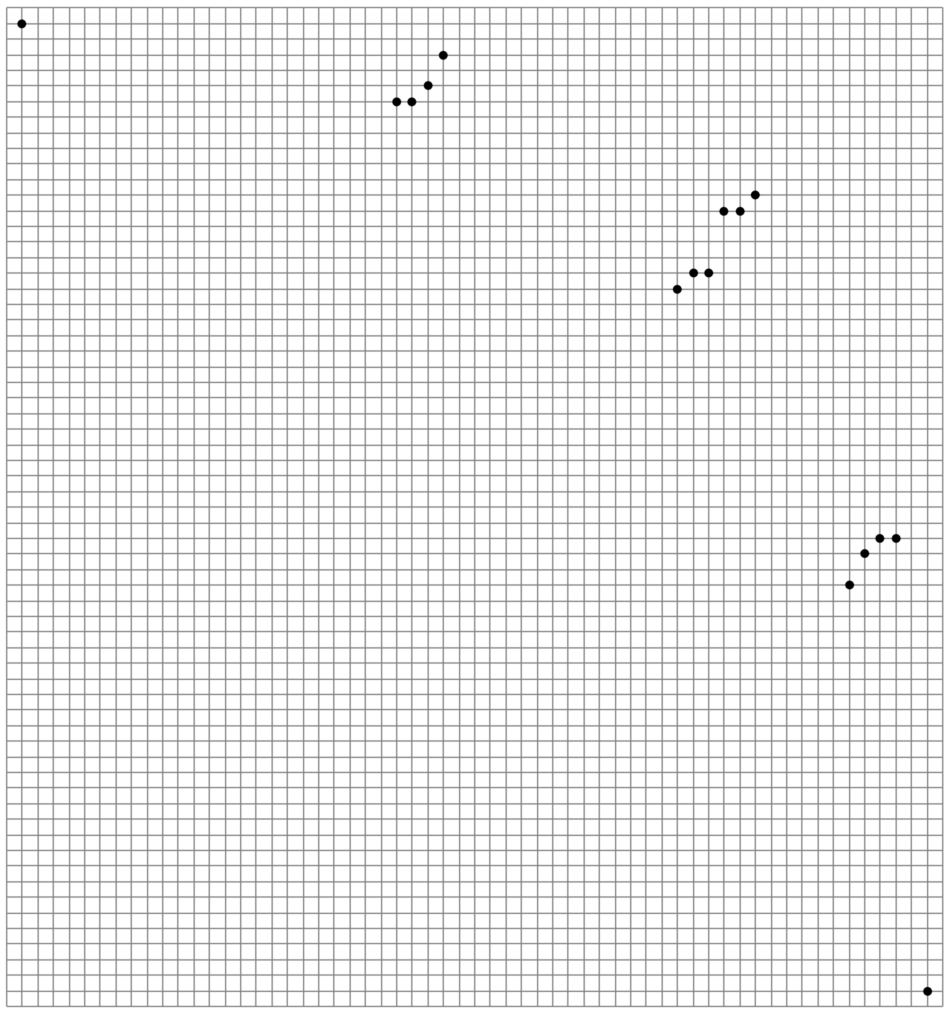


Figure 3.6: S_6 in a 58×62 integer grid.

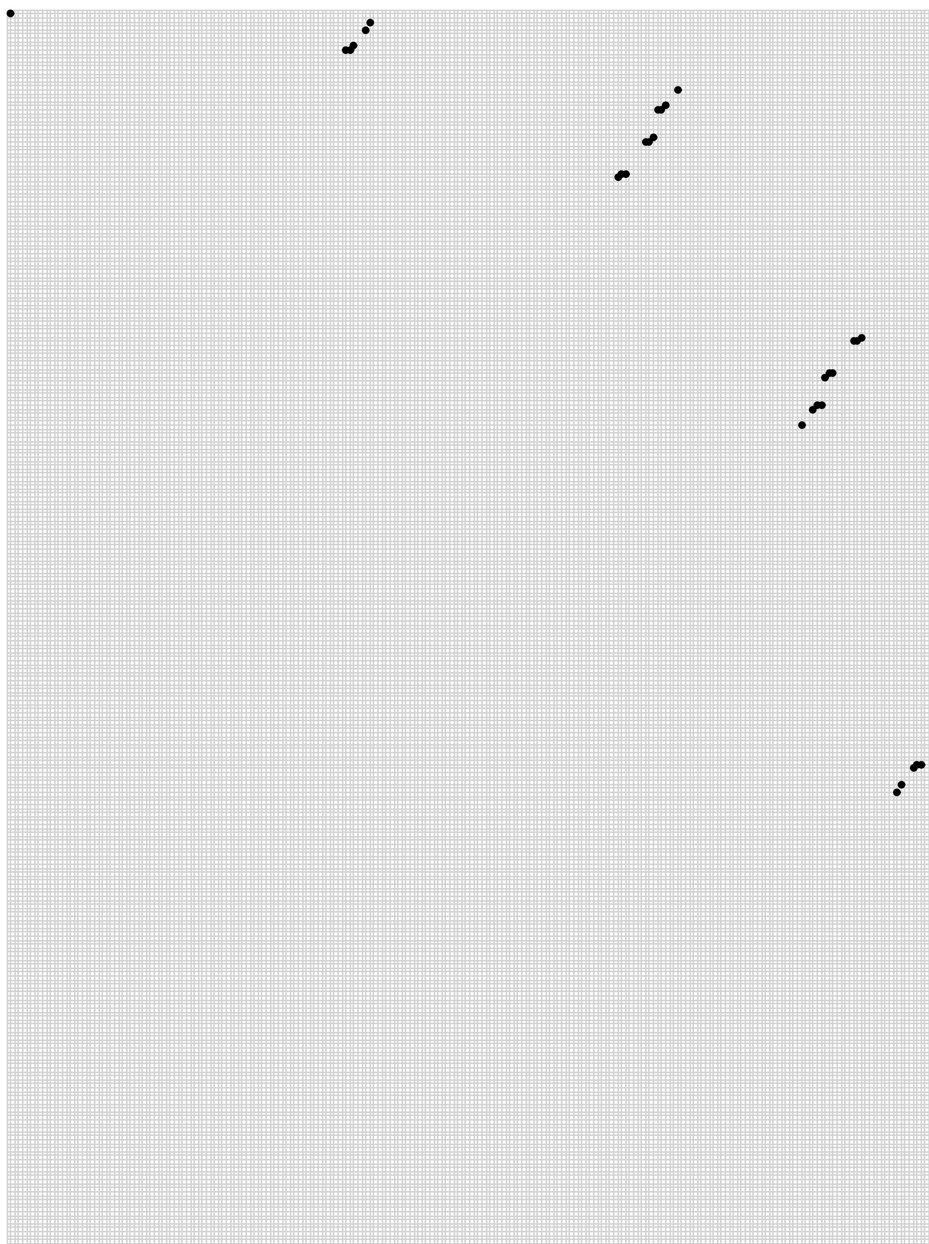


Figure 3.7: S_7 in a 230×310 integer grid.

3.3.2 Open problems

In this section we propose some open Erdős-Szekeres type problems on point sets in integer grids. Let $\operatorname{diam}(S)$ be the maximum distance between a pair of points of S, and let $\operatorname{mindist}(S)$ be the minimum distance between a pair of points of S. Alon, Katchalski and Pulleyblank [4], showed that if for some constant $\alpha > 0$, S satisfies

$$\frac{\operatorname{diam}(S)}{\operatorname{mindist}(S)} \leq \alpha n^{\frac{1}{2}},$$

then S contains a convex k-gon of $\Omega\left(n^{\frac{1}{4}}\right)$ vertices; in [44], Valtr improved this bound to $\Omega\left(n^{\frac{1}{3}}\right)$. He also showed that if for some constant $\alpha > 0$, S satisfies

$$\frac{\operatorname{diam}(S)}{\operatorname{mindist}(S)} \leq \alpha\sqrt{n},$$

then S contains a convex k-gon of $\Omega\left(n^{\frac{1}{3}}\right)$ vertices. That is, metric restrictions on S may force large convex polygons.

This prompted the following two problems in [12].

Problem 6. *Does there exist, for every $\beta \geq 1$, a suitable constant $\varepsilon(\beta) > 0$ with the following property: any set of S of n points in general position in the plane with $\frac{\operatorname{diam}(S)}{\operatorname{mindist}(S)} < n^{\beta}$ contains a convex $n^{\varepsilon(\beta)}$-gon?*

Problem 7. *Does there exist, for every $\gamma \geq 1$, a suitable constant $\varepsilon(\gamma) > 0$ with the following property: any set of n points in the general position in the plane with positive integer coordinates that do not exceed n^{γ} contains a convex $n^{\varepsilon(\gamma)}$-gon?*

Valtr in his PhD thesis [45] showed that the answer for Problem 6 is **"yes"** for $\beta < 1$. He also noted, in passing, in page 55 of his thesis the following.

> "If $\tau = 1/2$ then it is possible to construct an $(n^{\tau} = \sqrt{n})$-dense set of size n which contains no more than $O(\log n)$ vertices of a convex polygon. Such a set can be obtained by an

affine transformation from the construction of Erdős and Szekeres [19]..."

S is said to be α-*dense* if $\frac{\mathrm{diam}(S)}{\mathrm{mindist}(S)} \leq \alpha\sqrt{n}$. So this observation solves Problem 6 for $\beta \geq 1$ as well. Problem 7 appeared first in Valtr's thesis (Problem 10), where it is attributed to Welzl.

In this Chapter we have shown that the answer to Problem 7 is **"No"** for all $\gamma \geq 2$. We conjecture that the answer to Problem 7 is **"No"** for all $\gamma \geq 1$. However, we conjecture that there exists a $\gamma \in (1, 2)$ such that the Erdős-Szekeres construction cannot be realized in an $n^\gamma \times n^\gamma$ integer grid. We propose the following alternative to Problem 7.

Problem 8. *Does there exist, for every $\gamma \geq 1$ and every $n > 0$, a suitable constant $\varepsilon(\gamma) > 0$ and a set S of n points in general position in the plane with the following property: S has positive integer coordinates not exceeding n^γ, and S does not contain a convex $\varepsilon(\gamma)\log_2(n)$-gon?*

Empty k-gons

A convex k-gon of S is *empty* if it does not contain a point of S in its interior. In 1978, Erdős [20] asked whether an analogue of the Erdős-Szekeres theorem holds for empty convex k-gons. That is, if for every k, every sufficiently large point set in general position in the plane contains an empty k-gon.

Every set of at least three points contains an empty triangle; Esther Klein [18] proved that every set of five points contains an empty convex 4-gon; Harborth [28] proved that every set of 10 points contains an empty convex 5-gon; and Horton [29] constructed arbitrarily large point sets without empty convex 7-gons. (His construction is now known as the *Horton Set.*) The question for convex 6-gons remained open for more than a quarter of a century until Nicolás [38] and independently Gerken [22] showed that every sufficiently large set of points contains a convex empty 6-gon.

Alon et al. posed a problem in [4], similar to Problem 6, but for empty convex k-gons. They asked whether if for some constant $\alpha > 0$ every sufficiently large α-dense set of points contains an empty convex 7-gon. Valtr in [44] showed that there exist arbitrarily large $\sqrt{2\sqrt{3}/\pi}$-dense point sets not containing an empty convex 7-gon. His construction is based on the Horton set.

The same question can be asked for point sets in an integer grid.

Problem 9. *Does the following holds for every constant $\gamma \geq 0$? Every sufficiently large set of n points in the general position in the plane with positive integer coordinates that do not exceed n^γ contains an empty convex 7-gon.*

As far as we know all constructions without an empty convex 7-gons are based on the Horton set. This is particularly relevant for Problem 9 for the following reason. In [9], Barba, Duque, Fabila-Monroy and Hidalgo-Toscano proved that the Horton set cannot be realized in an integer grid of polynomial size (See Chapter 2).

Chapter 4

The Nested Almost Convex Sets

In this chapter, we obtain a characterization of when a set of points is a nested almost convex set. This is done by first defining a family of trees. If there exists a map, that satisfies certain properties, from the point set to the nodes of a tree in the family, then the point set is a nested almost convex set. This map encodes a lot of information about the point set. For example, it determines the location of any given point with respect to the convex hull; we use this information to obtain an $O(n \log n)$-time algorithm to decide whether a set of points is a nested almost convex set. This map also determines the orientation of any given triplet of points. This implies that for every n there exists essentially at most one nested almost convex set. We further apply this information to obtain a linear-time algorithm that produces a representation of a nested almost convex set of n points on a small integer grid of size $O(n^{\log_2 5})$.

In Section 4.1 we introduce our characterization of nested almost convex sets. As concequense of such characterization we obtain the following.

Theorem 21. *If $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$ there is exactly one order type that correspond to a nested almost convex set with n points; for other values of n, nested almost convex sets with n points do not exist.*

In Section 4.2, we prove that a nested almost convex set of n points (if it exists), can be drawn in an integer grid of size $O(n^{\log_2 5}) \simeq O(n^{2.322})$. Furthermore, we provide a linear time algorithm to find this drawing. A lower bound of $\Omega(n^{1.5})$ on the size of any drawing of a nested almost convex set of n points can be derived from the following observations. Any drawing of an n-point set in convex position has size $\Omega(n^{1.5})$ [30]; and every nested almost convex set of n points has a $\Theta(n)$ points in convex position (this is presented in detail in Section 4.1).

In Section 4.3, we are interested in finding an algorithm to decide whether a given point set is a nested almost convex set. A straightforward $O(n^4)$-time algorithm for this problem can be given using Definition 2. This can be improved to $O(n^2)$ as follows. Using the algorithm presented in Section 4.2 an instance of nested almost convex set can be constructed. Recently in [5], Aloupis, Iacono, Langerman, Öskan and Wuhrer gave an $O(n^2)$-time algorithm to decide whether two given sets of n points have the same order type. Thus, using their algorithm and our instance solves the decision problem in $O(n^2)$ time. We further improve on this by presenting $O(n \log n)$ time algorithm.

4.1 Characterization of Nested Almost Convex Sets.

In this section we prove Theorem 22, in which the nested almost convex sets are characterized. First we introduce some definitions.

Throughout this section: $\mathcal{X}$ will denote a set of n points in general position; k will denote the number of convex layers of $\mathcal{X}$; R_j will denote the set of points in the j-th convex layer of $\mathcal{X}$, R_1 being the most internal; and $\mathcal{X}_j$ will denote the set of points in $\mathcal{X}$, that are in R_j or in the interior of its convex hull.

$\mathbf{T_1(k)}$: We define $T_1(k)$ as the complete binary tree with $2^{k+1}-1$ nodes. The *j-level* of $T_1(k)$ is defined as the set of the nodes at distance j from the root.

Type 1: We say that $\mathcal{X}$ is of *type 1* if $|R_j| = 2^j$ for $1 \leq j \leq k-1$. Note that if $\mathcal{X}$ is of Type 1, then for every $1 \leq j \leq k$, the number of points in R_j is equal to the number of nodes in the j-level of $T_1(k)$.

Type 1 labeling: An injective function $\psi : \mathcal{X} \to T_1(k)$ is a *type 1 labeling*, if $\mathcal{X}$ is Type 1 and ψ labels the nodes (different to the root) of $T_1(k)$ with different points of $\mathcal{X}$.

$\mathbf{T_2(k)}$: We define $T_2(k)$ as the tree that, its root has three children, and each child is the root of a complete binary tree with $2^{k-1} - 1$ nodes. The *j-level* of $T_2(k)$ is defined as the set of the nodes at distance $j - 1$ from the root.

Type 2: We say that $\mathcal{X}$ is of *type 2* if $|R_1| = 1$ and $|R_j| = 3 \cdot 2^{j-2}$ for $2 \leq j \leq k$. Note that if $\mathcal{X}$ is of Type 2, then for every $1 \leq j \leq k$, the number of points in R_j is equal to the number of nodes in the j-level of $T_2(k)$.

Type 2 labeling: An injective function $\psi : \mathcal{X} \to T_2(k)$ is a *Type 2 labeling*, if $\mathcal{X}$ is Type 2 and ψ labels the nodes (also the root) of $T_2(k)$ with different points of $\mathcal{X}$.

Labeling: Let T be equal to $T_1(k)$ or $T_2(k)$. We say that a map $\psi : \mathcal{X} \to T$ is a *labeling*, if ψ is a Type 1 labeling or a Type 2 labeling. Note that, if $\mathcal{X}$ admits a labeling then $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$.

In the following, when the map $\psi : \mathcal{X} \to T$ is clear from the context, we say that a point is the *label* of a node of T if the point is mapped to the node by ψ. This way, given a node u of T, we denote by x_u its label. We denote by $u(l)$ and $u(r)$ the left and right children of u in T, respectively.

Nested: We say that a labeling is nested if, for $1 \leq j \leq k$, the left to right order of labels of the nodes in the j-level of T, corresponds to the counterclockwise order of the points in R_j.

Adoptable: Given a point p in R_j and two points q_1, q_2 in R_{j+1}, we say that q_1, q_2 are *adoptable from p* if, for every other point q_3 in R_{j+1}, p is in the interior of the triangle determined by q_1, q_2, q_3. We say that a nested labeling is *adoptable* if, for every node u in T, $x_{u(l)}$ and $x_{u(r)}$ are adoptable from x_u.

We denote by $R_j(u)$ the set of points in R_j that label a descendant of u. With respect to the counterclockwise order, we denote by: first$[R_j(u)]$, the first point in $R_j(u)$; last$[R_j(u)]$, the last point in $R_j(u)$; previous$[R_j(u)]$, the point in R_j previous to first$[R_j(u)]$; and next$[R_j(u)]$, the point in R_j next to last$[R_j(u)]$. See Figure 4.1.

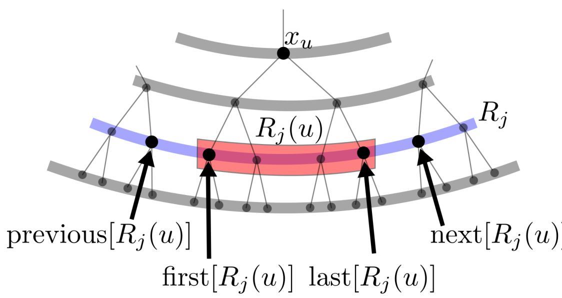


Figure 4.1: Illustration of $R_j(u)$, first$[R_j(u)]$, last$[R_j(u)]$, previous$[R_j(u)]$, and next$[R_j(u)]$.

Well laid: We say that a nested labeling is *well laid* if, for every u in T, x_u is in the intersection of the triangle determined by previous$[R_k(u)]$, first$[R_k(u)]$, last$[R_k(u)]$ and the triangle determined by first$[R_k(u)]$, last$[R_k(u)]$,
next$[R_k(u)]$.

Let u be a node of T. We denote by $\mathcal{X}_u$ the set of points x_v such that v is descendant of u in T. We denote by $\overline{\mathcal{X}_u}$ the set $\mathcal{X}_u \cup \{x_u\}$. Given two sets of points A and B, we call any directed line from a point in A to a point in B, an (A, B)-line.

Internal separation: We say that a nested labeling is an internal separation if for every node u of T, every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line ℓ.

External separation: We say that a nested labeling is an external separation if for every node u of T, every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line and to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$-line.

Theorem 22. *Let $\mathcal{X}$ be a point set in general position. Then the following statements are equivalent:*

1. *$\mathcal{X}$ is a nested almost convex set.*
2. *$\mathcal{X}$ admits a labeling that is nested, adoptable and well laid.*
3. *$\mathcal{X}$ admits a labeling that is an internal separation and an external separation.*

Proof of Theorem 22

The proof of Theorem 22 is divided into three parts: first we prove that $1 \implies 2$; afterwards we prove that $2 \implies 3$; and finally we prove that $3 \implies 1$.

4.1.1 The nested almost convex Set admits a labeling that is nested, adoptable and well laid ($1 \implies 2$).

In this part we assume that $\mathcal{X}$ is a nested almost convex set, and we introduce a labeling ψ' that is nested, adoptable and well laid.

It is clear from the definition of labeling that a necessary condition for $\mathcal{X}$ to admit a labeling is that $\mathcal{X}$ must be type 1 or type 2. In the following lemma we prove that, if $\mathcal{X}$ is a nested almost convex, then $\mathcal{X}$ is type 1 or type 2.

Lemma 23. *If $\mathcal{X}$ is a nested almost convex set then we have one of the following cases:*

1. *$|R_j| = 2^j$ for $1 \leq j \leq k-1$.*
2. *$|R_1| = 1$ and $|R_j| = 3 \cdot 2^{j-2}$ for $2 \leq j \leq k$.*

Proof. Suppose that R_1 has three or more points. In this case, the interior of the convex hull of R_1 has at least one point of $\mathcal{X}$; this contradicts that R_1

is the first convex layer of $\mathcal{X}$. Thus $R_1 = \mathcal{X}_1$, and $\mathcal{X}_1$ has one or two points. This proves the lemma for $j = 1$.

Any triangulation of R_{j+1}, has $|R_{j+1}| - 2$ triangles and each triangle has exactly one point of $\mathcal{X}_j$ in its interior; thus $|\mathcal{X}_j| = |R_{j+1}| - 2$. In particular, if $|\mathcal{X}_1| = 2$ or $|\mathcal{X}_1| = 1$ then $|\mathcal{X}_2| = 4$ or $|\mathcal{X}_2| = 3$, respectively. This proves the lemma for $j = 2$.

For the other cases, note that

$$|R_{j+1}| = |\mathcal{X}_j| + 2 = |R_j| + |\mathcal{X}_{j-1}| + 2 = 2|R_j|.$$

$\square$

First we define ψ' on a subset of nodes of T depending on whether $\mathcal{X}$ is of type 1 or type 2.

- If $\mathcal{X}$ is of type 1: ψ' labels the two nodes in the 1-level of $T_1(k)$, with the two points in R_1.
- If $\mathcal{X}$ is of type 2: ψ' labels the node in the 1-level of $T_2(k)$, with the point in R_1; ψ' labels the three nodes in the 1-level of $T_2(k)$, with the three points in R_2 (such that, the left to right order of labels of the nodes in the 2-level of T, coincides to the counterclockwise order of the points in R_2).

To define ψ' on the other nodes of T, we use the following Lemma.

Lemma 24. *Let $p_0, \ldots, p_t$ be the set of points in R_j in counterclockwise order. Then, the points in R_{j+1} can be listed in counterclockwise order as $q_0, q_1, \ldots, q_{2t+1}$, where the points q_{2i}, q_{2i+1} are adoptable from p_i for $0 \leq i \leq t$.*

Proof. Let $\mathcal{T}$ be the set of triangles determined by three consecutive points of R_{j+1} in counterclockwise order. We first show that:

Claim 24.1. *Each point of R_j is in exactly two consecutive triangles of $\mathcal{T}$.*

Assume that $j \geq 2$ (and note that Claim 24.1 holds for $j = 1$). Let $\triangle$ be the interior of a triangle of $\mathcal{T}$. By the almost convex set definition, there is one point of $\mathcal{X}_j$ in $\triangle$. This point must be in R_j, since the convex hull of R_{j+1} without $\triangle$ (and its boundary) is convex. Thus, there is one point of R_j in the interior of each triangle of $\mathcal{T}$. As the triangles of $\mathcal{T}$ are defined by consecutive points of R_{j+1}, each point of R_j is in at most two triangles of $\mathcal{T}$. Thereby Claim 24.1 follows from $|\mathcal{T}| = |R_{j+1}| = 2|R_j|$.

The two triangles of $\mathcal{T}$ that contain p_0, are defined by four consecutive points of R_{j+1}; let q_0 be the second of these points. Let $q_0, q_1, \dots q_{2t+1}$ be the points of R_{j+1} in counterclockwise order. Note that, for each p_i, the middle two points of the four points that define the two triangles that contain p_i, are q_{2i} and q_{2i+1}. Thus q_{2i} and q_{2i+1} are adoptable from p_i.

$\square$

Now we define ψ' on the other nodes of T recursively. For each labeled node u, ψ' labels $u(l)$ and $u(r)$ with the two points adoptable from the label of u. We do this so that, the left to right order of the labels of the nodes in the $(j+1)$-level of T, correspond to the counterclockwise order of the points in R_{j+1}. Note that ψ' is nested and adoptable. It remains to prove that ψ' is well laid. We prove this in Lemma 26.

Lemma 25. *If u is a node of T, the label of every descendant of u is contained in the convex hull of $R_k(u)$.*

Proof. We claim that every set $R_{j-1}(u)$, with at least two points, is contained in the convex hull of $R_j(u)$. Let p be a point in $R_{j-1}(u)$ and let q and q' be the labels of the children of the node labeled by p. By construction of ψ', q and q' are adoptable from p. As $R_{j-1}(u)$ has at least two points, $R_j(u)$ has at least four points. Let $\triangle$ be a triangle determined by q, q' and another point of $R_j(u)$. By definition of adoptable, p is in the interior of $\triangle$ and in consequence in the interior of the convex hull of R_j. An inductive application of the previous claim proves this lemma. $\square$

Lemma 26. *Let u be a node of T. Then x_u is in the intersection of the triangle determined by* $\text{previous}[R_k(u)]$, $\text{first}[R_k(u)]$ *and* $\text{last}[R_k(u)]$ *and the triangle determined by* $\text{first}[R_k(u)]$, $\text{last}[R_k(u)]$ *and* $\text{next}[R_k(u)]$.

Proof. Let j be the index such that the j-level of T contains u. Let R'_k be the set that contains first$(R_k(v))$ and last$(R_k(v))$ for all nodes v in the j-level of T. Let $\mathcal{T}$ be the set of triangles determined by three consecutive points of R'_k in counterclockwise order. We first show the following claim.

Claim 26.1. *Each point of R_j is in exactly two consecutive triangles of $\mathcal{T}$.*

Note that every point of $\mathcal{X} \setminus \mathcal{X}_j$, is the label of some descendant of a node v in the j-level of T. Thus, by Lemma 25, every point of $\mathcal{X} \setminus \mathcal{X}_j$ is in the convex hull of $R_k(v)$ for some node v in the j-level of T. Let $\mathcal{A}$ be the region obtained from the convex hull of $\mathcal{X}$, by removing the convex hull of $R_k(v)$ for each v in the j-level of T. Note that the set of points of $\mathcal{X}$ that are in $\mathcal{A}$ is $\mathcal{X}_j$.

Let $\triangle$ be the interior of a triangle of $\mathcal{T}$. By the nested almost convex set definition, there is one point of $\mathcal{X}$ in $\triangle$. As $\triangle$ is contained in $\mathcal{A}$, this point must be in $\mathcal{X}_j$. This point must also be in R_j, since $\mathcal{A}$ without $\triangle$ (and its boundary) is convex. Thus, there is one point of R_j in the interior of each triangle of $\mathcal{T}$. As the triangles of $\mathcal{T}$ are defined by consecutive points of R'_k, each point of R_j is in at most two triangles of $\mathcal{T}$. Thereby Claim 26.1 follows from $|\mathcal{T}| = |R'_k| = 2|R_j|$.

Let $\triangle'$ be the intersection of the triangle determined by previous$[R_{j+1}(u)]$, first$[R_{j+1}(u)]$ and last$[R_{j+1}(u)]$, with the triangle determined by first$[R_{j+1}(u)]$, last$[R_{j+1}(u)]$ and next$[R_{j+1}(u)]$. Note that first$[R_{j+1}(u)]$ and last$[R_{j+1}(u)]$ are the labels of the children of u. By definition of ψ', x_u is in the interior of every triangle determined by first$[R_{j+1}(u)]$, last$[R_{j+1}(u)]$ and every other point of R_{j+1}; thus x_u is in $\triangle'$. By Claim 26.1, x_u is in the interior of two triangles of $\mathcal{T}$, but there are only two triangles of $\mathcal{T}$ that intersect $\triangle'$; these are the triangles determined by previous$[R_k(u)]$, first$[R_k(u)]$ and last$[R_k(u)]$, and the triangle determined by first$[R_k(u)]$, last$[R_k(u)]$, next$[R_k(u)]$. $\square$

4.1.2 Sets that admit a labeling that is nested, adoptable and well laid, admit a labeling that is an internal separation and an external separation ($2 \implies 3$).

In this part we assume that there is a labeling ψ' of $\mathcal{X}$ that is nested, adoptable and well laid; and we prove that ψ' is an internal separation and an external separation.

Lemma 27. *ψ' is an internal separation.*

Proof. Let u be a node of T and recall that $u(l)$, $u(r)$ are the left and right children of u, respectively. We need to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line.

Let ℓ be the directed segment from first$[R_k(u(l))]$ to last$[R_k(u(r))]$. By Lemma 26, each point in $\mathcal{X}/\mathcal{X}_u$ is in the interior of a triangle whose vertices are to the left of, or on ℓ; thus every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of ℓ. By Lemma 25, every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$ is to the right of ℓ. We claim that:

Claim 27.1. *No $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line intersects ℓ.*

As the end points of ℓ, first$[R_k(u(l))]$ and last$[R_k(u(r))]$, are in the boundary of the convex hull of $\mathcal{X}$; to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line, it is enough to show Claim 27.1.

Let P_1 be the polygonal chain that starts at $q_1 :=$ first$[R_k(u(l))]$, follows the points of $R_k(u(l))$ in counterclockwise order, and ends at $q_2 :=$ last$[R_k(u(l))]$. Similarly, let P_2 be the polygonal chain that starts at $q_3 :=$ first$[R_k(u(r))]$, follows the points of $R_k(u(r))$ in counterclockwise order, and ends at $q_4 :=$ last$[R_k(u(r))]$. To prove Claim 27.1 it is enough to show that every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line intersects both P_1 and P_2.

Let q be the intersection point of the diagonals of the quadrilateral defined by q_1, q_2, q_3 and q_4. By Lemma 25 and Lemma 26, $\overline{\mathcal{X}_{u(l)}}$ is contained in the convex hull of $P_1 \cup \{q\}$, and $\overline{\mathcal{X}_{u(r)}}$ is contained in the convex hull of $P_2 \cup \{q\}$. Let ℓ' be an $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line. Note that the slope of ℓ', is in the range from the slope of the line define by q_1 and q_3, to the slope of the line define by q_2 and q_4, in counterclockwise order. Thus ℓ' intersects both P_1 and P_2.

□

Lemma 28. *ψ' is an external separation.*

Proof. Let u be a node of T. We need to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$, is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line and to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$-line. We prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line. That every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$-line can be proven in a similar way.

Let P be the polygonal chain that starts at next$[R_k(u)]$, follows the points of R_k in counterclockwise order, and ends at previous$[R_k(u)]$. Note that, by Lemma 26, $\mathcal{X}/\overline{\mathcal{X}_u}$ is contained in the convex hull of P. Thus, to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line, it is enough to show that x_u is to the right of the directed line from last$[R_k(u(l))]$ to next$[R_k(u)]$. See Figure 4.2.

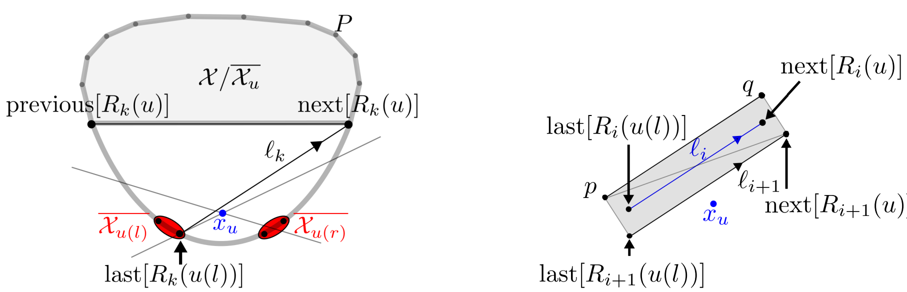


Figure 4.2: Illustration of the proof of Lemma 28

Let j be the index such that the j-level of T contains u. For $j < i \leq k$, let ℓ_i be the directed line from last$[R_i(u(l))]$ to next$[R_i(u)]$. We show that x_u is to the right of ℓ_i by induction. As $x_{u(l)}$ and $x_{u(r)}$ are adoptable from x_u, and $x_{u(l)} =$ last$[R_{j+1}(u(l))]$; x_u is in the interior of the triangle determined by last$[R_{j+1}(u(l))]$, $x_{u(r)}$ and next$[R_{j+1}(u)]$. Thus the induction holds for $i = j+1$. Suppose that x_u is to the right of ℓ_i. Let last$[R_{i+1}(u(l))]$ and p be the two children of last$[R_i(u(l))]$. Let next$[R_{i+1}(u)]$ and q be the two children of next$[R_i(u)]$. Let □ be the quadrilateral determined by last$[R_{i+1}(u(l))]$, p, q and next$[R_{i+1}(u)]$. As last$[R_{i+1}(u(l))]$, p, q and next$[R_{i+1}(u)]$ are in

R_{i+1}, and any triangulation of $\square$ has two triangles; there are two points of $\mathcal{X}_i$ in $\square$. As those points are last$[R_i(u(l))]$ and next$[R_i(u)]$, x_u is not in the interior of $\square$. Thus x_u is not between ℓ_i and ℓ_{i+1}, and therefore x_u is to the right of ℓ_{i+1}.

$\square$

4.1.3 Sets that admit a labeling that is an internal separation and an external separation, are nested almost convex sets (3 $\implies$ 1).

In this part we finish the proof of Theorem 22. We assume that there is a labeling ψ' of $\mathcal{X}$ that is an internal separation and an external separation, and we prove that $\mathcal{X}$ is a nested almost convex set. For this it is enough to prove Lemma 29. As consequence of Lemma 29 and Theorem 22, Theorem 21 holds.

Lemma 29. *Let $\mathcal{X}$ be an n-point set that admits a labeling $\psi : \mathcal{X} \to T$ that is an internal separation and an external separation. Then the order type of $\mathcal{X}$ is determined by T and:*

- *If $n = 2^{k-1} - 2$, then $\mathcal{X}$ has the same order type than any n-point set obtained from Construction 1.*
- *If $n = 3 \cdot 2^{k-1} - 2$, then $\mathcal{X}$ has the same order type than any n-point set obtained from Construction 2.*

Proof. The labeling that $\mathcal{X}$ admits can be a type 1 labeling or a type 2 labeling. If $\mathcal{X}$ admits a type 1 labeling, $|\mathcal{X}| = 2^{k+1} - 2$ for some integer k; in this case, an almost convex set with the same cardinality than $\mathcal{X}$ can be obtained using Construction 1. If $\mathcal{X}$ admits a type 2 labeling, $|\mathcal{X}| = 3 \cdot 2^{k-1} - 2$ for some integer k; in this case, an almost convex set with the same cardinality than $\mathcal{X}$ can be obtained using Construction 2. Let $\mathcal{Y}$ be an almost convex set with $|\mathcal{X}|$ points obtained from Construction 1 or Construction 2. We prove that $\mathcal{X}$ and $\mathcal{Y}$ have the same order type, and that this order type is determined by T.

Assume that $\mathcal{X}$ admits a type 1 labeling. The case when $\mathcal{X}$ admits a type 2 labeling can be proven in a similar way. As $\mathcal{Y}$ is an almost convex set, $\mathcal{Y}$ admits a labeling that is an internal separation and an external separation. Let $\psi_Y : \mathcal{Y} \to T$ be such type 1 labeling.

Let $f := \psi_Y^{-1}(\psi')$. We prove that $f : \mathcal{X} \to \mathcal{Y}$ is a bijection that preserves the orientation of all triplets. Let x_1, x_2, x_3 be different points in $\mathcal{X}$, let u_1, u_2, u_3 be the nodes of T that x_1, x_2, x_3 label in ψ', and let y_1, y_2, y_3 be the labels of u_1, u_2, u_3 in ψ_Y. Note that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$. To prove that (x_1, x_2, x_3) and (y_1, y_2, y_3) have the same orientation, we show that the position of u_1, u_2 and u_3 in T determines the orientation of any labeling of u_1, u_2 and u_3.

Given a node w of T, denote by T_w the subtree of T that contains every descendant of w. Let w be the farthest node from the root of T, such that at least two of u_1, u_2, u_3 are in T_w. If two of u_1, u_2, u_3 are in the left subtree of T_w or, two of u_1, u_2, u_3 are in the right subtree of T_w; the orientation of the labels of u_1, u_2, u_3 is determined by an external separation. If there are not two of u_1, u_2, u_3 in the left subtree of T_w or in the right subtree of T_w; there is one of u_1, u_2, u_3 in the left subtree of T_w, one u_1, u_2, u_3 in the right subtree of T_w, and the other one is not in the left or right subtree of T_w. In this case, the orientation of the labels of u_1, u_2, u_3 is determined by an internal separation.

□

4.2 Drawings of Nested Almost Convex Sets with Small Size.

Let $\mathcal{X}'$ be a nested almost convex set with n points, and let k be the number of convex layers of $\mathcal{X}'$. In this section we construct a drawing of $\mathcal{X}'$ of size $O(n^{\log_2 5})$. This section is divided into three parts. First, we construct a $2^{k+1} - 2$ point set $\mathcal{X}$ with integer coordinates and size $2 \cdot 5^{k+1}$. Afterwards, we prove that $\mathcal{X}$ is a nested almost convex set. Finally, we obtain a subset of $\mathcal{X}$ that is a drawing of $\mathcal{X}'$.

Construction of $\mathcal{X}$.

Recall that $T_1(k)$ is the complete binary tree with $2^{k+1}-1$ nodes, and the j-level of $T_1(k)$ is the set of nodes at distance j from the root of $T_1(k)$. Before defining $\mathcal{X}$, we will construct a point set $\mathcal{Y}$ in convex position, and for each node u in $T_1(k)$, we will define a set $\mathcal{Y}_u \subset \mathcal{Y}$ of consecutive points of $\mathcal{Y}$ in counterclockwise order. The point x_u will denote the midpoint between the first and last points of $\mathcal{Y}_u$ in counterclockwise order. The set $\mathcal{X}$ will be the set of points x_u such that u is a node of $T_1(k)$ different from the root.

Let p, o and q be points in the plane and let $c \in [0,1]$. We denote by $\overline{op}$ and $\overline{oq}$ the segments from o to p and from o to q, respectively. We say that $\alpha = (q,o,p)$ is a *corner*, if the angle from $\overline{op}$ to $\overline{oq}$ counterclockwise is less than π. Let $\alpha := (q,o,p)$ be a corner. We denote by $\mathrm{LeftPoint}(\alpha,c)$ the point in the segment $\overline{oq}$ at distance $c|\overline{oq}|$ from o. We denote by $\mathrm{RightPoint}(\alpha,c)$ the point in the segment $\overline{op}$ at distance $c|\overline{op}|$ from o. See Figure 4.3.

Recursively, we define a corner α_u for each node u of $T_1(k)$. The corner of the root of $T_1(k)$ is defined as $((0, 2\cdot 5^{k+1}),(0,0),(2\cdot 5^{k+1},0))$. Let u be a node for which its corner α_u has been defined; the corners of its left and right children, $u(l)$ and $u(r)$, are defined as follows (See Figure 4.3):

$$\alpha_{u(l)} = (\mathrm{LeftPoint}(\alpha_u,2/5), \mathrm{LeftPoint}(\alpha_u,1/5), \mathrm{RightPoint}(\alpha_u,1/5))$$
$$\alpha_{u(r)} = (\mathrm{LeftPoint}(\alpha_u,1/5), \mathrm{RightPoint}(\alpha_u,1/5), \mathrm{RightPoint}(\alpha_u,2/5))$$

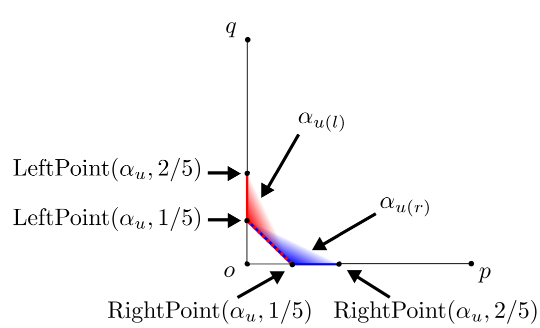


Figure 4.3: Illustration of corners α_u, $\alpha_{u(l)}$ and $\alpha_{u(r)}$, where $\alpha_u = (q,o,p)$.

Let v be a leaf of $T_1(k+1)$. Note that v is a child of a leaf u of $T_1(k)$. If v is the left child of u, let $y_v := \text{LeftPoint}(\alpha_u, 1/5)$. If v is the right child of u, let $y_v := \text{RightPoint}(\alpha_u, 1/5)$. We define $\mathcal{Y}$ as the set of points y_v such that v is a leaf of $T_1(k+1)$. Given a node u of $T_1(k)$, we define $\mathcal{Y}_u$ as the set of points y_v such that v is a descendant of u, and v is a leaf of $T_1(k+1)$. With respect to the counterclockwise order, we denote by: first$[\mathcal{Y}_u]$, the first point in $\mathcal{Y}_u$; last$[\mathcal{Y}_u]$, the last point in $\mathcal{Y}_u$; previous$[\mathcal{Y}_u]$, the point in $\mathcal{Y}_u$ previous to first$[\mathcal{Y}_u]$; and next$[\mathcal{Y}_u]$, the point in $\mathcal{Y}_u$ next to last$[\mathcal{Y}_u]$.

Lemma 30. *Let u be a node of $T_1(k)$. Let $v_1, v_2, \ldots, v_t$ be the leaves of $T_1(k+1)$, that are descendant of u, ordered from left to right. Then $y_{v_1}, y_{v_2}, \ldots, y_{v_t}$ are in convex position, and are the points in $\mathcal{Y}_u$ in counterclockwise order.*

Proof. Let $(q, o, p) := \alpha_u$; $q' := \text{LeftPoint}(\alpha_u, 2/5)$; and $p' := \text{RightPoint}(\alpha_u, 2/5)$. Let $\triangle(u)$ be the triangle determined by q', o and p'. inductively from the leaves to the root of $T_1(k)$, it can be proven that:

1. The set of points of $\mathcal{Y}$ in $\triangle(u)$ is $\mathcal{Y}_u$; from which: first$[\mathcal{Y}_u]$ is on the segment from o to q', last$[\mathcal{Y}_u]$ is on the segment from o to p', and the other points are in the interior of $\triangle(u)$.

2. The points $q', y_{v_1}, y_{v_2}, \ldots, y_{v_t}, p'$ are in convex position, and appear in this order counterclockwise.

This proof follows from 2. $\square$

By Lemma 30, $\mathcal{Y}$ is in convex position, and for each node u in $T_1(k)$, $\mathcal{Y}_u$ is a subset of consecutive points of $\mathcal{Y}$ in counterclockwise order. We denote by x_u the midpoint between first$[\mathcal{Y}_u]$ and last$[\mathcal{Y}_u]$. Let $\mathcal{X}$ be the set of points x_u such that u is a node of $T_1(k)$ different from the root.

Let u be a node of $T_1(k)$ at distance j from the root, let $(q, o, p) := \alpha_u$ and let v be a leaf of $T_1(k+1)$. Recursively note that, the coordinates of q, o and p are divisible by $2 \cdot 5^{k+1-j}$. Thus, the coordinates of y_v are divisible by 2, x_u has integer coordinates, and $\mathcal{X}$ has size $2 \cdot 5^{k+1}$.

$\mathcal{X}$ is a nested Almost Convex Set.

In this subsection we prove that $\mathcal{X}$ is a nested almost convex set. By Theorem 22, it is enough to prove that $\mathcal{X}$ admits a labeling that is an internal separation and an external separation. Let $\psi : \mathcal{X} \to T_1(k)$ be the type 1 labeling that labels each node u of $T_1(k)$ different from the root, with x_u. We prove that ψ is both an internal separation and an external separation.

Lemma 31. *If u is a node of $T_1(k)$ at distance j from the root, then* $\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j)$ *and* $\text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j)$*, where*

$$c_j = \frac{1}{4}\left(1 - 5^{(j-k-1)}\right).$$

Proof. Note that

$$c_j = \sum_{i=k}^{j} \left(\frac{1}{5}\right)^{k+1-j}.$$

If $j = k$, then: u is a leaf of $T_1(k)$; $c_j = 1/5$; and $\text{first}[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_u, c_j)$ and $\text{last}[\mathcal{Y}_u] = \text{RightPoint}(\alpha_u, c_j)$. Suppose that $j < k$, and that this lemma holds for larger values of j. Let $u(l)$ and $u(r)$ be the left and right children of u. Note that by induction,

$$first[\mathcal{Y}_u] = \text{LeftPoint}(\alpha_{u(l)}, c_{j+1}) = \text{LeftPoint}(\alpha_u, c*)$$

where $c* = (1/5)c_{j+1} + 1/5 = c_j$; thus $\text{first}[\mathcal{Y}_u] := \text{LeftPoint}(\alpha_u, c_j)$. In a similar way $\text{last}[\mathcal{Y}_u] := \text{RightPoint}(\alpha_u, c_j)$. □

Lemma 32. *ψ is an internal separation.*

Proof. Let u be a node of $T_1(k)$ different from the root, and let $u(l)$, $u(r)$ be the left and right children of u, respectively. We need to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line.

Let ℓ be the directed segment from $\text{first}[\mathcal{Y}_{u(l)}]$ to $\text{last}[\mathcal{Y}_{u(r)}]$. As each point in $\mathcal{X}/\mathcal{X}_u$, is the midpoint between two points that are not to the right of ℓ, every point in $\mathcal{X}/\mathcal{X}_u$ is not to the right of ℓ. As every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$, is the midpoint between a point to the right of ℓ and a point that is not to the left of ℓ, every point in $\overline{\mathcal{X}_{u(l)}} \cup \overline{\mathcal{X}_{u(r)}}$ is to the right of ℓ. We claim that:

Claim 32.1. *No $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line intersects ℓ.*

As the endpoints of ℓ, $\text{first}[\mathcal{Y}_{u(l)}]$ and $\text{last}[\mathcal{Y}_{u(r)}]$, are in the boundary of the convex hull of $\mathcal{Y}$; to prove that every point in $\mathcal{X}/\mathcal{X}_u$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line, it is enough to show Claim 32.1.

Let P_1 be the polygonal chain that starts at $\text{first}[\mathcal{Y}_{u(l)}]$, follows the points of $\mathcal{Y}_{u(l)}$ in counterclockwise order, and ends at $\text{last}[\mathcal{Y}_{u(l)}]$. Similarly, let P_2 be the polygonal chain that starts at $\text{first}[\mathcal{Y}_{u(r)}]$, follows the points of $\mathcal{Y}_{u(r)}$ in counterclockwise order, and ends at $\text{last}[\mathcal{Y}_{u(r)}]$. To prove Claim 32.1 it is enough to show that every $(\overline{\mathcal{X}_{u(l)}}, \overline{\mathcal{X}_{u(r)}})$-line intersects P_1 and P_2. This follows from the fact that $\overline{\mathcal{X}_{u(l)}}$ is contained in the convex hull of P_1, and $\overline{\mathcal{X}_{u(r)}}$ is contained in the convex hull of P_2. $\square$

Lemma 33. *Let u be a node of $T_1(k)$ at distance j from the root, and let $(q, o, p) := \alpha_u$. Suppose that the nodes in the j-level of $T_1(k)$, are ordered from left to right.*

1. *If u is not the first node, then the points o, $\text{first}[\mathcal{Y}_u]$, $\text{previous}[\mathcal{Y}_u]$ and q are collinear, and $\text{previous}[\mathcal{Y}_u] = \text{LeftPoint}(u, c)$ for some $c > 3/5$.*

2. *If u is not the last node, then the points o, $\text{last}[\mathcal{Y}_u]$, $\text{next}[\mathcal{Y}_u]$ and p are collinear, and $\text{next}[\mathcal{Y}_u] = \text{RightPoint}(u, c)$ for some $c > 3/5$.*

Proof. To prove 1 and 2, note that, for any two consecutive nodes in the j-level of $T_1(k)$, there is a segment that contains one side of each the corners corresponding to these nodes; then apply Lemma 31. $\square$

Lemma 34. *ψ is an external separation.*

Proof. Let u be a node of $T_1(k)$ and $u(l)$, $u(r)$ be the left and right children of u, respectively. We need to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$, is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line and to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$-line. We prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line. That every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\{x_u\}, \overline{\mathcal{X}_{u(r)}})$-line can be proven in a similar way.

Let P be the polygonal chain that starts at next$[\mathcal{Y}_u]$, follows the points of $\mathcal{Y}$ in counterclockwise order, and ends at previous$[\mathcal{Y}_u]$. Note that $\mathcal{X}/\overline{\mathcal{X}_u}$ is contained in the convex hull of P. Thus, to prove that every point in $\mathcal{X}/\overline{\mathcal{X}_u}$ is to the left of every $(\overline{\mathcal{X}_{u(l)}}, \{x_u\})$-line, it is enough to show that next$[\mathcal{Y}_u]$ is to the left of the directed line from last$[\mathcal{Y}_{u(l)}]$ to x_u.

Let ℓ be the directed line from last$[\mathcal{Y}_{u(l)}]$ to x_u and let $(q, o, p) := \alpha_u$. Note that x_u and last$[\mathcal{Y}_{u(l)}]$ are in the interior of the wedge determined by α_u, from $\overline{op}$ to $\overline{oq}$ in counterclockwise order. By Lemma 33-2, next$[\mathcal{Y}_u]$ is on $\overline{op}$ and next$[\mathcal{Y}_u] = \mathrm{RightPoint}(u, c)$ for some $c > 3/5$. To finish this proof we show that ℓ intersects $\overline{op}$ at a point $\mathrm{RightPoint}(u, c')$ for some $c' < 3/5$.

Consider the following coordinate system, o is the origin, p has coordinates $(1, 0)$ and q has coordinates $(0, 1)$. Assume that this is the new coordinate system. Let t be such that the intersection point between ℓ and the abscissa is the point $(t, 0)$; thereby, we need to prove that $t < 3/5$.

By Lemma 31, first$[\mathcal{Y}_u]$ and last$[\mathcal{Y}_u]$ have coordinates $(0, c_j)$ and $(c_j, 0)$; thus, x_u has coordinates $(c_j/2, c_j/2)$. By construction of $\alpha_{u(l)}$ and Lemma 31, last$[\mathcal{Y}_{u(l)}]$ is in the segment from $(0, 1/5)$ to $(1/5, 0)$ in $\mathrm{RightPoint}(u(l), c_{j+1})$. Thus last$[\mathcal{Y}_{u(l)}]$ has coordinates $(\frac{1}{5}c_{j+1}, \frac{1}{5}(1 - c_{j+1}))$ and the equation of ℓ is

$$x = \frac{c_{j+1}/5 - c_j/2}{(1 - c_{j+1})/5 - c_j/2}(y - c_j/2) + c_j/2$$

taking $y = 0, s = k - j$, and replacing c_j and c_{j+1}, we have that

$$t = -\frac{1}{40 \cdot 5^s} - \frac{1}{40(1 + 3/5^s)} - \frac{1}{40(3 \cdot 5^s + 5^{2s})} + \frac{3}{8(3/5^s + 1)} + \frac{1}{8(3 + 5^s)} + \frac{1}{8}$$

finally, as $5^s \geq 1$

$$t < \frac{3}{8} + \frac{1}{8(4)} + \frac{1}{8} = \frac{17}{32} < \frac{3}{5}.$$

□

Construction of a Drawing of $\mathcal{X}$.

In this subsection we find a subset of $\mathcal{X}$ that is a drawing of $\mathcal{X}'$. By Theorem 21, there are two cases: $\mathcal{X}'$ is of type 1 and has $n = 2^{k+1} - 2$ points; or $\mathcal{X}'$ is of type 2 and has $n = 3 \cdot 2^{k-1} - 2$ points. By Theorem 21, if $\mathcal{X}'$ is type 1, $\mathcal{X}'$ and $\mathcal{X}$ have the same order type and $\mathcal{X}$ is a drawing of $\mathcal{X}'$. Assume that $\mathcal{X}'$ is type 2.

Let w be the root of $T_1(k)$; u and u' be the children of w; $u(l)$ and $u(r)$ be the children of u; and $u'(l)$ and $u'(r)$ be the children of u'. We define T as the tree obtained from $T_1(k)$, by making $u'(l)$ the third child of u and removing w, u', $u'(r)$ and every descendant of $u'(r)$. Recall that $T_2(k)$ is a tree such that, its root has three children, and each child is the root of a complete binary tree with $2^{k-1} - 1$ points. Note that T and $T_2(k)$ are isomorphic.

Let $\mathcal{X}_2$ be the set of points x_u such that u is in T. Let $\psi' : \mathcal{X}_2 \to T$ be such that $\psi'(x_u) = u$. Note that: as ψ is an internal separation, ψ' is an internal separation; and as ψ is an external separation, ψ' is external separation. Thus by Theorem 22, $\mathcal{X}_2$ is a nested almost convex set.

By Theorem 21, as $\mathcal{X}_2$ has $3 \cdot 2^{k-1} - 2$ points, $\mathcal{X}_2$ and $\mathcal{X}'$ have the same order type and $\mathcal{X}_2$ is a drawing of $\mathcal{X}'$.

4.3 Decision Algorithm for Nested Almost Convexity.

Let $\mathcal{X}$ be a set of n points. In this section, we present an $O(n \log n)$ time algorithm, to decide whether $\mathcal{X}$ is a nested almost convex set. This algorithm is based in Theorem 22-2 and consists of four steps. At each step, it is verified if $\mathcal{X}$ satisfies a certain property; $\mathcal{X}$ is a nested almost convex set if and only if $\mathcal{X}$ satisfies each of these properties.

By Theorem 21, if $\mathcal{X}$ is a nested almost convex set, then $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$ for some integer k. The first step is to verify whether $\mathcal{X}$ has one of those cardinalities. If $n = 2^{k-1} - 2$ let $T := T_1(k)$. If $n = 3 \cdot 2^{k-1} - 2$ let $T := T_2(k)$. Recall that: the j-level of $T_1(k)$ is defined as the set of the nodes at distance j from the root; and the j-level of $T_2(k)$ is defined as the set of the nodes at distance $j-1$ from the root.

By Lemma 23, if $\mathcal{X}$ is a nested almost convex set then: for $1 \leq j \leq k$, the number of nodes in the j-level of T is equal to the number of nodes in the j-th convex layer of $\mathcal{X}$. The second step is to verify whether $\mathcal{X}$ satisfies Lemma 23. Chazelle [15] showed that, the convex layers of a given an n-point set can be found in $O(n \log n)$ time; thus the second step can be done in $O(n \log n)$ time. We denote by R_j the set of points in the j-th convex layer of $\mathcal{X}$.

The third step is to verify whether $\mathcal{X}$ satisfies Lemma 24. For $1 \leq j \leq k-1$, we do the following. Let $p_0, \dots, p_t$ be the points in R_j in counterclockwise order. We search for two consecutive points in R_{j+1} that are adoptable by p_0. If those points exist, they are the only pair of consecutive points in R_{j+1} that are adoptable by p_0. Let $q_0, q_1, \dots, p_{2t+1}$ be the points in R_{j+1} in counterclockwise order, such that q_0 and q_1 are adoptable by p_0. Then we verify whether q_{2i}, q_{2i+1} are adoptable by p_i for $0 \leq i \leq t$.

Let p be in R_j, and let q_r, q_{r+1}, q_{r+2}, q_{r+3} be four consecutive points in R_{j+1}. Note that q_{r+1} and q_{r+2} are adoptable by p, if and only if, p is in the intersection of the triangle determined by q_r, q_{r+1} and q_{r+2}, and the triangle determined by q_{r+1}, q_{r+2} and q_{r+3}. Thus, we can verify whether q_{2i}, q_{2i+1} are adoptable by p_i in constant time; the third step hence requires linear time.

If $\mathcal{X}$ satisfies Lemma 24, we can define a labeling $\psi : \mathcal{X} \to T$ like the one defined in Section 4.1-4.1.1. The fourth step is to verify if ψ is well laid, this requires linear time.

According to the proof of Theorem 22, $\mathcal{X}$ is a nested almost convex set if and only if $\mathcal{X}$ verifies the properties in previous four steps. This can be done in $O(n \log n)$ time.

Bibliography

[1] O. Aichholzer, F. Aurenhammer, and H. Krasser. Enumerating order types for small point sets with applications. *Order*, 19(3):265–281, 2002.

[2] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, and J. Urrutia. Empty monochromatic triangles. *Comput. Geom.*, 42(9):934–938, 2009.

[3] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *Computational Geometry*, 47(6):644–650, 2014.

[4] N. Alon, M. Katchalski, and W. R. Pulleyblank. The maximum size of a convex polygon in a restricted set of points in the plane. *Discrete Comput. Geom.*, 4(3):245–251, 1989.

[5] G. Aloupis, J. Iacono, S. Langerman, O. Özkan, and S. Wuhrer. The complexity of order type isomorphism. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '14, pages 405–415. SIAM, 2014.

[6] I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canad. Math. Bull.*, 30(4):436–445, 1987.

[7] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Studia Sci. Math. Hungar.*, 41(2):243–266, 2004.

[8] L. Barba, F. Duque, R. Fabila-Monroy, and C. Hidalgo-Toscano. Drawing the horton set in an integer grid of minimum size. In

Proceedings of the 26th Canadian Conference on Computational Geometry, CCCG 2014, Halifax, Nova Scotia, Canada, August 11-13, 2014, 2014.

[9] L. Barba, F. Duque, R. Fabila-Monroy, and C. Hidalgo-Toscano. Drawing the Horton set in an integer grid of minimum size. *CoRR*, abs/1506.05505, 2015.

[10] S. Bereg, R. Fabila-Monroy, D. Flores-Peñaloza, M. Lopez, and P. Pérez-Lantero. Drawing the double circle on a grid of minimum size. In *Proceedings of XV Spanish Meeting on Computational Geometry*, EGC '13, pages 65–68, 2013.

[11] A. Bialostocki, P. Dierker, and B. Voxman. Some notes on the Erdős-Szekeres theorem. *Discrete Mathematics*, 91(3):231 – 238, 1991.

[12] P. Brass, W. Moser, and J. Pach. *Research problems in discrete geometry*. Springer, New York, 2005.

[13] J. Cano, A. G. Olaverri, F. Hurtado, T. Sakai, J. Tejel, and J. Urrutia. Blocking the k-holes of point sets in the plane. *Graphs and Combinatorics*, 31(5):1271–1287, 2015.

[14] Y. Caro. On the generalized Erdős-Szekeres conjecture — a new upper bound. *Discrete Mathematics*, 160(1–3):229 – 233, 1996.

[15] B. Chazelle. On the convex layers of a planar set. *Information Theory, IEEE Transactions on*, 31(4):509–517, Jul 1985.

[16] O. Devillers, F. Hurtado, G. Károlyi, and C. Seara. Chromatic variants of the Erdős-Szekeres theorem on points in convex position. *Comput. Geom.*, 26(3):193–208, 2003.

[17] A. Dumitrescu. Planar sets with few empty convex polygons. *Studia Sci. Math. Hungar.*, 36(1-2):93–109, 2000.

[18] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[19] P. Erdős and G. Szekeres. On some extremum problems in elementary geometry. In *Annales Univ. Sci. Budapest*, pages 3–4, 1960.

[20] P. Erdős. Some more problems on elementary geometry. *Austral. Math. Soc. Gaz.*, 5(2):52–54, 1978.

[21] EUROCORES and EuroGIGA. Compose. `http://www.eurogiga-compose.eu/posezo.php`, May 2016.

[22] T. Gerken. Empty convex hexagons in planar point sets. *Discrete Comput. Geom.*, 39(1-3):239–272, 2008.

[23] J. E. Goodman and R. Pollack. On the combinatorial classification of nondegenerate configurations in the plane. *J. Combin. Theory Ser. A*, 29(2):220–235, 1980.

[24] J. E. Goodman and R. Pollack. Multidimensional sorting. *SIAM J. Comput.*, 12(3):484–507, 1983.

[25] J. E. Goodman and R. Pollack. Semispaces of configurations, cell complexes of arrangements. *J. Combin. Theory Ser. A*, 37(3):257–293, 1984.

[26] J. E. Goodman and R. Pollack. The complexity of point configurations. *Discrete Appl. Math.*, 31(2):167–180, 1991. First Canadian Conference on Computational Geometry (Montreal, PQ, 1989).

[27] J. E. Goodman, R. Pollack, and B. Sturmfels. Coordinate representation of order types requires exponential storage. In *Proceedings of the Twenty-first Annual ACM Symposium on Theory of Computing*, STOC '89, pages 405–410, New York, NY, USA, 1989. ACM.

[28] H. Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elem. Math.*, 33(5):116–118, 1978.

[29] J. Horton. Sets with no empty convex 7-gons. *Canad. Math. Bull.*, 26(4):482–484, 1983.

[30] V. Jarník. Über die gitterpunkte auf konvexen kurven. *Mathematische Zeitschrift*, 24(1):500–518, 1926.

[31] J. G. Kalbfleisch and R. G. Stanton. On the maximum number of coplanar points containing no convex n-gons. *Utilitas Math.*, 47:235–245, 1995.

[32] G. Károlyi, J. Pach, and G. Tóth. A modular version of the Erdős–Szekeres theorem. *Studia Scientiarum Mathematicarum Hungarica*, 38(1-4):245–260, 2001.

[33] M. Katchalski and A. Meir. On empty triangles determined by points in the plane. *Acta Math. Hungar.*, 51(3-4):323–328, 1988.

[34] V. Koshelev. On erdos-szekeres problem and related problems. *arXiv preprint arXiv:0910.2700*, 2009.

[35] G. Kun and G. Lippner. Large empty convex polygons in k-convex sets. *Periodica Mathematica Hungarica*, 46(1):81–88, 2003.

[36] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[37] W. Morris and V. Soltan. The Erdős-Szekeres problem on points in convex position—a survey. *Bull. Amer. Math. Soc. (N.S.)*, 37(4):437–458 (electronic), 2000.

[38] C. M. Nicolás. The empty hexagon theorem. *Discrete Comput. Geom.*, 38(2):389–397, 2007.

[39] M. Overmars. Finding sets of points without empty convex 6-gons. *Discrete & Computational Geometry*, 29(1):153–158, 2002.

[40] J. Pach and G. Tóth. Monochromatic empty triangles in two-colored point sets. *Discrete Appl. Math.*, 161(9):1259–1261, 2013.

[41] G. Pick. Geometrisches zur Zahlenlehre. *Sitzungsber. Lotos (Prague)*, 19:311–319, 1899.

[42] A. Suk. On the erdos-szekeres convex polygon problem. *ARXIV*, eprint arXiv:1604.08657, 2016.

[43] G. Szekeres and L. Peters. Computer solution to the 17-point erdős-szekeres problem. *The ANZIAM Journal*, 48(02):151–164, 2006.

[44] P. Valtr. Convex independent sets and 7-holes in restricted planar point sets. *Discrete Comput. Geom.*, 7(2):135–152, 1992.

[45] P. Valtr. *Planar point sets with bounded ratios of distances.* PhD thesis, Freie Universität Berlin, 1994.

[46] P. Valtr. On the minimum number of empty polygons in planar point sets. *Studia Sci. Math. Hungar.*, 30(1-2):155–163, 1995.

[47] P. Valtr. Open caps and cups in planar point sets. *Discrete and Computational Geometry*, 37(4):565–576, 2007.

[48] P. Valtr, G. Lippner, and G. Károlyi. Empty convex polygons in almost convex sets. *Periodica Mathematica Hungarica*, 55(2):121–127, 2007.