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Vertical and radial Toeplitz operators

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Introduction

This dissertation is about two types of Toeplitz operators: radial and vertical. We present a description of the C^*-algebra generated by Toeplitz operators with radial symbols by its eigenvalues sequence. It is about vertical Toeplitz operators and its corresponding spectral functions. It is shown a description of the C^*-algebra generated by vertical Toeplitz operators by means of its spectral functions and the relation among them.

The motivation for this description is as follows:

Toeplitz operators

Let $\mathbb{D}\colon = \{z \in \mathbb{C} |\ |z| < 1\}$ be the unit disc. Let $L_2(\mathbb{D}, dA)$ be the space of square integrable functions defined on the unit disc.

Let $\mathcal{A}^2(\mathbb{D}) \subset L_2(\mathbb{D}, dA)$ be the Bergman space which consists of analytic functions on the disc.

Denote by $\mathcal{L}(\mathcal{A}^2(\mathbb{D}))$ the space of bounded operators acting on the Bergman space.

The Bergman space is a reproducing kernel space:

$$f(z) = \langle f, K_z \rangle, \text{ con } z \in \mathbb{D}.$$

where

$$K_z(w) = \frac{1}{(1 - w\overline{z})^2}.$$

The Bergman projection $P\colon L_2(\mathbb{D}, dA) \to \mathcal{A}^2(\mathbb{D})$ has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} dA(w)$$

The Toeplitz operator T_a, with symbol $a \in L_\infty(\mathbb{D})$ has the integral representation

$$Taf(z) = \int_{\mathbb{D}} \frac{a(w)f(w)}{(1 - z\overline{w})^2} dA(w).$$

Next we consider radial Toeplitz operators.

Radial operators

The set $\{e_n\}_{n\in\mathbb{N}}$, where

$$e_n(z) = \sqrt{n+1} z^n$$

is an orthonormal basis for $\mathcal{A}^2(\mathbb{D})$.

Among other equivalences, an operator $S \in \mathcal{L}(\mathcal{A}^2(\mathbb{D}))$ is said to be radial if and only if there exist $\{\mu_n\}_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$ such that

$$Se_n = \mu_n e_n, \qquad \forall n \in \mathbb{N}.$$

For Toeplitz operators, T_a is radial if and only if $a(z) = a(|z|)$ is a radial function.

The eigenvalues of the Toeplitz operator T_a are calculated in an explicit form

$$\mu_n = (n+1)\int_0^1 a(r)r^n dr.$$

Hausdorff moment problem

Given a sequence $\mu \in \ell^\infty(\mathbb{N})$ we wish to determine if there exist $a \in L_\infty(0,1)$ such that $\mu(T_a) = \mu$.

Such a problem is known as the "Hausdorff moment problem" and for its solution we recall the following definition: let m be a natural number and let $x = \{x_n\}_{n\in\mathbb{N}}$ be a complex number sequence. The m-difference of x is denoted by $\triangle_n^m(x) = (-1)^m \sum_{j=0}^m \binom{m}{j}(-1)^j x_{n+j}$.

(Hausdorff moment problem) μ is the corresponding eigenvalue sequence of the Toeplitz operator T_a if and only if $(k+1)\binom{k}{m}|\triangle_{k-m}^m \sigma| \leq C$ for $0 \leq m \leq k$ where $\sigma_n = \frac{\mu_n}{n+1}$[42, p.101].

The equation $\mu_n = (n+1)\int_0^1 a(r)r^n dr$ is related as a transformation of $a(r)$ into the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. This transformation is strongly related to the Laplace transform, and in fact is a discrete analogous to this one. We will take advantage of this fact latter.

First description of the C^*-algebra generated by radial Toepllitz operators

We consider the C^*-algebra generated by

$$\{T_a\colon\ a \text{ is bounded and radial}\}.$$

Given that the radial operators are determined by its eigenvalues sequence it is possible to give a description of this C^*-algebra describing the C^*-algebra generated by

$$\{\mu\colon\ \mu = \mu(T_a) \text{ for } a \text{ bounded and radial}\}.$$

Suárez [37] gives a description as we describe next. We denote by $\mathfrak{T}$ the C^*-algebra generated by Toeplitz operators with bounded defining symbols

$$\{T_a\colon\ a \in L_\infty(\mathbb{D})\}.$$

He considers two set of sequences

$$d_1 = \{x \in \ell^\infty(\mathbb{N})\colon\ \sup_n n|\triangle_n^1(x)| < \infty\},$$

$$d_2 = \{x \in \ell^\infty(\mathbb{N}) \colon \ \sup_n n^2 | \triangle_n^2 (x)| < \infty\}.$$

d_1 is a self-adjoint subalgebra of ℓ^∞ and therefore $\overline{d_1}^{\ell\infty}$ is a C^*-algebra. Suárez proves that $d_2 \subset d_1$ y $\overline{d_1}^{\ell\infty} = \overline{d_2}^{\ell\infty}$.

Let $S \in \mathcal{L}(\mathcal{A}^2(\mathbb{D}))$ be a radial operator.

$S \in \mathfrak{T}$ if and only if $\mu(S) \in \overline{d_1}^{\ell\infty}$.

$S \in \mathfrak{T}$ if and only if $\mu(S) \in \overline{d_2}^{\ell\infty}$.

This equivalence characterizes the eigenvalues sequence of operators belonging to the Toeplitz algebra. At first it's difficult to decide if a sequence belongs to the $\ell_\infty-$closure of sequences that satisfies the Hausdorff condition. Introducing d_1 and d_2 simplifies the situation. The disadvantage is neither d_1 and d_2 are closed.

Radial operator and the set d_2

To prove the equivalence $S \in \mathfrak{T}$ if and only if $\mu(S) \in \overline{d_2}^{\ell\infty}$, Suárez introduces an operators set for which its eigenvalue sequence are characterized by d_2.

For such a task he uses the Berezin transform, which plays an important role from this point forward.

The Berezin transform is defined by $B_0 \colon \mathcal{L}(\mathcal{A}^2(\mathbb{D})) \to C^\infty(\mathbb{D})$,

$$B_0(S)(z) = \frac{\langle SK_z, K_z\rangle}{\langle K_z, K_z\rangle}$$

The invariant Laplacian is

$$\tilde{\triangle} = (1 - |z|^2)\triangle$$

with $\triangle = \partial\overline{\partial}$.

Suárez makes use of

$$\mathcal{D} = \{S \in \mathcal{L}(\mathcal{A}^2(\mathbb{D})) \colon \ \exists\, T \in \mathcal{L}(\mathcal{A}^2(\mathbb{D})) \text{ tal que } \ \tilde{\triangle} B_0(S) = B_0(T)\}.$$

Given that the Berezin transform is one to one, the invariant Laplacian of an operator $S \in \mathcal{D}$ is defined by

$$\tilde{\triangle}(S) = T.$$

Therefore another eigenvalue characterization is obtained: $S \in \mathcal{D}$ if and only if $\mu(S) \in d_2$.

The set of sequences $\mathrm{VSO}(\mathbb{N})$: second description of the C^*-algebra generated by radial Toeplitz operators

Grudsky, Maximenko and Vasilevski use another set of sequences

$$\mathrm{VSO}(\mathbb{N}) = \left\{ x \in \ell^\infty(\mathbb{N}) \colon \ \lim_{\frac{j}{k} - 1} |x_j - x_k| = 0 \right\}.$$

VSO($\mathbb{N}$) is a subalgebra of $\ell^\infty(\mathbb{N})$. The relation between d_1 and VSO($\mathbb{N}$) is as follows

$$d_1 \subset \mathrm{VSO}(\mathbb{N}),$$

and

$$\overline{d_1}^{\ell^\infty} = \mathrm{VSO}(\mathbb{N}).$$

To prove the preceding equality, they use De la Vallé-Poussin mean.

It is, let $x \in \mathrm{VSO}(\mathbb{N})$ and $\epsilon > 0$. There exists $\delta \in (0,1)$ such that

$$y_j = \frac{1}{1 + \lfloor j\delta \rfloor} \sum_{k=j}^{j+\lfloor j\delta \rfloor} x_k,$$

with $y \in d_1$ and $\|y - x\| < \epsilon$.

This way is easier to check if a sequence belongs to VSO($\mathbb{N}$). In other words, the C^*-algebra generated by

$$\{T_a \colon \ a \text{ radial and bounded}\}$$

is isometrically isomorphic to VSO($\mathbb{N}$).

Results

The main of the work is to extend this description to the case of weighted Bergman spaces over the unit ball $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, where the weight parameter $\lambda \in (-1, \infty)$. The development of the dissertation is stated in the following paragraphs.

Chapter 1 and Chapter 2 are based on the joint work with Wolfram Bauer and Nikolai Vasilevski.

Chapter 1 it is about the (m, λ)-Berezin transform

$$B_{m,\lambda} \colon \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n)) \to C^\infty(\mathbb{B}^n),$$

which is defined by Daniel Suárez for the Bergman space over the unit disc. Nam, Zheng and Zhong extend this definition to the unit ball.

So, in this chapter we present the weighted generalization of two approximation theorems, i.e., we establish conditions under which the convergence of the sequence

$$T_{B_{m,\lambda}(S)} \to S1$$

happens in operator norm.

For the first Theorem 1.3.7, the Schur test is used for guarantee convergence in operator norm (Lema 1.3.2).

The second Theorem 1.3.9, uses the invariant Laplacian (1.29).

In Chapter 2 we consider radial operators and its eigenvalue sequences. A bounded radial operator S belonging to the Toeplitz algebra satisfies Theorems 1.3.7 and 1.3.9. It is
$$T_{B_{m,\lambda}(S)} \to S$$
in operator norm.

Definition 2.1.1 established S is radial if and only if $S = Rad(S)$.

In Lemma 2.1.6 it is shown radialization commutes with the (m, λ)-Berezin transform, i.e.,
$$Rad \circ B_{m,\lambda}(S) = B_{m,\lambda} \circ Rad(S)$$

This way $B_{m,\lambda}(S)$ is radial and therefore $T_{B_{m,\lambda}}$ is radial. Particularly this shows that
$$Rad \cap \mathfrak{T} = \mathfrak{T}(\{T_a\colon\ a \text{ is bounded and radial}\}).$$

If $a \in L_\infty(0,1)$ is a radial symbol a straightforward calculation shows that the eigenvalue sequence of T_a is written as

$$\beta^{(n)}_{a,\lambda}(k) = \frac{1}{\mathrm{B}(n+k-1, \lambda+1)} \int_0^1 a(\sqrt{r}) r^{k+n-2}(1-r)^\lambda dr,$$

and Proposition 2.3.1 establishes $\beta^{(n)}_{a,\lambda} \in d_1$ and $\beta^{(n)}_{a,\lambda} \in d_2$.

In proposition 2.3.2 it is shown that if $\mu \in d_2$ then $S \in \mathcal{D}$ and the following bound holds
$$\|\tilde{\triangle}(S)\| \leq (6 + 4|\lambda|) \sup_n n^2 | \triangle_n^2 (\mu)|.$$

The most important result is Theorem 2.3.4, which establishes the ℓ^∞-norm closure of $\mathrm{B}^{(n)}_\lambda$ is equal to VSO($\mathbb{N}$) with

$$\mathrm{B}^{(n)}_\lambda = \{\beta^{(n)}_{a,\lambda}\colon\ a \in L_\infty(0,1)\}.$$

Chapter 3 is based on the joint work with Ondrej Hutník, Egor Maximenko and Nikolai Vasilevski. In Chapter 3 the so called vertical Toeplitz operators are treated.

For $h \in \mathbb{R}$, $H_h\colon \mathcal{A}^2_\lambda(\Pi) \to \mathcal{A}^2_\lambda(\Pi)$ the shift operator is $H_h f(z) = f(z-h)$.

An operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\Pi))$ is vertical if and only if

$$H_h S = S H_h, \quad \forall h \in \mathbb{R}.$$

For the Toeplitz operator T_b, $b \in L_\infty(\Pi)$, T_b is vertical if and only if $b(z) = b(\mathrm{Im}(z))$.

According to [39, Theorem 3.1], $\mathcal{A}^2_\lambda(\Pi)$ is isometrically isomorphic to $L_2(\mathbb{R}_+)$ by means of a unitary operator R. This way the vertical Toeplitz operator T_b is unitarily equivalent to a multiplication operator $\gamma_{b,\lambda} I$ acting on $L_2(\mathbb{R}_+)$, i.e.,

$$R^* T_b R = \gamma_{b,\lambda} I, \quad \text{where}$$

$$\gamma_{b,\lambda}(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty b(y)\, \mathrm{e}^{-xy}\, y^\lambda dy.$$

The purpose of this chapter is to describe the C^*-algebra generated by

$$\Gamma_\lambda = \{T_b\colon\ b \in L_\infty(\mathbb{R}_+)\}.$$

To this end we use similar techniques to those applied in the radial case. In order to consider the corresponding (m,λ)-Berezin transform for the vertical case, we introduce the functions sequence $\{\psi_{n,\lambda}\}_{n\in\mathbb{N}}$ which is an approximation to the identity. In explicit form

$$\psi_{n,\lambda}(x) = \frac{1}{\mathrm{B}(n+\lambda, n+\lambda)} \frac{x^{n+\lambda}}{(1+x)^{2(n+\lambda)}}, \quad x \in \mathbb{R}.$$

This sequence can be obtained from the usual Berezin transform with some modifications in order to facilitate calculations. Here we take advantage of the multiplicative group structure of $\mathbb{R}_+$.

The logarithmic metric $\rho\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\rho(x,y) = |\ln(x) - \ln(y)|.$$

In fact we obtain that Γ_λ is isometrically isomorphic to

$$\begin{gathered}\mathrm{VSO}(\mathbb{R}_+) = \{f \in L_\infty(\mathbb{R}_+)\colon\ f \text{ is uniformlly continuous}\\ \text{with respect to the logarithmic metric}\}.\end{gathered}$$

Equivalently

$$\mathrm{VSO}(\mathbb{R}_+) = \left\{ f \in L_\infty(\mathbb{R}_+)\colon\ \lim_{\frac{x}{y}\to 1} |f(x) - f(y)| \to 0 \right\}.$$

This way we give a characterization of the C^*-algebra generated by vertical Toeplitz operators by means of spectral functions. Similarly to the radial case, note this description is better than $\overline{d_1}^{\ell_\infty}$−type description. It is easier to check if a sequence belongs to $\mathrm{VSO}(\mathbb{R}_+)$.

Using known results for shift invariant operators it is shown that a bounded operator T is vertical if there exist $\sigma \in L_\infty(\mathbb{R}_+)$ such that

$$R^*TR = M_\sigma.$$

Chapter 4 is a joint work with Egor Maximenko and Nikolai Vasilevski. In Chapter 4 we show the relation among radial and vertical cases. To this point we have that the C^*-algebra generated by

$$\Gamma_\lambda = \{\gamma_{b,\lambda}\colon b \in L_\infty(\mathbb{R}_+)\}$$

is equal to $\mathrm{VSO}(\mathbb{R}_+)$.

We also have that the C^*-algebra generated by

$$\mathrm{B}_\lambda^{(n)} = \{\beta_{a,\lambda}^{(n)} \colon\ a \in L_\infty(0,1)\}$$

is equal to $\mathrm{VSO}(\mathbb{N})$.

In the calculation of the sequence $\beta_{a,\lambda}^{(n)}$ (some $a \in L_\infty(0,1)$) we apply a discrete analogous of the Laplace transform. Because of the appearance of the Laplace transform in the formula for $\gamma_{b,\lambda}$ (some $b \in L_\infty(\mathbb{R}_+)$) and the description for the C^*-algebras above, one might expect a relation between radial and vertical symbols.

We establish the relation among radial and vertical symbols in Lemma 4.2.1 So we give a description of the radial case by means of vertical case techniques. This is done without using d_1 and d_2 sets neither the (m,λ)-Berezin transform. The main result again is Theorem 4.2.5: the ℓ_∞-closure of

$$\{\beta_{a,\lambda}^{(n)} \colon\ a \in L_\infty(0,1)\}$$

is equal to $\mathrm{VSO}(\mathbb{N})$.

Chapter 1

(m, λ)-Berezin transform and approximation of operators on weighted Bergman spaces over the unit ball.

1.1 Preliminaries

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z|^2 := |z_1|^2 + \cdots + |z_n|^2 < 1\}$ be the open unit ball in $\mathbb{C}^n$ equipped with the standard weighted measure

$$dv_\lambda(z) = c_\lambda(1 - |z|^2)^\lambda dv(z), \tag{1.1}$$

where $\lambda > -1$ is fixed. Here c_λ is given by

$$c_\lambda := \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}, \tag{1.2}$$

so that $v_\lambda(\mathbb{B}^n) = 1$. We write $L_2(\mathbb{B}^n, dv_\lambda)$ for the Hilbert space of all functions that are square-integrable with respect to dv_λ. The corresponding norm and inner product are denoted by $\|\cdot\|_\lambda$ and $\langle \cdot, \cdot \rangle_\lambda$, respectively.

Let $\mathbb{Z}_+ := \{0, 1, \cdots\}$ be the set of non-negative integers. With $\alpha \in \mathbb{Z}_+^n$ we use the standard notations $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\alpha! := \alpha_1! \cdots \alpha_n!$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

By a straightforward calculation one verifies that

$$\|w^\alpha\|_\lambda = \sqrt{\frac{\alpha! \Gamma(n + \lambda + 1)}{\Gamma(n + |\alpha| + \lambda + 1)}}. \tag{1.3}$$

The Bergman (orthogonal) projection $\mathbf{B}_\lambda$ from $L_2(\mathbb{B}^n, dv_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ can be expressed as an integral operator in the explicit form

$$\big[\mathbf{B}_\lambda\varphi\big](z) = \int_{\mathbb{B}^n} \frac{\varphi(w)}{(1-\langle z,w\rangle)^{n+\lambda+1}} dv_\lambda(w) \qquad \text{with} \qquad \varphi \in L_2(\mathbb{B}^n, dv_\lambda),$$

where $\langle z,w\rangle := z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ denotes the Euclidean inner product on $\mathbb{C}^n$. The reproducing kernel of the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is given by

$$K^\lambda_z(w) = \frac{1}{(1-\langle w,z\rangle)^{n+\lambda+1}} = \sum_{|\alpha|=0}^{\infty} \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} \overline{z}^\alpha w^\alpha. \tag{1.4}$$

We frequently use the normalized version of the Bergman kernel and write

$$k^\lambda_z(w) = \frac{K^\lambda_z(w)}{\|K^\lambda_z\|_\lambda} = \frac{(1-|z|^2)^{\frac{n+\lambda+1}{2}}}{(1-\langle w,z\rangle)^{n+\lambda+1}}.$$

By $\phi_z(w)$ denote a biholomorphism of $\mathbb{B}^n$ that interchanges 0 and z. More precisely, we choose the explicit form of $\phi_z(w)$ given, for example, in [46, p.5] such that $\phi_0(w) = -w$. Recall [46, p.37] that the complex Jacobian $\det(\phi_z')$ of ϕ_z has the form

$$\det(\phi_z'(w)) = (-1)^n \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-\langle w,z\rangle)^{n+1}} = (-1)^n k^0_z(w).$$

It is standard that the kernel K^λ_z transforms under the biholomorphisms ϕ_u as

$$K^\lambda_z(w) = \overline{k^\lambda_u(z)} K^\lambda_{\phi_u(z)}\big(\phi_u(w)\big) k^\lambda_u(w). \tag{1.5}$$

Given $z \in \mathbb{B}^n$ we introduce the unitary operator U_z on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ which acts as the weighted composition

$$\begin{aligned}\big(U_z f\big)(w) :=& \big[\det(\phi_z'(w))\big]^{\frac{n+\lambda+1}{n+1}} (f\circ\phi_z)(w)\\ =& (-1)^{\frac{n(n+\lambda+1)}{n+1}} \frac{(1-|z|^2)^{\frac{n+\lambda+1}{2}}}{(1-\langle w,z\rangle)^{n+\lambda+1}} (f\circ\phi_z)(w)\\ =& (-1)^{\frac{n(n+\lambda+1)}{n+1}} k^\lambda_z(w)\cdot f\circ\phi_z(w).\end{aligned}$$

It is easy to check that U_z is self-adjoint and so $U_z^2 = I$. Since ϕ_0 induces a reflection at the origin we have

$$\big(U_0 f\big)(w) = (-1)^{\frac{n(n+\lambda+1)}{n+1}} f(-w).$$

If we fix $z \in \mathbb{B}^n$, then we can define an automorphism on the algebra $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ of all bounded operator on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ by

$$\mathcal{L}\big(\mathcal{A}^2_\lambda(\mathbb{B}^n)\big) \ni S \;\longmapsto\; S_z := U_z S U_z \in \mathcal{L}\big(\mathcal{A}^2_\lambda(\mathbb{B}^n)\big). \tag{1.6}$$

In particular, if $S = T_a$ is a Toeplitz operator it can be verified that $(T_a)_z = T_{a\circ\phi_z}$.

Finally, we introduce a convenient convention for simplifying the notations. In various of estimates throughout this chapter we will denote by C a positive constant whose value may change from place to place.

1.2 The (m,λ)-Berezin transform

Recall that the m-Berezin transform for the unweighted Bergman space over the unit disk and over the unit ball were defined in [35] and [29], respectively. In the case where $\lambda \neq 0$ the notion of the (k,α)-Berezin transform for measures on the weighted p-Bergman space over $\mathbb{B}^n$ was introduced in [28].

A generalization of the concept of the m-Berezin transform to an arbitrary bounded operator on the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ requires a modification of the definition in [28]. We will follow the recipe in [29] and first introduce some notation. Put

$$C_{m,\alpha} := \binom{m}{|\alpha|}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha_1!\cdots\alpha_n!}, \tag{1.7}$$

so that

$$\sum_{|\alpha|=0}^{m} C_{m,\alpha}\, z^\alpha \overline{w}^\alpha = (1-\langle z,w\rangle)^m. \tag{1.8}$$

Definition 1.2.1. For any $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$, we define its (m,λ)-*Berezin transform* by

$$(B_{m,\lambda}S)(z) := \frac{c_{\lambda+m}}{c_\lambda}\sum_{|\alpha|=0}^{m} C_{m,\alpha}\left\langle S_z w^\alpha, w^\alpha\right\rangle_\lambda. \tag{1.9}$$

Note that a direct application of the Cauchy-Schwarz inequality gives the following pointwise estimate

$$\big|(B_{m,\lambda}S)(z)\big| \le \|S\|\frac{c_{\lambda+m}}{c_\lambda}\sum_{|\alpha|=0}^{m}\big|C_{m,\alpha}\big|\|w^\alpha\|^2_\lambda =: C(\lambda,m,n)\,\|S\|,$$

where the constant $C(\lambda,m,n) > 0$ is independent of $z \in \mathbb{B}^n$. That is, $B_{m,\lambda}S$ is a bounded function on $\mathbb{B}^n$ with

$$\|B_{m,\lambda}S\|_\infty \le C(\lambda,m,n)\,\|S\|. \tag{1.10}$$

As usual we can define the (m,λ)-Berezin transform of a functions $a \in L_\infty(\mathbb{B}^n)$ by

$$
\begin{aligned}
B_{m,\lambda}(a)(z) &:= B_{m,\lambda}(T_a)(z) = \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \left\langle (a \circ \phi_z) w^\alpha, w^\alpha \right\rangle_\lambda \\
&= \frac{c_{\lambda+m}}{c_\lambda} \int_{\mathbb{B}^n} (a \circ \phi_z)(w)\, c_\lambda (1-|w|^2)^{\lambda+m} dv(w) \\
&= \int_{\mathbb{B}^n} (a \circ \phi_z)(w)\, dv_{\lambda+m}(w). \qquad (1.11)
\end{aligned}
$$

As was mentioned earlier Definition 1.2.1 is different from the one in [28], where the (m,λ)-Berezin transform $\widetilde{B}_{m,\lambda}$ for finite, complex valued, regular measures ν on $\mathbb{B}^n$ was introduced. In fact, in the special case of $\nu := a dv_\lambda$ with $a \in L_\infty(\mathbb{B}^n)$ the last one gives

$$\widetilde{B}_{m,\lambda}(\nu)(z) = \int_{\mathbb{B}^n} (a \circ \phi_z)(w)\, dv_m(w),$$

where the different from (1.11) right hand side is independent of the weight parameter λ. This seems to be inadequate as the initial data (measures and, more generally, operators) are defined on the specific weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

The next two propositions give alternative formulas for the (m,λ)-Berezin transform that, from time to time, are more suitable to work with. Note that the formula of the second proposition, in the particular case when $n = 1$ and $\lambda = 0$, coincides with the definition of the m-Berezin transform on the unit disk by Suárez [35].

Proposition 1.2.2. *Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$, $m \geq 0$ and $z \in \mathbb{B}^n$. Then*

$$
\begin{aligned}
\left(B_{m,\lambda} S\right)(z) = \frac{c_{\lambda+m}}{c_\lambda} \left(1-|z|^2\right)^{m+\lambda+n+1} \times \\
\int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \left(1-\langle u,w\rangle\right)^m K_z^{m+\lambda}(u) \overline{K_z^{m+\lambda}(w) S^* K_w^\lambda(u)} dv_\lambda(u) dv_\lambda(w).
\end{aligned}
$$

Proof. We have

$$(B_{m,\lambda}S)(z) = \frac{c_{\lambda+m}}{c_\lambda}\sum_{|\alpha|=0}^{m} C_{m,\alpha}\langle S_z w^\alpha, w^\alpha\rangle_\lambda$$

$$= \frac{c_{\lambda+m}}{c_\lambda}\sum_{|\alpha|=0}^{m} C_{m,\alpha}\int_{\mathbb{B}^n} S\left(\phi_z^\alpha k_z^\lambda\right)(w)\overline{\phi_z^\alpha(w)k_z^\lambda(w)}dv_\lambda(w) \tag{1.12}$$

$$= \frac{c_{\lambda+m}}{c_\lambda}\sum_{|\alpha|=0}^{m} C_{m,\alpha}\times \tag{1.13}$$
$$\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\phi_z^\alpha(u)k_z^\lambda(u)\overline{\phi_z^\alpha(w)k_z^\lambda(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w).$$

In the last equality we use that

$$S\left(\phi_z^\alpha k_z^\lambda\right)(w) = \langle S\left(\phi_z^\alpha k_z^\lambda\right), K_w^\lambda\rangle_\lambda = \langle \phi_z^\alpha k_z^\lambda, S^*K_w^\lambda\rangle_\lambda.$$

Then, by (1.8) and (1.5), the expression (1.12) equals to

$$\frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\left(1-\langle\phi_z(u),\phi_z(w)\rangle\right)^m k_z^\lambda(u)\overline{k_z^\lambda(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w)$$
$$= \frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\left(\frac{k_z^\lambda(u)\overline{k_z^\lambda(w)}}{K_w^\lambda(u)}\right)^{\frac{m}{\lambda+n+1}} k_z^\lambda(u)\overline{k_z^\lambda(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w)$$
$$= \frac{c_{\lambda+m}}{c_\lambda}\left(1-|z|^2\right)^{m+\lambda+n+1}\times$$
$$\times\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\left(1-\langle u,w\rangle\right)^m K_z^{m+\lambda}(u)\overline{K_z^{m+\lambda}(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w),$$

which finishes the proof. □

Proposition 1.2.3. *Let $S\in\mathcal{L}\left(\mathcal{A}_\lambda^2(\mathbb{B}^n)\right)$, $m\geq 0$ and $z\in\mathbb{B}^n$. Then*

$$\left(B_{m,\lambda}S\right)(z) = \frac{c_{\lambda+m}}{c_\lambda}\left(1-|z|^2\right)^{m+\lambda+n+1}\sum_{|\alpha|=0}^{m} C_{m,\alpha}\left\langle S(w^\alpha K_z^{m+\lambda}), w^\alpha K_z^{m+\lambda}\right\rangle_\lambda. \tag{1.14}$$

Proof. We have

$$\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\left(1-\langle u,w\rangle\right)^m K_z^{m+\lambda}(u)\overline{K_z^{m+\lambda}(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w)$$
$$= \sum_{|\alpha|=0}^{m} C_{m,\alpha}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n} u^\alpha\overline{w^\alpha}K_z^{m+\lambda}(u)\overline{K_z^{m+\lambda}(w)S^*K_w^\lambda(u)}dv_\lambda(u)dv_\lambda(w)$$
$$= \sum_{|\alpha|=0}^{m} C_{m,\alpha}\int_{\mathbb{B}^n} S(u^\alpha K_z^{m+\lambda})(w)\overline{w^\alpha K_z^{m+\lambda}(w)}dv_\lambda(w).$$

Thus the result follows from Proposition 1.2.2. □

Lemma 1.2.4. *Given $z, w \in \mathbb{B}^n$ the automorphism $\mathcal{U} := \phi_{\phi_w(z)} \circ \phi_w \circ \phi_z$ of $\mathbb{B}^n$ extends to a unitary transformation of $\mathbb{C}^n$ and*

$$U_z U_w = V_{\mathcal{U}} U_{\phi_w(z)},$$

where the operator $V_{\mathcal{U}}$ is given by

$$\big(V_{\mathcal{U}} f\big)(u) = (\det \mathcal{U})^{\frac{n+\lambda+1}{n+1}} \cdot f\big(\mathcal{U}u\big).$$

Proof. Since $\mathcal{U}$ in an automorphism of the unit ball having 0 as a fixed point it follows by the Cartan theorem that $\mathcal{U}$ acts by multiplication with a unitary matrix. This matrix will also be denoted by $\mathcal{U}$, i.e., $\mathcal{U}(u) = \mathcal{U}u$.

Differentiating the equality $\phi_{\phi_w(z)} \circ \mathcal{U} = \phi_w \circ \phi_z$ we have

$$\phi'_{\phi_w(z)}(\mathcal{U}(u))\, \mathcal{U}'(u) = \phi'_w(\phi_z(u))\, \phi'_z(u),$$

which implies

$$(-1)^n k^0_{\phi_w(z)}(\mathcal{U}u) \det \mathcal{U} = (-1)^n k^0_w(\phi_z(u)) \cdot (-1)^n k^0_z(u).$$

As $k^\lambda_z = (k^0_z)^{\frac{n+\lambda+1}{n+1}}$ and $(U_z f)(w) = (-1)^{\frac{n(n+\lambda+1)}{n+1}} k^\lambda_z(w) \cdot (f \circ \phi_z)(w)$, the application of the last formula gives

$$\begin{aligned}
(U_z U_w f)(u) &= k^\lambda_z(u) \cdot k^\lambda_w(\phi_z(u)) \cdot (f \circ \phi_w \circ \phi_z)(u) \\
&= (\det \mathcal{U})^{\frac{n+\lambda+1}{n+1}} \cdot (-1)^{\frac{n(n+\lambda+1)}{n+1}} k^\lambda_{\phi_w(z)}(\mathcal{U}u) \cdot (f \circ \phi_{\phi_w(z)} \circ \mathcal{U})(u) \\
&= (V_{\mathcal{U}} U_{\phi_w(z)} f)(u).
\end{aligned}$$

Note that $(\det \mathcal{U})^{\frac{n+\lambda+1}{n+1}}$ is a complex number of modulus one. □

Theorem 1.2.5. *Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$, $m \geq 0$ and $z \in \mathbb{B}^n$. Then $B_{m,\lambda} S_z = (B_{m,\lambda} S) \circ \phi_z$.*

Proof. By definition

$$\begin{aligned}
(B_{m,\lambda} S_z)(0) &= \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \langle U_0 S_z U_0 w^\alpha, w^\alpha \rangle_\lambda \\
&= \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \langle S_z (-w)^\alpha, (-w)^\alpha \rangle_\lambda \\
&= \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \langle S_z w^\alpha, w^\alpha \rangle_\lambda = B_{m,\lambda} S(z) = (B_{m,\lambda} S) \circ \phi_z(0).
\end{aligned}$$

For any $\eta \in \mathbb{B}^n$, by Proposition 1.2.2 and Lemma 1.2.4 we have

$$
\begin{aligned}
&(B_{m,\lambda}S_z)\circ\phi_\eta(0)\\
&= B_{m,\lambda}((S_z)_\eta)(0)\\
&= \frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}(1-\langle u,w\rangle)^m\,\overline{((S_z)_\eta)^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w)\\
&= \frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}(1-\langle u,w\rangle)^m\,\overline{U_\eta U_z S^* U_z U_\eta K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w)\\
&= \frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}(1-\langle u,w\rangle)^m\,\overline{V_{\mathcal{U}}U_{\phi_z(\eta)}S^*U_{\phi_z(\eta)}V_{\mathcal{U}}^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w)\\
&= B_{m,\lambda}S_{\phi_{z(\eta)}}(0),
\end{aligned}
$$

where $V_{\mathcal{U}}$ is the unitary operator of Lemma 1.2.4. This implies the lemma statement. □

Next two lemmas are preparatory for Proposition 1.2.8, which states the commutativity of the (m,λ)-Berezin transforms for different values of the parameter m.

Lemma 1.2.6. *Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$ and $m, j \geq 0$. If $|S^*K_z^\lambda(w)| \leq C$ for any $w \in \mathbb{B}^n$. Then*

$$(B_{m,\lambda}B_{j,\lambda})(S) = (B_{j,\lambda}B_{m,\lambda})(S).$$

Proof. Due to Theorem 1.2.5, we need to check only that $(B_{m,\lambda}B_{j,\lambda})S(0) = (B_{j,\lambda}B_{m,\lambda})S(0)$.
Property (1.11), Proposition 1.2.2, and Fubini theorem imply that

$$
\begin{aligned}
&B_{m,\lambda}(B_{j,\lambda}S)(0) = B_{m,\lambda}(T_{B_{j,\lambda}S})(0)\\
&= c_{m+\lambda}\int_{\mathbb{B}^n} B_{j,\lambda}S(z)(1-|z|^2)^{m+\lambda}dv(z)\\
&= \int_{\mathbb{B}^n}\frac{c_{m+\lambda}c_{j+\lambda}}{c_\lambda}(1-|z|^2)^{m+j+2\lambda+n+1}\times\\
&\quad\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}(1-\langle u,w\rangle)^j K_z^{j+\lambda}(u)\overline{K_z^{j+\lambda}(w)S^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w)dv(z)\\
&= \int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\frac{c_{m+\lambda}c_{j+\lambda}}{c_\lambda}(1-|z|^2)^{m+j+2\lambda+n+1}(1-\langle u,w\rangle)^j K_z^{j+\lambda}(u)\times\\
&\quad\overline{K_z^{j+\lambda}(w)S^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w)dv(z)\\
&= \int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\frac{c_{m+\lambda}c_{j+\lambda}}{c_\lambda}(1-\langle u,w\rangle)^j\times\\
&\quad\int_{\mathbb{B}^n}(1-|z|^2)^{m+j+2\lambda+n+1}K_z^{j+\lambda}(u)\overline{K_z^{j+\lambda}(w)}dv(z)\overline{S^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w).
\end{aligned}
$$

Introduce

$$F_{m,j}(u,w) = (1-\langle u,w\rangle)^j \int_{\mathbb{B}^n} (1-|z|^2)^{m+j+2\lambda+n+1} K_z^{j+\lambda}(u)\overline{K_z^{j+\lambda}(w)}dv(z),$$

and observe that it can be represented as a finite sum

$$F_{m,j}(u,w) = \sum_{i=1}^{l} H_i(u)\overline{G_i(w)}$$

for certain holomorphic functions H_i and G_i. By [8, Lemma 9], it is sufficient to show that $F_{m,j}(w,w) = F_{j,m}(w,w)$, where $w \in \mathbb{B}^n$, which can be easily verified by changing the variables:

$$\begin{aligned}
&F_{m,j}(w,w)\\
&= (1-|w|^2)^j \int_{\mathbb{B}^n} (1-|z|^2)^{m+j+2\lambda+n+1} |K_z^{j+\lambda}(w)|^2 dv(z)\\
&= (1-|w|^2)^j \int_{\mathbb{B}^n} (1-|\phi_w(z)|^2)^{m+j+2\lambda+n+1} |K_w^{j+\lambda}(\phi_w(z))|^2 |k_w^0(z)|^2 dv(z)\\
&= (1-|w|^2)^m \int_{\mathbb{B}^n} (1-|z|^2)^{m+j+2\lambda+n+1} |K_z^{m+\lambda}(w)|^2 dv(z)\\
&= F_{j,m}(w,w).
\end{aligned}$$

□

Denote by $S_1 = S_1(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ the set of all trace class operators acting on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$. Given $A \in S_1$, we write tr$[A]$ for its trace, and recall that the trace norm of A is given by

$$\|A\|_{S_1} := \text{tr}\left[\sqrt{A^*A}\right].$$

Given $f, g \in \mathcal{A}_\lambda^2(\mathbb{B}^n)$, the one-dimensional operator $f \otimes g$, acting on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ by the formula $(f \otimes g)h = \langle h, g\rangle_\lambda f$ obviously belongs to S_1. Furthermore,

$$\|f \otimes g\|_{S_1} = \|f\|_\lambda \cdot \|g\|_\lambda$$

and tr $[f \otimes g] = \langle f, g\rangle_\lambda$. Recall as well that if $A \in S_1$ has rank m, then one has the inequality

$$\|A\|_{S_1} \le \sqrt{m}\left(\text{tr }[\sqrt{A^*A}]\right)^{\frac{1}{2}}.$$

Lemma 1.2.7. *For any $S \in \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$, there exists sequences $\{S_\alpha\}$, satisfying the property*

$$|S_\alpha^* K_z^\lambda(w)| \le C(\alpha), \tag{1.15}$$

such that $B_{m,\lambda}S_\alpha$ converges to $B_{m,\lambda}S$ point-wise.

Proof. Both the density of H^∞ in $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ and the density of finite rank operators in the ideal $\mathcal{K}$ of compact operators on $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ imply that the set

$$\mathcal{F} := \left\{ \sum_{j=1}^{l} f_j \otimes g_j : f_j, g_j \in H^\infty \right\}$$

is dense in the ideal $\mathcal{K}$ in the norm topology. At the same time the ideal $\mathcal{K}$ is dense in $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ with respect to the strong operator topology. Thus, for each $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$ there exists a sequence $\{S_\alpha\}$ of finite rank operators

$$S_\alpha = \sum_{j=1}^{l(\alpha)} f_{\alpha,j} \otimes g_{\alpha,j}$$

converging strongly to S. The representation (1.14) shows that $B_{m,\lambda}S_\alpha$ converges to $B_{m,\lambda}S$ point-wise. To finish the proof we estimate

$$\begin{aligned} |S_\alpha^* K_z^\lambda(w)| &= \left| \sum_{j=1}^{l(\alpha)} (g_{\alpha,j} \otimes f_{\alpha,j}) K_z^\lambda(w) \right| = \left| \sum_{j=1}^{l(\alpha)} \langle K_z^\lambda(w), f_{\alpha,j}(w) \rangle_\lambda g_{\alpha,j}(w) \right| \\ &\leq \sum_{j=1}^{l(\alpha)} |f_{\alpha,j}(z)||g_{\alpha,j}(w)| \leq \sum_{j=1}^{l(\alpha)} \|f_{\alpha,j}\|_\infty \|g_{\alpha,j}\|_\infty < C(\alpha). \end{aligned}$$

□

Proposition 1.2.8. *Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$ and $m, j \geq 0$. Then*

$$\left(B_{m,\lambda} B_{j,\lambda}\right)(S) = \left(B_{j,\lambda} B_{m,\lambda}\right)(S).$$

Proof. Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$. By Lemma 1.2.7 there exists a sequence $\{S_\alpha\}$ of operators that satisfy (1.15) and the point-wise convergence $B_{m,\lambda}S_\alpha \to B_{m,\lambda}S$ holds. Lemma 1.2.6 implies that

$$B_{m,\lambda}(B_{j,\lambda}S_\alpha)(z) = B_{j,\lambda}(B_{m,\lambda}S_\alpha)(z). \tag{1.16}$$

By representation (1.11),

$$B_{m,\lambda}(B_{j,\lambda}S_\alpha)(z) = \int_{\mathbb{B}^n} (B_{j,\lambda}S_\alpha) \circ \phi_z(u) dv_{m+\lambda}(u).$$

Then, as the sequence $\{S_\alpha\}$ converges in the strong operator topology to S, by its construction, we have

$$\|(B_{j,\lambda}S_\alpha) \circ \phi_z\|_\infty = \|(B_{j,\lambda}S_\alpha)\|_\infty \leq \|B_{j,\lambda}\| \cdot \|S_\alpha\| \leq C(j,\lambda) \cdot \|S\|.$$

Furthermore $(B_{j,\lambda}S_\alpha) \circ \phi_z(u)$ converges to $(B_{j,\lambda}S) \circ \phi_z(u)$, thus $B_{m,\lambda}(B_{j,\lambda}S_\alpha)(z)$ converges to $B_{m,\lambda}(B_{j,\lambda}S)(z)$. Analogously, $B_{j,\lambda}(B_{m,\lambda}S_\alpha)(z)$ converges to $B_{j,\lambda}(B_{m,\lambda}S)(z)$. Thus passing to the limit in (1.16) finishes the proof. □

Corollary 1.2.9. *For all $\lambda > -1$ and $m \in \mathbb{Z}_+$ the (m,λ)-Berezin transform is one-to-one on bounded operators on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.*

Proof. Since $B_{m,\lambda}$ restricted to functions coincides with the usual Berezin transform on $\mathcal{A}^2_{\lambda+m}(\mathbb{B}^n)$ (cf. (1.11)) it is one-to-one on functions (or Toeplitz operators). Now assume that $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ such that $B_{m,\lambda}(S) \equiv 0$. Then we obtain from Proposition 1.2.8 that

$$0 = B_{0,\lambda}B_{m,\lambda}(S) = B_{m,\lambda}B_{0,\lambda}(S)$$

and from the last remark we see that $B_{0,\lambda}(S) \equiv 0$. Since $B_{0,\lambda}$ is known to be one-to-one on bounded operators over $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ we conclude that $S = 0$, which finishes the proof. □

Recall that the pseudo-hyperbolic metric on the unit ball is defined as

$$\rho(z,w) := |\phi_z(w)| = |\phi_w(z)|.$$

As is well known $\rho(\cdot,\cdot)$ is invariant under the automorphisms ϕ_u of $\mathbb{B}^n$. The next result shows the Lipschitz continuity of $B_{0,\lambda}S$ with respect to this metric.

Theorem 1.2.10. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$. Then there exists a constant $C(n,\lambda) > 0$ such that*

$$|B_{0,\lambda}S(z) - B_{0,\lambda}S(w)| \leq C(n,\lambda)\|S\|\,\rho(z,w).$$

Proof. By definition and the above properties of the trace class operators we have

$$\begin{aligned}|B_{0,\lambda}S(z) - B_{0,\lambda}S(w)| &= |\langle S_z 1, 1\rangle_\lambda - \langle S_w 1, 1\rangle_\lambda| \\ &= |\mathrm{tr}\ [S_z(1\otimes 1)] - tr[S_w(1\otimes 1)]| \\ &= |\mathrm{tr}\ [S_z(1\otimes 1) - SU_w(1\otimes 1)U_w]| \\ &= |\mathrm{tr}\ [S_z(1\otimes 1) - SU_z(U_zU_w1\otimes U_zU_w1)U_z]| = D.\end{aligned}$$

By Lemma 1.2.4,

$$\begin{aligned}|B_{0,\lambda}S(z) - B_{0,\lambda}S(w)| = D &< \|S_z\|\|1\otimes 1 - U_{\phi_w(z)}1\otimes U_{\phi_w(z)}1\|_{S_1} \\ &\leq \sqrt{2}\|S_z\|\big(2 - 2|\langle 1, k^\lambda_{\phi_w(z)}\rangle_\lambda|^2\big)^{1/2} \\ &= 2\|S\|\big[1-(1-|\phi_w(z)|^2)^{n+\lambda+1}\big]^{1/2} \\ &\leq C(n,\lambda)\|S\||\phi_w(z)|,\end{aligned}$$

which according to the definition of the pseudo-hyperbolic metric shows the result □

Now representation (1.11) yields

Corollary 1.2.11. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$, and $a(z) = B_{0,\lambda}S(z)$. Then*

$$\lim_{m\to\infty} \|B_{m,\lambda}(a) - a\|_\infty = 0.$$

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ with $|a(z) - a(w)| < \varepsilon$ whenever $z, w \in \mathbb{B}^n$ with $\rho(z,w) < \delta$. If $w \in \mathbb{B}^n$ and $m \in \mathbb{N}$, then according to (1.11) we have that

$$\begin{aligned}
&|B_{m,\lambda}(a)(w) - a(w)| \leq \\
\leq& c_{\lambda+m} \int_{\mathbb{B}^n} \big|a\circ\phi_w(z) - a\circ\phi_w(0)\big|(1-|z|^2)^{\lambda+m} dv(z)\\
\leq& c_{\lambda+m} \left\{ \int_{0\leq|z|<\delta} + \int_{1>|z|\geq\delta} \right\} \big|a\circ\phi_w(z) - a\circ\phi_w(0)\big|(1-|z|^2)^{\lambda+m} dv(z).
\end{aligned}$$

Since $\rho(\cdot,\cdot)$ is invariant under the automorphisms ϕ_w and $\rho(z,0) < |z|$ (see, for example, [46, page 28]), we have $\rho(\phi_w(z), \phi_z(0)) = \rho(z,0) < \delta$ in the first integral, and therefore by the Lipschitz continuity of a:

$$c_{\lambda+m} \int_{0\leq|z|<\delta} \big|a\circ\phi_w(z) - a\circ\phi_w(0)\big|(1-|z|^2)^{\lambda+m} dv(z) < \varepsilon. \tag{1.17}$$

Now, we estimate the second integral above.

$$\begin{aligned}
&c_{\lambda+m} \int_{1>|z|\geq\delta} \big|a\circ\phi_w(z) - a\circ\phi_w(0)\big|(1-|z|^2)^{\lambda+m} dv(z)\\
\leq& 2c_{\lambda+m}\|a\|_\infty \int_{1>|z|\geq\delta} (1-|z|^2)^{\lambda+m} dv(z)\\
\leq& 2c_{\lambda+m}\|a\|_\infty (1-\delta)^{\lambda+m} \mathrm{vol}(\mathbb{B}^n).
\end{aligned} \tag{1.18}$$

Since the normalizing constant $c_{\lambda+m}$ has at most polynomial growth as $m \to \infty$ (see the definition (1.2) and [13, Formula 8.328.2]) it is clear that the right hand side converges to zero as $m \to \infty$. The assertion follows by combining the estimates (1.17) and (1.18). □

1.3 Approximation by Toeplitz operators

We start this section with a technical statement which is due to [30, Proposition 1.4.10] and also stated as Lemma 3.1 in [29].

Lemma 1.3.1. *Suppose $a < 1$ and $a + b < n + 1$. Then*

$$\sup_{z\in\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{dv(w)}{(1-|w|^2)^a\,|1-\langle w, z\rangle|^b} < \infty.$$

Let $1 < q < \infty$ and p be the conjugate exponent of q. Note that the inequality

$$q = 1 + \frac{1}{p-1} < \frac{n+2(1+\lambda)}{n+1+\lambda} = 1 + \frac{1+\lambda}{n+1+\lambda} =: R \tag{1.19}$$

is equivalent to

$$p > 2 + \frac{n}{1+\lambda}.$$

In what follows we use the norm $\|\cdot\|_{p,\lambda}$, which is defined in the standard way,

$$\|f\|_{p,\lambda} = \left(\int_{\mathbb{B}^n} |f(z)|^p dv_\lambda(z)\right)^{\frac{1}{p}}.$$

Lemma 1.3.2. *Let $S \in \mathcal{L}\left(\mathcal{A}^2_\lambda(\mathbb{B}^n)\right)$, $p > \frac{n}{1+\lambda} + 2$, and let $h(z) = (1-|z|^2)^{-a}$ with*

$$a = \frac{(1+\lambda)(n+1+\lambda)}{n+2(1+\lambda)} = \frac{1+\lambda}{R}.$$

Then there exists $C(n,p,\lambda) > 0$ such that

$$\int_{\mathbb{B}^n} |(SK_z^\lambda)(w)| h(w) dv_\lambda(w) \le C(n,p,\lambda)\|S_z 1\|_{p,\lambda} h(z), \tag{1.20}$$

for all $z \in \mathbb{B}^n$, and

$$\int_{\mathbb{B}^n} |(SK_z^\lambda)(w)| h(z) dv_\lambda(z) \le C(n,p,\lambda)\|S_w^* 1\|_{p,\lambda} h(w), \tag{1.21}$$

for all $w \in \mathbb{B}^n$.

Proof. Given $z \in \mathbb{B}^n$, the equality

$$U_z 1 = (1-|z|^2)^{\frac{n+\lambda+1}{2}} K_z^\lambda,$$

implies

$$\begin{aligned} SK_z^\lambda &= \frac{1}{(1-|z|^2)^{\frac{n+\lambda+1}{2}}} SU_z 1 \\ &= \frac{1}{(1-|z|^2)^{\frac{n+\lambda+1}{2}}} U_z S_z 1 = (S_z 1 \circ \phi_z) K_z^\lambda. \end{aligned}$$

Then we change the variable $u = \phi_z(w)$ and apply the Hölder inequality:

$$\int_{\mathbb{B}^n} \frac{|(SK_z^\lambda)(w)|}{(1-|w|^2)^a} dv_\lambda(w)$$

$$
\begin{aligned}
&= c_\lambda \int_{\mathbb{B}^n} \frac{|S_z 1 \circ \phi_z(w)||K_z^\lambda(w)|(1-|w|^2)^\lambda}{(1-|w|^2)^a} dv(w) \\
&= \frac{1}{(1-|z|^2)^a} \int_{\mathbb{B}^n} \frac{|S_z 1(u)|}{|1-\langle u, z\rangle|^{n+\lambda+1-2a}(1-|u|^2)^a} dv_\lambda(u) \\
&\le \frac{\|S_z 1\|_{p,\lambda}}{(1-|z|^2)^a} \left(c_\lambda \int_{\mathbb{B}^n} \frac{dv(u)}{(1-|u|^2)^{aq-\lambda}|1-\langle u, z\rangle|^{(n+\lambda+1-2a)q}} \right)^{1/q}.
\end{aligned}
$$

According to (1.19) we have $aq-\lambda<1$ and $aq-\lambda+(n+\lambda+1-2a)q<n+1$, and inequality (1.20) follows from Lemma 1.3.1.

The second inequality (1.21) follows from (1.20) after replacing S by S^*, interchange w and z, and making use of the next equality

$$
\left(S^* K_w^\lambda\right)(z) = \left\langle S^* K_w^\lambda, K_z^\lambda \right\rangle_\lambda = \left\langle K_w^\lambda, S K_z^\lambda \right\rangle_\lambda = \overline{S K_z^\lambda}(w), \tag{1.22}
$$

which holds for all $z, w \in \mathbb{B}^n$. □

Lemma 1.3.3. *Let $S \in \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ and $p > 2 + \frac{n}{1+\lambda}$. Then*

$$
\|S\| \le C(n,p,\lambda) \left(\sup_{z \in \mathbb{B}^n} \|S_z 1\|_{p,\lambda} \right)^{1/2} \left(\sup_{z \in \mathbb{B}^n} \|S_z^* 1\|_{p,\lambda} \right)^{1/2},
$$

where $C(n,p,\lambda)$ is the constant of Lemma 1.3.2.

Proof. By (1.22) we have that

$$
\begin{aligned}
(Sf)(w) &= \left\langle Sf, K_w^\lambda \right\rangle_\lambda \\
&= \int_{\mathbb{B}^n} f(z) \overline{(S^* K_w^\lambda)}(z) dv_\lambda(z) \\
&= \int_{\mathbb{B}^n} f(z)(S K_z^\lambda)(w) dv_\lambda(z),
\end{aligned}
$$

for $f \in \mathcal{A}_\lambda^2(\mathbb{B}^n)$ and $w \in \mathbb{B}^n$. Now Lemma 1.3.2 and the Schur theorem (see, for example, [45, Corollary 3.2.3]) imply the result. □

Lemma 1.3.4. *Let $\{S_m\}$ be a bounded sequence in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ with*

$$
\lim_{m\to\infty} \|B_{0,\lambda} S_m\|_\infty = 0.
$$

Then

$$
\sup_{z \in \mathbb{B}^n} |\langle (S_m)_z 1, f \rangle_\lambda| \to 0
$$

as $m \to \infty$ for any $f \in \mathcal{A}_\lambda^2(\mathbb{B}^n)$, and

$$
\sup_{z \in \mathbb{B}^n} |(S_m)_z 1(\cdot)| \to 0 \tag{1.23}
$$

uniformly on compact subsets of $\mathbb{B}^n$ as $m \to \infty$.

Proof. To prove the first statement it is sufficient to check that for each multi-index k

$$\sup_{z\in\mathbb{B}^n} |\langle (S_m)_z 1, w^k\rangle_\lambda| \to 0 \quad \text{as} \quad m\to\infty.$$

Using (1.4) we calculate

$$\begin{aligned} B_{0,\lambda}S_m(\phi_z(u)) &= B_{0,\lambda}(S_m)_z(u) = (1-|u|^2)^{n+\lambda+1}\,\langle (\,S_m)_z K_u^\lambda, K_u^\lambda\rangle_\lambda \\ &= (1-|u|^2)^{n+\lambda+1}\sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\beta|+\lambda+1)}{\beta!\Gamma(n+\lambda+1)} \\ &\qquad\times \langle (S_m)_z w^\alpha, w^\beta\rangle_\lambda \overline{u}^\alpha u^\beta. \end{aligned}$$

Given a multi-index k and $r\in(0,1)$, we calculate

$$\begin{aligned} &\int_{|u|<r}\frac{B_{0,\lambda}S_m(\phi_z(u))\overline{u}^k}{(1-|u|^2)^{n+\lambda+1}}dv_\lambda(u) \\ &= \sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\beta|+\lambda+1)}{\beta!\Gamma(n+\lambda+1)} \\ &\qquad\qquad \times \left\langle (S_m)_z w^\alpha, w^\beta\right\rangle_\lambda \int_{|u|<r}\overline{u}^{\alpha+k}u^\beta dv_\lambda(u) \\ &= \sum_{|\alpha|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{(\alpha+k)!\Gamma(n+\lambda+1)} \\ &\qquad\qquad \times \left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \int_{|u|<r}|u^{\alpha+k}|^2 dv_\lambda(u) \\ &= \sum_{|\alpha|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{(\alpha+k)!\times\pi^n\Gamma(\lambda+1)}\times \\ &\qquad\qquad \times \left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \int_{|u|<r}|u^{\alpha+k}|^2(1-|u|^2)^\lambda dv(u). \end{aligned}$$

Passing in the last integral to the polar coordinates $u=s\xi$, where $s\in\mathbb{R}_+$ and $\xi\in S^{2n-1}$, and making use of the formulas (where dS is the surface measure on S^{2n-1})

$$\int_{\mathbb{B}^n} f(u)dv(u) = \int_0^1 s^{2n-1}dr\int_{S^{2n-1}} f(s\xi)dS(\xi),$$

$$\int_{S^{2n-1}}|\xi^m|^2 dS(\xi) = \frac{2\pi^n m!}{(n-1+|m|)!}$$

the last expression is equal to

$$\sum_{|\alpha|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{\Gamma(\lambda+1)\Gamma(n+|\alpha|+|k|)}\times$$
$$\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda 2\int_0^r s^{2n+2|\alpha|+2|k|-1}(1-s^2)^\lambda ds$$
$$=\sum_{|\alpha|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{\Gamma(\lambda+1)\Gamma(n+|\alpha|+|k|)}\times$$
$$\times\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \int_0^{r^2} s^{n+|\alpha|+|k|-1}(1-s)^\lambda ds$$
$$=\sum_{|\alpha|=0}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda I_{r^2}(n+|\alpha|+|k|,\lambda+1)$$
$$=\left\langle (S_m)_z 1, w^k\right\rangle_\lambda I_{r^2}(n+|k|,\lambda+1)+$$
$$\sum_{|\alpha|=1}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda I_{r^2}(n+|\alpha|+|k|,\lambda+1).$$

Here the function $I_x(a,b)$ is defined in the standard way (see, for example, [13, Formula 8.392]

$$I_x(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x t^{a-1}(1-t)^{b-1}dt.$$

Then we have

$$|\left\langle (S_m)_z 1, w^k\right\rangle_\lambda| \le \frac{1}{I_{r^2}(n+|k|,\lambda+1)}\left|\int_{|u|<r}\frac{B_{0,\lambda}S_m(\phi_z(u))\overline{u}^k}{(1-|u|^2)^{n+\lambda+1}}dv_\lambda(u)\right|$$
$$+\left|\sum_{|\alpha|=1}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\left\langle (S_m)_z w^\alpha, w^{\alpha+k}\right\rangle_\lambda \frac{I_{r^2}(n+|\alpha|+|k|,\lambda+1)}{I_{r^2}(n+|k|,\lambda+1)}\right|$$
$$\le \frac{1}{I_{r^2}(n+|k|,\lambda+1)}\|B_{0,\lambda}S_m\|_\infty c_\lambda \int_{|u|<r}\frac{|u^k|}{(1-|u|^2)^{n+1}}dv(u)$$
$$+\sum_{|\alpha|=1}^{\infty}\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\|(S_m)_z\|\|w^\alpha\|_{2,\lambda}\|w^{\alpha+k}\|_{2,\lambda}\frac{I_{r^2}(n+|\alpha|+|k|,\lambda+1)}{I_{r^2}(n+|k|,\lambda+1)}$$
$$\le \|B_{0,\lambda}S_m\|_\infty \frac{c_\lambda}{I_{r^2}(n+|k|,\lambda+1)}\int_{|u|<r}\frac{|u^k|}{(1-|u|^2)^{n+1}}dv(u)$$
$$+C\sum_{|\alpha|=1}^{\infty}\frac{I_{r^2}(n+|\alpha|+|k|,\lambda+1)}{I_{r^2}(n+|k|,\lambda+1)}=I+\Sigma,$$

where $C > 0$ is a constant independent of m and z. In the last line estimating Σ we used the boundedness of the sequence $\{S_m\}$ and the easily verified inequality

$$\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}\,\|w^\alpha\|_{2,\lambda}\|w^{\alpha+k}\|_{2,\lambda} < 1.$$

The first summand I above tends to zero as $m \to \infty$ due to the assumptions of the lemma. We estimate now the series in the second summand Σ. By [13, Formula 8.328.2]

$$\lim_{|\alpha|\to\infty} \frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{\Gamma(n+|\alpha|+|k|)}\frac{1}{(n+|\alpha|+|k|)^{\lambda+1}} = 1,$$

thus there exists $C > 0$ such that

$$\frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{\Gamma(n+|\alpha|+|k|)}\frac{1}{(n+|\alpha|+|k|)^{\lambda+1}} < C.$$

Then

$$\begin{aligned}
\Sigma_1 &:= \sum_{|\alpha|=1}^{\infty} \frac{I_{r^2}(n+|\alpha|+|k|,\lambda+1)}{I_{r^2}(n+|k|,\lambda+1)} \\
&= \frac{\Gamma(n+|k|)\Gamma(\lambda+1)}{\Gamma(n+|k|+\lambda+1)}\left(\int_0^{r^2} t^{n+|k|-1}(1-t)^\lambda dt\right)^{-1} \\
&\times \sum_{|\alpha|=1}^{\infty} \frac{\Gamma(n+|\alpha|+|k|+\lambda+1)}{\Gamma(n+|\alpha|+|k|)\Gamma(\lambda+1)} \int_0^{r^2} t^{n+|\alpha|+|k|-1}(1-t)^\lambda dt \\
&\le C\,\frac{\Gamma(n+|k|)}{\Gamma(n+|k|+\lambda+1)}\left(\int_0^{r^2} t^{n+|k|-1}(1-t)^\lambda dt\right)^{-1} \times \\
&\qquad \times \sum_{|\alpha|=1}^{\infty} (n+|\alpha|+|k|)^{\lambda+1} \int_0^{r^2} t^{n+|\alpha|+|k|-1}(1-t)^\lambda dt.
\end{aligned}$$

Estimating the multiple $(1-t)^\lambda$ in both integrals:

$$\begin{array}{lllll}
(1-r^2)^\lambda & \le & (1-t)^\lambda & \le & 1, \quad \text{for} \quad \lambda \ge 0, \\
1 & \le & (1-t)^\lambda & \le & (1-r^2)^\lambda, \quad \text{for} \quad \lambda \in (-1,0),
\end{array}$$

we come to the following estimate

$$\begin{aligned}
\Sigma_1 &\le C\,\frac{\Gamma(n+|k|+1)}{\Gamma(n+|k|+\lambda+1)}(1-r^2)^{-|\lambda|} \sum_{|\alpha|=1}^{\infty} (n+|\alpha|+|k|)^\lambda r^{2|\alpha|} \\
&= C\,\frac{\Gamma(n+|k|+1)}{\Gamma(n+|k|+\lambda+1)}(1-r^2)^{-|\lambda|} \sum_{m=1}^{\infty} \binom{m+n-1}{n}(n+m+|k|)^\lambda\, r^{2m}.
\end{aligned}$$

The power series in r in the last line has the radius of convergence equal to 1 and the value 0 at 0, thus the value of Σ can be made as small as needed taking r sufficiently closed to 0.

Both above estimates, on I and on Σ, are independent of $z \in \mathbb{B}^n$, which proves the first statement of the lemma.

To prove the second statement of the lemma we use the series representation (1.4),

$$\begin{aligned}
|(S_m)_z 1(u)| &= |\langle (S_m)_z 1, K_u^\lambda \rangle_\lambda| \\
&\leq \sum_{|\alpha|=0}^{\infty} \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} |\langle (S_m)_z 1, w^\alpha \rangle_\lambda| \cdot |u^\alpha| \\
&\leq \sum_{|\alpha|=0}^{l-1} \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} |\langle (S_m)_z 1, w^\alpha \rangle_\lambda| + \\
&\qquad \sum_{|\alpha|=l}^{\infty} \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} \|S_m\| \cdot \|w^\alpha\|_\lambda \cdot |u^\alpha| \\
&\leq \sum_{|\alpha|=0}^{l-1} \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} |\langle (S_m)_z 1, w^\alpha \rangle_\lambda| + \\
&\qquad C \sum_{|\alpha|=l}^{\infty} \left(\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} \right)^{\frac{1}{2}} |u^\alpha| \\
&= \Sigma_1 + \Sigma_2.
\end{aligned}$$

To estimate Σ_2 we use the Cauchy-Schwarz inequality,

$$\begin{aligned}
\Sigma_2 &= C \sum_{j=l}^{\infty} \left(\frac{\Gamma(n+j+\lambda+1)}{j!\Gamma(n+\lambda+1)} \right)^{\frac{1}{2}} \sum_{|\alpha|=j} \left[\frac{j!}{\alpha!} \right]^{\frac{1}{2}} |u^\alpha| \\
&\leq C \sum_{j=l}^{\infty} \left(\frac{\Gamma(n+j+\lambda+1)}{j!\Gamma(n+\lambda+1)} \right)^{\frac{1}{2}} \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} |u^\alpha|^2 \right)^{\frac{1}{2}} \left(\sum_{|\alpha|=j} 1 \right)^{\frac{1}{2}} \\
&= C \sum_{j=l}^{\infty} \left(\frac{\Gamma(n+j+\lambda+1)}{j!\Gamma(n+\lambda+1)} \right)^{\frac{1}{2}} \left(\frac{(n+j-1)!}{j!(n-1)!} \right)^{\frac{1}{2}} \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} |u^\alpha|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Let now $|u| \leq r < 1$, using the multi-nomial theorem for the expression in the last brackets

$$\sum_{|\alpha|=j} \frac{j!}{\alpha!} |u^\alpha|^2 = |u|^{2j},$$

we finely have

$$\Sigma_2 \leq C \sum_{j=l}^{\infty} \left(\frac{\Gamma(n+j+\lambda+1)}{j!\Gamma(n+\lambda+1)} \right)^{\frac{1}{2}} \left(\frac{(n+j-1)!}{j!(n-1)!} \right)^{\frac{1}{2}} r^j .$$

Choosing l sufficiently large we can make Σ_2 as small as needed, Σ_1, with l already fixed, tends uniformly to zero as $m \to \infty$ by the first statement of the lemma. This ends the proof. □

Lemma 1.3.5. *Let $\{S_m\}$ be a sequence in $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ such that $\|B_0 S_m\|_\infty \to 0$ as $m \to \infty$ and that for some $p > 2 + \frac{n}{1+\lambda}$*

$$\sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|_{p,\lambda} \leq C \qquad \text{and} \qquad \sup_{z\in\mathbb{B}^n} \|(S_m^*)_z 1\|_{p,\lambda} \leq C, \tag{1.24}$$

where $C > 0$ is independent of m. Then $S_m \to 0$ as $m \to \infty$ in the $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$-norm.

Proof. By Lemma 1.3.3 and (1.23) we have

$$\|S_m\| \leq C(n,p,\lambda) \left(\sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|_{p,\lambda} \right)^{1/2} \left(\sup_{z\in\mathbb{B}^n} \|(S_m^*)_z 1\|_{p,\lambda} \right)^{1/2} \leq C(n,p,\lambda).$$

Then, for $2 + \frac{n}{1+\lambda} < s < p$, Hölder's inequality gives

$$\begin{aligned}
\sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|^s_{s,\lambda} &\leq \sup_{z\in\mathbb{B}^n} \int_{|w|>r} |(S_m)_z 1(w)|^s dv_\lambda(w) + \\
&\qquad \sup_{z\in\mathbb{B}^n} \int_{|w|\leq r} |(S_m)_z 1(w)|^s dv_\lambda(w) \\
&\leq \sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|^s_{p,\lambda} \left(\int_{|w|>r} dv_\lambda(w) \right)^{1-s/p} + \\
&\qquad \sup_{z\in\mathbb{B}^n} \int_{|w|\leq r} |(S_m)_z 1(w)|^s dv_\lambda(w),
\end{aligned}$$

where, by (1.23), the second term tends to 0 as $m \to \infty$. By the first inequality in (1.24), the first term above can be made arbitrarily small by taking r sufficiently close to 1. Finally Lemma 1.3.3 yields

$$\begin{aligned}
\|S_m\| &\leq C(n,s,\lambda) \left(\sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|_{s,\lambda} \right)^{1/2} \left(\sup_{z\in\mathbb{B}^n} \|(S_m^*)_z 1\|_{s,\lambda} \right)^{1/2} \\
&\leq C(n,s,\lambda) \left(\sup_{z\in\mathbb{B}^n} \|(S_m)_z 1\|_{s,\lambda} \right)^{1/2} \to 0.
\end{aligned}$$

□

Corollary 1.3.6. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ such that for some $p > 2 + \frac{n}{1+\lambda}$,*

$$\sup_{z\in\mathbb{B}^n} \|S_z 1 - (T_{B_{m,\lambda}(S)})_z 1\|_{p,\lambda} \le C \qquad \text{and} \qquad \sup_{z\in\mathbb{B}^n} \|S^*_z 1 - (T_{B_{m,\lambda}(S^*)})_z 1\|_{p,\lambda} \le C, \tag{1.25}$$

where $C > 0$ is independent of m. Then $T_{B_{m,\lambda}(S)} \to S$ as $m \to \infty$ in $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$-norm.

Proof. We set $S_m = S - T_{B_{m,\lambda}(S)}$. By Proposition 1.2.8 we have

$$B_{0,\lambda}(S_m) = B_{0,\lambda}S - B_{0,\lambda}(T_{B_{m,\lambda}(S)}) = B_{0,\lambda}S - B_{m,\lambda}(B_{0,\lambda}S),$$

which, by Corollary 1.2.11, tends uniformly to 0 as $m \to \infty$, hence

$$\|B_{0,\lambda}(S_m)\|_\infty \to 0.$$

To finish the proof we use Lemma 1.3.5. □

Theorem 1.3.7. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$. If there is $p > 2 + \frac{n}{1+\lambda}$ such that*

$$\sup_{z\in\mathbb{B}^n} \|T_{(B_{m,\lambda}S)\circ\phi_z} 1\|_{p,\lambda} \le C \qquad \text{and} \qquad \sup_{z\in\mathbb{B}^n} \|T^*_{(B_{m,\lambda}S)\circ\phi_z} 1\|_{p,\lambda} \le C, \tag{1.26}$$

where $C > 0$ is independent of m. Then $T_{B_{m,\lambda}(S)} \to S$ as $m \to \infty$ in $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$-norm.

Proof. We prove first that

$$\sup_{z\in\mathbb{B}^n} \|S_z 1\|_{p,\lambda} < \infty. \tag{1.27}$$

The equality $T_{(B_{m,\lambda}S)\circ\phi_z} = (T_{B_{m,\lambda}S})_z$, together with Lemma 1.3.3, implies

$$\begin{aligned} &\|T_{B_{m,\lambda}S}\| \\ &\le C(n,p,\lambda) \left(\sup_{z\in\mathbb{B}^n} \|T_{(B_{m,\lambda}S)\circ\phi_z} 1\|_{p,\lambda}\right)^{1/2} \left(\sup_{z\in\mathbb{B}^n} \|T^*_{(B_{m,\lambda}S)\circ\phi_z} 1\|_{p,\lambda}\right)^{1/2} \\ &< C, \end{aligned} \tag{1.28}$$

where C is independent of m. Let $S_m = S - T_{B_{m,\lambda}S}$, then by arguments in the proof of Corollary 1.3.6 we have

$$\|B_{0,\lambda}S_m\|_\infty \to 0, \qquad \text{as} \quad m \to \infty.$$

By (1.28) the sequence $\{S_m\}$ is bounded; thus taking a polynomial f with $\|f\|_{q,\lambda} = 1$, by Lemma 1.3.4 we have

$$\sup_{z\in\mathbb{B}^n} |\langle (S_m)_z 1, f\rangle_\lambda| \to 0, \qquad \text{as} \quad m \to \infty.$$

Then for any $z_0 \in \mathbb{B}^n$ and any $\varepsilon > 0$, there is (a sufficiently large) m such that

$$|\langle S_{z_0}1, f\rangle_\lambda| \le \sup_{z\in\mathbb{B}^n} |\langle (S_m)_z 1, f\rangle_\lambda| + |\langle (T_{B_{m,\lambda}S})_{z_0}1, f\rangle_\lambda| \le \varepsilon + C,$$

with C being independent of m and z_0. This proves (1.27). Further, the equality

$$T^*_{(B_{m,\lambda}S)\circ\phi_z} = T_{\overline{B_{m,\lambda}S_z}} = T_{B_{m,\lambda}(S_z^*)} = T_{(B_{m,\lambda}(S^*))\circ\phi_z},$$

together with (1.26) and (1.27), implies (1.25), and Corollary 1.3.6 finishes the proof. □

Another approach to approximation theorems involves the invariant Laplacian and its application to the (m,λ)-Berezin transform.
Recall that the invariant Laplacian $\widetilde{\Delta}$ on $\mathbb{B}^n$, defined for $u \in C^2(\mathbb{B}^n)$ and $z \in \mathbb{B}^n$, is given by

$$(1.29) \qquad \big(\widetilde{\Delta}u\big)(z) := \Delta(u\circ\phi_z)(0), \quad \text{where} \quad \Delta := 4\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j}.$$

Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$ denote the Cauchy-Riemann operators with respect to the complex coordinate z_j, $j = 1, \cdots, n$ and Δ is the standard Laplacian on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ and $m \in \mathbb{Z}_+$, then we wish to calculate the function

$$\widetilde{\Delta} B_{m,\lambda}S \in C^\infty(\mathbb{B}^n).$$

Note that in the case $\lambda = 0$ and $n = 1$ this calculation was done in Proposition 2.4. of [35]. According to Theorem 1.2.5 we have

$$\widetilde{\Delta}\big[B_{m,\lambda}S\big](z) = \Delta\big(B_{m,\lambda}S\circ\phi_z\big)(0) = \Delta\big(B_{m,\lambda}S_z\big)(0)$$

and therefore we can assume that $z = 0$. We intend to use the form of $B_{m,\lambda}S$ in Proposition 1.2.2. We apply Δ to the z-dependent part of $B_{m,\lambda}S$ in the integral representation given there. Hence we have to evaluate the derivative

$$\Delta\left[\frac{(1-|z|^2)^{m+\lambda+n+1}}{(1-\langle u,z\rangle)^{n+m+\lambda+1}(1-\langle z,w\rangle)^{n+m+\lambda+1}}\right](0) = \\ = -4(m+n+\lambda+1) + 4(m+n+\lambda+1)^2\langle u,w\rangle.$$

Inserting this relation into the expression of $B_{m,\lambda}S$ given in Proposition 1.2.2 shows

$$(1.30) \quad \Delta(B_{m,\lambda}S)(0) = -4(m+n+\lambda+1)(B_{m,\lambda}S)(0) + \\ + 4(m+n+\lambda+1)^2\frac{c_{\lambda+m}}{c_\lambda}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\big(1-\langle u,w\rangle\big)^m\langle u,w\rangle\overline{S^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w).$$

On the other hand the same proposition shows that

$$\frac{c_\lambda}{c_{\lambda+m}}\left(B_{m,\lambda}S\right)(0)-\frac{c_\lambda}{c_{\lambda+m+1}}\left(B_{m+1,\lambda}S\right)(0)= \\ =\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\big(1-\langle u,w\rangle\big)^m\langle u,w\rangle\overline{S^*K_w^\lambda(u)}\,dv_\lambda(u)dv_\lambda(w). \tag{1.31}$$

Combining the equations (1.30) and (1.31) now implies

$$\Delta(B_{m,\lambda}S)(0)=4(m+n+\lambda+1)(m+n+\lambda)\big(B_{m,\lambda}S\big)(0) \\ -4\frac{c_{\lambda+m}}{c_{\lambda+m+1}}(m+n+\lambda+1)^2\big(B_{m+1,\lambda}S)(0).$$

According to (1.2) we have

$$\frac{c_{\lambda+m}}{c_{\lambda+m+1}}(m+n+\lambda+1)^2=(n+m+\lambda+1)(\lambda+m+1)$$

and we have shown the following relation, which in the case of $\lambda=0$ and $n=1$ is found in [35]:

Proposition 1.3.8. *Let $S\in\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$. For all $m\in\mathbb{Z}_+$ and $\lambda>-1$ it holds*

$$\widetilde{\Delta}\big[B_{m,\lambda}S\big]=4(m+n+\lambda+1)\Big[(m+n+\lambda)\big(B_{m,\lambda}S\big)-(m+\lambda+1)\big(B_{m+1,\lambda}S\big)\Big]. \tag{1.32}$$

Moreover, for all k,m we have

$$\widetilde{\Delta}B_{m,\lambda}\big(B_{k,\lambda}S\big)=B_{m,\lambda}\big(\widetilde{\Delta}B_{k,\lambda}S\big). \tag{1.33}$$

Proof. It suffices to prove (1.33). According to Proposition 1.2.8 and using (1.32) we have

$$\begin{aligned}&\widetilde{\Delta}B_{m,\lambda}\big(B_{k,\lambda}S\big)=\\ =&\widetilde{\Delta}B_{k,\lambda}\big(B_{m,\lambda}S\big)\\ =&4(k+n+\lambda+1)\big[(k+n+\lambda+1)B_{k,\lambda}B_{m,\lambda}S-(k+\lambda+1)B_{k+1,\lambda}B_{m,\lambda}S\big]\\ =&4(k+n+\lambda+1)\big[(k+n+\lambda+1)B_{m,\lambda}B_{k,\lambda}S-(k+\lambda+1)B_{m,\lambda}B_{k+1,\lambda}S\big]\\ =&B_{m,\lambda}\big(\widetilde{\Delta}B_{k,\lambda}S\big).\end{aligned}$$

which shows the assertion. □

For the remaining part of the section we specialize to the case of dimension $n=1$. Proposition 1.3.8) then implies

$$B_{m,\lambda}(S)-B_{m+1,\lambda}(S)=\frac{\widetilde{\Delta}[B_{m,\lambda}(S)]}{4(m+\lambda+2)(m+\lambda+1)} \tag{1.34}$$

and we can prove an analogue of Lemma 4.1 in [37]. We write $\mathbb{D}:=\mathbb{B}^1\subset\mathbb{C}$ for the open unit disc.

Proposition 1.3.9. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{D}))$ where $\lambda > -1$. Assume that*

$$\|T_{\widetilde{\Delta}(B_{m,\lambda}S)}\| \leq C$$

independently of $m \geq m_0$ and for some $m_0 \in \mathbb{Z}_+$. Then we have

$$\lim_{m\to\infty} T_{B_{m,\lambda}S} = S, \tag{1.35}$$

where the convergence is with respect to the norm topology of $\mathcal{A}^2_\lambda(\mathbb{D})$.

Proof. According to (1.34) we can write

$$\begin{aligned} T_{B_{m+1,\lambda}S} &= T_{B_{0,\lambda}S} - \sum_{k=0}^{m} \Big\{ T_{B_{k,\lambda}S} - T_{B_{k+1,\lambda}S} \Big\} \\ &= T_{B_{0,\lambda}S} - \sum_{k=0}^{m} \frac{T_{\widetilde{\Delta}(B_{k,\lambda}S)}}{4(k+\lambda+2)(k+\lambda+1)}. \end{aligned}$$

>From the boundedness assumption on the norms $\|T_{\widetilde{\Delta}(B_{k,\lambda}S)}\|$ we conclude that the right hand side of the equation converges in norm to some operator $R \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{D}))$. The continuity property of the usual Berezin transfrom $B_{0,\lambda}$, cf. (1.10) implies that

$$\lim_{m\to\infty} B_{0,\lambda} T_{B_{m,\lambda}S} = B_{0,\lambda} R.$$

On the other hand note that Proposition 1.2.8 and Corollary 1.2.11 imply the pointwise convergence

$$B_{0,\lambda}(T_{B_{m,\lambda}S}) = B_{0,\lambda} B_{m,\lambda}(S) = B_{m,\lambda} B_{0,\lambda}(S) \longrightarrow B_{0,\lambda} S$$

and it follows that $B_{0,\lambda} S = B_{0,\lambda} R$. Finally the injectivity of $B_{0,\lambda}$ shows that $S = R$. □

Chapter 2

Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball.

2.1 Radial operators

Denote by $\mathfrak{U}(n)$ the compact group of all $n \times n$ complex unitary matrices equipped with the Haar measure $d\mathcal{U}$. Recall that for each $\mathcal{U} \in \mathfrak{U}(n)$, the operator

$$V_{\mathcal{U}} f(w) = (\det \mathcal{U})^{\frac{n+\lambda+1}{n+1}} f(\mathcal{U}w)$$

is unitary on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Definition 2.1.1. An operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ is called *radial* if $SV_{\mathcal{U}} = V_{\mathcal{U}}S$, for all $\mathcal{U} \in \mathfrak{U}(n)$. The *radialization* of S is defined by

$$\mathrm{Rad}(S) := \int_{\mathfrak{U}(n)} V^*_{\mathcal{U}} S V_{\mathcal{U}} d\mathcal{U},$$

where the integral is taken in the weak sense.

We mention that the operator $\mathrm{Rad}(S)$ is radial, and that $\mathrm{Rad}(S) = S$ for each radial operator S.

With $a \in L_\infty(\mathbb{B}^n)$ and $z \in \mathbb{B}^n$ the *radialization* of a in z is defined by

$$\mathrm{rad}(a)(z) := \int_{\mathfrak{U}(n)} a(\mathcal{U}z) d\mathcal{U}.$$

Note that $\mathrm{rad}(a)$ is a radial function, i.e., $a(z) = a(|z|)$, and that $\mathrm{Rad}(T_a) = T_{\mathrm{rad}(a)}$. We need the following result.

Lemma 2.1.2. *The set of Toeplitz operators with bounded measurable symbols is dense in the algebra of all bounded operators on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ with respect to strong operator topology.*

Proof. In case of the unweighted Bergman space $\lambda = 0$ the proof can be found in [10]. However, the arguments almost literally serve for any $\lambda \in (-1, \infty)$. □

Recall that the standard monomial basis $[e_\alpha \,:\, \alpha \in \mathbb{Z}^n_+]$ of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is given by

$$e_\alpha(z) := \sqrt{\frac{\Gamma(n + |\alpha| + \lambda + 1)}{\alpha!\Gamma(n + \lambda + 1)}}\, z^\alpha. \tag{2.1}$$

The next result gives an independent characterization of the radial operators.

Proposition 2.1.3. *An operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ is radial if and only if it is diagonal with respect to the basis (2.1) and its eigenvalue sequence $\mu = \{\mu_\alpha\}_{\alpha \in \mathbb{Z}^n_+}$ is of the form $\mu_\alpha = \tilde{\mu}_{|\alpha|}$ for some bounded sequence $\{\tilde{\mu}_\ell\}_{\ell \geq 0}$, that is $Se_\alpha = \tilde{\mu}_{|\alpha|} e_\alpha$, for all $\alpha \in \mathbb{Z}^n_+$.*

Proof. Let S be a diagonal operator with $Se_\alpha = \tilde{\mu}_{|\alpha|} e_\alpha$, for all $\alpha \in \mathbb{Z}^n_+$. For each $m \in \mathbb{Z}_+$ consider the finite dimensional subspace H_m of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ defined by

$$H_m = \operatorname{span}\{e_\alpha \,:\, |\alpha| = m\}.$$

Then for all $f \in H_m$ we have that $Sf = \tilde{\mu}_m f$. Furthermore each subspace H_m is invariant under the operators $V_\mathcal{U}$ with $\mathcal{U} \in \mathfrak{U}(n)$. Thus $SV_\mathcal{U} = V_\mathcal{U} S$, and S is radial.

Conversely, assume that S is radial. Using Lemma 2.1.2 select a sequence $\{a_k\}_{k \in \mathbb{Z}_+} \subset L_\infty(\mathbb{B}^n)$ such that

$$\lim_{k \to \infty} T_{a_k} = S \qquad \text{(in SOT)}.$$

An application of the Banach Steinhaus theorem in combination with the Lebesgue's dominated convergence theorem shows that the radialization "Rad" is continuous with respect to the SOT and therefore we have convergence in SOT.

$$T_{\mathrm{rad}(a_k)} = \mathrm{Rad}\,(T_{a_k}) \longrightarrow \mathrm{Rad}(S) = S, \qquad (\text{as } \ k \to \infty).$$

As a consequence we can assume that a_k is a radial function for each $k \in \mathbb{Z}_+$ and therefore T_{a_k} is diagonal with $T_{a_k} e_\alpha = \mu^{(k)}_{|\alpha|} e_\alpha$. For all $\alpha \in \mathbb{Z}^n_+$ it follows

$$Se_\alpha = \lim_{k \to \infty} T_{a_k} e_\alpha = \tilde{\mu}_{|\alpha|} e_\alpha, \quad \text{with} \quad \tilde{\mu}_{|\alpha|} := \lim_{k \to \infty} \mu^{(k)}_{|\alpha|},$$

showing that S is diagonal with respect to the orthonormal basis $[e_\alpha \,:\, \alpha \in \mathbb{Z}^n_+]$ and that its eigenvalue sequence only depends on $|\alpha|$ for each $\alpha \in \mathbb{Z}^n_+$. □

Corollary 2.1.4. *The set of all bounded radial operators acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is a C^*-algebra which is isomorphic and isometric to $\ell_\infty(\mathbb{Z}_+)$. The isomorphism is given by the mapping*

$$S \quad \longmapsto \quad \tilde{\mu}(S),$$

where $\tilde{\mu}(S)$ is the eigenvalue sequence of the radial operator S in Proposition 2.1.3.

An interesting and important class of radial operators is provided by Toeplitz operators T_a with bounded measurable radial symbols $a = a(|z|)$. In Proposition 2.3.1 we will show that the eigenvalue sequences of such operators obey very specific properties. This implies that the class of radial Toeplitz operators is a quite restricted subset of the algebra of all radial operators.

Let $T_{a,n,\lambda}$ be the Toeplitz operator with radial generating symbol a acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. It is well known (see [24] for the one-dimensional case and [16] for the general case) that $T_{a,n,\lambda}$ is diagonal with respect to the basis $(e_\alpha)_{\alpha\in\mathbb{Z}^n_+}$:

$$T_{a,n,\lambda}e_\alpha = \beta^{(n)}_{a,\lambda}(|\alpha|+1)e_\alpha, \qquad \alpha \in \mathbb{Z}^n_+,$$

where the corresponding eigenvalues depend only on the norm of the multi-indices and are of the form

$$\beta^{(n)}_{a,\lambda}(k) = \frac{1}{\mathrm{B}(n+k-1,\lambda+1)} \int_0^1 a(\sqrt{r})r^{k+n-2}(1-r)^\lambda \,\mathrm{d}r, \qquad k \in \mathbb{N}. \tag{2.2}$$

We analyze now the (m,λ)-Berezin transform of radial operators. Given $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$, we have

$$\mathrm{rad} \circ B_{m,\lambda}(S)(z) = \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^m C_{m,\alpha} \int_{\mathfrak{U}(n)} \langle S_{\mathcal{U}z}w^\alpha, w^\alpha\rangle_\lambda \, d\mathcal{U}. \tag{2.3}$$

Then $S_{\mathcal{U}z} = U_{\mathcal{U}z}SU_{\mathcal{U}z}$ for all $\mathcal{U} \in \mathfrak{U}(n)$ and with $f \in L_2(\mathbb{B}^n, dv_\lambda)$ it follows

$$\big(U_{\mathcal{U}z}f\big)(w) = (-1)^{\frac{n(n+\lambda+1)}{n+1}} \frac{\big(1-|\mathcal{U}z|^2\big)^{\frac{n+\lambda+1}{2}}}{(1-\langle w,\mathcal{U}z\rangle)^{n+\lambda+1}} f \circ \phi_{\mathcal{U}z}(w) = (*).$$

By using the relation $\phi_{\mathcal{U}z} = \mathcal{U} \circ \phi_z \circ \mathcal{U}^*$ we find

$$(*) = (-1)^{\frac{n(n+\lambda+1)}{n+1}} \frac{\big(1-|z|^2\big)^{\frac{n+\lambda+1}{2}}}{(1-\langle \mathcal{U}^*w, z\rangle)^{n+\lambda+1}} f \circ \mathcal{U} \circ \phi_z \circ \mathcal{U}^*(w) = \big[V_{\mathcal{U}^*} \circ U_z \circ V_{\mathcal{U}} f\big](w),$$

which shows that $S_{\mathcal{U}z} = V_{\mathcal{U}^*} \circ U_z \circ V_{\mathcal{U}} \circ S \circ V_{\mathcal{U}^*} \circ U_z \circ V_{\mathcal{U}}$. Plugging these relation into (2.3) yields:

$$\begin{aligned}
\mathrm{rad} \circ B_{m,\lambda}(S)(z) &= \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \int_{\mathfrak{U}(n)} \Big\langle (V_{\mathcal{U}} \circ S \circ V_{\mathcal{U}^*})_z (\mathcal{U}w)^\alpha, (\mathcal{U}w)^\alpha \Big\rangle_\lambda d\mathcal{U} \\
&= \frac{c_{\lambda+m}}{c_\lambda} \sum_{|\alpha|=0}^{m} C_{m,\alpha} \int_{\mathfrak{U}(n)} \Big\langle (V_{\mathcal{U}} \circ S \circ V_{\mathcal{U}^*})_z w^\alpha, w^\alpha \Big\rangle_\lambda d\mathcal{U} \\
&= B_{m,\lambda} \circ \mathrm{Rad}\ (S)(z).
\end{aligned}$$

In the second equality we have used the following simple observation:

Lemma 2.1.5. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ and $\mathcal{U} \in \mathfrak{U}(n)$. Then it follows for all $m \in \mathbb{Z}_+$*

$$\sum_{|\alpha|=0}^{m} C_{m,\alpha} \big\langle S(\mathcal{U}w)^\alpha, (\mathcal{U}w)^\alpha \big\rangle_\lambda = \sum_{|\alpha|=0}^{m} C_{m,\alpha} \big\langle S w^\alpha, w^\alpha \big\rangle_\lambda .$$

Proof. Recall that any bounded operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ can be written as an integral operator with kernel $K_S(w,v) := \overline{[S^* K^\lambda_w(v)]} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$. In fact, let $g \in \mathcal{A}^2_\lambda(\mathbb{B}^n)$, then

$$[Sg](w) = \big\langle Sg, K^\lambda_w \big\rangle_\lambda = \big\langle g, S^* K^\lambda_w \big\rangle_\lambda = \int_{\mathbb{B}^n} g(v) \overline{[S^* K^\lambda_w(v)]}\, dv_\lambda(v).$$

Let $\mathcal{U} \in \mathfrak{U}(n)$ be fixed, then we find

$$\begin{aligned}
&\sum_{|\alpha|=0}^{m} C_{m,\alpha} \big\langle S(\mathcal{U}w)^\alpha, (\mathcal{U}w)^\alpha \big\rangle_\lambda \\
&= \sum_{|\alpha|=0}^{m} C_{m,\alpha} \int_{\mathbb{B}^n \times \mathbb{B}^n} K_S(w,v) \big(\mathcal{U}v\big)^\alpha \overline{\big(\mathcal{U}w\big)^\alpha} dv_\lambda(v) dv_\lambda(w) \\
&= \int_{\mathbb{B}^n \times \mathbb{B}^n} K_S(w,v) \big(1 - \langle \mathcal{U}v, \mathcal{U}w \rangle\big)^m\, dv_\lambda(v) dv_\lambda(w) \\
&= \int_{\mathbb{B}^n \times \mathbb{B}^n} K_S(w,v) \big(1 - \langle v, w \rangle\big)^m\, dv_\lambda(v) dv_\lambda(w) \\
&= \sum_{|\alpha|=0}^{m} C_{m,\alpha} \big\langle S w^\alpha, w^\alpha \big\rangle_\lambda,
\end{aligned}$$

and the assertion follows. □

Summarizing the above remark shows:

Lemma 2.1.6. *The "radialization" commutes with the (m, λ)-Berezin transform for all $\lambda > -1$ and $m \in \mathbb{Z}_+$, i.e.*

$$\text{rad} \circ B_{m,\lambda}(S) = B_{m,\lambda} \circ \text{Rad}\ (S), \qquad S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n)). \tag{2.4}$$

In particular, S is a radial operator if and only if $B_{m,\lambda}(S)$ is a radial function.

Proof. If $B_{m,\lambda}(S)$ is a radial function then it follows from (2.4) that

$$B_{m,\lambda}(S) = \text{rad} \circ B_{m,\lambda}(S) = B_{m,\lambda} \circ \text{Rad}(S).$$

Since $B_{m,\lambda}$ is one-to-one on bounded operators (cf. Proposition 1.2.8, (iii)) we have $S = \text{Rad}(S)$ and S is a radial operator.

On the other hand, if S is a radial operator, then we obtain $\text{rad} \circ B_{m,\lambda}(S) = B_{m,\lambda}(S)$ showing that $B_{m,\lambda}(S)$ is a radial function. □

We note that the (m, λ)-Berezin transform of a radial operator can be expressed in terms of its eigenvalue sequence. We need first the next preparatory formula:

Lemma 2.1.7. *Let $\alpha, \beta \in \mathbb{Z}^n_+$, then*

$$S_n(j, \beta) := \sum_{|\alpha|=j} \frac{(\alpha+\beta)!}{\alpha!\beta!} = \binom{n+j+|\beta|-1}{j}.$$

Proof. Let $\ell \in \mathbb{Z}_+$ and with $t \in (-1, 1)$ consider the power series

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(j+\ell)!}{j!\ell!} t^j &= \frac{1}{\ell!} \frac{d^\ell}{dt^\ell} \sum_{j=0}^{\infty} t^{\ell+j} = \frac{1}{\ell!} \frac{d^\ell}{dt^\ell} \left(\frac{t^\ell}{1-t} \right) \\ &= \frac{1}{\ell!} \frac{d^\ell}{dt^\ell} \left[\frac{1}{1-t} - \sum_{r=0}^{\ell-1} t^r \right] = (1-t)^{-\ell-1}.\end{aligned}$$

Put $x = (t, t, \cdots, t) \in (-1, 1)^n$, then it follows from the last identity

$$\begin{aligned}\sum_{\alpha \in \mathbb{Z}^n_+} \frac{(\alpha+\beta)!}{\alpha!\beta!} x^\alpha &= \prod_{k=1}^{n} \sum_{j=0}^{\infty} \frac{(j+\beta_k)!}{j!\beta_k!} t^j \\ &= \frac{1}{(1-t)^{|\beta|+n}} \\ &= \sum_{j=0}^{\infty} \binom{n+|\beta|+j-1}{j} t^j.\end{aligned}$$

Since $x^\alpha = t^{|\alpha|}$ the result follows by comparing coefficients. □

Now we are ready to prove

Proposition 2.1.8. *Let S be a radial operator with the eigenvalue sequence $\{\mu_{|\alpha|}\}_{\alpha\in\mathbb{Z}_+}$. Then its (m,λ)-Berezin transform has the form*

$$\begin{aligned}
&(B_{m,\lambda}S)(z)\\
=\;&\frac{c_{\lambda+m}}{c_\lambda}(1-|z|^2)^{m+\lambda+n+1}\sum_{j=0}^{m}(-1)^j\frac{m!}{(m-j)!}\sum_{k=0}^{\infty}\left[\frac{\Gamma(n+k+m+\lambda+1)}{\Gamma(n+m+\lambda+1)}\right]^2\\
\times\;&\frac{\Gamma(n+\lambda+1)}{\Gamma(n+j+k+\lambda+1)}\binom{n+j+k-1}{j}\mu_{j+k}\frac{|z|^{2k}}{k!}.
\end{aligned}$$

Proof. By (1.4) and then by (1.3), we have

$$\begin{aligned}
&\langle S(w^\alpha K_z^{m+\lambda}), w^\alpha K_z^{m+\lambda}\rangle_\lambda\\
=\;&\sum_{|\beta|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+m+\lambda+1)}{\beta!\,\Gamma(n+m+\lambda+1)}\right]^2|z^\beta|^2\,\langle Sw^{\alpha+\beta}, w^{\alpha+\beta}\rangle_\lambda\\
=\;&\sum_{|\beta|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+m+\lambda+1)}{\beta!\,\Gamma(n+m+\lambda+1)}\right]^2\frac{(\alpha+\beta)!\,\Gamma(n+\lambda+1)}{\Gamma(n+|\alpha|+|\beta|+\lambda+1)}\mu_{|\alpha|+|\beta|}\,|z^\beta|^2.
\end{aligned}$$

Using (1.14) and (1.7) we calculate then

$$\begin{aligned}
&(B_{m,\lambda}S)(z)=\\
=&\frac{c_{\lambda+m}}{c_\lambda}(1-|z|^2)^{m+\lambda+n+1}\sum_{|\alpha|=0}^{m}C_{m,\alpha}\langle S(w^\alpha K_z^{m+\lambda}), w^\alpha K_z^{m+\lambda}\rangle_\lambda\\
=&\frac{c_{\lambda+m}}{c_\lambda}(1-|z|^2)^{m+\lambda+n+1}\sum_{|\alpha|=0}^{m}\binom{m}{|\alpha|}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}\\
&\times\sum_{|\beta|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+m+\lambda+1)}{\Gamma(n+m+\lambda+1)}\right]^2\frac{(\alpha+\beta)!\,\Gamma(n+\lambda+1)}{\Gamma(n+|\alpha|+|\beta|+\lambda+1)}\\
&\times\mu_{|\alpha|+|\beta|}\frac{(\alpha+\beta)!}{[\beta!]^2}|z^\beta|^2\\
=&\frac{c_{\lambda+m}}{c_\lambda}(1-|z|^2)^{m+\lambda+n+1}\sum_{j=0}^{m}\binom{m}{j}(-1)^j j!\sum_{k=0}^{\infty}\left[\frac{\Gamma(n+k+m+\lambda+1)}{\Gamma(n+m+\lambda+1)}\right]^2\\
&\times\frac{\Gamma(n+\lambda+1)}{\Gamma(n+j+k+\lambda+1)}\frac{\mu_{j+k}}{k!}\sum_{|\beta|=k}\frac{k!}{\beta!}|z^\beta|^2\sum_{|\alpha|=j}\frac{(\alpha+\beta)!}{\alpha!\,\beta!}.
\end{aligned}$$

Finally, the statement follows by the multinomial theorem and Lemma 2.1.7. □

Corollary 2.1.9. *Let S be a radial operator with the eigenvalue sequence $\{\mu_{|\alpha|}\}_{\alpha\in\mathbb{Z}_+}$. Then its Berezin transform $B_\lambda(S) := B_{0,\lambda}(S)$ is given by*

$$(B_\lambda S)(z) = (1-|z|^2)^{n+\lambda+1} \sum_{k=0}^{\infty} \frac{\Gamma(n+k+\lambda+1)}{\Gamma(n+\lambda+1)}\, \mu_k \, \frac{|z|^{2k}}{k!}.$$

2.2 Approximation of radial operators

We specify here the results of the previous sections to the case of radial operators. Given a symbol $f \in L_\infty(\mathbb{B}^n)$ and $\mathcal{U} \in \mathfrak{U}(n)$ we have for all $g, h \in \mathcal{A}^2_\lambda(\mathbb{B}^n)$

$$\begin{aligned} \left\langle V_{\mathcal{U}}^* T_f V_{\mathcal{U}} g, h \right\rangle_\lambda &= \int_{\mathbb{B}^n} f(w) V_{\mathcal{U}} g(w) \overline{V_{\mathcal{U}} h(w)} dv_\lambda(w) \\ &= \int_{\mathbb{B}^n} f(\mathcal{U}^* w) g(w) \overline{h(w)} dv_\lambda(w). \end{aligned}$$

Hence it follows that $V_{\mathcal{U}}^* T_f V_{\mathcal{U}} = T_{f\circ\mathcal{U}^*}$, and, more generally, for any finite number of L_∞-symbols $f_1, \cdots, f_l$, we have

$$V_{\mathcal{U}}^* T_{f_1} \cdots T_{f_l} V_{\mathcal{U}} = T_{f_1\circ\mathcal{U}^*} \cdots T_{f_l\circ\mathcal{U}^*}.$$

Lemma 2.2.1. *Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ be a radial operator and $m \in \mathbb{Z}_+$. Then we have an integral representation of the Toeplitz operator with symbol $B_{m,\lambda}(S)$:*

$$T_{B_{m,\lambda}(S)} = \int_{\mathbb{B}^n} S_w dv_{m+\lambda}(w) \tag{2.5}$$

(in the weak sense). In particular, one obtains the norm estimate

$$\|T_{B_{m,\lambda}(S)}\| \le \|S\|. \tag{2.6}$$

Proof. For any $z \in \mathbb{B}^n$, definition (1.9) and Lemma 1.2.4 yield

$$\begin{aligned} B_{0,\lambda}\Big(\int_{\mathbb{B}^n} S_w dv_{m+\lambda}(w)\Big)(z) &= \Big\langle \Big(\int_{\mathbb{B}^n} S_w dv_{m+\lambda}(w)\Big)_z 1, 1\Big\rangle_\lambda \\ &= \int_{\mathbb{B}^n} \langle U_z U_w S U_w U_z 1, 1\rangle_\lambda \, dv_{m+\lambda}(w) \\ &= \int_{\mathbb{B}^n} \left\langle U_{\phi_z(w)} V_{\mathcal{U}}^* S V_{\mathcal{U}} U_{\phi_z(w)} 1, 1\right\rangle_\lambda dv_{m+\lambda}(w) = I_0, \end{aligned}$$

where the unitary matrix $\mathcal{U}$ has been defined in Lemma 1.2.4. The operator S is radial, thus by Proposition 1.2.8 (ii), (1.11), and Proposition 1.2.8, (i) we have that

$$
\begin{aligned}
I_0 &= \int_{\mathbb{B}^n} \left\langle U_{\phi_z(w)} S U_{\phi_z(w)} 1, 1 \right\rangle_\lambda dv_{m+\lambda}(w) \\
&= \int_{\mathbb{B}^n} (B_{0,\lambda} S) \circ \phi_z(w) dv_{m+\lambda}(w) \\
&= B_{m,\lambda} B_{0,\lambda} S(z) = B_{0,\lambda} B_{m,\lambda} S(z) = B_{0,\lambda} \left(T_{B_{m,\lambda}(S)} \right)(z).
\end{aligned}
$$

The injectivity of $B_{0,\lambda}$ on $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ completes the proof of the integral representation (2.5). The norm inequality (2.6) is an immediate consequence. □

By $\mathfrak{T}(L_\infty(\mathbb{B}^n))$ we denote the C^*-algebra generated by all Toeplitz operators T_a, with symbols $a \in L_\infty(\mathbb{B}^n)$, acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Theorem 2.2.2. *Let $S \in \mathfrak{T}(L_\infty(\mathbb{B}^n))$ be a radial operator. Then $T_{B_{m,\lambda}(S)} \to S$ as $m \to \infty$ in $\mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$-norm.*

Proof. As $S \in \mathfrak{T}(L_\infty(\mathbb{B}^n))$, there is a sequence of operators $\{S_k\}$ which converges in norm to S, and such that each operator S_k is a finite sum of finite products of Toeplitz operators with L_∞-symbols. Since the radialization is continuous and S is radial we have

$$\operatorname{Rad}(S_k) \to \operatorname{Rad}(S) = S, \qquad \text{as} \quad k \to \infty.$$

Using (2.6) of Lemma 2.2.1 shows

$$
\begin{aligned}
\|S - T_{B_{m,\lambda}(S)}\| &\leq \|S - \operatorname{Rad}(S_k)\| + \|\operatorname{Rad}(S_k) - T_{B_{m,\lambda}(\operatorname{Rad}(S_k))}\| \\
&\quad + \|T_{B_{m,\lambda}(\operatorname{Rad}(S_k))} - T_{B_{m,\lambda}(S)}\| \\
&\leq 2\|S - \operatorname{Rad}(S_k)\| + \|\operatorname{Rad}(S_k) - T_{B_{m,\lambda}(\operatorname{Rad}(S_k))}\|,
\end{aligned}
$$

and thus it is sufficient to prove that $T_{B_{m,\lambda}(\operatorname{Rad}(S_k))} \to \operatorname{Rad}(S_k)$. Then, as each S_k is a finite sum of finite products of Toeplitz operators with L_∞-symbols, it is sufficient to prove the convergence for the radialization of a finite product of Toeplitz operators. That is, it is sufficient to prove that if

$$Q := \int_{\mathfrak{U}(n)} T_{f_1 \circ \mathcal{U}^*} \cdots T_{f_l \circ \mathcal{U}^*} \, d\mathcal{U} \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n)),$$

with $f_1, \cdots, f_l \in L_\infty(\mathbb{B}^n)$, then $T_{B_{m,\lambda}(Q)} \to Q$ as $m \to \infty$ and with respect to the norm topology. By Lemma 2.2.1,

$$
\begin{aligned}
(2.7) \quad T_{(B_{m,\lambda}(Q)) \circ \phi_z} &= \int_{\mathbb{B}^n} (Q_z)_w dv_{m+\lambda}(w) \\
&= \int_{\mathbb{B}^n} \int_{\mathfrak{U}(n)} T_{f_1 \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} \cdots T_{f_l \circ \mathcal{U}^* \circ \phi_z \circ \phi_w} d\mathcal{U} dv_{m+\lambda}(w).
\end{aligned}
$$

Since $d\mathcal{U}$ and $dv_{m+\lambda}(w)$ are probability measures, and each Toeplitz operator T_f with bounded measurable symbol, considered as operator on $\mathcal{A}^p_\lambda(\mathbb{B}^n)$, with p of Theorem 1.3.7, obeys the estimate

$$\|T_f\|_{\mathcal{L}(\mathcal{A}^p_\lambda(\mathbb{B}^n))} \leq C_{p,\lambda}\|f\|_\infty,$$

where $C_{p,\lambda}$ is the norm of the Bergman projection from $L_p(\mathbb{B}^n, dv_\lambda)$ into $\mathcal{A}^p_\lambda(\mathbb{B}^n)$, we have the following norm estimate for (2.7) for all $m \in \mathbb{Z}_+$

$$\|T_{(B_{m,\lambda}(Q))\circ\phi_z}\|_{\mathcal{L}(\mathcal{A}^p_\lambda(\mathbb{B}^n))} \leq C^l_{p,\lambda}\|f_1\|_\infty \cdots \|f_l\|_\infty,$$

and analogously $\|T^*_{(B_{m,\lambda}(Q))\circ\phi_z}\|_{\mathcal{L}(\mathcal{A}^p_\lambda(\mathbb{B}^n))} \leq C^l_{p,\lambda}\|f_1\|_\infty \cdots \|f_l\|_\infty$. Finally, Theorem 1.3.7 yields the uniform convergence $T_{B_{m,\lambda}(Q)} \to Q$. □

2.3 Eigenvalue sequences of radial Toeplitz operators

Given a radial symbol $a = a(|z|) \in L_\infty(0,1)$, consider the corresponding Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. Recall that T_a is a radial operator and diagonal with respect to standard monomial basis (2.1). The corresponding eigenvalue sequence $\beta^{(n)}_{a,\lambda} = \{\beta^{(n)}_{a,\lambda}(m)\}_{m\in\mathbb{N}}$ has the form ((2.2)). Given n (the dimension of $\mathbb{B}^n$) and λ (the weight parameter), we denote by

$$\mathrm{B}^{(n)}_\lambda = \mathrm{B}^{(n)}_\lambda(L_\infty(0,1)) := \left\{\beta^{(n)}_{a,\lambda} \ : \ a \in L_\infty(0,1)\right\} \subset l_\infty(\mathbb{N}) \tag{2.8}$$

the set of all eigenvalue sequences of Toeplitz operators T_a, with $a \in L_\infty(0,1)$, acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. We introduce now several subsets on $l_\infty = l_\infty(\mathbb{N})$. Following [37] we denote by $d_1 = d_1(\mathbb{N})$ the set of all bounded sequences $x = \{x_m\}_{m\in\mathbb{N}}$ such that

$$\sup_{m\in\mathbb{N}} m|x_m - x_{m+1}| < \infty,$$

and we write $d_2 = d_2(\mathbb{N})$ for the set of all bounded sequences $x = \{x_m\}_{m\in\mathbb{N}}$ such that

$$\sup_{m\in\mathbb{N}} m^2|x_m - 2x_{m+1} + x_{m+2}| < \infty.$$

Finally we denote by VSO($\mathbb{N}$) the set of all bounded sequences that *slowly oscillate* in the sense of Schmidt [32] (see also Landau [25] and Stanojević and Stanojević [33]):

$$\mathrm{VSO}(\mathbb{N}) = \left\{x \in \ell_\infty : \ \lim_{\frac{j}{k}\to 1} |x_j - x_k| = 0\right\}.$$

Alternatively, VSO($\mathbb{N}$) consists of all bounded functions $\mathbb{N} \to \mathbb{C}$ that are uniformly continuous with respect to the "logarithmic metric" $\rho(j,k) = |\ln(j) - \ln(k)|$.

It is known [37, Proposition 2.4] that both d_1 and d_2 have the same closure in l_∞, and [17, Proposition 4.5] that this closure coincides with VSO($\mathbb{N}$). Furthermore, [17, Proposition 3.8], VSO($\mathbb{N}$) is a C^*-subalgebra of $l_\infty(\mathbb{N})$.

Proposition 2.3.1. *Given any $n \in \mathbb{N}$, $\lambda \in (-1,\infty)$, and a radial symbol $a \in L_\infty(0,1)$, the corresponding eigenvalue sequence $\beta^{(n)}_{a,\lambda}$ belongs to both d_1 and d_2.*

Proof. We start with the case of d_1.

$$\begin{aligned}
\beta^{(n)}_{a,\lambda+1}(m) &= \frac{\Gamma(m+n+\lambda+1)}{\Gamma(\lambda+2)\Gamma(n+m-1)} \int_0^1 a(\sqrt{r}) r^{m+n-2}\,(1-r)^\lambda (1-r)\,dr \\
&= \frac{m+n+\lambda}{\lambda+1}\,\beta^{(n)}_{a,\lambda}(m) - \frac{n+m-1}{\lambda+1}\,\beta^{(n)}_{a,\lambda}(m+1) \\
&= \beta^{(n)}_{a,\lambda}(m) + \frac{m+n-1}{\lambda+1}\left(\beta^{(n)}_{a,\lambda}(m) - \beta^{(n)}_{a,\lambda}(m+1)\right).
\end{aligned}$$

Thus

$$m|\beta^{(n)}_{a,\lambda}(m) - \beta^{(n)}_{a,\lambda}(m+1)| \le 2(\lambda+1)\|a\|_\infty$$

showing that $\beta^{(n)}_{a,\lambda} \in d_1$. Consider now the case of d_2.

$$\begin{aligned}
\beta^{(n)}_{a,\lambda+2}(m) &= \frac{\Gamma(m+n+\lambda+2)}{\Gamma(\lambda+3)\Gamma(m+n-1)} \int_0^1 a(\sqrt{r}) r^{m+n-2}\,(1-r)^\lambda (1-2r+r^2)\,dr \\
&= \frac{(m+n+\lambda+2)(n+m+\lambda)}{(\lambda+2)(\lambda+1)}\,\beta^{(n)}_{a,\lambda}(m) \\
&- 2\frac{(m+n+\lambda+1)(n+m-1)}{(\lambda+2)(\lambda+1)}\,\beta^{(n)}_{a,\lambda}(m+1) \\
&+ \frac{(m+n)(m+n-1)}{(\lambda+2)(\lambda+1)}\,\beta^{(n)}_{a,\lambda}(m+2).
\end{aligned}$$

Then

$$\begin{aligned}
&(\lambda+2)(\lambda+1)\beta^{(n)}_{a,\lambda+2}(m) \\
&= (m+n)(m+n-1)\left(\beta^{(n)}_{a,\lambda}(m+2) - 2\beta^{(n)}_{a,\lambda}(m+1) + \beta^{(n)}_{a,\lambda}(m)\right) \\
&+ [(\lambda+1)(m+n-1) + (\lambda+1)(m+n+\lambda+1)]\,\beta^{(n)}_{a,\lambda}(m) \\
&- 2(\lambda+1)(m+n-1)\beta^{(n)}_{a,\lambda}(m+1) \\
&= (m+n)(m+n-1)\left(\beta^{(n)}_{a,\lambda}(m+2) - 2\beta^{(n)}_{a,\lambda}(m+1) + \beta^{(n)}_{a,\lambda}(m)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2(\lambda+1)(m+n-1)\left[\beta^{(n)}_{a,\lambda}(m) - \beta^{(n)}_{a,\lambda}(m+1)\right] \\
&+ \quad \beta^{(n)}_{a,\lambda}(m)
\end{aligned}
$$

Or

$$
\begin{aligned}
& (m+n)(m+n-1)\left(\beta^{(n)}_{a,\lambda}(m) - 2\beta^{(n)}_{a,\lambda}(m+1) + \beta^{(n)}_{a,\lambda}(m+2)\right) \\
= \quad & \left(\beta^{(n)}_{a,\lambda}(m) - \beta^{(n)}_{a,\lambda}(m+1)\right) - 2(\lambda+1)(m+n-1)\left[\beta^{(n)}_{a,\lambda}(m) - \beta^{(n)}_{a,\lambda}(m+1)\right]
\end{aligned}
$$

which together with $\beta^{(n)}_{a,\lambda} \in d_1$ implies uniform boundedness of

$$
m^2 \left|\beta^{(n)}_{a,\lambda}(m) - 2\beta^{(n)}_{a,\lambda}(m+1) + \beta^{(n)}_{a,\lambda}(m+2)\right|.
$$

It follows that $\beta^{(n)}_{a,\lambda} \in d_2$. $\square$

Recall that the spaces d_1 and d_2 carry semi-norms $\|\cdot\|_{d_1}$ and $\|\cdot\|_{d_2}$ in a natural way (see [37]). As was shown in Proposition 2.4 of [37] the norm inequality $\|\cdot\|_{d_1} \leq \|\cdot\|_{d_2}$ holds proving that $d_2 \subset d_1$.

Observe now that the sequences $\beta^{(n)}_{a,\lambda}$, for $n > 1$, are nothing but the shifted sequences $\beta^{(1)}_{a,\lambda}$. To formalize this we introduce two unilateral shift operators, the left shift operator $\tau_L(x)$ and the right shift operator $\tau_R(x)$

$$
\tau_L \;:\; x \;\longmapsto\; (x_1, x_2, x_3, \cdots), \qquad \tau_R \;:\; x \;\longmapsto\; (0, x_0, x_1, \cdots).
$$

Due to [17, Propositions 3.10 and 3.11] both of them are bounded on VSO($\mathbb{N}$), and have norm one. Now $\beta^{(n)}_{a,\lambda} = \tau_L^{n-1}(\beta^{(1)}_{a,\lambda})$.

The last observation permits us to reduce our analysis to the set $\mathrm{B}^{(1)}_{\lambda}$ only. We already know that $\mathrm{B}^{(1)}_{\lambda} \subset d_2$, and our next aim is to prove that $\mathrm{B}^{(1)}_{\lambda}$ is dence in d_2. This will be done in Theorem 2.3.3.

We define first the invariant Laplacian of an operator $A \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$. With the notation in (1.29) put

$$
\mathcal{D}_\lambda := \left\{ S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n)) \;:\; \exists\, T \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n)) \text{ such that } \widetilde{\Delta} B_{0,\lambda}(S) = B_{0,\lambda}(T) \right\}.
$$

Note that T is uniquely defined since the Berezin transform $B_{0,\lambda}$ is one-to-one and therefore we can define $\widetilde{\Delta} : \mathcal{D}_\lambda \longrightarrow \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ by $\widetilde{\Delta} S = T$.

From Lemma 2.1.6 and Proposition 1.3.8 it is clear that $\widetilde{\Delta}$ maps radial operator to radial operators. Let $\mathbb{D} = \mathbb{B}^1$ be the open unit disc in $\mathbb{C}$ and for each $k \in \mathbb{Z}_+$ denote by P_k the rank one projection of $\mathcal{A}^2_\lambda(\mathbb{D})$ onto the subspace

$\{\rho e_k \,:\, \rho \in \mathbb{C}\} \subset \mathcal{A}^2_\lambda(\mathbb{D})$. If $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{D}))$ is a radial operator with bounded eigenvalue sequence $\theta = \{\theta_k\}_{k\in\mathbb{Z}_+}$ with respect to the standard orthonormal bases (2.1) of $\mathcal{A}^2_\lambda(\mathbb{D})$, then we can write

$$S = \sum_{k=0}^{\infty} \theta_k P_k. \tag{2.9}$$

where the convergence is in the strong sense. By a method analogue to the proof of Lemma 3.2 in [37] one can check that

$$B_{0,\lambda}(S) = \sum_{k=0}^{\infty} \theta_k B_{0,\lambda}(P_k) \quad \text{and} \quad \widetilde{\Delta}(B_{0,\lambda}S) = \sum_{k=0}^{\infty} \theta_k \widetilde{\Delta} B_{0,\lambda}(P_k). \tag{2.10}$$

Now we prove:

Proposition 2.3.2. *Let S be the radial operator in* (2.9). *If $\theta \in d_2$ then $S \in \mathcal{D}_\lambda$ and*

$$\|\widetilde{\Delta} S\| \leq \big(6 + 4|\lambda|\big)\|\theta\|_{d_2}. \tag{2.11}$$

Proof. By a direct calculation one verifies that

$$\begin{aligned} B_{0,\lambda}(P_k)(z) &= \big\langle P_k k_z^\lambda, k_z^\lambda \big\rangle_\lambda = \big|\big\langle k_z^\lambda, e_k\big\rangle_\lambda\big|^2 \\ &= (1-|z|^2)^{\lambda+2}|e_k(z)|^2 = \frac{\Gamma(2+k+\lambda)}{k!\Gamma(2+\lambda)}(1-|z|^2)^{\lambda+2}|z|^{2k}. \end{aligned} \tag{2.12}$$

We calculate

$$\begin{aligned} \Delta(1-|z|^2)^{\lambda+2}|z|^{2k} = (1-|z|^2)^{\lambda}\Big\{&(\lambda+2+k)^2|z|^{2(k+1)} \\ &- \big[(\lambda+2+k)(1+k) + k(k+\lambda+1)\big]|z|^{2k} + k^2|z|^{2(k-1)}\Big\}. \end{aligned}$$

With $\theta_{-1} := 0$ and by using (2.9) we find

(2.13)

$$\begin{aligned} \widetilde{\Delta}\big(B_{0,\lambda}S\big)(z) =& (1-|z|^2)^2\Delta\big(B_{0,\lambda}S\big)(z) \\ =& (1-|z|^2)^2\sum_{k=0}^{\infty}\theta_k\Delta\big(B_{0,\lambda}P_k\big)(z) \\ =& \frac{(1-|z|^2)^{2+\lambda}}{\Gamma(2+\lambda)}\sum_{k=0}^{\infty}|z|^{2k}\Big[\theta_{k-1}\frac{(\lambda+k+1)^2\Gamma(1+k+\lambda)}{(k-1)!} - \\ &- \theta_k\frac{\Gamma(2+k+\lambda)\big[(\lambda+2+k)(1+k)+k(k+\lambda+1)\big]}{k!} + \end{aligned}$$

$$+ \theta_{k+1}\frac{(k+1)^2\Gamma(3+k+\lambda)}{(k+1)!}\Big]$$
$$=(1-|z|^2)^{2+\lambda}\sum_{k=0}^{\infty}|e_k(z)|^2\zeta_k(\lambda).$$

In the last equality we have rearranged the summation and with $k \in \mathbb{Z}_+$ we write

$$\zeta_k(\lambda) := \Big[\theta_{k-1}k(\lambda+k+1) - \theta_k\big[(\lambda+2+k)(1+k)+k(k+\lambda+1)\big]+ \\ + \theta_{k+1}(\lambda+2+k)(k+1)\Big].$$

A straightforward calculation shows that $\zeta_k(\lambda) = \zeta_{k,1} + \lambda\zeta_{k,2} + \lambda\zeta_{k,3}$ can be decomposed into three parts where $\zeta_{k,j}$ for $j = 1, 2, 3$ are independent of λ and given by

$$\begin{aligned}
\zeta_{k,1} &= (k+1)\big[\theta_{k+1}(2+k) - 2(k+1)\theta_k + k\theta_{k-1}\big]\\
\zeta_{k,2} &= (k+1)\big[\theta_{k+1} - 2\theta_k + \theta_{k-1}\big]\\
\zeta_{k,3} &= \theta_k - \theta_{k-1}.
\end{aligned}$$

Consider the sequences $\zeta^{(j)} := \{\zeta_{k,j}\}_k$ for $j = 1, 2, 3$. In Lemma 3.3 of [37] it has been shown that $\|\zeta^{(1)}\|_\infty \leq 6\|\theta\|_{d_2}$ and in particular $\zeta^{(1)}$ is bounded in case of $\theta \in d_2$. Moreover, using $\|\cdot\|_{d_1} \leq \|\cdot\|_{d_2}$ (cf. Proposition 2.4. in [37]) we clearly have the estimates

$$\|\zeta^{(2)}\|_\infty \leq 2\|\theta\|_{d_2} \qquad \text{and} \qquad \|\zeta^{(3)}\|_\infty \leq 2\|\theta\|_{d_1} \leq 2\|\theta\|_{d_2}.$$

Hence $\theta \in d_2$ implies that $\zeta(\lambda) := \{\zeta_k(\lambda)\}_k$ is bounded with

$$\|\zeta(\lambda)\|_\infty \leq \|\zeta^{(1)}\|_\infty + |\lambda|\left(\|\zeta^{(2)}\|_\infty + \|\zeta^{(3)}\|_\infty\right) \leq \big(6+4|\lambda|\big)\|\theta\|_{d_2}. \tag{2.14}$$

Consider now the diagonal operator $T = \sum_{k=0}^{\infty}\zeta_k(\lambda)P_k$. From the previous remark it follows that $T \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{D}))$ for all $\theta \in d_2$. An application of (2.12) now shows that

$$\big(B_{0,\lambda}T\big)(z) = \sum_{k=0}^{\infty}\zeta_k(\lambda)\big(B_{0,\lambda}P_k\big)(z) = (1-|z|^2)^{\lambda+2}\sum_{k=0}^{\infty}\zeta_k(\lambda)|e_k(z)|^2.$$

Comparison with (2.13) shows that

$$B_{0,\lambda}T = \widetilde{\Delta}\big(B_{0,\lambda}S\big).$$

It follows that $S \in \mathcal{D}_\lambda$ with $\widetilde{\Delta}S = T$. The identity $\|\zeta(\lambda)\|_\infty = \|T\| = \|\widetilde{\Delta}S\|$ together with the estimate (2.14) implies (2.11). $\square$

Following the ideas in [37] we now can show:

Theorem 2.3.3. *The closure of* $\Gamma_\lambda^{(1)}$ *in* l_∞ *coincides with the closure of* d_2 *in* l_∞.

Proof. It follows from Proposition 2.3.1 that $\gamma_{a,\lambda}^{(1)} \in d_2$ for all $\lambda > -1$ and $a \in L_\infty(0,1)$, which shows that the closure of the eigenvalue sequences $\gamma_{a,\lambda}^{(1)}$ is contained in the closure of d_2. So it is sufficient to show that d_2 is contained in the closure of eigenvalue sequences $\gamma_{a,\lambda}^{(1)}$ where $a \in L_\infty(0,1)$.

Let $s \in d_2$ and denote by S the radial operator on $\mathcal{A}_\lambda^2(\mathbb{D})$ which has s as eigenvalue sequence (with respect to the standard orthonormal basis (2.1)). We show that we can approximate S in norm by Toeplitz operators with symbols in $L_\infty(0,1)$. According to Proposition 2.3.2 it holds $S \in \mathcal{D}_\lambda$ and

$$\|\widetilde{\Delta} S\| \leq \big(6 + 4|\lambda|\big)\|s\|_{d_2}. \tag{2.15}$$

Using Proposition 1.2.8, (i) and (1.33) gives for all $m \in \mathbb{Z}_+$:

$$\begin{aligned} B_{0,\lambda}\widetilde{\Delta} B_{m,\lambda}(S) &= \widetilde{\Delta} B_{0,\lambda} B_{m,\lambda}(S) = \widetilde{\Delta} B_{m,\lambda} B_{0,\lambda}(S) \\ &= B_{m,\lambda}\widetilde{\Delta} B_{0,\lambda}(S) = B_{m,\lambda} B_{0,\lambda}(\widetilde{\Delta} S) = B_{0,\lambda} B_{m,\lambda}(\widetilde{\Delta} S). \end{aligned}$$

Since the Berezin transform $B_{0,\lambda}$ is one-to-one we conclude that $\widetilde{\Delta} B_{m,\lambda}(S) = B_{m,\lambda}(\widetilde{\Delta} S)$. Therefore from (2.15) and from (2.6) of Lemma 2.2.1 we find the estimate

$$\|T_{\widetilde{\Delta}(B_{m,\lambda} S)}\| = \|T_{B_{m,\lambda}(\widetilde{\Delta} S)}\| \leq \|\widetilde{\Delta} S\| \leq \big(6 + 4|\lambda|\big)\|s\|_{d_2}.$$

Hence, Theorem 1.3.9 shows that $S = \lim_{m\to\infty} T_{B_{m,\lambda}(S)}$ with respect to the norm topology and the assertion follows. □

We can characterize now the C^*-algebra that is generated by Toeplitz operators with bounded radial symbols.

Theorem 2.3.4. *For each* $n \in \mathbb{N}$ *and* $\lambda \in (-1,\infty)$, *the* l_∞*-closure of* $\mathrm{B}_\lambda^{(n)}$ *coincides with* $\mathrm{VSO}(\mathbb{N})$.

Proof. For $n = 1$ the result follows from Theorem 2.3.3 and the density of d_2 in $\mathrm{VSO}(\mathbb{N})$.
Let now $n > 1$. Consider any $\gamma \in \mathrm{VSO}(\mathbb{Z}_+)$ and any $\varepsilon > 0$. Then $\widetilde{\gamma} = \tau_R^{n-1}(\gamma) \in \mathrm{VSO}(\mathbb{N})$. According to the case of $n = 1$, there is a function $a \in L_\infty(0,1)$ such that $\|\widetilde{\gamma} - \gamma_{a,\lambda}^{(1)}\| < \varepsilon$. Finally

$$\|\gamma - \gamma_{a,\lambda}^{(n)}\| = \|\tau_L^{n-1}(\widetilde{\gamma} - \gamma_{a,\lambda}^{(1)})\| \leq \|\widetilde{\gamma} - \gamma_{a,\lambda}^{(1)}\| < \varepsilon,$$

which proves the density of $\Gamma_\lambda^{(n)}$ in $\mathrm{VSO}(\mathbb{N})$. □

We denote by $\mathfrak{T}_{\mathrm{rad}} = \mathfrak{T}_{\mathrm{rad}}(L_\infty)$ the C^*-algebra generated by all Toeplitz operators, with radial symbols $a \in L_\infty(0,1)$, acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. Let $\mathrm{Rad}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ be the set of all radial operators acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. The proof of Theorem 2.2.2, in particular, shows that

$$\mathfrak{T}(L_\infty(\mathbb{B}^n)) \cap \mathrm{Rad}(\mathcal{A}^2_\lambda(\mathbb{B}^n)) = \mathfrak{T}_{\mathrm{rad}}(L_\infty),$$

which together with the next corollary affirmatively answers problem (i) in the final section of [5].

Corollary 2.3.5. *The algebras $\mathfrak{T}_{rad}$ for any $n \in \mathbb{N}$ and any $\lambda \in (-1, \infty)$ are all isomorphic and isometric among each other, being isomorphic and isometric to* $\mathrm{VSO}(\mathbb{Z}_+)$.

In each case the isomorphism is generated by the following mapping

$$T_a \;\longmapsto\; \beta^{(n)}_{a,\lambda}.$$

The set of initial generators $T = \{T_a \,:\, a \in L_\infty(0,1)\}$ is dense in $\mathfrak{T}_{rad}$, that is two different types of closures, the C^*-algebraic closure and topological (norm) closure of the set T give the same result $\mathfrak{T}_{rad}$.

Remark 2.3.6. As was mentioned in Corollary 2.3.5 the algebraic structure of $\mathfrak{T}_{\mathrm{rad}}$ does not depend on the dimension n of the unit ball, but the operators themselves, the *multiplicity of their eigenvalues*, do depend on n.

The eigenvalue $\beta^{(n)}_{a,\lambda}(m)$, $m \in \mathbb{N}$, has the multiplicity $\binom{n+m-1}{n-1}$.

Chapter 3

Verical Toeplitz operators

3.1 Vertical operators

This chapter is devoted to the description of a certain class of Toeplitz operators acting on the Bergman space over the upper half-plane and of the C^*-algebra generated by them.

Let $\Pi = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane, and let $d\mu = dxdy$ be the standard Lebesgue plane measure on Π. For $\lambda \in (-1, \infty)$ consider the weight measure $d\mu_\lambda = (\lambda+1)(2\text{Im}(z))^\lambda d\mu$. Recall that the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is the (closed) subspace of $L_2(\Pi, d\mu_\lambda)$ which consists of all function analytic in Π. It is well known that $\mathcal{A}^2_\lambda(\Pi)$

$$K^{(\lambda)}_{\Pi,w}(z) = \frac{i^{\lambda+2}}{\pi(z-\overline{w})^{\lambda+2}};$$

thus the Bergman (orthogonal) projection of $L_2(\Pi, d\mu_\lambda)$ onto $\mathcal{A}^2(\Pi)$ is given by

$$(Pf)(w) = \langle f, K^{(\lambda)}_{\Pi,w}\rangle.$$

In this chapter we write $K^{(\lambda)}_{\Pi,w}$ as K_w.

Given a function $g \in L_\infty(\Pi)$, the Toeplitz operator $T_g\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ with generating symbol g is defined by $T_g f = P(gf)$.

Let $\mathcal{L}(\mathcal{A}^2_\lambda(\Pi))$ be the algebra of all linear bounded operators acting on the Bergman space $\mathcal{A}^2_\lambda(\Pi)$. Given $h \in \mathbb{R}$, let $H_h \in \mathcal{L}(\mathcal{A}^2_\lambda(\Pi))$ be the *horizontal translation operator* defined by

$$H_h f(z) := f(z-h).$$

We call an operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\Pi))$ *vertical* (or *horizontal translation invariant*) if it commutes with all horizontal translation operators:

$$\forall h \in \mathbb{R}, \qquad H_h S = S H_h.$$

In this section we find a criterion for an operator from $\mathcal{A}_\lambda^2(\Pi)$ to be vertical. First we recall some known facts on translation invariant operators on the real line.

Introduce the standard Fourier transform

$$(Ff)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-ist} f(t)\,dt,$$

being a unitary operator on $L_2(\mathbb{R})$.

For each $h \in \mathbb{R}$, the translation operator $\tau_h\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined by

$$\tau_h f(s) := f(s-h).$$

An operator S on $L_2(\mathbb{R})$ is called *translation invariant* if $\tau_h S = S\tau_h$, for all $h \in \mathbb{R}$. It is well known (see, for example, [20, Theorem 2.5.10]) that an operator S on $L_2(\mathbb{R})$ is translation invariant if and only if it is a convolution operator, i.e., if and only if there exists a function $\sigma \in L_\infty(\mathbb{R})$ such that

$$S = F^{-1} M_\sigma F. \tag{3.1}$$

We introduce as well the associated *multiplication by a character operator* $M_{\Theta_h} f(s) := \Theta_h(s) f(s)$, where $\Theta_h(s) := \mathrm{e}^{ish}$.

Note that τ_h and $M_{\Theta_{-h}}$ are related via the Fourier transform,

$$M_{\Theta_{-h}} F = F\tau_h. \tag{3.2}$$

Lemma 3.1.1. *Let $M \in \mathcal{L}(L_2(\mathbb{R}))$. The following conditions are equivalent:*

(a) M is invariant under multiplication by Θ_h for all $h \in \mathbb{R}$:

$$M M_{\Theta_h} = M_{\Theta_h} M.$$

(b) M is the multiplication operator by a bounded measurable function:

$$\exists\, \sigma \in L_\infty(\mathbb{R}) \quad \text{such that} \quad M = M_\sigma.$$

Proof. The part (b)⇒(a) is trivial: $M_\sigma M_{\Theta_h} = M_{\sigma\Theta_h} = M_{\Theta_h} M_\sigma$. The implication (a)⇒(b) follows from the relation (3.2) and the result on the translation invariant operators cited above. □

Let Θ_h^+ denote the restriction of Θ_h to $\mathbb{R}_+$. The following lemma states that an operator on $L_2(\mathbb{R}_+)$ commutes with $M_{\Theta_h^+}$ if and only if it is a multiplication operator.

Lemma 3.1.2. *Let $M \in \mathcal{L}(L_2(\mathbb{R}_+))$. The following conditions are equivalent:*

(a) M is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$MM_{\Theta_h^+} = M_{\Theta_h^+}M.$$

(b) M is the multiplication operator by a bounded function:

$$\exists\, \sigma \in L_\infty(\mathbb{R}_+) \quad \text{such that} \quad M = M_\sigma.$$

Proof. To prove that (a) implies (b), define the *restriction* operator

$$P\colon L_2(\mathbb{R}) \to L_2(\mathbb{R}_+), \qquad g \mapsto g|_{\mathbb{R}_+},$$

and the *zero extension* operator

$$J\colon L_2(\mathbb{R}_+) \to L_2(\mathbb{R}), \qquad Jf(x) := \begin{cases} f(x) & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

For every $h \in \mathbb{R}$ the following equalities hold:

$$JM_{\Theta_h^+} = M_{\Theta_h}J, \qquad PM_{\Theta_h} = M_{\Theta_h^+}P.$$

If (a) holds, then the operator JMP is invariant under multiplication by Θ_h, for all $h \in \mathbb{R}$:

$$JMPM_{\Theta_h} = JMM_{\Theta_h^+}P = JM_{\Theta_h^+}MP = M_{\Theta_h}JMP,$$

and by Lemma 3.1.2 there exists a function $\sigma_1 \in L_\infty(\mathbb{R})$ such that $JMP = M_{\sigma_1}$. Set $\sigma = \sigma_1|_{\mathbb{R}_+}$. Then for all $f \in L_2(\mathbb{R}_+)$ and all $x \in \mathbb{R}_+$,

$$\begin{aligned}(M_\sigma f)(x) &= \sigma(x)f(x) = \sigma_1(x)(Jf)(x) = (M_{\sigma_1}Jf)(x) \\ &= (JMPJf)(x) = (JMf)(x) = (Mf)(x),\end{aligned}$$

and (b) holds. The implication (b)⇒(a) follows directly, as in the previous lemma. □

The *Berezin transform* [6, 7] of an operator $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$ is the function $\Pi \to \mathbb{C}$ defined by

$$\mathcal{B}(S)(w) := \frac{\langle SK_{\Pi,w}^{(\lambda)}, K_{\Pi,w}^{(\lambda)}\rangle}{\langle K_{\Pi,w}^{(\lambda)}, K_{\Pi,w}^{(\lambda)}\rangle}.$$

Following [14, Section 2] (see also [41, Section 3.1]), we introduce the isometric isomorphism $R\colon \mathcal{A}^2(\Pi) \to L_2(\mathbb{R}_+)$,

$$(R\phi)(x) := \frac{x^{(\lambda+1)/2}}{\sqrt{\Gamma(\lambda+2)}} \int_\Pi \phi(w)\, \mathrm{e}^{-i\overline{w}x}\, d\mu_\lambda(w).$$

The operator R is unitary, and its inverse $R^*\colon L_2(\mathbb{R}_+) \to \mathcal{A}^2(\Pi)$ is given by

$$(R^* f)(z) = \frac{1}{\sqrt{\Gamma(\lambda+2)}} \int_{\mathbb{R}_+} \xi^{(\lambda+2)/2} f(\xi)\, \mathrm{e}^{iz\xi}\, d\xi.$$

The next theorem gives a criterion for an operator to be vertical, and is an analogue of the Zorboska result [47] for radial operators.

Theorem 3.1.3. *Let $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$. The following conditions are equivalent:*

(a) S is invariant under horizontal shifts:

$$\forall h \in \mathbb{R} \qquad SH_h = H_h S.$$

(b) RSR^* is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$\forall h \in \mathbb{R} \qquad RSR^* M_{\Theta_h^+} = M_{\Theta_h^+} RSR^*.$$

(c) There exists a function $\sigma \in L_\infty(\mathbb{R}_+)$ such that

$$S = R^* M_\sigma R.$$

(d) The Berezin transform of S is a vertical function, i.e., depends on $\mathrm{Im}(w)$ *only.*

Proof. $(a) \Rightarrow (b)$. Follows from the formulas $R^* M_{\Theta_h^+} = H_{-h} R^*$ and $RH_h = M_{\Theta_{-h}^+} R$.

$(b) \Rightarrow (c)$. Follows from Lemma 3.1.2.

$(c) \Rightarrow (d)$. Using the residue theorem we get

$$(RK_{\Pi,w}^{(\lambda)})(x) = \frac{x^{\frac{\lambda+1}{2}}}{\sqrt{\Gamma(\lambda+2)}} e^{-ix\overline{w}}.$$

Therefore

$$\begin{aligned} \mathcal{B}(S)(w) &= \frac{\langle M_\sigma RK_{\Pi,w}^{(\lambda)}, RK_{\Pi,w}^{(\lambda)}\rangle}{\langle K_{\Pi,w}^{(\lambda)}, K_{\Pi,w}^{(\lambda)}\rangle} \\ &= \int_0^{+\infty} \sigma(x) \frac{(2x\,\mathrm{Im}(w))^{\lambda+2}}{\Gamma(\lambda+2)}\, \mathrm{e}^{-2\,\mathrm{Im}(w) x}\, \frac{dx}{x}, \end{aligned}$$

and $\mathcal{B}(S)(w)$ depends only on $\mathrm{Im}(w)$.

$(d) \Rightarrow (a)$. Compute the Berezin transform of $H_{-h} S H_h$ using the formula $H_h K_{\Pi,w} = K_{\Pi,w+h}$:

$$\begin{aligned} \mathcal{B}(H_{-h} S H_h)(w) &= \frac{\langle S H_h K_{\Pi,w}^{(\lambda)}, H_h K_{\Pi,w}^{(\lambda)}\rangle}{\|K_{\Pi,w}^{(\lambda)}\|^2} = \frac{\langle S K_{\Pi,w+h}^{(\lambda)}, K_{\Pi,w+h}^{(\lambda)}\rangle}{\|K_{\Pi,w+h}^{(\lambda)}\|^2} \\ &= \mathcal{B}(S)(w+h) = \mathcal{B}(S)(w). \end{aligned}$$

Since the Berezin transform is injective [34], $H_{-h} S H_h = S$. □

Corollary 3.1.4. *The set of all vertical operators on $\mathcal{L}(\mathcal{A}^2_\lambda(\Pi))$ is a commutative C^*-algebra which is isometrically isomorphic to $L_\infty(\mathbb{R}_+)$.*

3.2 Vertical Toeplitz operators

In this section we establish necessary and sufficient conditions for a Toeplitz operator to be vertical.

Lemma 3.2.1. *Let $g \in L_\infty(\Pi)$. Then T_g is zero if and only if $g = 0$ almost everywhere.*

Proof. The corresponding result for Toeplitz operators on the Bergman space on the unit disk is well known, see, for example, [41, Theorem 2.8.2]. To extend it to the upper half-plane case, we introduce the Cayley transform

$$\psi\colon \Pi \to \mathbb{D}, \qquad w \longmapsto \frac{w-i}{w+i},$$

the corresponding unitary operator

$$U\colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\Pi), \qquad f \longmapsto (f \circ \psi)\psi',$$

and observe that $U^* T_g U = T_{g\circ\psi^{-1}}$. □

The next elementary lemma gives a criterion for a function on $\mathbb{R}$ to be almost everywhere constant. We use there the Lebesgue measure in $\mathbb{R}^n$ for various dimensions ($n = 1, 2, 3$), indicating the dimension as a subindex: μ_n.

Lemma 3.2.2. *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a measurable function. Then the following conditions are equivalent:*

(a) There exists a constant $c \in \mathbb{C}$ such that $f(x) = c$ for almost all $x \in \mathbb{R}$.

(b) $\mu_2(D) = 0$, where $D := \big\{(x, y) \in \mathbb{R}^2 \mid f(x) \neq f(y)\big\}$.

(c) $\mu_1(D_x) = 0$ for almost all $x \in \mathbb{R}$, where $D_x := \big\{y \in \mathbb{R}^2 \mid f(x) \neq f(y)\big\}$.

Proof. (a)⇒(b). Let $C = \{x \in \mathbb{R} \mid f(x) \neq c\}$. The condition (a) means that $\mu_1(C) = 0$. Since $D \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times C)$, we obtain $\mu_2(D) = 0$.

(b)⇒(c). Follows from an application of Tonelli's theorem to the characteristic function of D.

(c)⇒(a). Choose a point $x_0 \in \mathbb{R}$ such that $\mu_1(D_{x_0}) = 0$ and set $c := f(x_0)$. Then $f = c$ almost everywhere. □

Proposition 3.2.3. *Let $g \in L_\infty(\Pi)$. The operator T_g is vertical if and only if there exists a function $b \in L_\infty(\mathbb{R}_+)$ such that $g(w) = b(\mathrm{Im}(w))$ for almost every $w \in \Pi$.*

Proof. Sufficiency. For every $h \in \mathbb{R}$, define $g_h \colon \Pi \to \mathbb{C}$ by $g_h(w) = g(w+h)$. Then for almost all $w \in \mathbb{C}$

$$g_h(w) = g(w+h) = b(\operatorname{Im}(w+h)) = b(\operatorname{Im}(w)) = g(w).$$

Applying the formula $H_{-h} T_g H_h = T_{g_h}$ we see that T_g is invariant with respect to horizontal translations.

Necessity. Since T_g is vertical, for every $h \in \mathbb{R}$ we have $T_g = H_{-h} T_g H_h = T_{g_h}$. By Lemma 3.2.1, $g = g_h$ almost everywhere. It means that for all $h \in \mathbb{R}$ the equality $\mu_2(E_h) = 0$ holds where

$$E_h := \big\{(u,v) \in \mathbb{R}^2 \mid g(u+h+iv) \neq g(u+iv)\big\}.$$

Define $\Lambda \colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{C}$ by

$$\Lambda(u,x,v) := \begin{cases} 0, & g(x+iv) = g(u+iv); \\ 1, & g(x+iv) \neq g(u+iv). \end{cases}$$

Then for all $h \in \mathbb{R}$

$$\big\{(u,v) \in \Pi \mid \Lambda(u, u+h, v) \neq 0\big\} = E_h$$

and by Tonelli's theorem

$$\begin{aligned} \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,x,v)\, d\mu_3(u,x,v) &= \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,u+h,v)\, d\mu_3(u,h,v) \\ &= \int_{\mathbb{R}} \left(\int_{\Pi} \Lambda(u,u+h,v)\, d\mu_2(u,v) \right) dh = \int_{\mathbb{R}} \mu_2(E_h)\, dh = 0. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^2} \Lambda(u,x,v)\, d\mu_2(u,x) \right) dv = \int_{\mathbb{R}^2\times\mathbb{R}_+} \Lambda(u,x,v)\, d\mu_3(u,x,v) = 0,$$

and for almost $v \in \mathbb{R}_+$

$$\mu_2(\{(u,x) \in \mathbb{R}^2 \mid g(x+iv) \neq g(u+iv)\}) = \int_{\mathbb{R}^2} \Lambda(u,x,v)\, d\mu(u,x) = 0.$$

For such v, by Lemma 3.2.2, there exists a constant $c(v)$ such that $g(u+iv) = c(v)$. Then for $b \colon \mathbb{R}_+ \to \mathbb{C}$ defined by

$$b(v) = \begin{cases} c(v), & \text{if } \mu_2(\{(u,x) \in \mathbb{R}^2 \mid g(x+iv) \neq g(u+iv)\}) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

we have $g(w) = b(\operatorname{Im}(w))$ for almost all $w \in \Pi$. □

We say that a measurable function $g\colon \Pi \to \mathbb{C}$ is *vertical* if there exists a measurable function $b\colon \mathbb{R}_+ \to \mathbb{C}$ such that $g(w) = b(\operatorname{Im}(w))$ for almost all w in Π.

The next result was proved in [39, Theorem 3.1] (see also [41, Theorem 5.2.1]).

Theorem 3.2.4. *Let $g(w) = b(\operatorname{Im}(w)) \in L_\infty$ be a vertical symbol. Then the Toeplitz operator T_g acting on $\mathcal{A}^2_\lambda(\Pi)$ is unitary equivalent to the multiplication operator $M_{\gamma_a} = RT_gR^*$ acting on $L_2(\mathbb{R}_+)$. The function $\gamma_b = \gamma_b(s)$ is given by*

$$\gamma_{b,\lambda}(s) := \int_0^\infty b(t)\frac{(2st)^{\lambda+1}}{\Gamma(\lambda+1)}\,\mathrm{e}^{-2ts}\,\frac{dt}{t}, \qquad s \in \mathbb{R}_+. \tag{3.1}$$

In particular, this implies that the C^*-algebra generated by vertical Toeplitz operators with bounded symbols is commutative and is isometrically isomorphic to the C^*-algebra generated by the set

$$\Gamma_\lambda := \bigl\{\gamma_{b,\lambda} \mid b \in L_\infty(\mathbb{R}_+)\bigr\}.$$

3.3 Very slowly oscillating functions on $\mathbb{R}_+$

In this section we introduce and discuss the algebra $\mathrm{VSO}(\mathbb{R}_+)$ of very slowly oscillating functions, and show that for any vertical symbol $a \in L_\infty(\mathbb{R}_+)$, the associated "spectral function" γ_a belongs to $\mathrm{VSO}(\mathbb{R}_+)$.

We introduce the logarithmic metric on the positive half-line by

$$\rho(x,y) := \bigl|\ln(x) - \ln(y)\bigr| \;:\; \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty).$$

It is easy to see that ρ is indeed a metric and that ρ is *invariant under dilations*: for all $x, y, t \in \mathbb{R}_+$,

$$\rho(tx, ty) = \rho(x, y).$$

Recall that the *modulus of continuity* of a function $f\colon \mathbb{R}_+ \to \mathbb{C}$ with respect to the metric ρ is defined for all $\delta > 0$ as

$$\omega_{\rho,f}(\delta) := \sup\bigl\{|f(x) - f(y)| \mid \quad x, y \in \mathbb{R}_+, \quad \rho(x,y) \le \delta\bigr\}.$$

Definition 3.3.1. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a bounded function. We say that f is *very slowly oscillating* if it is uniformly continuous with respect to the metric ρ or, equivalently, if the composition $f \circ \exp$ is uniformly continuous with respect the usual metric on $\mathbb{R}$. Denote by $\mathrm{VSO}(\mathbb{R}_+)$ the set of such functions.

Proposition 3.3.2. $\mathrm{VSO}(\mathbb{R}_+)$ *is a closed C^*-algebra of the C^*-algebra $C_b(\mathbb{R}_+)$ of bounded continuous functions $\mathbb{R}_+ \to \mathbb{C}$ with pointwise operations.*

Proof. Using the following elementary properties of the modulus of continuity one can see that $\mathrm{VSO}(\mathbb{R}_+)$ is closed with respect to the pointwise operations:

$$\omega_{\rho,f+g} \le \omega_{\rho,f} + \omega_{\rho,g}, \qquad \omega_{\rho,fg} \le \|f\|_\infty \omega_{\rho,g} + \|g\|_\infty \omega_{\rho,f},$$
$$\omega_{\rho,\lambda f} = |\lambda| \omega_{\rho,f}, \qquad \omega_{\rho,f^*} = \omega_{\rho,f}.$$

The inequality $\omega_{\rho,f}(\delta) \le 2\|f-g\|_\infty + \omega_{\rho,g}(\delta)$ and the usual "$\frac{\varepsilon}{3}$-argument" show that $\mathrm{VSO}(\mathbb{R}_+)$ is topologically closed in $C_b(\mathbb{R}_+)$. □

Note that instead of the logarithmic metric ρ we can use an alternative one:
Let $\rho_1 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ be defined by

$$\rho_1(x,y) := \frac{|x-y|}{\max(x,y)}.$$

It is easy to see that ρ_1 is a metric. To prove the triangle inequality $\rho_1(x,z) + \rho_1(z,y) - \rho_1(x,y) \ge 0$, use the symmetry between x and y and consider three cases: $x < y < z$, $x < z < y$, $z < x < y$. For example, if $x < y < z$, then

$$\rho_1(x,z) + \rho_1(z,y) - \rho_1(x,y) = \frac{(z-y)(x+y)}{yz} > 0.$$

The other two cases are considered analogously.

Lemma 3.3.3. *For every $x, y \in \mathbb{R}_+$ the following inequality holds*

$$\rho_1(x,y) \le \rho(x,y). \tag{3.1}$$

Proof. The metrics ρ and ρ_1 can be written in terms of max and min as shown below:

$$\rho(x,y) = \ln \frac{\max(x,y)}{\min(x,y)}, \qquad \rho_1(x,y) = 1 - \frac{\min(x,y)}{\max(x,y)}.$$

Since $\ln(u) \ge 1 - \dfrac{1}{u}$ for all $u \ge 1$, the substitution $u = \dfrac{\max(x,y)}{\min(x,y)}$ yields (3.1). □

It can be proved that $\rho(x,y) \le 2\ln(2)\rho_1(x,y)$ if $\rho_1(x,y) < 1/2$. Thus $\mathrm{VSO}(\mathbb{R}_+)$ could be defined alternatively as the class of all bounded functions that are uniformly continuous with respect to ρ_1.

Theorem 3.3.4. *Let $b \in L_\infty(\mathbb{R}_+)$. Then $\gamma_{b,\lambda} \in \mathrm{VSO}(\mathbb{R}_+)$. More precisely,*

$$\|\gamma_{b,\lambda}\|_\infty \le \|b\|_\infty,$$

and $\gamma_{b,\lambda}$ is Lipschitz continuous with respect to the distance ρ:

$$|\gamma_{b,\lambda}(y) - \gamma_{b,\lambda}(x)| \le 2(\lambda+1)\rho(x,y)\|b\|_\infty, \tag{3.2}$$

that is

$$\omega_{\gamma_{b,\lambda}}(\delta) \le 2\delta\|b\|_\infty. \tag{3.3}$$

Proof. The upper bound $\|\gamma_{b,\lambda}\|_\infty \leq \|b\|_\infty$ follows directly from the definition (3.1) of $\gamma_{b,\lambda}$. Firs, we add and sustract

$$\frac{b(v)v^\lambda}{\Gamma(\lambda+1)}(2x)^{\lambda+1}e^{-2yv}$$

in the integrand and use the inequality $|b(v)| \leq \|b\|_\infty$, to obtain

$$|\gamma_{b,\lambda}(x) - \gamma_{b,\lambda}(y)| \leq \frac{\|b\|_\infty}{\Gamma(\lambda+1)} \int_0^\infty v^\lambda \left((2x)^{\lambda+1}|e^{-2xv} - e^{-2yv} + e^{-2yv}|(2x)^{\lambda+1} - (2y)^{\lambda+1}|\right) \frac{dv}{v}.$$

Without lost of generality assume $y > x$, and solving the integrals we get the inequality

$$|\gamma_{b,\lambda}(x) - \gamma_{b,\lambda}(y)| \leq 2\|b\|_\infty \rho_1(x^{\lambda+1}, y^{\lambda+1}) \leq 2(\lambda+1)\|b\|_\infty \rho(x,y),$$

where the last inequality uses Lemma 3.3.3. □

3.4 Density of Γ_λ in VSO($\mathbb{R}_+$)

The set $\mathbb{R}_+$ provided with the standard multiplication and topology is a commutative locally compact topological group, whose Haar measure is given by $d\nu(s) := \frac{ds}{s}$.

For each $n \in \mathbb{N} := \{1, 2, \ldots\}$, we define a function $\psi_{n,\lambda} \colon \mathbb{R}_+ \to \mathbb{C}$ by

$$\psi_{n,\lambda}(s) = \frac{1}{\mathrm{B}(n+\lambda, n+\lambda)} \frac{s^{n+\lambda}}{(1+s)^{2(n+\lambda)}},$$

where B is the Beta function.

Proposition 3.4.1. *The sequence $(\psi_{n,\lambda})_{n=1}^\infty$ is a Dirac sequence, i.e., it satisfies the following three conditions:*

(a) For each $n \in \mathbb{N}$ and every $s \in \mathbb{R}_+$,

$$\psi_{n,\lambda}(s) \geq 0.$$

(b) For each $n \in \mathbb{N}$,

$$\int_0^\infty \psi_{n,\lambda}(s)(s)\frac{ds}{s} = 1.$$

(c) For every $\delta > 0$,

$$\lim_{n\to\infty} \int_{\rho(s,1)>\delta} \psi_{n,\lambda}(s)(s)\frac{ds}{s} = 0.$$

Proof. The property (a) is obvious, and (b) follows from the formula for the Beta function:

$$B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,ds.$$

We prove (c). Fix a $\delta > 0$. The function $s \mapsto \dfrac{s^{n+\lambda-1}}{(1+s)^{2(n+\lambda)}}$ reaches its maximum at the point $s_n := \frac{n+\lambda-1}{n+\lambda+1}$. It increases on the interval $[0, s_n]$ and decreases on the interval $[s_n, \infty)$. Since $s_n \to 1$, there exists a number $N \in \mathbb{N}$ such that $\mathrm{e}^{-\delta} < s_N$. Let $n \in \mathbb{N}$ with $n \geq N$. Then $\mathrm{e}^{-\delta} \leq s_N \leq s_n$, and for all $s \in (0, \mathrm{e}^{-\delta}]$ we obtain

$$\frac{s^{n+\lambda-1}}{(1+s)^{2(n+\lambda)}} \leq \frac{(\mathrm{e}^{-\delta})^{n+\lambda-1}}{(1+\mathrm{e}^{-\delta})^{2(n+\lambda)}}.$$

Integration of both sides from 0 to $\mathrm{e}^{-\delta}$ yields

$$\int_0^{\mathrm{e}^{-\delta}} \frac{s^{n+\lambda-1}}{(1+s)^{2(n+\lambda)}}\,ds \leq \left(\frac{\mathrm{e}^{-\delta}}{(1+e^{-\delta})^2}\right)^{n+\lambda} = \left(\frac{1}{4\cosh^2(\delta/2)}\right)^{n+\lambda}.$$

Applying Stirling's formula ([13, formula 8.327]), we have

$$\frac{1}{\mathrm{B}(n+\lambda, n+\lambda)} = \frac{\Gamma(2(n+\lambda))}{(\Gamma(n+\lambda))^2} \sim \frac{4^\lambda \Gamma(2n)}{(\Gamma(n))^2} \sim \frac{1}{2\sqrt{\pi n}} 4^{n+\lambda}.$$

Since $\cosh(\delta/2) > 1$,

$$\int_0^{\mathrm{e}^{-\delta}} \psi_{n,\lambda}(t)\frac{dt}{t} \leq \frac{1}{\mathrm{B}(n+\lambda, n+\lambda)}\left(\frac{1}{4\cosh^2(\delta/2)}\right)^{n+\lambda} \sim \frac{1}{2\sqrt{\pi n}\cosh^{2(n+\lambda)}(\delta/2)} \to 0.$$

To prove a similar result for the integral from e^{δ} to ∞, make the change of variable $s = 1/t$:

$$\lim_{n\to\infty} \int_{\mathrm{e}^\delta}^\infty \psi_{n,\lambda}(t)\,\frac{dt}{t} = \lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_{n,\lambda}(s)\,\frac{ds}{s}.$$

Let

$$R_{n,\delta} := \int_{\rho(s,1)>\delta} \psi_{n,\lambda}(s)\,\frac{ds}{s}, \tag{3.1}$$

then

$$\lim_{n\to\infty} R_{n,\delta} = \lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_{n,\lambda}(s)\,\frac{ds}{s} + \lim_{n\to\infty} \int_{\mathrm{e}^\delta}^\infty \psi_{n,\lambda}(s)\frac{ds}{s} = 0.$$ □

Introduce now the standard *Mellin convolution* of two functions a and b from $L_1(\mathbb{R}_+, d\nu)$:

$$(a * b)(x) := \int_0^\infty a(y) b\left(\frac{x}{y}\right) \frac{dy}{y}, \qquad x \in \mathbb{R}_+, \tag{3.2}$$

being a commutative and associative binary operation on $L_1(\mathbb{R}_+, d\nu)$.

Note that (3.2) is well defined also if one of the functions a or b belongs to $L_\infty(\mathbb{R}_+)$ and the other belongs to $L_1(\mathbb{R}_+, d\nu)$. In that case $a*b \in L_\infty(\mathbb{R}_+)$ and $a*b = b*a$. The associativity law also holds for any three functions a, b, c such that one of them belongs to $L_\infty(\mathbb{R}_+)$ and the other two belong to $L_1(\mathbb{R}_+, d\nu)$.

The next result is a special case of a well–known general fact on Dirac sequences and uniformly continuous functions on locally compact groups. For the reader's convenience we write a proof for our case.

Theorem 3.4.2. *Let* $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$. *Then*

$$\lim_{n\to\infty} \|\sigma * \psi_n^{(\lambda)} - \sigma\|_\infty = 0. \tag{3.3}$$

Proof. For every $n \in \mathbb{N}$, $\delta > 0$ and $x \in \mathbb{R}_+$,

$$\begin{aligned} |(\sigma * \psi_{n,\lambda})(x) - \sigma(x)| &= \left| \int_0^\infty \sigma\left(\frac{x}{y}\right) \psi_{n,\lambda}(y) \frac{dy}{y} - \int_0^\infty \sigma(x) \psi_{n,\lambda}(y) \frac{dy}{y} \right| \\ &\le \int_0^\infty \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_{n,\lambda}(y) \frac{dy}{y} = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\rho(y,1)\le\delta} \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_{n,\lambda}(y) \frac{dy}{y}, \\ I_2 &= \int_{\rho(y,1)>\delta} \left| \sigma\left(\frac{x}{y}\right) - \sigma(x) \right| \psi_{n,\lambda}(y) \frac{dy}{y}. \end{aligned}$$

If $\rho(y,1) \le \delta$, then $\rho(x/y, x) = \rho(x, xy) = \rho(y,1) \le \delta$. Thus

$$I_1 \le \omega_{\rho,\sigma}(\delta) \int_{\mathbb{R}} \psi_{n,\lambda}(y) \frac{dy}{y} = \omega_{\rho,\sigma}(\delta).$$

For the term I_2 we obtain an upper bound in terms of $R_{n,\delta}$, see (3.1):

$$I_2 \le 2\|\sigma\|_\infty \int_{\rho(y,1)>\delta} \psi_n(y) \frac{dy}{y} = 2\|\sigma\|_\infty R_{n,\delta}.$$

Therefore

$$\|\sigma * \psi_n - \sigma\|_\infty \le \omega_{\rho,\sigma}(\delta) + 2\|\sigma\|_\infty R_{n,\delta}.$$

Given $\varepsilon > 0$, first apply the hypothesis that $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and choose $\delta > 0$ such that $\omega_{\rho,\sigma}(\delta) < \frac{\varepsilon}{2}$. Then use the fact that $R_{n,\delta} \to 0$ and find a number $N \in \mathbb{N}$ such that $R_{n,\delta} < \frac{\varepsilon}{4\|\sigma\|_\infty}$ for all $n \geq N$. Then for all $n \geq N$

$$\|\sigma * \psi_{n,\lambda} - \sigma\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Recall now that, for each $m, n \in \mathbb{N}$, the *generalized Laguerre polynomial* (called also *associated Laguerre polynomial*) is defined by

$$L_n^{(\alpha)}(t) = \frac{1}{n!}\, t^{-\alpha}\, \mathrm{e}^t\, \frac{d^n}{dt^n}\Big(\mathrm{e}^{-t}\, t^{n+\alpha}\Big), \quad t \in \mathbb{R}_+.$$

Then, for each $n \in \mathbb{N}$, we introduce the function $\phi_{n,\lambda} \colon \mathbb{R}_+ \to \mathbb{C}$ by

$$\phi_{n,\lambda}(t) = \frac{\Gamma(\lambda+1)\Gamma(n)}{\Gamma^2(n+\lambda)}\, t^{n+\lambda}\, \mathrm{e}^{-t}\, L_{n-1}^{(n+2\lambda)}(t). \tag{3.4}$$

Each function $\phi_{n,\lambda}$ is obviously bounded and continuous on $\mathbb{R}_+$, and admits the following alternative representation

$$\frac{t^\lambda}{\Gamma(\lambda+1)}\phi_{n,\lambda}(t) = \frac{1}{\Gamma^2(n+\lambda)}\, \frac{d^{n-1}}{dt^{n-1}}\left(\mathrm{e}^{-t}\, t^{2(n+\lambda)-1}\right).$$

and the explicit representation

$$\phi_{n,\lambda}(t) = \frac{\Gamma(\lambda+1)}{\mathrm{B}(n+\lambda, n+\lambda)}\, \mathrm{e}^{-t}\, t^{n+\lambda} \sum_{k=0}^{n-1} \binom{n-1}{k} (-t)^k \frac{1}{\Gamma(k+n+2\lambda+1)}.$$

The next lemma relates the functions ψ_n and ϕ_n via the Laplace transform $\mathcal{L}$, which is defined by

$$\mathcal{L}(f)(s) := \int_0^\infty f(t)\, \mathrm{e}^{-st}\, dt.$$

Lemma 3.4.3. *For each $n \in \mathbb{N}$,*

$$\psi_{n,\lambda}(s) = s^{\lambda+1}\mathcal{L}\left(\frac{t^\lambda \phi_{n,\lambda}(t)}{\Gamma(\lambda+1)}\right)(s), \qquad s \in \mathbb{R}_+. \tag{3.5}$$

Proof. We can write

$$\mathcal{L}\left(t^{2(n+\lambda)-1} e^{-t}\right) = \frac{\Gamma(2(n+\lambda))}{(1+s)^{2(n+\lambda)}}$$

(see [13, Formula 17.13.26]), use the propertie

$$\mathcal{L}(F^{(n)}(t)) = s^n f(s) - s^{n-1}F(0) - \cdots - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

of the Laplace transform. Since the function $t^{2(n+\lambda)-1}e^{-t}$ and its first $n-1$ derivatives have zero limit in 0 and ∞ it follows that

$$\mathcal{L}\left(\frac{d^{n-1}}{dt^{n-1}}\left(t^{2(n+\lambda)-1}e^{-t}\right)\right) = \Gamma(2(n+\lambda))\frac{s^{n-1}}{(1+s)^{2(n+\lambda)}}$$

by multiplying both sides by $\frac{t^{\lambda+1}}{\Gamma^2(n+\lambda)}$ the equality(3.5) holds. □

Given a function $a\colon \mathbb{R}_+ \to \mathbb{C}$, we define $\widetilde{a}\colon \mathbb{R}_+ \to \mathbb{C}$ as $\widetilde{a}(t) = a(1/t)$.
The mapping $a \mapsto \widetilde{a}$ is obviously an involution:

$$\widetilde{\widetilde{a}} = a, \tag{3.6}$$

and, for all $a \in L_\infty(\mathbb{R}_+)$ and $b \in L_1(\mathbb{R}_+, d\nu)$, we have

$$\widetilde{a * b} = \widetilde{a} * \widetilde{b}. \tag{3.7}$$

The change of variable $t = \frac{1}{u}$ yields

$$\int_0^\infty a(t)b(st)\,\frac{dt}{t} = (\widetilde{a} * b)(s). \tag{3.8}$$

The next lemma relates "spectral functions" γ_a with Mellin convolutions.

Lemma 3.4.4. *Let* $\alpha(u) = 2u\,\mathrm{e}^{-2u}$, *then for each* $b \in L_\infty(\mathbb{R}_+)$,

$$\gamma_{b,\lambda} = \widetilde{b} * \alpha. \tag{3.9}$$

Proof. Rewrite $\gamma_{b,\lambda}$ in the form

$$\gamma_{b,\lambda}(s) = \int_0^\infty b(t)\,\left(\frac{2(st)^{(\lambda+1)}\,\mathrm{e}^{-2st}}{\Gamma(\lambda+1)}\right)\frac{dt}{t}$$

and apply (3.8). □

Introduce the function $m_2(s) := 2s$, then (3.5) and (3.9) imply that the elements ψ_n of the Dirac sequence are in fact certain "spectral functions":

$$\psi_{n,\lambda} = (\widetilde{\phi_{n,\lambda} \circ m_2}) * \alpha = \gamma_{\phi_{n,\lambda}\circ m_2}.$$

Now we are ready to prove the main result.
Recall first that, by Theorem 3.2.4, the C^*-algebra generated by vertical Toeplitz operators with bounded symbols is isometrically isomorphic to the C^*-algebra generated by the set

$$\Gamma_\lambda = \left\{\gamma_{b,\lambda} \mid b \in L_\infty(\mathbb{R}_+)\right\}. \tag{3.10}$$

Theorem 3.4.5. *We have that* $\overline{\Gamma_\lambda} = \mathrm{VSO}(\mathbb{R}_+)$.

Proof. Let $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$. For each $n \in \mathbb{N}$, we define $b_n \colon \mathbb{R}_+ \to \mathbb{C}$ by

$$b_n := \widetilde{\sigma} * (\phi_{n,\lambda} \circ m_2).$$

From (3.4) it follows that $\phi_n \in L_1(\mathbb{R}_+, d\nu)$, and thus $b_n \in L_\infty(\mathbb{R}_+)$. Then equations (3.7), (3.6) and the associativity of Mellin convolutions yield

$$\gamma_{b_n,\lambda} = \widetilde{b_n} * \alpha = \left(\widetilde{\widetilde{\sigma}} * \widetilde{(\phi_{n,\lambda} \circ m_2)}\right) * \alpha = \sigma * \left(\widetilde{(\phi_{n,\lambda} \circ m_2)} * \alpha\right) = \sigma * \psi_{n,\lambda},$$

which means that $\sigma_n * \psi_{n,\lambda} \in \Gamma_\lambda$. To finish the proof apply Theorem 3.4.2. □

Let us mention some important corollaries of the theorem. First of all it implies that the C^*-algebra $\mathcal{VT}(L_\infty)$ generated by Toeplitz operators with bounded vertical symbols is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$. Moreover it shows that the set of initial generators of $\mathcal{VT}(L_\infty)$ (i.e., the Toeplitz operators with bounded vertical symbols) is dense in $\mathcal{VT}(L_\infty)$. That is, the two quite different types of the closures, the C^*-algebraic closure and the topological closure, of the set of initial generators end up with the same result: the C^*-algebra $\mathcal{VT}(L_\infty)$ generated by Toeplitz operators with bounded vertical symbols.

Then, the theorem permits us to compare and realize the difference between the algebra generated by general vertical operators and its subalgebra generated by special vertical operators, Toeplitz operators with bounded vertical symbols. The first one is isomorphic to $L_\infty(\mathbb{R}_+)$, while the second, its subalgebra, is isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

In this connection it is interesting to consider "intermediate", in a sense, operators, the *bounded* vertical Toeplitz operators whose defining symbols are *unbounded*. As it turns out such operators *do not* necessarily belong to the algebra $\mathcal{VT}(L_\infty)$ generated by vertical Toeplitz operators with *bounded* symbols.

The next section is devoted to an example of such an operator.

3.5 Example

Write here γ_b for $\gamma_{b,\lambda}$ and $\lambda = 0$. Note that γ_b can be defined by the formula (3.1) not only if $b \in L_\infty(\mathbb{R}_+)$, but also if $b \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\, dt)$ for all $\eta > 0$.

In this section we construct a non-bounded function $b \colon \mathbb{R}_+ \to \mathbb{C}$ such that $b \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\, dt)$ for all $\eta > 0$ and $\gamma_b \in L_\infty(\mathbb{R}_+)$, but $\gamma_b \notin \mathrm{VSO}(\mathbb{R}_+)$. This implies that the corresponding vertical Toeplitz operator is bounded, but it does not belong to the C^*-algebra generated by vertical Toeplitz operators with bounded generating symbols.

The idea of this example is taken from [17].

Proposition 3.5.1. *Define* $f\colon \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\} \to \mathbb{C}$ *by*

$$f(z) := \frac{1}{z+1} \exp\left(\frac{i}{3\pi} \ln^2(z+1)\right), \tag{3.1}$$

where ln *is the principal value of the natural logarithm (with imaginary part in* $(-\pi, \pi]$*). Then there exists a unique function* $A\colon \mathbb{R}_+ \to \mathbb{C}$ *such that* $A \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta u}\, du)$ *for all* $\eta > 0$ *and* f *is the Laplace transform of* A*:*

$$f(z) = \int_0^{+\infty} A(u)\, \mathrm{e}^{-zu}\, du.$$

Proof. For every $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ we write $\ln(z+1)$ as $\ln|z+1| + i \arg(z+1)$ with $-\frac{\pi}{2} < \arg(z+1) < \frac{\pi}{2}$. Then

$$\begin{aligned} |f(z)| &= \frac{1}{|z+1|} \left|\exp\left(\frac{i}{3\pi}\bigl(\ln|z+1| + i\arg(z+1)\bigr)^2\right)\right| \\ &= \frac{1}{|z+1|} \exp\left(-\frac{2\arg(z+1)}{3\pi} \ln|z+1|\right) \\ &= \frac{1}{|z+1|^{1+\frac{2\arg(z+1)}{3\pi}}}. \end{aligned}$$

Since $|z+1| \geq 1$ and $-\frac{1}{3} < -\frac{2\arg(z+1)}{3\pi} < \frac{1}{3}$,

$$|f(z)| \leq \frac{1}{|z+1|^{2/3}}.$$

Therefore for every $x > 0$,

$$\int_{\mathbb{R}} |f(x+iy)|^2\, dy \leq \int_{\mathbb{R}} \frac{dy}{((x+1)^2 + y^2)^{2/3}} < \int_{\mathbb{R}} \frac{dy}{(1+y^2)^{2/3}} < +\infty,$$

and f belongs to the Hardy class H^2 on the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. By Paley–Wiener theorem (see, for example, Rudin [31, Theorem 19.2]), there exists a function $A \in L_2(\mathbb{R}_+)$ such that for all $x > 0$

$$f(x) = \int_0^{+\infty} A(u)\, \mathrm{e}^{-ux}\, du.$$

The uniqueness of A follows from the injective property of the Laplace transform. Applying Hölder's inequality we easily see that $A \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta u}\, du)$ for all $\eta > 0$:

$$\int_0^{+\infty} |A(u)|\, \mathrm{e}^{-\eta u}\, du \leq \|A\|_2 \left(\int_0^{+\infty} \mathrm{e}^{-2\eta u}\, du\right)^{1/2} = \frac{\|A\|_2}{\sqrt{2\eta}}. \qquad \square$$

Proposition 3.5.2. *The function* $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ *defined by*

$$\sigma(s) := \frac{s}{s+1} \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right), \tag{3.2}$$

belongs to $L_\infty(\mathbb{R}_+) \setminus \mathrm{VSO}(\mathbb{R}_+)$. *Moreover there exists a function* $b\colon \mathbb{R}_+ \to \mathbb{C}$ *such that* $b \in L_1(\mathbb{R}_+, \mathrm{e}^{-\eta t}\, dt)$ *for all* $\eta > 0$ *and* $\sigma = \gamma_b$.

Proof. The function σ is bounded since $|\sigma(s)| \le \frac{s}{s+1} \le 1$ for all $s \in \mathbb{R}_+$. Let A be the function from Proposition 3.5.1. Define $b\colon \mathbb{R}_+ \to \mathbb{C}$ by

$$b(s) = A(2s).$$

Then for all $\eta > 0$

$$\int_0^{+\infty} |a(t)|\, \mathrm{e}^{-\eta t}\, du = \frac{1}{2} \int_0^{+\infty} |A(t)|\, \mathrm{e}^{-\eta t/2}\, dt < +\infty,$$

and

$$\begin{aligned}
\gamma_b(s) &= 2s \int_0^{+\infty} b(t)\, \mathrm{e}^{-2st}\, dt = 2s \int_0^{+\infty} A(2t)\, \mathrm{e}^{-2st}\, dt \\
&= s \int_0^{+\infty} A(t)\, \mathrm{e}^{-st}\, dt = \frac{s}{s+1} \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) = \sigma(s).
\end{aligned}$$

Let us prove that $\sigma \notin \mathrm{VSO}(\mathbb{R}_+)$. For all $s, t \in \mathbb{R}_+$

$$\begin{aligned}
|\sigma(s) - \sigma(t)| &= \Bigg| \left(1 - \frac{1}{s+1}\right) \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) \\
&\qquad - \left(1 - \frac{1}{t+1}\right) \exp\left(\frac{i}{3\pi} \ln^2(t+1)\right) \Bigg| \\
&\ge \left| \exp\left(\frac{i}{3\pi} \ln^2(s+1)\right) - \exp\left(\frac{i}{3\pi} \ln^2(t+1)\right) \right| \\
&\qquad - \frac{1}{s+1} - \frac{1}{t+1} \\
&= \left| \exp\left(\frac{i}{3\pi} \big(\ln^2(s+1) - \ln^2(t+1)\big)\right) - 1 \right| - \frac{1}{s+1} - \frac{1}{t+1}.
\end{aligned}$$

Replace s by the following function of t:

$$s(t) := t + \frac{t+1}{\ln^{1/2}(t+1)}.$$

Then

$$\begin{aligned}
\ln(s(t)+1) &= \ln(t+1) + \ln\left(1 + \frac{1}{\ln^{1/2}(t+1)}\right) \\
&= \ln(t+1) + \frac{1}{\ln^{1/2}(t+1)} - \frac{1}{2\ln(t+1)} + \mathcal{O}\left(\frac{1}{\ln^{3/2}(t+1)}\right).
\end{aligned}$$

Denote $\ln^2(s(t)+1) - \ln^2(t+1)$ by L_t and consider the asymptotic behavior of L_t as $t \to +\infty$:

$$L_t := \ln^2(s(t)+1) - \ln^2(t+1) = -1 + 2\ln^{1/2}(t+1) + \mathcal{O}\left(\frac{1}{\ln(t+1)}\right).$$

Since L_t is continuous and tends to $+\infty$ as $t \to +\infty$, for every $T > 40$ there exists an integer $t \geq T$ such that $L_t + 1$ is equal to an integer multiple of $6\pi^2$, say to $6m\pi^2$:

$$L_t + 1 = 6m\pi^2.$$

For such t,

$$\begin{aligned}\left|\exp\left(\frac{i}{3\pi}L_t\right) - 1\right| &= \left|\exp\left(\frac{i}{3\pi}(6m\pi^2 - 1)\right) - 1\right| \\ &= \left|\exp\left(-\frac{i}{3\pi}\right) - 1\right| \approx 0.106 > \frac{1}{10}\end{aligned}$$

and

$$|\sigma(s(t)) - \sigma(t)| \geq \left|\exp\left(\frac{i}{3\pi}L_t\right) - 1\right| - \frac{2}{T+1} > \frac{1}{10} - \frac{1}{20} = \frac{1}{20}.$$

It means that $|\sigma(s(t)) - \sigma(t)|$ does not converge to 0 as t goes to infinity. On the other hand,

$$\rho(s(t), t) = \ln\frac{s(t)}{t} \leq \frac{t+1}{t\ln^{1/2}(t+1)} \to 0.$$

Thus $\sigma \notin \mathrm{VSO}(\mathbb{R}_+)$. $\square$

Chapter 4

Radial revisited

4.1 Very slowly oscillating functions and sequences

To describe the relations between VSO($\mathbb{N}$) and VSO($\mathbb{R}_+$), we introduce the piecewise-linear extensions of sequences as follows.

Let $\sigma\colon \mathbb{N} \to \mathbb{C}$. Denote by f the function $\mathbb{R}_+ \to \mathbb{C}$ obtained from σ by the piecewise-linear interpolation:

$$f(x) := \begin{cases} \sigma_1, & x \in (0,1); \\ (j+1-x)\sigma_j + (x-j)\sigma_{j+1}, & x \in [j, j+1),\ j \in \mathbb{N}. \end{cases} \tag{4.1}$$

In what follows $\lfloor x \rfloor$ stand for the integer part of $x \in \mathbb{R}_+$.

Lemma 4.1.1. *Given $\sigma\colon \mathbb{N} \to \mathbb{C}$, we define f by* (2.5)*. Then $\|f\|_\infty = \|\sigma\|_\infty$,*

$$|f(x) - f(y)| \le (y-x)\omega_{\rho,\sigma}(1), \qquad 0 < x < y \tag{4.2}$$

and

$$|f(x) - f(y)| \le 2\omega_{\rho,\sigma}(\rho(\lfloor x \rfloor, \lfloor y \rfloor + 1)), \qquad 1 \le x < y. \tag{4.3}$$

Proof. Put $\sigma_0 = \sigma_1$. Then (4.1) can be rewritten as

$$f(x) = (\lfloor x \rfloor + 1 - x)\sigma_{\lfloor x \rfloor} + (x - \lfloor x \rfloor)\sigma_{\lfloor x \rfloor + 1}.$$

Since the value of f at every point $x > 0$ is a convex combination of two values of the original sequence σ, the inequality $\|f\|_\infty \le \|\sigma\|_\infty$ holds. On the other hand, f is an extension of σ, therefore the inverse inequality is also true.

An elementary computation shows that if $s, t > 0$ and s, t belong to the same interval $[j, j+1]$ for some $j \in \{0, 1, 2, \ldots\}$, then

$$|f(s) - f(t)| = |t - s|\,|\sigma_j - \sigma_{j+1}|.$$

Since $|\sigma_j - \sigma_{j+1}| \le \omega_{\rho,\sigma}(\rho(j,j+1)) \le \omega_{\rho,\sigma}(1)$ for every $j \in \mathbb{N}$ and $\sigma_0 = \sigma_1$,

$$|f(s) - f(t)| \le |t-s|\,\omega_{\rho,\sigma}(1), \qquad \lfloor t\rfloor = \lfloor s\rfloor = j \in \mathbb{Z}_+. \tag{4.4}$$

To prove (4.2), assume that $0 < x < y$. The case $\lfloor x\rfloor = \lfloor y\rfloor$ is already covered by (4.4). If $\lfloor x\rfloor < \lfloor y\rfloor$, then insert intermediate integer points between x and y and apply (4.4) in each segment of this division:

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - \sigma_{\lfloor x\rfloor+1}| + \sum_{j=\lfloor x\rfloor+1}^{\lfloor y\rfloor-1} |\sigma_j - \sigma_{j+1}| + |\sigma_{\lfloor y\rfloor} - f(y)| \\ &\le (\lfloor x\rfloor + 1 - x)\omega_{\rho,\sigma}(1) + (\lfloor y\rfloor - \lfloor x\rfloor - 1)\omega_{\rho,\sigma}(1) \\ &\quad + (y - \lfloor y\rfloor)\omega_{\rho,\sigma}(1) \\ &= (y-x)\omega_{\rho,\sigma}(1).\end{aligned}$$

To prove (4.3), suppose that $1 \le x < y$. Then

$$\begin{aligned}|f(x) - f(y)| &= |(1-u)\sigma_j + u\sigma_{j+1} - (1-v)\sigma_k - v\sigma_{k+1}| \\ &\le (1-u)|\sigma_j - \sigma_k| + u|\sigma_{j+1} - \sigma_{k+1}| + |u-v|\,|\sigma_k - \sigma_{k+1}| \\ &\le 2\omega_{\rho,\sigma}(\rho(j,k+1)).\end{aligned}$$

□

For every function $f \in \mathrm{VSO}(\mathbb{R}_+)$ we denote by $R(f)$ its restriction onto $\mathbb{N}$, and for every sequence $\sigma \in \mathrm{VSO}(\mathbb{N})$ we denote by $E(\sigma)$ its piecewise-linear extension defined in (4.1). Note that $R(E(\sigma)) = \sigma$ for every $\sigma \in \mathrm{VSO}(\mathbb{N})$.

Theorem 4.1.2. *The mapping* $R\colon \mathrm{VSO}(\mathbb{R}_+) \to \mathrm{VSO}(\mathbb{N})$ *is an epimorphism of C^*-algebras. In particular, the set* $\mathrm{VSO}(\mathbb{N})$ *of sequences coincides with the set of the restrictions of functions from* $\mathrm{VSO}(\mathbb{R}_+)$:

$$\mathrm{VSO}(\mathbb{N}) = \{R(f)\colon\ f \in \mathrm{VSO}(\mathbb{R}_+)\}.$$

Proof. It is easy to see that $R(\mathrm{VSO}(\mathbb{R}_+)) \subseteq \mathrm{VSO}(\mathbb{N})$ and that R is a homomorphism. In order to prove that R is surjective, we start with $\sigma \in \mathrm{VSO}(\mathbb{N})$ and construct $f = E(\sigma)$, then $\|f\|_\infty = \|\sigma\|_\infty$. Considering two cases: $y - x < \sqrt{\delta}$ and $y - x \ge \sqrt{\delta}$, we prove first that for every $\delta \in (0, 1/4)$

$$\Omega_{\rho,f}(\delta) \le \max\bigl(\sqrt{\delta}\,\omega_{\rho,\sigma}(1), 2\omega_{\rho,\sigma}(6\sqrt{\delta})\bigr). \tag{4.5}$$

Let $y - x < \sqrt{\delta}$, then by (4.2)

$$|f(x) - f(y)| \le \sqrt{\delta}\,\omega_{\rho,\sigma}(1). \tag{4.6}$$

If $y - x \ge \sqrt{\delta}$, then

$$\delta \ge \rho(x,y) = \ln\frac{y}{x} \ge \frac{y-x}{y} \quad \text{and} \quad y \ge \frac{y-x}{\delta} \ge \frac{1}{\sqrt{\delta}}.$$

Moreover

$$x \geq y - y\delta \geq \frac{3y}{4} \geq \frac{3}{4\sqrt{\delta}}.$$

Therefore

$$x - 1 \geq \frac{3}{4\sqrt{\delta}} - 1 = \frac{3 - 4\sqrt{\delta}}{4\sqrt{\delta}} \geq \frac{1}{4\sqrt{\delta}}.$$

Finally

$$\begin{aligned}\rho(\lfloor x\rfloor, \lfloor y\rfloor + 1) = \ln\frac{\lfloor y\rfloor + 1}{\lfloor x\rfloor} &\leq \ln\frac{y+1}{x-1} = \ln\frac{y}{x} + \ln\frac{y+1}{y} + \ln\frac{x}{x-1} \\ &\leq \delta + \sqrt{\delta} + 4\sqrt{\delta} \leq 6\sqrt{\delta}.\end{aligned}$$

Applying (4.3) we conclude that if $y - x > \sqrt{\delta}$, then

$$|f(x) - f(y)| \leq \omega_{\rho,\sigma}(6\sqrt{\delta}). \tag{4.7}$$

Combining both cases $y - x < \sqrt{\delta}$ and $y - x \geq \sqrt{\delta}$, we obtain from (4.6) and (4.7) that

$$|f(x) - f(y)| \leq \max\big(\sqrt{\delta}\,\omega_{\rho,\sigma}(1),\, \omega_{\rho,\sigma}(6\sqrt{\delta})\big),$$

which implies (4.5). Inequality (4.5) guarantees that $\Omega_{\rho,f}(\delta) \to 0$ as $\delta \to 0$. □

Remark 4.1.3. *Theorem 4.1.2 was stated for the algebras of* bounded *very slowly oscillating sequences and functions, but the proof of* (4.5) *does not use the condition of boundedness. Therefore a result analogous to* $\mathrm{VSO}(\mathbb{N}) = \{R(f)\colon\ f \in \mathrm{VSO}(\mathbb{R}_+)\}$ *holds also for the corresponding classes of sequences and functions without the condition of boundedness.*

4.2 From vertical to radial case

For the reader's convenience we recall briefly the results of [18, 19] for the vertical Toeplitz operators.

We denote by Γ_λ the set of all spectral functions (3.10) for $b \in L_\infty(\mathbb{R}_+)$.

Remember here Theorem 3.4.5

$$\Gamma_\lambda \text{ is a dense subset of } \mathrm{VSO}(\mathbb{R}_+).$$

Passing to the radial case, we observe that in the non-weighted one-dimensional case ($\lambda = 0$, $n = 1$) the sequence $\beta_{a,0}$ is just the restriction to $\mathbb{N}$ of the function $\gamma_{b,0}$, where a and b are related by $a(r) = b(-\ln(r))$. In the weighted case the situation is a bit more complicated: in addition to the variable change $v = -\ln(r)$, two "correcting factors", an inner factor ξ_λ and an outer factor $\eta_{n,\lambda}$ are needed.

Lemma 4.2.1. *Let* $b \in L_\infty(\mathbb{R}_+)$*. Define*

$$a(\sqrt{r}) = \xi_{n,\lambda}(r)\, b\left(\frac{-\ln(r)}{2}\right), \qquad 0 < r < 1, \tag{4.8}$$

where

$$\xi_{n,\lambda}(r) = \left(\frac{-\ln(r)}{1-r}\right)^{\lambda} \frac{1}{r^{n-1}}, \qquad 0 < r < 1. \tag{4.9}$$

Then

$$\beta_{a,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k), \qquad k \in \mathbb{N}, \tag{4.10}$$

where

$$\eta_{n,\lambda}(k) = \frac{\Gamma(k+n+\lambda)}{k^{\lambda+1}\,\Gamma(k+n-1)}. \tag{4.11}$$

Proof. Direct computation. We start with (2.2), substitute (4.8), and make change of variables $v = -\ln(r)$:

$$\begin{aligned}
\beta_{a,n,\lambda}(k) &= \frac{1}{\mathrm{B}(k+n-1,\lambda+1)} \int_0^1 a(\sqrt{r}) r^{k+n-2}(1-r)^{\lambda}\,\mathrm{d}r \\
&= \frac{\Gamma(k+n+\lambda)}{\Gamma(k+n-1)\Gamma(\lambda+1)} \int_0^1 b\left(-\frac{\ln(r)}{2}\right) (-\ln(r))^{\lambda} r^{k-1}\,\mathrm{d}r \\
&= \frac{\Gamma(k+n+\lambda)}{k^{\lambda+1}\,\Gamma(k+n-1)}\,\frac{k^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} b\left(\frac{v}{2}\right) v^{\lambda}\,\mathrm{e}^{-kv}\,\mathrm{d}v \\
&= \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k).
\end{aligned}$$

□

Note that the function a defined by (4.8) can be unbounded, in general.

Lemma 4.2.2. *Let* $b \in L_\infty(\mathbb{R}_+)$*. For every* $L > 0$ *denote by* $\chi_{(0,L)}$ *the characteristic function of* $(0,L)$*. Then*

$$\lim_{L\to+\infty} \sup_{x\geq 1} |\gamma_{b,\lambda}(x) - \gamma_{b\chi_{(0,L)},\lambda}(x)| = 0.$$

Proof. For every $x \geq 1$,

$$\begin{aligned}
|\gamma_{b,\lambda}(x) - \gamma_{b\chi_{(0,L)},\lambda}(x)| &\leq \frac{\|b\|_\infty\, x^{\lambda+1}}{\Gamma(\lambda+1)} \int_L^{+\infty} \mathrm{e}^{-xv}\, v^{\lambda}\,\mathrm{d}v \\
&= \frac{\|b\|_\infty}{\Gamma(\lambda+1)} \int_{Lx}^{+\infty} \mathrm{e}^{-t}\, t^{\lambda}\,\mathrm{d}t \\
&\leq \frac{\|b\|_\infty}{\Gamma(\lambda+1)} \int_{L}^{+\infty} \mathrm{e}^{-t}\, t^{\lambda}\,\mathrm{d}t.
\end{aligned}$$

The integrability of the function $t \mapsto \mathrm{e}^{-t}\, t^{\lambda}$ ensures that the last expression tends to 0 as $L \to +\infty$. □

Lemma 4.2.3. *The sequence* $\eta_{n,\lambda} = (\eta_{n,\lambda}(k))_{k\in\mathbb{N}}$ *defined by* (4.11) *tends to* 1 *as* $k \to \infty$, *and, in particular, it is bounded.*

Proof. We write

$$\eta_{n,\lambda}(k) = \left(\frac{k+n-1}{k}\right)^{\lambda+1} \frac{\Gamma(k+n-1+\lambda+1)}{\Gamma(k+n-1)(k+n-1)^{\lambda+1}},$$

then using [13, Formula 8.328.2] we obtain required

$$\lim_{k\to\infty} \eta_{n,\lambda}(k) = 1. \qquad \square$$

As already was proved, the set Γ_λ is dense in $\mathrm{VSO}(\mathbb{R}_+)$. Now we are going to deduce from this fact that $\mathrm{B}_{n,\lambda}$ is a dense subset of $\mathrm{VSO}(\mathbb{N})$.

Theorem 4.2.4. *For each* $a \in L_\infty(0,1)$, $\beta_{a,n,\lambda}$ *belongs to* $\mathrm{VSO}(\mathbb{N})$.

Proof. We start from a function $a \in L_\infty(0,1)$, and introduce $a_1 = a \cdot \chi_{[\frac{1}{2},1)}$ and $a_2 = a - a_1 = a \cdot \chi_{[0,\frac{1}{2}]}$. We have that $\beta_{a_2,n,\lambda} \in c_0 \subset \mathrm{VSO}(\mathbb{N})$. Thus it is sufficient to show that $\beta_{a_1,n,\lambda} \in \mathrm{VSO}(\mathbb{N})$. Reverting (4.8) we define

$$b(\tfrac{v}{2}) :\, = a_1\left(e^{-\frac{v}{2}}\right)\left(\frac{1-e^{-v}}{v}\right)^{\lambda} e^{-v(n-1)}.$$

As

$$\lim_{v\to 0} \frac{1-e^{-v}}{v} = 1,$$

the function b is bounded, thus $\gamma_{b,\lambda}$ belongs to $\mathrm{VSO}(\mathbb{R}_+)$ and $\gamma_{b,\lambda}|_{\mathbb{N}} \in \mathrm{VSO}(\mathbb{N})$.

By (4.10), we have $\beta_{a_1,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{b,\lambda}(k)$, $k \in \mathbb{N}$, and thus $\beta_{a_1,n,\lambda} \in \mathrm{VSO}(\mathbb{N})$ as a product of two $\mathrm{VSO}(\mathbb{N})$-sequences. $\square$

Theorem 4.2.5. *The set* $\mathrm{B}_{n,\lambda}$ *is dense in* $\mathrm{VSO}(\mathbb{N})$.

Proof. We start from a sequence $\nu \in \mathrm{VSO}(\mathbb{N})$, and define the sequence σ as

$$\sigma(k) := \frac{\nu(k)}{\eta_{n,\lambda}(k)}, \qquad k \in \mathbb{N}.$$

By Lemma 4.2.3, $\sigma \in \mathrm{VSO}(\mathbb{N})$. Using Theorem 4.1.2 we construct a function f in $\mathrm{VSO}(\mathbb{R}_+)$ such that σ is the restriction of f to $\mathbb{N}$. Since $f \in \mathrm{VSO}(\mathbb{R}_+)$, by Theorem 3.4.5, for each $\varepsilon > 0$ there exists $g \in L_\infty(\mathbb{R}_+)$ such that

$$\|f - \gamma_{g,\lambda}\|_\infty < \frac{\varepsilon}{2\|\eta_{n,\lambda}\|_\infty}.$$

By Lemma 4.2.2, we take $L > 0$ such that

$$\sup_{x\geq 1} |\gamma_{g,\lambda}(x) - \gamma_{g\chi_{(0,L)},\lambda}(x)| < \frac{\varepsilon}{2\|\eta_{n,\lambda}\|_\infty}.$$

Define

$$a(\sqrt{r}) = \chi_{(0,L)}\left(\frac{-\ln r}{2}\right)\xi_{n,\lambda}(r) g\left(\frac{-\ln r}{2}\right), \qquad 0 < r < 1.$$

Factor $\chi_{(0,L)}$ insures that the function a vanishes near zero and is bounded. By Lemma 4.2.1

$$\beta_{a,n,\lambda}(k) = \eta_{n,\lambda}(k)\gamma_{g\chi_{(0,L)},\lambda}(k).$$

Therefore for every $k \in \mathbb{N}$

$$\begin{aligned} &|\nu(k) - \beta_{a,n,\lambda}(k)| = \eta_{n,\lambda}(k)|\sigma(k) - \gamma_{g\chi_{(0,L)},\lambda}(k)| \\ &\leq \|\eta_{n,\lambda}\|_\infty \left(\|f - \gamma_{g,\lambda}\|_\infty + \sup_{x\geq 1} |\gamma_{g,\lambda}(x) - \gamma_{g\chi_{(0,L)},\lambda}(x)| \right) < \varepsilon. \end{aligned}$$

□

Corollary 4.2.6. *For every $n \in \mathbb{N}$ and $\lambda > -1$ the C^*-algebra generated by Toeplitz operators $T_{a,n,\lambda}$ with bounded measurable radial symbols a is isometrically isomorphic to the algebra* VSO($\mathbb{N}$)*. The isomorphism is generated by the following assignment*

$$T_{a,n,\lambda} \quad \longmapsto \quad \beta_{a,n,\lambda}.$$

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