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Planeadores motrices multitareas en algunos espacios de configuración y productos poliédricos

T E S I S

Que presenta

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Multitasking motion planning in some configuration spaces and polyhedral products

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A mis padres : *Ignacia y Raúl.*

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Resumen

En esta tesis calculamos un invariante homotópico llamado "Complejidad Topológica Secuencial" para algunos productos poliédricos y para algunos espacios de configuraciones. Este invariante está relacionado a un problema particular de planeación motriz multitareas en un espacio conexo por arcos. En el Capítulo 2, calculamos este invariante para subcomplejos de productos de esferas, dando una descripción explícita en términos de información combinatoria asociada al subcomplejo. Los siguientes espacios para los cuales este invariante fue calculado son los espacios de configuraciones de n puntos distintos en una superficie orientable de género g, denotado por $\mathrm{Conf}(\Sigma_g, n)$, en el capítulo 4. Además, en el capítulo 3 estudiamos el comportamiento asintótico de este invariante para una familia aleatoria de subcomplejos de productos de círculos, dicho estudio es posible por los cálculos hechos en el caso determinístico presentados en el Capítulo 2. El valor de la complejidad topológica de Farber de todos estos espacios ha sido calculado previamente. Por lo tanto, los cálculos presentados aquí son generalizaciones de trabajos anteriores, pero es importante mencionar que no son consecuencias inmediatas, de hecho, se incluyen correcciones a las pruebas originales (en los Capítulos 2 y 4).

Abstract

In this thesis we compute a homotopy invariant called "Higher (or Sequential) Topological Complexity" for some polyhedral product spaces and some configuration spaces. This invariant is related to the problem of solving a particular multitasking motion planning problem in a path connected space. In Chapter 2 we compute this invariant for subcomplexes of products of spheres by giving an explicit description just in combinatorial terms associated to the subcomplex. The second kind of spaces for which we compute this invariant are the configuration spaces of n distinct ordered points in a orientable surface of genus g, denoted by $\mathrm{Conf}(\Sigma_g, n)$, in Chapter 4. Moreover, In Chapter 3 we study the asymptotic behavior of this invariant for a particular random family of subcomplexes of products of circles by using results in the deterministic case presented in Chapter 2. The value of Farber's topological complexity of all spaces we work with has already been computed. Thus, all computations presented here are generalizations of previous computations but they do not follow from straightforward arguments since, for instance, corrections (in Chapters 2 and 4) to the originals proofs are also provided.

Contents

Introduction

The higher or sequential topological complexity (higher or sequential TC) is a concept introduced by Yuli Rudyak in 2010 as a generalization of Farber's topological complexity introduced in [11] as a model to study the continuity instabilities in the motion planning of an autonomous system (robot). The term "higher" (or "sequential") comes from the consideration of a multitasking motion of a robot, and not only of initial-final tasks as in Farber's original concept. Roughly speaking, given a path connected space X and a positive integer s, the multitasking motion planning problem that the higher topological complexity is concerned with consists of connecting any s different points of our space X by a continuous path, with the additional requirement of doing this in a robust way. Here, we can think of X as the configuration space of a system. As the usual topological complexity, its higher version arises as the Schwarz genus of a fibration, thus it is a homotopy invariant. The homotopy invariance of the different versions of TC is a central feature that has captured much attention from topologists in recent years. In particular, standard obstruction theory can be used to obtain a general upper bound for $\mathrm{TC}_s(X)$ in terms of $\mathrm{hdim}(X)$, the homotopy dimension of X—that is, the minimal dimension of CW complexes having the homotopy type of X.

Concretely, for a positive integer s, the s-th (higher or sequential) topological complexity of X, $\mathrm{TC}_s(X)$, is defined as the reduced Schwarz genus (or the sectional category, *secat*) of the fibration

$$e_s = e_s^X : PX \to X^s$$

given by $e_s(\gamma) = \Big(\gamma(0), \gamma(\frac{1}{s-1}), \ldots, \gamma(\frac{s-2}{s-1}), \gamma(1)\Big)$, where $PX = \{\gamma : [0,1] \to X\}$ is the path space of X. Thus, $\mathrm{TC}_s(X) + 1$ is by definition the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. We write TC for TC_2, the standard topological complexity.

In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X) + 1$ local domains.

In this thesis, we compute TC_s for subcomplexes of products of spheres and for configuration spaces $\mathrm{Conf}(\Sigma_g, n)$ of n distinct ordered points in a orientable surface of genus g. The computations presented here are generalizations of the computations in [5] and [4] where the standard topological complexity was described for these spaces. It is worth mentioning that our proofs are not straightforward generalizations, in fact, in some cases we fixed some arguments in previous proofs. These mistakes will be pointed out carefully in Chapters 2 and 4. In Chapter 3 we use the description of TC_s of subcomplexes of products of odd-dimensional spheres (given in Chapter 2), for computing its asymptotic value for a particular family of random subcomplexes of products of circles (whose random nature will be inherited from the Erdős-Rényi model on graphs). The asymptotic behavior of the standard TC was studied in [9] .

In Chapter 1 we introduce some equivalent definitions of the higher TC and describe some of its standard properties such as homotopy invariance and standard lower and upper bounds.

In Chapter 2 we study subcomplexes of products of spheres, $X \subseteq \mathbb{S}(k_1, \ldots, k_n) := S^{k_1} \times \cdots \times S^{k_n}$, which can be thought as polyhedral product spaces of a family of pointed spheres. These spaces arise in several presentations depending on the constraints imposed either on the combinatorics of the subcomplex or in the dimension of the spheres. For instance, a particular family coming from taking subcomplexes of products of circles is related to complements of complex hyperplane arrangements in general position (see Example 2.1.2). Also, for each graph in n vertices we can get a subcomplex of a product of n circles that turns out to be an Eilenberg-MacLane space corresponding to the right-angled Artin group of the graph (see Example 2.1.3).

All our efforts in Chapter 2 are directed to prove

Theorem 0.0.1. *A subcomplex X of $\mathbb{S}(k_1, \ldots, k_n)$ has* $\mathrm{TC}_s(X) = \mathrm{zcl}_s(H^*(X; \mathbb{Q}))$.

The right side in the equality is the lower bound described in Definition 1.3.2. We provide an explicit description of $\mathrm{zcl}_s(H^*(X; \mathbb{Q}))$. The answer turns out to depend exclusively on the parity of the sphere dimensions k_i and on the combinatorics of the abstract simplicial complex underlying X. In order to better appreciate the phenomenon, it is convenient to focus first on the case where all the k_i have the same parity. The corresponding descriptions, in Theorems 2.2.5 and 2.2.13 as well as Corollary 2.4.4, generalize those in [5, 29]. The unrestricted description is given in Subsection 2.5.1 (see Theorem 2.5.1). In either case, the optimality of the cohomological lower bound will be a direct consequence of the fact that we actually construct an optimal motion planner. Our construction generalizes, in a highly non-trivial way, the one given first in ([30]) for $s = 2$ when X is an arrangement complement, and then independently by Cohen-Pruidze ([5], as corrected in [15]) in a more general case. The material presented in this chapter has been published in [17].

At the beginning of Chapter 3 we introduce the Erdős-Rényi model on graphs with n vertices and probability parameter p $(0 < p < 1)$, denoted by $\mathcal{G}(n, p)$, and define the random clique variable C which assigns to each random graph the maximum cardinality of a set of vertices inducing a complete subgraph in the graph. In other words, C assigns to each graph the cardinality of its largest possible clique. In the 1970s Matula studied the behavior of the random clique variable when n (the number of vertices in the graph) tends to infinity (see Theorem 3.1.1). In [9], Costa and Farber showed that, with probability tending to 1 as n tends to infinity, a random graph in $\mathcal{G}(n, p)$ has a pair of disjoint asymptotically-largest-possible cliques. In our first main result (Theorem 3.2.1) we show, more generally, that for any fixed positive integer s, and with probability tending to 1 as n tends to infinity, a random graph in $\mathcal{G}(n, p)$ has s pairwise-disjoint such asymptotically-largest-possible cliques. The topological spaces studied in this chapter, arise as follows: Given $\Gamma \in \mathcal{G}(n, p)$, we consider its random clique complex K_Γ, an abstract simplicial complex whose k-simplices correspond to complete subgraphs in Γ of $k + 1$ vertices. Then, we take the product of n circles and a subcomplex,

$$X_\Gamma \subseteq \mathbb{S}(\underbrace{1, 1, \ldots, 1}_{n \text{ factors}})$$

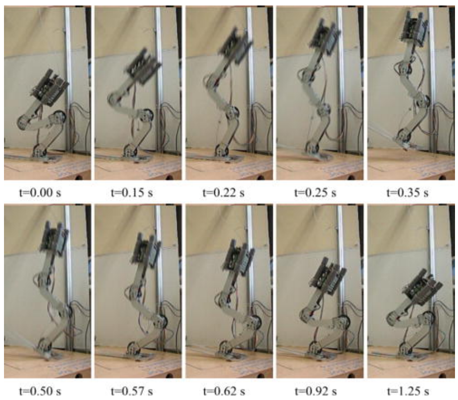
whose cells are indexed by K_Γ. In this way, the random nature of Γ induces a random behavior in X_Γ. In this chapter we give an estimation of $\mathrm{TC}_s(X_\Gamma)$ when n tends to infinity. For this purpose, we use Theorem 3.2.1, which will be enough in view of Theorem 2.2.5 in Chapter 2. Our computations extend the ones given in [9] for the usual TC. This chapter is the result of a collaboration with Hugo Mas and professor Jesús González.

In Chapter 4 we introduce the configuration space of n ordered distinct points of a orientable surface of genus g, $\mathrm{Conf}(\Sigma_g, n)$. These spaces play an important role in a number of settings in mathematics, see [6] for instance . Farber's topological complexity of $\mathrm{Conf}(\Sigma_g, n)$ has been described in [4]. The purpose of this chapter is to extend Cohen-Farber's results by describing (in Theorem 4.0.4 below) the higher topological complexity of $\mathrm{Conf}(\Sigma_g, n)$. Upper bounds are obtained by basically dimension reasons and lower bounds by cohomological computations. The main result in this chapter is Theorem 4.3.2 where we assert the nontriviality (and linearly independence) of two products of s-th zero-divisors in a quotient of a subalgebra of $H^*(\mathrm{Conf}(\Sigma_g, n), \mathbb{Q})^{\otimes s}$, which arises from the Totaro spectral sequence. Finally, this chapter is a collaboration with professor Jesús González, and has been accepted for publication in a special volume of the AMS series Contemporary Mathematics, [14] .

1 Higher topological complexity.

1.1 The multitasking motion planning problem

With the aim of motivating the "higher (or sequential) topological complexity" it will be useful to introduce the problem dealt by this invariant, called the multitasking motion planning problem (*MMP* problem). Roughly speaking, given a positive integer s and a path connected space X, the problem in question consists of finding a way to connect (through a path) s points in X, but we are interested in doing this in a way that is robust to noise.



Several states of a robot arm
From http://mechanismsrobotics.asmedigitalcollection.asme.org/article.aspx?articleid=1484860

In other words, let us think of X as the configuration space of a system. Given s states in the system, $x_1, \ldots, x_s \in X$:

- we need to find a path in X that passes through these states

and, once we have a path connecting the points $\{x_1, \ldots, x_s\} \in X$ and a path connecting the points $\{x'_1, \ldots, x'_s\} \in X$ (two different sets of s states in the space X),

- if the set of points are "near" each other, we have to guarantee that the corresponding paths connecting them will be "near" each other too.

Previous ideas can be formalized as follows:

Let PX be the path space of X, that is, $PX = \{\gamma \,|\, \gamma : [0,1] \to X,\ \gamma \text{ continuous map}\}$ with the compact-open topology, and consider the map

$$e_s = e_s^X : PX \to X^s$$

given by $e_s(\gamma) = \Big(\gamma(0), \gamma(\frac{1}{s-1}), \ldots, \gamma(\frac{s-2}{s-1}), \gamma(1)\Big)$. In these terms, the existence of a solution of the *MMP* problem is interpreted as the existence of a continuous section of e_s, that is, a continuous function $\sigma : X^s \to PX$, such that, $e_s \circ \sigma = Id_{X^s}$.

Proposition 1.1.1. *In previous terms, such a section σ exists iff X is contractible.*

For a proof see [11, Theorem 1].

1.2 Higher topological complexity

The higher (or sequential) topological complexity is an approach to study the instabilities of the *MMP* problem. This concept was introduced by Rudyak in [26] (as a generalization of Farber's topological complexity) and is defined as follows:
For a positive integer s, the s-th (higher or sequential) topological complexity of X, $\mathrm{TC}_s(X)$, is defined as the reduced Schwarz genus (or the sectional category, *secat*) of the fibration

$$e_s = e_s^X : PX \to X^s$$

given by $e_s(\gamma) = \Big(\gamma(0), \gamma(\frac{1}{s-1}), \ldots, \gamma(\frac{s-2}{s-1}), \gamma(1)\Big)$. Thus, $\mathrm{TC}_s(X) + 1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. We let TC stand for TC_2, the standard topological complexity.

In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X) + 1$ local domains.

Thus, Proposition 1.1.1 can be rewritten as follows:

Proposition 1.2.1. *Let $s \geq 2$ be a positive integer and X a path connected space. Then,* $\mathrm{TC}_s(X) = 0$ *if and only if X is contractible.*

For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numerical value of $\mathrm{TC}_s(X)$) by the requirement that local domains are pairwise disjoint Euclidean neighborhood retracts (ENR).

1.3 General properties

This section will be dedicated to some standard facts about the number $\mathrm{TC}_s(X)$.

One of the properties of the higher TC that has attracted the attention of topologists is its homotopy invariance.

Proposition 1.3.1. *For $s \geq 2$ a positive integer,* TC_s *is a homotopy invariant.*

For a proof see [11, Theorem 3].

Now, consider the fibration $e'_s : X^{\mathcal{J}_s} \to X^s$ given by $e'_s(f) = (f_1(1), \ldots, f_s(1))$, where $\mathcal{J}_s$ denotes the wedge of s copies of the closed interval $[0,1]$, in all of which $0 \in [0,1]$ is the base point, and we think of an element f in the function space $X^{\mathcal{J}_s}$ as an s-tuple $f = (f_1, \ldots, f_s)$ of paths in X all of which start at a common point. Since e_s and e'_s are homotopy equivalent as fibrations (see for instance [26, Remarks 3.2]), then, $\mathrm{TC}_s(X) = \mathrm{secat}(e'_s : X^{\mathcal{J}_s} \to X^s)$.

Moreover, note that e'_s is the standard fibrational substitute of the diagonal inclusion

$$d_s = d_s^X : X \hookrightarrow X^s,$$

and so $\mathrm{TC}_s(X)$ agrees with the reduced Schwarz genus of d_s.

Thinking of TC_s as the reduced Schwarz genus of the iterated diagonal, d_s, allows us to understand the lower bound in Proposition 1.3.3.

Definition 1.3.2. *Let X be a connected space and $\mathbb{F}$ a field.*

(a) *Given a positive integer s we denote by $\operatorname{zcl}_s(H^*(X;\mathbb{F}))$ the cup-length of elements in the kernel of the map induced by d_s in cohomology (with coefficients in $\mathbb{F}$). Explicitly, $\operatorname{zcl}_s(H^*(X;\mathbb{F}))$ is the largest integer m for which there exist m cohomology classes $u_i \in H^*(X^s;\mathbb{F})$, such that $d_s^*(u_i) = 0$ for $i = 1,\ldots,m$ and $0 \neq u_1 \cdot \cdots \cdot u_m \in H^*(X^s;\mathbb{F})$.*

(b) *The connectivity of X, $\operatorname{conn}(X)$, is the largest integer c such that all the homotopy groups of X of dimension at most c vanish. We set $\operatorname{conn}(X) = \infty$ when no such c exists.*

(c) *The homotopy dimension of X, $\operatorname{hdim}(X)$, is the minimal dimension of CW complexes having the homotopy type of X.*

Proposition 1.3.3. *For a path connected space X and any field $\mathbb{F}$,*

$$\operatorname{zcl}_s(H^*(X;\mathbb{F})) \leq \operatorname{TC}_s(X) \leq \frac{s\ \operatorname{hdim}(X)}{\operatorname{conn}(X)+1}.$$

In particular for every path connected space X,

$$\operatorname{TC}_s(X) \leq s\operatorname{hdim}(X).$$

For a proof see [2, Theorem 3.9] or, more generally, [27, Theorems 4 and 5].

The lower bound presented above is commonly optimal, that is, so far most of the computations reveal that the lower bound reaches the value of the higher TC. This phenomenon will be present in the spaces we work with.

Like Farber's topological complexity, its higher analog has a connection with the Lusternik-Schnirelmann category:

Theorem 1.3.4. *For a path-connected space X,*

$$\operatorname{cat}(X^{s-1}) \leq \operatorname{TC}_s(X) \leq \operatorname{cat}(X^s).$$

For a proof see [2, Corollary 3.3].

2 TC_s of subcomplexes of products of spheres.

2.1 Subcomplexes of products of spheres

For a positive integer k_i consider the minimal cellular structure on the k_i-dimensional sphere $S^{k_i} = e^0 \cup e^{k_i}$. Here e^0 is the base point. Take the product (therefore minimal) cell decomposition in

$$\mathbb{S}(k_1, \ldots, k_n) := S^{k_1} \times \cdots \times S^{k_n} = \bigsqcup_J e_J$$

whose cells e_J, indexed by subsets $J \subseteq [n] = \{1, \ldots, n\}$, are defined as $e_J = \prod_{i=1}^n e^{d_i}$ where $d_i = 0$ if $i \notin J$ and $d_i = k_i$ if $i \in J$. Explicitly,

$$e_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{S}(k_1, \ldots, k_n) \, | x_i = e^0 \text{ if and only if } i \notin J \right\}.$$

Note that, while $\mathbb{S}(k_1, \ldots, k_n)$ can be thought of as the configuration space of a mechanical robot arm whose i-th node moves freely in k_i dimensions, a subcomplex X of $\mathbb{S}(k_1, \ldots, k_n)$ encodes the information of the configuration space that results by imposing restrictions on the possible combinations of simultaneously moving nodes of the robot arm. Moreover, these spaces are examples of polyhedral product spaces associated to the family $\{(S^{k_i}, e^0)\}_{i=1}^n$ of pointed spheres. Concretely, given $X \subseteq \mathbb{S}(k_1, \ldots, k_n)$ a subcomplex, if we let $\mathcal{K}_X$ stand for the abstract simplicial complex associated to X (called *the index* of X), that is,

$$\mathcal{K}_X = \{J \mid e_J \text{ is a cell of } X\}, \tag{2.1}$$

we can write (using notation in [1]) $X = \mathcal{Z}(\mathcal{K}_X, (\underline{\mathbb{S}}, e^0))$, where $(\underline{\mathbb{S}}, e^0) = \{(S^{k_i}, e^0)\}_{i=1}^n$, that means,

$$X = \mathcal{Z}(\mathcal{K}_X, (\underline{\mathbb{S}}, e^0)) = \bigcup_{J \in \mathcal{K}_X} C_J$$

with $C_J = \prod_{i=1}^n Y_i$, where $Y_i = S^{k_i}$ if $i \in J$ and $Y_i = e^0$ if $i \notin J$.

Additional structure appears when some constraints are imposed on the index or on the dimension of the spheres.

Definition 2.1.1. *Let $X \subseteq \mathbb{S}(k_1, \ldots, k_n)$ be a subcomplex. We say that X is $d-$**pure** if all maximal sets in $\mathcal{K}_X$ have cardinality d.*

Example 2.1.2. Consider the case where all k_i are 1 (that is, the case when we consider a product of circles) and

$$X \subseteq \mathbb{S}(\underbrace{1, 1, \ldots, 1}_{n \text{ factors}})$$

is a subcomplex whose index $\mathcal{K}_X = \{J \in [n] \mid |J| \leq d\}$, with $d + 1 \leq n$ (in particular, X is d-pure). Then X has the same homotopy type as $\mathbb{C}^d - (\, L_1 \cup \ldots \cup L_n \,)$, where $L_1, \ldots, L_n$ is a set of affine hyperplanes in general position, see [20] for instance.

Example 2.1.3. Let $\Gamma = (V, E)$ a graph with vertex set $V = [n]$ (here and below, for a positive integer m, $[m]$ stands for the initial integer interval $\{1, 2, \ldots, m\}$, while $[m]_0$ stands for $[m] \cup \{0\}$) and edge set E. We let $\mathcal{K}_\Gamma$ stand for the clique complex of the graph Γ, thus $\mathcal{K}_\Gamma$ is the abstract simplicial complex whose k-simplices are the $(k+1)$-cliques of Γ. In other words, $J \subseteq [n]$ is a simplex of $\mathcal{K}_\Gamma$ if and only if this set induces a complete subgraph in Γ. Now consider,

$$X = \bigcup_{J \in \mathcal{K}_\Gamma} e_J \subseteq \mathbb{S}(\underbrace{1, 1, \ldots, 1}_{n \text{ factors}}).$$

Then, X is a subcomplex and it is also an Eilenberg-MacLane space of type $K(\pi, 1)$, where, $\pi = A_\Gamma$, is the right-angled Artin group of Γ, that is,

$$A_\Gamma = \langle\ \nu \in V \ \mid\ \nu\omega = \omega\nu \text{ iff } \{\nu, \omega\} \in E\ \rangle,$$

see [21, Theorem 10].

2.2 Optimal motion planners

In this section we construct optimal motion planners for a subcomplex X of $\mathbb{S}(k_1, \ldots, k_n)$ when all the k_i's have the same parity. We start by setting up some basic notation.

We think of an element $(b_1, b_2, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj}) \in X \subseteq \mathbb{S}(k_1, \ldots, k_n)$, as a matrix of size $n \times s$ whose entry b_{ij} belongs to S^{k_i} for all $(i, j) \in [n] \times [s]$. Let

$$\mathcal{P} = \{(P_1, \ldots, P_n) \mid P_i \text{ is a partition of } [s] \text{ for each } i \in [n]\}$$

be the set of n-tuples of partitions of the set $[s]$. We assume that the partitions P_i ($i = 1, \ldots, n$) are "ordered" in the sense that, if $P_i = \{\alpha_1^i, \ldots, \alpha_{|P_i|}^i\}$, then $L(\alpha_k^i) < L(\alpha_{k+1}^i)$ for $k \in [|P_i|-1]$ where $L(\alpha_k^i)$ is defined as the smallest element of the set α_k^i. In particular $1 \in \alpha_1^i$. The norm of each such $P = (P_1, \ldots, P_n) \in \mathcal{P}$ is defined as

$$|P| := \sum_{i=1}^{n} (|P_i| - 1) = \sum_{i=1}^{n} |P_i| - n, \tag{2.2}$$

the sum of all cardinalities of the partitions P_i minus n. We let

$$X_P^s = \left\{ (b_1, b_2, \ldots, b_s) \in X^s \;\middle|\; \begin{array}{l} \text{for each } i \in [n],\ b_{ik} = \pm b_{i\ell} \text{ if and only if} \\ \text{both } k \text{ and } \ell \text{ belong to the same part of } P_i \end{array} \right\},$$

and say that an element $(b_1, b_2, \ldots, b_s) \in X_P^s$ has type P. Note that, if $G := \mathbb{Z}_2 = \{1, -1\}$ acts antipodally on each sphere S^k and, for $x \in S^k$, $G \cdot x$ stands for the G-orbit of x, then

$$|P_i| = |\{G \cdot b_{ij} \mid j \in [s]\}| \tag{2.3}$$

for $(b_1, \ldots, b_s) \in X_P^s$ and $i \in [n]$. In addition, we consider n-tuples $\beta = (\beta^1, \ldots, \beta^n)$ of (possibly empty) subsets $\beta^i \subseteq \alpha_1^i - \{1\}$ for $i \in [n]$, and set

$$X_{P,\beta}^s = X_P^s \cap \left\{ (b_1, b_2, \ldots, b_s) \in X^s \mid b_{i1} = b_{ik} \Leftrightarrow k \in \beta^i, \forall\, (i, k) \in [n] \times ([s] - \{1\}) \right\}.$$

Note that the disjoint union decomposition

$$X_P^s = \bigsqcup_{\beta} X_{P,\beta}^s, \tag{2.4}$$

running over all n-tuples $\beta = (\beta^1, \ldots, \beta^n)$ as above, is topological, that is, the subspace topology in X_P^s agrees with the so called *disjoint union topology* determined by the subspaces $X_{P,\beta}^s$. In other words, a subset $U \subseteq X_P^s$ is open if and only if each of its pieces $U \cap X_{P,\beta}^s$ (for β as above) is open in $X_{P,\beta}^s$. Indeed, with the previous notation, consider $\beta = (\beta^1, \ldots, \beta^n)$ and $\beta' = (\beta'^1, \ldots, \beta'^n)$ two n-tuples, such that $\beta^i, \beta'^i \subseteq \alpha_1^i - \{1\}$ for all $i = 1, \ldots, n$, and $\beta \neq \beta'$. Then, there exists i_0 such that $\beta^{i_0} \neq \beta'^{i_0}$, without loss of generality, suppose that there exists $j_0 \in \beta^{i_0}$ and $j_0 \notin \beta'^{i_0}$. Then, for all $b = (b_1, \ldots, b_s) \in X_{P,\beta}^s$, one has $b_{i_0 1} = b_{i_0 j_0}$, and for all $b' = (b'_1, \ldots, b'_s) \in X_{P,\beta}^s$ one gets $b'_{i_0 1} = -b'_{i_0 j_0}$. Thus, $\overline{X_{P,\beta}^s} \cap \overline{X_{P,\beta'}^s} = \emptyset$.

Needless to say, the relevance of this property comes from the fact that the continuity of a local rule on X_P^s is equivalent to the continuity of the restriction of the local rule to each $X_{P,\beta}^s$.

2.2.1 Odd case

Throughout this subsection we assume that all k_i are odd. We start by recalling an optimal motion planner for the sphere $\mathbb{S}(2d+1) = S^{2d+1}$—for which $\mathrm{TC}_s(\mathbb{S}(2d+1)) = s-1$ as is well known.

Example 2.2.1. Local domains for $\mathbb{S}(2d+1)$ in the case $s = 2$ are given by

$$A_0 = \{(x, -x) \in \mathbb{S}(2d+1) \times \mathbb{S}(2d+1)\}$$

and

$$A_1 = \{(x, y) \in \mathbb{S}(2d+1) \times \mathbb{S}(2d+1) \,|\, x \neq -y\}$$

with corresponding local rules ϕ_i ($i = 0, 1$) described as follows: For $(x, -x) \in A_0$, $\phi_0(x, -x)$ is the path at constant speed from x to $-x$ along the semicircle determined by $\nu(x)$, where ν is some fixed non-zero tangent vector field of $\mathbb{S}(2d+1)$. For $(x, y) \in A_1$, $\phi_1(x, y)$ is the path at constant speed along the geodesic arc connecting x with y. To deal with the case $s > 2$, we consider the domains B_j, $j \in [s-1]_0$, consisting of s-tuples $(x_1, \ldots, x_s) \in \mathbb{S}(2d+1)^s$ for which

$$\{\, k \in \{2, \ldots, s\} \mid x_1 \neq -x_k \,\}$$

has cardinality j, with local rules $\psi_j : B_j \to \mathbb{S}(2d+1)^{\mathcal{J}_s}$ given by

$$\psi_j((x_1, \ldots, x_s)) = (\psi_{j1}(x_1, x_1), \ldots, \psi_{js}(x_1, x_s))$$

where $\psi_{ji}(x_1, x_i) = \phi_r(x_1, x_i)$ if $(x_1, x_i) \in A_r$, with $r = 0, 1$. As shown in [26, Section 4], the family $\{(B_j, \psi_j)\}$ is an optimal (higher) motion planner for $\mathbb{S}(2d+1)$.

A well known chess-board combination of the domains B_j in Example 2.2.1 yields domains for an optimal motion planner for the product $\mathbb{S}(k_1, \ldots, k_n)$ (see for instance the proof of Proposition 22 in page 84 of [27]). But the situation for an arbitrary subcomplex $X \subseteq \mathbb{S}(k_1, \ldots, k_n)$ is much more subtle. Actually, as it will be clear from the discussion below, $\mathrm{TC}_s(X)$ is determined by the combinatorics of X which we define next.

First, for a given integer $s > 1$, the s-norm of a finite (abstract) simplicial complex $\mathcal{K}$ is the integer invariant

$$\mathrm{N}^s(\mathcal{K}) := \max\{\, \mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) \mid J_j \text{ is a simplex of } \mathcal{K} \text{ for all } j \in [s]\},$$

where

$$\mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) := \sum_{\ell=2}^{s} \left(\Big| \bigcap_{m=1}^{\ell-1} J_m - J_\ell \Big| + \Big| J_\ell \Big| \right). \tag{2.5}$$

Now we notice some properties of the above formulas and give a simpler and more symmetric definition of $\mathrm{N}_{\mathcal{K}}$. Start by observing that $\mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) \leq \mathrm{N}_{\mathcal{K}}(J'_1, J'_2, \ldots, J'_s)$ provided $J_i \subseteq J'_i$ for $i \in [s]$. Consequently

$$\mathrm{N}^s(\mathcal{K}) = \max \{ \mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) \mid J_j \text{ is a } \textit{maximal} \text{ simplex of } \mathcal{K} \text{ for all } j \in [s] \},$$

a formula that is well suited for the computation of $\mathrm{N}^s(\mathcal{K})$ in concrete cases. Also let us put $I_\ell = \bigcap_{m=1}^{\ell-1} J_m - J_\ell$ for $\ell = 2, 3, \ldots, s$. Since $\bigcup_{\ell=2}^{s} I_\ell \subseteq J_1$ with $I_m \cap I_{m'} = \emptyset$ for every $m \neq m'$, we have:

Lemma 2.2.2. *For (not necessarily maximal) simplices $J_1, J_2, \ldots, J_s$ of $\mathcal{K}$,*

$$\mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) = \sum_{\ell=2}^{s} \Big| I_\ell \Big| + \sum_{\ell=2}^{s} \Big| J_\ell \Big| \leq \sum_{\ell=1}^{s} \Big| J_\ell \Big|.$$

Proposition 2.2.3. *For $J_1, J_2, \ldots, J_s$ as above*

$$\mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s) = \sum_{\ell=1}^{s} |J_\ell| - \Big| \bigcap_{\ell=1}^{s} J_\ell \Big|. \tag{2.6}$$

Proof. Due to Lemma 2.2.2 it suffices to prove the equality

$$\bigcup_{\ell=2}^{s} I_\ell = J_1 - \bigcap_{\ell=1}^{s} J_\ell.$$

An element x on the left hand side (LHS) satisfies $x \in I_\ell$ for some $\ell \geq 2$ whence $x \notin J_\ell$. Thus x lies on the right hand side (RHS). Conversely, for an element x on the RHS choose the smallest $\ell \geq 2$ such that $x \notin J_\ell$. By the choice of ℓ and definition of I_ℓ we have $x \in I_\ell$ whence x lies on LHS. □

Corollary 2.2.4. $\mathrm{N}_{\mathcal{K}}(J_1, J_2, \ldots, J_s)$ *does not depend on the ordering of the set of simplices.*

Recall that given X subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$, $\mathcal{K}_X$ denotes the index of X. We use the notation $\mathrm{N}_X(J_1, J_2, \ldots, J_s)$ and $\mathrm{N}^s(X)$ for $\mathrm{N}_{\mathcal{K}_X}(J_1, J_2, \ldots, J_s)$ and $\mathrm{N}^s(\mathcal{K}_X)$ respectively.

In these terms, we have our first theorem:

Theorem 2.2.5. *Assume all of the k_i are odd. A subcomplex X of the minimal CW cell structure on $\mathbb{S}(k_1, \ldots, k_n)$ satisfies*

$$\mathrm{TC}_s(X) = \mathrm{N}^s(X).$$

This subsection is devoted to establishing the inequality $\mathrm{TC}_s(X) \leq \mathrm{N}^s(X)$ by proving that the domains

$$D_j := \bigcup X^s_P, \quad j \in [\mathrm{N}^s(X)]_0, \tag{2.7}$$

where the union runs over those $P \in \mathcal{P}$ with $|P| = j$ as defined in (2.2), give a cover of X^s by pairwise disjoint ENR subspaces each of which admits a local rule—a section for e_s.

It is easy to see that the D_j's are pairwise disjoint. On the other hand, it follows from Proposition 2.2.7 below that (2.7) is a topological disjoint union, so that [10, Proposition IV.8.10] and the obvious fact that each X^s_P is an ENR imply the corresponding assertion for each D_j.

Lemma 2.2.6.

$$X^s = \bigcup_{j=0}^{\mathrm{N}^s(X)} D_j.$$

Proof. Let $b \in X^s$, say $b = (b_1, \ldots, b_s) \in e_{J_1} \times e_{J_2} \times \cdots \times e_{J_s} \subseteq X^s$, where $J_j \subseteq [n]$ for all $j \in [s]$. Recall $G = \mathbb{Z}_2$ acts antipodally on each sphere S^{k_i}. Note that

$$\sum_{i=1}^{n} |\{G \cdot b_{ij} \mid j \in [2]\}| - n = |\{i \in [n] \mid b_{i1} \neq \pm b_{i2}\}| \leq |J_1 - J_2| + |J_2|$$

where the last inequality holds since $\{i \in [n] \mid b_{i1} \neq \pm b_{i2}\} \subseteq J_1 \cup J_2$. More generally,

$$\sum_{i=1}^{n} |\{G \cdot b_{ij} \mid j \in [s]\}| - n = \sum_{\ell=2}^{s} |\{i \in [n] \mid b_{it} \neq \pm b_{i\ell} \text{ for all } 1 \leq t < \ell\}| \tag{2.8}$$

where, for each $2 \leq \ell \leq s$,

$$|\{i \in [n] \mid b_{it} \neq \pm b_{i\ell} \text{ for all } 1 \leq t < \ell\}| \leq |\bigcap_{t=1}^{\ell-1} J_t - J_\ell| + |J_\ell| \tag{2.9}$$

since in fact

$$\{i \in [n] \mid b_{it} \neq \pm b_{i\ell} \text{ for all } 1 \leq t < \ell\} \subseteq \Big(\bigcap_{t=1}^{\ell-1} J_t\Big) \cup J_\ell.$$

Therefore, if $P = (P_1, \ldots, P_n) \in \mathcal{P}$ is the type of b, and we set $j = |P|$, then $b \in X^s_P \subseteq D_j$. The inequality $j \leq \mathrm{N}^s(X)$ holds in view of (2.3), (2.8), and (2.9). $\square$

Next, in order to construct a (well defined and continuous) local section of e_s over each D_j, $j \in [\mathrm{N}^s(X)]_0$, we prove that (2.7) is a topological disjoint union.

Proposition 2.2.7. *For any pair of elements $P, P' \in \mathcal{P}$ with $|P| = |P'|$ and $P \neq P'$ we have*

$$\overline{X^s_P} \cap X^s_{P'} = \emptyset = X^s_P \cap \overline{X^s_{P'}}. \tag{2.10}$$

Proof. Write $P = (P_1, \ldots, P_n)$ and $P' = (P'_1, \ldots, P'_n)$ so that

$$\sum_{i=1}^{n} |P_i| = \sum_{i=1}^{n} |P'_i|.$$

If there exists an integer $j_1 \in [n]$ with $|P_{j_1}| > |P'_{j_1}|$ (or $|P_{j_1}| < |P'_{j_1}|$), then the hypothesis forces the existence of another integer $j_2 \in [n]$ with $|P_{j_2}| < |P'_{j_2}|$ ($|P_{j_2}| > |P'_{j_2}|$, respectively). In this case, in virtue of equation (2.3), $|P_{j_1}| > |P'_{j_1}|$ implies that there exist $m_1, m_2 \in [s]$, $m_1 < m_2$, such that

$$b'_{j_1 m_1} = \pm b'_{j_1 m_2} \tag{2.11}$$

for all $b' = (b'_1, \ldots, b'_s) \in X^s_{P'}$, and

$$b_{j_1 m_1} \neq \pm b_{j_1 m_2} \tag{2.12}$$

for all $b = (b_1, \ldots, b_s) \in X^s_P$. Since, condition (2.11) is inherited on elements of $\overline{X^s_{P'}}$, we see $X^s_P \cap \overline{X^s_{P'}} = \emptyset$. Analogously, one proves that

$$|P_{j_2}| < |P'_{j_2}|,$$

implies the another desired inequality, i.e., $\overline{X^s_P} \cap X^s_{P'} = \emptyset$. Now, let's assume $|P_i| = |P'_i|$, for all $i \in [n]$. Since $P \neq P'$, there exists $k \in [n]$ such that $P_k \neq P'_k$. Write $P_k = \{\alpha_1, \dots, \alpha_{\ell_0}\}$ and $P'_k = \{\alpha'_1, \dots, \alpha'_{\ell_0}\}$, both ordered in the sense indicated at the beginning of the section.

Assume there are integers $t \in [\ell_0]$ with $L(\alpha_t) < L(\alpha'_t)$, and let t_0 be the first such t (necessarily $t_0 > 1$). Then any $(b_1, \dots, b_s) \in X^s_{P'}$ must satisfy

$$b_{kL(\alpha_{t_0})} = \pm b_{kj_0}$$

for some $1 \leq j_0 \leq L(\alpha'_{t_0-1}) \leq L(\alpha_{t_0-1}) < L(\alpha_{t_0})$, condition that is then inherited by elements in $\overline{X^s_{P'}}$. However, by definition, any $(b_1, \dots, b_s) \in X^s_P$ satisfies

$$b_{kL(\alpha_{t_0})} \neq \pm b_{kj}$$

for all $1 \leq j < L(\alpha_{t_0})$. Therefore $X^s_P \cap \overline{X^s_{P'}} = \emptyset$. A symmetric argument shows $\overline{X^s_P} \cap X^s_{P'} = \emptyset$ whenever there are integers $t \in [\ell_0]$ with $L(\alpha'_t) < L(\alpha_t)$. As a consequence, we can assume, without loss of generality, that $L(\alpha_j) \leq L(\alpha'_j)$ for all $j \in [\ell_0]$ —this loses the symmetry, so we now have to show that both equations in (2.10) hold.

Case 1. Assume there are integers $t \in [\ell_0]$ such that $L(\alpha_t) < L(\alpha'_t)$, and let t_0 be the largest such t. We have already noticed that $X^s_P \cap \overline{X^s_{P'}} = \emptyset$ is forced. Moreover, note that either $t_0 = \ell_0$ or, else, $L(\alpha_{t_0}) < L(\alpha'_{t_0}) < L(\alpha'_{t_0+1}) = L(\alpha_{t_0+1})$, but in any case we have

- if $(b_1, \dots, b_s) \in X^s_P$, then $b_{kL(\alpha'_{t_0})} = \pm b_{kj_0}$ for some $1 \leq j_0 < L(\alpha'_{t_0})$, and
- if $(b_1, \dots, b_s) \in X^s_{P'}$, then $b_{kL(\alpha'_{t_0})} \neq \pm b_{kj}$ for all $1 \leq j < L(\alpha'_{t_0})$.

Since the former condition is inherited on elements of $\overline{X^s_P}$, we see $\overline{X^s_P} \cap X^s_{P'} = \emptyset$.

Case 2. Assume $L(\alpha_j) = L(\alpha'_j)$ for all $j \in [\ell_0]$. (Note that the symmetry is now restored.) Since $P_k \neq P'_k$, there is an integer $j_0 \in [\ell_0]$ with $\alpha_{j_0} \neq \alpha'_{j_0}$. Without loss of generality we can further assume there is an integer $m_0 \in \alpha_{j_0} - \alpha'_{j_0}$ (note $m_0 \neq L(\alpha_{j_0})$, but once again the symmetry has been destroyed). Under these conditions we have

- if $(b_1, \dots, b_s) \in X^s_P$, then $b_{kL(\alpha_{j_0})} = \pm b_{km_0}$, and
- if $(b_1, \dots, b_s) \in X^s_{P'}$ then $b_{kL(\alpha_{j_0})} = b_{kL(\alpha'_{j_0})} \neq \pm b_{km_0}$.

Since the former condition is inherited on elements of $\overline{X^s_P}$, we see $\overline{X^s_P} \cap X^s_{P'} = \emptyset$. Moreover, since $m_0 \notin \alpha'_{j_0}$, there is $d_0 \in [\ell_0]$ with $m_0 \in \alpha'_{d_0}$. Necessarily $d_0 \neq j_0$ and $m_0 \notin \alpha_{d_0}$, so we now have

- if $(b_1, \dots, b_s) \in X^s_{P'}$, then $b_{kL(\alpha'_{d_0})} = \pm b_{km_0}$, and
- if $(b_1, \dots, b_s) \in X^s_P$, then $b_{kL(\alpha'_{d_0})} = b_{kL(\alpha_{d_0})} \neq \pm b_{km_0}$,

implying $X^s_P \cap \overline{X^s_{P'}} = \emptyset$. □

Our only remaining task in this subsection is the construction of a local rule over D_j for each $j \in [\mathrm{N}^s(X)]_0$. Actually, by (2.4), (2.7), and Proposition 2.2.7, the task can be simplified to the construction of a local rule over each $X^s_{P,\beta}$. To fulfill such a goal, it will be convenient to normalize each sphere S^{k_i} so to have great semicircles of length 1/2. Then, for $x, y \in S^{k_i}$, we let $d(x, y)$ stand for the length of the shortest geodesic in S^{k_i} between x and y (e.g. $d(x, -x) = 1/2$). Likewise, the local rules ϕ_0 and ϕ_1 for each S^{k_i} defined at Example 2.2.1 need to be adjusted—but the domains A_i, $i = 0, 1$, remain unchanged—as follows: For $i = 0, 1$ and $(x, y) \in A_i$ we set

$$\tau_i(x, y)(t) = \begin{cases} \phi_i(x, y)\left(\frac{1}{d(x,y)}t\right), & 0 \leq t < d(x, y); \\ y, & d(x, y) \leq t \leq 1. \end{cases}$$

Thus, τ_i reparametrizes ϕ_i so to perform the motion at speed 1, keeping still at the final position once it is reached—which happens at most at time 1/2.

In what follows it is helpful to keep in mind that, as before, elements $(b_1,\dots,b_s) \in X^s$, with $b_j = (b_{1j},\dots,b_{nj})$ for $j \in [s]$, can be thought of as matrices $(b_{i,j})$ whose columns represent the various stages in X through which motion is to be planned (necessarily along rows). Actually, we follow a "pivotal" strategy: starting at the first column, motion spreads to all other columns—keeping still in the direction of the first column. In terms of the notation set in Section 1.3 for elements in the function space $X^{\mathcal{J}_s}$, consider the map

$$\varphi\colon X^s \to \mathbb{S}(k_1,\dots,k_n)^{\mathcal{J}_s} \tag{2.13}$$

given by $\varphi\left((b_1,\dots,b_s)\right) = (\varphi_1(b_1,b_1),\dots,\varphi_s(b_1,b_s))$ where, for $j \in [s]$,

$$\varphi_j(b_1,b_j) = (\varphi_{1j}(b_{11},b_{1j}),\dots,\varphi_{nj}(b_{n1},b_{nj}))$$

is the path in $\mathbb{S}(k_1,\dots,k_n)$, from b_1 to b_j, whose i-th coordinate $\varphi_{ij}(b_{i1},b_{ij})$, $i \in [n]$, is the path in S^{k_i}, from b_{i1} to b_{ij}, defined by

$$\varphi_{i,j}(b_{i1},b_{ij})(t) = \begin{cases} b_{i1}, & 0 \le t \le t_{b_{i1}}, \\ \sigma(b_{i1},b_{ij})(t - t_{b_{i1}}), & t_{b_{i1}} \le t \le 1. \end{cases}$$

Here $t_{b_{i1}} = \frac{1}{2} - d(b_{i1},e^0)$ and

$$\sigma(b_{i1},b_{ij}) = \begin{cases} \tau_1(b_{i1},b_{ij}), & (b_{i1},b_{ij}) \in A_1; \\ \tau_0(b_{i1},b_{ij}), & (b_{i1},b_{ij}) \in A_0. \end{cases} \tag{2.14}$$

Fix n-tuples $P = (P_1,\dots,P_n) \in \mathcal{P}$ and $\beta = (\beta^1,\dots,\beta^n)$, with $P_i = \{\alpha^i_1,\dots,\alpha^i_{n(P_i)}\}$ and $\beta^i \subseteq \alpha^i_1 - \{1\}$ for all $i \in [n]$. Although φ is not continuous, its restriction $\varphi_{P,\beta}$ to $X^s_{P,\beta}$ is, for then (2.14) takes the form

$$\sigma = \begin{cases} \tau_1, & j \notin \alpha^i_1 \text{ or } j \in \beta^i \cup \{1\}; \\ \tau_0, & j \in \alpha^i_1 \text{ and } j \notin \beta^i \cup \{1\}. \end{cases}$$

Since $\varphi_{P,\beta}$ is clearly a section for the end-points evaluation map $e_s^{\mathbb{S}(k_1,\dots,k_n)}$, we only need to check that $\varphi_{P,\beta}$ actually takes values in $X^{\mathcal{J}_s}$, i.e. that our proposed motion planner does not leave X.

Remark 2.2.8. An attempt to verify the analogous assertion in [5, proof of Proposition 3.5] for $s = 2$, and the eventual realizing and fixing of the problems with that assertion, led to the work in [15]. The verification in the current more general setting (i.e. proof of Proposition 2.2.9 below) is inspired by the one carefully explained in [15, page 7], and here we include full details for completeness.

Proposition 2.2.9. *The image of φ is contained in $X^{\mathcal{J}_s}$.*

Proof. Choose $(b_1,b_2,\dots,b_s) \in X^s$ where, as above, $b_j = (b_{1j},b_{2j},\dots,b_{nj}) \in X$. We need to check that, for all $j \in [s]$, the image of $\varphi_j(b_1,b_j)\colon [0,1] \to \mathbb{S}(k_1,\dots,k_n)$ lies inside X. By construction, the path $\varphi_j(b_1,b_j)$ runs coordinate-wise, from b_1 to b_j, according to the instructions $\tau_k(b_{i1},b_{ij})$ ($k = 0,1,\ i \in [n]$), except that, in the i-th coordinate, the movement is delayed by time $t_{b_{i1}} \le 1/2$. The closer b_{i1} gets to e^0, the closer the delaying time $t_{b_{i1}}$ gets to 1/2. It is then convenient to think of the path $\varphi_j(b_1,b_j)$ as running in two sections. In the first section ($t \le 1/2$) all initial coordinates $b_{i1} = e^0$ keep still, while the rest of the coordinates (eventually) start traveling to their corresponding final position b_{ij}. Further, when the second section starts ($t = 1/2$), any final coordinate $b_{ij} = e^0$ will already have been reached, and will keep still throughout the rest of the motion. As a result, the image of $\varphi_j(b_1, b_j)$ is forced to be contained in X. In more detail, let $e(J_1, \ldots, J_s) := e_{J_1} \times e_{J_2} \times \cdots \times e_{J_s} \subseteq X^s$ be the product of cells of X containing $(b_1, b_2, \ldots, b_s)$. Then, coordinates corresponding to indexes $i \in [n] - J_1$ keep their initial position $b_{i1} = e^0$ through time $t \leq 1/2$. Therefore $\varphi_j(b_1, b_j)[0, 1/2]$ stays within $\overline{e_{J_1}} \subseteq X$. On the other hand, by construction, $\varphi_{ij}(b_{i1}, b_{ij})(t) = b_{ij} = e^0$ whenever $t \geq 1/2$ and $i \in [n] - J_j$. Thus, $\varphi_j(b_1, b_j)[1/2, 1]$ stays within $\overline{e_{J_j}} \subseteq X$. $\square$

2.2.2 Even case

We now turn our attention to the case when X is a subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$ with all the k_i even—an assumption that will be in force throughout this subsection. As above, the goal is the construction of an optimal motion planner for the s-th topological complexity of X. We start with the following analogue of Example 2.2.1:

Example 2.2.10. Local domains for the sphere $\mathbb{S}(2d) = S^{2d}$ in the case $s = 2$ are given by

$$\begin{aligned} B_0 &= \{(e^0, -e^0), (-e^0, e^0)\} \subseteq \mathbb{S}(2d) \times \mathbb{S}(2d), \\ B_1 &= \{(x, -x) \in \mathbb{S}(2d) \times \mathbb{S}(2d) \,|\, x \neq \pm e^0\}, \text{ and} \\ B_2 &= \{(x, y) \in \mathbb{S}(2d) \times \mathbb{S}(2d) \mid x \neq -y\} \;=\; \mathbb{S}(2d) \times \mathbb{S}(2d) - (B_0 \cup B_1), \end{aligned}$$

with corresponding local rules $\lambda_i \colon B_i \to \mathbb{S}(2d)^{[0,1]}$ ($i = 0, 1, 2$) described as follows:

- $\lambda_0(e^0, -e^0)$ and $\lambda_0(-e^0, e^0)$ are the paths, at constant speed, from e^0 to $-e^0$ and from $-e^0$ to e^0, respectively, along some fixed meridian—thinking of e^0 and $-e^0$ as the poles of $\mathbb{S}(2d)$.
- For a fixed nowhere zero tangent vector field υ on $\mathbb{S}(2d) - \{\pm e^0\}$, $\lambda_1(x, -x)$ (with $x \neq \pm e^0$) is the path at constant speed from x to $-x$ along the great semicircle determined by the tangent vector $\upsilon(x)$.
- For $x \neq -y$, $\lambda_2(x, y)$ is the path from x to y, at constant speed, along the shortest geodesic arc determined by x and y.

The generalization of Example 2.2.10 to the higher topological complexity of a subcomplex of a product of even dimensional spheres is slightly more elaborate than the corresponding generalization of Example 2.2.1 in the previous section due, in part, to the additional local domain in Example 2.2.10. So, before considering the general situation (Theorem 2.2.13 below), and in order to illustrate the essential points in our construction, it will be convenient to give full details in the case of $\mathrm{TC}_s(\mathbb{S}(2d))$.

Consider the sets

$$\begin{aligned} T_0 &= \{(x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \mid x_j \neq \pm e^0, \text{ for all } j \in [s]\}, \\ T_1 &= \{(x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \mid x_j = \pm e^0, \text{ for some } j \in [s]\} \end{aligned}$$

and, for each partition P_1 of $[s]$ and each $i \in \{0, 1\}$,

$$\mathbb{S}(2d)^s_{P_1,i} = \left\{ (x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \;\middle|\; \begin{array}{l} x_l = \pm x_k \text{ if and only if } k \text{ and } l \\ \text{belong to the same part in } P_1 \end{array} \right\} \cap T_i.$$

The norm of the pair (P_1, i) above is defined as $\mathrm{N}(P_1, i) = |P_1| - i$. Lastly, for $k \in [s]_0$, consider the set

$$H_k = \bigcup_{\mathrm{N}(P_1,i)=k} \mathbb{S}(2d)^s_{P_1,i}. \tag{2.15}$$

Proposition 2.2.11. *There is an optimal motion planner for* $\mathbb{S}(2d)$ *with local domains* H_k, $k \in [s]_0$.

Proof. The optimality of such a motion planner follows by the fact that the s-th topological complexity of an even sphere is s (see for instance [2, Corollary 3.12]). On the other hand, it is obvious that $H_0, \ldots, H_s$ form a pairwise disjoint covering of $\mathbb{S}(2d)^s$. Since each $\mathbb{S}(2d)^s_{P_1,i}$ is clearly an ENR, it suffices to show that (2.15) is a topological disjoint union (so H_k is also an ENR), and that each $\mathbb{S}(2d)^s_{P_1,i}$ admits a local rule (all of which, therefore, determine a local rule on H_k).

Topology of H_k: For pairs (P_1, i) and (P'_1, i') as above, with $\mathrm{N}(P_1, i) = \mathrm{N}(P'_1, i')$ and $(P_1, i) \neq (P'_1, i')$, we prove

$$\overline{\mathbb{S}(2d)^s_{P_1,i}} \cap \mathbb{S}(2d)^s_{P'_1,i'} = \emptyset = \mathbb{S}(2d)^s_{P_1,i} \cap \overline{\mathbb{S}(2d)^s_{P'_1,i'}}. \tag{2.16}$$

If $i \neq i'$, say $i = 1$ and $i' = 0$, then the first equality in (2.16) is obvious, whereas the second equality follows since $|P_1| > |P'_1|$. On the other hand, if $i = i'$, then $|P_1| = |P'_1|$ with $P_1 \neq P'_1$, and the argument starting in the second paragraph of the proof of Proposition 2.2.7 gives (2.16).

Local section on $\mathbb{S}(2d)^s_{P_1,i}$: We assume the partition $P_1 = \{\alpha_1, \ldots, \alpha_n\}$ is ordered in the sense indicated at the beginning of this section. For each $\beta \subseteq \alpha_1 - \{1\}$, let

$$\mathbb{S}(2d)^s_{P_1,i,\beta} = \mathbb{S}(2d)^s_{P_1,i} \cap \{(x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \,|\, x_1 = x_j \Leftrightarrow j \in \beta,\ \forall j \in [s] - 1\}.$$

Since

$$\mathbb{S}(2d)^s_{P_1,i} = \bigsqcup_{\beta \subseteq \alpha_1 - \{1\}} \mathbb{S}(2d)^s_{P_1,i,\beta}$$

is a topological disjoint union, it suffices to construct a local section on each $\mathbb{S}(2d)^s_{P_1,i,\beta}$.

Case $i = 0$. As in the previous subsection, the required local section can be defined by the formula $\sigma(x_1, \ldots, x_s) = (\sigma_1(x_1, x_1), \ldots, \sigma_s(x_1, x_s))$ where

$$\sigma_j = \begin{cases} \lambda_2, & \text{if } j \in ([s] - \alpha_1) \cup \beta \cup \{1\}; \\ \lambda_1, & \text{otherwise.} \end{cases}$$

Case $i = 1$. The required local section is now defined in terms of the decomposition

$$\mathbb{S}(2d)^s_{P_1,i,\beta} = \big(\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_0(\alpha_1)\big) \sqcup \big(\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_1(\alpha_1)\big) \tag{2.17}$$

which will be shown in Lemma 2.2.12 below to be a topological disjoint union. Here

$$T_0(\alpha_1) = \{(x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \mid x_j \neq \pm e^0, \text{ for all } j \in \alpha_1\}$$

and

$$T_1(\alpha_1) = \{(x_1, \ldots, x_s) \in \mathbb{S}(2d)^s \mid x_j = \pm e^0, \text{ for some } j \in \alpha_1\}.$$

A local section on $\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_0(\alpha_1)$ is defined just as in the case $i = 0$, whereas a local section on $\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_1(\alpha_1)$ is defined by the formula

$$\mu(x_1, \ldots, x_s) = (\mu_1(x_1, x_1), \ldots, \mu_s(x_1, x_s))$$

where

$$\mu_j = \begin{cases} \lambda_2, & \text{if } j \in ([s] - \alpha_1) \cup \beta \cup \{1\}; \\ \lambda_0, & \text{otherwise.} \end{cases}$$

$\square$

Lemma 2.2.12. *The decomposition* (2.17) *is a topological disjoint union* (*recall* $i = 1$).

Proof. The condition "$x_j = \pm e^0$ for some $j \in \alpha_1$" in $T_1(\alpha_1)$ is inherited by elements in its closure, in particular

$$\left(\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_0(\alpha_1)\right) \cap \overline{\left(\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_1(\alpha_1)\right)} = \emptyset.$$

On the other hand, since $i = 1$, the condition "$x_j = \pm e^0$ for some $j \notin \alpha_1$" is forced on elements of $\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_0(\alpha_1)$ and, consequently, on elements of its closure. But the latter condition is not fulfilled by any element in $\mathbb{S}(2d)^s_{P_1,i,\beta} \cap T_1(\alpha_1)$. ☐

We now focus on the general situation.

Theorem 2.2.13. *Assume all of the* k_i *are even. A subcomplex* X *of the minimal CW structure on* $\mathbb{S}(k_1, \ldots, k_n)$ *has*

$$\mathrm{TC}_s(X) = s(1 + \dim(\mathcal{K}_X)).$$

The inequality $s(1 + \dim(\mathcal{K}_X)) \leq \mathrm{TC}_s(X)$ will be dealt with in Section 2.3 using cohomological methods; in the rest of this subsection we prove the inequality $\mathrm{TC}_s(X) \leq s(1 + \dim(\mathcal{K}_X))$ by constructing an explicit motion planner with $1 + s(1 + \dim(\mathcal{K}_X))$ local domains—given by the sets in (2.18) below.

As in previous constructions, we think of an element $(b_1, \ldots, b_s) \in X^s$ with $b_j = (b_{1j}, \ldots, b_{nj})$, $j \in [s]$, as an $n \times s$ matrix whose (i, j) coordinate is $b_{ij} \in \mathbb{S}(k_i)$. For $P \in \mathcal{P}$ and $k \in [n]_0$, set $\mathrm{N}(P, k) := \sum_{i=1}^n |P_i| - k$, the norm of the pair (P, k), and

$$X^s_{P,k} := X^s_P \cap \left\{ (b_1, \ldots, b_s) \in \mathbb{S}(k_1, \ldots, k_n)^s \;\middle|\; \begin{array}{l} (b_{i1}, \ldots, b_{is}) \in T_{1,k_i} \text{ for} \\ \text{exactly } k \text{ indexes } i \in [n] \end{array} \right\}$$

where $T_{1,k_i} = \left\{(x_1, \ldots, x_s) \in \mathbb{S}(k_i)^s \mid x_j = \pm e^0, \text{ for some } j \in [s]\right\}$. The local domains we propose are given by

$$W_r = \bigcup_{\mathrm{N}(P,k)=r} X^s_{P,k}. \tag{2.18}$$

By (2.3), the norm $\mathrm{N}(P, k)$ is the number of "row" G-orbits different from that of e^0 in any matrix $(b_1, \ldots, b_s) \in X^s_{P,k}$. Therefore the sets W_r with $r \in [s(1 + \dim(\mathcal{K}_X))]_0$ yield a pairwise disjoint cover of X^s. Our task then is to show:

Proposition 2.2.14. *Each* W_r *is an ENR admitting a local rule.*

Our proof of Proposition 2.2.14 depends on showing that (2.18) is a topological disjoint union (Lemma 2.2.15 below) and that each piece $X^s_{P,k}$ admits a suitably finer topological decomposition ((2.19), (2.21), and Proposition 2.2.16 below).

Lemma 2.2.15. *For* $P, P' \in \mathcal{P}$ *and* $k, k' \in [n]_0$ *with* $N(P, k) = N(P', k')$ *and* $(P, k) \neq (P', k')$,

$$\overline{X^s_{P,k}} \cap X^s_{P',k'} = \emptyset = X^s_{P,k} \cap \overline{X^s_{P',k'}}.$$

Proof. Write $P = (P_1, \ldots, P_n)$ and $P' = (P'_1, \ldots, P'_n)$ so that, by hypothesis, $\sum_{i=1}^n |P_i| - k = \sum_{i=1}^n |P'_i| - k'$. If $k > k'$, then $\overline{X^s_{P,k}} \cap X^s_{P',k'} = \emptyset$, and since $\sum_{i=1}^n |P_i| > \sum_{i=1}^n |P'_i|$ is forced, we also get $X^s_{P,k} \cap \overline{X^s_{P',k'}} = \emptyset$. If $k = k'$, then $|P| = |P'|$ with $P \neq P'$ and, just as for (2.16), the argument starting in the second paragraph of the proof of Proposition 2.2.7 yields the conclusion. ☐

Next we work with a fixed pair $(P,k)\in\mathcal{P}\times[n]_0$ with $P=(P_1,\ldots,P_n)$ and where each $P_i=\{\alpha^i_1,\ldots,\alpha^i_{n(P_i)}\}$ is ordered as described at the beginning of this section. For a subset $I\subseteq[n]$ consider the set $T_I=\{(b_1,\ldots,b_s)\in X^s \mid (b_{i1},\ldots,b_{is})\in T_{1,k_i}$ if and only if $i\in I\}$. Then (2.4) yields a topological disjoint union

$$X^s_{P,\,k}=\bigsqcup_{\beta,I}\left(X^s_{P,\beta}\cap T_I\right) \tag{2.19}$$

running over subsets $I\subseteq[n]$ of cardinality k, and n-tuples $\beta=(\beta^1,\ldots,\beta^n)$ of (possibly empty) subsets $\beta^i\subseteq\alpha^i_1-\{1\}$. Besides, as suggested by (2.17) in the proof of Proposition 2.2.11, it is convenient to decompose even further each piece in (2.19). For each $i\in[n]$, let

$$\begin{aligned} T_0(\alpha^i_1) &= \{(b_1,\ldots,b_s)\in X^s \mid b_{ij}\neq\pm e^0 \text{ for all } j\in\alpha^i_1\},\\ T_1(\alpha^i_1) &= \{(b_1,\ldots,b_s)\in X^s \mid b_{ij}=\pm e^0 \text{ for some } j\in\alpha^i_1\} \end{aligned} \tag{2.20}$$

and, for $I=\{\ell_1,\ldots,\ell_{|I|}\}\subseteq[n]$ and $\varepsilon=(t_1,\ldots,t_{|I|})\in\{0,1\}^{|I|}$,

$$T_\varepsilon(I)=T_I\cap\bigcap_{i=1}^{|I|}T_{t_i}(\alpha^{\ell_i}_1).$$

In these terms there is an additional topological disjoint union decomposition

$$X^s_{P,\beta}\cap T_I=\bigsqcup_{\varepsilon\in\{0,1\}^{|I|}}\left(X^s_{P,\beta}\cap T_\varepsilon(I)\right). \tag{2.21}$$

Proposition 2.2.14 is now a consequence of (2.19), (2.21), Lemma 2.2.15, and the following result:

Proposition 2.2.16. *For P, β, I, and ε as above, $X^s_{P,\beta}\cap T_\varepsilon(I)$ is an ENR admitting a local rule.*

Proof. The ENR property follows since, in fact, $X^s_{P,\beta}\cap T_\varepsilon(I)$ is homeomorphic to the Cartesian product of a finite discrete space and a product of punctured spheres. Indeed, the information encoded by P and β produces the discrete factor, as coordinates in a single G-orbit are either repeated (e.g. in the case of β) or sign duplicated. Besides, after ignoring such superfluous information as well as all e^0-coordinates (determined by I and ε), we are left with a product of punctured spheres.

The needed local rule will be defined as follows. Let ρ_i $(i=0,1,2)$ denote the local rules obtained by normalizing the corresponding λ_i (defined in Example 2.2.10) in the same manner as the local rules τ_i were obtained right after the proof of Proposition 2.2.7 from the corresponding ϕ_i. Then consider the (discontinuous) global section $\varphi\colon X^s\to\mathbb{S}(k_1,\ldots,k_n)^{\mathcal{J}_s}$ defined through the algorithm following (2.13), except that (2.14) gets replaced by

$$\sigma(b_{i1},b_{ij})=\rho_m(b_{i1},b_{ij}),\ \text{if } (b_{i1},b_{ij})\in B_m \text{ for } m\in\{0,1,2\}$$

where the domains B_m are now those defined in Example 2.2.10. As in the previous subsection, the point is that the restriction of φ to $X^s_{P,\beta}\cap T_\varepsilon(I)$ is continuous since, in that domain, the latter equality can be written as

$$\sigma=\begin{cases}\rho_2, & \text{if } j\in\left([s]-\alpha^i_1\right)\cup\beta^i\cup\{1\};\\ \rho_1, & \text{if } j\in\alpha^i_1-\left(\beta^i\cup\{1\}\right) \text{ and } t_i=0;\\ \rho_0, & \text{if } j\in\alpha^i_1-\left(\beta^i\cup\{1\}\right) \text{ and } t_i=1.\end{cases}$$

In addition, the proof of Proposition 2.2.9 applies word for word to show that the image of φ is contained in $X^{\mathcal{J}_s}$. □

Example 2.2.17. *The gap noted in Remark 2.2.8 also holds in [5] when all the k_i are even. The new situation is subtler in view of an additional gap (pinpointed in [15, Remark 2.3]) in the proof of [5, Theorem 6.3]. Of course, the detailed constructions in this section fix the problem and generalize the result.*

2.3 Zero-divisors cup-length

We now show that, for a subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ where all the k_i have the same parity, the cohomological lower bound for $\mathrm{TC}_s(X)$ in Proposition 1.3.3 is optimal and agrees with the upper bound coming from our explicit motion planners in the previous section. Throughout this section we use cohomology with rational coefficients, writing $H^*(X)$ as a shorthand of $H^*(X;\mathbb{Q})$.

Recall that $H^*(\mathbb{S}(k_1,\ldots,k_n))$ is the graded tensor product,

$$H^*(\mathbb{S}(k_1,\ldots,k_n)) = \bigotimes_{i=1}^{n} E_i,$$

where

$$E_i = \begin{cases} \bigwedge_{\mathbb{Q}}[\epsilon_i] & \text{if } k_i \text{ odd} \\ \mathbb{Q}[\epsilon_i]/(\epsilon_i^2) & \text{if } k_i \text{ even} \end{cases},$$

and $\deg(\epsilon_i) = k_i$ for all $i = 1,\ldots,n$. That is, E_i is the exterior (when k_i is odd) or the truncated polynomial algebra of height 2 (when k_i is even) generated over the rationals by a degree k_i element ϵ_i.

For $J = \{j_1,\ldots,j_k\} \subseteq [n]$, let $\epsilon_J = \epsilon_{j_1}\cdots\epsilon_{j_k}$. In previous terms, we let $E(\epsilon_1,\ldots,\epsilon_n)$ stand for the cohomology ring $H^*(\mathbb{S}(k_1,\ldots,k_n))$.

The cohomology ring $H^*(X)$ is a quotient of $E(\epsilon_1,\ldots,\epsilon_n)$:

Proposition 2.3.1. *For a subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ (the latest considered with the minimal CW-decomposition), the cohomology ring $H^*(X)$ is the quotient of the algebra $E(\epsilon_1,\ldots,\epsilon_n)$ by the monomial ideal I_X generated by those ϵ_J for which e_J is not a cell of X.*

For a proof (in a more general context) of this proposition see [1, Theorem 2.35]. In particular, an additive basis for $H^*(X)$ is given by the products ϵ_J with e_J a cell of X. We will work with the corresponding tensor power basis for $H^*(X^s)$.

Remark 2.3.2. *In the next two results, the hypothesis of having a fixed parity for all the k_i will be crucial when handling products of zero divisors in $H^*(X^s)$. Indeed, a typical such element has the form*

$$z = c_1\cdot\epsilon_i\otimes 1\otimes\cdots\otimes 1 + c_2\cdot 1\otimes\epsilon_i\otimes 1\otimes\cdots\otimes 1 + \cdots + c_s\cdot 1\otimes\cdots\otimes 1\otimes\epsilon_i$$

for $i\in[n]$ and $c_1,\ldots,c_s\in\mathbb{Q}$ with $c_1+\cdots+c_s=0$. Then, by graded commutativity, z^2 is forced to vanish when k_i is odd. However $z^s\neq 0$ if k_i is even and $c_j\neq 0$ for all $j\in[s]$.

Proposition 1.3.3 and the following result complete the proof of Theorem 2.2.5.

Proposition 2.3.3. *Let X be as in Proposition* 2.3.1*. If all of the k_i are odd, then*

$$\mathrm{N}^s(X) \le \mathrm{zcl}_s(H^*(X)).$$

Proof. Let $H_X = H^*(X^s) = [H^*(X)]^{\otimes s}$. For $u \in H^*(X)$ and $2 \leq \ell \leq s$, let

$$u(\ell) = \underbrace{u \otimes 1 \otimes \cdots \otimes 1}_{s \text{ factors}} - \underbrace{1 \otimes \cdots \otimes 1 \otimes \overset{\ell}{u} \otimes 1 \otimes \cdots \otimes 1}_{s \text{ factors}} \in H_X$$

where an ℓ on top of a tensor factor indicates the coordinate where the factor appears. Take a cell $e_{J_1} \times e_{J_2} \times \cdots \times e_{J_s} \subseteq X^s$, $J_1, \ldots, J_s \subseteq [n]$. For $2 \leq \ell \leq s$, let

$$\begin{aligned} \gamma(J_1, \ldots, J_\ell) &= \prod_{j \in \left(\bigcap_{m=1}^{\ell-1} J_m - J_\ell \right) \cup J_\ell} \epsilon_j(\ell) \\ &= \sum_{\phi_\ell \subseteq \left(\bigcap_{m=1}^{\ell-1} J_m - J_\ell \right) \cup J_\ell} \pm \epsilon_{\phi_\ell^c} \otimes 1 \otimes \cdots \otimes 1 \otimes \overset{\ell}{\epsilon_{\phi_\ell}} \otimes 1 \otimes \cdots \otimes 1 \end{aligned}$$

where ϕ_ℓ^c stands for the complement of ϕ_ℓ in $\left(\bigcap_{m=1}^{\ell-1} J_m - J_\ell \right) \cup J_\ell$. It suffices to prove the non-triviality of the product of $\mathrm{N}_X(J_1, \ldots, J_s)$ zero-divisors

$$\gamma(J_1, J_2) \cdots \gamma(J_1, \ldots, J_s) = \sum_{\phi_2, \ldots, \phi_s} \pm \epsilon_{\phi_2^c} \cdots \epsilon_{\phi_s^c} \otimes \epsilon_{\phi_2} \otimes \cdots \otimes \epsilon_{\phi_s} \tag{2.22}$$

where the sum runs over all $\phi_\ell \subseteq \left(\bigcap_{m=1}^{\ell-1} J_m - J_\ell \right) \cup J_\ell$ with $2 \leq \ell \leq s$. With this in mind, note that the term

$$\pm \epsilon_{J_1 - J_2} \cdots \epsilon_{(J_1 \cap \cdots \cap J_{\ell-1}) - J_\ell} \cdots \epsilon_{(J_1 \cap \cdots \cap J_{s-1}) - J_s} \otimes \epsilon_{J_2} \otimes \cdots \otimes \epsilon_{J_\ell} \otimes \cdots \otimes \epsilon_{J_s}, \tag{2.23}$$

which appears in (2.22) with $\phi_\ell = J_\ell$ for $2 \leq \ell \leq s$, is a basis element because

$$\epsilon_{J_1 - J_2} \cdots \epsilon_{(J_1 \cap \cdots \cap J_{\ell-1}) - J_\ell} \cdots \epsilon_{(J_1 \cap \cdots \cap J_{s-1}) - J_s} = \epsilon_{J_0}$$

with $J_0 \subseteq J_1$. The non-triviality of (2.22) then follows by observing that (2.23) cannot arise when other summands in (2.22) are expressed in terms of the basis for H_X. In fact, each summand

$$\pm \epsilon_{\phi_2^c} \cdots \epsilon_{\phi_s^c} \otimes \epsilon_{\phi_2} \otimes \cdots \otimes \epsilon_{\phi_s} \tag{2.24}$$

in (2.22) is either zero or a basis element and, in the latter case, (2.24) agrees (up to sign) with (2.23) only if $\phi_\ell = J_\ell$ for $\ell = 2, \ldots, s$. ☐

Likewise, the proof of Theorem 2.2.13 is complete by Proposition 1.3.3 and the following result:

Proposition 2.3.4. *Let X be as in Proposition* 2.3.1. *If all of the k_i are even, then*

$$s\,(1 + \dim(\mathcal{K}_X)) \leq \mathrm{zcl}_s(H^*(X)).$$

Proof. For $u \in H^*(X)$, set

$$\overline{u} = \left(\sum_{i=1}^{s-1} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{u} \otimes 1 \otimes \cdots \otimes 1 \right) - 1 \otimes \cdots \otimes 1 \otimes (s-1)u \in H_X.$$

Fix a maximal cell e_L of X where $L = \{\delta_1, \ldots, \delta_\ell\} \subseteq [n]$ (so $\ell = 1 + \dim(\mathcal{K}_X)$). A straightforward calculation yields, for $i \in [\ell]$,

$$(\overline{\epsilon_{\delta_i}})^s = (1-s)s!(\underbrace{\epsilon_{\delta_i} \otimes \cdots \otimes \epsilon_{\delta_i}}_{s \text{ factors}}),$$

so

$$\prod_{i=1}^{\ell} (\overline{\epsilon_{\delta_i}})^s = ((1-s)s!)^\ell \underbrace{\epsilon_L \otimes \cdots \otimes \epsilon_L}_{s \text{ factors}}$$

which is a nonzero product of $s\ell$ zero-divisors in H_X. ☐

Remark 2.3.5. *The estimate* $s(1+\dim(\mathcal{K}_X)) \leq \mathrm{TC}_s(X)$ *can also be obtained by noticing that, in the notation of the proof of Proposition 2.3.4,* $\mathbb{S}(k_{\delta_1},\ldots,k_{\delta_\ell}) \cong \overline{e_L}$ *is a retract of* X *(cf. [13, proof of Proposition 4]).*

2.4 Explicit computations

In this section we analyze some consequences of Theorem 2.2.13 and 2.2.5 for interesting special instances.

Corollary 2.4.1. *Suppose all of the* k_i *are odd and* X *is* d*-pure. Then*

$$\mathrm{TC}_s(X) = sd - \min\left|\bigcap_{i=1}^{s} J_i\right|$$

where the minimum is taken over all sets $\{J_1,\ldots,J_s\}$ *of maximal simplices of* $\mathcal{K}_X$*. In particular* $\mathrm{TC}_s(X) \leq sd$ *with equality if and only if* $\bigcap_{i=1}^{s} J_i$ *is empty for some choice of maximal simplices* J_i*'s.*

Corollary 2.4.1 implies that, for X d-pure, $\mathrm{TC}_s(X)$ grows linearly on s provided s is large enough. More precisely, if $w = w(\mathcal{K}_X)$ denotes the number of maximal simplices in $\mathcal{K}_X$, then

$$\mathrm{TC}_s(X) = d(s-w) + \mathrm{TC}_w(X) \tag{2.25}$$

for $s \geq w$. More generally we have:

Proposition 2.4.2. *Let* w *be as above, and set* $d = 1+\dim(\mathcal{K}_X)$*. Equation* (2.25) *holds for any (pure or not) subcomplex* X *of* $\mathbb{S}(k_1,\ldots,k_n)$ *(where all* k_i *are odd) as long as* $s \geq w$*.*

The proof of Proposition 2.4.2 uses the following auxiliary result:

Lemma 2.4.3. *In the setting of Proposition* 2.4.2*, if* $J_1,\ldots,J_w$ *are maximal simplices of* $\mathcal{K}_X$ *such that* $\mathrm{TC}_w(X) = \sum_{i=1}^{w}|J_i| - \left|\bigcap_{i=1}^{w} J_i\right|$*, then* $\max\{\,|J_i| \mid i \in [w]\} = d$*.*

Proof. Assume for a contradiction that $J_1,\ldots,J_w$ are maximal simplices of $\mathcal{K}_X$ such that $\mathrm{TC}_w(X) = \sum_{i=1}^{w}|J_i| - |\bigcap_{i=1}^{w} J_i|$ with $|J_i| < d$ for all $i \in [w]$. Choose a simplex J_0 of $\mathcal{K}_X$ with $|J_0| = d$, and indexes $i_1, i_2 \in [w]$, $i_1 < i_2$, with $J_{i_1} = J_{i_2}$. Set

$$(J'_1,\ldots,J'_w) := (J_0, J_1,\ldots,J_{i_1-1}, J_{i_1+1},\ldots,J_w).$$

The contradiction comes from

$$\mathrm{N}_X(J'_1,\ldots,J'_w) = \sum_{i=1}^{w}|J'_i| - \left|\bigcap_{i=1}^{w} J'_i\right| > \sum_{i=1}^{w}|J_i| - \left|\bigcap_{i=1}^{w} J'_i\right| \geq \sum_{i=1}^{w}|J_i| - \left|\bigcap_{i=1}^{w} J_i\right| = \mathrm{TC}_w(X)$$

where the last inequality holds because $\bigcap_{i=1}^{w} J'_i \subseteq \bigcap_{i=2}^{w} J'_i = \bigcap_{i=1}^{w} J_i$. □

Proof of Proposition 2.4.2. Let $s \geq w$. Choose maximal simplices J'_i and J_j (with $i = 1,\ldots,s$ and $j = 1,\ldots,w$) of $\mathcal{K}_X$ with

$$\mathrm{N}^s(X) = \sum_{i=1}^{s}|J'_i| - \left|\bigcap_{i=1}^{s} J'_i\right| \quad \text{and} \quad \mathrm{N}^w(X) = \sum_{i=1}^{w}|J_i| - \left|\bigcap_{i=1}^{w} J_i\right|.$$

Assume without loss of generality (since $s \geq w$) that $\{J'_1, \ldots, J'_s\} = \{J'_1, \ldots, J'_w\}$. Then

$$\begin{aligned}
\mathrm{TC}_s(X) &= \sum_{i=1}^{s} |J'_i| - \Big|\bigcap_{i=1}^{s} J'_i\Big| = \sum_{i=1}^{w} |J'_i| + \sum_{i=w+1}^{s} |J'_i| - \Big|\bigcap_{i=1}^{w} J'_i\Big| \\
&\leq \mathrm{TC}_w(X) + \sum_{i=w+1}^{s} |J'_i| \leq \mathrm{TC}_w(X) + (s-w)d
\end{aligned}$$

where, as before, $d = 1 + \dim(\mathcal{K}_X)$. On the other hand, Lemma 2.4.3 yields an integer $i_0 \in [w]$ with $|J_{i_0}| = d$. Set $J_j := J_{i_0}$ for $w+1 \leq j \leq s$. Then

$$\mathrm{TC}_w(X) + (s-w)d = \sum_{i=1}^{w} |J_i| - \Big|\bigcap_{i=1}^{w} J_i\Big| + \sum_{i=w+1}^{s} |J_i| = \sum_{i=1}^{s} |J_i| - \Big|\bigcap_{i=1}^{s} J_i\Big| \leq \mathrm{TC}_s(X),$$

completing the proof. □

A more precise description of $\mathrm{TC}_s(X)$ can be obtained by imposing conditions on X which are stronger than purity. For instance, let $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$ stand for the d-pure subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$ with index $\Delta[n-1]^{d-1}$, the $(d-1)$-skeleton of the full simplicial complex on n vertices. For instance, when $k_i = 1$ for all $i \in [n]$, $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$ is the d-dimensional skeleton in the minimal CW structure of the n-torus—the n-fold Cartesian product of S^1 with itself.

Corollary 2.4.4. *If all of the k_i are odd, then* $\mathrm{TC}_s\left(\mathbb{S}(k_1, \ldots, k_n)^{(d)}\right) = \min\{sd, (s-1)n\}$.

In view of Hattori's theorem ([20], see also [25, Theorem 5.21]), Corollary 2.4.4 specializes, with $k_i = 1$ for all $i \in [n]$, to the assertion in [29, page 8] describing the higher topological complexity of complements of complex hyperplane arrangements that are either linear generic, or affine in general position (cf. [30, Section 3]). It is also interesting to highlight that the "min" part in Corollary 2.4.4 (with $d = 1$) can be thought of as a manifestation of the fact that, while the s-th topological complexity of an odd sphere is $s-1$, wedges of at least two spheres have $\mathrm{TC}_s = s$.

Proof of Corollary 2.4.4. Let X stand for $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$. For simplices $J_1, \ldots, J_s$ of $\Delta[n-1]^{d-1}$, the inequality $\mathrm{N}_X(J_1, \ldots, J_s) \leq \min\{sd, (s-1)n\}$ follows from Corollary 2.4.1 and Lemma 2.2.2 since $|I_\ell| + |J_\ell| \leq n$. Thus $\mathrm{TC}_s(X) \leq \min\{sd, (s-1)n\}$ (notice this holds for any d-pure X). To prove the opposite inequality suppose first that $sd \leq (s-1)n$, equivalently $n \leq s(n-d)$. Then there exist a covering $\{C_1 \ldots, C_s\}$ of $[n]$ with $|C_k| = n - d$ for every $k \in [s]$. Put $J_k = [n] - C_k$ and notice that J_k is a maximal simplex of $\Delta[n-1]^{d-1}$ for every k. Further $\bigcap_{k=1}^{s} J_k = \emptyset$, so that Corollary 2.4.1 yields

$$\mathrm{TC}_s(X) = sd = \min\{sd, (s-1)n\}.$$

Finally assume that $(s-1)n \leq sd$, i.e., $s(n-d) \leq n$. Then there exists a collection $\{C_1 \ldots, C_s\}$ of mutually disjoint subsets of $[n]$ with $|C_k| = n - d$ for every k. Again put $J_k = [n] - C_k$. We have

$$\mathrm{TC}_s(X) \geq \sum_{k=1}^{s} |J_k| - \Big|\bigcap_{k=1}^{s} J_k\Big| = sn - \sum_{k=1}^{s} |C_k| - \Big|\bigcap_{k=1}^{s} J_k\Big| = sn - \sum_{k=1}^{s} |C_k| - n + \Big|\bigcup_{k=1}^{s} C_k\Big|.$$

The result follows since the latter term simplifies to $(s-1)n = \min\{sd, (s-1)n\}$. □

The higher topological complexity of a subcomplex X of $\mathbb{S}(k_1, \ldots, k_n)$ whose index is pure but not a skeleton depends heavily on the combinatorics of $\mathcal{K}_X$ —and not just on its dimension. To illustrate the situation, we offer the following example.

Example 2.4.5. Suppose the parameters are $n = 4$, $d = 2$, $s = 3$; $\mathcal{K}_1$ has the set of maximal simplices $\{\{1,2\},\{2,3\},\{3,4\}\}$ while $\mathcal{K}_2$ the set $\{\{1,2\},\{1,3\},\{1,4\}\}$. Fix positive odd integers k_1, k_2, k_3, k_4, and let X_i $(i = 1,2)$ be the CW subcomplex of $\mathbb{S}(k_1,k_2,k_3,k_4)$ having $\mathcal{K}_i$ as its index. Then Corollary 2.4.1 gives $\mathrm{TC}_3(X_1) = 6$ while $\mathrm{TC}_3(X_2) = 5$.

Interesting phenomena can arise if X is not pure. This can be demonstrated by the following examples:

Example 2.4.6. Take $s = n$. For $i \in [n]$, let $K_i = [n] - \{i\}$, and for $I \subseteq [n]$, let

$$W_I = \mathbb{S}(k_1,\ldots,k_n)^{(n-1)} - \bigcup_{i\in I} e_{K_i},$$

the subcomplex obtained from the fat wedge after removing the facets corresponding to vertices $i \in I$. As before, we assume that all of the k_i are odd. Note that W_I is $(n-1)$-pure if $|I| \le 1$, in which case Corollary 2.4.1 gives

$$\mathrm{TC}_n(W_I) = n(n-1) - |I|. \tag{2.26}$$

But the situation is slightly subtler when $2 \le |I| < n$ because, although the corresponding W_I all have the same dimension, they fail to be pure, in fact:

$$\mathrm{TC}_n(W_I) = \begin{cases} n(n-1) - (\delta+1), & \text{if } |I| = 2\delta + 1; \\ n(n-1) - \delta, & \text{if } |I| = 2\delta. \end{cases} \tag{2.27}$$

Note however that, by Corollary 2.4.4, once all maximal simplices have been removed from the fat wedge, we find the rather smaller value $\mathrm{TC}_n(W_{[n]}) = n(n-2)$, back in accordance to (2.26). The straightforward counting argument verifying (2.27) is left as an exercise for the interested reader; we just provide the hint that the set of maximal simplices of $\mathcal{K}_{W_I}$ is

$$\{\, K_i \mid i \notin I \,\} \cup \{\, J \mid [n] - J \subseteq I \text{ and } |J| = n-2 \,\}.$$

Example 2.4.7. Let $c_1 > c_2$ be positive integers and $n = c_1 + c_2$. Consider the simplicial complex $\mathcal{K} = \mathcal{K}^{c_1,c_2}$ with vertices $[n]$ determined by two disjoint maximal simplices K_1 and K_2 with $|K_1| = c_1$ and $|K_2| = c_2$. Then, for any collection $J_1,\ldots,J_s$ of maximal simplices of $\mathcal{K}$, where precisely s_1 sets among $J_1,\ldots,J_s$ are equal to K_1 and $0 \le s_1 \le s$, Proposition 2.2.3 yields

$$\mathrm{N}_{\mathcal{K}}(J_1,\ldots,J_s) = \begin{cases} (s-1)c_2, & s_1 = 0; \\ s_1 c_1 + (s-s_1)c_2, & 0 < s_1 < s; \\ (s-1)c_1, & s_1 = s. \end{cases}$$

This function of s_1 reaches its largest value when $s_1 = s-1$ whence $\mathrm{N}^s(\mathcal{K}) = (s-1)c_1 + c_2 = sc_1 - (c_1 - c_2)$. The latter formula shows that, as $c_1 - c_2$ runs through the integers $1, 2, \ldots, c_1 - 1$, $\mathrm{N}^s(\mathcal{K})$ runs through $sc_1 - 1, sc_1 - 2, \ldots, (s-1)c_1 + 1$. Whence, due to Theorem 2.2.5, the same is true for $\mathrm{TC}_s(X)$ where $X = X_{c_1,c_2}$ is the subcomplex of some $\mathbb{S}(k_1,\ldots,k_n)$ (with all k_i odd) whose index equals $\mathcal{K}$.

Remark 2.4.8. The previous example should be compared with the fact (proved in [2, Corollary 3.3]) that the s-th topological complexity of a given path connected space X is bounded by $\mathrm{cat}(X^{s-1})$ from below, and by $\mathrm{cat}(X^s)$ from above. Example 2.4.7 implies that not only can both bounds be attained (by allowing $c_2 = 0$ and $c_1 = c_2$, respectively) but any possibility in between can occur.

It is well known that, under suitable normality conditions, the higher topological complexity of a Cartesian product can be estimated by

$$\mathrm{zcl}_s(H^*(X)) + \mathrm{zcl}_s(H^*(Y)) \leq \mathrm{zcl}_s(H^*(X \times Y)) \leq \mathrm{TC}_s(X \times Y) \leq \mathrm{TC}_s(X) + \mathrm{TC}_s(Y), \tag{2.28}$$

see [2, Proposition 3.11] and [4, Lemma 2.1]. Of course, these inequalities are sharp provided $\mathrm{TC}_s = \mathrm{zcl}_s$ for both X and Y. In particular, for subcomplexes of products of spheres, TC_s is additive in the sense that the higher topological complexity of a Cartesian product is the sum of the higher topological complexities of the factors. This generalizes the known TC_s-behavior of products of spheres, see [2, Corollary 3.12]. However, if Cartesian products are replaced by wedge sums, the situation becomes much subtler. To begin with, we remark that Theorem 3.6 and Remark 3.7 in [8], together with [12, Theorem 19.1], give evidence suggesting that a reasonable wedge-substitute of (2.28) (for $s = 2$) would be given by

$$\max\{\mathrm{TC}_2(X), \mathrm{TC}_2(Y), \mathrm{cat}(X \times Y)\} \leq \mathrm{TC}_2(X \vee Y),$$

and

$$\mathrm{TC}_2(X \vee Y) \leq \max\{\mathrm{TC}_2(X), \mathrm{TC}_2(Y), \mathrm{cat}(X) + \mathrm{cat}(Y)\}.$$

We show that both of these inequalities hold as equalities for the spaces dealt with in the previous section (cf. [5, Proposition 3.10]). More generally:

Proposition 2.4.9. *Let X and Y be subcomplexes of $\mathbb{S}(k_1, \ldots, k_n)$ and $\mathbb{S}(k_{n+1}, \ldots, k_{n+m})$ respectively. If $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$ and all the k_i have the same parity, then*

$$\mathrm{TC}_s(X \vee Y) = \max\{\mathrm{TC}_s(X), \mathrm{TC}_s(Y), \mathrm{cat}(X^{s-1}) + \mathrm{cat}(Y)\}.$$

Proof. If all the k_i are even, then the conclusion holds, since $\mathrm{TC}_s(X \vee Y) = \mathrm{TC}_s(X)$ under the present hypothesis. Assume now that all the k_i are odd, and think of $X \vee Y$ as a subcomplex of $X \times Y$ inside $\mathbb{S}(k_1, \ldots, k_n, k_{n+1}, \ldots, k_{n+m})$, so that $\mathcal{K}_{X\vee Y}$ is the disjoint union of $\mathcal{K}_X$ and $\mathcal{K}_Y$. Since $\mathrm{cat}(X) = \dim(\mathcal{K}_X) + 1 \geq \mathrm{cat}(Y) = \dim(\mathcal{K}_Y) + 1$, for maximal simplices $J_1, \ldots, J_s$ of $\mathcal{K}_{X\vee Y}$ we see

$$\mathrm{N}_{X\vee Y}(J_1, \ldots, J_s) \leq \begin{cases} \mathrm{TC}_s(X), & \text{if } J_1, \ldots, J_s \subseteq [n]; \\ \mathrm{TC}_s(Y), & \text{if } J_1, \ldots, J_s \subseteq \{n+1, \ldots, n+m\}; \\ (s-1)\,\mathrm{cat}(X) + \mathrm{cat}(Y), & \text{otherwise.} \end{cases} \tag{2.29}$$

Therefore $\mathrm{TC}_s(X \vee Y) \leq \max\{\mathrm{TC}_s(X), \mathrm{TC}_s(Y), (s-1)\,\mathrm{cat}(X) + \mathrm{cat}(Y)\}$. The reverse inequality holds since each of $\mathrm{TC}_s(X)$, $\mathrm{TC}_s(Y)$, and $(s-1)\,\mathrm{cat}(X) + \mathrm{cat}(Y)$ can be achieved as a $\mathrm{N}_{X\vee Y}(J_1, \ldots, J_s)$ for a suitable combination of maximal simplices J_i of $\mathcal{K}_{X\vee Y}$. □

2.5 The unrestricted case

We now prove Theorem 0.0.1 in the general case, that is, for X a subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$ where all the k_i are positive integers with no restriction on their parity. As in previous cases, we start by establishing the upper bound.

2.5.1 Motion planner

Consider the disjoint union decomposition $[n] = J_E \sqcup J_O$ where J_E is the collection of indices $i \in [n]$ for which k_i is even (thus $i \in J_O$ if and only if k_i is odd). For a subset $K \subseteq J_E$ and $P \in \mathcal{P}$, let $X^s_{P,K} \subseteq X^s$ and $\mathrm{N}(P, K)$, the norm of (P, K), be defined by

- $X^s_{P,K} = X^s_P \cap \left\{ (b_1,\dots,b_s) \in X^s \;\middle|\; \begin{array}{l} \text{for each } (i,j) \in K \times [s],\ b_{ij} \neq \pm e^0, \text{ while} \\ \text{for each } i \in J_E - K \text{ there is } j \in [s] \text{ with } b_{ij} = \pm e^0 \end{array} \right\}$
- $\mathrm{N}(P,K) = |P| + |K|$ where $|P|$ is defined in (2.2).

This extends and refines the definitions of $X^s_{P,k}$ and $\mathrm{N}(P,k)$ made when all the k_i are even.

As in the cases where all the k_i have the same parity, the higher topological complexity of a subcomplex X of $\mathbb{S}(k_1,\dots,k_n)$, now with no restrictions on the parity of the sphere factors, is encoded just by the combinatorial information on the cells of X. Consider

$$\mathcal{N}^s(X) = \max\left\{ \mathrm{N}_X(J_1,\dots,J_s) + \left|\bigcap_{i=1}^s J_i \cap J_E\right| \;\middle|\; J_1,\dots,J_s \in \mathcal{K}_X \right\} \tag{2.30}$$

where $\mathrm{N}_X(J_1,\dots,J_s)$ is defined in (2.5) for $\mathcal{K} = \mathcal{K}_X$. Since both $\mathrm{N}_X(J_1,\dots,J_s)$ and $|\bigcap_{i=1}^s J_i \cap J_E|$ are monotonically non-decreasing functions of the J_i's, the definition of $\mathcal{N}^s(X)$ can equally well be given using only maximal simplices $J_i \in \mathcal{K}_X$. Further, by (2.6), $\mathcal{N}^s(X)$ can be rewritten as

$$\mathcal{N}^s(X) = \max\left\{ \sum_{i=1}^s |J_i| - \left|\bigcap_{i=1}^s J_i \cap J_O\right| \;\middle|\; J_1,\dots,J_s \in \mathcal{K}_X \right\}. \tag{2.31}$$

Theorem 2.5.1. *For a subcomplex X of $\mathbb{S}(k_1,\dots,k_n)$,*

$$\mathrm{TC}_s(X) = \mathcal{N}^s(X).$$

Theorem 2.5.1 also generalizes Theorems 2.2.5 and 2.2.13. This is obvious when all the k_i are odd for then both $\mathcal{N}^s(X)$ and $\mathrm{N}^s(X)$ agree with

$$\max\left\{ \sum_{i=1}^s |J_i| - \left|\bigcap_{i=1}^s J_i\right| \;\middle|\; J_1,\dots,J_s \in \mathcal{K}_X \right\},$$

whereas if all the k_i are even,

$$\mathcal{N}^s(X) = \max\left\{ \sum_{i=1}^s |J_i| \;\middle|\; J_1,\dots,J_s \in \mathcal{K}_X \right\} = s(1+\dim \mathcal{K}_X).$$

The estimate $\mathcal{N}^s(X) \leq \mathrm{TC}_s(X)$ in Theorem 2.5.1 will be proved in the next subsection by extending the cohomological methods in Section 2.5.2. Here we prove the estimate $\mathrm{TC}_s(X) \leq \mathcal{N}^s(X)$ by constructing an optimal motion planner with $\mathcal{N}^s(X)+1$ local rules. The corresponding local domains will be obtained by clustering subsets $X^s_{P,K}$ for which the pair $(P,K) \in \mathcal{P} \times 2^{J_E}$ has a fixed norm. In detail, for $j \in [\mathcal{N}^s(X)]_0$ let

$$G_j := \bigcup_{\mathrm{N}(P,K)=j} X^s_{P,K}. \tag{2.32}$$

Lemma 2.5.2. *The sets $G_0,\dots,G_{\mathcal{N}^s(X)}$ yield a pairwise disjoint covering of X^s.*

Proof. It is easy to see that $G_j \cap G_{j'} = \emptyset$ for $j \neq j'$. Let $b = (b_1,\dots,b_s) \in e_{J_1} \times \cdots \times e_{J_s} \subseteq X^s$, where $J_j \subseteq [n]$ for $j \in [s]$. As in Lemma 2.2.6, we have

$$\sum_{i=1}^n |\{G \cdot b_{ij} \mid j \in [s]\}| - n \leq \sum_{j=1}^s |J_j| - \left|\bigcap_{j=1}^s J_j\right| = \mathrm{N}_X(J_1,\dots,J_s). \tag{2.33}$$

Moreover, it is clear that

$$\Big|\{i \in J_E \mid b_{ij} \neq \pm e^0,\ \forall j \in [s]\}\Big| \leq \Big|\bigcap_{i=1}^{s} J_i \cap J_E\Big|. \tag{2.34}$$

Thus, if $P \in \mathcal{P}$ is the type of b, and $K \subseteq J_E$ is determined by the condition that $b \in X^s_{P,K}$, then $N(P,K) = |P| + |K| \leq \mathcal{N}^s(X)$ in view of (2.3), (2.33) and (2.34). □

Lemma 2.5.3. (2.32) *is a topological disjoint union. Indeed,*

$$X^s_{P,K} \cap \overline{X^s_{P',K'}} = \emptyset = \overline{X^s_{P,K}} \cap X^s_{P',K'} \tag{2.35}$$

for $(P,K),(P',K') \in \mathcal{P} \times 2^{J_E}$ *provided that* $(P,K) \neq (P',K')$ *and* $\mathrm{N}(P,K) = \mathrm{N}(P',K')$.

The following observation will be useful in the proof of Lemma 2.5.3:

Remark 2.5.4. *Let* $K, K' \subseteq 2^{J_E}$ *and* $P, P' \in \mathcal{P}$. *If there exists an index* $i \in K - K'$, *then*

- $b_{ij} \neq \pm e^0$ *for all* $j \in [s]$ *provided* $b = (b_1, \ldots, b_s) \in X^s_{P,K}$.
- $b_{ij_0} = \pm e^0$ *for some* $j_0 \in [s]$ *provided* $b = (b_1, \ldots, b_s) \in X^s_{P',K'}$.

Therefore, $X^s_{P,K} \cap \overline{X^s_{P',K'}} = \emptyset$.

Proof of Lemma 2.5.3. There are three possibilities:

Case $K = K'$. In this case, we conclude that $P \neq P'$ with $|P| = |P'|$, since $(P,K) \neq (P',K')$ and $\mathrm{N}(P,K) = \mathrm{N}(P',K')$. The desired equalities follow from Proposition 2.2.7.

Case $P = P'$. In this case we have $K \neq K'$ with $|K| = |K'|$. Then, there exist indexes $i, i' \in [n]$ such that $i \in K - K'$ and $i' \in K' - K$. Therefore, equalities (2.35) follow from Remark 2.5.4.

Case $P \neq P'$ and $K \neq K'$. Without loss of generality we can assume $|P| > |P'|$. Then there exists $i \in [n]$ such that $|P_i| > |P'_i|$, thus $X^s_{P,K} \cap \overline{X^s_{P',K'}} = \emptyset$. Moreover, since $|K| < |K'|$ is forced, there exists $i \in K' - K$, so that $\overline{X^s_{P,K}} \cap X^s_{P',K'} = \emptyset$ by Remark 2.5.4. □

Lemmas 2.5.2 and 2.5.3 reduce the proof of Theorem 2.5.1 to checking that each $X^s_{P,K}$ is an ENR admitting a local rule. Thus, throughout the remainder of this subsection we fix a pair $(P,K) \in \mathcal{P} \times 2^{J_E}$ with $P = (P_1, \ldots, P_n)$ and where each $P_i = \{\alpha^i_1, \ldots, \alpha^i_{n(P_i)}\}$ is assumed to be ordered as indicated at the beginning of Section 2.2.

Our analysis of $X^s_{P,K}$ depends on establishing a topological decomposition of $X^s_{P,K}$. To start with, note the topological disjoint union decomposition

$$X^s_{P,K} = \bigsqcup_{\beta} X^s_{P,K} \cap X^s_{P,\beta}$$

where the union runs over all $\beta = (\beta^1, \ldots, \beta^n)$ as in (2.4). But we need a further splitting of each term $X^s_{P,K} \cap X^s_{P,\beta}$.

Let $I = \{\ell_1, \ldots, \ell_{|I|}\}$ stand for $J_E - K$ and, for each $i \in [n]$, consider the subsets $T_0(\alpha^i_1)$ and $T_1(\alpha^i_1)$ defined in (2.20). For each $\epsilon = (t_1, \ldots, t_{|I|}) \in \{0,1\}^{|I|}$ define

$$T_\epsilon = \bigcap_{i=1}^{|I|} T_{t_i}(\alpha^{\ell_i}_1).$$

We then get a topological disjoint union decomposition

$$X^s_{P,K} \cap X^s_{P,\beta} = \bigsqcup_{\epsilon \in \{0,1\}^{|I|}} X^s_{P,K} \cap X^s_{P,\beta} \cap T_\epsilon.$$

Therefore, the updated task is the proof of:

Lemma 2.5.5. *Each $X^s_{P,K,\beta,\epsilon} := X^s_{P,K} \cap X^s_{P,\beta} \cap T_\epsilon$ is an ENR admitting a local rule.*

Proof. The ENR assertion follows just as in the first paragraph of the proof of Proposition 2.2.16. The construction of the local rule is also similar to those at the end of Subsections 2.2.1 and 2.2.2, and we provide the generalized details for completeness.

For $i = 0, 1$ and $j = 0, 1, 2$, let τ_i and ρ_j be the local rules, with corresponding local domains A_i and B_j, obtained in Subsections 2.2.1 and 2.2.2 by normalizing the local rules ϕ_i and λ_j given in Examples 2.2.1 and 2.2.10 —see the proof of Proposition 2.2.16 and the considerations following the proof of Proposition 2.2.7.

As before, it is useful to keep in mind that elements $(b_1, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj})$ for $j \in [s]$, can be thought of as matrices $(b_{i,j})$ whose columns represent the various stages in X through which motion is to be planned (necessarily along rows). Again, we follow a pivotal strategy. In detail, in terms of the notation set at the beginning of the introduction for elements in the function space $X^{\mathcal{J}_s}$, consider the map

$$\varphi\colon X^s \to \mathbb{S}(k_1, \ldots, k_n)^{\mathcal{J}_s} \tag{2.36}$$

given by $\varphi\left((b_1, \ldots, b_s)\right) = (\varphi_1(b_1, b_1), \ldots, \varphi_s(b_1, b_s))$ where, for $j \in [s]$,

$$\varphi_j(b_1, b_j) = (\varphi_{1j}(b_{11}, b_{1j}), \ldots, \varphi_{nj}(b_{n1}, b_{nj}))$$

is the path in $\mathbb{S}(k_1, \ldots, k_n)$, from b_1 to b_j, whose i-th coordinate $\varphi_{ij}(b_{i1}, b_{ij})$, $i \in [n]$, is the path in S^{k_i}, from b_{i1} to b_{ij}, defined by

$$\varphi_{i,j}(b_{i1}, b_{ij})(t) = \begin{cases} b_{i1}, & 0 \le t \le t_{b_{i1}}, \\ \sigma(b_{i1}, b_{ij})(t - t_{b_{i1}}), & t_{b_{i1}} \le t \le 1. \end{cases}$$

Here $t_{b_{i1}} = \frac{1}{2} - d(b_{i1}, e^0)$ and

$$\sigma(b_{i1}, b_{ij}) = \begin{cases} \tau_0(b_{i1}, b_{ij}), & \text{if } i \in J_O \text{ and } (b_{i1}, b_{ij}) \in A_0; \\ \tau_1(b_{i1}, b_{ij}), & \text{if } i \in J_O \text{ and } (b_{i1}, b_{ij}) \in A_1; \\ \rho_0(b_{i1}, b_{ij}), & \text{if } i \in J_E \text{ and } (b_{i1}, b_{ij}) \in B_0; \\ \rho_1(b_{i1}, b_{ij}), & \text{if } i \in J_E \text{ and } (b_{i1}, b_{ij}) \in B_1; \\ \rho_2(b_{i1}, b_{ij}), & \text{if } i \in J_E \text{ and } (b_{i1}, b_{ij}) \in B_2. \end{cases} \tag{2.37}$$

Although φ is not continuous, its restriction $\varphi_{P,K,\beta,\epsilon}$ to $X^s_{P,K,\beta,\epsilon}$ is, for then (2.37) takes the form

$$\sigma = \begin{cases} \tau_1, & i \in J_O,\ j \notin \alpha_1^i \text{ or } j \in \beta^i \cup \{1\}; \\ \tau_0, & i \in J_O,\ j \in \alpha_1^i \text{ and } j \notin \beta^i \cup \{1\}; \\ \rho_2, & i \in J_E,\ j \notin \alpha_1^i \text{ or } j \in \beta^i \cup \{1\}; \\ \rho_1, & i \in J_E,\ j \in \alpha_1^i - \left(\beta^i \cup \{1\}\right) \text{ and } t_i = 0; \\ \rho_0, & i \in J_E,\ j \in \alpha_1^i - \left(\beta^i \cup \{1\}\right) \text{ and } t_i = 1. \end{cases}$$

Moreover, $\varphi_{P,K,\beta,\epsilon}$ is clearly a section for $e_s^{\mathbb{S}(k_1,\ldots,k_n)}$, while the fact that $\varphi_{P,K,\beta,\epsilon}$ actually takes values in $X^{\mathcal{J}_s}$ is verified with an argument identical to the one proving Proposition 2.2.9. ☐

2.5.2 Zero-divisors cup-length

We next show that, for a subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ (with no restrictions on the parity of the k_i, $i\in[n]$), the cohomological lower bound for $\mathrm{TC}_s(X)$ in Proposition 1.3.3 is optimal and agrees with the upper bound coming from our explicit motion planner in the previous subsection. Here we use the same considerations and notation as in Section 2.3.

Proposition 2.5.6. *A subcomplex X of $\mathbb{S}(k_1,\ldots,k_n)$ has*

$$\mathcal{N}^s(X)\leq \mathrm{zcl}_s(H^*(X)).$$

Proof. We use the tensor product ring H_X, and the elements $u(\ell)\in H_X$ for $u\in H^*(X)$, as well as the elements $\gamma(J_1,\ldots,J_\ell)\in H_X$ for $J_1,\ldots,J_\ell\in\mathcal{K}_X$ defined for $2\leq\ell\leq s$ at the beginning of the proof of Proposition 2.3.3 (but this time we will only need the latter elements in the range $3\leq\ell\leq s$). In addition, let $J'=\bigcap_{j=1}^s J_j\cap J_E$ and consider

$$\begin{aligned}\bar{\epsilon}_{J'} &= \prod_{j\in J'}(\epsilon_j\otimes 1\otimes\cdots\otimes 1-1\otimes\epsilon_j\otimes 1\otimes\cdots\otimes 1)^2 \\ &= (-2)^{|J'|}\epsilon_{J'}\otimes\epsilon_{J'}\otimes 1\otimes\cdots\otimes 1\end{aligned}\tag{2.38}$$

and

$$\begin{aligned}\bar{\gamma}(J_1,J_2) &= \prod_{j\in(J_1-J_2)\cup(J_2-J')}\epsilon_j(2) \\ &= \sum_{\phi_2\subseteq(J_1-J_2)\cup(J_2-J')}\pm\epsilon_{\phi_2^c}\otimes\epsilon_{\phi_2}\otimes 1\otimes\cdots\otimes 1\end{aligned}\tag{2.39}$$

where, as in the proof of Proposition 2.3.3, ϕ_2^c stands for the complement of ϕ_2 in $(J_1-J_2)\cup(J_2-J')$. Then

$$\bar{\epsilon}_{J'}\cdot\bar{\gamma}(J_1,J_2)\cdot\prod_{\ell=3}^s\gamma(J_1,\ldots,J_\ell)=\sum_{\phi_2,\ldots,\phi_s}\pm 2^{|J'|}\epsilon_{J'}\epsilon_{\phi_2^c}\cdots\epsilon_{\phi_s^c}\otimes\epsilon_{J'}\epsilon_{\phi_2}\otimes\epsilon_{\phi_3}\otimes\cdots\otimes\epsilon_{\phi_s}\tag{2.40}$$

where, for $3\leq\ell\leq s$,

$$\phi_\ell\subseteq\Big(\bigcap_{m=1}^{\ell-1}J_m-J_\ell\Big)\cup J_\ell$$

with ϕ_ℓ^c standing for the complement of ϕ_ℓ in $\left(\bigcap_{m=1}^{\ell-1}J_m-J_\ell\right)\cup J_\ell$ —here we are using the notation in Proposition 2.3.3. Recalling that

$$\mathrm{N}_X(J_1,\ldots,J_s)=\sum_{\ell=2}^s\left(\Big|\bigcap_{m=1}^{\ell-1}J_m-J_\ell\Big|+\Big|J_\ell\Big|\right),$$

we easily see that the left-hand side of (2.40) is a product of $\mathrm{N}_X(J_1,\ldots,J_s)+|\bigcap_{j=1}^s J_j\cap J_E|$ zero-divisors. Thus, by (2.30), it suffices to prove the non-triviality of the right-hand side of (2.40). With this in mind, note that the term

$$\pm 2^{|J'|}\,\epsilon_{J'}\,\epsilon_{J_1-J_2}\,\epsilon_{(J_1\cap J_2)-J_3}\cdots\epsilon_{(J_1\cap\cdots\cap J_{s-1})-J_s}\otimes\epsilon_{J_2}\otimes\cdots\otimes\epsilon_{J_s},\tag{2.41}$$

which appears in (2.40) with $\phi_\ell=J_\ell$ for $3\leq\ell\leq s$ and $\phi_2=J_2-J'$, is a basis element because

$$\epsilon_{J'}\cdot\epsilon_{J_1-J_2}\cdots\epsilon_{(J_1\cap\cdots\cap J_{\ell-1})-J_\ell}\cdots\epsilon_{(J_1\cap\cdots\cap J_{s-1})-J_s}=\epsilon_{J'}\cdot\epsilon_{(J_1-\cap_{j=1}^s J_j)}=\epsilon_{J_0}$$

with $J_0 \subseteq J_1$. The non-triviality of (2.40) then follows by observing that (2.41) cannot arise when other summands in (2.40) are expressed in terms of the basis for H_X. In fact, each summand

$$\pm 2^{|J'|}\epsilon_{J'}\epsilon_{\phi_2^c}\cdots\epsilon_{\phi_s^c}\otimes\epsilon_{J'}\epsilon_{\phi_2}\otimes\epsilon_{\phi_3}\otimes\cdots\otimes\epsilon_{\phi_s} \tag{2.42}$$

in (2.40) is either zero or a basis element and, in the latter case, (2.42) agrees (up to sign) with (2.41) only if $\phi_\ell = J_\ell$ for $\ell = 3, \ldots, s$, and $\phi_2 = J_2 - J'$. □

Remark 2.5.7. *The factors (2.38) and (2.39) adjust the product (2.22) of zero divisors in the proof of Proposition 2.3.3 so to account for the differences noted in Remark 2.3.2.*

We close this section by noticing that Proposition 2.4.9 holds without restriction on the parity of the sphere dimensions $k_1, \ldots, k_{n+m}$. That is:

Proposition 2.5.8. *Let X and Y be subcomplexes of $\mathbb{S}(k_1 \ldots, k_n)$ and $\mathbb{S}(k_{n+1}, \ldots, k_{n+m})$ respectively. If $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$, then*

$$\mathrm{TC}_s(X \vee Y) = \max\{\mathrm{TC}_s(X), \mathrm{TC}_s(Y), \mathrm{cat}(X^{s-1}) + \mathrm{cat}(Y)\}.$$

The argument given in the second paragraph of the proof of Proposition 2.4.9 applies word for word in the unrestricted case (replacing, of course, $\mathrm{N}_{X\vee Y}(J_1, \ldots, J_s)$ by $\sum_{i=1}^s |J_i| - |\bigcap_{i=1}^s J_i \cap J_O|$ in (2.29) and in the last line of that proof).

3 Asymptotic behavior of the higher TC of random models of a family of subcomplexes of products of spheres.

3.1 The Erdős-Rényi model and the random clique variable

For a positive integer n and probability parameter p, $0 < p < 1$, consider the Erdős-Rényi model $\mathcal{G}(n,p)$ of random graphs Γ in which each edge of the complete graph on the n vertices $[n] = \{1, 2, \ldots, n\}$ is included in Γ with probability p independently of all other edges. In other words, the random variables e_{ij}, $1 \leq i < j \leq n$, defined by

$$e_{i,j}(\Gamma) = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge in } \Gamma; \\ 0, & \text{otherwise,} \end{cases}$$

are independent and have $P(e_{i,j} = 1) = p$. Recall that, $\mathcal{C} \subseteq [n]$ is a *clique* of Γ if every pair of vertices in $\mathcal{C}$ are adjacent in Γ. In other words, $\mathcal{C} \subseteq [n]$ is a clique of Γ if the induced subgraph of $\mathcal{C}$ in Γ is a complete graph.

In this context, the clique random variable $C = C_{n,p}$,

$$C(\Gamma) = \max\{r \in \mathbb{N} : \Gamma \text{ admits a complete subgraph with } r \text{ vertices}\},$$

has been the subject of intensive research since the 1970's. Matula provided in [23] numerical evidence suggesting that C has a very peaked density around $2\log_q n$ where $q = 1/p$. Such a property was established in [19] by Grimmett and McDiarmid who proved that, as $n \to \infty$, $\frac{C}{\log_q n} \to 2$. A much finer result, Theorem 3.1.1 below, was proved by Matula. From now on, $\lfloor x \rfloor$ stands for the integral part of the real number x, and we set

$$z = z(n,p) = 2\log_q n - 2\log_q \log_q n + 2\log_q(e/2) + 1.$$

Theorem 3.1.1 ([24, Equation (2)])**.** *For $0 < p < 1$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} \operatorname{Prob}\Big(\lfloor z - \epsilon \rfloor \leq C \leq \lfloor z + \epsilon \rfloor\Big) = 1.$$

It should be stressed that the probability parameter p is fixed throughout the limiting process. In common parlance, Theorem 3.1.1 can be stated by the assertion that, for a fixed $p \in (0,1)$, the inequalities $\lfloor z - \epsilon \rfloor \leq C(\Gamma) \leq \lfloor z + \epsilon \rfloor$ hold *asymptotically almost surely* for random graphs $\Gamma \in \mathcal{G}(n,p)$. Alternatively, since $0 \leq \lfloor z + \epsilon \rfloor - \lfloor z - \epsilon \rfloor \leq 1$ when $\epsilon \leq 1/2$, C is asymptotically almost surely determined by z with spikes of at most a unit whose appearance depend on the "resolution" parameter ϵ used.

Definition 3.1.2. *An s-th multi-clique of size r of a (random) graph $\Gamma \in \mathcal{G}(n,p)$ is an ordered s-tuple $(V_1, \ldots, V_s)$ of pairwise disjoint subsets $V_i \subseteq [n]$, each of cardinality r, such that each of the induced subgraphs $\Gamma_{|V_i}$ is complete.*

Note that, we do not require in Definition 3.1.2 that each V_i be a maximal clique of $\Gamma \in \mathcal{G}(n,p)$, nor that r is related to the clique number $C(\Gamma)$. The following results concerns with the existence, with high probability, of s-th multi-clique of size $r = \lfloor z - \epsilon \rfloor$ some (small but fixed) $\epsilon > 0$.

3.2 Maximal disjoint cliques

Our main goal in this section is proving the following result:

Theorem 3.2.1. *Fix a positive integer s, a positive real number ϵ, and a probability parameter $p \in (0,1)$. Then, with probability tending to 1 as $n \to \infty$, a random graph in $\mathcal{G}(n,p)$ has an s-th multi-clique of size $\lfloor z-\epsilon \rfloor$.*

Throughout this section we let $r := \lfloor z-\epsilon \rfloor$, a function on n, p, and ϵ. Although n will indeed vary, in what follows the parameters p and ϵ (as well as s) will be kept fixed. We will assume $s \geq 2$, as the case $s=1$ in Theorem 3.2.1 is covered by Theorem 3.1.1.

Let $X_{r,s} : \mathcal{G}(n,p) \to \mathbb{Z}$ be the random variable that assigns to each random graph the number of its s-th multi-cliques of size r. Note that $X_{r,s}(\Gamma)$ is divisible by $s!$, for an s-th multi-clique is an *ordered* s-tuple of disjoint sets. We could of course normalize by dividing by $s!$, but the unnormalized setting yields slightly simpler formulas in the arguments below.

By the second moment method,

$$\mathrm{Prob}\left(X_{r,s} > 0\right) \geq \frac{E(X_{r,s})^2}{E(X_{r,s}^2)}, \tag{3.1}$$

so it suffices to show that the ratio on the right hand side of (3.1) tends to 1 as $n \to \infty$.

Let $\mathcal{W}(s)$ stand for the set of s-tuples $(W_1,\dots,W_s)$ of pairwise disjoint subsets W_i of $[n]$, each having cardinality r. Each $\mathbf{W} \in \mathcal{W}(s)$ determines a random variable $I_{\mathbf{W}} : \mathcal{G}(n,p) \to \{0,1\}$ given by

$$I_{\mathbf{W}}(\Gamma) = \begin{cases} 1, & \text{if } \mathbf{W} \text{ is an } s\text{-th multi-clique of size } r \text{ of } \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

In these terms, $X_{r,s}$ can be written as

$$X_{r,s} = \sum_{\mathbf{W} \in \mathcal{W}(s)} I_{\mathbf{W}},$$

and since $E(I_{\mathbf{W}}) = p^{s\binom{r}{2}}$ for each $\mathbf{W} \in \mathcal{W}(s)$, linearity of the expectation yields

$$E(X_{r,s}) = \binom{n}{\underbrace{r,\dots,r}_{s}} p^{s\binom{r}{2}}$$

where $\binom{a}{b_1,\dots,b_k}$ stands for the multinomial coefficient

$$\binom{a}{b_1,\dots,b_k} = \frac{a!}{\left(\prod_{i=1}^k b_i!\right)\left(a - \sum_{i=1}^k b_i\right)!}$$

determined by non-negative integers $a, b_1, \dots, b_k$ with $k \in \mathbb{N}$ and $a \geq \sum_{i=1}^k b_i$. On the other hand, in order to deal with $E(X_{r,s}^2)$, write $X_{r,s}^2 = \sum I_{\mathbf{W}} \cdot I_{\mathbf{W}'}$ and note that

$$E(I_{(W_1,\dots,W_s)} \cdot I_{(W'_1,\dots,W'_s)}) = p^{2s\binom{r}{2} - \sum \binom{a_{ij}}{2}}$$

where we set $a_{ij} := |W_i \cap W'_j|$. We say that the pair $(\mathbf{W}, \mathbf{W}')$ has intersection type given by the matrix $A = (a_{ij})$.

Before using the previous considerations to estimate the right hand side term of (3.1), it is convenient to introduce some auxiliary notation. Given an $(s \times s)$-matrix $A = (a_{ij})$ with integer coefficients, let A_i and A^i $(1 \le i \le s)$ denote the s-tuples determined by the i-th row and the i-th column of A, respectively. Moreover, let $\Sigma(c_1, \ldots, c_s) := \Sigma_{i=1}^s c_i$. In these terms, we get

$$\frac{E(X_{r,s}^2)}{E(X_{r,s})^2} = \sum_{A \in D} F_A \cdot q^{L(A)} = \sum_{A \in D} T_A. \tag{3.2}$$

Here the summations run over the set D of $(s \times s)$-matrices $A = (a_{ij})$ with non-negative integer coefficients satisfying $\max_{1 \le i \le s} \{\Sigma A_i, \Sigma A^i\} \le r$ (since $\Sigma A_i = \sum_{k=1}^s |W_i \cap W_k'| \le |W_i| = r$ and $\Sigma A^j = \sum_{k=1}^s |W_k \cap W_j'| \le |W_j'| = r$, for $i, j = 1, \ldots, s$), and we have set

$$F_A = \frac{\binom{r}{A^1}\binom{r}{A^2}\cdots\binom{r}{A^s}\binom{n-sr}{r-\Sigma A_1, r-\Sigma A_2, \ldots, r-\Sigma A_s}}{\binom{n}{\underbrace{r, \ldots, r}_{s}}}, \quad L(A) = \sum_{i,j=1}^{s} \binom{a_{ij}}{2},$$

and

$$T_A = F_A \cdot q^{L(A)}.$$

Remark 3.2.2. Note that

$$\sum_{A \in D} F_A = \frac{\sum_{A \in D} \binom{n}{r, \ldots, r}\binom{r}{A^1}\binom{r}{A^2}\cdots\binom{r}{A^s}\binom{n-sr}{r-\Sigma A_1, r-\Sigma A_2, \ldots, r-\Sigma A_s}}{\binom{n}{r, \ldots, r}^2} = 1,$$

as both the numerator and denominator in the quotient give the cardinality of $\mathcal{W}(s)^2$.

Our updated task is to show that $\sum_{A \in D} T_A \to 1$ as $n \to \infty$. In fact, Lemma 3.2.3 below implies that it suffices to show

$$\lim_{n \to \infty} \left(\sum_{A \in D - \{A_0\}} T_A \right) = 0 \tag{3.3}$$

where $A_0 \in D$ is the 0-matrix.

Lemma 3.2.3. $\lim_{n \to \infty} T_{A_0} = 1$.

Proof. We have

$$\begin{aligned} T_{A_0} &= F_{A_0} = \frac{\binom{n-rs}{\underbrace{r, \ldots, r}_{s}}}{\binom{n}{\underbrace{r, \ldots, r}_{s}}} = \frac{\frac{(n-rs)!}{(r!)^s (n-2rs)!}}{\frac{n!}{(r!)^s (n-rs)!}} \\ &= \frac{(n-rs)!(n-rs)!}{n!(n-2rs)!} = \frac{(n-2rs+1)\cdots(n-rs)}{(n-rs+1)\cdots n} \\ &= \prod_{k=0}^{sr-1} \left(\frac{n-k-sr}{n-k} \right) = \prod_{k=0}^{sr-1} \left(1 - \frac{sr}{n-k} \right) \\ &\ge \left(1 - \frac{sr}{n-sr+1} \right)^{sr} = \left(\left(1 - \frac{sr}{n-sr+1} \right)^{2r} \right)^{s/2}. \end{aligned}$$

Further, since $\left(1 - \frac{sr}{n-sr+1}\right)^{2r}$ can be written as

$$\left(1 - \frac{2sr^2}{n-sr+1}\right) + \left[\binom{2r}{2}\left(\frac{sr}{n-sr+1}\right)^2 - \binom{2r}{3}\left(\frac{sr}{n-sr+1}\right)^3\right] + \cdots + \left[\left(\frac{sr}{n-sr+1}\right)^{2r}\right],$$

we see that

$$T_{A_0} \geq \left(1 - \frac{2sr^2}{n - sr + 1}\right)^{s/2}$$

for n large enough[1]. The result then follows from Remark 3.2.2 and from the fact that the term on the right hand side of the latter inequality tends to 1 as $n \to \infty$. □

The rest of this section is devoted to the proof of (3.3), which requires a number of technical preliminary results. Our first goal is Proposition 3.2.5 below, a generalization of [9, Lemma 6].

Lemma 3.2.4. *For each positive integer m, there is a positive integer $N(m)$ and a positive real number $\alpha(m)$ such that the number c_n defined through the formula*

$$\binom{n}{mr} = c_n \left(\frac{n}{mr}\right)^{mr} e^{mr} (mr)^{-1/2}$$

satisfies $c_n \geq \alpha(m) > 0$ whenever $n \geq N(m)$.

Proof. Using Stirling's formula for factorials (see for instance formula (1.4) in [3])

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{\alpha_n}, \qquad \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}, \tag{3.4}$$

we have

$$\begin{aligned} \binom{n}{mr} &= \frac{1}{\sqrt{2\pi}} \left(\frac{n}{mr}\right)^{mr} \left(\frac{n}{n-mr}\right)^{n-mr} \sqrt{\frac{n}{mr(n-mr)}}\ \ell_n \\ &= c_n \left(\frac{n}{mr}\right)^{mr} e^{mr} (mr)^{-\frac{1}{2}} \end{aligned}$$

where $\ell_n = \frac{e^{\alpha_n}}{e^{\alpha_{mr}} e^{\alpha_{n-mr}}} \to 1$ as $n \to \infty$ and

$$c_n = \frac{1}{\sqrt{2\pi}} \left(\frac{n}{n-mr}\right)^{n-mr} \sqrt{\frac{n}{n-mr}}\ e^{-mr}\, \ell_n.$$

In order to check that, for large enough n, c_n is bounded from below by a fixed positive real number α (which in general depends on m), we use the inequality

$$\left(\frac{a-b}{a}\right)^x \leq e^{-\frac{b}{a}x},$$

which holds for any positive integers a, b, and x with $b < a$. Taking in particular $a = n$, $b = mr$ and $x = n - mr - 1$, we get

$$\left(\frac{n-mr}{n}\right)^{n-mr-1} \leq e^{-\frac{mr}{n}(n-mr-1)} = e^{-mr} e^{\frac{m^2r^2}{n}} e^{\frac{mr}{n}}$$

or, equivalently,

$$\left(\frac{n-mr}{n}\right)^{-1} e^{-\frac{m^2r^2}{n}} e^{-\frac{mr}{n}} \leq \left(\frac{n}{n-mr}\right)^{n-mr} e^{-mr}.$$

Since the left hand side of the latter inequality approaches 1 as $n \to \infty$, there exists a positive real number α such that

$$c_n = \frac{1}{\sqrt{2\pi}} \left(\frac{n}{n-mr}\right)^{n-mr} \sqrt{\frac{n}{n-mr}}\ e^{-mr}\, \ell_n > \alpha > 0,$$

for n large enough. □

[1]Here and in what follows we use without further notice the easily checked fact that $r^k = o(n)$ for any positive integer k.

Proposition 3.2.5. *Fix non-negative integers k and m with $m > 0$. Then*

$$\lim_{n\to\infty}\left(r^{-k}\binom{n}{\underbrace{r,\dots,r}_{m}} p^{m\binom{r}{2}} \right) = \infty.$$

Proof. Recall $r = \lfloor z-\epsilon \rfloor \le z-\epsilon$, so

$$p^{m\binom{r}{2}} \ge \left(p^{\frac{z-\epsilon-1}{2}}\right)^{mr} = \left(p^{\log_q n - \log_q \log_q n + \log_q(e/2) - \frac{\epsilon}{2}}\right)^{mr} = \left(\frac{2C\log_q n}{en}\right)^{mr}$$

where $C = q^{\frac{\epsilon}{2}} > 1$. Note also that

$$\binom{n}{\underbrace{r,\dots,r}_{m}} = \frac{n!}{r!^m(n-mr)!} = \binom{n}{mr}\frac{(mr)!}{(r!)^m}.$$

By Lemma 3.2.4, there is a positive real number $\alpha(m)$ and a large positive integer $N(m)$ so that

$$\binom{n}{mr} = c_n\left(\frac{n}{mr}\right)^{mr} e^{mr}(mr)^{-1/2}$$

holds with $c_n \ge \alpha(m) > 0$ for $n \ge N(m)$. For such large values of n we then have

$$\begin{aligned} r^{-k}\binom{n}{\underbrace{r,\dots,r}_{m}} p^{m\binom{r}{2}} &\ge r^{-km}c_n\left(\frac{n}{mr}\right)^{mr} e^{mr}(mr)^{-1/2}\frac{(mr)!}{r!^m}\left(\frac{2C\log_q n}{en}\right)^{mr} \\ &= m^{-1/2}r^{-km}r^{-\frac{1}{2}}c_n\frac{(mr)!}{r!^m m^{mr}}\left(\frac{2C\log_q n}{r}\right)^{mr}. \end{aligned} \tag{3.5}$$

Using Stirling's formula (3.4), we get

$$\frac{(mr)!}{r!^m m^{mr}} = \frac{\sqrt{2\pi mr}\left(\frac{mr}{e}\right)^{mr}}{\sqrt{2\pi r}^{\,m}\left(\frac{r}{e}\right)^{mr} m^{mr}}\,d_n = \frac{\sqrt{2\pi mr}}{\sqrt{2\pi r}^{\,m}}\,d_n$$

where $d_n = e^{\alpha_{mr}}/e^{m\alpha_r} \to 1$ as $n \to \infty$. Therefore, we can rewrite (3.5) as

$$\begin{aligned} r^{-k}\binom{n}{\underbrace{r,\dots,r}_{m}} p^{m\binom{r}{2}} &\ge m^{-1/2}r^{-km}r^{-\frac{1}{2}}\frac{\sqrt{2\pi mr}}{\sqrt{2\pi r}^{\,m}}\left(\frac{2C\log_q n}{r}\right)^{mr} c_n d_n \\ &= (2\pi)^{\frac{1-m}{2}} r^{-km}r^{-\frac{m}{2}}\left(\frac{2C\log_q n}{r}\right)^{mr} c_n d_n \\ &= (2\pi)^{\frac{1-m}{2}}\left[r^{-\frac{2k+1}{2}}\left(\frac{2C\log_q n}{r}\right)^{r}\right]^{m} c_n d_n \\ &\ge (2\pi)^{\frac{1-m}{2}}\left[r^{-\frac{2k+1}{2}}C^r\right]^{m} c_n d_n \end{aligned}$$

for $n \ge N(m)$, where the last inequality holds for n large enough (condition that can be incorporated by increasing $N(m)$ if needed) in view of the definition of r. The proof is complete in view of the noted characterization of the sequences $\{c_n\}$ and $\{d_n\}$, and since $r^{-\frac{2k+1}{2}}C^r$ tends to infinity as $n \to \infty$, for

$$\log_q(r^{-\frac{2k+1}{2}}C^r) = \frac{\epsilon}{2}r - \left(\frac{2k+1}{2}\right)\log_q r = \frac{\epsilon}{2}r - \left(\frac{2k+1}{2\ln q}\right)\ln r$$

tends to infinity as $n \to \infty$. $\square$

The next step toward the proof of (3.3) is an analysis of the asymptotic behavior of T_A for certain matrices $A \in D - \{A_0\}$. In more detail, recall that the set D depends on n. Using subindices to stress the dependence, we have $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$. In Proposition 3.2.6 below we will be concerned with sequences of matrices $\{A_n \in D_n\}_{n\geq 1}$ whose only non-zero entries lie on the main diagonal and are either 1 or r. Such a sequence $\{A_n \in D_n\}_{n\geq 1}$ as above will simply be referred to as a *diagonal sequence* and, by abuse of notation, will be denoted by $A \in D$. In addition, by a diagonal sequence $A \in D - \{A_0\}$ we mean one for which no A_n is the zero matrix.

Proposition 3.2.6. *Any diagonal sequence $A \in D - \{A_0\}$ satisfies $Q(r)T_A = o(1)$ for any polynomial Q with real coefficients.*

Proof. For each $n \geq 1$, let $m = m(n)$ and $m' = m'(n)$ be the integers in $\{0, 1, \ldots, s\}$ such that A_n has m entries with value r and m' entries with value 1 (all of these in the main diagonal of A_n). In these terms, the generic T_A is

$$\begin{aligned}
T_A &= \frac{r^{m'}\binom{n-sr}{\underbrace{r,\ldots,r}_{s-(m+m')},\underbrace{r-1,\ldots,r-1}_{m'}}}{\binom{n}{\underbrace{r,\ldots,r}_{s}}}\, q^{m\binom{r}{2}} \\
&= \frac{r^{m'}(n-sr)!r!^s(n-sr)!q^{m\binom{r}{2}}}{n!r!^{s-(m+m')}(r-1)!^{m'}(n-2sr+mr+m')!} \\
&= \frac{r^{2m'}(n-sr)!r!^s(n-sr)!q^{m\binom{r}{2}}}{n!r!^{s-m}(n-2sr+mr+m')!} = \frac{r^{2m'}(n-sr)!r!^m(n-sr)!q^{m\binom{r}{2}}}{n!(n-2sr+mr+m')!} \\
&= \left[\binom{n}{\underbrace{r,\ldots,r}_{m}}p^{m\binom{r}{2}}\right]^{-1} \frac{r^{2m'}(n-sr)!(n-sr)!}{(n-mr)!(n-2sr+mr+m')!}.
\end{aligned}$$

The multinomial coefficient $\binom{n-sr}{r,\ldots,r,r-1,\ldots,r-1}$ in the first line of the above equalities should be ignored if $m = s$ and $m' = 0$. Likewise, the multinomial coefficient $\binom{n}{r,\ldots,r}$ in the last line of the above equalities should be ignored if $m = 0$. Note that

$$\begin{aligned}
\frac{r^{2m'}(n-sr)!(n-sr)!}{(n-mr)!(n-2sr+mr+m')!} &= \frac{r^{2m'}(n-sr)!}{(n-sr+1)\cdots(n-mr)(n-2sr+mr+m')!} \\
&\leq \frac{r^{2m'}(n-2sr+mr)!}{(n-2sr+mr+m')!} \leq \frac{r^{2m'}}{(n-2sr)^{m'}}.
\end{aligned}$$

Thus, for n large enough,

$$T_A \leq \left[\binom{n}{\underbrace{r,\ldots,r}_{m}}p^{m\binom{r}{2}}\right]^{-1} \frac{r^{2m'}}{(n/2)^{m'}} = \left[\binom{n}{\underbrace{r,\ldots,r}_{m}}p^{m\binom{r}{2}}\right]^{-1} \frac{2^{m'}r^{2m'}}{n^{m'}},$$

and the desired conclusion follows from Proposition 3.2.5 if $m > 0$, whereas the conclusion is obvious if $m = 0$ for, then, m' is positive. Note that Proposition 3.2.5 has to be applied for each possible value of (m, m'), but this is not a problem as there are at most $(s+1)^2$ such pairs. □

Choose λ with

$$0 < \lambda < \frac{1}{1 + 2seq}$$

and consider the partition of $\{0, 1, \ldots, r\}$ into the three sets

$$\begin{aligned} S_\lambda &= \{x \in \mathbb{Z} : 0 \le x \le (1-\lambda)\log_q n\}, \\ I_\lambda &= \{x \in \mathbb{Z} : (1-\lambda)\log_q n < x < (1+\lambda)\log_q n\}, \quad \text{and} \\ L_\lambda &= \{x \in \mathbb{Z} : (1+\lambda)\log_q n \le x \le r\}. \end{aligned}$$

An integer will be referred as *small*, *intermediate*, or *large*, depending on whether it lies in S_λ, I_λ, or L_λ, respectively.

Propositions 3.2.7–3.2.9 below will enable us to bound from above each term in (3.3) by a term T_A for a suitable diagonal matrix A as those in Proposition 3.2.6.

Proposition 3.2.7. *There is a large integer N (which depends only on the fixed parameters s, ϵ, p, and λ) such that, for $n \ge N$:*

i) If $A' \in D$ arises by adding 1 to a small entry in $A \in D$, then $T_{A'} < T_A$.

ii) If $A' \in D$ arises by adding 1 to a large entry in $A \in D$, then $T_A < T_{A'}$.

Proof. Suppose $A' \in D$ arises by increasing by 1 an entry $a_{ij} = a$ in $A \in D$ (in particular $\Sigma A_i < r$ and $\Sigma A^j < r$). Then

$$\frac{T_{A'}}{T_A} = \frac{(r - \Sigma A^j)(r - \Sigma A_i)\, q^a}{(a+1)\left(n - 2sr + \displaystyle\sum_{k=1}^{s} \Sigma A^k + 1\right)}$$

Since $r = o(n)$, we have

$$\frac{n}{2} \le n - 2sr + \sum_{k=1}^{s} \Sigma A^k + 1 \le n$$

for n large enough (depending only on s, ϵ, and p), so that for those large values of n we have

$$Bq^a \le \frac{T_{A'}}{T_A} \le 2Bq^a \tag{3.6}$$

where

$$B = \frac{(r - \Sigma A^j)(r - \Sigma A_i)}{(a+1)n} \le \frac{r^2}{n}. \tag{3.7}$$

Case $a \in S_\lambda$: We have $q^a \le q^{(1-\lambda)\log_q n} = n^{1-\lambda}$, so that

$$Bq^a \le \frac{r^2 n^{1-\lambda}}{n} = \frac{r^2}{n^\lambda}.$$

But $\lim\limits_{n\to\infty}(r^2/n^\lambda) = 0$, so that the second inequality in (3.6) gives $T_{A'} < T_A$ for n large enough (depending now on s, ϵ, p, and λ).

Case $a \in L_\lambda$: Now $q^a \ge n^{1+\lambda}$ and, since $a + 1 \le r < 2\log_q n$ for n large enough (depending only on s, ϵ, and p), we have

$$B \ge \frac{1}{(a+1)n} \ge \frac{1}{2n\log_q n}.$$

Therefore

$$Bq^a \ge \frac{n^{1+\lambda}}{2n\log_q n} = \frac{n^\lambda}{2\log_q n}.$$

This time the quotient $n^\lambda/(2\log_q n)$ tends to ∞ as $n \to \infty$, so that the first inequality in (3.6) gives $T_{A'} > T_A$ for n large enough (depending now on s, ϵ, p, and λ). $\square$

Proposition 3.2.8. *There is a large integer N (which depends only on the fixed parameters s, ϵ, p, and λ) such that, for $n \geq N$, the following assertion holds: If $A' \in D$ arises by increasing by 1 some entry $a_{ij} = a$ in $A \in D$ with $0 < a \leq r/2$, then $T_A > T_{A'}$ for n large enough provided the following two conditions hold:*

(i) All small entries in A_i and in A^j are zero.

(ii) There exists either an entry $a_{ij'} \neq 0$ with $j' \neq j$, or an entry $a_{i'j} \neq 0$ with $i' \neq i$.

Proof. Let B be defined as in (3.7) so that (3.6) applies if n is large enough. The fact that $r < 2\log_q n$ (for large enough n's) together with *(i)* and *(ii)* yield

$$(r-\Sigma A_i)(r-\Sigma A^j) \leq (r-2(1-\lambda)\log_q n)\, r \leq (2\log_q n-2(1-\lambda)\log_q n)\, 2\log_q n \leq 4\lambda \log_q^2 n.$$

Moreover, since $a \leq r/2 \leq \log_q n - \log_q \log_q n + \log_q(e/2) + 1$, we have

$$q^a \leq \frac{eqn}{2\log_q n}.$$

Since $a+1 \geq (1-\lambda)\log_q n$, the previous considerations amount to $2Bq^a \leq \frac{4\lambda eq}{1-\lambda} < 1$ where the last inequality comes from the definition of λ. The desired conclusion then follows from (3.6). □

Proposition 3.2.9. *There is a large integer N (which depends only on the fixed parameters s, ϵ, p, and λ) such that, for $n \geq N$: If $A = (a_{i,j})$ is a matrix in D with a non-zero entry $a_{ij} = a$ such that $a_{ij'} = a_{i'j} = 0$ whenever $i \neq i'$ and $j \neq j'$, then $T_A \leq \max\{T_{A'}, T_{A''}\}$. Here $A' \in D$ is the matrix whose (i,j)-entry is 1, whereas all its remaining entries agree with the corresponding entries of A. Likewise, $A'' \in D$ is the matrix whose (i,j)-entry is r, whereas all its remaining entries agree with the corresponding entries of A.*

Proof. We can assume $a \in I_\lambda$—otherwise the result follows by repeated used of Proposition 3.2.7. Let B_ω ($\omega = 1, 2$) arise by adding ω to the (i,j)-entry in A. Note that both B_1 and B_2 belong to D if n is large enough. Direct calculation yields

$$\frac{T_{B_2} T_A}{T_{B_1}^2} = \left[\frac{(r-a-1)^2}{(r-a)^2} \cdot \frac{a+1}{a+2} \cdot \frac{n - 2sr + \displaystyle\sum_{k=1}^{s} \Sigma A^k + 1}{n - 2sr + \displaystyle\sum_{k=1}^{s} \Sigma A^k + 2} \right] \cdot q.$$

Each of the three quotients inside the bracket tends (uniformly on a) to 1 as $n \to \infty$. This holds for the first quotients because $a \in I_\lambda$. Since $q > 1$, we get for large enough n that $T_{B_2} T_A > T_{B_1}^2$ or, equivalently, that $\log_q T_A$ is a convex function on the interval I_λ—and even two units to the right of this open interval. The result now follows from Proposition 3.2.7. □

Equation (3.3) and, therefore, Theorem 3.2.1 now follow from Proposition 3.2.6, Corollary 3.2.10 below, and the fact that the size of $D - \{A_0\}$ increases (as $n \to \infty$) polynomially on r.

Corollary 3.2.10. *There is a large integer N (which depends only on the fixed parameters s, ϵ, p, and λ) such that, for $n \geq N$, the term T_A of any matrix $A \in D - \{A_0\}$ is bounded from above by a term $T_{A'}$ where A' is a diagonal matrix in D whose non-zero entries are either 1 or r. (In general, the matrix A' above depends on the given matrix A.)*

Remark 3.2.11. *Before proving Corollary 3.2.10, it is useful to note that, from its bare definition, the term T_A does not change after permuting the columns of $A \in D$. In other words, if A^σ is obtained by permuting the columns of $A \in D$ according to a permutation σ, then the rule*

$$\Big((W_1, \ldots, W_s), (W'_1, \ldots, W'_s)\Big) \mapsto \Big((W_{\sigma(1)}, \ldots, W_{\sigma(s)}), (W'_1, \ldots, W'_s)\Big)$$

sets a 1-1 correspondence between pairs in $\mathcal{W}(s)$ with intersection type A, and pairs in $\mathcal{W}(s)$ with intersection type A^σ. In particular we can assume without loss of generality that the matrix $A \in D - \{A_0\}$ in Corollary 3.2.10 has non-zero entries on its main diagonal.

Proof of Corollary 3.2.10. The following arguments hold for values of n large enough so that Propositions 3.2.7–3.2.9 apply. If all entries in A are small, then by Proposition 3.2.7 there exists a diagonal matrix $A' \in D$, with zeros and ones on its main diagonal, satisfying $T_A \leq T_{A'}$. So we can assume that A has at least one entry which is either intermediate or large, and that such an entry lies on the main diagonal. In addition, using again Proposition 3.2.7, we can assume that all small entries in A are zero.

At this point, if on a given row (or column) of A there are two non-zero entries, then Proposition 3.2.8 implies that one of them (the one which is at most $r/2$) can be lowered down to zero at the price of increasing the value of T_A—which is all right for the purposes of this proof. We can thus assume that each row (as well as each column) of A has at most one non-zero entry. By Remark 3.2.11, this amounts to assuming that A is a diagonal matrix. The proof is then completed by Proposition 3.2.9. □

3.3 Higher topological complexity

In this section, we work with the spaces in Example 2.1.3. That is, we consider $\Gamma = (V, E)$ a graph with vertex set $V = [n]$ and edge set E. We let K_Γ stand for the clique complex of the graph Γ, thus K_Γ is the abstract simplicial complex whose k-simplices are the $(k+1)$-cliques of Γ. In other words, $J \subseteq [n]$ is a simplex of K_Γ if and only if this set induces a complete subgraph in Γ. Now consider,

$$X_\Gamma = \bigcup_{J \,\in\, K_\Gamma} e_J \subseteq \mathbb{S}(\underbrace{1, 1, \ldots, 1}_{n \text{ factors}}).$$

Then, we have

Theorem 3.3.1. *For a random graph $\Gamma \in \mathcal{G}(n, p)$, let X_Γ stand for the (random) Eilenberg-MacLane space associated to the right-angled Artin group defined by Γ (as above). Then, for any positive real constant ϵ, positive integer s, and probability parameter $p \in (0, 1)$, the random variable TC_s given by $\mathrm{TC}_s(\Gamma) = \mathrm{TC}_s(X_\Gamma)$ satisfies*

$$\lim_{n \to \infty} \mathrm{Prob}\Big(s\lfloor z - \epsilon \rfloor \leq \mathrm{TC}_s \leq s\lfloor z + \epsilon \rfloor\Big) = 1.$$

The relevance of Matula's Theorem 3.1.1 for Theorem 3.3.1 can already be seen from Proposition 1.3.3: By definition, X_Γ comes equipped with a CW structure having a d-dimensional cell for each complete subgraph of Γ with d vertices. In particular

$$\mathrm{hdim}(X_\Gamma) \leq C(\Gamma) \tag{3.8}$$

and, consequently,

$$\mathrm{Prob}(\mathrm{TC}_s \leq s\lfloor z + \epsilon \rfloor) \geq \mathrm{Prob}(C \leq \lfloor z + \epsilon \rfloor). \tag{3.9}$$

As $n \to \infty$, the left hand side in (3.9) tends to 1 since the right hand side does too in view of Matula's theorem. This gives half of Theorem 3.3.1. Before proving the other half, namely the equality

$$\lim_{n\to\infty} \operatorname{Prob}\Big(s\lfloor z-\epsilon\rfloor \leq \mathrm{TC}_s\Big) = 1, \tag{3.10}$$

we pause to remark that (3.8) is in fact an equality, as follows easily from the description of the cohomology ring of X_Γ (see Proposition 2.3.1). In particular, the cohomological dimension of X_Γ, $\mathrm{cd}(X_\Gamma)$, agrees with the Lusternik-Schnirelmann category of X_Γ, $\mathrm{cat}(\Gamma)$. Indeed,

$$C(\Gamma) = \mathrm{cd}(X_\Gamma) \leq \mathrm{cat}(X_\Gamma) \leq \mathrm{hdim}(X_\Gamma) \leq C(\Gamma).$$

We now explain how (3.10) follows from our previous work. By the previous paragraph, the case $s = 1$ reduces to Matula's Theorem 3.1.1. On the other hand, the case $s \geq 2$ follows at once from Theorem 3.2.1 and Theorem 2.2.5 that we rewrite for convenience as follows:

Theorem 3.3.2. *For $s \geq 2$,*

$$\mathrm{TC}_s(X_\Gamma) = \max\left\{ \sum_{\ell=1}^{s} \big|V_\ell\big| - \left|\bigcap_{\ell=1}^{s} V_\ell\right| : \textit{each } V_i \subseteq [n] \textit{ yields a complete induced subgraph } \Gamma_{|V_i} \right\}.$$

We close this chapter by noticing that the s-th higher topological complexity of X_Γ is asymptotically almost surely within an s-neighborhood of the upper bound given in Proposition 1.3.3. Indeed, by Matula's Theorem 3.1.1 (with $\epsilon < 1/2$), the number $r = \lfloor z-\epsilon\rfloor$ in the previous section satisfies $r \geq \mathrm{hdim}-1$ asymptotically almost surely. So Theorems 3.2.1 and 3.3.2 yield:

Corollary 3.3.3. $\lim_{n\to\infty} \operatorname{Prob}\Big(s(\mathrm{hdim}-1) \leq \mathrm{TC}_s \leq s\,\mathrm{hdim}\Big) = 1.$

4 TC_s of configuration spaces of orientable surfaces.

The configuration space of n distinct ordered points of a space X, $\mathrm{Conf}(X, n)$, is the subspace of the n-fold cartesian power $X^{\times n}$ given by

$$\mathrm{Conf}(X, n) = \left\{(x_1, \dots, x_n) \in X^{\times n} \,:\, x_i \neq x_j \text{ whenever } i \neq j\right\}.$$

Our interest to study these spaces lies in topological robotics, where $\mathrm{Conf}(X, n)$ arises as the model for the state space of a system consisting of n distinct particles moving without collisions on X. We focus on the case $X = \Sigma_g$, an orientable surface of genus g. Farber's topological complexity of $\mathrm{Conf}(\Sigma_g, n)$ has been described in [4].

We now state our main result.

Theorem 4.0.4. *The s-th topological complexity of* $\mathrm{Conf}(\Sigma_g, n)$ *is given by*

$$\mathrm{TC}_s(\mathrm{Conf}(\Sigma_g, n)) = \begin{cases} s, & \text{if } g = 0 \text{ and } n \leq 2; \\ sn - 3, & \text{if } g = 0 \text{ and } n \geq 3; \\ s(n+1) - 2, & \text{if } g = 1 \text{ and } n \geq 1; \\ s(n+1), & \text{if } g \geq 2 \text{ and } n \geq 1. \end{cases}$$

Note that the value of $\mathrm{TC}_s(\mathrm{Conf}(\Sigma_g, n))$ stabilizes for $g \geq 2$. The case $n = 1$ in Theorem 4.0.4 has been noted in previous works; see [2, Corollary 3.12] for the case $g \leq 1$, [12, Example 16.4] for case $g \geq 2$ with $s = 2$, and [16, Proposition 5.1] for the case $g \geq 2$ with $s \geq 3$. This also covers the case $g = 0$ with $n = 2$ since $\mathrm{Conf}(S^2, 2)$ has the homotopy type of S^2. Indeed, by the Gram-Schmidt process, S^2 sits inside $F(S^2, 2)$ via the map $x \mapsto (x, -x)$ as a strong deformation retract. Therefore, in what follows we restrict ourselves to the case $n \geq 2$, and in fact $n \geq 3$ if $g = 0$.

4.1 Upper bounds

Genus 0. For $n \geq 3$ the ordered configuration space of n distinct points on the 2-dimensional sphere S^2 admits a homotopy decomposition

$$\mathrm{Conf}(S^2, n) \simeq \mathrm{SO}(3) \times \mathrm{Conf}(\mathbb{R}^2 - Q_2, n - 3) \tag{4.1}$$

where Q_2 is a set of two fixed points on $\mathbb{R}^2$ (see [4, Theorem 3.1], for instance). The higher topological complexity of both factors is known: The topological group $\mathrm{SO}(3) \simeq \mathbb{R}\mathrm{P}^3$ has

$$\mathrm{TC}_s(\mathrm{SO}(3)) = \mathrm{cat}((\mathbb{R}\mathrm{P}^3)^{s-1}) = (s-1)\,\mathrm{cat}(\mathbb{R}\mathrm{P}^3) = 3(s-1) \tag{4.2}$$

in view of [22], whereas [18, Theorem 1.3] gives

$$\mathrm{TC}_s(\mathrm{Conf}(\mathbb{R}^2 - Q_2, n - 3)) = s(n-3). \tag{4.3}$$

Then [2, Proposition 3.11] gives $\mathrm{TC}_s(\mathrm{Conf}(S^2, n)) \leq sn - 3$.

Genus 1. Since $T = S^1 \times S^1$ is a topological group, there is a topological decomposition

$$\mathrm{Conf}(T, n) \cong T \times \mathrm{Conf}(T - Q_1, n-1)$$

where Q_1 is a fixed point in T, see [6, Example 2.6] for instance. It has been noted that

$$\mathrm{TC}_s(T) = 2(s-1).$$

On the other hand, $\mathrm{Conf}(T - Q_1, n-1)$ has the homotopy type of a cell complex of dimension $n-1$ (see [4, proof of Theorem 4.1]). So [2, Theorem 3.9] gives

$$\mathrm{TC}_s(\mathrm{Conf}(T - Q_1, n-1)) \leq s(n-1),$$

and we get $\mathrm{TC}_s(\mathrm{Conf}(T, n)) \leq s(n+1) - 2$.

Genus at least 2. As noted in the proof of [4, Theorem 5.1], $\mathrm{Conf}(\Sigma_g, n)$ has the homotopy type of a cell complex of dimension $n+1$. We thus immediately obtain $\mathrm{TC}_s(\mathrm{Conf}(\Sigma_g, n)) \leq s(n+1)$.

4.2 Zero divisors via the Totaro spectral sequence

We use Proposition 1.3.3 to show that each of the upper bounds described in the previous section are sharp. The simplest situation, i.e. that for S^2, is based on the obvious generalization of item (iii) of [4, Lemma 2.1] that, for algebras A' and A'', $A := A' \otimes A''$ is a (graded-) commutative unital algebra with multiplication

$$(a_1' \otimes a_1'')(a_2' \otimes a_2'') := (-1)^{\deg(a_1'')\deg(a_2')} a_1' a_2' \otimes a_1'' a_2''$$

and, in these conditions,

$$\mathrm{zcl}_s(A) \geq \mathrm{zcl}_s(A') + \mathrm{zcl}_s(A'').$$

For instance, (4.1) yields

$$\mathrm{zcl}_s(H^*(\mathrm{Conf}(S^2, n), \mathbb{F})) \geq \mathrm{zcl}_s(H^*(\mathbb{R}\mathrm{P}^3, \mathbb{F})) + \mathrm{zcl}_s(H^*(\mathrm{Conf}(\mathbb{R}^2 - Q_2, n-3), \mathbb{F})), \quad (4.4)$$

where $\mathbb{F}$ is a field.

Proof of Theorem 4.0.4 for $g = 0$ and $n \geq 3$. In view of the proof of [18, Theorem 5.1], the assertion in (4.3) can be strengthened to

$$\mathrm{TC}_s(\mathrm{Conf}(\mathbb{R}^2 - Q_2, n-3)) = s(n-3) = \mathrm{zcl}_s(H^*(\mathrm{Conf}(\mathbb{R}^2 - Q_2, n-3), \mathbb{Z}_2)),$$

whereas the corresponding equality

$$\mathrm{TC}_s(\mathbb{R}\mathrm{P}^3) = 3(s-1) = \mathrm{zcl}_s(H^*(\mathbb{R}\mathrm{P}^3, \mathbb{Z}_2)),$$

extending (4.2), is an easy exercise. Together with (4.4) and Proposition 1.3.3 we then get

$$\mathrm{TC}_s(\mathrm{Conf}(S^2, n)) \geq sn - 3,$$

which completes the proof in view of the upper bound given in Section 4.1 for $g = 0$ and $n \geq 3$. □

Proving that the upper bounds in Section 4.1 are also optimal for Σ_g with $g \geq 1$ (and, thus, completing the proof of Theorem 4.0.4) depends on Proposition 1.3.3 and a rather explicit calculation to estimate $\mathrm{zcl}_s(H^*(\mathrm{Conf}(\Sigma_g, n), \mathbb{Q}))$. We will show

$$\mathrm{zcl}_s(H^*(\mathrm{Conf}(\Sigma_g, n), \mathbb{Q})) \geq \begin{cases} s(n+1) - 2, & g = 1; \\ s(n+1), & g \geq 2. \end{cases} \tag{4.5}$$

As suggested in (4.5), all cohomology rings in the remainder of this chapter will have rational coefficients.

The Leray spectral sequence of the inclusion $\mathrm{Conf}(M, n) \hookrightarrow M^{\times n}$ is a central tool for computing the rational cohomology ring of the ordered configuration space $\mathrm{Conf}(M, n)$ when M is an orientable manifold. As shown by Cohen-Taylor ([7]) and Totaro ([28]), the spectral sequence is particularly amenable when M is a complex projective manifold (e.g. $M = \Sigma_g$). We do not need the whole spectral sequence $\{E(g)_i^{*,*}\}_{i\geq 2}$ for $M = \Sigma_g$, only the subalgebra $E(g)_\infty^{*,0}$ of $H^*(\mathrm{Conf}(\Sigma_g, n))$ detected on the base axis of the spectral sequence, which is described next.

Recall that the rational cohomology algebra $H^*(\Sigma_g)$ is the polynomial ring on $2g$ generators $a(p), b(p) \in H^1(\Sigma_g)$ with $1 \leq p \leq g$, and an additional generator $\omega \in H^2(\Sigma_g)$ subject to the relations

$$a(p)a(q) = b(p)b(q) = 0, \qquad \text{and} \qquad a(p)b(q) = \begin{cases} \omega, & p = q; \\ 0, & p \neq q, \end{cases}$$

for any $p, q \in \{1, \ldots, g\}$. Consequently, $H^*(\Sigma_g^{\times n})$ is generated by 1-dimensional classes $a_i(p)$ and $b_i(p)$ ($1 \leq i \leq n$ and $1 \leq p \leq g$) and by 2-dimensional classes ω_i ($1 \leq i \leq n$), where the subindex i indicates the cartesian factor where the classes come from, subject to the relations

$$a_i(p)a_i(q) = b_i(p)b_i(q) = 0 \quad \text{and} \quad a_i(p)b_i(q) = \begin{cases} \omega_i, & p = q; \\ 0, & p \neq q, \end{cases} \tag{4.6}$$

for $p, q \in \{1, \ldots, g\}$ and $i \in \{1, \ldots, n\}$. In particular, an additive basis for $H^*(\Sigma_g^{\times n})$ is given by the set β_1 consisting of the (tensor) products $\mathbf{u} = u_1 \cdots u_n$ satisfying

$$u_i \in \{1, a_i(p), b_i(p), \omega_i \colon 1 \leq p \leq g\}, \quad \text{for each } i \in \{1, \ldots, n\}. \tag{4.7}$$

Let D_g be the ideal of $H^*(\Sigma_g^{\times n})$ generated by the elements

$$\omega_i + \omega_j + \sum_{p=1}^{g} (b_i(p)a_j(p) - a_i(p)b_j(p)) \tag{4.8}$$

for $1 \leq i < j \leq n$. In the spectral sequence, $H^*(\Sigma_g^{\times n})$ corresponds to the base $E_2^{*,0}$, and D_g corresponds to the image of the only differentials landing on the base. Therefore:

Lemma 4.2.1 ([28, Theorem 4]). *The quotient $E(g)_\infty^{*,0} = H^*(\Sigma_g^{\times n})/D_g$ is a subalgebra of $H^*(\mathrm{Conf}(\Sigma_g, n))$.*

In particular, (4.5) will follow once we prove

$$\mathrm{zcl}_s(H^*(E(g)_\infty^{*,0})) \geq \begin{cases} s(n+1) - 2, & g = 1; \\ s(n+1), & g \geq 2. \end{cases} \tag{4.9}$$

Actually, a more explicit statement (in terms of a suitably large non-trivial product of s-th zero-divisors of $E(g)_\infty^{*,0}$) is given in Theorem 4.2.3 below, which requires some preparatory notation.

For $1 \leq i \leq n$ and $1 \leq p \leq g$, consider the elements $x_i(p), y_i(p) \in E(g)_\infty^{*,0}$ defined by

- $x_i(p) = a_i(p)$ and $y_i(p) = b_i(p)$, if $p \geq 2$, or if $p = 1$ with $i = 1$;
- $x_i(1) = a_i(1) - x_1(1)$ and $y_i(1) = b_i(1) - y_1(1)$, if $i \geq 2$.

In order to simplify notation, it will be convenient to write x_i and y_i as alternatives for $x_i(1)$ and $y_i(1)$, respectively. Likewise, a_i and b_i will be used as substitutes of $a_i(1)$ and $b_i(1)$, respectively.

Note that the substitution of generators $a_i(p)$ and $b_i(p)$ by generators $x_i(p)$ and $y_i(p)$ allows us to replace the basis β_1 of $H^*(\Sigma_g^{\times n})$ considered in (4.7) by the basis β_1' consisting of the products $\mathbf{v} = v_1 \cdots v_n$ satisfying

$$v_i \in \{1, x_i(p), y_i(p), \omega_i \colon 1 \leq p \leq g\}, \quad \text{for each } i \in \{1, \dots, n\}. \tag{4.10}$$

Example 4.2.2. The relations (4.6) do not hold in $H^*(\Sigma_g^{\times n})$ if the letters a and b are replaced, respectively, by the letters x and y. For instance, $a_j(p)a_j(1) = 0$, but if $j, p \geq 2$,

$$x_j(p)x_j(1) = a_j(p)(a_j(1) - a_1(1)) = -a_j(p)a_1(1) \neq 0.$$

Likewise, $a_j(1)b_j(1) = \omega_j$, while for $2 \leq j \leq n$,

$$\begin{aligned} x_j(1)y_j(1) &= (a_j(1) - a_1(1))(b_j(1) - b_1(1)) = \omega_j + \omega_1 + b_1(1)a_j(1) - a_1(1)b_j(1) && (4.11)\\ &= \omega_j + \omega_1 + y_1(1)(x_j(1) + x_1(1)) - x_1(1)(y_j(1) + y_1(1)) \\ &= \omega_j + \omega_1 + y_1(1)x_j(1) - \omega_1 - x_1(1)y_j(1) - \omega_1 \\ &= \omega_j - \omega_1 + y_1(1)x_j(1) - x_1(1)y_j(1). && (4.12) \end{aligned}$$

We are now in a position to define the s-th zero-divisors of $E(g)_\infty^{*,0}$ we need. In fact, we start by describing four types of s-zero-divisors of $H^*(\Sigma_g^{\times n})$.

(I) For an element $u \in H^*(\Sigma_g^{\times n})$ of positive degree (so $u^2 = 0$), consider the product $\bar{u} \in H^*(\Sigma_g^{\times n})^{\otimes s}$ given by

$$\begin{aligned} \bar{u} \;&:=\; \prod_{\ell=2}^{s} (u \otimes 1 \otimes \cdots \otimes 1 \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes \overset{\ell}{u} \otimes 1 \otimes \cdots \otimes 1) \\ &= \sum_{\ell=1}^{s} \pm\, u \otimes u \otimes \cdots \otimes \overset{\ell}{1} \otimes u \otimes \cdots \otimes u. \end{aligned}$$

Here, the index on top of a tensor factor indicates the coordinate where such a factor appears. Note that $\bar{u}$ is a product of $s-1$ s-th zero-divisors. We are interested in the product

$$\prod_{i=1}^{n} \bar{x}_i = \sum \pm x_{J_1} \otimes x_{J_2} \otimes \cdots \otimes x_{J_s} \tag{4.13}$$

where the sum is taken over all subsets $J_1, J_2, \dots, J_s \subseteq \{1, \dots, n\}$ with the property that every $i \in \{1, \dots, n\}$ belongs to exactly $s-1$ subsets J_k ($1 \leq k \leq s$), and where

$$x_{J_t} := \prod_{i \in J_t} x_i$$

for $t \in \{1, \dots, s\}$.

(II) For $i \in \{1, \ldots, n\}$, consider the s-th zero-divisor

$$\widetilde{y_i} := y_i \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes y_i \in H^*(\Sigma_g^{\times n})^{\otimes s}$$

and the product

$$\prod_{i=1}^{n} \widetilde{y_i} = \sum_{J \subseteq \{1,\ldots,n\}} \pm\, y_{J^c} \otimes 1 \otimes \cdots \otimes 1 \otimes y_J, \tag{4.14}$$

where J^c stands for the complement of J in $\{1, \ldots, n\}$.

(III) For $i \in \{2, \ldots, s-1\}$, consider the s-th zero-divisor

$$y_{1,i} := y_1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes \overset{i}{y_1} \otimes 1 \otimes \cdots \otimes 1 \in H^*(\Sigma_g^{\times n})^{\otimes s}$$

and the product

$$\prod_{i=2}^{s-1} y_{1,i} = \sum_{(\epsilon_1,\ldots,\epsilon_s) \in M_s} \pm y_1^{\epsilon_1} \otimes y_1^{\epsilon_2} \otimes \cdots \otimes y_1^{\epsilon_{s-1}} \otimes 1, \tag{4.15}$$

where $M_s := \{(\varepsilon_1, \ldots, \varepsilon_{s-1}) : \exists!\ j \in \{1, \ldots, s-1\}$ with $\varepsilon_j = 0$ and $\varepsilon_i = 1$ for $i \neq j\}$.

(IV) If $g \geq 2$, consider the s-th zero divisors $c, d \in H^*(\Sigma_g^{\times n})^{\otimes s}$ given by

$$\begin{aligned} c &= a_1(2) \otimes 1 \otimes 1 \cdots \otimes 1 - 1 \otimes a_1(2) \otimes 1 \cdots \otimes 1, \\ d &= \begin{cases} b_1(2) \otimes 1 - 1 \otimes b_1(2), & \text{if } s = 2; \\ b_1(2) \otimes 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes 1 \otimes b_1(2) \otimes 1 \otimes \cdots \otimes 1, & \text{if } s \geq 3. \end{cases} \end{aligned}$$

The inequality in (4.9) and, therefore, Theorem 4.0.4 for $g > 0$ are immediate consequences of the following result, whose proof is the central goal in the remainder of this chapter.

Theorem 4.2.3. *(i) The image of $\left(\prod_{i=2}^{s-1} y_{1,i}\right) \cdot \left(\prod_{i=1}^{n} (\bar{x}_i \widetilde{y_i})\right)$ in $\left(E(1)_\infty^{*,0}\right)^{\otimes s}$ is non-zero.*

(ii) If $g \geq 2$, the image of $c \cdot d \cdot \left(\prod_{i=2}^{s-1} y_{1,i}\right) \cdot \left(\prod_{i=1}^{n} (\bar{x}_i \widetilde{y_i})\right)$ in $\left(E(g)_\infty^{*,0}\right)^{\otimes s}$ is non-zero.

4.3 A subquotient of the cohomology of $\mathrm{Conf}(\Sigma_g, n)$

The proof of the non-vanishing of the products indicated in Theorem 4.2.3 is greatly simplified by actually working on the quotient of $E(g)_\infty^{*,0}$ obtained by modding out by the ideal generated by the elements

$$x_i(p)x_j(q), \quad x_i(p)y_j(q), \quad y_i(p)y_j(q) \tag{4.16}$$

with $p, q \in \{2, \ldots, g\}$ and $i, j \in \{1, \ldots, n\}$, $i \neq j$, and by the elements

$$x_i y_j \tag{4.17}$$

with $i, j \in \{2, \ldots, n\}$, $i \neq j$. Our strategy has two main steps:

S1. We first get a full additive description of the quotient A_g of $H^*(\Sigma_g^{\times n})$ by the ideal generated by the elements in (4.16).

S2. Then we prove that the products indicated in Theorem 4.2.3 are in fact non-trivial in the quotient B_g of A_g by the A_g-ideal generated by the elements in (4.8) and (4.17).

Furthermore, when dealing with the second step, and in view of the relations coming from (4.16), the elements in (4.8) can safely be replaced by the elements

$$\omega_i + \omega_j + b_i(1)a_j(1) - a_i(1)b_j(1) \tag{4.18}$$

for $1 \leq i < j \leq n$. It follows that the identity maps on generators induce ring morphisms $B_1 \to B_2 \to B_3 \to \cdots$. In particular, item *(i)* in Theorem 4.2.3 becomes a direct consequence of the proof of item *(ii)* in Theorem 4.2.3 sketched in steps S1 and S2 above. Accordingly, we assume $g \geq 2$ in the remainder of the section.

Step S1 above is accomplished in the next result.

Proposition 4.3.1. *An additive basis of A_g is given by the set β_2' consisting of the images in A_g of the monomials $v_1 \cdots v_n \in H^*(\Sigma_g^{\times n})$ satisfying the following two conditions:*

(i) For each $i \in \{1, \ldots, n\}$, the factor v_i belongs to $\{1, x_i(p), y_i(p), \omega_i \colon 1 \leq p \leq g\}$.

(ii) At most one of $v_1, \ldots, v_n$ lies in $\{x_i(p), y_i(p), \omega_i \colon 1 \leq i \leq n \text{ and } 2 \leq p \leq g\}$.

Proof. We first observe that an additive basis of A_g is given by the set β_2 consisting of the images in A_g of the monomials $u_1 \cdots u_n \in H^*(\Sigma_g^{\times s})$ satisfying the following two conditions:

(iii) For each $i \in \{1, \ldots, n\}$, the factor u_i belongs to $\{1, a_i(p), b_i(p), \omega_i \colon 1 \leq p \leq g\}$.

(iv) At most one of $u_1, \ldots, u_n$ lies in $\{a_i(p), b_i(p), \omega_i \colon 1 \leq i \leq n \text{ and } 2 \leq p \leq g\}$.

Indeed, in terms of the additive basis β_1 of $H^*(\Sigma_g^{\times n})$ in (4.7), the defining relations for A_g coming from the elements in (4.16) take the form

$$a_i(p)a_j(q) = a_i(p)b_j(q) = b_i(p)b_j(q) = 0 \tag{4.19}$$

for $p, q \in \{2, \ldots, g\}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Thus, an additive basis for the ideal generated by relations (4.16) is given by the monomials in β_1 which fail to satisfy condition (iv). Indeed, note that for any $\mathbf{u} = u_1 \cdots u_n \in \beta_1$, the $\mathbf{u}$-multiple (in $H^*(\Sigma_g^{\times n})$) of any of the elements

$$a_i(p)a_j(q), \quad a_i(p)b_j(q), \quad b_i(p)b_j(q)$$

as in (4.19) either vanishes or, else, reduces (up to a sign) to an element of β_1 for which (iv) fails. For instance, if we consider an element of the form $a_i(p_0)b_j(q_0)$ with $p_0, q_0 \geq 2$ and $i \neq j$,

$$u_1 \cdots u_n \cdot a_i(p_0)b_j(q_0) = \pm u_1 \cdots \widehat{u}_i \cdots \widehat{u}_j \cdots u_n \left(u_i a_i(p_0)\right)\left(u_j b_j(q_0)\right)$$

then, in view of (iii), is either zero (in the case where $u_i \notin \{1,\, b_i(p_0)\}$ or $u_j \notin \{1,\, a_j(q_0)\}$) or, else, an element of β_1 of the form:

- Case $u_i = 1$ and $u_j = 1$.

$$u_1 \cdots u_n \cdot a_i(p_0)b_j(q_0) = \pm u_1 \cdots \widehat{u}_i \cdots \widehat{u}_j \cdots u_n \cdot a_i(p_0) \cdot b_j(q_0).$$

- Case $u_i = b_i(p_0)$ and $u_j = 1$.

$$u_1 \cdots u_n \cdot a_i(p_0)b_j(q_0) = \pm u_1 \cdots \widehat{u}_i \cdots \widehat{u}_j \cdots u_n \cdot \omega_i \cdot b_j(q_0).$$

- Case $u_i = 1$ and $u_j = a_j(q_0)$.

$$u_1 \cdots u_n \cdot a_i(p_0) b_j(q_0) = \pm u_1 \cdots \widehat{u}_i \cdots \widehat{u}_j \cdots u_n \cdot a_i(p_0) \cdot \omega_j.$$

- Case $u_i = b_i(p_0)$ and $u_j = a_j(q_0)$.

$$u_1 \cdots u_n \cdot a_i(p_0) b_j(q_0) = \pm u_1 \cdots \widehat{u}_i \cdots \widehat{u}_j \cdots u_n \cdot \omega_i \cdot \omega_j.$$

Thus, the set β_2 determines an additive basis of A_g.
To complete the proof it now suffices to observe, on the one hand, that β_2 and β_2' have the same cardinality and, on the other hand, that (just as in the β_1 vs. β_2 situation just discussed) any element in β_1' not satisfying (ii) vanishes in A_g. □

We now start working toward the completion of step S2. Recall that B_g is the quotient of A_g by the ideal generated by the elements in (4.17) and (4.18). As noted in (4.11), the case $1 = i < j \leq n$ of the latter generators is given by $x_j y_j$, whereas for the case $2 \leq i < j \leq n$ we have

$$\begin{aligned} \omega_i + \omega_j + b_i a_j - a_i b_j &= (a_i - a_j)(b_i - b_j) \\ &= (x_i - x_j)(y_i - y_j) \\ &= x_i y_i + x_j y_j - x_i y_j - x_j y_i. \end{aligned}$$

Consequently we will work with the simplified presentation

$$B_g = A_g / \mathcal{J}_g \tag{4.20}$$

where $\mathcal{J}_g$ is the A_g-ideal generated by the products $x_i y_j$ with $i, j \in \{2, \ldots, n\}$.

A key ingredient for step S2 is given by the next result, whose proof is deferred to the next section of the chapter.

Theorem 4.3.2. *The images in B_g of the two elements $\omega_1 x_2 \cdots x_n, \omega_1 y_2 \cdots y_n \in H^*(\Sigma_g^{\times n})$ are distinct and, in fact, linearly independent.*

Proof of item (ii) of Theorem 4.2.3 for $s = 2$. As advertised at the beginning of this section, it suffices to work in B_g. Direct calculation gives $c\, d\, \bar{x}_1 \widetilde{y}_1 = 2\omega_1 \otimes \omega_1$ and (by induction on $n \geq 2$, keeping in mind the relations in B_g coming from the ideal $\mathcal{J}_g$)

$$cd \left(\prod_{i=1}^{n} (\bar{x}_i \widetilde{y}_i) \right) = 2\omega_1 \otimes \omega_1 \Big(\pm x_2 \cdots x_n \otimes y_2 \cdots y_n \pm y_2 \cdots y_n \otimes x_2 \cdots x_n \Big),$$

which is non-zero in B_g in view of Theorem 4.3.2. (Note that the factor (4.15) degenerates to 1.) □

The proof of item *(ii)* of Theorem 4.2.3 for $s \geq 3$ is slightly more involved, partly due to the presence of the factor (4.15), and partly because of the resulting larger combinatorial objects to deal with. Actually, the main reason for the s-th zero-divisor d to be slightly different for $s \geq 3$ is to simplify the proof argument.

Proof of item (ii) of Theorem 4.2.3 for $s \geq 3$. Up to a sign, the product under consideration, $cd \left(\prod_{i=2}^{s-1} y_{1,i} \right) \left(\prod_{i=1}^{n} (\bar{x}_i \widetilde{y}_i) \right)$, is a sum running over the subsets $J, J_1, J_2, \ldots, J_s$ of $\{1, \ldots, n\}$ specified in (4.13) and (4.14), over the tuples $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{s-1}) \in M_s$ specified in (4.15), and over the pairs (α_1, α_2) and (β_1, β_3) satisfying $\{\alpha_1, \alpha_2\} = \{0, 1\} = \{\beta_1, \beta_3\}$. The term T corresponding to such a data takes the form indicated below, depending on the value of s.

• If $s \geq 5$,

$$\pm a_1(2)^{\alpha_1} b_1(2)^{\beta_1} y_1^{\epsilon_1} y_{J^c} x_{J_1} \otimes a_1(2)^{\alpha_2} y_1^{\epsilon_2} x_{J_2} \otimes b_1(2)^{\beta_3} y_1^{\epsilon_3} x_{J_3} \otimes y_1^{\epsilon_4} x_{J_4} \otimes \cdots \otimes y_1^{\epsilon_{s-1}} x_{J_{s-1}} \otimes y_J x_{J_s}.$$

• If $s = 4$,

$$\pm a_1(2)^{\alpha_1} b_1(2)^{\beta_1} y_1^{\epsilon_1} y_{J^c} x_{J_1} \otimes a_1(2)^{\alpha_2} y_1^{\epsilon_2} x_{J_2} \otimes b_1(2)^{\beta_3} y_1^{\epsilon_3} x_{J_3} \otimes y_J x_{J_4}.$$

• If $s = 3$,

$$\pm a_1(2)^{\alpha_1} b_1(2)^{\beta_1} y_1^{\epsilon_1} y_{J^c} x_{J_1} \otimes a_1(2)^{\alpha_2} y_1^{\epsilon_2} x_{J_2} \otimes b_1(2)^{\beta_3} y_J x_{J_3}.$$

In any case, such a term T vanishes in B_g unless each of the following conditions holds:

1. $J = \{1\}$ or $J = \{1, \ldots, n\}$.

   Indeed, if $1 \notin J$, then the non-triviality of T in B_g forces $\alpha_1 = \beta_1 = \epsilon_1 = 0$, so that $\alpha_2 = \beta_3 = 1$ and $\epsilon_i = 1$ for $2 \leq i \leq s-1$, which is impossible since $a_1(2)y_1 = 0$. Thus $1 \in J$ must hold. Furthermore, 2 lies in $s-1$ of the sets $J_1, \ldots, J_s$ so, in particular, x_2 shows up either in the first tensor factor of T (where y_{J^c} appears), or in the last tensor factor of T (where y_J appears). Therefore, the reduced form of the defining relations in B_g and the non-triviality of T in B_g force either $J - \{1\} = \varnothing$, or $J - \{1\} = \{2, \ldots, n\}$.

2. $1 \notin J_1$, so that $1 \in J_i$ for $2 \leq i \leq s$.

   Indeed, if $1 \in J_1$, the non-triviality of T in B_g forces $\alpha_1 = 0 = \beta_1$, so $\alpha_2 = 1 = \beta_3$. But this is incompatible with the non-triviality of T in B_g and the fact that 1 must lie in either J_2 or J_3.

3. $\alpha_2 = 0 = \beta_3$, so that $\alpha_1 = 1 = \beta_1$.

   For we have just noted that $1 \in J_2 \cap J_3$.

4. $\epsilon_1 = 0$, so that $\epsilon_i = 1$ for $2 \leq i \leq s-1$.

   For we have just noted that $\alpha_1 = 1 = \beta_1$.

Further, when $J = \{1\}$, the term T vanishes in B_g unless $J_1 = \varnothing$ (the inclusion $J_1 \subseteq \{1\}$ follows by looking at the first tensor factor of T and the relations defining B_g, whereas the actual equality $J_1 = \varnothing$ follows from condition 2 above) and, therefore, $J_i = \{1, \ldots, n\}$ for $2 \leq i \leq s$. Thus, the only such T with (potentially) non-vanishing image in B_g is, up to a sign,

$$\begin{aligned} &a_1(2)b_1(2)y_2 \cdots y_n \otimes y_1 x_1 \cdots x_n \otimes \cdots \otimes y_1 x_1 \cdots x_n \\ &\qquad\qquad = \pm \omega_1 y_2 \cdots y_n \otimes \omega_1 x_2 \cdots x_n \otimes \cdots \otimes \omega_1 x_2 \cdots x_n. \end{aligned} \tag{4.21}$$

Likewise, when $J = \{1, \ldots, n\}$, the term T vanishes in B_g unless $J_s = \{1\}$ (the inclusion $J_s \subseteq \{1\}$ follows by looking at the last tensor factor of T and the relations defining B_g, whereas the actual equality $J_s = \{1\}$ follows from condition 2 above) and $J_1 = \{2, \ldots, n\}$ while $J_i = \{1, \ldots, n\}$ for $2 \leq i \leq s-1$ (in view of condition 2 above and the properties of the J_i's). Thus, the only such T with (potentially) non-vanishing image in B_g is, up to a sign,

$$\begin{aligned} &a_1(2)b_1(2)x_2 \cdots x_n \otimes y_1 x_1 \cdots x_n \otimes \cdots \otimes y_1 x_1 \cdots x_n \otimes y_1 \cdots y_n x_1 \\ &\qquad\qquad = \pm \omega_1 x_2 \cdots x_n \otimes \cdots \otimes \omega_1 x_2 \cdots x_n \otimes \omega_1 y_2 \cdots y_n. \end{aligned} \tag{4.22}$$

Consequently, the image in B_g of the product under consideration is the sum of the term in (4.21) and the term in (4.22), which is non-zero by Theorem 4.3.2. $\square$

4.4 Proof of Theorem 4.3.2

In view of the particularly simple presentation (4.20) of B_g, it might be tempting to guess the form of an additive basis for B_g which, in addition, could easily imply Theorem 4.3.2. However, a few unexpected relations holding in B_g are hidden in $\mathcal{J}_g$. It is the purpose of this section to uncover, in the most efficient way (for the purpose of proving Theorem 4.3.2), some of these unexpected relations.

Recall the additive basis β_2' of A_g in Corollary 4.3.1, that is, the set of products $v_1 \cdots v_n$ satisfying the two conditions:

(i) For each $i \in \{1, \ldots, n\}$, the factor v_i belongs to $\{1, x_i(p), y_i(p), \omega_i \colon 1 \leq p \leq g\}$.

(ii) At most one of $v_1, \ldots, v_n$ belongs to $\{x_i(p), y_i(p), \omega_i \colon 1 \leq i \leq n \text{ and } 2 \leq p \leq g\}$.

The verification of the following two lemmas is a straightforward and, thus, omitted task.

Lemma 4.4.1. *Let $2 \leq j \leq n$. For $v_1 \cdots v_n \in \beta_2'$, the product $v_1 \cdots v_n \cdot x_j y_j$ vanishes in A_g provided any one of the following conditions holds:*

(i) $v_j \in \{x_j(p), y_j(p), \omega_j : 1 \leq p \leq g\}$.

(ii) $v_1 \in \{x_1(p), y_1(p), \omega_1 : 2 \leq p \leq g\}$.

(iii) $v_1 \in \{x_1, y_1\}$ *and* $v_k \in \{x_k(p), y_k(p), \omega_k : 2 \leq p \leq g\}$ *for some* $k \notin \{1, j\}$.

Furthermore, the following relations hold in A_g:

(iv) $x_1 \cdot x_j y_j = x_1 \omega_j + \omega_1 x_j$.

(v) $y_1 \cdot x_j y_j = y_1 \omega_j + \omega_1 y_j$.

(vi) $z_k \cdot x_j y_j = z_k y_1 x_j - z_k x_1 y_j$, *for* $z_k \in \{x_k(p), y_k(p), \omega_k \colon 2 \leq p \leq g\}$ *with* $k \notin \{1, j\}$.

Lemma 4.4.2. *Let $i, j \in \{2, \ldots, n\}$ with $i \neq j$. Then, in A_g:*

(1) The only non-trivial products $z_i \cdot x_i y_j$ with $z_i \in \{x_i(p), y_i(p), \omega_i \colon 1 \leq p \leq g\}$ are

- *(i)* $y_i \cdot x_i y_j = -\omega_i y_j + \omega_1 y_j - y_1 x_i y_j + x_1 y_i y_j$.
- *(ii)* $z_i \cdot x_i y_j = -z_i x_1 y_j$, *for* $z_i \in \{x_i(p), y_i(p), \omega_i \colon 2 \leq p \leq g\}$.

(2) The only non-trivial products $z_j \cdot x_i y_j$ with $z_j \in \{x_j(p), y_j(p), \omega_j \colon 1 \leq p \leq g\}$ are

- *(iii)* $x_j \cdot x_i y_j = -x_i \omega_j + x_i \omega_1 + y_1 x_i x_j - x_1 x_i y_j$.
- *(iv)* $z_j \cdot x_i y_j = -z_j x_i y_1$, *for* $z_j \in \{x_j(p), y_j(p), \omega_j \colon 2 \leq p \leq g\}$.

(3) The only non-trivial product $z_i z_j \cdot x_i y_j$ with z_i and z_j as in (1) and (2) above is

- *(v)* $y_i x_j \cdot x_i y_j = y_1 \omega_i x_j + y_1 x_i \omega_j - x_1 \omega_i y_j - x_1 y_i \omega_j + \omega_1 y_i x_j - \omega_1 x_i y_j$.

(4) The only non-trivial products $z_1 z_i \cdot x_i y_j$ with $z_1 \in \{x_1(p), y_1(p), \omega_1 \colon 1 \leq g \leq p\}$ and z_i as in (1) above are

- *(vi)* $x_1 y_i \cdot x_i y_j = -x_1 \omega_i y_j - \omega_1 x_i y_j$.
- *(vii)* $y_1 y_i \cdot x_i y_j = -y_1 \omega_i y_j - \omega_1 y_i y_j$.

(5) The only non-trivial products $z_1 z_j \cdot x_i y_j$ with z_j and z_1 as in (2) and (4) above are

- *(viii)* $x_1 x_j \cdot x_i y_j = -x_1 x_i \omega_j + \omega_1 x_i x_j$.
- *(ix)* $y_1 x_j \cdot x_i y_j = -y_1 x_i \omega_j + \omega_1 x_i y_j$.

(6) All products $z_1 z_i z_j \cdot x_i y_j$ with z_i, z_j and z_1 as in (1), (2) and (4) vanish.

Set $\gamma_2 = \beta_2' - \gamma_1$, where $\gamma_1 \subseteq \beta_2'$ consists of the products $v_1 \cdots v_n$ satisfying either one of the following two conditions:

(iii) There is a unique $i \in \{1, \cdots, n\}$ for which $v_i = \omega_i$ and $v_j = x_j$ for $j \neq i$.

(iv) There is a unique $i \in \{1, \cdots, n\}$ for which $v_i = \omega_i$ and $v_j = y_j$ for $j \neq i$.

There is an obvious additive splitting $A_g = C_{g,1} \oplus C_{g,2}$, where $C_{g,\epsilon}$ is the additive span of γ_ϵ ($\epsilon = 1, 2$). The final technical task in this section, the proof of Theorem 4.3.2, will be accomplished below by arguing first that the ideal $\mathcal{J}_g$ defining B_g preserves the above splitting, i.e. by giving an additive decomposition

$$\mathcal{J}_g = \mathcal{J}_{g,1} \oplus \mathcal{J}_{g,2}, \tag{4.23}$$

where $\mathcal{J}_{g,\epsilon}$ is a vector subspace of $C_{g,\epsilon}$ ($\epsilon = 1, 2$), and then by giving a description of the (additive structure of the) quotient $C_{g,1}/\mathcal{J}_{g,1}$, for which a basis will clearly be given by the two elements in the statement of Theorem 4.3.2.

In what follows, an element $\mathbf{v} = v_1 \cdots v_n \in \beta_2'$, will be denoted as

- $\mathbf{v}(0)$ to indicate that $v_k \in \{1, x_k, y_k\}$ for all $k = 1, \ldots, n$;
- $\mathbf{v}(i_1, \ldots, i_t)$, for $i_1, \ldots, i_t \in \{1, \ldots, n\}$, to indicate that $v_{i_k} = 1$ for $k \in 1, \ldots, t$.

These two conventions will also be combined. For instance, by writing $\mathbf{v}(0, 1, j)$ we mean that the element $\mathbf{v} \in \beta_2'$ satisfies $v_k \in \{1, x_k, y_k\}$ for all $k = 1, \ldots, n$, as well as $v_1 = v_j = 1$.

Proof of Theorem 4.3.2. A set of additive generators of $\mathcal{J}_g$ is given by the products $\mathbf{v} \cdot r$ with $\mathbf{v} = v_1 \cdots v_n \in \beta_2'$ and $r \in \{x_i y_j : i, j \in \{2, \ldots, n\}\}$. The additive decomposition (4.23) will follow once we check that

the expression of each such product $\mathbf{v} \cdot r = v_1 \cdots v_n \cdot x_i y_j$ (in terms of the basis β_2') involves either only elements of γ_1 or, else, only elements of γ_2. (4.24)

Case $i = j \geq 2$. By Lemma 4.4.1(i), we only need to consider products $\mathbf{v}(j) \cdot x_j y_j$. Recalling from (4.12) that $x_j y_j = \omega_j - \omega_1 + y_1 x_j - x_1 y_j$, it is clear that (4.24) holds, with γ_2 being the relevant basis, if $\mathbf{v} = \mathbf{v}(1, j)$ —in checking this type of assertions, the reader might find it convenient to consider first the case $\mathbf{v} = \mathbf{v}(0, 1, j)$. Thus, by Lemma 4.4.1(ii) and (iii), we can assume $v_1 \in \{x_1, y_1\}$ and $\mathbf{v} = \mathbf{v}(0)$. In other words, it remains to consider products of the form

$$x_1 \mathbf{v}(0, 1, j) \cdot x_j y_j \quad \text{and} \quad y_1 \mathbf{v}(0, 1, j) \cdot x_j y_j.$$

It is clear from Lemma 4.4.1(iv) and (v) that (4.24) holds true for the two types of products just described, and that the only such products whose expression in terms of the basis β_2' involves only elements from γ_1 can actually be written, up to a sign, as

$$\omega_1 x_2 \cdots x_n + (-1)^j x_1 x_2 \cdots x_{j-1} \omega_j x_{j+1} \cdots x_n \tag{4.25}$$

and

$$\omega_1 y_2 \cdots y_n + (-1)^j y_1 y_2 \cdots y_{j-1} \omega_j y_{j+1} \cdots y_n. \tag{4.26}$$

Case $i, j \in \{2, \ldots, n\}$ **with** $i \neq j$. It is obvious that (4.24) holds, with γ_2 being the relevant basis, provided $\mathbf{v} = \mathbf{v}(i, j)$. The rest of the possibilities can be analyzed on a term-by-term basis, depending on the values of z_i and z_j in a product $z_i z_j \mathbf{v}(i, j) \cdot x_i y_j$, where $z_t \in \{1, x_t(p), y_t(p), \omega_t\colon 1 \leq p \leq g\}$. Actually, by Lemma 4.4.2, the only factors involved in the expression of any $z_i z_j \cdot x_i y_j$ can come from the coordinates 1, i and j. Therefore it is convenient to split the analysis by considering the products

$$z_i z_j \mathbf{v}(1,i,j) \cdot x_i y_j \quad \text{and} \quad z_1 z_i z_j \mathbf{v}(1,i,j) \cdot x_i y_j. \tag{4.27}$$

Lemma 4.4.2 describes the expression of the corresponding factors $z_i z_j \cdot x_i y_j$ and $z_1 z_i z_j \cdot x_i y_j$ in terms of the basis β_2'. In all such cases one checks, by direct inspection, that

- (4.24) holds true for all products in (4.27),
- the only products in (4.27) whose expression in terms of β_2' involves elements from γ_1 are those arising from instances (vii) and (viii) of Lemma 4.4.2, in which case
- the resulting expressions in terms of the basis β_2' coincide with those in (4.25) and (4.26) —note that signs in items (vii) and (viii) of Lemma 4.4.2 are important here!

The proof is complete since the above considerations imply that the decomposition (4.23) holds in such a way that an additive basis for the resulting additive summand $C_{g,1}/\mathcal{J}_{g,1}$ of B_g is given by the two elements in the statement of Theorem 4.3.2. □

4.5 The case $s = 2$

The case $s = 2$ in Theorem 4.0.4 reduces to Theorem A in [4]. We have given full proof details for that case too because we believe that there are a couple of weak points and, most critically, at least one flawed argument in the homological part of Cohen-Farber's argument. This section describes such potential problems. The reader is assumed to be familiar with the notation in [4].

The main problem happens at the end of the fourth paragraph of the proof of [4, Theorem 5.1], where the authors assert that the proof of the case for genus $g \geq 2$ can be reduced to the consideration of the $g = 2$ case by "annihilating all generators of the form $1 \times \cdots \times u \times \cdots \times 1$ where $u \in \{a(q), b(q)\colon 3 \leq q \leq g\}$". (Note the typo "$3 \leq q \leq n$" in [4].) Such an argument does not work because if, for instance, we set $a(3) = 0$ in the i-th axis, then $w = a(3)b(3)$ would also be zero in that axis. But this interferes (for $i = 1$) with Cohen and Farber's later calculation using the non-triviality of ω_1 (see the last displayed formula in the proof of [4, Theorem 5.1]).

In addition, we believe that a weak argument arises at the end of the proof of [4, Theorem 5.1], where the authors assert that

> the non-zero term $\pm 2\omega_1 y_2 y_3 \cdots y_n \otimes \omega_1 x_2 x_3 \cdots x_n$ arises in the expansion of the product $\bar{a}_1 \bar{b}_1 \bar{c}_1 \bar{d}_1 \prod_{j=2}^n \bar{x}_j \bar{y}_j$ in such a way that *no other summand in the expansion involves this (non-zero) tensor product.*
>
> (4.28)

The (apparently implicit) argument supporting (4.28) is based on two facts noted in earlier parts of Cohen-Farber's paper:

(I) On the one hand, as indicated at the end of the proof of [4, Theorem 4.1] (i.e. when dealing with the algebra A_T in the genus-1 case), the expansion (in terms of basis elements) of $\prod_{j=1}^n \bar{x}_j \bar{y}_j$ uses (with coefficient ± 1) the basis element $y_1 y_2 y_3 \cdots y_n \otimes x_1 x_2 x_3 \cdots x_n$.

(II) On the other hand, near the bottom of page 656 of [4], it is observed (without further explanation, though) that "The subalgebra of B_Σ generated by $\{a_i, b_i\colon 1 \leq i \leq n\}$ is isomorphic to the subalgebra A_T arising in the genus one case".

The problem is that the latter two facts do not really support (4.28) for, although A_T were a honest subalgebra of B_Σ, nothing is said about the (potential) injectivity of the obvious map $(2\omega_1 \otimes \omega_1) \cdot A_T \to B_\Sigma$. In the Cohen-Farber approach, fixing these problems requires, in principle, an explicit description of additive bases for the subquotient algebras they deal with. Such a task tends to become combinatorially involved, especially in the case of Rudyak's higher TC. We have greatly simplified the job by working in a much smaller subquotient—small enough to detect just the minimal needed information.

It is also worth remarking what appears to us to be a weak statement of item (ii) in [4, Lemma 2.1], namely, the assertion that an epimorphic image B of an algebra A over a field has $\mathrm{zcl}(A) \geq \mathrm{zcl}(B)$. The verification of such a property is left as a "straightforward exercise" in [4] and, as in the case of the dual statement in item (i), its proof should naturally start by picking zero-divisors $b_1, \ldots, b_t \in B \otimes B$ with $b_1 \cdots b_t \neq 0$. With these conditions it is certainly obvious that, for any choice of preimages $a_i \in A \otimes A$ of each b_i, the product $a_1 \cdots a_n$ is forced to be non-zero. But the point is to make sure that each a_i can be chosen to be a zero-divisor in A, which does not seem to be accomplishable in the stated generality. Nonetheless, what can certainly be done (and has been done in this thesis) is to argue the non-triviality of some given product of zero-divisors in $A \otimes A$ by exhibiting the non-triviality of the image of the product in $B \otimes B$.

Conclusions.

In this thesis we computed the higher topological complexity of:

- subcomplexes of products of spheres,
- configuration spaces of orientable surfaces,

and we studied the asymptotic behavior of this invariant for an explicit random family of subcomplexes of products of circles.

In Chapter 2, Theorem 2.5.1 described the higher topological complexity of any subcomplex of a product of spheres expressed just in combinatorial terms associated to the subcomplex. We also include computations for some particular examples using this theorem.

In Chapter 3 we used Theorem 2.2.5 (that is a particularization of Theorem 2.5.1, for the case where all spheres involved in the product are odd dimensional), to give an estimation of the value of the TC_s in a limiting process for a specific family of random subcomplexes of products of circles. All computations in this chapter were done by considering a fixed probability parameter p $(0 < p < 1)$. So a possible improvement to this work consists of varying this probability parameter in the limiting process in order to get not just an estimation of the higher TC, but an exact value.

Finally, in Chapter 4 we computed the higher TC of configuration spaces of orientable surfaces, see Theorem 4.0.4. For this purpose, we computed the lower bound given in Proposition 1.3.3 for these spaces, that turns out to be optimal by dimensional considerations. Our main challenge in this Chapter was Theorem 4.3.2 where we asserted the nontriviality (and linearly independence) of two products of s-th zero-divisors in a quotient of a subalgebra of $H^*(\mathrm{Conf}(\Sigma_g, n), \mathbb{Q})^{\otimes s}$.

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