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Propiedades asintóticas de ideales monomiales

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Asymptotic properties of monomial ideals

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Resumen

Un ideal I tiene la propiedad de persistencia fuerte si $\left(I^{k+1}\colon I\right) = I^k$ para $k \geq 1$. En esta tesis estudiamos esta propiedad para varias familias de ideales. En particular, probamos que los ideales monomiales, cuyo conjunto mínimo de generadores son homogéneos de grado 2, tienen la propiedad de persistencia fuerte. Decimos que una hipergráfica simple $\mathcal{C}$ tiene la propiedad de persistencia fuerte, si su ideal de aristas $I(\mathcal{C})$ tiene la propiedad de persistencia fuerte. Mostramos que una hipergráfica simple tiene la propiedad de persistencia fuerte si y sólo si alguna de sus componentes conexas la tiene. También demostramos que una hipergráfica simple con a lo más 4 vértices y una hipergráfica simple no mezclada Köning sin 4-ciclos tienen la propiedad de persistencia fuerte. Además demostramos que $I(\mathcal{C})$ tiene la propiedad de persistencia fuerte si y sólo si el ideal pesado $I_w(\mathcal{C})$ tiene la propiedad de persistencia fuerte. El resultado anterior también se obtuvo para la propiedad de persistencia. Finalmente, introducimos y estudiamos la propiedad de persistencia fuerte simbólica.

Otra propiedad que se estudia en esta tesis es la propiedad Gorenstein para el subanillo monomial homogéneo S_G asociado a una gráfica G. Mostramos que si S_G es normal, entonces S_G es Gorenstein si y sólo si G es no mezclada y su número de cubierta es $\lceil \frac{|V(G)|}{2} \rceil$. También demostramos que si $|V(G)|$ es par y S_G es Gorenstein, entonces G es bipartita.

Además introducimos el ideal monomial $I(D)$ asociado a una gráfica orientada pesada D. Determinamos la descomposición irredundante irreducible de $I(D)$. En particular, caracterizamos los primos asociados de $I(D)$. También caracterizamos cuando $I(D)$ es no mezclado y damos una caracterización explícita (combinatoria) de esta propiedad cuando D es bipartita, un whiskers o un ciclo. Finalmente estudiamos la propiedad Cohen-Macaulay de $I(D)$ para algunos grafos orientados pesados.

ABSTRACT

An ideal I has the strong persistence property if $\left(I^{k+1} \colon I\right) = I^k$ for $k \geq 1$. In this thesis we study this property for some families of ideals. In particular, we prove that the monomial ideals whose minimal set of generators has degree two have the strong persistence property. We say a clutter $\mathcal{C}$ has the strong persistence property if its edge ideal $I(\mathcal{C})$ has the strong persistence property. We show a clutter has the strong persistence property if and only if at least one of its connected components has the strong persistence property. Also, we prove that a clutter with at most 4 vertices and an unmixed König clutter without 4-cycles have the strong persistence property. Furthermore, we show $I(\mathcal{C})$ has the strong persistence property if and only if its weighted ideal $I_w(\mathcal{C})$ has the strong persistence property. We prove the last result for the persistence property. Finally, we introduce and study the symbolic strong persistence.

Another property studied in this thesis is the Gorenstein property for the homogeneous monomial subrings S_G associated to a graph G. We prove that if S_G is normal, then S_G is Gorenstein if and only if G is unmixed and its cover number is $\lceil \frac{n}{2} \rceil$. Also, if $|V(G)|$ is even and S_G is Gorenstein, then we show that G is bipartite.

Furthermore, we introduce the edge ideal $I(D)$ associated to a weighted oriented graph D. We determine irredundant irreducible decomposition of $I(D)$. In particular, we characterize the associated primes. Also, we characterize the unmixed property for $I(D)$ and we give an explicit (combinatorial) characterization, for this property when D is bipartite, D is a whisker or D is a cycle. Finally, we study the Cohen-Macaulay property of $I(D)$, for some weighted oriented graphs D.

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PREFACE

The structure of the dissertation is as follows. In Chapter 1 we introduce the general concepts and results used in this thesis, we study the monomial ideals, their minimal sets of generators and their primary decomposition. Also, we characterised when a monomial ideal is prime, irreducible or primary.

In Chapter 2, we work with two of the most studied asymptotic properties, the strong persistence property and the persistence property. An ideal I has the strong persistence property if $(I^{k+1} \colon I) = I^k$ for each k. Furthermore, ideal I has the persistence property if $\operatorname{Ass}(I^k) \subseteq \operatorname{Ass}(I^{k+1})$ for each k. We start studying the case of monomial ideals whose minimal set of generators consists of monomials of degree 2, we prove that these ideals have the strong persistence property. Thus, we obtain that the strong persistence property is satisfied for a more general class that the edge ideals of graphs. Another class that generalises to the edges ideals of graphs are the squarefree monomial ideals. There ideals are associated to clutters. furthermore, we say that a clutter has the strong persistence property if its squarefree monomial ideal has the strong persistence property. There are some squarefree monomial ideals without the strong persistence property. In this chapter we find subfamilies and examples of squarefree monomial ideal with the strong persistence property and we give tools and results that permit to verify this property. In particular, we prove that a clutter has the strong persistence property if and only if any of its connected components has the strong persistence property. This result helps us to study the strong persistence property in a clutter from its connected components. Another examples of these results is: if $\mathcal{C}$ contains an edge f such that the set $\{g \cap f \mid g \in E(\mathcal{C})\}$ is a chain, then we show $I(\mathcal{C})$ has the strong persistence property. Also, if $|V(\mathcal{C})| \leq 4$ or $\mathcal{C}$ is an unmixed König clutter without 4-cycles, then we prove that $\mathcal{C}$ has the strong persistence property. In addition, we introduce the weight ideal I_w of a squarefree monomial ideal I, and we prove that I has the strong persistence property if and only if I_w has this property. This result permits to find non-squarefree monomial ideals that satisfy the persistence property. Finally, we introduce the concept of symbolic strong persistence as a tool for the study of asymptotic properties.

Let G be a simple graph, whose vertex set is $V = \{x_1, \dots, x_n\}$. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and we take $R' = K[t, x_1 t, \dots, x_n t]$ the subring of $R[t]$ where t is a new variable. Hence, the homogeneous monomial subring of

G is the ring $S_G = R'[\{x_i x_j t \mid \{x_i, x_j\} \in E(G)\}] \subseteq R[t]$. Thus, S_G is a standard k-algebra. A standard k-algebra is called Gorenstein if it is Cohen-Macaulay and its canonical module is a principal ideal. In [9] is proven that if G is bipartite, then S_G is Gorenstein if and only if G is unmixed. In Chapter 3 we prove that if S_G is normal, then S_G is Gorenstein if and only if G is unmixed and $\tau(G) = \lceil \frac{n}{2} \rceil$. This generalizes the result given in [9], since if G is bipartite, then G is normal. Furthermore, we prove that if n is even and S_G is Gorenstein, then G is bipartite.

A weighted oriented graph is a triple $D = (V, E, w)$ where $V = \{x_1, \ldots, x_n\}$, $E \subseteq V \times V$ and w is a function $w : V \to \mathbb{N}$. The underlying graph of D is the simple graph G whose vertex set is V and whose edge set is $\{\{x, y\} \mid (x, y) \in E\}$. In Chapter 4, we introduce the edge ideal $I(D)$ of D, given by $I(D) = (x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D))$ in $R = K[x_1, \ldots, x_n]$. We study the vertex covers of D. In particular, we introduce the notion of strong vertex cover. We characterize the irredundant irreducible decomposition of $I(D)$ and we show that each irreducible ideal of this decomposition is associated with a strong vertex cover of D. Furthermore, we prove that $I(D)$ is unmixed if and only if the underlying graph of D is unmixed and every strong vertex cover of D is minimal. When D is bipartite, D is a whisker of D is a cycle, we give an effective (combinatorial) characterization of the unmixed property of $I(D)$. Also, we study the Cohen-Macaulay property of $I(D)$. In particular, we show that unmixed property and Cohen-Macaulayness are equivalent when D is a path or D is complete and in both cases we give a combinatorial characterization of these properties. Finally, we give an example where Cohen-Macaulay property depend of the field K.

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CHAPTER 1

INTRODUCTION

1.1 MONOMIAL IDEALS

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. If $B \subseteq R$, in this thesis we denoted by $\mathrm{Mon}(B)$ the set of monomials of B. Given $E \subseteq [n]$, x_E denotes the squarefree monomial $\prod_{i \in E} x_i$ and x denotes the monomial $x_1 \cdots x_n = x_{[n]}$.

Definition 1.1 An ideal I of R is called **monomial** if it is generated by a set of monomials.

Definition 1.2 Given a polynomial $f = \sum_{m \in \mathrm{Mon}(R)} a_m m$ in R, the **support** of f is $\{m \in \mathrm{Mon}(R) \mid a_m \neq 0\}$ and it is denoted by $\mathrm{supp}(f)$.

Proposition 1.3 An ideal I of R is monomial if and only if $\mathrm{supp}(f) \subseteq I$ for each polynomial $f \in I$.

Proof. $\Rightarrow$) I is generated by a set of monomials $\mathcal{M}$. Now, given $f \in I$ we have that $f = f_1 g_1 + \ldots + f_k g_k$, where $g_i \in \mathcal{M}$ and $f_i \in R$ for $i = 1, \ldots, k$. Consequently, $\mathrm{supp}(f) \subseteq \cup_{i=1}^{k} \mathrm{supp}(f_i)\{g_i\}$, where $\mathrm{supp}(f_i)\{g_i\} = \{m g_i \mid m \in \mathrm{supp}(f_i)\}$. Therefore $\mathrm{supp}(f) \subseteq I$, since $\mathrm{supp}(f_i)\{g_i\} \subseteq I$.

$\Leftarrow$) We take $\mathcal{M} = \cup_{f \in I} \mathrm{supp}(f)$, then $\mathcal{M} \subseteq I$ and $(\mathcal{M}) \subseteq I$. Furthermore if $f \in I$, then $f = \sum_{m \in \mathrm{supp}(f)} a_m m$. Thus, $f \in (\mathcal{M})$. Hence $I = (\mathcal{M})$, therefore I monomial. □

Proposition 1.4 A monomial ideal is generated by a finite set of monomials.

Proof. Let I a monomial ideal. Since R is Noetherian, I is generated by a finite number of polynomials $f_1, \ldots, f_k$. Hence, I is generated by $\cup_{i=1}^{k} \mathrm{supp}(f_i)$. □

Remark 1.5 Let I be a monomial ideal generated by $\mathcal{M} \subseteq \text{Mon}(R)$, then a monomial $m \in I$ if and only if there is $v \in \mathcal{M}$ such that $v|m$.

Lemma 1.6 If $I \in \text{Mon}(R)$ and $\mathcal{M}$ is a set of monomial generators of I, then $\mathcal{M}$ is minimal (among the sets of monomial generators of I) if and only if any two distinct monomials of $\mathcal{M}$ are not divided.

Proof. $\Rightarrow$) Suppose $m, n \in \mathcal{M}$, $m \neq n$ such that $m|n$, then $\mathcal{M} \setminus \{n\}$ generates I. This contradicts the minimality of $\mathcal{M}$.

$\Leftarrow$) By Remark 1.5, each proper subset of $\mathcal{M}$ does not generate I. □

Proposition 1.7 If $I \in \text{Mon}(R)$, then I has a unique minimal monomial generating set.

Proof. Let G, H be minimal generating sets of I. If $u \in G$, then by Remark 1.5, $v|u$ for some $v \in H$. Furthermore $u'|v$ for some $u' \in G$, hence $u'|u$. Thus, by Lemma 1.6, $u = u'$ so $u = v$. Consequently, $G \subseteq H$. Similarly we obtain the other inclusion. □

Definition 1.8 We say that $\mathcal{M} \subseteq \text{Mon}(R)$ is a **minimal set of monomials** if it does not have divisibility relations.

Corollary 1.9 Each minimal set of monomials is finite.

Proof. If $\mathcal{M}$ is a minimal set of monomials, then $\mathcal{M}$ is the minimal monomial generating set of $I = (\mathcal{M})$. Hence, by Proposition 1.4, $\mathcal{M}$ is finite. □

We have a natural bijection between the set of monomial ideals of R and the collection of minimal sets of monomials given by $I \longmapsto G(I)$.

Definition 1.10 Given $\mathcal{M} \subseteq \text{Mon}(R)$, $\mathcal{M}^{\min}$ is the set of monomials v of $\mathcal{M}$ such that if $u \in \mathcal{M}$ and $u|v \Rightarrow u = v$. Also, $\mathcal{M}^{\max}$ is the set of monomials v of $\mathcal{M}$ such that if $u \in \mathcal{M}$ and $v|u \Rightarrow u = v$.

Remark 1.11 If $\mathcal{M} \subseteq \text{Mon}(R)$, then $(\mathcal{M}^{\min}) = (\mathcal{M})$.

Proof. Now, we take $v \in \mathcal{M} \subseteq \text{Mon}(R)$. Since $\{m \in \text{Mon}(R) \mid m|v\}$ is finite, there is a minimal u (in the sense of divisibility) in $\mathcal{M}$ such that $u|v$. Hence, $u \in \mathcal{M}^{\min}$ so $u \in (\mathcal{M}^{\min})$. Therefore, $(\mathcal{M}) = (\mathcal{M}^{\min})$.

Lemma 1.12 If $\mathcal{M} \subseteq \mathrm{Mon}(R)$, then $\mathcal{M}^{\min}$ and $\mathcal{M}^{\max}$ are finite.

Proof. $\mathcal{M}^{\min}$ and $\mathcal{M}^{\max}$ are minimal sets of monomials then by Corollary 1.9, they are finite. □

Corollary 1.13 Let $\mathcal{M} \subseteq \mathrm{Mon}(R)$ be an infinity set, then there exist a sequence $m_1, m_2, \ldots$ in $\mathcal{M}$ with $m_i \neq m_j$ if $i \neq j$ such that $m_i | m_{i+1}$ for each i.

Proof. By Lemma 1.12, $\mathcal{M}^{\max}$ is finite. Hence, $A = \{m \in \mathcal{M} \mid m|m' \text{ for some } m' \in \mathcal{M}^{\max}\}$ is finite. Consequently, $M = \mathcal{M} \setminus A$ is infinite. Since $\mathcal{M}^{\max} \subseteq A$, if $u \in M$, then there is $v \in \mathcal{M}$ such that $u \neq v$ and $u|v$. Furthermore, if $v \in A$, then there is $m' \in \mathcal{M}^{\max}$ such that $v|m'$. So $u|m'$, a contradiction. Therefore, $v \notin A$ implies $v \in \mathcal{M}$. □

Definition 1.14 A monomial ideal is called **squarefree**, if $G(I)$ consists of squarefree monomials (i.e., monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha_i \in \{0, 1\}$).

Definition 1.15 A **clutter** $\mathcal{C}$ is a pair (V, E) where V is a set and E consists of subsets of V such that $f \nsubseteq g$ and $g \nsubseteq f$ for each $f, g \in E$. The elements of V and E are called **vertices** and **edges**, respectively.

Definition 1.16 Let $\mathcal{C}$ be a clutter with vertex set $\{x_1, \ldots, x_n\}$. The **edge ideal** of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the squarefree monomial ideal $I(\mathcal{C}) = \big(\{x_{i_1} \cdots x_{i_s}\} \mid \{x_{i_1}, \ldots, x_{i_s}\} \in E(\mathcal{C})\}\big)$ in the polynomial ring $K[x_1, \ldots, x_n]$.

Remark 1.17 There is a bijection between squarefree monomial ideals in $K[x_1, \ldots, x_n]$ and clutters with vertices in $\{x_1, \ldots, x_n\}$, since inclusion in $\{x_1, \ldots, x_n\}$ implies divisibility in $K[x_1, \ldots, x_n]$.

Proposition 1.18 A monomial ideal P is prime if and only if $G(P) \subseteq \{x_1, \ldots, x_n\}$.

Proof. $\Rightarrow$) We take $u \in G(P)$. Since $u \in \mathrm{Mon}(R)$, there is $x_i \in \{x_1, \ldots, x_n\}$ such that $u = x_i v$ and $v \in \mathrm{mon}(R)$. Thus, $x_i \in P$ or $v \in P$ since P is prime. If $v \in P$, then there is $v' \in G(P)$ such that $v'|v$. Hence, $v'|u$ and $v' \neq u$. A contradiction by Lemma 1.6. Consequently, $x_i \in P$. So $x_i \in G(P)$, since P is prime. Therefore $u = x_i$, by Lemma 1.6.

$\Leftarrow$) Since $G(P) \subseteq \{x_1, \ldots, x_n\}$, $R/(G(I)) \cong K[x_i \mid x_i \notin G(I)]$. Thus, $R/(G(I))$ is a domain. Therefore $(G(I))$ is prime. □

Proposition 1.19 Let $\{I_\alpha\}_{\alpha\in\Phi}$ be a family of monomial ideals of R. Then, $\sum_{\alpha\in\phi} I_\alpha$ and $\cap_{\alpha\in\phi} I_\alpha$ are monomial ideals.

Proof. $\sum_{\alpha\in\Phi} I_\alpha$ is generated by $\cup_{\alpha\in\Phi} G(I_\alpha)$, then the sum is a monomial ideal. Now, if $f \in \cap_{\alpha\in\Phi} I_\alpha$, then $f \in I_\alpha$. Thus, $\text{supp}(f) \subseteq I_\alpha$ for each α. Hence, $\text{supp}(f) \subseteq \cap_{\alpha\in\Phi} I_\alpha$. Therefore, $\cap_{\alpha\in\phi} I_\alpha$ is a monomial ideal, by Proposition 1.3. □

Remark 1.20 If I, J are monomial ideals of R, then $I \cap J$ is generated by $B = \{\text{lcm}(u, v) \mid u \in G(I),\ v \in G(J)\}$.

Proof. If $m \in \beta$, then $m = \text{lcm}(u, v)$ for $u \in G(I)$ and $v \in G(J)$. Thus $u|m$ and $v|m$, so $m \in I \cap J$. Hence $B \subseteq I \cap J$. By Proposition 1.19, $I \cap J$ is monomial. Consequently, if $m' \in G(I \cap J) \subseteq I \cap J$, then there are $u' \in G(I)$ and $v' \in G(J)$ such that $u'|m'$ and $v'|m'$. Consequently $\gcd(u', v')|m'$. Therefore $I \cap J \subseteq (B)$. □

Lemma 1.21 If $I, J \in \text{Mon}(R)$, then $(I\colon J) = \cap_{u\in G(J)}(I\colon u)$.

Proof. If $m \in (I\colon J)$, then $mJ \in I$. In particular $mu \in I$ for $u \in G(J)$. So, $(I\colon J) \subseteq (I\colon u)$. Now, we take $m'' \in \cap_{u\in G(J)}(I\colon u)$. If $f \in J$, then $f = \sum_{m'\in\text{supp}(f)} a_{m'} m'$, By Proposition 1.3, if $m' \in \text{supp}(f)$, then $m' \in J$. Consequently, there is $u \in G(J)$ such that $u|m'$. Furthermore $m''u \in I$, so $m'' \cdot m' \in I$. This implies $m''f \in I$. Therefore $m'' \in (I\colon J)$ and $\cap_{u\in G(J)}(I\colon u) \subseteq (I\colon J)$. □

Definition 1.22 Let A be a ring. If I is an ideal and L a subset of A, then the **quotient ideal** of I by L, denoted by $(I\colon L)$, is the ideal $\{x \in A \mid xL \subseteq I\}$.

Proposition 1.23 If I and J are monomial ideals, then IJ and $(I\colon J)$ are also monomial ideals.

Proof. Since $\{uv \mid u \in G(I),\ v \in G(J)\}$ generated IJ, IJ is a monomial ideal. By Lemma 1.21 and by Proposition 1.19 it is sufficient to prove $(I\colon m) \in \text{Mon}(R)$ for each $m \in \text{Mon}(R)$. Now, if $f \in (I\colon m)$, then $\text{supp}(fm) = \{um \mid u \in \text{supp}(f)\}$. Since I is monomial, $um \in I$ for each $u \in \text{supp}(f)$. Consequently $\text{supp}(f) \subseteq (I\colon m)$. Therefore $(I\colon m)$ is monomial, by Proposition 1.3. □

Definition 1.24 Let $\leq$ be a total order on $\text{Mon}(R)$, we say that $\leq$ is a **monomial order** if

i) $1 \leq u\ \forall u \in \text{Mon}(R)$

ii) if $u \leq v$, then $um \leq vm$ for each $m \in \mathrm{Mon}(R)$.

Example 1.25 The **lexicographical order** $\leq_{\text{lex}}$ is the follow monomial order:

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq_{\text{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n} \text{if and only if } \alpha_j < \beta_j \text{ where } j = \min\{i \in [n] \mid \alpha_i \neq \beta_i\}.$$

Remark 1.26 If $\leq$ is a monomial order of $K[x_1, \ldots, x_n]$, then $\tilde{\leq}$ is a monomial order in $K[x_1, \ldots, x_n, x_{n+1}]$ given by $x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} \tilde{\leq} x_1^{b_1} \cdots x_{n+1}^{b_{n+1}}$ if $a_{n+1} < b_{n+1}$ or $a_{n+1} = b_{n+1}$ and $x_1^{a_1} \cdots x_n^{a_n} \leq x_1^{b_1} \cdots x_n^{b_n}$.

Example 1.27 Given a permutation σ of $[n]$, the σ-**lexicographical order** is the following monomial order $\leq_{\sigma-\text{lex}}$ given by $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq_{\sigma-\text{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n}$ if and only if $\alpha_{\sigma(j)} < \beta_{\sigma(j)}$ where $j = \min\{i \in [n] \mid \alpha_{\sigma(i)} \neq \beta_{\sigma(i)}\}$.

Remark 1.28 If $\mathrm{Div}(R) = \{(u, v) \in \mathrm{Mon}(R) \times \mathrm{Mon}(R) \mid u|v\}$ and $\leq$ is a monomial order in R, then $\mathrm{Div}(R) \subseteq \{(u, v) \in \mathrm{Mon}(R) \mid u \leq v\}$.

Lemma 1.29 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring with $n \geq 2$. If $u, v \in \mathrm{Mon}(R)$ with $(u, v), (v, u) \notin \mathrm{Div}(R)$, then there is a monomial order $\leq$ on R such that $u \leq v$.

Proof. We can suppose $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{n_n}$. Thus, there are $i, j \in [n]$ such that $a_i < b_i$ and $b_j < a_j$, since (u, v), $(v, u) \notin \mathrm{Div}(R)$. We take σ a permutation of $[n]$ such that $\sigma(1) = i$. Hence, $u <_{\sigma-\text{lex}} v$. □

Proposition 1.30 $\mathrm{Div}(R)$ is the intersection of all monomial orders over R.

Proof. Let $\{\leq_\alpha\}_{\alpha \in \Lambda}$ be the family of every monomial orders over R. By Remark 1.28, $\mathrm{Div}(R) \subseteq \cap_{\alpha \in \Lambda} \leq_\alpha$. Now if (u, v), $(v, u) \notin \mathrm{Div}(R)$, then by Lemma 1.29 there exist monomial orders $\leq_1$ and $\leq_2$, such that $u \leq_1 v$ and $v \leq_2 u$. Hence, (u, v), $(v, u) \notin \cap_{\alpha \in \Lambda} \leq_\alpha$. □

Proposition 1.31 Let $\leq$ be a monomial order over R, then each descendent chain $m_1 > m_2 > \cdots$ in $\mathrm{Mon}(R)$ is finite.

Proof. Since $T = \big(\{m_i \mid i \in \mathbb{N}\}\big)^{\min}$ is finite. We take $k = \min\{i \in \mathbb{N} \mid m_i \in T\}$. If there is m_{k+1}, then $m_{k+1} \notin T$. So, there is $m_i \in T$ such that $m_i | m_{k+1}$. Thus, $m_i < m_{k+1}$, implies $i > k + 1$. A contradiction, since $m_i \in T$. Therefore, the descendent chain $m_1 > m_2 > \cdots$ has k elements. □

Definition 1.32 If $\leq$ is a monomial order and $A \subseteq \mathrm{Mon}(R)$, then $\mathbf{max}_{\leq}(A) = \{m \in A \mid m \geq n \text{ for each } n \in A\}$ and $\mathbf{min}_{\leq}(A) = \{m \in A \mid m \leq n \text{ for each } n \in A\}$.

Proposition 1.33 Let I be a monomial ideal of R, then $\mathrm{rad}(I)$ is monomial.

Proof. If $f \in \mathrm{rad}(I)$, then $f^k \in I$ for some k. If $\{u\} = \max(\mathrm{supp}(f))$, then $u^k \in \mathrm{supp}(f^k)$. Since I is monomial, $u^k \in I$. Hence, $u \in \mathrm{rad}(I)$ and $f_1 = f - a_u u \in \mathrm{rad}(I)$, where a_u is the coefficient of u in f. If we continue with this process, then we obtain that $\mathrm{supp}(f) \subseteq \mathrm{rad}(I)$. Therefore, $\mathrm{rad}(I)$ is monomial by Proposition 1.3. □

Remark 1.34 If $u \in \mathrm{Mon}(R)$, then $\mathrm{lib}(u) = \prod_{x_i \mid u} x_i$. Furthermore, if I is a monomial ideal, then $\mathrm{rad}(I)$ is generated by $\{\mathrm{lib}(u) \mid u \in G(I)\}$.

Lemma 1.35 If $f \in R$ such that $|\mathrm{supp}(f)| \geq 2$, then $|\mathrm{supp}(fg)| \geq 2$ for each $g \in R$.

Proof. We take a monomial order $\leq$. If $\max(\mathrm{supp}(f)) = \{u\}$ and $\min(\mathrm{supp}(f)) = \{v\}$, then $u > v$, since $|\mathrm{supp}(f)| \geq 2$. We take $\{g_1\} = \max(\mathrm{supp}(g))$ and $\{g_2\} = \min(\mathrm{supp}(g))$. Thus, $vg_2 < mn \leq ug_1$ for $(m, n) \in \mathrm{supp}(f) \times \mathrm{supp}(g) \setminus \{(v, g_2), (u, g_1)\}$. Therefore $|\mathrm{supp}(fg)| \geq 2$. □

Remark 1.36 An element a of a ring is said **squarefree** if $b^2 | a$ implies b is a unit. By Lemma 1.35, a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is squarefree if and only if $\alpha_i \in \{0, 1\}$. In particular, we recover Definition 1.14.

Example 1.37 If $J = \langle x_1x_2 + x_1x_3, x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2 \rangle$, then $\mathrm{rad}(J) = \langle x_1x_2, x_2x_3, x_1x_3 \rangle$. Observe $\mathrm{rad}(J)$ is a monomial ideal and J is not a monomial ideal.

Proposition 1.38 If $I, J \in \mathrm{Mon}(R)$, then $I \neq IJ$ and $J \neq IJ$.

Proof. We take a monomial order $\leq$. Furthermore, we take $\{u\} = \min\big(\mathrm{Mon}(I)\big)$ and $\{v\} = \min\big(\mathrm{Mon}(J)\big)$. If $m \in \mathrm{Mon}(IJ)$, then $m = u_1v_1$ with $u_1 \in \mathrm{Mon}(I)$ and $v_1 \in \mathrm{Mon}(J)$. Thus, $u_1 \geq u$ so $m > u$, since $v > 1$. Hence, $u \in I \setminus IJ$. Similarly $v \in J \setminus IJ$, $I \neq IJ$ and $J \neq IJ$. So, $u \in I \setminus IJ$ and $v \in J \setminus IJ$. □

Definition 1.39 Let I be an ideal of a ring A and $a \in A$ we say that a is **integer** over I if there is a polynomial $h = x^k + a_1x^{k-1} + \cdots + a_k \in A[x]$ where $a_i \in I^i$ such that $h(a) = 0$. The set of all elements of A integers on I is called the **integer closure** of I and denoted by $\overline{I}$.

Remark 1.40 If $a \in I$, then $h(a) = 0$, where $h(x) = x - a$. Consequently $I \subseteq \overline{I}$. Furthermore, if $a \in \overline{I}$, then there is $f(x) \in A[x]$ such that $f(a) = a^k + a_1a^{k-1} + \cdots + a_k = 0$. Consequently $a^k = -a_1a^{k-1} - \cdots - a_k \in I$. Hence $\overline{I} \subseteq \mathrm{rad}(I)$.

Proposition 1.41 If I is an ideal of a ring A, then $\overline{I}$ is an ideal of A.

Proof. By Lemma 8.2.2 and Proposition 8.2.3 in [ ], $\overline{I}$ is a subgroup. Now, we take $a \in A$ and $b \in \overline{I}$, then there is $g(x) = x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots + a_k$ with $a_i \in I^i$ such that $g(b) = 0$. Thus, if $h(x) = x^k + aa_1x^{k-1} + a^2a_2x^{k-2} + \cdots + a^ka_k$, then $a^ia_i \in I^i$ and $h(ab) = 0$. Therefore, $ab \in \overline{I}$. □

Proposition 1.42 If I is a monomial ideal of R, then $\overline{I}$ is a monomial ideal generated by $\{u \in \mathrm{Mon}(R) \mid u^k \in I^k\}$.

Proof. See [12, Theorem 1.4.2]. □

Definition 1.43 Given a monomial ideal I, we define the **Newton's polyhedron** of I as the intersection of the convex closure in $\mathbb{R}^n$ of the set $\{v \in \mathbb{Z}^n_+ \mid x^v \in I\}$. The Newton's polyhedron is denoted by $N(I)$.

Remark 1.44 If P is a prime ideal of a ring A, then $\overline{P} = P$.

Definition 1.45 Let A be a commutative ring, we say that an ideal I of A is **normal** if $I^i = \overline{I^i}$ for each i.

1.2 PRIMARY DECOMPOSITION

Definition 1.46 An ideal I of a ring A is **primary** if for each $a, b \in A$ such that $ab \in I$ implies $a \in I$ or $b^k \in I$ for some k.

Proposition 1.47 If I is primary, then $\mathrm{rad}(I)$ is prime.

Proof. If $ab \in \text{rad}(I) = P$, then $(ab)^k \in I$ for some k. Thus, $a^k b^k \in I$ consequently $a^k \in I$ or $b^{kt} \in I$ for every t. Therefore $a \in \text{rad}(I)$ or $b \in \text{rad}(I)$. □

Example 1.48 The converse affirmation of the above proposition is not true. If $I = (x_1^2, x_1 x_2)$, then $\text{rad}(I) = (x_1)$. Consequently $\text{rad}(I)$ is prime. But I is not primary.

Proposition 1.49 If $\text{rad}(I)$ is maximal, then I is primary.

Proof. If $M = \text{rad}(I)$, then M is the unique prime ideal containing I, and A/I is local with maximal ideal M/I. So, if $b \in A \setminus M$, then $\bar{b} \in A/I$ is a unit. Let $a, b \in A$ such that $ab \in I$, if $b \notin M$, then I is primary. □

Definition 1.50 A primary ideal I is P-**primary** if $\text{rad}(I) = P$ and P is prime.

Proposition 1.51 The intersection of a finite family of P-primary ideals is P-primary.

Proof. Let $I_1, \ldots, I_k$ be P-primary ideals. Now, we take $a, b \in A$ such that $ab \in \cap_{i=1}^k I_i \subseteq P$. If $a \notin I_j$ for some j, then $b^s \in I_j$ for some s. Thus, $b \in P$. So, then there is r_i such that $b^{r_i} \in I_i$ for each i, since $P = \text{rad}(I_i)$. We take $r = \max\{r_1, \ldots, r_k\}$, then $b^k \in \cap_{i=1}^k I_i$. Hence, $\cap_{i=1}^k I_i$ is primary. Furthermore, $\text{rad}(\cap_{i=1}^k I_i) \subseteq P$. Now, if $a \in P$, then there is u_i such that $a^{u_i} \in I_i$ for each i. If $u = \max\{u_1, \ldots, u_k\}$, then $a^u \in \cap_{i=1}^k I_i$ implies $a \in \text{rad}(\cap_{i=1}^k I_i)$. Therefore $P = \text{rad}(\cap_{i=1}^k I_i)$. □

Definition 1.52 Given I an ideal, we say that a collection of primary ideals $\{Q_1, \ldots, Q_k\}$ is a **primary decomposition** of I if $I = \cap_{i=1}^k Q_i$.

Lemma 1.53 Let $I_1, \ldots, I_k$ be ideals and P a prime ideal such that $\cap_{i=1}^k I_i \subseteq P$, then $I_j \subseteq P$ for some j.

Proof. By induction over k. For $k = 2$, we take I_1, I_2 ideals such that $I_1 \cap I_2 \subseteq P$. If $I_1 \nsubseteq P$, then there is $a \in I_1 \setminus P$. Furthermore, if $b \in I_2$, then $ab \in I_1 \cap I_2 \subseteq P$. Consequently, $b \in P$ implies $I_2 \subseteq P$. Now, we take $I_1, \ldots, I_k, I_{k+1}$ ideals such that $\cap_{i=1}^{k+1} I_i \subseteq P$. If $J = \cap_{i=2}^{t+1} I_i$, then $I_1 \cap J \subseteq P$. Consequently $I_1 \subseteq P$ or $J \subseteq P$. In the second case, by induction hypothesis, there is $I_i \subseteq P$ for $2 \leq i \leq k+1$. □

Corollary 1.54 If P is a prime ideal such that $P = \cap_{i=1}^k I_i$, where I_i is an ideal, then $P = I_j$ for some j.

Proof. By Lemma 1.53, $I_j \subseteq P$ for some j. Furthermore, $P \subseteq \cap_{i=1}^{k} I_i \subseteq I_j$, then $P = I_j$. □

Definition 1.55 A primary decomposition $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ of an ideal I is **minimal** if no proper subset of $\mathcal{Q}$ is a primary decomposition of I.

Lemma 1.56 A primary decomposition $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ of I is minimal if and only if $\cap_{i \in S} Q_i \subset Q_j$ for every $S \subsetneq [k]$ with $j \notin S$.

Proof. $\Rightarrow$) If $\cap_{i \in S} Q_i \subseteq Q_j$, then $\mathcal{Q} \setminus \{Q_j\}$ is a decomposition primary of I, a contradiction.

$\Leftarrow$) If $\mathcal{Q}$ is not minimal then there is $S \subsetneq [k]$ such that $\cap_{i \in S} Q_i = I = \cap_{j=1}^{k} Q_j \subseteq Q_j$ for $j \notin S$. □

Corollary 1.57 If $\{Q_1, \dots, Q_k\}$ is a minimal primary decomposition of I. Then for any $S \subseteq [k]$ we have $\{Q_i \mid i \in S\}$ is a minimal primary decomposition of $J = \cap_{i \in S} Q_i$.

Proof. If $S' \subseteq S$ and $j \in S \setminus S'$, then by Lemma 1.56 $\cap_{i \in S'} Q_i \subsetneq Q_j$. Hence, $\{Q_i \mid i \in S\}$ is a monomial primary decomposition of J. □

Proposition 1.58 Let A be a ring, with $\mathcal{Q}$ a minimal primary decomposition of I. Hence, $P = \mathrm{rad}(Q)$ with $Q \in \mathcal{Q}$ if and only if $P = (I \colon a)$ for some $a \in A$.

Proof. See [1, Theorem 4.5]. □

Corollary 1.59 Let $\mathcal{Q}$ and $\mathcal{Q}'$ minimal primary decompositions of I. Then

$$\{\mathrm{rad}(Q) \mid Q \in \mathcal{Q}\} = \{\mathrm{rad}(Q') \mid Q' \in \mathcal{Q}'\}.$$

Proof. By Proposition 1.58. □

Let A be a ring. If M is a A-module and $x \in M$, then $\mathrm{ann}(x) = \{a \in A \mid a \cdot x = 0\}$. A prime ideal P of A is an **associated prime** of M if exist $x \in M$ such that $P = \mathrm{ann}(x)$. The set of associated primes M is denoted by $\mathrm{Ass}(M)$. If I is an ideal of A, then A/I is an A-module. Furthermore $\mathrm{ann}(\overline{x}) = (I \colon x)$. Also $\mathrm{Ass}(A/I)$ is denoted by $\mathrm{Ass}(I)$.

Corollary 1.60 If Q is a minimal primary decomposition of I, then $\{\mathrm{rad}(Q) \mid Q \in$

$\mathcal{Q}\} = \text{Ass}(A/I)$.

Proof. By Proposition 1.58. □

Corollary 1.61 Let I be an ideal with a primary decomposition and P a prime ideal such that $I \subseteq P$. Then P is minimal prime containing I if and only if P is minimal in $\text{Ass}(I)$.

Proof. Suppose that P is minimal containing I and $\mathcal{Q}$ is a minimal primary decomposition of I. By Lemma 1.53, there is some ideal $T \in \mathcal{Q}$ such that $T \subseteq P$. Thus, $I \subseteq \text{rad}(T) \subseteq P$ and $\text{rad}(T)$ is prime so, $\text{rad}(T) = P$. Hence P is minimal of $\text{Ass}(I)$. Now, if P is minimal of $\text{Ass}(I)$ and Q is a minimal prime such that $I \subseteq Q \subseteq P$, then $Q \in \text{Ass}(I)$. Consequently $Q = P$, so P is a minimal prime of containing I. □

Remark 1.62 In a Noetherian ring, the radical of an ideal is the intersection of its associated primes.

Definition 1.63 An ideal I is **irreducible** if $I = J \cap L$ for some ideals I and J, then $J = I$ or $L = I$. Furthermore, I is an **irreducible monomial ideal** if $I = J \cap L$ with J and L monomial ideals, then $J = I$ or $L = R$.

Proposition 1.64 In a Noetherian ring each irreducible ideal is primary.

Proof. See [1, Lemma 7.12]. □

Proposition 1.65 Let A be a Noetherian ring, then each ideal I of A has a primary decomposition consisting by irreducible ideals.

Proof. See [1, Lemma 7.11]. □

Proposition 1.66 An ideal is primary if and only if has a single associated prime.

Proof. By Corollary 1.59. □

Remark 1.67 By Corollary 1.54, every prime ideal is irreducible.

Example 1.68 If $I = \langle x_1^4x_2^3, x_1^3x_2^4\rangle$, then I is a monomial ideal and $I = \langle x_1^6x_2 + x_1^5x_2^2, x_1^4x_2^3, x_1^3x_2^4\rangle \cap \langle x_1^3x^4x_2, x_1^4x^3x_2, x_1^5 + x_1x_22x_2^6\rangle$. Hence I is the intersection of two non-monomial ideals.

Proposition 1.69 A monomial ideal I is irreducible-monomial if and only if $G(I)$ consists of powers of variables.

Proof. $\Rightarrow$) Assume there is $u \in G(I)$ such that $x_i x_j | u$ for some $i, j \in [n]$ and $i \neq j$. If $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then $I = (G(I) \cup \{x_i^{\alpha_i}\}) \cap (G(I) \cup \{\frac{u}{x_i^{\alpha_i}}\})$. Consequently I is not irreducible, since $x_i^{\alpha_i}, \frac{u}{x_i^{\alpha_i}} \notin I$.

$\Leftarrow$) Let J, H be monomial ideals such that $J \cap H = I$. Suppose $J \neq I$, then there is $u \in G(J)$ such that $x_i^{\alpha_i} \nmid u$ for each $x_i^{\alpha_i} \in G(I)$. On the other hand, $\operatorname{lcm}(u,v) \in J \cap H = I$ for each $v \in H$. Thus, $x_j^{\alpha_j} \mid \operatorname{lcm}(u,v)$ for some $x_j^{\alpha_j} \in G(I)$. So, $\alpha_j \leq \max\{u_j, v_j\}$, where $u = x_1^{u_1} \cdots x_n^{u_n}$ and $v = x_1^{v_1} \cdots x_n^{v_n}$. Since $x^{\alpha_j} \nmid u$, $u_j < \alpha_j$. Hence, $\alpha_j = v_j$ and $v \in I$. Therefore $H = I$, so I is irreducible. □

Corollary 1.70 Let I be a monomial ideal. Then the following conditions are equivalent:

1) I is primary.

2) If $x_i | m$ for some $m \in G(I)$, then there is k such that $x_i^k \in G(I)$.

Proof. 1) $\Rightarrow$ 2) We assume $m = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_m}^{\alpha_{i_m}}$. If $m = m' x_{i_j}^{\alpha_j}$ for some $j \in [m]$, then $m' \in I$, since $m \in G(I)$. Thus, $(x_{i_j}^{\alpha_j})^r \in I$, for some r. Therefore $x_{i_j}^k \in G(I)$ for some k.

2) $\Rightarrow$ 1) We consider $A = \{x_i \mid x_i^s \in G(I) \text{ for some } s\}$ and $A' = \{x_1, \ldots, x_n\} \setminus A$. We take $f, g \in R$ such that $fg \in I$ and $f \notin I$. If $\operatorname{supp}(f)$ is minimal, then $\operatorname{supp}(f) \cap I = \emptyset$, since in otherwise $f = f_1 + f_2$ where $\operatorname{supp}(f_1) \cap I = \emptyset$, $\operatorname{supp}(f_2) \subseteq I$, $f_1 g = fg - f_2 g \in I$ and $f_1 \in I$. If $g \notin \operatorname{rad}(I)$, then there is $u \in \operatorname{supp}(g)$ such that $x_i \nmid u$ for each $x_i \in A$. Now, we take a monomial order $\leq$ and $v = \min_{\leq}\{u \in \operatorname{supp}(g) \mid u \notin \operatorname{rad}(I)\}$, then $vm \in I$ where $m = \min_{\leq} \operatorname{supp}(f)$. Thus $h | vm$ for some $h \in G(I)$. Furthermore $\gcd(h, v) = 1$, so $h | m$, a contradiction. Hence $g \in \operatorname{rad}(I)$ so, $g^k \in I$ for some k. Therefore I is primary. □

Example 1.71 The ideal $(x_1^2, x_1 x_2, x_2^2)$ is a monomial primary but it is not an irreducible-monomial ideal.

Corollary 1.72 If I is an irreducible-monomial ideal, then I is primary.

Proof. By Proposition 1.69, $G(I)$ consists of powers of variables. Therefore I is primary, by Corollary 1.70. □

Example 1.73 By Corollary 1.70, (x_1) and (x_1^2, x_1, x_2, x_2^k) are primary ideal for each k. Furthermore $(x_1^2, x_1x_2) = (x_1) \cap (x_1^2, x_1x_2, x_2^k)$, then (x_1^2, x_1x_2) has an infinite number of primary decompositions consisting of monomial ideals.

Proposition 1.74 If I is a monomial ideal, then I has a minimal primary decomposition formed by irreducible-monomial ideals.

Proof. We suppose $G(I) = \{u_1, \ldots, u_k\}$ where $u_i = x_1^{a_{i_1}} \cdots x_n^{a_{i_n}}$. We take $U_i = \{x_i^{a_{i_j}} \mid a_{i_j} \neq 0\}$ for each $i \in [k]$. Now, if $e = (m_1, \ldots, m_k) \in \prod_{i=1}^k U_i$, then we define the monomial ideal $I_e = (m_1, \ldots, m_k)$. I_e is an irreducible-monomial and $I = \cap_{e\in\Omega} I_e$ with $\Omega = \prod_{i=1}^k U_i$. Therefore I has a minimal primary decomposition. □

Definition 1.75 An irreducible-monomial ideal J is I-**minimal** if J is minimal in the set of irreducible-monomial ideals that contains I.

Proposition 1.76 If I is a monomial ideal and Ω is the set of the I-minimal ideal, then $I = \cap_{J\in\Omega} J$.

Proof. If $J \in \Omega$, then $I \subseteq J$. Hence, $I \subseteq \cap_{J\in\Omega} J$. Now, we consider $G(I) = \{m_1, \ldots, m_r\}$. If $m \notin I$, then $m_i \nmid m$ for each $1 \leq i \leq r$. Thus, for each i there is $\alpha_{j_i}^i$ such that $\alpha_{j_i}^i > \beta_{j_i}$ where $m_i = x_1^{\alpha_1^i} \cdots x_n^{\alpha_n^i}$. We take $A = \{x_{j_1}^{\alpha_{j_1}}, \ldots, x_{j_y}^{\alpha_{j_r}}\}$ and $\mathcal{A} = \{B \subseteq A \mid I \subseteq (B)\}$. If D is a minimal set in $\mathcal{A}$, then $J = (D)$ is I-minimal. Therefore $m \notin \cap_{J\in\Omega} J$. □

Remark 1.77 A minimal irreducible decomposition of a monomial ideal is unique, see [12, Theorem 1.3.1].

Proposition 1.78 If A is a finite set of irreducible-monomial ideals without contention relation, then A is a minimal primary decomposition of $\cap_{I\in A} I$.

Proof. We consider $A = \{I_1, \ldots, I_r\}$, then $\cap_{i=1}^r I_i$ is a primary decomposition, since I_i is irreducible for $1 \leq i \leq r$. Now, $I_i \nsubseteq I_j$ for each $i \neq j$, then there exist $m_i \in G(I_i)$ such that $m_i \notin I_j$. Thus, if $m = \operatorname{lcm}(m_1, \ldots, m_{j-1}, m_{j+1}, \ldots)$, then $m \in \cap_{i\neq j} I_i$ and $m \notin I_j$. Hence, $\cap_{i\neq j} I_i \nsubseteq I_j$ for each j. □

Corollary 1.79 The set of I-minimal ideals is a minimal primary decomposition of I.

Corollary 1.80 Each associated prime ideal of a monomial ideal is a monomial ideal.

Proof. By Proposition 1.74, Proposition 1.33 and Corollary 1.60. □

Corollary 1.81 If P is an associated prime of a monomial ideal I, then there is $u \in \mathrm{Mon}(R)$ such that $P = (I\colon u)$.

Proof. Since P is an associated prime of I, $P = (I\colon f)$ for some $f \in R$. Since $(I\colon f) = \cap_{u\in \mathrm{supp}(f)}(I\colon u)$ and by Corollary 1.54, $P = (I\colon u)$ for some $u \in \mathrm{supp}(f)$. □

Definition 1.82 Let I be a monomial ideal of R, I has the **persistence property** if $\mathrm{Ass}(A/I^k) \subseteq \mathrm{Ass}(A/I^{k+1})$ for all k.

Definition 1.83 Let $G = (V, E)$ be a graph. If $f \in E(G)$, then we take the monomial $\widetilde{f} = x_1 \cdots x_j$ where $f = \{x_1, \ldots, x_j\}$. In this context the **edge ideal** of G is the ideal $I(G) = \big(\widetilde{f} \mid f \in E(G)\big)$.

Lemma 1.84 Let I be the edge ideal of a graph G. If P is a monomial ideal, then P is a prime ideal containing I if and only if $G(P)$ is a vertex cover of G.

Proof. $\Rightarrow$) We assume $E(G) = \{f_1, \ldots, f_k\}$. Since $I \subseteq P$, for each $1 \le i \le k$ there is $x_{j_i} \in G(P)$ such that $x_{j_i} | \widetilde{f_i}$. So $\{x_{j_i}\} = x_{j_i} \cap f_i \subseteq G(P) \cap f_i$. Hence, $G(P)$ is a vertex cover of G.

$\Leftarrow$) Since $G(P)$ is a vertex cover, $G(P) \subseteq \{x_1, \ldots, x_n\}$. Thus, P is a prime ideal. Furthermore, if $g \in E(G)$, then there is $x_i \in G(P)$ such that $x_i | \widetilde{g}$ so, $x_i \in g$. Therefore, $G(P)$ is a vertex cover of G. □

Corollary 1.85 Let I be the edge ideal of a graph G, then P is an associate prime of I if and only if $G(P)$ is a minimal vertex cover of G.

Proposition 1.86 Let G be a graph and $I(G)$ the edge ideal, then $I(G)$ is normal if and only if any two odd cycles in G can be joined by an edge of G.

Proof. See [28, p.322]. □

CHAPTER 2

ON THE STRONG PERSISTENCE PROPERTY FOR MONOMIAL IDEALS

2.1 INTRODUCTION

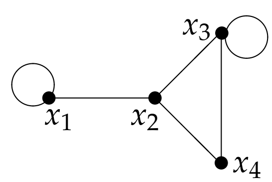
Let R be a commutative Noetherian ring. The **associated primes set** of an ideal I is $\text{Ass}(I) = \{P \in \text{Spec}(R) \mid P = (I\colon a) \text{ for some } a \in R\}$. If $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I, then $\text{Ass}(I) = \{\text{rad}(Q_1), \ldots, \text{rad}(Q_s)\}$ where $\text{rad}(Q_i)$ is the radical of Q_i. I has the **persistence property** if $\text{Ass}(I^k) \subseteq \text{Ass}(I^{k+1})$ for each k. In [18] is showed that the edge ideal of a simple graph has the persistence property, and they use that these edge ideals satisfy $(I^{k+1}\colon I) = I^k$ for each k. Recently was proved that this concept implies the persistence property (see [13]) and it is called the **strong persistence property**. These concepts are not equivalent, in [18, Example 2.18] is given a squarefree monomial ideal with the persistence property, but it does not have the strong persistence property. Assuming this terminology, in [18, Lemma 2.12] was proved that the edge ideal of a simple graph has the strong persistence property. In this chapter we study the strong persistence property for edge ideals of graphs with loops, weighted graphs, and clutters.

This chapter is organized as follow: in Sect. 2.2 we prove the edge ideals of graphs with loops have the strong persistence property. In Sect. 2.3 we prove that the edge ideal of a vertex–weighted graph (G, w) has the strong persistence property. Furthermore, we prove that $I(G)^k$ and $I(G, w)^k$ have the same associated primes. In Sect. 2.4 we study the edge ideals of clutters. In particular, we show that a clutter has the strong persistence property if and only if at less one of its connected component has the strong persistence property. Also, we prove that a König unmixed clutter without 4-cycles and squarefree monomials in four variables have the strong persistence property. Furthermore, we show that $(I^2\colon I) = I$, if I is a squarefree monomial ideal. Finally we prove that the strong persistence property is closed under c-minor and cones. In Sect. 2.5 we give some properties of the strong persistence property. Also, we introduce the symbolic persistence property and we show that an ideal has this property if it has the strong persistence property.

2.2 Graphs with loops

A **graph with loops** is a triplet $\mathcal{G} = (V, E, L)$ where $G = (V, E)$ is a simple graph with $V = \{x_1, \ldots, x_n\}$ and $L \subseteq \big\{(x_i, x_i) \mid x_i \in V\big\}$, L is called the **set of loops** of $\mathcal{G}$. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, as usual we use x^a as abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n)$ is an integer vector with $a_i \geq 0$. If $f = \{x_i, x_j\} \in E$ or $f = (x_i, x_i) \in L$, then we take the monomial $\widetilde{f} = x_i x_j$ or $\widetilde{f} = x_i^2$, respectively. The **edge ideal** of a graph with loops $\mathcal{G} = (V, E, L)$ is the ideal $I(\mathcal{G}) = \big(\{f_i \mid f_i \in E \cup L\}\big) = I(G) + \big(\{x_i^2 \mid (x_i, x_i) \in L\}\big)$ where $I(G) = \big(\{x_i x_j \mid \{x_i, x_j\} \in E\}\big)$ is the edge ideal of $G = (V, E)$.

Example 2.1 Graph with loops, where $L = \big\{(x_1, x_1),\ (x_3, x_3)\big\}$.



For an integer vector $a = (a_1, \ldots, a_n)$ with $a_i \geq 0$, we define the simple graph $\mathcal{G}^a$ with vertex set is $V^a = \{x_1^1, \ldots, x_1^{a_1}, \ldots, x_i^1, \ldots, x_i^{a_i}, \ldots, x_n^1, \ldots, x_n^{a_n}\}$, and whose edge set

$$E^a = \left\{ \{x_i^{k_i}, x_j^{k_j}\} \mid \begin{array}{c} \{x_i, x_j\} \in E, k_i \leq a_i, \text{and } k_j \leq a_j; \text{or} \\ (x_i, x_j) \in L \text{ and } 1 \leq k_i < k_j \leq a_i \end{array} \right\},$$

where j in x_i^j is only an index. Furthermore, if $x_i \in V(\mathcal{G})$, then we define the **duplication** of x_i in $\mathcal{G}^a$ as the simple graph $\mathcal{G}^{ax_i} = \mathcal{G}^{a+e_i}$ where e_i is the i-th unit vector in $\mathbb{R}^n$. This operation is commutative, that is $(\mathcal{G}^{ax_i})^{x_j} = (\mathcal{G}^{ax_j})^{x_i}$ for each $x_i, x_j \in V$. Furthermore, if $f = \{x_i, x_j\} \in E$, then we denote by $(\mathcal{G}^a)^f = \mathcal{G}^{a+e_i+e_j}$; and if $f = (x_i, x_i) \in L$, then $(\mathcal{G}^a)^f = \mathcal{G}^{a+e_i+e_i}$.

Definition 2.2 Let G a simple graph. A **matching** of G is a set of pairwise disjoint edges. The **matching number** of G, denoted by $\nu(G)$, is the size of any maximum matching of G. A matching that covers all vertices of $V(G)$ is called a **perfect matching** of G.

Notation: $\mathrm{Mon}(R)$ is the set of monomials in $R = K[x_1, \ldots, x_n]$. If $I = (m_1, \ldots, m_s)$ with $m_i \in \mathrm{Mon}(R)$, then $G(I)$ is the minimal monomial generating set of I.

Proposition 2.3 Let $\mathcal{G}$ be a graph with loops whose vertex set is $V = \{x_1, \ldots, x_n\}$. If $a = (a_1, \ldots, a_n)$ is an integer vector where $a_i \geq 0$, then $\mathcal{G}^a$ has a matching of size l if and only if $x^a \in I(\mathcal{G})^l$.

Proof. $\Rightarrow$) Let $P = \{g_1, \ldots, g_l\}$ be a matching of $\mathcal{G}^a$ where $g_j = \{x_{i_j}^{k_{i_j}}, x_{r_j}^{s_{r_j}}\}$. Now, we consider the monomial $x^b = \prod_{j=1}^{l} x_{i_j} x_{r_j} \in I(\mathcal{G})^l$. If $b = (b_1, \ldots, b_n)$, then $b_i = \left|\{r \mid x_i^r \in \bigcup_{j=1}^{l} g_j\}\right|$ for each $1 \leq i \leq n$. Since P is a matching, $b_i \leq a_i$. Therefore, $x^b \mid x^a$ and $x^a \in I(\mathcal{G})^l$.

$\Leftarrow$) We take $E \cup L = \{f_1, \ldots, f_q\}$. If $x^a \in I(\mathcal{G})^l$, then there exist an integer vector $\alpha = (\alpha_1, \ldots, \alpha_q)$ such that $\alpha_1 + \cdots + \alpha_q = l$ and $x^a = m\widetilde{f}_1^{\alpha_1} \cdots \widetilde{f}_q^{\alpha_q}$ with $m \in \mathrm{Mon}(R)$. We can assume that $\alpha_1 > 0$. If $f_1 = \{x_r, x_s\} \in E$, then $\alpha_1 \leq a_r$ and $\alpha_1 \leq a_s$ since $\widetilde{f}_1^{\alpha_1} \mid x^a$. If $P_1 = \{g_1, \ldots, g_{\alpha_1}\}$, where $g_j = \{x_r^{a_r - j + 1}, x_s^{a_s - j + 1}\}$ for $j \leq \alpha_1$, then P_1 is a matching of $\mathcal{G}^a$ of size α_1. If $f_1 = (x_r, x_r) \in L$, then $2\alpha_1 \leq a_r$ since $\widetilde{f}_1^{\alpha_1} \mid x^a$. Consequently, $P_1 = \{g_1, \ldots, g_{\alpha_1}\}$, where $g_j = \{x_r^{a_r - 2j + 2}, x_r^{a_r - 2j + 1}\}$ for $j \leq \alpha_1$, is a matching of $\mathcal{G}^a$ of size α_1. Hence,

$$\mathcal{G}^b = \mathcal{G}^a \setminus \bigcup_{j=1}^{\alpha_1} g_j \text{ and } x^b = \frac{x^a}{\widetilde{f}_1^{\alpha_1}} = m\widetilde{f}_2^{\alpha_2} \cdots \widetilde{f}_q^{\alpha_q} \in I(\mathcal{G})^{l - \alpha_1},$$

where $b = a - \alpha_1(e_r + e_s)$ if $f_1 \in E$ or $b = a - 2\alpha_1 e_r$ if $f_1 \in L$. Following with the processes, we obtain matchings $P_1, \ldots, P_q$ such that

$$V(P_{i+1}) \cap \Big(\bigcup_{j=1}^{i} V(P_j)\Big) = \emptyset \text{ since } V(P_{i+1}) \subseteq V(\mathcal{G}^a) \setminus \bigcup_{j=1}^{i} V(P_j).$$

Therefore, $\cup_{j=1}^{q} P_j$ is a matching of $\mathcal{G}^a$ of size l. □

Corollary 2.4 $x^a \in I(\mathcal{G})^k \setminus I(\mathcal{G})^{k+1}$ if and only if $k = \nu(\mathcal{G}^a)$.

Definition 2.5 The **deficiency** of a simple graph G is given by

$$\mathrm{def}(G) = |V(G)| - 2\nu(G).$$

Theorem 2.6 ([18]) If G is a simple graph, then

$$\mathrm{def}(G) = \max\{c_0(G \setminus S) - |S| \mid S \subseteq V(G)\},$$

where $c_0(G)$ denotes the number of odd components (components with an odd number of vertices) of G.

Proposition 2.7 Let $\mathcal{G} = (V, E, L)$ be a graph with loops, so $\mathrm{def}(\mathcal{G}^{af}) = \delta$ for all $f \in F = E \cup L$ if and only if $\mathrm{def}(\mathcal{G}^{a}) = \delta$ and $\nu(\mathcal{G}^{af}) = \nu(\mathcal{G}^{a}) + 1$ for all $f \in F$.

Proof. We take a maximum matching $g_1, \ldots, g_\ell$ of $\mathcal{G}^a$. If $f \in F$, then $g_1, \ldots, g_\ell, g$ is a matching of $\mathcal{G}^{af}$, where $g = \{x_i^{a_i+1}, x_j^{a_j+1}\}$ when $f = \{x_i, x_j\} \in E$ and $g = \{x_i^{a_i+1}, a_i^{a_i+2}\}$ when $f = (x_i, x_i) \in L$. Hence, $\nu(\mathcal{G}^{af}) \geq \nu(\mathcal{G}^a) + 1$. This implies $\mathrm{def}(\mathcal{G}^a) = |V(\mathcal{G}^a)| - 2\nu(\mathcal{G}^a) \geq |V(\mathcal{G}^{af})| - 2\nu(\mathcal{G}^{af})$ since $|V(\mathcal{G}^{af})| = |V(\mathcal{G}^a)| + 2$. Therefore, $\mathrm{def}(\mathcal{G}^a) \geq \mathrm{def}(\mathcal{G}^{af})$.

$\Rightarrow$) By contradiction, suppose $\mathrm{def}(\mathcal{G}^a) > \delta$. Thus, by Theorem 2.6, there is an $S \subseteq V(\mathcal{G}^a)$ such that $c_0(\mathcal{G}^a \setminus S) - |S| > \delta$. We set $r = c_0(\mathcal{G}^a \setminus S)$ and $H_1, \ldots, H_r$ the odd components of $\mathcal{G}^a \setminus S$. We take $x_i^{k_i} \in H_k$ for some $1 \leq k \leq r$ and $k_i \leq a_i$. If $f = (x_i, x_i) \in L$, then we take the subgraph H'_k of $\mathcal{G}^{af} \setminus S$ induced by $V(H_k) \cup \{x_i^{a_i+1}, x_i^{a_i+2}\}$. We obtain that the odd connected components of $\mathcal{G}^{af} \setminus S$ are $H_1, H_2, \ldots, H_{k-1}, H'_k, H_{k+1} \ldots, H_r$. Consequently,

$$c_0(\mathcal{G}^{af} \setminus S) - |S| > \delta = \mathrm{def}(\mathcal{G}^{af}).$$

A contradiction. Now, assume $f = \{x_i, x_j\} \in E(\mathcal{G})$. If $\{x_i^{k_i}, x_j^{k_j}\} \in E(H_k)$, then we consider the subgraph $H_{k'}$ of $\mathcal{G}^{af} \setminus S$ induced by $V(H_k) \cup \{x_i^{a_i+1}, x_j^{a_j+1}\}$. We obtain that the odd connected components of $\mathcal{G}^{af} \setminus S$ are $H_1, H_2, \ldots, H_{k-1}, H'_k, H_{k+1} \ldots, H_r$. So,

$$c_0(\mathcal{G}^{af} \setminus S) - |S| > \delta = \mathrm{def}(\mathcal{G}^{af}).$$

This implies $V(H_k) = \{x_i^{k_i}\}$ and $a_j = 0$ or $x_j^{k_j} \in S$ for each $k_j \leq a_j$. Hence, the odd components of $\mathcal{G}^{af} \setminus (S \cup \{x_j^{a_j+1}\})$ are $H_1, \ldots, H_r, \{x_i^{a_i+1}\}$. Thus,

$$c_0(\mathcal{G}^{af} \setminus (S \cup \{x_j^{a_j+1}\})) - |S \cup \{x_j^{a_j+1}\}| = c_0(\mathcal{G}^a \setminus S) - |S| > \delta = \mathrm{def}(\mathcal{G}^{af}).$$

A contradiction, therefore $\mathrm{def}(\mathcal{G}^a) = \mathrm{def}(\mathcal{G}^{af})$ for all $f \in F$. Therefore, $\nu(G^{af}) = \nu(G^a) + 1$, since $|V(\mathcal{G}^{af})| = |V(\mathcal{G}^a)| + 2$ for all $f \in F$.

$\Leftarrow$) $\mathrm{def}(\mathcal{G}^{af}) = |V(\mathcal{G}^{af})| + 2\nu(\mathcal{G}^{af}) = |V(\mathcal{G}^a)| + 2 - 2(\nu(\mathcal{G}^a) + 1) = \mathrm{def}(\mathcal{G}^a) = \delta$.
$\square$

Theorem 2.8 $I(G)$ has the strong persistence property if G is a simple graph.

Proof. See [18, Lemma 2.12]. □

Theorem 2.9 If $\mathcal{G}$ is a graph with loops, then $\left(I^{k+1}\colon I\right) = I^k$ with $I = I(\mathcal{G})$.

Proof. We take a monomial $m = x^a \in (I^{k+1}\colon I)$. If $mf \in I^{k+2}$ for some $f = x_i x_j \in G(I)$, then $m(x_i x_j) = m' g_1 \cdots g_{k+2}$ with $g_i \in G(I)$ and $m' \in \mathrm{Mon}(R)$. Thus, $m \in I^k$. So, we can assume that $mf \in I^{k+1} \setminus I^{k+2}$ for each $f \in G(I)$. Consequently, by Corollary 2.4, $\nu(\mathcal{G}^{af}) = k+1$ for each $f \in G(I)$. Hence, $\mathrm{def}(\mathcal{G}^{af}) = |V(\mathcal{G}^a)| + 2 - 2(k+1) = |V(\mathcal{G}^a)| - 2k$ for each $f \in G(I)$. Furthermore, by Proposition 2.7, $\mathrm{def}(\mathcal{G}^{af}) = \mathrm{def}(\mathcal{G}^a) = |V(\mathcal{G}^a)| - 2\nu(\mathcal{G}^a)$, then $\nu(\mathcal{G}^a) = k$. Therefore, by Proposition 2.3, $m = x^a \in I^k$. □

Corollary 2.10 $I(\mathcal{G})$ has the persistence property if $\mathcal{G}$ is a graph with loops.

Proof. By Theorem 2.9 and [13, Lemma 2.12]. □

2.3 WEIGHTED MONOMIAL IDEALS

Let I be a monomial ideal, recall that an irreducible monomial ideal J is I-minimal if J is minimal in the set of irreducible monomial ideals (with the form $\{x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_s}^{\alpha_{i_s}}\}$) such that $I \subseteq J$. The set of I-minimal ideals is a minimal primary decomposition of I.

Definition 2.11 For $m_1, m_2 \in \mathrm{Mon}(R)$, $m_1^s \mid\mid m_2$ if $m_1^s | m_2$ and $m_1^{s+1} \nmid m_2$.

Proposition 2.12 Let I be a monomial ideal. If $(x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_s}^{\alpha_{i_s}})$ is a I-minimal ideal, then for each $1 \le t \le s$ there is $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} \mid\mid m$.

Proof. Since $J = (x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_s}^{\alpha_{i_s}})$ is an I-minimal ideal, then $I \subseteq J$. Thus, if $x_{i_t}^{\alpha_{i_t}} \nmid u$ for each $u \in G(I)$, then $I \subseteq \left(\{x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_s}^{\alpha_{i_s}}\} \setminus \{x_{i_t}^{\alpha_{i_t}}\}\right)$. This contradicts the minimality of J. Hence, $x_{i_t}^{\alpha_{i_t}} | u$ for some $u \in G(I)$. Now, if $x_{i_t}^{\alpha_{i_t}+1} | m$ for each $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} | m$, then $I \subseteq (x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_t}^{\alpha_{i_t+1}}, \ldots, x_{i_s}^{\alpha_{i_s}}) \not\subseteq J$. A contradiction, therefore there is $m \in G(I)$ such that $x_{i_t}^{\alpha_{i_t}} \mid\mid m$. □

Definition 2.13 A weight over a polynomial ring $R = K[x_1, \ldots, x_n]$ is a function $w\colon \{x_1, \ldots, x_n\} \to \mathbb{N}$, $w_i = w(x_i)$ is called the **weight** of the variable x_i. Given a monomial ideal I and a weight w, the **weighted ideal** of I and w is $I_w = \big(h(m) | m \in G(I)\big)$ where h is the isomorphism $h : R \to K[x_1^{w_1}, \ldots, x_n^{w_n}]$ given by $x_i \mapsto x_i^{w_i}$.

Remark 2.14 Since h is an isomorphism, $G((I_w)^k) = G((I^k)_w)$, so $(I_w)^k = (I^k)_w$.

Theorem 2.15 Let I be a monomial ideal and w a weight over R, then

i) $\mathrm{Ass}(I_w^k) = \mathrm{Ass}(I^k)$ for each k;

ii) I has the persistence property if and only if I_w has the persistence property;

iii) I has the strong persistence property if and only if I_w has the strong persistence property.

Proof. i) If $(x_{i_1}^{\beta_{i_1}}, \ldots, x_{i_s}^{\beta_{i_s}})$ is an I_w^k-minimal ideal, then by Proposition 2.12 there is $m' \in G(I_w^k)$ such that $x_{i_j}^{\beta_{i_j}} \mid\mid m'$, so there is r_{i_j} such that $\beta_{i_j} = w_{i_j} r_{i_j}$ for $1 \leq j \leq s$. If $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in G(I^k)$, then $h(m) = x_1^{\alpha_1 w_1} \cdots x_n^{\alpha_n w_n} \in G(I_w^k) \subseteq (x_{i_1}^{\beta_{i_1}}, \ldots, x_{i_s}^{\beta_{i_s}})$. Hence, there exist $t \leq s$, such that $x_{i_t}^{\beta_{i_t}} | h(m)$. Thus, $w_{i_t} r_{i_t} = \beta_{i_k} \leq w_{i_t} \alpha_{i_t}$ implies $r_{i_t} \leq \alpha_{i_t}$ and $x_{i_t}^{r_{i_t}} | m$. Consequently $I^k \subseteq (x_{i_1}^{r_1}, \ldots, x_{i_s}^{r_s})$. Now, if $(x_{j_1}^{\alpha_{j_1}}, \ldots, x_{j_l}^{\alpha_{j_l}})$ is an I^k-minimal, then $I_w^k \subseteq (x_{j_1}^{w_{j_1}\alpha_{j_1}}, \ldots, x_{j_l}^{w_{j_l}\alpha_{j_l}})$. So, $(x_{i_1}^{r_1}, \ldots, x_{i_s}^{r_s}$ is I^k-minimal and $\big(x_{j_1}^{w_{j_1}\alpha_{j_1}}, \ldots, x_{j_l}^{w_{j_l}\alpha_{j_l}}\big)$ is I_w^k-minimal. Therefore, $(x_{i_1}^{\alpha_{i_1}}, \ldots, x_{i_s}^{\alpha_{i_s}})$ is I^k-minimal if and only if $(x_{i_1}^{w_{i_1}\alpha_{i_1}}, \ldots, x_{i_s}^{w_{i_s}\alpha_{i_s}})$ is I_w^k-minimal. Taking radicals of the I_w^k-minimal and I^k-minimal ideals we obtain $\mathrm{Ass}(I_w^k) = \mathrm{Ass}(I^k)$.

ii) By i).

iii) $\Rightarrow$) Since h is an isomorphism of k-algebras between $R' = K[x_1^{w_1}, \ldots, x_n^{w_n}]$ and R, $h(I)$ has the strong persistence property in $K[x_1^{w_1}, \ldots, x_n^{w_n}]$. Also, $m = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in R'$ if and only if $w_i | \lambda_i$ for each i. Thus, $I_w \cap K[x_1^{w_1}, \ldots, x_n^{w_n}] = h(I)$. Now, if $m \in (I_w^{k+1}\colon I_w)$, then $gm = \ell g_1 \cdots g_{k+1}$ for each $g \in G(I_w)$ where $g_i \in G(I_w)$. We take $m = x_1^{a_1} \cdots x_1^{a_n}$ and $\ell = x_1^{b_1} \cdots x_n^{b_n}$. If r_i and t_i are the remainders obtained by dividing a_i and b_i by w_i respectively, then $w_i | r_i - t_i$, since $G(I_w) \subseteq K[x_1^{w_1}, \ldots, x_n^{w_n}]$. So, $r_i = t_i$ and we take $m' = x_1^{a_1 - r_1} \cdots x_n^{a_n - r_n}$ and $\ell' = x_1^{b_1 - r_1} \cdots x_n^{b_n - r_r}$. Hence, $m', \ell' \in K[x_1^{w_1}, \ldots, x_n^{w_n}]$ and $gm' = \ell' g_1 \cdots g_{k+1}$. Since $G(I_w) = G(h(I))$, $m' \in \big(h(I)^{k+1}\colon h(I)\big) = h(I)^k$ implies $m' \in I_w^k$. Therefore $m \in I_w^k$, since $m' | m$.

$\Leftarrow$) We take $m \in (I^{k+1}\colon I) \cap \mathrm{Mon}(R)$, then $mf = \ell f_1 \cdots f_{k+1}$ with $f, f_1, \ldots, f_{k+1} \in G(I)$. So $h(m)h(f) = h(\ell)h(f_1)\cdots h(f_{k+1}) \in I_w^{k+1}$. Thus, $h(m) \in (I_w^{k+1}\colon I_w) = I_w^k$ since $G(I_w) = G(h(I))$. This implies, $h(m) = \ell g_1 \cdots g_k$ with $g_i = h(g_i') \in G(I_w)$ where $g_i' \in G(I)$. Since $h(m) \in R' = K[x_1^{w_1}, \ldots, x_n^{w_n}]$, $\ell \in R'$. Therefore, $m = h^{-1}(\ell)g_1' \cdots g_k' \in I^k$, since h is an isomorphism. □

Definition 2.16 A **weighted graph** (G, w) consists of a simple graph G and a function $w : V(G) \to \mathbb{N}$. The **weight** of $x \in V(G)$ is $w(x)$.

Definition 2.17 The edge ideal of the weighted graph (G, w) denoted by $I(G, w)$ is the ideal generated by $\{x_i^{w_i} x_j^{w_j} \mid x_i x_j \in E(G)\}$, where $w_k = w(x_k)$.

Corollary 2.18 If $I = I(G)$ and $J = I(G, w)$, then $\mathrm{Ass}(J^k) = \mathrm{Ass}(I^k)$ for all k.

Proof. By Theorem 2.15, since $J = I_w$. □

Theorem 2.19 The edge ideal $I(G, w)$ has the strong persistence property.

Proof. By Theorem 2.15, since $I(G)$ has the strong persistence property. □

2.4 SQUAREFREE MONOMIAL IDEAL

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring and let $\mathcal{C}$ be a clutter where $V(\mathcal{C}) = \{x_1, \ldots, x_n\}$. If $f = \{x_{i_1}, \ldots, x_{i_r}\} \in E(\mathcal{C})$, then we denote by $\widetilde{f}$ the squarefree monomial $x_{i_1} \cdots x_{i_r}$. Hence, if $f_1 \subseteq f_2 \subseteq X = \{x_1, \ldots, x_n\}$, then $\widetilde{f_1} | \widetilde{f_2}$. We say that a clutter $\mathcal{C}$ has the strong persistence property if its edge ideal $I(\mathcal{C})$ has the strong persistence property.

Lemma 2.20 Let f, g be squarefree monomials, if there exists an integer $k \geq 2$ such that $f^k | mg$, then $f^{k-1} | m$.

Proof. Since $f^k | mg$, $mg = f^k \ell$ with $\ell \in \mathrm{Mon}(R)$. We take $m' = \gcd(f, g)$, then $f = m'f'$ and $g = m'g'$ with $\gcd(f', g') = 1$. Hence, $\gcd(f, g') = \gcd(m', g') = u$. Consequently $u^2 | g$. But g is a squarefree monomial, so $\gcd(f, g') = 1$. Thus $g' | \ell$, since $mg' = f' f^{k-1} \ell$. Therefore $m = f^{k-1}(f'u')$ where $\ell = u'g'$ implies $f^{k-1} | m$. □

Corollary 2.21 Let I be a squarefree monomial ideal. If $G(I)$ has at most two elements, then I has the strong persistence property.

Proof. Let m be a monomial in $(I^{k+1}\colon I)$. So, for each $f \in G(I)$ there are monomials $\ell, g_1, \ldots, g_{k+1}$ with $g_i \in G(I)$, such that $mf = \ell g_1 \cdots g_{k+1}$. If $f = g_i$ for some i, then $m \in I^k$. Now, if $f \neq g_i$ for each i, then $g_i = g_1$ since $|G(I)| \leq 2$. Thus $g_1^{k+1}|mf$. Hence, by Lemma 2.20, $g_1^k|m$ and $m \in I^k$. □

Theorem 2.22 If I is a squarefree monomial ideal, then $(I^2\colon I) = I$.

Proof. Let m be a monomial in $(I^2\colon I)$, then for each $f_1 \in G(I)$ there are $h_1, g_1 \in G(I)$ and a monomial ℓ_1 such that $mf_1 = \ell_1 g_1 h_1$. Consequently,

$$m^2 f_1 = \ell_1 (m g_1) h_1 = \ell_1 \ell_2 g_2 h_2 h_1$$

where $mg_1 = \ell_2 g_2 h_2$ and $g_2, h_2 \in G(I)$. Follows, multiplying by m we obtain

$$m^r f_1 = \ell_1 \cdots \ell_r g_r h_r \cdots h_2 h_1,$$

where $mg_{i-1} = \ell_i g_i h_i$ and $g_i, h_i \in G(I)$ for $2 \leq i \leq r$. If $r \geq |G(I)|$, then $g_r = h_j$ or $h_j = h_i$ for some $1 \leq i < j \leq r$. Hence, $h_j^2|m^r f_1$ and by Lemma 2.20, $h_j|m^r$. Thus, $h_j|m$, since h_j is squarefree. Therefore $m \in I$. □

Corollary 2.23 If I is a squarefree monomial ideal and $k \geq 2$, then $(I^k\colon I) \subseteq I$.

Proof. By Theorem 2.22, $(I^k\colon I) \subseteq I$. Hence, $I^k \subseteq I^2$ and $(I^k\colon I) \subseteq (I^2\colon I)$. □

Theorem 2.24 A clutter has the strong persistence property if and only if some of its connected components has the strong persistence property.

Proof. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ the connected components of $\mathcal{C}$ with $V_i = V(\mathcal{C}_i)$.

$\Leftarrow$) We can suppose that $\mathcal{C}_1$ has the strong persistence property. We take a monomial $m \in (I^{k+1}\colon I)$. We can write $m = m_1 \cdots m_r$ where $m_i \in \mathrm{Mon}(K[V_i])$ and we take a_i such that $m_i \in I_i^{a_i} \setminus I_i^{a_i+1}$. For each $f \in \mathcal{C}_1$ we consider s_f such that $m_1 f \in I_1^{s_f} \setminus I_1^{s_f+1}$ and $s_1 = \min\{s_f \mid f \in \mathcal{C}_1\}$. Thus $m_1 f \in I_1^{s_1}$ for each $f \in \mathcal{C}_1$, so $m_1 \in \left(I_1^{s_1}\colon I_1\right) = I_1^{s-1}$. Hence,

$$m \in I^{s_1 - 1 + \sum_{i=2}^{r} a_i} \text{ and } mf \in I^{s_f + \sum_{i=2}^{r} a_i} \setminus I^{s_f + 1 + \sum_{i=2}^{r} a_i} \text{ for each } f \in \mathcal{C}_1.$$

Since $mf \in I^{k+1}$, $s_f + \sum_{i=2}^{r} a_i \geq k+1$. Then, $s_1 + \sum_{i=2}^{r} a_i \geq k+1$. Therefore $m \in I^k$.

$\Rightarrow$) If $I_i = I(\mathcal{C}_i)$ has no the strong persistence property, then there is k_i and a monomial $m_i \in (I_i^{k_i+1} \colon I_i) \setminus I_i^{k_i}$. We take a_i such that $m_i \in I_i^{a_i} \setminus I_i^{a_i+1}$, then $a_i \leq k_i - 1$. Now, we consider $m = m_1 \cdots m_r$, then $m \in I^b \setminus I^{b+1}$, for $b = \sum_{i=1}^{r} a_i$. If we take $f_i \in E(\mathcal{C}_i)$, then $mf_i \in I^{s_i}$, where $s_i = a_1 + \cdots + k_i + 1 + \cdots + a_r$. But $s_i \geq \sum_{j=1}^{r} a_j + 2$, thus $s = \min\{s_1, \ldots, s_r\} \geq \sum_{j=1}^{r} a_j + 2$. Therefore $m \in (I^s \colon I) \setminus I^{s-1}$. □

Example 2.25 Let $\mathcal{C}$ be a clutter. If $f_1, f_2 \in \{A \subseteq V(G) \mid A \cap f = \emptyset \text{ if } f \in E(\mathcal{C})\}$, then by Theorem 2.24 and Corollary 2.21, $\mathcal{C} \cup \{f_1, f_2\}$ has the strong persistence property.

Lemma 2.26 Let $\mathcal{C}$ be a clutter. If there exists an edge $f \in E(\mathcal{C})$ such that $A = \{g \cap f \mid g \in E(\mathcal{C})\}$ is a chain, then $I(\mathcal{C})$ has the strong persistence property.

Proof. If m is a monomial in $(I^{k+1} \colon I)$, then $m\widetilde{f} = \widetilde{g}\widetilde{f_1} \cdots \widetilde{f}_{k+1}$ where $f_i \in E(\mathcal{C})$ and $g \subseteq V(\mathcal{C})$. So, $f \subseteq g \cup f_1 \cup \cdots \cup f_{k+1}$. Since A is a chain, we can assume $f_{k+1} \cap f \subseteq f_k \cap f \subseteq \cdots \subseteq f_1 \cap f$. Thus, $f \subseteq g \cup f_1$ and $\widetilde{f} | \widetilde{g}\widetilde{f_1}$. Therefore $m \in I^k$. □

Corollary 2.27 If $\mathcal{C}$ is a clutter without the strong persistence property, then for $f \in E(\mathcal{C})$ there are $f_1, f_2 \in E(\mathcal{C})$ such that $f \cap f_1 \nsubseteq f \cap f_2$ and $f \cap f_2 \nsubseteq f \cap f_1$.

Definition 2.28 Let $\mathcal{C}$ be a clutter, $A \subseteq V(\mathcal{C})$ is a vertex cover if $A \cap e \neq \emptyset$ for each $e \in E(\mathcal{C})$. The cover number of $\mathcal{C}$ is $\tau(\mathcal{C}) = \min\{|A| \mid A \text{ is a vertex cover}\}$. $\mathcal{C}$ is unmixed if $|B| = \tau(\mathcal{C})$ for each minimal vertex cover B. A **matching** is a set of disjoint edges $\{e_1, \ldots, e_s\}$ of $\mathcal{C}$. It is **perfect** if $\cup_{i=1}^{s} e_i = V(\mathcal{C})$. Furthermore, $\mathcal{C}$ is König if there is a matching with $\tau(\mathcal{C})$ edges.

Proposition 2.29 Let $\mathcal{C}$ be a König clutter, then $\mathcal{C}$ is unmixed if and only if there is a perfect matching $e_1, \ldots, e_g$ with $g = \tau(\mathcal{C})$, such that for any two edges $e \neq e'$ and for any two distinct vertices $x \in e$, $y \in e'$ contained in some e_i, one has that $(e \setminus \{x\}) \cup (e' \setminus \{y\})$ contains an edge.

Proof. See Corollary 2.11 in [19]. □

Definition 2.30 The incidence matrix of a clutter $\mathcal{C}$, denoted by $A_{\mathcal{C}}$, is the matrix whose columns are the characteristic vectors of the edges of $\mathcal{C}$. A r-cycle of $\mathcal{C}$ is a

$r \times r$-submatrix of $A_{\mathcal{C}}$ with exactly two 1's in each row and each column.

Theorem 2.31 Let $\mathcal{C}$ be a König unmixed clutter. If $\mathcal{C}$ does not contain 4-cycles, then $\mathcal{C}$ has the strong persistence property.

Proof. By Proposition 2.29, $\mathcal{C}$ has a perfect matching $e_1, \ldots, e_s$ where $s = \tau(\mathcal{C})$. If $\mathcal{C}$ does not have the strong persistence property, then by Corollary 2.27 there exist $f_1, f_2 \in E(\mathcal{C})$ and vertices $x_1 \in (f_1 \cap e_1) \setminus f_2$ and $x_2 \in (f_2 \cap e_1) \setminus f_1$. Now by Proposition 2.29, there exist $f \in E(\mathcal{C})$ such that $f \subseteq (f_1 \setminus x_1) \cup (f_2 \setminus x_2)$. We can assume $e_1 \cap (f_2 \cup f_1)$ is minimal in

$$B = \{e_1 \cap (g_2 \cup g_1) \mid g_1, g_2 \in E(\mathcal{C}), g_2 \cap e_1 \nsubseteq g_1 \cap e_1, \text{and } g_1 \cap e_1 \nsubseteq g_2 \cap e_1\}.$$

Thus, $(e_1 \cap f) \subseteq e_1 \cap ((f_1 \setminus x_1) \cup (f_2 \setminus x_2)) = e_1 \cap (f_1 \cup f_2 \setminus x_1x_2)$. Hence, $e_1 \cap (f_i \cup f) \subseteq (e_1 \cap (f_1 \cup f_2)) \setminus x_j$ where $\{i, j\} = \{1, 2\}$. Since $e_1 \cap (f_1 \cap f_2)$ is monomial in B, $e_1 \cap f \subseteq e_1 \cap f_2$ or $e_1 \cap f_2 \subseteq e_1 \cap f$. But $x_2 \in (e_1 \cap f_2) \setminus (e_1 \cap f)$, then $e_1 \cap f \subseteq e_1 \cap f_2$. Now, if $(f_1 \cap f) \subseteq (e_1 \cup f_2)$, then $f \subseteq (f_1 \cup f_2) \cap f \subseteq (f_1 \cap f) \cup (f_2 \cap f) \subseteq (e_1 \cup f_2) \cup (f_2 \cap f) \subseteq (e_1 \cup f_2)$. So, $f \subseteq (e_1 \cap f) \cup f_2 \subseteq (e_1 \cap f_2) \cup f_2 \subseteq f_2$. But $x_2 \in f_2 \setminus f$, a contradiction. Hence, there is $y_1 \in (f_1 \cap f) \setminus (e_1 \cup f_2)$. Similarly there is $y_2 \in (f_2 \cap f) \setminus (e_1 \cup f_1)$. Consequently, the matrix

$$\begin{array}{c} \\ f_1 \\ f_2 \\ e_1 \\ f \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

is a 4-cycle. A contradiction, therefore $\mathcal{C}$ has the strong persistence property. $\square$

Example 2.32 ([13]) Let $\mathcal{C}_0$ be the clutter with vertex set $\{x_1, \ldots, x_6\}$ whose edges are $x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5$ and $x_3x_4x_6$. $\mathcal{C}_0$ is an unmixed shellable clutter. But $\left(I(\mathcal{C}_0)^3 \colon I(\mathcal{C}_0)\right) \neq I(\mathcal{C}_0)^2$, then $\mathcal{C}_0$ does not have the strong persistence property.

Definition 2.33 The **cone** over a clutter $\mathcal{C}$, denoted by $\mathcal{C}x$, is the clutter whose vertex set is $V(\mathcal{C}) \cup \{x\}$ and edge set $\{f \cup \{x\} \mid f \in E(\mathcal{C})\}$, where x is a new vertex.

Proposition 2.34 $\mathcal{C}$ has the strong persistence property if and only if $\mathcal{C}x$ has the strong persistence property.

Proof. $\Rightarrow$) If $m = x^{\alpha} m' \in \big(I(\mathcal{C}x)^{k+1} \colon I(\mathcal{C}x)\big)$ with $\gcd(m', x) = 1$, then $\widetilde{f} m \in I(\mathcal{C}x)^{k+1}$ for $f \in E(\mathcal{C}x)$. Furthermore $\widetilde{f} = \widetilde{g}x$ with $g \in E(G)$ then $x^{k+1} | \widetilde{g}xm$ implying $x^k | m$. Thus, $\alpha \geq k$ and $\widetilde{g}m' \in I(\mathcal{C})^{k+1}$. Hence $m' \in \big(I(\mathcal{C})^{k+1} \colon I(\mathcal{C})\big) = I(\mathcal{C})^k$, i.e., $m' = \ell \widetilde{f_1} \cdots \widetilde{f_k}$ where $f_i \in E(\mathcal{C})$. Therefore, $m = x^{\alpha} \ell \widetilde{f_1} \cdots \widetilde{f_k} = x^{\alpha - k} \ell (\widetilde{f_1}x) \cdots (\widetilde{f_k}x)$, so $m \in I(\mathcal{C}x)^k$.

$\Leftarrow$) If $m \in \big(I(\mathcal{C})^{k+1} \colon I(\mathcal{C})\big)$, then $fm = \ell g_1 \cdots g_{k+1}$ for each $f \in I(\mathcal{C})$ and $g_i \in I(\mathcal{C})$. Thus, $(fx)(mx^k) = \ell(xg_1) \cdots (xg_{k+1}) \in I(\mathcal{C}x)^{k+1}$. So, $mx^k \in \big(I(\mathcal{C}x)^{k+1} \colon I(\mathcal{C}x)\big) = I(\mathcal{C}x)^k$. Hence, $mx^k = \ell(f_1 x) \cdots (f_k x)$ for $f_i \in I(\mathcal{C})$. Therefore $m \in I(\mathcal{C})^k$. □

Proposition 2.35 $\mathcal{C}$ has the persistence property if and only if $\mathcal{C}x$ has the persistence property.

Proof. If $Q_1, \ldots, Q_r$ is the monomial minimal primary decomposition of $I(\mathcal{C})^k$ and $Q_i' = R[x] \cdot Q_i$, then $Q_1', \ldots, Q_r', (x^k)$ is the monomial minimal primary decomposition of $I(\mathcal{C}x)^k$. Hence, $\mathrm{Ass}(I(\mathcal{C}x)^k) = \mathrm{Ass}(I(\mathcal{C})^k) \cup \{(x)\}$. □

Proposition 2.36 $\mathcal{C} = (V, E)$ has the strong persistence property if and only if $\mathcal{C}' = (V, E')$ has the strong persistence property, where $E' = \{f \setminus \cap_{g \in E} g \mid f \in E\}$.

Proof. Set $A = \cap_{g \in E} g$. By induction on $k = |A|$. If $k = 0$, then $\mathcal{C} = \mathcal{C}'$. Now if $k \geq 1$ and $x \in A$, then $\mathcal{C} = \mathcal{C}_1 x$ where $\mathcal{C}_1 = \mathcal{C} \setminus x$. So, by induction hypothesis $\mathcal{C}_1$ has the strong persistence property if and only if $\mathcal{C}'$ has the strong persistence property. Therefore, we obtain the result by Proposition 2.34. □

Proposition 2.37 A clutter $\mathcal{C}$ with 3 edges has the strong persistence property.

Proof. We assume $E(\mathcal{C}) = \{f_1, f_2, f_3\}$ and $V(X) = \{x_1, \ldots, x_n\}$. By Proposition 2.36, we can suppose that $f_1 \cap f_2 \cap f_3 = \emptyset$. If $\mathcal{C}$ is not connected, then it has a component with one edge. Hence, by Corollary 2.21 and Theorem 2.24, $\mathcal{C}$ has the strong persistence property. Now, we assume that $\mathcal{C}$ is connected. If $f_i \cap f_j = \emptyset$ for some $i \neq j$, then $\mathcal{C}$ has the strong persistence property by Lemma 2.26. Consequently, we suppose $a_{ij} = f_i \cap f_j \neq \emptyset$ for $i \neq j$. We set b_i such that $f_i = a_{ij} \uplus b_i \uplus a_{ir}$ for $\{i, j, r\} = \{1, 2, 3\}$. So, each pair of $b_1, b_2, b_3, a_{12}, a_{13}, a_{23}$ are disjoint. We take $m \in (I^{k+1} \colon I)$ where $I = I(\mathcal{C})$, then $m\widetilde{f_1} = \ell \widetilde{f}_1^{\alpha_1} \widetilde{f}_2^{\alpha_2} \widetilde{f}_3^{\alpha_3}$ with $\alpha_1 + \alpha_2 + \alpha_3 = k + 1$. If $\alpha_1 > 0$, then $m \in I^k$. Now, if $\alpha_1 = 0$, then $b_1 | \ell$ since b_1, f_1, f_3 are disjoint pairs. This implies, $\ell = b_1 \ell'$ and $m a_{12} a_{13} = \ell' \widetilde{f}_2^{\alpha_2} \widetilde{f}_3^{\alpha_3}$. If $\alpha_2 = 0$, then $\ell' = u_1 a_{12}$ and $m = u_1 u_2 \widetilde{f}_3^{\alpha_k - 1}$ where $\widetilde{f_3} = u_2 a_{13}$. Thus, $m \in I^k$. Similarly if $\alpha_3 = 0$, then we suppose $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Consequently $m = \ell'(b_2 b_3 a_{23}^2) \widetilde{f}_2^{\alpha_2 - 1} \widetilde{f}_3^{\alpha_3 - 1}$, implying $a_{23}^{k+1}|m$ and $b_2^{\alpha_2}b_3^{\alpha_3}|m$. Similarly, we can assume $a_{12}^{k+1}|m$ and $a_{13}^{k+1}|m$. Hence, $(a_{12}a_{13}a_{23})^{k+1}b_2^{\alpha_2}b_3^{\alpha_3}|m$ so $\widetilde{f}_2^{\alpha_2}\widetilde{f}_3^{\alpha_3}|m$, since $\alpha_2+\alpha_3=k+1$. Therefore, $m \in I^{k+1} \subseteq I^k$. □

Proposition 2.38 If X is a set $A \subseteq X$ and $x \notin X$, then the clutter $\mathcal{C}$ whose edge set is $\{X\} \cup \{xx_i \mid x_i \in A\}$ has the strong persistence property.

Proof. We set $A = \{x_1, \ldots, x_r\}$, $f_0 = X$ and $f_i = \{x, x_i\}$. Since $\mathcal{C}$ is clutter, $r > 1$. We take $m \in (I^{k+1} \colon I)$ where $I = I(\mathcal{C})$, then $m\widetilde{f_i} = \ell_i \widetilde{f}_0^{\alpha_{0i}} \widetilde{f}_1^{\alpha_{1i}} \cdots \widetilde{f}_r^{\alpha_{ri}}$ where $\sum_{j=0}^{r} \alpha_{ji} = k+1$. If $\alpha_{0i} = 0$ for each $i \geq 1$, then $m \in \left(J^{k+1} \colon J\right)$, where $J = (\widetilde{f}_1, \ldots, \widetilde{f}_r)$. But J is an edge ideal of a graph so, by Theorem 2.8, $m \in J^k \subseteq I^k$. Thus, we can assume $\alpha_{01} > 0$ and we take $\alpha_i = \alpha_{i1}$. If $\alpha_1 = 0$ and $x \nmid \ell_1$, then $x^{k-\alpha_0} \mid\mid m$ and $\widetilde{f}_0^{\alpha_0 - 1} \mid\mid m$, since

$$m = \ell_1 \frac{\widetilde{f}_0^{\alpha_2}}{x_1} \cdot \frac{\widetilde{f}_2^{\alpha_2} \cdots \widetilde{f}_n^{\alpha_n}}{x} \text{ and } \widetilde{f}_0 \nmid \widetilde{f}_2^{\alpha_2} \cdots \widetilde{f}_n^{\alpha_n}.$$

So, $x^{k-\alpha_0+1} \mid\mid m\widetilde{f}_j$ and $\widetilde{f}_0^{\alpha_0-1} \mid\mid m\widetilde{f}_j$ for $j \neq 1$. Hence, $m\widetilde{f}_j \notin I^{k+1}$ a contradiction. Now if $\alpha_1 \neq 0$ or $x|\ell_1$, then $m = \ell_1 \widetilde{f}_0^{\alpha_0} \widetilde{f}_1^{\alpha_1 - 1} \widetilde{f}_2^{\alpha_2} \cdots \widetilde{f}_n^{\alpha_n}$ or $m = ab\widetilde{f}_0^{\alpha_0-1}\widetilde{f}_1^{\alpha_1} \cdots \widetilde{f}_n^{\alpha_n}$, where $\ell_1 = xa$ and $\widetilde{f}_0 = x_1 b$. Therefore $m \in I^k$. □

Theorem 2.39 If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4]$, then I has the strong persistence property.

Proof. Let $\mathcal{C}$ be the clutter associated to I. By Proposition 2.37 and Theorem 2.24 we can assume that $|E(\mathcal{C})| > 3$ and $\mathcal{C}$ has no edges of cardinality 1. If $\mathcal{C}$ has only edges of cardinality 3, then $4 \leq |E(\mathcal{C})| \leq \binom{4}{3} = 4$. Hence, $\mathcal{C}$ is a complete clutter, implies $\mathcal{C}$ is a base set of a polymatroid. Consequently, by [13, Proposition 2.4] $\mathcal{C}$ has the strong persistence property. If $\mathcal{C}$ has only one edge of cardinality 2, then $|E(\mathcal{C})| \leq 3$. A contradiction, so there are $f_1, f_2 \in E(\mathcal{C})$ such that $|f_1| = |f_2| = 2$. By Theorem 2.8 we can suppose $f = \{x_1, x_2, x_3\} \in E(\mathcal{C})$. So, if $f' \in E(\mathcal{C}) \setminus \{f\}$, then $x_4 \in f'$. Hence, we can assume $f_1 = \{x_1, x_4\}$ and $f_2 = \{x_2, x_4\}$. Thus, if $f' \in E(\mathcal{C}) \setminus \{f_1 f_1, f_2\}$, then $f' = \{x_3, x_4\}$. Therefore, by Proposition 2.38, $\mathcal{C}$ has the strong persistence property. □

Definition 2.40 If $Y \subseteq \{x_1, \ldots, x_n\}$ and $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[x_1, \cdots, x_n]$, we set $\deg_Y(m) = \sum_{x_i \in Y} \alpha_i$.

Proposition 2.41 If $\mathcal{C}$ has an edge f_0 such that $|f_0 \cap f| = 1$ for each $f \in E(\mathcal{C}) \setminus \{f_0\}$, then $\mathcal{C}$ has the strong persistence property.

Proof. We assume $E(\mathcal{C}) = \{f_0, f_1, \ldots, f_r\}$. If $q \in I^k$ is a monomial, then $\deg_{f_0}(q) \geq k$, since $|f_i \cap f_0| = 1$ for $0 \leq i \leq r$. We take $m \in (I^{k+1} \colon I)$, then

$$m\widetilde{f_0} = l\widetilde{f_0}^{\alpha_0}\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}$$

where l is a monomial and $\alpha_0 + \alpha_1 + \cdots + \alpha_r = k+1$. If $\alpha_0 > 0$,then $m \in I^k$. Now we assume $\alpha_0 = 0$. We consider $|f_0| = t$. If $\deg(\gcd(\widetilde{f_0}, l)) \geq t-1$, then $\deg(\frac{\widetilde{f_0}}{\gcd(\widetilde{f_0},l)}) \leq 1$. Consequently, $\widetilde{f_0} \mid l\widetilde{f_i}$ for some $i \geq 1$ with $\alpha_i > 0$. Thus, $m = \frac{l\widetilde{f_i}}{\widetilde{f_0}} \frac{\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}}{\widetilde{f_i}} \in I^k$. Now, we suppose $\deg(\gcd(\widetilde{f_0}, l)) \leq t-2$ and we consider $\frac{\widetilde{f_0}}{\gcd(\widetilde{f_0},l)} = x_{i_1} \cdots x_{i_s}$, then $s \geq 2$ and

$$mx_{i_1} \cdots x_{i_s} = l'\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}$$

where $l' = \frac{l}{\gcd(f_0,l)}$. This implies $\gcd(\widetilde{f_0}, l') = 1$, since $\widetilde{f_0}$ is squarefree. Now, we take $f_{j_1}, \cdots, f_{j_s}$ such that $\alpha_{j_a} > 0$ and $x_{i_a} \mid \widetilde{f_{j_a}}$ for each $1 \leq a \leq s$, then $\deg_{f_0}\left(\frac{\widetilde{f_{j_a}}}{x_{i_a}}\right) = 0$, since $|f_0 \cap f_{j_a}| = 1$. Consequently, $m = l'\frac{\widetilde{f_{j_1}}}{x_{i_1}} \cdots \frac{\widetilde{f_{j_s}}}{x_{i_s}} \frac{\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}}{\widetilde{f_{j_1}} \cdots \widetilde{f_{j_s}}}$, and $\deg_{f_0}(m) = k+1-s$, since $\deg_{f_0}(\widetilde{f_i}) = 1$ for $1 < i \leq r$. Hence $\deg_{f_0}(m) \leq k-1$, since $s \geq 2$. Thus, if $f_i \neq f_0$, then $\deg_{f_0}(m\widetilde{f_i}) = \deg_{f_0}(m) + \deg_{f_0}(\widetilde{f_i}) \leq k$. This is a contradiction, since $m\widetilde{f_i} \in I^{k+1}$ and $|f_0 \cap f_j| = 1$ for $1 \leq j \leq r$. □

Proposition 2.42 If $V(\mathcal{C}) = \{x_1, x_2, x_3, x_4, x_5\}$ with $f_0 \in E(\mathcal{C})$ such that $|f_0| = 2$ and $V(\mathcal{C}) \setminus f_0 \in E(\mathcal{C})$, then $\mathcal{C}$ has the strong persistence property.

Proof. We can assume $E(\mathcal{C}) = \{f, f_0, f_1, \ldots, f_r\}$ where $f_0 = \{x_1, x_2\}$ and $f = \{x_3, x_4, x_5\}$. We take $m \in (I^{k+1} \colon I)$, then

$$mx_1x_2 = m\widetilde{f_0} = l\widetilde{f_0}^{\alpha_0}\widetilde{f}^{\alpha}\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}$$

where $\alpha + \alpha_0 + \cdots + \alpha_r = k+1$. If $\gcd(\widetilde{f_0}, l) \neq 1$ or $\alpha_0 > 0$ or l contains some edge, then $m \in I^k$. Thus, we assume that $\gcd(\widetilde{f_0}, l) = 1$, $\alpha_0 = 0$ and l does not contain an edge. We take f_i, f_j such that $x_1 \mid \widetilde{f_i}$ and $x_2 \mid \widetilde{f_j}$ with $\alpha_i > 0$ and $\alpha_j > 0$. If $i = j$, then $m = l\frac{\widetilde{f_i}}{x_1x_2} \cdot \frac{\widetilde{f}^{\alpha}\widetilde{f_1}^{\alpha_1} \cdots \widetilde{f_r}^{\alpha_r}}{\widetilde{f_i}} \in I^k$. Now, we assume $i \neq j$, $x_2 \notin f_i$ and $x_1 \notin f_j$. We take $g = (f_i \cup f_j) \setminus \{x_1, x_2\}$. Thus, $g \subseteq \{x_3, x_4, x_5\} = f$. If $g = f$, then $\widetilde{f}^{\alpha} \mid \frac{\widetilde{f_i}}{x_1} \cdot \frac{\widetilde{f_j}}{x_2}$ implying

$m = l\frac{\widetilde{f_i}}{x_1} \cdot \frac{\widetilde{f_j}}{x_2} \cdot \frac{\widetilde{f}^{\alpha} \widetilde{f}_1^{\alpha_1} \ldots \widetilde{f}_r^{\alpha_r}}{\widetilde{f_i}\widetilde{f_j}} \in I^k$. So we assume $|g| < 2$. If $\widetilde{f}^{\alpha} \mid \frac{\widetilde{f_i} \cdot \widetilde{f_j}}{x_1 x_2} l$, then $m \in I^k$. So, we can suppose $\widetilde{f}^{\alpha} \nmid \frac{\widetilde{f_i} \cdot \widetilde{f_j}}{x_1 x_2} l$, then $\deg(\gcd(l, g)) \leq 2$. We write $m' = \widetilde{f}^{\alpha} \widetilde{f}_1^{\beta_1} \cdots \widetilde{f}_r^{\beta_r}$ where $\beta_t = \alpha_t$ if $t \notin \{i, j\}$ and $\beta_t = \alpha_t - 1$ if $t \in \{i, j\}$. If there exist $a_1, a_2, a_3 \in \{0, 1\}$ and $0 \leq b_i \leq \beta_i$ such that $l^{a_1} \left(\frac{\widetilde{f_i}}{x_1}\right)^{a_2} \left(\frac{\widetilde{f_j}}{x_2}\right)^{a_3} \widetilde{f}_1^{b_1} \cdots \widetilde{f}_r^{b_r} \in I^{b+1}$, where $b = b_1 + \cdots + b_r$ then $m \in I^k$, so we assume that for each sequence $a_1, a_2, a_3 \in \{0, 1\}$ and $0 \leq b_i \leq \beta_i$ we have $l^{a_1} \left(\frac{\widetilde{f_i}}{x_1}\right)^{a_2} \left(\frac{\widetilde{f_j}}{x_2}\right)^{a_3} f_1^{b_1} \cdots f_r^{b_r} \in I^b \setminus I^{b+1}$, then we have $m\widetilde{f_i} \notin I^{k+1}$. This is a contradiction. □

Proposition 2.43 If I is a square free monomial ideal of $K[x_1, x_2, x_3, x_4, x_5]$, such that $G(I)$ has an edges of cardinality 2, then I has the strong persistence property.

Proof. Let $\mathcal{C}$ be the clutter associated to I. We assume that f_0 is an edge of cardinality 2. If $\{x_1, x_2, x_3, x_4, x_5\} \setminus f_0$ does not contain some edge, then $|f \cap f_0| = 1$ for each $f \in E(\mathcal{C}) \setminus \{f_0\}$. Thus, by Proposition 2.41, $\mathcal{C}$ has the strong persistence property.

If $g = \{x_1, x_2, x_3, x_4, x_5\} \setminus f_0$ is an edge, then by Proposition 2.42, $\mathcal{C}$ has the strong persistence property. Hence, we can suppose that g contains a proper edge. We assume $f_0 = \{x_1, x_2\}$, $f_1 = \{x_3, x_4\}$, $f_2 = \{x_1, x_4, x_5\}$, $f_3 = \{x_1, x_3\}$, $f_4 = \{x_2, x_4\}$, $f_5 = \{x_2, x_3, x_5\}$ are the edges of $\mathcal{C}$. We take $m \in (I^{k+1} \colon I)$, then

$$m\widetilde{f}_0 = l\widetilde{f}_0^{\alpha_0} \widetilde{f}_1^{\alpha_1} \widetilde{f}_2^{\alpha_2} \widetilde{f}_3^{\alpha_3} \widetilde{f}_4^{\alpha_4} \widetilde{f}_5^{\alpha_5}$$

where l a monomial and $\alpha_0 + \alpha_1 + \cdots + \alpha_5 = k + 1$. If $\gcd(l, \widetilde{f}_0) \neq 1$, or $\alpha_0 > 0$ or l contains some edges, we obtain $m \in I^k$. So, we assume $\gcd(l, \widetilde{f}_0) = 1$ and $\alpha_0 = 0$. We take f_i, f_j such that $\alpha_i > 0$, $\alpha_j > 0$ and $x_1 \mid \widetilde{f}_i, x_2 \mid \widetilde{f}_j$, then one of the following condition holds:

1. $\{f_i, f_j\} = \{f_2, f_5\}$
2. $\{f_i, f_j\} = \{f_3, f_4\}$
3. $\{f_i, f_j\} = \{f_2, f_4\}$
4. $\{f_i, f_j\} = \{f_3, f_5\}$.

If $f_i = f_2 = \{x_1, x_4, x_5\}$ and $f_j = f_5 = \{x_2, x_3, x_5\}$, then $\frac{\widetilde{f_i}}{x_1} \frac{\widetilde{f_j}}{x_2} = x_3 x_4 x_5^2$ and $\widetilde{f}_1 \mid \frac{\widetilde{f_i}}{x_1} \frac{\widetilde{f_j}}{x_2}$, since $m = l\frac{\widetilde{f_i}}{x_1} \frac{\widetilde{f_j}}{x_2} m'$ where $m' = \frac{\widetilde{f}_1^{\alpha_1} \widetilde{f}_2^{\alpha_2} \widetilde{f}_3^{\alpha_3} \widetilde{f}_4^{\alpha_4} \widetilde{f}_5^{\alpha_5}}{\widetilde{f_i}\widetilde{f_j}} \in I^{k-1}$. Hence, $m \in i^k$. If $f_i = f_3 = \{x_1, x_3\}$ and $f_j = f_4 = \{x_2, x_4\}$ then $\frac{\widetilde{f_i}}{x_1} \frac{\widetilde{f_j}}{x_2} = x_3 x_4$ and $x_3 x_4 \mid \frac{\widetilde{f_i}}{x_1} \frac{\widetilde{f_j}}{x_2}$, since $m = l\widetilde{f}_1 m'$. This implies $m \in i^k$. Now, we consider $f_i = f_2 = \{x_1, x_4, x_5\}$ and $f_j = f_4 = \{x_2, x_4\}$. We take $m' = f_1^{\beta_1} \cdots f_5^{\beta_5}$ where $\beta_t = \alpha_t$ if $t \notin \{i, j\}$ and $\beta_t = \alpha_t - 1$, so $\frac{\widetilde{f}_i}{x_1}\frac{\widetilde{f}_j}{x_2} = x_4^2 x_5$. If $x_1 \mid l$ or $x_2 \mid l$ or $x_3 \mid l$, then $\widetilde{f}_2 \mid l x_4^2 x_5$ or $\widetilde{f}_4 \mid l x_4^2 x_5$ or $\widetilde{f}_1 \mid l x_4^2 x_5$ respectively. Consequently, $m \in I^k$. Similarly, if $\beta_3 > 0$ or $\beta_5 > 0$ we obtain $\widetilde{f}_1 \widetilde{f}_2 \mid x_4^2 x_5 \widetilde{f}_3$ or $\widetilde{f}_1 \widetilde{f}_4 \mid x_4^2 x_5 \widetilde{f}_5$ respectively. This implies, $x_4^2 x_5 \widetilde{f}_3 \in I^2$ or $x_4^2 x_5 \widetilde{f}_5 \in I^2$. Since $\frac{m'}{\widetilde{f}_3}$ or $\frac{m'}{\widetilde{f}_5} \in I^{k-2}$, we have $m \in I^k$. So, we assume $l = x_4 x_5^b$, $\beta_3 = 0$ and $\beta_5 = 0$, implying $m f_1 \notin I^{k+1}$. This is a contradiction, hence $m \in I^k$. Similarly we obtain $m \in I^k$, if $f_i = \{x_1, x_3\}$ and $f_j = \{x_2, x_3, x_5\}$. □

Corollary 2.44 If $I \subseteq K[x_1, \ldots, x_n]$ is a squarefree monomial ideal without the strong persistence property, then $n \geq 5$ and there is $k \geq 3$ such that $(I^k : I) \neq I^{k-1}$.

Proof. By Theorem 2.39 and Theorem 2.22. □

Definition 2.45 Let $\mathcal{C} = (V, E)$ be a clutter with $x \in V$, the deleting of x is the clutter $\mathcal{C} \setminus x$ with vertex set $V \setminus \{x\}$ and edge set $\{f \in E \mid x \notin f\}$. Furthermore, the contraction of x is the clutter $\mathcal{C}/x$ with vertex set $V \setminus \{x\}$ and whose edges are $f \setminus \{x\}$ with $f \in E$ and there is not $f' \in E$ such that $f' \setminus \{x\} \subset f \setminus \{x\}$.

Example 2.46 We consider the clutter $\mathcal{C}$ with vertex set $V(\mathcal{C}_0) \cup \{x\}$ and edge set $E(\mathcal{C}_0) \cup \{x x_1\}$, where $\mathcal{C}_0$ is the clutter in Example 2.32. By Theorem 2.24, $I(\mathcal{C})$ has the strong property but $\mathcal{C} \setminus x = \mathcal{C}_0$ has no the strong persistence property.

Proposition 2.47 Let $\mathcal{C}$ be a clutter and $x \in V(\mathcal{C})$. If $\mathcal{C}$ has the (strong) persistence property, then $\mathcal{C}/x$ has the (strong) persistence property.

Proof. We set $E(\mathcal{C}) = \{f_1, \ldots, f_r\}$. We can suppose $\{f_i \mid x \in f_i\} = \{f_1, \ldots, f_{r_1}\}$ and $\{f_i \mid f_j \setminus \{x\} \nsubseteq f_i \text{ for each } j \leq r_1\} = \{f_{r_1+1}, \ldots, f_{r_2}\}$. We define $f_i' = f_i \setminus \{x\}$ for $i \leq r_2$ and $A = \bigcup_{i \leq r_2} f_i'$. Also, we set $I = I(\mathcal{C}/x)$ and $J = I(\mathcal{C})$. Thus, $f_1', \ldots, f_{r_2}'$ are the edges of $\mathcal{C}/x$ and $f_i' = f_i$ for $r_1 + 1 \leq i$. Furthermore, if $i > r_2$, then $f_j' \subseteq f_i$ for some j. So, for each $1 \leq i \leq r$ there is $j \leq r_2$ such that $\widetilde{f}_j'|f_i$. Consequently, if $m \in G(J^k)$, then there is $m' \in G(I^k)$ such that $m'|m$. We take $\mathcal{L} = (z_1, \ldots, z_s)$ where $z_j = x_j^{\beta_{i_j}}$. Hence if $\mathcal{L}$ is an I^k-minimal ideal, then $J^k \subseteq \mathcal{L}$. Furthermore, $G(\mathrm{rad}(\mathcal{L})) = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq A$ since $\mathcal{L}$ is I^k-minimal. Now, we suppose $\mathcal{L}$ is J^k-minimal and $G(\mathrm{rad}(\mathcal{L})) \subseteq A$. If $m \in G(I^k)$, then $m = \widetilde{f}_1'^{a_1} \cdots \widetilde{f}_{r_2}'^{a_{r_2}}$ with $a_1 + \cdots + a_{r_2} = k$. So, $x^\alpha m = \widetilde{f}_1^{a_1} \cdots \widetilde{f}_{r_2}^{a_{r_2}} \in J^k$, where $\alpha = a_1 + \cdots + a_{r_1}$. Thus,

$z_j|x^\alpha m$ for some $j \leq s$. Since $x \notin A$, $\gcd(x, z_j) = 1$, and $z_j|m$. Therefore $I^k \subseteq \mathcal{L}$. Now, we will prove that $\mathcal{L}$ is an I^k-minimal ideal if and only if $\mathcal{L}$ is an J^k-minimal ideal and $G(\text{rad}(\mathcal{L})) \subseteq A$. Assume $\mathcal{L}$ is I^k-minimal so, $\text{rad}(\mathcal{L}) \subseteq A$. If $\mathcal{L}$ is not J^k-minimal, then there is $\mathcal{L}'$ such that $J^k \subseteq \mathcal{L}' \subset \mathcal{L}$ and $\text{rad}(\mathcal{L}') \subseteq \text{rad}(\mathcal{L}) \subseteq A$. Consequently, $I^k \subseteq \mathcal{L}'$. A contradiction, therefore $\mathcal{L}$ is J^k-minimal. Now suppose $\mathcal{L}$ is J^k-minimal and $\mathcal{L}$ is not I^k-minimal, then there is $\mathcal{L}'$ such that $I^k \subseteq \mathcal{L}' \subset \mathcal{L}$. This implies $J^k \subseteq \mathcal{L}'$, a contradiction, since $\mathcal{L}$ is J^k-minimal.

Hence, $\text{Ass}(I^k) = \{P \in \text{Ass}(J^k) \mid G(P) \subseteq A\}$ for each k. Since J has the persistence property, if $P \in \text{Ass}(I^k)$, then $P \in \text{Ass}(J^{k+1})$ and $G(P) \subseteq A$. Thus, $P \in \text{Ass}(I^{k+1})$. Therefore, I has the persistence property.

(Strong). Now, we set $m \in (I^{k+1} \colon I)$. If $1 \leq i \leq r_2$, then $mf'_i = \ell_i f_1'^{\alpha_{i1}} \cdots f_r'^{\alpha_{ir_2}}$ where $\ell \in \text{Mon}(R)$ and $\alpha_{i1} + \cdots + \alpha_{ir_2} = k+1$. We take $u_i = \alpha_{i1} + \cdots + \alpha_{ir_1}$. If $i \leq r_1$, then

$$x^{k+1} m f_i = x^{k+2} m f'_i = x^{k+2} \ell_i (f'_1)^{\alpha_{i1}} \cdots (f'_{r_2})^{\alpha_{ir_2}} = x^{k+2-u_i} \ell_i f_1^{\alpha_{i1}} \cdots f_{r_2}^{\alpha_{ir_2}}.$$

Now if $r_1 + 1 \leq i \leq r_2$, then $x^{k+1} m f_i = x^{k+1} m f'_i = x^{k+1-u} \ell f_1^{\alpha_{i1}} \cdots f_{r_2}^{\alpha_{ir_2}}$. Finally if $r_2 + 1 \leq i \leq r$, then there exist $j \leq r_1$ such that $f'_j | f_i$. So,

$$x^{k+1} m f_i = \frac{f_i}{f'_j} x^{k+1} m f'_j = \frac{f_i}{f'_j} x^{k+1-u_j} \ell_j f_1^{\alpha_{j1}} \cdots f_{r_2}^{\alpha_{jr_2}}.$$

Consequently, $x^{k+1} m \in (J^{k+1} \colon J) = J^k$. This implies $x^{k+1} m = \ell f_1^{\beta_1} \cdots f_r^{\beta_r}$ with $\ell \in \text{Mon}(R)$ and $\beta_1 + \cdots + \beta_r = k$. Since $x \nmid f_j$ for $j \geq r_1 + 1$, $x^w | \ell$, where $w = k + 1 - (\beta_1 + \cdots + \beta_{r_1})$. Therefore, $\ell = x^w \ell'$ where $\ell' \in \text{Mon}(R)$ and $m = \ell' (f'_1)^{\beta_1} \cdots (f'_{r_1})^{\beta_{r_1}} (f_{r_1+1})^{\beta_{r_1}+1} \cdots f_r^{\beta_r} \in I^k$. □

Remark 2.48 The converse affirmation of Proposition 2.47 is not true. We take $\mathcal{C}_0$ as in Example 2.32. So, $\mathcal{C}_0 \diagup \{x_i\}$ is a simple graph for each i. Hence, by Theorem 2.9, $\mathcal{C}_0 \diagup \{x_i\}$ has the strong persistence property.

Definition 2.49 Let $\mathcal{C} = (V, E)$ be a clutter and $\sigma \in S_V$ a permutation. We consider the clutter $\sigma(\mathcal{C}) = (V, E')$ where $E' = \{x_{\sigma(i_1)} \cdots x_{\sigma(i_s)} \mid x_{i_1} \cdots x_{i_s} \in E\}$.

Proposition 2.50 If $\mathcal{C}$ has the strong persistence property and $\sigma \in S_{V(\mathcal{C})}$, then $\sigma(\mathcal{C})$ also has the strong persistence property.

Proof. We take a morphism of k-algebras $\phi : R = K[x_1, \ldots, x_n] \to R$ given by $\phi(x_i) = x_{\sigma(i)}$. Hence, ϕ is an automorphism of R, with $\phi(I(\mathcal{C})) = I(\sigma(\mathcal{C}))$. Therefore, $I(\mathcal{C})$ and $I(\sigma(\mathcal{C}))$ are isomorphic. □

2.5 THE SYMBOLIC STRONG PERSISTENCE PROPERTY

In this section we study some properties of the strong persistence property in a general ring. Furthermore, we introduce the symbolic strong persistence property and we prove that the strong persistence property implies the symbolic strong persistence property.

Theorem 2.51 An ideal I has the strong persistence property if and only if $(I^t : I^s) = I^{t-s}$ for all $s \leq t$.

Proof. We proceed by induction on s. For $s = 1$ we recover the strong persistence property. Now, we take $a \in (I^t : I^{s+1})$ with $t \geq s+1$ and $x \in I$, then $axb \in I^t$ for all $b \in I^s$. Hence $ax \in (I^t : I^s)$. By induction hypothesis $ax \in I^{t-s}$. Consequently $a \in (I^{t-s} : I)$ and, by induction, $a \in I^{t-s-1}$. Therefore $(I^t : I^{s+1}) = I^{t-s-1}$. □

Corollary 2.52 If I has the strong persistence property, then I^t has the strong persistence property.

Proof. By Theorem 2.51 $(I^{kt} : I^t) = I^{kt-t} = I^{t(k-1)}$ for all $k \geq 1$. Therefore, I^t has the strong persistence property. □

By [23] normal ideals in an integer domain satisfy $(I^r : I^s) = I^{r-s}$ for all $s \leq r$. Hence, by Theorem 2.51 a normal ideal has the strong persistence property, but the converse affirmation is not true.

Example 2.53 ([28]) Let G be a simple connected graph, the $I(G)$ has the strong persistence property but if $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $E(G) = \{x_1x_2, x_2x_3, x_1x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_5x_7\}$, then $I(G)$ is not normal.

Definition 2.54 Let $P_1, \ldots, P_r$ be the minimal primes of I. The k-th symbolic power of I is $I^{(k)} = q_1 \cap \cdots \cap q_r$, where q_i is the P_i-primary component of I^k.

Remark 2.55 $I^{(i)} \subseteq (I^{(i+1)} : I^{(1)})$ for each i.

Definition 2.56 I has the symbolic strong persistence property if $\left(I^{(i+1)} : I^{(1)}\right) = I^{(i)}$ for each i.

Theorem 2.57 Strong persistence property implies the symbolic strong persistence property.

Proof. Let $\text{Min}(I) = \{P_1, \dots, P_r\}$ be the set of minimal primes containing I. We take $I^d = Q_{1d} \cap \cdots \cap Q_{s_d d}$ a minimal primary decomposition of I^d for each d. We can suppose that there exists $r_d \leq s_d$ such that $\text{rad}(Q_{id}) \in \text{Min}(I)$ if and only if $i \leq r_d$. Now for $j > r_{k+1}$, then $\text{rad}(Q_{j\,k+1})$ is not minimal. Consequently, $\text{rad}(Q_{j\,k+1}) \nsubseteq \text{rad}(Q_{i\,k+1})$ with $i \leq r_{k+1}$. This implies $\text{rad}(Q_{j\,k+1}) \nsubseteq B$, where $B = \bigcup_{i=1}^{r_{k+1}} \text{rad}(Q_{i\,k+1})$. Thus, there is $a_j \in \text{rad}(Q_{j\,k+1}) \setminus B$. So, $b_j = a_j^{s_1} \in Q_{j\,k+1}$ for some s_j. Hence, $b_j \in Q_{jk+1} \setminus B$. Now, we take $a \in (I^{(k+1)} \colon I^{(1)})$, then $ax \in I^{(k+1)}$ for all $x \in I^{(1)}$. Consequently, if $c = \prod_{j \geq r_{k+1}} b_j$, then $axc \in I^{k+1}$ for all $x \in I$ since $I \subseteq I^{(1)}$. So, $ac \in I^k$, since I has the strong persistence property. Furthermore, if $j > r_{k+1}$, then $b_j \notin \text{rad}(Q_{ik})$ for $i \leq r_k$. Thus, $a \in Q_{ik}$ for $1 \leq i \leq r_k$, since $ac \in Q_{ik}$ and Q_{ik} is primary. Therefore, $a \in I^{(k)}$. □

Proposition 2.58 An ideal I has the symbolic strong persistence property if and only if $\left(I^{(r)} \colon I^{(s)}\right) = I^{(r-s)}$ for all $s \leq r$.

Proof. Similar to proof of Theorem 2.51. □

CHAPTER 3

ON GORENSTEIN HOMOGENOEOUS MONOMIAL SUBRINGS OF GRAPHS

3.1 INTRODUCTION

Let $G = (V(G), E(G))$ be a graph whose vertex set and edge set are $V(G) = \{x_1, \ldots, x_n\}$ and $E(G) = \{y_1, \ldots, y_q\}$, respectively. Let $y = \{x_i, x_j\}$ be an edge of G, the characteristic vector of y is the vector $v_y \in \{0,1\}^n$ such that its i-th entry is 1, its j-th entry is 1, and the remaining entries are zero, i.e., $v_y = e_i + e_j$. We denote by $v_1, \ldots, v_q$ the characteristic vector of $y_1, \ldots, y_q$, respectively. We consider the set $w_1, \ldots, w_r$ of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some $i \in \{1, \ldots, r\}$. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, the **homogeneous monomial** subring of G is the ring:

$$S_G = K[x^{w_1}t, \ldots, x^{w_r}t] \subset R[t]$$

where t is a new variable. Since $(w_i, 1)$ lies in the hyperplane $x_{n+1} = 1$ for each i then, S_G is a standard K-algebra, where a monomial $x^a t^b$ has degree b. We assume that S_G has this grading. If S_G is normal, then according to Danilov-Stanley formula (see [4, 6]), the canonical module of S_G is the ideal given by

$$w_S = \left(\{x^a t^b \mid (a,b) \in \mathbb{N}B \cap (\mathbb{R}_+ B)^\circ\}\right),$$

where $B = \{(w_1, 1), \ldots, (w_r, 1)\}$ and $(\mathbb{R}_+ B)^o$ is the interior of $\mathbb{R}_+ B$ relative to $\text{aff}(\mathbb{R}_+ B)$ (the affine hull of $\mathbb{R}_+ B$). Furthermore, $\text{aff}(\mathbb{R}_+ B) = \mathbb{R}^{n+1}$. In [9] is proven that if G is connected then S is normal if and only if there exists a edge between every two vertex disjoint odd cycles. A vertex cover is a subset A of $V(G)$ such that $A \cap e \neq \emptyset$ for each $e \in E(G)$. The cover number of G, $\tau(G)$, is the cardinality of a minimum vertex cover. G is called unmixed if every minimal vertex cover has $\tau(G)$ elements.

A monomial algebra $\mathcal{A}$ is called **Gorenstein** if $\mathcal{A}$ is Cohen-Macaulay and its canonical module $w_{\mathcal{A}}$ is a principal ideal. In [15] Hochsther proved that if $\mathcal{A}$ is normal then $\mathcal{A}$ is Cohen-Macaulay. Hence, if $\mathcal{A}$ is normal then $\mathcal{A}$ is Gorenstein if and only if $w_{\mathcal{A}}$ is principal. In [9] is proven that if G is bipartite then S_G is Gorenstein if and only if G is unmixed. In this chapter we prove that if S_G is normal then S_G is Gorenstein if and only if G is unmixed and $\tau(G) = \lceil \frac{n}{2} \rceil$. Furthermore we prove that if n is even and S_G is Gorenstein, then G is bipartite.

3.2 Some properties of unmixed graphs

A subset F of $V(G)$ is a **stable set** if $y \not\subseteq F$ for each $y \in E(G)$. The cardinality of a maximum stable set is denoted by $\beta(G)$. Furthermore, G is called well-covered if every maximal stable set has $\beta(G)$ elements. F is a (maximal) stable set if and only if $V(G) \setminus F$ is a (minimal) vertex cover. Hence, $\tau(G) + \beta(G) = |V(G)|$. Furthermore, G is **unmixed** if and only if G is well-covered. A set of induced subgraphs $G_1, \ldots, G_s$ of G is a τ**-reduction** of G if $\{V(G_1), \ldots, V(G_s)\}$ is a partition of $V(G)$ and $\tau(G) = \sum_{i=1}^{s} \tau(G_i)$.

Proposition 3.1 Let G be a bipartite graph. G is unmixed if and only if there is a τ-reduction $y_1, \ldots, y_r$ such that $y_i \in E(G)$.

Proposition 3.2 ([22]) If G is an unmixed graph, with $\tau(G) = \frac{n+1}{2}$, then there exist a τ-reduction $\{H_1, \ldots, H_s\}$ of G such that $H_i \in E(G)$ for $1 \leq i \leq s-1$ and H_s is an j-cycle with $j \in \{3, 5, 7\}$. Furthermore, if $V(H_i) = \{a, a'\}$ and $\{a, b\}, \{a', b'\} \in E(G)$, then $\{b, b'\} \in E(G)$.

Lemma 3.3 If $G_1, \ldots, G_s$ is a τ-reduction of G, then $\sum_{i=1}^{s} \beta(G_i) = \beta(G)$.

Proof. Since $G_1, \ldots, G_s$ is a τ-reduction of G, $\sum_{i=1}^{s} \tau(G_i) = \tau(G)$. Hence,

$$\sum_{i=1}^{s} \beta(G_i) = \sum_{i=1}^{s} |V(G_i)| - \tau(G_i) = \sum_{i=1}^{s} |V(G_i)| - \sum_{i=1}^{s} \tau(G_i) = |V(G)| - \tau(G) = \beta(G),$$

since $\{V(G_1), \ldots, V(G_s)\}$ is a partition of $V(G)$. □

Lemma 3.4 If G is an unmixed graph with a τ-reduction $G_1, \ldots, G_s$, then $\beta(G_i) = |F \cap V(G_i)|$ for each F maximal stable of G.

Proof. We take F a maximal stable set of G. Hence $|F \cap V(G_i)| \leq \beta(G_i)$. Consequently, by Lemma 3.3, $|F| = \sum_{i=1}^{s} |F \cap V(G_i)| \leq \sum_{i=1}^{s} \beta(G_i) = \beta(G)$, since $\{V(G_1), \ldots, V(G_s)\}$ is a partition of $V(G)$. But G is well-covered, then $|F| = \beta(G)$. Therefore $|F \cap V(G_i)| = \beta(G_i)$. □

Proposition 3.5 Let G be an unmixed graph with a τ-reduction $G_1, \ldots, G_s$ such that $G_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is a 7-cycle.

1) If $\{x_1, y_1\} \in E(G)$ with $y_1 \notin V(H_1)$, then $\mathrm{N}_G(y_1) \cap \{x_3, x_6\} \neq \emptyset$.

2) If there is not a 4-cycle C such that $V(C) \cap V(G_1)$ is an edge, then there is a stable set $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ where $1 \leq i_1 < i_2 < i_3 \leq 7$, and $\deg(x_{i_j}) = 2$ for $j = 1, 2, 3$.

Proof. 1) By contraction, suppose $\mathrm{N}_G(y_1) \cap \{x_3, x_6\} = \emptyset$. Hence $A = \{y_1, x_3, x_6\}$ is a stable set. Consequently, there is a maximal stable set F of G such that $A \subseteq F$. Thus, $F \cap V(G_1) = \{x_3, x_6\}$. A contradiction by Lemma 3.4, since $\beta(G_1) = 3$.

2) We can suppose there is $\{x_1, y_1\} \in E(G)$ with $y_1 \notin V(C)$. By 1) assume $\{y_1, x_3\} \in E(G)$. If $\deg_G(x_2) \neq 2$, then there is $\{x_2, y_2\} \in E(G)$ with $y_2 \notin V(C)$. By 1) we can suppose $\{x_4, y_2\} \in E(G)$. By hypothesis $\{y_1, y_2\} \notin E(G)$. Consequently, there is a maximal stable set $F \supseteq \{y_1, y_2\}$. Hence, $F \cap V(G_1) \subseteq \{x_5, x_6, x_7\}$. Therefore $|F \cap V(G_1)| \leq 2$. A contradiction by Lemma 3.4, since $\beta(G_1) = 3$. This implies $\deg_G(x_2) = 2$. If $\deg_G(x_7) = \deg_G(x_4) = 2$, then $\{x_3, x_4, x_7\}$ is a stable set. Now, we can assume $\deg_G(x_4) \neq 2$. So, there is $\{x_4, y_2\} \in E(G)$ with $y_2 \notin V(G)$. Thus, by 1) $\{y_2, x_6\}$, since $\deg_G(x_2) = 2$. Hence, by the last argument $\deg_G(x_5) = 2$. Furthermore by 1), $\deg_G(x_7) = 2$ since $\deg_G(x_2) = \deg_G(x_5) = 2$. Therefore $\{x_2, x_5, x_7\}$ is a stable set. □

Proposition 3.6 Let G be an unmixed graph with a τ-reduction $G_1, \ldots, G_s$ such that $G_1 = (x_1, x_2, x_3, x_4, x_5)$ is a 5-cycle. If G has no a 4-cycle C such that $V(C) \cap V(G_1)$ is an edge, then there is a stable set $\{x_{i_1}, x_{i_2}\} \subseteq V(G_1)$ such that $\deg_G(x_{i_1}) = \deg_G(x_{i_2}) = 2$.

Proof. Assume $\deg_G(x_1) \geq 3$. If $\deg_G(x_2) = \deg_G(x_5) = 2$ we obtain the result. So, we can suppose there are $\{x_1, y_1\}, \{x_2, y_2\} \in E(G)$ such that $y_1, y_2 \notin V(G_1)$. By hypothesis $\{y_1, y_2\} \notin E(G)$. If $A = \{y_1, y_2, x_4\}$ is a stable set, then there is a maximal stable set F such that $A \subseteq F$. But $F \cap V(G_1) = \{x_4\}$ a contradiction by Lemma 3.4, since $\beta(G_1) = 2$. So, we can assume $\{y_1, x_4\} \in E(G)$. If $\{x_3, y_3\} \in E(G)$ with $y_3 \notin V(G_1)$, then $A_1 = \{y_1, y_2, y_3\}$ is a stable set by hypothesis. Hence, if F_1 is a maximal stable set with $A_1 \subseteq F_1$, then $F_1 \cap V(G_1) \subseteq \{x_1, x_5\}$.

A contradiction by Lemma 3.4. Thus, $\deg_G(x_3) = 2$. Now if $\{x_5, y_5\} \in E(G)$ with $y_5 \notin V(G_1)$, then $A_2 = \{y_5, y_1\} \notin E(G)$. So, there is a maximal stable set F_2 with $A_2 \subseteq F_2$ and $F_2 \cap V(G_1) \subseteq \{x_2, x_3\}$. A contradiction, therefore $\deg_G(x_5) = 2$. $\square$

3.3 GORENSTEIN HOMOGENEOUS SUBRING OF GRAPHS

Let S_G be the homogeneous monomial subring of G, then $\mathbb{R}_+B = H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H^+_{(-\ell_m,1)}$ for some $\ell_1, \ldots, \ell_m \in \mathbb{R}^n$, where $H^+_w = \{v \in \mathbb{R}^{n+1} \mid v \cdot w \geq 0\}$. Hence, if S_G is normal, then:

$$\begin{aligned} \omega_S &= \left(\{x^a t^b \mid (a,b) \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ\}\right) \\ &= \left(\left\{ x^a t^b \mid \begin{array}{l} (a,b)\cdot(-l_j,1) > 0 \text{ for } j = 1,\ldots,m \\ (a,b)\cdot e_i > 0 \text{ for } i = 1,\ldots,n \end{array} \right\}\right). \end{aligned}$$

Notation. In this chapter we take $|v| = v \cdot \mathbf{1} = \sum_{i=1}^n v_i$, where $v = (v_1, \ldots, v_n)$.

Lemma 3.7 If $(w, a) \in \mathbb{N}B$ with $w \in \mathbb{R}^n$ and $a \in \mathbb{R}$, then $|w| \leq 2a$.

Proof. Since $(w, a) \in \mathbb{N}B$ then $(w, a) = \sum_{i=1}^r \lambda_i (w_i, 1)$, where $\lambda_i \in \mathbb{N}$. Thus, $|w| = |\sum_{i=1}^r \lambda_i w_i| = \sum_{i=1}^r \lambda_i |w_i| \leq 2 \sum_{i=1}^r \lambda_i$. Hence $|w| \leq 2a$, since $a = \sum_{i=1}^r \lambda_i$. $\square$

Proposition 3.8 Let G be a connected graph. If τ is a generating tree of G and $\tilde{e}_\tau = \sum_{v_i \in E(\tau)} (v_i, 1) + e_{n+1}$, then $\tilde{e}_\tau \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ$.

Proof. Since $\mathbb{R}_+B = H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H^+_{(-\ell_m,1)}$, it is sufficient to show that $\tilde{e}_\tau \cdot e_i > 0$ and $\tilde{e}_\tau \cdot (-\ell_j, 1) > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Since τ is a generating tree, we have $\tilde{e}_\tau \cdot e_i > 0$ for $i = 1, \ldots, n$. On the other hand, $(v_i, 1) \cdot (-\ell_j, 1) \geq 0$, since $(v_i, 1) \in \mathbb{R}_+B$ for $v_i \in E(\tau)$. Furthermore, $e_{n+1} \cdot (-\ell_j, 1) = 1 > 0$. Therefore $\tilde{e}_\tau \cdot (-\ell_j, 1) > 0$ for $j = 1, \ldots, m$, since $\tilde{e}_\tau = \sum_{v_i \in E(\tau)} (v_i, 1) + e_{n+1}$. $\square$

Proposition 3.9 If $I = \{x_{j_1}, \ldots, x_{j_d}\}$ is a maximal stable set and $\ell' = \sum_{i=1}^d e_{j_i}$, then $H_{(-\ell',1)}$ is a supporting hyperplane of $\mathbb{R}_+B$.

Proof. Since I is an independent set then $\ell' \cdot v_i \leq 1$ for $i = 1, \ldots, q$. Hence, $\ell' \cdot w_j \leq 1$ and $0 \leq (-\ell', 1) \cdot (w_j, 1)$ for $j = 1, \ldots, r$. We can assume without loss of generality that $I = \{x_1, \ldots, x_d\}$ then $\mathcal{C} = V(G) \setminus I = \{x_{d+1}, \ldots, x_n\}$ is a minimal vertex cover. Hence, there exists $v_1, \ldots, v_{n-d} \in E(G)$ such that $v_i \cap \mathcal{C} = \{x_{d+i}\}$ for $i = 1, \ldots, n-d$. Thus, $(e_1, 1), \ldots, (e_d, 1), (e_{v_1}, 1), \ldots, (e_{v_{n-d}}, 1)$ are independent vector in $H_{(-\ell',1)} \cap \mathbb{R}_+ B$. Therefore $H_{(-\ell',1)}$ is a supporting hyperplane of $\mathbb{R}_+ B$. $\square$

Let $\mathcal{C}$ be a minimal vertex cover of G, we suppose, without loss of generality, that $\mathcal{C} = \{x_1, \ldots, x_c\}$. Since $\mathcal{C}$ is minimal, there exist $y_1, \ldots, y_c \in E(G)$ such that $y_i \cap \mathcal{C} = \{x_i\}$. We can suppose, without loss of generality, that $y_i = \{x_i, x_{j_i}\}$ for $i = 1, \ldots, c$, where $\{x_{j_1}, \ldots, x_{j_c}\} = \{x_{c+1}, \ldots, x_{c+s}\}$ (some x_{j_i} can be equal to each other). We define

$$\tilde{e}(\mathcal{C}) = e_{n+1} + \sum_{i=1}^{c} (v_i, 1) + \sum_{j=s+c+1}^{n} (e_j, 1)$$

where $v_i = e_i + e_{j_i}$ (the characteristic vector of y_i). Then,

$$\begin{aligned} \tilde{e}(\mathcal{C}) &= e_{n+1} + \sum_{i=1}^{c} (e_i + e_{j_i} + e_{n+1}) + \sum_{i=s+c+1}^{n} (e_i + e_{n+1}) \\ &= \sum_{i=1}^{c} e_i + \sum_{i=1}^{c} e_{j_i} + \sum_{i=s+c+1}^{n} (e_i) + (1 + c + n - (s+c)) e_{n+1} \\ &= (\underbrace{1, \ldots, 1}_{c}, a_{c+1}, \ldots, a_{c+s}, \underbrace{1, \ldots, 1}_{n-(s+c)}, n - s + 1) \end{aligned} \tag{3.1}$$

where $a_{c+i} \geq 1$ for $i = 1, \ldots, s$. Furthermore, $c = \sum_{i=1}^{c} |e_i| = \sum_{i=1}^{c} |e_{j_i}| = \sum_{i=1}^{s} a_{c+i}$.

Proposition 3.10 If S is normal and $\mathcal{C}$ is a minimal vertex cover of G then $x^{\tilde{e}(\mathcal{C})} \in \omega_S$

Proof. By definition $\tilde{e}(\mathcal{C}) \in \mathbb{N}B$. Furthermore, $(v_i, 1) \cdot (-\ell_u, 1) \geq 0$ and $(e_j, 1) \cdot (-\ell_u, 1) \geq 0$ for $1 \leq u \leq m$, since $(v_i, 1), (e_j, 1) \in \mathbb{R}_+ B$. Also, $e_{n+1} \cdot (-\ell_u, 1) = 1 > 0$. Hence, $\tilde{e}(\mathcal{C}) \cdot (-\ell_u, 1) > 0$ for $1 \leq u \leq m$. By (3.1), $\tilde{e}(\mathcal{C}) \cdot e_i > 0$ for $1 \leq i \leq n$. Therefore, $x^{\tilde{e}(\mathcal{C})} \in \omega_S$. $\square$

Proposition 3.11 Let G be a connected graph. If S_G is normal and w_S is principal, then $w_S = (x^{\mathbf{1}} t^{\beta})$ where $\beta \leq \lfloor \frac{n}{2} \rfloor + 1$.

Proof. If ω_S is principal then $w_S = (x^v t^\beta)$ with $(v, \beta) \in \mathbb{N}B \cap (\mathbb{R}_+ B)^\circ$. But $\mathbb{R}_+ B = H_{e_1}^+ \cap \cdots \cap H_{e_n}^+ \cap H_{(-\ell_1,1)}^+ \cap \cdots \cap H_{(-\ell_m,1)}^+$. Hence, if $v = (v_1, \ldots, v_n)$ then $v_i = v \cdot e_i > 0$. Furthermore $(v, \beta) \in \mathbb{N}B$ then $v_i \geq 1$. On the other hand if $b_1 = \max\{|\ell_1|, |\ell_2|, \ldots, |\ell_m|\}$, then $(\mathbf{1}, b_1) \in (\mathbb{R}_+ B)^\circ$. Also $(\mathbf{1}, n) \in \mathbb{N}B$.

Thus, if $b_2 = \max\{b_1, n\}$ then $(\mathbf{1}, b_2) \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ$ and $x^{\mathbf{1}}t^{b_2} \in w_S$. Hence, $x^{\mathbf{1}}t^{b_2} = (x^v t^\beta)(x^{v'} t^{\beta'})$ but $\mathbf{1} \leq v$, then $\mathbf{1} = v$. Now by Proposition 3.8 $x^{\tilde{e}_\tau} \in \omega_S$, thus $x^{\tilde{e}_\tau} = (x^{\mathbf{1}}t^\beta)x^u$ where $x^u \in S_G$. So, $\tilde{e}_\tau - (\mathbf{1}, \beta) = u \in \mathbb{N}B$. Since τ is a generating tree of G, $|E(\tau)| = n-1$. Hence, $\tilde{e}_\tau = (v, n)$ where $v = \sum_{v_i \in E(\tau)} v_i$. Thus, $u = (v - \mathbf{1}, n - \beta)$ and by Lemma 3.7

$$2(n-\beta) \geq |v - \mathbf{1}| = |v| - |\mathbf{1}| = 2(n-1) - n = n - 2.$$

Therefore, $\beta \leq \lfloor \frac{n}{2} \rfloor + 1$. □

Proposition 3.12 If G is a connected not bipartite graph and $\ell = (\frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{R}^n$, then $H_{(-\ell,1)}$ is a supporting hyperplane of $\mathbb{R}_+B$.

Proof. If $(w_i, 1) \in B$, then $w_i \cdot \ell = 1$ or $w_i \cdot \ell = \frac{1}{2}$ or $w_i \cdot \ell = 0$. Hence, $w_i \cdot \ell \leq 1$ and we have that $(w_i, 1) \cdot (-\ell, 1) \geq 0$. Let C be an odd cycle of G, we take an edge e and a generating tree τ such that C is the unique cycle of $\tau \cup e$. We can assume that $E(\tau \cup e) = \{y_1, \ldots, y_n\}$ with $E(C) = \{y_1, \ldots, y_k\}$ where k is odd. If v_i is the characteristic vector of t_i, then we can suppose $v_i = e_i + e_{i+1}$ for $i = 1, \ldots, k-1$ and $v_k = e_k + e_1$. Thus,

$$\textstyle\sum_{i=1}^k (-1)^{i+1} v_i = (e_1 + e_2) - (e_2 + e_3) + (e_3 + e_4) - \cdots + (e_k + e_1) = 2e_1.$$

Hence, $e_1 \in \mathbb{R}(v_1, \ldots, v_k)$. In the same form we have that $e_i \in \mathbb{R}(v_1, \ldots, v_k)$ for $i = 1, \ldots, k$. Thus $\mathbb{R}(v_1, \ldots, v_k) = \mathbb{R}^k$. Therefore, $v_1, \ldots, v_k$ are linearly independent. Since τ is a generating tree, we can index its edges such that if $A_j = \{y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+j}\}$, then the induced subgraph $G_j = [A_j]_\tau$ is a connected graph. We will prove that A_j is a linearly independent set, by induction on j. For $j = 0$ it already has been proven. Now, we take

$$A_{j+1} = \{y_1, \ldots, y_k, y_{k+1}, \ldots, y_{k+j}, y_{k+j+1}\}.$$

Since $G_{j+1} = [A_{j+1}]_\tau$ is connected and $\tau \cup e$ has only one cycle, $|V(G_j) \cap y_{k+j+1}| = 1$. So, we can suppose $V(G_j) = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+j}\}$ and $y_{k+j+1} = \{x_s, x_{k+j+1}\}$ with $s \in \{1, \ldots, k+j\}$. Hence, $\mathbb{R}(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+j}) \subseteq \mathbb{R}(e_1, \ldots, e_{k+j})$ and by induction hypothesis $\mathbb{R}(v_1, \ldots, v_k, \ldots, v_{k+j}) = \mathbb{R}(e_1, \ldots, e_{k+j})$. Furthermore, since $v_{k+j+1} = e_{k+j+1} + e_s$ then

$$\mathbb{R}(v_1, \ldots, v_k, \ldots, v_{k+j}, v_{k+j+1}) = \mathbb{R}(e_1, \ldots, e_{k+j+1}).$$

Hence, $(v_1, 1), \ldots, (v_n, 1)$ are linearly independent of B and these are ortogonal to $(-\ell, 1)$. Therefore, $H_{(-\ell,1)}$ is a supporting hyperplane of $\mathbb{R}_+B$. □

Lemma 3.13 If G is a connected not bipartite graph, S_G is normal and $w_S = (x^{\mathbf{1}}t^{b_0})$, then G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and $b_0 = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. By Proposition 3.11, $b_0 \leq \lfloor \frac{n}{2} \rfloor + 1$. Furthermore, by Proposition 3.12 $(\mathbf{1}, b_0) \cdot (-\ell, 1) > 0$ where $\ell = (\frac{1}{2}, \ldots, \frac{1}{2})$. Thus, $b_0 > \ell \cdot \mathbf{1} = \frac{n}{2}$. Therefore, $b_0 = \lfloor \frac{n}{2} \rfloor + 1$.

Now, let $\mathcal{C}$ be a minimal vertex cover of G. By Proposition 3.10 $x^{\tilde{e}(\mathcal{C})} \in \omega_S$, then $x^{\tilde{e}(\mathcal{C})} = (x^{\mathbf{1}}t^{b_0})x^v$, where $x^v \in S_G$. Consequently, $x^v = x^{\tilde{e}(\mathcal{C})-(\mathbf{1},b_0)}$ so $\tilde{e}(\mathcal{C}) - (\mathbf{1}, b_0) = v \in \mathbb{N}B$. Hence, by (3.1)

$$v = (\underbrace{0, \ldots, 0}_{c}, a_{c+1} - 1, \ldots, a_{c+s} - 1, \underbrace{0, \ldots, 0}_{n-(s+c)}, n - s + 1 - b_0).$$

On the other hand, $\{x_{c+1}, \ldots, x_n\}$ is an independent vertex set, since $C = \{x_1, \ldots, x_c\}$ is a vertex cover. Also, $v \in \mathbb{N}B$ and the only possible entries of v different to zero are $c+1, \ldots, c+s+1$ and $n+1$, then $v = \sum_{i=c+1}^{c+s} \lambda_i(e_i + e_{n+1}) + \lambda_{n+1}e_{n+1}$. Thus, $v \cdot (\sum_{i=1}^{n} e_i) = \sum_{i=c+1}^{c+s} \lambda_i$ and $v \cdot e_{n+1} = (\sum_{i=c+1}^{c+s} \lambda_i) + \lambda_{n+1}$. This implies $v \cdot (\sum_{i=1}^{n} e_i) \leq v \cdot e_{n+1}$, i.e. $\sum_{i=1}^{s}(a_{c+i} - 1) \leq n - s + 1 - b_0$. Since $\sum_{i=1}^{s} a_{c+i} = c$, $\sum_{i=1}^{s}(a_{c+i} - 1) = c - s$. So, $c - s \leq n - s + 1 - b_0$. Consequently $c \leq \lceil \frac{n}{2} \rceil$, since $b_0 = \lfloor \frac{n}{2} \rfloor + 1$. By Proposition 3.9, $H_{(-\ell',1)}$ is a supporting hyperplane of $\mathbb{R}^+B$ where $\ell' = \sum_{i=c+1}^{n} e_i$. Hence, $(-\ell', 1) \cdot (\mathbf{1}, b_0) > 0$ and $b_0 > \ell' \cdot \mathbf{1} = n - c$. Furthermore $b_0 = \lfloor \frac{n}{2} \rfloor + 1$, then $c > n - b_0 = \lceil \frac{n}{2} \rceil - 1$. Therefore $c \geq \lceil \frac{n}{2} \rceil$, so $c = \lceil \frac{n}{2} \rceil$. □

Theorem 3.14 ([9]) Let G be a bipartite graph, then S_G is Gorenstein if and only if G is unmixed

Theorem 3.15 If G is connected, S_G is normal and n is even, then S_G is Gorenstein if and only if G is an unmixed bipartite graph.

Proof. $\Rightarrow$) Suppose that G is not bipartite, then there exist an odd cycle C. We can suppose $C = (x_1, \ldots, x_{2l+1})$. By Proposition 3.11 and Lemma 3.13, $w_s = (x^{\mathbf{1}}t^{b_0})$ where $b_0 = \lfloor \frac{n}{2} \rfloor + 1$, $\tau(G) = \frac{n}{2}$ and G is unmixed. By Proposition 3.1 there is a τ-reduction $\{y_1, \ldots, y_r\}$ with $y_i \in E(G)$. So, $r = \frac{n}{2}$. We take $u_i = v_{y_i}$ and $u = \sum_{i=1}^{r}(u_i, 1) + \sum_{j=1}^{l}(v_j, 1) + (e_1, 1)$ where $v_j = e_{2j} + e_{2j+1}$ for $j = 1, \ldots, l$. So, $u = (\mathbf{1} + \mathbf{1}_C, r + l + 1)$ where $\mathbf{1}_C = \sum_{i=1}^{2l+1} e_i$. Then, $u \in H_{e_1}^+ \cap \cdots \cap H_{e_n}^+$.

We will prove that $(\mathbf{1}_C, l+1) \in (\mathbb{R}_+B)^\circ$. By contradiction suppose there exist a hyperplane $H_{(-q,1)}$ such that $(-q, 1) \cdot (v_j, 1) = 0$ and $(e_1, 1) \cdot (-q, 1) = 0$, since $(\mathbf{1}_C, l+1) = \sum_{j=1}^{l}(v_j, 1) + (e_1, 1)$. Consequently, if $q = (q_1, \ldots, q_n)$, then $q_1 = 1$ and $(v_e, 1) \in H_{(-q,1)}^+$ where $v_e = e_1 + e_2$, since $e = \{x_1, x_2\} \in E(C)$. This implies,

$0 \leq -v_e \cdot q + 1$, then $q_1 + q_2 = q \cdot v_e \leq 1$, so $q_2 = 0$. Also $q_3 = 1$, since $v_2 \cdot q = 1$. Similarly we prove that $q_j = 0$ if j is even and $q_j = 1$ if j is odd, for $1 \leq j \leq 2l+1$. This implies $q_{2l+1} = 1$ and $(-q, 1)(e_1 + e_{2l+1}, 1) = -2 + 1 = -1$. A contradiction, since $\{x_1, x_{2l+1}\} \in E(C)$. Consequently, $(\mathbf{1}_C, l+1) \in (\mathbb{R}_+B)^\circ$. Therefore, $u \in (\mathbb{R}_+B)^\circ \cap \mathbb{N}B$. Thus, $(x,t)^u = x^{\mathbf{1}+\mathbf{1}_C} t^{r+l+1} \in w_s$. Furthermore $w_s = (x^{\mathbf{1}} t^{r+1})$, then $(\mathbf{1}_C, l) \in \mathbb{N}B$. Hence, by Lemma 3.7, $2l \geq |\mathbf{1}_C| = |V(C)| = 2l+1$. A contradiction, therefore G is bipartite. Furthermore by Theorem 3.14 G is unmixed.
$\Leftarrow$) By Theorem 3.14, S_G is Gorenstein. □

Proposition 3.16 If $C = (x_1, \ldots, x_r)$ is a cycle of G and $\mathbb{R}_+B = H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H_{(-\ell_m,1)}$, then $\sum_{i=1}^r \ell_j^i \leq \frac{s}{2}$ where $\ell_j = (\ell_j^1, \ldots, \ell_j^n)$.

Proof. Since $e = \{x_i, x_{i+1}\} \in E(C)$, then $e \cdot \ell_j \leq 1$. So $\ell_j^i + \ell_j^{i+1} \leq 1$ for $1 \leq i \leq s-1$ and $\ell_j^s + \ell_j^1 \leq 1$. Consequently,

$$2\sum_{i=1}^{r} \ell_j^i = (\ell_j^1 + \ell_j^s) + \sum_{i=1}^{r-1}(\ell_j^i + \ell_j^{i+1}) \leq \sum_{i=1}^{r} 1 = r.$$

Hence, $\sum_{i=1}^r \ell_j^i \leq \frac{s}{2}$. □

Definition 3.17 Let $w = (v, b)$ be an element of $\mathbb{R}_+B$, then $w = \sum_{i=1}^q \alpha_i(v_i, 1) + \sum_{i=1}^n \beta_i(e_i, 1) + \lambda e_{n+1}$ is a minimal representation of w in $\mathbb{R}_+B$ if $\sum_{i=1}^q \alpha_i + \sum_{i=1}^n \beta_i$ is minimal and $\{i \mid \lambda_i \neq 0\} \cup \{i \mid \beta_i \neq 0\}$ is minimal.

Remark 3.18 If $w = (w_1, \ldots, w_{n+1}) \in \mathbb{R}_+B$ and $w = \sum_{i=1}^q \alpha_i(v_i, 1) + \sum_{i=1}^n \beta_i(e_i, 1) + \lambda e_{n+1}$ is a minimal representation, then $w_{n+1} = \sum_{i=1}^q \alpha_i + \sum_{i=1}^n \beta_i + \lambda$. Hence λ is maximal, since $\sum_{i=1}^q \alpha_i + \sum_{i=1}^n \beta_i$ is minimal.

Lemma 3.19 Let $w = \sum_{i=1}^q \alpha_i(v_i, 1) + \sum_{i=1}^n \beta_i(e_i, 1) + \lambda e_{n+1}$ be a minimal representation of w in $\mathbb{R}_+B$. Hence,

1) $\mathcal{A} = \{x_i \in V(G) \mid \beta_i \neq 0\}$ is a stable set.

2) If G_w is the graph with $V(G_w) = V(G)$ and $E(G_w) = \{y_i \in E(G) \mid \alpha_i \neq 0\}$, then G_w has no even cycles.

3) If H is a connected component of G_w, then $|V(H) \cap \mathcal{A}| \leq 1$. Let $H_1, \ldots, H_s$ be connected components of G_w such that $V(H_i) \cap \mathcal{A} = \{z_i\}$. Hence, $H_1, \ldots, H_s$ are trees; furthermore if A_i is the chromatic class of z_i in H_i, then $A_1 \cup \cdots \cup A_s$ is a stable set in G.

Proof. 1) Suppose $x_{i_1}, x_{i_2} \in \mathcal{A}$ such that $y_j = \{x_{i_1}, x_{i_2}\} \in E(G)$. Hence,

$$w = \sum_{i=1}^{q} \alpha_i^1 (v_i, 1) + \sum_{i=1}^{n} \beta_i^1 (e_i, 1) + (\lambda + \gamma_1) e_{n+1}$$

where $\gamma_1 = \min\{\beta_{i_1}, \beta_{i_2}\}$; $\alpha_i^1 = \alpha_i$ if $i \neq j$ and $\alpha_j^1 = \alpha_j + \gamma_1$; $\beta_i^1 = \beta_i$ if $i \notin \{i_1, i_2\}$, $\beta_{i_1}^1 = \beta_{i_1} - \gamma_1$ and $\beta_{i_2}^1 = \beta_{i_2} - \gamma_1$. Thus, $\sum_{i=1}^{q} \alpha_i^1 + \sum_{i=1}^{n} \beta_i^1 < \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{n} \beta_i$. A contradiction, since the representation of w is minimal. Therefore, $\mathcal{A}$ is a stable set.

2) Suppose C is an even cycle of G_w whose edges are $y_{j_1}, \ldots, y_{j_{2k}}$. Thus,

$$w = \sum_{i=1}^{q} \alpha_i^2 (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1) + \lambda e_{n+1}$$

where $\gamma_2 = \min(\alpha_{j_1}, \ldots, \alpha_{j_{2k}})$; $\alpha_i^2 = \alpha_i + (-1)^i \gamma_2$ for $1 \leq i \leq 2k$ and $\alpha_i^2 = \alpha_i$ if $i \notin \{j_1, \ldots, j_{2k}\}$. A contradiction, since $|\{i \mid \alpha_i^2 \neq 0\}| < |\{i \mid \alpha_i \neq 0\}|$. Therefore, G_w has no even cycles.

3) For facility we take $w' = \sum_{i=1}^{q} \alpha_i (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1)$. Suppose $x_{i_1}, x_{i_2} \in V(H) \cap \mathcal{A}$, then $\beta_{i_1} \neq 0$ and $\beta_{i_2} \neq 0$. Since H is connected, there exist a path $\mathcal{L}$ of H between x_{i_1} and x_{i_2}. We can assume $E(\mathcal{L}) = \{y_{j_1}, \ldots, y_{j_s}\}$. We take $\gamma_3 = \min\{\alpha_{j_1}, \ldots, \alpha_{j_s}, \beta_{i_1}, \beta_{i_2}\}$, thus $\gamma_3 > 0$. If s is odd, then

$$w = w' + \gamma_3 \sum_{i=1}^{s} (-1)^{i+1} (v_{j_i}, 1) - \gamma_3 \sum_{j=1}^{2} (e_{i_j}, 1) + (\lambda + \gamma_3) e_{n+1}.$$

A contradiction, by Remark 3.18, since $\gamma_3 > 0$. Hence, s is even. If $\gamma_3 = \beta_{i_\ell}$ for $\ell \in \{1, 2\}$, then we can suppose $\gamma_3 = \beta_{i_1}$. Also, if $\gamma_3 = \alpha_{i_\ell}$ for some $\ell \in \{1, \ldots, s\}$, then we can assume ℓ is even, since in other case we take $z_{i_j} = y_{i_{s-j}}$ and $E(\mathcal{L}) = \{z_{i_1}, \ldots, z_{i_s}\}$. So,

$$w = w' + \gamma_3 \sum_{i=1}^{s} (-1)^{i+1} (v_{j_i}, 1) + \gamma_3 \sum_{j=1}^{2} (-1)^j (e_{i_j}, 1) + \lambda e_{n+1}.$$

A contradiction, since $\beta_{i_1} - \gamma_3 = 0$ or $\alpha_{i_\ell} - \gamma_3 = 0$. Therefore $|V(H) \cap \mathcal{A}| \leq 1$.

Now, suppose C is a cycle of H_i. Since H_i is connected, there is a path $\mathcal{L}$ between C and z_i. We can assume $z_i = x_1$, $E(\mathcal{L}) = \{y_1, \ldots, y_\ell\}$, and $E(C) = \{y_{\ell+1}, \ldots, y_{\ell+k}\}$. By 2) k is odd. We take $\gamma = \min\{\alpha_1, \ldots, \alpha_\ell, 2\alpha_{\ell+1}, \ldots, 2\alpha_{\ell+k}, \beta_1\}$. Consequently,

$$w = w' - 2\gamma(e_1, 1) + \gamma \sum_{i=1}^{\ell} (-1)^{i+1} (v_i, 1) + \frac{\gamma}{2} \sum_{i=1}^{k} (-1)^{\ell+1+i} (v_i, 1) + (\lambda + \frac{\gamma}{2}) e_{n+1}.$$

A contradiction, by Remark 3.18. Therefore H_i is a tree. Now, we take $a_1, b_1 \in A_i$. Suppose $\{a_1, b_1\} \in E(G)$, then there exist an even path $\mathcal{L}_1$ in H_i between z_i and a_1. We can assume $E(\mathcal{L}_1) = \{y_1, \ldots, y_{\ell_1}\}$ and $\{a_1, b_1\} = y_{\ell_1+1}$. Furthermore, there is a path $\mathcal{L}_2$ in G_w between b_1 and $\mathcal{L}_1$. We can assume $E(\mathcal{L}_2) = \{y_{\ell_1+2}, \ldots, y_{\ell_2}\}$. If $\{u\} = V(\mathcal{L}_1) \cap V(\mathcal{L}_2)$, then we can assume that $u \in y_{j-1} \cap y_j$ for some $2 \leq j \leq \ell_1$. Hence, there is an odd cycle C' with $E(C') = \{y_j, \ldots, y_{\ell_2}\}$, since $a_1, b_1 \in A_i$.

$$w = w' + \gamma' \sum_{i=1}^{j-1} (-1)^{i+1}(v_i, 1) + \frac{\gamma'}{2} \sum_{i=j}^{\ell_2} (-1)^{i+1}(v_i, 1) - \gamma'(e_1, 1) + (\lambda + \frac{\gamma'}{2})(e_{n+1})$$

where $\gamma' = \min\{\alpha_1, \ldots, \alpha_{j-1}, 2\alpha_j, \ldots, 2\alpha_{\ell_2}, \beta_1\}$. This is a contradiction, hence A_i is a stable set in G. Now, assume $a_i \in A_i$, $a_j \in A_j$, and $\{a_i, a_j\} \in E(G)$. Thus, there are even paths $\mathcal{L}_3$ and $\mathcal{L}_4$ in G_w between z_i, a_i and z_j, a_j, respectively. We can suppose $E(\mathcal{L}_3) = \{y_1, \ldots, y_{k_1}\}$, $y_{k_1+1} = \{a_i, a_j\}$, $E(\mathcal{L}_4) = \{y_{k_1+2}, \ldots, y_{k_2}\}$, $z_i = x_1$ and $z_j = x_2$. Hence,

$$w = w' + \textstyle\sum_{i=1}^{k_2+1} (-1)^{i+1} \gamma_4 (v_i, 1) - \sum_{i=1}^{2} \gamma_4 (e_i, 1) + (\lambda + \gamma_4) e_{n+1}$$

where $\gamma_4 = \min\{\alpha_1, \ldots, \widehat{\alpha_{k_1+1}}, \ldots, \alpha_{k_2}, \beta_1, \beta_2\}$. A contradiction by Remark 3.18, therefore $A_1 \cup \cdots \cup A_s$ is a stable set in G. □

Proposition 3.20 Let $H_1, \ldots, H_r$ be the components of G_w, where $H_i \cap \mathcal{A} = \{z_i\}$ and A_i is the chromatic class of z_i in H_i, for $i = 1, \ldots, s$. Furthermore, $H_i \cap \mathcal{A} = \emptyset$ for $i > s$. We take H_j with $j > s$, hence

1) If H_j is not bipartite and $e \in E(G)$ such that $e \cap H_j \neq \emptyset$, then $e \cap (A_1 \cup \cdots \cup A_s) = \emptyset$.

2) Suppose H_j is bipartite whose chromatic classes are D_j^1 and D_j^2, and $e \in E(G)$ such that $e \cap (A_1 \cup \cdots \cup A_s) \neq \emptyset$ and $b \in e \cap V(H_j)$. If $b \in D_j^1$, then D_j^2 is a stable set in G. Furthermore, if there are $e_1, e_2 \in E(G)$ such that $e_1 \cap A_{i_1} \neq \emptyset$, $e_2 \cap A_{i_2} \neq \emptyset$, and $e_1 \cap D_j^1 \neq \emptyset$, then $e_2 \cap V(H_j) \subseteq D_j^1$.

Proof. 1) There is an odd cycle $C \subseteq H_j$. By contradiction we can assume $z \in e \cap A_1$. Consequently there is an even path $\mathcal{L}$ in H_1 between z_1 and z. Furthermore, if $e = \{z, z'\}$, then there is a path $\mathcal{L}'$ in H_j between C and z'. We can suppose $E(\mathcal{L}) = \{y_1, \ldots, y_{\ell_1}\}$ with ℓ_1 odd, $E(\mathcal{L}') = \{y_{\ell_1+2}, \ldots, y_{\ell_2}\}$, $E(C) = \{y_{\ell_2+1}, \ldots, y_{\ell_3}\}$ and $y_{\ell_1+1} = e$, where $z \in y_{\ell_1}$, $z' \in y_{\ell_1+2}$, and $y_{\ell_2} \cap V(C) = y_{\ell_2} \cap y_{\ell_2+1} \neq \emptyset$. Hence $\ell_1 + 2$ is even and

$$w = w' - \gamma(e_1, 1) + \gamma \sum_{i=1}^{\ell_2} (-1)^{i+1}(v_i, 1) + \frac{\gamma}{2} \sum_{i=\ell_2+1}^{\ell_3} (-1)^{i+1}(v_i, 1) + (\lambda + \frac{\gamma}{2}) e_{n+1}$$

where $\gamma = \min\{\alpha_1, \ldots, \alpha_{\ell_2}, 2\alpha_{\ell_2+1}, \ldots, 2\alpha_{\ell_3}, \beta_1\}$. A contradiction by Remark 3.18.

2) We can assume $e = \{a, b\}$ such that $a \in A_1$ and $b \in D_j^1$. By contradiction suppose there is $e' = \{a', b'\} \subseteq D_j^2$. Consequently, there is a even path $\mathcal{L}$ in H_1 between z_1 and a. We can assume $z_1 = x_1$, $E(\mathcal{L}) = \{y_1, \ldots, y_{\ell_1}\}$, and $e = y_{\ell_1+1}$, then ℓ_1 is even. Furthermore, there is an odd path $\mathcal{L}_1 = \{a_1 = b, a_2, \ldots, a_{s_1} = a'\}$ in H_j between b and a', then s_1 is even. Also there is a path $\mathcal{L}_2$ in H_j between $\mathcal{L}_1$ and b'. We can assume $\mathcal{L}_2 = \{a_k, b_1, \ldots, b_{s_2} = b'\}$. Consequently, $C = (a_k, a_{k+1}, \ldots, a_{s_1}, b_{s_2}, b_{s_2-1}, \ldots, b_1)$ is an odd cycle. We take $\mathcal{L}' = (a_1 = b, a_2, \ldots, a_k)$. We can assume $E(\mathcal{L}') = \{y_{\ell_1+2}, y_{\ell_2+3}, \ldots, y_{\ell_1+k}\}$ and $E(C) = \{y_{\ell_1+k+1}, \ldots, y_u\}$. Furthermore, $e = y_{\ell_1} + 1$ and $e' = y_{\ell_1+s_1+1}$ where ℓ_1 and s_1 are even. Hence $(-1)^{\ell_1+2} = (-1)^{\ell_1+s_1+2} = 1$ and

$$w = w' - \gamma(e_1, 1) + \gamma \sum_{i=2}^{\ell_1+k} (-1)^{i+1}(v_i, 1) + \frac{\gamma}{2} \sum_{i=\ell_1+k+1}^{u} (-1)^{i+1}(v_i, 1) + (\lambda + \frac{\gamma}{2})e_{n+1}$$

where $\gamma = \min\{\alpha_1, \ldots, \widehat{\alpha_{\ell_1+1}}, \ldots, \alpha_{\ell_1+k}, \alpha_{\ell_1+k+1}, \ldots, \widehat{\alpha_{\ell_1+s_1+1}}, \ldots, \alpha_u, \beta_1\}$. A contradiction by Remark 3.18. □

Now, suppose $e_1, e_2 \in E(G)$ such that $e_1 \cap A_{i_1} = \{a_1\}$, $e_2 \cap A_{i_2} = \{a_2\}$, $e_1 \cap D_j' = \{b_1\}$ and $e_2 \cap D_j^2 = \{b_2\}$. Hence, there are even paths $\mathcal{L}_1$ and $\mathcal{L}_2$ in G_w between z_{i_1} and a_1; and z_{i_2} and a_2, respectively. Since $b_1 \in D_j^1$ and $b_2 \in D_j^2$, there is odd path $\mathcal{L}_3$ in H_j between b_1 and b_2. We can assume $E(\mathcal{L}_1) = \{y_1, \ldots, y_{k_1}\}$, $e_1 = y_{k_1+1}$, $E(\mathcal{L}_3) = \{y_{k_1+2}, \ldots, y_{k_2}\}$, $e_2 = y_{k_2+1}$ and $E(\mathcal{L}_2) = \{y_{k_2+2}, \ldots, y_u\}$. Also we can suppose $x_1 = z_{i_1}$ and $x_2 = z_{i_2}$. Thus, k_1 and k_2 are even; furthermore u is odd. Consequently $(-1)^{k_1+2} = (-1)^{k_2+2} = 1$ and

$$w = w' - \gamma'(e_1, 1) - \gamma'(e_2, 1) + \gamma' \sum_{i=1}^{u} (-1)^{i+1}(v_i, 1) + (\lambda + \gamma')e_{n+1},$$

where $\gamma' = \min\{\beta_1, \beta_2, \alpha_1, \ldots, \widehat{\alpha_{k_1+1}}, \ldots, \alpha_{k_2}, \widehat{\alpha_{k_2+1}}, \ldots, \alpha_u\}$.

Definition 3.21 Let $H_1, \ldots, H_{s_1}, H_{s_1+1}, \ldots, H_{s_2}, H_{s_2+1}, \ldots, H_{s_3}$ be the connected components of G_w such that $|H_i \cap \mathcal{A}| = 1$ if and only if $i \leq s_1$, H_i is bipartite for $s_1 < i \leq s_2$ and H_i is not bipartite if $i > s_2$. Furthermore, D_i^1 and D_i^2 are the chromatic classes of H_i for $s_1 < i \leq s_2$ such that if $e \in E(G)$ with $e \cap (A_1 \cup \cdots \cup A_{s_1}) \neq \emptyset$, then $e \cap (H_{s_1+1} \cup \cdots \cup H_{s_2}) \subseteq D_{s_1+1}^1 \cup \cdots \cup D_{s_2}^1$.

Proposition 3.22 If $w = \sum_{i=1}^{q} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1) + \lambda e_{n+1}$ is a minimal representation of w in $\mathbb{R}_+B$, then $H_{(-\lambda,1)}$ is a support hyperplane of $\mathbb{R}_+B$, where

$\lambda = (\lambda_1, \ldots, \lambda_n)$ and

$$\lambda_i = \begin{cases} \frac{1}{2} & \text{if } x_i \in H_{s_2+1} \cup \cdots \cup H_r \\ 1 & \text{if } x_i \in (A_1 \cup \cdots \cup A_{s_1}) \cup (D^2_{s_1+1} \cup \cdots \cup D^2_{s_2}) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.19 and Proposition 3.20, $(A_1 \cup \cdots \cup A_{s_1}) \cup (D^2_{s_1+1} \cup \cdots \cup D^2_{s_2})$ is a stable set. Furthermore, if $e \in E(G)$ with $e \cap (A_1 \cup \cdots \cup A_{s_1})$, then $e \cap (V(H_{s_1+1}) \cup \cdots \cup V(H_r)) = \emptyset$. Hence, $\lambda \cdot v_j \leq 1$ so $(-\lambda, 1) \cdot (v_j, 1) \geq 0$ for $j = 1, \ldots, q$. Furthermore, $\lambda \cdot e_i \leq 1$ implying $(-\lambda, 1) \cdot e_{n+1} = 1 \geq 0$. Thus, $\mathbb{R}_+ B \subseteq H^+_{(-\lambda,1)}$. We take $B_1 = \{e_i \mid x_i \in (A_1 \cup \cdots \cup A_{s_1}) \cup (D^2_{s_1+1} \cup \cdots \cup D^2_{s_2})\}$ and $B_3 = \{v_i \mid y_i \in E(H_1) \cup \cdots \cup E(H_{s_2})$ such that $x_j \in y_i$ with $e_j \in B_2\}$. Consequently, $B_1 \cup B_2 \cup B_3 \subseteq H_{(-\lambda,1)}$. Furthermore, $\dim \mathbb{R}(B_1 \cup B_2 \cup B_3) = n$, therefore $H_{(-\lambda,1)}$ is a support hyperplane. □

Proposition 3.23 Let $w = \sum_{i=1}^{q} \alpha_i (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1) + \lambda e_{n+1}$ is a minimal representation of $w \in \mathbb{R}_+ B$. Hence, $w = (w_1, \ldots, w_{n+1}) \in (\mathbb{R}_+ B)^\circ$ if and only if $w_i \geq 0$ for $i = 1, \ldots, n$ and $\lambda > 0$.

Proof. We have $\mathbb{R}_+ B = H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H_{(-\ell_m,1)}$.

$\Rightarrow$) Since $w \in (\mathbb{R}_+ B)^\circ$, $w \in H^+_{e_i} \setminus H_{e_i}$ for $1 \leq i \leq n$, so $w_i > 0$. By Proposition 3.22 $H_{(-\lambda,1)}$ is a support hyperplane of $\mathbb{R}_+ B$. Hence, if $\beta_j \neq 0$, then $x_j \in A_1 \cup \cdots \cup A_{s_1}$. Consequently, $\lambda_j = 1$ and $(e_j, 1) \cdot (-\lambda, 1) = 0$. Now, we take $\alpha_j \neq 0$. If $y_j \in E(H_{s_2+1}) \cup \cdots \cup E(H_r)$, then $(v_j, 1) \cdot (-\lambda, 1) = 0$, since $\lambda_{j_1} = \lambda_{j_2} = \frac{1}{2}$ where $y_j = \{x_{j_1}, x_{j_2}\}$. Now, if $y_j \in E(H_1) \cup \cdots \cup E(H_{s_2})$, then $|y_j \cap (A_1 \cup \cdots \cup A_{s_1}) \cup (D^2_{s_1+1} \cup \cdots \cup D^2_{s_2})| = 1$, since $H_1, \cup \cdots \cup H_{s_2}$ are bipartite. Thus, $(v_j, 1) \cdot (-\lambda, 1) = 0$. This implies $(-\lambda, 1) \cdot (\sum_{i=1}^{q} \alpha_i (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1)) = 0$. Therefore $\lambda > 0$, since $w \in \left(\mathbb{R}_+ B\right)^\circ$ and $H_{(-\lambda,1)}$ is a support hyperplane of $\mathbb{R}_+ B$.

$\Leftarrow$) Since $w_i > 0$, $w \in H^+_{e_i} \setminus H_{e_i}$ for $1 \leq i \leq n$. Furthermore, $(v_i, 1) \cdot (-\ell_j, 1) \geq 0$ and $(e_j, 1) \cdot (-\ell_1, 1) \geq 0$, since $(v_i, 1), (e_s, 1) \in \mathbb{R}_+ B$ for $1 \leq i \leq q$ and $1 \leq s \leq n$. Hence $w \cdot (-\ell_j, 1) = 1$. Therefore $w \in H^+_{(-\ell_j,1)} \setminus H_{(-\ell_j,1)}$ so $w \in (\mathbb{R}_+ B)^\circ$. □

We assume G is well-covered graph with a reduction $\{H, y_{s_1+1}, \ldots, y_{s_1+s}\}$ where $H \in \{C_3, C_5, C_7\}$ and $E(H) = \{y_1, \ldots, y_{s_1}\}$.

Proposition 3.24 Let $w = \sum_{i=1}^{q} \alpha_i (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1) + \lambda e_{n+1}$ be a minimal representation of w in $\mathbb{R}_+ B$ such that $w_i > 0$, $\sum_{i=1}^{s_1} \alpha_i$ is maximal, $\beta(H') < \beta(H)$

where $E(H') = \{y_i \in E(H) \mid \alpha_i > 1\}$. Furthermore, $y_{i_1}, y_{i_2} \in E(G_w)$ such that $y_{i_1} = \{a_1, b_1\}, y_{i_2} = \{a_2, b_2\}$ with $\{b_j\} = y_{i_j} \cap V(H)$ for $j = 1, 2$. Hence,

1) If $x_j \in V(H)$ such that $y_{i_3} = \{b_1, x_j\} \in E(H)$, then $\beta_j = 0$.

2) If $\{b_1, b_2\}$ or $\{b_1, c, b_2\}$ is a path of $E(H)$, then $\{a_1, a_2\} \notin E(G)$.

Proof. 1) Suppose $\beta_j > 0$, then

$$w = w' + \lambda e_{n+1} + \gamma(v_{i_3}, 1) - \gamma(v_{i_1}, 1) + \gamma(e_{j'}, 1) - \gamma(e_j, 1),$$

where $\delta = \min\{\alpha_{i_1}, \beta_j\}$, $w' = \sum_{i=1}^{q} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1)$ and $b_1 = y_{j'}$. A contradiction, therefore $\beta_j = 0$.

2) By contradiction, assume $y_{i_4} = \{a_1, a_2\} \in E(G)$. If $y_{i_5} = \{b_1, b_2\} \in E(G)$, then

$$w = w' + \lambda e_{n+1} + \gamma(v_{i_4}, 1) - \gamma(v_{i_1}, 1) - \gamma(v_{i_2}, 1) + \gamma(v_{i_5}, 1),$$

where $\gamma = \min = \{\alpha_{i_1}, \alpha_{i_2}\}$ and $w' = \sum_{i=1}^{q} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1)$. But $\sum_{i=1}^{s_1} \alpha_i < \sum_{i=1}^{s_1} \alpha_1 + \gamma$. A contradiction, then $\{b_1, b_2\} \notin E(G)$. Consequently $H \neq C_3$ and there is path $\{b_1, c, b_2\}$. We can suppose $y_{i_1} = \{b_1, c\}$, $y_{i_2} = \{c, b_2\} \in E(H)$. By Proposition 3.5, $H = C_5$. Thus,

$$w = w' + \lambda e_{n+1} + \sum_{i=3}^{5} (-1)^{i+1} \gamma(v_i, 1) - \gamma(v_{i_1}, 1) - \gamma(v_{i_2}, 1) + \gamma(v_{i_4}, 1),$$

where $\gamma = \min\{\alpha_{i_1}, \alpha_{i_2}, \alpha_4\}$. Since $\sum_{i=1}^{s} \alpha_i$ is maximal, $\alpha_4 = 0$. We can suppose $H = (c, b_2, x_2, x_1, b_1)$. Since $\alpha_{i_1} > 0$ and $\alpha_{i_2} > 0$, then by 1) $\beta_1 = \beta_2 = 0$. By hypothesis, $\alpha_3 < 1$ or $\alpha_5 < 1$. We assume $\alpha_5 < 1$, then there is $y_{j_1} = \{x_1, z_1\} \in E(G_w)$. By the last argument $\{a_1, z_1\} \notin E(G)$. Suppose $\alpha_3 \geq 1$. Since G is well-covered, $\{a_1, z_1, b_2\}$ is not a stable set. If $y_{j_6} = \{z_1, b_2\} \in E(G)$, then

$$w = w' + \lambda e_{n+1} + \gamma(v_4, 1) - \gamma(v_{j_1}, 1) + \gamma(v_{j_6}, 1) - \gamma(v_3, 1)$$

where $\gamma = \min\{\alpha_3, \alpha_{j_1}\}$. A contradiction, since in this representation of w the coefficient of $(v_4, 1)$ is $\gamma > 0$. Hence, $y_{j_7} = \{a_1, b_2\} \in E(G)$. We can assume α_1 is maximal. We have

$$w = w' + \lambda e_{n+1} + \gamma(v_1, 1) - \gamma(v_{j_1}, 1) + \gamma(v_{j_7}, 1) - \gamma(v_{i_7}, 1)$$

where $\gamma = \min\{\alpha_2, \alpha_{i_1}\}$. Since α_1 is maximal, $\gamma = \alpha_2 = 0$. This implies there is $\{c, c'\} \in E(G)$. By the last argument $\{c', a_1\} \notin E(G)$. A contradiction since G is well-covered. Therefore $\alpha_3 < 1$ and there is $y_{j_2} = \{x_2, z_2\} \in E(G_w)$ such that $z_1, z_2 \in V(G) \setminus V(H)$. By the last argument $\{z_1, z_2\}$, $\{z_2, a_2\} \notin E(G)$. Since

G is well-covered $\{a_1, z_1, z_2\}$ and $\{z_1, z_2, a_2\}$ are not stable sets. Hence $y_{k_1} = \{a_1, z_2\}$, $y_{k_2} = \{z_1, a_2\} \in E(G)$. Consequently,

$$w = w' + \lambda e_{n+1} - \gamma \sum_{s=1}^{2} \big((v_{i_s}, 1) + (v_{j_s}, 1)\big) + \gamma \sum_{s=1}^{2} \big((v_{k_s}, 1) + (v_{q_s}, 1)\big)$$

where $\gamma = \min\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{j_1}, \alpha_{j_2}\}$ and $y_{q_s} = \{x_s, b_s\}$ for $s = 1, 2$. A contradiction. $\square$

Proposition 3.25 Let G be an unmixed graph with $\tau(G) = \frac{n+1}{2}$. If $w \in (\mathbb{R}_+ B)^\circ$, then $u = w - (\mathbf{1}, \frac{n+1}{2}) \in \mathbb{R}_+ B$.

Proof. By Proposition 3.2, G has a reduction $\{H, y_{s+1}, \ldots, y_n\}$, where $H \in \{C_3, C_5, C_7\}$ and $E(H) = \{y_1, \ldots, y_s\}$. First assume G satisfies

1) There is a minimal representation of w such that if $\{x_{i_1}, x_{j_1}\}, \{x_{i_2}, x_{j_2}\} \in E(G_w)$ with $x_{i_1}, x_{i_2} \in V(H)$ and $x_{j_1}, x_{j_2} \notin V(H)$, then $\{x_{i_1}, x_{i_2}\} \notin E(H)$.

If $H = C_3$, we can assume $\deg_{G_w}(x_2) = \deg_{G_w}(x_3) = 2$, then $\alpha_2 + \alpha_3$, $\alpha_2 + \alpha_1 \in \mathbb{N}$ since $w \in \mathbb{N}^{n+1}$. Hence,

$$u = w' - \sum_{i=4}^{m} (v_i, 1) - \sum_{i=1}^{3} \gamma_i (v_i, 1) - \gamma_{i_1}(v_{i_1}, 1) + \gamma_{i_1}(e_{i_1}, 1) + (\lambda + \gamma_1 - 1)e_{n+1}$$

where $\gamma_2 = \alpha_2 - \lceil \alpha_2 \rceil + 1$, $\gamma_1 = \gamma_3 = 1 - \gamma_2$, and $\gamma_{i_1} = \gamma_2$ if $y_{i_1} = \{x_1, x_{k_1}\} \in E(G)$ or $\gamma_{i_1} = 0$ in other case. This implies $u \in \mathbb{R}_+ B$. Now suppose $H = C_5$, so we can assume $\deg_{G_w}(x_2) = \deg_{G_w}(x_3) = \deg_{G_w}(x_4) = 2$. Consequently $\alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_4 + \alpha_5 \in \mathbb{N}$, since $w \in \mathbb{N}^{n+1}$. Thus,

$$u = w' - \sum_{i=6}^{m} (v_{i_1}, 1) - \sum_{i=1}^{5} \gamma_i (v_i, 1) - \gamma_{i_1}(v_{i_1} - e_{i_1}, 0) - \gamma_{i_4}(v_{i_4} - e_{i_4}, 0) + (\lambda + \gamma_1 - 1)e_{n+1}$$

where $\gamma_2 = \alpha_2 - \lceil \alpha_2 \rceil + 1$, $\gamma_1 = \gamma_3 = 1 - \gamma_2$, $\gamma_4 = \alpha_4 - \lceil \alpha_4 \rceil + 1$, $\gamma_5 = 1 - \gamma_4$, and $\gamma_{i_1} = 1 - \alpha_1 - \alpha_5$ and $\gamma_{i_4} = 1 - \alpha_3 - \alpha_5$. Finally suppose $H = C_7$. By Proposition 3.5, we can suppose $\deg_G(x_3) = \deg_G(x_5) = \deg_G(x_7) = 2$. Also, $\deg_G(x_2) = 2$, since G satisfies 1) of Proposition 3.25. Now assume $\deg_{G_w}(x_6) = 2$. This implies

$$u = w' - \sum_{i=8}^{m} (v_i, 1) - \sum_{i=1}^{7} \gamma_i (v_i, 1) - \gamma_{i_1}(v_{i_1} - e_{i_1}, 0) - \gamma_{i_4}(v_{i_4} - e_{i_4}, 0) + (\lambda - \gamma_1 - 1)e_{n+1}$$

where $\gamma_2 = \alpha_2 - \lceil \alpha_2 \rceil + 1$, $\gamma_1 = \gamma_3 = 1 - \gamma_2$, $\gamma_4 = \alpha_4 - \lceil \alpha_4 \rceil + 1 = \gamma_6$, $\gamma_5 = \gamma_7 = 1 - \gamma_4$, $\gamma_{i_1} = 1 - \gamma_1 - \gamma_7$, and $\gamma_{i_4} = 1 - \gamma_3 - \gamma_4$. Finally if there is $y_{i_6} =$

$(x_6, x_{i_6}) \in G_w$ with $x_{i_6} \notin V(H)$, then by Proposition 3.5 $\{x_4, x_1\} \cap \mathrm{N}_G(x_{i_6}) \neq \emptyset$. We can suppose $y_{i_7} = \{x_{i_6}, x_4\} \in E(G)$. We also assume α_{i_6} is minimal in the representation of w. Hence,

$$w = w' - \gamma(v_4, 1) + \gamma(v_5, 1) - \gamma(v_{i_6}, 1) + \gamma(v_{i_7}, 1) + \lambda e_{n+1}$$

where $\gamma = \min\{\alpha_4, \alpha_{i_6}\}$. Since in the representation of w α_{i_6} is minimal, $\alpha_4 = 0$. So, $\alpha_5 \in \mathbb{N}$ and

$$w'' = w - \alpha_{i_6}(v_{i_6}, 1) + \gamma'(e_7, 1) + (\lambda - \gamma' + \alpha_{i_6})e_{n+1}$$

where $\gamma' = \min\{0, 1 - \alpha_7\}$. Therefore $w'' \in (\mathbb{R}_+ B)^\circ$, $w'' \cdot e_i \geq 1$ and $\alpha_{i_6} = 0$, we are in the last case.

Now, suppose w does not satisfy 1). We take a minimal representation of $w = \sum_{i=1}^{q} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(v_i, 1) + \lambda e_{n+1}$. By Proposition 3.23 we can assume $\lambda \geq 1$ and $\alpha_i \geq 1$ for each $i \in \{s+1, \ldots, m\}$. We take $V(H) = \{x_1, \ldots, x_s\}$ and $y_i = \{x_i, x_{i+1}\}$ for $i = 1, \ldots, s-1$ and $y_s = \{x_s, x_1\}$. Furthermore $y_{i_1} = \{x_1, x_{k_1}\}$, $y_{i_2} = \{x_2, x_{k_2}\} \in E(G_w)$ with $x_{k_1}, x_{k_2} \notin V(H)$. Also we can assume that the representation of w satisfies $\alpha_{i_1} + \alpha_{i_2}$ is minimal. First suppose $H = C_3$. By Proposition 3.1 $\deg_G(x_3) = 2$, since G is unmixed. Hence,

$$u = w' - \sum_{i=4}^{m}(v_i, 1) - \sum_{i=1}^{3}\gamma_i(v_i, 1) - \sum_{j=1}^{2} u_j(v_{i_j}, 1) + \sum_{j=1}^{2} u_j(e_{k_j}, 1) + (\lambda - u_1 - u_2)e_{n+1}$$

where $\gamma_2 + \gamma_3 = 1$, $\gamma_1 = \min\{\alpha_1, 1-\gamma_2, 1-\gamma_3\}$, $u_1 + \gamma_3 = u_2 + \gamma_2 = 1 - \gamma_1$. Now assume $H = C_5$. By Proposition 3.6 we have that $\deg_G(x_3) = \deg_G(x_5) = 2$. By Proposition 3.1 $\{x_{k_1}, x_{k_2}\} \notin E(G)$. Since G is unmixed $\{x_{k_1}, x_{k_2}, x_4\}$ is not a stable set, we can suppose $y_{i_3} = \{x_{i_1}, x_4\} \in E(G)$. Thus,

$$w = w' + \gamma(v_5, 1) - \gamma(v_4, 1) + \gamma(v_{i_3}, 1) - \gamma(v_{i_1}, 1) + \lambda e_{n+1}$$

where $\gamma = \min\{\alpha_4, \alpha_{i_1}\}$. Since $\alpha_{i_1} + \alpha_{i_2}$ is minimal, $\gamma = 0$ and $\alpha_4 = 0$. Hence, $\alpha_5 \geq 1$ and $w'' = w - \alpha_{i_1}(v_{i_1}, 1) \in (\mathbb{R}_+ B)^\circ$, $w'' \cdot e_i \geq 1$ and w'' satisfies 1). Finally if $H = C_7$, by Proposition 3.5 there is $y_{i_4} = \{x_{i_1}, x_6\}$, $y_{i_5} = \{x_{i_5}, x_4\} \in E(G)$ and $\deg_G(x_3) = \deg_G(x_5) = \deg_G(x_7) = 2$. We take

$$w = w' + \gamma(v_7, 1) - \gamma(v_6, 1) + \gamma(v_{i_4}, 1) - \gamma(v_{i_1}, 1) + \lambda e_{n+1}$$

where $\gamma = \min\{\alpha_{i_1}, \alpha_6\}$. Since $\alpha_{i_1} + \alpha_{i_2}$ is minimal, $\gamma = 0$ and $\alpha_6 = 0$. Hence $\alpha_7 \geq 1$ and $w'' = w - \alpha_{i_1}(v_{i_1}, 1) \in (\mathbb{R}_+ B)^\circ$, $w'' \cdot e_i \geq 1$ and w'' satisfies 1). □

Theorem 3.26 If S is normal and G is connected not bipartite graph, then S_G is Gorenstein if and only if G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and $b_0 = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. $\Leftarrow$) By Theorem 3.15, we can assume that n is odd. Since G is unmixed and $\tau(G) = \frac{n+1}{2}$, then by Proposition 3.2, there exist a τ-reduction $\{H_1, \ldots, H_s\}$ of G, where $H_i \in E(G)$ for $1 \leq i \leq s-1$ and H_s is a j-cycle with $j \in \{3,5,7\}$. We can assume $V(H_i) = \{x_{2i-1}, x_{2i}\}$ for $i = 1, \ldots, s-1$ and $V(H_s) = \{x_{2s-1}, \ldots, x_n\}$. We take $\mathbb{R}_+B = H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H^+_{(-\ell_m,1)}$. Consequently, $\ell_j^{2i-1} + \ell_j^{2i} \leq 1$ for $1 \leq i \leq s-1$, where $\ell_j = (\ell_j^1, \ldots, \ell_J^n)$. Hence, by Proposition 3.16, $\mathbf{1} \cdot \ell_j = \sum_{i=1}^n \ell_j^i \leq \frac{n}{2}$. Thus, $(\mathbf{1}, \frac{n+1}{2}) \cdot (-\ell_j, 1) > 0$. This implies, $(\mathbf{1}, \frac{n+1}{2}) \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ$. Now, if $(v,b) \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ$, then $(v,b) \cdot e_i > 0$ for $1 \leq i \leq n$. This implies $v \geq \mathbf{1}$. Also, by Proposition 3.12, we have that $b \geq \frac{n+1}{2}$. Thus, $u = (v,b) - (\mathbf{1}, \frac{n+1}{2}) \in \mathbb{Z}_+^{n+1}$. By Proposition 3.25 $u \in \mathbb{R}_+B$. So, $u \in \mathbb{Z}_+^{n+1} \cap \mathbb{R}_+B = \mathbb{N}B$, since S_G is normal. Therefore $w_S = (x^{\mathbf{1}} t^{\frac{n+1}{2}})$ and S_G is Gorenstein.
$\Rightarrow$) By Theorem 3.13. □

Theorem 3.27 Let G be a graph such that S_G is normal then S_G is Gorenstein if and only if G is unmixed and $\tau(G) = \lceil \frac{n}{2} \rceil$.

Proof. By Theorem 3.14 and Theorem 3.26. □

CHAPTER 4

MONOMIAL IDEALS OF WEIGHTED ORIENTED GRAPHS

4.1 INTRODUCTION

A **weighted oriented graph** is a triplet $D = (V(D), E(D), w)$, where $V(D)$ is a finite set, $E(D) \subseteq V(D) \times V(D)$ and w is a function $w : V(D) \to \mathbb{N}$. The vertex set of D and the edge set of D are $V(D)$ and $E(D)$, respectively. Some times for short we denote these sets by V and E respectively. The **weight** of $x \in V$ is $w(x)$. If $e = (x, y) \in E$, then x is the tail of e and y is the head of e. The **underlying graph** of D is the simple graph G whose vertex set is V and whose edge set is $\{\{x, y\} \mid (x, y) \in E\}$. If $V(D) = \{x_1, \dots, x_n\}$, then we consider the polynomial ring $R = K[x_1, \dots, x_n]$ in n variables over a field K. In this paper we introduce and study the edge ideal of D given by $I(D) = (x_i x_j^{w(x_j)} : (x_i, x_j) \in E(D))$ in R, (see Definition 4.15).

In Sect. 4.2 we study the vertex covers of D. In particular we introduce the notion of strong vertex cover (Definition 4.6) and we prove that a minimal vertex cover is strong. In Sect. 4.3 we characterize the irredundant irreducible decomposition of $I(D)$. In particular we show that the minimal monomial irreducible ideals of $I(D)$ are associated with the strong vertex covers of D. In Sect. 4.4 we give the following characterization of the unmixed property of $I(D)$.

$I(D)$ is unmixed $\Longleftrightarrow$ G is unmixed and D has the minimal strong propert

$\Updownarrow$ $\quad$ $\Updownarrow$ $\quad$ $\Updownarrow$

All strong vertex covers have the same cardinality	All minimal vertex covers have the same cardinality	All strong vertex covers are minimals

Furthermore, if D is bipartite, D is a whisker or D is a cycle, we give an effective (combinatorial) characterization of the unmixed property. Finally in Sect. **??** we study the Cohen-Macaulayness of $I(D)$. In particular we characterize the Cohen-Macaulayness when D is a path or D is complete. Also, we give an example where this property depend of the characteristic of the field K.

4.2 WEIGHTED ORIENTED GRAPHS AND THEIR VERTEX COVERS

In this section we define the weighted oriented graphs and we study their vertex covers. Furthermore, we define the strong vertex covers and we characterize when $V(D)$ is a strong vertex cover of D. In this paper we denote the set $\{x \in V \mid w(x) \neq 1\}$ by V^+.

Definition 4.1 A **vertex cover** C of D is a subset of V, such that if $(x, y) \in E$, then $x \in C$ or $y \in C$. A vertex cover C of D is **minimal** if each proper subset of C is not a vertex cover of D.

Definition 4.2 Let x be a vertex of a weighted oriented graph D, the sets $N^+_D(x) = \{y \mid (x, y) \in E(D)\}$ and $N^-_D(x) = \{y \mid (y, x) \in E(D)\}$ are called the **out-neighbourhood** and the **in-neighbourhood** of x, respectively. Furthermore, the **neighbourhood** of x is the set $N_D(x) = N^+_D(x) \cup N^-_D(x)$.

Definition 4.3 Let C be a vertex cover of a weighted oriented graph D, we define

$$L_1(C) = \{x \in C \mid N^+_D(x) \cap C^c \neq \emptyset\},$$
$$L_2(C) = \{x \in C \mid x \notin L_1(C) \text{ and } N^-_D(x) \cap C^c \neq \emptyset\} \text{ and}$$
$$L_3(C) = C \setminus (L_1(C) \cup L_2(C)),$$

where C^c is the complement of C, i.e. $C^c = V \setminus C$.

Proposition 4.4 If C is a vertex cover of D, then

$$L_3(C) = \{x \in C \mid N_D(x) \subset C\}.$$

Proof. If $x \in L_3(C)$, then $N^+_D(x) \subseteq C$, since $x \notin L_1(C)$. Furthermore $N^-_D(x) \subseteq C$, since $x \notin L_2(C)$. Hence $N_D(x) \subset C$, since $x \notin N_D(x)$. Now, if $x \in C$ and $N_D(x) \subset C$, then $x \notin L_1(C) \cup L_2(C)$. Therefore $x \in L_3(C)$. □

Proposition 4.5 If C is a vertex cover of D, then $L_3(C) = \emptyset$ if and only if C is a minimal vertex cover of D.

Proof. $\Rightarrow$) If $x \in C$, then by Proposition 4.4 we have $N_D(x) \not\subset C$, since $L_3(C) = \emptyset$. Thus, there is $y \in N_D(x) \setminus C$ implying $C \setminus \{x\}$ is not a vertex cover. Therefore, C is a minimal vertex cover.

$\Leftarrow$) If $x \in L_3(C)$, then by Proposition 4.4, $N_D(x) \subseteq C \setminus \{x\}$. Hence, $C \setminus \{x\}$ is a vertex cover. A contradiction, since C is minimal. Therefore $L_3(C) = \emptyset$.

Definition 4.6 A vertex cover C of D is **strong** if for each $x \in L_3(C)$ there is $(y, x) \in E(D)$ such that $y \in L_2(C) \cup L_3(C)$ with $y \in V^+$ (i.e. $w(y) \neq 1$).

Remark 4.7 Let C be a vertex cover of D. Hence, by Proposition 4.4 and since $C = L_1(C) \cup L_2(C) \cup L_3(C)$, we have that C is strong if and only if for each $x \in C$ such that $N(x) \subset C$, there exist $y \in N^-(x) \cap (C \setminus L_1(C))$ with $y \in V^+$.

Corollary 4.8 If C is a minimal vertex cover of D, then C is strong.

Proof. By Proposition 4.5, we have $L_3(C) = \emptyset$, since C. Hence, C is strong. □

Remark 4.9 The vertex set V of D is a vertex cover. Also, if $z \in V$, then $N_D(z) \subseteq V \setminus z$. Hence, by Proposition 4.4, $L_3(V) = V$. Consequently, $L_1(V) = L_2(V) = \emptyset$. By Proposition 4.5, V is not a minimal vertex cover of D. Furthermore since $L_3(V) = V$, V is a strong vertex cover if and only if $N_D^-(x) \cap V^+ \neq \emptyset$ for each $x \in V$.

Definition 4.10 If G is a cycle with $E(D) = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ and $V(D) = \{x_1, \ldots, x_n\}$, then D is called **oriented cycle**.

Definition 4.11 D is called **unicycle oriented graph** if it satisfies the following conditions:

1) The underlying graph of D is connected and it has exactly one cycle C.
2) C is an oriented cycle in D. Furthermore for each $y \in V(D) \setminus V(C)$, there is an oriented path from C to y in D.
3) $w(x) \neq 1$ if $\deg_G(x) \geq 1$.

Lemma 4.12 If $V(D)$ is a strong vertex cover of D and D_1 is a maximal unicycle oriented subgraph of D, then $V(D')$ is a strong vertex cover of $D' = D \setminus V(D_1)$.

Proof. We take $x \in V(D')$. Thus, by Remark 4.9, there is $y \in N_D^-(x) \cap V^+(D)$. If $y \in D_1$, then we take $D_2 = D_1 \cup \{(y,x)\}$. Hence, if C is the oriented cycle of D_1, then C is the unique cycle of D_2, since $\deg_{D_2}(x) = 1$. If $y \in C$, then (y,x) is an oriented path from C to x. Now, if $y \notin C$, then there is an oriented path $\mathcal{L}$ form C to y in D_1. Consequently, $\mathcal{L} \cup \{(y,x)\}$ is an oriented path form C to x. Furthermore, $\deg_{D_2}(x) = 1$ and $w(y) \neq 1$, then D_2 is an unicycle oriented graph. A contradiction, since D_1 is maximal. This implies $y \in V(D')$, so $y \in N_{D'}^-(x) \cap V^+(D')$. Therefore, by Remark 4.9, $V(D')$ is a strong vertex cover of D'. $\square$

Lemma 4.13 If $V(D)$ is a strong vertex cover of D, then there is an unicycle oriented subgraph of D.

Proof. Let y_1 be a vertex of D. Since $V = V(D)$ is a strong vertex cover, there is $y_2 \in V$ such that $y_2 \in N^-(y_1) \cap V^+$. Similarly, there is $y_3 \in N^-(y_2) \cap V^+$. Consequently, (y_3, y_2, y_1) is an oriented path. Continuing this process, we can assume there exist $y_2, y_3, \dots, y_k \in V^+$ where $(y_k, y_{k-1}, \dots, y_2, y_1)$ is an oriented path and there is $1 \leq j \leq k-2$ such that $(y_j, y_k) \in E(D)$, since V is finite. Hence, $C = (y_k, y_{k-1}, \dots, y_j, y_k)$ is an oriented cycle and $\mathcal{L} = (y_j, \dots, y_1)$ is an oriented path form C to y_1. Furthermore, if $j = 1$, then $w(y_1) \neq 1$. Therefore, $D_1 = C \cup \mathcal{L}$ is an unicycle oriented subgraph of D. $\square$

Proposition 4.14 Let $D = (V, E, w)$ be a weighted oriented graph, hence V is a strong vertex cover of D if and only if there are $D_1, \dots, D_s$ unicycle oriented subgraphs of D such that $V(D_1), \dots, V(D_s)$ is a partition of $V = V(D)$.

Proof. $\Rightarrow$) By Lemma 4.13, there is a maximal unicycle oriented subgraph D_1 of D. Hence, by Lemma 4.12, $V(D')$ is a strong vertex cover of $D' = D \setminus V(D_1)$. So, by Lemma 4.13, there is D_2 a maximal unicycle oriented subgraph of D'. Continuing this process we obtain unicycle oriented subgraphs $D_1, \dots, D_s$ such that $V(D_1), \dots, V(D_s)$ is a partition of $V(D)$.

$\Leftarrow$) We take $x \in V(D)$. By hypothesis there is $1 \leq j \leq s$ such that $x \in V(D_j)$. We assume C is the oriented cycle of D_j. If $x \in V(C)$, then there is $y \in V(C)$ such that $(y,x) \in E(D_j)$ and $w(y) \neq 1$, since $\deg_{D_j}(y) \geq 2$ and D_j is a unicycle oriented subgraph. Now, we assume $x \notin V(C)$, then there is an oriented path $\mathcal{L} = (z_1, \dots, z_r)$ such that $z_1 \in V(C)$ and $z_r = x$. Thus, $(z_{r-1}, x) \in E(D)$. Furthermore,

$w(z_{r-1}) \neq 1$, since $\deg_{D_j}(z_{r-1}) \geq 2$. Therefore V is a strong vertex cover. □

4.3 EDGE IDEALS AND THEIR PRIMARY DECOMPOSITION

As is usual if I is a monomial ideal of a polynomial ring R, we denote by $\mathcal{G}(I)$ the minimal monomial set of generators of I. Furthermore, there exists a unique decomposition, $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are irreducible monomial ideals such that $I \neq \bigcap_{i \neq j} \mathfrak{q}_i$ for each $j = 1, \ldots, r$. This is called the irredundant irreducible decomposition of I. Furthermore, $\mathfrak{q}_i$ is an irreducible monomial ideal if and only if $\mathfrak{q}_i = (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$ for some variables x_{i_j}. Irreducible ideals are primary, then a irreducible decomposition is a primary decomposition. For more details of primary decomposition of monomial ideals see [28, Chapter 6]. In this section, we define the edge ideal $I(D)$ of a weighted oriented graph D and we characterize its irredundant irreducible decomposition. In particular we prove that this decomposition is an irreducible primary decomposition, i.e, the radicals of the elements of the irredundant irreducible decomposition of $I(D)$ are different.

Definition 4.15 Let $D = (V, E, w)$ be a weighted oriented graph with $V = \{x_1, \ldots, x_n\}$. The **edge ideal of** D, denote by $I(D)$, is the ideal of $R = K[x_1, \ldots, x_n]$ generated by $\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E\}$.

Definition 4.16 A **source** of D is a vertex x, such that $N_D(x) = N_D^+(x)$. A **sink** of D is a vertex y such that $N_D(y) = N_D^-(y)$.

Remark 4.17 Let $D = (V, E, w)$ be a weighted oriented graph. We take $D' = (V, E, w')$ a weighted oriented graph such that $w'(x) = w(x)$ if x is not a source and $w'(x) = 1$ if x is a source. Hence, $I(D) = I(D')$. For this reason in this paper, we will always assume that if x is a source, then $w(x_i) = 1$.

Definition 4.18 Let C be a vertex cover of D, the **irreducible ideal associated to** C is the ideal

$$I_C = \big(L_1(C) \cup \{x_j^{w(x_j)} \mid x_j \in L_2(C) \cup L_3(C)\}\big).$$

Lemma 4.19 $I(D) \subseteq I_C$ for each vertex cover C of D.

Proof. We take $I = I(D)$ and $m \in \mathcal{G}(I)$, then $m = xy^{w(y)}$, where $(x, y) \in D$. Since

C is a vertex cover, $x \in C$ or $y \in C$. If $y \in C$, then $y \in I_C$ or $y^{w(y)} \in I_C$. Thus, $m = xy^{w(y)} \in I_C$. Now, we assume $y \notin C$, then $x \in C$. Hence, $y \in N_D^+(x) \cap C^c$, so $x \in L_1(C)$. Consequently, $x \in I_C$ implying $m = xy^{w(y)} \in I_C$. Therefore $I \subseteq I_C$. $\square$

Definition 4.20 Let I be a monomial ideal. An irreducible monomial ideal $\mathfrak{q}$ that contains I is called a **minimal irreducible monomial ideal of** I if for any irreducible monomial ideal $\mathfrak{p}$ such that $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ one has that $\mathfrak{p} = \mathfrak{q}$.

Lemma 4.21 Let D be a weighted oriented graph. If $I(D) \subseteq (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$, then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D.

Proof. We take $J = (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$. If $(a, b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_j}^{a_j} | ab^{w(b)}$ for some $1 \leq j \leq s$. Hence, $x_{i_j} \in \{a, b\}$ and $\{a, b\} \cap \{x_{i_1} \ldots x_{i_s}\} \neq \emptyset$. Therefore $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D. $\square$

Lemma 4.22 Let J be a minimal irreducible monomial ideal of $I(D)$ where $\mathcal{G}(J) = \{x_{i_1}^{a_s}, \ldots, x_{i_s}^{a_s}\}$. If $a_j \neq 1$ for some $1 \leq j \leq s$, then there is $(x, x_{i_j}) \in E(D)$ where $x \notin \mathcal{G}(J)$.

Proof. By contradiction suppose there is $a_j \neq 1$ such that if $(x, x_{i_j}) \in E(D)$, then $x \in M = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. We take the ideal $J' = (M \setminus \{x_{i_j}^{a_j}\})$. If $(a, b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Consequently, $x_{i_k}^{a_k} | ab^{w(b)}$ for some $1 \leq k \leq s$. If $k \neq j$, then $ab^{w(b)} \in J'$. Now, if $k = j$, then by hypothesis $a_j \neq 1$. Hence, $x_{i_j}^{a_j} | b^{w(b)}$ implying $x_{i_j} = b$. Thus, $(a, x_{i_j}) \in E(D)$, so by hypothesis $a \in M \setminus \{x_{i_j}^{a_j}\}$. This implies $ab^{w(b)} \in J'$. Therefore $I(D) \subseteq J' \subsetneq J$. A contradiction, since J is minimal. $\square$

Lemma 4.23 Let J be a minimal irreducible monomial ideal of $I(D)$ where $\mathcal{G}(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. If $a_j \neq 1$ for some $1 \leq j \leq s$, then $a_j = w(x_{i_j})$.

Proof. By Lemma 4.22, there is $(x, x_{i_j}) \in E(D)$ with $x \notin M = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. Also, $xx_{i_j}^{w(x_{i_j})} \in I(D) \subseteq J$, so $x_{i_k}^{a_k} | xx_{i_j}^{w(x_{i_j})}$ for some $1 \leq k \leq s$. Hence, $x_{i_k}^{a_k} | x_{i_j}^{w(x_{i_j})}$, since $x \notin M$. This implies, $k = j$ and $a_j \leq w(x_{i_j})$. If $a_j < w(x_{i_j})$, then we take $J' = (M')$ where $M' = \{M \setminus \{x_{i_j}^{a_j}\}\} \cup \{x_{i_j}^{w(x_{i_j})}\}$. So, $J' \subsetneq J$. Furthermore, if $(a, b) \in E(D)$, then $m = ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_k}^{a_k} | ab^{w(b)}$ for some $1 \leq k \leq s$. If $k \neq j$, then $x_{i_k}^{a_k} \in M'$

implying $ab^{w(b)} \in J'$. Now, if $k = j$ then $x_{i_j}^{a_j} | b^{w(b)}$, since $a_j > 1$. Consequently, $x_{i_j} = b$ and $x_{i_j}^{w(x_{i_j})} | m$. Then $m \in J'$. Hence $I(D) \subseteq J' \subsetneq J$, a contradiction since J is minimal. Therefore $a_j = w(x_{i_j})$. □

Theorem 4.24 The following conditions are equivalent:

1) J is a minimal irreducible monomial ideal of $I(D)$.

2) There is a strong vertex cover C of D such that $J = I_C$.

Proof. *2) ⇒ 1)* By definition $J = I_C$ is a monomial irreducible ideal. By Lemma 4.19, $I(D) \subseteq J$. Now, suppose $I(D) \subseteq J' \subseteq J$, where J' is a monomial irreducible ideal. We can assume $\mathcal{G}(J') = \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$. If $x \in L_1(C)$, then there is $(x, y) \in E(D)$ with $y \notin C$. Hence, $xy^{w(y)} \in I(D)$ and $y^r \notin J$ for each $r \in \mathbb{N}$. Consequently $y^r \notin J'$ for each r, implyinig $y \notin \{x_{j_1}, \ldots, x_{j_s}\}$. Furthermore $x_{j_i}^{b_i} | xy^{w(y)}$ for some $1 \leq i \leq s$, since $xy^{w(y)} \in I(D) \subseteq J'$. This implies, $x = x_{j_i}^{b_i} \in J'$. Now, if $x \in L_2(C)$, then there is $(y, x) \in E(D)$ with $y \notin C$. Thus $y \notin J$, so $y \notin \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$. Also, $x^{w(x)}y \in I(D) \subseteq J'$, then $x_{j_i}^{b_i} | x^{w(x)}y$ for some $1 \leq i \leq s$. Consequently, $x_{j_i}^{b_i} | x^{w(x)}$ implies $x^{w(x)} \in J'$. Finally if $x \in L_3(C)$, then there is $(y, x) \in E(D)$ where $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$, since C is a strong vertex cover. So, $x^{w(x)}y \in I(D) \subseteq J'$ implies $x_{j_i}^{b_i} | x^{w(x)}y$ for some i. Furthermore $y \notin J = I_C$, since $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$. This implies $y \notin J'$ so, $x_{j_i}^{b_i} | x^{w(x)}$ then $x^{w(x)} \in J'$. Hence, $J = I_C \subseteq J'$. Therefore, J is a minimal monomial irreducible of $I(D)$.

1) ⇒ 2) Since J is irreducible, we can suppose $\mathcal{G}(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. By Lemma 4.23, we have $a_j = 1$ or $a_j = w(x_{i_j})$ for each $1 \leq j \leq s$. Also, by Lemma 4.21, $C = \{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D. We can assume $\mathcal{G}(I_C) = \{x_{i_1}^{b_1}, \ldots, x_{i_s}^{b_s}\}$, then $b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq s$. Now, suppose $b_k = 1$ and $w(x_{i_k}) \neq 1$ for some $1 \leq k \leq s$. Consequently $x_{i_k} \in L_1(C)$. Thus, there is $(x_{i_k}, y) \in E(D)$ where $y \notin C$. So, $x_{i_k}y^{w(y)} \in I(D) \subseteq J$ and $x_{i_r}^{a_r} | x_{i_k}y^{w(y)}$ for some $1 \leq r \leq s$. Furthermore $y \notin C$, then $r = k$ and $a_k = a_r = 1$. Hence, $I_C \cap V(D) \subseteq J \cap V(D)$. This implies, $I_C \subseteq J$, since $a_j, b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq s$. Therefore $J = I_C$, since J is minimal. In particular $a_i = b_i$ for each $1 \leq i \leq s$.

Now, assume C is not strong, then there is $x \in L_3(C)$ such that if $(y, x) \in E(D)$, then $w(y) = 1$ or $y \in L_1(C)$. We can assume $x = x_{i_1}$, and we take J' the monomial ideal with $\mathcal{G}(J') = \{x_{i_2}^{a_2}, \ldots, x_{i_s}^{a_s}\}$. We take $(z_1, z_2) \in E(D)$. If $x_{i_j}^{a_j} | z_1 z_2^{w(z_2)}$ for some

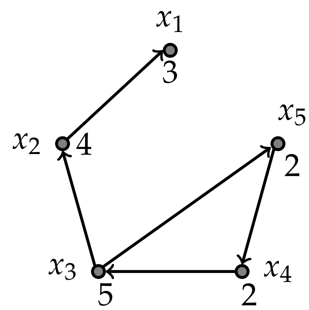
$2 \leq j \leq s$, then $z_1 z_2^{w(z_2)} \in J'$. Now, assume $x_{i_j}^{a_j} \nmid z_1 z_2^{w(z_2)}$ for each $2 \leq j \leq s$. Consequently $z_2 \notin \{x_{i_2} \dots x_{i_s}\}$, since $a_j \in \{1, w(x_{i_j})\}$. Also $z_1 z_2^{w(z_2)} \in I(D) \subseteq J$, then $x_{i_1}^{a_1} | z_1 z_2^{w(z_2)}$. But $x_{i_1} \in L_3(C)$, so $z_1, z_2 \in N_G[x_{i_1}] \subseteq C$. If $x_{i_1} = z_1$, then there is $2 \leq r \leq s$ such that $z_2 = x_{i_r}$. Thus $x_{i_r}^{a_r} \mid z_1 z_2^{w(z_2)}$. A contradiction, then $x_{i_1} = z_2$, $z_1 \in C$ and $(z_1, x_{i_1}) \in E(D)$. Then, $w(z_1) = 1$ or $z_1 \in L_1(C)$. In both cases $z_1 \in \mathcal{G}(I_C)$. Furthermore $z_1 \neq z_2$ since $(z_1, z_2) \in E(D)$. This implies $z_1 \in \mathcal{G}(J')$. So, $z_1 z_2^{w(z_2)} \in J'$. Hence, $I(D) \subseteq J'$. This is a contradiction, since J is minimal. Therefore C is strong. □

Theorem 4.25 If $\mathcal{C}_s$ is the set of strong vertex covers of D, then the irredundant irreducible decomposition of $I(D)$ is given by $I(D) = \bigcap_{C \in \mathcal{C}_s} I_C$.

Proof. By [12, Theorem 1.3.1], there is a unique irredundant irreducible decomposition $I(D) = \bigcap_{i=1}^{m} I_i$. If there is an irreducible ideal I'_j such that $I(D) \subseteq I'_j \subseteq I_j$ for some $j \in \{1, \dots, m\}$, then $I(D) = (\bigcap_{i \neq j} I_i) \cap I'_j$ is an irreducible decomposition. Furthermore this decomposition is irredundant. Thus, $I'_j = I_j$. Hence, $I_1, \dots, I_m$ are minimal irreducible ideals of $I(D)$. Now, if there is $C \in \mathcal{C}_s$ such that $I_C \notin \{I_1, \dots, I_m\}$, then there is $x_{j_i}^{\alpha_i} \in I_i \setminus I_C$ for each $i \in \{1, \dots, m\}$. Consequently, $m = lcm(x_{j_1}^{\alpha_1}, \dots, x_{j_m}^{\alpha_m}) \in \bigcap_{i=1}^{m} I_i = I(D) \subseteq I_C$. Furthermore, if $C = \{x_{i_1}, \dots, x_{i_k}\}$, then $I_C = (x_{i_1}^{\beta_1}, \dots, x_{i_k}^{\beta_k})$ where $\beta_j \in \{1, w(x_{i_j})\}$. Hence, there is $j \in \{1, \dots, k\}$ such that $x_{i_j}^{\beta_j} | m$. So, there is $1 \leq u \leq m$ such that $x_{i_j}^{\beta_j} \mid x_{j_u}^{\alpha_u}$. A contradiction, since $x_{j_u}^{\alpha_u} \notin I_C$. Therefore $I(D) = \bigcap_{C \in \mathcal{C}_s} I_C$ is the irredundant irreducible decomposition of $I(D)$. □

Remark 4.26 If $C_1, \dots, C_s$ are the strong vertex covers of D, then by Theorem 4.25, $I_{C_1} \cap \dots \cap I_{C_s}$ is the irredundant irreducible decomposition of $I(D)$. Furthermore, if $P_i = \text{rad}(I_{C_i})$, then $P_i = (C_i)$. So, $P_i \neq P_j$ for $1 \leq i < j \leq s$. Thus, $I_{C_1} \cap \dots \cap I_{C_s}$ is an irredundant primary decomposition of $I(D)$. In particular we have $\text{Ass}(I(D)) = \{P_1, \dots, P_s\}$.

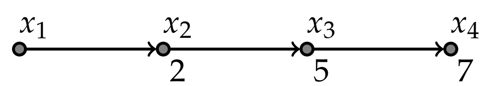
Example 4.27 Let D be the following oriented weighted graph whose edge ideal is $I(D) = (x_1^3 x_2, x_2^4 x_3, x_3^5 x_4, x_3 x_5^2, x_4^2 x_5)$.



From Theorem 4.24 and Theorem 4.25, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1^3, x_3, x_4^2) \cap (x_1^3, x_3, x_5) \cap (x_2, x_3, x_4^2) \cap (x_2, x_3^5, x_5) \cap (x_2, x_4, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_5) \cap (x_1^3, x_2^4, x_4, x_5^2) \cap (x_2, x_3^5, x_4^2, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_4^2, x_5^2).$$

Example 4.28 Let D be the following oriented weighted graph



Hence, $I(D) = (x_1x_2^2, x_2x_3^5, x_3x_4^7)$. By Theorem 4.24 and Theorem 4.25, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1, x_3) \cap (x_2^2, x_3) \cap (x_2, x_4^7) \cap (x_1, x_3^5, x_4^7) \cap (x_2^2, x_3^5, x_4^7).$$

In Example 4.27 and Example 4.28, $I(D)$ has embedding primes. Furthermore the monomial ideal $(V(D))$ is an associated prime of $I(D)$ in Example 4.27. Proposition 4.14 and Remark 4.26 give a combinatorial criterion for to decide when $(V(D)) \in \text{Ass}(I(D))$.

4.4 UNMIXED WEIGHTED ORIENTED GRAPHS

Let $D = (V, E, w)$ be a weighted oriented graph whose underlying graph is $G = (V, E)$. In this section we characterize the unmixed property of $I(D)$ and we prove that this property is closed under c-minors. In particular if G is a bipartite graph or G is a whisker or G is a cycle, we give an effective (combinatorial) characterization of this property.

Definition 4.29 An ideal I is **unmixed** if each one of its associated primes has the same height.

Theorem 4.30 The following conditions are equivalent:

1) $I(D)$ is unmixed.

2) Each strong vertex cover of D has the same cardinality.

3) $I(G)$ is unmixed and $L_3(C) = \emptyset$ for each strong vertex cover C of D.

Proof. Let $C_1, \ldots, C_\ell$ be the strong vertex covers of D. By Remark 4.26, the associated primes of $I(D)$ are $P_1, \ldots, P_\ell$, where $P_i = \text{rad}(I_{C_i}) = (C_i)$ for $1 \leq i \leq \ell$.

1) $\Rightarrow$ 2) Since $I(D)$ is unmixed, $|C_i| = \text{ht}(P_i) = \text{ht}(P_j) = |C_j|$ for $1 \leq i < j \leq \ell$.

2) $\Rightarrow$ 3) If C is a minimal vertex cover, then by Corollary 4.8, $C \in \{C_1, \ldots, C_\ell\}$. By hypothesis, $|C_i| = |C_j|$ for each $1 \leq i \leq j \leq \ell$, then C_i is a minimal vertex cover of D. Thus, by Lemma 4.5, $L_3(C_i) = \emptyset$. Furthermore $I(G)$ is unmixed, since $C_1, \ldots, C_\ell$ are the minimal vertex covers of G.

3) $\Rightarrow$ 1) By Proposition 4.5, C_i is a minimal vertex cover, since $L_3(C_i) = \emptyset$ for each $1 \leq i \leq \ell$. This implies $C_1, \ldots, C_\ell$ are the minimal vertex covers of G. Since G is unmixed, we have $|C_i| = |C_j|$ for $1 \leq i < j \leq \ell$. Therefore $I(D)$ is unmixed. □

Definition 4.31 A weighted oriented graph D has the **minimal-strong property** if each strong vertex cover is a minimal vertex cover.

Remark 4.32 Using Proposition 4.5, we have that D has the minimal-strong property if and only if $L_3(C) = \emptyset$ for each strong vertex cover C of D.

Definition 4.33 D' is a **c-minor** of D if there is a stable set S of D, such that $D' = D \setminus N_G[S]$.

Lemma 4.34 If D has the minimal-strong property, then $D' = D \setminus N_G[x]$ has the minimal-strong property, for each $x \in V$.

Proof. We take a strong vertex cover C' of $D' = D \setminus N_G[x]$ where $x \in V$. Thus, $C = C' \cup N_D(x)$ is a vertex cover of D. If $y' \in L_3(C')$, then by Proposition 4.4, $N_{D'}(y') \subseteq C'$. Consequently, $N_D(y') \subseteq C' \cup N_D(x) = C$ implying $y' \in L_3(C)$. Hence, $L_3(C') \subseteq L_3(C)$. Now, we take $y \in L_3(C)$, then $N_D(y) \subseteq C$. This implies $y \notin N_D(x)$, since $x \notin C$. Then, $y \in C'$ and $N_{D'}(y) \cup (N_D(y) \cap N_D(x)) = N_D(y) \subseteq C = C' \cup N_D(x)$. So, $N_{D'}(y) \subseteq C'$ implies $y \in L_3(C')$. Therefore $L_3(C) = L_3(C')$.

Now, if $y \in L_3(C) = L_3(C')$, then there is $z \in C' \setminus L_1(C')$ with $w(z) \neq 1$, such that $(z, y) \in E(D')$. If $z \in L_1(C)$, then there exist $z' \notin C$ such that $(z, z') \in E(D)$. Since $z' \notin C$, we have $z' \notin C'$, then $z \in L_1(C')$. A contradiction, consequently $z \notin L_1(C)$. Hence, C is strong. This implies $L_3(C) = \emptyset$, since D has the minimal-strong property. Thus, $L_3(C') = L_3(C) = \emptyset$. Therefore D' has the minimal-strong property. □

Proposition 4.35 If D is unmixed and $x \in V$, then $D' = D \setminus N_G[x]$ is unmixed.

Proof. By Theorem 4.30, G is unmixed and D has the minimal-strong property. Hence, by [28], $G' = G \setminus N_G[x]$ is unmixed. Also, by Lemma 4.34 we have that D' has the minimal-strong property. Therefore, by Theorem 4.30, D' is unmixed. □

Theorem 4.36 If D is unmixed, then a c-minor of D is unmixed.

Proof. If D' is a c-minor of D, then there is a stable set $S = \{a_1, \ldots, a_s\}$ such that $D' = D \setminus N_G[S]$. Since S is a stable set, $D' = (\cdots((D \setminus N_G[a_1]) \setminus N_G[a_2]) \setminus \cdots) \setminus N_G[a_s]$. Hence, by induction and Proposition 4.35, D' is unmixed. □

Proposition 4.37 If $V(D)$ is a strong vertex cover of D, then $I(D)$ is mixed.

Proof. By Proposition 4.4 $V(D)$ is not minimal, since $L_3(V(D)) = V(D)$. Therefore, by Theorem 4.30, $I(D)$ is mixed. □

Remark 4.38 If $V^+ = V$, then $I(D)$ is mixed.

Proof. If $x_i \in V$, then by Remark 4.17 $N_D^-(x_i) \neq \emptyset$, since $V = V^+$. Thus, there is $x_j \in V$ such that $(x_j, x_i) \in E(D)$. Also, $w(x_j) \neq 1$ and $x_j \in V = L_3(V)$. So, V is a strong vertex cover. Hence, by Proposition 4.37, $I(D)$ is mixed. □

In the following three results we assume that $D_1, \ldots, D_r$ are the connected components of D. Furthermore G_i is the underlying graph of D_i.

Lemma 4.39 Let C be a vertex cover of D, then $L_1(C) = \bigcup_{i=1}^r L_1(C_i)$ and $L_3(C) = \bigcup_{i=1}^r L_3(C_i)$, where $C_i = C \cap V(D_i)$.

Proof. We take $x \in C$, then $x \in C_j$ for some $1 \leq j \leq r$. Thus, $N_D(x) = N_{D_j}(x)$. In particular $N_D^+(x) = N_{D_j}^+(x)$, so $C \cap N_D^+(x) = C_j \cap N_{D_j}^+(x)$. Hence, $L_1(C) = \bigcup_{i=1}^r L_1(C_i)$. On the other hand,

$$x \in L_3(C) \Leftrightarrow N_D(x) \subseteq C \Leftrightarrow N_{D_j}(x) \subseteq C_j \Leftrightarrow x \in L_3(C_j).$$

Therefore, $L_3(C) = \bigcup_{i=1}^{r} L_3(C_i)$. □

Lemma 4.40 Let C be a vertex cover of D, then C is strong if and only if each $C_i = C \cap V(D_i)$ is strong with $i \in \{1, \dots, r\}$.

Proof. $\Rightarrow$) We take $x \in L_3(C_j)$. By Lemma 4.39, $x \in L_3(C)$ and there is $z \in N_D^-(x) \cap V^+$ with $z \in C \setminus L_1(C)$, since C is strong. So, $z \in N_{D_j}^-(x)$ and $z \in V(D_j)$, since $x \in D_j$. Consequently, by Lemma 4.39, $z \in C_j \setminus L_1(C_j)$. Therefore C_j is strong.

$\Leftarrow$) We take $x \in L_3(C)$, then $x \in C_i$ for some $1 \leq i \leq r$. Then, by Lemma 4.39, $x \in L_3(C_i)$. Thus, there is $a \in N_{D_i}^-(x)$ such that $w(a) \neq 1$ and $a \in C_i \setminus L_1(C_i)$, since C_i is strong. Hence, by Lemma 4.39, $a \in C \setminus L_1(C)$. Therefore C is strong. □

Corollary 4.41 $I(D)$ is unmixed if and only if $I(D_i)$ is unmixed for each $1 \leq i \leq r$.

Proof. $\Rightarrow$) By Theorem 4.36, since D_i is a c-minor of D.

$\Leftarrow$) By Theorem 4.30, G_i is unmixed thus G is unmixed. Now, if C is a strong vertex cover, then by Lemma 4.39, $C_i = C \cap V(D_i)$ is a strong vertex cover. Consequently, $L_3(C_i) = \emptyset$, since $I(D_i)$ is unmixed. Hence, by Lemma 4.39, $L_3(C) = \bigcup_{i=1}^{r} L_3(C_i) = \emptyset$. Therefore, by Theorem 4.30, $I(D)$ is unmixed. □

Definition 4.42 Let G be a simple graph whose vertex set is $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. A **whisker** of G is a graph H whose vertex set is $V(H) = V(G) \cup \{y_1, \dots, y_n\}$ and whose edge set is $E(H) = E(G) \cup \{\{x_1, y_1\}, \dots, \{x_n, y_n\}\}$.

Definition 4.43 Let D and H be weighted oriented graphs. H is a **weighted oriented whisker** of D if $D \subseteq H$ and the underlying graph of H is a whisker of the underlying graph of D.

Theorem 4.44 Let H a weighted oriented whisker of D, where $V(D) = \{x_1, \dots, x_n\}$ and $V(H) = V(D) \cup \{y_1, \dots, y_n\}$, then the following conditions are equivalents:

1) $I(H)$ is unmixed.

2) If $(x_i, y_i) \in E(H)$ for some $1 \leq i \leq n$, then $w(x_i) = 1$.

Proof. 2) $\Rightarrow$ 1) We take a strong vertex cover C of H. Suppose $x_j, y_j \in C$, then $y_j \in L_3(C)$, since $N_D(y_j) = \{x_j\} \subseteq C$. Consequently, $(x_j, y_j) \in E(G)$ and $w(x_j) \neq 1$, since C is strong. This is a contradiction by condition 2). This implies, $|C \cap \{x_i, y_i\}| = 1$ for each $1 \leq i \leq n$. So, $|C| = n$. Therefore, by Theorem 4.30, $I(H)$ is unmixed.

1) $\Rightarrow$ 2) By contradiction suppose $(x_i, y_i) \in E(H)$ and $w(x_i) \neq 1$ for some i. Since $w(x_i) \neq 1$ and by Remark 4.17, we have that x_i is not a source. Thus, there is $x_j \in V(D)$, such that $(x_j, x_i) \in E(H)$. We take the vertex cover $C = \{V(D) \setminus x_j\} \cup \{y_j, y_i\}$, then by Proposition 4.4, $L_3(C) = \{y_i\}$. Furthermore $N_D(x_i) \setminus C = \{x_j\}$ and $(x_j, x_i) \in E(H)$, then $x_i \in L_2(C)$. Hence C is strong, since $L_3(C) = \{y_i\}$, $(x_i, y_i) \in E(G)$ and $w(x_i) \neq 1$. A contradiction by Theorem 4.30, since $I(H)$ is unmixed. □

Theorem 4.45 Let D be a bipartite weighted oriented graph, then $I(D)$ is unmixed if and only if

1) G has a perfect matching $\{\{x_1^1, x_1^2\}, \ldots, \{x_s^1, x_s^2\}\}$ where $\{x_1^1, \ldots, x_s^1\}$ and $\{x_1^2, \ldots, x_s^2\}$ are stable sets. Furthermore if $\{x_j^1, x_i^2\}$, $\{x_i^1, x_k^2\} \in E(G)$ then $\{x_j^1, x_k^2\} \in E(G)$.

2) If $w(x_j^k) \neq 1$ and $N_D^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$ where $\{k, k'\} = \{1, 2\}$, then $N_D(x_{i_\ell}^k) \subseteq N_D^+(x_j^k)$ and $N_D^-(x_{i_\ell}^k) \cap V^+ = \emptyset$ for each $1 \leq \ell \leq r$.

Proof. $\Leftarrow$) By 1) and [10, Theorem 2.5.7], G is unmixed. We take a strong vertex cover C of D. Suppose $L_3(C) \neq \emptyset$, thus there exist $x_i^k \in L_3(C)$. Since C is strong, there is $x_j^{k'} \in V^+$ such that $(x_j^{k'}, x_i^k) \in E(D)$, $x_j^{k'} \in C \setminus L_1(C)$ and $\{k, k'\} = \{1, 2\}$. Furthermore $N_D^+(x_j^{k'}) \subseteq C$, since $x_j^{k'} \notin L_1(C)$. Consequently, by 3), $N_D(x_i^{k'}) \subseteq N_D^+(x_j^{k'}) \subseteq C$ and $N_D^-(x_i^{k'}) \cap V^+ = \emptyset$. A contradiction, since $x_i^{k'} \in L_3(C)$ and C is strong. Hence, $L_3(C) = \emptyset$ and D has the strong-minimal property. Therefore $I(D)$ is unmixed, by Theorem 4.30.

$\Rightarrow$) By Theorem 4.30, G is unmixed. Hence, by [10, Theorem 2.5.7], G satisfies 1). If $w(x_j^k) \neq 1$, then we take $C = N_D^+(x_j^k) \cup \{x_i^k \mid N_D(x_i^k) \not\subseteq N_D^+(x_j^k)\}$ and k' such that $\{k, k'\} = \{1, 2\}$. If $\{x_i^k, x_{i'}^{k'}\} \in E(G)$ and $x_i^k \notin C$, then $x_{i'}^{k'} \in N_D(x_i^k) \subseteq N_D^+(x_j^k) \subseteq C$. This implies, C is a vertex cover of D. Now, if $x_{i_1}^k \in L_3(C)$, then $N_D(x_{i_1}^k) \subseteq C$. Consequently $N_D(x_{i_1}^k) \subseteq N_D^+(x_j^k)$ implies $x_{i_1}^k \notin C$. A contradiction, then $L_3(C) \subseteq N_D^+(x_j^k)$. Also, $N_G^-(x_j^k) \neq \emptyset$, since $w(x_j^k) \neq 1$. So $x_j^k \in L_2(C)$, since $N_G^-(x_j^k) \cap C = \emptyset$. Hence C is strong, since $L_3(C) \subseteq N_D^+(x_j^k)$ and $x_j^k \in V^+$. Furthermore

$\{x'_1, \ldots, x'_s\}$ is a minimal vertex cover, then by Theorem 4.30 $|C| = s$, since D is unmixed. We assume $N^+_D(x^k_j) = \{x^{k'}_{i_1}, \ldots x^{k'}_{i_r}\}$. Since C is minimal, $x^k_{i_\ell} \notin C$ for each $1 \leq \ell \leq r$. Thus, $N_D(x^k_{i_\ell}) \subseteq N^+_D(x^k_j)$. Now, suppose $z \in N^-_D(x^k_{i_\ell}) \cap V^+$, then $z = x^{k'}_{i_{\ell'}}$ for some $1 \leq \ell' \leq r$, since $N_D(x^k_{i_\ell}) \subseteq N^+_D(x^k_j)$. We take $C' = N^+_D(x^k_j) \cup \{x^k_i \mid i \notin \{i_1, \ldots, i_r\}\} \cup N^+_D(x^{k'}_{i_{\ell'}})$. Since $N_D(x^k_{i_u}) \subseteq N^+_D(x^k_j)$ for each $1 \leq u \leq r$, we have that C' is a vertex cover. If $\{x^k_q, x^{k'}_q\} \cap L_3(C) \neq \emptyset$, then $\{x^k_q, x^{k'}_q\} \subseteq C'$. So, $x^{k'}_q \in N^+_D(x^k_j)$ implies $q \in \{i_1, \ldots, i_r\}$. Consequently, $x^k_q \in N^+_D(x^{k'}_{i_{\ell'}})$, since $x^k_q \in C'$. This implies, $(x^k_j, x^{k'}_q), (x^{k'}_{i_{\ell'}}, x^k_q) \in E(D)$. Also, $N^+_D(x^{k'}_{i_{\ell'}}) \cup N^+_D(x^k_j) \subseteq C'$, then $x^{k'}_{i_{\ell'}} \notin L_1(C')$ and $x^k_j \notin L_1(C')$. Thus, C' is strong, since $x^k_j, x^{k'}_{i_{\ell'}} \in V^+$. Furthermore, by Theorem 4.30, $|C'| = s$. But $x^{k'}_{i_\ell} \in N^+_D(x^k_j)$ and $x^k_{i_\ell} \in N^+_D(x^{k'}_{i_{\ell'}})$, hence $x^{k'}_{i_\ell}, x^k_{i_\ell} \in C'$. A contradiction, so $N^-_D(x^k_{i_\ell}) \cap V^+ = \emptyset$. Therefore D satisfies 2). □

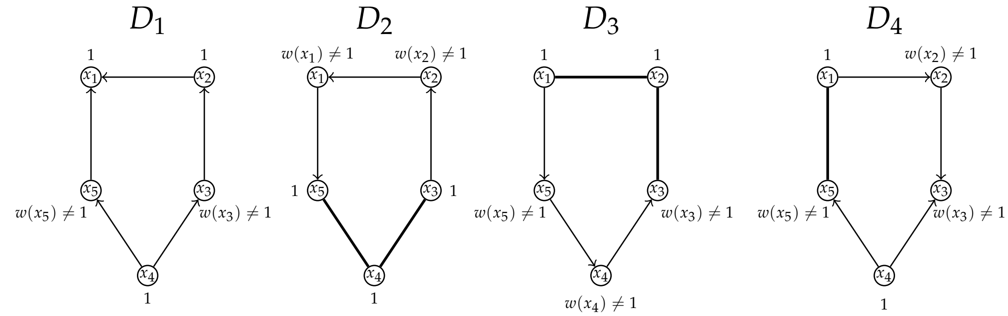
Lemma 4.46 If the vertices of V^+ are sinks, then D has the minimal-strong property.

Proof. We take a strong vertex cover C of D. Hence, if $y \in L_3(C)$, then there is $(z, y) \in E(D)$ with $z \in V^+$. Consequently, by hypothesis, z is a sink. A contradiction, since $(z, y) \in E(D)$. Therefore, $L_3(C) = \emptyset$ and C is a minimal vertex cover. □

Lemma 4.47 Let D be a weighted oriented graph, where $G \simeq C_n$ with $n \geq 6$. Hence, D has the minimal-strong property if and only if the vertices of V^+ are sinks.

Proof. $\Leftarrow$) By Lemma 4.46.

$\Rightarrow$) By contradiction, suppose there is $(z, y) \in E(D)$, with $z \in V^+$. We can assume $G = (x_1, x_2, \ldots, x_n, x_1) \simeq C_n$, with $x_2 = y$ and $x_3 = z$. We take a strong vertex cover C in the following form: $C = \{x_1, x_3, \ldots, x_{n-1}\} \cup \{x_2\}$ if n is even or $C = \{x_1, x_3, \ldots, x_{n-2}\} \cup \{x_2, x_{n-1}\}$ if n is odd. Consequently, if $x \in C$ and $N_D(x) \subseteq C$, then $x = x_2$. Hence, $L_3(C) = \{x_2\}$. Furthermore $(x_3, x_2) \in E(D)$ with $x_3 \in V^+$. Thus, x_3 is not a source, so, $(x_4, x_3) \in E(D)$. Then, $x_3 \in L_2(C)$. This implies C is a strong vertex cover. But $L_3(C) \neq \emptyset$. A contradiction, since D has the minimal-strong property. □



Theorem 4.48 If $G \simeq C_n$, then $I(D)$ is unmixed if and only if one of the following conditions hold:

1) $n = 3$ and there is $x \in V(D)$ such that $w(x) = 1$.

2) $n \in \{4, 5, 7\}$ and the vertices of V^+ are sinks.

3) $n = 5$, there is $(x, y) \in E(D)$ with $w(x) = w(y) = 1$ and $D \not\simeq D_1, D \not\simeq D_2, D \not\simeq D_3$.

4) $D \simeq D_4$.

Proof. $\Rightarrow$) By Theorem 4.30, D has the minimal- strong property and G is unmixed. Then, by [10, Exercise 2.4.22], $n \in \{3, 4, 5, 7\}$. If $n = 3$, then by Remark 4.38, D satisfies 1). If $n = 7$, then by Lemma 4.47, D satisfies 2). Now suppose $n = 4$ and D does not satisfies 2), then we can assume $x_1 \in V^+$ and $(x_1, x_2) \in E(D)$. Consequently, $(x_4, x_1) \in E(G)$, since $w(x_1) \neq 1$. Furthermore, $\mathcal{C} = \{x_1, x_2, x_3\}$ is a vertex cover with $L_3(\mathcal{C}) = \{x_2\}$. Thus, $x_1 \in L_2(\mathcal{C})$ and $(x_1, x_2) \in E(D)$ so $\mathcal{C}$ is strong. A contradiction, since $\mathcal{C}$ is not minimal. This implies D satisfies 2). Finally suppose $n = 5$. If $D \simeq D_1$, then $\mathcal{C}_1 = \{x_1, x_2, x_3, x_5\}$ is a vertex cover with $L_3(\mathcal{C}_1) = \{x_1, x_2\}$. Also $(x_5, x_1), (x_3, x_2) \in E(D)$ with $x_5, x_3 \in V^+$. Consequently, $\mathcal{C}_1$ is strong, since $x_5, x_3 \in L_2(\mathcal{C}_1)$. A contradiction, since $\mathcal{C}_1$ is not minimal. If $D \simeq D_2$, then $\mathcal{C}_2 = \{x_1, x_2, x_4, x_5\}$ is a vertex cover where $L_3(\mathcal{C}_2) = \{x_1, x_5\}$ and $(x_2, x_1), (x_1, x_5) \in E(D)$ with $x_2, x_1 \in V^+$. Hence, $\mathcal{C}_2$ is strong, since $x_2, x_1 \notin L_1(\mathcal{C}_2)$. A contradiction, since $\mathcal{C}_2$ is not minimal. If $D \simeq D_3$, $\mathcal{C}_3 = \{x_2, x_3, x_4, x_5\}$ is a vertex cover where $L_3(\mathcal{C}_3) = \{x_3, x_4\}$ and $(x_4, x_3), (x_5, x_4) \in E(D)$ with $x_4, x_5 \in V^+$. Thus, $\mathcal{C}_3$ is strong, since $x_4, x_5 \notin L_1(\mathcal{C}_3)$. A contradiction, since $\mathcal{C}_3$ is not minimal. Now, since $n = 5$ and by 3) we can assume $(x_2, x_3) \in E(D)$, $x_2, x_3 \in V^+$ and there are not two adjacent vertices with weight 1. Since $x_2 \in V^+$, $(x_1, x_2) \in E(D)$. Suppose there are not 3 vertices z_1, z_2, z_3 in V^+ such that (z_1, z_2, z_3) is a path in G, then $w(x_4) = w(x_1) = 1$. Furthermore, $w(x_5) \neq 1$, since there are not adjacent vertices with weight 1. So, $\mathcal{C}_4 = \{x_2, x_3, x_4, x_5\}$ is a vertex cover of D, where $L_3(\mathcal{C}_4) = \{x_3, x_4\}$. Also $(x_2, x_3) \in E(G)$ with $w(x_2) \neq 1$. Hence, if $(x_3, x_4) \in E(D)$ or $(x_5, x_4) \in E(D)$, then $\mathcal{C}_4$ is strong, since $x_3, x_5 \in V^+$. But $\mathcal{C}_4$ is not minimal. Consequently, $(x_4, x_3), (x_4, x_5) \in E(D)$ and $D \simeq D_4$. Now, we can assume there is a path (z_1, z_2, z_3) in D such that $z_1, z_2, z_3 \in V^+$. Since there are not adjacent vertices with weight 1, we can suppose there is $z_4 \in V^+$ such that $\mathcal{L} = (z_1, z_2, z_3, z_4)$ is a path. We take $\{z_5\} = V(D) \setminus V((L))$ and we can assume $(z_2, z_3) \in E(D)$. This implies, (z_1, z_2), $(z_5, z_1) \in E(D)$, since $z_1, z_2 \in V^+$. Thus, $\mathcal{C}_5 = \{z_1, z_2, z_3, z_4\}$ is a vertex cover with $L_3(\mathcal{C}_5) = \{z_2, z_3\}$. Then $\mathcal{C}_5$ is strong, since $(z_1, z_2), (z_2, z_3) \in E(D)$ with $z_2 \in L_3(\mathcal{C}_5)$ and $z_1 \in L_2(\mathcal{C}_5)$. A contradiction, since $\mathcal{C}_5$ is not minimal.

$\Leftarrow$) If $n \in \{3, 4, 5, 7\}$, then by [10, Exercise 2.4.22] G is unmixed. By Theorem 4.30, we will only prove that D has the minimal-strong property. If D satisfies 2), then by Lemma 4.46, D has the minimal-strong property. If D satisfies 1) and $\mathcal{C}$ is a strong vertex cover, then by Proposition 4.14, $|\mathcal{C}| \leq 2$. This implies $\mathcal{C}$ is minimal. Now, suppose $n = 5$ and $\mathcal{C}'$ is a strong vertex cover of D with $|\mathcal{C}'| \geq 4$. If $D \simeq D_4$, then $x_2, x_5 \notin L_3(\mathcal{C}')$, since $(N_D^-(x_2) \cup N_D^-(x_5)) \cap V^+ = \emptyset$. So $N_D(x_2) \nsubseteq \mathcal{C}'$ and $N_D(x_5) \nsubseteq \mathcal{C}'$. Consequently, $x_1 \notin \mathcal{C}'$ implies $\mathcal{C}' = \{x_2, x_3, x_4, x_5\}$. But $x_4 \in L_3(\mathcal{C}')$ and $N_D^-(x_4) = \emptyset$. A contradiction, since $\mathcal{C}'$ is strong. Now assume D satisfies 3). Suppose there is a path $\mathcal{L} = (x_1, x_2, x_3)$ in G such that $w(x_1) = w(x_2) = w(x_3) = 1$. We can suppose $(x_4, x_5) \in E(D)$ where $V(D) \setminus V(\mathcal{L}) = \{x_4, x_5\}$. Since $w(x_1) = w(x_3) = 1$, $x_2 \notin L_3(\mathcal{C}')$. If $x_2 \notin \mathcal{C}'$, then $\mathcal{C}' = \{x_1, x_3, x_4, x_5\}$ and $x_4 \in L_3(\mathcal{C}')$. But $N_D^-(x_4) = \{x_3\}$ and $w(x_3) = 1$. A contradiction, hence $x_2 \in \mathcal{C}'$. We can assume $x_3 \notin \mathcal{C}'$, since $x_2 \notin L_3(\mathcal{C}')$. This implies $\mathcal{C}' = \{x_1, x_2, x_4, x_5\}$ and $L_3(\mathcal{C}') = \{x_1, x_5\}$. Thus, $(x_5, x_1) \in E(D)$ and $x_5, x_4 \in V^+$. Consequently $(x_3, x_4) \in E(D)$, since $x_4 \in V^+$. A contradiction, since $D \not\simeq D_2$. Hence, there are not three consecutive vertices whose weights are 1. Consequently, since D satisfies 3), we can assume $w(x_1) = w(x_2) = 1$, $w(x_3) \neq 1$ and $w(x_5) \neq 1$. If $w(x_4) = 1$, then $x_3, x_5 \notin L_3(\mathcal{C}')$ since $N_D(x_3, x_5) \cap V^+ = \emptyset$. This implies $N_D(x_3) \nsubseteq \mathcal{C}'$ and $N_D(x_5) \nsubseteq \mathcal{C}'$. Then, $x_4 \notin \mathcal{C}'$ and $\mathcal{C}' = \{x_1, x_2, x_3, x_5\}$. Thus, $(x_5, x_1), (x_3, x_2) \in E(D)$, since $L_3(\mathcal{C}') = \{x_1, x_2\}$. Consequently, $(x_4, x_5), (x_4, x_3) \in E(D)$, since $x_5, x_3 \in V^+$. A contradiction, since $D \not\simeq D_1$. So, $w(x_4) \neq 1$ and we can assume $(x_5, x_4) \in E(D)$, since $x_4 \in V^+$. Furthermore $(x_1, x_5) \in E(D)$, since $x_5 \in V^+$. Hence, $(x_3, x_4) \in E(D)$, since $D \not\simeq D_3$. Then $(x_2, x_3) \in E(D)$, since $x_3 \in V^+$. This implies $x_1, x_2, x_3, x_5 \notin L_3(\mathcal{C}')$, since $N_D^-(x_i) \cap V^+ = \emptyset$ for $i \in \{1, 2, 3, 5\}$. A contradiction, since $|\mathcal{C}'| \geq 4$. Therefore D has the minimal-strong property. □

4.5 COHEN-MACAULAY WEIGHTED ORIENTED GRAPHS

In this section we study the Cohen-Macaulayness of $I(D)$. In particular we give a combinatorial characterization of this property when D is a path or D is complete. Furthermore, we show the Cohen-Macaulay property depends of the characteristic of K.

Definition 4.49 The weighted oriented graph D is **Cohen-Macaulay** over the field K if the ring $R/I(D)$ is Cohen-Macaulay.

Remark 4.50 If G is the underlying graph of D, then $\text{rad}(I(D)) = I(G)$.

Proposition 4.51 If $I(D)$ is Cohen-Macaulay, then $I(G)$ is Cohen-Macaulay and D has the minimal-strong property.

Proof. By Remark 4.50, $I(G) = \text{rad}(I(D))$, then by [14, Theorem 2.6], $I(G)$ is Cohen-Macaulay. Furthermore $I(D)$ is unmixed, since $I(D)$ is Cohen-Macaulay. Hence, by Theorem 4.30, D has the minimal-strong property.

Example 4.52 In Example 4.27 and Example 4.28 $I(D)$ is mixed. Hence, $I(D)$ is not Cohen-Macaulay, but $I(G)$ is Cohen-Macaulay.

Conjecture 4.53 $I(D)$ is Cohen- Macaulay if and only if $I(G)$ is Cohen-Macaulay and D has the minimal-strong property. Equivalently $I(D)$ is Cohen-Macaulay if and only if $I(D)$ is unmixed and $I(G)$ is Cohen-Macaulay.

Proposition 4.54 Let D be a weighted oriented graph such that $V = \{x_1, \ldots, x_k\}$ and whose underlying graph is a path $G = (x_1, \ldots, x_k)$. Then the following conditions are equivalent.

1) $R/I(D)$ is Cohen-Macaulay.

2) $I(D)$ is unmixed.

3) $k = 2$ or $k = 4$. In the second case, if $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, then $w(x_2) = 1$ or $w(x_3) = 1$ respectively.

Proof. *1) $\Rightarrow$ 2)* By [10, Corollary 1.5.14].

2) $\Rightarrow$ 3) By Theorem 4.45, G has a perfect matching, since D is bipartite. Consequently k is even and $\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{k-1}, x_k\}$ is a perfect matching. If $k \geq 6$, then by Theorem 4.45, we have $\{x_2, x_5\} \in E(G)$, since $\{x_2, x_3\}$ and $\{x_4, x_5\} \in E(G)$. A contradiction since $\{x_2, x_5\} \notin E(G)$. Therefore $k \in \{2, 4\}$. Furthermore by Theorem 4.45, $w(x_2) = 1$ or $w(x_3) = 1$ when $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, respectively.

3) ⇒ 1) We take $I = I(D)$. If $k = 2$, then we can assume $(x_1, x_2) \in E(D)$. So, $I = (x_1 x_2^{w(x_2)}) = (x_1) \cap (x_2^{w(x_2)})$. Thus, by Remark 4.26, $\mathrm{Ass}(I) = \{(x_1), (x_2)\}$. This implies, $\mathrm{ht}(I) = 1$ and $\dim(R/I) = k - 1 = 1$. Also, $\mathrm{depth}(R/I) \geq 1$, since $(x_1, x_2) \notin \mathrm{Ass}(I)$. Hence, R/I is Cohen-Macaulay. Now, if $k = 4$, then $\mathrm{ht}(I) = \mathrm{ht}(\mathrm{rad}(I)) = \mathrm{ht}(I(G)) = 2$. Consequently, $\dim(R/I) = k - 2 = 2$. Furthermore one of the following sets $\{x_2 - x_1^{w(x_1)}, x_3 - x_4^{w(x_4)}\}$, $\{x_2 - x_1^{w(x_1)}, x_4 - x_3^{w(x_3)}\}$, $\{x_1 - x_2^{w(x_2)}, x_4 - x_3^{w(x_3)}\}$ is a regular sequence of R/I, then $\mathrm{depth}(R/I) \geq 2$. Therefore, I is Cohen-Macaulay. □

Theorem 4.55 If G is a complete graph, then the following conditions are equivalent.

1) $I(D)$ is unmixed.

2) $I(D)$ is Cohen-Macaulay.

3) There are not $D_1, \ldots, D_s$ unicycles orientes subgraphs of D such that $V(D_1), \ldots, V(D_s)$ is a partition of $V(D)$

Proof. We take $I = I(D)$. Since $I(G) = \mathrm{rad}(I)$ and G is complete, $\mathrm{ht}(I) = \mathrm{ht}(I(G)) = n - 1$.

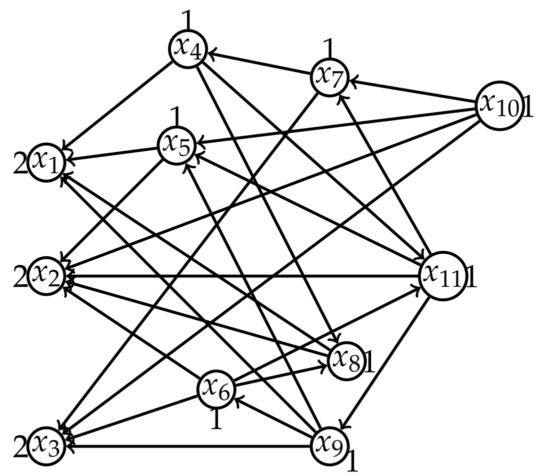
1) ⇒ 3) Since $\mathrm{ht}(I) = n - 1$ and I is unmixed, $(x_1, \ldots, x_n) \notin \mathrm{Ass}(I)$. Thus, by Remark 4.26, $V(D)$ is not a strong vertex cover of D. Therefore, by Proposition 4.14, D satisfies 3).

3) ⇒ 2) By Proposition 4.14, $V(D)$ is not a strong vertex cover of D. Consequently, by Remark 4.26, $(x_1, \ldots, x_n) \notin \mathrm{Ass}(I)$. This implies, $\mathrm{depth}(R/I) \geq 1$. Furthermore, $\dim(R/I) = 1$, since $\mathrm{ht}(I) = n - 1$. Therefore I is Cohen-Macaulay.

2) ⇒ 1) By [10, Corollary 1.5.14]. □

If D is complete or D is a path, then the Cohen-Macaulay property does not depend of the field K. It is not true in general, see the following example.

Example 4.56 Let D be the following weighted oriented graph:



Hence,

$$I(D) = (x_1^2x_4, x_1^2x_8, x_1^2x_5, x_1^2x_9, x_2^2x_{10}, x_2^2x_5, x_2^2x_{11}, x_2^2x_8, x_2^2x_6, x_3^2x_7, x_3^2x_{10}, x_3^2x_6,$$
$$x_3^2x_9, x_4x_8, x_4x_7, x_4x_{11}, x_5x_{10}, x_5x_9, x_5x_{11}, x_6x_8, x_6x_9, x_6x_{11}, x_7x_{10}, x_7x_{11}, x_9x_{11}).$$

By [17, Example 2.3], $I(G)$ is Cohen- Macaulay when the characteristic of the field K is zero but it is not Cohen-Macaulay in characteristic 2. Consequently, $I(D)$ is not Cohen-Macaulay when the characteristic of K is 2. Also, $I(G)$ is unmixed. Furthermore, by Lemma 4.46, $I(D)$ has the minimal-strong property, then $I(D)$ is unmixed. Using Macaulay2 [11] we show that $I(D)$ is Cohen-Macaulay when the characteristic of K is zero.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[2] S. Bayati, J. Herzog, and G. Rinaldo, On the stable set of associated prime ideals of a monomial ideal. Preprint, 2011.

[3] J. Brennan, L. A. Dupont, and R. H. Villarreal, Duality, a-invariants and canonical modules of rings arising from linear optimization problems, *Bull. Math. Soc. Sci. Math. Roumanie* (N.S.) **51** (2008) no. 4, 279–305.

[4] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, Cambridge, Revised Edition, 1998.

[5] J. Chen, S. Morey, A. Sung, and A. Sung, The stable set of associated primes of the ideal of a graph, *Rocky Mountain J. Math.* **32** (2002) 71–89.

[6] V. I. Danilov, The geometry of toric varieties, *Russian Math Surveys* **33** (1978) 97–154.

[7] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics **173**, Springer-Verlag, New York, 2nd ed., 2000.

[8] G. A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg.* **38** (1961) 71–76.

[9] L. A. Dupont, C. Rentería-Márquez and R. H. Villarreal, Systems with the integer rounding property in normal monomial subrings, *An. Acad. Brasil. Ciênc.* **82** (2010) no. 4, 801–811.

[10] I. Gitler and R. H. Villarreal, *Graphs, Rings and Polyhedral*, Texto Nivel Avanzado **35**, Aportaciones Matemáticas, Sociedad Matematica Mexicana, 2011.

[11] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, 2017. Available at `http://www.math.uiuc.edu/Macaulay2/`

[12] J. Herzog and T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics **260**, Springer–Verlag, 2011.

[13] J. Herzog and A. A. Qureshi, Persistence and stability properties of powers of ideals. *J. Pure Appl. Algebra* **219** (2015) no. 3, 530–542.

[14] J. Herzog, Y. Takayama, and N. Terai, On the radical of a monomial ideal, *Arch. Math.* **85** (2005) 397–408.

[15] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, *Ann. of Math.* **96** (1972) 318–337.

[16] Huy Tai Ha and Mengyao Sun, Squarefree monomial ideals that fail the persistence property and non-increasing depth, *Acta Math. Vietnam* **40** (2015) no. 1, 125–137.

[17] A. Madadi and R. Zaare-Nahandi, Cohen-Macaulay r-partite graphs with minimal clique cover, *Bull. Iranian Math. Soc.* **40** (2014) no. 3, 609–617.

[18] S. Morey, J. Martínez-Bernal, and R. H. Villarreal, Associated primes of powers of edges ideals, *Collect. Math.* **63** (2012) no. 3, 361–374.

[19] S. Morey, E. Reyes, and R. H. Villarreal, Cohen–Macaulay, shellable and unmixed clutters with a perfect matching of Köning type, *J. Pure Appl. Algebra* **212** (2008) no. 7, 1770–1786.

[20] S. Morey and R. H. Villarreal, *Edge ideals: algebraic and combinatorial properties*, Progress in Commutative Algebra 1, 85–126, De Gruyter, Berlin, 2012.

[21] Y. Pitones, E. Reyes, and J. Toledo, Monomial ideals of weighted oriented graphs. Preprint, 2017.

[22] B. Randerath and P. D. Vestergaard, Well–covered graphs and factors, *Discrete Appl. Math.* **154** (2006) 1416–1428.

[23] L. J. Ratliff, On prime divisors of I^n, n large, *Michigan Math. J.* **23** (1976) no. 4, 337–352.

[24] E. Reyes and J. Toledo, On the strong persistence property for monomial ideals, *Bull. Math. Soc. Sci. Math. Roumaine* **60** (2017) no. 3, 293–305.

[25] E. Reyes, J. Toledo, and L. Zárate, On Gorenstein homogeneous monomial subrings of graphs. Preprint, 2017.

[26] W. V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer-Verlag, 1998.

[27] R. H. Villarreal, Cohen-Macaulay graphs, Manuscripta Math. **66** (1990) 277–293.

[28] R. H. Villarreal, *Monomial Algebras*, Second Edition, Monographs and Research Notes in Mathematics, Chapman and Hall/CRC, 2015.