CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE MATEMÁTICAS

Funciones Zeta Locales de Igusa y amplitudes de cuerdas p-ádicas

Tesis que presenta

M. en C. Miriam Bocardo Gaspar

para obtener el Grado de

Doctora en Ciencias

en la Especialidad de

Matemáticas

Director de la Tesis:

Dr. Wilson Álvaro Zúñiga Galindo

Ciudad de México Noviembre del 2017

CENTER FOR RESEARCH AND ADVANCED STUDIES
OF THE NATIONAL POLYTECHNIC INSTITUTE

CAMPUS ZACATENCO
DEPARTMENT OF MATHEMATICS

Igusa's Local Zeta Functions and p-adic string amplitudes

A dissertation presented by

Msc. Miriam Bocardo Gaspar

to obtain the Degree of

Doctora en Ciencias en la Especialidad de Matemáticas

Thesis advisor:
Prof. Dr. Wilson Álvaro Zúñiga Galindo

Ciudad de México November, 2017

Doctoral Dissertation Jury

Dr. Xavier Gómez Mont
Centro de Investigación en Matemáticas
Guanajuato

Dr. Rafael H. Villarreal
Departamento de Matemáticas
Cinvestav-Zacatenco

Dr. Juan Manuel Burgos
Departamento de Matemáticas
Cinvestav-Zacatenco

Dr. Hugo García Compeán
Departamento de Física
Cinvestav-Zacatenco

Dr. Wilson Álvaro Zúñiga Galindo
Departamento de Matemáticas
Cinvestav-Zacatenco

Reviewers

Dr. Alejandro Melle Hernández

Departamento de Algebra, Facultad de Matemáticas

Universidad Complutense de Madrid

Dr. Willem Veys

Department of Mathematics, Section of Algebra

University of Leuven (KULeuven).

Dedication

To my mother Ma. Trinidad Gaspar.

“A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs; and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.”

Stefan Banach

Acknowledgements

I want to express my gratitude to my supervisor Prof. Dr. Wilson A. Zúñiga Galindo for his helpful guidance, enthusiastic encouragement advice, and his kind motivational comments. My thanks are also to Prof. Dr. Hugo Compeán for his helpful and numerous discussions to improve my knowledge on this subject. I would also like to thank Prof. Dr. Willem Veys, for his exhaustive comments and suggestions about this dissertation. His recommendations were useful to improve this work. My thanks to Prof. Dr. Alejandro Melle for his valuable comments and to the doctoral dissertation jury for their careful reading of my dissertation. I take this opportunity to thank Prof. Dr. Pedro Luis del Ángel for his invaluable advice and support.

I want to thank the Consejo Nacional de Ciencia y Tecnología (CONACYT) of México for the financial support for this academic project.

Finally, my special thanks to my family and friends for their support and encouragement throughout my PhD studies. Particularly, to my mother and María de la Paz Argüelles Martínez.

Abstract

This dissertation is divided into two parts. The first part is dedicated to the study of the p-adic string amplitudes and the limit when p approaches to one of such amplitudes using techniques of local zeta functions. We prove that the p-adic Koba-Nielsen type string amplitudes are bona fide integrals. We attach to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then we use these functions to regularize the Koba-Nielsen amplitudes, which was an open problem. These results were obtained in collaboration with Prof. Dr. Wilson A. Zúñiga Galindo and Prof. Dr. Hugo García Compeán, see [8]. In p-adic string theory the limit when p approaches to one plays an important role. There is an empirical evidence that the p-adic strings are related to the ordinary strings in the $p \to 1$ limit. In [8], we established that p-adic Koba-Nielsen string amplitudes are finite sums of Igusa's local zeta functions. Denef and Loeser established that the limit $p \to 1$ of Igusa's local zeta functions give rise to new objects, that they called topological zeta functions. By using Denef-Loeser's theory of topological zeta functions, we show that limit $p \to 1$ of a tree-level p-adic open strings amplitudes give rise to new amplitudes, which we have called string amplitudes underlying topological zeta functions. We expect that these amplitudes will be related with the theory derived of one given by Gerasimov and Shatashvili.

The second part is dedicated to the study of p-adic local zeta functions attached to certain rational functions. These objects are very alike to Feynman parametric integrals. These results were obtained in collaboration with Dr. W. A. Zúñiga Galindo in [7]. In this part, we introduce a new non-degeneracy condition for rational functions with respect to a certain Newton polyhedra, and study local zeta functions attached to non-degenerate rational functions. We obtained explicit formulas for these local zeta functions in terms of some data associated to the corresponding Newton polyhedra.

Resumen

Este trabajo se divide en dos partes. La primera parte es dedicada al estudio de las amplitudes de cuerdas p-ádicas y su límite cuando p se aproxima a uno usando técnicas de funciones zeta locales. Demostramos que las amplitudes de cuerdas p-ádicas del tipo Koba-Nielsen son integrales convergentes. Asociamos a estas amplitudes integrales de tipo Igusa dependiendo de varios parámetros complejos y mostramos que estas integrales admiten continuaciones meromorfas como funciones racionales. Entonces, usamos estas funciones para regularizar las amplitudes Koba-Nielsen, el cuál era un problema abierto. Estos resultados se obtuvieron en colaboración con el Dr. Wilson A. Zúñiga Galindo y el Dr. Hugo García Compeán, ver [8]. En la teoría de cuerdas p-ádicas, el límite cuando p se aproxima a uno juega un rol importante. Existe evidencia empírica de que las cuerdas p-ádicas están relacionadas con las cuerdas ordinarias en el límite $p \to 1$. En [8], demostramos que las amplitudes de cuerdas p-ádicas Koba-Nielsen son sumas finitas de funciones zeta locales de Igusa. Denef y Loeser establecieron que el límite $p \to 1$ de funciones zeta locales de Igusa genera nuevos objetos, llamados funciones zeta topológicas. Usando la teoría de Denef-Loeser de funciones zeta topológicas, mostramos que el límite $p \to 1$ de una amplitud p-ádica de cuerdas abiertas a nivel árbol genera nuevas amplitudes, las cuáles llamamos amplitudes de cuerdas topológicas. Esperamos que estas amplitudes estén relacionadas con la teoría derivada del trabajo de Gerasimov y Shatashvili.

La segunda parte está dedicada al estudio de funciones zeta locales p-ádicas asociadas a ciertas funciones racionales. Estos objetos son muy semejantes a las integrales paramétricas de Feynman. Estos resultados se obtuvieron en colaboración con el Dr. W. A. Zúñiga Galindo en [7]. En esta parte, introducimos una nueva condición de no-degeneración para funciones racionales con respecto a un cierto poliedro de Newton y estudiamos funciones zeta locales asociadas a funciones racionales no-degeneradas. Obtuvimos fórmulas explícitas para estas funciones zeta locales en términos de algunos datos asociados al correspondiente poliedro de Newton.

Contents

Overview of the Dissertation

This dissertation is dedicated to the study of the connections between local zeta functions and p-adic string amplitudes. The dissertation is divided into two parts. The first part (Chapters 1, 2) is dedicated to the study of the p-adic string amplitudes and the limit when p approaches to one of such amplitudes using techniques of local zeta functions. The second part (Chapter 3) is dedicated to the study of p-adic local zeta functions attached to certain rational functions.

In Chapter 1, we prove that the p-adic Koba-Nielsen type string amplitudes are bona fide integrals. We attach to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then we use these functions to regularize the Koba-Nielsen amplitudes. The regularization of the Koba-Nielsen string amplitudes was an open problem in Archimedean and non-Archimedean settings. As far as we now, there is no a similar result to the one established here in the Archimedean setting. The results presented in Chapter 1 were obtained in collaboration with Prof. Dr. Wilson A. Zúñiga Galindo and Prof. Dr. Hugo García Compeán, see [8].

In p-adic string theory the limit when p approaches to one plays an important role. It seems that in the limit $p \to 1$ the p-adic strings approximate ordinary strings, see e.g. [28], [30]. A central motivation for this dissertation is to understand the above mentioned calculations from a mathematical perspective. In Chapter 2, by using the topological zeta functions introduced by Denef and Loeser, we introduce topological string amplitudes. We are writing an article, based in Chapter 2, that aims to explain the calculations done by Gerasimov and Shatashvili in [28], see [9].

In Chapter 3, we present some new results about the meromorphic continuation of local zeta functions attached to rational functions over non-Archimedean local fields. These objects are very alike to Feynman parametric integrals. These results were obtained in collaboration with Dr. W. A. Zúñiga Galindo, in this chapter, we introduce a new non-degeneracy condition for rational functions with respect to a certain Newton polyhedra, and study local zeta functions attached to non-degenerate rational functions. We obtain explicit formulas for these local zeta functions and a geometric description for the poles in terms of some data associated to the corresponding Newton polyhedra.

In the '60s, the local zeta functions were introduced by Israel Gel'fand and André Weil. In the Archimedean setting, i.e. in $\mathbb{R}$, $\mathbb{C}$, the local zeta functions were studied by Gel'fand and Shilov in [27]. A central motivation was that the meromorphic continuation of the Archimedean local zeta functions implies the existence of fundamental solutions for differential operators with constant coefficients. The meromorphic continuation of local zeta functions was conjectured by I. Gel'fand, and this result was proved, independently, by Atiyah [2] and Bernstein [5]. On the other hand, Weil studied local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [64]. In the '70s, Igusa developed a uniform theory for local zeta functions in characteristic zero.

Nowadays, there are several types of local zeta functions, for instance p-adic, Archimedean, topological, motivic, among others, see e.g. [39], [18], [22], [21] and references therein. The topological zeta functions were introduced, in the '90s, by Denef and Loeser, and recently they also introduced the motivic ones, which constitute a vast generalization of the p-adic local zeta functions as well as of the topological zeta functions. The local zeta functions have deep connections with number theory, algebraic geometry, singularity theory, and other branches of mathematics. In the p-adic setting, they are connected with the number of solutions of polynomial congruences $\bmod\, p^m$ and with exponential sums $\bmod\, p^m$, see e.g. [39].

This dissertation is focused on the study of non-Archimedean and topological zeta functions and their relations with p-adic string amplitudes. From a more general perspective, our work, is motivated by the connections between non-Archimedean analysis and mathematical physics. There are two main forces behind this interaction. First, in the '80s, Volovich posed the conjecture that the space-time has a non-Archimedean structure at the level of the Planck scale and initiated the p-adic string theory [62], see also [56, Chapter 6], [63]. Volovich noted that the integral expression for the Veneziano amplitude of the open bosonic string can be generalized to a p-adic integral and to an adelic integral giving rise to non-Archimedean Veneziano amplitudes. Then Freund and Witten established (formally) that the ordinary Veneziano and Virasoro-Shapiro four-particle scattering amplitudes can be factored in terms of an infinite product of non-Archimedean string amplitudes [26], see also [3]. As a consequence of the interest on p-adic models of quantum field theory, which is motivated by the fact that these models are exactly solvable, there is a large list of p-adic type Feynman and string amplitudes that are related with local zeta functions of Igusa-type, and it is interesting to mention that seems that the mathematical community working on local zeta functions is not aware of this fact, see e.g. [3], [4], [6], [16], [15], [10], [25], [24], [26], [37], [44], [45], [46], [49], [51], [52], and the references therein.

Second, p-adic strings seems to have many properties in common with the ordinary strings. We recall that "$\lim_{p\to 1}$" already appeared in several calculations in p-adic string theory, see e.g. [28], [29], but the limit $p \to 1$ does not seem to have sense for the discrete variable p. As a consequence of the connections between p-adic string amplitudes and local zeta functions, it is possible to use the theory of topological zeta functions due to Denef and Loeser [21] to give sense to this limit by producing topological string amplitudes, which should be string analogues of the topological zeta functions. We developed this idea in this dissertation.

Another interesting problem is the study of local zeta functions for rational functions. The study of these new local zeta functions is a recent mathematical problem and it is motivated by their relations with parametric Feynman integrals. In [61], W. Veys and W. A. Zúñiga-Galindo extended Igusa's theory to the case of rational functions, or, more generally, meromorphic functions f/g, with coefficients in a local field of characteristic zero. From a physical perspective, the local zeta functions attached to meromorphic functions are very alike to parametric Feynman integrals and to p-adic string amplitudes, see e.g. [4], [10], [15], [49]. For instance in [49, Section 3.15], M. Marcolli pointed out explicitly that the motivic Igusa zeta function constructed by J. Denef and F. Loeser may provide the right tool for a motivic formulation of the dimensionally regularized parametric Feynman integrals. In this dissertation we studied the local zeta functions attached to certain non-degenerate rational functions with coefficients in a non-Archimedean local field of arbitrary characteristic.

We now describe briefly our contributions.

Regularizations of p -adic string amplitudes

Take $N \geq 4$ and $s_{ij} \in \mathbb{C}$ satisfying $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. In this thesis we study the following multivariate Igusa-type zeta function:

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right) = \int\limits_{\mathbb{Q}_p^{N-3} \smallsetminus \Lambda} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} \, |1-x_i|_p^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

where $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, and

$$\Lambda := \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3};\ \prod_{i=2}^{N-2} x_i \left(1 - x_i\right) \prod_{2 \leq i < j \leq N-2} (x_i - x_j) = 0 \right\}.$$

We call this type of integrals *p-adic open string N-point zeta functions* because they appeared in connection with the p-adic open string N-tachyon tree amplitudes, see e.g. [15], [16], [25], [26], [37], and the references therein. These amplitudes are defined as

$$\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}\right)$$
$$=\int\limits_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}\left|x_i\right|_p^{\boldsymbol{k}_1\boldsymbol{k}_i}\left|1-x_i\right|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i}\prod_{2\leq i<j\leq N-2}\left|x_i-x_j\right|_p^{\boldsymbol{k}_i\boldsymbol{k}_j}\prod_{i=2}^{N-2}dx_i,$$

where $\prod_{i=2}^{N-2}dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$,

$$\underline{\boldsymbol{k}}=\left(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N\right),\boldsymbol{k}_i=\left(k_{0,i},\ldots,k_{25,i}\right),i=1,\ldots,N,N\geq 4,$$

(with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j=-k_{0,i}k_{0,j}+k_{1,i}k_{1,j}+\cdots+k_{25,i}k_{25,j}$) obeying

$$\sum_{i=1}^{N}\boldsymbol{k}_i=\boldsymbol{0},\ \boldsymbol{k}_i\boldsymbol{k}_i=2\text{ for }i=1,\ldots,N,$$

In all the published literature about p-adic string amplitudes have been used without considering the convergence of them, i.e. the problem of the regularization of p-adic open string N-tachyon amplitudes has not been considered before. In the light of the theory of local zeta functions, the possible convergence of integrals of type $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ is a new and remarkable aspect. In this thesis, we proved that

Theorem 1.29 *The p-adic open string N-point zeta function, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$, gives rise to a holomorphic function on $H(\mathbb{C})$, which contains an open and connected subset of $\mathbb{C}^D$. Furthermore, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ admits an analytic continuation to $\mathbb{C}^D$, denoted also as $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$, as a rational function in the variables $p^{-s_{ij}}, i,j\in\{1,\ldots,N-1\}$. The real parts of the poles of $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C6 with the symbols '$<$', '$>$' replaced by '$=$'. (2) If $\underline{\boldsymbol{s}}=(s_{ij})\in\mathbb{C}^D$, with $\operatorname{Re}(s_{ij})\geq 0$ for $i,j\in\{1,\ldots,N-1\}$, then $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)=+\infty$.*

Here $H(\mathbb{C})$ is as in Definition 1.27 and C1-C6 are as in Remarks 1.21, 1.25.

Take $\phi\left(x_2,\ldots,x_{N-2}\right)$ a locally constant function with compact support, then

$$Z_\phi^{(N)}(\underline{\boldsymbol{s}})$$
$$=\int\limits_{\mathbb{Q}_p^{N-3}\smallsetminus\Lambda}\phi\left(x_2,\ldots,x_{N-2}\right)\prod_{i=2}^{N-2}\left|x_i\right|_p^{s_{1i}}\left|1-x_i\right|_p^{s_{(N-1)i}}\prod_{2\leq i<j\leq N-2}\left|x_i-x_j\right|_p^{s_{ij}}\prod_{i=2}^{N-2}dx_i,$$

is a multivariate Igusa local zeta function. A general theory for this type of local zeta functions was elaborated by Loeser in [47]. In particular, these local zeta functions admit analytic continuations as rational functions of the variables $p^{-s_{ij}}$. If we take ϕ to be the characteristic function of B_r^{N-3}, the ball centered at the origin with radius p^r, the dominated convergence theorem and Theorem 1.29, imply that $\lim_{r\to\infty} Z^{(N)}_{B_r^{N-3}}(\underline{\boldsymbol{s}}) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ for any $\underline{\boldsymbol{s}}$ in the natural domain of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$.

A central problem is to know whether or not integrals of type $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ converge for some values $\boldsymbol{k}_i\boldsymbol{k}_j \in \mathbb{C}$. Our Theorem 1.29 allows us to solve this problem. We take the p-adic open string N-point tree integrals $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ as regularizations of the amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$. More precisely, we define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) \mid_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1, \ldots, N-1\}, \, j \in T \text{ or } i, j \in T,$$

where $T = \{2, \ldots, N-2\}$. By Theorem 1.29, $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ are well-defined rational functions of the variables $p^{-\boldsymbol{k}_i\boldsymbol{k}_j}$, i, $j \in \{1, \ldots, N-1\}$, which agree with amplitudes used by the physicists, when they converge. This definition allows us to recover all the calculations made in [15] and other similar publications. At this point, it is relevant to mention that there is no similar result for the Archimedean string amplitudes at the three level, as Witten pointed out in [66, p. 4]. We notice that the string amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ are limits of local zeta functions when they are considered as distributions, by a slight abuse of notation, this means that

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = \lim_{r\to\infty} \boldsymbol{Z}^{(N)}_{B_r^{N-3}}(\underline{\boldsymbol{k}}),$$

for $\underline{\boldsymbol{k}}$ in the natural domain of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{k}})$. Another important problem is to determine the existence of (in the sense of quantum field theory) ultraviolet and infrared divergencies for $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$. If we use the Euclidean product instead of the Minkowski product to define $s_{ij} = \boldsymbol{k}_i\boldsymbol{k}_j$, then $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ has infrared divergencies ($\boldsymbol{A}^{(N)}(0) = +\infty$) and ultraviolet divergencies ($\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = +\infty$ for $\boldsymbol{k}_i\boldsymbol{k}_j > 0$).

Lerner and Missarov studied a class of p-adic integrals that includes certain type of Feynman integrals and Koba-Nielsen amplitudes. They showed, see [44, Theorem

2], that this type of integrals can be computed recursively by using hierarchies, but they did no investigate the convergence, or more generally the holomorphy, of the Koba-Nielsen amplitudes, which is a delicate matter.

At this point, it is worth to mention that the typical approach for establishing that an integral of Igusa-type admits an analytic continuation is via Hironaka's resolution of singularities theorem, see e.g. [39, Chapters 3, 5, 8]. Roughly speaking Hironaka's resolution theorem provides a finite sequence of changes of variables (blow-ups) that allows to express an Igusa-type integral as a linear combination of integrals involving monomials, for this type of integrals the existence of an analytic continuation is easy to show. If the initial Igusa-type integral is a holomorphic function in a certain domain, then by using any suitable sequence of blow-ups the existence of an analytic continuation can be established. If the convergence of the original integral is unknown then, in principle, by using Hironaka's theorem is possible to find an analytic continuation, i.e. a regularization, of the given integral, but this regularization depends on the sequence of blow-ups used, which is not unique. This method gives infinitely many regularizations of the original integral. The problem of choosing a specific definition of the problem or the problem of showing uniqueness of the regularized integral is highly non-trivial. For this reason, our approach is not based on resolution of singularities, instead of this, we use an approach inspired in the calculations presented in [15] and in the Igusa's p-adic stationary phase formula, see [39, Theorem 10.2.1], [67]-[69]. As a consequence of this approach, all of our results are still valid if we replace $\mathbb{Q}_p$ by $\mathbb{F}_q((t))$, the field of formal Laurent series over a finite field $\mathbb{F}_q$.

Topological string amplitudes

Physicists have related the p-adic string amplitudes with classical string amplitudes by taking "$\lim p \to 1$" in certain calculations in p-adic string theory, see [27], [30], and references therein. In this dissertation, using Denef-Loeser formalism of topological zeta functions, we show the existence of topological string amplitudes at the tree level that are obtained from the corresponding p-adic amplitudes by taking "lim $p \rightarrow 1$". We explain here briefly how the topological string amplitudes are constructed, for further details, see Chapter 2.

Consider $\boldsymbol{f} = (f_1, \ldots, f_r)$ where each $f_i(\boldsymbol{x})$ is a non-constant polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$, for $i = 1, \ldots, r$. Put $f = \prod_{i=1}^r f_i(\boldsymbol{x})$. Let (Y, φ) be an embedded resolution of singularities for $D = Spec\, \mathbb{Q}[\boldsymbol{x}] / (f)$ with $\{E_i\}_{i \in T}$ the irreducible components of $\varphi^{-1}(f^{-1}(0))_{red}$. Let $\{(N_{i1}, \ldots, N_{ir}, v_i); i \in T\}$ the numerical data of φ. There exists a finite set of primes $S \subseteq \mathbb{Z}$ such that for p not in S, and any finite extension K of $\mathbb{Q}_p$, the formula

$$\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, K) = q_K^{-n} \sum_{I \subset T} c_I(K) \prod_{i \in I} \frac{(q_K - 1)\, q_K^{-v_i - \sum_{j=1}^r N_{ij} s_j}}{1 - q_K^{-v_i - \sum_{j=1}^r N_{ij} s_j}},$$

where

$$c_I(K) = Card\left\{a \in \overline{Y}\left(\overline{K}\right); a \in \overline{E}_i\left(\overline{K}\right) \Leftrightarrow i \in I\ \right\}$$

is valid. Here $\bar{\cdot}$ denotes the reduction mod P_K for which we refer to [17]. This explicit formula is a simple variation of the one given by Denef in [17].

In [21] Denef and Loeser introduced the topological zeta function

$$\boldsymbol{Z}_{top}(\boldsymbol{s}) = \sum_{I \subset T} \chi\left(\overset{\circ}{E}_I\right) \prod_{i \in I} \frac{1}{v_i + \sum_{j=1}^r N_{ij} s_j},$$

where for any scheme V of finite type over a field $L \subset \mathbb{C}$, $\chi(V)$ denotes the Euler characteristic of the $\mathbb{C}$-analytic space associated with V. We mention that in arbitrary dimension there is not a canonical way of picking an embedded resolution of singularities for a divisor. Then, it is necessary to show that definition (2.4.3) is independent of the resolution of singularities chosen, this fact was established by Denef and Loeser in [47]. By using the explicit formula given by Denef for $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, K)$, Denef and Loeser showed that

$$\boldsymbol{Z}_{top}(\boldsymbol{s}) = \lim_{e \rightarrow 0} \boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, K_e),$$

where K_e is the unramified extension of $\mathbb{Q}_p$ (for almost all prime number p of $\mathbb{Z}$) of degree e. The limit $e \to 0$ makes sense because one can l-adically interpolate $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, K_e)$ as a function of e. Furthermore, they gave a description of the poles of the local zeta functions in terms of the poles of the topological zeta function:

Theorem *If $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}_{top}(\boldsymbol{s})$, then for almost all P there exists infinitely many unramified extensions L of K for which $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, L)$.*

The techniques used to prove 1.29 over $\mathbb{Q}_p$ also work for any extension of $\mathbb{Q}_p$, in particular for K_e the unique unramified extension of $\mathbb{Q}_p$ of degree e, see Theorem 2.7. We denote this N-point zeta function as $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, K_e)$, by replacing the p-adic norms $|\cdot|_p$ by the norm $|\cdot|_{K_e}$ over K_e and $\mathbb{Z}_p$ by $\mathcal{O}_{K_e}$ in $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$. Let

$$M(\underline{\boldsymbol{s}}) := |T \smallsetminus I| + \sum_{i \in T \smallsetminus I} (s_{1i} + s_{(N-1)i}) + \sum_{\substack{2 \le i < j \le N-2 \\ i \in T \smallsetminus I, j \in T}} s_{ij} + \sum_{\substack{2 \le i < j \le N-2 \\ i \in I, j \in T \smallsetminus I}} s_{ij},$$

as in the case of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$, we show that

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, K_e) = \sum_{I \subseteq T} q_{K_e}^{M(\underline{\boldsymbol{s}})} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, K_e)\, \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, K_e),$$

see Section 2.2.

Since $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, K_e)$ and $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, K_e)$ are multivariate local zeta functions of type $\boldsymbol{Z}(\boldsymbol{s}, \boldsymbol{f}, K_e)$ for suitable $\boldsymbol{f}$, for any $I \subseteq T = \{2, .., N-3\}$, we can define, as above,

$$\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; I, 0) := \lim_{e \to 0} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0, K_e) \quad \text{and}$$

$$\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1) := \lim_{e \to 0} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, K_e),$$

which are elements of $\mathbb{Q}(s_{ij}, i, j \in \{1, \ldots, N-1\})$, the field of rational functions in the variables s_{ij}, $i, j \in \{1, \ldots, N-1\}$, with coefficients in $\mathbb{Q}$.

Then, we define the *open string N-point topological zeta functions* as

$$\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}) = \sum_{I \subseteq T} \boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; I, 0)\, \boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}; T \smallsetminus I, 1) \in \mathbb{Q}(s_{ij}, i, j \in \{1, \ldots, N-1\})$$

Now, by applying Theorems 2.7, 2.5, we obtain that the possible poles of $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ belong to a finite union of hyperplanes. Formally, we have the following result:

Theorem[Theorem 2.1] *The open string N-point topological zeta function $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ is a rational function from $\mathbb{Q}(s_{ij}, i,j \in \{1,\ldots,N-1\})$ defined as (2.5.1). The real parts of the possible poles of $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C6 with the symbols '$<$', '$>$' replaced by '$=$'. (2) If $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, with* $\mathrm{Re}(s_{ij}) \geq 0$ *for $i,j \in \{1,\ldots,N-1\}$, then $\boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}}) = +\infty$.*

where *C1-C6* are as in Remarks 2.3, 2.4. And finally, we define the *topological open string N-point tree amplitudes* as

$$\boldsymbol{A}_{top}^{(N)}(\underline{\boldsymbol{k}}) = \boldsymbol{Z}_{top}^{(N)}(\underline{\boldsymbol{s}})\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1,\ldots,N-1\}\,,\, j \in T \text{ or } i,j \in T,$$

where $T = \{2,\ldots,N-2\}$, which are rational functions of the variables $\boldsymbol{k}_i\boldsymbol{k}_j$.

Local zeta functions for non-degenerate rational functions

Let K be a non-Archimedean local field of arbitrary characteristic and let $\mathcal{O}_K$ be the ring of valuation of K,

$$\mathcal{O}_K := \{x \in K : |x|_K \leq 1\}\,,$$

and P_K the maximal ideal of $\mathcal{O}_K$; this ideal is formed by the non-units of $\mathcal{O}_K$. In terms of the absolute value $|\cdot|_K$, this maximal ideal can be described as

$$P_K = \{x \in K : |x|_K < 1\}\,.$$

Let $\overline{K} = \mathcal{O}_K/\ P_K$ the residue field of K. Thus $\overline{K} = \mathbb{F}_q$, the finite field with q elements. Let π be fixed generator of P_K, π is called a uniformizing parameter of K, then $P_K = \pi\mathcal{O}_K$, furthermore, we assume that $|\pi|_K = q^{-1}$. For $z \in K$, ord$(z)\in$

$\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, and $|z|_K = q^{-ord(z)}$. If $z\in K\setminus\{0\}$, then $\mathrm{ac}(z)=z\pi^{-ord(z)}$ denotes the angular component of z.

Let $\boldsymbol{h}$ be a polynomial mapping $\boldsymbol{h}=(h_1,\ldots,h_r): K^n\to K^r$ such that each $h_i(\boldsymbol{x})$ is a non-constant polynomial in $\mathcal{O}_K[x_1,\ldots,x_n]\setminus\pi\mathcal{O}_K[x_1,\ldots,x_n]$, $\boldsymbol{x}=(x_1,\ldots,x_n)$, $r\leq n$, and let $\boldsymbol{s}=(s_1,\ldots,s_r)\in\mathbb{C}^r$. We attach to these data the multivariate local zeta function

$$Z(\boldsymbol{s},\boldsymbol{h}) := \int\limits_{\mathcal{O}_K^n\setminus D_K}\prod_{i=1}^{r}|h_i(\boldsymbol{x})|_K^{s_i}\,|d\boldsymbol{x}|_K$$

for $\mathrm{Re}(s_i)>0$ for all i, where $D_K := \cup_{i\in\{1,\ldots,r\}}\{\boldsymbol{x}\in K^n; h_i(\boldsymbol{x})=0\}$.

If $[K:\mathbb{Q}_p]<\infty$, i.e. if K is a p-adic field, $Z(\boldsymbol{s},\boldsymbol{h})$ were first studied by Loeser, see [47]. He showed that $Z(\boldsymbol{s},\boldsymbol{h})$ has a meromorphic continuation to whole $\mathbb{C}^r$ as a rational function. In Chapter 3 we introduced a new non-degeneracy condition for polynomial mappings, see Definition 3.3, and established an explicit formula for the meromorphic continuation of $Z(\boldsymbol{s},\boldsymbol{h})$ over any non-Archimedean local field K of arbitrary characteristic when $\boldsymbol{h}$ is non-degenerate. In the case $K=\mathbb{Q}_p$ and $r=1$, this non-degeneracy condition coincides with the one given in [20].

We now introduce some notation. Let $\Gamma(\boldsymbol{h})$ be the Newton polyhedron associated to $\boldsymbol{h}$, see section 3.3. Denote by $\mathcal{F}(\boldsymbol{h})$ the simplicial polyhedral subdivision subordinate to $\Gamma(\boldsymbol{h})$. Let $\Delta\in\mathcal{F}(\boldsymbol{h})$, then, there exist vectors $\boldsymbol{w}_i\in\mathbb{N}^n$, $i=1,..,e_\Delta$ with relatively prime coordinates such that

$$\Delta=\left\{\sum\lambda_i\boldsymbol{w}_i:\lambda_i\in\mathbb{R}_+,\ \lambda_i>0\right\}.$$

Set $b(\Delta):=\sum_{i=1}^{e_\Delta}\boldsymbol{w}_i$ and $b(\{\boldsymbol{0}\}):=\boldsymbol{0}$. For $I\subseteq\{1,\ldots,r\}$, we put

$$\overline{V}_{\Delta,I} := \left\{\overline{\boldsymbol{z}}\in(\mathbb{F}_q^\times)^n;\ \overline{h}_{i,b(\Delta)}(\overline{\boldsymbol{z}})=0\ \Leftrightarrow\ i\in I\right\}.$$

If $\Delta=\boldsymbol{0}$, we set

$$\overline{V}_I := \left\{\overline{\boldsymbol{z}}\in(\mathbb{F}_q^\times)^n;\ \overline{h}_i(\overline{\boldsymbol{z}})=0\ \Leftrightarrow\ i\in I\right\}.$$

Let $\boldsymbol{a}=(a_1,\ldots,a_n)\in\mathbb{N}^n$, we put $\sigma(\boldsymbol{a})=a_1+a_2+\ldots+a_n$ and $d(\boldsymbol{a},\Gamma(\boldsymbol{h}))=\min_{x\in\Gamma(\boldsymbol{h})}\langle\boldsymbol{a},\boldsymbol{x}\rangle$.

In Chapter 3, we showed that the multivariate local zeta function $Z(\boldsymbol{s},\boldsymbol{h})$ has a meromorphic continuation as a rational function in the variables q^{-s_i} when $\boldsymbol{h}$ is a non-degenerated polynomial mapping with respect to the Newton polyhedra $\Gamma(\boldsymbol{h})$, see Definition 3.3. More precisely,

Theorem *Assume that $\boldsymbol{h}=(h_1,\ldots,h_r)$ is non-degenerated polynomial mapping over $\mathbb{F}_q$ with respect to $\Gamma(\boldsymbol{h})$, with $r\leq n$ as before. Fix a simplicial polyhedral subdivision $\mathcal{F}(\boldsymbol{h})$ subordinate to $\Gamma(\boldsymbol{h})$. Then $Z(\boldsymbol{s},\boldsymbol{h})$ has a meromorphic continuation to $\mathbb{C}^r$ as a rational function in the variables q^{-s_i}, $i=1,\ldots,r$. In addition, the following explicit formula holds:*

$$Z(\boldsymbol{s},\boldsymbol{h})=L_{\{\mathbf{0}\}}(\boldsymbol{s},\boldsymbol{h})+\sum_{\Delta\,\in\,\mathcal{F}(\boldsymbol{h})}L_{\Delta}(\boldsymbol{s},\boldsymbol{h})S_{\Delta},$$

where

$$L_{\{\mathbf{0}\}}=q^{-n}\sum_{I\subseteq\{1,\ldots,r\}}Card(\overline{V}_I)\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$
$$L_{\Delta}=q^{-n}\sum_{I\subseteq\{1,\ldots,r\}}Card(\overline{V}_{\Delta,I})\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$

with the convention that for $I=\varnothing$, $\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}:=1$, and

$$S_{\Delta}=\sum_{\boldsymbol{k}\in\mathbb{N}^n\cap\Delta}q^{-\sigma(\boldsymbol{k})-\sum_{i=1}^{r}d(\boldsymbol{k},\Gamma(h_i))s_i}.$$

Let Δ be the cone strictly positively generated by linearly independent vectors $\boldsymbol{w}_1,\ldots,\,\boldsymbol{w}_l\in\mathbb{N}^n\setminus\{\mathbf{0}\}$, then

$$S_{\Delta}=\frac{\sum_{\boldsymbol{t}}q^{-\sigma(\boldsymbol{t})-\sum_{i=1}^{r}d(\boldsymbol{t},\Gamma(h_i))s_i}}{\left(1-q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^{r}d(\boldsymbol{w}_1,\Gamma(h_i))s_i}\right)\cdots\left(1-q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^{r}d(\boldsymbol{w}_l,\Gamma(h_i))s_i}\right)},$$

where $\boldsymbol{t}$ runs through the elements of the set

$$\mathbb{Z}^n\cap\left\{\sum_{i=1}^{l}\lambda_i\boldsymbol{w}_i;\ 0<\lambda_i\leq 1\text{ for }i=1,\ldots,l\right\}.$$

This theorem extends some results due to Hoornaert and Denef [20], and Bories [13]. Also, we applied this theorem to the study of local zeta functions attached to a rational function f/g with coefficients in a non-Archimedean local field of arbitrary characteristic, when f/g is non-degenerate with respect to a certain Newton polyhedron. In [43] E. León-Cardenal and W. A. Zúñiga-Galindo studied similar matters. In our results, we present a more suitable and general notion of non-degeneracy which allows us to study the local zeta functions attached to much larger class of rational functions. In this case, we extend the condition of non-degeneracy for polynomial mappings to rational functions f/g. Let f, g be relatively prime polynomials. We say that f/g is non-degenerate with respect to the Newton polyhedra $\Gamma\left(f/g\right) := \Gamma\left((f,g)\right)$ if the polynomial mapping (f,g) is non-degenerate with respect to $\Gamma\left((f,g)\right)$. Thus, by using the meromorphic continuation of $Z((s,-s),(f,g))$, see Theorem 3.1, we obtain the convergence and the explicit formula for the meromorphic continuation of the *local zeta function attached to the rational function* f/g

$$Z(s,f/g) = \int\limits_{\mathcal{O}_K^n \setminus D_K} \left|\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}\right|_K^s |d\boldsymbol{x}|_K$$

where $D_K = f^{-1}(0) \cup g^{-1}(0)$, $n \geq 2$, $s \in \mathbb{C}$, and $|d\boldsymbol{x}|_K$ is the normalized Haar measure on K^n, see Theorem 3.1

In Chapter 3, it is given an explicit list for the possible poles of $Z(s,f/g)$, including the smallest and largest one, in terms of the normal vectors to the supporting hyperplanes of a Newton polyhedra attached to (f,g). In contrast with the classical local zeta functions, these objects have poles with positive and negative real parts.

The study of local zeta functions associated to meromorphic functions is motivated by the fact that these objects can be considered 'toy versions' of parametric Feynman integrals.

Chapter 1

Regularization of p-adic string amplitudes and multivariate local zeta functions

This chapter aims to discuss some connections between p-adic string amplitudes and p-adic local zeta functions (also called Igusa's local zeta functions). We prove that the p-adic Koba-Nielsen type string amplitudes are bona fide integrals. We attach to these amplitudes Igusa-type integrals depending on several complex parameters and show that these integrals admit meromorphic continuations as rational functions. Then we use these functions to regularize the Koba-Nielsen amplitudes. As far as we know, there is no a similar result for the Archimedean Koba-Nielsen amplitudes. We also discuss the existence of divergencies and the connections with multivariate Igusa's local zeta functions.

The chapter is organized as follows. In section 1.1 we present the basic aspects of the p-adic analysis needed in this chapter, and in section 1.3, we prove the main result, Theorem 1.29.

1.1 Essential Ideas of p-Adic Analysis

In this section, we review some ideas and results on p-adic analysis that we will use along this chapter. For an in-depth exposition, the reader may consult [1], [55], [63].

1.1.1 The field of p-adic numbers

Along this chapter p will denote a prime number. As we mentioned in Section 2.1, the field of p-adic numbers $\mathbb{Q}_p$ is a non-Archimedean local field, it is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } \ x = 0 \\ \\ p^{-\gamma} & \text{if } \ x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of x*. We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||\boldsymbol{x}||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(\boldsymbol{x}) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||\boldsymbol{x}||_p = p^{-ord(\boldsymbol{x})}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a complete ultrametric space. As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected and locally compact. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [63, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{i=0}^{\infty} x_i p^i,$$

where $x_i \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$.

For $r \in \mathbb{Z}$, denote by $B_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{Q}_p^n; ||\boldsymbol{x} - \boldsymbol{a}||_p \leq p^r\}$ *the ball of radius p^r with center at* $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(\boldsymbol{0}) := B_r^n$. Note that $B_r^n(\boldsymbol{a}) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of p-adic integers.* In addition, $B_r^n(\boldsymbol{a}) = \boldsymbol{a} + (p^{-r}\mathbb{Z}_p)^n$. We also denote by $S_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{Q}_p^n; ||\boldsymbol{x} - \boldsymbol{a}||_p = p^r\}$ *the sphere of radius* p^r *with center at* $\boldsymbol{a} \in \mathbb{Q}_p^n$, and take $S_r^n(\boldsymbol{0}) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^n \subsetneq S_0^n = \left\{\boldsymbol{x} \in \mathbb{Q}_p^n; ||\boldsymbol{x}|| = 1\right\}$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

Remark 1.1 *There is a natural map, called the reduction* $\bmod p$ *and denoted as* $\bar{\cdot}$, *from* $\mathbb{Z}_p$ *onto* $\mathbb{F}_p$, *the finite field with* p *elements. More precisely, if* $x = \sum_{j=0}^{\infty} x_j p^j \in \mathbb{Z}_p$, *then* $\overline{x} = \overline{x}_0 \in \mathbb{F}_p = \left\{\overline{0}, \overline{1}, \ldots, \overline{p-1}\right\}$. *If* $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_p^n$, *then* $\overline{\boldsymbol{a}} = (\overline{a}_1, \ldots, \overline{a}_n)$.

1.1.2 Integration on $\mathbb{Q}_p^n$

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a Borel measure dx, called the Haar measure of $(\mathbb{Q}_p, +)$, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non-empty Borel open set $U \subset \mathbb{Q}_p$, and satisfying $\int_{E+z} dx = \int_E dx$ for every Borel set $E \subset \mathbb{Q}_p$, see e.g. [35, Chapter XI]. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique. From now on we denote by dx the normalized Haar measure of $(\mathbb{Q}_p, +)$ and by $d^n\boldsymbol{x}$ the product measure on $(\mathbb{Q}_p^n, +)$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be *locally constant* if for every $\boldsymbol{x} \in \mathbb{Q}_p^n$ there exists an open compact subset U, $\boldsymbol{x} \in U$, such that $\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{u})$ for all $\boldsymbol{u} \in U$. Any locally constant function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form $\varphi(\boldsymbol{x}) = \sum_{k=1}^{\infty} c_k 1_{U_k}(\boldsymbol{x})$, where $c_k \in \mathbb{C}$ and $1_{U_k}(\boldsymbol{x})$ is the characteristic function of U_k, an open compact subset of $\mathbb{Q}_p^n$, for every k. If φ has compact support, then $\varphi(\boldsymbol{x}) = \sum_{k=1}^{L} c_k 1_{U_k}(\boldsymbol{x})$ and in this case

$$\int_{\mathbb{Q}_p^n} \varphi(\boldsymbol{x})\, d^n\boldsymbol{x} = c_1 \int_{U_1} d^n\boldsymbol{x} + \ldots + c_L \int_{U_L} d^n\boldsymbol{x}.$$

A locally constant function with compact support is called a *Bruhat-Schwartz function*. These functions form a $\mathbb{C}$-vector space denoted as $\mathcal{S}\left(\mathbb{Q}_p^n\right)$. By using the Stone-Weierstrass theorem, $\mathcal{S}\left(\mathbb{Q}_p^n\right)$ is a dense subspace of $C_0\left(\mathbb{Q}_p^n\right)$, the space of continuous functions with compact support, and consequently the functional $\varphi \rightarrow \int_{\mathbb{Q}_p^n} \varphi\left(\boldsymbol{x}\right) d^n\boldsymbol{x}$, $\varphi \in \mathcal{S}\left(\mathbb{Q}_p^n\right)$ has a unique extension to $C_0\left(\mathbb{Q}_p^n\right)$. For integrating more general functions, say locally integrable functions, the following notion of improper integral will be used.

Definition 1.2 *A function $\varphi \in L^1_{loc}$ is said to be integrable in $\mathbb{Q}_p^n$ if*

$$\lim_{m\rightarrow+\infty} \int\limits_{B^n_m(0)} \varphi\left(\boldsymbol{x}\right) d^n\boldsymbol{x} = \lim_{m\rightarrow+\infty} \sum_{j=-\infty}^{m} \int\limits_{S^n_j(0)} \varphi\left(\boldsymbol{x}\right) d^n\boldsymbol{x}$$

exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n}\varphi\left(\boldsymbol{x}\right) d^n\boldsymbol{x}$, and we say that the (improper) integral exists.

1.1.3 Analytic change of variables

A function $h : U \rightarrow \mathbb{Q}_p$ is said to be *analytic* on an open subset $U \subset \mathbb{Q}_p^n$, if for every $\boldsymbol{b} \in U$ there exists an open subset $\widetilde{U} \subset U$, with $\boldsymbol{b} \in \widetilde{U}$, and a convergent power series $\sum_i a_i\left(\boldsymbol{x}-\boldsymbol{b}\right)^{\boldsymbol{i}}$ for $\boldsymbol{x} \in \widetilde{U}$, such that $h\left(\boldsymbol{x}\right) = \sum_{i\in\mathbb{N}^n} a_i\left(\boldsymbol{x}-\boldsymbol{b}\right)^{\boldsymbol{i}}$ for $\boldsymbol{x} \in \widetilde{U}$, with $\boldsymbol{x}^{\boldsymbol{i}} = x_1^{i_1}\cdots x_n^{i_n}$, $\boldsymbol{i} = (i_1, \ldots, i_n)$. In this case, $\frac{\partial}{\partial x_l}h\left(\boldsymbol{x}\right) = \sum_{i\in\mathbb{N}^n} a_i\frac{\partial}{\partial x_l}\left(\boldsymbol{x}-\boldsymbol{b}\right)^{\boldsymbol{i}}$ is a convergent power series. Let U, V be open subsets of $\mathbb{Q}_p^n$. A mapping $\boldsymbol{h} : U \rightarrow V$, $\boldsymbol{h} = (h_1, \ldots, h_n)$ is called *analytic* if each h_i is analytic.

Let $\varphi : V \rightarrow \mathbb{C}$ be a continuous function with compact support, and let $\boldsymbol{h} : U \rightarrow V$ be an analytic mapping. Then

$$\int\limits_V \varphi\left(\boldsymbol{y}\right) d^n\boldsymbol{y} = \int\limits_U \varphi\left(\boldsymbol{h}(\boldsymbol{x})\right) \left|Jac(\boldsymbol{h}(\boldsymbol{x}))\right|_p d^n\boldsymbol{x},$$

where $Jac(\boldsymbol{h}(\boldsymbol{z})) := \det\left[\frac{\partial h_i}{\partial x_j}\left(\boldsymbol{z}\right)\right]_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}$, see e.g. [14, Section 10.1.2].

1.2 The p-adic multivariate Igusa zeta functions

Let $f_i(\boldsymbol{x}) \in \mathbb{Q}_p[x_1, \ldots, x_n]$ be a non-constant polynomial for $i = 1, \ldots, l$, and let Φ be a Bruhat-Schwartz function. The multivariate local zeta function attached to $(f_1, \ldots, f_l, \Phi)$ (also called Igusa local zeta function) is defined by the integral

$$\boldsymbol{Z}_\Phi(s_1, \ldots, s_l, ; f_1, \ldots, f_l,) = \int\limits_{\mathbb{Q}_p^n \smallsetminus \cup_{i=1}^l f_i^{-1}(\boldsymbol{0})} \Phi(\boldsymbol{x}) \prod_{i=1}^l |f_i(\boldsymbol{x})|_p^{s_i} \, d^n\boldsymbol{x}$$

for $(s_1, \ldots, s_l) \in \mathbb{C}^l$ with $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, l$. This integral defines a holomorphic function of $(s_1, \ldots, s_l)$ in the half-space $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, l$. In the case $l = 1$, this assertion corresponds to Lemma 5.3.1 in [39]. For the general case, we recall that a continuous complex-valued function defined in an open set $A \subseteq \mathbb{C}^l$, which is holomorphic in each variable separately, is holomorphic in A. The multivariate local zeta functions admit analytic continuations to the whole $\mathbb{C}^l$ as rational functions of the variables p^{-s_i}, $i = 1, \ldots, l$, see [47]. The Igusa local zeta functions are related with the number of solutions of polynomial congruences mod p^m and with exponential sums mod p^m, there are many intriguing conjectures relating the poles of local zeta functions with the topology of complex singularities, see e.g. [18], [39].

We want to highlight that the convergence of the local zeta functions depends crucially on the fact that Φ has compact support. Consider the following integral:

$$\boldsymbol{J}(s) = \int\limits_{\mathbb{Q}_p} |x|_p^s \, dx, \; s \in \mathbb{C}.$$

Assume that $\boldsymbol{J}(s_0)$ exists for some $s_0 \in \mathbb{R}$, then necessarily the integrals

$$\boldsymbol{J}_0(s_0) = \int\limits_{\mathbb{Z}_p} |x|_p^{s_0} \, dx \quad \text{and } \boldsymbol{J}_1(s_0) = \int\limits_{\mathbb{Q}_p \smallsetminus \mathbb{Z}_p} |x|_p^{s_0} \, dx$$

exist. The first integral is well-known, $\boldsymbol{J}_0(s_0) = \frac{1-p^{-1}}{1-p^{-1-s_0}}$ for $s_0 > -1$. For the second integral, we use that $|x|_p^{s_0}$ is locally integrable, and thus

$$\boldsymbol{J}_1(s_0) = \sum_{j=1}^{\infty} \int\limits_{p^{-j}\mathbb{Z}_p^\times} |x|_p^{s_0} \, dx = \sum_{j=1}^{\infty} p^{j+js_0} \int\limits_{\mathbb{Z}_p^\times} dx = \left(1 - p^{-1}\right) \sum_{j=1}^{\infty} p^{j(1+s_0)} < \infty$$

if and only if $s_0 < -1$. Then, integral $\boldsymbol{J}(s)$ does not exist for any $s \in \mathbb{R}$ and consequently $\boldsymbol{J}(s)$ does not exist for any complex value s.

For an in-depth discussion on local zeta functions the reader may consult [18], [39] and the references therein.

1.3 p-adic String Zeta Functions

We fix an integer $N \geq 4$. To each pair (i,j) with $i,j \in \{1,\ldots,N-1\}$ we attach a complex number $s_{(i,j)}$ such that $s_{(i,j)} = s_{(j,i)}$. To simplify the notation we will use ij, respectively s_{ij}, instead of (i,j), respectively, instead of $s_{(i,j)}$. We set $T := \{2,\ldots,N-2\}$, $D = \frac{(N-3)(N-4)}{2} + 2(N-3)$ and $\mathbb{C}^D$ as

$$\begin{cases} \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T\} & \text{if} \quad N=4 \\ \\ \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T \text{ or } i,j \in T \text{ with } i<j\} & \text{if} \quad N \geq 5. \end{cases}$$

We set $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, $\boldsymbol{x} = (x_2,\ldots,x_{N-2}) \in \mathbb{Q}_p^{N-3}$, and

$$F(\underline{\boldsymbol{s}}, \boldsymbol{x}; N) = \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_p^{s_{ij}}.$$

Definition 1.1 *The p-adic open string N-point zeta function is defined as*

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) := \int_{\mathbb{Q}_p^{N-3}\smallsetminus\Lambda} F(\underline{\boldsymbol{s}}, \boldsymbol{x}; N) \prod_{i=2}^{N-2} dx_i \tag{1.3.1}$$

for $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, *where*

$$\Lambda := \left\{ (x_2,\ldots,x_{N-2}) \in \mathbb{Q}_p^{N-3}; \prod_{i=2}^{N-2} x_i(1-x_i) \prod_{2\leq i<j\leq N-2} (x_i - x_j) = 0 \right\}$$

and $\prod_{i=2}^{N-2} dx_i$ *is the normalized Haar measure of* $\mathbb{Q}_p^{N-3}$.

Remark 1.2 *We notice that the domain of integration in (1.3.1) is taken to be* $\mathbb{Q}_p^{N-3} \smallsetminus \Lambda$ *in order to use* $a^s = e^{s\ln a}$, *with* $a>0$ *and* $s \in \mathbb{C}$, *as the definition* *of the complex power function. The convergence of integral (1.3.1), as well as its holomorphy, will be discussed later on.*

We define for $I \subseteq T$, *the sector attached to* I as

$$Sect(I) = \left\{(x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3}; |x_i|_p \leq 1 \ \Leftrightarrow i \in I\right\}$$

and

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I\right) = \int\limits_{Sect(I)} F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) \prod_{i=2}^{N-2} dx_i.$$

Hence

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right) = \sum_{I \subseteq T} \boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I\right). \tag{1.3.2}$$

Notation 1.3 *(i) The cardinality of a finite set A will be denoted as $|A|$. (ii) We will use the symbol $\bigsqcup$ to denote the union of disjoint sets. (iii) Given a non-empty subset I of $\{2, \ldots, N-2\}$ and B a non-empty subset of $\mathbb{Q}_p$, we set*

$$B^{|I|} = \left\{(x_i)_{i \in I}; x_i \in B\right\}.$$

(iv) By convention, we define $\prod_{i \in \varnothing} \cdot := 1$, $\sum_{i \in \varnothing} \cdot := 0$, and if $J = \varnothing$, then $\int_{B^{|J|}} \cdot :=$ 1. (v) The indices i, j will run over subsets of T, if we do not specify any subset, we will assume that is T.

Lemma 1.4 *With the above notation the following formulas hold:*
(i) $F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right)|_{Sect(I)} = F_0\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) F_1\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right)$, where

$$F_0\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) := \prod_{i \in I} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} |x_i - x_j|_p^{s_{ij}}$$

and

$$F_1\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) := \prod_{i \in T \smallsetminus I} |x_i|_p^{s_{1i} + s_{(N-1)i} + \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i,\ j \in I}} s_{ij}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in T \smallsetminus I}} |x_i - x_j|_p^{s_{ij}}.$$

(ii) If $\mathrm{Re}\,(s_{1i})+\mathrm{Re}\left(s_{(N-1)i}\right)+\sum_{2\leq j\leq N-2, j\neq i}\mathrm{Re}\,(s_{ij})+2<1$ *for* $i\in T\smallsetminus I$*, and* $\mathrm{Re}\,(s_{ij})>-1$ *for* $i,j\in T\smallsetminus I$*, then*

$$\int\limits_{(\mathbb{Q}_p\smallsetminus\mathbb{Z}_p)^{|T\smallsetminus I|}} F_1\left(\underline{\boldsymbol{s}},\boldsymbol{x};N\right)\prod_{i\in T\smallsetminus I}dx_i$$

$$= \quad p^{M(\underline{\boldsymbol{s}})}\int\limits_{\mathbb{Z}_p^{|T\smallsetminus I|}}\frac{\prod\limits_{\substack{2\leq i<j\leq N-2\\ i,j\in T\smallsetminus I}}|y_i-y_j|_p^{s_{ij}}}{\prod\limits_{i\in T\smallsetminus I}|y_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i}s_{ij}}}\prod_{i\in T\smallsetminus I}dy_i,$$

where $M(\underline{\boldsymbol{s}}):=|T\smallsetminus I|+\sum_{i\in T\smallsetminus I}(s_{1i}+s_{(N-1)i})+\sum_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I, j\in T}}s_{ij}+\sum_{\substack{2\leq i<j\leq N-2\\ i\in I, j\in T\smallsetminus I}}s_{ij}$*.*

(iii) If $\mathrm{Re}\,(s_{1i})+\mathrm{Re}\left(s_{(N-1)i}\right)+\sum_{2\leq j\leq N-2, j\neq i}\mathrm{Re}\,(s_{ij})+2<1$ *for* $i\in T\smallsetminus I$*,* $\mathrm{Re}\,(s_{ij})>-1$ *for* $i,j\in T\smallsetminus I$*,* $\mathrm{Re}\,(s_{1i})>-1$ *for* $i\in I$ *and* $\mathrm{Re}\left(s_{(N-1)i}\right)>-1$ *for* $i\in I$*, then*

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I\right)=p^{M(\underline{\boldsymbol{s}})}\left\{\int\limits_{\mathbb{Z}_p^{|I|}}F_0\left(\boldsymbol{s},\boldsymbol{x};N\right)\prod_{i\in I}dx_i\right\}$$

$$\times\left\{\int\limits_{\mathbb{Z}_p^{|T\smallsetminus I|}}\frac{\prod\limits_{\substack{2\leq i<j\leq N-2\\ i,j\in T\smallsetminus I}}|x_i-x_j|_p^{s_{ij}}}{\prod\limits_{i\in T\smallsetminus I}|x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i}s_{ij}}}\prod_{i\in T\smallsetminus I}dx_i\right\}$$

$$=: p^{M(\underline{\boldsymbol{s}})}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right). \tag{1.3.3}$$

Remark 1.5 *Later on we will show that the integrals in the right-hand side in the formulas given in (ii) and (iii) are convergent and holomorphic functions on a certain subset of* $\mathbb{C}^D$ *for all* $I\subseteq T$*.*

Proof. (i) Notice that $F\left(\underline{\boldsymbol{s}},\boldsymbol{x};N\right)|_{Sect(I)}$ equals

$$\prod_{i\in I}|x_i|_p^{s_{1i}}\,|1-x_i|_p^{s_{(N-1)i}}\prod_{i\in T\smallsetminus I}|x_i|_p^{s_{1i}+s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{s_{ij}}\ \times$$

$$\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in T\smallsetminus I}}|x_i-x_j|_p^{s_{ij}}\prod_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I, j\in I}}|x_i|_p^{s_{ij}}\prod_{\substack{2\leq i<j\leq N-2\\ i\in I, j\in T\smallsetminus I}}|x_j|_p^{s_{ij}}. \tag{1.3.4}$$

Now, by using that $s_{ij} = s_{ji}$,

$$\prod_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I, j\in I}} |x_i|_p^{s_{ij}} \prod_{\substack{2\leq i<j\leq N-2\\ i\in I, j\in T\smallsetminus I}} |x_j|_p^{s_{ij}} = \prod_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I, j\in I}} |x_i|_p^{s_{ij}} \prod_{\substack{2\leq j<i\leq N-2\\ j\in I, i\in T\smallsetminus I}} |x_i|_p^{s_{ij}}$$

$$= \prod_{\substack{2\leq j,i\leq N-2\\ i\neq j,\ i\in T\smallsetminus I, j\in I}} |x_i|_p^{s_{ij}} = \prod_{i\in T\smallsetminus I} |x_i|_p^{\sum_{\substack{2\leq j\leq N-2\\ j\neq i,\ j\in I}} s_{ij}}. \tag{1.3.5}$$

The announced formula follows from (1.3.4)-(1.3.5).

(ii) For $|T \setminus I| \geq 1$, we set

$$\boldsymbol{I}\left(\underline{\boldsymbol{s}}; T\setminus I\right) := \int\limits_{(p\mathbb{Z}_p)^{|T\smallsetminus I|}} F_1\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) \prod_{i\in T\smallsetminus I} dx_i,$$

and for $l \in \mathbb{N}\setminus\{0\}$,

$$(p\mathbb{Z}_p)_{-l}^{|T\smallsetminus I|} := \left\{(x_i)_{i\in T\smallsetminus I} \in (\mathbb{Q}_p \smallsetminus \mathbb{Z}_p)^{|T\smallsetminus I|}; -l \leq ord(x_i) \leq -1 \text{ for } i \in T\smallsetminus I\right\},$$

$$(p\mathbb{Z}_p)_{l}^{|T\smallsetminus I|} := \left\{(x_i)_{i\in T\smallsetminus I} \in (p\mathbb{Z}_p)^{|T\smallsetminus I|}; 1 \leq ord(x_i) \leq l \text{ for } i \in T\smallsetminus I\right\},$$

and

$$\boldsymbol{I}_{-l}\left(\underline{\boldsymbol{s}}; T\setminus I\right) := \int\limits_{(\mathbb{Q}_p\smallsetminus\mathbb{Z}_p)_{-l}^{|T\smallsetminus I|}} F_1\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N\right) \prod_{i\in T\smallsetminus I} dx_i.$$

Notice that $(\mathbb{Q}_p \smallsetminus \mathbb{Z}_p)_{-l}^{|T\smallsetminus I|}$, $(p\mathbb{Z}_p)_{l}^{|T\smallsetminus I|}$ are compact sets and that

$$\begin{array}{ccc} (\mathbb{Q}_p \smallsetminus \mathbb{Z}_p)_{-l}^{|T\smallsetminus I|} & \rightarrow & (p\mathbb{Z}_p)_{l}^{|T\smallsetminus I|} \\ & & \\ (x_i)_{i\in T\smallsetminus I} & \rightarrow & \left(\sigma\left(x_i\right)\right)_{i\in T\smallsetminus I}, \end{array}$$

with $\sigma\left(x_i\right) = \frac{1}{y_i}$ is an analytic change of variables satisfying

$$\prod\nolimits_{i\in T\smallsetminus I} dx_i = \prod\nolimits_{i\in T\smallsetminus I} \frac{dy_i}{|y_i|_p^2},$$

then by using this change of variables and the fact that

$$\prod_{i\in T\smallsetminus I}|y_i|_p^{s_{1i}+s_{(N-1)i}+\sum_{\substack{2\leq j\leq N-2\\ j\neq i,j\in I}}s_{ij}}\prod_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I,j\in T\smallsetminus I}}|y_i|_p^{s_{ij}}\prod_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I,j\in T\smallsetminus I}}|y_j|_p^{s_{ij}}$$
$$=\prod_{i\in T\smallsetminus I}|y_i|_p^{s_{1i}+s_{(N-1)i}+\sum_{\substack{2\leq j\leq N-2\\ j\neq i,j\in I}}s_{ij}}\prod_{i\in T\smallsetminus I}|y_i|_p^{\sum_{\substack{2\leq j\leq N-2\\ j\neq i,j\in T\smallsetminus I}}s_{ij}}$$
$$=\prod_{i\in T\smallsetminus I}|y_i|_p^{s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2,j\neq i}s_{ij}},$$

we have

$$\boldsymbol{I}_{-l}\left(\underline{\boldsymbol{s}};T\setminus I\right)=\int_{(p\mathbb{Z}_p)_l^{|T\smallsetminus I|}}\frac{\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in T\smallsetminus I}}|y_i-y_j|_p^{s_{ij}}\prod_{i\in T\smallsetminus I}dy_i}{\prod_{i\in T\smallsetminus I}|y_i|_p^{s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2,j\neq i}s_{ij}+2}}. \tag{1.3.6}$$

Then $\lim_{l\to\infty}\boldsymbol{I}_{-l}\left(\underline{\boldsymbol{s}};T\setminus I\right)=\boldsymbol{I}\left(\underline{\boldsymbol{s}};T\setminus I\right)$. Indeed, the formula follows from the dominated convergence theorem, by using that $|y_i-y_j|_p^{\mathrm{Re}(s_{ij})}<1$ for $y_i,\ y_j\in p\mathbb{Z}_p$, and the fact that $\int_{p\mathbb{Z}_p}\frac{1}{|y|_p^s}dy$ converges for $\mathrm{Re}\,(s)<1$. Finally, the announced formula follows from (1.3.6) by a change of variables.

(iii) It is a consequence of (i)-(ii). ∎

Remark 1.6 *From Lemma 1.4, we have*

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)=\sum_{I\subseteq T}p^{M(\underline{\boldsymbol{s}})}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right). \tag{1.3.7}$$

By convention $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};\varnothing,0\right)=1$, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};\varnothing,1\right)=1$. A central goal of this article is to show that $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ has an analytic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{-s_{ij}}$. To establish this result, we show that all functions appearing on the right-hand side of formula (1.3.7) admit analytic continuations to the whole $\mathbb{C}^D$ as rational functions in the variables $p^{-s_{ij}}$, and that each of these functions is holomorphic on certain domain, and that the intersection of all these domains contains an open and connected subset of $\mathbb{C}^D$, which allows us to use the principle of analytic continuation. We will show that each of the integrals $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)$ and

$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; T \smallsetminus I, 1\right)$ satisfies several recursive formulas, and that by using them, the problem of finding analytic continuations is reduced to case of certain simple integrals.

1.3.1 Some p-adic integrals

We compute some p-adic integrals needed for calculating

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0\right) \text{ and } \boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 1\right).$$

Let J be a subset of T with $|J| \geq 2$. We define

$$\boldsymbol{L}_0^{(N)}\left(\left(s_{ij}\right)_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}; J\right) := \boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}}; J\right) = \int\limits_{(\mathbb{Z}_p^{\times})^{|J|}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in J} dx_i \tag{1.3.8}$$

for $\operatorname{Re}\left(s_{ij}\right) > 0$ for any ij, and

$$\boldsymbol{L}_1^{(N)}\left(\left(s_{ij}\right)_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}; J, K\right) := \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; J, K\right) = \int\limits_{\mathbb{Z}_p^{|J|}} \prod_{(i,j)\in K} |x_i - x_j|_p^{s_{ij}} \prod_{i\in J} dx_i \tag{1.3.9}$$

where $K \subseteq T_J := \{(i,j) \in T \times T; 2 \leq i < j \leq N-2, i,j \in J\}$ and $\operatorname{Re}\left(s_{ij}\right) > 0$ for any ij. Notice that if $|J| = 1$, then $\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}}; J\right) = 1 - p^{-1}$ and $K = \varnothing$ which implies $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; J, K\right) = 1$. A precise definition of integrals $\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}}; J\right)$ requires to integrate on

$$(\mathbb{Z}_p^{\times})^{|J|} \smallsetminus \left\{ x \in (\mathbb{Z}_p^{\times})^{|J|}; \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}} (x_i - x_j) = 0 \right\}.$$

A similar consideration is required for $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; J, K\right)$. However, for the sake of simplicity we use definitions (1.3.8)-(1.3.9). We will use this simplified notation later on for similar integrals. The integrals $\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}}; J\right)$, $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; J, K\right)$ are p-adic multivariate local zeta function, these functions were studied by Loeser in [47]. In particular, it is known that these functions have an analytic continuation to $\mathbb{C}^D$ as rational functions in the variables $p^{-s_{ij}}$ and that they are holomorphic functions on $\mathrm{Re}(s_{ij}) > 0$ for any ij.

Remark 1.7 *Let J be subset of T, with $|J| \geq 2$. Set*

$$T_J = \{(i,j) \in T \times T; 2 \leq i < j \leq N-2, i,j \in J\}$$

as before. For $\overline{\boldsymbol{a}} = (\overline{a}_i)_{i \in J} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$, with

$$\overline{\Delta}(J) := \left\{ \overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|}; \overline{a}_i \neq \overline{a}_j \text{ for } i \neq j, \text{ with } i,j \in J \right\},$$

we set

$$K(\overline{\boldsymbol{a}}) := \{(i,j) \in T_J; \overline{a}_i = \overline{a}_j\}.$$

Now, we introduce on $(\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$, the following equivalence relation:

$$\overline{\boldsymbol{a}} \sim \overline{\boldsymbol{b}} \Leftrightarrow K(\overline{\boldsymbol{a}}) = K(\overline{\boldsymbol{b}}).$$

We denote by $\overline{A}(\overline{\boldsymbol{a}}) = \left\{ \overline{\boldsymbol{b}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J); \overline{\boldsymbol{a}} \sim \overline{\boldsymbol{b}} \right\}$, the equivalence class defined by $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$. For instance, if $\overline{\boldsymbol{a}} = \overline{\boldsymbol{1}} = \left(\overline{1}\right)_{i \in J}$, then $\overline{A}\left(\overline{\boldsymbol{1}}\right) = \bigsqcup_{\overline{b} \in \mathbb{F}_p^{\times}} \left\{ \overline{b}\left(\overline{1}\right)_{i \in J} \right\}$. By taking a unique representative in each equivalence class, we obtain $\mathcal{R}(J) \subset (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$ such that

$$(\mathbb{F}_p^{\times})^{|J|} = \bigsqcup_{\overline{\boldsymbol{a}} \in R(J)} \overline{A}(\overline{\boldsymbol{a}}) \bigsqcup \overline{\Delta}(J).$$

Given a subset $K \subseteq T_J$ with $K = \{(i_1, j_1), \ldots, (i_m, j_m)\}$, we define

$$K_{list} = \{i_1, j_1, \ldots, i_m, j_m\} \subset J.$$

We will use the notation $K_{list}(\overline{\boldsymbol{a}})$ to mean $K(\overline{\boldsymbol{a}})_{list}$, for $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|}$. Notice that $K(\overline{\boldsymbol{a}}) \subset K_{list}(\overline{\boldsymbol{a}}) \times K_{list}(\overline{\boldsymbol{a}})$, $|K_{list}(\overline{\boldsymbol{a}})| \geq 2$ for any $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$ and that $K_{list}(\overline{\boldsymbol{1}}) = J$.

Lemma 1.8 *If $|J| \geq 2$, then, with the notation of Remark 1.7, the following formula holds:*

$$\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) = \sum_{\overline{\boldsymbol{a}} \in \mathcal{R}(J)} \left|\overline{A}(\overline{\boldsymbol{a}})\right| p^{-|J| - \sum_{(i,j) \in K(\overline{\boldsymbol{a}})} s_{ij}} \boldsymbol{L}_1^{(N)}(\underline{\boldsymbol{s}}; K_{list}(\overline{\boldsymbol{a}}), K(\overline{\boldsymbol{a}})) + \left|\overline{\Delta}(J)\right| p^{-|J|}$$

for $\mathrm{Re}(s_{ij}) > 0$ *for all* $i,\ j \in J$.

Proof. For $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$, set $A(\overline{\boldsymbol{a}}) := \left\{\boldsymbol{b} + p\boldsymbol{x}; \overline{\boldsymbol{b}} \in \overline{A}(\overline{\boldsymbol{a}})\right\}$, and for $\overline{\Delta}(J)$, $\Delta(J) := \left\{\boldsymbol{a} + p\boldsymbol{x}; \overline{\boldsymbol{a}} \in \overline{\Delta}(J)\right\}$. Now

$$
\begin{aligned}
\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) &= \sum_{\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|}} \int_{\boldsymbol{a}+(p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \le i < j \le N-2 \\ i,j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\
&= \sum_{\overline{\boldsymbol{a}} \in \mathcal{R}(J)} \sum_{\overline{\boldsymbol{b}} \in \overline{A}(\overline{\boldsymbol{a}})} \int_{\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \le i < j \le N-2 \\ i,j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i + \\
&\qquad \sum_{\overline{\boldsymbol{a}} \in \overline{\Delta}(J)} \int_{\boldsymbol{a}+(p\mathbb{Z}_p)^{|J|}} \prod_{\substack{2 \le i < j \le N-2 \\ i,j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \\
&= \sum_{\overline{\boldsymbol{a}} \in \mathcal{R}(J)} \left|\overline{A}(\overline{\boldsymbol{a}})\right| p^{-|J| - \sum_{(i,j) \in K(\overline{\boldsymbol{a}})} s_{ij}} \int_{(\mathbb{Z}_p)^{|K_{\text{list}}(\overline{\boldsymbol{a}})|}} \prod_{(i,j) \in K(\overline{\boldsymbol{a}})} |x_i - x_j|_p^{s_{ij}} \prod_{i \in K_{\text{list}}(\overline{\boldsymbol{a}})} dx_i \\
&\qquad + \left|\overline{\Delta}(J)\right| p^{-|J|}.
\end{aligned}
$$

■

Lemma 1.9 *We use all the notation introduced in Remark 1.7. Given* $\overline{\boldsymbol{a}} = (\overline{a}_i)_{i \in J} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Delta}(J)$ *and* $(i,j) \in K(\overline{\boldsymbol{a}})$, *we set*

$$
K((i,j), \overline{\boldsymbol{a}}) := \left\{\left(\widetilde{i}, \widetilde{j}\right) \in K(\overline{\boldsymbol{a}}); \overline{a}_i = \overline{a}_{\widetilde{i}}\right\}
$$

and use $K_{list}((i,j), \overline{\boldsymbol{a}}) := K((i,j), \overline{\boldsymbol{a}})_{list}$. *Then the following assertions hold:*
(i)

$$
K((i,j), \overline{\boldsymbol{a}}) = T_{K_{list}((i,j),\overline{\boldsymbol{a}})} = \{(r,s); 2 \le r < s \le N-2, r,s \in K_{list}((i,j), \overline{\boldsymbol{a}})\};
$$

(ii) the subsets $K((i,j), \overline{\boldsymbol{a}})$ *form a partition of* $K(\overline{\boldsymbol{a}})$, *i.e. there exists a finite set* $\mathcal{R}(\overline{\boldsymbol{a}})$ *of elements* $(i,j) \in K(\overline{\boldsymbol{a}})$, *such that* $K(\overline{\boldsymbol{a}}) = \bigsqcup_{(i,j) \in \mathcal{R}(\overline{\boldsymbol{a}})} K((i,j), \overline{\boldsymbol{a}})$.

Proof. (i) By definition $K((i,j), \overline{\boldsymbol{a}}) \subseteq T_{K_{\text{list}}((i,j),\overline{\boldsymbol{a}})}$. Conversely, let $\left(\widetilde{i}_m, \widetilde{j}_l\right) \in T_{K_{\text{list}}((i,j),\overline{\boldsymbol{a}})}$, then there exists $\widetilde{j}_m \in K_{\text{list}}((i,j), \overline{\boldsymbol{a}})$ such that $\left(\widetilde{i}_m, \widetilde{j}_m\right) \in K((i,j), \overline{\boldsymbol{a}})$ or $\left(\widetilde{j}_m, \widetilde{i}_m\right) \in K((i,j), \overline{\boldsymbol{a}})$. In any case, either $\left(\widetilde{i}_m, \widetilde{j}_m\right)$ or $\left(\widetilde{j}_m, \widetilde{i}_m\right)$ belongs to $K(\overline{\boldsymbol{a}})$ and

$\overline{a}_i = \overline{a}_{\widetilde{i}_m} = \overline{a}_{\widetilde{j}_m}$. Similarly, there exists $\widetilde{i}_l \in K_{\text{list}}((i,j), \overline{\boldsymbol{a}})$ such that either $\left(\widetilde{i}_l, \widetilde{j}_l\right)$ or $\left(\widetilde{j}_l, \widetilde{i}_l\right)$ belongs to $K((i,j), \overline{\boldsymbol{a}})$ and $\overline{a}_i = \overline{a}_{\widetilde{i}_l} = \overline{a}_{\widetilde{j}_l}$. Therefore $\overline{a}_{\widetilde{i}_m} = \overline{a}_{\widetilde{j}_l}$ i.e. $\left(\widetilde{i}_m, \widetilde{j}_l\right) \in K(\overline{\boldsymbol{a}})$, furthermore $\left(\widetilde{i}_m, \widetilde{j}_l\right) \in K((i,j), \overline{\boldsymbol{a}})$. Hence $K((i,j), \overline{\boldsymbol{a}}) = T_{K_{\text{list}}((i,j),\overline{\boldsymbol{a}})}$.

(ii) Let $(i_m, j_m) \in K((i,j), \overline{\boldsymbol{a}}) \cap K(\left(\widetilde{i}, \widetilde{j}\right), \overline{\boldsymbol{a}})$, then $\overline{a}_i = \overline{a}_{i_m} = \overline{a}_{\widetilde{i}}$ and $\left(\widetilde{i}, \widetilde{j}\right) \in K((i,j), \overline{\boldsymbol{a}})$, and consequently $K(\left(\widetilde{i}, \widetilde{j}\right), \overline{\boldsymbol{a}}) \subseteq K((i,j), \overline{\boldsymbol{a}})$. Similarly, one verifies that $K((i,j), \overline{\boldsymbol{a}}) \subseteq K(\left(\widetilde{i}, \widetilde{j}\right), \overline{\boldsymbol{a}})$. ■

Remark 1.10 *As a consequence of Lemmas 1.8- 1.9, we have*

$$\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; K_{list}(\overline{\boldsymbol{a}}), K(\overline{\boldsymbol{a}})\right) = \prod_{(i,j)\in\mathcal{R}(\overline{\boldsymbol{a}})} \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; K_{list}((i,j), \overline{\boldsymbol{a}}), T_{K_{list}((i,j),\overline{\boldsymbol{a}})}\right).$$

Example 1.11 *Take $p \geq 3$, $\overline{\boldsymbol{a}} = \left(\overline{1}, \overline{2}, \overline{1}, \overline{2}, \overline{2}\right) \in \mathbb{F}_p^5$, and $J = \{2,3,4,5,6\}$. Hence*

$$T_J = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\},$$

and by Lemma 1.9,

$$K(\overline{\boldsymbol{a}}) = \{(2,4), (3,5), (3,6), (5,6)\} = K((2,4), \overline{\boldsymbol{a}}) \bigsqcup K((3,5), \overline{\boldsymbol{a}}),$$

where $K((2,4), \overline{\boldsymbol{a}}) = \{(2,4)\}$, $K((3,5), \overline{\boldsymbol{a}}) = \{(3,5), (3,6), (5,6)\}$. Thus

$$K_{list}((2,4), \overline{\boldsymbol{a}}) = \{2,4\} \text{ and } K_{list}((3,5), \overline{\boldsymbol{a}}) = \{3,5,6\}.$$

With this notation, $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; K_{list}(\overline{\boldsymbol{a}}), K(\overline{\boldsymbol{a}})\right)$ equals

$$\int_{\mathbb{Z}_p^5} |x_2 - x_4|_p^{s_{24}} |x_3 - x_5|_p^{s_{35}} |x_3 - x_6|_p^{s_{36}} |x_5 - x_6|_p^{s_{56}} dx_2 dx_3 dx_4 dx_5 dx_6$$

$$= \left\{\int_{\mathbb{Z}_p^2} |x_2 - x_4|_p^{s_{24}} dx_2 dx_4\right\} \left\{\int_{\mathbb{Z}_p^3} |x_3 - x_5|_p^{s_{35}} |x_3 - x_6|_p^{s_{36}} |x_5 - x_6|_p^{s_{56}} dx_3 dx_5 dx_6\right\}$$

$$= \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; K_{list}((2,4), \overline{\boldsymbol{a}}), T_{K_{list}((2,4),\overline{\boldsymbol{a}})}\right) \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; K_{list}((3,5), \overline{\boldsymbol{a}}), T_{K_{list}((3,5),\overline{\boldsymbol{a}})}\right).$$

Lemma 1.12 *Set $F(s_1, s_2, s_3, x, y) := |x|_p^{s_1} |y|_p^{s_2} |x - y|_p^{s_3}$, s_1, s_2, $s_3 \in \mathbb{C}$, and*

$$\boldsymbol{Z}(s_1, s_2, s_3) := \int_{\mathbb{Z}_p^2} F(s_1, s_2, s_3, x, y) dx dy \text{ for } \operatorname{Re}(s_i) > 0,\ i = 1, 2, 3.$$

Then $\boldsymbol{Z}(s_1, s_2, s_3)$ *is a holomorphic function on*

$$\left\{(s_1, s_2, s_3) \in \mathbb{C}^3; \mathrm{Re}(s_i) > -1 \text{ for } i = 1, 2, 3 \text{ and } \mathrm{Re}(s_1) + \mathrm{Re}(s_2) + \mathrm{Re}(s_3) > -2\right\}.$$

In addition,

$$\boldsymbol{Z}(s_1, s_2, s_3) := \frac{Q\left(p^{-s_1}, p^{-s_2}, p^{-s_3}\right)}{\left(1 - p^{-2-s_1-s_2-s_3}\right) \prod_{i=1}^{3} \left(1 - p^{-1-s_i}\right)},$$

where $Q\left(p^{-s_1}, p^{-s_2}, p^{-s_3}\right)$ *denotes a polynomial with rational coefficients in the variables* p^{-s_1}, p^{-s_2}, p^{-s_3}.

Remark 1.13 *If* $s_1 = s_2 = 0$, *then the denominator of* $\boldsymbol{Z}(s_1, s_2, s_3)$ *is* $1 - p^{-1-s_3}$.

Proof. By using that $\mathbb{Z}_p^2 = (p\mathbb{Z}_p)^2 \sqcup S_0^2$ with $S_0^2 = p\mathbb{Z}_p \times \mathbb{Z}_p^\times \sqcup \mathbb{Z}_p^\times \times p\mathbb{Z}_p \sqcup \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$, and then by changing variables, we get

$$\boldsymbol{Z}(s_1, s_2, s_3) = \frac{\int_{S_0^2} F(s_1, s_2, s_3, x, y) dx dy}{1 - p^{-2-s_1-s_2-s_3}} =: \frac{\boldsymbol{Z}_0(s_1, s_2, s_3)}{1 - p^{-2-s_1-s_2-s_3}}.$$

On the other hand,

$$\begin{aligned} \boldsymbol{Z}_0(s_1, s_2, s_3) &= \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} F(s_1, s_2, s_3, x, y) dx dy \\ &+ \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} F(s_1, s_2, s_3, x, y) dx dy + \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} F(s_1, s_2, s_3, x, y) dx dy \\ &=: \boldsymbol{Z}_{0,1}(s_1, s_2, s_3) + \boldsymbol{Z}_{0,2}(s_1, s_2, s_3) + \boldsymbol{Z}_{0,3}(s_1, s_2, s_3). \end{aligned}$$

First, we compute $\boldsymbol{Z}_{0,1}(s_1, s_2, s_3)$. By a change of variables, we get

$$\boldsymbol{Z}_{0,1}(s_1, s_2, s_3) = p^{-1-s_1}(1 - p^{-1}) \int_{\mathbb{Z}_p} |x|_p^{s_1} dx = \frac{\left(1 - p^{-1}\right)^2 p^{-1-s_1}}{1 - p^{-1-s_1}}$$

for $\mathrm{Re}(s_1) > -1$. By a similar computation we obtain

$$\boldsymbol{Z}_{0,2}(s_1, s_2, s_3) = \frac{\left(1 - p^{-1}\right)^2 p^{-1-s_2}}{1 - p^{-1-s_2}} \text{ for } \mathrm{Re}(s_2) > -1.$$

In order to compute

$$\boldsymbol{Z}_{0,3}(s_1, s_2, s_3) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} |x - y|_p^{s_3} dx dy,$$

we use that $(\mathbb{Z}_p^\times)^2 = \sqcup_{\overline{a}_0,\overline{a}_1\in\mathbb{F}_p^\times} a_0 + p\mathbb{Z}_p \times a_1 + p\mathbb{Z}_p$, where $\mathbb{F}_p^\times = \{1,2,...,p-1\}$ as sets, to get

$$\boldsymbol{Z}_{0,3}(s_1,s_2,s_3) = \sum_{\overline{a}_0,\overline{a}_1\in\mathbb{F}_p^\times} \int_{a_0+p\mathbb{Z}_p\times a_1+p\mathbb{Z}_p} |x-y|_p^{s_3} dxdy$$

$$= p^{-2} \sum_{\substack{\overline{a}_0,\overline{a}_1\in\mathbb{F}_p^\times \\ \overline{a}_0\neq\overline{a}_1}} \int_{\mathbb{Z}_p\times\mathbb{Z}_p} |a_0+px-a_1-py|_p^{s_3} dxdy + p^{-2} \sum_{\substack{\overline{a}_0,\overline{a}_1\in\mathbb{F}_p^\times \\ \overline{a}_0=\overline{a}_1}} \int_{\mathbb{Z}_p\times\mathbb{Z}_p} |x-y|_p^{s_3} dxdy$$

$$= p^{-2}(p-1)(p-2) + p^{-2-s_3}(p-1)\frac{1-p^{-1}}{1-p^{-1-s_3}}.$$

■

Lemma 1.14 *Let I be a subset of T satisfying $|I|\geq 2$. Then $\boldsymbol{L}_1^{(N)}(\underline{\boldsymbol{s}}; I, T_I)$ admits an analytic continuation as a rational function of the form*

$$\boldsymbol{L}_1^{(N)}(\underline{\boldsymbol{s}}; I, T_I) = \frac{Q_I\left(\{p^{-s_{ij}}\}_{i,j\in I}\right)}{\prod_{J\in\mathcal{F}(I)}\left(1-p^{-\left(|J|-1+\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in J}} s_{ij}\right)}\right)^{e_J} \prod_{ij\in S_I}\left(1-p^{-1-s_{ij}}\right)^{e_{ij}}}, \tag{1.3.10}$$

where $Q_I\left(\{p^{-s_{ij}}\}_{i,j\in I}\right)$ is a polynomial with rational coefficients in the variables $\{p^{-s_{ij}}\}_{i,j\in I}$, $\mathcal{F}(I)$ is a family of subsets of I, with $I\in\mathcal{F}(I)$, S_I is a non-empty subset of

$$\{2\leq i<j\leq N-2, i,j\in I\},$$

and the e_J, e_{ij}'s are positive integers.

Proof. By using the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$, where $\mathbb{Z}_p^{|I|} = \{(x_i)_{i\in I}; x_i\in\mathbb{Z}_p\}$, $(p\mathbb{Z}_p)^{|I|} = \{(x_i)_{i\in I}; x_i\in p\mathbb{Z}_p\}$, and $S_0^{|I|} = \left\{(x_i)_{i\in I}\in\mathbb{Z}_p^{|I|}; \max_{i\in I}\left\{|x_i|_p\right\}=1\right\}$. By a change of variables, we get

$$\begin{aligned}
\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; I, T_I\right) &= \frac{\displaystyle\int_{S_0^{|I|}} \prod_{\substack{2\le i<j\le N-2\\ i,j\in I}} \left|x_i-x_j\right|_p^{s_{ij}} \prod_{i\in J} dx_i}{1-p^{-|I|-\sum_{\substack{2\le i<j\le N-2\\ i,j\in I}} s_{ij}}} \\
&=: \frac{\boldsymbol{A}_0\left(\underline{\boldsymbol{s}}; I\right)}{1-p^{-|I|-\sum_{\substack{2\le i<j\le N-2\\ i,j\in I}} s_{ij}}}.
\end{aligned}$$

For every non-empty subset $J\subseteq I$, we define

$$S_J^{|I|} = \left\{ (x_i)_{i\in I} \in \mathbb{Z}_p^{|I|}; \left|x_i\right|_p = 1 \Leftrightarrow i\in J \right\},$$

then $S_0^{|I|} = \sqcup_{J\subseteq I, J\neq\varnothing} S_J^{|I|}$ and $\boldsymbol{A}_0\left(\underline{\boldsymbol{s}}; I\right) = \sum_{J\subseteq I, J\neq\varnothing} \boldsymbol{A}_{0,J}\left(\underline{\boldsymbol{s}}\right)$ where

$$\boldsymbol{A}_{0,J}\left(\underline{\boldsymbol{s}}\right) := \int_{S_J^{|I|}} \prod_{\substack{2\le i<j\le N-2\\ i,j\in I}} \left|x_i-x_j\right|_p^{s_{ij}} \prod_{i\in I} dx_i,$$

for this reason

$$\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}}; I, T_I\right) = \frac{\boldsymbol{A}_{0,I}\left(\underline{\boldsymbol{s}}\right) + \sum_{J\subsetneq I, J\neq\varnothing} \boldsymbol{A}_{0,J}\left(\underline{\boldsymbol{s}}\right)}{1-p^{-|I|-\sum_{\substack{2\le i<j\le N-2\\ i,j\in I}} s_{ij}}}. \tag{1.3.11}$$

On the other hand,

$$\left|x_i-x_j\right|_p^{s_{ij}} \Big|_{S_J^{|I|}} = \begin{cases} \left|x_i-x_j\right|_p^{s_{ij}} & \text{if } \ i,j\in J \\ \\ \left|x_i-x_j\right|_p^{s_{ij}} & \text{if } \ i,j\in I\smallsetminus J \\ \\ 1 & \text{if } \ i\in J, j\in I\smallsetminus J \\ \\ 1 & \text{if } \ j\in J, i\in I\smallsetminus J. \end{cases} \tag{1.3.12}$$

Then

$$\boldsymbol{A}_{0,I}\left(\underline{\boldsymbol{s}}\right) = \boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}}; I\right),$$

and if $J \subsetneqq I$,

$$\begin{aligned}\mathbf{A}_{0,J}\left(\underline{\boldsymbol{s}}\right) &= \left\{\int\limits_{(p\mathbb{Z}_p)^{|I\smallsetminus J|}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}} |x_i-x_j|_p^{s_{ij}} \prod_{i\in I\smallsetminus J} dx_i\right\} \boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};J\right) \quad (1.3.13)\\ &= p^{-|I\smallsetminus J|-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}} s_{ij}} \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I\smallsetminus J,T_{I\smallsetminus J}\right)\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};J\right).\end{aligned}$$

Therefore, from (1.3.11)-(1.3.13), $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I,T_I\right)$ equals

$$\frac{\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};I\right)+\sum_{J\subsetneqq I,J\neq\varnothing} p^{-|I\smallsetminus J|-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}} s_{ij}} \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I\smallsetminus J,T_{I\smallsetminus J}\right)\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};J\right)}{1-p^{-|I|-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} s_{ij}}}. \tag{1.3.14}$$

Now, by Lemma 1.8 and the fact that $\overline{A}\left(\overline{\mathbf{1}}\right)=\bigsqcup_{\overline{b}\in\mathbb{F}_p^\times}\left\{\left(\overline{b}\right)_{i\in I}\right\}$, $K_{\text{list}}(\overline{\mathbf{1}})=I$, see Remark 1.7,

$$\begin{aligned}\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};I\right) = &\sum_{\overline{\boldsymbol{a}}\in\mathcal{R}(I)\smallsetminus\left\{\overline{\mathbf{1}}\right\}} \left|\overline{A}(\overline{\boldsymbol{a}})\right| p^{-|I|-\sum_{(i,j)\in K(\overline{\boldsymbol{a}})} s_{ij}} \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};K_{\text{list}}(\overline{\boldsymbol{a}}),K(\overline{\boldsymbol{a}})\right) \quad (1.3.15)\\ &+(p-1)\,p^{-|I|-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} s_{ij}} \boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I,T_I\right)+\left|\overline{\Delta}\left(I\right)\right| p^{-|I|},\end{aligned}$$

with $|K_{\text{list}}(\overline{\boldsymbol{a}})|\geq 2$, hence from (1.3.14)-(1.3.15),

$$\begin{aligned}&\left(1-p^{1-|I|-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} s_{ij}}\right)\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I,T_I\right)\\ &= \sum_{\overline{\boldsymbol{a}}\in\mathcal{R}(I)\smallsetminus\left\{\overline{\mathbf{1}}\right\}} d_{\overline{\boldsymbol{a}}}\left(\underline{\boldsymbol{s}}\right)\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};K_{\text{list}}(\overline{\boldsymbol{a}}),K(\overline{\boldsymbol{a}})\right)\\ &+\sum_{\substack{J\subsetneqq I\\ J\neq\varnothing}} c_J(\underline{\boldsymbol{s}})\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I\smallsetminus J,T_{I\smallsetminus J}\right)\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}},J\right)+\left|\overline{\Delta}\left(I\right)\right| p^{-|I|}.\end{aligned}$$

This formula and Lemmas 1.8-1.12 give a recursive algorithm for computing integrals $\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};I,T_I\right)$, from which we get (1.3.10). ■

From Lemmas 1.8-1.14, we obtain the following result:

Corollary 1.15 *If $|I| \geq 2$, then*

$$\boldsymbol{L}_0^{(N)}\left(\underline{\boldsymbol{s}};I\right)=\frac{R_I\left(\left\{p^{-s_{ij}}\right\}_{i,j\in I}\right)}{\prod\limits_{J\in\mathcal{G}(I)}\left(1-p^{-\left(|J|-1+\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}s_{ij}\right)}\right)^{f_J}\prod\limits_{ij\in G_I}\left(1-p^{-1-s_{ij}}\right)^{f_{ij}}},$$

where $R_I\left(\left\{p^{-s_{ij}}\right\}_{i,j\in I}\right)$ is a polynomial with rational coefficients in the variables $\left\{p^{-s_{ij}}\right\}_{i,j\in I}$, $\mathcal{G}(I)$ is a family of non-empty subsets of I, with $I\in\mathcal{G}(I)$, G_I is a non-empty subset of $\{2\leq i<j\leq N-2, i,j\in I\}$, and the f_J, f_{ij} 's are positive integers.

Given $I\subseteq T$, with $|I|\geq 2$, and $K\subseteq I$, with $|K|\geq 1$, and $M\subseteq T_I$, with $|M|\geq 1$, we define

$$\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I,K,M\right)=\int\limits_{\mathbb{Z}_p^{|I|}}\prod_{i\in K}|x_i|_p^{s_{ti}}\prod_{(i,j)\in M}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx_i$$

for $\mathrm{Re}\left(s_{ij}\right)>0$ for any ij. If $|M|=0$, then

$$\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I,K,M\right)=\int_{\mathbb{Z}_p^{|I|}}\prod\nolimits_{i\in K}|x_i|_p^{s_{ti}}\prod\nolimits_{i\in I}dx_i.$$

Lemma 1.16 *Let $t\in\{1,N-1\}$. Then*

$$\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I,K,T_I\right)=\int\limits_{\mathbb{Z}_p^{|I|}}\prod_{i\in K}|x_i|_p^{s_{ti}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx_i$$

admits an analytic continuation as a rational function of the form

$$\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I,K,T_I\right)=\frac{Q_{I,K}\left(\left\{p^{-s_{ij}}\right\}_{i,j\in I},\left\{p^{-s_{ti}}\right\}_{t\in\{1,N-1\},i\in I}\right)}{R_0(\underline{\boldsymbol{s}};I,K)R_1(\underline{\boldsymbol{s}};I,K)R_2(\underline{\boldsymbol{s}};I,K)},\qquad(1.3.16)$$

where

$$R_0(\underline{\boldsymbol{s}};I,K)=\prod_{J\in\mathcal{G}_1(I)}\left(1-p^{-\left(|J|-1+\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}s_{ij}\right)}\right)^{f_J}\prod_{ij\in S_I}\left(1-p^{-1-s_{ij}}\right)^{g_{ij}},$$

$$R_1(\underline{\boldsymbol{s}}; I, K) = \prod_{i \in U_K} \left(1 - p^{-1-s_{ti}}\right)^{h_i},$$

$$R_2(\underline{\boldsymbol{s}}; I, K) = \prod_{(J,R) \in \mathcal{G}_2(I \times I)} \left(1 - p^{-|J| - \sum_{i \in R} s_{ti} - \sum_{2 \leq i < j \leq N-2,\ i,j \in J} s_{ij}}\right),$$

where $Q_{I,K}\left(\{p^{-s_{ij}}\}_{i,j \in I}, \{p^{-s_{ti}}\}_{t \in \{1,N-1\}, i \in I}\right)$ *denotes a polynomial with rational coefficients in the variables* $\{p^{-s_{ij}}\}_{i,j \in I}, \{p^{-s_{ti}}\}_{t \in \{1,N-1\}, i \in I}$*,* $\mathcal{G}_1(I)$ *is a non-empty family of subsets of* I*, with* $I \in \mathcal{G}_1(I)$*,* $\mathcal{G}_2(I \times I)$ *is a non-empty family of subsets* $J \times R$ *of* $I \times I$*, with* $R \subseteq J$ *and* $(I, K) \in \mathcal{G}_2(I \times I)$*,* U_K *is a non-empty subset of* K*,* S_I *is a non-empty subset of* $\{2 \leq i < j \leq N-2, i, j \in I\}$*, and the* f_J*'s,* g_{ij}*'s, and the* h_i*'s are positive integers.*

Remark 1.17 *The integral* $\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, M)$ *is also a multivariate* p*-adic local zeta function. If* $|I| \geq 2$ *and* $|K| = 0$*, then* $\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, M) = \boldsymbol{L}_1^{(N)}(\underline{\boldsymbol{s}}; I, M)$*.*

Proof. We use the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$ as in the proof of Lemma 1.14 and a change of variables, to get

$$\begin{aligned}
\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, T_I) &= \frac{\displaystyle\int_{S_0^{|I|}} \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{2 \leq i < j \leq N-2, i,j \in I} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2, i,j \in I} s_{ij}}} \\
&=: \frac{\boldsymbol{B}_0(\underline{\boldsymbol{s}}; I, K, T_I)}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2,\ i,j \in I} s_{ij}}}.
\end{aligned}$$

We now use the partition $S_0^{|I|} = \sqcup_{J \subseteq I, J \neq \varnothing} S_J^{|I|}$ to obtain

$$\boldsymbol{B}_0(\underline{\boldsymbol{s}}; I, K, T_I) = \sum_{J \subseteq I, J \neq \varnothing} \boldsymbol{B}_{0,J}(\underline{\boldsymbol{s}}),$$

where

$$\boldsymbol{B}_{0,J}(\underline{\boldsymbol{s}}) := \int_{S_J^{|I|}} \prod_{i \in K} |x_i|_p^{s_{ti}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in I} dx_i.$$

Consequently

$$\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, T_I) = \frac{\boldsymbol{B}_{0,I}(\underline{\boldsymbol{s}}) + \sum_{J \subsetneqq I, J \neq \varnothing} \boldsymbol{B}_{0,J}(\underline{\boldsymbol{s}})}{1 - p^{-|I| - \sum_{i \in K} s_{ti} - \sum_{2 \leq i < j \leq N-2,\ i,j \in I} s_{ij}}}.$$

On the other hand, $|x_i - x_j|_p^{s_{ij}} \mid_{S_J^{|I|}}$ is given in (1.3.12) and

$$\prod_{i\in K} |x_i|_p^{s_{ti}} \mid_{S_J^{|I|}} = \prod_{i\in K} |x_i|_p^{s_{ti}} \mid_{(p\mathbb{Z}_p)^{|K\setminus J|}} .$$

Then $\boldsymbol{B}_{0,I}(\underline{\boldsymbol{s}}) = \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; I)$, and if $J \subsetneqq I$, $\boldsymbol{B}_{0,J}(\underline{\boldsymbol{s}})$ equals

$$\left\{ \int\limits_{(p\mathbb{Z}_p)^{|I\setminus J|}} \prod_{i\in K\setminus J} |x_i|_p^{s_{ti}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I\setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I\setminus J} dx_i \right\} \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) =$$

$$p^{-|I\setminus J| - \sum_{i\in K\setminus J} s_{ti} - \sum_{2\leq i<j\leq N-2, i,j\in I\setminus J} s_{ij}} \times$$

$$\left\{ \int\limits_{\mathbb{Z}_p^{|I\setminus J|}} \prod_{i\in K\setminus J} |x_i|_p^{s_{ti}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I\setminus J}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I\setminus J} dx_i \right\} \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) =$$

$$p^{-|I\setminus J| - \sum_{i\in K\setminus J} s_{ti} - \sum_{2\leq i<j\leq N-2, i,j\in I\setminus J} s_{ij}} \boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I\setminus J, K\setminus J, T_{I\setminus J}) \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) .$$

Hence $\left(1 - p^{-|I| - \sum_{i\in K} s_{ti} - \sum_{2\leq i<j\leq N-2,\ i,j\in I} s_{ij}}\right) \boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, T_I)$ equals

$$\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; I) + \tag{1.3.17}$$

$$\sum_{J\subsetneqq I,\ J\neq\varnothing} p^{-|I\setminus J| - \sum_{i\in K\setminus J} s_{ti} - \sum_{2\leq i<j\leq N-2, i,j\in I\setminus J} s_{ij}} \boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I\setminus J, K\setminus J, T_{I\setminus J}) \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J) .$$

By using that $|I \setminus J| < |I|$ if $J \subsetneqq I$, $J \neq \varnothing$, and that integrals $\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; I)$, $\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J)$ can be computed effectively, see Corollary 1.15, formula (1.3.17) gives a recursive algorithm for computing $\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, T_I)$, by using it, we obtain (1.3.16). Notice the integrals of type $\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}}; I, K, T_I)$, with $|I| = 1$ and $K = \{i\}$ contribute with terms of the form $\frac{1-p^{-1}}{1-p^{-1-s_{ti}}}$. ■

Lemma 1.18 *Given J a non-empty subset of T, with $|J| \geq 2$, we define*

$$\boldsymbol{M}_{\boldsymbol{J}}(\underline{\boldsymbol{s}}; 1) = \int\limits_{(\mathbb{Z}_p^{\times})^{|J|}} \prod_{i\in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in J} dx_i$$

for $\mathrm{Re}\left(s_{(N-1)i}\right)>0$, $i\in J$, *and* $\mathrm{Re}(s_{ij})>0$, *for* $i,j\in J$. *Then,* $\boldsymbol{M_J}(\underline{\boldsymbol{s}};1)$ *admits an analytic continuation as a rational function of the form*

$$\boldsymbol{M_J}(\underline{\boldsymbol{s}};1)=\frac{Q_J\left(\{p^{-s_{ij}}\}_{i,j\in J},\{p^{-s_{(N-1)i}}\}_{i\in J}\right)}{\prod_{i=1}^{3}U_i(\underline{\boldsymbol{s}};J)}, \tag{1.3.18}$$

where

$$U_1(\underline{\boldsymbol{s}};J)=\prod_{M\in\mathcal{F}_1(J)}\left(1-p^{-\left(|M|-1+\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in M}}s_{ij}\right)}\right)^{e_M}\prod_{ij\in S_J^{(1)}}\left(1-p^{-1-s_{ij}}\right)^{f_{ij}},$$

$$U_2(\underline{\boldsymbol{s}};J)=\prod_{(M,S)\in\mathcal{F}_2(J)}\left(1-p^{-|M|-\sum_{i\in S}s_{(N-1)i}-\sum_{2\leq i<j\leq N-2,\ i,j\in M}s_{ij}}\right)^{g_{(M,S)}},$$

and

$$U_3(\underline{\boldsymbol{s}};J)=\prod_{i\in S_J^{(2)}}\left(1-p^{-1-s_{(N-1)i}}\right)^{h_i},$$

where $\mathcal{F}_1(J)$ *is a non-empty family of subsets of* J, *with* $J\in\mathcal{F}_1(J)$, $\mathcal{F}_2(J)$ *is a non-empty family of subsets* $M\times S\subseteq J\times J$, *with* $S\subseteq M$, $S_J^{(1)}$ *and* $S_J^{(2)}$ *are non-empty subsets of* T, *and the* e_M*'s,* f_{ij}*'s,* $g_{(M,S)}$*'s and the* h_i*'s are positive integers.*

Remark 1.19 *If* $|J|=1$, *then* $\boldsymbol{M_J}(\underline{\boldsymbol{s}};1)=p^{-1}\left(\frac{1-p^{-1}}{1-p^{-1-s_{(N-1)i}}}+p-2\right)$.

Proof. To compute $\boldsymbol{M_J}(\underline{\boldsymbol{s}};1)$, we proceed as follows. We set

$$T_J=\{(i,j)\in T\times T;2\leq i<j\leq N-2,i,j\in J\}$$

as before, and for $\overline{\boldsymbol{a}}=(\overline{a}_i)_{i\in J}\in(\mathbb{F}_p^{\times})^{|J|}\setminus\overline{\Pi}(J)$, with

$$\overline{\Pi}(J):=\left\{\overline{\boldsymbol{a}}\in(\mathbb{F}_p^{\times})^{|J|};\overline{a}_i\neq\overline{a}_j\text{ if }i\neq j,\text{ for }i,\,j\in J\text{ and }\overline{a}_s\neq 1\text{ for any }s\in J\right\},$$

we define

$$K(\overline{\boldsymbol{a}})=\{(i,j)\in T_J;\overline{a}_i=\overline{a}_j\},\quad K^{(1)}(\overline{\boldsymbol{a}})=\{(i,j)\in T_J;\overline{a}_i=\overline{a}_j=1\},$$

and

$$K^{(2)}(\overline{\boldsymbol{a}})=\{i\in J;\overline{a}_i=1\text{ and }\overline{a}_i\neq\overline{a}_s\ \text{ for any }\ (i,s)\in T_J\}.$$

Notice that $K^{(1)}(\overline{\boldsymbol{a}}) \subseteq K(\overline{\boldsymbol{a}})$ and $K^{(2)}(\overline{\boldsymbol{a}}) \cap K_{\text{list}}(\overline{\boldsymbol{a}}) = \varnothing$. Now, we introduce on $(\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Pi}\,(J)$, the following equivalence relation:

$$\overline{\boldsymbol{a}} \sim \overline{\boldsymbol{b}} \;\Leftrightarrow K(\overline{\boldsymbol{a}}) = K(\overline{\boldsymbol{b}}) \text{ and } K^{(1)}(\overline{\boldsymbol{a}}) = K^{(1)}(\overline{\boldsymbol{b}}) \text{ and } K^{(2)}(\overline{\boldsymbol{a}}) = K^{(2)}(\overline{\boldsymbol{b}}).$$

We denote by $\overline{A}(\overline{\boldsymbol{a}}) = \left\{\overline{\boldsymbol{b}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Pi}\,(J)\,; \overline{\boldsymbol{a}} \sim \overline{\boldsymbol{b}}\right\}$, the equivalence class defined by $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Pi}\,(J)$. By taking a unique representative in each equivalence class, we obtain $\mathcal{R}(J) \subset (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Pi}\,(J)$ such that

$$(\mathbb{F}_p^{\times})^{|J|} = \bigsqcup_{\overline{\boldsymbol{a}} \in \mathcal{R}(J)} \overline{A}(\overline{\boldsymbol{a}}) \bigsqcup \overline{\Pi}\,(J)\,. \tag{1.3.19}$$

Given a subset $K \subseteq T_J$ with $K = \{(i_1, j_1)\,, \ldots, (i_m, j_m)\}$, we define

$$K_{\text{list}} = \{i_1, j_1, \ldots, i_m, j_m\} \subseteq J$$

as before. With this notation, $\boldsymbol{M_J}(\underline{\boldsymbol{s}}; 1)$ equals

$$\sum_{\overline{\boldsymbol{a}} \in \mathcal{R}(J)} \sum_{\overline{\boldsymbol{b}} \in \overline{A}(\overline{\boldsymbol{a}})} \int_{\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i \tag{1.3.20}$$

$$+ \sum_{\overline{\boldsymbol{b}} \in \overline{\Pi}(J)} \int_{\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}} \prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} \prod_{\substack{2 \leq i < j \leq N-2 \\ i,j \in J}} |x_i - x_j|_p^{s_{ij}} \prod_{i \in J} dx_i$$

$$: \;= \boldsymbol{M}(\underline{\boldsymbol{s}}; J, 1) + \boldsymbol{M}(\underline{\boldsymbol{s}}; J, 2).$$

We now use that for each $\overline{\boldsymbol{a}} \in (\mathbb{F}_p^{\times})^{|J|} \setminus \overline{\Pi}\,(J)$,

$$T_J = K(\overline{\boldsymbol{a}}) \bigsqcup \{(i,j) \in T_J; \overline{a}_i \neq \overline{a}_j\}$$

and

$$J = K_{\text{list}}^{(1)}(\overline{\boldsymbol{a}}) \bigsqcup K^{(2)}(\overline{\boldsymbol{a}}) \bigsqcup \left\{i \in J; \overline{a}_i \neq \overline{1}\right\},$$

to obtain

$$\prod_{i \in J} |1 - x_i|_p^{s_{(N-1)i}} = \prod_{i \in K_{\text{list}}^{(1)}(\overline{\boldsymbol{a}})} |1 - x_i|_p^{s_{(N-1)i}} \prod_{i \in K^{(2)}(\overline{\boldsymbol{a}})} |1 - x_i|_p^{s_{(N-1)i}}$$

on $\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}$, and

$$\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}|x_i-x_j|_p^{s_{ij}}=\prod_{(i,j)\in K(\overline{\boldsymbol{a}})}|x_i-x_j|_p^{s_{ij}}$$

on $\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}$. With $J(\overline{\boldsymbol{a}}):=K^{(2)}(\overline{\boldsymbol{a}})\bigsqcup K_{\text{list}}(\overline{\boldsymbol{a}})$, we have

$$\begin{aligned}\boldsymbol{M}(\underline{\boldsymbol{s}};J,1)&=\sum_{\overline{\boldsymbol{a}}\in\mathcal{R}(J)}\left|\overline{A}(\overline{\boldsymbol{a}})\right|p^{-|J|-\sum_{i\in K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}})\sqcup K^{(2)}(\overline{\boldsymbol{a}})}s_{(N-1)i}-\sum_{(i,j)\in K(\overline{\boldsymbol{a}})}s_{ij}}\times\\&\int_{(\mathbb{Z}_p)^{|J(\overline{\boldsymbol{a}})|}}\prod_{i\in K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}})\sqcup K^{(2)}(\overline{\boldsymbol{a}})}|x_i|_p^{s_{(N-1)i}}\prod_{(i,j)\in K(\overline{\boldsymbol{a}})}|x_i-x_j|_p^{s_{ij}}\prod_{i\in J(\overline{\boldsymbol{a}})}dx_i\\&=\sum_{\overline{\boldsymbol{a}}\in\mathcal{R}(J)}\left|\overline{A}(\overline{\boldsymbol{a}})\right|p^{-|J|-\sum_{i\in K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}})\sqcup K^{(2)}(\overline{\boldsymbol{a}})}s_{(N-1)i}-\sum_{(i,j)\in K(\overline{\boldsymbol{a}})}s_{ij}}\times\\&\left\{\int_{(\mathbb{Z}_p)^{\left|K^{(2)}(\overline{\boldsymbol{a}})\right|}}\prod_{i\in K^{(2)}(\overline{\boldsymbol{a}})}|x_i|_p^{s_{(N-1)i}}\prod_{i\in K^{(2)}(\overline{\boldsymbol{a}})}dx_i\right\}\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};K_{\text{list}}(\overline{\boldsymbol{a}}),K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}}),K(\overline{\boldsymbol{a}})\right).\end{aligned}\tag{1.3.21}$$

Now, by using the partition of $K(\overline{\boldsymbol{a}})$ given in Lemma 1.9, we obtain

$$\begin{aligned}\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};K_{\text{list}}(\overline{\boldsymbol{a}}),K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}}),K(\overline{\boldsymbol{a}})\right)&=\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}}),K^{(1)}_{\text{list}}(\overline{\boldsymbol{a}}),T_{K^{(1)}_{list}(\overline{\boldsymbol{a}})}\right)\\&\times\prod_{(i,j)\in\mathcal{R}(\overline{\boldsymbol{a}})\setminus K^{(1)}(\overline{\boldsymbol{a}})}\boldsymbol{L}_1^{(N)}\left(\underline{\boldsymbol{s}};K_{\text{list}}((i,j),\overline{\boldsymbol{a}}),T_{K_{list}((i,j),\overline{\boldsymbol{a}})}\right)\end{aligned}\tag{1.3.22}$$

with the convention that $\boldsymbol{L}_2^{(N)}(\underline{\boldsymbol{s}},\varnothing,\varnothing,\varnothing):=1$. Finally,

$$\boldsymbol{M}(\underline{\boldsymbol{s}};J,2)=\sum_{\overline{\boldsymbol{b}}\in\overline{\Pi}(J)}\int_{\boldsymbol{b}+(p\mathbb{Z}_p)^{|J|}}\prod_{i\in J}dx_i=p^{-|J|}\left|\overline{\Pi}(J)\right|.\tag{1.3.23}$$

Hence, formula (1.3.18) follows from (1.3.20)-(1.3.23) by using Lemma 1.16 and Remark 1.17. ■

1.3.2 Computation of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$

Proposition 1.20 *Let I be a non-empty subset of T. Then, the integral*

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1) = \begin{cases} \displaystyle\int_{\mathbb{Z}_p^{|I|}} \frac{\prod_{\substack{2\leq i<j\leq N-2 \\ i,j\in I}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i\in I} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} \prod_{i\in I} dx_i & \text{if } |I| \geq 2 \\ \displaystyle\int_{\mathbb{Z}_p} \frac{1}{|x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} dx_i & \text{if } |I| = 1 \end{cases}$$

converges on the set

$$\left\{(s_{ij}) \in \mathbb{C}^D; \operatorname{Re}(s_{ij}) > -1 \text{ for } 2 \leq i < j \leq N-2, i, j \in I\right\} \cap$$
$$\left\{(s_{ij}) \in \mathbb{C}^D; 1 + \operatorname{Re}(s_{1i} + s_{(N-1)i}) + \sum_{2\leq j\leq N-2, j\neq i} \operatorname{Re}(s_{ij}) < 0 \text{ for } i \in I\right\},$$

which is an open and connected subset of $\mathbb{C}^D$. In addition, $\boldsymbol{Z}_p^{(N)}(\underline{\boldsymbol{s}}; I, 1)$ admits an analytic continuation to $\mathbb{C}^D$ as a rational function of the form

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1) = \frac{Q_{I,1}(\{p^{-s_{ij}}; i, j \in \{1, \ldots, N-1\}\})}{S_1(\underline{\boldsymbol{s}}; I) S_2(\underline{\boldsymbol{s}}; I) S_3(\underline{\boldsymbol{s}}; I) S_4(\underline{\boldsymbol{s}}; I)}, \tag{1.3.24}$$

where $Q_{I,1}(\{p^{-s_{ij}}; i, j \in \{1, \ldots, N-1\}\})$ denotes a polynomial with rational coefficients in the variables $p^{-s_{ij}}$, i, $j \in \{1, \ldots, N-1\}$,

$$S_1(\underline{\boldsymbol{s}}; I) = \prod_{J\in\mathcal{H}_1(I)} \left(1 - p^{|J| + \sum_{i\in J}(s_{1i} + s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in J}} s_{ij} + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus J, j\in J}} s_{ij}}\right),$$

where $\mathcal{H}_1(I)$ is a family of non-empty subsets of I, with $I \in \mathcal{H}_1(I)$,

$$S_2(\underline{\boldsymbol{s}}; I) := \prod_{\substack{J\subseteq I \\ J\neq\varnothing}} \prod_{K\in\mathcal{H}_2(J)} \left(1 - p^{-\left(|K| - 1 + \sum_{\substack{2\leq i<j\leq N-2 \\ i,j\in K}} s_{ij}\right)}\right)^{e_K},$$

where $\mathcal{H}_2(J)$ is a family of non-empty subsets of J, with $J \in \mathcal{H}_2(J)$, and the e_K's are positive integers,

$$S_3(\underline{\boldsymbol{s}}; I) := \prod_{\substack{J\subseteq I \\ J\neq\varnothing}} \prod_{ij\in G_J^{(0)}} \left(1 - p^{-1-s_{ij}}\right),$$

where $G_J^{(0)}$ is a non-empty subset $\{2 \leq i < j \leq N-2, i,j \in J\}$,

$$S_4(\underline{\boldsymbol{s}}; I) := \prod_{i \in G_I^{(1)}} \left(1 - p^{1+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}\right),$$

where $G_I^{(1)}$ is a non-empty subset $\{2 \leq i < j \leq N-2, i,j \in I\}$.

Proof. By using the partition $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$ as in the proof of Lemma 1.14, and a change of variables, we get

$$\begin{aligned}
\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1) &= \frac{\displaystyle\int_{S_0^{|I|}} \frac{\prod_{\substack{2\leq i<j\leq N-2 \\ i,j\in I}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i\in I} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} \prod_{i\in I} dx_i}{1 - p^{|I|+\sum_{i\in I}(s_{1i}+s_{(N-1)i})+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in I}} s_{ij}+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus I, j\in I}} s_{ij}}} \\
&=: \frac{\boldsymbol{C}_0(\underline{\boldsymbol{s}})}{1 - p^{|I|+\sum_{i\in I}(s_{1i}+s_{(N-1)i})+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in I}} s_{ij}+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus I, j\in I}} s_{ij}}}.
\end{aligned}$$

We now use the partition $S_0^{|I|} = \sqcup_{J\subseteq I, J\neq\varnothing} S_J^{|I|}$ to obtain

$$\boldsymbol{C}_0(\underline{\boldsymbol{s}}) = \sum_{J\subseteq I, J\neq\varnothing} \boldsymbol{C}_{0,J}(\underline{\boldsymbol{s}}),$$

where

$$\boldsymbol{C}_{0,J}(\underline{\boldsymbol{s}}) := \int_{S_J^{|I|}} \frac{\prod_{\substack{2\leq i<j\leq N-2 \\ i,j\in I}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i\in I} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} \prod_{i\in I} dx_i,$$

and consequently,

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1) = \frac{\boldsymbol{C}_{0,I}(\underline{\boldsymbol{s}}) + \sum_{J\subsetneq I, J\neq\varnothing} \boldsymbol{C}_{0,J}(\underline{\boldsymbol{s}})}{1 - p^{|I|+\sum_{i\in I}(s_{1i}+s_{(N-1)i})+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in I}} s_{ij}+\sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus I, j\in I}} s_{ij}}}.$$

On the other hand, by using (1.3.12), we have $\boldsymbol{C}_{0,I}(\underline{\boldsymbol{s}}) = \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}, I)$, and if $J \subsetneq I$,

$$\mathbf{C}_{0,J}(\underline{\boldsymbol{s}}) = \left\{ \int_{(p\mathbb{Z}_p)^{|I\smallsetminus J|}} \frac{\prod_{\substack{2\leq i<j\leq N-2 \\ i,j\in I\smallsetminus J}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i\in I\smallsetminus J} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{\substack{2\leq j\leq N-2 \\ j\neq i}} s_{ij}}} \prod_{i\in I\smallsetminus J} dx_i \right\} \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}, J)$$

$$= p^{|I \smallsetminus J| + \sum_{i\in I\smallsetminus J}(s_{1i}+s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in I\smallsetminus J}} s_{ij} + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus(I\smallsetminus J), j\in I\smallsetminus J}} s_{ij}} \times$$

$$\left\{ \int_{\mathbb{Z}_p^{|I\smallsetminus J|}} \frac{\prod_{\substack{2\leq i<j\leq N-2 \\ i,j\in I\smallsetminus J}} |x_i - x_j|_p^{s_{ij}}}{\prod_{i\in I\smallsetminus J} |x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{\substack{2\leq j\leq N-2 \\ j\neq i}} s_{ij}}} \prod_{i\in I\smallsetminus J} dx_i \right\} \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}, J)$$

$$= p^{|I \smallsetminus J| + \sum_{i\in I\smallsetminus J}(s_{1i}+s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in I\smallsetminus J}} s_{ij} + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus(I\smallsetminus J), j\in I\smallsetminus J}} s_{ij}} \times$$

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I\smallsetminus J, 1)\, \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}, J).$$

Therefore

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1) = \frac{\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; I) + \sum_{J\subsetneq I, J\neq\varnothing} p^{M(\underline{\boldsymbol{s}},J)} \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I\smallsetminus J, 1)\, \boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J)}{1 - p^{|I| + \sum_{i\in I}(s_{1i}+s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in I}} s_{ij} + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus I, j\in I}} s_{ij}}}, \quad (1.3.25)$$

where

$$\begin{aligned} M(\underline{\boldsymbol{s}}, J) \ : \ &= |I\smallsetminus J| + \sum_{i\in I\smallsetminus J}(s_{1i}+s_{(N-1)i}) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in I\smallsetminus J}} s_{ij} \\ &+ \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus(I\smallsetminus J), j\in I\smallsetminus J}} s_{ji}. \end{aligned}$$

Notice that in (1.3.25), $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I\smallsetminus J, 1)$ may occur with $|I\smallsetminus J| = 1$, say $I\smallsetminus J = \{i\}$, in this case $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$ becomes

$$\int_{\mathbb{Z}_p} \frac{1}{|x_i|_p^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} dx_i = \frac{1-p^{-1}}{1-p^{1+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2, j\neq i} s_{ij}}} \quad (1.3.26)$$

for $\mathrm{Re}(s_{1i}) + \mathrm{Re}(s_{(N-1)i}) + \sum_{2\leq j\leq N-2, j\neq i} \mathrm{Re}(s_{ij}) < -1$.

Finally, formula (1.3.25) gives a recursive algorithm for computing $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$, since $I\smallsetminus J \subsetneq I \subseteq T$ and $\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; I)$, $\boldsymbol{L}_0^{(N)}(\underline{\boldsymbol{s}}; J)$ can be effectively computed, see Corollary 1.15, by using this algorithm and (1.3.26), we obtain (1.3.24). ■

Remark 1.21 *Given positive integers N_i, $i \in I \subseteq T$, v, and complex numbers s_i for $i \in I$, we notice that the function $\frac{1}{1-p^{-v-\sum_{i\in I} N_i s_i}}$ gives rise to a holomorphic function of the s_i on the half-plane $\sum_{i\in I} N_i \,\mathrm{Re}\,(s_i) + v > 0$. As a consequence of Proposition*

1.20 there exist families $\mathfrak{F}_1$, $\mathfrak{F}_2$ of non-empty subsets of T, and a non-empty subset $\mathcal{G}$ of $\{ij; 2 \leq i < j \leq N-2, i, j \in T\}$, such that all the integrals $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$ for all $I \subseteq T$ are holomorphic functions of $\underline{\boldsymbol{s}}$ on the solution set of the conditions:

$$|J| + \sum_{i \in J} (\operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i})) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J}} \operatorname{Re}(s_{ij}) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \smallsetminus J, j \in J}} \operatorname{Re}(s_{ij}) < 0 \text{ for } J \in \mathfrak{F}_1; \tag{C1}$$

$$|K| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in K}} \operatorname{Re}(s_{ij}) > 0 \text{ for } K \in \mathfrak{F}_2; \tag{C2}$$

$$1 + \operatorname{Re}(s_{ij}) > 0 \text{ for } ij \in \mathcal{G} \subseteq \{ij; 2 \leq i < j \leq N-2\}. \tag{C3}$$

Notice that the condition

$$1 + \operatorname{Re}(s_{1i}) + \operatorname{Re}(s_{(N-1)i}) + \sum_{2 \leq j \leq N-2, j \neq i} \operatorname{Re}(s_{ij}) < 0$$

is included in Condition C1 taking $|J| = 1$. This fact follows from the following identities:

$$\sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \smallsetminus J, j \in J}} s_{ij} = \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T, j \in J}} s_{ij} - \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} =$$

$$\sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \smallsetminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T, j \in J}} s_{ij} = \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \smallsetminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \smallsetminus J, j \in J}} s_{ij} =$$

$$\sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J, j \in T \smallsetminus J}} s_{ij} + \sum_{\substack{2 \leq j < i \leq N-2 \\ i \in J, j \in T \smallsetminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij} = \sum_{\substack{2 \leq j \leq N-2 \\ j \neq i, i \in J, j \in T \smallsetminus J}} s_{ij} + \sum_{\substack{2 \leq i < j \leq N-2 \\ i, j \in J}} s_{ij}.$$

Finally, by taking $J = \{i\}$, the last formula becomes $\sum_{\substack{2 \leq j \leq N-2 \\ j \neq i}} s_{ij}$.

Denote by $D_{I,1}$ the natural domain of definition of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$, i.e. $D_{I,1}$ is an open and connected subset of $\mathbb{C}^D$ in which $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 1)$ is holomorphic and there no exists a larger domain where this property holds.

Lemma 1.22 *Take I to be a non-empty subset of T and set $H_{I,1}(\mathbb{C})$ to be the solution set in $\mathbb{C}^D$ of the following conditions:*

$$1+\mathrm{Re}\,(s_{1i})+\mathrm{Re}\,\big(s_{(N-1)i}\big)+\sum_{2\leq j\leq N-2, j\neq i}\mathrm{Re}\,(s_{ij})<0,\ \textit{for } i\in I. \tag{1.3.27}$$

Then $D_{I,1}$ is contained in $H_{I,1}(\mathbb{C})$.

Proof. Denote by $H_{I,1}(\mathbb{R})$ the solution set of (1.3.27) in $\mathbb{R}^D$. Set

$$\mathrm{Re}\,(D_{I,1})=\left\{\mathrm{Re}(s_{ij})\in\mathbb{R}^D;(s_{ij})\in D_{I,1}\right\}.$$

With this notation, it is sufficient to show that $\mathrm{Re}\,(D_{I,1})\subset H_{I,1}(\mathbb{R})$. In order to do this, we show that $\boldsymbol{Z}^{(N)}\left(\widetilde{\underline{\boldsymbol{s}}};I,1\right)$ diverges to $+\infty$ for any $\widetilde{\underline{\boldsymbol{s}}}\in\mathbb{R}^D\smallsetminus H_{I,1}(\mathbb{R})$. We prove this last assertion by contradiction. Assume that $\boldsymbol{Z}^{(N)}\left(\widetilde{\underline{\boldsymbol{s}}};I,1\right)<+\infty$ for $\widetilde{\underline{\boldsymbol{s}}}=(\widetilde{s}_{ij})\in\mathbb{R}^D$ with $\widetilde{s}_{ij}\geq 0$ for $2\leq i<j\leq N-2$, i, $j\in I$ and that $\widetilde{\underline{\boldsymbol{s}}}\notin H_{I,1}(\mathbb{R})$. This last condition implies that at least a condition of the form

$$1+\widetilde{s}_{1i_0}+\widetilde{s}_{(N-1)i_0}+\sum_{2\leq j\leq N-2, j\neq i_0}\widetilde{s}_{ij}\geq 0 \tag{1.3.28}$$

for some $i_0\in I$, holds. Then, from $\boldsymbol{Z}^{(N)}\left(\widetilde{\underline{\boldsymbol{s}}};I,1\right)<+\infty$, we have

$$\boldsymbol{I}\left(\widetilde{\underline{\boldsymbol{s}}};A\right):=\int_A\frac{\prod\limits_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{\widetilde{s}_{ij}}}{\prod\limits_{i\in I}|x_i|_p^{2+\widetilde{s}_{1i}+\widetilde{s}_{(N-1)i}+\sum_{2\leq j\leq N-2,j\neq i}\widetilde{s}_{ij}}}\prod_{i\in I}dx_i<+\infty$$

for any $A\subset\mathbb{Z}_p^{|I|}$. Take

$$A_0=\left\{(x_i)_{i\in I}\in\mathbb{Z}_p^{|I|};|x_{i_0}|_p<1\text{ and }|x_i|_p=1\text{ for }i\in I\smallsetminus\{i_0\}\right\}.$$

Then, by (1.3.28) and some $\epsilon\geq 0$,

$$\boldsymbol{I}\left(\widetilde{\underline{\boldsymbol{s}}};A_0\right)=\int_{A_0}\frac{\prod\limits_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus\{i_0\}}}|x_i-x_j|_p^{\widetilde{s}_{ij}}}{|x_{i_0}|_p^{1+\epsilon}}\prod_{i\in I}dx_i=+\infty.$$

Therefore, if $\boldsymbol{Z}^{(N)}\left(\widetilde{\underline{\boldsymbol{s}}};I,1\right)<+\infty$, necessarily $\widetilde{\underline{\boldsymbol{s}}}\in H_{I,1}(\mathbb{R})$. ■

Corollary 1.23 *If $\underline{\boldsymbol{s}}=(s_{ij})\in\mathbb{R}^D$, with $s_{ij}\geq 0$ for $i,j\in\{1,\ldots,N-1\}$, then $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,1\right)=+\infty$, for any non-empty subset I of T.*

1.3.3 Computation of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0)$

Proposition 1.24 *Let I be a subset of T satisfying $|I| \geq 2$. Then, the integral*

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0) = \int\limits_{\mathbb{Z}_p^{|I|}} \prod_{i\in I} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I} dx_i$$

gives rise to a holomorphic function on

$$H_{I,0} := \left\{(s_{ij}) \in \mathbb{C}^D; \mathrm{Re}\,(s_{ij}) > 0 \text{ for } i,j\in I\right\} \cap \left\{(s_{ij}) \in \mathbb{C}^D; \mathrm{Re}(s_{1i}) > 0 \text{ for } i\in I\right\}$$
$$\cap \left\{(s_{ij}) \in \mathbb{C}^D; \mathrm{Re}(s_{(N-1)i}) > 0 \text{ for } i\in I\right\},$$

which is an open and connected subset of $\mathbb{C}^D$. Furthermore $\boldsymbol{Z}_p^{(N)}(\underline{\boldsymbol{s}}; I, 0)$ has an analytic continuation as a rational function of the form

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I, 0) = \frac{Q_{I,0}(\{p^{-s_{1i}}, p^{-s_{(N-1)i}}, p^{-s_{ij}}; i,j\in T\})}{\prod\limits_{i=0}^{2} R_i(\underline{\boldsymbol{s}}; I, I) \prod_{i=1}^{3} U_i(\underline{\boldsymbol{s}}; I)},$$

where $Q_{I,0}(\{p^{-s_{1i}}, p^{-s_{(N-1)i}}, p^{-s_{ij}}; i,j\in T\})$ is a polynomial in the variables $p^{-s_{1i}}$,, $p^{-s_{ij}}$, $p^{-s_{(N-1)i}}$ for i, $j\in T$, $U_i(\underline{\boldsymbol{s}}; I)$, $i = 1$, 2, 3 are as in Lemma 1.18,

$$R_1(\underline{\boldsymbol{s}}; I, I) = \left(1 - p^{-1-s_{1i}}\right)^{h_1},$$

$$R_2(\underline{\boldsymbol{s}}; I, K) = \prod_{(J,R)\in\mathcal{G}_2(I\times I)} \left(1 - p^{-|J| - \sum_{i\in R} s_{1i} - \sum_{2\leq i<j\leq N-2,\ i,j\in J} s_{ij}}\right)^{l_{(J,R)}},$$

$R_0(\underline{\boldsymbol{s}}; I, I)$, $\mathcal{G}_2(I\times I)$ are as in Lemma 1.16, and the $l_{(J,R)}$'s are positive integers.

Proof. By using that $\mathbb{Z}_p^{|I|} = (p\mathbb{Z}_p)^{|I|} \sqcup S_0^{|I|}$, we have

$$\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}; I; 0) = \boldsymbol{M}_1^{(N)}(\underline{\boldsymbol{s}}; I) + \boldsymbol{M}_2^{(N)}(\underline{\boldsymbol{s}}; I), \tag{1.3.29}$$

where

$$\boldsymbol{M}_1^{(N)}(\underline{\boldsymbol{s}}; I) := \int\limits_{(p\mathbb{Z}_p)^{|I|}} \prod_{i\in I} |x_i|_p^{s_{1i}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I} dx,$$

$$\boldsymbol{M}_2^{(N)}(\underline{\boldsymbol{s}}; I) := \int\limits_{S_0^{|I|}} \prod_{i\in I} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}} |x_i - x_j|_p^{s_{ij}} \prod_{i\in I} dx.$$

Now, by changing variables and using Lemma 1.16 with $t=1$, $\boldsymbol{M}_1^{(N)}\left(\underline{\boldsymbol{s}};I\right)$ equals

$$\begin{aligned}
&p^{-|I|-\sum_{i\in I}s_{1i}-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}s_{ij}}\int\limits_{\mathbb{Z}_p^{|I|}}\prod_{i\in I}|x_i|_p^{s_{1i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx_i \qquad (1.3.30)\\
&\qquad = p^{-|I|-\sum_{i\in I}s_{1i}-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}s_{ij}}\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I,I,T_I\right).
\end{aligned}$$

To compute $\boldsymbol{M}_2^{(N)}\left(\underline{\boldsymbol{s}};I\right)$, we use the partition $S_0^{|I|}=\sqcup_{J\subseteq I,J\neq\varnothing}S_J^{|I|}$, with

$$S_J^{|I|}=\left\{(x_i)_{i\in I}\in\mathbb{Z}_p^{|I|};|x_i|_p=1\Leftrightarrow i\in J\right\},$$

then $\boldsymbol{M}_2^{(N)}\left(\underline{\boldsymbol{s}};I\right)$ equals

$$\begin{aligned}
&\sum_{\substack{J\subseteq I\\ J\neq\varnothing}}\int\limits_{S_J^{|I|}}\prod_{i\in I}|x_i|_p^{s_{1i}}\left|1-x_i\right|_p^{s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx \qquad (1.3.31)\\
= &\sum_{\substack{J\subseteq I\\ J\neq\varnothing}}M_J\left(\underline{\boldsymbol{s}}\right),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{M}_J\left(\underline{\boldsymbol{s}}\right) &= \int\limits_{S_J^{|I|}}\prod_{i\in I\smallsetminus J}|x_i|_p^{s_{1i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in J}|1-x_i|_p^{s_{(N-1)i}}\times\\
&\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I}dx=\int\limits_{(p\mathbb{Z}_p)^{|I\smallsetminus J|}}\prod_{i\in I\smallsetminus J}|x_i|_p^{s_{1i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in I\smallsetminus J}dx_i\\
&\times\int\limits_{(\mathbb{Z}_p^{\times})^{|J|}}\prod_{i\in J}|1-x_i|_p^{s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}|x_i-x_j|_p^{s_{ij}}\prod_{i\in J}dx_i:=\boldsymbol{M}_{I\smallsetminus J}\left(\underline{\boldsymbol{s}};0\right)\boldsymbol{M}_J\left(\underline{\boldsymbol{s}};1\right).
\end{aligned}$$

We notice that if $J=I$, then, by convention, $\boldsymbol{M}_{I\smallsetminus J}\left(\underline{\boldsymbol{s}};0\right)=1$. Now suppose that $J\subsetneqq I$. From Lemma 1.16 with $t=1$, we have

$$\boldsymbol{M}_{I\smallsetminus J}\left(\underline{\boldsymbol{s}};0\right)=p^{-|I\smallsetminus J|-\sum_{i\in I\smallsetminus J}s_{1i}-\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in I\smallsetminus J}}s_{ij}}\boldsymbol{L}_2^{(N)}\left(\underline{\boldsymbol{s}};I\smallsetminus J,I\smallsetminus J,T_{I\smallsetminus J}\right). \qquad (1.3.32)$$

The announced result follows from formulas (1.3.29)-(1.3.32), and $\boldsymbol{M}_J\left(\underline{\boldsymbol{s}};1\right)$ by using Lemmas 1.16-1.18 and Remark 1.19. ∎

Remark 1.25 *As a consequence of Proposition 1.24 all the integrals* $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0\right)$ *for all* $I \subseteq T$ *are holomorphic functions of* $\underline{\boldsymbol{s}}$ *on the solution set in* $\mathbb{C}^D$ *of the following conditions:*

$$|J| + \sum_{i \in S} \mathrm{Re}(s_{ti}) + \sum\nolimits_{2 \leq i < j \leq N-2,\ i,j \in J} \mathrm{Re}(s_{ij}) > 0 \text{ for } J \times S \in \mathfrak{F}_3, \tag{C4}$$

with $S \subseteq J$, $t \in \{1, N-1\}$,*and* $\mathfrak{F}_3$ *a family of non-empty subsets of* $I \times I$*;*

$$|K| - 1 + \sum_{\substack{2 \leq i < j \leq N-2 \\ i,j \in K}} \mathrm{Re}(s_{ij}) > 0 \text{ for } K \in \mathfrak{F}_4, \tag{C5}$$

where $\mathfrak{F}_4$ *s a family of non-empty subsets of* I*;*

$$1 + \mathrm{Re}(s_{ij}) > 0 \text{ for } ij \in \mathcal{H}, \tag{C6}$$

where $\mathcal{H}$ *is a non-empty subset of* $\{2 \leq i < j \leq N-2,\ i,j \in J\}$ *with* $(N-1)i$, $1i \in \mathcal{H}$.

Remark 1.26 *If* $\underline{\boldsymbol{s}} = (0)_{ij}$ *for* i, $j \in \{1, \ldots, N-1\}$, *then* $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{0}}; I, 0\right) = 1$, *for any non-empty subset* I *of* T.

Definition 1.27 *Denote by* $H(\mathbb{R})$, *respectively by* $H(\mathbb{C})$, *the solution set of conditions C1-C6 in* $\mathbb{R}^D$, *respectively in* $\mathbb{C}^D$.

1.3.4 Main Theorem

To show the holomorphy of the N-point zeta function $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$, we need to show that the intersection of all of the domains where all of the functions $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0\right)$, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 1\right)$ are holomorphic contains a connected open subset of $\mathbb{C}^D$. This allows to use the principle of analytic continuation.

Lemma 1.28 *Consider the following conditions:*

$$\begin{aligned} |J| + \sum_{i \in J} \left(\mathrm{Re}\left(s_{1i}\right) + \mathrm{Re}\left(s_{(N-1)i}\right)\right) + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in J}} \mathrm{Re}\left(s_{ij}\right) \\ + \sum_{\substack{2 \leq i < j \leq N-2 \\ i \in T \smallsetminus J, j \in J}} \mathrm{Re}\left(s_{ij}\right) < 0 \text{ for } J \subseteq T,\ |J| \geq 1; \end{aligned} \tag{C'1}$$

$$|J|-1+\sum_{\substack{2\leq i<j\leq N-2\\ i,j\in J}}\operatorname{Re}(s_{ij})>0 \textit{ for } J\subseteq T,\ |J|\geq 2; \tag{C'2}$$

$$|J|+\sum_{i\in S}\operatorname{Re}(s_{ti})+\sum\nolimits_{2\leq i<j\leq N-2,\ i,j\in J}\operatorname{Re}(s_{ij})>0 \tag{C'3}$$

$$\textit{for } t\in\{1,N-1\}\,,\ J\times S\subseteq T\times T \textit{ with } |J|\geq 2 \textit{ or } |S|\geq 1, S\subseteq J;$$

$$1+\operatorname{Re}(s_{ij})>0 \textit{ for } ij\in\{(i,j);1\leq i<j\leq N-1\}\,. \tag{C'4}$$

Denote by $H_0(\mathbb{R})$, respectively by $H_0(\mathbb{C})$, the solution set of conditions C'1-C'4 in $\mathbb{R}^D$, respectively in $\mathbb{C}^D$. Then $H_0(\mathbb{R})$ is convex and bounded set with non-empty interior, and $H_0(\mathbb{C})$ contains an open and connected subset of $\mathbb{C}^D$. Furthermore, $H_0(\mathbb{R})\subset H(\mathbb{R})$ and $H_0(\mathbb{C})\subset H(\mathbb{C})$.

Proof. We first notice that for all $N\geq 4$, the solution set $H_0(\mathbb{R})$ is an open convex set because it is a finite intersection of open half-spaces.

Claim $H_0(\mathbb{R})$ is a non-empty bounded subset**.** We consider the case $N\geqslant 5$ in which $|T|\geqslant 2$. Set $N_1=\frac{(N-4)(N-3)}{2}$. We define, for $i,j\in\{2,...,N-2\}$, the following conditions:

$$-\frac{2}{3N_1}<\operatorname{Re}(s_{ij})<0\ , \tag{C"1}$$

$$-\frac{2}{3}<\operatorname{Re}(s_{1i})<-\frac{1}{2}, \tag{C"2}$$

$$-\frac{2}{3}<\operatorname{Re}(s_{(N-1)i})<-\frac{1}{2}. \tag{C"3}$$

We notice that the solution set of conditions C"1–C"3 is a non-empty open and connected subset in $\mathbb{R}^D$. We now verify that the conditions C"1–C"2 imply conditions C'1-C'4. First, consider $J\subseteq T$ such that $|J|=1$. We can assume that $J=\{i_0\}$ for some $i_0\in T$. By conditions C"1–C"3, we have

$$1+\operatorname{Re}\left(s_{1i_0}\right)+\operatorname{Re}s_{(N-1)i_0}<1-1/2-1/2=0, \tag{1.3.33}$$

$$\sum_{2\leqslant i_0<j\leq N-2}\operatorname{Re}\left(s_{i_0j}\right)+\sum_{\substack{2\leq i<i_0\leq N-2,\\ i\in T\setminus J}}\operatorname{Re}\left(s_{ii_0}\right)<0, \tag{1.3.34}$$

thus, C'1 follows from (1.3.33) and (1.3.34). Conditions C'2, C'3 and C'4 follow directly from C"1–C"3.

We now consider $J \subseteq T$ such that $|J| \geq 2$. Condition C'1 is obtained with a similar calculation to (1.3.33) and (1.3.34). Now, by condition C"1, we get

$$|J|-1+\sum_{2\leq i<j\leq N-2,\ i,j\in J}\operatorname{Re}(s_{ij})>|J|-1-\frac{2}{3}>|J|-\frac{5}{3}>0,$$

which implies C'2. We now verify Condition C'3. Let $t \in \{1, N-1\}$, by using conditions C"2 and C"3,

$$\begin{aligned} & |J|+\sum_{i\in S}\operatorname{Re}(s_{ti})+\sum_{2\leq i<j\leq N-2,\ i,j\in J}\operatorname{Re}(s_{ij}) \\ > \ & |J|-\frac{2}{3}|S|-\frac{2\left|(i,j)\,;2\leq i<j\leq N-2,\ i,j\in J\right|}{3N_1} \\ \geq \ & |J|-\frac{2}{3}|S|-\frac{2}{3}. \end{aligned}$$

There are two cases. First, $|S| = 1$. In this case $|J| - \frac{2}{3}|S| - \frac{2}{3} > 0$. If $|S| \geq 2$, by using $\frac{-2}{3}|S| - \frac{2}{3} \geq -|S|$ and $|J| \geq |S|$, then $|J| - \frac{2}{3}|S| - \frac{2}{3} \geq |J| - |S| \geq 0$.

Finally, conditions C'4 follows from conditions C"1-C"3. Therefore, $H_0(\mathbb{R})$ is convex and bounded set with non-empty interior, and $H_0(\mathbb{C})$ contains an open and connected subset of $\mathbb{C}^D$. Finally, since conditions C'1-C'4 imply conditions C1-C6, we conclude that $H_0(\mathbb{R}) \subset H(\mathbb{R})$ and that $H_0(\mathbb{C}) \subset H(\mathbb{C})$.

In the case $N = 4$, $|T| = 1$, the verification of the claim is straightforward. ■

Theorem 1.29 *(1) The p-adic open string N-point zeta function, $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$, gives rise to a holomorphic function on $H(\mathbb{C})$, which contains an open and connected subset of $\mathbb{C}^D$. Furthermore, $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ admits an analytic continuation to $\mathbb{C}^D$, denoted also as $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$, as a rational function in the variables $p^{-s_{ij}}, i, j \in \{1, \ldots, N-1\}$. The real parts of the poles of $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C6 with the symbols '$<$', '$>$' replaced by '$=$'.*
(2) If $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, with $\operatorname{Re}(s_{ij}) \geq 0$ for $i, j \in \{1, \ldots, N-1\}$, then the integral p-adic open string N-point zeta function $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}) = +\infty$.

Proof. (1) We recall that

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)=\sum_{I\subseteq T}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I\right)=\sum_{I\subseteq T}p^{M(\underline{\boldsymbol{s}})}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right),\qquad(1.3.35)$$

see Remark 1.6. Now, by Propositions 1.20-1.24 and Lemma 1.28, for any $I\subseteq T$, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)$ and $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right)$ are holomorphic functions of $\underline{\boldsymbol{s}}\in H_0(\mathbb{C})$, which is an open and connected subset, and consequently the analytic continuations of the integrals $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)$ and $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right)$ and formula (1.3.35) give rise to an analytic continuation of $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ with the announced properties.

(2) It follows from formula (1.3.35) by Corollary 1.23 and Remark 1.26. ■

Chapter 2

p-adic string amplitudes in the limit p approaches to one

In this chapter we use the theory of topological zeta functions introduced by Denef and Loeser in [21], to define topological open string N-point tree amplitudes, which should be string analogues of the topological zeta functions. This chapter is organized as follows. In Section 2.1, we introduced some aspects of the non-Archimedean local fields. In Sections 2.2, 2.3 we generalized the results obtained in Chapter 1, to the case of the unramified finite extensions of any non-Archimedean local field of characteristic zero and we define the p-adic string amplitudes over these extensions. In 2.4, we summarize some results used to define the topological zeta function in the multivariate case. In 2.5, we define the topological open string N-point zeta functions and the topological open string N-point tree amplitudes. Finally, in 2.6 we give the calculation for $N = 4, 5$ of the topological N-point zeta function and topological N-point amplitude.

2.1 Non-Archimedean local fields

A non-archimedean local field K is a locally compact topological field with respect to a non-discrete topology with an absolute value $|\cdot|_K$ satisfying

$$|x+y|_K \leq \max\{|x|_K, |y|_K\} \text{ for } x, y \in K,$$

i.e. $|\cdot|_K$ is ultrametric. For an in-depth exposition, the reader may consult [64], [55], see also [1], [63].

Let K be a non-Archimedean local field of arbitrary characteristic and let $\mathcal{O}_K$ be the ring of valuation of K,

$$\mathcal{O}_K := \{x \in K : |x|_K \leq 1\},$$

and P_K the maximal ideal of $\mathcal{O}_K$; this ideal is formed by the non-units of $\mathcal{O}_K$. In terms of the absolute value $|\cdot|_K$, this maximal ideal can be described as

$$P_K = \{x \in K : |x|_K < 1\}.$$

Let $\overline{K} = \mathcal{O}_K / P_K$ the residue field of K. Thus $\overline{K} = \mathbb{F}_q$, the finite field with q elements. Let π be a fixed generator of P_K, π is called a uniformizing parameter of K, then $P_K = \pi\mathcal{O}_K$. Furthermore, we assume that $|\pi|_K = q^{-1}$. For $z \in K$, ord$(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, and $|z|_K = q^{-ord(z)}$. If $z \in K \setminus \{0\}$, then ac(z)=$z\pi^{-ord(z)}$ denotes the angular component of z.

The natural map $\mathcal{O}_K \to \mathcal{O}_K/P_K \simeq \mathbb{F}_q$ is called the reduction mod P_K, and it will be denoted as . We fix $\mathfrak{S} \subset \mathcal{O}_K$ a set of representatives of $\mathbb{F}_q$ in $\mathcal{O}_K$, i.e. $\mathfrak{S}$ is mapped bijectively into $\mathbb{F}_q$ by the reduction mod P_K. We assume that $0 \in \mathfrak{S}$. Any non-zero element x of K can be written as

$$x = \pi^{ord(x)} \sum_{i=0}^{\infty} x_i \pi^i,\ x_i \in \mathfrak{S}, \text{ and } x_0 \neq 0.$$

This series converges in the norm $|\cdot|_K$.

Example 2.1 *We now fix a prime number p. A basic example of non-Archimedean local field is the field of p-adic numbers $\mathbb{Q}_p$, which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as*

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ \\ p^{-\gamma} & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the p-adic order of x.

Any non-Archimedean local field K of characteristic zero is isomorphic (as topological field) to a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers. In this case we say that K is a p-adic field. In case of positive characteristic, K is isomorphic to a finite extension of the field of formal Laurent series $\mathbb{F}_q((T))$ with coefficients in a finite field $\mathbb{F}_q$ with q elements.

Remark 2.2 *As we mentioned above, any finite extension of $\mathbb{Q}_p$ is a non-Archimedean local field. Let K_e denote the unique unramified extension of $\mathbb{Q}_p$ of degree e, with π a local uniformizing parameter of K_e. Then $p\mathcal{O}_{K_e} = \pi\mathcal{O}_{K_e}$ and $\mathcal{O}_{K_e}/P_{K_e} \simeq \mathbb{F}_{p^e}$. Notice that $|\pi|_{K_e} = p^{-e}$.*

We extend the norm $|\cdot|_K$ to K^n by taking

$$||\boldsymbol{x}||_K := \max_{1\leq i\leq n} |x_i|_K, \qquad \text{for } \boldsymbol{x} = (x_1, \dots, x_n) \in K^n.$$

We define $ord(\boldsymbol{x}) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||\boldsymbol{x}||_K = q^{-ord(\boldsymbol{x})}$. The metric space $(K^n, ||\cdot||_K)$ is a complete ultrametric space.

For $r \in \mathbb{Z}$, denote by $B_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in K^n; ||\boldsymbol{x} - \boldsymbol{a}||_K \leq q^r\}$ *the ball of radius q^r with center at* $\boldsymbol{a} = (a_1, \dots, a_n) \in K^n$, and take $B_r^n(\boldsymbol{0}) := B_r^n$. Note that $B_r^n(\boldsymbol{a}) = B_r(a_1)\times\cdots\times B_r(a_n)$, where $B_r(a_i) := \{x \in K; |x_i - a_i|_K \leq q^r\}$ is the one-dimensional ball of radius q^r with center at $a_i \in K$. The ball B_0^n equals the product of n copies of $B_0 = \mathcal{O}_K$. In addition, $B_r^n(\boldsymbol{a}) = \boldsymbol{a} + (\pi^{-r}\mathcal{O}_K)^n$. We also denote by

$S_r^n(\boldsymbol{a}) = \{\boldsymbol{x} \in K^n; ||\boldsymbol{x} - \boldsymbol{a}||_K = q^r\}$ *the sphere of radius* q^r *with center at* $\boldsymbol{a} \in K^n$, and take $S_r^n(\mathbf{0}) := S_r^n$. We notice that $S_0^1 = \mathcal{O}_K^\times$ (the group of units of $\mathcal{O}_K$), but $\left(\mathcal{O}_K^\times\right)^n \subsetneq S_0^n$, for $n \geq 2$. The balls and spheres are both open and closed subsets in K^n. In addition, two balls in K^n are either disjoint or one is contained in the other.

As a topological space $(K^n, ||\cdot||_K)$ is totally disconnected, i.e. the only connected subsets of K^n are the empty set and the points. A subset of K^n is compact if and only if it is closed and bounded in K^n. The balls and spheres are compact subsets. Thus $(K^n, ||\cdot||_K)$ is a locally compact topological space.

2.2 p-adic String Zeta Functions

In this section, we review the main result of our publication [8]. In this article the results were stated over $\mathbb{Q}_p$ but they are still valid in K_e, the unique unramified extension of $\mathbb{Q}_p$ of degree e. We fix an integer $N \geq 4$. To each pair (i,j) with $i,j \in \{1,\ldots,N-1\}$ we attach a complex number $s_{(i,j)}$ such that $s_{(i,j)} = s_{(j,i)}$. To simplify the notation we will use ij, respectively s_{ij}, instead of (i,j), respectively, instead of $s_{(i,j)}$. We set $T := \{2,\ldots,N-2\}$, $D = \frac{(N-3)(N-4)}{2} + 2\,(N-3)$ and $\mathbb{C}^D$ as

$$\begin{cases} \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T\} & \text{if} \quad N = 4 \\ \\ \{s_{ij} \in \mathbb{C}; i \in \{1, N-1\}, j \in T \text{ or } i,j \in T \text{ with } i < j\} & \text{if} \quad N \geq 5. \end{cases}$$

We set $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, $\boldsymbol{x} = (x_2, \ldots, x_{N-2}) \in K_e^{N-3}$, and

$$F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, K_e\right) = \prod_{i=2}^{N-2} |x_i|_{K_e}^{s_{1i}} \, |1 - x_i|_{K_e}^{s_{(N-1)i}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{K_e}^{s_{ij}}.$$

Definition 2.1 *The open string N-point zeta function is defined as*

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right) := \int\limits_{K_e^{N-3} \smallsetminus \Lambda} F\left(\underline{\boldsymbol{s}}, \boldsymbol{x}; N, K_e\right) \prod_{i=2}^{N-2} dx_i \tag{2.2.1}$$

for $\underline{\boldsymbol{s}}=(s_{ij})\in\mathbb{C}^{D}$, *where*

$$\Lambda:=\left\{(x_2,\ldots,x_{N-2})\in K_e^{N-3};\prod_{i=2}^{N-2}x_i\left(1-x_i\right)\prod_{2\leq i<j\leq N-2}(x_i-x_j)=0\right\}$$

and $\prod_{i=2}^{N-2}dx_i$ *is the Haar measure of* K_e^{N-3} *normalized so that the measure of* $\mathcal{O}_{K_e}^{N-3}$ *is* 1.

Notation 2.2 *(i) The cardinality of a finite set* A *will be denoted as* $|A|$. *(ii) We will use the symbol* $\bigsqcup$ *to denote the union of disjoint sets. (iii) Given a non-empty subset* I *of* $\{2,\ldots,N-2\}$ *and* B *a non-empty subset of* K_e, *we set*

$$B^{|I|}=\left\{(x_i)_{i\in I};x_i\in B\right\}.$$

(iv) By convention, we define $\prod_{i\in\varnothing}\cdot:=1$, $\sum_{i\in\varnothing}\cdot:=0$, *and if* $J=\varnothing$, *then* $\int_{B^{|J|}}\cdot:=1$. *(v) The indices* i, j *will run over subsets of* T, *if we do not specify any subset, we will assume that is* T.

Let p^e the cardinality of the residue field $\overline{K_e}$, see Section 2.1. We define for $I\subseteq T$, *the sector attached to* I as

$$Sect(I)=\left\{(x_2,\ldots,x_{N-2})\in K_e^{N-3};|x_i|_{K_e}\leq 1\ \Leftrightarrow i\in I\right\}$$

and

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,K_e\right)=\int\limits_{Sect(I)}F\left(\underline{\boldsymbol{s}},\boldsymbol{x};N,K_e\right)\prod_{i=2}^{N-2}dx_i.$$

Then $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}},K_e\right)=\sum_{I\subseteq T}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,K_e\right)$. In addition, we have

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}},K_e\right)=\sum_{I\subseteq T}p^{eM(\underline{\boldsymbol{s}})}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0,K_e\right)\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1,K_e\right),\qquad(2.2.2)$$

where

$$M(\underline{\boldsymbol{s}}):=|T\smallsetminus I|+\sum_{i\in T\smallsetminus I}(s_{1i}+s_{(N-1)i})+\sum_{\substack{2\leq i<j\leq N-2\\ i\in T\smallsetminus I,j\in T}}s_{ij}+\sum_{\substack{2\leq i<j\leq N-2\\ i\in I,j\in T\smallsetminus I}}s_{ij},$$

and

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0,K_e\right)=\int\limits_{\mathcal{O}_{K_e}^{|I|}}F_0\left(\boldsymbol{s},\boldsymbol{x};N,K_e\right)\prod_{i\in I}dx_i$$

where

$$F_0\left(\underline{\boldsymbol{s}},\boldsymbol{x};N,K_e\right):=\prod_{i\in I}|x_i|_{K_e}^{s_{1i}}\,|1-x_i|_{K_e}^{s_{(N-1)i}}\prod_{\substack{2\leq i<j\leq N-2\\ i,j\in I}}|x_i-x_j|_{K_e}^{s_{ij}}$$

and

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1,K_e\right)=\int\limits_{\mathcal{O}_{K_e}^{|T\smallsetminus I|}}\frac{\prod\limits_{\substack{2\leq i<j\leq N-2\\ i,j\in T\smallsetminus I}}|x_i-x_j|_{K_e}^{s_{ij}}}{\prod\limits_{i\in T\smallsetminus I}|x_i|_{K_e}^{2+s_{1i}+s_{(N-1)i}+\sum_{2\leq j\leq N-2,j\neq i}s_{ij}}}\prod_{i\in T\smallsetminus I}dx_i.$$

By convention $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};\varnothing,0,K_e\right)=1$, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};\varnothing,1,K_e\right)=1$. In [8] we showed that $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}},K_e\right)$ has an analytic continuation to the whole $\mathbb{C}^D$ as a rational function in the variables $p^{-es_{ij}}$. More precisely, we showed that all functions appearing on the right-hand side of formula (2.2.2) admit analytic continuations to the whole $\mathbb{C}^D$ as rational functions in the variables $p^{-es_{ij}}$, and that each of these functions is holomorphic on certain domain, and that the intersection of all of these domains contains an open and connected subset of $\mathbb{C}^D$, which allows to use the principle of analytic continuation. In Propositions 1.20 and 1.24 in Chapter 1, we gave algorithms for computing recursively the integrals $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0,K_e\right)$, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1,K_e\right)$. These algorithms reduced the calculation of any of these integrals to the calculation of certain integrals in one or two variables, that can be computed directly. These simple integrals are rational functions in the variables $p^{-es_{ij}}$ with possible poles depending on combinatorial data but not on the residue field of K_e.

Remark 2.3 *As a consequence of Proposition 1 in [8] there exist families $\mathfrak{F}_1$, $\mathfrak{F}_2$ of non-empty subsets of T, and a subset $\mathcal{G}$ of $\{ij;2\leq i<j\leq N-2,i,j\in T\}$ such that all the integrals $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,1,K_e\right)$ for all $I\subseteq T$ are holomorphic functions of $\underline{\boldsymbol{s}}$ on the*

solution set of the conditions:

$$|J| + \sum_{i\in J}(\mathrm{Re}\,(s_{1i}) + \mathrm{Re}\,(s_{(N-1)i})) + \sum_{\substack{2\leq i<j\leq N-2 \\ i\in J}} \mathrm{Re}\,(s_{ij}) \tag{C1}$$

$$+ \sum_{\substack{2\leq i<j\leq N-2 \\ i\in T\smallsetminus J, j\in J}} \mathrm{Re}\,(s_{ij}) < 0 \text{ for } J \in \mathfrak{F}_1;$$

$$|M| - 1 + \sum_{\substack{2\leq i<j\leq N-2 \\ i,j\in M}} \mathrm{Re}(s_{ij}) > 0 \text{ for } M \in \mathfrak{F}_2; \tag{C2}$$

$$1 + \mathrm{Re}(s_{ij}) > 0 \text{ for } ij \in \mathcal{G} \subseteq \{ij; 2 \leq i < j \leq N-2\}. \tag{C3}$$

The sets $\mathfrak{F}_1$, $\mathfrak{F}_2$ *and* $\mathcal{G}$ *do not depend on* K_e.

Remark 2.4 *As a consequence of Proposition 2 in [8] all the integrals*

$$\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0, K_e\right)$$

for all $I \subseteq T$ *are holomorphic functions of* $\underline{\boldsymbol{s}}$ *on the solution set in* $\mathbb{C}^D$ *of the following conditions:*

$$|J| + \sum_{i\in S} \mathrm{Re}(s_{ti}) + \sum\nolimits_{2\leq i<j\leq N-2,\ i,j\in J} \mathrm{Re}(s_{ij}) > 0 \text{ for } J \times S \in \mathfrak{F}_3, \tag{C4}$$

with $S \subseteq J$, $t \in \{1, N-1\}$,*and* $\mathfrak{F}_3$ *a family of non-empty subsets of* $I \times I$;

$$|M| - 1 + \sum_{\substack{2\leq i<j\leq N-2 \\ i,j\in M}} \mathrm{Re}(s_{ij}) > 0 \text{ for } M \in \mathfrak{F}_4, \tag{C5}$$

where $\mathfrak{F}_4$ *is a family of non-empty subsets of* I;

$$1 + \mathrm{Re}(s_{ij}) > 0 \text{ for } ij \in \mathfrak{H}, \tag{C6}$$

where $\mathfrak{H}$ *is a non-empty subset of* $\{2 \leq i < j \leq N-2,\ i, j \in J\}$ *with* $(N-1)i$, $1i \in \mathfrak{H}$.

The sets $\mathfrak{F}_3$, $\mathfrak{F}_4$ *and* $\mathfrak{H}$ *do not depend on* K_e.

Remark 2.5 *If* $\underline{\boldsymbol{s}} = (0)_{ij}$ *for* i, $j \in \{1, \ldots, N-1\}$, *then* $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{0}}; I, 0, K_e\right) = 1$, *for any non-empty subset* I *of* T.

Definition 2.6 *Denote by $H(\mathbb{R})$, respectively by $H(\mathbb{C})$, the solution set of conditions C1-C6 in $\mathbb{R}^D$, respectively in $\mathbb{C}^D$.*

Due to the method used to calculate the main result in [8], we can extend our results to the non-Archimedean local fields of characteristic zero. The following theorem is a generalization of this fact.

Theorem 2.7 ([8, Theorem 1]) *(1)The open string N-point zeta function, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right)$, gives rise to a holomorphic function on $H(\mathbb{C})$, which contains a non-empty open and connected subset of $\mathbb{C}^D$. Furthermore, $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right)$ admits an analytic continuation to $\mathbb{C}^D$, denoted also as $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right)$, as a rational function in the variables $p^{-es_{ij}}, i, j \in \{1, \ldots, N-1\}$. The real parts of the poles of $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right)$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C6 with the symbols '$<$', '$>$' replaced by '$=$'. (2) If $\underline{\boldsymbol{s}} = (s_{ij}) \in \mathbb{C}^D$, with $\mathrm{Re}(s_{ij}) \geq 0$ for $i, j \in \{1, \ldots, N-1\}$, then the integral open string N-point zeta function $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}, K_e\right) = +\infty$.*

2.3 p-Adic String Amplitudes

The open string $N-$point tree amplitudes over K_e are defined as

$$\begin{aligned} &\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}, K_e\right) \qquad (2.3.1)\\ &= \int_{K_e^{N-3}} \prod_{i=2}^{N-2} |x_i|_{K_e}^{\boldsymbol{k}_1\boldsymbol{k}_i} |1-x_i|_{K_e}^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_{K_e}^{\boldsymbol{k}_i\boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i, \end{aligned}$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of K_e^{N-3},

$$\underline{\boldsymbol{k}} = \left(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N\right), \ \boldsymbol{k}_i = \left(k_{0,i}, \ldots, k_{25,i}\right), \ i = 1, \ldots, N, \ N \geq 4,$$

(with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{25,i}k_{25,j}$) obeying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}, \ \boldsymbol{k}_i\boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N.$$

In this case, it is a central problem to know whether or not integrals of type (2.3.1) converge for some values $\boldsymbol{k}_i\boldsymbol{k}_j \in \mathbb{C}$. Theorem 2.7 allows us to solve this problem.

We take the open string N-point tree integrals $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, K_e)$ as regularizations of the amplitudes $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}, K_e)$. More precisely, we define

$$\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}; K_e) = \boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}}, K_e)\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1,\ldots,N-1\}, \ j \in T \text{ or } i,j \in T,$$

where $T = \{2,\ldots,N-2\}$. By Theorem 2.7, $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}, K_e)$ are well-defined rational functions of the variables $p^{-e\boldsymbol{k}_i\boldsymbol{k}_j}$, $i,\ j \in \{1,\ldots,N-1\}$, which agree with integrals (2.3.1) when they converge. This definition allows us to recover all the calculations made in [15] and other similar publications.

2.4 Igusa zeta functions and topological zeta functions

In this section we present some results, which are variations of well-known results, that we will use to define the topological string amplitudes.

2.4.1 Multivariate local zeta functions

Let K be a non-Archimedean local field and let $\boldsymbol{f}$ be a polynomial mapping $\boldsymbol{f} = (f_1,\ldots,f_r) : K^n \to K^r$ such that each $f_i(\boldsymbol{x})$ is a non-constant polynomial in $K[x_1,\ldots,x_n]$, $i = 1,..,r$. Let Φ a Bruhat-Schwartz function and let $\boldsymbol{s} = (s_1,\ldots,s_r) \in \mathbb{C}^r$. The multivariate local zeta function associated to Φ and $\boldsymbol{f}$ is defined as

$$Z_\Phi(\boldsymbol{s},\boldsymbol{f},K) = \int_{K^n\setminus D_K} \Phi(\boldsymbol{x}) \prod_{i=1}^{r} |f_i(\boldsymbol{x})|_K^{s_i}\, |d\boldsymbol{x}|_K$$

for $\mathrm{Re}(s_i) > 0$ for all i, where $D_K := \cup_{i\in\{1,\ldots,r\}} \{\boldsymbol{x} \in K^n; f_i(\boldsymbol{x}) = 0\}$.

This integral defines a holomorphic function of $(s_1,\ldots,s_r)$ in the half-space $\mathrm{Re}(s_i) > 0$, $i = 1,\ldots,r$. In the case of characteristic zero and $r = 1$, this assertion corresponds to Lemma 5.3.1 in [39]. For $r > 1$, we recall that a continuous complex-valued function defined in an open set $A \subseteq \mathbb{C}^r$, which is holomorphic in each variable separately, is holomorphic in A. In the case of the p-adic fields, the multivariate local zeta functions admit analytic continuations to the whole $\mathbb{C}^r$ as rational functions in the variables q^{-s_i}, $i = 1, \ldots, r$, see Theorem [47].

Notation 2.1 *If Φ is the characteristic function of $\mathcal{O}_K^n$ we denote $Z(\boldsymbol{s}, \boldsymbol{f}, K)$ by $Z_\Phi(\boldsymbol{s}, \boldsymbol{f}, K)$.*

2.4.2 Embedded resolution of singularities

In this subsection L is an arbitrary field of characteristic zero and $f_i(\boldsymbol{x}) \in L[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a non-constant polynomial for $i = 1, \ldots, r$. Put $X = Spec$ $L[\boldsymbol{x}]$ (the n-dimensional affine space over L), $D = Spec\ L[\boldsymbol{x}] / (\prod_{i=1}^r f_i(\boldsymbol{x}))$ (the divisor attached to the polynomials $f_1, \ldots, f_r$). (the divisor attached to polynomials $f_1, ..., f_r$). An *embedded resolution of singularities for D* over L consists of a pair (Y, h), where Y is a smooth algebraic variety (an integral smooth closed subscheme of the projective space over X), and the morphism $h : Y \longrightarrow X$ is the natural map, which satisfies that the restriction $h : Y \backslash h^{-1}(D) \longrightarrow X \backslash D$ is an isomorphism, and the reduced scheme $(h^{-1}(D))_{red}$ has only normal crossings, *i.e.* its irreducible components are smooth and intersect transversally.

Let E_i, $i \in T$, be the irreducible components of $(h^{-1}(D))_{red}$. For each $i \in T$, let N_{ij} be the multiplicity of E_i in the divisor $f_j \circ h$ on Y, and $v_i - 1$ the multiplicity of E_i in the divisor $h^*(dx_1 \wedge \ldots \wedge dx_n)$. The $(N_{i1}, \ldots, N_{ir}, v_i)$, $i \in T$, are called the numerical data of the resolution (Y, h). For $i \in T$ and $I \subseteq T$ we define

$$\overset{\circ}{E}_i = E_i \smallsetminus \bigcup_{j \neq i} E_j, \quad E_I = \bigcap_{i \in I} E_i, \quad \overset{\circ}{E}_I = E_I \smallsetminus \bigcup_{j \in T \smallsetminus I} E_j.$$

If $I = \emptyset$, we put $E_\emptyset = Y$.

Theorem 2.2 (Loeser, [47, Theorem 1.1.4]) *Let K be a p-adic field. The local zeta function $Z_\Phi(\boldsymbol{s}, \boldsymbol{f}, K)$ admits a meromorphic continuation to the whole $\mathbb{C}^r$ as a* *rational function of* $q^{-s_1},\ldots,q^{-s_r}$*, more precisely,*

$$Z_{\Phi}\left(\boldsymbol{s},\boldsymbol{f},K\right)=\frac{P\left(q^{-s_1},\ldots,q^{-s_r}\right)}{\prod\limits_{i\in T}\left(1-q^{-v_i-\sum_{j=1}^{r}N_{ij}s_j}\right)},$$

where P *is a polynomial in the variables* $q^{-s_1},\ldots,q^{-s_r}$ *and the real parts of the poles of* $Z_{\Phi}\left(\boldsymbol{s},\boldsymbol{f},K\right)$ *belong to a union of hyperplanes of the form*

$$v_i+\sum_{j=1}^{r}N_{ij}\operatorname{Re}\left(s_j\right)=0,\ i\in T.$$

The following theorem is a variation of Theorem 3.1 in [17].

Theorem 2.3 (Denef) *Let* $f_i(\boldsymbol{x})\in\mathbb{Z}\left[\boldsymbol{x}\right]$*,* $\boldsymbol{x}=(x_1,\ldots,x_n)$*, be a non-constant polynomial for* $i=1,\ldots,r$*. We assume that* $\overline{f}_i(\boldsymbol{x})\in\mathbb{F}_p\left[\boldsymbol{x}\right]\setminus\{\boldsymbol{0}\}$ *for almost all prime numbers p. Let* (Y,h) *be an embedded resolution of singularities for* $D=Spec$ $\mathbb{Q}\left[\boldsymbol{x}\right]/\left(\prod_{i=1}^{r}f_i(\boldsymbol{x})\right)$ *over* $\mathbb{Q}$ *, with numerical data* $\{(N_{i1},\ldots,N_{ir},v_i)\,;i\in T\}$*. Then, there exists a finite set of primes* $S\subset\mathbb{Z}$ *such that for any non-Archimedean local field* $K\supset\mathbb{Q}$ *with* $P_K\cap\mathbb{Z}\notin S$*, we have*

$$\boldsymbol{Z}\left(\boldsymbol{s},\boldsymbol{f},K\right)=q^{-n}\sum_{I\subset T}c_I\left(K\right)\prod_{i\in I}\frac{\left(q-1\right)q^{-v_i-\sum_{j=1}^{r}N_{ij}s_j}}{1-q^{-v_i-\sum_{j=1}^{r}N_{ij}s_j}},\tag{2.4.1}$$

where $q=q\left(K\right)$ *denotes the cardinality of the residue field* $\overline{K}$ *and*

$$c_I(K)=Card\left\{a\in\overline{Y}\left(\overline{K}\right);a\in\overline{E}_i\left(\overline{K}\right)\Leftrightarrow i\in I\ \right\}.\tag{2.4.2}$$

Here $\bar{\cdot}$ *denotes the reduction* mod P_K *for which we refer to [17, Sec. 2].*

2.4.3 Topological zeta functions

For any scheme V of finite type over a field $L\subset\mathbb{C}$, we denote by $\chi\left(V\right)$ the Euler characteristic of the $\mathbb{C}$-analytic space associated with V. Let $f_i(\boldsymbol{x})\in\mathbb{Q}\left[\boldsymbol{x}\right]$, $\boldsymbol{x}=(x_1,\ldots,x_n)$, be a non-constant polynomial for $i=1,\ldots,r$. To $\prod_{i=1}^{r}f_i(\boldsymbol{x})$ Denef and Loeser associated the topological zeta function

$$\boldsymbol{Z}_{top}\left(\boldsymbol{s}\right)=\sum_{I\subset T}\chi\left(\overset{\circ}{E}_I\right)\prod_{i\in I}\frac{1}{v_i+\sum_{j=1}^{r}N_{ij}s_j},\tag{2.4.3}$$

where the notation is an in Section 2.4.2 for a resolution of D over $\mathbb{Q}$. We mention that in arbitrary dimension there is not a canonical way of picking an embedded resolution of singularities for a divisor. Then, it is necessary to show that definition (2.4.3) is independent of the resolution of singularities chosen, this fact was established by Denef and Loeser in [47]. By using the explicit formula (2.4.1)-(2.4.2), Denef and Loeser showed that

$$\boldsymbol{Z}_{top}\left(\boldsymbol{s}\right) = \lim_{e\longrightarrow 0} \boldsymbol{Z}\left(\boldsymbol{s}, \boldsymbol{f}, K_e\right), \tag{2.4.4}$$

where K_e is the unramified extension of $\mathbb{Q}_p$ of degree e. The limit $e \rightarrow 0$ makes sense because one can l-adically interpolate $\boldsymbol{Z}\left(\boldsymbol{s}, \boldsymbol{f}, K_e\right)$ as a function of e. This means that there exist $\kappa \in \mathbb{N} \smallsetminus \{0\}$ and a meromorphic function in the variables e and $\boldsymbol{s}$, $\boldsymbol{Z}_l\left(\boldsymbol{s}, \boldsymbol{f}, e\right)$ on $\mathbb{Z}_l \times \mathbb{Z}_l^r$ such that for any $\boldsymbol{s} \in \mathbb{Z}^r$ and $e \in \mathbb{N}$ holds

$$\boldsymbol{Z}_l\left(\boldsymbol{s}, \boldsymbol{f}, e\right) = \boldsymbol{Z}\left(\boldsymbol{s}, \boldsymbol{f}, K_e\right).$$

In particular

$$\lim_{e\longrightarrow 0} c_I(K_e) = \chi_c\left(\overset{\circ}{E}_I \otimes \mathbb{F}^a_{p^e}, \mathbb{Q}_l\right) = \chi\left(\overset{\circ}{E}_I\right),$$

for almost all p, where χ_c denotes the Euler characteristic with respect to l-adic cohomology with compact support, and $\mathbb{F}^a_{p^e}$ denotes an algebraic closure of $\mathbb{F}_{p^e}$.

Remark 2.4 *The uniqueness of the topological zeta function $\boldsymbol{Z}_{top}\left(\boldsymbol{s}\right)$ follows from the theory established in [58], which is a generalization of the theory of topological zeta functions given by Denef and Loeser to the multivariate case.*

Theorem 2.5 *If $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}_{top}\left(\boldsymbol{s}\right)$, then for almost all p there exists infintely many unramified extensions L of $\mathbb{Q}_p$ for which $\boldsymbol{\rho}$ is a pole of $\boldsymbol{Z}\left(\boldsymbol{s}, \boldsymbol{f}, L\right)$.*

2.5 Topological String Zeta Functions and Topological string amplitudes

Since $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; I, 0, K_e\right)$ and $\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}}; T \smallsetminus I, 1, K_e\right)$ are 'both' multivariate local zeta functions of type $\boldsymbol{Z}\left(\boldsymbol{s}, \boldsymbol{f}, K_e\right)$ for suitable $\boldsymbol{f}$, for any $I \subseteq T = \{2, .., N-3\}$ we can apply (2.4.4), to define

$$\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right):=\lim_{e\rightarrow 0}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};I,0,K_{e}\right)\quad\text{and}$$

$$\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right):=\lim_{e\rightarrow 0}\boldsymbol{Z}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1,K_{e}\right),$$

which are elements of $\mathbb{Q}\left(s_{ij},i,j\in\{1,\ldots,N-1\}\right)$, the field of rational functions in the variables s_{ij}, $i,j\in\{1,\ldots,N-1\}$, with coefficients in $\mathbb{Q}$. Then by using (2.2.2) we define the *open string N-point topological zeta function as*

$$\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}\right)=\sum_{I\subseteq T}\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}};I,0\right)\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}};T\smallsetminus I,1\right)\in\mathbb{Q}\left(s_{ij},i,j\in\{1,\ldots,N-1\}\right) \tag{2.5.1}$$

Now, by applying Theorems 2.7, 2.5, we obtain that the possible poles of $\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ belong to a finite union of hyperplanes. Formally we have the following result:

Theorem 2.1 *The open string N-point topological zeta function $\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ is a rational function from $\mathbb{Q}\left(s_{ij},i,j\in\{1,\ldots,N-1\}\right)$ defined as (2.5.1). The real parts of the possible poles of $\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}\right)$ belong to a finite union of hyperplanes, the equations of these hyperplanes have the form C1-C6 with the symbols '$<$', '$>$' replaced by '$=$'.*

Definition 2.2 *We define the* topological string N-point tree amplitudes *as*

$$\boldsymbol{A}_{top}^{(N)}\left(\underline{\boldsymbol{k}}\right)=\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}\right)|_{s_{ij}=\boldsymbol{k}_{i}\boldsymbol{k}_{j}}\ \text{with } i\in\{1,\ldots,N-1\}\,,\ j\in T\ \text{or } i,j\in T,$$

where $T=\{2,\ldots,N-2\}$, which are rational functions of the variables $\boldsymbol{k}_i\boldsymbol{k}_j$.

2.6 The four and five-point topological zeta functions

For a prime number p, let K_e be the unique unramified extension of $\mathbb{Q}_p$ of degree e, let $\mathcal{O}_{K_e}$ denote the valuation ring of K_e and P_{K_e} its unique maximal ideal. Then $\overline{K}_e\cong\mathbb{F}_{p^e}$ the finite field with p^e elements. Thus,

$$\lim_{e\rightarrow 0}\frac{1-p^{-e}}{p^{e(1+s_{ij})}-1}=\frac{1}{1+s_{ij}}.$$

Let

$$\underline{\boldsymbol{k}} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N), \ \boldsymbol{k}_i = (k_{0,i}, \ldots, k_{25,i}), \ i = 1, \ldots, N, \ N \geq 4,$$

(with Minkowski product $\boldsymbol{k}_i \boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{25,i}k_{25,j}$) obeying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}, \ \boldsymbol{k}_i \boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N.$$

With the algorithms introduced in Chapter 1 and the relations between $\boldsymbol{k}_i's$ we compute the open string N-point topological zeta functions and the topological string N-point amplitudes for $N = 4, 5$.

2.6.1 Topological string 4-point tree amplitudes

In this section, we compute the open string 4-point topological zeta function, which is defined as

$$\boldsymbol{Z}^{(4)}(\underline{s}, K_e) = \int_{K_e} |x_2|_{K_e}^{s_{12}} |1 - x_2|_{K_e}^{s_{32}} dx_2.$$

By using formulas 1.3.2, 1.3.3, with $I \subseteq T = \{2\}$, we have

$$\begin{aligned}
\boldsymbol{Z}^{(4)}(\underline{s}, K_e) &= \int_{\mathcal{O}_{K_e}} |x_2|_{K_e}^{s_{12}} |1 - x_2|_{K_e}^{s_{32}} dx_2 + p^{1+s_{12}+s_{32}} \int_{\mathcal{O}_{K_e}} |x_2|_{K_e}^{-2-s_{12}-s_{32}} dx_2 \\
&= \boldsymbol{Z}^{(4)}(\underline{s}; \{2\}, 0, K_e) \boldsymbol{Z}^{(4)}(\underline{s}; \{\varnothing\}, 1, K_e) + \\
&\quad p^{e(1+s_{12}+s_{32})} \boldsymbol{Z}^{(4)}(\underline{s}; \{\varnothing\}, 0, K_e) \boldsymbol{Z}^{(4)}(\underline{s}; \{2\}, 1, K_e).
\end{aligned}$$

Recall that $\boldsymbol{Z}^{(4)}(\underline{s}; \{\varnothing\}, 1, K_e) = 1$, $\boldsymbol{Z}^{(4)}(\underline{s}; \{\varnothing\}, 0, K_e) = 1$.

By using the algorithms given in Propositions 1.20, 1.24, and Theorem 2.7, we obtain

$$\begin{aligned}
\boldsymbol{Z}^{(4)}(\underline{s}; \{2\}, 0, K_e) &= 1 - 2p^{-e} + \frac{(1 - p^{-e}) p^{e(-1-s_{12})}}{1 - p^{e(-1-s_{12})}} + \frac{(1 - p^{-e}) p^{e(-1-s_{32})}}{1 - p^{e(-1-s_{32})}} \\
\boldsymbol{Z}^{(4)}(\underline{s}; \{2\}, 1, K_e) &= \frac{(1 - p^{-e}) p^{e(1+s_{12}+s_{32})}}{1 - p^{e(1+s_{12}+s_{32})}}.
\end{aligned}$$

Applying the limit when p approaches to one,

$$\begin{aligned}\boldsymbol{Z}_{top}^{(4)}(\underline{s};\{2\},0) &= -1+\frac{1}{s_{12}+1}+\frac{1}{s_{32}+1}\\ \boldsymbol{Z}_{top}^{(4)}(\underline{s};\{2\},1) &= -\frac{1}{s_{12}+s_{32}+1},\end{aligned}$$

and

$$\boldsymbol{Z}_{top}^{(4)}(\underline{\boldsymbol{s}})=-1+\frac{1}{s_{12}+1}+\frac{1}{s_{32}+1}+\frac{1}{s_{12}+s_{32}+1}.$$

By using the relation $\boldsymbol{k}_1+...+\boldsymbol{k}_4=0$ and $\boldsymbol{k}_i^2=2$ we get $1+\boldsymbol{k}_1\boldsymbol{k}_2+\boldsymbol{k}_3\boldsymbol{k}_2=-1-\boldsymbol{k}_2\boldsymbol{k}_4$, thus the topological string 4-point tree amplitude

$$\begin{aligned}\boldsymbol{A}_{top}^{(4)}(\underline{\boldsymbol{k}}) &= Z_{top}^{(4)}(\underline{\boldsymbol{k}})\\ &= -1+\frac{1}{\boldsymbol{k}_1\boldsymbol{k}_2+1}+\frac{1}{\boldsymbol{k}_3\boldsymbol{k}_2+1}+\frac{1}{\boldsymbol{k}_2\boldsymbol{k}_4+1}.\end{aligned}$$

2.7 Topological string 5-point tree amplitudes

The open string 5-point topological zeta function is defined as

$$\boldsymbol{Z}^{(5)}(\underline{\boldsymbol{s}},K_e)=\int_{K_e^2}|x_2|_{K_e}^{s_{12}}|x_3|_{K_e}^{s_{13}}|1-x_2|_{K_e}^{s_{42}}|1-x_3|_{K_e}^{s_{43}}|x_2-x_3|_{K_e}^{s_{23}}dx_2dx_3.$$

The sector attached to $I\subseteq T=\{1,2\}$ is defined as

$$Sect(I)=\left\{(x_2,x_3)\in K_e^2:|x_i|_p=1\Longleftrightarrow i\in I\right\}.$$

I	I^c	$Sect(I)$
$\{2\}$	$\{3\}$	$\mathcal{O}_{K_e}\times K_e\backslash\mathcal{O}_{K_e}$
$\{3\}$	$\{2\}$	$K_e\backslash\mathcal{O}_{K_e}\times\mathcal{O}_{K_e}$
$\{2,3\}$	$\varnothing$	$\mathcal{O}_{K_e}\times\mathcal{O}_{K_e}$
$\varnothing$	$\{2,3\}$	$K_e\backslash\mathcal{O}_{K_e}\times K_e\backslash\mathcal{O}_{K_e}$

Then, the open string 5-point topological zeta function equals

$$\boldsymbol{Z}_{top}^{(5)}(\underline{\boldsymbol{s}})=\sum_{I\subseteq T}\boldsymbol{Z}_{top}^{(5)}(\underline{\boldsymbol{s}};I,0)\,\boldsymbol{Z}_{top}^{(5)}(\underline{\boldsymbol{s}};T\smallsetminus I,1)$$

where

I	$\boldsymbol{Z}_{top}^{(5)}\left(\underline{\boldsymbol{s}}; I, 0\right)$	$\boldsymbol{Z}_{top}^{(N)}\left(\underline{\boldsymbol{s}}; T \setminus I, 1\right)$
$\{2\}$	$-1 + \frac{1}{1+s_{12}} + \frac{1}{1+s_{42}}$	$-\frac{1}{1+s_{13}+s_{43}+s_{23}}$
$\{3\}$	$-1 + \frac{1}{1+s_{13}} + \frac{1}{1+s_{43}}$	$-\frac{1}{1+s_{12}+s_{42}+s_{23}}$
$\{2,3\}$	$\left[\frac{1}{1+s_{12}} + \frac{1}{1+s_{13}} + \frac{1}{1+s_{23}} - 1\right] \frac{1}{2+s_{12}+s_{13}+s_{23}}$ $+\frac{1}{1+s_{12}}\left[\frac{1}{1+s_{43}} - 1\right] + \frac{1}{1+s_{13}}\left[\frac{1}{1+s_{42}} - 1\right] +$ $2 - \frac{1}{1+s_{23}} - \frac{1}{1+s_{42}} - \frac{1}{1+s_{43}} +$ $\frac{1}{2+s_{42}+s_{43}+s_{23}}\left[\frac{1}{1+s_{42}} + \frac{1}{1+s_{43}} + \frac{1}{1+s_{23}} - 1\right]$	1
$\{\varnothing\}$	1	$-\frac{1}{2+s_{52}+s_{53}+s_{23}} \times$ $\left[\frac{1}{1+s_{12}+s_{42}+s_{23}} + \frac{1}{1+s_{13}+s_{43}+s_{23}} + \frac{1}{1+s_{23}} - 1\right]$

Thus, the topological string 5-point tree amplitude is

$$\boldsymbol{A}_{top}^{(5)}(\underline{\mathbf{k}}) = \left[\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_2} + \frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_2} - 1\right]\left[-\frac{1}{1+\boldsymbol{k}_3\boldsymbol{k}_5}\right] + \left[-\frac{1}{1+\boldsymbol{k}_2\boldsymbol{k}_5}\right]\left[\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_3} + \frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_3} - 1\right] +$$

$$\left[\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_2} + \frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_3} + \frac{1}{1+\boldsymbol{k}_2\boldsymbol{k}_3} - 1\right]\frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_5} + \frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_2}\left[\frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_3} - 1\right] +$$

$$\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_3}\left[\frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_2} - 1\right] + 2 - \frac{1}{1+\boldsymbol{k}_2\boldsymbol{k}_3} - \frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_2} - \frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_3} +$$

$$\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_5}\left[\frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_2} + \frac{1}{1+\boldsymbol{k}_4\boldsymbol{k}_3} + \frac{1}{1+\boldsymbol{k}_2\boldsymbol{k}_3} - 1\right] -$$

$$\frac{1}{1+\boldsymbol{k}_1\boldsymbol{k}_4}\left[\frac{1}{1+\boldsymbol{k}_5\boldsymbol{k}_2} + \frac{1}{1+\boldsymbol{k}_3\boldsymbol{k}_5} + \frac{1}{1+\boldsymbol{k}_2\boldsymbol{k}_3} - 1\right].$$

Chapter 3

Local zeta functions for rational functions and Newton polyhedra

In this chapter, we introduce a notion of non-degeneracy with respect to certain Newton polyhedra for polynomial mappings over non-Archimedean locals fields of arbitrary characteristic. Furthermore, we use this non-degeneracy to define non-degenerate rational functions over the same class of local fields. This definition allows us to study the local zeta functions attached to non-degenerate rational functions, we show the existence of a meromorphic continuation for these zeta functions as rational functions of q^{-s}, and give explicit formulas. In contrast with the classical local zeta functions, the meromorphic continuation of zeta functions for rational functions have poles with positive and negative real parts.

In Section 3.2 we compute some integrals that are needed in the chapter. In Section 3.3 we review some basic aspects about polyhedral subdivisions and Newton polyhedra, we also introduce a notion of non-degeneracy for polynomials mappings. It seems that our notion of non-degeneracy is a new one. In Section 3.4 we study the meromorphic continuation for multivariate local zeta functions attached to non-degenerate mappings.

3.1 Multivariate local zeta functions

A non-archimedean local field K is a locally compact topological field with respect to a non-discrete topology with an absolute value $|\cdot|_K$ satisfying

$$|x+y|_K \leq \max\{|x|_K, |y|_K\} \text{ for } x, y \in K,$$

i.e. $|\cdot|_K$ is ultrametric. For an in-depth exposition, the reader may consult [64], [55], see also [1], [63].

Let K be a non-Archimedean local field of arbitrary characteristic i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, and let $\mathcal{O}_K$ be the ring of valuation of K,

$$\mathcal{O}_K := \{x \in K : |x|_K \leq 1\},$$

and P_K the maximal ideal of $\mathcal{O}_K$; this ideal is formed by the non-units of $\mathcal{O}_K$. In terms of the absolute value $|\cdot|_K$, this maximal ideal can be described as

$$P_K = \{x \in K : |x|_K < 1\}.$$

Let $\overline{K} = \mathcal{O}_K / P_K$ the residue field of K. Thus $\overline{K} = \mathbb{F}_q$, the finite field with q elements. Let π be a fixed generator of P_K, π is called a uniformizing parameter of K, then $P_K = \pi\mathcal{O}_K$. Furthermore, we assume that $|\pi|_K = q^{-1}$. For $z \in K$, ord$(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, and $|z|_K = q^{-ord(z)}$. If $z \in K \setminus \{0\}$, then ac(z)=$z\pi^{-ord(z)}$ denotes the angular component of z.

With the above notation, let $\boldsymbol{h} = (h_1, \ldots, h_r) : K^n \to K^r$ be a polynomial mapping such that each $h_i(\boldsymbol{x})$ is a non-constant polynomial in $\mathcal{O}_K[\boldsymbol{x}] \setminus \pi\mathcal{O}_K[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $r \leq n$. Let $\boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. We define the multivariate local zeta function attached to $(\boldsymbol{s}, \boldsymbol{h})$ as

$$Z(\boldsymbol{s}, \boldsymbol{h}) = \int\limits_{\mathcal{O}_K^n \setminus D_K} \prod_{i=1}^{r} |h_i(\boldsymbol{x})|_K^{s_i} \, |d\boldsymbol{x}|_K$$

for $\mathrm{Re}(s_i) > 0$ for all i, where $D_K := \cup_{i \in \{1,\ldots,r\}} \{\boldsymbol{x} \in K^n; h_i(\boldsymbol{x}) = 0\}$. The multivariate local zeta functions were studied by Loeser in the case of local fields of characteristic zero. He showed that they admit analytic continuations to the whole $\mathbb{C}^r$ as rational functions of the variables q^{-s_i}, $i = 1, \ldots, r$, see Theorem [47].

Notation

Along this chapter, vectors will be written in boldface, so for instance we will write $\mathbf{b} := (b_1, \ldots, b_l)$ where l is a positive integer. For polynomials we will use $\boldsymbol{x} = (x_1, \ldots, x_n)$, thus $h(\boldsymbol{x}) = h(x_1, \ldots, x_n)$. For each n-tuple of natural numbers $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we will denote by $\sigma(\boldsymbol{k})$ the sum of all its components i.e. $\sigma(\boldsymbol{k}) = k_1 + k_2 + \ldots + k_n$. Furthermore, we will use the notation $|d\boldsymbol{x}|_K$ for the Haar measure on $(K^n, +)$ normalized so that the measure of $\mathcal{O}_K^n$ is equal to one. In dimension one, we will use the notation $|dx|_K$.

By $\overline{\boldsymbol{x}}$ we mean the image of an element of $\mathcal{O}_K^n$ under the canonical homomorphism $\mathcal{O}_K^n \rightarrow \mathcal{O}_K^n/(\pi\mathcal{O}_K)^n \cong \mathbb{F}_q^n$, we call $\overline{\boldsymbol{x}}$ *the reduction modulo* π of $\boldsymbol{x}$. Given $h(\boldsymbol{x}) \in \mathcal{O}_K[x_1, \ldots, x_n]$, we denote by $\overline{h}(\boldsymbol{x})$ the polynomial obtained by reducing modulo π the coefficients of $h(\boldsymbol{x})$. Furthermore if $\boldsymbol{h} = (h_1, \ldots, h_r)$ is a polynomial mapping with $h_i \in \mathcal{O}_K[x_1, \ldots, x_n]$ for all i, then $\overline{\boldsymbol{h}} := (\overline{h}_1, \ldots, \overline{h}_r)$ denotes the polynomial mapping obtained by reducing modulo π all the components of $\boldsymbol{h}$.

3.2 Some π-adic integrals

Let K be a non-Archimedean local field of arbitrary characteristic. Before we prove the meromorphic continuation of $Z(\boldsymbol{s}, \boldsymbol{h})$ as a rational function we present here some result that will be used later on. With the notation in Section 3.1, let $\boldsymbol{h} = (h_1, h_2, \ldots, h_r)$ be a polynomial mapping as above. For $\boldsymbol{a} \in (\mathcal{O}_K^\times)^n$, we set

$$J_{\boldsymbol{a}}(\boldsymbol{s}, \boldsymbol{h}) := \int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n \smallsetminus D_K} \prod_{i=1}^{r} |h_i(\boldsymbol{x})|_K^{s_i} |d\boldsymbol{x}|_K, \tag{3.2.1}$$

where $D_K := \cup_{i\in\{1,\ldots,r\}} \{\boldsymbol{x} \in K^n; h_i(\boldsymbol{x}) = 0\}$, $\boldsymbol{s} = (s_1, \ldots s_r) \in \mathbb{C}^r$ with $\mathrm{Re}(s_i) > 0$, $i = 1, \ldots, r$.

The Jacobian matrix of $\boldsymbol{h}$ at $\boldsymbol{a}$ is $Jac(\boldsymbol{h}, \boldsymbol{a}) = \left[\frac{\partial h_i}{\partial x_j}(\boldsymbol{a})\right]_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$ with $r \leq n$. In a similar way we define the Jacobian matrix of $\overline{\boldsymbol{h}}$ at $\overline{\boldsymbol{a}}$. For every non-empty subset I of $\{1, \ldots, r\}$ we set $Jac\left(\overline{\boldsymbol{h}}_I, \overline{\boldsymbol{a}}\right) := \left[\frac{\partial \bar{h}_i}{\partial x_j}(\overline{\boldsymbol{a}})\right]_{\substack{i\in I\\ 1\leq j\leq n}}$.

Lemma 3.1 *Let I be the subset of $\{1,\dots,r\}$ such that $\overline{h}_i(\overline{\boldsymbol{a}})=0 \Leftrightarrow i\in I$. Assume that $\boldsymbol{a}\notin D_K$ and that $Jac\left(\overline{\boldsymbol{h}}_I,\overline{\boldsymbol{a}}\right)$ has rank $m=Card(I)$ for $I\neq\varnothing$. Then $J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})$ equals*

$$\begin{cases} q^{-n} & \text{if } I=\varnothing \\ \\ q^{-n}\prod\limits_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} & \text{if } I\neq\varnothing. \end{cases}$$

Proof. By change of variables we get

$$J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})=q^{-n}\int\limits_{\mathcal{O}_K^n\smallsetminus\cup_{i\in\{1,\dots,r\}}\{\boldsymbol{x}\in K^n;h_i(\pi\boldsymbol{x}+\boldsymbol{a})=0\}}\prod_{i=1}^{r}|h_i(\pi\boldsymbol{x}+\boldsymbol{a})|_K^{s_i}|d\boldsymbol{x}|_K.$$

We first consider the case $I=\varnothing$. Then $h_i(\boldsymbol{a})\not\equiv 0 \bmod \pi$, thus $|h_i(\pi\boldsymbol{x}+\boldsymbol{a})|_K=1$, and $J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})=q^{-n}$. In the case $I\neq\varnothing$, by reordering I (if necessary) we can suppose that $I=\{1,\dots,m\}$ with $m\leq r$. Integral $J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})$ is computed by changing variables as $\boldsymbol{y}=\phi(\boldsymbol{x})$ with

$$\boldsymbol{y}_i=\phi_i(\boldsymbol{x}):=\begin{cases} \frac{h_i(\boldsymbol{a}+\pi\boldsymbol{x})-h_i(\boldsymbol{a})}{\pi} & \text{if } i=1,\dots,m \\ \\ x_i & \text{if } i\geq m+1. \end{cases}$$

By using that rank of $Jac(\overline{\boldsymbol{h}}_I,\overline{\boldsymbol{a}})$ is m we get that $det\left[\frac{\partial\phi_i}{\partial x_j}(\boldsymbol{0})\right]_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\not\equiv 0 \bmod \pi$, which implies that $\boldsymbol{y}=\phi(\boldsymbol{x})$ gives a measure-preserving map from $\mathcal{O}_K^n$ to itself (see e.g. [39, Lemma 7.4.3]), hence

$$J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})=q^{-n}\prod_{i=1}^{m}\int\limits_{\mathcal{O}_K\setminus\{\pi y_i+h_i(\boldsymbol{a})=0\}}|\pi y_i+h_i(\boldsymbol{a})|_K^{s_i}|dy_i|_K=:q^{-n}\prod_{i=1}^{m}J'_{\boldsymbol{a}}(y_i).$$

To prove the announced formula we compute integrals $J'_{\boldsymbol{a}}(y_i)$. Now, since $h_i(\boldsymbol{a})\equiv 0 \bmod \pi$, by taking $z_i=\pi y_i+h_i(\boldsymbol{a})$ in $J'_{\boldsymbol{a}}(y_i)$, we obtain

$$J'_{\boldsymbol{a}}(y_i)=q^{-s_i}\int\limits_{\mathcal{O}_K\setminus\{0\}}|z_i|_K^{s_i}|dz_i|_K=\frac{(q-1)q^{-1-si}}{1-q^{-1-s_i}}.$$

Therefore

$$J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})=\begin{cases}q^{-n} & I=\varnothing\\ q^{-n}\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} & I\neq\varnothing.\end{cases} \tag{3.2.2}$$

■

Remark 3.2 *If in integral (3.2.1), we replace $h_i(\boldsymbol{x})$ by $h_i(\boldsymbol{x})+\pi g_i(\boldsymbol{x})$, where each $g_i(\boldsymbol{x})$ is a polynomial with coefficients in $\mathcal{O}_K$, then the formulas given in Lemma 3.1 are valid.*

For every subset $I\subseteq\{1,\ldots,r\}$ we set

$$\overline{V}_I:=\left\{\overline{\boldsymbol{z}}\in(\mathbb{F}_q^{\times})^n;\ \overline{h}_i(\overline{\boldsymbol{z}})=0\Leftrightarrow i\in I\right\}. \tag{3.2.3}$$

To simplify the notation we will denote $\overline{V}_{\{1,\ldots,r\}}$ as $\overline{V}$.

Lemma 3.3 *Let $\boldsymbol{h}=(h_1,\ldots,h_r)$ with $r\leq n$, be as before. Assume that for all $I\neq\varnothing$ if $\overline{V}_I\neq\varnothing$, then*

$$rank_{\mathbb{F}_q}\left[\frac{\partial\overline{h}_i}{\partial x_j}(\overline{\boldsymbol{a}})\right]_{i\in I,\ j\in\{1,\ldots,n\}}=Card(I),\ \textit{for any}\ \overline{\boldsymbol{a}}\in\overline{V}_I.$$

Set

$$L(\boldsymbol{s},\boldsymbol{h}):=\int\limits_{(\mathcal{O}_K^{\times})^n\setminus D_K}\prod_{i=1}^{r}|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K,\ \ \boldsymbol{s}=(s_1,\ldots s_r)\in\mathbb{C}^r,$$

for $\operatorname{Re}(s_i)>0$ *for all i. Then, with the convention that $\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}=1$ when $I=\varnothing$, we have*

$$L(\boldsymbol{s},\boldsymbol{h})=q^{-n}\sum_{I\subseteq\{1,\ldots,r\}}Card(\overline{V}_I)\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}.$$

Proof. Note that $L(\boldsymbol{s},\boldsymbol{h})$ can be expressed as a finite sum of integrals

$$J_{\boldsymbol{a}}(\boldsymbol{s},\boldsymbol{h})=\int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\prod_{i=1}^{r}|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K,$$

where $\boldsymbol{a}$ runs through a fixed set of representatives $\mathcal{R}$ in $\left(\mathcal{O}_K^\times\right)^n$ of $(\mathbb{F}_q^\times)^n$. Then $L(\boldsymbol{s},\boldsymbol{h})$ is equals

$$\begin{aligned}
&\sum_{\overline{\boldsymbol{a}}\in\overline{V}_\varnothing}\ \int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\ \prod_{i=1}^r|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K\\
&+\sum_{\substack{I\subsetneq\{1,\dots,r\}\\ I\neq\varnothing}}\ \sum_{\overline{\boldsymbol{a}}\in\overline{V}_I}\ \int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\ \prod_{i=1}^r|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K\\
&+\sum_{\overline{\boldsymbol{a}}\in\overline{V}}\ \int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\ \prod_{i=1}^r|h_i(\boldsymbol{x})|_K^{s_i}\,|d\boldsymbol{x}|_K\\
&=: J(\boldsymbol{s},\overline{V}_\varnothing)+\sum_{\substack{I\subsetneq\{1,\dots,r\}\\ I\neq\varnothing}} J(\boldsymbol{s},\overline{V}_I)+J(\boldsymbol{s},\overline{V}),
\end{aligned}$$

with the convention that if $\overline{V}_I=\varnothing$, then $\sum_{\overline{\boldsymbol{a}}\in\overline{V}_I}\int_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\cdot=0$. Notice that

$$J(\boldsymbol{s},\overline{V}_\varnothing)=q^{-n}Card(\overline{V}_\varnothing). \tag{3.2.4}$$

Thus we may assume that $I\neq\varnothing$. In the calculation of $J(\boldsymbol{s},\overline{V}_I)$ we use the following result:

Claim.

$$\sum_{\overline{\boldsymbol{a}}\in\overline{V}_I}\ \int\limits_{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K}\ \prod_{i=1}^r|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K=\sum_{\overline{\boldsymbol{a}}\in\overline{V}_I}\ \int\limits_{\substack{\boldsymbol{a}+(\pi\mathcal{O}_K)^n\setminus D_K\\ \boldsymbol{a}\notin D_K}}\ \prod_{i=1}^r|h_i(\boldsymbol{x})|_K^{s_i}|d\boldsymbol{x}|_K.$$

The Claim follows from the following reasoning. The analytic mapping $h_1\cdots h_r:\boldsymbol{a}+(\pi\mathcal{O}_K)^n\to K$ is not identically zero, otherwise by [39, Lemma 2.1.3], the polynomial $(h_1\cdots h_r)(\boldsymbol{x})$ would be the constant polynomial zero contradicting the hypothesis that all the h_i's are non-constant polynomials. Hence there exists an element $\boldsymbol{b}\in\boldsymbol{a}+(\pi\mathcal{O}_K)^n$ such that $(h_1\cdots h_r)(\boldsymbol{b})\neq 0$. Finally, we use the fact that every point of a ball is its center, which implies that $\boldsymbol{a}+(\pi\mathcal{O}_K)^n=\boldsymbol{b}+(\pi\mathcal{O}_K)^n$.

By using Lemma 3.1,

$$J(\boldsymbol{s},\overline{V}_I)=q^{-n}Card(\overline{V}_I)\prod_{i\in I}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}. \tag{3.2.5}$$

The formula for $J(\boldsymbol{s},\overline{V})$ is a special case of formula (3.2.5):

$$J(\boldsymbol{s},\overline{V})=q^{-n}Card(\overline{V})\prod_{i\in\{1,\ldots,r\}}\frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}}. \tag{3.2.6}$$

■

Remark 3.4 *In integral $L(\boldsymbol{s},\boldsymbol{h})$ we can replace $\boldsymbol{h}$ by $\boldsymbol{h}+\pi\boldsymbol{g}$, where $\boldsymbol{g}$ is a polynomial mapping over $\mathcal{O}_K$, and the formulas given in Lemma 3.3 remain valid.*

3.3 Polyhedral Subdivisions of $\mathbb{R}_+^n$ and Non-degeneracy conditions

In this section we review, without proofs, some well-known results about Newton polyhedra and non-degeneracy conditions that we will use along this chapter. Our presentation follows closely [70], [57].

3.3.1 Newton polyhedra

We set $\mathbb{R}_+ := \{x\in\mathbb{R}; x\geqslant 0\}$. Let G be a non-empty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma=\Gamma(G)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{\boldsymbol{m}\in G}\left(\boldsymbol{m}+\mathbb{R}_+^n\right)$. For instance classically one associates a *Newton polyhedron* $\Gamma(h)$ *(at the origin)* to a polynomial function $h(\boldsymbol{x})=\sum_{\boldsymbol{m}}c_{\boldsymbol{m}}\boldsymbol{x}^{\boldsymbol{m}}$ ($\boldsymbol{x}=(x_1,\ldots,x_n)$, $h(\boldsymbol{0})=\boldsymbol{0}$), where $G=$supp$(h):=\{\boldsymbol{m}\in\mathbb{N}^n; c_{\boldsymbol{m}}\neq 0\}$. Further we will associate more generally a Newton polyhedron to a polynomial mapping.

We fix a Newton polyhedron Γ as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

Let H be the hyperplane $H=\{\boldsymbol{x}\in\mathbb{R}^n;\langle\boldsymbol{x},\mathbf{b}\rangle=c\}$, H determines two closed half-spaces

$$H^+=\{\boldsymbol{x}\in\mathbb{R}^n;\langle\boldsymbol{x},\mathbf{b}\rangle\geq c\}\text{ and }H^-=\{\boldsymbol{x}\in\mathbb{R}^n;\langle\boldsymbol{x},\mathbf{b}\rangle\leq c\}.$$

We say that H is a *supporting hyperplane* of $\Gamma(h)$ if $\Gamma(h) \cap H \neq \varnothing$ and $\Gamma(h)$ is contained in one of the two closed half-spaces determined by H. By a *proper face* τ of $\Gamma(h)$, we mean a non-empty convex set τ obtained by intersecting $\Gamma(h)$ with one of its supporting hyperplanes. By the *faces* of $\Gamma(h)$ we will mean the proper faces of $\Gamma(h)$ and the whole the polyhedron $\Gamma(h)$. By *dimension of a face* τ of $\Gamma(h)$ we mean the dimension of the affine hull of τ, and its *codimension* is $\mathrm{cod}(\tau) = n - \dim(\tau)$, where $\dim(\tau)$ denotes the dimension of τ. A face of codimension one is called a *facet*.

For $\boldsymbol{a} \in \mathbb{R}_+^n$, we define

$$d(\boldsymbol{a}, \Gamma) = \min_{x \in \Gamma} \langle \boldsymbol{a}, \boldsymbol{x} \rangle ,$$

and *the first meet locus* $F(\boldsymbol{a}, \Gamma)$ of $\boldsymbol{a}$ as

$$F(\boldsymbol{a}, \Gamma) := \{\boldsymbol{x} \in \Gamma; \langle \boldsymbol{a}, \boldsymbol{x} \rangle = d(\boldsymbol{a}, \Gamma)\}.$$

The first meet locus is a face of Γ. Moreover, if $\boldsymbol{a} \neq \boldsymbol{0}$, $F(\boldsymbol{a}, \Gamma)$ is a proper face of Γ.

If $\Gamma = \Gamma(h)$, we define the *face function* $h_{\boldsymbol{a}}(\boldsymbol{x})$ *of* $h(\boldsymbol{x})$ *with respect to* $\boldsymbol{a}$ as

$$h_{\boldsymbol{a}}(\boldsymbol{x}) = h_{F(\boldsymbol{a},\Gamma)}(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in F(\boldsymbol{a},\Gamma)} c_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}}.$$

In the case of functions having subindices, say $h_i(\boldsymbol{x})$, we will use the notation $h_{i,\boldsymbol{a}}(\boldsymbol{x})$ for the face function of $h_i(\boldsymbol{x})$ with respect to $\boldsymbol{a}$. Notice that

$$h_{\boldsymbol{0}}(\boldsymbol{x}) = h_{F(\boldsymbol{0},\Gamma)}(\boldsymbol{x}) = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}}.$$

3.3.2 Polyhedral Subdivisions Subordinate to a Polyhedron

We define an equivalence relation in $\mathbb{R}_+^n$ by taking $\boldsymbol{a} \sim \boldsymbol{a}' \Leftrightarrow F(\boldsymbol{a}, \Gamma) = F(\boldsymbol{a}', \Gamma)$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{\boldsymbol{a} \in \mathbb{R}_+^n; F(\boldsymbol{a}, \Gamma) = \tau\},$$

where τ is a face of Γ.

We recall that the *cone strictly spanned* by the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l \in \mathbb{R}_+^n \backslash \{0\}$ is the set $\Delta = \{\lambda_1 \boldsymbol{a}_1 + ... + \lambda_l \boldsymbol{a}_l; \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$. If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone.* If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l \in \mathbb{Z}^n$, we say Δ is a *rational cone.* If $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_l\}$ is a subset of a basis of the $\mathbb{Z}$-modulo $\mathbb{Z}^n$, we call Δ a *simple cone.*

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* γ of Γ has a unique vector $\boldsymbol{a}(\gamma) = (a_{\gamma,1}, \ldots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime, which is perpendicular to γ, it is called primitive vector associated to γ. We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \{\sum_{i=1}^{r} \lambda_i \boldsymbol{a}(\gamma_i); \lambda_i \in \mathbb{R}_+,\ \lambda_i > 0\},$$

where τ runs through the set of faces of Γ, and γ_i, $i = 1, \ldots, r$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ, is a partition of $\mathbb{R}^n_+ \setminus \{0\}$; we call this partition a *polyhedral subdivision of* $\mathbb{R}^n_+$ *subordinate* to Γ. We call $\left\{\overline{\Delta_\tau}\right\}_\tau$, the family formed by the topological closures of the Δ_τ, a *fan subordinate* to Γ.

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}^n_+ \setminus \{0\}$:

$$\mathbb{R}^n_+ \setminus \{0\} = \bigcup_\tau \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right),$$

where τ runs over the proper faces of Γ, and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ. We will say that $\{\Delta_{\tau,i}\}$ is a *simplicial polyhedral subdivision of* $\mathbb{R}^n_+$ *subordinate* to Γ, and that $\left\{\overline{\Delta_{\tau,i}}\right\}$ is a *simplicial fan subordinate* to Γ.

By adding new rays , each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision* of $\mathbb{R}^n_+$ *subordinate* to Γ, and a *simple fan subordinate* to Γ (or a *complete regular fan*) (see e.g. [40]).

3.3.3 The Newton polyhedron associated to a polynomial mapping

Let $\boldsymbol{h} = (h_1, \ldots, h_r)$, $\boldsymbol{h}(\mathbf{0}) = 0$, be a non-constant polynomial mapping. In this section we associate to $\boldsymbol{h}$ a Newton polyhedron $\Gamma(\boldsymbol{h}) := \Gamma\left(\prod_{i=1}^r h_i(\boldsymbol{x})\right)$. From a geometrical point of view, $\Gamma(\boldsymbol{h})$ is the Minkowski sum of the $\Gamma(h_i)$, for $i = 1, \cdots, r$, (see e.g. [57], [59]). By using the results previously presented, we can associate to $\Gamma(\boldsymbol{h})$ a simplicial polyhedral subdivision $\mathcal{F}(\boldsymbol{h})$ of $\mathbb{R}_+^n$ subordinate to $\Gamma(\boldsymbol{h})$.

Remark 3.1 *A basic fact about the Minkowski sum operation is the additivity of the faces. From this fact follows:*
(1) $F(\boldsymbol{a}, \Gamma(\boldsymbol{h})) = \sum_{j=1}^r F(\boldsymbol{a}, \Gamma(h_j))$, *for* $\boldsymbol{a} \in \mathbb{R}_+^n$ *;*
(2) $d(\boldsymbol{a}, \Gamma(\boldsymbol{h})) = \sum_{j=1}^r d(\boldsymbol{a}, \Gamma(h_j))$, *for* $\boldsymbol{a} \in \mathbb{R}_+^n$ *;*
(3) let τ *be a proper face of* $\Gamma(\boldsymbol{h})$, *and let* τ_j *be proper face of* $\Gamma(h_j)$, *for* $i = 1, \cdots, r$. *If* $\tau = \sum_{j=1}^r \tau_j$, *then* $\Delta_\tau \subseteq \overline{\Delta}_{\tau_j}$, *for* $i = 1, \cdots, r$.

Remark 3.2 *Note that the equivalence relation,*

$$\boldsymbol{a} \sim \boldsymbol{a}' \Leftrightarrow F(a, \Gamma(\boldsymbol{h})) = F(\boldsymbol{a}', \Gamma(\boldsymbol{h})),$$

used in the construction of a polyhedral subdivision of $\mathbb{R}_+^n$ *subordinate to* $\Gamma(\boldsymbol{h})$ *can be equivalently defined in the following form:*

$$\boldsymbol{a} \sim \boldsymbol{a}' \Leftrightarrow F(\boldsymbol{a}, \Gamma(h_j)) = F(\boldsymbol{a}', \Gamma(h_j)), \text{ for each } j = 1, \ldots, r.$$

This last definition is used in Oka's book [57].

3.3.4 Non-degeneracy Conditions

For $K = \mathbb{Q}_p$, Denef and Hoornaert in [20, Theorem 4.2] gave an explicit formula for $Z(\boldsymbol{s}, \boldsymbol{h})$, in the case $r = 1$ with $\boldsymbol{h}$ a non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(\boldsymbol{h})$. This explicit formula can be generalized to the case $r \geq 1$ by using the condition of non-degeneracy for polynomial mappings introduced here.

Definition 3.3 *Let $\boldsymbol{h} = (h_1, \ldots, h_r)$, $\boldsymbol{h}(\mathbf{0}) = 0$, be a polynomial mapping with $r \leq n$ as in Section 3.1 and let $\Gamma(\boldsymbol{h})$ be the Newton polyhedron of $\boldsymbol{h}$ at the origin. The mapping $\boldsymbol{h}$ is called non-degenerate over $\mathbb{F}_q$ with respect to $\Gamma(\boldsymbol{h})$, if for every vector $\boldsymbol{k} \in \mathbb{R}_+^n$ and for any non-empty subset $I \subseteq \{1, \ldots, r\}$, it verifies that*

$$rank_{\mathbb{F}_q} \left[\frac{\partial \overline{h}_{i,\boldsymbol{k}}}{\partial x_j} (\overline{\boldsymbol{z}}) \right]_{i \in I,\ j \in \{1,\ldots,n\}} = Card(I) \tag{3.3.1}$$

for any

$$\overline{\boldsymbol{z}} \in \left\{ \overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^\times \right)^n ; \overline{h}_{i,\boldsymbol{k}}(\overline{\boldsymbol{z}}) = 0 \Leftrightarrow i \in I \right\}. \tag{3.3.2}$$

We notice that above notion is different to the those introduced in [60], [70]. The notion introduced here is similar to the usual notion given by Khovansky, see [42], [57]. For a discussion about the relation between Khovansky's non-degeneracy notion and other similar notions we refer the reader to [60].

Let Δ be a rational simplicial cone spanned by $\boldsymbol{w}_i$, $i = 1, \ldots, e_\Delta$. We define the *barycenter* of Δ as $b(\Delta) = \sum_{i=1}^{e_\Delta} \boldsymbol{w}_i$. Set $b(\{\mathbf{0}\}) := \mathbf{0}$.

Remark 3.4 *(i)Let $\mathcal{F}(\boldsymbol{h})$ be a simplicial polyhedral subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma(\boldsymbol{h})$. Then, it is sufficient to verify the condition given in Definition 3.3 for $\boldsymbol{k} = b(\Delta)$ with $\Delta \in \mathcal{F}(\boldsymbol{h}) \cup \{\mathbf{0}\}$.*

(ii) Notice that our notion of non-degeneracy agrees, in the case $K = \mathbb{Q}_p$, $r = 1$, with the corresponding notion in [20].

Example 3.5 *Set $\boldsymbol{h} = (h_1, h_2)$ with $h_1(x, y) = x^2 - y$, $h_2(x, y) = x^2 y$ polynomials in $\mathcal{O}_K[x, y]$. Then a simplicial polyhedral subdivision subordinate to $\Gamma(\boldsymbol{h})$ is given by*

Cone	$h_{1,b(\Delta)}$	$h_{2,b(\Delta)}$
$\Delta_1 := (1,0)\mathbb{R}_{>0}$	y	x^2y
$\Delta_2 := (1,0)\mathbb{R}_{>0} + (1,2)\mathbb{R}_{>0}$	y	x^2y
$\Delta_3 := (1,2)\mathbb{R}_{>0}$	$x^2 - y$	x^2y
$\Delta_4 := (1,2)\mathbb{R}_{>0} + (0,1)\mathbb{R}_{>0}$	x^2	x^2y
$\Delta_5 := (0,1)\mathbb{R}_{>0}$	x^2	x^2y,

where $\mathbb{R}_{>0} := \mathbb{R}_+ \smallsetminus \{\mathbf{0}\}$. *Notice that for every* $\boldsymbol{k} \in \mathbb{R}_+^n \smallsetminus (\{\mathbf{0}\} \cup \Delta_3)$ *and every non-empty subset* $I \subseteq \{1,2\}$, *the subset defined in (3.3.2) is empty, thus (3.3.1) is always satisfied. In the case* $\boldsymbol{k} = \mathbf{0}$ *and* $\boldsymbol{k} \in \Delta_3$, $h_{1,\boldsymbol{k}} = x^2 - y$, $h_{2,\boldsymbol{k}} = x^2 y$, *the conditions (3.3.2)-(3.3.1) are also verified. Hence* $\boldsymbol{h}$ *is non-degenerate over* $\mathbb{F}_q$ *with respect to* $\Gamma(\boldsymbol{h})$.

Example 3.6 *Let* $\boldsymbol{h} = (h_1(\boldsymbol{x}), \ldots, h_r(\boldsymbol{x}))$ *be a monomial mapping. In this case,* $\Gamma(\boldsymbol{h}) = \boldsymbol{m}_0 + \mathbb{R}_+^n$ *for some nonzero vector* $\boldsymbol{m}_0$ *in* $\mathbb{N}^n$. *Then for every vector* $\boldsymbol{k} \in \mathbb{R}_+^n$ $h_{i,\boldsymbol{k}}(\boldsymbol{x}) = h_i(\boldsymbol{x})$ *for* $i = 1, \ldots, r$, *and thus the subset in (3.3.2) is always empty, which implies that condition (3.3.1) is always satisfied. Therefore any monomial mapping (with* $r \leq n$*) is non-degenerate over* $\mathbb{F}_q$ *with respect to its Newton polyhedron.*

Example 3.7 $f(\boldsymbol{x}),\ g(\boldsymbol{x}) \in \mathcal{O}_K[x_1, ..., x_n] \backslash \pi\mathcal{O}_K[x_1, ..., x_n]$ *such that* $g(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{m}_0}$, *with* $\boldsymbol{m}_0 \neq \mathbf{0}$, *is a monomial. Suppose that* f *is non-degenerate with respect to* $\Gamma(f)$ *over* $\mathbb{F}_q$. *In this case,* $\Gamma((f,g)) = \boldsymbol{m}_0 + \Gamma(f)$. *Then the subset in (3.3.2) can take three different forms:*

$$\textit{(i)}\ \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}_{\boldsymbol{k}}(\overline{\boldsymbol{z}}) = \overline{g}\left(\overline{\boldsymbol{z}}\right) = 0\right\} = \varnothing,\ \textit{(ii)} \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}_{\boldsymbol{k}}(\overline{\boldsymbol{z}}) = 0\right\},$$
$$\textit{(iii)} \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{g}\left(\overline{\boldsymbol{z}}\right) = 0, \overline{f}_{\boldsymbol{k}}(\overline{\boldsymbol{z}}) \neq 0\right\} = \varnothing.$$

In the second case, conditions (3.3.2)-(3.3.1) are verified due to the hypothesis that f *is non-degenerate with respect* $\Gamma(f)$ *over* $\mathbb{F}_q$. *Hence,* (f,g) *is a non-degenerate mapping over* $\mathbb{F}_q$ *with respect to* $\Gamma((f,g))$ *over* $\mathbb{F}_q$.

3.4 Meromorphic continuation of multivariate local zeta functions

Along this section, we work with a fix simplicial polyhedral subdivision $\mathcal{F}(\boldsymbol{h})$ subordinate to $\Gamma(\boldsymbol{h})$. Let $\Delta \in \mathcal{F}(\boldsymbol{h}) \cup \{\mathbf{0}\}$ and $I \subseteq \{1, \ldots, r\}$, we put

$$\overline{V}_{\Delta,I} := \left\{\overline{\boldsymbol{z}} \in (\mathbb{F}_q^{\times})^n;\ \overline{h}_{i,b(\Delta)}(\overline{\boldsymbol{z}}) = 0\ \Leftrightarrow\ i \in I\right\}.$$

We use the convention $\overline{V}_{\Delta,\{1,\ldots,r\}} = \overline{V}_{\Delta}$. If $\Delta = \mathbf{0}$, then

$$\overline{V}_{\mathbf{0},I} = \left\{\overline{\boldsymbol{z}} \in (\mathbb{F}_q^{\times})^n;\ \overline{h}_i(\overline{\boldsymbol{z}}) = 0 \ \Leftrightarrow\ i \in I\right\} = \overline{V}_I,$$

where $\overline{V}_I$ is the set defined in (3.2.3). In particular, $\overline{V}_{\mathbf{0},\{1,\ldots,r\}} = \overline{V}$ and

$$\overline{V}_{\mathbf{0},\varnothing} = \left\{\overline{\boldsymbol{z}} \in (\mathbb{F}_q^{\times})^n;\ \overline{h}_i(\overline{\boldsymbol{z}}) \neq 0,\ i = 1,\ldots,r\right\} = \overline{V}_{\varnothing}.$$

If $\boldsymbol{h} = (h_1,\ldots,h_r)$ is non-degenerated polynomial mapping over $\mathbb{F}_q$ with respect to $\Gamma(\boldsymbol{h})$, then Lemma 3.3 is true for $\boldsymbol{h}_{b(\Delta)} = (h_{1,b(\Delta)},\ldots,h_{r,b(\Delta)})$.

Theorem 3.1 *Assume that $\boldsymbol{h} = (h_1,\ldots,h_r)$ is non-degenerated polynomial mapping over $\mathbb{F}_q$ with respect to $\Gamma(\boldsymbol{h})$, with $r \leq n$ as before. Fix a simplicial polyhedral subdivision $\mathcal{F}(\boldsymbol{h})$ subordinate to $\Gamma(\boldsymbol{h})$. Then $Z(\mathbf{s},\boldsymbol{h})$ has a meromorphic continuation to $\mathbb{C}^r$ as a rational function in the variables q^{-s_i}, $i = 1,\ldots,r$. In addition, the following explicit formula holds:*

$$Z(\boldsymbol{s},\boldsymbol{h}) = L_{\{\mathbf{0}\}}(\boldsymbol{s},\boldsymbol{h}) + \sum_{\Delta\,\in\,\mathcal{F}(\boldsymbol{h})} L_{\Delta}(\boldsymbol{s},\boldsymbol{h}) S_{\Delta},$$

where

$$L_{\{\mathbf{0}\}} = q^{-n} \sum_{I\subseteq\{1,\ldots,r\}} Card(\overline{V}_I) \prod_{i\in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$

$$L_{\Delta} = q^{-n} \sum_{I\subseteq\{1,\ldots,r\}} Card(\overline{V}_{\Delta,I}) \prod_{i\in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}},$$

with the convention that for $I = \varnothing$, $\prod_{i\in I} \frac{(q-1)q^{-1-s_i}}{1-q^{-1-s_i}} := 1$, and

$$S_{\Delta} = \sum_{\boldsymbol{k}\in\mathbb{N}^n\cap\Delta} q^{-\sigma(\boldsymbol{k})-\sum_{i=1}^{r} d(\boldsymbol{k},\Gamma(h_i))s_i}.$$

Let Δ be the cone strictly positively generated by linearly independent vectors $\boldsymbol{w}_1,\ldots,$ $\boldsymbol{w}_l \in \mathbb{N}^n\setminus\{\mathbf{0}\}$, then

$$S_{\Delta} = \frac{\sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t})-\sum_{i=1}^{r} d(\boldsymbol{t},\Gamma(h_i))s_i}}{\left(1-q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^{r} d(\boldsymbol{w}_1,\Gamma(h_i))s_i}\right)\cdots\left(1-q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^{r} d(\boldsymbol{w}_l,\Gamma(h_i))s_i}\right)},$$

where $\boldsymbol{t}$ runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{\sum_{i=1}^{l} \lambda_i \boldsymbol{w}_i;\ 0 < \lambda_i \leq 1 \text{ for } i = 1,\ldots,l\right\}. \tag{3.4.1}$$

Proof. By using the simplicial polyhedral subdivision $\mathcal{F}(\boldsymbol{h})$, we have

$$\mathbb{R}_+^n = \{\mathbf{0}\} \bigsqcup \bigsqcup_{\Delta \in \mathcal{F}(\boldsymbol{h})} \Delta.$$

We set for $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$,

$$E_{\boldsymbol{k}} := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n; ord(x_i) = k_i, i = 1, \ldots, n\}.$$

Hence

$$Z(\boldsymbol{s}, \boldsymbol{h}) = \int\limits_{(\mathcal{O}_K^\times)^n \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |d\boldsymbol{x}|_K + \sum_{\Delta\, \in\, \mathcal{F}(\boldsymbol{h})} \sum_{\boldsymbol{k} \in \mathbb{N}^n \cap \Delta} \int\limits_{E_{\boldsymbol{k}} \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |d\boldsymbol{x}|_K.$$

For $\Delta \in \mathcal{F}(\boldsymbol{h})$, $\boldsymbol{k} \in \mathbb{N}^n \cap \Delta$, and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E_{\boldsymbol{k}}$, we put $x_j = \pi^{k_j} u_j$ with $u_j \in \mathcal{O}_K^\times$. Then

$$|d\boldsymbol{x}|_K = q^{-\sigma(\boldsymbol{k})} |d\boldsymbol{u}|_K \text{ and } \boldsymbol{x}^{\boldsymbol{m}} = x_1^{m_1} \cdots x_n^{m_n} = \pi^{\langle \boldsymbol{k}, \boldsymbol{m} \rangle} u_1^{m_1} \cdots u_n^{m_n}.$$

Fix $i \in \{1, \ldots, r\}$ and $\boldsymbol{k}$. For $\boldsymbol{m} \in supp(h_i)$, the scalar product $\langle \boldsymbol{k}, \boldsymbol{m} \rangle$ attains its minimum $d(\boldsymbol{k}, \Gamma(h_i))$ exactly when $\boldsymbol{m} \in F(\boldsymbol{k}, \Gamma(h_i))$, and thus $\langle \boldsymbol{k}, \boldsymbol{m} \rangle \geq d(\boldsymbol{k}, \Gamma(h_i)) + 1$ for $\boldsymbol{m} \in supp(h_i) \setminus supp(h_i) \cap F(\boldsymbol{k}, \Gamma(h_i))$. This fact implies that

$$\begin{aligned} h_i(\boldsymbol{x}) &= \pi^{d(\boldsymbol{k}, \Gamma(h_i))} (h_{i,\boldsymbol{k}}(\boldsymbol{u}) + \pi \widetilde{h}_{i,\boldsymbol{k}}(\boldsymbol{u})) \\ &= \pi^{d(\boldsymbol{k}, \Gamma(h_i))} (h_{i,b(\Delta)}(\boldsymbol{u}) + \pi \widetilde{h}_{i,\boldsymbol{k}}(\boldsymbol{u})), \end{aligned}$$

where $\widetilde{h}_{i,\boldsymbol{k}}(\boldsymbol{u})$ is a polynomial over $\mathcal{O}_K$ in the variables $u_1, \ldots, u_n$. Note that $h_{i,\boldsymbol{k}}(\boldsymbol{u})$ does not depend on $\boldsymbol{k} \in \Delta$, for this reason we take $h_{i,\boldsymbol{k}}(\boldsymbol{u}) = h_{i,b(\Delta)}(\boldsymbol{u})$. Therefore

$$Z(\boldsymbol{s}, \boldsymbol{h}) = L_{\{\mathbf{0}\}}(\boldsymbol{s}, \boldsymbol{h}) + \sum_{\Delta\, \in\, \mathcal{F}(\boldsymbol{h})} L_\Delta(\boldsymbol{s}, \boldsymbol{h}) \sum_{\boldsymbol{k} \in \mathbb{N}^n \cap \Delta} q^{-\sigma(\boldsymbol{k}) - \sum_{i=1}^r d(\boldsymbol{k}, \Gamma(h_i)) s_i}$$

where

$$L_{\{\mathbf{0}\}}(\boldsymbol{s}, \boldsymbol{h}) := \int\limits_{(\mathcal{O}_K^\times)^n \setminus D_K} \prod_{i=1}^r |h_i(x)|_K^{s_i} |d\boldsymbol{x}|_K,$$

$$L_\Delta(\boldsymbol{s}, \boldsymbol{h}) := \int\limits_{(\mathcal{O}_K^\times)^n \setminus D_\Delta} \prod_{i=1}^r |h_{i,b(\Delta)}(\boldsymbol{u}) + \pi \widetilde{h}_{i,\boldsymbol{k}}(\boldsymbol{u})|_K^{s_i} |d\boldsymbol{u}|_K$$

with $D_\Delta = \bigcup_{i=1}^r \left\{ \boldsymbol{x} \in (\mathcal{O}_K^\times)^n; h_{i,b(\Delta)}(\boldsymbol{u}) + \pi \widetilde{h}_{i,\boldsymbol{k}}(\boldsymbol{u}) = 0 \right\}$. By using the non-degeneracy condition, integrals $L_{\{\boldsymbol{0}\}}(\boldsymbol{s},\boldsymbol{h})$, $L_\Delta(\boldsymbol{s},\boldsymbol{h})$ can be computed using Lemma 3.3 and Remarks 3.2, 3.4.

Let Δ be the cone strictly positively generated by linearly independent vectors $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_l \in \mathbb{N}^n \setminus \{\boldsymbol{0}\}$. If Δ is a simple cone then $\mathbb{N}^n \cap \Delta = (\mathbb{N}\setminus\{0\})\,\boldsymbol{w}_1 + \cdots + (\mathbb{N}\setminus\{0\})\,\boldsymbol{w}_l$. By using that the functions $d(\cdot,\Gamma(h_i)$ are linear over each cone Δ, and that

$$\sigma(\boldsymbol{w}_m) + \sum_{i=1}^r d(\boldsymbol{w}_m,\Gamma(h_i))\operatorname{Re}(s_i) > 0, m = 1,\ldots,l,$$

since $\operatorname{Re}(s_1),\ldots,\operatorname{Re}(s_r) > 0$, we obtain

$$\begin{aligned} S_\Delta &= \sum_{\lambda_1,\ldots,\lambda_l \in \mathbb{N}\setminus\{0\}} q^{-\sigma(\lambda_1\boldsymbol{w}_1+\ldots+\lambda_l\boldsymbol{w}_l)-\sum_{i=1}^r d(\lambda_1\boldsymbol{w}_1+\ldots+\lambda_l\boldsymbol{w}_l,\Gamma(h_i))s_i} \\ &= \sum_{\lambda_1=1}^\infty \left(q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^r d(\boldsymbol{w}_1,\Gamma(h_i))s_i}\right)^{\lambda_1} \cdots \sum_{\lambda_l=1}^\infty \left(q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^r d(\boldsymbol{w}_l,\Gamma(h_i))s_i}\right)^{\lambda_l} \\ S_\Delta &= \frac{q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^r d(\boldsymbol{w}_1,\Gamma(h_i))s_i}}{1-q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^r d(\boldsymbol{w}_1,\Gamma(h_i))s_i}} \cdots \frac{q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^r d(\boldsymbol{w}_l,\Gamma(h_i))s_i}}{1-q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^r d(\boldsymbol{w}_l,\Gamma(h_i))s_i}} \\ &= \frac{\sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t})-\sum_{i=1}^r d(\boldsymbol{t},\Gamma(h_i))s_i}}{\left(1-q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^r d(\boldsymbol{w}_1,\Gamma(h_i))s_i}\right)\cdots\left(1-q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^r d(\boldsymbol{w}_l,\Gamma(h_i))s_i}\right)}, \end{aligned}$$

where $\boldsymbol{t}$ runs through the elements of the set (3.4.1), which consists exactly of one element: $\boldsymbol{t} = \sum_{i=1}^l \boldsymbol{w}_i$. We now consider the case in which Δ is a simplicial cone. Note that $\mathbb{N}^n \cap \Delta$ is the disjoint union of the sets

$$\boldsymbol{t} + \mathbb{N}\boldsymbol{w}_1 + \cdots + \mathbb{N}\boldsymbol{w}_l,$$

where $\boldsymbol{t}$ runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{i=1}^l \lambda_i \boldsymbol{w}_i;\ 0 < \lambda_i \leq 1 \text{ for } i = 1,\ldots,l \right\}.$$

Hence S_Δ equals

$$\sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t})-\sum_{i=1}^r d(\boldsymbol{t},\Gamma(h_i))s_i} \sum_{\lambda_1,\ldots,\lambda_l \in \mathbb{N}} q^{-\sigma(\sum_{j=1}^l \lambda_j\boldsymbol{w}_j)-\sum_{i=1}^r d(\lambda_1\boldsymbol{w}_1+\ldots+\lambda_l\boldsymbol{w}_l,\Gamma(h_i))s_i},$$

and since $\mathrm{Re}(s_1), \ldots, \mathrm{Re}(s_r) > 0$,

$$S_\Delta = \frac{\sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t})-\sum_{i=1}^{r} d(\boldsymbol{t},\Gamma(h_i))s_i}}{\left(1-q^{-\sigma(\boldsymbol{w}_1)-\sum_{i=1}^{r} d(\boldsymbol{w}_1,\Gamma(h_i))s_i}\right)\cdots\left(1-q^{-\sigma(\boldsymbol{w}_l)-\sum_{i=1}^{r} d(\boldsymbol{w}_l,\Gamma(h_i))s_i}\right)}.$$

■

Remark 3.2 *In the p-adic case, $K = \mathbb{Q}_p$, Theorem 3.1 is a generalization of Theorem 4.2 in [20] and Theorem 4.3 in [13].*

3.5 Local zeta function for rational functions

From now on, we fix two non-constant relatively prime polynomials $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ in n variables, $n \geq 2$, with coefficients in $\mathcal{O}_K[x_1, \ldots, x_n] \backslash \pi\mathcal{O}_K[x_1, \ldots, x_n]$ and set

$$D_K := \{\boldsymbol{x} \in K^n; f(\boldsymbol{x}) = 0\} \cup \{\boldsymbol{x} \in K^n; g(\boldsymbol{x}) = 0\}$$

and

$$\frac{f}{g} : K^n \smallsetminus D_K \to K.$$

Furthermore, we define the *Newton polyhedron* $\Gamma\left(\frac{f}{g}\right)$ of $\frac{f}{g}$ to be $\Gamma(fg)$, and assume that the mapping $(f, g) : K^n \to K^2$ is non-degenerate over $\mathbb{F}_q$ with respect to $\Gamma\left(\frac{f}{g}\right)$ as before. In this case we will say that $\frac{f}{g}$ *is non-degenerate over $\mathbb{F}_q$ with respect to* $\Gamma\left(\frac{f}{g}\right)$. We fix a simplicial polyhedral subdivision $\mathcal{F}\left(\frac{f}{g}\right)$ of $\mathbb{R}_+^n$ subordinate to $\Gamma\left(\frac{f}{g}\right)$. For $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\boldsymbol{0}\}$, we put

$$N_{\Delta,\{f\}} := Card\left\{\overline{\boldsymbol{a}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\boldsymbol{a}}) = 0 \text{ and } \overline{g}_{b(\Delta)}(\overline{\boldsymbol{a}}) \neq 0\right\},$$

$$N_{\Delta,\{g\}} := Card\left\{\overline{\boldsymbol{a}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\boldsymbol{a}}) \neq 0 \text{ and } \overline{g}_{b(\Delta)}(\overline{\boldsymbol{a}}) = 0\right\},$$

$$N_{\Delta,\{f,g\}} := Card\left\{\overline{\boldsymbol{a}} \in (\mathbb{F}_q^\times)^n; \overline{f}_{b(\Delta)}(\overline{\boldsymbol{a}}) = 0 \text{ and } \overline{g}_{b(\Delta)}(\overline{\boldsymbol{a}}) = 0\right\},$$

with the convention that if $b(\Delta) = b(\boldsymbol{0}) = \boldsymbol{0}$, then $f_{b(\Delta)} = f$ and $g_{b(\Delta)} = g$. We also define $\mathfrak{D}\left(\frac{f}{g}\right) = \mathfrak{D}(f, g)$, which is the set of primitive vectors in $\mathbb{N}^n \backslash \{\boldsymbol{0}\}$ perpendicular to the facets of $\Gamma\left(\frac{f}{g}\right)$. We set

$$T_{+} := \left\{ \boldsymbol{w} \in \mathfrak{D}\left(\frac{f}{g}\right);\ d(\boldsymbol{w}, \Gamma(g)) - d(\boldsymbol{w}, \Gamma(f)) > 0 \right\},$$
$$T_{-} := \left\{ \boldsymbol{w} \in \mathfrak{D}\left(\frac{f}{g}\right);\ d(\boldsymbol{w}, \Gamma(g)) - d(\boldsymbol{w}, \Gamma(f)) < 0 \right\},$$

$$\alpha := \alpha\left(\frac{f}{g}\right) = \begin{cases} \min_{\boldsymbol{w}\in T_{+}} \left\{ \frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))} \right\} & \text{if } \ T_{+} \neq \varnothing \\ \\ +\infty & \text{if } \ T_{+} = \varnothing, \end{cases}$$

$$\beta := \beta\left(\frac{f}{g}\right) = \begin{cases} \max_{\boldsymbol{w}\in T_{-}} \left\{ \frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))} \right\} & \text{if } \ T_{-} \neq \varnothing \\ \\ -\infty & \text{if } \ T_{-} = \varnothing, \end{cases}$$

and

$$\widetilde{\alpha} := \widetilde{\alpha}\left(\frac{f}{g}\right) = \min\{1, \alpha\},\ \ \widetilde{\beta} := \widetilde{\beta}\left(\frac{f}{g}\right) = \max\{-1, \beta\}.$$

Notice that $\alpha > 0$ and $\beta < 0$.

We define the local zeta function attached to $\frac{f}{g}$ as

$$Z\left(s, \frac{f}{g}\right) = Z(s, -s, f, g),\ \ s \in \mathbb{C},$$

where $Z(s_1, s_2, f, g)$ denotes the meromorphic continuation of the local zeta function attached to the polynomial mapping (f, g), see Theorem 3.1.

Theorem 3.1 *Assume that* $\frac{f}{g}$ *is non-degenerate over* $\mathbb{F}_q$ *with respect to* $\Gamma\left(\frac{f}{g}\right)$, *with* $n \geq 2$ *as before. We fix a simplicial polyhedral subdivision* $\mathcal{F}\left(\frac{f}{g}\right)$ *of* $\mathbb{R}_{+}^{n}$ *subordinate to* $\Gamma\left(\frac{f}{g}\right)$. *Then the following assertions hold:*
(i) $Z\left(s, \frac{f}{g}\right)$ *has a meromorphic continuation to the whole complex plane as a rational function of* q^{-s} *and the following explicit formula holds:*

$$Z\left(s, \frac{f}{g}\right) = \sum_{\Delta\, \in\, \mathcal{F}(\frac{f}{g}) \cup \{\mathbf{0}\}} L_{\Delta}\left(s, \frac{f}{g}\right) S_{\Delta}(s),$$

where for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}$,

$$L_{\Delta}(s, \frac{f}{g}) = q^{-n}\left[(q-1)^n - N_{\Delta,\{f\}}\frac{1-q^{-s}}{1-q^{-1-s}} - N_{\Delta,\{g\}}\frac{1-q^{s}}{1-q^{-1+s}} \right. \\ \left. -N_{\Delta,\{f,g\}}\frac{(1-q^{-s})(1-q^{s})}{q(1-q^{-1-s})(1-q^{-1+s})}\right]$$

and

$$S_{\Delta}(s) = \frac{\sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t})-(d(\boldsymbol{t},\Gamma(f))-d(\boldsymbol{t},\Gamma(g)))s}}{\prod_{i=1}^{l}(1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))s})},$$

for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right)$ *a cone strictly positively generated by linearly independent vectors* $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l \in \mathfrak{D}\left(\frac{f}{g}\right)$, *and where* $\boldsymbol{t}$ *runs through the elements of the set*

$$\mathbb{Z}^n \cap \left\{\sum_{i=1}^{l} \lambda_i \boldsymbol{w}_i;\ 0 < \lambda_i \leq 1 \textit{ for } i = 1, \ldots, l\right\}.$$

By convention $S_{\mathbf{0}}(s) := 1$.

(ii) $Z\left(s, \frac{f}{g}\right)$ *is a holomorphic function on* $\widetilde{\beta} < \mathrm{Re}(s) < \widetilde{\alpha}$, *and on this band it verifies that*

$$Z\left(s, \frac{f}{g}\right) = \int_{\mathcal{O}_K^n \setminus D_K} \left|\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}\right|_K^s |d\boldsymbol{x}|_K; \tag{3.5.1}$$

(iii) the poles of the meromorphic continuation of $Z\left(s, \frac{f}{g}\right)$ *belong to the set*

$$\bigcup_{k\in\mathbb{Z}}\left\{1+\frac{2\pi\sqrt{-1}k}{\ln q}\right\} \cup \bigcup_{k\in\mathbb{Z}}\left\{-1+\frac{2\pi\sqrt{-1}k}{\ln q}\right\} \cup \\ \bigcup_{\boldsymbol{w}\in\,\mathfrak{D}\left(\frac{f}{g}\right)} \bigcup_{k\in\mathbb{Z}}\left\{\frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))} + \frac{2\pi\sqrt{-1}k}{\{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))\}\ln q}\right\}.$$

Proof. (i) The explicit formula for $Z(s, \frac{f}{g})$ follows from Theorem 3.1 as follows: we take $r = 2$, $s_1 = s$, $s_2 = -s$, $h_1 = f_{b(\Delta)}$ and $h_2 = g_{b(\Delta)}$ for $\Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\}$, with the convention that if $b(\Delta) = b(\mathbf{0}) = \mathbf{0}$, then $h_1 = f$ and $h_2 = g$. Now

$$\overline{V}_{\Delta} = \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n; \overline{f}_{b(\Delta)}(\overline{\boldsymbol{z}}) = \overline{g}_{b(\Delta)}(\overline{\boldsymbol{z}}) = 0\right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

and thus $Card(\overline{V}_{\Delta}) = N_{\Delta,\{f,g\}}$. Now, with $I=\{1,2\}$, by using (3.2.6), we have

$$J\left(s,-s,\overline{V}_{\Delta}\right) = \frac{q^{-n}\left(1-q^{-1}\right)^{2} N_{\Delta,\{f,g\}}}{\left(1-q^{-1-s}\right)\left(1-q^{-1+s}\right)}. \tag{3.5.2}$$

We now consider the case $I \neq \varnothing$, $I \subsetneqq \{1,2\}$, thus there are two cases: $I=\{1\}$ or $I=\{2\}$. Note that

$$\overline{V}_{\Delta,\{1\}} = \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}_{b(\Delta)}\left(\overline{\boldsymbol{z}}\right) = 0 \text{ and } \overline{g}_{b(\Delta)}\left(\overline{\boldsymbol{z}}\right) \neq 0\right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

and that $Card\left(\overline{V}_{\Delta,\{1\}}\right) = N_{\Delta,\{f\}}$, with the convention that

$$\overline{V}_{\mathbf{0},\{1\}} = \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}\left(\overline{\boldsymbol{z}}\right) = 0 \text{ and } \overline{g}\left(\overline{\boldsymbol{z}}\right) \neq 0\right\}.$$

In this case, by using (3.2.5),

$$J\left(s,-s,\overline{V}_{\Delta,\{1\}}\right) = \frac{q^{-n-s}\left(1-q^{-1}\right) N_{\Delta,\{f\}}}{1-q^{-1-s}}. \tag{3.5.3}$$

Analogously,

$$J\left(s,-s,\overline{V}_{\Delta,\{2\}}\right) = \frac{q^{-n+s}\left(1-q^{-1}\right) N_{\Delta,\{g\}}}{1-q^{-1+s}}. \tag{3.5.4}$$

We now consider the case $I=\varnothing$, then

$$\overline{V}_{\Delta,\varnothing} = \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}_{b(\Delta)}\left(\overline{\boldsymbol{z}}\right) \neq 0 \text{ and } \overline{g}_{b(\Delta)}\left(\overline{\boldsymbol{z}}\right) \neq 0\right\} \text{ for } \Delta \in \mathcal{F}\left(\frac{f}{g}\right) \cup \{\mathbf{0}\},$$

with the convention that

$$\overline{V}_{\mathbf{0},\varnothing} = \left\{\overline{\boldsymbol{z}} \in \left(\mathbb{F}_q^{\times}\right)^n ; \overline{f}\left(\overline{\boldsymbol{z}}\right) \neq 0 \text{ and } \overline{g}\left(\overline{\boldsymbol{z}}\right) \neq 0\right\}.$$

Notice that $Card(\overline{V}_{\Delta,\varnothing}) = (q-1)^n - N_{\Delta,\{f\}} - N_{\Delta,\{g\}} - N_{\Delta,\{f,g\}}$. Then, by using (3.2.4),

$$J\left(s,-s,\overline{V}_{\Delta,\varnothing}\right) = q^{-n} Card(\overline{V}_{\Delta,\varnothing}). \tag{3.5.5}$$

Then from Theorem 3.1 and (3.5.2)-(3.5.5), we get

$$\begin{aligned} L_{\Delta}(s,\frac{f}{g}) = {} & \frac{q^{-n}\left(1-q^{-1}\right)^{2} N_{\Delta,\{f,g\}}}{\left(1-q^{-1-s}\right)\left(1-q^{-1+s}\right)} + \frac{q^{-n-s}\left(1-q^{-1}\right) N_{\Delta,\{f\}}}{1-q^{-1-s}} + \\ & \frac{q^{-n+s}\left(1-q^{-1}\right) N_{\Delta,\{g\}}}{1-q^{-1+s}} + q^{-n}\left\{(q-1)^n - N_{\Delta,\{f\}} - N_{\Delta,\{g\}} - N_{\Delta,\{f,g\}}\right\}. \end{aligned}$$

The announced formula for $L_\Delta(s, \frac{f}{g})$ is obtained from the above formula after some simple algebraic manipulations.

(ii) Notice that for $\boldsymbol{w} \in \mathfrak{D}\left(\frac{f}{g}\right)$, $\frac{1}{1-q^{-\sigma(\boldsymbol{w})-(d(\boldsymbol{w},\Gamma(f))-d(\boldsymbol{w},\Gamma(g)))s}}$ is holomorphic on $\sigma(\boldsymbol{w}) + (d(\boldsymbol{w}, \Gamma(f)) - d(\boldsymbol{w}, \Gamma(g)))\,\mathrm{Re}(s) > 0$, and that $\frac{1}{1-q^{-1-s}}$ is holomorphic on $\mathrm{Re}(s) > -1$, and $\frac{1}{1-q^{-1+s}}$ is holomorphic on $\mathrm{Re}(s) < 1$, then, from the explicit formula for $Z(s, \frac{f}{g})$ given in (i) follows that it is holomorphic on the band $\widetilde{\beta} < \mathrm{Re}(s) < \widetilde{\alpha}$. Now, since $Z(s, \frac{f}{g}) = Z(s, -s, f, g)$, $Z(s, \frac{f}{g})$ is given by integral (3.5.1) because $Z(s_1, s_2, f, g)$ agrees with an integral on its natural domain.

(iii) It is a direct consequence of the explicit formula. ■

3.6 The largest and smallest real part of the poles of $Z(s, \frac{f}{g})$

In this section we use all the notation introduced in Section 3.5. We work with a fix simplicial polyhedral subdivision $\mathcal{F}\left(\frac{f}{g}\right)$ of $\mathbb{R}_+^n$ subordinate to $\Gamma\left(\frac{f}{g}\right)$. We recall that in the case $T_- \neq \varnothing$,

$$\beta = \max_{\boldsymbol{w} \in T_-} \left\{ \frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w}, \Gamma(g)) - d(\boldsymbol{w}, \Gamma(f))} \right\}$$

is the largest possible 'non-trivial' negative real part of the poles of $Z(s, \frac{f}{g})$. We set

$$\mathcal{P}(\beta) := \left\{ \boldsymbol{w} \in T_-;\, \frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w}, \Gamma(g)) - d(\boldsymbol{w}, \Gamma(f))} = \beta \right\},$$

and for $m \in \mathbb{N}$ with $1 \leq m \leq n$,

$$\mathcal{M}_m(\beta) := \left\{ \Delta \in \mathcal{F}\left(\frac{f}{g}\right);\, \Delta \text{ has exactly } m \text{ generators belonging to } \mathcal{P}(\beta) \right\},$$

and $\rho := \max\{m; \mathcal{M}_m(\beta) \neq \varnothing\}$.

Theorem 3.1 *Suppose that $\frac{f}{g}$ is non-degenerated over $\mathbb{F}_q$ with respect to $\Gamma(\frac{f}{g})$ and that $T_- \neq \varnothing$. If $\beta > -1$, then β is a pole of $Z(s, \frac{f}{g})$ of multiplicity ρ.*

Proof. In order to prove that β is a pole of $Z(s, \frac{f}{g})$ of order ρ, it is sufficient to show that

$$\lim_{s \to \beta} (1 - q^{\beta - s})^{\rho} Z\left(s, \frac{f}{g}\right) > 0.$$

This assertion follows from the explicit formula for $Z(s, \frac{f}{g})$ given in Theorem 3.1, by the following claim:

Claim. *$Res\,(\Delta, \beta) := \lim_{s \to \beta} (1 - q^{s-\beta})^{\rho} L_{\Delta}(s, \frac{f}{g}) S_{\Delta}(s) \geq 0$ for every cone $\Delta \in \mathcal{F}(\frac{f}{g})$. Furthermore, there exists a cone $\Delta_0 \in \mathcal{M}_{\rho}(\beta)$ such that $Res\,(\Delta_0, \beta) > 0$.*

We show that for at least one cone Δ_0 in $\mathcal{M}_{\rho}(\beta)$, $Res\,(\Delta_0, \beta) > 0$, because for any cone $\Delta \notin \mathcal{M}_{\rho}(\beta)$, $Res\,(\Delta, \beta) = 0$. This last assertion can be verified by using the argument that we give for the cones in $\mathcal{M}_{\rho}(\beta)$. We first note that there exists at least one cone Δ_0 in $\mathcal{M}_{\rho}(\beta)$. Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\rho}, \boldsymbol{w}_{\rho+1}, \ldots, \boldsymbol{w}_l$ its generators with $\boldsymbol{w}_i \in \mathcal{P}(\beta) \Leftrightarrow 1 \leq i \leq \rho$.

On the other hand,

$$\lim_{s \to \beta} L_{\Delta}\left(s, \frac{f}{g}\right) > 0 \tag{3.6.1}$$

for all cones $\Delta \in \mathcal{F}(\frac{f}{g}) \cup \{\mathbf{0}\}$. Inequality (3.6.1) follows from

$$L_{\Delta}\left(\beta, \frac{f}{g}\right) > q^{-n}((q-1)^n - N_{\Delta,\{f\}} - N_{\Delta,\{g\}} - N_{\Delta,\{f,g\}}) \geq 0$$

for all cones $\Delta \in \mathcal{F}(\frac{f}{g}) \cup \{\mathbf{0}\}$. We prove this last inequality in the case $N_{\Delta,\{f\}} > 0$, $N_{\Delta,\{g\}} > 0$, $N_{\Delta,\{f,g\}} > 0$ since the other cases are treated in similar form. In this case, the inequality follows from the formula for $L_{\Delta}(\beta, \frac{f}{g})$ given in Theorem 3.1 , by using that

$$N_{\Delta,\{f\}} \frac{1 - q^{-\beta}}{1 - q^{-1-\beta}} < N_{\Delta,\{f\}}, \; N_{\Delta,\{g\}} \frac{1 - q^{\beta}}{1 - q^{-1+\beta}} < N_{\Delta,\{g\}},$$
$$N_{\Delta,\{f,g\}} \frac{(1 - q^{-\beta})(1 - q^{\beta})}{q(1 - q^{-1-\beta})(1 - q^{-1+\beta})} < N_{\Delta,\{f,g\}} \quad \text{when } \beta > -1.$$

We also notice that

$$\lim_{s \to \beta} \sum_{\boldsymbol{t}} q^{-\sigma(\boldsymbol{t}) - (d(\boldsymbol{t}, \Gamma(f)) - d(\boldsymbol{t}, \Gamma(g)))s} > 0.$$

Hence in order to show that $Res\,(\Delta_0, \beta) > 0$, it is sufficient to show that

$$\lim_{s\to\beta} \frac{(1-q^{s-\beta})^{\rho}}{\prod_{i=1}^{l}(1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))s})} > 0.$$

Now, notice that there are positive integer constants c_i such that

$$\prod_{i=1}^{\rho}(1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))s}) = \prod_{i=1}^{\rho}(1-q^{(s-\beta)c_i})$$
$$= (1-q^{s-\beta})^{\rho}\prod_{i=1}^{\rho}\prod_{\varsigma^{c_i}=1,\varsigma\neq 1}\left(1-\varsigma q^{s-\beta}\right).$$

In addition, for $i = \rho+1, \ldots, l$,

$$1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))\beta} > 0$$

because $-\sigma(\boldsymbol{w}_i) - (d(\boldsymbol{w}_i, \Gamma(f)) - d(\boldsymbol{w}_i, \Gamma(g)))\beta \leq 0$ for any $\boldsymbol{w}_i \in T_+ \cup T_-$ with $i = \rho+1, \ldots, l$. From these observations, we have

$$\lim_{s\to\beta} \frac{(1-q^{s-\beta})^{\rho}}{\prod_{i=1}^{l}(1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))s})} =$$
$$\lim_{s\to\beta} \frac{(1-q^{s-\beta})^{\rho}}{(1-q^{s-\beta})^{\rho}\prod_{i=1}^{\rho}\prod_{\varsigma^{c_i}=1,\varsigma\neq 1}(1-\varsigma q^{s-\beta})} \times$$
$$\lim_{s\to\beta} \frac{1}{\prod_{i=\rho+1}^{l}(1-q^{-\sigma(\boldsymbol{w}_i)-(d(\boldsymbol{w}_i,\Gamma(f))-d(\boldsymbol{w}_i,\Gamma(g)))s})} > 0.$$

■

In the case $T_+ \neq \varnothing$,

$$\alpha = \min_{\boldsymbol{w}\in T_+}\left\{\frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))}\right\}.$$

is the smallest possible 'non-trivial' positive real part of the poles of $Z(s, \frac{f}{g})$. We set

$$\mathcal{P}(\alpha) := \left\{\boldsymbol{w}\in T_+; \frac{\sigma(\boldsymbol{w})}{d(\boldsymbol{w},\Gamma(g))-d(\boldsymbol{w},\Gamma(f))} = \alpha\right\},$$

and for $m \in \mathbb{N}$ with $1 \leq m \leq n$,

$$\mathcal{M}_m(\alpha) := \left\{\Delta \in \mathcal{F}\left(\frac{f}{g}\right); \Delta \text{ has exactly } m \text{ generators belonging to } \mathcal{P}(\alpha)\right\},$$

and $\kappa := \max\{m; \mathcal{M}_m(\alpha) \neq \varnothing\}$

The proof of the following result is similar to the proof of Theorem 3.1.

Theorem 3.2 *Suppose that $\frac{f}{g}$ is non-degenerated over $\mathbb{F}_q$ with respect to $\Gamma(\frac{f}{g})$ and that $T_+ \neq \varnothing$. If $\alpha < 1$, then α is a pole of $Z(s, \frac{f}{g})$ of multiplicity κ.*

Example 3.3 *We compute the local zeta function for the rational function given in Example 3.5. With the notation of Theorem 3.1, one verifies that*

Cone	L_Δ	S_Δ
$\{\mathbf{0}\}$	$q^{-2}((q-1)^2 - (q-1)\frac{1-q^{-s}}{1-q^{-1-s}})$	1
Δ_1	$q^{-2}(q-1)^2$	$\frac{q^{-1+2s}}{1-q^{-1+2s}}$
Δ_2	$q^{-2}(q-1)^2$	$\frac{q^{-2+2s}+q^{-4+4s}}{(1-q^{-1+2s})(1-q^{-3+2s})}$
Δ_3	$q^{-2}((q-1)^2 - (q-1)\frac{1-q^{-s}}{1-q^{-1-s}})$	$\frac{q^{-3+2s}}{1-q^{-3+2s}}$
Δ_4	$q^{-2}(q-1)^2$	$\frac{q^{-4+3s}}{(1-q^{-3+2s})(1-q^{-1+s})}$
Δ_5	$q^{-2}(q-1)^2$	$\frac{q^{-1+s}}{(1-q^{-1+s})}.$

Therefore

$$Z(s, \frac{f}{g}) = \frac{\frac{(q-1)}{q^2} L(q^{-s})}{(1-q^{s-1})(1-q^{-1-s})(1-q^{2s-1})(1-q^{2s-3})},$$

where

$$\begin{aligned} L(q^{-s}) = {} & q - q^{-1} - 2 - q^{2s-4} + q^{s-3} - q^{s-2} + q^{2s-2} + q^{3s-3} \\ & + 2q^{2s-1} - q^{3s-2} - q^{3s-1} + q^{-s-1}. \end{aligned}$$

Furthermore, $Z(s, \frac{f}{g})$ has poles with real parts belonging to $\{-1, 1/2, 1, 3/2\}$.

Chapter 4

Final remarks and some open problems

From a mathematical perspective, there are several open problems involving string amplitudes and parametric Feynman integrals. The following are some open problems that we expect to study in the near future.

1) **Determination of the divergencies of p-adic string amplitudes.**

In [8], we find the divergencies of the p-adic amplitude $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ using the Euclidean product instead of the Minkowski product to define $s_{ij} = \boldsymbol{k}_i \boldsymbol{k}_j$. We showed that $\boldsymbol{A}^{(N)}(0) = +\infty$ and $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}}) = +\infty$ for $\boldsymbol{k}_i \boldsymbol{k}_j > 0$. The determination of the ultraviolet and infrared divergencies, in the signature $- + + \ldots +$ for $\boldsymbol{A}^{(N)}(\underline{\boldsymbol{k}})$ is an open problem. This problem requires the determination of the geometry of the natural domain of function $\boldsymbol{Z}^{(N)}(\underline{\boldsymbol{s}})$. This type of problems has been not studied in the case of multivariate local zeta functions.

2) **Motivic amplitudes**

A natural problem consists in developing motivic string amplitudes (motivic in the sense of motivic integration), these objects should specialize to the p-adic and topological string amplitudes. Some connections between motives and quantum field theory are considered in [48].

3) **Archimedean string amplitudes.**

In the real case the string amplitudes at the tree level, are defined as

$$\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}}\right)$$
$$= \int_{\mathbb{R}^{N-3}} \prod_{i=2}^{N-2} |x_i|_\infty^{\boldsymbol{k}_1\boldsymbol{k}_i} |1-x_i|_\infty^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i} \prod_{2\leq i<j\leq N-2} |x_i - x_j|_\infty^{\boldsymbol{k}_i\boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i,$$

$N \geq 4$. Except for $\boldsymbol{A}^{(4)}\left(\underline{\boldsymbol{k}}\right)$, the integrals have not been computed analytically as in the p-adic case. In the light of the theory of local zeta functions, it is also natural to conjecture that local zeta functions corresponding to string amplitudes over $\mathbb{R}$ and $\mathbb{C}$ have meromorphic continuations of $\mathbb{C}^D$.

4) **Connections between local zeta functions of rational functions and monodromy conjectures.**

S. Gusein-Zade, I. Luengo and A. Melle-Hernández have studied the complex monodromy (and A'Campo zeta functions attached to it) of meromorphic functions, see e.g. [34]. Our work [7] drives naturally to ask about the existence of local zeta functions with poles related with the monodromies studied by the mentioned authors.

5) **Local zeta functions for rational functions over K-analytic submanifolds.**

Let K be a locally compact local field of characteristic zero, i.e. $K = \mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. Let X_K be a K-analytic closed submanifold of K^n, let Φ be a test function in $\mathcal{S}(K^n)$, and γ be a Gel'fand-Leray differential form along X_K. Consider $f, g \in K[x_1, .., x_n]$. To study the convergence and meromorphic continuation of local zeta functions:

$$Z_\Phi(s; X_K, f/g) := \int_{X_K \setminus D_K} \Phi(x) \frac{|f(x)|_K^{-n+sl/2}}{|g(x)|_K^{-n+(l+1)s/2}} |\gamma|_K$$

where $D_K = f^{-1}(0) \cup g^{-1}(0)$, $s \in \mathbb{C}$, $l > 0$, and $|\gamma|_K$ the measure along X_K induced by γ. To establish, in the case $K = \mathbb{R}$, a relation with the classical parametric Feynman

Integrals. In general the study in detail of the Archimedean and non-Archimedean parametric Feynman integrals as local zeta functions is still an open problem since these integrals are not completely covered by theory developed in [7], [60].

References

[1] Albeverio S., Khrennikov A. Yu., Shelkovich V. M., Theory of p-adic distributions: linear and nonlinear models. London Mathematical Society Lecture Note Series, 370. Cambridge University Press, Cambridge, 2010.

[2] Atiyah M. F., Resolution of Singularities and Division of Distributions, Comm. pure Appl. Math. 23 (1970),145–150.

[3] Aref'eva I. Ya., Dragović B. G., Volovich I. V., On the adelic string amplitudes, Phys. Lett. B 209 (1988), no. 4, 445–450.

[4] Belkale Prakash, Brosnan Patrick, Periods and Igusa local zeta functions, Int. Math. Res. Not. 2003, no. 49, 2655–2670.

[5] Bernstein I. N., Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients, Functional Analysis and its Applications 5, No.2, 1-16 (1972).

[6] Bleher P. M., Analytic continuation of massless Feynman amplitudes in the Schwartz space $\mathcal{S}'$, Rep. Math. Phys. 19 (1984), no. 1, 117–142.

[7] Bocardo-Gaspar, Miriam, W. A. Zúñiga-Galindo, Local zeta functions for rational functions and Newton polyhedra, arXiv:1702.06938 (2017), submitted to appear in Proceedings Second Conference Brazil-Mexico in Singularity Theory, Salvador-Bahia, 2015, Springer Proceedings in Mathematics & statistics.

[8] Bocardo-Gaspar, Miriam, García-Compeán, H., W. A. Zúñiga-Galindo, Regularization of p-adic String Amplitudes, and Multivariate Local Zeta Functions, arXiv:1611.03807 (2017). Submitted to Letters in Mathematical Physics.

[9] Bocardo-Gaspar, Miriam, García-Compeán, H., W. A. Zúñiga-Galindo, p-adic string amplitudes in the limit p approaches to one. Article in progress.

[10] Bogner Christian, Weinzierl Stefan, Periods and Feynman integrals, J. Math. Phys. 50 (2009), no. 4, 042302, 16 pp.

[11] Bollini C.G., Giambiagi J. J. , González Domínguez A., Analytic regularization and the divergencies of quantum field theories, Il Nuovo Cimiento XXXI, no. 3 (1964) 550-561.

[12] Borevich, Z. I. & Shafarevich, I. R., Number Theory, Academic Press, New York, (1966).

[13] Bories Bart, Igusa's p-adic local zeta function associated to a polynomial mapping and a polynomial integration measure, Manuscripta Math. (2012), vol. 138, Issue 3, 395-417.

[14] Bourbaki N., Éléments de mathématique. Fasc. XXXVI. Variétés différentielles et analytiques. Fascicule de résultats (Paragraphes 8 à 15). (French) Actualités Scientifiques et Industrielles, No. 1347 Hermann, Paris 1971 99 pp.

[15] Brekke Lee, Freund, Peter G. O., Olson Mark, Witten Edward, Non-Archimedean string dynamics, Nuclear Phys. B 302 (1988), no. 3, 365–402.

[16] Brekke Lee, Freund Peter G. O., p-adic numbers in physics, Phys. Rep. 233 (1993), no. 1, 1–66.

[17] Denef J., On the degree of Igusa's local zeta function, Amer. J. Math. 109 (1987), no. 6, 991–1008.

[18] Denef J., Report on Igusa's Local Zeta Function, Séminaire Bourbaki 43 (1990-1991), exp. 741; Astérisque 201-202-203 (1991), 359-386. Available at http://www.wis.kuleuven.ac.be/algebra/denef.html.

[19] Denef J., Poles of p-adic complex powers and Newton polyhedra, Nieuw. Arch. Wisk. 13 (1995), 289-295.

[20] Denef J., Hoornaert K., Newton polyhedra and Igusa's local zeta function, J. Number Theory 89 (2001), 31-64.

[21] Denef Jan, Loeser François, Caractéristiques D'Euler-Poincaré, Fonctions $\widehat{\text{z}}$locales et modifications analytiques, (1998), no. 3, 505–537.

[22] Denef Jan, Loeser François, Motivic Igusa zeta functions, J. Algebraic Geom. 7 (1998), no. 3, 505–537.

[23] Denef Jan, Loeser François, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. 135 (1999), no. 1, 201–232.

[24] Frampton Paul H., Okada Yasuhiro, p-adic string N-point function, Phys. Rev. Lett. 60 (1988), no. 6, 484–486.

[25] Freund Peter G. O., Olson Mark, Non-Archimedean strings, Phys. Lett. B 199 (1987), no. 2, 186–190.

[26] Freund Peter G. O., Witten Edward, Adelic string amplitudes, Phys. Lett. B 199 (1987), no. 2, 191–194.

[27] Gel'fand I. M. and Shilov G.E., Generalized Functions, vol 1., Academic Press, New York and London, 1977.

[28] Gerasimov Anton A., Shatashvili Samson L., On exact tachyon potential in open string field theory, J. High Energy Phys. (2000), no. 10, Paper 34, 12 pp.

[29] Ghoshal D., Noncommutative p-tachyon, Tr. Mat. Inst. Steklova 245 (2004), Izbr. Vopr. p-adich. Mat. Fiz. i Anal., 91–98; translation in Proc. Steklov Inst. Math. 2004, no. 2 (245), 83–90.

[30] Ghoshal D., p-adic string theories provide lattice discretization of the ordinary string worldsheet Phys. Rev. Lett. 97, 151601, (2006).

[31] I. P. Goulden and D. M. Jackson, Combinatorial enumeration, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, (1983).

[32] Gusein-Zade S., Luengo, I., Melle Hernández A., Bifurcations and topology of meromorphic germs. New developments in singularity theory (Cambridge, 2000), 279–304, NATO Sci. Ser. II Math. Phys. Chem., 21, Kluwer Acad. Publ., Dordrecht, 2001.

[33] Gusein-Zade S., Luengo I., Melle-Hernández A., On the topology of germs of meromorphic functions and its applications. (Russian) Algebra i Analiz 11 (1999), no. 5, 92–99; translation in St. Petersburg Math. J. 11 (2000), no. 5, 775–780.

[34] Gusein-Zade S., Luengo, I., Melle-Hernández A., Zeta functions of germs of meromorphic functions, and the Newton diagram. (Russian) Funktsional. Anal. i Prilozhen. 32 (1998), no. 2, 26–35, 95; translation in Funct. Anal. Appl. 32 (1998), no. 2, 93–99.

[35] Halmos Paul R., Measure Theory. D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[36] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. Math., 79 (1969), 109–326.

[37] Hloušek Zvonimir, Spector Donald, p-adic string theories, Ann. Physics 189 (1989), no. 2, 370–431.

[38] Igusa J.-I., Forms of higher degree. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 59. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[39] Igusa J.-I., An introduction to the theory of local zeta functions, AMS/IP Studies in Advanced Mathematics, 2000.

[40] Kempf G., Knudsen F., Mumford D., Saint-Donat B., Toroidal embeddings, Lectures notes in Mathematics vol. 339, Springer-Verlag, 1973.

[41] Kleinert Hagen, Schulte-Frohlinde Verena, Critical Properties of ϕ^4-Theories, World Scientific, Singapore 2001.

[42] Khovanskii A. G., Newton polyhedra, and toroidal varieties, Funkcional. Anal. i Prilozhen. 11 (1977), no. 4, 56–64.

[43] León-Cardenal E., Zuniga-Galindo W. A., Local zeta functions for non-degenerate Laurent polynomials over p-adic fields, J. Math. Sci. Univ. Tokyo 20 (2013), no. 4, 569-595.

[44] Lerner È. Yu., Missarov M. D., p-adic Feynman and string amplitudes, Comm. Math. Phys. 121 (1989), no. 1, 35–48.

[45] Lerner È. Yu, Feynman integrals of a p-adic argument in a momentum space. I. Convergence. Theoret. and Math. Phys. 102 (1995), no. 3, 267–274.

[46] Lerner È. Yu, Feynman integrals of a p-adic argument in a momentum space. II. Explicit formulas. Theoret. and Math. Phys. 104 (1995), no. 3, 1061–1077.

[47] Loeser F., Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants. Ann. Sci. École Norm. Sup. (4) 22 (1989), no. 3, 435–471.

[48] Marcolli Matilde, Connes A., Quantum fieds and motives, J. Geom. Phys. 56 (1), (2005) 55-85.

[49] Marcolli Matilde, Feynman motives. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[50] Serre Jean-Pierre, Local fields. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979.

[51] Smirnov V. A., Renormalization in p-adic quantum field theory, Modern Phys. Lett. A 6 (1991), no. 15, 1421–1427.

[52] Smirnov V. A., Calculation of general p-adic Feynman amplitude, Comm. Math. Phys. 149 (1992), no. 3, 623–636.

[53] Speer Eugene R., Generalized Feynman amplitudes, Annals of Mathematics Studies, No. 62 Princeton University Press,1969.

[54] Speer Eugene R., Ultraviolet and infrared singularity structure of generic Feynman amplitudes,Ann. Inst. H. Poincaré Sect. A (N.S.) 23 (1975), no. 1, 1–21.

[55] Taibleson M. H., Fourier analysis on local fields. Princeton University Press, 1975.

[56] Varadarajan V. S., Reflections on quanta, symmetries, and supersymmetries. Springer, New York, 2011.

[57] Oka Mutsuo, Non-degenerate Complete Intersection Singularity. Actualités Mathématiques. Hermann, Paris, (1997).

[58] Rossmann T., Computing topological zeta functions of groups, algebras, and modules, I, arXiv:1405.5711, (2014).

[59] Sturmfels Bernd, Gröbner bases and convex polytopes. American Mathematical Society, Providence, RI, (1996).

[60] Veys Willem, Zúñiga-Galindo W. A., Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra, Trans. Amer. Math. Soc. 360 (2008), no. 4, 2205–2227.

[61] Veys, W. and Zuñiga Galindo, W.A., Zeta functions and oscillatory integrals for meromorphic functions, arXiv:1510.03622v1, (2015).

[62] Volovich I. V., p-adic string, Classical Quantum Gravity 4 (1987), no. 4, L83–L87.

[63] Vladimirov V. S., Volovich I. V., Zelenov E. I., p-adic analysis and mathematical physics. World Scientific, 1994.

[64] Weil A., Sur la formule de Siegel dans la théorie des groupes classiques, Acta Math. 113 (1965), 1-87.

[65] Weil A., Basic Number Theory, Springer-Verlag, Berlin, 1967.

[66] Witten Edward, The Feynman $i\varepsilon$ in string theory, J. High Energy Phys. 2015, no. 4, 055, front matter +24 pp.

[67] Zúñiga-Galindo W. A., Igusa's local zeta functions of semiquasihomogeneous polynomials, Trans. Amer. Math. Soc. 353 (2001), no. 8, 3193–3207.

[68] Zúñiga-Galindo W. A., Local zeta functions and Newton polyhedra, Nagoya Math. J., 172 (2003), 31-58.

[69] Zúñiga-Galindo W. A., Local zeta function for nondegenerate homogeneous mappings, Pacific J. Math. 218 (2005), no. 1, 187–200.

[70] Zuñiga Galindo W.A., Local Zeta Functions Supported on Analytic Submanifolds and Newton Polyhedra, Int. Math. Res. Not. IMRN (2009), no. 15, 2855-2898.