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Spectral decompositions for diffusions

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Dedicado a mi familia,
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Resumen

Trabajamos con el primer tiempo de llegada del proceso generalizado de Cox-Ingersoll-Ross. Además estudiamos el operador de Green asociado al operador infinitesimal del operador de Brox, en un sentido débil, usando un producto interno. Mostraremos una caracterización de las eigenfunciones y de los eigenvalores asociados al operador infinitesimal del operador de Brox con muerte, usando el operador de Green. Finalmente, estudiamos la teoría clásica del potencial.

Abstract

We work with the first hitting time of the generalized Cox-Ingersoll-Ross process. We also study the Green operator associated to the infinitesimal operator of Brox diffusion defined in weak sense, using an inner product. We show a characterization of the eigenfunctions and eigenvalues associated to the infinitesimal operator of Brox with killing, using the Green operator. Finally, we study the classical potential theory.

Contents

Introduction

The theory of stochastic processes is a tool that allow us to explain many natural phenomena, in particular if these phenomena have a random character.

A stochastic process is a mathematical model that permits us to study the evolution of natural phenomena which evolves in time. For example, the trajectory of a molecule.

It is important to note that the trajectories are not necessarily continuous. Nevertheless in this work we study stochastic processes that have continuous trajectories. This kind of stochastic processes are called diffusions.

A particular case of diffusions are the solutions of stochastic differential equations. However there exit diffusions that are not solutions of an Itô equation. In general, a diffusion may be defined through of the infinitesimal operator.

In this work we study two diffusions. The so-called generalized Cox-Ingersoll-Ross process, that is solution of a stochastic differential equation, and the so-called Brox diffusion with killing, that is not solution of a stochastic differential equation. This last process is defined through of the infinitesimal operator.

The generalized Cox-Ingersoll-Ross process has several applications, for example in finance and biology. We found some interesting applications in the context of the so-called membrane potential between two neurons, see for instance [42]. In this context, it is very important the study of the first hitting time, therefore we will work with a spectral decomposition of the density of such random variable.

In another study we consider the Brox diffusion, which is a stochastic process in random environment that may be studied for each trajectory with a fixed environment. For this diffusion it is not known the density function. A reason that makes difficult the study of this process, is that the state space is $(-\infty, \infty)$, and the end-points $-\infty$ and ∞ are not regular, see [10]. For this reason, we study the Brox diffusion with killing in a, b. This new process has two advantages: the end-points a and b are regulars, and the state space is a finite interval, then we can apply the Sturm-Liouville theory to the infinitesimal operator associated. Our main goal is to study the density function of this diffusion with killing, analyzing the eigenfunctions and eigenvalues of the infinitesimal operator.

In the coming sections we give an account about some important concepts for our work, as the definition of diffusion, the infinitesimal operator, the scale function, the speed measure and the Green operator. We also give a brief summary of the main results of the Sturm-Liouville theory. At the end we present a summary of the work done in this thesis.

0.1 Markov Processes and Diffusions

A **stochastic process** is a collection of random variables defined on a common probability space $(\Omega, \mathfrak{F}, P)$, where Ω is the sample space, $\mathfrak{F}$ is a $\sigma-$algebra and P is a probability measure. The random variable is denoted by X_t, where t belongs to some set T contained in $\mathbb{R}$, that admits a convenient interpretation as time. Usually $T = [0, \infty)$ or $T = (-\infty, \infty)$.

In particular, a stochastic process X is a **Markov process** if for all $t, s \geq 0$ and any A Borel we have

$$P(X_{t+s} \in A \mid \mathfrak{F}_\mathfrak{t}) = P(X_{t+s} \in A \mid X_t),$$

where $\mathfrak{F}_\mathfrak{s} := \sigma\{X_u : u \leq s\}$.

Another equivalent condition is the following. If S represents the state space of X, then for all $t, s \geq 0$, and any bounded measurable function $F : S \to \mathbb{R}$ we have

$$E(F(X_{t+s}) \mid \mathfrak{F}_\mathfrak{t}) = E(F(X_{t+s}) \mid X_t).$$

A way of study the Markov processes is using the **infinitesimal operator** associated to the process X. This operator is defined in the following way

$$Lf := \lim_{t \to 0} \frac{P_t f - f}{t},$$

where f is a function such that the previous limit exists, and P_t represents the **semigroup** associated to the process X. It is well known that if we define $P_t(f)(x) := E_x(f(X_t))$, then P_t is the semigroup associated to this process.

In particular, in this work we will study **diffusions**, which are Markov processes that satisfy the followings two conditions

1. The paths of X are continuous,
2. X enjoys the strong Markov property.

In this case, it is known that there exists a function s called the **scale function**, and a measure m called the **speed measure**, such that the infinitesimal operator associated to the diffusion may be written as

$$Lf(x) = \frac{d}{dm} \frac{d}{ds} f(x),$$

where x is in the domain of f, usually is an interval. Also dm and ds represent the derivatives with respect to a measure or a functions, that is, for $h > 0$: (see e.g. [60])

$$\frac{df}{dm}(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{m([x, x+h])},$$

and

$$\frac{df}{ds}(x) := \lim_{h\to 0}\frac{f(x+h)-f(x)}{s(x+h)-s(x)}.$$

In general it is difficult to calculate the function s and the measure m. However, if we consider the case when the diffusion X takes values on $\mathbb{R}$, and is solution of a stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \tag{1}$$

then the function s and the measure m may be written using the functions μ and σ in the following way

$$s(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{y}\frac{2\mu(z)}{\sigma^2(z)}dz},$$

and

$$m(A) = \int_A \frac{2dx}{s'(x)\sigma^2(x)},$$

where A is a measurable set and x_0 is a fixed point in the state space. In this case, the infinitesimal operator associated to the process X that is solution of (1) is the following

$$Lf(x) = \frac{\sigma^2(x)}{2}f''(x) + \mu(x)f'(x). \tag{2}$$

In our work, the infinitesimal operator is going to be an important tool and we will calculate the eigenvalues and eigenfunctions associated, i.e. we calculate the values λ and the functions ϕ such that $L\phi + \lambda\phi = 0$. We will use this to make a spectral representation of objects such as the density function and the density of the first hitting time.

A way to study the infinitesimal operator is considering the Green operator. From the theory of differential equations we know that this operator is defined as

$$Tf(x) := \int_a^b G(x,y)f(y)dy, \tag{3}$$

where G is in terms of two independent solutions of $Lf = 0$. This operator T also satisfies $L(Tf) = f$. See Chapter 3.

0.2 Sturm-Liouville theory

The Sturm-Liouville equation is a second order linear differential equation of the following form

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) + \lambda w(x)y(x) = 0, \quad a < x < b, \tag{4}$$

where y is a function of the variable x, and the functions $p, q, w > 0$ are continuous on the finite interval $[a, b]$. The function w is called the weight or density function. Usually in the text books the function p is considered smooth, however this is not necessary, it is sufficient to consider that $\frac{1}{p}$ is measurable in (a, b), see [3].

To find the values of λ for which there exists a non-trivial solution of 4, satisfying boundary conditions is known as the SturmLiouville problem.

A Sturm-Liouville problem is called regular if p, q, w are continuous on [a,b], and also the boundary conditions are of the form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0, \quad \alpha_1^2 + \alpha_2^2 > 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0, \quad \beta_1^2 + \beta_2^2 > 0. \end{aligned}$$

In this work, we will use the conditions $y(a) = y(b) = 0$. These conditions, as we will see in the chapter 1, are associated with the killing of the process at that points, and the conditions $y'(a) = y'(b) = 0$ are associated with the reflection of the process at the points a and b. See [10].

We now mention same theorems that are very important for our work. The proofs of the these theorems may be found in [3].

Theorem 0.2.1. *The eigenvalues of the regular Stum-Liouville problem are real numbers and form an increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 ...,$$

such that

$$\lim_{n \to \infty} \lambda_n = \infty.$$

The previous theorem is important because it allows us to know that the eigenvalues are countable. On the other hand, if in the equation (4), the function q is zero and the boundary conditions are $y(a) = y(b) = 0$, then the eigenvalues are positives, and using the previous theorem then we have

$$0 < \lambda_1 < \lambda_2 < \lambda_3 ...$$

In particular this case occurs when we work with the Brox diffusion with killing in chapter 4.

Other important theorem that will allows us to make use of the spectral decomposition is the following

Theorem 0.2.2. *The normalized eigenfunctions associated to the Sturm-Liouville problem form an orthonormal basis in the Hilbert space $L^2([a, b], w(x)dx)$, with the inner product*

$$\int_a^b \phi_n(x) \phi_m(x) w(x) dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta.

Furthermore, the orthonormal sequence ϕ_n of eigenfunctions is complete in $L^2([a,b], w(x)dx)$, i.e. for any $f \in L^2([a,b], w(x)dx)$ we have that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \quad c_n := \int_a^b f(x)\phi_n(x)w(x)dx.$$

It is important to note that the convergence to f in the previous theorem is in the norm of $L^2([a,b], w(x)dx)$.

The followings two theorems, the Green identity Theorem and the Oscillation Theorem, are useful tools in this work, see [3]. This is so because they allow us to find a relation between the zeros of a special solution ψ of $L\psi + \lambda\psi = 0$, and the number of eigenvalues in an interval.

Theorem 0.2.3. (Green identity) *Suppose that y_1 and y_2 are two solutions of the Sturm-liouville equation associated with λ_1 and λ_2 respectively. If α and β are two points in $[a,b]$, then*

$$[p(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x))]_\alpha^\beta = (\lambda_1 - \lambda_2)\int_\alpha^\beta y_1(x)y_2(x)w(x)dx.$$

Theorem 0.2.4. (Oscillation) *Let $L_i(y) := (py')' + g_i y = 0$, where $g_2 > g_1$. If $L_1(u_1) = 0$ and $L_2(u_2) = 0$, then between any two consecutive zeros of u_1 there exists a zero of u_2.*

Another theorem that is very important is the following. It states how many zeros has each eigenfunction in the interval (a,b). The proof the this theorem use the Prüfer method.

Theorem 0.2.5. *Corresponding to each eigenvalue λ_n of the Sturm-Liouville problem, there is a unique eigenfunction ϕ_n, up to a normalization constant, which has exactly $n-1$ zeros in (a,b).*

0.3 Summary of the work and main contributions

In the chapter 1 we study the so-called generalized Cox-Ingersoll-Ross process. This process is solution of the stochastic differential equation

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}dB_t.$$

We work with a spectral decomposition of the density of the first hitting time of this process. The main interest to study this density is because has applications in the area of biology, as a model to study neuronal activity, see [42].

In the literature, for example [24], [44], [45] the authors work with the spectral decomposition of the density of the first hitting time of the processes Cox-Ingersoll-Ross and Ornstein Uhlenbeck. In this work we propose slight extension. We work with the density of the first hitting time of the generalized Cox-Ingersoll-Ross process and we present explicit formulae considering all the cases depending on the parameters involved.

We also find explicitly the eigenfunctions associated to the infinitesimal operator, using the so-called Kummer's equation, and we compute the eigenfunctions in terms of the so-called confluent hypergeometric functions. Moreover we do approximations of the eigenvalues using approximations of the confluent hypergeometric functions.

In the chapter 2 and 3 we work with the operator

$$Lf(x) = \frac{f''(x)}{2} - \frac{W'(x)f'(x)}{2}.$$

Informally speaking L is the infinitesimal operator associated to the Brox diffusion. In particular, in the Chapter 2 we study this operator in a weak sense, because in a rigorous way the operator L is not well defined. Therefore we consider the operator L defined in a weak sense using an inner product. Our main contribution in this chapter is the construction of the Green operator associated to L. This new operator is the right inverse of the operator L.

In the chapter 3 we study the spectral decomposition of the density function of the Brox diffusion with killing at a and b. Actually the Brox process has been studied by many authors, for instance see [12], [4], [14], [15]. There are many work that present asymptotic properties of this process when the time is large, however in the literature not exists an explicit formula for the density function of this process. For this reason, in this chapter we study the spectral decomposition of the density function of the killed Brox process; the main advantage of this is that the state space is a bounded interval. We then apply the Sturm-Liouville theory to study the eigenvalues and eigenfunctions associated to the infinitesimal operator of the killed Brox process.

The details of this chapter are the following, we work with the infinitesimal operator

$$Lf(x) = \frac{e^{W(x)}}{2}\left(e^{-W(x)}f'(x)\right)',$$

where f satisfies $f(a) = f(b) = 0$. We give a characterization of the eigenvalues and eigenfunctions of this operator. To do this, we construct the Green operator T, and we obtain

$$Tf(x) := \int_a^b 2e^{-W(z)}g(z,x)f(z)dz,$$

where g is in terms of two independent solutions of $Lf = 0$. We use this operator T and the Riccati transform to obtain that the eigenfunctions satisfies a stochastic differential equation 2-dimensional. Also we use the Sturm-Liouville theory to characterize the eigenvalues of the operator L. Our contribution in this chapter is the construction of the Green operator, and the characterization of the eigenfunctions and the eigenvalues associated to L.

Finally, in the chapter 4, we carry out a summary of the classical potential theory. We present the mains known results and theorems in $\mathbb{R}^n$, for example, the Frostman's theorem, the uniqueness of the equilibrium measure and relations with the Brownian motion. We use the following references [28], [7], [52], [48], [58]. We also work with the concept of capacity, and we give a known geometric interpretation of it. At the end of this chapter we show a known relation between the problem of minimum energy and the Dirichlet problem.

The reason we studied the potential theory is because we are interested on some connections between operators with random environment, like those in Chapters 2 and 3, and the theory of random matrices. In particular, there exist operators with random environment that can be seen as the limit of random matrices, see [57], [56], and the distribution of the eigenvalues of these random matrices can be written in terms of the energy of an electrostatic problem. Thus, we found out that it is relevant to know about potential theory to begin to analyze these connections.

In this chapter, we believe our only contribution is that we gathered ideas from different texts that our allowed to restate the Dirichlet principle.

Chapter 1

Hitting times for the generalized CIR

We give explicitly the spectral decomposition for the first hitting time density of the so-called generalized Cox-Ingersoll-Ross process for all the cases depending on the parameters. In doing so, we solve associated boundary value problems which are set using the generator. In addition, we also compute the eigenfunctions in terms of the confluent hypergeometric functions. It turns out that a common tool in the analysis is the so-called Kummer equation, from which we make use of the known solutions. This chapter is based on [24].

1.1 Introduction

As described in [42], the changes of the so-called membrane potential between two neurons in the human brain can be well modeled using an Itô's diffusion

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

see also [21]. It turns out that this kind of models is a good approximation of the real phenomenon, according to [42]. Furthermore, one of the main interests in these applications is what people in neuroscience call the interspike intervals, which is assumed to be a random variable of the form

$$\tau = \inf\{t : X_t \geq f(t)\},$$

where f is a deterministic time function. The importance of this random variable relies in the hypothesis that the flow of information in the nervous system is encoded in the timing of spikes. A model considered in the literature for this kind of phenomenon is the diffusion

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}dB_t,$$

where $\mu,\ a,\ \sigma,\ S$ are constants.

Coincidentally, this model is also used in financial mathematics to model interest rates. In fact, according to the theory of pricing, the price of a so-called zero-coupon bond is given by the expectation

$$E\left[e^{-\int_0^t X_s ds}\right],$$

see e.g. [9]. Again, knowing information of the first time when X_t surpasses a function is of importance in this context. When $S = 0$, the process X is called the Cox-Ingersoll-Ross (CIR) process, see also [39] for more details on this. Yet again in the context of financial mathematics, specifically in the so-called risk theory, an important paradigm is modelling the time of default of a bond based on the so-called intensity models, where it is used a function $\lambda(x)$ to study the random time

$$\tau = \inf\{t : \int_0^t \lambda(X_s)ds \geq E\},$$

for some constant E, see e.g. [41].

These are some reasons that motivate us to work with a diffusion that is a generalization of the CIR process. Precisely, the solution of

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}dB_t.$$

In particular, we study the first hitting time of the process X, defined as

$$\tau_y := \inf\{t > 0 : X_t = y\},$$

where y is a given constant.

Such a task is done by using the theory of spectral decompositions, in particular we use the results in [44]. The case $S = 0$ is well known and studied, see e.g. [45] . Notice that when S is not zero one might try to apply a space transformation and use Itô's lemma to go back to the case $S = 0$. In this paper however we do not do that. Instead we work directly with the differential equations and the spectral decomposition, this helps to see the right connection with the so-called Kummer differential equation, which is a key tool for the whole analysis. This way of working has the benefit to see how the spectral theory works when dealing hitting times.

Let us mention how this chapter is organized. In the coming Section 2 we present the basic theory that we use along the chapter. Basic concepts such as the speed measure, the scale function and the killing measure are introduced. We also recall the classification of the end-points in the state space.

In the Section 3 we will state the result of V. Linetsky [44] and we present some tables which summarizes our results. Sections 4, 5,6 and 7 provide the proofs of the results.

Finally, in Section 8 we study the first hitting time density of the process Y that is solution of the stochastic differential equation

$$dY_t = (\beta - bY_t)dt + \sigma\sqrt{S - Y_t}dB_t.$$

Such a process is the reflected analogous of X. To this end, we will apply the Itô's formula to recycle the formulas obtained for X. To finish, in the Section 9 we present a numerical illustration of one of the expressions we obtain.

1.2 Preliminaries

Let $\{X_t : t \geq 0\}$ be a one-dimensional diffusion whose state space is some interval $I \subseteq \mathbb{R}$ with end-points e_1 and e_2.

Every diffusion has three basic characteristics that determine the process: speed measure, scale function and killing measure, see [10] for more details. We consider here the special case in which the three basic characteristics are absolutely continuous with respect to Lebesgue measure in the interior of I, i.e.

$$m(dx) = m(x)dx, \quad k(dx) = k(x)dx, \quad s(x) = \int_{x_0}^{x} s'(y)dy, \quad x \in (e_1, e_2),\ x_0 \in (e_1, e_2) \text{ fixed.} \tag{1.1}$$

Then the infinitesimal generator is the following second order differential operator

$$Lf(x) = \frac{1}{2}a^2(x)f''(x) + b(x)f'(x) - c(x)f(x), \quad x \in (e_1, e_2). \tag{1.2}$$

The functions a, b, c are called the infinitesimal parameters of X, and are related to m, k, s through the following formulas

$$s'(x) = e^{-\int_{x_0}^{x} \frac{2b(y)}{a^2(y)}dy}, \quad m(x) = \frac{2}{a^2(x)s'(x)}, \quad k(x) = \frac{2c(x)}{a^2(x)s'(x)}.$$

The speed measure, the scale function and the killing measure determine the behavior of the diffusion in the interior of the state space I. However the behavior of the diffusion at the boundary points is characterized by boundary conditions.

In [10] it is presented a classification of the end-points of I according to the behavior of the diffusion in the neighborhood of these end-points. To explain this, let z be fixed such that $e_1 < z < e_2$. According with (1.1) we have

$$m((x, z)) := \int_{x}^{z} m(y)dy, \text{ and } s(x) := \int_{z}^{x} s'(y)dy. \tag{1.3}$$

Now define

$$A := \int_{e_1}^{z} [m((x, z)) + k(x)]\, s'(x)dx, \quad B := \int_{e_1}^{z} [s(z) - s(x)]\, (m(x) + k(x))dx.$$

Then the end-point e_1 is classified in the following manner, classifying the end-point e_2 is similar, see [49]. (The classification does not depend of the value z)

i. The end-point e_1 is called **exit** if

$$A < \infty \text{ and } B = \infty.$$

ii. The end-point e_1 is called **entrance** if

$$A = \infty \text{ and } B < \infty.$$

iii. The end-point e_1 is called **regular** if

$$A < \infty \text{ and } B < \infty.$$

In this case, one additionally has the following subclassification:

- if $m(\{e_1\}) = k(\{e_1\}) = 0$ then e_1 is called **regular reflecting**,
- if $m(\{e_1\}) < \infty$, and $k(\{e_1\}) = \infty$ then e_1 is called **regular killing**,
- if $0 < m(\{e_1\}) < \infty$, and $k(\{e_1\}) = 0$ then e_1 is called **regular sticky**,
- if $m(\{e_1\}) = 0$, and $k(\{e_1\}) > 0$ then e_1 is called **regular elastic**,
- if $m(\{e_1\}) = \infty$, and $k(\{e_1\}) \geq 0$ then e_1 is called **regular absorbing**.

iv. The end-point e_1 is called **natural** if

$$A = \infty \text{ and } B = \infty.$$

Remark 1.2.1. *In this paper we consider the case when the boundary condition at a regular boundary are only reflecting or killing, for more details see [49]. Also, in this paper we do not consider the case when e_1 is natural, because in our examples this case is not presented.*

Note that if $k((e_1, e_2)) = 0$, then using the previous classification and the function s, it is known that for $e_1 < x < y$

$$P_x(\tau_y < \infty) = \begin{cases} 1, & \text{if } e_1 \text{ is entrance or regular reflecting;} \\ \dfrac{\int_{e_1}^{x} s'(z)dz}{\int_{e_1}^{y} s'(z)dz}, & \text{if } e_1 \text{ is exit or regular killing,} \end{cases} \tag{1.4}$$

where $\tau_y := \inf\{t > 0 : X_t = y\}$; see for instance [44].

1.3 Spectral expansion for first hitting time density

Let us present the spectral decomposition theorem of V. Linetsky found in [44]; see also [43]. Consider a diffusion X which is solution of the stochastic differential equation

$$dX_t = (\mu - aX_t)dt + \sigma\sqrt{X_t - S}\, dB_t, \tag{1.5}$$

where $\mu, a, S \in \mathbb{R}$ and $\sigma > 0$. It is known that the infinitesimal operator is

$$Lf(x) = \frac{\sigma^2(x-S)}{2} f''(x) + (\mu - ax) f'(x), \quad x \in (S, \infty), \tag{1.6}$$

acting on twice differentiable functions $f : (S, \infty) \to \mathbb{R}$. As we will see, to specify completely the process X, we also need to specify the behavior of X at S; this fact is important for the following theorem.

Theorem 1.3.1. (Linetsky [44]) *Let X be a diffusion that is solution of $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ and whose state space I has the end-points e_1 and e_2. Define $I^y := [e_1, y]$ if e_1 is regular reflecting, and $I^y := (e_1, y]$ in any other case. Fix $X_0 = x$ and $y \in I$ such that $e_1 < x < y < e_2$, and suppose that e_1 is either regular, entrance or exit. For $\lambda \in \mathbb{C}$ and $x \in I^y$, let $\psi(x, \lambda)$ be the unique non trivial solution (up to a multiple independent of x) of the Sturm-Liouville equation*

$$L\psi + \lambda\psi = 0, \quad \text{with boundary condition at } e_1 \text{ given by } \lim_{x \to e_1^+} \psi(x, \lambda) = 0 \text{ or } \lim_{x \to e_1^+} \frac{\psi_x(x, \lambda)}{s'(x)} = 0. \tag{1.7}$$

Then the spectral expansion of $P_x(t < \tau_y < \infty)$, with $e_1 < x < y$ takes the form

$$P_x(t < \tau_y < \infty) = -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x, \lambda_n)}{\lambda_n \psi_\lambda(y, \lambda_n)}, \tag{1.8}$$

where $0 < \lambda_1 < \lambda_2 < \ldots$ are the simple positive zeros of $\psi(y, \lambda)$, i.e. $\psi(y, \lambda_n) = 0$. Note that each λ_n depends of y.

Remark 1.3.2. *The function $\psi(x, \lambda)$ of the previous theorem is square-integrable with respect to m in a neighborhood of e_1; and $\psi(x, \lambda)$ and $\psi_x(x, \lambda)$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each $x \in I^y$ fixed. For more details see [43].*

If we apply the identity

$$P_x(\tau_y < \infty) = P_x(\tau_y < t) + P_x(t < \tau_y < \infty), \tag{1.9}$$

by Theorem 1.3.1 and (1.4) we obtain that for $e_1 < x < y$

$$P_x(\tau_y \le t) = P_x(\tau_y < \infty) + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x, \lambda_n)}{\lambda_n \psi_\lambda(y, \lambda_n)}. \tag{1.10}$$

As we noticed, an important ingredient for the methodology is solving the equation $L\psi + \lambda\psi = 0$. It turns out that the boundary condition for ψ at $e_1 = S$ depends on the classification of S (see [10] for details):

i. If e_1 is exit or regular killing then the boundary condition is $\lim_{x \to e_1^+} \psi(x, \lambda) = 0$.

ii. If e_1 is entrance or regular reflecting then the boundary condition is $\lim_{x\to e_1^+} \dfrac{\psi_x(x,\lambda)}{s'(x)} = 0$, where $\psi_x(x,\lambda) := \frac{\partial}{\partial x}\psi(x,\lambda)$.

Remark 1.3.3. *When $e_1 < y < x < e_2$, the first hitting time problem is treated similarly. In this case we have $I^y := [y, e_2]$ if e_2 is regular reflecting, and $I^y := [y, e_2)$ in any other case. It is also known that*

$$P_x\left(\tau_y < \infty\right) = \begin{cases} 1, & \text{if } e_2 \text{ is entrance or regular reflecting;} \\ \dfrac{\int_x^{e_2} s'(z)dz}{\int_y^{e_2} s'(z)dz}, & \text{if } e_2 \text{ is exit or regular killing.} \end{cases} \tag{1.11}$$

Then the spectral expansion of $P_x(t < \tau_y < \infty)$, with $y < x < e_2$ is

$$P_x(t < \tau_y < \infty) = -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\phi(x,\lambda_n)}{\lambda_n \phi_\lambda(y,\lambda_n)}, \tag{1.12}$$

where the solution $\phi(x,\lambda)$ of $Lu + \lambda u = 0$ is square-integrable with respect to m in a neighborhood of e_2 and satisfying the appropriate boundary condition at e_2, i.e.

i. $\lim_{x\to e_2^-} \phi(x,\lambda) = 0$ *if e_2 is exit or regular killing, or*

ii. $\lim_{x\to e_2^-} \dfrac{\phi_x(x,\lambda)}{s'(x)} = 0$ *if e_2 is entrance or regular reflecting.*

Also, $\phi(x,\lambda)$ and $\phi_x(x,\lambda)$ are continuous in x and λ in $I^y \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each $x \in I^y$ fixed. The λ_n are all the simple positive zeros of $\phi(y,\lambda)$.

We will apply the Theorem 1.3.1 to find the first hitting time density of the process X that is solution of (1.5). We summarize our results in the following tables. The Table 1 shows the nature of the end-point S (second column) and the type of boundary condition (third column), both depending on certain regions for the parameters μ, a, S and σ (first column).

Table 2 shows the solution of the equation (1.7) and Table 3 shows precisely the formula (1.10). Note that in the end there are four types, I, II, III, IV. The proofs of the first two columns of Table 1 are give in Section 4. Section 5 deals with cases I and III of these tables. Section 6 deals with the case II, and the Section 7 presents the case IV.

Let us give the notation used in the tables:

- $F(a,b,x) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$, where $(a)_n := a \cdot (a+1) \cdots (a+n-1)$.

- F_λ represents the derivative of F with respect to λ.

- $J_v(x) := \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{v+2n}}{n!\Gamma(v+n+1)}$ is the Bessel function, where Γ is the Gamma function.

- $J_{v,n}$ represents a positive zero of the Bessel function J_v.

Table 1.

Parameters	S	Boundary Condition
I. $-1 < \frac{2aS - 2\mu}{\sigma^2} < 0$ with $a \neq 0$	Regular reflecting	$\lim_{x \to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$
II. $\frac{2aS - 2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	Exit	$\lim_{x \to S^+} \psi(x,\lambda) = 0$
III. $\frac{2aS - 2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	Entrance	$\lim_{x \to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	Regular killing	$\lim_{x \to S^+} \psi(x,\lambda) = 0$

Table 2.

Parameters	Solution of $L\psi + \lambda\psi = 0$
I. $-1 < \frac{2aS-2\mu}{\sigma^2} < 0$ with $a \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
II. $\frac{2aS-2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	$\psi(x,\lambda) = \left(\frac{2a(x-S)}{\sigma^2}\right)^{1-\frac{2\mu-2aS}{\sigma^2}} F\left(\frac{-\lambda}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
III. $\frac{2aS-2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	$\psi(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)} \cdot J_{\left(1-\frac{2\mu}{\sigma^2}\right)}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)$

Table 3.

Parameters	Spectral decomposition for the first hitting time density
I. $-1 < \frac{2aS-2\mu}{\sigma^2} < 0$ with $a \neq 0$	$P_x(\tau_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}$
II. $\frac{2aS-2\mu}{\sigma^2} \geq 0$ with $a \neq 0$	$P_x(\tau_y \leq t) = \frac{\int_S^x s'(z)dz}{\int_S^y s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)}$
III. $\frac{2aS-2\mu}{\sigma^2} \leq -1$ with $a \neq 0$	$P_x(\tau_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}$
IV. $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$	$P_x(\tau_y \leq t) = \left(\frac{x-S}{y-S}\right)^v - 2 \cdot \left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \cdot \sum_{n=0}^{\infty} e^{-\frac{\sigma^2 J_{v,n} t}{8(y-S)}} \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{v,n} \cdot J_{1+v}\left(J_{v,n}\right)}$

1.4 Classification of the end-point S

We consider the process X that is solution of (1.5) with state space given by the interval $I = [S, \infty)$ if S is regular reflecting or $I = (S, \infty)$ in any other case. We suppose that $X_0 = x > S$ and we consider $y \in I$ fixed such that $x < y$. To study this process we first classify the end-point S.

Proposition 1.4.1. *The end-point S obeys the following classification:*

i. If $-1 < \dfrac{2aS - 2\mu}{\sigma^2} < 0$ *then S is regular.*

ii. If $\dfrac{2aS - 2\mu}{\sigma^2} \geq 0$ *then S is exit.*

iii. If $\dfrac{2aS - 2\mu}{\sigma^2} \leq -1$ *then S is entrance.*

Proof. We prove the first case, the other two cases are similar.

Suppose that $-1 < \dfrac{2aS - 2\mu}{\sigma^2} < 0$, and let z be a fixed value such that $S < z < \infty$. To verify that S is regular, we calculate the following integrals to see that they are finite:

$$\begin{aligned}
\int_S^z m((x,z))s'(x)dx &= \int_S^z \left(\int_x^z 2\sigma^{-2} e^{\frac{-2ay}{\sigma^2}} (y-S)^{\frac{2\mu - 2aS}{\sigma^2} - 1} dy \right) e^{\frac{2ax}{\sigma^2}} (x-S)^{\frac{2aS - 2\mu}{\sigma^2}} dx \\
&\leq M \int_S^z \left(\int_S^z (y-S)^{\frac{2\mu - 2aS}{\sigma^2} - 1} dy \right) (x-S)^{\frac{2aS - 2\mu}{\sigma^2}} dx \\
&= M \int_S^z (y-S)^{\frac{2\mu - 2aS}{\sigma^2} - 1} dy \cdot \int_S^z (x-S)^{\frac{2aS - 2\mu}{\sigma^2}} dx \\
&< \infty,
\end{aligned}$$

where M is a constant. In the same way we obtain

$$\int_S^z s(x)m(x)dx = \int_S^z \left(\int_x^z e^{\frac{2ay}{\sigma^2}} (y-S)^{\frac{2aS - 2\mu}{\sigma^2}} dy \right) 2\sigma^{-2} e^{\frac{-2ax}{\sigma^2}} (x-S)^{\frac{2\mu - 2aS}{\sigma^2} - 1} dx < \infty.$$

We used, in both cases, the fact that $-1 < \dfrac{2aS - 2\mu}{\sigma^2} < 0$, to obtain that the integrals are finite. $\square$

Corollary 1.4.2. *If $a = 0$ and $-1 < \frac{-2\mu}{\sigma^2} < 0$ then the end-point S is regular.*

The condition $a \neq 0$ presented in tables shows up in the following sections.

1.5 First hitting time for the cases I and III

According to Theorem 1.3.1, for the cases I and III (in Tables 1, 2 and 3) we have to find a function $\psi(x,\lambda)$ such that it satisfies

$$\frac{\sigma^2(x-S)}{2}\psi''(x)+(\mu-ax)\psi'(x)+\lambda\psi(x)=0,\quad \text{and} \quad \lim_{x\to S^+}\frac{\psi_x(x,\lambda)}{s'(x)}=0. \tag{1.13}$$

In turn we have the following

Proposition 1.5.1. *If $a\neq 0$, then the function*

$$\psi(x,\lambda)=F\left(\frac{-\lambda}{a},\frac{2\mu-2aS}{\sigma^2},\frac{2a(x-S)}{\sigma^2}\right), \tag{1.14}$$

is solution of (1.13), with $F(a,b,x):=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}$ *and* $(a)_n := a\cdot(a+1)\cdots(a+n-1)$.

Proof. First consider $z=\frac{\sigma^2(x-S)}{2}$, and define $g(z):=f\left(\frac{2z}{\sigma^2}+S\right)$. Then from (1.13) we obtain the equation

$$zg''(z)+\left(\frac{2\mu}{\sigma^2}-\frac{4az}{\sigma^4}-\frac{2aS}{\sigma^2}\right)g'(z)+\frac{4\lambda}{\sigma^4}g(z)=0.$$

Now consider $w=\frac{4az}{\sigma^4}$ and define $h(w):=g\left(\frac{\sigma^4 w}{4a}\right)$. Since $a\neq 0$, we arrive at the Kummer equation (see [68]):

$$wh''(w)+(\frac{2\mu-2aS}{\sigma^2}-w)h'(w)-\left(\frac{-\lambda}{a}\right)h(w)=0,$$

and according with [68, p.2] the solution is

$$h(w)=F\left(\frac{-\lambda}{a},\frac{2\mu-2aS}{\sigma^2},w\right).$$

Returning to the variable x we obtain the result. □

Applying the Proposition 1.5.1 and the formula (1.10) we obtain the following theorem

Theorem 1.5.2. *Let X be the process that is solution of (1.5) with $S<x<y$, $a\neq 0$, and such that S is regular reflecting or entrance. Then the first hitting time distribution of the process X is*

$$P_x(\tau_y\le t)=1+\sum_{n=1}^{\infty}e^{-\lambda_n t}\frac{F\left(\frac{-\lambda_n}{a},\frac{2\mu-2aS}{\sigma^2},\frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a},\frac{2\mu-2aS}{\sigma^2},\frac{2a(y-S)}{\sigma^2}\right)}, \tag{1.15}$$

where the derivative $F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)$ *is equal to*

$$\frac{-1}{a}\sum_{k=0}^{\infty}\frac{\left(\frac{-\lambda_n}{a}\right)_k}{\left(\frac{2\mu-2aS}{\sigma^2}\right)_k}\phi\left(\frac{-\lambda_n}{a}+k\right)\frac{\left(\frac{2a(y-s)}{\sigma^2}\right)^k}{k!}+\frac{1}{a}\phi\left(\frac{-\lambda_n}{a}\right)F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right), \tag{1.16}$$

and $\phi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$, *where* Γ *is the Gamma function. See [44] for more details.*

From Theorem 1.5.2 we obtain a formula for the first hitting time density for the cases I and III

$$P_x(\tau_y \in dt) = -\sum_{n=1}^{\infty} e^{-\lambda_n t}\frac{F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)} = \sum_{n=1}^{\infty} e^{-\lambda_n t}\lambda_n c_n, \tag{1.17}$$

where

$$c_n := \frac{-F\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{a}, \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}. \tag{1.18}$$

Remark 1.5.3. *To be able to plug previous formulas into the computer, we propose a naive but effective approximation of* λ_n *and* c_n*. First consider the following formula found in [68],*

$$F(a,b,x) = \pi^{\frac{-1}{2}}\Gamma(b)e^{\frac{x}{2}}\left[x\left(\frac{b}{2}-a\right)\right]^{\frac{1}{4}-\frac{b}{2}} cos\left(2\sqrt{x\left(\frac{b}{2}-a\right)}-\frac{b\pi}{2}+\frac{\pi}{4}\right)\left\{1+O\left(\mid a\mid^{\frac{-1}{2}}\right)\right\}. \tag{1.19}$$

Since the λ_n *are the zeros of* $\psi(y,\lambda)$*, with* y *fixed, we will use the formula (1.19) to find an approximation of* λ_n *for* n *large given by*

$$\lambda_n \approx \left[\frac{\sigma^2}{2a(y-S)}\left(\frac{\pi(\mu-aS)}{2\sigma^2}+\frac{n\pi}{2}-\frac{3\pi}{8}\right)^2-\frac{\mu-aS}{\sigma^2}\right]\times a. \tag{1.20}$$

Then applying again the formula (1.19) we obtain an approximation for c_n *in (1.18) with* n *large given by*

$$\begin{aligned} c_n \quad &\approx \quad \frac{(-1)^{n+1}2\pi\left(n+\frac{\mu-aS}{\sigma^2}-\frac{3}{4}\right)\cdot e^{\frac{2a(x-y)}{\sigma^2}}}{\pi^2\left(n+\frac{\mu-aS}{\sigma^2}-\frac{3}{4}\right)^2-\frac{4\mu-4aS}{\sigma^2}\cdot\frac{2a(y-S)}{\sigma^2}}\cdot\left(\frac{x-S}{y-S}\right)^{\frac{1}{4}-\frac{\mu-aS}{\sigma^2}} \\ &\times \quad cos\left(\pi\left(n+\frac{\mu-aS}{\sigma^2}-\frac{3}{4}\right)\sqrt{\frac{x-S}{y-S}}-\frac{\pi(\mu-aS)}{\sigma^2}+\frac{\pi}{4}\right). \end{aligned} \tag{1.21}$$

1.6 First hitting time for the case II

For the case II, in the tables of section 3, we have to find a function $\psi(x,\lambda)$ such that it satisfies

$$L\psi + \lambda\psi = 0 \text{ and } \lim_{x\to S^+} \psi(x,\lambda) = 0. \tag{1.22}$$

We have the

Proposition 1.6.1. *If $a \neq 0$, then the function*

$$\psi(x,\lambda) = \left(\frac{2a(x-S)}{\sigma^2}\right)^{1-\frac{2\mu-2aS}{\sigma^2}} F\left(\frac{-\lambda}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right), \tag{1.23}$$

is solution of (1.22).

Proof. The proof is similar to the Proposition 1.5.1, with the same changes of variable we obtain the Kummer equation. $\square$

Applying the formula (1.10) and (1.4) we arrive at

$$P_x(\tau_y \le t) = \frac{\int_S^x s'(z)dz}{\int_S^y s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)}, \text{ where } s'(z) = e^{\frac{2az}{\sigma^2}} (z-S)^{\frac{2aS-2\mu}{\sigma^2}}. \tag{1.24}$$

Therefore the first hitting time density for the case II is

$$P_x(\tau_y \in dt) = -\sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{(x-S)^{1-\frac{2\mu-2aS}{\sigma^2}} F\left(\frac{-\lambda_n}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(x-S)}{\sigma^2}\right)}{(y-S)^{1-\frac{2\mu-2aS}{\sigma^2}} F_\lambda\left(\frac{-\lambda_n}{a} - \frac{2\mu-2aS}{\sigma^2} + 1, 2 - \frac{2\mu-2aS}{\sigma^2}, \frac{2a(y-S)}{\sigma^2}\right)}. \tag{1.25}$$

To find an approximation of the λ_n (with n large), we use again the formula (1.19), thus

$$\lambda_n \approx -a \times \left[y\left(1 - \frac{\mu - aS}{\sigma^2}\right) - \left\{\frac{\pi}{8} - \frac{\pi}{2}\left(\frac{\mu-aS}{\sigma^2} - n\right)\right\}^2 + \frac{2\mu-2aS}{\sigma^2} - 1\right]. \tag{1.26}$$

We apply again the formula (1.19) if we wish to approximate c_n.

1.7 First hitting time for the case IV

Note that in the previous two sections we consider $a \neq 0$. Now we present an example of a particular situation when $a = 0$. In this case the process X is the solution of

$$dX_t = \mu dt + \sigma\sqrt{X_t - S}dB_t.$$

We consider only the case when $-1 < \frac{-2\mu}{\sigma^2} < 0$ (The other cases in Proposition 1.4.1 with $a = 0$ are similar). From Corollary 1.4.2 we have that the end-point S is regular, and we now assume that S is killing. Therefore we have to find a function $\psi(x, \lambda)$ such that it satisfies

$$\frac{\sigma^2(x-S)}{2}\psi''(x) + \mu\psi'(x) + \lambda\psi(x) = 0 \text{ and } \lim_{x\to S^+}\psi(x,\lambda) = 0. \tag{1.27}$$

Proposition 1.7.1. *The function*

$$\psi(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{(\frac{1}{2}-\frac{\mu}{\sigma^2})} \cdot J_{(1-\frac{2\mu}{\sigma^2})}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right), \tag{1.28}$$

is solution of (1.27), where $J_v(x) := \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{v+2n}}{n!\Gamma(v+n+1)}$.

Proof. It follows by using the formula in Table 15 of [54]. □

Remark 1.7.2. *For y fixed such that $S < x < y$, we find the λ_n such that $\psi(y, \lambda_n) = 0$ in the following manner, let $J_{v,n}$ be the positive zeros of the Bessel function J_v, where $v := 1 - \frac{2\mu}{\sigma^2}$. Then by (1.28) the values λ_n must satisfies the equation*

$$2\sqrt{\frac{2\lambda_n(y-S)}{\sigma^2}} = J_{v,n}. \tag{1.29}$$

Thus

$$\lambda_n = \frac{\sigma^2 J_{v,n}^2}{8(y-S)}. \tag{1.30}$$

Lemma 1.7.3. *Let ψ be the function in (1.28), then*

$$\psi_\lambda(y,\lambda_n) = -\left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{3}{4}-\frac{\mu}{\sigma^2}} \cdot \frac{2\sqrt{2}\sqrt{y-S}}{\sigma\cdot J_{v,n}} \cdot J_{2-\frac{2\mu}{\sigma^2}}\left(J_{v,n}\right). \tag{1.31}$$

Proof. We first compute the derivative of ψ with respect to λ, where ψ is (1.28). Then we arrive at

$$\psi_\lambda(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}} \cdot \sqrt{\frac{2(x-S)}{\sigma^2}} \cdot \frac{1}{2\sqrt{\lambda}} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (v+2k)\left(\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)^{v+2k-1}}{k! \cdot \Gamma(k+v+1)}. \quad (1.32)$$

On the other hand, notice that

$$J_v'(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (v+2n)\left(\frac{x}{2}\right)^{v+2n-1}}{n!\Gamma(v+n+1)}. \quad (1.33)$$

Then by evaluating the function in (1.33) at $\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}$, then (1.32) reads as

$$\psi_\lambda(x,\lambda) = \left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{1}{\sqrt{\lambda}} \cdot J_v'\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right). \quad (1.34)$$

Now we use the following identity founded in [44]:

$$J_v'(z) = -J_{v+1}(z) + \frac{v}{z}J_v(z). \quad (1.35)$$

Applying the identity (1.35) we arrive at

$$\begin{aligned}
\psi_\lambda(x,\lambda) &= \left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{1}{\sqrt{\lambda}} \cdot J_v'\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right) \\
&= \frac{\left(\frac{2(x-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}}}{\sqrt{\lambda}} \left[\frac{\sigma v}{2\sqrt{2\lambda(x-S)}} \cdot J_v\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right) - J_{v+1}\left(2\sqrt{\frac{2\lambda(x-S)}{\sigma^2}}\right)\right].
\end{aligned}$$

Now using (1.30), and the fact that $J_v(J_{v,n}) = 0$, we obtain

$$\begin{aligned}
\psi_\lambda(y,\lambda_n) &= \left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{2\sqrt{2(y-S)}}{\sigma \cdot J_{v,n}} \cdot \left[\frac{\sigma v}{2\sqrt{2\lambda_n(y-S)}} \cdot J_v\left(J_{v,n}\right) - J_{v+1}\left(J_{v,n}\right)\right] \\
&= \left(\frac{2(y-S)}{\sigma^2}\right)^{\frac{v}{2}+\frac{1}{2}} \cdot \frac{2\sqrt{2(y-S)}}{\sigma \cdot J_{v,n}} \cdot \left[-J_{v+1}\left(J_{v,n}\right)\right].
\end{aligned}$$

$\square$

Note that applying the Lemma 1.7.3 we have

$$\frac{\psi(x,\lambda_n)}{\lambda_n \psi_\lambda(y,\lambda_n)} = \frac{-2}{J_{v,n}} \cdot \left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \cdot \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{1+v}\left(J_{v,n}\right)}. \tag{1.36}$$

By joining everything, we obtain:

Theorem 1.7.4. *Let X be the process that is solution of $dX_t = \mu dt + \sigma\sqrt{X_t - S}dB_t$. Suppose that $X_0 = x$ and $S < x < y$, for y fixed. If $-1 < \frac{-2\mu}{\sigma^2} < 0$ and S is killing, then*

$$P_x(\tau_y \leq t) = \left(\frac{x-S}{y-S}\right)^v - 2\cdot\left(\frac{x-S}{y-S}\right)^{\frac{v}{2}} \cdot \sum_{n=0}^{\infty} e^{-\frac{\sigma^2 J_{v,n} t}{8(y-S)}} \frac{J_v\left(J_{v,n}\sqrt{\frac{x-S}{y-S}}\right)}{J_{v,n}\cdot J_{1+v}\left(J_{v,n}\right)}. \tag{1.37}$$

Remark 1.7.5. *When $\sigma = 2$, $\mu = 2v+2$ and $S = 0$, then the process X is the Squared Bessel process. Also note that if $\tau_y^R :=inf\{t > 0 : R_t = y\}$ where R represents the Bessel process, then*

$$\begin{aligned} P_x\left(\tau_y^R \leq t\right) &= P_{x^2}\left(inf\{s > 0 : R_s^2 = y^2\} \leq t\right) \\ &= P_{x^2}\left(inf\{s > 0 : X_s = y^2\} \leq t\right) \\ &= P_{x^2}\left(\tau_{y^2}^X \leq t\right). \end{aligned}$$

Therefore using the formula (1.37), we can recover the formula for the first hitting time density of the Bessel process when $S = 0$ is killing. Indeed, in [44, p.391] one can see such formula:

$$P_x(\tau_y^R \leq t) = \left(\frac{x}{y}\right)^{2v} - 2\left(\frac{x}{y}\right)^v \sum_{n=0}^{\infty} e^{\frac{-J_{v,n}^2 t}{2y^2}} \frac{J_v\left(\frac{x}{y}J_{v,n}\right)}{J_{v,n}J_{1+v}(J_{v,n})}.$$

1.8 The reflected generalized CIR process

In this section we will consider another process Y that is solution of

$$dY_t = (\beta - bY_t)dt + \sigma\sqrt{S - Y_t}\, dB_t, \tag{1.38}$$

with state space $I = (-\infty, S]$ if S is regular reflecting, or $I = (-\infty, S)$ if S is not regular reflecting. The infinitesimal generator is given by

$$Lf(x) = \frac{\sigma^2(S-x)}{2} f''(x) + (\beta - bx)f'(x). \tag{1.39}$$

We want to find the density of $\zeta_y :=\inf\{t>0 : Y_t = y\}$. To this end, we will use the formula of the first hitting time density of the process X that is solution of (1.5). We first present three tables with the results of this section. Similar to the Tables 1, 2, and 3, Table 4 shows the nature of the end-point S and the type of boundary condition, Table 5 shows the solution of equation (1.7), and Table 6 shows the formula (1.10).

Table 4.

Parameters	S	Boundary Condition
I′. $-1 < \frac{2\beta-2bS}{\sigma^2} < 0$ with $b \neq 0$	Regular reflecting	$\lim\limits_{x\to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$
II′. $\frac{2\beta-2bS}{\sigma^2} \geq 0$ with $b \neq 0$	Exit	$\lim\limits_{x\to S^+} \psi(x,\lambda) = 0$
III′. $\frac{2\beta-2bS}{\sigma^2} \leq -1$ with $b \neq 0$	Entrance	$\lim\limits_{x\to S^+} \frac{\psi_x(x,\lambda)}{s'(x)} = 0$

Table 5.

Parameters	Solution of $L\psi + \lambda\psi = 0$
I′. $-1 < \frac{2\beta-2bS}{\sigma^2} < 0$ with $b \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{b}, \frac{2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$
II′. $\frac{2\beta-2bS}{\sigma^2} \geq 0$ with $b \neq 0$	$\psi(x,\lambda) = \left(\frac{2b(x-S)}{\sigma^2}\right)^{1-\frac{2bS-2\beta}{\sigma^2}} F\left(\frac{-\lambda}{b} - \frac{2bS-2\beta}{\sigma^2} + 1, 2 - \frac{2bS-2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$
III′. $\frac{2\beta-2bS}{\sigma^2} \leq -1$ with $b \neq 0$	$\psi(x,\lambda) = F\left(\frac{-\lambda}{b}, \frac{2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right)$

Table 6.

Parameters	Spectral decomposition for the first hitting time density
I′. $-1 < \frac{2\beta-2bS}{\sigma^2} < 0$ with $b \neq 0$	$P_x(\zeta_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-y)}{\sigma^2}\right)}$
II′. $\frac{2\beta-2bS}{\sigma^2} \geq 0$ with $b \neq 0$	$P_x(\zeta_y \leq t) = \frac{\int_S^{2S-x} s'(z)dz}{\int_S^{\overline{y}} s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(2S-x,\lambda_n)}{\lambda_n \psi_\lambda(\overline{y},\lambda_n)}$
III′. $\frac{2\beta-2bS}{\sigma^2} \leq -1$ with $b \neq 0$	$P_x(\zeta_y \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-y)}{\sigma^2}\right)}$

In order to analyze process Y, we use the Itô's formula to construct a new process Z, which will allow us to use the results of previous sections. Suppose that $Y_0 = x$ and let y be fixed such that $S > x > y$. Consider the function $g(x,t) := -x + 2S$. Applying the Itô's formula we arrive at

$$dZ_t = (2bS - \beta - bZ_t)dt + \sigma\sqrt{Z_t - S}\, dB_t, \text{ where } Z_t = -Y_t + 2S. \tag{1.40}$$

This process is (1.5) with $\mu := 2bS - \beta$ and $a := b$. Note that $Z_0 = 2S - x$ and define $\overline{y} := 2S - y$. Then using the Proposition 1.4.1 for the end-point S, we obtain the

Proposition 1.8.1. *The end-point S obeys the following classification*

i. If $-1 < \dfrac{2\beta - 2bS}{\sigma^2} < 0$ *then S is regular.*

ii. If $\dfrac{2\beta - 2bS}{\sigma^2} \geq 0$ *then S is exit.*

iii. If $\dfrac{2\beta - 2bS}{\sigma^2} \leq -1$ *then S is entrance.*

Proof. Notice that the nature of the end-point S for process Y is the same as for process Z. Then we can use Proposition 1.4.1. □

For the cases I′ and III′ in Proposition 1.8.1 we have to find a function $\psi(x,\lambda)$ such that it satisfies

$$\frac{\sigma^2(x-S)}{2}\psi''(x) + (2bS - \beta - bx)\psi'(x) + \lambda\psi(x) = 0, \text{ and } \lim_{x\to S^+}\frac{\psi_x(x,\lambda)}{s'(x)} = 0. \tag{1.41}$$

In turn we have the following

Proposition 1.8.2. *If $b \neq 0$, then the function*

$$\psi(x,\lambda) = F\left(\frac{-\lambda}{b}, \frac{2bS - 2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right), \tag{1.42}$$

is solution of (1.41).

Proof. Is similar to Proposition 1.5.1, with the same changes of variable to obtain the Kummer equation. □

If we define $\eta_y := \inf\{t > 0 : Z_t = y\}$, then

$$P_x(\zeta_y \leq t) = P_{2S-x}(\eta_{\overline{y}} \leq t). \tag{1.43}$$

Then applying the formula (1.10) we arrive at

$$P_x\left(\zeta_y \leq t\right) = P_{2S-x}(\eta_{\overline{y}} \leq t) = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{F\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-x)}{\sigma^2}\right)}{\lambda_n F_\lambda\left(\frac{-\lambda_n}{b}, \frac{2bS-2\beta}{\sigma^2}, \frac{2b(S-y)}{\sigma^2}\right)}. \tag{1.44}$$

Remark 1.8.3. *To find an approximation for λ_n (with n large), we use the formula (1.20)*

$$\lambda_n \approx \left[\frac{\sigma^2}{2b(\overline{y}-S)}\left(\frac{\pi(bS-\beta)}{2\sigma^2} + \frac{n\pi}{2} - \frac{3\pi}{8}\right)^2 - \frac{bS-\beta}{\sigma^2}\right] \times b. \tag{1.45}$$

Remark 1.8.4. *For the case II′, we have to find a function $\psi(x,\lambda)$ such that it satisfies $L\psi + \lambda\psi = 0$ and $\lim\limits_{x \to S^+} \psi(x,\lambda) = 0$. Then, if $b \neq 0$, we obtain that the solution is*

$$\psi(x,\lambda) = \left(\frac{2b(x-S)}{\sigma^2}\right)^{1-\frac{2bS-2\beta}{\sigma^2}} F\left(\frac{-\lambda}{b} - \frac{2bS-2\beta}{\sigma^2} + 1, 2 - \frac{2bS-2\beta}{\sigma^2}, \frac{2b(x-S)}{\sigma^2}\right). \tag{1.46}$$

Then applying the formula (1.10) and (1.4) we arrive at

$$P_x\left(\zeta_y \leq t\right) = P_{2S-x}(\eta_{\overline{y}} \leq t) = \frac{\int_S^{2S-x} s'(z)dz}{\int_S^{\overline{y}} s'(z)dz} + \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\psi(2S-x,\lambda_n)}{\lambda_n \psi_\lambda(\overline{y},\lambda_n)}, \tag{1.47}$$

where $s'(z) = e^{\frac{2bz}{\sigma^2}}(z-S)^{\frac{2\beta-2bS}{\sigma^2}}$. To find an approximation for λ_n (with n large), we use again the formula (1.20). Then we have

$$\lambda_n \approx -b \times \left[\overline{y}\left(1 - \frac{bS-\beta}{\sigma^2}\right) - \left\{\frac{\pi}{8} - \frac{\pi}{2}\left(\frac{bS-\beta}{\sigma^2} - n\right)\right\}^2 + \frac{2bS-2\beta}{\sigma^2} - 1\right]. \tag{1.48}$$

1.9 Numerical example

In this section, using the package Wolfram Mathematica 10.1, we give a illustration of a numerical approximation for the case III of Tables 1, 2 and 3. Consider the process X that is solution of

$$dX_t = (3 - 2X_t)dt + \sqrt{X_t - 1}\, dB_t. \tag{1.49}$$

Suppose that $X_0 = x = \frac{9}{8}$ and $y = \frac{5}{4}$. In this case, applying the Proposition 1.4.1, we have that the end-point $S = 1$ is entrance because

$$\frac{2aS - 2\mu}{\sigma^2} = \frac{2 \cdot 2 \cdot 1 - 2 \cdot 3}{1} = -2 < -1. \tag{1.50}$$

Then the function ψ that we consider is

$$\psi(x, \lambda) = F\left(-4\lambda,\ 1,\ 2x - 1\right). \tag{1.51}$$

Using the formula (1.19) we obtain an approximation for the function ψ, and we also have approximations for the eigenvalues λ_n ($\psi(y, \lambda) \approx 0$ with $y = \frac{5}{4}$) for $n = 1, 2, 3....$ The approximation of the graph of ψ is drawn in Figure 1.

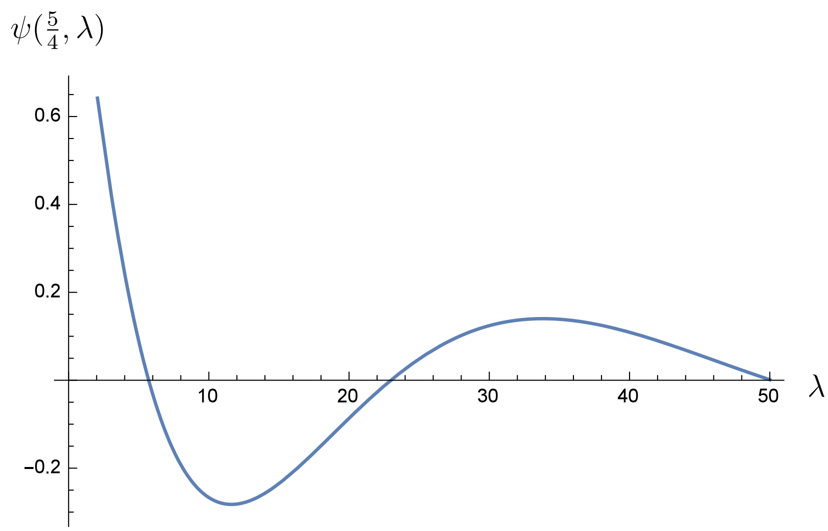

Figure 1.

Now we give an approximation for c_n using the formula (1.21) to have one picture of an approximation the first hitting time density. For our estimation, we have truncated the series (1.17) at the first 100 terms. The graph of this approximation is presented in the Figure 2.

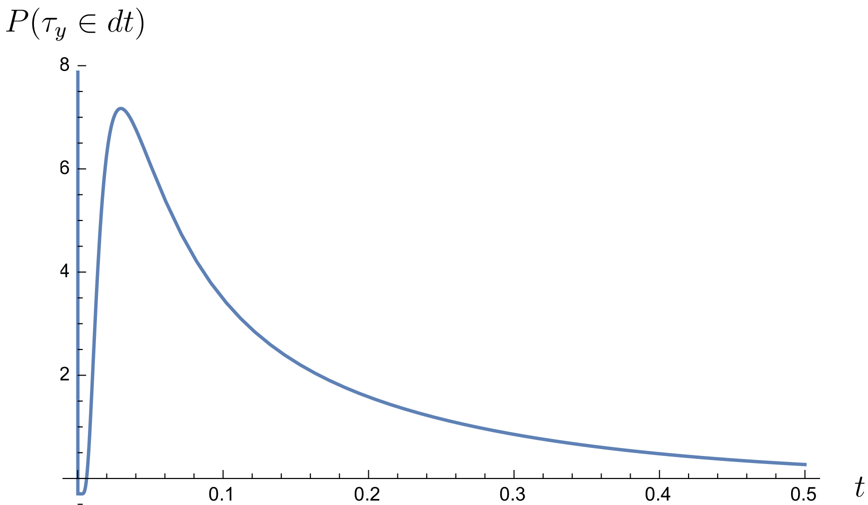

Figure 2.

Chapter 2

Green kernel for random operators

We find explicitly the Green kernel associated to certain differential operators with random coefficients. This task is done using recent ideas on random operators where we appropriately define these operators using the inner product. Important tools that we use are the Sturm-Liouville theory and the stochastic calculus. This chapter is based on [25].

2.1 Introduction

There are plenty of examples of probabilistic models where there is an operator that resembles differential operator with coefficients symbolically given in terms of the derivative of the Brownian motion. The so-called stochastic heat equation or certain random Schrödinger equations are very well known cases studied in the literature. In this paper we work with two examples of random operators, which can be seen as second order differential operators. One of our aims is to find the inverse of such random operators.

This kind of models are instances of the so-called Schrödinger operators with random potential. They have been important in theoretical physics, in particular in the theory of disorder systems, e.g. [53]. The importance of these models is well documented, see for instance [13].

As we mentioned, we find the inverse operator of Schrödinger operators. Let us mention two examples of applications.

In [27] it is consider the Schrödinger operators with random potential given informally by the expression

$$Lf(t) = -f''(t) + W'(t)f(t),\ t \in [0,1],$$

where W' is white noise and can be thought as the derivative of the Brownian motion. One very first task is to give a proper meaning of the operator L. As shown in [20] such operator has a discrete spectrum given by a set of eigenvalues. It turns out one can give expressions of the inverse operator, see [51], which leads to spectral information.

In the context of random processes with random environment an important model is the so-called Brox diffusion, see [12], amply studied in the literature. This process can be worked out as a Markov process, and informally speaking the generator has the form

$$Lf(t) = \frac{1}{2}\left(-f''(t) + W'(t)f'(t)\right), \; t \in \mathbb{R}.$$

It turns out that one can analyze L by finding its inverse, as done in the companion paper [26], where a bounded version of the Brox diffusion is studied. Moreover, there is a remarkable similarity with an operator arising in the theory of random matrices, see [57]. Loosely speaking, such operator plays the role of the infinite random matrix, and the spectrum helps to characterize the limiting eigenvalues of a random matrix

As it is traditionally thought, knowing spectral information of the inverse helps to analyze the differential operator. As demonstrated in [26], the inverse of L helps to obtain spectral information which eventually leads to information of the probability density function. From a more theoretical point of view, one can see that is possible to deal with the inverse in fairly friendly way, with out, for instance, making use of machinary such as the theory of distributions. This is so, courtesy of well-known tools in the Sturm-Liouville theory and the stochastic calculus.

In this paper, the two operators that we consider are:

$$(Lf)(t) := \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2},$$

and

$$(Lf)(t) := f''(t) - W(t)f'(t) - W'(t)f(t).$$

In order to make sense of the term W', we will define these operators in a weak sense using the inner product. In that way, we can make sense of the term $\int_a^b W'(t)h(t)dt$ by rewriting it as

$$\int_a^b h(t)dW(t). \tag{2.1}$$

After specifying the domains, our goal is to find the inverse of these two operators, called also the Green operator. In the classical Sturm-Liouville theory, to tackle this problem one should consider the solutions of the homogeneous problem $Lf = 0$. Here we will also consider the solutions of the homogeneous equation but in a weak sense, again using the inner product. It turns out that the homogeneous solutions are explicit functions of the Brownian motions.

We start the following section with some preliminaries. Then in Section 3 we deal with the first operator and find explicitly the solutions of $Lf = 0$; in addition we show the construction of the solutions using approximations of the Brownian motion. In Section 4 we work with the second operator, and we also find explicit solutions of the homogeneous equation.

2.2 Preliminaries

We will work with two random operators whose domain are functions defined on an interval $[a, b]$. More precisely, the domain is the set of functions $f \in L_2[a, b]$ that satisfies the Dirichlet conditions $f(a) = 0 = f(b)$. Our first goal is to give a rigorous definitions of the operators we work with. Next we find solutions for the homogeneous equation which eventually will lead to the inverse operator.

The first operator that we consider has the following formal expression:

$$(Lf)(t) = \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2},$$

where $W := \{W(t) : t \in \mathbb{R}\}$ is a two sided Brownian motion, and W' denotes its derivative, sometimes called the white noise. Originally the domain of this operator are the functions in $C^2(\mathbb{R})$, however we will work with this operator on an interval (a, b) in a week sense, and with functions in $C^1((a, b))$

The second operator that we consider can be expressed as follows:

$$(Lf)(t) = f''(t) - W(t)f'(t) - W'(t)f(t).$$

To define properly the domain of our operators, we need to introduce the following Hilbert spaces:

$$H := \{h \in L_2[a, b] : h(a) = h(b) = 0\},$$

and

$$H_1 := \{h \in H : h \text{ is absolutely continuous}\}.$$

With these ingredients, one may define the operator L through the inner product. Consider

$$\langle Lf, h\rangle = \int_a^b Lf(t)h(t)dt, \quad \text{for all } \ f, h \in H_1. \tag{2.2}$$

Then, we can propose a specific expression for $\langle Lf, h\rangle$ to define how L acts on f. On the other hand, as we mentioned in the Introduction, we need to find the solutions of the homogeneous equation $Lf = 0$, then we consider the following definition. At this point, we should mention that this way of thinking can be found in the work of Anatolii Vladimirovich Skorohod, see [67].

Keeping in mind the random operators L described above, we have the following concept.

Definition 2.2.1. *We say that a stochastic process $\{u(t) : t \in [a, b]\}$ is a solution of the equation $Lf = 0$, if for all $h \in H_1$,*

$$\langle Lu, h\rangle = 0 \ \textit{ almost surely.} \tag{2.3}$$

In turns out that it is possible to find solutions of this problem for the operators we consider.

2.3 With random potential

Informally speaking, we consider the following stochastic operator

$$(Lf)(t) = \frac{f''(t)}{2} - \frac{W'(t)f'(t)}{2}. \tag{2.4}$$

Taking into account equation (2.1) and (2.2) we define the operator (2.4) in the following weak sense

$$\langle Lf, h\rangle := \int_a^b \frac{f''(t)h(t)}{2}dt - \int_a^b \frac{f'(t)h(t)}{2}dW(t). \tag{2.5}$$

We go an step further and instead of (2.5), we use integration by parts to obtain the following definition.

Definition 2.3.1. *The stochastic operator L is such that for any pair $f, h \in H_1$*

$$\langle Lf, h\rangle := -\int_a^b \frac{f'(t)h'(t)}{2}dt - \int_a^b \frac{f'(t)h(t)}{2}dW(t). \tag{2.6}$$

This definition can be applied for functions $f \in C^1((a,b))$, not necessarily in H_1.

Our main aim is to construct the so-called Green operator associated to L. To do this task, we notice that we need to find two linearly independent solutions of the problem $Lf = 0$. We say that two solutions u and v are **linearly independent** if the paths are almost always linearly independent functions.

In this case, a way to construct two linearly independent solutions is to find a solution u such that $u(a) = 0$ and $u(b) = 1$ almost surely, and other solution v such that $v(a) = 1$ and $v(b) = 0$ almost surely. This guarantees the desired property.

It turns out that the two linearly independent solutions always exist; we will prove this fact later on. For the moment, let us suppose that we already have the two solutions u and v of homogeneous equation. With these functions, we are going to construct an operator T, called the Green operator, which will be the inverse operator of L.

The following theorem shows how to use the two solutions of the homogeneous problem to construct T. We take the idea of this constructions from the Sturm-Liouville theory.

Theorem 2.3.2. *Let u, v be solutions of $Lf = 0$, such that $u(a) = 1$ and $u(b) = 0$ a.s., and $v(a) = 0$ and $v(b) = 1$ a.s. The stochastic Green operator associated to L is given by*

$$(Tf)(t) := \int_a^b G(t,s)f(s)ds, \tag{2.7}$$

where

$$G(t,s) := \begin{cases} \dfrac{2u(t)v(s)}{\alpha(s)}, & a \le s \le t \le b; \\ \dfrac{2u(s)v(t)}{\alpha(s)}, & a \le t \le s \le b. \end{cases},$$

and

$$\alpha(t) := u'(t)v(t) - v'(t)u(t).$$

The operator T in (2.7) is the right inverse of L in the sense that for all $h \in H_1$

$$\langle LTf, h\rangle = \langle f, h\rangle \ \text{ almost surely.}$$

Proof. Let u, v be solutions of $Lf = 0$, such that $u(a) = 0$ and $u(b) = 1$ always, and that $v(a) = 1$ and $v(b) = 0$ always as well.

Note that

$$(Tf)(t) = 2u(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + 2v(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds. \tag{2.8}$$

On calculating the derivative of (2.8) we obtain

$$\frac{d[(Tf)(t)]}{dt} = 2u'(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + \frac{2u(t)v(t)f(t)}{\alpha(t)} + 2v'(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds - \frac{2u(t)v(t)f(t)}{\alpha(t)}. \tag{2.9}$$

Note that in the above expression the first and last term are canceled. Now, by using Definition 2.3.1

$$\langle L(Tf), h\rangle = \frac{-1}{2}\left[\int_a^b (Tf(t))'h'(t)dt + \int_a^b (Tf(t))'h(t)dW(t)\right]. \tag{2.10}$$

Inserting (2.9) into (2.10) we arrive at

$$\begin{aligned} \langle L(Tf), h\rangle &= -\int_a^b u'(t)\left[\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]h'(t)dt - \int_a^b u'(t)\left[\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]h(t)dW(t) \\ &- \int_a^b v'(t)\left[\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]h'(t)dt - \int_a^b v'(t)\left[\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]h(t)dW(t). \end{aligned} \tag{2.11}$$

Therefore, if we add and subtract in (2.11) the following two terms

$$\int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)}dt, \quad \text{and} \quad \int_a^b \frac{u(t)v'(t)f(t)h(t)}{\alpha(t)}dt,$$

and we use the fact that

$$\left[h(t)\int_a^t \frac{v(s)f(s))}{\alpha(s)}ds\right]' = h'(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + h(t)\frac{v(t)f(t)}{\alpha(t)}, \tag{2.12}$$

we arrive at

$$\begin{aligned}\langle L(Tf),h\rangle &= -\int_a^b u'(t)\left[h(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]' dt - \int_a^b u'(t)\left[h(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right] dW(t)\\ &- \int_a^b v'(t)\left[h(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]' dt - \int_a^b v'(t)\left[h(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right] dW(t)\\ &+ \int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)}dt - \int_a^b \frac{v'(t)u(t)f(t)h(t)}{\alpha(t)}dt. \end{aligned}\tag{2.13}$$

Now, using the fact that u and v are solutions of $Lf = 0$ in the sense of Definition 2.2.1, we see that only the last two terms in (2.13) survive. Thus we finally arrive at

$$\begin{aligned}\langle L(Tf),h\rangle &= \int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)}dt - \int_a^b \frac{v'(t)u(t)f(t)h(t)}{\alpha(t)}dt\\ &= \int_a^b \left[\frac{u'(t)v(t) - u(t)v'(t)}{\alpha(t)}\right] f(t)h(t)dt\\ &= \langle f,h\rangle,\end{aligned}$$

where we have substitute the very definition of α. This concludes the proof. □

To use this theorem we need to find the two solutions of $Lf = 0$. We do so by using approximations of Brownian motion.

First, to obtain intuitively such so solutions we consider the followings approximations of W, sometime called approximation by transport processes, see [22].

$$W_n(t) := n\left[\left(\frac{j+1}{n} - t\right)W\left(\frac{j}{n}\right) + \left(t - \frac{j}{n}\right)W\left(\frac{j+1}{n}\right)\right],$$

where $t \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$, and $j = 0, \pm1, \pm2...$ Therefore, the random function W_n is almost everywhere differentiable.

Then the following equation is valid for almost every $t \in [a,b]$

$$V_n''(t) = W_n'(t)V_n'(t).$$

We want to use $V_n(t)$ to find heuristically a solution of

$$V''(t) = W'(t)V'(t).$$

We consider the change of variable $Z_n(t) := V_n'(t)$. Then we obtain the new equation

$$Z_n'(t) = W_n'(t) Z_n(t). \tag{2.14}$$

From the Corollary of Theorem 7.3 of [32], we have that there exist a sequence $Z_n(t)$ of solutions of (2.14) such that, with probability one

$$Z_n(t) \to Z(t), \quad \text{as } n \to \infty, \tag{2.15}$$

where $Z(t)$ is solution of the stochastic differential equation

$$dZ(t) = Z(t) dW(t). \tag{2.16}$$

Then we obtain that with probability one

$$V_n'(t) \to Z(t), \quad \text{as } n \to \infty. \tag{2.17}$$

On the other hand, the equation (2.16) has unique solution, and this solution is

$$Z(t) = e^{W(t) - \frac{t}{2}}. \tag{2.18}$$

Hence

$$V_n'(t) \to e^{W(t) - \frac{t}{2}}, \quad \text{as } n \to \infty. \tag{2.19}$$

This implies that

$$V_n(t) \to \int_a^t e^{W(s) - \frac{s}{2}} ds, \quad \text{as } n \to \infty. \tag{2.20}$$

In the following theorem we verify rigourously that $v(t) := C \cdot \int_a^t e^{W(s)-\frac{s}{2}} ds$ satisfies $Lu = 0$, where C is an appropriate constant. We also consider other solution v that we need to construct the Green operator.

Theorem 2.3.3. *Two linearly independent solutions of the problem $Lf = 0$ are the following integrals of Geometric Brownian motion*

$$u(t) := \frac{\displaystyle\int_t^b e^{W(s)-\frac{s}{2}} ds}{\displaystyle\int_a^b e^{W(s)-\frac{s}{2}} ds}. \tag{2.21}$$

$$v(t) := \frac{\displaystyle\int_a^t e^{W(s)-\frac{s}{2}} ds}{\displaystyle\int_a^b e^{W(s)-\frac{s}{2}} ds}. \tag{2.22}$$

Furthermore, they satisfy $u(a) = 1$, $u(b) = 0$, $v(a) = 0$ and $v(b) = 1$.

Proof. We verify that v is solution of $Lf = 0$. For u is similar. To do that, according to Definition 2.3.1 we want to show that $\langle Lv, h\rangle = 0$ for all $h \in H_1$, i.e.

$$\int_a^b \frac{v'(t)h'(t)}{2}dt + \int_a^b \frac{v'(t)h(t)}{2}dW(t) = 0. \tag{2.23}$$

From the definition of u in (2.21), we have

$$v'(t) = \frac{e^{W(t)-\frac{t}{2}}}{\displaystyle\int_a^b e^{W(s)-\frac{s}{2}}ds}. \tag{2.24}$$

Let $\beta := \left[\int_a^b e^{W(s)-\frac{s}{2}}ds\right]^{-1}$, then

$$\langle Lv, h\rangle = \frac{-\beta}{2}\left[\int_a^b e^{W(t)-\frac{t}{2}}h'(t)dt + \int_a^b e^{W(t)-\frac{t}{2}}h(t)dW(t)\right]. \tag{2.25}$$

On the other hand, applying the Itô's formula we obtain

$$\int_a^b e^{W(s)-\frac{s}{2}}h(s)dW(s) = h(b)e^{W(b)-\frac{b}{2}} - h(a)e^{W(a)-\frac{a}{2}} - \int_a^b e^{W(s)-\frac{s}{2}}h'(s)ds. \tag{2.26}$$

Substituting (2.26) in (2.25), and recalling that $h \in H_1$, we arrive at

$$\langle Lv, h\rangle = \frac{-\beta}{2}\left[\int_a^b e^{W(t)-\frac{t}{2}}h'(t)dt + h(b)e^{W(b)-\frac{b}{2}} - h(a)e^{W(a)-\frac{a}{2}} - \int_a^b e^{W(t)-\frac{t}{2}}h'(t)dt\right] = 0. \tag{2.27}$$

□

2.4 With random potential and random coefficient

In this section we consider an operator with the following formal expression:

$$(Lf)(t) = f''(t) - W(t)f'(t) - W'(t)f(t). \tag{2.28}$$

We define the operator (2.28) in the following weak sense, using the inner product

$$\langle Lf, h\rangle := \int_a^b f''(t)h(t)dt - \int_a^b f'(t)W(t)h(t)dt - \int_a^b f(t)h(t)dW(t). \tag{2.29}$$

Now, we use integration by parts in the first term of (2.29) and the Itô's formula in the third term of (2.29) to obtain the following definition

Definition 2.4.1. *The stochastic operator L is such that for any pair $f, h \in H_1$:*

$$\langle Lf, h\rangle := -\int_a^b f'(t)h'(t)dt + \int_a^b f(t)h'(t)W(t)dt. \tag{2.30}$$

We want to construct the Green operator associated to L. To this end, we need to find two solutions linearly independent of the homogeneous equation. Intuitively we have

$$f''(t) - W(t)f'(t) - W'(t)f(t) = 0.$$

This equation can be rewritten as

$$f''(t) = [W(t)f(t)]'.$$

Moreover, integrating both side we arrive at

$$f'(t) = W(t)f(t) + C, \quad \text{where C is a constant.}$$

This equation is easy to solve, and we exhibit the solutions in the following theorem. However, we rigourously verify that the solutions satisfies the equation $Lf = 0$.

Theorem 2.4.2. *Two linearly independent solutions of the problem $Lf = 0$ are the following*

$$v(t) := \frac{e^{\int_a^t W(s)ds} \displaystyle\int_a^t e^{-\int_a^s W(r)dr} ds}{e^{\int_a^b W(s)ds} \displaystyle\int_a^b e^{-\int_a^s W(r)dr} ds}, \tag{2.31}$$

$$u(t) := \frac{e^{\int_a^t W(s)ds} \displaystyle\int_t^b e^{-\int_a^s W(r)dr} ds}{e^{\int_a^b W(s)ds} \displaystyle\int_a^b e^{-\int_a^s W(r)dr} ds}. \tag{2.32}$$

Furthermore, they satisfy $u(a) = 1$, $u(b) = 0$, $v(a) = 0$ and $v(b) = 1$.

Proof. Let us verify that v is solution. For u is similar. According to the Definition 2.4.1 we need to show that $\langle Lv, h\rangle = 0$ for all $h \in H_1$, i.e.

$$-\int_a^b v'(t)h'(t)dt + \int_a^b v(t)h'(t)W(t)dt = 0. \tag{2.33}$$

Note that

$$v'(t) = \frac{W(t)e^{\int_a^t W(s)ds} \displaystyle\int_a^t e^{-\int_a^s W(r)dr} ds + 1}{e^{\int_a^b W(s)ds} \displaystyle\int_a^b e^{-\int_a^s W(r)dr} ds}. \tag{2.34}$$

Substituting (2.34) and the definition of v in (2.31), we end up with $\langle Lv, h\rangle = 0$. □

Using previous two solutions, we construct the Green operator. The following theorem shows the construction.

Theorem 2.4.3. *Let u, v two solution of $Lf = 0$, such that $u(a) = 1$ and $u(b) = 0$ always, and $v(a) = 0$ and $v(b) = 1$ always. The stochastic Green operator associated to L is given by*

$$(Tf)(t) := \int_a^b G(t,s)f(s)ds, \tag{2.35}$$

where

$$G(t,s) := \begin{cases} \dfrac{u(t)v(s)}{\alpha(s)}, & a \le s \le t \le b; \\ \dfrac{u(s)v(t)}{\alpha(s)}, & a \le t \le s \le b. \end{cases},$$

and

$$\alpha(t) := u'(t)v(t) - v'(t)u(t).$$

This operator T is the right inverse of the operator L in the sense that for all $h \in H_1$

$$\langle LTf, h\rangle = \langle f, h\rangle \quad \textit{almost surely}$$

Proof. We want to proof that $\langle L(Tf), h\rangle = \langle f, h\rangle$. First note that

$$(Tf)(t) = u(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + v(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds. \tag{2.36}$$

Calculating the derivative of (Tf) and simplifying yield

$$\frac{d[(Tf)(t)]}{dt} = u'(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds + v'(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds. \tag{2.37}$$

From the Definition 2.30 we have

$$\langle L(Tf), h\rangle = -\int_a^b (Tf)'(t)h'(t)dt + \int_a^b (Tf)(t)h'(t)W(t)dt. \tag{2.38}$$

After plugging (2.37) into (2.38) one arrives at

$$\begin{aligned} \langle L(Tf), h\rangle &= -\int_a^b u'(t)\left[\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]h'(t)dt + \int_a^b u(t)\left[\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]h'(t)W(t)dt \\ &- \int_a^b v'(t)\left[\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]h'(t)dt + \int_a^b v(t)\left[\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]h'(t)W(t)dt. \end{aligned} \tag{2.39}$$

Now, we add and subtract in (2.39) the following three terms:

$$\int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)}dt, \tag{2.40}$$

$$\int_a^b \frac{u(t)v'(t)f(t)h(t)}{\alpha(t)}dt, \tag{2.41}$$

$$\int_a^b \frac{u(t)v(t)f(t)h(t)W(t)}{\alpha(t)}dt. \tag{2.42}$$

Hence, after calculations,

$$\begin{aligned}\langle L(Tf), h\rangle &= -\int_a^b u'(t)\left[h(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]' dt + \int_a^b u(t)\left[h(t)\int_a^t \frac{v(s)f(s)}{\alpha(s)}ds\right]' W(t)dt \\ &- \int_a^b v'(t)\left[h(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]' dt + \int_a^b v(t)\left[h(t)\int_t^b \frac{u(s)f(s)}{\alpha(s)}ds\right]' W(t)dt \\ &+ \int_a^b \frac{u'(t)v(t)f(t)h(t)}{\alpha(t)}dt - \int_a^b \frac{v'(t)u(t)f(t)h(t)}{\alpha(t)}dt. \end{aligned} \tag{2.43}$$

Using the fact that $Lu = 0$ and $Lv = 0$, we obtain the result. $\square$

Chapter 3

The killed Brox diffusion

We give explicit expressions of the probability density function of the Brox diffusion with killing in terms of the spectral decomposition. Such decomposition is in terms of the eigenvalues and eigenfuntions of the infinitesimal generator. To achieve this task we find in close form the Green operator associated to the generator. With such operator we use the Itô's formula to find stochastic differential equations for to characterize the eigenfunctions. In addition, we recast the theory of Sturm-Liouville to analyze the generator of the Brox process. This chapter is based on [26].

3.1 Introduction

Probabilistic models with random environment are important models to capture some kind of external randomness that there exists in the medium. From this type of modelling different mathematical tools have been deviced to analyze their complexities; probably a good starting point to initiate being acquainted with the random-environment phenomena is the book by B.D. Hughes [31]. A popular model of the sort is what some people call the Sinai's random walk, or also called the Temkin's model, which run in discrete time and discrete state space. The continuous time-space idealization of the Temkin's model is what we call the Brox diffusion, precisely because TH. Brox carefully studied it in [12] in 1989, and proved the same asymptotic behavior as Y.G. Sinai did for the Temkin's model in 1982. In [63], this model was studied as a diffusion with random coefficients.

Different studies have been carried out to have more understanding of the Brox diffusion, let us give a brief account. For instance in [37] it is analyzed the limit behavior of the process as time evolves. Asymptotic behavior regarding the first passage time has been also studied in [38]. In [65] it is studied some asymptotics of the local time, and in [29] more ideas on sample path asymptotic are carried out. Interesting formulas were discovered in [14] in connection with functional of the environment. Yet, without conditioning on the environment more asymptotic information about the process was found in [71] and more understanding on the paths behavior was found in [15]. In [36], limit behavior about occupation time is worked out. A relevant detail analysis on asymptotic dynamics of the local time is done in [4], and more recently in [30] it is proposed stochastic differential equations driven by the Brox process.

Strictly speaking the Brox diffusion becomes a diffusion only after freezing an environment (i.e. conditioning on one realization of the environment), which gives rise to the so-called quenched case, and without conditioning it is called the annealed case. Informally speaking, the generator L acting on f has the form

$$[Lf](x) = \frac{1}{2}\left\{f''(x) - W'(x)f'(x)\right\},$$

where W represents the Brownian motion. Since in the quenched case one is dealing with a bona fide diffusion, the whole apparatus of diffusions, for instance the one manufactured by K. Itô and H.P. McKean [33], can be used for this model. Moreover, one knows that behind the theory of diffusions there is lurking the Sturm-Liouville theory of second order linear operators.

In case of the Brox diffussion the generator is somewhat different to the traditional ones with differentiable coefficients. However, as it turns out, one can addapt many of the results in second order operators, which helps to say a lot about the generator, and ultimately on the stochastic process.

What we do in this chapter is to build the killed version of the Brox diffusion, which ends up having a finite state space. This version of the Brox process allows us to recast results from the theory of Sturm-Liouville and analyze the generator in such a way that it becomes feasible to write down an spectral representation of the probability density function. Such representation, as one might expect, is in terms of the eigenfunction and eigenvalues of the generator. It should be mention that an important tool in this analysis is the so-called Green operator, that is the inverse of the generator. We are able to find explicitly the Green operator, which becomes of tremendous help at the time to conclude about the eigenfunctions of the generator. Important to mention is that our results are stated in the quenched case.

Let us explain how this work is organized. In the coming section we present the original construction of the Brox diffusion, making emphasis on the domain of the generator; something we need to know for our purposes. In section 3 we build the killed Brox process and give results on its generator, in particular we make use of what people call the Wronskian to analyze solutions of certain related equation. In Section 4 we find the Green operator; additionally we need to establish the validity of the Lagrange identity in this context. Section 5 has the spectral representation of the density function, which is in terms of the eigenvalues and eigenfunctions. Precisely, Section 6 has a dissection of the eigenfunctions and eigenvalues, providing a way to obtain these elements. In doing so, we establish a version of an oscillation theorem suited for our generator. At the end of Section 6, we also use the Ricatti transformation to provide yet another stochastic differential equation which is of interest. There is one appendix at the end with the proof using the classic Prüfer method to study zeros of eigenfunctions, again, such proof suited for our generator.

3.2 Brox diffusion

The Brox diffusion is usually described with the following stochastic differential equation

$$dX_t = -\frac{1}{2}W'(X_t)dt + dB_t, \tag{3.1}$$

where $B := \{B_t : t \geq 0\}$ is the standard Brownian motion, and $W := \{W(x) : x \in \mathbb{R}\}$ is a two sided Brownian motion, and they both are independent from each other. Here W' denotes the derivative of W, sometimes called the white noise.

The expression (3.1) needs to have a rigorous meaning. It happens that one can use the associated generator in order to construct properly the process. When looking the equation (3.1) one could say that the process $X := \{X_t : t \geq 0\}$ has associated the infinitesimal operator given by

$$Lf(x) := \frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{df(x)}{dx}\right).$$

This corresponds to considering the scale function

$$s(x) := \int_0^x e^{W(y)}dy, \tag{3.2}$$

and the speed measure

$$m(A) := \int_A 2e^{-W(y)}dy, \text{ for Borel sets } A \subseteq \mathbb{R}. \tag{3.3}$$

Then one considers rigourously the operator

$$\frac{d}{dm}\frac{d}{ds}f.$$

To learn about the derivatives with respect functions or measures we refer the reader to [60, 33, 23]. We want to see that the process X associated with this operator is indeed a diffusion, when leaving fixed W. One can appeal to the theory in [33] to build a process Y in natural scale through the speed measure m_Y and the local time. The procedure is done in the following way:

$$Y_t = B_{T_t^{-1}},$$

where

$$T_t := \frac{1}{2}\int_{-\infty}^{\infty} L_t(x)m_Y(dx).$$

The process $Y_t := s(X_t)$ is in natural scale. Now, we consider the problem 3.18, p.310 from [60], and we obtain that the new scale function and speed measure of the process $Y := s(X)$ are

$$s_Y(x) = x, \text{ and}$$

$$m_Y(dx) = 2e^{-2W(s^{-1}(x))}dx.$$

Then we define the Brox process as

$$X_t = s^{-1}(B_{T_t^{-1}}),$$

where

$$T_t := \int_0^t e^{-2W(s^{-1}(B_u))}du.$$

Let us now describe the domain of the generator.

Theorem 3.2.1. *For any environment W, the domain $D(L)$ is contained in the space of differential functions $C^1(\mathbb{R})$. Moreover the derivative of a function f in $D(L)$ takes the form $f'(x) = e^{W(x)}g(x)$ with $g \in C^1(\mathbb{R})$.*

Proof. According to Mandl [46], p.22, if $h(x) := Lf(x)$ for $f \in D(L)$, then

$$f(x) = \int_a^x \int_a^y h(z)dm(z)ds(z) + f(a) + [s(x) - s(a)] \frac{df}{ds}(a), \tag{3.4}$$

where

$$\frac{df}{ds}(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{s(a+h) - s(a)} = \frac{f'(a)}{e^{W(a)}}. \tag{3.5}$$

Thus, using (3.4) and (3.5), as well as the speed measure (3.3) and the scale function (3.2), we have that

$$f(x) = 2\int_a^x \int_a^y h(z)e^{-W(z)}e^{W(y)}dzdy + f(a) + \frac{f'(a)}{e^{W(a)}} \int_a^x e^{W(z)}dz. \tag{3.6}$$

Therefore we obtain that $f \subseteq C^1(\mathbb{R})$. Now, if we calculate explicitly the derivative of f, we arrive to

$$f'(x) = e^{W(x)} \cdot \left[2\int_a^x h(z)e^{-W(z)}dz + \frac{f'(a)}{e^{W(a)}}\right].$$

Then we have that $f'(x) = e^{W(x)}g(x)$ with $g \in C^1(\mathbb{R})$. $\square$

We use the previous construction of the Brox process for construct the Brox process with killing in a and b.

3.3 Brox process with killing

Let $a < b$. We first consider the Brownian motion with killing in $s(a)$ and $s(b)$, where s is the scale function of the Brox process, i.e.

$$s(x) := \int_0^x e^{W(y)}dy.$$

Since s is non-drecreasing, $s(a) < s(b)$.

Hence, according to [10], p.105, the infinitesimal operator associated to the Brownian motion with killing at $s(a)$ and $s(b)$ is

$$L_B f(x) = \frac{1}{2} f''(x), \quad s(a) < x < s(b),$$

where the domain of this operator is

$$D(L_B) = \{f : L_B f \in C_b([a,b]), f(s(a)) = f(s(b)) = 0\}.$$

We use this Brownian with killing and the construction in the previous section to construct the Brox process with killing. We consider the following idea, taken from [12], p.1216.

Start with the Brownian motion with killing in $s(a)$ and $s(b)$, denote it by $\bar{B}$. Now, we consider

$$\bar{T}_t := \int_0^t e^{-2W(s^{-1}(\bar{B}_u))} du,$$

Then the new process $\bar{X}$ defined as

$$\bar{X}_t = s^{-1}(\bar{B}_{\bar{\gamma}_t}),$$

is the Brox process with killing in a and b. Where s is the scale function associated to the Brox process, and $\bar{\gamma}$ is the inverse of $\bar{T}$.

Thus, the infinitesimal operator associated to $\bar{X}$ is

$$\bar{L} f(x) = \frac{d}{dm} \frac{d}{ds} f(x),$$

where m and s are the speed measure and the scale function of the Brox process, and from general theory of diffusions, see e.g. [33, 10], we know that the domain of $\bar{L}$ is the set of functions in the domain of the generator of the Brox process which are zero at a and b. Therefore, the domain $D(\bar{L})$ satisfies the following conditions:

- $f(a) = 0$,
- $f(b) = 0$,
- $f'(x) = e^{W(x)} g(x), \; g \in C^1(\mathbb{R})$.

In general, $\bar{L} f$ is well defined whenever $e^{-W(x)} f'(x)$ is still differentiable, we denote this as

$$D_0 = \left\{ f \;\; \text{differentiable} : e^{-W(x)} f'(x) \text{ is differentiable} \right\}.$$

Now we present a couple of results regarding the generator and solutions of the so-called eigenfunction equation. These results will be very useful for the rest of the paper. In the following result we regard $\bar{L}$ as an operator acting not just on $D(\bar{L})$ but on any function in D_0. Abusing the notation, sometimes we write $f(x, \lambda)$ in lieu of $f(x)$ to emphasize the dependence with the spectral parameter λ.

Proposition 3.3.1. *i) The operator $\bar{L}$ can be applied as*

$$\bar{L}f(x) = \frac{e^{W(x)}}{2}\left(e^{-W(x)}f'(x)\right)',$$

for any differentiable functions $f \in D_0$.
ii) For any $\lambda > 0$, the problem

$$\bar{L}g + \lambda g = 0, \text{ given } g(a,\lambda) = 0 \text{ and } g'(a,\lambda) = 1,$$

admits a solution in D_0, where g' means the derivative with respect to x.Moreover, such solution satisfies the equation

$$g(x,\lambda) = -2\lambda \int_a^x \int_a^y g(z,\lambda)e^{-W(z)}e^{W(y)}dzdy + \frac{1}{e^{W(a)}}\int_a^x e^{W(z)}dz. \tag{3.7}$$

iii) The converse of ii) is also true. That is, if g satisfies (3.7), then g solves the problem $\bar{L}g + \lambda g = 0$, $g(a,\lambda) = 0$ and $g'(a,\lambda) = 1$.

Proof. i) We simply calculate $\frac{d}{dm}\frac{d}{ds}f(x)$ using the speed measure m and the scale function s. Then we have

$$\begin{aligned}
\frac{d}{dm}\frac{d}{ds}f(x) &= \frac{d}{dm}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{\int_x^{x+h} e^{W(y)}dy}\right)\\
&= \frac{d}{dm}\left(e^{-W(x)}f'(x)\right)\\
&= \lim_{h\to 0}\frac{e^{-W(x+h)f'(x+h)} - e^{-W(x)f'(x)}}{\int_x^{x+h} 2e^{-W(y)}dy}\\
&= \frac{e^{W(x)}}{2}\left(e^{-W(x)}f'(x)\right)'.
\end{aligned}$$

ii) The existence of a solution comes from general theory of diffusions, see e.g. [33], specifically in Section 4.6, page 128. To prove the second statement, we apply twice a fundamental theorem of calculus adapted to derivatives with respect a measure, see e.g. [60, 23]. Indeed, using the definition of $\bar{L}f(x) = \frac{d}{dm}\frac{d}{ds}f(x)$ in the equation $\bar{L}g + \lambda g = 0$, we calculate first the integral with respect to m and later with respect to s, from where equation (3.7) arises.
iii) The proof of this fact is simply to calculate on the equation the derivative with respect to s and later with respect a m. □

In the theory of differential equations, the the so-called Wronskian is an important tool to detect whether there is dependance among solutions of a differential equation. As one might expect, it turns out that in this context with a random coefficients the Wronskian can be calculated and it is used as well for the same purpose. It is defined as the following determinant,

$$w_{f,g}(x) := \begin{vmatrix} g(x) & f(x) \\ \frac{dg}{ds}(x) & \frac{df}{ds}(x) \end{vmatrix} = \frac{f'(x)g(x)}{e^{W(x)}} - \frac{f(x)g'(x)}{e^{W(x)}}.$$

where f and g are two functions, and $e^{W(x)}$ is the density of the scale function associated to the Brox process.

Proposition 3.3.2. *Let f and g be two solutions of $\bar{L}\psi + \lambda\psi = 0$, with $x \in [a,b]$, such that $f(a,\lambda) = 0$ and $g(a,\lambda) = 0$. Then $\frac{dw_{f,g}(x)}{dx} = 0$, which implies that there exists a constant C such that $f = Cg$.*

Proof. Using Proposition 3.3.1, if f and g are solution of $\bar{L}\psi + \lambda\psi = 0$, then

$$\frac{e^{W(x)}}{2}\left[\left(e^{-W(x)}\psi'(x)\right)' + \lambda\psi(x)\right] = 0, \tag{3.8}$$

which implies

$$\left(\frac{\psi'(x)}{e^{W(x)}}\right)' = (-2)\frac{\lambda\psi(x)}{e^{Wx}}.$$

Hence, we can substitute into the derivative of the Wronskian to see that

$$\begin{aligned} \frac{dw_{f,g}(x)}{dx} &= \left(\frac{f'(x)g(x)}{e^{W(x)}} - \frac{f(x)g'(x)}{e^{W(x)}}\right)' \\ &= \left(\frac{f'(x)}{e^{W(x)}}\right)' g(x) - \left(\frac{g'(x)}{e^{W(x)}}\right)' f(x) \\ &= 0 \end{aligned}$$

This implies that $w_{f,g}(x) = M$ for some constant M, so

$$f'(x)g(x) - g'(x)f(x) = Me^{W(x)},$$

where M is a constant. Using that $f(a,\lambda) = 0$ and $g(a,\lambda) = 0$, we have that $M = 0$, then for all $x \in [a,b]$

$$f'(x)g(x) - g'(x)f(x) = 0. \tag{3.9}$$

This implies that

$$f(x) = \frac{f'(x)}{g'(x)}g(x).$$

We now show that $\dfrac{f'(x)}{g'(x)}$ is a constant. To do that, we calculate the derivative. Using again the fact that f and g are solutions of (3.8) and taking into account (3.9), we have

$$\begin{aligned}\left(\frac{f'(x)}{g'(x)}\right)' &= \left(\frac{f'(x)e^{-W(x)}}{g'(x)e^{-W(x)}}\right)' \\ &= \frac{g'(x)e^{-W(x)}\cdot(f'(x)e^{-W(x)})' - f'(x)e^{-W(x)}\cdot(g'(x)e^{-W(x)})'}{(g'(x)e^{-W(x)})^2} \\ &= \frac{g'(x)e^{-W(x)}\cdot(-2\lambda f(x)e^{-W(x)}) - f'(x)e^{-W(x)}\cdot(-2\lambda g(x)e^{-W(x)})}{(g'(x)e^{-W(x)})^2} \\ &= 0.\end{aligned}$$

And the proof is done. □

To facilitate the notation, from now on we use L alone instead of $\bar{L}$.

3.4 Green Operator

From the previous section we have that the infinitesimal operator associated with the Brox diffusion with killing on a and b is given by

$$Lf(x) = \frac{e^{W(x)}}{2}\left(e^{-W(x)}f'(x)\right)',$$

whose domain are functions $f \in D_0$ such that $f(a) = f(b) = 0$.

We want to construct the so-called Green operator, which is actually the inverse operator of L. First, consider the following identity known as the Lagrange identity.

Lemma 3.4.1. *Let f, g in D_0, then*

$$2e^{-W(x)}\left[g(x)Lf(x) - f(x)Lg(x)\right] = \left[e^{-W(x)}\left(f'(x)g(x) - f(x)g'(x)\right)\right]'.$$

Proof. Let $h_1(x) = e^{-W(x)}f'(x)$ and $h_2(x) = e^{-W(x)}g'(x)$. Then

$$\begin{aligned}\left[e^{-W(x)}\left(f'(x)g(x) - f(x)g'(x)\right)\right]' &= [h_1(x)g(x) - h_2(x)f(x)]' \\ &= h_1'(x)g(x) + g'(x)h_1(x) - h_2'(x)f(x) - f'(x)h_2(x) \\ &= h_1'(x)g(x) - h_2'(x)f(x) \\ &= 2e^{-W(x)}\left[g(x)Lf(x) - f(x)Lg(x)\right].\end{aligned}$$

□

It is important to notice in this formula that it is not necessary to differentiate $e^{W(x)}$. From previous identity, after integrating we also obtain

Corollary 3.4.2. *Let α, β be such that $a \leq \alpha < \beta \leq b$. And λ_1 and λ_2 with $Lf + \lambda_1 f = 0$, $Lg + \lambda_2 g = 0$. Then*

$$\left[e^{-W(x)}\left(f'(x)g(x) - f(x)g'(x)\right)\right]_{\alpha}^{\beta} = 2(\lambda_2 - \lambda_1)\int_{\alpha}^{\beta} e^{-W(x)} f(x)g(x)dx.$$

Using methods developed in the companion paper [25], see also [51], we construct the Green operator. Consider f in the domain of L. We propose the following kernel,

$$g(x,\xi) := \begin{cases} -Cu(x)v(\xi), & a \leq x \leq \xi; \\ -Cu(\xi)v(x), & \xi \leq x \leq b, \end{cases} \tag{3.10}$$

where

$$C := \int_a^b e^{W(z)}dz,\ u(x) := \frac{1}{C}\int_a^x e^{W(z)}dz,\ v(x) := \frac{1}{C}\int_x^b e^{W(z)}dz.$$

Let ξ be fixed. Note that g satisfies

$$g(a,\xi) = 0, \quad g(b,\xi) = 0.$$

Let us also see that $Lg(x,\xi) = 0$ for any fixed value ξ, with the understanding that $g'(x,\xi)$ is the derivative of $g(x,\xi)$ with respect to the first argument x. One can see that

$$g'(x,\xi) := \begin{cases} -e^{W(x)}v(\xi), & a \leq x \leq \xi; \\ e^{W(x)}u(\xi), & \xi \leq x \leq b, \end{cases}$$

Then, after multiplying by $e^{-W(x)}$ there is no more dependance on x, therefore the second derivative gives 0. This implies that $Lg(x,\xi) = 0$.

Now we use the Lagrange identity with the function f and g, using that $Lg(x,\xi) = 0$. Then

$$2e^{-W(x)}\left[g(x,\xi)Lf(x)\right] = \left[e^{-W(x)}\left(f'(x)g(x,\xi) - f(x)g'(x,\xi)\right)\right]'. \tag{3.11}$$

On integrating both sides of (3.11) on the intervals (a,ξ^-) and (ξ^+,b), where $\xi^- := \xi - \epsilon$ and $\xi^+ := \xi + \epsilon$. Then we have the following two equalities

$$\int_a^{\xi^-} 2e^{-W(x)}g(x,\xi)Lf(x)dx = \left[e^{-W(x)}\left(f'(x)g(x,\xi) - f(x)g'(x,\xi)\right)\right]_a^{\xi^-}, \tag{3.12}$$

$$\int_{\xi^+}^{b} 2e^{-W(x)}g(x,\xi)Lf(x)dx = \left[e^{-W(x)}\left(f'(x)g(x,\xi) - f(x)g'(x,\xi)\right)\right]_{\xi^+}^{b}. \tag{3.13}$$

By adding (3.12) and (3.13), and using that $f(a) = f(b) = g(a,\xi) = g(b,\xi) = 0$, we have

$$\int_a^b 2e^{-W(x)}g(x,\xi)Lf(x)dx - \int_{-\epsilon}^{\epsilon} 2e^{-W(x)}g(x,\xi)Lf(x)dx$$

$$
\begin{aligned}
&= e^{-W(\xi^-)}\left(f'(\xi^-)g(\xi^-,\xi) - f(\xi^-)g'(\xi^-,\xi)\right)\\
&- e^{-W(\xi^+)}\left(f'(\xi^+)g(\xi^+,\xi) - f(\xi^+)g'(\xi^+,\xi)\right).
\end{aligned}
$$

After expanding we end up with four terms. From the continuity of W, f' and g, the first and third terms cancel each other when $\epsilon \to 0$.

Since g' is not continuous, the second and fourth terms do not vanish. These terms are

$$-e^{-W(\xi^-)}f(\xi^-)g'(\xi^-,\xi) + e^{-W(\xi^+)}f(\xi^+)g'(\xi^+,\xi).$$

Taking the discontinuity into account, previous display is

$$e^{-W(\xi^-)}f(\xi^-)e^{W(\xi^-)}v(\xi) + e^{-W(\xi^+)}f(\xi^+)e^{W(\xi^+)}u(\xi).$$

Then, when $\epsilon \to 0$, it becomes

$$e^{-W(\xi)}f(\xi)e^{W(\xi)}u(\xi) + e^{-W(\xi)}f(\xi)e^{W(\xi)}v(\xi).$$

Since $u(\xi) + v(\xi) = 1$,

$$\int_a^b 2e^{-W(x)}g(x,\xi)Lf(x)dx = f(\xi).$$

The conclusion is given as follows.

Theorem 3.4.3. *Let g given in (3.10). Define*

$$Tf(x) := \int_a^b 2e^{-W(z)}g(z,x)f(z)dz,$$

which is called the Green operator. Then T satisfies $T(Lf)(x) = L(Tf)(x) = f(x)$, for every x and for all f in the domain of L.

Proof. We have already shown that $T(Lf)(x) = f(x)$. Using that

$$Tf(x) = -2Cv(x)\int_a^x e^{-W(z)}u(z)f(z)dz - 2Cu(x)\int_x^b e^{-W(z)}v(z)f(z)dz,$$

One can apply L to verify that $LTf = f$. □

3.5 Toward density

In the theory of Markov processes, it is well known that spectral information of the generator helps to study the transition probability functions of the stochastic process. In turn, one can use the eigenvalues and eigenfunctions to give expressions for the probability density. In fact, we can identify the eigenvalues of the generator of the killed Brox process with the eigenvalues of the Green operator T of Theorem 3.4.3, precisely because T is the inverse of L.

Corollary 3.5.1. *Operator L has almost surely a countable set of eigenvalues.*

Proof. This comes from the fact that for almost every trajectory of W, the operator T is a compact operator, thus it has a countable set of eigenvalues. Then, if (λ, f) is an eigenpair of T, then $Tf + \lambda f = 0$. Thus, $f = LTf = -\lambda Lf$, i.e. $(1/\lambda, f)$ is an eigenpair of L. □

Recall the Brox process $X := \{X_t : t \geq 0\}$ defined at the beginning of Section 3.2. We know that the generator of X has a discrete spectrum given by the eigenvalues λ_n, and each one has associated an eigenfunction ϕ_n. Thus, at a theoretical point of view, it is just a matter to join pieces to have the spectral decomposition of the probability transition function.

Notice that apriori we do not know if the transition probabilities are absolutely continuous with respect to the Lebesgue measure, however it is indeed the case.

Theorem 3.5.2. *If we leave fixed an environment W, then for all $x, y \in (a, b)$ we have*

$$p(t, x, y) = 2e^{-W(y)} \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x)\phi_n(y), \tag{3.14}$$

where $p(t, x, y)$ is the density function of X_t given that $X_0 = x$, and $\{\lambda_n,\ \phi_n\}_{n=1}^{\infty}$ are the eigenvalues and eigenfunctions of L.

Some properties known in the classic case, where the parameters of the operator L are differentiable functions, are also known in the case the same parameters are not necessarily differentiable.

Proof. The set $\{\phi_n\}_{n=1}^{\infty}$ forms a basis for the space $L^2\left([a, b], 2e^{-W(x)}\right)$, where the inner product is given by $\langle f, g\rangle := \displaystyle\int_a^b f(x)g(x)2e^{-W(x)}dx$, see e.g. Theorem 4.6.2 point (5) in [72].

Define $u_f := \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle f, \phi_n\rangle\phi_n(x)$. Note that u_f satisfies the following equation

$$\begin{cases} \dfrac{du_f}{dt} = Lu_f, \\ u_f(0, x) = f(x). \end{cases} \tag{3.15}$$

The solution of this problem is unique, see for instance [8] page 500. On the other hand, if we define for W fixed, the semigroup $P_t(f(x)) := E_x\left(f\left(X_t\right)\right)$, where X_t is the Brox Process with killing on a and b, then we have from the general theory of Markov processes that $P_t f(x)$ also satisfies (3.15). Therefore we have

$$u_f(t, x) = E_x\left(f\left(X_t\right)\right).$$

Now, since $P_t f(x) := E(f(X_t)|X_0 = x) = \int_{-\infty}^{\infty} f(y)p(t,x,dy)$, using dominated convergence theorem we arrive at

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y)p(t,x,dy) &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\langle f, \phi_n\rangle \phi_n(x) \\
&= \sum_{n=1}^{\infty} e^{-nt}\left(\int_{-\infty}^{\infty} f(y)\phi_n(y)2e^{-W(y)}dy\right)\phi_n(x) \\
&= \int_{-\infty}^{\infty} f(y)\left(\sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(y)\phi_n(x)2e^{-W(y)}\right)dy.
\end{aligned}$$

This proves the absolutely continuity and formula (3.14). □

3.6 Spectral analysis of the generator

In previous section we have shown, at least at a theoretical level, how one can give an spectral decomposition for the densities of X. Let us go further to try to find or characterize the components of such representation, that is to say: the eigenvalues and the eigenfunctions. We deal first with the eigenfunctions and after with the eigenvalues. We will keep noticing how the Green operator T of Theorem 3.4.3 will be useful for our analysis.

3.6.1 Eigenfunctions

Let ϕ be an eigenfunction and λ an eigenvalue of the generator L, then it holds

$$L\phi + \lambda\phi = 0 \text{ with } \phi(a) = \phi(b) = 0.$$

The Green operator gives the identity $TL\phi = \phi = -\lambda T\phi$, that is

$$\phi(x) = -2\lambda \int_a^b e^{-W(z)}\phi(z)g(z,x)dz.$$

From the definition of g previous display becomes

$$\phi(x) = 2C\lambda v(x)\int_a^x u(z)e^{-W(z)}\phi(z)dz + 2C\lambda u(x)\int_x^b v(z)e^{-W(z)}\phi(z)dz.$$

After taking the derivative, a cancelation occurs that yields

$$\phi'(x) = 2\lambda e^{W(x)}\left[\int_x^b v(z)e^{-W(z)}\phi(z)dz - \int_a^x u(z)e^{-W(z)}\phi(z)dz\right].$$

Consider the following function, a trick borrow from [34] page 269, which is previous display writing t in lieu of x, and x in lieu of $W(x)$:

$$h(t,x) := 2\lambda e^{x}\left[\int_t^b v(z)e^{-W(z)}\phi(z)dz - \int_a^t u(z)e^{-W(z)}\phi(z)dz\right].$$

Applying the Itô's formula to function h one has

$$\begin{aligned} h(t,W(t)) - h(a,W(a)) &= 2\lambda\int_a^t e^{-W(s)}\left(-v(s)e^{-W(s)}\phi(s) - u(s)e^{-W(s)}\phi(s)\right)ds \\ &+ \frac{1}{2}\int_a^t h(s,W(s))ds + \int_a^t h(s,W(s))ds. \end{aligned}$$

Since $\phi'(t) = h(t,W(t))$, using $u(s)+v(s) = 1$, we have that ϕ satisfies the following stochastic differential equation.

Proposition 3.6.1. *Let ϕ be an eigenfunction of L associated to the eigenvalue λ. Then, ϕ is solution of*

$$d\phi'(t) = \left[-2\lambda\phi(t) + \frac{1}{2}\phi'(t)\right]dt + \phi'(t)dW(t),$$

with conditions $\phi(a) = 0$ and $\phi(b) = 0$.

3.6.2 Eigenvalues

In this section we give a method to deal with the eigenvalues. To do that, with the aid of the Sturm-Liouville theory, we develop few results suited to work with our operator L.

Theorem 3.6.2. *Consider functions $f \in D_0$. Define the following two operators for $a < x < b$,*

$$L_1 f(x) := \left(e^{-W(x)}f'(x)\right)' + 2\lambda_1 e^{-W(x)}f(x),$$

$$L_2 f(x) := \left(e^{-W(x)}f'(x)\right)' + 2\lambda_2 e^{-W(x)}(x)f(x),$$

where $\lambda_2 > \lambda_1$. Let ϕ_1 and ϕ_2 such that $L_1\phi_1 = L_2\phi_2 = 0$. Then, between two zeros of ϕ_1 there is a zero of ϕ_2. Thus ϕ_2 has at least as many zeros as ϕ_1 on $[a,b]$.

Proof. Suppose that x_1 and x_2 are two successive zeros of ϕ_1, and that $\phi_2(x) \neq 0$ for any $x \in (x_1,x_2)$. Without loss of generality we assume that $\phi_1(x) > 0$ and $\phi_2(x) > 0$ for any $x \in (x_1,x_2)$.

The Lagrange's identity in Lemma 3.4.2 gives

$$\left[e^{-W(x)}\left(\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x)\right)\right]_{x_1}^{x_2} = 2(\lambda_2-\lambda_1)\int_{x_1}^{x_2} e^{-W(x)}\phi_1(x)\phi_2(x)dx.$$

Note that the right hand side is strictly positive. However, the left hand side reduces to

$$e^{-W(x_2)}\phi_1'(x_2)\phi_2(x_2) - e^{-W(x_1)}\phi_1'(x_1)\phi_2(x_1).$$

Using the assumptions of ϕ on x_1 and x_2, we observe that $\phi_2(x_2) \geq 0$, $\phi_1'(x_2) < 0$, $\phi_2(x_1) \geq 0$ and $\phi_1'(x_1) > 0$, then the above expression is less or equal to 0, giving a contradiction. Therefore ϕ_2 has a zero between x_1 and x_2.

In particular, since $\phi_1(a) = \phi_2(a) = 0$ and $\phi_1(x_1) = 0$ with $a < x_1 < b$, then there exists z, with $a < z < x_1$ such that $\phi_2(z) = 0$. Thus ϕ_2 has at least as many zeros as ϕ_1 on $[a, b]$. □

Proposition 3.6.3. *If ϕ_n is an eigenfunction of L associated with λ_n, with $n = 1, 2...$, then ϕ_n has exactly $n - 1$ zeros in the interval (a, b).*

We left the proof in the Appendix. The classic proof of this result uses the so-called method of Prüfer which is based in a change of coordinates. The original proof for the standard equation is difficult to find in the literature, one can find it thought in [17], from where we adapted it to our situation.

Using previous two results we can to show the following theorem.

Theorem 3.6.4. *Let $\lambda \in \mathbb{R}$ be fixed, and let $\psi(x, \lambda)$ be solution of $L\psi(x, \lambda) + \lambda\psi(x, \lambda) = 0$, $x \in (a, b)$, that satisfies $\psi(a, \lambda) = 0$ and $\psi'(a, \lambda) = 1$. Then the number of zeros of $x \mapsto \psi(x, \lambda)$ on $(a, b]$ equals the number of eigenvalues of L less or equal to λ.*

Proof. First, from Proposition 3.3.1, we know that such function ψ really exists.

The proof relies on Theorem 3.6.2 and Corollary 3.6.3. In what follows ϕ_n is the eigenfunction associated with the n-eigenvalue λ_n, i.e. $L\phi_n + \lambda_n\phi_n = 0$, with $\phi_n(a) = \phi_n(b) = 0$.

Fix λ. We first suppose that there exist only n eigenvalues less or equal to λ, i.e. $\lambda_1 < ... < \lambda_n \leq \lambda < \lambda_{n+1}$, and let us prove that the map $x \mapsto \phi(x, \lambda)$ has exactly n zeros in $(a, b]$. By the Corollary 3.6.3, ϕ_n has exactly $n - 1$ zeros on the open interval (a, b), thus it has $n + 1$ zeros in $[a, b]$. Since $\lambda_n \leq \lambda$, by Theorem 3.6.2 we know that between two consecutive zeros of ϕ_n there is one zero for ψ. Then ψ has at least n zeros on $(a, b]$, i.e. it has at least $n + 1$ zeros on $[a, b]$. However, if ψ had $n + 1$ zeros on $(a, b]$, again using Theorem 3.6.2 with $\lambda < \lambda_{n+1}$, the $n + 2$ zeros of ψ on $[a, b]$ would imply that ϕ_{n+1} had $n + 1$ zeros on (a, b), which is not the case. We conclude that ψ has exactly n zeros on $(a, b]$.

On the other hand, we now suppose that ψ has exactly n zeros in $(a, b]$. Let us now show that there exist only n eigenvalues less or equal that λ. Suppose by contradiction that the eigenvalue $n + 1$ is less or equal to λ, i.e. $\lambda_{n+1} \leq \lambda$. If $\lambda = \lambda_{n+1}$ we have that $\psi = \phi_{n+1}$, and by the Corollary 3.6.3 ψ has n zeros in (a, b), and since $\phi_{n+1}(b) = 0$, we obtain that ψ has $n + 1$ zeros in $(a, b]$, which is a contradiction. If $\lambda_{n+1} < \lambda$, by the Corollary 3.6.3 we know that ϕ_{n+1} has $n + 2$ zeros in $[a, b]$. Now, by Theorem 3.6.2 we have that ψ should have at least $n + 2$ in $[a, b]$, this implies that ψ has $n + 1$ zeros or more in $(a, b]$, which is again a contradiction.

We know now that if ψ has n zeros in $(a, b]$, the $n+1$ eigenvalue satisfies $\lambda < \lambda_{n+1}$. We will now show that $\lambda_n \leq \lambda$, i.e. there exist only n eigenvalues less or equal to λ.

Suppose that $\lambda < \lambda_n$. Recall that we are supposing that ψ has n zeros in $(a, b]$, since $\psi(a) = 0$, it has $n + 1$ zeros in $[a, b]$. Again, appealing to the Theorem 3.6.2, since $\phi_n(a) = \phi_n(b) = 0$, we have that ϕ_n has at least $n + 2$ zeros in $[a, b]$. We are saying that ϕ_n has n zeros or more in (a, b), which contradicts Corollary 3.6.3. And the proof is completed. □

Remark 3.6.5. *Let us give a characterization of function ψ of previous theorem, i.e. ψ such that*

$$L\psi(x, \lambda) + \lambda\psi(x, \lambda) = 0, \ x \in (a, b)$$

with $\psi(a, \lambda) = 0$ and $\psi'(a, \lambda) = 1$.

From ii) of Proposition 3.3.1, ψ is solution of the equation

$$\psi(x, \lambda) = -2\lambda \int_a^x \int_a^y \psi(z, \lambda) e^{-W(z)} e^{W(y)} dz dy + \psi(a, \lambda) + \frac{\psi'(a, \lambda)}{e^{W(a)}} \int_a^x e^{W(z)} dz.$$

By differentiating and taking into account the initial conditions, we have the following two equations,

$$\psi(x, \lambda) = -2\lambda \int_a^x \int_a^y \psi(z, \lambda) e^{-W(z)} e^{W(y)} dz dy + \frac{1}{e^{W(a)}} \int_a^x e^{W(z)} dz, \tag{3.16}$$

$$\psi'(x, \lambda) = e^{W(x)} \cdot \left[-2\lambda \int_a^x \psi(z, \lambda) e^{-W(z)} dz + \frac{1}{e^{W(a)}} \right].$$

We finally arrive to the point where it is possible to identify the eigenvalues of the generator of X.

Theorem 3.6.6. *Considering the function ψ in (3.16), then we have that λ is an eigenvalue of L if and only if $\psi(b, \lambda) = 0$.*

Proof. Since it holds $L\psi + \lambda\psi = 0$ and $\psi(a, \lambda) = 0$, if we are told that $\psi(b, \lambda) = 0$, then ψ would be an eigenfunction, consequently λ would be an eigenvalue.

Let us now suppose that λ is an eigenvalue of L. If that is the case, then there exists an eigenfunction φ, thus it holds that $L\varphi + \lambda\varphi = 0$, $\varphi(a, \lambda) = 0$, $\varphi(b, \lambda) = 0$. From iii) of Proposition 3.3.1, we know that ψ also satisfies $L\psi + \lambda\psi = 0$, $\psi(a, \lambda) = 0$ and $\psi'(a, \lambda) = 1$. And by Proposition 3.3.2, there exists a constant C such that

$$\varphi(x, \lambda) = C\psi(x, \lambda),$$

which implies that $\psi(b, \lambda) = 0$. □

At this point, we are providing grounds to produce a procedure to generate eigenvalues and eigenfunctions, thus one may be able to approximate the transition probability densities in (3.14).

Now, let us produce another stochastic equation which may additionally help to deal with eigenvalues. The ideas comes from [50] (see also [56]).

For λ fixed, consider the Riccati transform

$$P_t := \frac{\psi'(t,\lambda)}{\psi(t,\lambda)},$$

which is valid whenever ψ is not zero.

If we define

$$g(t,x) := e^x \cdot \left[-2\lambda \int_a^t \psi(z,\lambda) e^{-W(z)} dz + \frac{1}{e^{W(a)}} \right],$$

then $\psi'(t,\lambda) = g(t, W(t))$.

Applying the Itô's formula to the function

$$h(t,x) := \frac{g(t,x)}{\psi(t,\lambda)},$$

and we obtain that P_t satisfies the following stochastic differential equation

$$dP_t = \left(-2\lambda - P_t^2 + \frac{1}{2} P_t \right) dt + P_t dW_t.$$

The relevance of the diffusion P_t is given for the following theorem.

Theorem 3.6.7. *Consider the diffusion P_t started at $+\infty$ at $x = a$, and restarted at $+\infty$ immediately after any passage to $-\infty$. Then the number of eigenvalues of L less that λ is equal to the number of explosions of P_t on $(a, b]$.*

Proof. The proof is using the Theorem 3.6.4 and the Riccati Transform. □

3.7 Appendix. Proof of Proposition 3.6.3

We want to analyze the equation $Lx + \lambda x = 0$. We consider the equation

$$2e^{-W(t)} [Lx(t) + \lambda x(t)] = (e^{-W(t)} x'(t))' + 2\lambda e^{-W(t)} x(t) = 0, \quad a < t < b, \tag{3.17}$$

We are going to use the Prüfer method, which we adapt following the arguments in [17]. In this method, one first defines $y(t) = e^{-W(t)} x'(t)$. Using (3.17) we have

$$x'(t) = \frac{y(t)}{e^{-W(t)}}, \quad y'(t) = -2\lambda e^{-W(t)} x(t). \tag{3.18}$$

Notice that eventhough $y(t)$ is in terms of the Brownian motion $W(t)$, the derivative $y'(t)$ is well defined.

The big leap in this classic method is to propose the following change of coordinates

$$x(t) = r(t)\sin(w(t)), \quad y(t) = r(t)\cos(w(t)). \tag{3.19}$$

Differentiating the equations (3.19) with respect to t we have

$$x'(t) = r'(t)\sin(w(t)) + r(t)\cos(w(t))w'(t),$$

$$y'(t) = r'(t)\cos(w(t)) - r\sin(w(t))w'(t).$$

We now use (3.18), and solving for r' and w', we obtain

$$r'(t) = \left(\frac{1}{e^{-W(t)}} - 2\lambda e^{-W(t)}\right) r(t)\sin(w(t))\cos(w(t)), \tag{3.20}$$

and

$$w'(t) = \frac{1}{e^{-W(t)}}\cos^2(w(t)) + 2\lambda e^{-W(t)}\sin^2(w(t)). \tag{3.21}$$

These equations with initial conditions have unique solution. If ϕ is a solution of (3.17) then ϕ has the form $\phi(r) = r(t)\sin(w(t))$. Equation (3.20) is of the form $r'(t) = h(t)r(t)$, so the solution is non-negative on t. A consequence of this is that ϕ vanishes only when w is a multiple of π.

Taking into account the conditions

$$x(a) = 0, \; x(b) = 0, \tag{3.22}$$

let $\phi(t,\lambda)$ be a nontrivial solution of (3.17) and (3.22).

To analyze ϕ, we now give some properties of w, defined in (3.21).

First of all, it holds $w(a,\lambda) = 0$. This is because from formulas (3.18) and (3.19), one has

$$w(a,\lambda) = \tan^{-1}\left(\frac{\phi(a,\lambda)}{e^{-W(a)}\phi'(a,\lambda)}\right) = 0.$$

Second, it turns out that for λ fixed, w is increasing function of t. To prove this, let us see that, using equation (3.21), the derivative is positive. This is the case if the eigenvalues are positive, let us check this fact. Let (x,λ) be an eigenpair of the generator, thus

$$(e^{-W(t)}x'(t))' + 2\lambda e^{-W(t)}x(t) = 0. \tag{3.23}$$

Multiplying (3.23) by $x(t)$ we obtain

$$x(t)(e^{-W(t)}x'(t))' + 2\lambda e^{-W(t)}x^2(t) = 0.$$

Integrating and solving for λ we arrive at

$$\lambda = \frac{-\int_a^b x(t)(e^{-W(t)}x'(t))'dt}{\int_a^b 2e^{-W(t)}x^2(t)dt}.$$

Integrating by parts and using that $x(a) = x(b) = 0$ we have

$$\lambda = \frac{\int_a^b e^{-W(t)}(x'(t))^2 dt}{\int_a^b 2e^{-W(t)}x^2(t)dt} > 0.$$

Now, for fixed t, let us see that $w(t, \lambda)$ is monotone increasing function of λ. This is actually a consequence of the following theorem (for a proof see e.g. [17]).

Theorem 3.7.1. *Let $L_i x := (p_i x')' + g_i x$. And let p_i and g_i be continuous functions on $[a, b]$, such that*

$$0 < p_2(t) \leq p_1(t), \quad g_2(t) \geq g_1(t).$$

Let $L_1\phi_1 = 0$ and $L_2\phi_2 = 0$ and using the Prüfer method take w_1 and w_2 as in (3.21) with $w_2(a) > w_1(a)$. Then

$$w_2(t) > w_1(t), \quad a \leq t \leq b.$$

In our case, $p_1(t) = p_2(t) = e^{-W(t)}$ and the function $g(t) = 2\lambda e^{-W(t)}$ is increasing in λ.
Finally, we have this property of w:

Lemma 3.7.2. *$w(b, \lambda) \to \infty$ as $\lambda \to \infty$.*

Proof. We consider the equation (3.17). Let P, G be constants such that for all $t \in [a, b]$

$$e^{-W(t)} \leq P, \quad 2\lambda e^{-W(t)} \geq G.$$

Now we consider the equation

$$Px''(t) + 2\lambda G x(t) = 0, \tag{3.24}$$

and take v to be the analogous of w with the condition $v(a, \lambda) = w(a, \lambda)$. Hence, from Theorem 3.7.1 we have that

$$w(t, \lambda) \geq v(t, \lambda).$$

On the other hand, if f is a solution of the equation (3.24), then we have that for large λ, f is of the form

$$f(t) = A\cos\left(\sqrt{\frac{\lambda G}{P}}t\right) + B\sin\left(\sqrt{\frac{\lambda G}{P}}t\right),$$

where A and B are constants. This solution implies that the zeros of f increase in number when λ is large, because the periodicity increases. The only way to have that is because v hit the value π many times. In particular we have that $v(b, \lambda) \to \infty$ as $\lambda \to \infty$. Since $w(b, \lambda) \geq v(b, \lambda)$, then we obtain the result. $\square$

Let us see that $w(b,0) \neq \pi$. If that it not the case, the associated function $\phi(t,0)$ should be an eigenfunction of the eigenvalue $\lambda = 0$. However, as we mentioned before the eigenvalues are strictly positive. Now, since $w(t,0)$ is a continuous function strictly increasing and $w(a,0) = 0$, then we conclude that $0 < w(b,0) < \pi$.

On the other hand, since $w(b,\lambda)$ is increasing in λ and $w(b,\lambda) \to \infty$ as $\lambda \to \infty$, therefore there exists a first value $\lambda_1 > 0$ such that $w(b,\lambda_1) = \pi$.

Since w is increasing in t we have that

$$0 = w(a,\lambda_1) < w(t,\lambda_1) < w(b,\lambda_1) = \pi, \quad a < t < b.$$

Then we have that $w(t,\lambda_1)$ is not a multiple of π, hence the solution $\phi(t,\lambda_1)$ does not vanish in (a,b), and this function $\phi(t,\lambda_1)$ is the eigenfunction associated to the first eigenvalue λ_1.

In the same way, there exist $\lambda_2 > \lambda_1$ such that $w(b,\lambda_2) = 2\pi$. Then the function $\phi(t,\lambda_2)$ is the eigenfunction associated with the eigenvalue λ_2 and has only one zero in (a,b), precisely because w touches only once the value π. And the very same reasoning follows to conclude that the nth eigenfunction has exactly $n-1$ zeros in (a,b).

Chapter 4

Potential theory

In this chapter we study the classical potential theory, and we present the mains results of this theory, as the Frostman's theorem and the uniqueness of the equilibrium measure for compact sets. Also we work with the concept of capacity, and we show a known relation between the problem of minimum energy and the Dirichlet problem. Some proofs in this chapter about of the well-known results are different to those given in the literature. Also, the main contribution of this chapter is the Theorem 4.5.4, that provides the relation between the problem of minimum energy and a modification of the Dirichlet problem.

4.1 Introduction

The principle of minimum energy is essentially a restatement of the second law of thermodynamics. It states that for a close system, with constant external parameters and entropy, the internal energy will decrease and approach a minimum value at equilibrium. This minimum value is important in physics and mathematics.

In particular, in mathematics, the potential theory is a tool that permits to study this minimum value using the so-called electrostatic potential. Our goal is the following, if we have a set S compact, we want to find a measure μ such that μ minimizes the energy on S. We use the Coulomb's law to find a explicit formula for the energy and for the electrostatic potential.

We will see that this potential is in term of the integral of the fundamental solution of $\Delta u = 0$ with respect to a measure μ, where Δ is the Laplacian, with this connection, in the next sections we will show that there exist a relation between the electrostatic potential and the Brownian motion. Also we will see that there exists a relation between the Dirichlet problem and the problem of minimum energy.

4.2 Electrostatic Potential and Energy

In physics, the energy is defined as the work needed to bring a unit charge from infinity to a fixed point (x, q), where the electrostatic field is given by other fixed point (x_1, q_1). We use this definition to obtain an explicit mathematical formula. We first consider the definition of work as the following integral of the force

$$W := \int_\gamma \vec{F}, \tag{4.1}$$

where $\vec{F}$ represent the force and γ is a curve on $\mathbb{R}^3$. The integral (4.1) is a path integral.

We are now going to use the Coulomb's law to obtain a formula for the force. This law of physics describes the interacting force between electrically charged particles in the following way. The magnitude of the electrostatic force in the interaction between two charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. In this way the force is

$$\vec{F} = \frac{qq_1}{\|x - x_1\|^2} \cdot \frac{x - x_1}{\|x - x_1\|}, \tag{4.2}$$

where q and q_1 represent the magnitudes of the charges, and the scalar $\|x - x_1\|$ is the distance between the charges. Here we suppose that the charge q is located at x, and the charge q_1 is located at x_1.

In general, the force of interaction between n charges is calculated using the law of superposition. In this way, the force on a charge is the addition of the individual forces, i.e. if we have q_1,...,q_n charges located at x_1,...,$x_n \in \mathbb{R}^3$ respectively, then the force at a point $x \in \mathbb{R}^3$ with charge q is the following

$$\begin{aligned} \vec{F} &= \frac{qq_1}{\|x - x_1\|^2} \cdot \frac{x - x_1}{\|x - x_1\|} + \frac{qq_2}{\|x - x_2\|^2} \cdot \frac{x - x_2}{\|x - x_2\|} + ... + \frac{qq_n}{\|x - x_n\|^2} \cdot \frac{x - x_n}{\|x - x_n\|} \\ &= q \sum_{i=1}^{n} \frac{q_i(x - x_i)}{\|x - x_i\|^3}. \end{aligned} \tag{4.3}$$

We now introduce the so-called electrostatic field, that will be very important for our goal. But first we consider a charge distribution μ defined as

$$\mu := \{(x_1, q_1), ..., (x_n, q_n)\}.$$

Using this charge distribution we define the electrostatic field generated by μ on the point x in the following way

$$E^\mu(x) := \sum_{i=1}^{n} \frac{q_i(x - x_i)}{\|x - x_i\|^3}. \tag{4.4}$$

Note that if we define

$$G^{\mu}(x) := \sum_{i=1}^{n} \frac{q_i}{\|x - x_i\|}, \tag{4.5}$$

then we have the following equality

$$-q\nabla G^{\mu}(x) = qE^{\mu}(x) = \vec{F}, \tag{4.6}$$

where ∇ represent the gradient. The function G^{μ} is called the electrostatic potential.

Remark. Note that the electrostatic potential may be written as

$$G^{\mu}(x) := \sum_{i=1}^{n} \frac{q_i}{\|x - x_i\|} = \int \frac{1}{\|x - y\|} \rho_n(dy), \tag{4.7}$$

where $\rho_n(A) := \sum_{i=1}^{n} q_i \delta_{x_i}(A)$, and δ represent the Dirac's Delta.

Remark. If we denote $E(\mu)$ as the energy generated by a charge distribution μ, and the charge distribution is $\mu := \{(x_1, q_1), (x_2, q_2)\}$, then let us see that the energy of the system can be expressed as

$$E(\mu) = \frac{q_1 q_2}{\|x_1 - x_2\|}.$$

If we consider $\vec{F}$ as in the formula (4.2), we have that the work is

$$W = \int_{\gamma} \vec{F} = \int_{\gamma} \frac{q_1 q_2 (x_1 - x_2)}{\|x_1 - x_2\|^3},$$

where γ is a curve on $\mathbb{R}^3$ with finals points x_1, x_∞ and charges q_1, q_∞, and x_2 is a fix point with charge q_2. If ϕ is a parametrization of γ with $\phi : [a, b] \to \mathbb{R}^3$, then by formula (4.6) we have that

$$W = \int_{\gamma} \vec{F} = \int_{a}^{b} \vec{F}(\phi(t))\phi'(t)dt = -q_2 \int_{a}^{b} \nabla G^{\mu}(\phi(t))\phi'(t)dt. \tag{4.8}$$

On the other hand, it is easy to verify that G^{μ} satisfies

$$\nabla G^{\mu}(\phi(t))\phi'(t) = \frac{dG^{\mu}(\phi(t))}{dt}, \tag{4.9}$$

then substituting (4.9) in (4.8) we arrive at

$$W = -q_2 \int_{a}^{b} \nabla G^{\mu}(\phi(t))\phi'(t)dt = -q_2 \int_{a}^{b} \frac{dG^{\mu}(\phi(t))}{dt} dt = -q_2[G^{\mu}(x_\infty) - G^{\mu}(x_2)].$$

Remember that the energy $E(\mu)$ is defined as the work W that we need to bring a charge from infinity to a fixed point. Therefore if $\|x_\infty\| \to \infty$, then

$$E(\mu) = \lim_{\|x_\infty\| \to \infty} W = \lim_{\|x_\infty\| \to \infty} -q_2[G^{\mu}(x_\infty) - G^{\mu}(x_2)] = q_2 G^{\mu}(x_2),$$

because $G^{\mu}(x_{\infty}) = 0$. Thus we obtain that if $\mu := \{(x_1, q_1), (x_2, q_2)\}$, then the energy is

$$E(\mu) = q_2 G^{\mu}(x_2) = \frac{q_1 q_2}{\|x_1 - x_2\|}. \tag{4.10}$$

We now want to find a formula for the energy $E(\mu)$ when the charge distribution μ has n points, i.e. $\mu_n := \{(x_1, q_1), (x_2, q_2), ..., (x_n, q_n)\}$. Applying again the law of superposition, the energy is defined as

$$E(\mu_n) = \frac{1}{2} \sum_{i,j=1, i \neq j}^{n} \frac{q_i q_j}{\|x_i - x_j\|}. \tag{4.11}$$

This idea can be generalized in the following way, see also [28]. We consider a continuous function (called the charge density) $\rho : \mathbb{R}^3 \to \mathbb{R}$, with compact support S such that for all cube set $A \subseteq \mathbb{R}^3$, we have

$$\mu(A) = \int_A \rho(z) d\lambda(z),$$

where μ is interpreted as the charge distribution, and λ is the Lebesgue measure. Let x be a point in $\mathbb{R}^3$ and Q a cube small enough such that

$$\mu(Q) = \int_Q \rho(z) d\lambda(z) = \rho(x*)\lambda(Q),$$

for anything point $x* \in S$. Note that if Q is small enough, then $\mu(Q)$ can be interpreted as the charge q at the point x, because $\rho(x*) \approx \rho(x)$, then we have

$$q \approx \mu(Q) \approx \rho(x)\lambda(Q).$$

On the other hand, as S is compact, there exists a finite number of cubes $Q_1, ...Q_m$ small enough such that

$$S \subseteq \bigcup_{i=1}^{m} Q_i.$$

Using the formulas (4.5) and (4.11), we have

$$G^{\mu_m}(x) = \sum_{i=1}^{m} \frac{q_i}{\|x - x_i\|} = \sum_{i=1}^{m} \frac{\rho(x_i)\lambda(Q_i)}{\|x - x_i\|},$$

and

$$E(\mu_m) = \frac{1}{2} \sum_{i,j=1, i \neq j}^{m} \frac{q_i q_j}{\|x_i - x_j\|} = \frac{1}{2} \sum_{i,j=1, i \neq j}^{m} \frac{\rho(x_i)\lambda(Q_i)\rho(x_j)\lambda(Q_j)}{\|x_i - x_j\|}.$$

With this idea, let us consider the charge density ρ, then we can define the electrostatic potential and the energy as

$$G^\mu(x) = \lim_{\delta_m \to 0} G^{\mu_m}(x),$$

$$E(\mu) = \lim_{\delta_m \to 0} E(\mu_m),$$

where $\delta_m := \max_{i=1\ldots m} \operatorname{diam}(Q_i)$. Hence we have the following definition

Definition 4.2.1. *If $\rho : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function with compact support $S \subseteq \mathbb{R}^3$, then the electrostatic potential is*

$$G^\mu(x) := \int_S \frac{\rho(y)}{\|x - y\|} dy,$$

and the energy is

$$E(\mu) = \int\int \frac{\rho(x)\rho(y)}{\|x - y\|} dydx = \int_S G^\mu(x) d\mu(x),$$

where $\mu(A) := \displaystyle\int_A \rho(x) d\lambda(x)$.

Remark. On $\mathbb{R}^2$ if we assume that the repulsive force between two charged particles is

$$\vec{F} = \frac{q_1 q_2}{\|x_1 - x_2\|} \cdot \frac{x_1 - x_2}{\|x_1 - x_2\|}, \tag{4.12}$$

then we can apply the same construction for the case on $\mathbb{R}^3$. If we carry out the same analysis than before, we obtain a different expression for the electrostatic potential.

Definition 4.2.2. *If $\rho : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with compact support $S \subseteq \mathbb{R}^2$, then the electrostatic potential is*

$$G^\mu(x) := \int_S \rho(y) \log\left(\frac{1}{\|x - y\|}\right) dy.$$

Similarly we can define the energy for the case on $\mathbb{R}^2$ as

$$E(\mu) = \int\int \rho(x)\rho(y) \log \frac{1}{\|x - y\|} dxdy = \int_S G^\mu(y)\rho(y)dy = \int_S G^\mu(y) d\mu(y).$$

Remark. In general, if μ is a non-negative measure with support on S, then we define the energy by

$$E(\mu) := \int\int K(x, y) d\mu(x) d\mu(y),$$

where

$$K(x,y) := \begin{cases} \dfrac{1}{\|x-y\|}, & \text{for } n \geq 3; \\ -\log(\|x-y\|), & \text{for } n = 2. \end{cases}$$

Note that the initial idea is the same for all $\mathbb{R}^n$, however the electrostatic potential in the limit is different for $n \geq 3$ and for $n = 2$. The principal difference between these potentials is that the electrostatic potential for $n \geq 3$ is not negative, however for $n = 2$ the electrostatic potential may be negative, therefore the energy also may be negative. For this reason we will work both cases separately.

However all the potentials for $n \geq 2$ satisfies some common properties. We first show these properties in common y next we will work these cases separately. First, the potential electrostatic for all $n \geq 2$ is a function that satisfies a property called lower semi-continuous, that we will use next. This properties is the following

Definition 4.2.3. *Let $S \subseteq \mathbb{R}^n$, with $n \geq 2$. A function $f : S \to \mathbb{R}$ is lower semi-continuous at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) - \epsilon$ for all x in U. This can be expressed as*

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

It is easy to proof that the electrostatic potential is lower semi-continuous using the Fatou's Lemma.

Proposition 4.2.4. *Let $S \subseteq \mathbb{R}^n$, with $n \geq 2$. A function $f : S \to \mathbb{R}$ is lower semi-continuous if and only if $\{x \in S : f(x) > \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$.*

Alternatively, f is lower semi-continuous if and only if $\{x \in S : f(x) \leq \alpha\}$ is an closed set for every $\alpha \in \mathbb{R}$.

Proof. This is easy to proof of the definition. □

Continuing with the previous construction of the electrostatic potential and of the energy, our goal is to find a probability measure μ_0 such that μ_0 minimizer the energy on $S \subseteq \mathbb{R}^n$, for all n. Then we have the following definition for the minimum energy

Definition 4.2.5. *We denote by $P(S)$ the family of all Borel probability measures on $S \subseteq \mathbb{R}^n$, $n \geq 2$. If there exists $\mu_0 \in P(S)$ such that*

$$E(\mu_0) = \inf_{\mu \in P(S)} E(\mu),$$

then μ_0 is called the ***equilibrium measure****.*

The following theorem showed that if the set $S \subseteq \mathbb{R}^n$, $n \geq 2$, is compact, then the equilibrium measure always exists, see [40].

Theorem 4.2.6. *Let $S \subseteq \mathbb{R}^n$ be a compact set, $n \geq 2$. Then the equilibrium measure μ_0 always exists.*

Proof. By well known properties of $\mathbb{R}$, there exists a sequence $\mu_n \in P(S)$ such that $E(\mu_n) \to \inf_{\mu \in P(S)} E(\mu)$.

On the other hand, by the Alaoglu's Theorem we know that there exists a measure $\mu_0 \in P(S)$ such that $\mu_n \to \mu_0$ in the weak-topology.

Let $K_m(x,y) := \min\{K(x,y), m\}$, where m is a constant. Then we have that for each m fixed

$$\int\int K_m(x,y)d\mu_n(x)d\mu_n(y) \to \int\int K_m(x,y)d\mu_0(x)d\mu_0(y) \tag{4.13}$$

as $n \to \infty$. Also we have that

$$\int\int K_m(x,y)d\mu_n(x)d\mu_n(y) \leq \int\int K(x,y)d\mu_n(x)d\mu_n(y) = E(\mu_n).$$

If we let $n \to \infty$ and we use (4.13), then we get

$$\int\int K_m(x,y)d\mu_0(x)d\mu_0(y) \leq \liminf_{n\to\infty} E(\mu_n).$$

We now let $m \to \infty$. Then we arrive at

$$E(\mu_0) \leq \liminf_{n\to\infty} E(\mu_n).$$

Therefore we have that μ_0 is a equilibrium measure. □

Remark. Also this measure of equilibrium is unique, in the next sections, where the so-called capacity of a set play an important role we show this fact. A important tool for to show this fact is the called Frostman's Theorem, which states that the electrostatic potential defined with the equilibrium measure is constant in el support of the equilibrium measure. We will show this theorem in both cases, for $n \geq 3$ and $n = 2$.

In the coming theorems, we will show several properties of the electrostatic potential on $\mathbb{R}^n$, with $n \geq 3$, that we will use to prove that the equilibrium measure is unique.

4.3 Potential theory on $\mathbb{R}^n$, $n \geq 3$

First we show the relation between the electrostatic potential and the Brownian motion on $\mathbb{R}^3$, for more details see [7].

Theorem 4.3.1. *If $\rho \geq 0$, $\rho : \mathbb{R}^3 \to \mathbb{R}$ with compact support S, then*

$$G^\mu(x) := \int \frac{\rho(y)}{\|x-y\|}dy = 2\pi E_x\left(\int_0^\infty \rho(B_s)ds\right). \tag{4.14}$$

Proof. The right hand side is

$$E_x\left(\int_0^\infty \rho(B_s)ds\right) = \int_0^\infty E_x(\rho(B_s))ds = \int \rho(y)\left[\int_0^\infty \frac{1}{(2\pi s)^{-3/2}} e^{\frac{-\|y-x\|^2}{2s}} ds\right] dy.$$

If we make the substitution $t = \dfrac{\|y-x\|^2}{2s}$, then we arrive at

$$\int_0^\infty \frac{1}{(2\pi s)^{-3/2}} e^{\frac{-\|y-x\|^2}{2s}} ds = \frac{\|y-x\|^{-1}}{2\pi^{3/2}} \Gamma(1/2) = \frac{\|y-x\|^{-1}}{2\pi}.$$

Then we have

$$E_x\left(\int_0^\infty \rho(B_s)ds\right) = \frac{1}{2\pi}\int \rho(y)\|y-x\|^{-1}dy.$$

□

In the following theorem we show the relation between the electrostatic potential and the density function ρ. We suppose that $\rho \in C^2$, this implies that $G^\mu \in C^2$, but if this is not true, then la next identity is considered in the distributional sense, see [1]. This theorem is a very important tool to find the equilibrium measure.

Theorem 4.3.2. *Let*

$$G^\mu(x) = \int_S \frac{\rho(y)}{\|x-y\|} dy$$

be the electrostatic potential, and let $S \subseteq \mathbb{R}^3$ be the support of ρ. Then

$$\Delta G^\mu(x) = -4\pi\rho(x).$$

Proof. First note that if $\rho \in C^2$, then $G^\mu \in C^2$. Now, by Itô's formula we have that

$$G^\mu(B_t) - G^\mu(B_0) - \frac{1}{2}\int_0^t \Delta G^\mu(B_s)ds, \tag{4.15}$$

is a local martingale starting at 0. On the other hand, by the previous theorem and well know properties of the Brownian motion we have

$$\begin{aligned}
G^\mu(B_t) &= 2\pi E_{B_t}\int_0^\infty \rho(B_s)ds \\
&= 2\pi E_x\left(\int_0^\infty \rho(B_{t+s})ds \mid \mathfrak{F}_t\right) \\
&= 2\pi E_x\left(\int_t^\infty \rho(B_s)ds \mid \mathfrak{F}_t\right) \\
&= 2\pi E_x\left(\int_0^\infty \rho(B_s)ds \mid \mathfrak{F}_t\right) - 2\pi\int_0^t \rho(B_s)ds,
\end{aligned}$$

where $\mathfrak{F}_t := \sigma\{B_s : s \leq t\}$.

Since ρ has compact support, then the integral $\displaystyle\int_0^\infty \rho(B_s)ds$ is finite for almost every path. Hence we have that

$$G^\mu(B_t) - G^\mu(B_0) + 2\pi \int_0^t \rho(B_s)ds, \tag{4.16}$$

is also a local martingale starting at 0. Now, from (4.15) and (4.16), we obtain that

$$\int_0^t [\frac{1}{2}\Delta G^\mu(B_s) + 2\pi\rho(B_s)]ds,$$

is a local martingale that is zero at 0. Also is continuous and of bounded variation, hence is identically 0. Therefore for almost every s we have

$$\frac{1}{2}\Delta G^\mu(B_s) + 2\pi\rho(B_s) = 0, \ a.s.$$

Since B_s, $G^\mu(B_s)$ and ρ are continuous, then with probability one we have

$$\frac{1}{2}\Delta G^\mu(B_s) + 2\pi\rho(B_s) = 0, \ s \geq 0. \tag{4.17}$$

If $\frac{1}{2}\Delta G^\mu(z) + 2\pi\rho(z) > 0$ for some z, then by continuity, we have the inequality for a neighborhood V of z. With positive probability, B_t enters V, contradicting (4.17). Therefore we obtain that

$$\Delta G^\mu(z) \leq -4\pi\rho(z), \text{ for all } \ z.$$

If we apply the same argument with the inequality reversed, we obtain the equality for all z. □

The following theorem is also very important for our goal. We will show that the electrostatic potential is harmonic for all x in the complement of the support of μ.

Theorem 4.3.3. *Suppose that $\mu(dx) = \rho(x)dx$ is supported on a compact set $S \subseteq \mathbb{R}^3$. Then G^μ is harmonic on $\mathbb{R}^3 \setminus S$.*

Proof. First note that if f is a function defined on the boundary of $B(x,r)$, then by the strong Markov property we have

$$E_x\left(f(B_{\tau(B(x,r))})\right) = \int_{\partial B(x,r)} f(y)\sigma(dy), \tag{4.18}$$

where σ is normalized surface measure on the boundary of $B(x,r)$ and $\tau(A) := \inf\{t : B_t \notin A\}$.

In our case, we consider the function $f(x) = \frac{1}{\|x-y\|}$. Let $x \in S^c$ and let $r < dist(x, \partial S)$. Let $\tau_r := \inf\{t : B_t \notin B(x,r)\}$. Then by (4.18), and using the fact that f is harmonic on $\mathbb{R}^3 \setminus \{y\}$ we have

$$E_x\left(\frac{1}{\|B_{\tau_r} - y\|}\right) = \frac{1}{\|x-y\|}.$$

Then

$$\begin{aligned} E_x G^\mu(B_{\tau_r}) &= E_x\left(\int \frac{\rho(y)dy}{\|B_{\tau_r} - y\|}\right) \\ &= \int_S \rho(y) E_x\left(\frac{1}{\|B_{\tau_r} - y\|}\right) dy \\ &= \int_S \frac{\rho(y)dy}{\|x-y\|} \\ &= G^\mu(x) \end{aligned}$$

Using (4.18) and the previous equality we obtain that G^μ is harmonic on $\mathbb{R}^3 \setminus S$. □

In the following theorem we show the relation between the energy and the electrostatic potential, for more details see [1]. Using this theorem, we will show in the next sections the connection between the Dirichlet problem and the minimum energy problem.

Theorem 4.3.4. *Let μ be a measure supported on $S \subseteq \mathbb{R}^3$, associated with the electrostatic potential G^μ, then*

$$E(\mu) = \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla G^\mu(x)|^2 dx. \tag{4.19}$$

Proof. By Green's identity we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla G^\mu(x)|^2 dx &= \int_{\mathbb{R}^3} \nabla G^\mu(x) \cdot \nabla G^\mu(x) dx \\ &= -\int_{\mathbb{R}^3} G^\mu(x) \Delta G^\mu(x) dx. \end{aligned}$$

We know that $\Delta G^\mu(x) = -4\pi\rho(x)$ on S, and $\Delta G^\mu(x) = 0$ on $\mathbb{R}^3 \setminus S$, then we arrive at

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla G^\mu(x)|^2 dx &= -\int_{\mathbb{R}^3} G^\mu(x) \Delta G^\mu(x) dx \\ &= 4\pi \int_S G^\mu(x) \rho(x) dx \\ &= 4\pi \int_S G^\mu(x) d\mu(x) \\ &= 4\pi E(\mu). \end{aligned}$$

Therefore we obtain

$$E(\mu) = \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla G^\mu(x)|^2 dx. \tag{4.20}$$

□

In the following section we will work with the concept of capacity. This concept is very important, in particular because the sets of capacity zero play the role of negligible sets in potential theory, in the same way that sets of measure zero are negligible in measure theory.

4.3.1 Capacity on $\mathbb{R}^n$, $n \geq 3$

In the context of potential theory, the capacity of a set is a measure of the ability of the set to hold electric charge.

Definition 4.3.5. *Let $S \subseteq \mathbb{R}^n$ be a compact set, with $n \geq 3$. We denote $\Gamma(S)$ as the family of all non-negative Borel measure which are supported in S and satisfy $G^\mu \leq 1$ everywhere in S. We define the* ***capacity*** *of the compact $S \subseteq \mathbb{R}^n$, $n \geq 3$, by*

$$C(S) := \sup_{\mu\in\Gamma(S)} \mu(S).$$

Remark: We will see later that this definition of capacity is wrong for $n = 2$, then we will work with other definition called logarithmic capacity.

If the set A is Borel, not necessarily compact, then the capacity of A is defined as

$$C(A) := \sup\{C(E) : \text{E is compact subset of } A\}.$$

From the definition of capacity, it is clear that if $A \subseteq B$, then $C(A) \leq C(B)$.

All the proofs of the theorems of this section can be found in [52]. The following Lemma shows a relation between a set of zero capacity and the minimum energy.

Lemma 4.3.6. *Let $S \subseteq \mathbb{R}^n$ be a compact set, $n \geq 3$, and let μ_0 the equilibrium measure on S. Then $E(\mu_0) = \infty$ if and only if $C(S) = 0$.*

Proof. First suppose that $C(S) > 0$. We show that $E(\mu_0) < \infty$. As $C(S) > 0$, we take a measure μ supported on S such that $\mu(S) > 0$ and $G^\mu \leq 1$.

Then $\nu := \dfrac{1}{\mu(S)}\mu$ is a probability measure supported on S and satisfies

$$E(\nu) = \int G^\nu(x)d\nu(x) = \frac{1}{(\mu(S))^2}\int\int K(x,y)d\mu(y)d\mu(x) = \frac{1}{(\mu(S))^2}\int G^\mu(x)d\mu(x) < \infty,$$

therefore $E(\mu_0) \leq E(\nu) < \infty$.

On the other hand, suppose that $E(\mu_0) < \infty$. We show that $C(S) = 0$. Define $\mu := \frac{1}{E(\mu_0)}\mu_0$, then $\mu \in \Gamma(S)$ and we have

$$\mu(S) = \frac{1}{E(\mu_0)} > 0,$$

therefore $C(S) > 0$. $\square$

Usually a set that has zero capacity is called Polar, and by the previous lemma, these sets are characterized by $E(\mu_0) = \infty$. Also from previous lemma is easy to proof that a set A is Polar if $E(\mu) = \infty$ for all finite and compactly supported measures μ with support contain in A.

We now show a lemma that will use to proof that a set of zero capacity has zero Lebesgue measure.

Lemma 4.3.7. *Let $A \subseteq \mathbb{R}^n$, $n \geq 3$. If $C(A) = 0$ and μ is any non negative Borel measure supported in a compact set with G^μ being bounded from above in the set A, then $\mu(E) = 0$ for every compact $E \subseteq A$.*

Proof. Assume that $\mu(E) > 0$ for some compact $E \subseteq A$, and consider the restriction μ_E of μ in E.

Since the kernel defining G^μ is positive, then G^{μ_E} is also bounded from above in the set A and hence there exist a constant $M > 0$ such that

$$G^{\mu_E}(x) \leq M.$$

Now, define $\nu := \dfrac{1}{M}\mu_E$. Then $\nu \in \Gamma(E)$ and $\nu(E) > 0$.

Therefore

$$C(A) \geq C(E) \geq \nu(E) > 0.$$

But this is a contradiction. Thus we have the result. $\square$

Theorem 4.3.8. *If $A \subseteq \mathbb{R}^n$ is a Borel polar set, with $n \geq 3$, then A has zero Lebesgue measure.*

Proof. The proof is easy using the fact that the electrostatic potential defined by the Lebesgue measure restrict to disc $B(0, r)$ is bounded for all $r > 0$, i.e. $G^{m|_{B(0,r)}}(x) < \infty$ for all x. Later we use the Lemma 4.3.7. $\square$

The following lemma shows other important property of the capacity.

Lemma 4.3.9. *If all $A_k \subseteq \mathbb{R}^3$ are Borel sets and $C(A_k) = 0$ for all k, then*

$$C\left(\bigcup A_k\right) = 0.$$

Proof. Take an arbitrary non-negative Borel measure μ, supported in a compact subset of $\bigcup A_k$ with $G^\mu \leq 1$ everywhere.

By previous Lemma, we have that $\mu(E) = 0$ for all compact E such that $E \subseteq A_k$. This implies that for all k

$$\mu(A_k) = 0.$$

Note that $\mu\left(\bigcup A_k\right) \leq \sum \mu(A_k) = 0$, then

$$\mu\left(\bigcup A_k\right) = 0.$$

Therefore if E is compact, $E \subseteq \bigcup A_k$, then we have that $\mu(E) \leq \mu\left(\bigcup A_k\right) = 0$. This implies that $C(E) = \sup\limits_{\mu \in \Gamma(E)} \mu(E) = 0$.

Then, as $C\left(\bigcup A_k\right) = \sup\{C(E) : \text{E is compact}, E \subseteq \bigcup A_k\}$, we have the result. □

The followings two theorems are very important to show that the equilibrium measure is unique on $\mathbb{R}^n$, with $n \geq 3$.

Theorem 4.3.10. *Let $S \subseteq \mathbb{R}^n$ be compact, $n \geq 3$, and let μ_0 be the equilibrium measure. Then the electrostatic potential $G^{\mu_0}(x)$ satisfies*

$$G^{\mu_0}(x) \leq E(\mu_0) \ \textit{ for every } x \in S.$$

Proof. Consider an arbitrary $x \in S$ and assume that

$$G^{\mu_0}(x) > E(\mu_0).$$

Take an arbitrary γ so that

$$G^{\mu_0}(x) > \gamma > E(\mu_0).$$

By lower-semicontinuity we have

$$G^{\mu_0}(x) > \gamma,$$

everywhere in some open neighborhood $B(x, r_x)$.

On the other hand, the Borel set $A := \{x \in S : G^{\mu_0}(x) < E(\mu_0)\}$ has zero capacity, to prove this, first note that $A = \bigcup A_k$, where $A_k := \{x \in S : G^{\mu_0}(x) \leq E(\mu_0) - \frac{1}{k}\}$.

We will now prove that $C(A_k) = 0$. Suppose that $C(A_k) > 0$. By definition of capacity, there is some non negative Borel measure τ supported in A_k such that

$$\tau(A_k) > 0 \ \text{ and } \ G^\tau \leq 1.$$

We define the following probability measure

$$\sigma := \frac{1}{\tau(A_k)} \tau,$$

supported in A_k, and note that satisfies

$$G^\sigma \leq \frac{1}{\tau(A_k)}.$$

we now consider $0 < \delta < 1$ and we define

$$\mu_\delta := (1-\delta)\mu_0 + \delta\sigma.$$

Then μ_δ is a probability measure supported in S and hence

$$\begin{aligned} E(\mu_0) &\leq E(\mu_\delta) \\ &= \int\int K(x,y) d\mu_\delta(x) d\mu_\delta(y) \\ &= (1-\delta)^2 E(\mu_0) + 2\delta(1-\delta)\int_{A_k} G^{\mu_0}(x) d\sigma(x) + \delta^2 E(\sigma). \end{aligned}$$

From the previous inequality, it is easy to check that

$$0 \leq -2\frac{\delta}{k} + \delta^2\left(E(\sigma) + E(\mu_0) - 2\int_{A_k} G^{\mu_0}(x) d\sigma(x)\right).$$

This brings a contradiction for δ sufficiently small, thus

$$C(A_k) = 0.$$

Therefore $A := \{x \in S : G^{\mu_0}(x) < E(\mu_0)\}$ has zero capacity.
By Lemma 4.3.7, we have that

$$\mu_0(\{x \in S : G^{\mu_0}(x) < E(\mu_0)\}) = 0.$$

Then we have that $\mu_0(\{x \in S : G^{\mu_0} \geq E(\mu_0)\}) = 1$. This implies that

$$\begin{aligned} E(\mu_0) &= \int_S G^{\mu_0}(y) d\mu_0(dy) \\ &= \int_{S\bigcap B(x,r_x)} G^{\mu_0}(y) d\mu_0(y) + \int_{S\setminus B(x,r_x)} G^{\mu_0}(y) d\mu_0(y) \\ &\geq \gamma\mu_0(S\bigcap B(x,r_x)) + E(\mu_0)\mu_0(S\setminus B(x,r_x)). \end{aligned}$$

From the previous inequality we obtain

$$\mu_0(S\bigcap B(x,r_x)) = 0, \tag{4.21}$$

because if $\mu_0(S \bigcap B(x, r_x)) > 0$, and since we also know that $\gamma > E(\mu_0)$, then we have a contradiction:

$$E(\mu_0) \geq \gamma\mu_0(S \bigcap B(x, r_x)) + E(\mu_0)\mu_0(S \setminus B(x, r_x)) > E(\mu_0).$$

On the other hand, consider the open set

$$O = \bigcup\{B(x, r_x) : x \in S \text{ and } G^{\mu_0}(x) > E(\mu_0)\}.$$

If F is any compact subset of O, then we can cover F by finitely many balls of the form $B(x, r_x)$ with $x \in S$ and $G^{\mu_0}(x) > E(\mu_0)$. By (4.21) we have that $\mu_0(F) = 0$. Since F is arbitrary, then $\mu_0(O) = 0$. Note that this implies that μ_0 is supported in the compact set $S \setminus O$ and

$$G^{\mu_0}(x) \leq E(\mu_0) \text{ for every } x \in S.$$

□

Theorem 4.3.11. *(***Frostman's***) Let $S \subseteq \mathbb{R}^3$ be compact, and let μ_0 be the equilibrium measure. Then the electrostatic potential $G^{\mu_0}(x)$ satisfies*

$$G^{\mu_0}(x) = E(\mu_0) \text{ everywhere in } S \text{ perhaps except in a subset of } S \text{ of zero capacity.}$$

Proof. We consider again the set

$$A_k := \{x \in S : G^{\mu_0}(x) \leq E(\mu_0) - \frac{1}{k}\}.$$

By the previous theorem, we known that $C(A_k) = 0$ for all k. Then, $A := \{x \in S : G^{\mu_0}(x) < E(\mu_0)\}$ also have zero capacity, because $A = \bigcup A_k$.

Therefore $G^{\mu_0}(x) \geq E(\mu_0)$ perhaps except in a subset of S of zero capacity, and using the previous theorem, we obtain the result. □

With the following corollary we show an application of the this theorem.

Corollary 4.3.12. *Let $S \subseteq \mathbb{R}^n$ be with $n \geq 3$. Let $\nu \in P(S)$ be such that $G^\nu(x) = C$ perhaps except in a subset of S of zero capacity. Then $C = E(\mu_0)$ and $\nu = \mu_0$.*

Proof. Using the Lemma 4.3.7 and the previous theorem, this fact follows by simply integrating the equality $G^\nu = C$ with respect to μ_0, and interchanging order of integration. □

With the previous results, we now can to show that the equilibrium measure is unique. We have the following theorem

Theorem 4.3.13. *Let $S \subseteq \mathbb{R}^n$ be a compact set, with $n \geq 3$, and let μ_0 be the equilibrium measure such that $\gamma := E(\mu_0) < \infty$. Then the equilibrium measure μ_0 is unique.*

Proof. Suppose that μ and ν are two equilibrium measures. By Frostman's theorem we know that the electrostatic potential in the equilibrium measure is equal to γ everywhere in S except perhaps in a subset of S of zero capacity. Then we have $G^{\mu}(x) = \gamma$ and $G^{\nu}(x) = \gamma$ except perhaps in a subset of capacity zero. Also $G^{\mu}(x) \leq \gamma$ and $G^{\nu}(x) \leq \gamma$ everywhere.

Let us see that $\nu\left(\{x \in S : G^{\mu}(x) < \gamma\}\right) = 0$. This is because if we define $S^{\epsilon} := \{x \in S : G^{\mu}(x) \leq \gamma - \epsilon\}$, and we use the proof of the Frostman's theorem, then we have that S^{ϵ} is compact and has capacity zero.

By Lemma 4.3.7 we obtain that $\nu(S^{\epsilon}) = 0$, and since $\{x \in S : G^{\mu}(x) < \gamma\} = \bigcup S^{\frac{1}{n}}$, then

$$\nu\left(\{x \in S : G^{\mu}(x) < \gamma\}\right) = 0. \tag{4.22}$$

Let $\Gamma := \{x \in S : G^{\mu}(x) < \gamma\}$. Applying the Theorem 4.3.11 we have

$$\begin{aligned}
E(\mu - \nu) &= \int_S G^{\mu-\nu}(x) d(\mu - \nu)(x) \\
&= \int\int K(x, y) d(\mu - \nu)(y) d(\mu - \nu)(x) \\
&= \int_S G^{\mu}(x) d\mu(x) + \int_S G^{\nu}(x) d\nu(x) - 2\int_S G^{\mu}(x) d\nu(x) \\
&= E(\mu) + E(\nu) - 2\int_S G^{\mu}(x) d\nu(x) \\
&= 2\gamma - 2\int_S G^{\mu}(x) d\nu(x) \\
&= 2\gamma - 2\int_{S\setminus\Gamma} G^{\mu}(x) d\nu(x) - 2\int_{\Gamma} G^{\mu}(x) d\nu(x) \\
&= 2\gamma - 2\int_{S\setminus\Gamma} \gamma d\nu(x) \\
&= 0
\end{aligned}$$

We will show that if $E(\mu - \nu) = 0$ then $\mu = \nu$. To this end, we use the equality from Theorem 4.3.4

$$E(\mu - \nu) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mid \nabla G^{\mu-\nu}(x) \mid^2 dx.$$

If $E(\mu - \nu) = 0$ then we have that $\int_{\mathbb{R}^3} \mid \nabla G^{\mu-\nu}(x) \mid^2 dx = 0$, this implies that $\nabla G^{\mu-\nu}(x) = 0$. Since $G^{\mu} = \gamma$ and $G^{\nu} = \gamma$ except perhaps in a subset of zero capacity, then we arrive at

$$G^{\mu} = G^{\nu}, \text{ except perhaps in a subset of zero capacity.}$$

We will use this fact, that $G^{\mu} = G^{\nu}$, except perhaps in a subset of zero capacity, to show that $\mu = \nu$.

We suppose that $\mu(dx) = \rho(x)dx$ and $\nu(x) = \phi(x)dx$. Note that if we define the semigroup $P_t(f(x)) := E_x(f(B_t))$, where $p(t,x,y)$ is the transition density function of the Brownian motion, and we use the Theorem 4.3.8, then we obtain

$$P_h(G^\mu)(x) = E_x(G^\mu(B_h)) = \int G^\mu(y)p(h,x,y)dy = \int G^\nu(y)p(h,x,y)dy = P_h(G^\nu(x)). \tag{4.23}$$

On the other hand, by the Markov property and the Theorem 4.3.1 we have

$$\begin{aligned}
P_h(G^\mu(x)) &= E_x(G^\mu(B_h)) \\
&= 2\pi E_x\left(E_{B_h}\left(\int_0^\infty \rho(B_s)ds\right)\right) \\
&= 2\pi E_x\left(E_x\left(\int_0^{\ \infty} \rho(B_{s+h})ds\right) \mid \mathfrak{F}_h\right) \\
&= 2\pi E_x\left(\int_0^\infty \rho(B_{s+h})ds\right) \\
&= 2\pi E_x\left(\int_h^\infty \rho(B_s)ds\right) \\
&= 2\pi \int_h^\infty P_s\rho(x)ds,
\end{aligned}$$

where $\mathfrak{F}_h := \sigma\{B_s : s \leq h\}$. Then we have

$$P_h(G^\mu(x)) = 2\pi \int_h^\infty P_s\rho(x)ds. \tag{4.24}$$

Since (4.23) and (4.24) are also true for ν, then we have that for all h and x

$$\int_h^\infty P_s\rho(x)ds = \int_h^\infty P_s\phi(x)ds. \tag{4.25}$$

It follows that for each x, and for almost every s

$$P_s(\rho(x)) = P_s(\phi(x)). \tag{4.26}$$

Now we take f continuous with compact support, then applying Fubini we arrive at

$$\int f(x)P_t(\rho(x))dx = \int\int f(x)p(t,x,y)dx\mu(dy) = \int P_t(f(y))\mu(dy), \tag{4.27}$$

and similarly for ϕ, we obtain that

$$\int f(x)P_t(\phi(x))dx = \int\int f(x)p(t,x,y)dx\nu(dy) = \int P_t(f(y))\nu(dy). \tag{4.28}$$

Considering (4.26), (4.27) and (4.28) we end up with

$$\int P_t(f(y))\mu(dy) = \int P_t(f(y))\nu(dy).$$

Since it is easy to see that $P_t(f(y)) \to f(y)$ uniformly as $t \to 0$, then we have

$$\int f(y)\mu(dy) = \int f(y)\nu(dy), \tag{4.29}$$

for all f continuous with compact support. Then we have that $\mu = \nu$. An idea of why is this true is the following: if we take a rectangle $A \subseteq S$, then we can to define the following function

$$f(x) = \begin{cases} 1, & x \in A; \\ 0, & x \in S \backslash A. \end{cases}$$

We now choose continuous functions f_n such that $f_n \to f$ uniformly. Then using (4.29) we have

$$\mu(A) = \int f(x)\mu(dx) = \lim_{n\to\infty} \int f_n(x)\mu(dx) = \lim_{n\to\infty} \int f_n(x)\nu(dx) = \int f(x)\nu(dx) = \nu(A).$$

We can to generalize this equality for all Borel set. Therefore the equilibrium measure is unique.

□

The following theorem shows a explicit relation between the capacity and the minimum energy.

Theorem 4.3.14. *Suppose that $S \subseteq \mathbb{R}^3$ is compact and $E(\mu_0) > 0$. Then*

$$C(S) = \frac{1}{E(\mu_0)}.$$

Proof. Let μ_0 be the equilibrium measure of S, and define $\mu := \dfrac{1}{E(\mu_0)}\mu_0$.

Since $G^{\mu_0} \leq E(\mu_0)$ everywhere, then $\mu \in \Gamma(S)$, and we have

$$C(S) \geq \mu(S) = \frac{1}{E(\mu_0)}.$$

On the other hand, let ν be any measure Borel non negative supported in S with $G^{\nu} \leq 1$. Then by Lemma 4.3.7 we have

$$\nu(\{x \in S : G^{\mu} \leq 1\}) = 0.$$

Hence applying the previous equality and the Fubini's Theorem we have

$$\frac{1}{E(\mu_0)} = \mu(S) \geq \int_S G^\nu(x)d\mu(x) = \int_S G^\mu(x)d\nu(x) \geq \nu(S).$$

Then

$$\frac{1}{E(\mu_0)} \geq C(S),$$

Therefore

$$\frac{1}{E(\mu_0)} = C(S).$$

□

4.3.2 An interpretation of the minimum energy

Let $S \subseteq \mathbb{R}^n$ be compact, with $n \geq 3$. We define

$$M_n := \sup_{x_1,\ldots,x_n \in S} \inf_{x \in S} \frac{1}{n} \sum_{j=1}^{n} K(x, x_j),$$

and

$$D_n := \inf_{x_1,\ldots,x_n \in S} \frac{2}{n(n-1)} \sum_{i<j} K(x_i, x_j),$$

then we have the following theorem

Theorem 4.3.15. *Let M_n and D_n be as in the previous definition, then*

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} D_n = E(\mu_0) := \inf_{\mu \in P(E)} E(\mu).$$

Proof. We divide the proof into steps.

Step 1.

We consider $x_1, ..., x_{m+1} \in S$ such that

$$D_{m+1} = \frac{2}{m(m+1)} \sum_{i<j} K(x_i, x_j).$$

Note that D_{m+1} can be rewrite as

$$D_{m+1} = \frac{2}{(m+1)} \left(\frac{1}{m} \sum_{j=2}^{m+1} K(x_1, x_j) + \sum_{i=2}^{m} \frac{1}{m} \sum_{j=i+1}^{m+1} K(x_i, x_j) \right).$$

The second sum in the last equality do not depend of x_1, and hence x_1 minimizes

$$\frac{1}{m}\sum_{j=2}^{m+1} K(x, x_j).$$

Therefore

$$\frac{1}{m}\sum_{j=2}^{m+1} K(x_1, x_j) = \inf_{x\in S}\frac{1}{m}\sum_{j=2}^{m+1} K(x, x_j) \leq \sup_{y_1,\dots,y_m\in S}\inf_{x\in S}\frac{1}{m}\sum_{j=2}^{m+1} K(x, y_j) = M_m.$$

Similarly we apply this process for all i, then we have

$$\frac{1}{m}\sum_{j\neq i}^{m+1} K(x_i, x_j) \leq M_m. \tag{4.30}$$

Thus using (4.30) we have

$$\begin{aligned} D_{m+1} &= \frac{2}{m(m+1)}\sum_{i<j}^{m+1} K(x_i, x_j) \\ &= \frac{1}{m+1}\left(\frac{1}{m}\sum_{i\neq 1} K(x_i, x_1) + \frac{1}{m}\sum_{i\neq 2} K(x_i, x_2) + \dots + \frac{1}{m}\sum_{i\neq m+1} K(x_i, x_{m+1})\right) \\ &\leq \frac{1}{m+1}(M_m + M_m + \dots + M_m) \\ &= M_m. \end{aligned}$$

Step 2.

Let μ_0 be the equilibrium measure of S, then applying Theorem 4.3.10, we have that for all $x_1, \dots, x_m \in S$:

$$\inf_{x\in S}\frac{1}{m}\sum_{j=1}^{m} K(x, x_j) \leq \frac{1}{m}\sum_{j=1}^{m}\int_S K(x, x_j)d\mu_0(x) \leq \frac{1}{m}\sum_{j=1}^{m} E(\mu_0) = E(\mu_0).$$

Hence,

$$M_m \leq E(\mu_0).$$

Step 3.

Let $x_1, \dots x_m \in S$ be such that

$$D_m = \frac{2}{m(m-1)}\sum_{i<j} K(x_i, x_j).$$

Define $\nu_m := \frac{1}{m}\sum_{j=1}^{m} \delta_{x_j}$ and consider $K_N(x,y) := \min(K(x,y), N)$.
Then

$$\begin{aligned}
\int\int K_N(x,y)d\nu_m(x)d\nu_m(y) &= \frac{1}{m}\sum_{j=1}^{m}\int K_N(x,x_j)d\nu_m(x) \\
&= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i\neq j}\frac{1}{m}K_N(x_i,x_j) + \frac{1}{m}K_N(x_j,x_j)\right) \\
&\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i\neq j}\frac{1}{m}K(x_i,x_j) + \frac{N}{m}\right) \\
&= \frac{1}{m^2}\left(\sum_{i\neq 1}K(x_i,x_1) + \ldots + \sum_{i\neq m}K(x_i,x_m)\right) + \frac{N}{m} \\
&\leq \frac{2}{m(m-1)}\sum_{i<j}K(x_i,x_j) + \frac{N}{m}.
\end{aligned}$$

Now we pick a subsequence m_k such that

$$D_{m_k} \to \liminf_{m\to\infty} D_m, \quad \text{and} \quad \nu_{m_k} \to \nu \quad \text{weakly in S},$$

where ν is a probability measure with support on S.
By the weak convergence $\nu_{m_k} \times \nu_{m_k} \to \nu \times \nu$ in $S \times S$, we have

$$\begin{aligned}
\int\int K_N(x,y)d\nu(x)d\nu(y) &= \lim_{k\to\infty}\int\int K_N(x,y)d\nu_{m_k}(x)d\nu_{m_k}(y) \\
&\leq \lim_{k\to\infty}\frac{2}{m_k(m_k-1)}\sum_{i<j}K(x_i,x_j) + \frac{N}{m_k} \\
&= \liminf_{m\to\infty} D_m.
\end{aligned}$$

Now, letting $N \to \infty$ we find that

$$E(\mu_0) \leq E(\nu) \leq \liminf_{m\to\infty} D_m.$$

After $1, 2$ and 3 Steps we have

$$E(\mu_0) \leq \liminf_{m\to\infty} D_m \leq \limsup_{m\to\infty} D_m \leq \limsup_{m\to\infty} M_m \leq E(\mu_0),$$

and

$$E(\mu_0) \leq \liminf_{m\to\infty} D_m \leq \liminf_{m\to\infty} M_m \leq \limsup_{m\to\infty} M_m \leq E(\mu_0).$$

Therefore we arrive at

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} D_n = E(\mu_0)$$

□

4.4 Potential theory on $\mathbb{R}^2$

From the definition in (4.2.2) we have that the electrostatic potential on $\mathbb{R}^2$ is the following

$$G^{\mu}(x) := \int_S \rho(y) \log \frac{1}{\|x-y\|} dy, \tag{4.31}$$

where $\mu(A) := \displaystyle\int_A \rho(x)dx$, and the energy is defined as

$$E(\mu) := \int_S G^{\mu}(x)d\mu(x). \tag{4.32}$$

This potential satisfies some similar properties that the electrostatic potential on $\mathbb{R}^3$. For example, this potential is lower semi-continuous, G^{μ} is harmonic on $\mathbb{R}^2 \setminus supp(\mu)$ and satisfies the following property

$$\Delta G^{\mu}(x) = -2\pi\rho(x),$$

where ρ is the density of the measure μ. If ρ is not smooth, the previous property is considered in distributional sense. For more details see [48], [58], [62].

Also on $\mathbb{R}^2$, we have a similar identity as $\mathbb{R}^3$, that shows the relation between the energy and the electrostatic potential defined over all $\mathbb{R}^2$,

$$E(\mu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla G^{\mu}(x)|^2 dx.$$

The proof of this fact is the same as in $\mathbb{R}^3$, using one more time the Green's identity. We will use this identity later, when we work with the Dirichlet's problem.

On the other hand, our goal is to show the Frostman's theorem on $\mathbb{R}^2$, and that the equilibrium measure also is unique is this case. To do this, we need to consider again the concept of capacity.

First, it is necessary to consider other definition of capacity, called logarithmic capacity. This is by the following reason. If we consider on $\mathbb{R}^2$ the same definition of capacity on $\mathbb{R}^n$ with $n \geq 3$, then there is an inconsistency. The problem is that the energy in $\mathbb{R}^2$ may be negative, and if that happens one can prove that $C(S) = \infty$, and therefore the problem to find a measure $\mu \in \Gamma(S)$ such that $\sup \mu(S) = \infty$ has no solution.

For this reason it is used other definition of capacity on $\mathbb{R}^2$. Intuitively, the idea is the following. It is known that the capacity of a ball in $\mathbb{R}^3$ of radius r is r. We want a definition of capacity on $\mathbb{R}^2$ that keep this property.

Let us see the following, we consider the coming function, called the Green function with pole at infinity,

$$g_\Omega(x, \infty) := E(\mu_0) - G^{\mu_0}(x),$$

where Ω is the unbounded component of $\mathbb{C} \setminus S$, and μ_0 is the equilibrium measure on S.

This function satisfies $g_\Omega(x, \infty) - \log |x| \to E(\mu_0)$ as $|x| \to \infty$ on $\mathbb{R}^2$, see [18]. In particular if S is the ball with radius r we have that $g_\Omega(x, \infty) - \log |x| \to \log \left(\frac{1}{r}\right)$. Hence $E(\mu_0) = \log \left(\frac{1}{r}\right)$, this implies that $r = e^{-E(\mu_0)}$. Since r is the capacity of the ball of radius r on $\mathbb{R}^3$, then we consider the following definition.

4.4.1 Logarithmic capacity

Definition 4.4.1. *If $S \subseteq \mathbb{R}^2$ we define the **logarithmic capacity** by*

$$C(S) := e^{-E(\mu_0)},$$

and this new definition of capacity in $\mathbb{R}^2$ has several similar properties with the definition of capacity on $\mathbb{R}^3$.

If the set A is Borel, not necessarily compact, then the capacity of A is defined as

$$C(A) := \sup\{C(E) : \text{E is compact subset of } A\}.$$

From the definition of capacity, it is clear that if $A \subseteq B$, then $C(A) \leq C(B)$.

The following Lemma shows a relation between a set of zero capacity and the minimum energy.

Lemma 4.4.2. *Let $S \subseteq \mathbb{R}^2$ be a compact set. Then $E(\mu_0) = \infty$ if and only if $C(S) = 0$.*

Proof. The proof is easy from the definition. □

Lemma 4.4.3. *Let μ be a finite Borel measure on $\mathbb{R}^2$ with support compact, and suppose that $E(\mu) < \infty$. If $A \subseteq \mathbb{R}^2$ is a polar set, then $\mu(A) = 0$.*

Proof. We may suppose that $A \subseteq supp(\mu)$. We now argue by contraposition, i.e. let A be a Borel set such that $\mu(A) > 0$, we want to show that A cannot be polar, in other words, we want to show that there exists a measure compactly supported in A with finite energy.

To construct this measure notice first that any Borel measure is regular, so there exists a compact set $K \subseteq A$ such that $\mu(K) > 0$.

Restricting μ to K, we have a measure $\mu\mid_K$ with support contain in A. Now set $d := diam(supp(\mu))$ and note that $\frac{\|z-w\|}{d} < 1$ if $z, w \in supp(\mu)$. Then we have

$$\begin{aligned}
E(\mu\mid_K) &= -\int_K\int_K \log\|z-w\| d\mu\mid_K(z) d\mu\mid_K(w)\\
&= -\int_K\int_K (\log\|z-w\| + \log d - \log d)\, d\mu\mid_K(z) d\mu\mid_K(w)\\
&\le -\int_{supp(\mu)}\int_{supp(\mu)} \log\frac{\|z-w\|}{d} d\mu(z)d\mu(w) - \mu(K)^2\log d\\
&= E(\mu) + \mu(supp(\mu))^2\log d - \mu(K)^2\log d\\
&< \infty,
\end{aligned}$$

therefore we conclude that A is non polar. $\square$

Theorem 4.4.4. *If $A \subseteq \mathbb{R}^2$ is a Borel polar set, then A has zero Lebesgue measure.*

Proof. Let m be the Lebesgue measure. The idea is restrict the Lebesgue measure m to a disc $B(0,r)$ and show that $E(m\mid_{B(0,r)}) < \infty$ for all $r > 0$, and then we apply the previous lemma to deduce that $m(A\bigcap B(0,r)) = 0$ for any Borel polar set A and on letting $r \to \infty$ we obtain the result.

Then we have that show that $E(m\mid_{B(0,r)}) < \infty$. To this end, we show that the electrostatic potential is bounded, applying a similar procedure as the previous lemma. Then we have that for all $x \in B(0,r)$

$$\begin{aligned}
G^{m|_{B(0,r)}}(x) &= -\int_{B(0,r)} \log\frac{\|x-y\|}{2r} dm\mid_{B(0,r)}(y) - \pi r^2\log(2r)\\
&\le -\int_0^{2\pi}\int_0^{2r} \log\left[\frac{s}{2r}\right] sdsdt - \pi r^2\log(2r)\\
&= 2\pi r^2 - \pi r^2\log(2r)\\
&< \infty.
\end{aligned}$$

Then $E(m\mid_{B(0,r)}) = \int_{B(0.r)} G^{m|_{B(0,r)}}(x) d\mu\mid_{B(0,r)}(x) < \infty$, and we have the result. $\square$

Theorem 4.4.5. (Frostman's) *Let $S \subseteq \mathbb{R}^2$ be compact, not polar, and let μ_0 be the equilibrium measure. Then the electrostatic potential $G^{\mu_0}(x)$ satisfies*

$$G^{\mu_0}(x) = E(\mu_0), \quad \textit{everywhere in } S \textit{ perhaps except in a subset of } S \textit{ of zero capacity.}$$

Proof. We first show that

$$\int G^{\mu_0}(x)d(\nu-\mu_0)(x) \geq 0, \text{ for all } \nu \in P(S). \tag{4.33}$$

We have that S is not polar, then $E(\mu_0) < \infty$. Let $\epsilon \in [0,1]$, and μ, $\nu \in P(S)$, then we have

$$E(\epsilon\nu + (1-\epsilon)\mu) - E(\mu) = 2\epsilon \int G^{\mu}(x)d(\nu-\mu)(x) + \epsilon^2 E(\nu-\mu). \tag{4.34}$$

If we take $\mu = \mu_0$, then the left side is not negative because μ_0 is the equilibrium measure, then we arrive at

$$2\int G^{\mu_0}(x)d(\nu-\mu_0)(x) + \epsilon E(\nu-\mu_0) \geq 0.$$

If $\epsilon \to 0$ then we obtain (4.33).

We now show that $G^{\mu_0}(x) \geq E(\mu_0)$ everywhere in S perhaps except in a subset of S of zero capacity.

Suppose by contradiction that there exists a set $E \subseteq S$ such that $C(E) > 0$ and $G^{\mu_0}(x) < E(\mu_0)$ for all $x \in E$.

Integrating $G^{\mu_0}(x) < E(\mu_0)$ with respect to ν we obtain

$$\int G^{\mu_0}(x)d\nu(x) < E(\mu_0) = \int G^{\mu_0}(x)d\mu_0(x),$$

this is a contradiction with (4.33). Therefore $G^{\mu_0}(x) \geq E(\mu_0)$ everywhere in S perhaps except in a subset of S of zero capacity.

We now show that $G^{\mu_0}(x) \leq E(\mu_0)$ for all $x \in S = supp(\mu_0)$. We know that the set $\{x \in S : G^{\mu_0} > E(\mu_0)\}$ is open.

Define $E := \{x \in S : G^{\mu_0} > E(\mu_0)\} \bigcap \{supp(\mu_0)\}$, and suppose by contradiction that $E \neq \emptyset$. Then we have that $\mu_0(E) > 0$ and

$$E(\mu_0) = \int G^{\mu_0}(x)d\mu_0(x) = \int_E G^{\mu_0}(x)d\mu_0(x) + \int_{supp(\mu_0)\setminus E} G^{\mu_0}(x)d\mu_0(x) > E(\mu_0),$$

this is a contradiction, therefore $E = \emptyset$, and this implies that

$$supp(\mu_0) \subseteq \{x : G^{\mu_0}(x) > E(\mu_0)\}^c,$$

hence $G^{\mu_0}(x) \leq E(\mu_0)$ for all $x \in S = supp(\mu_0)$.

□

The following theorem shows that the equilibrium measure is unique.

Theorem 4.4.6. *Let $S \subseteq \mathbb{R}^2$ be a compact set, μ_0 the equilibrium measure, and let $\gamma := E(\mu_0)$. Then the equilibrium measure μ_0 is unique.*

Proof. Suppose that μ and ν are two measure such that $E(\mu) = E(\nu) = \gamma$.

Using the Frostman's Theorem we have that

$$E\left(\frac{\nu+\mu}{2}\right) \leq \gamma,$$

this implies that $\frac{\nu+\mu}{2}$ also is a equilibrium measure, therefore $E\left(\frac{\nu+\mu}{2}\right) = \gamma$.

On the other hand, is easy to verify that

$$E(\mu-\nu) = 2E(\mu) + 2E(\nu) - 4E\left(\frac{\mu+\nu}{2}\right),$$

this implies that $E(\mu-\nu) = 0$. We now use the lemma 1.8, pp. 29, in [62]. Using this lemma we have that $\mu = \nu$. □

4.4.2 An interpretation of logarithmic capacity

Place n points on a compact $S \subseteq \mathbb{R}^2$, such that they are as far apart as possible in the sense of the geometric mean of the pairwise distances the points. Since the number of different pairs of n points is $\frac{n(n-1)}{2}$, then we consider

$$\delta_n(S) := \max_{x_1,\dots,x_n \in S} \left[\prod_{i<j} \|x_i - x_j\|\right]^{\frac{2}{n(n-1)}}. \tag{4.35}$$

Any system of points $F_n := \{z_1, ..., z_n\}$ for which the maximum is attained is called an n-point **Fekete set** for S, see [62].

If we take the logarithm in (4.35), can we see that the maximization problem is equivalent to the minimization problem

$$\xi_n(S) := \min_{x_1,\dots,x_n} \sum_{i<j} -\log(\|x_i - x_j\|). \tag{4.36}$$

We also have the following relation

$$\xi_n(S) = \frac{n(n-1)}{2} \log\left[\frac{1}{\delta_n(S)}\right]. \tag{4.37}$$

Theorem 4.4.7. *The sequence* $\left\{\frac{\xi_n(S)}{n(n-1)}\right\}_{n=2}^{\infty}$ *is increasing, or equivalently, the sequence* $\{\delta_n(S)\}_{n=2}^{\infty}$ *is decreasing.*

Proof. Let $F_n = \{z_1, ..., z_n\}$ be the Fekete set for S. We have for each $k = 1, ..., n$ fixed

$$\begin{aligned}
\xi_n(S) &= \sum_{i\neq k} \log\left(\frac{1}{\|z_i - z_k\|}\right) + \sum_{i\neq k, j\neq k} \log\left(\frac{1}{\|z_i - z_j\|}\right) \\
&\geq \sum_{i\neq k} \log\left(\frac{1}{\|z_i - z_k\|}\right) + \xi_{n-1}(S).
\end{aligned}$$

We now add these n inequalities for each k, but first note that

$$\sum_{i\neq 1} \log\left(\frac{1}{\|z_i - z_1\|}\right) + \sum_{i\neq 2} \log\left(\frac{1}{\|z_i - z_2\|}\right) + ... + \sum_{i\neq n} \log\left(\frac{1}{\|z_i - z_n\|}\right) = 2\xi_n(S),$$

then we have

$$n\xi_n(S) \geq 2\xi_n(S) + n\xi_{n-1}(S),$$

and dividing by $n(n-1)(n-2)$ we get the result. □

By previous theorem, we obtain that the sequence $\delta_n(S)$ has a limit. We denote this limit as

$$\tau(S) := \lim_{n\to\infty} \delta_n(S),$$

which is called the **transfinite diameter** of S.

Theorem 4.4.8. *For any compact set $S \subseteq \mathbb{R}^2$, the logarithmic capacity is equal to the transfinite diameter, i.e.*

$$C(S) = \tau(S).$$

Proof. First we will show that

$$E(\mu_0) = \log\frac{1}{C(S)} \geq \log\frac{1}{\tau(S)}. \tag{4.38}$$

Let $F(y_1, ..., y_n) := \sum_{1\leq i<j\leq n} \log\frac{1}{\|y_i - y_j\|}$. Note that $F(y_1, ..., y_n) \geq \xi_n(S)$, then we have that

$$\frac{n(n-1)}{2}E(\mu_0) = \int ... \int F(y_1, ..., y_n) d\mu_0(y_1)...d\mu_0(y_n) \geq \xi_n(S) = \frac{n(n-1)}{2}\log\frac{1}{\delta_n(S)}.$$

Dividing by $\frac{n(n-1)}{2}$ and letting $n \to \infty$ we obtain (4.38).

To proof the other inequality, let $\hat{\mu}$ be a weak-star limit point of the measure $\nu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}$, where $\{z_1, ..., z_n\}$ is the Fekete set. Define $\log_M x := \min\{\log x, M\}$. By the monotone convergence theorem and the weak-star convergence of $\nu_n \times \nu_n$ to $\hat{\mu} \times \hat{\mu}$, we have

$$\begin{aligned}
E(\hat{\mu}) &= \int\int \log \frac{1}{\|z-t\|} d\hat{\mu}(z)d\hat{\mu}(t) \\
&= \lim_{M\to\infty} \int\int \log_M \frac{1}{\|z-t\|} d\hat{\mu}(z)d\hat{\mu}(t) \\
&= \lim_{M\to\infty}\lim_{n\to\infty} \int\int \log_M \frac{1}{\|z-t\|} d\nu_n(z)d\nu_n(t) \\
&= \lim_{M\to\infty}\lim_{n\to\infty} \frac{1}{n^2}\sum_{i\neq j}^{n} \log_M \frac{1}{\|z_i - z_j\|} + \lim_{M\to\infty}\lim_{n\to\infty} \frac{nM}{n^2} \\
&\leq \lim_{M\to\infty}\lim_{n\to\infty} \frac{2}{n^2}\xi_n(S) \\
&= \lim_{M\to\infty}\lim_{n\to\infty} \log \frac{1}{\delta_n(S)} \\
&= \log \frac{1}{\tau(S)}.
\end{aligned}$$

Now, if we apply (4.38), the previous inequality, and the minimality property of $E(\mu_0)$, we arrive at

$$E(\mu_0) \leq E(\hat{\mu}) \leq \log\frac{1}{\tau(S)} \leq E(\mu_0),$$

Then we obtain the result. □

In the next section, we show the relation between the minimum energy problem and a modification of the Dirichlet problem.

4.5 Dirichlet Problem

In this section we show the relation between the Dirichlet problem and a problem of minimization. To this end we use the identity obtained in the Theorem 4.3.4,

$$E(\mu) = \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla G^{\mu}(x)|^2 dx.$$

The proves of the theorems of this section are for $n = 3$. The prove for the case $n = 2$ is similar.

Definition 4.5.1. *Let U be an open, connected subset of $\mathbb{R}^3$ and let $f : \partial U \to \mathbb{R}$ be a bounded continue function. The Dirichlet problem is to find $u : \overline{U} \to \mathbb{R}$, where $u \in C^2(U)$ and u is continue on ∂U, and such that u satisfies*

$$\begin{aligned} \Delta u &= 0 \ \ on \ \ U, \\ u &= f \ \ on \ \ \partial U. \end{aligned}$$

This problem is associated with a problem of minimization. The following theorem show this equivalence.

Theorem 4.5.2. *(***Dirichlet's Principle***) Let U be an open and bounded subset. Then u is solution of Dirichlet problem if and only if*

$$\int_U |\nabla u|^2 = \min_{w \in A} \int_U |\nabla w|^2,$$

where $A := \{w \in C^2(U) : w = f \ \ on \ \ \partial U\}$.

Proof. First, suppose that u is solution of Dirichlet problem, and define

$$D(u) := \int_U |\nabla u|^2.$$

We need to show that $D(u) \leq D(w)$ for all $w \in A$. Let $w \in A$, then we have

$$\begin{aligned} 0 &= \int_U \Delta u(u - w) \\ &= -\int_U | \nabla u |^2 + \int_U \nabla u \cdot \nabla w \\ &\leq -\int_U | \nabla u |^2 + \frac{1}{2}\int_U | \Delta u |^2 + \frac{1}{2}\int_U | \Delta w |^2 \\ &= \frac{-1}{2}\int_U | \Delta u |^2 + \frac{1}{2}\int_U | \Delta w |^2 . \end{aligned}$$

Therefore we have that for all $w \in A$,

$$\int_U | \Delta u |^2 \leq \int_U | \Delta w |^2 .$$

Then

$$\int_U |\nabla u|^2 = \min_{w \in A} \int_U |\nabla w|^2.$$

Next, suppose u minimizes $D(w)$ for all $w \in A$. We will show that u is solution of Dirichlet problem. Let v be a C^2 function such that $v = 0$ for $x \in \partial U$. Therefore, for all t, $u + tv \in A$. Now we define

$$i(t) := D(u + tv).$$

By assumption, u is a minimizer of $D(w)$ for all $w \in A$. Therefore i must have a minimum at $t = 0$, and therefore $i'(0) = 0$. Note that

$$\begin{aligned} i(t) &= D(u+tv) \\ &= \int_U \mid \nabla(u+tv) \mid^2 \\ &= \int_U \mid \nabla u \mid^2 + t^2 \int_U \mid \nabla v \mid^2 + 2t \int_U \nabla u \cdot \nabla v. \end{aligned}$$

This implies

$$i'(t) = 2t \int_U \mid \nabla v \mid^2 + 2 \int_U \nabla u \cdot \nabla v.$$

Therefore by the Green identity we have

$$\begin{aligned} i'(0) &= \int_U \nabla u \cdot \nabla v \\ &= -\int_U v \Delta u + \int_{\partial U} v \frac{du}{dn} dS \\ &= -\int_U v \Delta u. \end{aligned}$$

Since $i'(0) = 0$, this implies

$$\int_U v \Delta u = 0.$$

This fact is true for all $v \in C^2(U)$ such that $v = 0$ for $x \in \partial U$. Therefore we can conclude that $\Delta u = 0$ on U. □

Now, we use the idea of previous theorem to find a relation between the problem of minimum energy and a modification of the Dirichlet problem. To this end, we first consider the following definition

Definition 4.5.3. *Let U be an open, connected subset of $\mathbb{R}^3$, and let $f : \partial U \to \mathbb{R}$ be a bounded continue function. The* ***modified Dirichlet problem*** *is to find $u : \mathbb{R}^3 \to \mathbb{R}$, where $u \in C^2(\mathbb{R}^3 \setminus \partial U)$, $u \in C_0(\mathbb{R}^3)$ and u is continuous on ∂U, also u satisfies*

$$\begin{aligned} \Delta u &= 0 \quad on \ \mathbb{R}^3 \setminus \partial U, \\ u &= f \quad on \ \partial U. \end{aligned}$$

Theorem 4.5.4. *Let U be an open and bounded subset. Then u is solution of modified Dirichlet problem if and only if*

$$\int_{\mathbb{R}^3} |\nabla u|^2 = \min_{w\in B} \int_{\mathbb{R}^3} |\nabla w|^2,$$

where $B := \{w \in C^2(\mathbb{R}^3 \setminus \partial U) : w = f \ \text{ on } \ \partial U, \ \text{ and } \ w \in C_0(\mathbb{R}^3)\}$

Proof. We use the idea of the Dirichlet's principle. Let u a solution of modified Dirichlet problem, and define

$$I(u) := \int_{\mathbb{R}^3} |\nabla u|^2.$$

We need to show that $I(u) \leq I(w)$ for all $w \in B$. Let $w \in B$, then by Green identity we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} \Delta u(u-w) \\
&= \int_U \Delta u(u-w) + \int_{U^c\setminus\partial U} \Delta u(u-w) + \int_{\partial U} \Delta u(u-w) \\
&= -\int_U \nabla(u-w)\cdot\nabla u + \int_{\partial U} (u-w)\cdot\frac{du}{dn}dS \\
&\quad -\int_{U^c\setminus\partial U} \nabla(u-w)\cdot\nabla u + \int_{\partial U\bigcup\{\infty\}} (u-w)\cdot\frac{du}{dn}dS \\
&= -\int_U \nabla u\cdot\nabla u + \int_U \nabla u\cdot\nabla w - \int_{U^c\setminus\partial U} \nabla u\cdot\nabla u + \int_{U^c\setminus\partial U} \nabla u\cdot\nabla w \\
&= -\int_U \mid\nabla u\mid^2 - \int_{U^c\setminus\partial U} \mid\nabla u\mid^2 + \int_U \nabla u\cdot\nabla w + \int_{U^c\setminus\partial U} \nabla u\cdot\nabla w.
\end{aligned}$$

Now, using the fact that $\nabla u\cdot\nabla w \leq \frac{|\nabla w|^2+|\nabla u|^2}{2}$, we arrive at

$$\int_{\mathbb{R}^3\setminus\partial U} \mid\nabla u\mid^2 \leq \int_{\mathbb{R}^3\setminus\partial U} \mid\nabla w\mid^2,$$

then for all $w \in B$ we have that

$$\int_{\mathbb{R}^3} \mid\nabla u\mid^2 \leq \int_{\mathbb{R}^3} \mid\nabla w\mid^2 .$$

Therefore

$$\int_{\mathbb{R}^3} |\nabla u|^2 = \min_{w\in B} \int_{\mathbb{R}^3} |\nabla w|^2.$$

Next, suppose u minimizes $I(w)$ for all $w \in B$. We will show that u is solution of modified Dirichlet problem. Let $v \in C^2(\mathbb{R}^3 \setminus \partial U)$ such that $v = 0$ for $x \in \partial U \bigcup \{\infty\}$. Note that $u + tv \in B$ for all $t \in \mathbb{R}$. Now we define

$$r(t) := I(u + tv).$$

By assumption, u is a minimizer of $I(w)$ for all $w \in B$. Therefore r must have a minimum at $t = 0$, and therefore $r'(0) = 0$. Note that

$$\begin{aligned} r(t) &= I(u + tv) \\ &= \int_{\mathbb{R}^3} \mid \nabla(u + tv) \mid^2 \\ &= \int_{\mathbb{R}^3} \mid \nabla u \mid^2 + t^2 \int_{\mathbb{R}^3} \mid \nabla v \mid^2 + 2t \int_{\mathbb{R}^3} \nabla u \cdot \nabla v, \end{aligned}$$

This implies

$$r'(t) = 2t \int_{\mathbb{R}^3} \mid \nabla v \mid^2 + 2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla v.$$

Therefore by the Green identity

$$\begin{aligned} r'(0) &= 2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \\ &= -2 \int_{\mathbb{R}^3} v \Delta u + 2 \int_{\partial \mathbb{R}^3} v \frac{du}{dn} dS \\ &= -2 \int_{\mathbb{R}^3} v \Delta u. \end{aligned}$$

Since $r'(0) = 0$, this implies

$$\int_{\mathbb{R}^3} v \Delta u = 0.$$

This is true for all $v \in C^2(\mathbb{R}^3 \setminus \partial U)$ such that $v = 0$ for $x \in \partial U \bigcup \{\infty\}$. Hence we can conclude that $\Delta u = 0$ on $\mathbb{R}^3 \setminus \partial U$. □

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