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Curvas y trenzas planas sin auto-intersecciones triples

T E S I S

QUE PRESENTA

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Plane curves and braids without triple self-intersections

A dissertation presented by

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A la guapita
Yadi

A mis padres
Maria del Carmen
y Edmundo

Resumen

Consideramos el espacio $\mathrm{Conf}_3(\mathbb{R}, n) := \mathbb{R}^n \setminus \{(x_1, \ldots, x_n) \,|\, x_i = x_j = x_k$ para distintos $i, j, k\}$, donde $PP_n = \pi_1(\mathrm{Conf}_3(\mathbb{R}, n))$ es una versión plana del grupo de trenzas puras. Estudiando la combinatoria del espacio $\mathrm{Conf}_3(\mathbb{R}, n)$, mostramos que su cubierta universal es un complejo cúbico CAT(0). Para el grupo PP_n obtenemos una presentación en términos de tuplas $(m_1, \ldots, m_{n-1})$, donde cada coordenada m_k es una regla a aplicar en el proceso de Reidemeister-Schreier para obtener los generadores asociados a la tupla. Para casos pequeños, pudimos reescribir la presentación del grupo PP_n de manera más sencilla en términos de posets. Por otro lado, considerando curvas planas sin triples intersecciones y una cerradura de trenzas puras planas, obtenemos un teorema de tipo Birman-Markov. Por último, describimos los invariantes de Vassiliev de curvas y trenzas planas, y utilizando integrales iteradas de Chen, definimos el invariante universal de Vassiliev para trenzas puras planas.

Abstract

We consider the space $\mathrm{Conf}_3(\mathbb{R}, n) := \mathbb{R}^n \setminus \{(x_1, \ldots, x_n) \,|\, x_i = x_j = x_k$ for distinct $i, j, k\}$, where $PP_n = \pi_1(\mathrm{Conf}_3(\mathbb{R}, n))$ is a planar version of the pure braid group. We study the combinatorics of the space $\mathrm{Conf}_3(\mathbb{R}, n)$ and show that its universal cover is a CAT(0) cubical complex. For the group PP_n we obtain a presentation in terms of tuples $(m_1, \ldots, m_{n-1})$, where the coordinate m_k stands for the rule applied in the Reidemeister-Schreier process in order to produce the generators associated with the tuple. For the cases of small n, we rewrite the presentation of the group PP_n in a simpler way in terms of posets. We also consider triple points free plane curves as closures of plane braids and obtain a Birman-Markov type theorem. Finally, we review Vassiliev invariants of plane curves and plane braids, and applying Chen's iterated integrals we define the universal Vassiliev invariant for plane pure braids.

Contents

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Introduction

The classical configuration space of n ordered points in $\mathbb{C}$, usually denoted by $\mathcal{F}(\mathbb{C}, n)$, is, by definition, the complement in $\mathbb{C}^n$ of the braid arrangement defined by the diagonals Δ_{ij}. The space $\mathcal{F}(\mathbb{C}, n)$ is the Eilenberg-MacLance space $K(P_n, 1)$ whose fundamental group is the pure braid group P_n. Configurations spaces and braid groups lead to a rich theory and appear in many different contexts in topology and other areas of mathematics. We are particularly interested in their connection with knot theory.

Braids and knots are closely related objects. By the Alexander Theorem, any knot or link can be represented as a closed braid, and the Markov Theorem gives conditions on when two closed braids represent the same knot or link. There are other closures of braids, such as the plat closure by Birman [7] and its modification, the short circuit closure by Mostovoy and Stanford [31]; the latter has the advantage that the closure of a pure braid is always a knot. The group structure on braids makes them more manageable than knots; often, one works with braids with the hope that similar constructions might work for knots: this is how the Jones polynomial and the Kontsevich integral were discovered.

The Kontsevich integral is one of the most powerful knot invariants; namely, it is as strong as the set of all rational-valued Vassiliev invariants [27]. The Vassiliev invariants include, among others, the coefficients of the Alexander-Conway polynomial and, after a certain renormalization, the coefficients of the Jones, HOMFLY and Kauffman polynomials. The construction of the Kontsevich integral has as an inspiration Kohno's construction [23] of an invariant of pure braids via Chen's iterated integrals; it has been one of the main motivations for the present work.

In chapter 1, we review the construction of the configurations spaces $\mathrm{Conf}_3(\mathbb{R}, n)$ of n-tuples of points $\mathbb{R}^n$ with multiplicity at most 2 and recall the definition of the planar version PP_n of the pure braid group on n strands. Let $\mathcal{A}_{n,3}$ be the collection of subspaces in $\mathbb{R}^n$ given by $x_i = x_j = x_k$ for all $1 \leq i < j < k \leq n$. Although these subspaces are of real codimension two, they cannot be viewed as a complex hyperplane arrangement. Set $\mathrm{Conf}_3(\mathbb{R}, n)$ to be the complement $\mathbb{R}^n \setminus \mathcal{A}_{n,3}$. Similarly to the classical case of the configurations spaces of distinct points, the space $\mathrm{Conf}_3(\mathbb{R}, n)$ is an Eilenberg-MacLane space $K(\pi, 1)$ with fundamental group the plane pure braid group PP_n. (In figure 1 we have the generator of the first non-trivial case $PP_3 \cong \pi_1(\mathrm{Conf}_3(\mathbb{R}, 3))$.) This fact was first proved by Khovanov in [22]. Our contribution is an analysis of the combinatorics of the space $\mathrm{Conf}_3(\mathbb{R}, n)$, based in a work of Barcelo, Severs and White [5] [34], in which we show that the universal cover of $\mathrm{Conf}_3(\mathbb{R}, n)$ is a cubical CAT(0) space and, hence, is contractible. As a corollary, we deduce that $\mathrm{Conf}_3(\mathbb{R}, n)$ is an Eilenberg-MacLane space $K(\pi, 1)$. Furthermore, we observe that

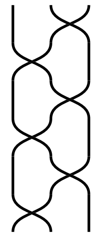
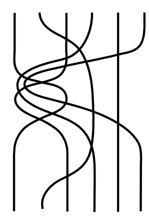


Figure 1: Generator $(\sigma_2\sigma_1)^3$ of PP_3

PP_n as the fundamental group of $\mathrm{Conf}_3(\mathbb{R}, n)$ acts properly and cocompactly on its universal cover; in other words PP_n is a cubed group.

Chapter 2 is dedicated to the problem of finding a presentation for the group of plane pure braids using the Reidemeister-Schreier process and to the problem of representing the group PP_n as an almost-direct product. In our attempt to calculate the presentation explicitly, we translate the generators into paths of a decision tree (see figure 2.3), represented as a tuple $(m_1, \ldots, m_{n-1})$, where the coordinate m_k stands for the rule applied in k-iteration of the Reidemeister-Schreier process. We obtain conditions on tuples which allow to avoid some of the redundancy in the resulting presentation. The presentation we obtain in terms of tuples $(m_1, \ldots, m_{n-1})$ is rather impractical. Despite that, for the cases of small n we obtain a more compact presentation in terms of so-called basic partitioner posets defined in the last section 1.4 of chapter 1. Partitioner posets are in correspondence with (co)homology classes of the space $\mathrm{Conf}_3(\mathbb{R}, n)$ found by Baryshnikov [6], and for each basic partitioner poset $P = (I)[J](K)$ we construct an associated plane pure braid b_P.

$$(46)[13](25) \quad \longmapsto \quad b_{(46)[13](25)} =$$



With this construction of plane pure braids from partitioner posets, we give an alternative presentation of PP_n for $n = 3, 4, 5, 6$, instead of the long list of generators and relations in tables 2.3.2, 2.3.3 at the end of chapter 2.

Theorem 2.2.7. For $n = 1, 2$, PP_n is the trivial group. For $3 \leq n < 6$, the generators of PP_n are plane pure braids associated (see construction 1) to basic partitioner posets of $[n]$. The presentation of PP_n for $3 \leq n \leq 5$ is

$$\begin{aligned} \textit{generators}: &\quad \{b_P \mid P \text{ is a basic partitioner poset of } [n]\} \\ \textit{relations}: &\quad \{ \text{ no relations } \} \end{aligned}$$

i.e., are free groups. For $n = 6$, the relations are by conjugation on some particular partitioner

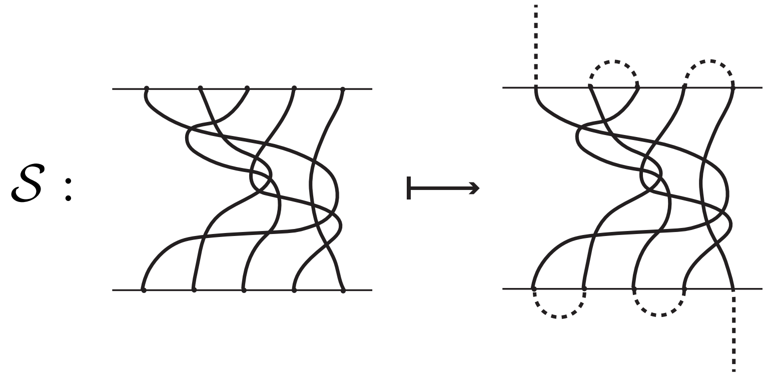


Figure 2: Short-circuit map

posets. Let P_{rel} a basic partitioner poset of [6] of the form $(k)[i,j](L)$ (hence $k > i, j$ and $|L| = 3$). The presentation of PP_6 is

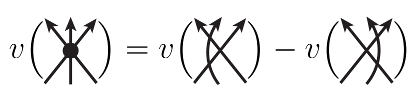
$$\begin{aligned} \textit{generators}: &\quad \{b_P \mid P \text{ is a basic partitioner poset of } [6]\} \\ \textit{relations}: &\quad \{(\widetilde{\nu_1}(P_{rel}) \cdot b_{P_{rel}})((\widetilde{\nu_2}(P_{rel}) \cdot b_{P_{rel}}))^{-1} \quad | \quad P_{rel} = (k)[i,j](L)\} \end{aligned}$$

where $\cdot$ is the action by conjugation and $\widetilde{\nu_i}(P_{rel})$ is a product of plane pure braids constructed from L for $i = 1, 2$.

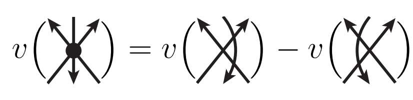
In the last section of the chapter we prove that although the homomorphism of forgetting one strand gives a splitting of the group PP_n as a semidirect product of two subgroups, this product is not almost-direct. This stands in contrast to the case of the usual pure braids, where the splitting as an almost-direct product plays a crucial role in the proof that the pure braid group is residually nilpotent. (We should note that PP_n is residually nilpotent and this can be established by different methods).

In chapter 3 we trace the connection of plane pure braids with a planar version of knots, that is, triple points free plane curves. A (long) triple points free plane curve is an immersion $C : \mathbb{R} \to \mathbb{R}^2$ which coincides with the linear embedding $x = 0$ outside a compact interval and all of whose multiple points are transversal double points. As in the usual case, every triple points free plane curve can be represented as the closure of a plane pure braid. We follow the work of Mostovoy and Standford [31] defining a short circuit map $\mathcal{S}_n : PP_{2n+1} \to \mathscr{C}$ as in figure 2. The main result of this chapter is a Birman-Markov-type theorem for this closure. We find subgroups H^T, H^B, such that the monoid of triple points free plane curves $\mathscr{C}$ is equivalent to the biquotient $H^T \backslash PP_\infty / H^B$.

Going back to our source of inspiration, in chapter 4 we define Vassiliev invariants for plane curves and plane pure braids in an axiomatic way by skein relations



and



Following Kohno's construction of an iterated integral invariant for braids, we apply the theory of

Chen's iterated integrals to propose a universal Vassiliev invariant for the plane pure braid group PP_n. For small n, we know $\mathrm{Conf}_3(\mathbb{R}, n)$ is a formal space and we conjecture it is true for any n. Since $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ is generated in degree 1 and assuming the formality of the space, there is a (co)multiplicative linear map

$$Z : \mathbb{R}[PP_n] \longrightarrow \mathbb{R}\langle\langle \mathbf{X} \rangle\rangle / \mathbf{J},$$

given by

$$Z(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots + \int_\gamma \omega \cdots \omega + \cdots \qquad \text{with } \omega = \sum_{P \in \mathcal{P}} \omega_P X_P.$$

where ω_P is an associated differential form to the corresponding cohmology class of a basic partitioner poset P, and $\mathcal{P}$ is the set of all basic partitioner posets. Despite the fact that we can not make calculations in absence of explicit differential forms representing the generators of the cohomology of $\mathrm{Conf}_3(\mathbb{R}, n)$, by a general result of Chen's theory and the fact that PP_n is residually nilpotent and torsion-free, we would obtain that Vassiliev invariants separate plane pure braids and the universal Vassiliev invariant is a Taylor expansion in the sense of Bar-Natan [3].

Chapter 1

$\mathrm{Conf}_3(\mathbb{R}, n)$ space and cubical $\mathrm{CAT}(0)$ spaces

In this chapter we study a real counterpart of the braid arrangement and a kind of configuration space $\mathrm{Conf}_3(\mathbb{R}, n)$. Analogously to the usual case in which the pure braid group is the fundamental group of a configuration space, we shall see in the first section that the fundamental group of $\mathrm{Conf}_3(\mathbb{R}, n)$ is a planar version of the pure braid group. Section two is based in [5], we review some properties of Coxeter groups in order to study the combinatorics and homotopy type of $\mathrm{Conf}_3(\mathbb{R}, n)$. In section three we have an unkwnon result in which we realize the universal cover of $\mathrm{Conf}_3(\mathbb{R}, n)$ as a cubical $\mathrm{CAT}(0)$ space, and as an immediate consequence $\mathrm{Conf}_3(\mathbb{R}, n)$ is a $K(\pi, 1)$ space, proved in other way firstly by Khovanov in [22]. In the last section, we study the cohomology ring of these spaces calculated by Baryshnikov in [6].

1.1 The space $\mathrm{Conf}_3(\mathbb{R}, n)$

The classical configuration space of n ordered points in $\mathbb{C}$ is by definition the complement of $\mathbb{C}^n$ by a hyperplane arrangement defined by the diagonals Δ_{ij}. This arrangement is a collection of subspaces of real codimension two. Here, we also study the complement of a subspace arrangement of real codimension two. However, this arrangement can not be viewed as a complex hyperplane arrangement.

Definition 1.1.1 (3-Equal Arrangement)**.** A 3-*equal arrangement* consists of the collection of all subspaces of $\mathbb{R}^n$ of the form $x_i = x_j = x_k$ for $1 \leq i < j < k \leq n$. We denote the 3-equal arrangement by $\mathcal{A}_{n,3}$.

Definition 1.1.2 (No 3-equal Manifold)**.** The *no 3-equal manifold* is the complement of the 3-equal arrangement $\mathcal{A}_{n,3}$, i.e.,

$$\mathbb{R}^n \backslash \{(x_1, \ldots, x_n) \,|\, x_i = x_j = x_k, 1 \leq i < j < k \leq n\}.$$

We denote by $\mathrm{Conf}_3(\mathbb{R}, n)$ the no 3-equal manifold by its similarities with usual configuration spaces, because by definition points in $\mathrm{Conf}_3(\mathbb{R}, n)$ are configurations of n ordered points in $\mathbb{R}$ without triple coincidences.

We only work with 3-equal arrangements but the definition make sense for any k leading to k-equal arrangements $\mathcal{A}_{n,k}$ and no k-equal manifolds $\mathrm{Conf}_k(\mathbb{R}, n)$ as in [6].

Example 1.1.3. For $n = 1, 2$ the space $\mathrm{Conf}_3(\mathbb{R}, n)$ is contractible.

Example 1.1.4. The first non-trivial but very simple case is $\mathrm{Conf}_3(\mathbb{R}, 3)$, which is $\mathbb{R}^3$ minus the diagonal $x_1 = x_2 = x_3$. Hence, $\mathrm{Conf}_3(\mathbb{R}, 3)$ is homotopy equivalent to the circle $\mathbb{S}^1$. If we take $(1, 0, -1) \in \mathrm{Conf}_3(\mathbb{R}, 3)$ as our base point, it is easy to see that the class of the based loop

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t, -\cos 2\pi t),$$

is a generator of $\pi_1(\mathrm{Conf}_3(\mathbb{R}, 3))$. Let $b : I_1 \sqcup I_2 \sqcup I_3 \to \mathbb{R}^2$ the immersion defined by $b(t_i) = (\gamma_i(t), t)$. Drawing the image of b (see figure 1.1) we have 3 ascending arcs which intersect with multiplicity at most 2. These is an example of what we call a planar pure braid. Furthermore, $\pi_1(\mathrm{Conf}_3(\mathbb{R}, 3))$ is equivalent to a planar version of the pure braid group on 3 strands, denoted by PP_3. Therefore, PP_3 is the infinite cyclic group with generator b as in figure 1.1.

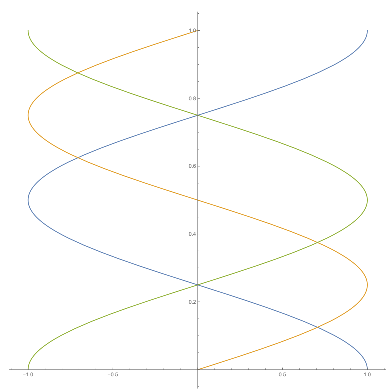


Figure 1.1: The generator of PP_3

In general, the fundamental group of $\mathrm{Conf}_3(\mathbb{R}, n)$ gives rise to planar pure braids. Before, we give the formal definition.

Definition 1.1.5 (Planar Braid)**.** A *geometric planar braid* on n strands is an immersion $b : I_1 \sqcup \cdots \sqcup I_n \to \mathbb{R}^2$ formed by n ascending arcs $b_i : I \to \mathbb{R}^2$ called the strands, such that

(a) the strands join n distinct points in the line $y = 0$ with n distinct points in the line $y = 1$.

(b) tangent vectors of the strands are never horizontal.

(c) three strands can not have a common point.

Two geometric planar braids b and b' are equivalent if there exists a smooth homotopy $F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_0 = \mathrm{id}_{\mathbb{R}^2}$, $F_1(b) = b'$ and $F_t \circ b$ is a planar braid for each $t \in [0, 1]$. A *planar braid* is a geometric planar braid up to smooth homotopies of this type (see figure 1.2).

In chapter 3, we see a planar version of Reidemeister moves (figure 3.2). In this terms, two planar braids are equivalent if there exists a sequence of diffeomorphisms of the plane and local moves $\Omega_2^{\pm}$, which transform one planar braid into the other.

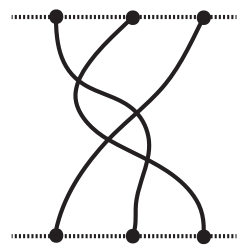


Figure 1.2: Planar braid

In the set of planar braids on n strands we can define the product $b_1 \cdot b_2$ by putting the bottom of b_2 on the top of b_1. By the equivalence of planar braids, we can choose b_1, b_2 in such a way the n points in the bottom of b_2 coincide with the n points in the top of b_1. The planar braid with all vertical strands works as an identity, and the inverses are reflection over horizontal line $y = 1$. With this product and identity, the set of planar braids on n strands forms a group and we denote it by PB_n . Every planar braid can be written as a product of σ_j's, where σ_j is the planar braid consisting of only one double point as in figure 1.3. Furthermore, PB_n can be written as an abstract group and is finitely presented.

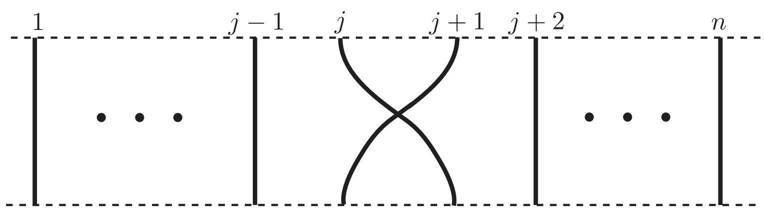


Figure 1.3: Generator σ_j

Proposition 1.1.6 ([22]). *PB_n has a presentation given by*

$$\textit{generators}: \quad \sigma_1, \ldots, \sigma_{n-1}, \tag{1.1.1}$$

$$\textit{relations}: \quad \sigma_j^2\,, \qquad 1 \le j \le n-1 \tag{1.1.2}$$

$$(\sigma_j \sigma_k)^2\,, \qquad |j-k| > 1 \tag{1.1.3}$$

Remark 1.1.7. Recall Σ_n has a presentation with generators the transpositions $s_j = (j, j+1)$ and relations $s_j^2 = 1$, $s_j s_k = s_k s_j$ when $|j-k| > 1$ and $s_{j+1} s_j s_{j+1} = s_j s_{j+1} s_j$. We recover the presentation of PB_n from the presentation of Σ_n by omitting the relation $s_{j+1} s_j s_{j+1} = s_j s_{j+1} s_j$.

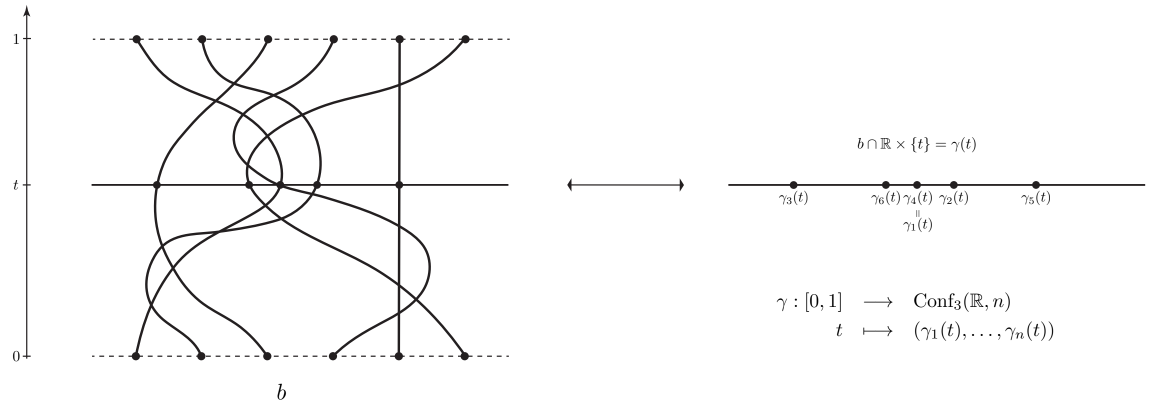


Figure 1.4: Correspondence between planar braids and loops on $\mathrm{Conf}_3(\mathbb{R}, n)$

In the same way as in the classical case, each planar braid on n strands induce a permutation of a set of n elements. Therefore, there is a natural homomorphism from the planar braid group to the symmetric group $\varphi : PB_n \to \Sigma_n$, defined in generators by sending σ_j to the transposition $s_j = (j, j+1)$.

Definition 1.1.8 (Planar Pure Braid Group)**.** The *planar pure braid group* on n strands denoted by PP_n is the kernel of the homomorphism $\varphi : PB_n \to \Sigma_n$. A geometric planar braid represent an element in PP_n if and only if the ith strand join the point $(i, 0)$ with the point $(i, 1)$, for all $i = 1, \ldots, n$.

Another names in the literature for the planar braid group and planar pure braid group are twin group and pure twin group, respectively [22]. In the same way as in the classical case, the planar pure braid group is the fundamental group of a kind of configuration space.

Proposition 1.1.9. *The fundamental group of the space* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is isomorphic to* PP_n*.*

Proof. A based loop $\gamma : I = [0,1] \to \mathrm{Conf}_3(\mathbb{R}, n)$, $t \mapsto (\gamma_1(t), \ldots, \gamma_n(t))$ induces a geometric planar braid $b : I_1 \sqcup \cdots \sqcup I_n \to \mathbb{R}^2$, defined by $b(t_i) = (\gamma_i(t_i), t_i)$. The condition of based loop, guarantees the planar braid is pure. Reciprocally, if b represents a planar pure braid, the intersection $b \cap \mathbb{R} \times \{t\}$ is a configuration of n points without triple coincidences; scanning along t we obtain the based loop in $\mathrm{Conf}_3(\mathbb{R}, n)$ (see figure 1.4). The correspondence respects products. $\square$

Furthermore, Khovanov proves in [22] that $\mathrm{Conf}_3(\mathbb{R}, n)$ is in fact the classifying space of PP_n. In section 1.3, we reach to the same fact as a consequence of the cubical CAT(0) structure of the universal cover of $\mathrm{Conf}_3(\mathbb{R}, n)$.

1.2 $\mathrm{Conf}_3(\mathbb{R}, n)$ and Coxeter groups

The theory of Coxeter groups allows us to understand the structure of the complement of the 3-equal arrangement. We give basic definitions and we derive elementary combinatorial facts. The material in this section is taken primarily from [10] and [5].

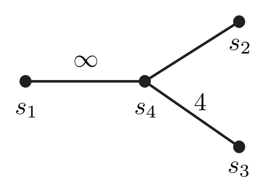
Definition 1.2.1 (Coxeter matrix)**.** Let S be a set. A matrix $m : S \times S \to \{1, 2, \dots, \infty\}$ is called a *Coxeter matrix* if it satisfies

$$m(s, s') = m(s', s) \tag{1.2.1}$$

$$m(s, s') = 1 \Leftrightarrow s = s' \tag{1.2.2}$$

Equivalently, m can be represented by a Dynkin diagram whose vertex set is S and whose edges are the unordered pairs $\{s, s'\}$ such that $m(s, s') \geq 3$. The edges with $m(s, s') \geq 4$ are labelled by that number. For instance,

$$\begin{pmatrix} 1 & 2 & 2 & \infty \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 1 & 4 \\ \infty & 3 & 4 & 1 \end{pmatrix} \Longleftrightarrow$$



Definition 1.2.2 (Coxeter Group)**.** A Coxeter matrix m defines a group W, called *Coxeter group*, with presentation:

$$W = \langle S \mid (ss')^{m(s,s')} \text{ for all } s, s' \in S \text{ and } m(s, s') < \infty \rangle$$

The pair (W, S) is called a *Coxeter system*, and $|S|$ is the rank of (W, S).

Remark 1.2.3. The condition $m(s, s') = \infty$ means there is no relation between s and s'. The relation $(ss')^m = 1$ is equivalent to

$$\underbrace{ss'ss's\cdots}_{m} = \underbrace{s'ss'ss'\cdots}_{m}.$$

In particular, $m(s, s') = 2$ means that s and s' commute, and in the Dynkin diagram there is no edge between s and s'.

Example 1.2.4. The group of planar braids PB_n is a Coxeter group. If we take $S = \{\sigma_1, \dots, \sigma_{n-1}\}$ and $m : S \times S \to \{1, 2, \dots, \infty\}$ defined by

$$m(\sigma_j, \sigma_k) := \begin{cases} 1 & \text{for } j = k, \\ 2 & \text{for } |j - k| > 1, \\ \infty & \text{otherwise.} \end{cases}$$

The Dynking diagram of PB_n is as in figure 1.5.

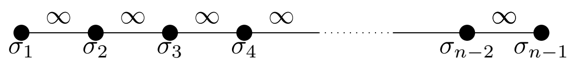


Figure 1.5: Dynkin diagram of PB_n

Another examples of Coxeter groups are finite reflection groups. For $\alpha \in \mathbb{R}^n \setminus \{0\}$, let s_α denote the reflection in the hyperplane H_α orthogonal to α. In particular $s_\alpha(\alpha) = -\alpha$, while fixes point wise the hyperplane H_α. A *finite reflection group* W is a finite group generated by a set of reflections. One way to obtain finite reflection groups is since root systems.

Definition 1.2.5 (Root System)**.** A collection Φ of vectors in $\mathbb{R}^n$ is a *root system* if:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

A *simple system* is a collection $\Pi \subset \Phi$ if spans $\mathbb{R}^n$ and each $\alpha \in \Phi$ is a linear combination of elements in Π whose all coefficients are either all non-negative, or all non-positive.

The group W generated by the reflections s_α for all $\alpha \in \Phi$ is a reflection group. Furthermore, the set S of reflections s_α for $\alpha \in \Pi$ called simple reflections, generates W and (W, S) is a Coxeter system of rank $|S| = |\Pi|$.

Example 1.2.6. The symmetric group Σ_n is a finite reflection group on $\mathbb{R}^n$ by permuting the standard basis $e_1, \ldots, e_n$. The collection of vectors $e_i - e_j$ for $i \neq j$ form a root system Φ in which a transposition (ij) acts as the reflection in the hyperplane $H_{ij} = \{x \in \mathbb{R}^n | x \perp e_i - e_j\}$. The collection of vectors $\alpha_j = e_j - e_{j+1}$, for all $1 \leq j \leq n-1$ is a simple system. If we take all the reflections s_{α_j}, corresponds to the transpositions $s_j = (j, j+1)$ of Σ_n, and produces the known presentation of Σ_n by generators $s_1, \ldots, s_{n-1}$ and relations:

$$
\begin{aligned}
&s_j^2 = 1, \quad 1 \leq j \leq n-1.\\
&s_j s_k = s_k s_j, \quad |j-k| > 1, \quad 1 \leq j < k \leq n-1,\\
&s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}, \quad 1 \leq j \leq n-1.
\end{aligned}
$$

Therefore, the symmetric group Σ_n has indeed a presentation as a finite Coxeter group.

When Σ_n acts on $\mathbb{R}^n$ as above, it fixes the line spanned by $e_1 + \cdots + e_n$ and leaves stable the orthogonal complement $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 + \cdots + x_n = 0\}$. Thus Σ_n acts on an $(n-1)$-dimensional vector space with no fixed points except the origin. An action of a reflection group W on an euclidean space V with no non-zero fixed points is called *essential.*

Note 1.2.7. Σ_n is also referred as the finite Coxeter group of type A_{n-1}.

Definition 1.2.8 (Irreducible)**.** A Coxeter group is *irreducible* if its Dynkin diagram is a connected graph.

The essential and irreducible finite Coxeter groups are completely classified in terms of Dynkin diagrams. All they are reflection groups of some finite-dimensional Euclidean space, then for each finite Coxeter group W we can associate an arrangement of hyperplanes. Let Φ be a root system of W, then the *Coxeter arrangement* $\mathcal{H}(W)$ is the collection of hyperplanes $H_\alpha = \{x | x \perp \alpha\}$ for all $\alpha \in \Phi$. Obviously $\mathcal{H}(W)$ could change if we change the root system, but the combinatorial information remains. We can associate a poset to $\mathcal{H}(W)$, consisting in all the intersections of hyperplanes, ordered by reverse inclusion. In fact, every pair of elements in the poset have a unique upper bound and lower bound, then this poset is a lattice. Given two root systems Φ, Φ' of W, the corresponding lattices are isomorphic. The poset is called the intersection lattice and is denoted by $\mathcal{L}(\mathcal{H}(W))$.

Example 1.2.9. For Σ_n the finite Coxeter group of type A_{n-1}, the Coxeter arrangement $\mathcal{H}(\mathrm{A}_{n-1})$ is the collection of hyperplanes $H_{i,j}$'s defined in example 1.2.6 (figure 1.6). If $x = (x_1, \ldots, x_n) \in H_{i,j} \cap H_{j,k}$ for some $i < j < k$ then $x_i = x_j = x_k$, hence $H_{i,j} \cap H_{j,k} \in \mathcal{A}_{n,3}$. More generally, we can also associate a poset to $\mathcal{A}_{n,3}$ in the same way, i.e., the intersection poset ordered by reverse inclusion. This poset denoted by $\mathcal{L}(\mathcal{A}_{n,3})$ is a subposet of $\mathcal{L}(\mathcal{H}(\mathrm{A}_{n-1}))$. There is already a well-known combintorial description of both posets. The poset of set partitions of $[n]$ ordered by refinement is isomorphic to $\mathcal{L}(\mathcal{H}(\mathrm{A}_{n-1}))$, and under this isomorphism, $\mathcal{L}(\mathcal{A}_{n,3})$ is the subposet of partitions in which at least one block has size at least 3.

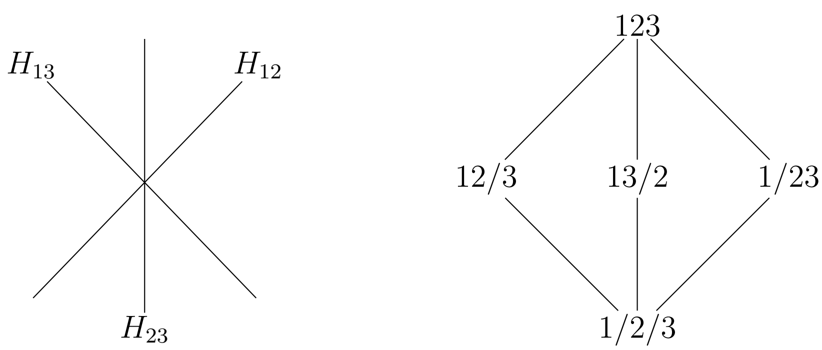


Figure 1.6: $\mathcal{H}(\mathrm{A}_2)$ and $\mathcal{L}(\mathcal{H}(\mathrm{A}_2))$

Another description of $\mathcal{L}(\mathcal{H}(\mathrm{A}_{n-1}))$ was given by Barcelo and Ihrig [4] in terms of parabolic subgroups of A_{n-1}, and for our purposes more useful.

Definition 1.2.10 (Parabolic Subgroup)**.** Let W a Coxeter group with set of simple reflections S. A subgroup $G \subset W$ is a parabolic subgroup if there exist a subset $I \subset S$ of simple reflections and an element $w \in W$ with $G = \langle wIw^{-1} \rangle$. If w is the identity G is a *standard parabolic subgroup.* The pair (G, wIw^{-1}) can be viewed as a Coxeter system, and G is irreducible if its Dynkin diagram is connected.

The correspondence between $\mathcal{L}(\mathcal{H}(\mathrm{A}_{n-1}))$ and the lattice of parabolic subgroups $\mathcal{P}(\mathrm{A}_{n-1})$ is given by sending a parabolic subgroup G to $Fix(G) = \{x \in \mathbb{R}^n | wx = x, \forall w \in G\}$, and the inverse is given by sending an intersection of hyperplanes X to $Gal(X) = \{w \in G | wx = x, \forall x \in X\}$. This correspondence gives rise to a generalization of k-equal arrangements, called k-parabolic arrangements, defined by Barcelo, Severs and White. In [5] they gave a new description of 3-equal arrangements.

Proposition 1.2.11. *The Galois Correspondence gives a bijection between subspaces of $\mathcal{A}_{n,3}$ and irreducible parabolic subgroups of A_{n-1} of rank 2.*

Proof. Let $X \subset \mathbb{R}^n$ the subspace such that $x_1 = x_2 = x_3$. The Galois correspondence sends X to $Gal(X) = \langle (1,2),(2,3) \rangle$ which is irreducible parabolic subgroup of rank 2. Any other subspace X' (for instance $x_i = x_j = x_k$) is in the orbit of X by the action of $\Sigma_n = \mathrm{A}_{n-1}$, i.e., exists $w \in \Sigma_n$ such that $wX = X'$, then $Gal(X') = Gal(wX) = wGal(X)w^{-1}$ and hence irreducible parabolic of rank 2.

Conversely, given G an irreducible parabolic subgroup of rank 2, there exists a subset $I \subset S$ and $w \in \Sigma_n$ such that $G = \langle wIw^{-1} \rangle$. Since is irreducible, $I = \{(j, j+1), (j+1, j+2)\}$ and $Fix(\langle I \rangle)$ is the subspace such that $x_j = x_{j+1} = x_{j+2}$. Therefore $Fix(G) = Fix(w\langle I \rangle w^{-1}) = wFix(\langle I \rangle)$ is given by $x_{w(j)} = x_{w(j+1)} = x_{w(j+2)}$ which is a subspace in the 3-equal arrangement. □

In the same direction, topological and combinatorial information of the space $\mathrm{Conf}_3(\mathbb{R}, n)$ can be described in terms of parabolic subgroups. Before that, we review the definition of the Coxeter complex and the associated W-permutahedron.

Let (W, S) be a finite Coxeter group with Π a simple system. For a given $I \subset S$, let $W_I = \langle I \rangle$, and $\Pi_I = \{\alpha \in \Pi | s_\alpha \in I\}$. We can associate the set

$$C_I = \{x \in \mathbb{R}^n | (x, \alpha) = 0 \text{ for all } \alpha \in \Pi_I, \text{ and } (x, \alpha) > 0 \text{ for all } \alpha \in \Pi \setminus \Pi_I\},$$

which is an intersection of certain hyperplanes H_α and certain open half-spaces. The sets C_I's partition a fundamental region in simplicial cones, where $C_S = \{0\}$ and $C_\emptyset$ is the interior of the fundamental region. The collection $\mathcal{C}(W)$ of all sets wC_I ($w \in W, I \subset S$), partitions $\mathbb{R}^n$. More precisely, for each fixed I the sets wC_I and $w'C_I$ are disjoint unless w and w' are in the same left coset in W/W_I. If $I, J \subset S$ are distinct, all sets wC_I, $w'C_J$ are disjoint. The sets wC_I are called cells, and is q-dimensional if $|S \setminus I| = q$. We define an order in $\mathcal{C}(W)$ given by inclusions in closed sets, i.e,

$$wC_I \leq w'C_J \iff \overline{wC_I} \subseteq \overline{w'C_J},$$

making $\mathcal{C}(W)$ into the poset of faces of a simplicial complex. For each $(q+1)$-dimensional cell wC_I, corresponds a q-simplex with vertex set all the 1-cells contained in wC_I. Note we are considering the 0-dimensional cell as the (-1)-simplex.

Definition 1.2.12 (Coxeter Complex)**.** As the face poset of a simplicial complex, the collection $\mathcal{C}(W)$ is the *Coxeter complex* of W. A geometric realization of the Coxeter complex is the simplicial decomposition of $\mathbb{S}^{n-1}$ by intersecting the sphere with the arrangement $\mathcal{H}(W)$.

From the definition of C_I it is clear that W_I is its isotropy group. In general, parabolic subgroups of W are the isotropy groups of the cells of $\mathcal{C}(W)$. On the other hand, if we have a standard parabolic subgroup W_I, the set of fixed points of W_I in C_I is exactly C_I. Furthermore, cells $C_I \leq C_J$ correspond to a standard parabolic subgroups $W_I \leq W_J$, where parabolic subgroups are ordered by reverse inclusion, i.e.,

$$wW_I \leq w'W_J \iff w'W_J \subseteq wW_I.$$

Extending this correspondence in both posets by the action of W, this gives a caracterization of cells of the Coxeter complex in terms of cosets wW_I. In particular, the vertex set of $\mathcal{C}(W)$ corresponds to $V = \cup_{s\in S} W/W_{S\setminus\{s\}}$.

Since the 3-equal arrangement is embedded in $\mathcal{H}(\mathrm{A}_{n-1})$, we can also define a subcomplex of $\mathcal{C}(\mathrm{A}_{n-1})$ by

$$\mathcal{C}_0 = \{F \in \mathcal{C}(\mathrm{A}_{n-1}) | \exists A \in \mathcal{A}_{n,3} \text{ such that } F \subset A\}. \tag{1.2.3}$$

Its geometric realization is given by intersecting $\mathcal{A}_{n,3}$ with $\mathbb{S}^{n-1}$. Defined all these, the following comes from general results of subspace arrangements having appeared in the literature before, for instance, see section 5.2 in [32].

Proposition 1.2.13. *The space* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is homotopy equivalent to* $|\mathcal{C}(A_{n-1})| \setminus |\mathcal{C}_0|$.

Proof. Since the intersection of all the subspaces in the 3-equal arrangement contains the origin, there is a map from $\mathrm{Conf}_3(\mathbb{R}, n)$ into the sphere, given by $r : x \mapsto \frac{x}{\|x\|}$, which gives the homotopy equivalence between $\mathrm{Conf}_3(\mathbb{R}, n)$ and $|\mathcal{C}(\mathrm{A}_{n-1})| \setminus |\mathcal{C}_0|$. We are assuming here that $|\mathcal{C}(\mathrm{A}_{n-1})|$ is the geometric realization as the simplicial decomposition in $\mathbb{S}^{n-1}$. $\square$

Next, we consider a polytope related with the Coxeter complex as its dual, called the Coxeter permutahedron.

Definition 1.2.14 (Coxeter Permutahedron)**.** Let (W, S) be a finite Coxeter group and C the interior of a fundamental region (for example $C = C_\emptyset$). Let $x \in C$ any element, the *Coxeter permutahedron* or *W-permutahedron* is the convex hull of the orbit $Wx = \{wx | w \in W\}$. We denote it by $Perm(W)$.

The face poset of the W-permutahedron is exactly dual of the face poset of the Coxeter complex, i.e., is defined by cosets wW_I for all $w \in W$, $I \subset S$, but ordered by inclusion, instead of the reversed inclusion as in the Coxeter complex. For example, by definition of the W-permutahedron, the vertex set is exactly the orbit Wx, so for each $w \in W$ corresponds a vertex, but we can see each w as the coset $wW_\emptyset$ which corresponds to maximal faces of the Coxeter complex. Edges correspond to cosets wW_s for each $s \in S$, dual to boundaries of maximal faces of the Coxeter complex which are parts of hyperplanes. What we have is a duality in faces between the W-permutahedron and the Coxeter complex.

There is a much deeper correspondence between the Coxeter complex and the W-permutahedron. Given any subomplex Δ_0 of the Coxeter complex $\mathcal{C}(W)$, there is a subcomplex $Perm_0(W)$ of the W-permutahedron such that $|\mathcal{C}(W)| \setminus |\Delta_0|$ is homotopy equivalent to $Perm_0(W)$. To prove this, we need the following specialization of Proposition 3.1 from Bj̈orner and Ziegler in [12].

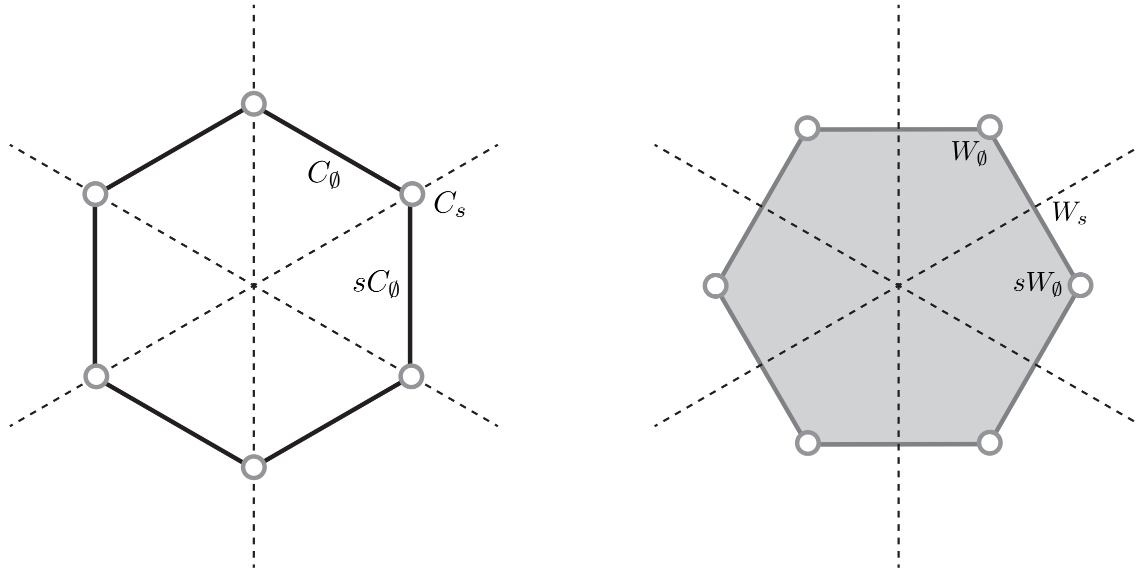


Figure 1.7: Coxeter complex and Permutahedron of type A_2

Proposition 1.2.15. *Let Δ be a simplicial decomposition of the k-sphere, and let Δ_0 be a subcomplex of Δ. Let P be the face poset of Δ, and let P_0 be the lower order ideal generated by Δ_0. Then $|\Delta|\setminus|\Delta_0|$ is homotopy equivalent to a regular CW-complex X, and moreover, the face poset of X is $(P \setminus P_0)^*$, where $*$ denotes taking the dual poset.*

In our case, Δ is the realization of the Coxeter complex $\mathcal{C}(\mathrm{A}_{n-1})$ as the simplicial decomposition of $\mathbb{S}^{n-1}$ by intersecting with $\mathcal{H}(\mathrm{A}_{n-1})$ and Δ_0 is the realization of the subcomplex $\mathcal{C}_0$ as the intersection of $\mathbb{S}^{n-1}$ with $\mathcal{A}_{n,3}$.

Since regular CW-complexes are determined by their face posets [9], we see that X has the same face poset as a subcomplex of $Perm(\mathrm{A}_{n-1})$ and hence, X is homeomorphic to a polyhedral subcomplex $Perm_0(\mathrm{A}_{n-1})$ with face poset in terms of face posets of $\mathcal{C}(\mathrm{A}_{n-1})$ and $\mathcal{C}_0$. We already know the face poset of $\mathcal{C}(\mathrm{A}_{n-1})$ in terms of parabolic subgroups. For $\mathcal{C}_0$ is useful to remember its definition (see 1.2.3). A cell $wC_I \in \mathcal{C}(\mathrm{A}_{n-1})$ is in $\mathcal{C}_0$ if and only if exists $X \in \mathcal{A}_{n,3}$ such that $C_I \subset w^{-1}X$. In terms of parabolic subgroups and w.l.g, we can only consider standard, $W_I \in \mathcal{C}(\mathrm{A}_{n-1})$ is in $\mathcal{C}_0$ if and only if exists W_J irreducible parabolic of rank 2 such that $J \subset I$. In conclusion,

Lemma 1.2.16 ([34])**.** *For the finite reflection group $W = A_{n-1}$, the face poset of the complex $\mathcal{C}_0$ is the collection of cosets wW_I where there exists $J \subset I$ such that W_J is an irreducible parabolic subgroup of rank* 2.

By the definition of irreducible parabolic subgroup, a cosets wW_I in the poset of $\mathcal{C}_0$ is such that I contains two simple reflections which as vertices in the Dynkin diagram are connected by an edge. In other words, I contains at least two transpositions $(j, j+1)$ and $(j+1, j+2)$ for some $j \in [1, n-2]$.

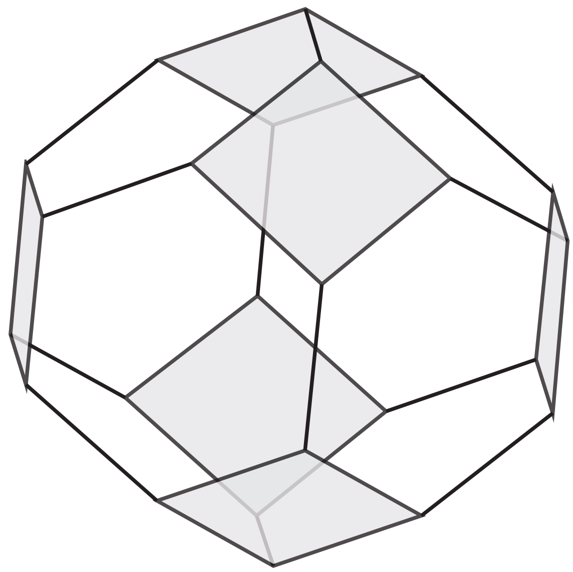


Figure 1.8: $Perm_0(\mathrm{A}_3)$

The following result is a corollary of proposition 1.2.15 and lemma 1.2.16, but by its importance will be a theorem.

Theorem 1.2.17 ([34]). *There is a subcomplex $Perm_0(A_{n-1})$ of $Perm(A_{n-1})$ such that the space* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is homotopy equivalent to $Perm_0(A_{n-1})$. Moreover, the face poset of $Perm_0(A_{n-1})$ corresponds to cosets wW_I ordered by inclusion and such that any $s, s' \in I$ commute.*

Proof. By propositions 1.2.13 and 1.2.15 we have that $\mathrm{Conf}_3(\mathbb{R}, n) \simeq |\mathcal{C}(\mathrm{A}_{n-1})| \backslash |\mathcal{C}_0| \simeq Perm_0(\mathrm{A}_{n-1})$. If P denotes the face poset of $\mathcal{C}(\mathrm{A}_{n-1})$ and P_0 the lower ideal generated by $\mathcal{C}_0$, when we take duals, we reversed the order and hence P_0^* is an upper ideal in P^*. The face poset P^* corresponds to all cosets wW_I of parabolic subgroups ordered by inclusion, and P_0^* is the upper ideal whose minimal elements are cosets wW_J where $J = \{(j, j+1), (j+1, j+2)\}$ for some $j \in [1, n-2]$. Using the fact that $(P \setminus P_0)^* = P^* \setminus P_0^*$, we have that the face poset of $Perm_0(\mathrm{A}_{n-1})$ correspond to cosets wW_I such that I doesn't contain consecutive transpositions, i.e., any $s, s' \in I$ commute. □

Remark 1.2.18. The 0-skeleton of $Perm_0(\mathrm{A}_{n-1})$ consist of cosets $wW_\emptyset$, equivalently, each permutation is a vertex. The 1-skeleton corresponds to cosets $wW_{\{s\}}$ with s a simple reflection, and where $wW_{\{s\}}$ as a 1-dimensional cell, contains the vertices $wW_\emptyset$ and $wsW_\emptyset$. The 2-skeleton corresponds to cosets $wW_{\{s,s'\}}$ where $ss' = s's$, i.e., $s = (j, j+1)$ and $s' = (k, k+1)$ where $|j-k| > 1$. In other words, a 2-dimensional cell correspond to a square with vertices $wW_\emptyset, wsW_\emptyset, wss'W_\emptyset, ws'W_\emptyset$ and edges $wW_{\{s\}}, wsW_{\{s'\}}, ws'W_{\{s\}}, wW_{\{s'\}}$. In general, cells correspond to cubes, a fact we need for the next section (see figure 1.8).

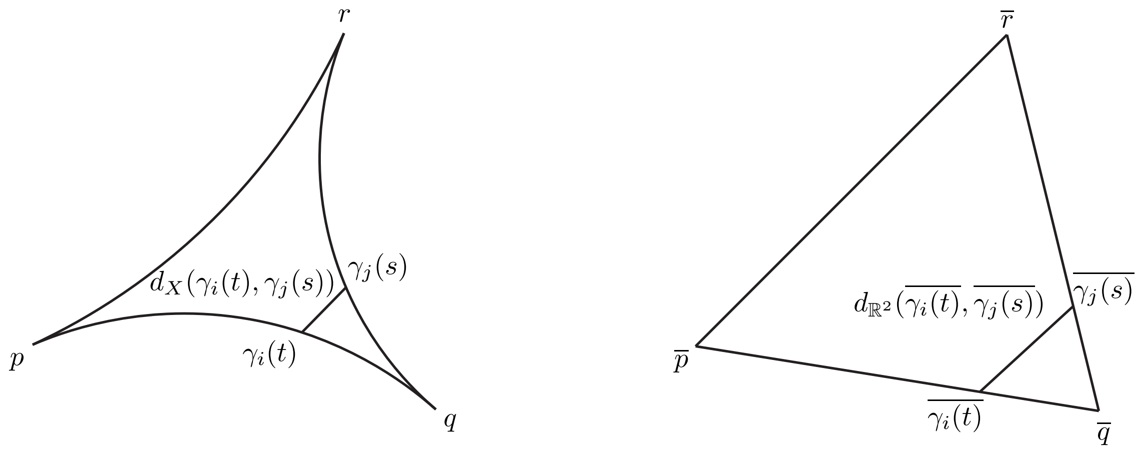


Figure 1.9: A geodesic triangle and a comparison triangle

1.3 Universal Cover of $\mathrm{Conf}_3(\mathbb{R}, n)$ as a $\mathrm{CAT}(0)$ cubical complex

Briefly, we turn our attention to cubical complexes and CAT(0) spaces. First, some definitions.

Definition 1.3.1 (Geodesic Metric Space)**.** Let (X, d) be a metric space. A geodesic joining $x \in X$ and $y \in X$ is a map $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$, and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for any $t_1, t_2 \in [0, d(x, y)]$. A metric space X is said to be a (uniquely) *geodesic metric space* if any two points can be joined by some (unique) geodesic.

Definition 1.3.2 (Geodesic Triangle)**.** Let (X, d) be a geodesic metric space. A *geodesic triangle* consists of three points $p, q, r \in X$ and three geodesics $\gamma_1, \gamma_2, \gamma_3$ in X joining p with q, q with r, and r with p respectively. We denote it by $\Delta(p, q, r)$. Then there exists a (unique up to isometry) geodesic triangle $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ with sides $\overline{\gamma_1}, \overline{\gamma_2}, \overline{\gamma_3}$ such that there exist isometries $\phi_i : \gamma_i \to \overline{\gamma_i}$ with $\phi_i(v) = \phi_j(v)$ whenever v is an endpoint of both γ_i and γ_j. We call $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ a *comparison triangle* for $\Delta(p, q, r)$.

Definition 1.3.3 (CAT(0) Space)**.** Let (X, d) be a geodesic space. If $\Delta(p, q, r)$ is a geodesic triangle and $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ a comparison triangle, then the CAT(0) *inequality* for such Δ is the inequality

$$d_X(\gamma_i(t), \gamma_j(s)) \leq \|\phi_i(\gamma_i(t)) - \phi_j(\gamma_j(s))\| \tag{1.3.1}$$

for any $i, j \in \{1, 2, 3\}$ and t, s in the domain of γ_i and γ_j respectively. We say that a geodesic metric space is CAT(0) if all geodesic triangles in X satisfy their CAT(0) inequality (see figure 1.9)

Some elementary consequence of the CAT(0) condition but essential for our purposes, is that CAT(0) spaces are contractible. To prove it, we need the following lemma.

Lemma 1.3.4. *A* CAT(0) *space is uniquely geodesic.*

Proof. Let $p, q \in X$ and $\gamma, \beta : [0, d(x, y)] \to X$ geodesics joining p and q. The geodesic triangle $\Delta(p, q, p)$, with sides γ, β and the constant geodesic at p, has a degenerate comparison triangle in $\mathbb{R}^2$. By the CAT(0) inequality, we have $d_X(\gamma(t), \beta(t)) \leq \|\overline{\gamma(t)} - \overline{\beta(t)}\| = 0$, then $\gamma = \beta$. □

Proposition 1.3.5. *A* CAT(0) *space is contractible.*

Proof. Let $p_0 \in X$. For each $p \in X$ there is a unique geodesic γ_p from p_0 to p. We define $H : X \times [0, 1] \to X$ by $H(p, t) = \gamma_p(td(p_0, p))$. Note that for $t = 0$, $H(p, 0) = \gamma_p(0) = p_0$ and for $t = 1$, $H(p, 1) = \gamma_p(d(p_0, p)) = p$ for any $p \in X$, also, $H(p_0, t) = \gamma_{p_0}(td(p_0, p_0)) = p_0$ the constant geodesic at p_0. The map H is the required homotopy, we only check the continuity. Let $\{(p_n, t_n)\}_{n\in\mathbb{N}}$ in $X \times [0, 1]$ such that converges to (p, t). By definition of geodesic and by the CAT(0) inequality for the triangle $\Delta(p_n, p, p_0)$, we can estimate

$$\begin{aligned}
d(H(p_n, t_n), H(p, t)) &\leq d(H(p_n, t_n), H(p_n, t)) + d(H(p_n, t), H(p, t)) \\
&\leq d(\gamma_{p_n}(t_n(d(p_0, p))), \gamma_{p_n}(t(d(p_0, p)))) + d(H(p_n, t), H(p, t)) \\
&\leq |t_n - t|d(p_0, p) + d(\gamma_{p_n}(t(d(p_0, p_n))), \gamma_p(t(d(p_0, p)))) \\
&\leq |t_n - t|d(p_0, p) + \|\overline{\gamma_{p_n}(t(d(p_0, p_n)))} - \overline{\gamma_p(t(d(p_0, p)))}\| \\
&\leq |t_n - t|d(p_0, p) + t\|\overline{p_n} - \overline{p}\| \longrightarrow 0
\end{aligned}$$

□

The intuitive picture of the CAT(0) inequality make us think in spaces of non-positive curvature, and is not wrong. The intuition is justified by a theorem in differential geometry that says that a Riemannian manifold has sectional curvature $\kappa \leq 0$ if and only if it is a locally CAT(0) space. A proof can be founded in the appendix of chapter II.1 of [13]. Furthermore, for some metric spaces, the condition of being a CAT(0) space can be characterized combinatorially. That is the case for cubical complexes, which are spaces built by gluing unit cubes along their faces by isometries.

The standard n-cube I^n is the n-fold product $[0, 1]^n$. By convention I^0 is a point. A face of I is either $\{0\}$, $\{1\}$ or I. A face of I^n is a subset of S of I^n such that is a product of faces $S_1 \times \cdots \times S_n$ where each S_i is a face of I. The dimension of S is the number of factors that are $[0, 1]$.

Definition 1.3.6 (Cubical Complex)**.** A *cubical complex* K is a regular cell complex which is the quotient of a disjoint union of cubes $X = \sqcup_\Lambda I^{n_\lambda}$ by an equivalence relation $\sim$. The restrictions $p_\lambda : I^{n_\lambda} \to K$ of the natural projection $p : X \to K = X/\sim$ are required to satisfy:

1. for every $\lambda \in \Lambda$, the map p_λ is injective.

2. if $p_\lambda(I^{n_\lambda}) \cap p_{\lambda'}(I^{n_{\lambda'}}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset I^{n_\lambda}$ onto a face $T_{\lambda'} \subset I^{n_{\lambda'}}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

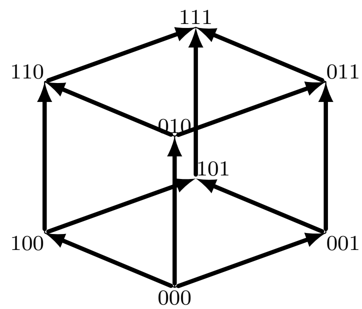


Figure 1.10: Faces of a 3-cube

One can define a very natural metric in a cubical complex K using the path length metric of each cube in the complex. Briefly, define a rectifiable path in K as one that can be broken into finitely many subpaths, each of which is contained in a cube of K and is rectifiable in the classical sense. Then we can define the length of the original path as the sum of the length of subpaths. Finally, the metric between two points $p, q \in K$ is then defined as the infimum of the lengths of the rectifiable paths joining p and q.

As we mentioned before, a regular cell complex is determined by its face poset, i.e., from the abstract poset of cells we can reconstruct a topological space. Furthermore, Bj̈orner proved that if we have two isomorphic posets, we obtain isomorphic regular complexes ([9], Proposition 3.1). For our special case in which the regular complex is a cubical complex is also true. First note who is the face poset of the standard n-cube.

Example 1.3.7. Let K be the standard n-cube $[0, 1]^n$. The set of vertices or 0-cells is the set $P = \{0, 1\}^n$. If we declare $0 < 1$ and order P componentwise, we obtain the Boolean lattice of subsets of $\{1, 2, \ldots, n\}$. If we orient the cube as in figure 1.10, it's clear how faces correspond to intervals in P ordered by inclusion, where an interval for $x, y \in P$ is the set $[x, y] = \{z \in P | x \leq z \leq y\}$. Hence, the complex associated is exactly the standard n-cube, and the face poset of $[0, 1]^n$ is exactly the poset of intervals in P.

Working with cubical complexes we have a specific structure in face posets. By proposition 3.1 in [9], we recover the cubical structure only looking the face poset. [1]

Proposition 1.3.8. *A poset P is isomorphic to the face poset of a cubical complex if and only if it satisfies the following two conditions:*

(1) For any $x, y \in P$ there is a lower bound, then they have a greatest lower bound.

(1′) For any $x, y \in P$ there is an upper bound, then they have a least upper bound.

(2) For any $x \in P$ the poset $P_{\leq x} = \{z \in P | z \leq x\}$ is isomorphic to the face poset of a cube.

The condition (1) is to guarantee that a non-empty intersection is a closed cell, while condition (2) identifies cells of the complex with cubes.

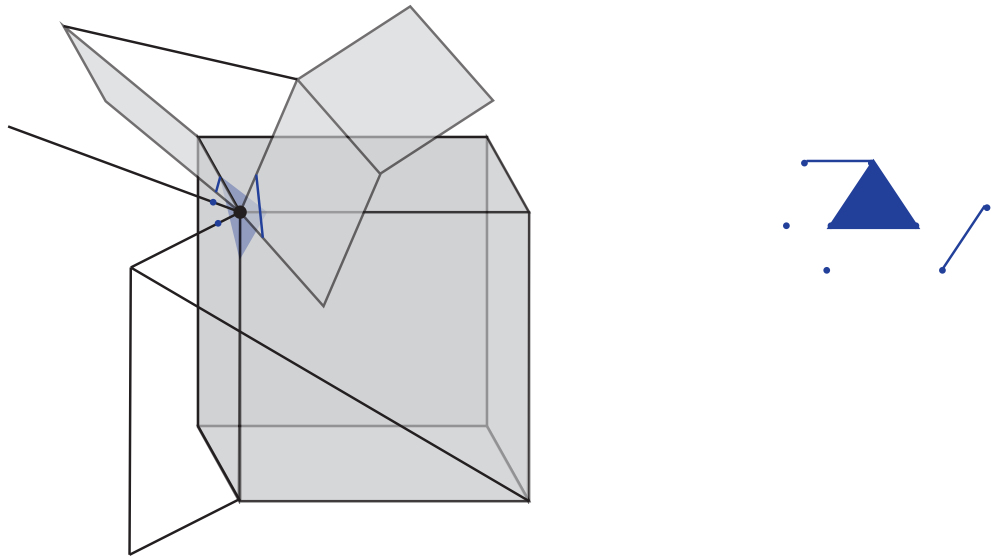


Figure 1.11: A cubical complex and a link of one vertex

Returning to CAT(0) spaces, we have that a cubical complex has a natural metric by minimizing lengths of rectifiable paths, and cubical complexes have a purely combinatorial structure. Combining both properties, Gromov established a necessary and sufficient condition which translates questions concerning the CAT(0) condition on cubical complexes into questions concerning the structure of links of vertices.

Definition 1.3.9 (Link of a vertex)**. Def.1** Let K be a cubical complex. If v is a vertex in K then the link $Lk(v, K)$ is the complex defined as follows. If K is the standard n-cube and v is the origin, then $Lk(v, K)$ is the $(n-1)$-simplex with vertices $\{\frac{1}{3}e_1, \ldots, \frac{1}{3}e_n\}$. More generally, if v is any vertex of the cube then $Lk(v, K)$ is the $(n-1)$-simplex with vertices the points on the edges of the n-cube that distance $\frac{1}{3}$ away from v. For a general cubical complex K and vertex v, we define $Lk(v, K)$ by gluing the links of each cube that v is a vertex, according to how the cubes are glued in the cubical complex.
Def.2 Let K be a cubical complex or equivalently its poset of cells. If v is a vertex in K then the link $Lk(v, K)$ is the simplicial complex $K_{\geq v} = \{e \in K | e \geq v\}$.

Remark 1.3.10. In the definition of simplicial complex, we are considering the empty set as the simplex of dimension -1. Then, for K a cubical complex, $K_{\geq v}$ is a simplicial complex with vertex set as all 1-cells which contain v, and v is identified with the empty simplex.

Definition 1.3.11 (Flag Complex)**.** A *flag complex* is a simplicial complex with no "missing" simplices. This means that for each complete graph in the 1-skeleton of the complex, there is a simplex in the complex whose 1-skeleton is the given complete graph.

We are ready to state the fundamental theorem which characterizes CAT(0) cubical complexes due to Gromov.

Theorem 1.3.12 ([18])**.** *Let K be a cubical complex. Then K is* CAT(0) *if and only if:*

(i) K is simply connected.

(ii) K satisfies the link condition, i.e., the link of each vertex is a flag complex.

Our goal in this chapter is to proof that the universal cover of $\mathrm{Conf}_3(\mathbb{R}, n)$ is a CAT(0) space. As a corollary, $\mathrm{Conf}_3(\mathbb{R}, n)$ is an Eilenberg-MacLane space because of the contractible universal cover (proposition 1.3.5).

Theorem 1.3.13. *The universal cover of* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is a cubical* CAT(0) *complex.*

Proof. By theorem 1.2.17 we know that $\mathrm{Conf}_3(\mathbb{R}, n) \simeq Perm_0(\mathrm{A}_{n-1})$, and its face poset corresponds to cosets wW_I ordered by inclusion, for all $w \in \mathrm{A}_{n-1}$ and $I \subset S$ such that any $s, s' \in I$ commute. In remark 1.2 we said that cells of $Perm_0(\mathrm{A}_{n-1})$ correspond to cubes, we can verify it by proposition 1.3.8. The condition (1) it's clear. We check condition (2). Let wW_I be a coset corresponding to a cell in $Perm_0(\mathrm{A}_{n-1})$. The poset $P_{\leq wW_I}$ of subcells of wW_I corresponds to all uW_J contained in wW_I, and subcells uW_J correspond to intervals in the poset of vertices of wW_I with the weak order, i.e., we have a Boolean lattice as in example 1.3.7. For instance, vertices $uW_\emptyset$ of wW_I correspond to intervals $[u, u]$ for all $u \in wW_I$; 1-cells $uW_{\{s\}}$ are intervals $[u, us]$ between two adjacent permutations which differ by multiplication by s; 2-cells $uW_{\{s,s'\}}$ correspond to intervals $[u, uss']$. Recall any element commute in I, therefore we can go from u to uss' multiplying first by s and then by s', or first s' and then s (see the example in figure 1.12). Then $P_{\leq wW_I}$ is the poset of faces of a cube and condition (2) is verified. Hence, $Perm_0(\mathrm{A}_{n-1})$ is a cubical complex. Denote by $X = Perm_0(\mathrm{A}_{n-1})$. If we take the universal cover $\widetilde{X}$ of X, the cubical structure of the complex is induced to $\widetilde{X}$. To proof that $\widetilde{X}$ is a CAT(0) space, by theorem 1.3.12, we only need to check the link condition. As $\widetilde{X}$ is locally the same as X, it's enough to check the link condition in X. Vertices corresponds to elements in A_{n-1}. Let w a permutation, by second definition of link we have $Lk(w, X)$ is the regular simplicial complex which corresponds to the cells that contain w as a vertex. For instance, the permutation w is contained in all 1-cells $wW_{\{s_i\}}$ ($s_i = (i, i+1)$ transpositions which generates A_{n-1}); w is contained in all 2-cells $wW_{\{s_i,s_j\}}$ for $|i-j| > 1$; in general w is contained in cells wW_I which all $s, s' \in I$ commute. The vertex set of the link consists of all 1-cells containing w. If we consider a complete graph of the 1-skeleton of $Lk(w, X)$, is the same as 2-cells containing w which pairwise share common 1-cells, i.e., $wW_{\{s_{i_1},s_{i_2}\}}, wW_{\{s_{i_2},s_{i_3}\}}, \ldots, wW_{\{s_{i_r},s_{i_{r+1}}\}}$ such that all s_{i_a}, s_{i_b} commute, then wW_T with $T = \{s_{i_1}, \ldots, s_{i_{r+1}}\}$ its a cell in X containing w, and whose link is the r-simplex with the complete graph as its 1-skeleton. Therefore, by proposition 1.3.12, the universal cover $\widetilde{X}$ is a cubical CAT(0) space. □

Remark 1.3.14. We know that the fundamental group of any space X acts on its universal cover $\widetilde{X}$. By the above proposition, we have that PP_n is acting on a cubical CAT(0) space. Furthermore,

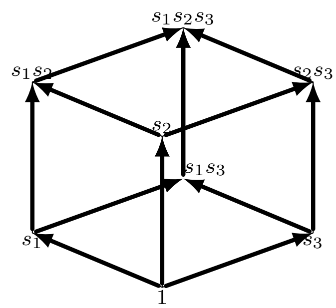


Figure 1.12: Poset for the coset W_{s_1,s_2,s_3}

is acting properly and cocompactly on a CAT(0) cubical complex, i.e., PP_n is a cubed group. In usual braids, it is known that B_n acts on a CAT(0) space for small n, namely, for $n \leq 6$ [19].

Corollary 1.3.15. *The space* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is an Eilenberg-MacLane space* $K(\pi, 1)$*. Therefore,* $\mathrm{Conf}_3(\mathbb{R}, n)$ *is the classifying space of the planar pure braid group on* n *strands* PP_n*.*

It is a known result that if we have a finite dimensional CW-complex which is also a $K(\pi, 1)$ space, then the group π is torsion-free. A proof can be seen in any introductory book in algebraic topology, for instance [21].

Proposition 1.3.16. PP_n *is torsion-free.*

Proof. By corollary 1.3.15 and theorem 1.2.17, $\mathrm{Conf}_3(\mathbb{R}, n)$ is a $K(PP_n, 1)$ space and homotopy equivalent to the finite dimensional CW-complex $Perm_0(\mathrm{A}_{n-1})$. If PP_n has torsion, there is a subgroup $T < PP_n$ such that $T \cong \mathbb{Z}/\mathbb{Z}_k$ for some $k > 1$. The corresponding covering space $\widetilde{X}$ of $\mathrm{Conf}_3(\mathbb{R}, n)$ with $\pi_1(\widetilde{X}) = \mathbb{Z}/\mathbb{Z}_k$ is also a $K(\mathbb{Z}/\mathbb{Z}_k, 1)$ space and it is a finite dimensional CW-complex, hence $H_k(\widetilde{X}) = H_k(\mathbb{Z}/\mathbb{Z}_m)$. On one hand $H_k(\widetilde{X})$ is zero for all $k > \dim(\widetilde{X})$, but on the other hand contradicts the known fact that $H_k(\mathbb{Z}/\mathbb{Z}_m)$ is non-zero for infinitely many k's. Therefore PP_n is torsion-free. $\square$

1.4 Cohomology of $\mathrm{Conf}_3(\mathbb{R}, n)$

In this section, we give a brief description of the cohomology ring of $\mathrm{Conf}_3(\mathbb{R}, n)$ in order to use it in chapter 4.

Björner and Welker in [11] were the first in compute the cohomology groups of no k-equal manifolds. As an special case of their results, they obtain that

(i) $H^i(\mathrm{Conf}_3(\mathbb{R}, n), \mathbb{Z})$ is free for all i,

(ii) $H^i(\mathrm{Conf}_3(\mathbb{R}, n)) \neq 0$ if and only if $0 \leq i \leq \frac{n}{3}$,

(iii) the rank of $H^1(\mathrm{Conf}_3(\mathbb{R}, n))$ is $\sum_{i=3}^{n} \binom{n}{i}\binom{i-1}{2}$.

Then, Baryshnikov compute the ring structure giving an explicit isomorphism of $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ to a quadratic ring $\mathcal{R}_n$, generated by certain posets on $[n] = \{1, 2, \ldots, n\}$. This result also was done in a more general setting. He computes the ring structure on the cohomologies of no k-equal manifolds $\mathrm{Conf}_k(\mathbb{R}, n)$. In our case, $k = 3$. All the material here is extracted from [6].

Definition 1.4.1 (Partitioner Poset)**.** Let $(P, \succeq)$ be a poset on a set of equivalence classes of $[n]$. We say that P is a *partitioner poset* on $[n]$ if exists a height function $h : [n] \to \mathbb{R}$, such that

(a) $i \succeq j$ implies $h(i) \geq h(j)$,

(b) $h(i) > h(j)$ implies $i \succeq j$,

(c) each fiber $h^{-1}(x)$ is the empty set, or a subset $I = \{i_1, i_2, \ldots, i_s\}$ of all unrelated elements, or all equivalent elements.

Every partitioner poset P can be written in blocks. Let $I = h^{-1}(x)$ non empty, by condition (c) I could be written in a ( )- or [ ]-block. If $I = \{i_1, i_2, \ldots, i_s\}$, the block $[I]$ means that all the elements are equivalent, i.e., $i_1 \approx i_2 \approx \cdots \approx i_s$; the block (I) means that all the elements are unrelated. By condition (b), the blocks are ordered decreasingly, i.e., the elements in a block on the left are $\succeq$-greater than the elements in a block on the right.

Example 1.4.2.

$$(6)[13](245)(9)[78] = \{6 \succ 1 \approx 3 \succ 2, 4, 5 \succeq 9 \succeq 7 \approx 8\}.$$

In this way, if we have a partition of $[n]$ in ( )- or [ ]-blocks, we can induce a partitioner poset in $[n]$. By simplicity, we use the notation of partitions and blocks, instead of the height function with the properties (a)-(c).

Definition 1.4.3 (Elementary Partitioner Poset)**.** A partitioner poset on $[n]$ written as $P = (I)[J](K)$ such that $|J| = 2$ is called *elementary*. We denote by $\mathcal{P}$ the set of all elementary partitioner posets on $[n]$.

To define the ring $\mathcal{R}_n$, we need some notation. For any posets $P_1, \ldots, P_k$ on $[n]$, denote the transitive closure of $P_1 \cup \cdots \cup P_k$ by $P_1 \circ \cdots \circ P_k$. Then by definition, $\circ$ is commutative and associative on the set of posets of $[n]$. Any transitive closure of partitioner posets is a partitioner poset, i.e., can be written in ( )-blocks and [ ]-blocks. However, note that a transitive closure of elementary partitioner posets is not always an elementary partitioner poset.

Example 1.4.4. Let be $P_1 = (56)[24](13)$ and $P_2 = (456)[12](3)$, then

$$P_1 \circ P_2 = \{6, 5 \succ 1 \approx 2 \approx 4 \succ 3\} = (65)[124](3),$$

where $P_1 \circ P_2$ is of the form $(I)[J](K)$ but $|J| \neq 2$.

Definition 1.4.5 (The ring $\mathcal{R}_n$)**.** Let $\bigwedge \mathcal{P}$ be the free exterior ring generated by elementary partitioner posets of $[n]$, in which have degree one. Let I_1 be the ideal generated by all the elements of the form

$$\delta(P) = \sum_{i \in I} (-1)^{|I|-1} (I \setminus \{i\})[ji](K) + \sum_{k \in K} (-1)^{|I|} (I)[jk](K \setminus \{k\})$$

where $P = I \sqcup \{j\} \sqcup K$ is an ordered partition of $[n]$. For $\mathcal{E} = \bigwedge \mathcal{P}/I_1$, consider the ideal I_2 generated by classes of products $P_1 \wedge P_2$ with P_1, P_2 partitioner posets such that the transitive closure $P_1 \circ P_2$ has a [ ]-block of size at least 3. Finally,

$$\mathcal{R}_n = \mathcal{E}/I_2.$$

In [6], Baryshnikov constructs a map between $\mathcal{R}_n$ and $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ and proves that in fact is an isomorphism. By the importance of some elements in the proof, we describe it briefly.

Definition 1.4.6 (Cells of Posets)**.** For each poset P on $[n]$ corresponds a subspace $C(P)$ in $\mathbb{R}^n$ in the following way. A point $(x_1, \dots, x_n)$ is in $C(P)$ if and only if the coordinates x_i's respect the order induced by P, i.e., $x_i \sim x_j$ with the canonical order in $\mathbb{R}$, if and only if $i \sim j$ with the order P on $[n]$. The subspace $C(P)$ is called the *cell associated with* P. If P is an elementary partitioner poset, we call $C(P)$ an *elementary cell.*

Example 1.4.7. Let be $P = (2)[14](3)$, then $C(P)$ is the set of points (x_1, x_2, x_3, x_4) such that

$$x_2 > x_1 = x_4 > x_3$$

Remark 1.4.8. The transitive closure of two partitioner posets P_1 and P_2, has a geometric interpretation as the poset defined by the intersection of their cells. In other words, $C(P_1 \circ P_2) = C(P_1) \cap C(P_2)$. In this way, the ideal I_2 in $\mathcal{E}$ generated by $P_1 \wedge P_2$ such that $P_1 \circ P_2$ has a [ ]-block of size at least 3 correspond to cells $C(P_1)$ and $C(P_2)$ such that the intersection cell $C(P_1 \circ P_2)$ has at least 3 coordinates which coincide, hence, $C(P_1 \circ P_2) \not\subseteq \mathrm{Conf}_3(\mathbb{R}, n)$. On the other hand, the generators of the ideal I_1 are boundaries of cells $C(P)$ with $P = I \sqcup \{j\} \sqcup K$.

Orienting and co-orienting elementary cells $C(P)$, there is a map

$$w : \mathcal{P} \to H^1(\mathrm{Conf}_3(\mathbb{R}, n))$$

via intersection product, which extends to a ring homomorphism

$$w : \mathcal{R}_n \to H^*(\mathrm{Conf}_3(\mathbb{R}, n)).$$

The map w is defined on a product $P_1 \wedge \cdots \wedge P_k \in \left(\bigwedge \mathcal{P}\right)^{(k)}$ as the cohomology class induced by intersection with the cell $C(P_1 \circ \cdots \circ P_k)$. The main result in [6] is the following.

Theorem 1.4.9. *The homomorphism* $w : \mathcal{R}_n \to H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ *is an isomorphism.*

To prove the theorem 1.4.9, Baryshnikov give a basis of $\mathcal{R}_n$ in which checks the injectivity and surjectivity of the map. Here is where a kind of partitioner posets are included, which are very important because we found a correspondence with planar pure braids.

Definition 1.4.10 (Basic Partitioner Posets)**.** A partitioner poset $P = (I_0)[J_1](I_1)\cdots[J_s](I_s)$ is called *basic of degree* s if

(a) all [ ]-blocks are of size 2, i.e., $|J_k| = 2$ for $k = 1, \ldots, s$,

(b) in the canonical order of $[n]$, $\max\{i \in I_{k-1} \cup J_k\} \in I_{k-1}$ for $k = 1, \ldots, s$.

By condition (b), we have that I_{k-1} is non empty for $k = 1, \ldots, s$.

Geometrically, basic partitioner posets of higher degree are nothing else than the product of basic partitioner posets of degree 1, in which their corresponding cells intersect transversally. This kind of posets are very relevant because generate the ring $\mathcal{R}_n$, hence $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$.

Theorem 1.4.11. *The set of classes of partitioner posets forms a basis in* $\mathcal{R}_n$*.*

By theorems 1.4.9 and 1.4.11, the cohomology $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ is a quadratic ring generated by the first cohomology group, which is generated by partitioner posets $P = (I)[J](K)$ with $|J| = 2$ and $\max\{i \in I \cup J\} \in J$. An immediate corollary is a formula for Betti numbers by counting basic partitioner posets.

Corollary 1.4.12. *The Betti numbers for* $\mathrm{Conf}_3(\mathbb{R}, n)$ *are*

$$H^s(\mathrm{Conf}_3(\mathbb{R}, n)) = \sum_{i_1,\ldots,i_s} \binom{n}{i_1 \ldots i_s} \binom{i_1 - 1}{2} \cdots \binom{i_s - 1}{2}$$

Proof. A basic partitioner poset $(I_0)[J_1](I_1)\cdots[J_s](I_s)$ is uniquely defined by (1) a choice of s subsets $I_{k-1} \cup J_k$ with $|I_{k-1} \cup J_k| = i_k \geq 3$, i.e., the multi-binomial $\binom{n}{i_1 \ldots i_s}$; and (2) choices of subsets J_k within maximal elements deleted, i.e., the binomial $\binom{i_k - 1}{2}$'s. □

Remark 1.4.13. This section is based on an unpublished work of Baryshnikov [6]. However, an open source to learn more about this, is in a work of Dobrinskaya and Turchin [15], in which they generalize the work of Baryshnikov, computing the (co)homology of non k-overlapping discs which are bimodules over the little discs operad.

In following chapters we use basic partitioner posets of degree 1 as indices for planar pure braids and we use some facts in the cohomology $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ to construct an universal invariant in planar pure braids.

Chapter 2

Planar Pure Braid Group

In this chapter we study the planar pure braid group as an abstract group. We obtain a description of the presentation of the planar pure braid group applying iteratively the Reidemeister-Schreier process. The presentation obtained is in terms of tuples $(m_1, \ldots, m_{n-1})$, where a coordinate m_i means the process to apply from a list of rules. Graphically, the iterative applying of the algorithm looks like a decision tree, and a tuple corresponds to a path in this rooted tree (see figure 2.3). The presentation of PP_n in terms of tuples is highly impractical, but despite that, for particular cases we obtain an easier presentation in terms of basic partitioner posets of degree 1 (see definition 1.4.10).

Theorem 2.2.7. For $n = 1, 2$, PP_n is the trivial group. For $3 \leq n < 6$, the generators of PP_n are planar pure braids associated (see construction 1) to basic partitioner posets of $[n]$. The presentation of PP_n for $3 \leq n \leq 5$ is

$$\begin{aligned} \textit{generators} : &\quad \{b_P \,|\, P \text{ is a basic partitioner poset of } [n]\} \\ \textit{relations} : &\quad \{ \text{ no relations } \} \end{aligned}$$

i.e., are free groups. For $n = 6$, the relations are by conjugation on some particular partitioner posets. Let P_{rel} a basic partitioner poset of [6] of the form $(k)[i,j](L)$ (hence $k > i, j$ and $|L| = 3$). The presentation of PP_6 is

$$\begin{aligned} \textit{generators} : &\quad \{b_P \,|\, P \text{ is a basic partitioner poset of } [6]\} \\ \textit{relations} : &\quad \{(\widetilde{\nu_1}(P_{rel}) \cdot b_{P_{rel}})((\widetilde{\nu_2}(P_{rel}) \cdot b_{P_{rel}}))^{-1} \quad | \quad P_{rel} = (k)[i,j](L)\} \end{aligned}$$

where $\cdot$ is the action by conjugation and $\widetilde{\nu_i}(P_{rel})$ is a product of planar pure braids constructed from L for $i = 1, 2$.

In the last section we see PP_n as a semidirect product of groups which is not almost-direct. This is a crucial difference between the classical pure braid group and its planar version.

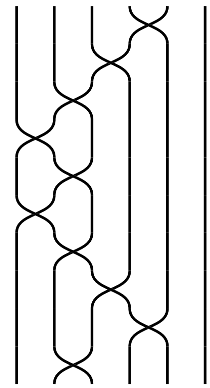


Figure 2.1: Example of an element of $D_6^{(3)}$

2.1 Presentation of PP_n

To find a presentation of PP_n, we shall apply the Reidemeister-Schreier process iteratively to a chain of subgroups of PB_n. Essentially, the Reidemeister-Schreier process tell us how to compute presentations of subgroups. We recall briefly the process, but for a full description or more references see the appendix A.

Let $G = \langle X \mid R \rangle$ and let H be a subgroup of G. Let $\lambda : F(X) \to G$ the natural projection given by the presentation of G. We denote by $L = \lambda^{-1}(H)$ the subgroup of $F(X)$, and $\mathscr{S}$ a Schreier transversal set of L in $F(X)$ (see definition A.1.1). Let $\overline{u} \in \mathscr{S}$ be the representative of $u \in F(X)$ such that $L\overline{u} = Lu$. If we set

$$Y = \{tx(\overline{tx})^{-1} \mid t \in \mathscr{S}, x \in X, tx \neq \overline{tx}\},$$

the Nielsen-Schreier theorem states that Y is a free basis of L (see theorem A.1.2). If we set

$$Q = \{trt^{-1} \mid t \in \mathscr{S}, r \in R\},$$

and regard the elements of R rewritten in terms of elements of Y, the Reidemeister-Schreier theorem states that H has a presentation $\langle Y \mid Q \rangle$ (see theorem A.1.3).

To apply the process to our case, we shall define the chain of subgroups of PB_n that we mentioned. Let $\varphi : PB_n \to \Sigma_n$ the homomorphism such that $\varphi(\sigma_j) = s_j$. We see Σ_{n-l} as the subgroup of Σ_n whose permutations map $n - i \mapsto n - i$ for $i = 0, 1, \ldots, l - 1$. We set

$$D_n^{(l)} = \varphi^{-1}(\Sigma_{n-l}) \qquad l = 1, \ldots, n$$

Geometrically, $D_n^{(l)}$ consists of planar braids whose last l strands, do not change the order (see figure 2.1). Note that $D_n^{(n-1)} = D_n^{(n)} = PP_n$ and by convention $D_n^{(0)} = PB_n$. The chain of subgroups in Σ_n

$$\Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_2 \supset \Sigma_1 = \Sigma_0 = \{1\},$$

induce the chain of subgroups in PB_n

$$PB_n \supset D_n^{(1)} \supset \cdots \supset D_n^{(n-2)} \supset D_n^{(n-1)} = D_n^{(n)} = PP_n. \tag{2.1.1}$$

Applying the Reidemeister-Schreier process iteratively, we obtain a presentation of $D_n^{(l)}$ until PP_n. First of all, we need a Schreier transversal set of $D_n^{(l)}$ in $D_n^{(l-1)}$, which coincides with the index $[\Sigma_{n-l+1} : \Sigma_{n-l}] = n - l + 1$. If we set

$$\mathscr{S}_l = \{M_{n-l+1,i_l} \,|\, 0 \leq i \leq n-l\},$$

where $M_{n-l+1,i_l} = \sigma_{n-l}\sigma_{n-l-1}\cdots\sigma_{n-l-i_l}$ for $1 \leq i \leq n-l-1$ and $M_{n-l+1,0} = 1$, then $\mathscr{S}_l$ is a Schreier transversal set of $D_n^{(l)}$ in $D_n^{(l-1)}$ for $l = 1, \ldots, n$. We include two first examples. The details of the calulations can be seen in the appendix A.

Proposition 2.1.1. *The subgroup $D_n^{(1)}$ has a presentation with*

$$\begin{array}{lll}
\textit{generators}: & \{\sigma_j \,|\, j \in [1, n-2]\} \quad \textit{and} \quad \{N_{n,j} \cdot \sigma_{j+1} \,|\, j \in [1, n-2]\} & \\
\textit{relations}: & (\sigma_j)^2 & \\
 & (N_{n,j} \cdot \sigma_{j+1})^2 & \\
 & (\sigma_j\sigma_k)^2 & |j-k| > 1 \\
 & [(N_{n,j} \cdot \sigma_{j+1})(N_{n,k} \cdot \sigma_{k+1})]^2 & |j-k| > 1 \\
 & [\sigma_j(N_{n,k} \cdot \sigma_{k+1})]^2 & k > j+1
\end{array}$$

In the above proposition we add new notation explained in the appendix A: $N_{n-l+1,j}$ means $\sigma_{n-l}\cdots\sigma_j$ for $l = 1, \ldots, n$, $j = 1, \ldots, n-l$ and $N_{l,l} = 1$ for any l; and $N \cdot \sigma$ means conjugation $N\sigma N^{-1}$. In proposition 2.1.1 $l = 1$, and by relation 1.1.2 we know $\sigma_j = \sigma_j^{-1}$, hence

$$N_{n,j} \cdot \sigma_{j+1} = \sigma_{n-1}\cdots\sigma_j\sigma_{j+1}\sigma_j\cdots\sigma_{n-1}.$$

Geometrically, $N_{n,j} \cdot \sigma_{j+1}$ is the planar braid in which its nth strand tangles in the generator σ_j. Going on with the applying of the process to the case $D_n^{(2)}$ in $D_n^{(1)}$ we obtain the next presentation.

Proposition 2.1.2. *The subgroup $D_n^{(2)}$ has a presentation with*

$$\begin{array}{ll}
\textit{generators}: & \{\sigma_j \quad | \quad j \in [1, n-3]\} \\
 & \{N_{n-1,j} \cdot \sigma_{j+1} \quad | \quad j \in [1, n-3]\} \\
 & \{M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1} \quad | \quad j \in [1, n-3], i_2 \in [0, n-4], i_2 < n-2-j\} \\
 & \{N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3 \quad | \quad j \in [1, n-2]\} \\
 & \{N_{n-1,j}N_{n,j+1} \cdot \sigma_{j+2} \quad | \quad j \in [1, n-3]\}
\end{array} \tag{2.1.2}$$

$$\begin{array}{ll}
\textit{relations}: & (\sigma_j)^2 \\
 & (N_{n-1,j} \cdot \sigma_{j+1})^2
\end{array}$$

$$
\begin{array}{lr}
(N_{n-1,j}N_{n,j+1} \cdot \sigma_{j+2})^2 & \\
(M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1})^2 & \\
(\sigma_j\sigma_k)^2 & |j-k|>1 \\
[\sigma_j(N_{n-1,k} \cdot \sigma_{k+1})]^2 & k>j+1 \\
[\sigma_j(N_{n-1,k}N_{n,k+1} \cdot \sigma_{k+2})]^2 & k>j+1 \\
[\sigma_j(M_{n-1,i}N_{n,k} \cdot \sigma_{k+1})]^2 & k>j+1 \\
[\sigma_j, N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3] & k>j+1 \\
[(N_{n-1,j} \cdot \sigma_{j+1})(N_{n-1,k} \cdot \sigma_{k+1})]^2 & |j-k|>1 \\
[(N_{n-1,j} \cdot \sigma_{j+1})(N_{n-1,k}N_{n,k+1} \cdot \sigma_{k+2})]^2 & k>j+1 \\
[(M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1})(M_{n-1,i_2}N_{n,k} \cdot \sigma_{k+1})]^2 & |j-k|>1 \\
[(M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k}N_{n,k+1} \cdot \sigma_{k+2})]^2 & k>j+1 \\
[(N_{n-1,j}N_{n,j+1} \cdot \sigma_{j+2})(N_{n-1,k}N_{n,k+1} \cdot \sigma_{k+2})]^2 & |j-k|>1 \\
(N_{n-1,k+1}N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3) & k>j+1 \\
\quad = (N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3)(N_{n-1,k}N_{n,j} \cdot \sigma_{j+1}) & \\
[N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3, N_{n-1,k}N_{n,k+1} \cdot \sigma_{k+2}] & k>j.
\end{array}
$$

Remark 2.1.3. The group $D_3^{(2)}$ is by definition the group on 3 strands in which the last 2 strands, do not change the order, and hence, the first neither does. In other words, $D_3^{(2)} = PP_3$. Taking $n = 3$ in the above proposition 2.1.2, the unique generator is $(\sigma_2\sigma_1)^3$ (see figure 2.2) and there are no relations, i.e, PP_3 is the free group in one generator as we saw in example 1.1.4 as the fundamental group of $\text{Conf}_3(\mathbb{R}, 3) \simeq \mathbb{S}^1$. Furthermore, for any n, the family of generators 2.1.2 induced the trivial permutation under the homomorphism $\varphi : PB_n \to \Sigma_n$. First note that $\varphi((\sigma_{j+1}\sigma_j)^3) = (s_{j+1}s_j)^3 = 1$. Therefore $\varphi(N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3) = \varphi(N_{n-1,j+1}N_{n,j+2})\cdot\varphi((\sigma_{j+1}\sigma_j)^3) = \varphi(N_{n-1,j+1}N_{n,j+2})\cdot 1 = 1$. In other words, the family of generators 2.1.2 is already in PP_n.

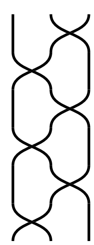


Figure 2.2: Generator $(\sigma_2\sigma_1)^3$ of PP_3

In PB_n we have $n-1$ generators σ_j's. In $D_n^{(1)}$ we have one less generators, i.e., $n-2$ generators σ_j's. In $D_n^{(2)}$ we have again one less the previous, i.e., $n-3$ generators σ_j's. That happens in general each time we apply the Reidemeister-Schreier process. In $D_n^{(l)}$ there are $n-1-l$ generators σ_j's. At the last, for $D_n^{(n-1)} = D_n^{(n)} = PP_n$ there are no more generators of type σ_j. A similar behaviour happens in all index sets of generators at each applying of the process. It turns out from the conditions on indices i's and j's at each rewriting, i.e., when we restrict to the conditions $i+j \sim n-l$ for some l and where $\sim$ could be $>, <, =$. We describe below how it works.

The Reidemeister-Schreier process is an algorithm to find presentations of subgroups, and as an algorithm it behaves in a very systematic way. What follows is a description of such systematic behaviour of the algorithm applied to our chain of subgroups 2.1.1

$$PB_n \supset D_n^{(1)} \supset \cdots \supset D_n^{(n-2)} \supset D_n^{(n-1)} = D_n^{(n)} = PP_n.$$

For instance, in the appendix A for the cases of $D_n^{(1)}$ and $D_n^{(2)}$, the Reidemeister-Schreier process on each family of generators produces four families of generators (sometimes families of trivial generators) depending in conditions on $i_l + j$. See for example the case A.2.9 for the family of generators $\{N_{n,j} \cdot \sigma_{j+1}\}_{j=1}^{n-2}$

$$\chi\begin{pmatrix} M_{n-1,i} \\ N_{n,j} \cdot \sigma_{j+1} \end{pmatrix} = \begin{cases} M_{n-1,i_2} N_{n,j} \cdot \sigma_{j+1} & \text{if } i+j < n-2, \quad (1 \le j \le n-3) \\ N_{n-1,j+1} N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3 & \text{if } i+j = n-2, \quad (1 \le j \le n-2) \\ N_{n-1,j+1} N_{n,j+2} \cdot (\sigma_j \sigma_{j+1})^3 & \text{if } i+j = n-1, \quad (1 \le j \le n-2) \\ N_{n-1,j-1} N_{n,j} \cdot \sigma_{j+1} & \text{if } i+j > n-1. \quad (2 \le j \le n-2). \end{cases}$$

An important point we should notice is that the fourth family is always re-indexed at the last, the index set instead of starts at 2, starts at 1. Summarizing, we denote by RS^l the Reidemeister-Schreier process applied to obtain a presentation of the group $D_n^{(l)}$ given the presentation of $D_n^{(l-1)}$. At each process RS^l we input a family of generators $\{x_j \,|\, j \in J_{l-1}\}$ of $D_n^{(l-1)}$ with information (1) an index set J_{l-1} and (2) indexed generators x_j's. The process RS^l outputs four families of generators, i.e., (1) four index sets and (2) their indexed generators. If we denote the shift mentioned above by $\mathrm{Sh}^{-1}(J_{l-1})$, the Reidemeister-Schreier process RS^l works on index sets as follows

1. If $i_l + j < n - l$, it produces the first family of generators where the new index set is $J_l^1 = J_{l-1} \cap [1, n-l-1]$ and to satisfy the condition, i_l requires $0 \le i_l \le n-l-2$.
2. If $i_l + j = n - l$, it produces the second family of generators where the new index set is $J_l^2 = J_{l-1} \cap [1, n-l]$.
3. When $i_l + j = n - l + 1$, it produces the third family of generators where the new index set is $J_l^3 = J_{l-1} \cap [1, n-l+1]$.
4. When $i_l + j > n - l + 1$, it produces the fourth family of generators where the new index set is $J_l^4 = \mathrm{Sh}^{-1}(J_{l-1} \cap [2, n-l+2])$.

Remark 2.1.4. At the last step RS^{n-1} produces only three families, $J_{n-1}^1 = \emptyset$ (see aslo theorem 2.1.11).

The process RS^l produces four families of generators (out of the exception above). However, there are two kinds of four families of generators from where we can choose as the output. The choice depends on the permutations induced by the family of generators at the input. If the family induces trivial permutations, i.e., if the family is already in PP_n as in remark 2.1, the new four families are constructed from conjugations of the input family. Otherwise, the rule is as follows.

$$\mathrm{RS}^l(\{\square_j\}_{j\in J_{l-1}}) = \begin{cases} \left.\begin{array}{llll} 1)\{M_{n-l+1,i_l} & \square_j & M^{-1}_{n-l+1,i_l} & \mid\ j \in J^1_l\} \\ 2)\{N_{n-l+1,j+1} & \square_j & N^{-1}_{n-l+1,j} & \mid\ j \in J^2_l\} \\ 3)\{N_{n-l+1,j} & \square_j & N^{-1}_{n-l+1,j+1} & \mid\ j \in J^3_l\} \\ 4)\{N_{n-l+1,j} & \square_{j+1} & N^{-1}_{n-l+1,j} & \mid\ j \in J^4_l\} \end{array}\right\} & \text{if } \square_j \notin PP_n \\ \\ \left.\begin{array}{llll} 1')\{M_{n-l+1,i_l} & \square_j & M^{-1}_{n-l+1,i_l} & \mid\ j \in J^1_l\} \\ 2')\{N_{n-l+1,j+1} & \square_j & N^{-1}_{n-l+1,j+1} & \mid\ j \in J^2_l\} \\ 3')\{N_{n-l+1,j} & \square_j & N^{-1}_{n-l+1,j} & \mid\ j \in J^3_l\} \\ 4')\{N_{n-l+1,j} & \square_{j+1} & N^{-1}_{n-l+1,j} & \mid\ j \in J^4_l\} \end{array}\right\} & \text{if } \square_j \in PP_n \end{cases} \tag{2.1.3}$$

where $\{\square_j\}_{j\in J_{l-1}}$ means the family of generators at the input of the Reidemeister-Schreier process. In the input square, a generator x_j is written as $\boxed{x}_j$. On the other hand for the output square, we consider the input x_j as the jth generator of type x, then the output square means the kth generator of type x, i.e.,

$$\boxed{x}_j \overset{\mathrm{RS}^l}{\rightsquigarrow} \boxed{x}_k.$$

For the first three families the index remains equal, but for the fourth family the index is added by one.

Example 2.1.5. If we take the family of generators 2.1.2 in proposition 2.1.2, we know that are already in PP_n. In this case $l = 3$, therefore, we have

$$\mathrm{RS}^3(\{N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3\}^{n-2}_{j=1}) = \begin{cases} 1')\{M_{n-2,i_3}N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3\}^{n-4}_{j=1} \\ 2')\{N_{n-2,j+1}N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3\}^{n-3}_{j=1} \\ 3')\{N_{n-2,j}N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3\}^{n-2}_{j=1} \\ 4')\{N_{n-2,j}N_{n-1,j+2}N_{n,j+3}\cdot(\sigma_{j+2}\sigma_{j+1})^3\}^{n-3}_{j=1} \end{cases}$$

Expanding the case $4')$. We have as input the generator $N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3$. In $4')$ the output square is $\square_{j+1}$, so we have to re-index j by $j+1$, resulting $N_{n-1,j+2}N_{n,j+3}\cdot(\sigma_{j+2}\sigma_{j+1})^3$. Next, we conjugate by $N_{n-2,j}$, obtaining

$$N_{n-2,j}[N_{n-1,j+2}N_{n,j+3}\cdot(\sigma_{j+2}\sigma_{j+1})^3]N^{-1}_{n-2,j} = N_{n-2,j}N_{n-1,j+2}N_{n,j+3}\cdot(\sigma_{j+2}\sigma_{j+1})^3$$

with the $\cdot$ notation for the conjugation action. For the index set, RS^3 works in the input $J_2 = [1, n-2]$ by the rules above:

$$\begin{aligned} J^4_3 &= \mathrm{Sh}^{-1}(J_2 \cap [2, n-1]) \\ &= \mathrm{Sh}^{-1}([2, n-2]) \\ &= [1, n-3]) \end{aligned}$$

Obtaining the family of generators $\{N_{n-2,j}N_{n-1,j+2}N_{n,j+3}\cdot(\sigma_{j+2}\sigma_{j+1})^3 | j \in [1, n-3]\}$.

Remark 2.1.6. Sometimes, the generators produced by rules 1) and 1′) depend on i_l, where $0 \leq i_l \leq n-l-2$. In example 2.1.5, the generators by the rule 1′), require their index i_3 be among $0 \leq i_3 \leq n-5$.

Diagrammatically the iterative Reidemeister-Schreier process in the chain of subgroups 2.1.1 is a branching rooted tree T. The root of the tree T is the family of generators $\sigma_1, \ldots, \sigma_{n-1}$ of PB_n. The height of T is the number of iterations of the Reidemesiter-Schreier process, and a vertex of height l is a family of generators of $D_n^{(l)}$. In figure 2.3 we have an example of the rooted tree of height 4. As we said before, at each application of the Reidemesiter-Schreier processs, we obtain four families of generators at the output for each family of generators at the input. In figure 2.3 at the first applying, we have only two branches, corresponding to the rules 1) and 4), instead of four branches. This is because generators by rules 2) and 3) are trivial. For the second application, if we take the generators $\{\sigma_j\}_{j=1}^{n-2}$ produced by the rule 1) in the first process as the family at the input, at the output we obtain again only two families of generators by rules 1) and 4), the others are trivial. We can identify these generators at the output for the second process as the generators of type $(1,1)$ and $(1,4)$, meaning the rules we applied at each process. If for the second application now we take the generators $\{N_{n,j} \cdot \sigma_{j+1}\}_{j=1}^{n-2}$ produced by the rule 4) in the first process, now at the output we obtain 4 non trivial families of generators in which families produced by rules 2) and 3) are inverses between them (see proposition 2.1.10). Similarly, these generators of the second process can be identified by tuples $(4,1)$, $4,2)$, $(4,3)$ and $(4,4)$, where $(4,2)$ $(4,3)$ are inverses between them. In general, each vertex in the tree, can be identified by an l-tuple $(m_1, m_2, \ldots, m_l)$ where l is the height of the vertex and $m_k \in \{1,2,3,4,1',2',3',4'\}$ is the rule applied to produce the family of generators at the output of the process RS^k.

Definition 2.1.7. If $\{x_j\}_{j\in J_l}$ is a family of generator of $D_n^{(l)}$ defined by the iterative Reidemeister-Schreier process following the rules $(m_1, m_2, \ldots, m_l)$ where $m_k \in \{1,2,3,4,1',2',3',4'\}$, we said the family of generators $\{x_j\}_{j\in J_l}$ is of type $(m_1, m_2, \ldots, m_l)$.

In rules 2.1.3 to apply the Reidemeister-Schreier process, we distinguish if the generators at the input are already in PP_n or not. For that, we will give conditions on the tuple $(m_1, m_2, \ldots, m_l)$, which corresponds to a family of generators in $D_n^{(l)}$, which tell us whether or not they are elements in PP_n.

Proposition 2.1.8. *Let $l \leqslant n-2$ and $\{x_j\}_{j\in J_l} \subset D_n^{(l)}$ a family of generators of type $(m_1, m_2, \ldots, m_l)$ where $m_k \in \{1,4\}$ for all $k = 1, \ldots, l$. For the natural map $\varphi : PB_n \to \Sigma_n$, we have $\varphi(x_j) = s_j$ for any $j \in J_l$.*

Proof. By induction over l. For $l = 1$, we have two family of generators, $\{\sigma_j\}_{j=1}^{n-2}$ of type (1) and $\{N_{n,j} \cdot \sigma_{j+1}\}_{j=1}^{n-2}$ of type (2). For type (1) it's clear, for type (2) we have:

$$\begin{aligned}\varphi(N_{n,j} \cdot \sigma_{j+1}) &= \varphi(N_{n,j}\sigma_{j+1}N_{n,j}^{-1}) \\ &= \varphi(\sigma_{n-1}\cdots\sigma_{j+1}\sigma_j)\varphi(\sigma_{j+1})\varphi(\sigma_j\sigma_{j+1}\cdots\sigma_{n-1})\end{aligned}$$

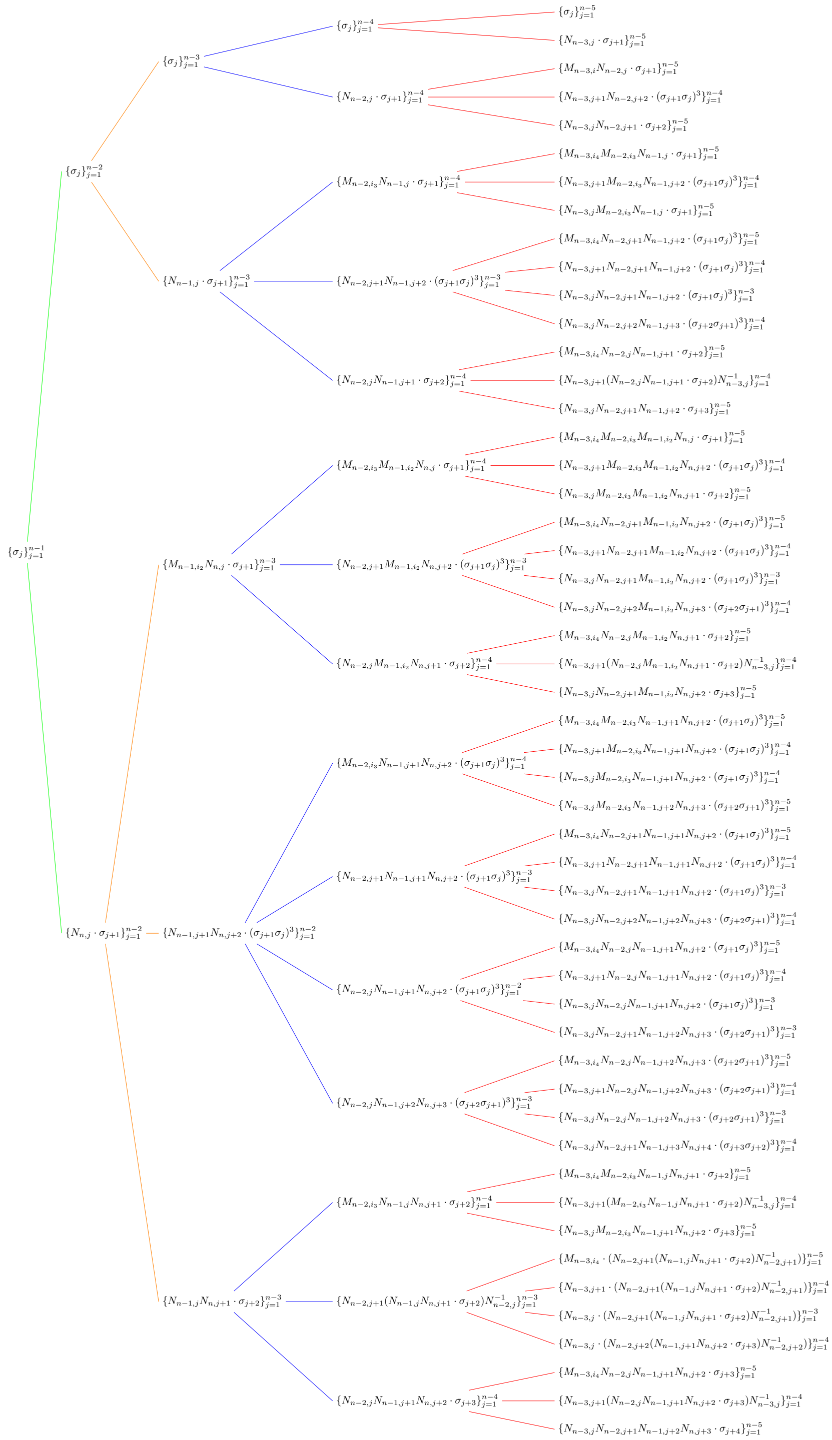


Figure 2.3: Tree of generators from PB_n to $D_n^{(4)}$.

$$
\begin{aligned}
&= (s_{n-1} \cdots s_{j+1} s_j) s_{j+1} (s_j s_{j+1} \cdots s_{n-1}) \\
&= (s_{n-1} \cdots s_{j+1})(s_j s_{j+1} s_j)(s_{j+1} \cdots s_{n-1}) \\
&= (s_{n-1} \cdots s_{j+1})(s_{j+1} s_j s_{j+1})(s_{j+1} \cdots s_{n-1}) \\
&= (s_{n-1} \cdots s_{j+1}^2) s_j (s_{j+1}^2 \cdots s_{n-1}) \\
&= (s_{n-1} \cdots s_{j+2})(s_{j+2} \cdots s_{n-1}) s_j \\
&= s_j
\end{aligned}
$$

by relations in the presentation of Σ_n (see remark 1.1). Let $\{y_j\}_{j\in J_l}$ be a family of generators of type $(m_1, \ldots, m_{l-1}, m_l)$ where $m_k \in \{1, 4\}$ for all k. If we set $\{x_j\}_{j\in J_{l-1}}$ as the family of generators of type $(m_1, \ldots, m_{l-1})$ such that $\mathrm{RS}^l(\{x_j\}_{j\in J_{l-1}}) = \{y_j\}_{j\in J_l}$ for some $m_l \in \{1, 4\}$, by the induction hypothesis, $\varphi(x_j) = s_j$ for all $j \in J_{l-1}$. If $m_l = 1$, y_j is conjugation of x_j by M_{n-l+1,i_l} where $0 \le i_l \le n-l-2$ and $i_l + j < n-l$, or equivalently, $1 < |(n-l+1-i_l) - j|$. If $i_l = 0$, $M_{n-l+1,0} = 1$ and it's clear. Otherwise

$$
\begin{aligned}
\varphi(y_j) &= \varphi(M_{n-l+1,i_l} x_j M_{n-l+1,i_l}^{-1}) \\
&= \varphi(\sigma_{n-l} \cdots \sigma_{n-l+1-i_l}) \varphi(x_j) \varphi(\sigma_{n-l+1-i_l} \cdots \sigma_{n-l}) \\
&= (s_{n-l} \cdots s_{n-l+1-i_l}) s_j (s_{n-l+1-i_l} \cdots s_{n-l}) \\
&= (s_{n-l} \cdots s_{n-l+1-i_l})(s_{n-l+1-i_l} \cdots s_{n-l}) s_j \\
&= (s_{n-l} \cdots s_{n-l+1-i_l}^2 \cdots s_{n-l}) s_j \\
&= s_j.
\end{aligned}
$$

If $m_l = 4$, y_j is conjugation of x_{j+1} by $N_{n-l+1,j}$, then

$$
\begin{aligned}
\varphi(y_j) &= \varphi(N_{n-l+1,j} x_{j+1} N_{n-l+1,j}^{-1}) \\
&= \varphi(\sigma_{n-l} \cdots \sigma_{j+1} \sigma_j) \varphi(x_{j+1}) \varphi(\sigma_j \sigma_{j+1} \cdots \sigma_{n-l}) \\
&= (s_{n-l} \cdots s_{j+1} s_j) s_{j+1} (s_j s_{j+1} \cdots s_{n-l}) \\
&\;\;\vdots \\
&= s_j.
\end{aligned}
$$

Similarly as in the case $l = 1$. Therefore, if we have a generator x_j of type $(m_1, \ldots, m_l)$ wihtout 2's or 3's in the tuple, its induced permutation is always s_j. □

Proposition 2.1.9. *Generators of type $(m_1, m_2, \ldots, m_l)$ such that $m_k \in \{2, 3\}$ for some k, induce the trivial permutation, hence they are in PP_n.*

Proof. Suppose $m_k = 2$ for some k. The case $m_k = 3$ is a consequence of proposition 2.1.10. Let k_0 be the smallest in which $m_{k_0} = 2$. The proof is by induction over l. If $k_0 = l = 1$, it is clear because the generators of type (2) in $\mathrm{RS}^1(\{\sigma_j\}_{j=1}^{n-1})$ are the trivial ones. If $k_0 < l$, by the induction hypothesis the generators of type $(m_1, \ldots, m_{l-1})$ are already in PP_n, then to obtain the generators of type $(m_1, \ldots, m_{l-1}, m_l)$ the rules in 2.1.3 say they are conjugation of the generators of type $(m_1, \ldots, m_{l-1})$ by some elements. It is clear that if $\varphi(x) = 1$ then $\varphi(NxN^{-1}) = 1$, hence the generators of type $(m_1, \ldots, m_l)$ induce the trivial permutation. If $k_0 = l$, the tuple $(m_1, \ldots, m_l)$ is such that $m_i \in \{1, 4\}$ for all $i < l$ and $m_l = 2$. By proposition 2.1.8, the family of generators $\{x_j\}_{j \in J_{l-1}}$ of type $(m_1 \ldots, m_{l-1})$ is such that $\varphi(x_j) = s_j$. Then the generators of type $(m_1, \ldots, m_{l-1}, 2)$ are written as $N_{n-l+1,j+1} x_j N^{-1}_{n-l+1,j+1}$, hence

$$\begin{aligned}
\varphi(N_{n-l+1,j+1} x_j N^{-1}_{n-l+1,j}) &= \varphi(N_{n-l+1,j+1})\varphi(x_j)\varphi(N^{-1}_{n-l+1,j}) \\
&= (s_{n-l} \cdots s_{j+1}) s_j (s_j s_{j+1} \cdots s_{n-l}) \\
&= s_{n-l} \cdots s_{j+1} s_j^2 s_{j+1} \cdots s_{n-l} \\
&\vdots \\
&= 1
\end{aligned}$$

Therefore, permutations induced by generators of type $(m_1, \ldots, m_l)$ with some $m_k = 2$ are trivial, which by definition are elements in PP_n. □

An immediate consequence of the last proposition is that if we have $(m_1, \ldots, m_{k-1}, 2, m_{k+1}, \ldots, m_l)$ with $k = \min\{i | m_i = 2\}$ then $m_j \in \{1', 2', 3', 4'\}$ for all $j > k$. From here, to simplify notation we write the set of rules $\{1', 2', 3', 4'\}$ without the prime notation. For instance, generators of type $(4, 2, 1', 3', 2')$ are the same as generators of type $(4, 2, 1, 3, 2)$.

Proposition 2.1.10. *For generators of type $(m_1, \ldots, m_{k-1}, 2, m_{k+1}, \ldots, m_l)$ such that $m_i \in \{1, 4\}$ for all $i < k$, the generators of type $(m_1, \ldots, m_{k-1}, 3, m_{k+1}, \ldots, m_l)$ are the inverses.*

Proof. Let $\{x_j\}_{j \in J_{k-1}}$ be generators of type $(m_1, \ldots, m_{k-1})$. It is an easy exercise that $J_k^2 = J_k^3$. For $m_k = 2$, the generators $\{y_j\}_{j \in J_k^2}$ of type $(m_1, \ldots, m_{k-1}, 2)$ are given by

$$y_j = N_{n-k+1,j+1} x_j N^{-1}_{n-k+1,j}.$$

For $m_k = 3$, the generators $\{y'_j\}_{j \in J_k^3}$ of type $(m_1, \ldots, m_{k-1}, 3)$ are given by

$$y'_j = N_{n-k+1,j} x_j N^{-1}_{n-k+1,j+1}.$$

In theorem 2.1.14 we will see the generators x_j's as before, satisfy the relation $x_j^2 = 1$. It follows that $y_j y'_j = 1 = y'_j y_j$, hence they are inverses at each $j \in J_k^2 = J_k^3$. By proposition 2.1.9, the generators of type $(m_1, \ldots, m_{k-1}, 2)$ and $(m_1, \ldots, m_{k-1}, 3)$ are already in PP_n. By rules in 2.1.3, the generators of type $(m_1, \ldots, m_{k-1}, 2, m_{k+1}, \ldots, m_l)$ and $(m_1, \ldots, m_{k-1}, 3, m_{k+1}, \ldots, m_l)$ are defined by $N_j y_j N_j^{-1}$ and $N_j y'_j N_j^{-1}$ for some $N_j \in PB_n$, which clearly are also inverses between them at each j. □

What we have now are conditions in tuples $(m_1, \ldots, m_l)$ which allow us to avoid a little bit of redundancy when we are considering all the generators, omitting their inverses or trivial ones.

Theorem 2.1.11. *Non trivial generators of PP_n are generators of type $(m_1, \ldots, m_{n-1})$ with $m_i \in \{1, 2, 3, 4\}$ for all $i = 1, \ldots, n-1$, such that*

(1) $m_1 \in \{1, 4\}$ *and* $m_{n-1} \in \{2, 3, 4\}$.

(2) $m_k = 2$ *for some* $k > 1$.

(3) If $k_0 = \min\{k \mid m_k = 2\}$, *then exists* $l < k_0$ *such that* $m_l = 4$.

(4) If $m_k = 1$ *for some* $k < n-1$, *let* $k_0 = \max\{k | m_k = 1\}$, *then exist* $l > k_0$ *such that* $m_l \in \{2, 3\}$.

Proof. For (1), the condition on m_1 is clear by proposition 2.1.1. For m_{n-1}, we know how it works RS^{n-1} on index sets, $m_{n-1} \neq 1$ because of $J^1_{n-1} = \emptyset$. Condition (2) follows by proposition 2.1.9 to be in PP_n. For (3), exists such l with $m_l = 4$, otherwise we have a sequence of 1's followed by a 2, i.e., $(1, \ldots, 1, 2, \ldots)$ which generators of this type are the trivial ones. For (4), if we have that $m_i = 4$ for all $i > k_0$, we have generators of type $(\ldots, 1, 4, \ldots, 4)$. On index sets, $J^1_{k_0} \subset [1, n - k_0 - 1]$, then $J^4_{k_0+1} \subset \text{Sh}^{-1}([1, n-k_0-1] \cap [2, n-k_0+2]) \subset [1, n-k_0-2]$, and at the last two steps, $J^4_{n-2} \subset [1, 1]$ and then $J^4_{n-1} \subset \text{Sh}^{-1}(J_{n-2} \cap [2, 3]) = \emptyset$. Therefore, exist at least one $l > k_0$ such that $m_l \in \{2, 3\}$. □

Remark 2.1.12. Conditions of theorem 2.1.11 are good conditions to consider non trivial generators of PP_n and remove a little of redundancy at the moment we count tuples of generators of PP_n. However, conditions of theorem 2.1.11 are not excluding other tuples to be generators of PP_n. For instance, tuple generators $(m_1, \ldots, m_{k_0-1}, 2, m_{k_0+1}, \ldots, m_{n-1})$ and its inverses $(m_1, \ldots, m_{k_0-1}, 3, m_{k_0+1}, \ldots, m_{n-1})$.

Example 2.1.13. For PP_4 is easy to find all the tuples (m_1, m_2, m_3) as in theorem 2.1.11. The generators of type $(1, 4, 2)$, $(4, 1, 2)$, $(4, 2, 2)$, $(4, 2, 3)$, $(4, 2, 4)$, $(4, 4, 2)$, generate PP_4. We need to determine the index sets from the generators $\{\sigma_1, \sigma_2, \sigma_3\}$ of PB_4.

- $(1, 4, 2)$: Applying the Reidemeister-Schreier process with rules 2.1.3, we have

$$\begin{array}{c}\sigma_j \\ \big\downarrow{\scriptstyle 1} \\ M_{4,i_1}[\sigma_j]M_{4,i_1}^{-1} = \sigma_j \qquad (i_1 + j < 3) \\ \big\downarrow{\scriptstyle 4} \\ N_{3,j}[\sigma_{j+1}]N_{3,j}^{-1} = N_{3,j} \cdot \sigma_{j+1} \\ \big\downarrow{\scriptstyle 2} \\ N_{2,j+1}[N_{3,j} \cdot \sigma_{j+1}]N_{2,j} = N_{2,j+1}N_{3,j+2} \cdot (\sigma_{j+1}\sigma_j)^3\end{array}$$

  and on index sets, we have

$$J_0 = [1, 3] \xrightarrow{1} J_1 = [1, 2] \xrightarrow{4} J_2 = \{1\} \xrightarrow{2} J_3 = \{1\}.$$

  Therefore, the family of generators of type $(1, 4, 2)$ consists of one element and is $N_{2,2}N_{3,3} \cdot (\sigma_2\sigma_1)^3 = (\sigma_2\sigma_1)^3$.

- $(4,2,3)$: Note that $(4,2,3)=(4,2,3')$. In the same way, we have

$$\begin{array}{c} \sigma_j \\ \downarrow 4 \\ N_{4,j}\sigma_{j+1}N_{4,j}^{-1} = N_{4,j}\cdot\sigma_{j+1} \\ \downarrow 2 \\ N_{3,j+1}[N_{4,j}\cdot\sigma_{j+1}]N_{3,j}^{-1} = N_{3,j+1}N_{4,j+2}\cdot(\sigma_{j+1}\sigma_j)^3 \\ \downarrow 3 \\ N_{2,j}[N_{3,j+1}N_{4,j+2}\cdot(\sigma_{j+1}\sigma_j)^3]N_{2,j} = N_{2,j}N_{3,j+1}N_{4,j+2}\cdot(\sigma_{j+1}\sigma_j)^3 \end{array}$$

and on index sets,

$$J_0=[1,3]\xrightarrow{1}J_1=[1,2]\xrightarrow{4}J_2=[1,2]\xrightarrow{2}J_3=[1,2].$$

Therefore, the family of generators of type $(4,2,3)$ consists of two elements. If $j=1$, we have the generator $N_{2,1}N_{3,2}N_{4,3}\cdot(\sigma_2\sigma_1)^3=\sigma_1\sigma_2\sigma_3(\sigma_2\sigma_1)^3\sigma_3\sigma_2\sigma_1$ and when $j=2$, we have the other generator $N_{2,2}N_{3,3}N_{4,4}\cdot(\sigma_3\sigma_2)^3=(\sigma_3\sigma_2)^3$. The other generators are calculated in the same way. The group PP_4 is a free group in 7 generators with presentation:

$$PP_4=\left\langle \begin{array}{ccc} (\sigma_2\sigma_1)^3, & \sigma_3(\sigma_2\sigma_1)^3\sigma_3, & (\sigma_3\sigma_2)^3, \\ & \sigma_2\sigma_3(\sigma_2\sigma_1)^3\sigma_3\sigma_2, & \sigma_1(\sigma_3\sigma_2)^3\sigma_1, \\ & \sigma_1\sigma_2\sigma_3(\sigma_2\sigma_1)^3\sigma_3\sigma_2\sigma_1, & (\sigma_2\sigma_1\sigma_3\sigma_2\cdot\sigma_3)\sigma_1 \end{array}\right\rangle \tag{2.1.4}$$

In 2.3.1 we have another presentation of PP_4 with the generator $x=(\sigma_2\sigma_1\sigma_3\sigma_2\cdot\sigma_3)\sigma_1$ replaced by the product $xy^{-1}=\sigma_2\sigma_1(\sigma_3\sigma_2)^3\sigma_1\sigma_2$ where $y=(\sigma_2\sigma_1)^2$.

We already know the generators of PP_n which are associated to $(n-1)$-tuples $(m_1,\ldots,m_{n-1})$. However, we need the relations as in propositions 2.1.1 and 2.1.2. The Reidemeister-Schreier process tell us how it works on relations. For instance, the relation σ_j^2 in PB_n after apply the process, we obtain 4 outputs which correspond to apply RS^1 to each letter, i.e., $\sigma_j\sigma_j\longrightarrow \mathrm{RS}^1(\sigma_j)\,\mathrm{RS}^1(\sigma_j)$ (see figure 2.4).

Theorem 2.1.14. *Let $(m_1^1,\ldots,m_l^1)$ and $(m_1^2,\ldots,m_l^2)$ two types of generators. A relation $(m_1^1,\ldots,m_l^1)|(m_1^2,\ldots,m_l^2)$ comes from applying the Reidemeister-Schreier process to the relation σ_j^2, if we have the following conditions on m_i's:*

(1) if $m_i^1\in\{1,4\}$ then $m_i^2=m_i^1$.

(2) if $m_i^1=2$ and $m_j\in\{1,4\}$ for all $j<i$, then $m_i^2=3$.

(2') if $m_i^1=3$ and $m_j\in\{1,4\}$ for all $j<i$, then $m_i^2=2$.

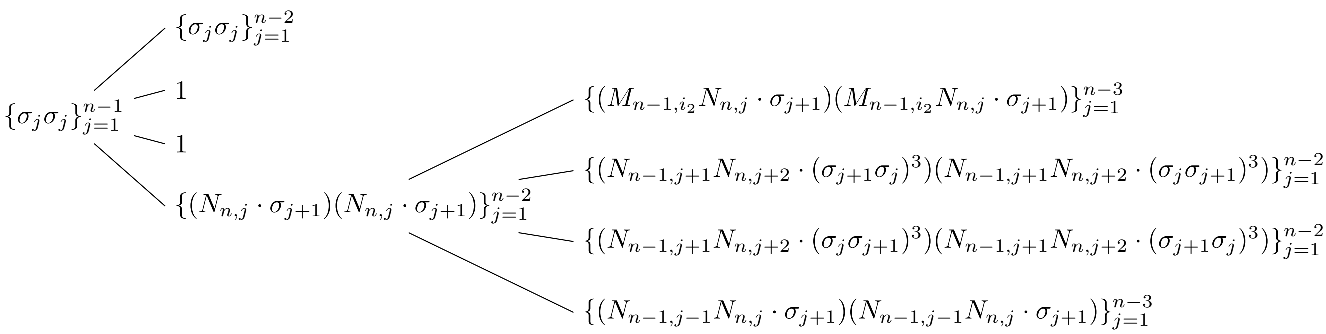


Figure 2.4: Part of the tree of relations of σ_j^2

(3) if $m_i^1 \in \{2,3\}$ and exists $j < i$ such that $m_j^1 \in \{2,3\}$, then $m_i^2 = m_i^1$.

Proof. We know abstractly how it works the Reisemeister Schreier process on relations (illustrated with examples in tables of proposition 2.1.1 and proposition 2.1.2). Primarily, we need to know the representatives on the Schreier transversal set and it is easy by the permutation induced as in 2.1.8 and 2.1.9. The rules follow by the conditions $i + j \sim n - l$ for some l and $\sim$ could be $>, <, =$. □

Let x_j generators of type $(m_1, \ldots, m_l)$ such that $m_i \in \{1, 4\}$ for all i. By theorem 2.1.14 and condition (1), the relation $(m_1, \ldots, m_l)|(m_1, \ldots, m_l)$ comes from applying the Reidemeister-Schreier process which means the relation $x_j \,|\, x_j$. Therefore, $(x_j)^2 = 1$ for all j, this is what we need in the proof of proposition 2.1.10. Furthermore, proposition 2.1.10 is a direct consequence of theorem 2.1.14. By (2) the relation $(m_1, \ldots, m_{k-1}, 2, m_{k+1}, \ldots, m_l)|(m_1, \ldots, m_{k-1}, 3, m_{k+1}, \ldots, m_l)$ comes from applying the Reidemeister-Schreier process, the corresponding generators to the tuples, are inverses between them. For instance, in figure 2.5 we have the relation $(4, 2)|(4, 3)$ corresponding to the relation in generators:

$$(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3)(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_j\sigma_{j+1})^3) = 1$$

On the other hand, applying the Reidmeister-Schreier process to the relation $(\sigma_j\sigma_k)^2$, it's a bit more complicated, because we have two indices. The idea is the same, we apply RS^l to each letter, first we fix the index j and we obtain 4 outputs for letters which depend on k. Then, by the initial condition $k > j + 1$, k it's fixed and we obtain 4 outputs which depend on j. In figure 2.6 we have the relations for the group $D_n^{(1)}$ as we can see in proposition 2.1.1

Theorem 2.1.15. *Let $(m_1^1, \ldots, m_l^1)$, $(m_1^2, \ldots, m_l^2)$, $(m_1^3, \ldots, m_l^3)$ and $(m_1^4, \ldots, m_l^4)$ four types of generators. We have conditions on m_i's to determine, when a relation*

$$(m_1^1, \ldots, m_l^1)|(m_1^2, \ldots, m_l^2)|(m_1^3, \ldots, m_l^3)|(m_1^4, \ldots, m_l^4)$$

comes from applying the Reidemeister-Schreier process to the relation $(\sigma_j\sigma_k)^2$.

(1) if $m_i^1 = 1$, then $m_i^3 = m_i^1$ and $m_i^2 \in \{1, 2, 3, 4\}$. If $m_i^2 \in \{1, 4\}$, then $m_i^4 = m_i^2$ and

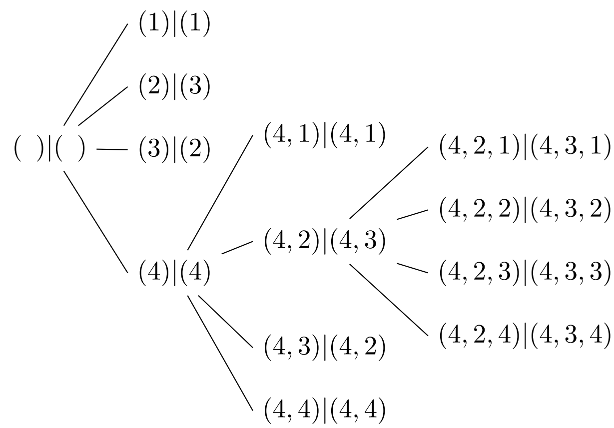


Figure 2.5: The tree of 2.4 embedded in a bigger tree but in terms of tuples

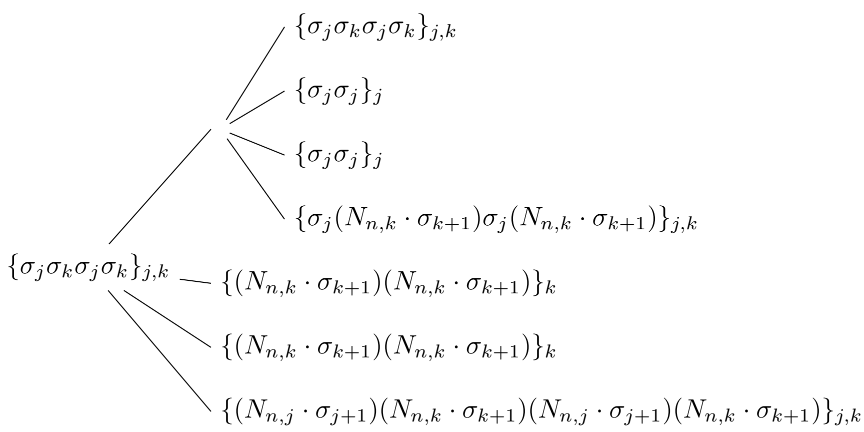


Figure 2.6: Part of the tree of relations of $(\sigma_j\sigma_k)^2$

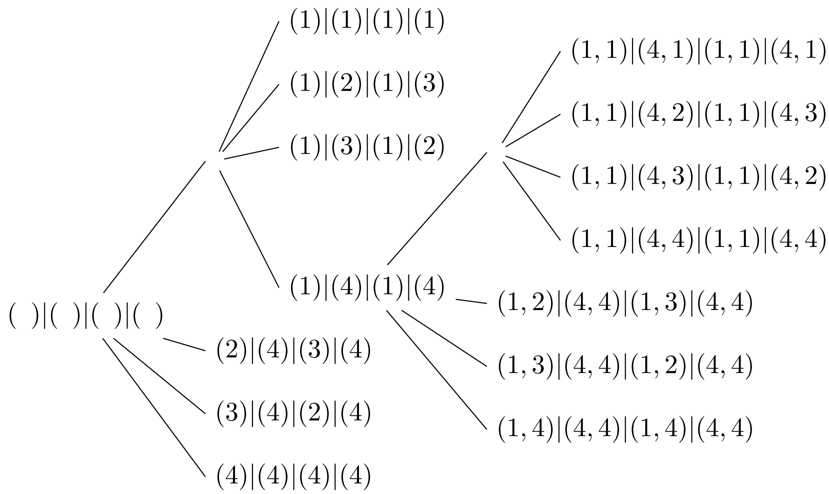


Figure 2.7: The tree of figure 2.6 embedded in a bigger tree but in terms of tuples

- *(1a) if $m_i^2 = 2$ and $m_j^2 \in \{1,4\}$ for all $j < i$, then $m_i^4 = 3$; or viceversa, if $m_i^2 = 3$ and $m_j^2 \in \{1,4\}$ for all $j < i$, then $m_i^4 = 2$.*
- *(1b) if $m_i^2 \in \{2,3\}$ and exists $j < i$ such that $m_j \in \{2,3\}$, then $m_i^4 = m_i^2$.*

(2) If $m_i^1 \in \{2,3,4\}$, then $m_i^2 = m_i^4 = 4$ and

- *(2a) if $m_i^1 = 2$ and $m_j^1 \in \{1,4\}$ for all $j < i$, then $m_i^3 = 3$; or viceversa, if $m_i^1 = 3$ and $m_j^1 \in \{1,4\}$ for all $j < i$, then $m_i^3 = 2$.*
- *(2b) if $m_i^1 \in \{2,3\}$ and exists $j < i$ such that $m_j^1 \in \{2,3\}$, then $m_i^3 = m_i^1$.*
- *(2c) if $m_i^1 = 4$, then $m_i^3 = m_i^1$.*

Proof. The proof is analogous as in theorem 2.1.14 following the algorithm of Reidemeister-Schreier process with the help of proposition 2.1.8 of induced permutations and proposition 2.1.9 of conditions to be in PP_n. □

Remark 2.1.16. When we have a relation of type

$$(m_1^1, \ldots, m_{l-1}^1, 1)|(m_1^2, \ldots, m_{l-1}^2, 2)|(m_1^3, \ldots, m_{l-1}^3, 1)|(m_1^4, \ldots, m_{l-1}^4, 3)$$

we need to be careful with some indices. If $(m_1^2, \ldots, m_{l-1}^2)$ is a type of a family of generators which is not in PP_n (see proposition 2.1.9), then

$$(\ldots, 1)|(\ldots, 2)|(\ldots, 1)|(\ldots, 3)$$

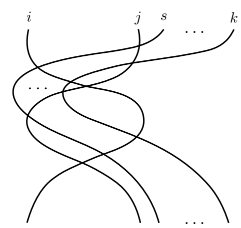


Figure 2.8: Corresponding planar pure braid for $P = (\dots k \dots s \dots)[i,j](\dots)$

$$\downarrow$$

$$M_{n-l+1,i_l}\square_j M^{-1}_{n-l+1,i_l}|N_{n-l+1,k+1}\square_k N^{-1}_{n-l+1,k}|M_{n-l+1,i_l+1}\square_j M^{-1}_{n-l+1,i_l+1}|N_{n-l+1,k}\square_k N^{-1}_{n-l+1,k+1}$$

and because of conditions, it means that $i_l + j < n - l$ and $i_l + k = n - l$, then $M_{n-l+1,i_l} = N_{n-l+1,k+1}$ and $M_{n-l+1,i_l+1} = N_{n-l+1,k}$. If $(m^2_1, \dots, m^2_{l-1})$ is a type of a family which is already in PP_n, then

$$(\dots,1)|(\dots,2)|(\dots,1)|(\dots,2)$$

$$\downarrow$$

$$M_{n-l+1,i_l}\square_j M^{-1}_{n-l+1,i_l}|N_{n-l+1,k+1}\square_k N^{-1}_{n-l+1,k+1}|M_{n-l+1,i_l}\square_j M^{-1}_{n-l+1,i_l}|N_{n-l+1,k+1}\square_k N^{-1}_{n-l+1,k+1}$$

and also by conditions, we have that $M_{n-l+1,i_l} = N_{n-l+1,k+1}$. Therefore, the generators of type $(m^1_1, \dots, m^1_{l-1}, 1)$ and $(m^3_1, \dots, m^3_{l-1}, 1)$ which were only indexed by j, now also are indexed by k. All these also happens if we exchange by 3's where appear 2's.

2.2 PP_n for $n = 3, 4, 5, 6$

Now, what we have is a way to obtain a presentation of PP_n. By theorem 2.1.11 we know which tuples to consider for generators. By theorems 2.1.14 and 2.1.15 we know which combinations of tuples to consider for relations. In particular, theorem 2.1.14 identify tuples of generators which are inverses between them and are exactly as in proposition 2.1.10, then at the moment we consider tuples of generators, we can omit their inverses. Even so, the presentation in terms of tuples has redundancy, there are relations which kill some generators. As an application of all these, we can calculate the presentation for the case of PP_6. The tuples associated to the generators of PP_6 are in table 2.3, and relations are in tables 2.4 and 2.5. Simplifying we can obtain an explicit presentation of PP_6 with generators and relations listed in 2.3.2 and 2.3.3. Unfortunately, it is not written in an easy way. For that, we resort to basic partitioner posets of degree 1 (see definition 1.4.10). For each basic partitioner poset $P = (K)[ij](K')$, corresponds a planar pure braid as follows.

Construction 1. If $P = (K)[i,j](K')$ is a basic partitioner poset, let $S := \{s \in K | s > i, j\}$, hence $k \in S$ where $k := \max\{i' \in K \sqcup \{i,j\}\}$. We define the planar pure braid such that the ith and jth

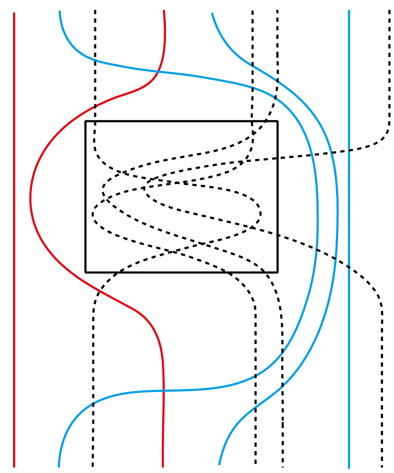


Figure 2.9: Strands corresponding to K and K' of $P = (K)[i,j](K')$ by construction 1

strands cross like in the second Reidemeister move (see figure 3.2), and the sth strand is "linked" with the ith and jth strands as in the figure 2.8, for all $s \in S$. We call it the ijS configuration, an we identify it in figure 2.9 as the tangle inside the square. The other strands are placed by the partitioner poset as follows:

(a) if $l \in K'$, the lth strand goes around the ijS configuration through the right satisfying condition (c) (strands in blue in figure 2.9);

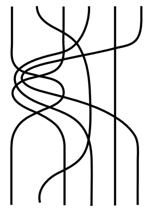
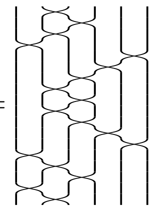
(b) if $l \in K \setminus S$, the lth strand goes around the ijS configuration through the left satisfying condition (c) (strands in red figure 2.9);

(c) we parametrize strands from the bottom to the top. Let $s \in \{i,j\} \cup S$. Going along the sth strand, intersection points with other strands are ordered as follows:

   1. points with lth strands for $l \in K'$ and $l < s$. An lth strand intersects first than the l'th strand if $l > l'$.
   2. points with lth strands for $l \in K \setminus S$ and $l > s$. An lth strand intersects first than the l'th strand if $l < l'$.
   3. points with lth strands in the ijS configuration.
   4. points with lth strands for $l \in K \setminus S$ and $l > s$. An lth strand intersects first than the l'th strand if $l > l'$.
   5. points with lth strands for $l \in K'$ and $l < s$. An lth strand intersects first than the l'th strand if $l < l'$.

We denote this planar pure braid by b_P.

For instance, the basic partitioner poset (46)[13](25) corresponds to the generator $F_{84} = \sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$. See the next planar pure braid of the construction compared with F_{84} in the list of generators in table 2.12.

$$(46)[13](25) \quad \longmapsto \quad b_{(46)[13](25)} =$$

$$=$$

$$= \sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$$

With this construction, we can give a 1-1 correspondence by hand with basic partitioner posets and the generators of PP_n for $3 \leq n \leq 6$. Now, we can express the relations in terms of basic partitioner posets. For instance, the resulting relation 11 from the Reidemeister-Schreier process in 2.3.3 is a conjugacy relation of F_{37}

$$F_{20}^{-1}F_{21}^{-1}F_4^{-1}F_{37}F_4F_{21}F_{20}F_{35}^{-1}F_8^{-1}F_{10}^{-1}F_{37}^{-1}F_{10}F_8F_{35},$$

and in terms of basic partitioner posets, $F_{37} = b_{(6)[12](345)}$, then

$$\begin{aligned} &b^{-1}_{(1256)[34]}b^{-1}_{(126)[35](4)}b^{-1}_{(1236)[45]}b_{(6)[12](345)}b_{(1236)[45]}b_{(126)[35](4)}b_{(1256)[34]}\cdots \\ &\qquad\cdots b^{-1}_{(126)[45](3)}b^{-1}_{(1246)[35]}b^{-1}_{(126)[34](5)}b^{-1}_{(6)[12](345)}b_{(126)[34](5)}b_{(1246)[35]}b_{(126)[45](3)} \end{aligned} \tag{2.2.1}$$

What we have to realize is that every relation in 2.3.3, is conjugacy relation of planar pure braids which come from a basic partitioner poset of the form $P = (k)[i,j](L)$. An easy count say that are $\binom{6}{3} = 20$ basic partitioner posets of this type, which coincides with the 20 relations we have. Denote by $\mathcal{P}_{rel}$ the set of all posets of this form. If $P = (k)[i,j](L) \in \mathcal{P}_{rel}$, the set L consists of three elements, and we can construct a sequence of partitioner posets from L. Denote by $\mathcal{P}$ the set of partitioner posets $P = (K)[i,j](K')$ (not necessarily basic). We define the function

$$\begin{array}{cccccccc} \nu_1: & \mathcal{P}_{rel} & \longrightarrow & \mathcal{P} & \times & \mathcal{P} & \times & \mathcal{P} \\ & (k)[ij](L) & \longmapsto & ((\ldots c \ldots)[ab] & , & (\ldots)[ac](b) & , & (\ldots a \ldots)[bc]) \end{array} \tag{2.2.2}$$

where $L = \{a < b < c\}$, and the dots are understood as the remaining elements, i.e., i, j, and k are always in the first ( )-block.

Example 2.2.1.

$$\nu_1: \quad (5)[24](136) \quad \longmapsto \quad ((2456)[13] \quad , \quad (245)[16](3) \quad , \quad (1245)[36])$$

Note 2.2.2. Partitioner posets generated by ν_1 are not necessarily basic, in the last example, is clear that (245)[16](3) and (1245)[36] are not basic.

Now we can define a product of planar pure braids from an element $P \in \mathcal{P}_{rel}$ in a very naive way. Let $\beta : \mathcal{P} \to PP_6$ defined by

$$\beta(P) = \begin{cases} b_P^{-1} & \text{if } P \text{ is a basic partitioner poset} \\ 1 & \text{otherwise.} \end{cases}$$

Then we can define $\widetilde{\nu_1}$ as the composition of

$$\begin{array}{ccccc} \mathcal{P}_{rel} & \xrightarrow{\nu_1} & \mathcal{P} \times \mathcal{P} \times \mathcal{P} & \xrightarrow{(\beta^1,\beta^2,\beta^3)} & PP_6 \\ P & \longmapsto & (P_1, P_2, P_3) & \longmapsto & \beta(P_1)\beta(P_2)\beta(P_3) \end{array}$$

Example 2.2.3. Let $P = (6)[12](345)$, the associated basic partitioner poset to the generator F_{37}.

$$\widetilde{\nu_1} : \ (6)[12](345) \longmapsto ((1256)[34], (126)[35](4), (1236)[45]) \longmapsto b^{-1}_{(1256)[34]} b^{-1}_{(126)[35](4)} b^{-1}_{(1236)[45]}$$

Hence, the first line in the 11th conjugacy relation 2.2.1 can be rewrite as

$$\begin{aligned} & b^{-1}_{(1256)[34]} b^{-1}_{(126)[35](4)} b^{-1}_{(1236)[45]} b_{(6)[12](345)} b_{(1236)[45]} b_{(126)[35](4)} b_{(1256)[34]} = \\ & \qquad \widetilde{\nu_1}((6)[12](345)) b_{(6)[12](345)} \widetilde{\nu_1}((6)[12](345))^{-1} = \widetilde{\nu_1}((6)[12](345)) \cdot b_{(6)[12](345)} \end{aligned} \tag{2.2.3}$$

where $\cdot$ is the action by conjugation $b \cdot a = bab^{-1}$.

Remark 2.2.4. In the definition of $\beta : \mathcal{P} \to PP_6$, we take inverses b_P^{-1} when P is a basic partitioner poset, in order to use the same notation $\cdot$ for the action by conjugation that we have used along this chapter.

In the same way, we can define another function

$$\begin{array}{cccccccc} \nu_2 : & \mathcal{P}_{rel} & \longrightarrow & \mathcal{P} & \times & \mathcal{P} & \times & \mathcal{P} \\ & (k)[ij](L) & \longmapsto & ((\dots)[bc](a) & , & (\dots b \dots)[ac] & , & (\dots)[ab](c)) \end{array} \tag{2.2.4}$$

where $L = \{a < b < c\}$, and the dots are understood as the remaining elements, i.e., i, j, and k are always in the first ( )-block.

Example 2.2.5.

$$\nu_2 : \ (5)[24](136) \longmapsto ((245)[36](1) \ , \ (2345)[16] \ , \ (245)[13](6))$$

Analogy, we can define a product of planar pure braids for any $P \in \mathcal{P}_{rel}$. Let $\widetilde{\nu_2}$ be the composition of

$$\begin{array}{ccccc} \mathcal{P}_{rel} & \xrightarrow{\nu_2} & \mathcal{P} \times \mathcal{P} \times \mathcal{P} & \xrightarrow{(\beta^1,\beta^2,\beta^3)} & PP_6 \\ P & \longmapsto & (P_1, P_2, P_3) & \longmapsto & \beta(P_1)\beta(P_2)\beta(P_3) \end{array}$$

Example 2.2.6. Let $P = (6)[12](345)$ as in example 2.2.3.

$$\widetilde{\nu_2}: \ (6)[12](345) \ \longmapsto \ ((126)[45](3), (1246)[35], (126)[34](5)) \ \longmapsto \ b^{-1}_{(126)[45](3)} b^{-1}_{(1246)[35]} b^{-1}_{(126)[34](5)}$$

Hence, the second line in the 11th conjugacy relation 2.2.1 can be rewrite as

$$b^{-1}_{(126)[45](3)} b^{-1}_{(1246)[35]} b^{-1}_{(126)[34](5)} b^{-1}_{(6)[12](345)} b_{(126)[34](5)} b_{(1246)[35]} b_{(126)[45](3)} = \widetilde{\nu_2}((6)[12](345)) b^{-1}_{(6)[12](345)} \widetilde{\nu_2}((6)[12](345))^{-1} = (\widetilde{\nu_2}((6)[12](345)) \cdot b_{(6)[12](345)})^{-1} \quad (2.2.5)$$

At last, by 2.2.3 and 2.2.5, we can rewrite all the relation 2.2.1 as

$$(\widetilde{\nu_1}((6)[12](345)) \cdot b_{(6)[12](345)})(\widetilde{\nu_2}((6)[12](345)) \cdot b_{(6)[12](345)})^{-1}$$

This happens for all the relations 2.3.3 of PP_6. As a consequence of this construction, we have a simplified presentation of some planar pure braid groups.

Theorem 2.2.7. *For $n = 1, 2$, PP_n is the trivial group. For $3 \leq n \leq 6$, the generators of PP_n are planar pure braids associated (by construction 1) to basic partitioner posets of $[n]$. The presentation of PP_n for $3 \leq n \leq 5$ is*

$$\begin{aligned} \textit{generators}: &\quad \{b_P \,|\, P \textit{ is a basic partitioner poset of } [n]\} \\ \textit{relations}: &\quad \{ \textit{ no relations } \} \end{aligned}$$

i.e., are free groups. For $n = 6$, the relations are by conjugation on some particular partitioner posets. Let P_{rel} a basic partitioner poset of $[n]$ of the form $(k)[i,j](L)$ (hence $k > i, j$ and $|L| = 3$). The presentation of PP_n ($n = 6$) is

$$\begin{aligned} \textit{generators}: &\quad \{b_P \,|\, P \textit{ is a basic partitioner poset of } [n]\} \\ \textit{relations}: &\quad \{(\widetilde{\nu_1}(P_{rel}) \cdot b_{P_{rel}})((\widetilde{\nu_2}(P_{rel}) \cdot b_{P_{rel}}))^{-1} \quad | \quad P_{rel} = (k)[i,j](L)\} \end{aligned}$$

where $\cdot$ is the action by conjugation and $\widetilde{\nu_i}(P_{rel})$ is a product of planar pure braids constructed from L for $i = 1, 2$.

2.3 PP_n as a semidirect product

Similarly to the classical pure braid groups, we can define a forgetting homomorphism $\rho_i^n : PP_n \to PP_{n-1}$, which omits the ith strand, producing a pure planar braid in $n-1$ strands. There is a natural inclusion of PP_n into PP_{n+1} adding a vertical strand. Here, we distinguish where we add a vertical strand. We denote by $\iota_i^n : PP_{n-1} \to PP_n$ as the homomorphism which adds a vertical strand in the middle of the $(i-1)$th strand and the ith strand. To simplify notation, we omit the index n in ρ_i^n and ι_i^n. For each n and i we have a splitting S.E.S.

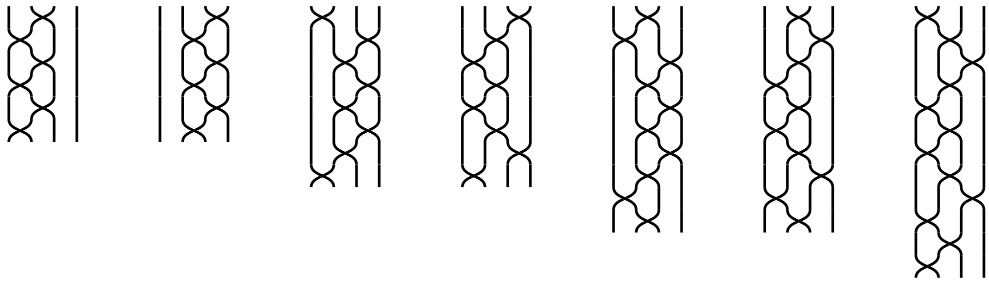


Figure 2.10: The 7 generators of PP_4

$$1 \longrightarrow K_{n,i} \hookrightarrow PP_n \underset{\rho_i}{\overset{\iota_i}{\rightleftarrows}} PP_{n-1} \longrightarrow 1$$

where $K_{n,i} = \ker \rho_i$. Therefore $PP_n = K_{n,i} \rtimes PP_{n-1}$ with the action of PP_{n-1} on $K_{n,i}$ by conjugation. If $i = n$ for each n, this can be repeated so that the planar pure braid group PP_n is an iterated semidirect product of groups.

$$PP_n = K_{n,n} \rtimes (K_{n-1,n-1} \rtimes (\cdots \rtimes (K_{5,5} \rtimes (K_{4,4} \rtimes PP_3) \cdots).$$

but it's hard to identify $K_{n,n}$ in general, unlike in the usual case for pure braids where $K_{n,n}$ is a free group in $n-1$ generators. We computed $K_{4,4}$ with the Reidemeister-Schreier process obtaining also a free group, but in contrast with the usual case, $K_{4,4}$ is free on a countable set of generators. Firstly, we write explicitly the generators of PP_4 as in 2.1.4 but replacing the generator $x = (\sigma_2\sigma_1\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$ by the product $xy^{-1} = \sigma_2\sigma_1(\sigma_3\sigma_2)^3\sigma_1\sigma_2$ where $y = (\sigma_2\sigma_1)^2$.

$$PP_4 = \left\langle \begin{array}{ccc} (\sigma_2\sigma_1)^3, & \sigma_3(\sigma_2\sigma_1)^3\sigma_3, & (\sigma_3\sigma_2)^3, \\ & \sigma_2\sigma_3(\sigma_2\sigma_1)^3\sigma_3\sigma_2, & \sigma_1(\sigma_3\sigma_2)^3\sigma_1, \\ & \sigma_1\sigma_2\sigma_3(\sigma_2\sigma_1)^3\sigma_3\sigma_2\sigma_1, & \sigma_2\sigma_1(\sigma_3\sigma_2)^3\sigma_1\sigma_2 \end{array} \right\rangle \tag{2.3.1}$$

If we forget the last strand in the generators of PP_4 (figure 2.10), we note that the only non trivial image comes from the generator $(\sigma_2\sigma_1)^3$ of PP_3 embedded in PP_4 adding a vertical strand in the 4th position. We have the S.E.S

$$1 \longrightarrow K_{4,4} \hookrightarrow PP_4 \underset{\rho_4}{\overset{\iota_4}{\rightleftarrows}} PP_3 \longrightarrow 1$$

where

$$\rho_4(x) = \begin{cases} x & \text{if } x = (\sigma_2\sigma_1)^3, \\ 1 & \text{otherwise.} \end{cases}$$

Let X be the set of generators of PP_4 and $b_0 = (\sigma_2\sigma_1)^3 \in X$. Applying the R-S process we obtain that $K_{4,4} = \langle b_0^k x b_0^{-k} | k \in \mathbb{Z}, x \in X \setminus \{b_0\} \rangle$. Recall that the action of PP_3 on $K_{4,4}$ is by conjugation and $\iota_4(b_0) = b_0$, i.e., $b_0 \cdot (b_0^k x b_0^{-k}) = b_0^{k+1} x b_0^{-k-1}$. Therefore $PP_4 = K_{4,4} \rtimes PP_3$ but in contrast with pure braids, it is not an almost-direct product.

Definition 2.3.1 (Almost-Direct)**.** A semi-direct product $A \rtimes B$ is almost-direct if the action of B on the abelianization of A is trivial.

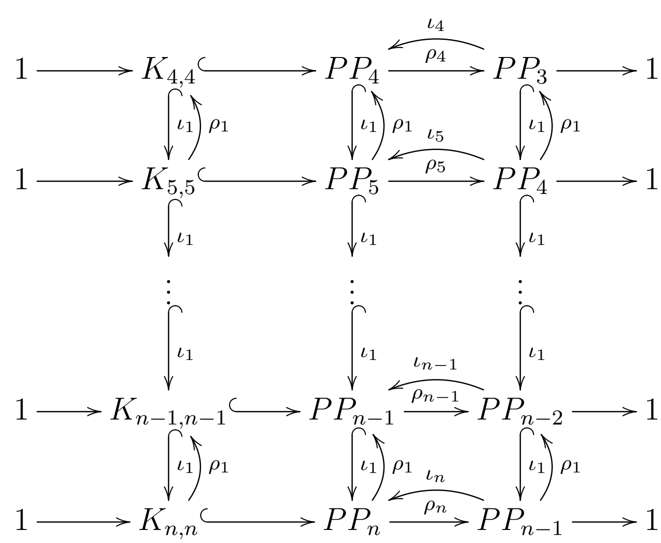
In general, almost-direct products are well-behaved taking lower central series, dimension series and augmentation ideals among other things. With the help of this fact, was calculated the Malcev Lie algebra and Hopf algebra for usual pure braids[23]. We can not do the same with planar braids, because $K_{n,n} \rtimes PP_{n-1}$ is not almost direct. For that we analyse the little case $K_{4,4} \rtimes PP_3$. The abelianization of the group $K_{4,4}$ is the free abelian group on generators $b_0^k x b_0^{-k}$, with $k \in \mathbb{Z}, x \in X \setminus \{b_0\}$, i.e., is the group $K_{4,4}$ written with sums instead of products. The action of PP_3 on $K_{4,4}$ is given by $b_0 \cdot (b_0^k x b_0^{-k}) = b_0^{k+1} x b_0^{-k-1}$. If it were trivial in the abelianization, it means

$$b_0^{k+1} x b_0^{-k-1} = b_0^k x b_0^{-k} \Longrightarrow b_0 x b_0^{-1} = 1$$

which clearly $b_0 x b_0^{-1}$ is not trivial. Therefore, $K_{4,4} \rtimes PP_3$ is not almost direct. In general we have the same.

Proposition 2.3.2. *PP_n is a semidirect product $K_{n,n} \rtimes PP_{n-1}$, which is not almost-direct for all n.*

Proof. Taking the forgetting homomorphism in the first and last strand, and their respective sections, we have the following commutative diagram with exact rows.



The homomorphisms in the left column are understood as the well-restriction of the middle ones to the kernels. It is an easy exercise to check the commutativity of the diagram. By the left column, we have the chain

$$K_{4,4} \underset{\rho_1}{\overset{\iota_1}{\rightleftarrows}} K_{5,5} \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_1} K_{n-1,n-1} \underset{\rho_1}{\overset{\iota_1}{\rightleftarrows}} K_{n,n}$$

where $\rho_1 \circ \iota_1 = \mathrm{id}_{K_{*,*}}$. The abelianization functor, not always preserves monomorphisms, but in our case we have that $\mathrm{Ab}(\rho_1) \circ \mathrm{Ab}(\iota_1) = \mathrm{id}_{\mathrm{Ab}(K_{*,*})}$ by naturality; therefore, $\mathrm{Ab}(\iota_1)$ is a monomorphism. Then

$$\mathrm{Ab}(K_{4,4}) \hookrightarrow \mathrm{Ab}(K_{5,5}) \hookrightarrow \cdots \hookrightarrow \mathrm{Ab}(K_{n-1,n-1}) \hookrightarrow \mathrm{Ab}(K_{n,n}).$$

Then $K_{n,n} \rtimes PP_{n-1}$ is not almost direct because of $\mathrm{Ab}(K_{4,4}) \hookrightarrow \mathrm{Ab}(K_{n,n})$ and $PP_3 \hookrightarrow PP_{n-1}$. $\square$

Remark 2.3.3. Other important property of these groups pointed and proved by Jacob Mostovoy is that planar pure braid groups are residually nilpotent [30]. This property will be important when we define Vassiliev invariants and their relation with Chen's theory of iterated integrals in chapter 4. As a consequence, Vassiliev invariants classify planar pure braids. The idea to prove it is using chord diagrams of planar pure braids to embeds the group PP_n in a residually nilpotent group G which elements consist of formal power series in chord diagrams with non-zero constant term.

Table 2.3: All the generators of PP_6

Type of generator	Size of the family	Type of generator	Size of the family
(1,1,1,4,2)	1	(4,2,2,2,4)	1
(1,1,4,4,2)	1	(4,2,2,3,2)	1
(1,4,1,4,2)	1	(4,2,2,3,3)	2
(1,4,4,4,2)	1	(4,2,2,3,4)	2
(4,1,1,4,2)	1	(4,2,2,4,2)	1
(4,1,4,4,2)	1	(4,2,2,4,3)	2
(4,4,1,4,2)	1	(4,2,2,4,4)	1
(4,4,4,4,2)	1	(4,2,3,1,2)	1
(1,1,4,2,2)	1	(4,2,3,1,3)	1
(1,1,4,2,3)	2	(4,2,3,2,2)	1
(1,1,4,2,4)	1	(4,2,3,2,3)	2
(1,4,4,2,2)	1	(4,2,3,2,4)	1
(1,4,4,2,3)	2	(4,2,3,3,2)	1
(1,4,4,2,4)	1	(4,2,3,3,3)	2
(4,1,4,2,2)	1	(4,2,3,3,4)	2
(4,1,4,2,3)	2	(4,2,3,4,2)	1
(4,1,4,2,4)	1	(4,2,3,4,3)	2
(4,4,4,2,2)	1	(4,2,3,4,4)	2
(4,4,4,2,3)	2	(4,2,4,1,2)	1
(4,4,4,2,4)	1	(4,2,4,1,3)	1
(1,4,2,1,2)	1	(4,2,4,2,2)	1
(1,4,2,1,3)	1	(4,2,4,2,3)	2
(1,4,2,2,2)	1	(4,2,4,2,4)	1
(1,4,2,2,3)	2	(4,2,4,3,2)	1
(1,4,2,2,4)	1	(4,2,4,3,3)	2
(1,4,2,3,2)	1	(4,2,4,3,4)	2
(1,4,2,3,3)	2	(4,2,4,4,2)	1
(1,4,2,3,4)	2	(4,2,4,4,3)	2
(1,4,2,4,2)	1	(4,2,4,4,4)	1
(1,4,2,4,3)	2	(1,1,4,1,2)	1
(1,4,2,4,4)	1	(1,4,4,1,2)	1
(4,4,2,1,2)	2	(4,1,4,1,2)	1
(4,4,2,1,3)	2	(4,4,4,1,2)	1
(4,4,4,2,2)	1	(1,4,1,2,2)	1
(4,4,4,2,3)	2	(1,4,1,2,3)	2
(4,4,4,2,4)	1	(1,4,1,2,4)	1
(4,4,2,3,2)	1	(4,4,1,2,2)	1
(4,4,2,3,3)	2	(4,4,1,2,3)	2
(4,4,2,3,4)	2	(4,4,1,2,4)	1
(4,4,2,4,2)	1	(4,1,2,1,2)	1
(4,4,2,4,3)	2	(4,1,2,1,3)	1
(4,4,2,4,4)	1	(4,1,2,2,2)	1
(4,2,1,1,2)	1	(4,1,2,2,3)	2
(4,2,1,1,3)	1	(4,1,2,2,4)	1
(4,2,1,2,2)	1	(4,1,2,3,2)	1
(4,2,1,2,3)	2	(4,1,2,3,3)	2
(4,2,1,2,4)	1	(4,1,2,3,4)	2
(4,2,1,3,2)	1	(4,1,2,4,2)	1
(4,2,1,3,3)	2	(4,1,2,4,3)	2
(4,2,1,3,4)	1	(4,1,2,4,4)	1
(4,2,1,4,2)	1	(1,4,1,1,2)	1
4(2,1,4,3,)	1	(4,4,1,1,2)	1
(4,2,2,1,2)	1	(4,1,1,2,2)	1
(4,2,2,1,3)	1	(4,1,1,2,3)	2
(4,2,2,2,2)	1	(4,1,1,2,4)	1
(4,2,2,2,3)	2	(4,1,1,1,2)	1

Table 2.4: All the relations of PP_6 (Part 1)

Relation of type $(m_1^1,\ldots,m_5^1)\|(m_1^2,\ldots,m_5^2)\|(m_1^3,\ldots,m_5^3)\|(m_1^4,\ldots,m_5^4)$	Conditions on indices j,k
$(1,1,1,4,2)\|(4,2,3,4,4)\|(1,1,1,4,3)\|(4,3,3,4,4)$	$k=2$
$(4,1,1,4,2)\|(4,2,3,4,4)\|(4,1,1,4,3)\|(4,3,3,4,4)$	$k=2$
$(1,1,4,2,2)\|(4,2,4,4,4)\|(1,1,4,3,2)\|(4,3,4,4,4)$	$\emptyset$
$(1,1,4,2,3)\|(4,2,4,4,4)\|(1,1,4,3,3)\|(4,3,4,4,4)$	$j=1$
$(4,1,4,2,2)\|(4,2,4,4,4)\|(4,1,4,3,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,4,2,3)\|(4,2,4,4,4)\|(4,1,4,3,3)\|(4,3,4,4,4)$	$j=1$
$(1,4,2,1,2)\|(1,4,4,2,4)\|(1,4,3,1,2)\|(1,4,4,3,4)$	$\emptyset$
$(1,4,2,1,2)\|(4,4,4,2,4)\|(1,4,3,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(1,4,2,1,3)\|(1,4,4,2,4)\|(1,4,3,1,3)\|(1,4,4,3,4)$	$\emptyset$
$(1,4,2,1,3)\|(4,4,4,2,4)\|(1,4,3,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,4,2,1,2)\|(4,4,4,2,4)\|(4,4,3,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(4,4,2,1,3)\|(4,4,4,2,4)\|(4,4,3,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,1,1,2)\|(4,4,1,2,4)\|(4,3,1,1,2)\|(4,4,1,3,4)$	$\emptyset$
$(4,2,1,1,2)\|(4,4,2,2,4)\|(4,3,1,1,2)\|(4,4,3,2,4)$	$\emptyset$
$(4,2,1,1,2)\|(4,4,2,3,4)\|(4,3,1,1,2)\|(4,4,3,3,4)$	$\emptyset$
$(4,2,1,1,2)\|(4,4,2,4,4)\|(4,3,1,1,2)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,1,2)\|(4,4,4,2,4)\|(4,3,1,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,1,1,3)\|(4,4,1,2,4)\|(4,3,1,1,3)\|(4,4,1,3,4)$	$\emptyset$
$(4,2,1,1,3)\|(4,4,2,2,4)\|(4,3,1,1,3)\|(4,4,3,2,4)$	$\emptyset$
$(4,2,1,1,3)\|(4,4,2,3,4)\|(4,3,1,1,3)\|(4,4,3,3,4)$	$\emptyset$
$(4,2,1,1,3)\|(4,4,2,4,4)\|(4,3,1,1,3)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,1,3)\|(4,4,4,2,4)\|(4,3,1,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,1,2,2)\|(4,4,2,4,4)\|(4,2,1,2,2)\|(4,4,2,4,4)$	$\emptyset$
$(4,2,1,2,3)\|(4,4,2,4,4)\|(4,3,1,2,3)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,2,4)\|(4,4,2,4,4)\|(4,2,1,2,4)\|(4,4,2,4,4)$	$\emptyset$
$(4,2,1,3,2)\|(4,4,2,4,4)\|(4,3,1,3,2)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,3,3)\|(4,4,2,4,4)\|(4,3,1,3,3)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,3,4)\|(4,4,2,4,4)\|(4,3,1,3,4)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,4,2)\|(4,4,2,4,4)\|(4,3,1,4,2)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,1,4,3)\|(4,4,2,4,4)\|(4,3,1,4,3)\|(4,4,3,4,4)$	$\emptyset$
$(4,2,2,1,2)\|(4,4,4,2,4)\|(4,3,2,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,2,1,3)\|(4,4,4,2,4)\|(4,3,2,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,3,1,2)\|(4,4,4,2,4)\|(4,3,3,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,3,1,3)\|(4,4,4,2,4)\|(4,3,3,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,2,4,1,2)\|(4,4,4,2,4)\|(4,3,4,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(1,1,4,1,2)\|(4,2,4,3,4)\|(1,1,4,1,3)\|(4,3,4,3,4)$	$k=2$
$(4,1,4,1,2)\|(4,2,4,3,4)\|(4,1,4,1,3)\|(4,3,4,3,4)$	$k=2$
$(1,4,1,2,2)\|(1,4,2,4,4)\|(1,4,1,3,2)\|(1,4,3,4,4)$	$\emptyset$
$(1,4,1,2,2)\|(4,4,2,4,4)\|(1,4,1,3,2)\|(4,4,3,4,4)$	$\emptyset$
$(1,4,1,2,3)\|(1,4,2,4,4)\|(1,4,1,3,3)\|(1,4,3,4,4)$	$j=1$
$(1,4,1,2,3)\|(4,4,2,4,4)\|(1,4,1,3,3)\|(4,4,3,4,4)$	$j=1$
$(4,4,1,2,2)\|(4,4,2,4,4)\|(4,4,1,3,2)\|(4,4,3,4,4)$	$\emptyset$
$(4,4,1,2,3)\|(4,4,2,4,4)\|(4,4,1,3,3)\|(4,4,3,4,4)$	$j=1$

Table 2.5: All the relations of PP_6 (Part 2)

Relation of type $(m^1_1,\ldots,m^1_5)\|(m^2_1,\ldots,m^2_5)\|(m^3_1,\ldots,m^3_5)\|(m^4_1,\ldots,m^4_5)$	Conditions on indices j,k
$(4,1,2,1,2)\|(4,1,4,2,4)\|(4,1,3,1,2)\|(4,1,4,3,4)$	$\emptyset$
$(4,1,2,1,2)\|(4,2,4,2,4)\|(4,1,3,1,2)\|(4,3,4,2,4)$	$\emptyset$
$(4,1,2,1,2)\|(4,2,4,3,4)\|(4,1,3,1,2)\|(4,3,4,3,4)$	$\emptyset$
$(4,1,2,1,2)\|(4,2,4,4,4)\|(4,1,3,1,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,1,2)\|(4,4,4,2,4)\|(4,1,3,1,2)\|(4,4,4,3,4)$	$\emptyset$
$(4,1,2,1,3)\|(4,1,4,2,4)\|(4,1,3,1,3)\|(4,1,4,3,4)$	$\emptyset$
$(4,1,2,1,3)\|(4,2,4,2,4)\|(4,1,3,1,3)\|(4,3,4,2,4)$	$\emptyset$
$(4,1,2,1,3)\|(4,2,4,3,4)\|(4,1,3,1,3)\|(4,3,4,3,4)$	$\emptyset$
$(4,1,2,1,3)\|(4,2,4,4,4)\|(4,1,3,1,3)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,1,3)\|(4,4,4,2,4)\|(4,1,3,1,3)\|(4,4,4,3,4)$	$\emptyset$
$(4,1,2,2,2)\|(4,2,4,4,4)\|(4,1,3,2,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,2,3)\|(4,2,4,4,4)\|(4,1,3,2,3)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,2,4)\|(4,2,4,4,4)\|(4,1,3,2,4)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,3,2)\|(4,2,4,4,4)\|(4,1,3,3,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,3,3)\|(4,2,4,4,4)\|(4,1,3,3,3)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,3,4)\|(4,2,4,4,4)\|(4,1,3,3,4)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,4,2)\|(4,2,4,4,4)\|(4,1,3,4,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,2,4,3)\|(4,2,4,4,4)\|(4,1,3,4,3)\|(4,3,4,4,4)$	$j=1$
$(1,4,1,1,2)\|(1,4,2,3,4)\|(1,4,1,1,3)\|(1,4,3,3,4)$	$k=2$
$(1,4,1,1,2)\|(4,4,2,3,4)\|(1,4,1,1,3)\|(4,4,3,3,4)$	$k=2$
$(4,4,1,1,2)\|(4,4,2,3,4)\|(4,4,1,1,3)\|(4,4,3,3,4)$	$k=2$
$(4,1,1,2,2)\|(4,1,2,4,4)\|(4,1,1,3,2)\|(4,1,3,4,4)$	$\emptyset$
$(4,1,1,2,2)\|(4,2,2,4,4)\|(4,1,1,3,2)\|(4,3,2,4,4)$	$\emptyset$
$(4,1,1,2,2)\|(4,2,3,4,4)\|(4,1,1,3,2)\|(4,3,3,4,4)$	$\emptyset$
$(4,1,1,2,2)\|(4,2,4,4,4)\|(4,1,1,3,2)\|(4,3,4,4,4)$	$\emptyset$
$(4,1,1,2,2)\|(4,4,2,4,4)\|(4,1,1,3,2)\|(4,4,3,4,4)$	$\emptyset$
$(4,1,1,2,3)\|(4,1,2,4,4)\|(4,1,1,3,3)\|(4,1,3,4,4)$	$j=1$
$(4,1,1,2,3)\|(4,2,2,4,4)\|(4,1,1,3,3)\|(4,3,2,4,4)$	$j=1$
$(4,1,1,2,4)\|(4,2,3,4,4)\|(4,1,1,3,4)\|(4,3,3,4,4)$	$j=1$ & $k=1,2$ or $j=2=k$
$(4,1,1,2,4)\|(4,2,4,4,4)\|(4,1,1,3,4)\|(4,3,4,4,4)$	$j=1$
$(4,1,1,2,4)\|(4,4,2,4,4)\|(4,1,1,3,4)\|(4,4,3,4,4)$	$j=1$
$(4,1,1,2,4)\|(4,2,3,4,4)\|(4,1,1,3,4)\|(4,3,3,4,4)$	$k=2$
$(4,1,1,1,2)\|(4,1,2,3,4)\|(4,1,1,1,3)\|(4,1,3,3,4)$	$k=2$
$(4,1,1,1,2)\|(4,2,2,3,4)\|(4,1,1,1,3)\|(4,3,2,3,4)$	$k=2$
$(4,1,1,1,2)\|(4,2,2,3,4)\|(4,1,1,1,3)\|(4,3,2,3,4)$	$k=2$
$(4,1,1,1,2)\|(4,2,3,3,4)\|(4,1,1,1,3)\|(4,3,3,3,4)$	$k=2$
$(4,1,1,1,2)\|(4,2,3,4,4)\|(4,1,1,1,3)\|(4,3,3,4,4)$	$k=2$
$(4,1,1,1,2)\|(4,2,4,3,4)\|(4,1,1,1,3)\|(4,3,4,3,4)$	$k=2$
$(4,1,1,1,2)\|(4,4,2,3,4)\|(4,1,1,1,3)\|(4,4,3,3,4)$	$k=2$

Generators of PP_6 : (2.3.2)

$F_1 = (\sigma_2\sigma_1)^3$
$F_2 = (\sigma_3\sigma_2)^3$
$F_3 = (\sigma_4\sigma_3)^3$
$F_4 = (\sigma_5\sigma_4)^3$
$F_5 = \sigma_1(\sigma_3\sigma_2)^3\sigma_1$
$F_6 = \sigma_2(\sigma_4\sigma_3)^3\sigma_2$
$F_7 = \sigma_3(\sigma_2\sigma_1)^3\sigma_3$
$F_8 = \sigma_3(\sigma_5\sigma_4)^3\sigma_3$
$F_9 = \sigma_4(\sigma_3\sigma_2)^3\sigma_4$
$F_{10} = \sigma_5(\sigma_4\sigma_3)^3\sigma_5$
$F_{11} = \sigma_1\sigma_2(\sigma_4\sigma_3)^3\sigma_2\sigma_1$
$F_{12} = \sigma_1\sigma_4(\sigma_3\sigma_2)^3\sigma_2\sigma_1$
$F_{13} = (\sigma_2\sigma_1\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$
$F_{14} = \sigma_2\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{15} = \sigma_2\sigma_3 \cdot (\sigma_5\sigma_4)^3$
$F_{16} = \sigma_2\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{17} = (\sigma_3\sigma_2\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2$
$F_{18} = \sigma_3\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{19} = \sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{20} = (\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_3$
$F_{21} = \sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{22} = \sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{23} = \sigma_1\sigma_2\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{24} = \sigma_1\sigma_2\sigma_3 \cdot (\sigma_5\sigma_4)^3$
$F_{25} = \sigma_1\sigma_2\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{26} = \sigma_1 \cdot ((\sigma_3\sigma_2\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{27} = \sigma_1\sigma_3\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{28} = \sigma_1\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{29} = (\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$
$F_{30} = \sigma_2\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_2)$
$F_{31} = \sigma_2\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{32} = \sigma_2 \cdot ((\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_3)$
$F_{33} = \sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{34} = (\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2$
$F_{35} = \sigma_3\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{36} = \sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{37} = \sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{38} = \sigma_1\sigma_2\sigma_3\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{39} = \sigma_1\sigma_2\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{40} = \sigma_1\sigma_2 \cdot ((\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_3)$
$F_{41} = \sigma_1\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{42} = \sigma_1 \cdot ((\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{43} = \sigma_1\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{44} = (\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1$
$F_{45} = \sigma_2\sigma_1\sigma_3\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{46} = \sigma_2\sigma_1\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{47} = \sigma_2\sigma_3\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{48} = \sigma_2\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{49} = \sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{50} = (\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$
$F_{51} = \sigma_3\sigma_2\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{52} = (\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_2)$
$F_{53} = \sigma_3\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{54} = (\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2$
$F_{55} = \sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{56} = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{57} = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{58} = \sigma_1\sigma_2\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{59} = \sigma_1\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{60} = \sigma_1\sigma_3\sigma_2\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{61} = \sigma_1 \cdot ((\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_2)$
$F_{62} = \sigma_1\sigma_3\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{63} = \sigma_1 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{64} = \sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{65} = (\sigma_2\sigma_1\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1$
$F_{66} = \sigma_2\sigma_1\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{67} = \sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{68} = \sigma_2\sigma_3\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{69} = \sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{70} = (\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$
$F_{71} = \sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{72} = \sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{73} = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$

$F_{74} = \sigma_1\sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{75} = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{76} = \sigma_1\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{77} = \sigma_1\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{78} = \sigma_1\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{79} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{80} = (\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot \sigma_5)\sigma_1$
$F_{81} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5(\sigma_4\sigma_3)^3$
$F_{82} = (\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1$
$F_{83} = \sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{84} = \sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{85} = \sigma_2\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{86} = \sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{87} = (\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1$
$F_{88} = \sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{89} = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5 \cdot (\sigma_4\sigma_3)^3$
$F_{90} = \sigma_1\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{91} = \sigma_1\sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{92} = \sigma_1\sigma_2\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{93} = \sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{94} = \sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3(\sigma_2\sigma_1)^3$
$F_{95} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$

$F_{96} = \sigma_2\sigma_1\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{97} = \sigma_2 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{98} = (\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1$
$F_{99} = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{100} = \sigma_1\sigma_2\sigma_1\sigma_3 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_2)$
$F_{101} = \sigma_1\sigma_2 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{102} = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{103} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{104} = \sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{105} = \sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1)$
$F_{106} = \sigma_3\sigma_2 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{107} = \sigma_1\sigma_2 \cdot ((\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot \sigma_4)\sigma_1)$
$F_{108} = \sigma_1\sigma_3\sigma_2 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$
$F_{109} = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4 \cdot (\sigma_3\sigma_2)^3$
$F_{110} = \sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3 \cdot (\sigma_2\sigma_1)^3$
$F_{111} = \sigma_2\sigma_1\sigma_3\sigma_2 \cdot ((\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2 \cdot \sigma_3)\sigma_1)$

Relations of PP_6 **:** (2.3.3)

1. $F_4F_1F_4^{-1}F_1^{-1}$,
2. $F_8F_7F_8^{-1}F_7^{-1}$,
3. $F_{15}F_{14}F_{15}^{-1}F_{14}^{-1}$,
4. $F_{24}F_{23}F_{24}^{-1}F_{23}^{-1}$,
5. $F_{20}^{-1}F_{19}F_{20}F_3^{-1}F_{19}^{-1}F_3$,
6. $F_{32}^{-1}F_{31}F_{32}F_6^{-1}F_{31}^{-1}F_6$,
7. $F_{40}^{-1}F_{39}F_{40}F_{11}^{-1}F_{39}^{-1}F_{11}$,
8. $F_{26}^{-1}F_{60}F_{26}F_{61}^{-1}F_{60}^{-1}F_{61}$,
9. $F_{52}^{-1}F_{51}F_{52}F_{17}^{-1}F_{51}^{-1}F_{17}$,
10. $F_{80}^{-1}F_{79}F_{80}F_{44}^{-1}F_{79}^{-1}F_{44}$,

11. $F_{20}^{-1}F_{21}^{-1}F_4^{-1}F_{37}F_4F_{21}F_{20}F_{35}^{-1}F_8^{-1}F_{10}^{-1}F_{37}^{-1}F_{10}F_8F_{35}$,
12. $F_{40}^{-1}F_{41}^{-1}F_4^{-1}F_{59}F_4F_{41}F_{40}F_{57}^{-1}F_{24}^{-1}F_{25}^{-1}F_{59}^{-1}F_{25}F_{24}F_{57}$,
13. $F_{75}^{-1}F_{24}^{-1}F_{42}^{-1}F_{77}F_{42}F_{24}F_{75}F_{61}^{-1}F_{62}^{-1}F_8^{-1}F_{77}^{-1}F_8F_{62}F_{61}$,
14. $F_{47}^{-1}F_{15}^{-1}F_{16}^{-1}F_{49}F_{16}F_{15}F_{47}F_{32}^{-1}F_{33}^{-1}F_4^{-1}F_{49}^{-1}F_4F_{33}F_{32}$,
15. $F_{85}^{-1}F_{32}^{-1}F_{54}^{-1}F_{88}F_{54}F_{32}F_{85}F_{52}^{-1}F_{72}^{-1}F_{20}^{-1}F_{88}^{-1}F_{20}F_{72}F_{52}$,
16. $F_{61}^{-1}F_{78}^{-1}F_{20}^{-1}F_{94}F_{20}F_{78}F_{61}F_{92}^{-1}F_{40}^{-1}F_{63}^{-1}F_{94}^{-1}F_{63}F_{40}F_{92}$,
17. $F_{80}^{-1}F_{96}^{-1}F_{32}^{-1}F_{104}F_{32}F_{96}F_{80}F_{100}^{-1}F_{40}^{-1}F_{82}^{-1}F_{104}^{-1}F_{82}F_{40}F_{100}$,
18. $F_{68}^{-1}F_{15}^{-1}F_{34}^{-1}F_{71}F_{34}F_{15}F_{68}F_{52}^{-1}F_{53}^{-1}F_8^{-1}F_{71}^{-1}F_8F_{53}F_{52}$,
19. $F_{80}^{-1}F_{81}^{-1}F_{15}^{-1}F_{95}F_{15}F_{81}F_{80}F_{89}^{-1}F_{24}^{-1}F_{65}^{-1}F_{95}^{-1}F_{65}F_{24}F_{89}$,
20. $F_{107}^{-1}F_{61}^{-1}F_{98}^{-1}F_{110}F_{98}F_{61}F_{107}F_{80}^{-1}F_{105}^{-1}F_{52}^{-1}F_{110}^{-1}F_{52}F_{105}F_{80}$,

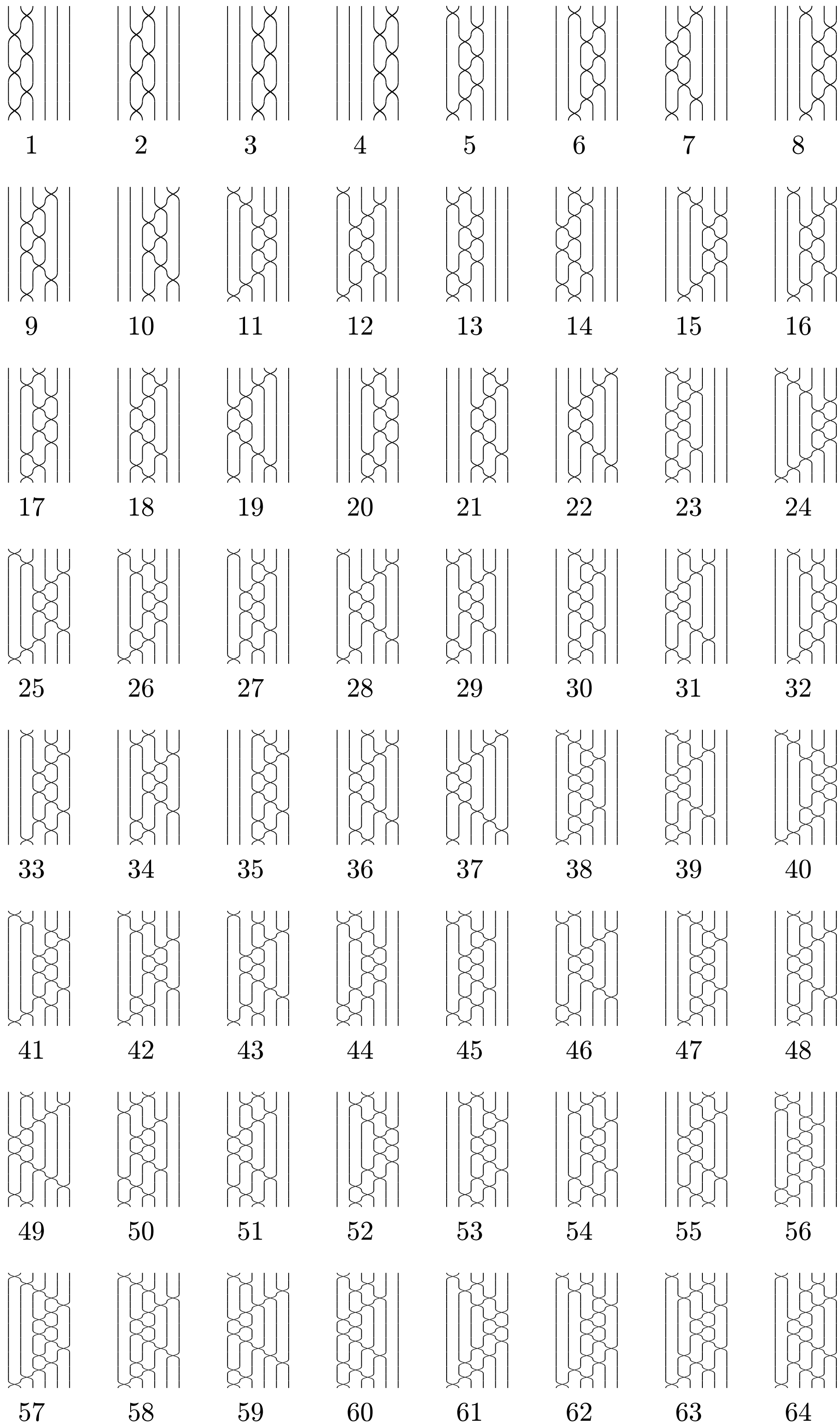


Figure 2.11: List of generators of PP_6 (part 1)

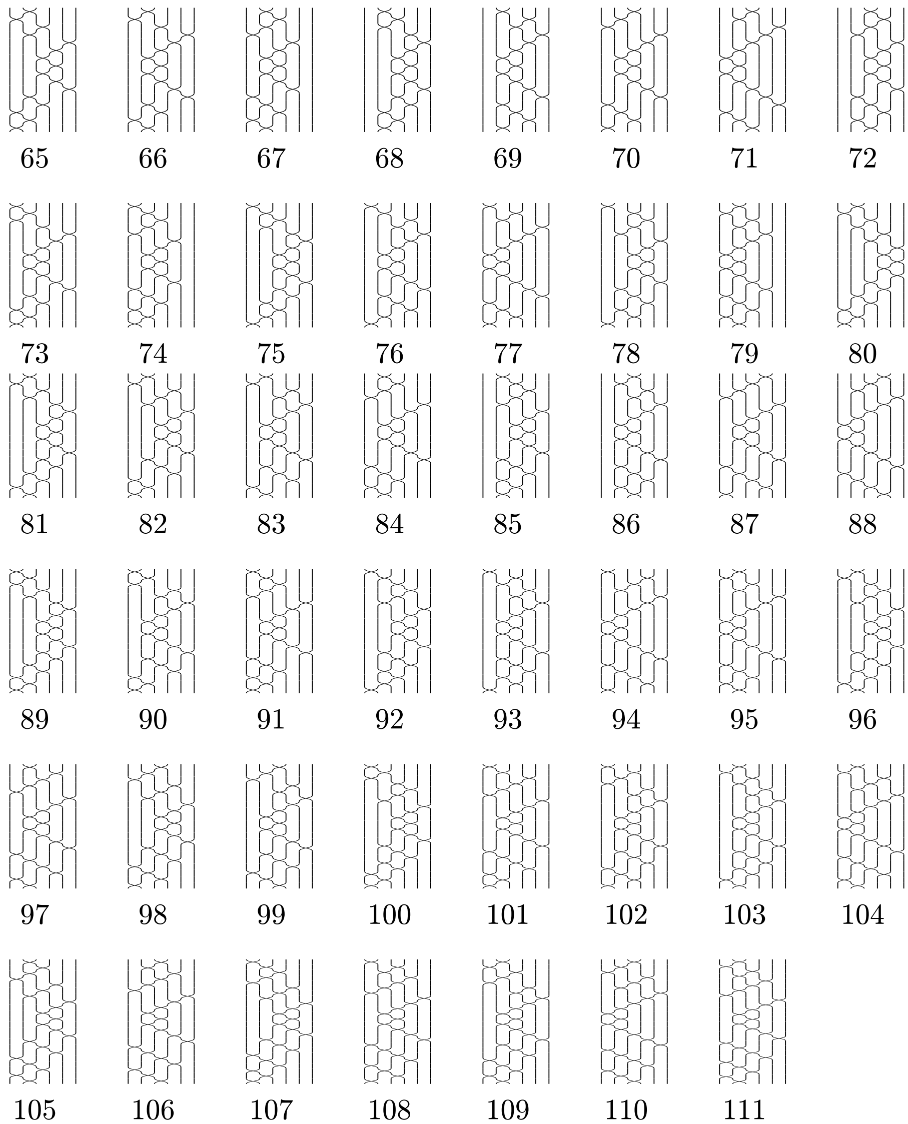


Figure 2.12: List of generators of PP_6 (part 2)

Chapter 3

Plane Curves

In this chapter we relate planar pure braids with the planar version of knots. Following the work of Mostovoy and Stanford [31], we give a certain type of closure for planar pure braids in which we obtain triple points free plane curves. The main result, is a Birman-Markov-type theorem for this closure. In the proof of the main theorem, we resort to partitioner posets of degree 1 and a modified construction to obtain planar pure braids.

3.1 Triple Points Free Plane Curves

Definition 3.1.1 (Plane curve)**.** A *plane curve* is an immersion $C : X \to S$ where X is a 1-dimensional manifold and S is a surface.

Arnold made a remarkable contribution to the theory of plane curves considering the point of view of singularity theory [2]. He studies the space of immersions of an oriented circle into the plane, via its discriminant. Considering three different strata of this discriminant, he introduces three invariants J^+, J^- and St. Such invariants are defined axiomatically via their values on some standard curves and their jumps under different deformations. In this work, we also concern the singularity theory point of view, but the discriminant is smaller, and the invariants we define come from Vassiliev theory (see chapter 4). We work with a particular type of plane curves.

Definition 3.1.2 (Triple points free plane curve)**.** A *triple points free plane curve* is an immersion of $\mathbb{R}$ into $\mathbb{R}^2$ which coincides with the linear embedding $x = 0$ outside a compact interval and all of whose multiple points are transversal double points (see figure 3.1).

Remark 3.1.3. Mainly we consider long triple points free plane curves, but of course exists the compact case of triple points free plane curves as immersion of $\mathbb{S}^1$ into de plane $\mathbb{R}^2$ or $\mathbb{S}^2$. In the literature, compact triple points free plane curves are known as *doodles* [22], and enclose immersions of any finite union of circles, being the planar version of links.

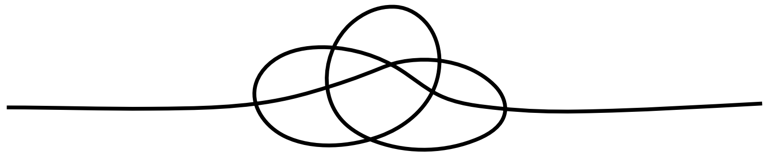


Figure 3.1: Triple points free plane curve

Definition 3.1.4 (Equivalent triple points free plane curves)**.** Two triple points free plane curves are considered to be *equivalent* if they can be obtained from each other by a finite sequence of

a) commutative squares of diffeomorphisms with compact support

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{C} & \mathbb{R}^2 \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R} & \xrightarrow[C']{} & \mathbb{R}^2 \end{array} \quad ,$$

b) local moves of type I or II (figure3.2).

Equivalently, two triple points free plane curves C and C' are equivalent if there exists a smooth homotopy $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_0 = \mathrm{id}_{\mathbb{R}^2}$, $F_1(C) = C'$ and $F_t \circ C$ is a triple points free plane curve for each $t \in [0,1]$.

Remark 3.1.5. Local moves in figure 3.2 are the planar version of Reidemeister moves in knot theory. Therefore we name them as the same way, the first, second and third Reidemeister moves for the local moves of type I, II and III, respectively. If we consider the local move of type III in the definition of equivalent plane curves, the theory becomes trivial. Moves $\Omega_3^{\pm}$ change the equivalence class of a curve, and plane curves defined in these jumps are what we call singular triple points free plane curves, which live in the discriminant (see definition 4.1.1).

Despite the theory of plane curves being parallel to knot theory, the absence of the third Reidemeister move leads to very different behaviours. For instance, a minimal representative for a triple points free plane curve is an equivalent curve with the minimum number of double points and each triple points free plane curve has a unique minimal representative, whereas in the case of knots does not. In general, plane curves give rise to rich combinatorial structures, although it doesn't mean that the theory is easier.

With a triple points free plane curve C with n double points we can associate a word on $2n$ letters as follows. Let $\{a_1, \ldots, a_n\}$ be the set of double points of C in $\mathbb{R}^2$. Going along the curve, we encounter each double point twice, so we can assign a word w in the alphabet $\{a_1, \ldots, a_n\}$ in order of appearance of double points. We can give an order to the set of double points in order of first appearance. From the beginning we can label double points such that $a_1 < a_2 < \cdots < a_n$. Observe

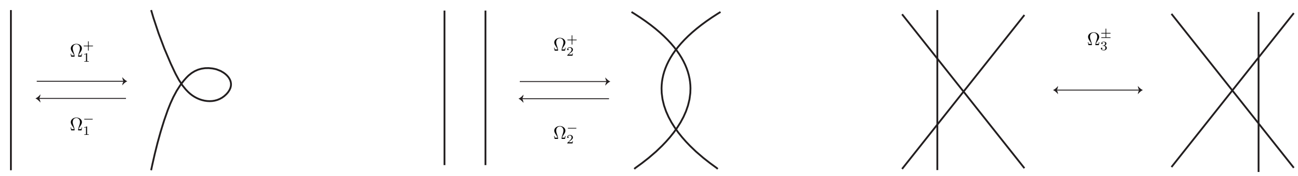


Figure 3.2: Reidemeister move of type I, II and III

that from the word we recover completely the curve up to diffeomporphism of the plane $\mathbb{R}^2$. The next proposition is a well known fact [22], [29], and essentially all the proofs have the same idea, ours is not the exception.

Proposition 3.1.6. *Each equivalence class of a triple points free plane curve has a unique minimal representative.*

Proof. Let C be a representative of the class, $a_1 < a_2 < \cdots < a_n$ the order in the set of double points and $w = w_1 \cdots w_{2n}$ the word associated with the plane curve that we described above. If $I \subset [n] = \{1, \ldots, n\}$, we can restrict the canonical order of $[n]$ to I. We say that $i, j \in I$ are neighbours in I if there's no elements in I between them. We define elementary reductions on words such that double points are going to be eliminated (as Reidemeister moves do in plane curves). We get a reduced word in a smaller set of double points $\{a_i | i \in I\}$ for a subset $I \subset [n]$, and then a corresponding new plane curve which is equivalent to the first. The elementary reductions are

$$w_1 \cdots w_l a_i a_i w_{l+3} \cdots w_{2n} \longmapsto w_1 \cdots w_l w_{l+3} \cdots w_{2n} \tag{3.1.1}$$

$$w_1 \cdots w_l a_i a_j w_{l+3} \cdots w_m a_i a_j w_{m+3} \cdots w_{2n} \longmapsto w_1 \cdots w_l w_{l+3} \cdots w_m w_{m+3} \cdots w_{2n} \tag{3.1.2}$$

$$w_1 \cdots w_l a_i a_j w_{l+3} \cdots w_m a_j a_i w_{m+3} \cdots w_{2n} \longmapsto w_1 \cdots w_l w_{l+3} \cdots w_m w_{m+3} \cdots w_{2n} \tag{3.1.3}$$

for i, j neighbours in I, and the corresponding set of double points $\{a_i | i \in I\}$. Geometrically, elementary reductions are nothing else than Reidemeister moves Ω_1^- and Ω_2^- expressed in words.
In the process we get a sequence of subsets $[n] \supset I_1 \supset \cdots \supset I_k$ such that the word in the alphabet $\{a_i | i \in I_k\}$ can not be reduced by elementary reductions. The triple points free plane curve associated with the last reduced word is the minimal representative.
Suppose there are two different minimal representatives C and C'. Then, they are connected by a sequence of triple points free plane curves $C = C_0, C_1, \ldots, C_k = C'$ by apply $\Omega_1^\pm$ and $\Omega_2^\pm$ moves. These sequence of Reidemeister moves always can be ordered in a sequence of Ω_i^- moves followed by Ω_j^+ moves. Note that Ω_i^- move delete i points, and the Ω_j^+ move create j points. We have the following cases:

(a) $\Omega_i^+ \circ \Omega_j^-$ such that created points are uncommon the deleted, hence moves commute and we can reorder to $\Omega_i^- \circ \Omega_j^+$.

The next cases always have common created and deleted points.

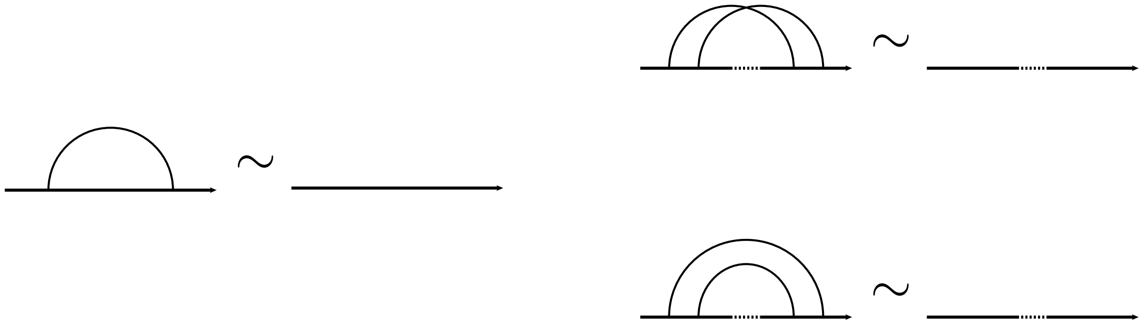


Figure 3.3: Moves in Gauss diagrams

(b) $\Omega_1^+ \circ \Omega_1^-$ is do nothing, hence we can omit them.

(c) $\Omega_1^+ \circ \Omega_2^-$ delete one point, hence is equivalent to an Ω_1^- move.

(d) $\Omega_2^+ \circ \Omega_1^-$ create one point, hence is equivalent to an Ω_1^+ move.

(e) $\Omega_2^+ \circ \Omega_2^-$ is do nothing, hence we can omit them.

Therefore, the sequence of Reidemeister moves can be reordered in a sequence of Ω_i^- followed by Ω_j^+ moves. By minimality of C, there's no Ω_i^- moves in the sequence, only Ω_j^+ moves which add points which contradicts the minimality of C'. Therefore $C = C'$. □

Another way to visualize these elementary reductions 3.1.1,3.1.2,3.1.3 are in diagrams which by analogy to knot theory, we called Gauss diagrams (see figure 3.3).

An important fact related with this minimal representative is that if C is a minimal representative plane curve, the corresponding Gauss diagram produces a Vassiliev invariant via Gauss diagram formula introduced by Polyak and Viro [33], [29]. In an Arnold's review for the bulletin of the IMU, he wrote:

> *The theory of smooth (possibly selfintersecting) curves in the plane is parallel to knot theory (the last being a simplified, commutative version of the theory of plane curves).*

Putting aside the complexity of each theory, what he might refer in one way is that knots forms a commutative monoid with the connected sum, while plane curves do not (see figure 3.4). Let C and C' be triple points free plane curves, then we can multiply them by concatenation. This product is natural if we think triple points free plane curves as a special case of plane tangles, where a plane tangle is a generalization of a triple points free plane curve, as well a tangle is a generalization of a knot in the usual case.

3.2 Closure of Planar Pure Braids

Analogously to [31], we study a certain type of closure from planar braids to obtain triple points free plane curves. We define the short circuit map from planar pure braids on $2n + 1$ strands into

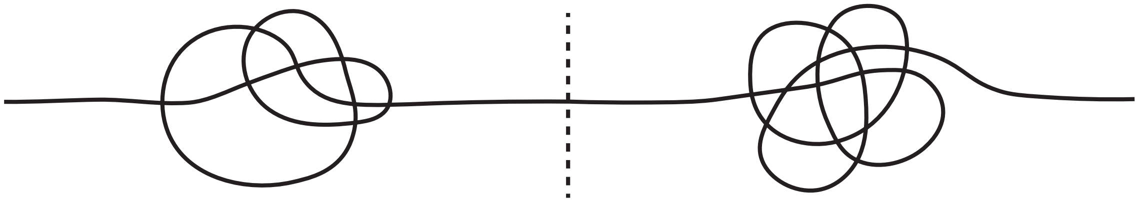


Figure 3.4: Product of triple point free plane curves

the monoid of triple points free plane curves. This map is surjective, i.e., each equivalence class of triple points free plane curve has a representative coming from a planar braid group. Furthermore, we prove a Birman-Markov type theorem, which states that the monoid of triple points free plane curves is equivalent to a biquotient of all planar pure braids. From here to the end of this chapter, we consider triple points free plane curves in vertical position, i.e., as an immersion of $\mathbb{R}$ into $\mathbb{R}^2$ which coincides with the linear embedding $y = 0$ outside a compact interval and all of whose multiple points are transversal double points.

Definition 3.2.1 (Short Circuit Closure)**.** Let be $b \in PP_{2n+1}$, p_i and q_i the ends of the ith strand of b. The *short-circuit closure* of b is defined as follows: at the top, join by pairs the end p_{2k} with the end p_{2k+1}; at the bottom, join the end q_{2k-1} with the end q_{2k}, for all $k = 1, \dots, n$ as pictured on figure 3.5. If $\mathscr{C}$ is the monoid of triple points free plane curves, the short-circuit closure can be thought as a map $\mathcal{S}_n : PP_{2n+1} \to \mathscr{C}$

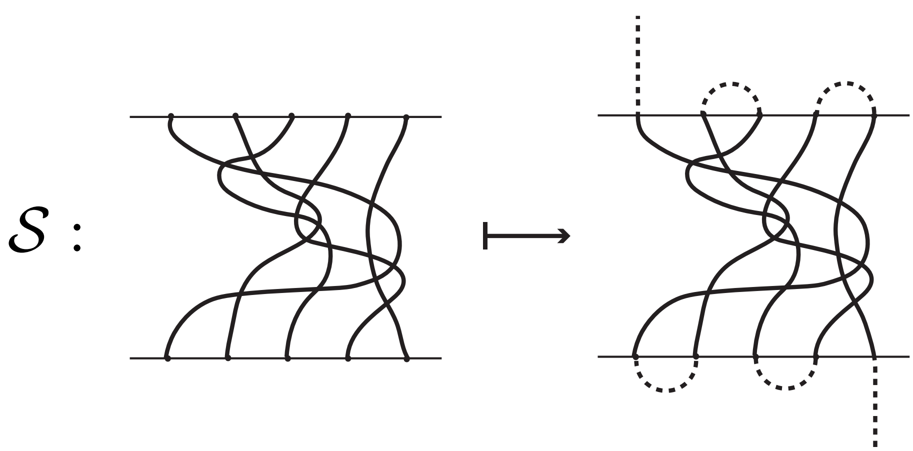


Figure 3.5: Short-circuit map

Each group PP_n can be included into any group PP_m for $n \leq m$ in the natural way adding vertical strands to complete the missing strands. Let PP_∞ be the direct limit of the system of inclusions

$PP_i \hookrightarrow PP_{i+1}$. Note that maps $\mathcal{S}_n$ are compatible with the inclusions $PP_{2n+1} \hookrightarrow PP_{2n+3}$ so they extend to a map $\mathcal{S} : PP_\infty \to \mathscr{C}$.

Lemma 3.2.2. *The map $\mathcal{S} : PP_\infty \to \mathscr{C}$ is onto.*

Proof. To make the proof easier, we use a plane curve that is in a good geometric position. Any equivalence class has a representative of that form. Let C be a triple points free plane curve. The height function of C is the composition

$$\mathbb{R} \xrightarrow{C} \mathbb{R}^2 \xrightarrow{\pi_2} \mathbb{R}$$

where π_2 is the projection to the second coordinate. We say that C is a Morse triple points free plane curve if the height function has only a finite number of critical points, all which are non-degenerate. Note that the number of critical points is even, one half are maxima and the other one are minima. Two Morse triple points free plane curves are equivalent if are equivalent as triple points free plane curves, preserving the number of maxima and minima. Let C be a Morse triple points free plane curve with n maxima, we say that C is well-positioned if for all $a \in \mathbb{R}$ such that $|a| \geq 1$, the line $y = a$ intersects C only once. For $y = 1$ and $y = -1$, the lines intersects the curve C at $(0, 1)$ and $(2n, -1)$, respectively. Denote by l_i the line segment which connects the ith maximum of the curve C with the point $(2i - \frac{1}{2}, 1)$. Similarly l'_i the line segment which connects the ith minimum of the curve C with the point $(2i - \frac{3}{2}, -1)$, see figure 3.6.

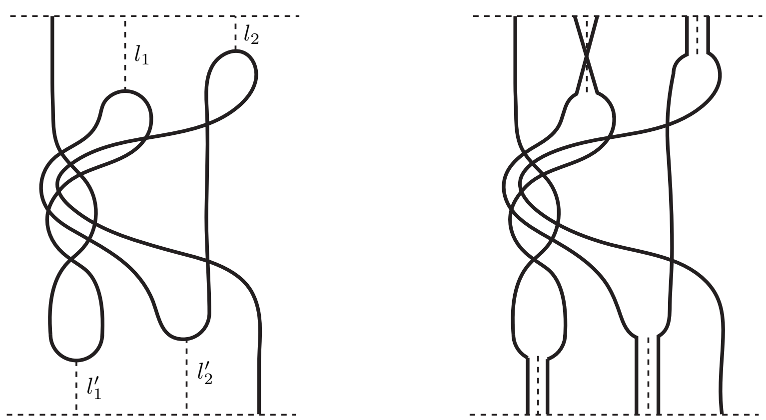


Figure 3.6: Well-positioned C and its corresponding planar pure braid

To a well-positioned triple points free plane curve corresponds a planar pure braid as follows. Removing the intsersection of the ith maximum of C with the segment l_i, we connect the ith maximum and the top line $t = 1$, with two strands at the points $(2i - 1, 1)$ and $(2i, 1)$ for all i. Similarly, removing the intersection of the ith minimum of C with the segment l'_i, we connect the ith minimum and the line $y = -1$, with two strands at the points $(2i - 2, -1)$ and $(2i - 1, -1)$ for all i. If it is necessary, the strands have to cross in order to produce a planar pure braid (see figure 3.6). As a

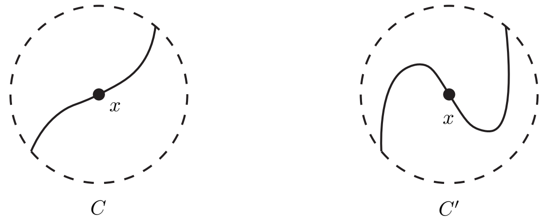


Figure 3.7: Inserting a hump

result, if we close by the short-circuit closure, we obtain C. □

The main theorem of this chapter is a Markov type theorem, defining an equivalence of the monoid of triple points free plane curves with a biquotient of PP_∞ using the short circuit map and an alternative construction of planar braids from basic partitioner posets. First, we need to observe a stability behaviour in Morse triple points free plane curves, i.e., two Morse triple points free plane curves equivalent as plane curves, become Morse equivalent after insertion of humps.

Let C a Morse triple points free plane curve and x a point in C which is not a critical point in the height function. A Morse triple points free plane curve C' is obtained from C by insertion of a hump at x if C and C' coincide outside a small neighbourhood of x and inside this neighbourhood they differ as in figure 3.7.

Lemma 3.2.3. *Any two triple points free plane curves obtained from the same Morse triple points free plane curve by insertion of a hump are Morse equivalent.*

Proof. If there are no critical points of the height function between the points x_1 and x_2 where we insert humps, clearly are Morse equivalent. If there is one critical point between x_1 and x_2 the lemma follows from the argument as in figure 3.8. This argument also proves the general case. □

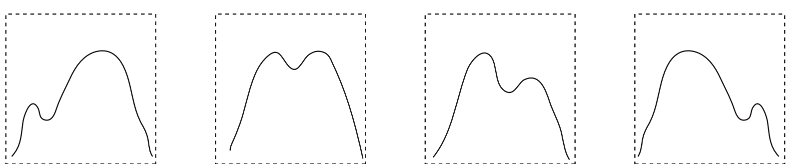


Figure 3.8: Passing a hump through a critical point

Let $b_1 \in PP_{2n+1}$ and $b_2 \in PP_{2m+1}$ and let $\iota(b_j)$ the image of b_j by the standard inclusion into P_{2N+1}, $N \geq n, m$.

Lemma 3.2.4. *If $\mathcal{S}_n(b_1)$ and $\mathcal{S}_m(b_2)$ are equivalent as plane curves, then there exists $N \geq n, m$ such that $\mathcal{S}_N(\iota(b_1))$ and $\mathcal{S}_N(\iota(b_2))$ are Morse equivalent.*

Proof.

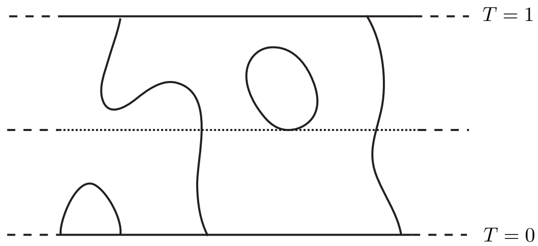


Figure 3.9: Tangential points

Think $C = \mathcal{S}_n(b_1)$ and $C' = \mathcal{S}_m(b_2)$ as immersions $\mathbb{R} \looparrowright \mathbb{R}^2$. By hypothesis are equivalent as plane curves, then there exists a smooth homotopy $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_0 = \mathrm{id}_{\mathbb{R}^2}$, $F_1(C) = C'$ and $F_T \circ C$ is a triple points free plane curve for each $T \in [0,1]$, say

$$(F_T \circ C)(t) = (x_T(t), y_T(t)).$$

To define an equivalence between Morse triple points free plane curves, first we identify the critical points which appear an disappear in the smooth homotopy between $\mathcal{S}_n(b_1)$ and $\mathcal{S}_m(b_2)$. In $[0,1] \times \mathbb{R}$ consider the subset W of pairs (T, t) such that $\frac{\partial}{\partial t} y_T(t) = 0$. Without loss of generality we can assume W is a union of finite compact 1-dimensional manifolds whose boundary is empty or belongs to $(\{0\} \cup \{1\}) \times \mathbb{R}$, and that there are only a finite number of points of tangency in W with horizontal lines $\{T\} \times \mathbb{R}$ (see figure 3.9). Points of tangency identify insertions or removals of humps.
If there are no points of tangency, $\mathcal{S}_n(b_1)$ and $\mathcal{S}_m(b_2)$ are Morse equivalent and $N = n = m$. Otherwise, choose the point of tangency in W with the horizontal line $\{T\} \times \mathbb{R}$ with the smallest value of T, which corresponds to an insertion of a hump. We can remove a small neighbourhood of the point of tangency and connect with two segements with the lower boundary $\{0\} \times \mathbb{R}$ and are disjoint from W as in figure 3.10. We do the same with all points of tangency, such that at the end there are no more points of tangency. In the process we insert new humps and the result is a smooth homotopy between $\mathcal{S}_N(\iota(b_1))$ and $\mathcal{S}_N(\iota(b_2))$ as Morse plane curves, where N is the number of boundary points in the horizontal line $\{0\} \times \mathbb{R}$.

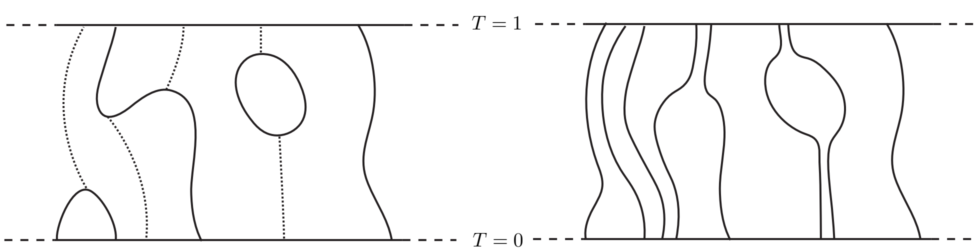


Figure 3.10: Bifurcation of tangential points

□

The alternative construction of planar braids from basic partitioner posets is as follows.

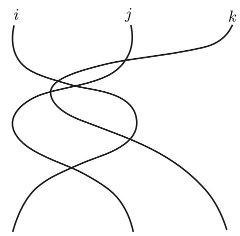


Figure 3.11: The planar pure braid corresponding to $P = (\dots k \dots)[i,j](\dots)$

Construction 2. If $P = (K)[i,j](K')$ is a basic partitioner poset and $k := \max\{i' \in K \sqcup \{i,j\}\}$, we define the planar pure braid by taking $S = \{k\}$ in construction 1. Explicitly, the ith and jth strands cross as in the second Reidemeister move (see figure 3.2), and the strand k is "linked" with the upper crossing of the ith and jth strands as in the figure 3.11. We call it the ijk configuration, an we identify it in figure 3.12 as the tangle inside the square. The other strands are placed by the partitioner poset as follows:

(a) if $l \in K'$, the lth strand goes around the ijS configuration through the right satisfying condition (c) (strands in blue in figure 3.12);

(b) if $l \in K \setminus S$, the lth strand goes around the ijS configuration through the left satisfying condition (c) (strands in red figure 3.12);

(c) we parametrize strands from the bottom to the top. Let $s \in \{i,j,k\}$. Going along the sth strand, intersection points with other strands are ordered as follows:

1. points with lth strands for $l \in K'$ and $l < s$. An lth strand intersects first than the l'th strand if $l > l'$.
2. points with lth strands for $l \in K \setminus \{k\}$ and $l > s$. An lth strand intersects first than the l'th strand if $l < l'$.
3. points with lth strands in the ijk configuration.
4. points with lth strands for $l \in K \setminus \{k\}$ and $l > s$. An lth strand intersects first than the l'th strand if $l > l'$.
5. points with lth strands for $l \in K'$ and $l < s$. An lth strand intersects first than the l'th strand if $l < l'$.

We denote by ϕ_i^n the homomorphism $PP_{2n} \to PP_{2n+1}$ which doubles the ith strand. Homomorphisms ϕ_i^n respect the standard inclusions of planar pure braid groups, so as n tends to infinity, the limit $\phi_i : PP_\infty \to PP_\infty$ is well defined.

Theorem 3.2.5. *There exist two subgroups H^T and H^B of PP_∞ such that the map $\mathcal{S} : PP_\infty \to \mathscr{C}$ is constant on the double cosets of the form $H^T x H^B$. Furthermore, fibres are of this type, hence the short-circuit map identifies the monoid of triple points free plane curves $\mathscr{C}$ with the biquotient $H^T \setminus PP_\infty / H^B$.*

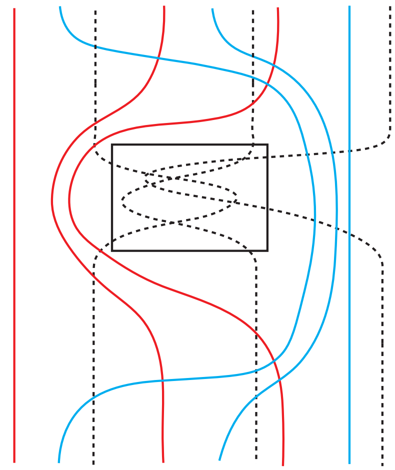


Figure 3.12: Strands corresponding to K and K' of $P = (K)[i, j](K')$ by construction 2

Proof. Let H_n^T be the subgroup of PP_{2n+1} generated by b_P's with P a partitioner poset (p.p.), such that:

1. $P = (K)[i, i+1](K')$ is a p.p. of $[2n+1]$ with i even.
2. $P = (K)[i, j](K')$ is a p.p of $[2n+1]$ such that $j+1$ concides with $\max\{i' \in K \sqcup \{i, j\}\}$ and j even.
3. $b_P = \phi_k(b_{P'})$, where $P' = (K)[i, j](K')$ is a p.p. of $[2n]$ and $k = \max\{i' \in K \sqcup \{i, j\}\}$ is even.
4. $b_P = \phi_i(b_{P'})$, where $P' = (K)[i, j](K')$ is a p.p. of $[2n]$, $j+1$ coincides with $\max\{i' \in K \sqcup \{i, j\}\}$ and i is even.

Similarly we define the subgroup H_n^B with the only difference that all is odd, instead of the even condition, i.e., H_n^B is the subgroup of PP_{2n+1} generated by b_P's with P a partitioner poset (p.p.), such that:

1. $P = (K)[i, i+1](K')$ is a p.p. of $[2n+1]$ with i odd.
2. $P = (K)[i, j](K')$ is a p.p of $[2n+1]$ such that $j+1$ concides with $\max\{i' \in K \sqcup \{i, j\}\}$ and j odd.
3. $b_P = \phi_k(b_{P'})$, where $P' = (K)[i, j](K')$ is a p.p. of $[2n]$ and $k = \max\{i' \in K \sqcup \{i, j\}\}$ is eodd.
4. $b_P = \phi_i(b_{P'})$, where $P' = (K)[i, j](K')$ is a p.p. of $[2n]$, $j+1$ coincides with $\max\{i' \in K \sqcup \{i, j\}\}$ and i is odd.

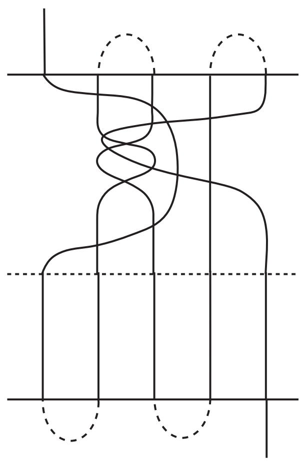
The subgroup H_n^T acts on PP_{2n+1} on the left (top) and H_n^B acts on the right (bottom). The sequence of inclusions $PP_{2n+1} \hookrightarrow PP_{2n+3}$ which adds new vertical strands, restrict well to a sequence of inclusions for the subgroups $H_n^T \hookrightarrow H_{n+1}^T$. Let H^T be the direct limit of the system, then H^T is a subgroup on PP_∞ which acts on the left. In the same way, let H^B be the direct limit of the system $H_n^B \hookrightarrow H_{n+1}^B$, then H^B is a subgroup on PP_∞ which acts on the right.

From the figures of b_P's generators of H^T (see 1-4), the closure in the top of any element in H^T produce a triple points free plane curve equivalent to the closure in the top of a trivial planar braid. The equivalence follows by retracting some of the maximum of the plane curve in order to unlink and obtain the closure of a trivial planar braid; the retraction is a sequence of Reidemeister moves I and II. Then, the map $\mathcal{S}$ is constant on the coset $H^T x$. The same holds for H^B, so that the map $\mathcal{S}$ is constant on the double cosets $H^T x H^B$.

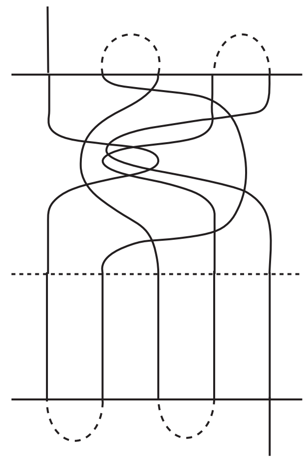
Let $b_0 \in PP_{2n+1}$ and $b_1 \in PP_{2n+1}$ such that $\mathcal{S}(b_0) = \mathcal{S}(b_1)$ as Morse triple points free plane curves of n maximums. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^2$ the homotopy such that for each $T \in [0,1]$, f^T is a Morse triple points free plane curve with $f^0 = \mathcal{S}(b_0)$ and $f^1 = \mathcal{S}(b_1)$. We can assume that the family of Morse triple points free plane curves takes place entirely between two horizontal lines, then are well-positioned. We're going to construct a family of planar pure braids $g : [0,1] \to PP_{2n+1}$, which is continuous except in a finite number of T's, these jumps are multiplication by some element of H^T or H^B.

By lemma 3.2.2, we obtain a family of planar pure braids by the construction of planar braids from well-positioned Morse triple points free plane curves. Fixing the end points of the rubber line segments l_i's and l_i''s and its corresponding strands, in the homotopy we have a continuous family of braids, except when a triple points intersection happens. For example, in $T = 0$ we obtain the planar braid from the well-positioned triple points free plane curve $f^0 = \mathcal{S}(b_0)$, and in $T = 1$ we obtain the planar braid from the well-positioned triple points free plane curve $f^1 = \mathcal{S}(b_1)$. The triple points intersection may happen when in the homotopy a line segment l_i intersects the plane curve or another line segment l_j, similarly with l_i'. Recall that for a line segment l_i corresponds the $(2i)$th and $(2i+1)$th strands, and for a line segment l_i' corresponds the $(2i-1)$th and $(2i)$th strands. If it is necessary, strands make a crossing 3.6. We analyse the different events which could happen in figures 3.13, 3.14, 3.15.

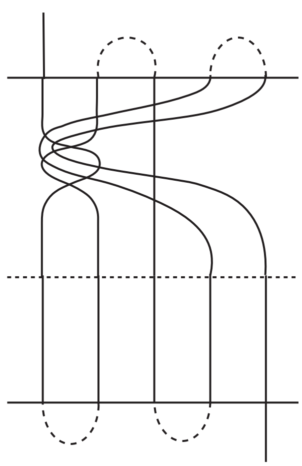
For the event of the figure 3.15, the top and bottom case, are a particular case of our generators of type 3 and 4. The middle case is not in our list of generators, but it's a product of the other cases. □



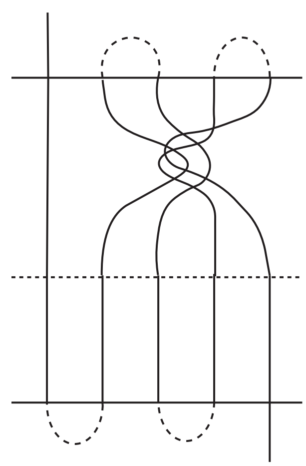
b_P of type 1.



b_P of type 2.



b_P of type 3.



b_P of type 4.

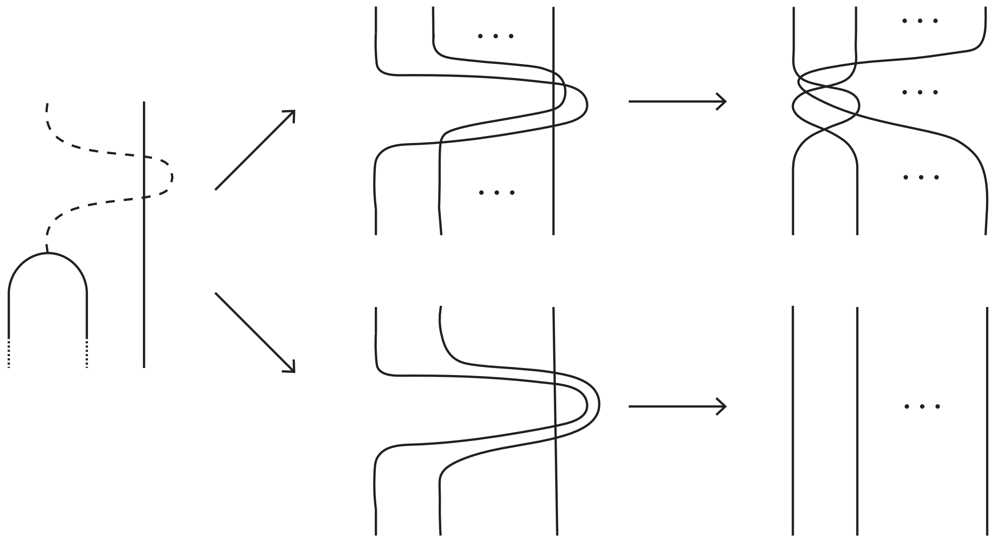


Figure 3.13: type 1

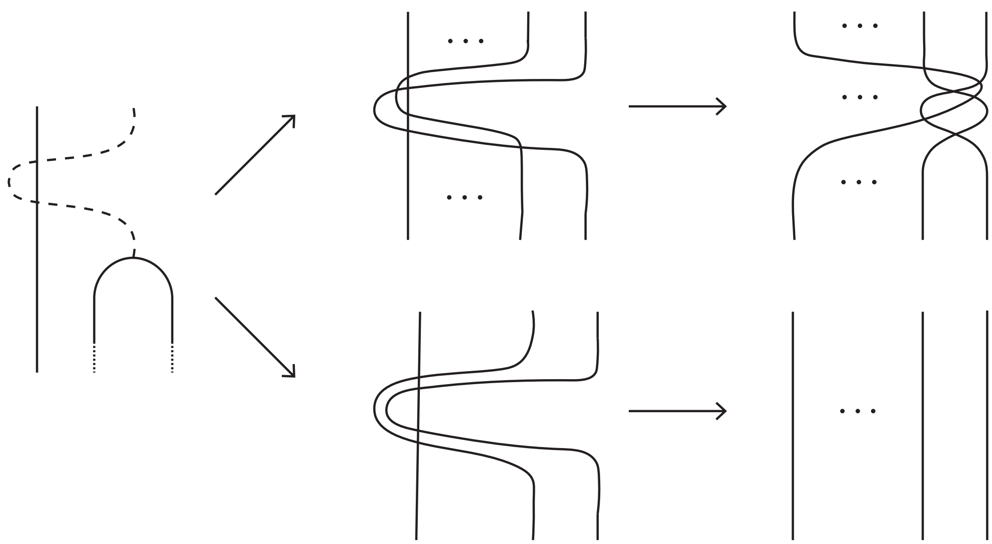


Figure 3.14: type 2

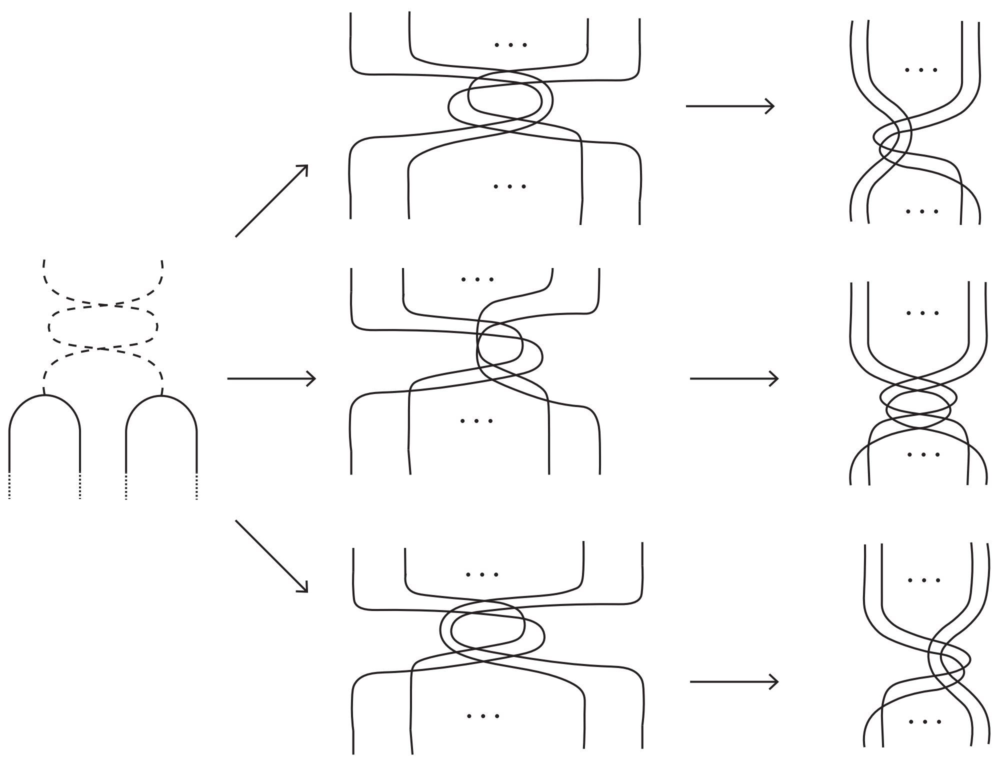


Figure 3.15: type 3 and 4

Chapter 4

Vassiliev Invariants of Triple Points Free Plane Curves and Planar Pure Braids

In this chapter we define finite type invariants, also called Vassiliev invariants, for triple points free plane curves and planar braids. In analogue to Kohno's construction [24], we sketch the construction of a universal Vassiliev invariant for planar pure braids following [25] and [26] . For that purpose, we review some of the theory of Chen's iterated integrals. In spite of differential forms are missing, we know the cohomology of $\mathrm{Conf}_3(\mathbb{R}, n)$ and the universal invariant can be expressed with some direct results by the general theory developed by Chen.

4.1 Finite type invariants

In 1990, Vassiliev introduced the notion of finite type invariant just as an application of his general machinery on the study of discriminants in spaces of maps [35]. Briefly, he consider the space of smooth maps

$$\mathcal{M} = \{f : \mathbb{S}^1 \to \mathbb{R}^3\},$$

and an element $f \in \mathcal{M}$ is a knot if it is an embedding. The complement of the set of all knots is the discriminant $\Sigma \subset \mathcal{M}$ and consists of all maps with singularities. Two knots are equivalent if they are connected by a path in $\mathcal{M}$ which does not intersect Σ. Therefore, a knot type is a connected component of the space $\mathcal{M} \setminus \Sigma$. Setting $\mathcal{K} = \mathcal{M} \setminus \Sigma$, the zero dimensional cohomology $H^0(\mathcal{K}; R)$ corresponds to R-valued knot invariants. Beyond a doubt, the cohomology of the space of knots is of importance. Vassiliev in his attempt to compute it, build a general machinery to study complements of discriminant. His contruction involves three techical tools: the Alexander duality, simplicial resolutions and stabilization. His method produces a spectral sequence, which contains in particular the finite type invariants. At the moment, there are (among others) three main equivalent definitions of finite type invariants, the "geometrical" and initial definition in terms of discriminants [35], the "axiomatic", in terms of differences of knot diagrams [8], and the "combinatorial", in terms of homomorphisms of chord diagrams [17].

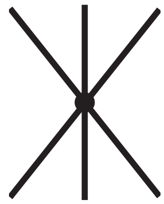
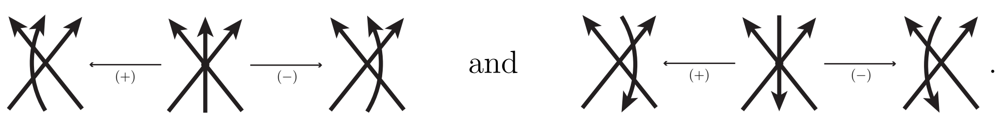


Figure 4.1: A triple point

In our case, we can apply the Vassiliev's machinery to the space of smooth maps $\mathcal{M} = \{f : \mathbb{R} \to \mathbb{R}^2\}$ and discriminant $\Sigma \subset \mathcal{M}$ as the complement of the set of all the triple points free plane curves. We will define Vassiliev invariants in the axiomatic way, but is equivalent to the geometrical and combinatorial [36].

Definition 4.1.1 (Singular Triple Points Free Plane Curve)**.** Let f be a map of a one-dimensional manifold to $\mathbb{R}^2$. A point $p \in \text{im}(f) \subset \mathbb{R}^2$ is a *triple point* of f if $f^{-1}(p)$ consists of three points t_1, t_2 and t_3 in the domain, and the tangent vectors $f'(t_1)$, $f'(t_1)$ and $f'(t_1)$ are pairwise linearly independent. A *singular triple points free plane curve* is an immersion of $\mathbb{R}$ into $\mathbb{R}^2$ which coincides with some fixed linear embedding outside a compact interval, and all of whose multiple points are transversal double points and triple points. Triple points are also referred as *singular points*, because is where the map fails to be a triple points free plane curve. Locally, a triple point looks like in figure 4.1.

We can degenerate a singular point to obtain only double points by perturbing the curve in a small neighbourhood of the triple point. Set the natural orientation as a parametrized curve by $\mathbb{R}$ and signs $(+)$ and $(-)$ for perturbing in a small neighbourhood as follows



Remark 4.1.2. The signs are determined by something called *vanishing triangles*. Such triangles are formed by three branches of the curve after and before perturbing the curve in a small neighbourhood of a singular point. The coorientation of the discriminant is determined by vanishing triangles [2].

Any invariant v of triple points free plane curves with values in an algebraic structure with at least of abelian group, can be extended to singular triple points free plane curves by the planar version of the *Vassiliev skein relation*:

(4.1.1)

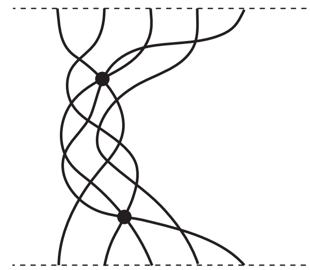


Figure 4.2: Singular planar pure braid

The process of applying the skein relations to each triple point is also referred to as resolving a singular point. Using the Vassiliev skein relation recursively, we can extend the invariant to singular triple points free plane curves with an arbitrary number of triple points. There are many ways to do it, only by changing the order in which we resolve triple points. However, it doesn't depend, the complete resolution of a singular triple points free plane curve C with n triple points yields an alternating sum

$$v(C) = \sum_{\epsilon_1,\ldots,\epsilon_n} (-1)^{|\epsilon|} v(C_{\epsilon_1,\ldots,\epsilon_n}),$$

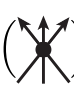
where $|\epsilon|$ is the number of -1's in the sequence $\epsilon_1, \ldots, \epsilon_n$, and $C_{\epsilon_1,\ldots,\epsilon_n}$ is the triple points free plane curve obtained by resolving triple points of C with the positive or negative resolution, according to the sign of ϵ_i for the singular point i.

Definition 4.1.3 (Vassiliev Invariant)**.** An invariant v of triple points free plane curve is a *Vassiliev invariant* (or finite type invariant) of order $\leq k$ if its extension vanishes on all singular triple points free plane curves with more than k triple points. A Vassiliev invariant is of order k if it is of order $\leq k$ but not of order $\leq k-1$.

In general, Vassiliev invariants can take values in an arbitrary abelian group. For our purpose all invariants will take values in real numbers $\mathbb{R}$.

For the case of planar pure braids, we can consider $\mathbb{R}^n$ as a space of functions, and the 3-equal arrangement $\mathcal{A}_{n,3}$ as the discriminant. A singular planar pure braid can be thought as a based loop $\gamma : I = [0,1] \to \mathrm{Conf}_3(\mathbb{R}, n) \sqcup \mathcal{A}_{n,3}$ such that $\gamma(t) \in \mathcal{A}_{n,3}$ for a finite number of t's such that $\gamma(t)$ has at most three equal coordinates. Representing the singular planar pure braid as a plane tangle, looks like a planar pure braid with some triple points in the same sense like before.

By the orientation of the strands of planar braids, the branches in a triple point never point down, then we obtain only one Vassiliev skein relation for planar braids:

v () $= v$ () $- v$ () .   (4.1.2)

Using the Vassiliev skein relation, we can extend an invariant of planar pure braids to singular planar pure braids.

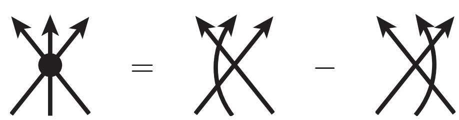
Definition 4.1.4 (Vassiliev Invariant of Planar Pure Braids)**.** An invariant of planar pure braids $v : PP_n \to \mathbb{R}$ is a Vassiliev invariant of order $\leq k$ if its extension vanishes on singular planar pure braids with more than k triple points. A Vassiliev invariant is of order k if it is of order $\leq k$ but not of order $\leq k-1$.

Let us denote by $\mathcal{V}_k^n$ the vector space of Vassiliev invariants of order $\leq k$ of PP_n with values in $\mathbb{R}$. It follows from the definition that $\mathcal{V}_k^n \subset \mathcal{V}_k^n$, so we have an increasing filtration

$$\mathcal{V}_0^n \subset \mathcal{V}_1^n \subset \cdots \subset \mathcal{V}_k^n \subset \cdots \subset \mathcal{V}^n := \bigcup_{k=0}^{\infty} \mathcal{V}_k^n.$$

We have $\mathcal{V}_0^n = \mathbb{R}$ is the set of constant invariants. If $v \in \mathcal{V}_0^n$, then v vanishes in every singular planar pure braid, and every planar pure braid becomes trivial after a finite sequence of the third Reidemeister moves which imply triple points, then it suffices to know $v(1)$ with 1 the trivial planar pure braid in n strands.

Singular planar pure braids can be identified as elements of the group algebra $\mathbb{R}[PP_n]$ via the relation



$\in \mathbb{R}[PP_n]$. (4.1.3)

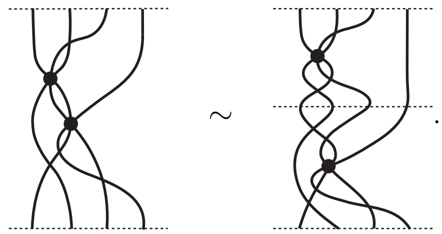
Let $I \subset \mathbb{R}[PP_n]$ the augmentation ideal, that is, the kernel of the homomorphism $\mathbb{R}[PP_n] \to \mathbb{R}$ that sends each $b \in PP_n$ to 1. Elements of I are linear combinations $\sum a_i b_i$ with $\sum a_i = 0$. The power I^k of the augmentation ideal forms a descending filtration

$$\mathbb{R}[PP_n] \supset I \supset I^2 \supset \cdots \supset I^k \supset \ldots.$$

Every planar pure braid can be transformed to the trivial planar braid by a sequence of deformations in which appear and disappear triple points., i.e., we allow the third Reidemeister move $\Omega_3^{\pm}$. Using 4.1.3 we have

$$\begin{aligned} I &= \langle (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_m - 1) | b_i, b_{i+1} \in PP_n \text{ differ by an } \Omega_3^{\pm} \text{ move} \rangle \\ &= \langle (b - b') | b, b' \text{ differ by an } \Omega_3^{\pm} \text{ move} \rangle \\ &= \langle \text{ singular planar pure braids } \rangle. \end{aligned}$$

It is easy to see that any singular planar pure braid with k triple points can be written as the product of k singular planar pure braids, each with one singular point. For instance,



Therefore, each power of the augmentation ideal can be described in a very simple form

$$I^k = \langle \text{planar pure braids with } k \text{ singular points} \rangle.$$

This gives a new identification of Vassiliev invariants as follows.

Proposition 4.1.5. *There is a canonical isomorphism*

$$\mathcal{V}_k^n \cong Hom(\mathbb{R}[PP_n]/I^{k+1}, \mathbb{R}).$$

Another approach to describe Vassiliev invariants for planar pure braids uses a general theory known as Chen's iterated integral [14]. We shall briefly review it.

4.2 Bar Complex and Chen's Iterated Integrals

Chen's method of iterated integrals generalizes the notion of integrating forms over cycles, and by generalizing the de Rham theorem provides homotopy information than just the abelianization of the fundamental group.

Let X be a smooth manifold, and let ΩX be the space of smooth based loops on X. For a 1-form $\omega \in A^1_{DR}(X)$, by integration we have a map

$$\int \omega : \Omega X \to \mathbb{R}, \qquad \gamma \mapsto \int_\gamma \omega := \int_0^1 \alpha(t)dt.$$

where $\gamma : [0,1] \to X$ and $\alpha = \gamma^*\omega$ is the pull-back. By the de Rham theorem, $\int_\gamma \omega$ only depends on the homotopy of the loop γ if and only if ω is closed. Then $\int \omega$ defines an homomorphism

$$\int \omega : \pi_1(X) \to \mathbb{R}$$

hence, with the composition of the Hurewicz homomorphism we obtain the de Rham isomorphism $Hom(\pi_1(X)/(\pi_1(X))^{ab}, \mathbb{R}) \cong H^1(X, \mathbb{R}) \cong H^1_{DR}(X)$. Chen generalized the integration map.

For 1-forms $\omega_1, \ldots, \omega_k \in A^1_{DR}(X)$, Chen defined a functional on the loop space

$$\int \omega_1 \cdots \omega_k : \Omega X \to \mathbb{R}, \qquad \gamma \mapsto \int_\gamma \omega_1 \cdots \omega_k.$$

The iterated integration for this particular case is defined by

$$\begin{aligned}\int_\gamma \omega_1\cdots\omega_k &:= \int\cdots\int_{0\le t_1<\cdots<t_k\le 1} \alpha_1(t_1)\alpha_2(t_2)\cdots\alpha_k(t_k)dt_1\cdots dt_k \\ &= \int_0^1\left(\int_0^{t_{k-1}}\cdots\left(\int_0^{t_3}\left(\int_0^{t_2}\alpha_1(t_1)dt_1\right)\alpha_2(t_2)dt_2\right)\cdots\right)\alpha_k(t_k)dt_k\end{aligned}$$

where $\alpha_i = \gamma^*\omega_i$ is the pull-back. Such functional is called a *basic iterated integral* of length k. An *iterated integral* will mean an $\mathbb{R}$-linear combination of basic iterated integrals and constant functionals on ΩX. The *length* of an iterated integral is the largest length of its summands (constant functionals have zero length). Chen's generalization of the integration map states that if 1-forms are "closed" in some sense, the integration map $\int \omega_1 \dots \omega_k$ is a homotopy functional or a 0-form of ΩX. Chen generalized the notion of manifolds to the infinite dimensional case with something called differentiable spaces. Chen defined the de Rham complex on differentiable spaces. In this setting, iterated integrals are differential forms of ΩX and with their exterior derivative we determine when are closed forms. In this sense the integration map as above $\int \omega_1 \dots \omega_k$ is a homotopy functional on ΩX. In order to define the exterior derivative, we recall the bar complex and iterated integrals for arbitrary differential forms.

We denote by $\pi_i : X^k \to X$ the projection on the ith factor. For differential forms $\omega_1, \dots, \omega_k$ on X, we write

$$\omega_1 \times \cdots \times \omega_k := \pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k.$$

Let

$$B^n = \bigoplus_{p-k=n} C^{p,-k}$$

where $C^{p,-k}$ is the vector space spanned by the symbols

$$[\omega_1|\cdots|\omega_k] := \omega_1 \times \cdots \times \omega_k,$$

with differential forms of positive degrees $\omega_1, \dots, \omega_k$ on X such that

$$\deg\omega_1 + \cdots + \deg\omega_k = p.$$

Let d_1 and d_2 be differentials as follows

$$\begin{aligned} d_1[\omega_1|\cdots|\omega_k] &= \sum_{i=1}^k (-1)^i [J\omega_1|\cdots|J\omega_{i-1}|d\omega_i|\omega_{i+1}|\cdots|\omega_k], \\ d_2[\omega_1|\cdots|\omega_k] &= \sum_{i=1}^k (-1)^{i-1}[J\omega_1|\cdots|J\omega_{i-1}|J\omega_i\wedge\omega_{i+1}|\omega_{i+2}|\cdots|\omega_k], \end{aligned}$$

where $J\omega = (-1)^{\deg\omega}\omega$. With these two differentials, define $d = d_1 + d_2$ and $B(A^*_{DR}(X)) = \bigoplus_{n=0}^\infty B^n$. The associated complex (B^*, d) is called the bar complex.

Definition 4.2.1 (Iterated Integral)**.** Let $\Delta_k = \{(t_1, \ldots, t_k) \in [0,1]^k | 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$ the standard k-simplex, and consider $\mathrm{ev} : \Omega X \times \Delta_k \to X^k$ the natural map defined by $\mathrm{ev}(\gamma, (t_1, \ldots, t_k)) = (\gamma(t_1), \ldots, \gamma(t_k))$. *Chen's iterated integral* of the differential forms $\omega_1, \ldots, \omega_k$ along the path γ is by definition

$$\int_\gamma \omega_1 \cdots \omega_k := \oint_{\Delta_k} \mathrm{ev}^*(\omega_1 \times \cdots \times \omega_k)$$

where $\oint$ is the integral along the fiber in the trivial bundle $\rho : \Omega X \times \Delta_k \to \Omega X$.

The iterated integral is considered to be a differential form of degree $p - k$ on the based loop space ΩX. Let $A^*_{DR}(\Omega X)$ the de Rham complex of differential forms on ΩX. The integration map in the above construction, defines a map

$$\int : B(A^*_{DR}(X)) \longrightarrow A^*_{DR}(\Omega X). \tag{4.2.1}$$

Chen showed that the integration map 4.2.1 determines a morphism of complexes. Furthermore, if the manifold X is simply connected, the bar complex computes the cohomology of the loop space ΩX, i.e.,

$$H^i(\Omega X) \cong H^i(B(A^*_{DR}(X))).$$

In the case X is non-simply connected, the 0-dimensional cohomology of the bar complex extracts information on the fundamental group of X in the following sense.

Theorem 4.2.2 ([14])**.** *The integration map 4.2.1 defines a pairing*

$$H^0(B(A^*_{DR}(X))) \otimes \mathbb{R}[\pi_1(X)] \longrightarrow \mathbb{R}.$$

Furthermore, for the increasing filtration on the bar complex $F_k(B(A^*_{DR}(X))) = \bigoplus_{q \leq k} C^{p,-q}$, *which induces an increasing filtration on the cohomology, the above pairing gives an isomorphism*

$$F_k(H^0(B(A^*_{DR}(X)))) \cong Hom(\mathbb{R}[\pi_1(X)]/I^{k+1}, \mathbb{R}),$$

which induces an isomorphism by taking projective limits

$$H^0(B(A^*_{DR}(X))) \cong Hom(\mathbb{R}\widehat{[\pi_1(X)]}, \mathbb{R}),$$

where $\mathbb{R}\widehat{[\pi_1(X)]}$ *is the completion of the group algebra with respect to the powers of the augmentation ideal.*

Remark 4.2.3. In [14], Chen proves that the map 4.2.1 $[\omega_1 | \cdots | \omega_k] \mapsto \int \omega_1 \cdots \omega_k$ induces an isomorphism of Hopf algebras between $B(A^*_{DR}(X))$ and a differential graded subalgebra of $A_{DR}(\Omega X)$ which consists of Chen's iterated integrals and denoted by $Ch^*(\Omega X)$. Furthermore, if $A \subset A^*_{DR}(X)$ is a differential graded subalgebra such that the inclusion $A \to A^*_{DR}(X)$ induces isomorphisms in cohomology in all dimensions, then all elements in $Ch^*(\Omega X)$ can be written as iterated integrals in differential forms of A. That is the case when the space X is formal in the sense of rational homotopy.

For the last in this section, we recall the Chen's formal connection. Let $X_1, \dots, X_m$ be a basis for $H_1(X, \mathbb{R})$ and $\omega_1, \dots, \omega_m$ be a set of real closed 1-forms on X representing the basis of $H^1(X, \mathbb{R})$ and dual to the basis $\{X_i\}$. We denote by $\mathbb{R}\langle\langle X_1, \dots, X_m\rangle\rangle$ the algebra of non-commutative formal power series with indeterminates X_i. Consider the expression

$$\alpha = \sum_i \alpha_i X_i + \sum_{i,j} \alpha_{i,j} X_i X_j + \sum_{i,j,k} \alpha_{i,j,k} X_i X_j X_k + \dots$$

where all coefficients are 1-forms on X. We shall say that α is an $\mathbb{R}\langle\langle X_1, \dots, X_m\rangle\rangle$-valued 1-form on X. We refer to $\sum_i \alpha_i X_i$ as the linear part of α. Chen proves the following.

Theorem 4.2.4 ([14])**.** *There exists an $\mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle$-valued 1-form ω on X whose linear part is $\sum_i \omega_i X_i$ and an ideal $\boldsymbol{J} \subset \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle$ such that there is a ring homomorphism*

$$Z : \mathbb{R}[\pi_1(X)] \longrightarrow \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle / \boldsymbol{J}$$

given by

$$Z(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \dots + \int_\gamma \omega \cdots \omega + \cdots .$$

If $\boldsymbol{I} \subset \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle$ the augmentation ideal which consists of formal series with zero constant term, the kernel of the composite map

$$Z : \mathbb{R}[\pi_1(X)] \longrightarrow \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle / \boldsymbol{J} \longrightarrow \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle / (\boldsymbol{J} + \boldsymbol{I}^n)$$

is precisely $I^n \otimes \mathbb{R}$ where I^n is the nth power of the augmentation ideal of $\mathbb{R}[\pi_1(X)]$.

Corollary 4.2.5. *The map $Z : \mathbb{R}[\pi_1(X)] \longrightarrow \mathbb{R}\langle\langle \boldsymbol{X}\rangle\rangle / \boldsymbol{J}$ is injective if $\pi_1(X)$ is torsion free and residually nilpotent.*

Definition 4.2.6 (Formal Homology Connection)**.** The map $Z : \mathbb{R}[\pi_1(X)] \longrightarrow \mathbb{R}\langle\langle \mathbf{X}\rangle\rangle / \mathbf{J}$ is called the *formal homology connection or Chen's expansion.*

Remark 4.2.7. If there exist the differential graded subalgebra as in remark 4.2, Chen shows that $\mathbf{J}$ is an homogeneous ideal. Consequently, $\mathbb{R}\langle\langle \mathbf{X}\rangle\rangle / \mathbf{J}$ is a graded algebra. Furthermore, the formal connection Z induces an injective map

$$\mathcal{A}(\pi_1(X)) \otimes \mathbb{R} \longrightarrow \mathbb{R}\langle\langle \mathbf{X}\rangle\rangle / \mathbf{J}$$

where $\mathcal{A}(\pi_1(X)) = \bigoplus_{k \geq 0} I^k / I^{k+1}$, and the image of the formal homology connection is contained in the graded completion of the image of this map. This means that the 1-form

$$\omega = \sum_i \omega_i X_i + \sum_{i,j} \omega_{i,j} X_i, X_j + \sum_{i,j,k} \omega_{i,j,k} X_i X_j X_k + \dots$$

is actually, $\widehat{\mathcal{A}(\pi_1(X))} \otimes \mathbb{R}$-valued and we can think of X_i as the generators of $\mathcal{A}(\pi_1(X))$.

4.3 The case of $\mathrm{Conf}_3(\mathbb{R}, n)$

Let us apply the construction of the last section to our configurations space $\mathrm{Conf}_3(\mathbb{R}, n)$. First of all, by proposition 4.1.5 and theorem 4.2.2 we have

$$\mathcal{V}_k^n \cong Hom(\mathbb{R}[PP_n]/I^{k+1}, \mathbb{R}) \cong F_k(H^0(B(A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))))),$$

for each k, and taking projective limits we have

$$\mathcal{V}^n \cong Hom(\widehat{\mathbb{R}[PP_n]}, \mathbb{R}) \cong H^0(B(A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n)))).$$

Now, we have an equivalence between Vassiliev invariants and the zero dimensional cohomology of $B(A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n)))$.

One good feature of the space $\mathrm{Conf}_3(\mathbb{R}, n)$ which allow us to give a better description of $H^0(B(A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))))$ is that $\mathrm{Conf}_3(\mathbb{R}, n)$ is a formal space in the sense of rational homotopy. In [34], Severs and White use discrete Morse theory to find a minimal Morse complex for $\mathrm{Conf}_3(\mathbb{R}, n)$, and by the discrete version of Witten deformation of Forman [16], this minimal Morse complex is a model of $\mathrm{Conf}_3(\mathbb{R}, n)$, hence, is formal. As a consequence, instead of take $A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))$ which is quite huge, we can use the cohomology $H_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))$ to compute $H^0(B(A_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))))$. In chapter 1, we review the cohomology ring $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$, which is generated by classes of basic partitioner posets in the first cohomology group. By the De Rham theorem, both cohomologies are isomorphic. If we denote by ω_P a differential form corresponding to the basic partitioner poset P, the cohomology $H_{DR}^*(\mathrm{Conf}_3(\mathbb{R}, n))$ is generated by the set of 1-forms ω_P for P basic partitioner posets of degree 1. We know by Björner and Welker in [11] that the cohomology groups are free, hence, dual to homology groups. In the same way, let be $X_P \in H_1(\mathrm{Conf}_3(\mathbb{R}, n))$ dual to the 1-form ω_P. Then the set of X_P's for all P basic partitioner poset of degree 1, forms a basis in $H_1(\mathrm{Conf}_3(\mathbb{R}, n))$. Let be $\mathbb{R}\langle\langle \mathbf{X}_{\mathcal{P}} \rangle\rangle$ be the algebra of non-commutative formal power series with indeterminates X_P. By theorem 4.2.4, by the formality of the space $\mathrm{Conf}_3(\mathbb{R}, n)$, and the fact that the cohomology $H^*(\mathrm{Conf}_3(\mathbb{R}, n))$ is generated in degree 1, exists an $\mathbb{R}\langle\langle \mathbf{X}_{\mathcal{P}} \rangle\rangle$-valued 1-form and is given by only the linear part, i.e.,

$$\omega = \sum_{P \in \mathcal{P}} \omega_P X_P. \tag{4.3.1}$$

Additionally, exists an ideal $\mathbf{J} \subset \mathbb{R}\langle\langle \mathbf{X}_{\mathcal{P}} \rangle\rangle$ and a ring homomorphism

$$Z : \mathbb{R}[PP_n] \longrightarrow \mathbb{R}\langle\langle \mathbf{X} \rangle\rangle / \mathbf{J},$$

given by

$$Z(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots + \int_\gamma \omega \cdots \omega + \cdots . \tag{4.3.2}$$

By proposition 1.3.16, remark 2.3 and corollary 4.2.5, this ring homomorphism is injective. In consequence we have the following direct results.

Theorem 4.3.1. *The formal homology connection 4.3.2 with* $\mathbb{R}\langle\langle \boldsymbol{X}_{\mathcal{P}} \rangle\rangle$*-valued 1-form given by 4.3.1, separates planar pure braids.*

Applying proposition 4.1.5, theorem 4.2.2 and by the formality of $\mathrm{Conf}_3(\mathbb{R}, n)$, we have isomorphisms

$$F_k(H^0(B(H^*_{DR}(\mathrm{Conf}_3(\mathbb{R}, n))))) \cong Hom(\mathbb{R}[PP_n]/I^{k+1}, \mathbb{R}) \cong \mathcal{V}^n_k$$

An the summands of the formal homology connection, contains all the homotopy iterated integrals of all lengths k. Hence,

Theorem 4.3.2. *The formal homology connection 4.3.2 contain all the Vassiliev invariants. Then,* Z *is the universal Vassiliev invariant.*

As an immediate consequence of theorems 4.3.1 and 4.3.2 we have the next corollary.

Corollary 4.3.3. *Vassiliev invariants classify planar pure braids.*

This last corollary was known by Merkov in [29], but his point of view is combinatorial. He represents Vassiliev invariants via Gauss diagram formulas, as Goussarov, Polyak and Viro do it for Vassiliev invariants for classical and virtual knots [17]. In this chapter, we obtain an universal Vassiliev invariant of planar pure braids as the Kontsevich integral is the universal Vassiliev invariant for knots and braids [27]. The big difference from the Kontsevich integral for planar braids and our universal Vassiliev invariant for planar pure braids is that we can not make computations in absence of differential forms. However, theoretically has a lot of consequences by the general theory of iterated integrals developed by Chen.

Remark 4.3.4. Bar-Natan in [3] has an expository note about Taylor expansions on groups. In that sense, the group PP_n we study, has a Taylor and faithful expansion given by the universal Vassiliev invariant 4.3.2.

Conclusions and Future Directions

The spaces $\mathrm{Conf}_3(\mathbb{R}, n)$ and groups PP_n are very similar to the usual configurations spaces $\mathcal{F}(\mathbb{C}, n)$ and the pure braid groups P_n in many aspects. In particular,

- $\mathrm{Conf}_3(\mathbb{R}, n)$ is an Eilenberg-MacLane space $K(\pi, 1)$ (a new proof of this is given in the present thesis);
- its fundamental group has a geometric representation as planar braids;
- it is a formal space;
- PP_n is iterated semidirect products of subgroups;
- PP_n is torsion free;
- PP_n is residually nilpotent;
- the cohomology ring of $\mathrm{Conf}_3(\mathbb{R}, n)$ is quadratic;
- PP_n has a universal Vassiliev invariant por small cases given by a Chen-Kohno integral;
- Vassiliev invariants separate planar pure braids;
- closures of elements in PP_n represent plane curves;
- PP_n acts on a CAT(0) space; same is known for the usual braids B_n for small values of n.

Things that we know are different:

- the forgetful map $\mathrm{Conf}_3(\mathbb{R}, n) \to \mathrm{Conf}_3(\mathbb{R}, n-1)$ is not a fibration;
- the semidirect product decomposition of PP_n is not almost-direct (as shown in the present thesis).

Things that work for braids and knots which we wish to know for planar braids and plane curves:

- whether there are connections with operator algebras;
- whether there is a normalization of the universal Vassiliev invariant of PP_n that works for plane curves;
- whether there exist polynomial invariants, categorifications, and homologies;

- explicit descriptions of the group presentation, Malcev completion and the corresponding Hopf algebra;
- whether there are versions of quantum groups and associators;
- what plays the role of the Chern-Simons theory;
- what plays the role of the weight systems coming from Lie algebras.

Appendix A

Reidemeister-Schreier Process

Given a presentation of a group G and a subgroup $H < G$, the Reidemeister-Schreier process shows how to compute a presentation for H in terms of certain knowledge of its right (or left) cosets. For more details about the process, see [28] and [20].

A.1 The Algorithm

We give a full description of the Reidemeister-Schreier process following the appendix of Gæde in [20].

Let us first consider the case when $G = F(X)$ is a free group generated by a set X. Let L be a subgroup of $F(X)$.

Definition A.1.1 (Schreier transversal)**.** A (right) Schreier transversal of L is a subset $\mathscr{S} \subset F(X)$, consisting of exactly one representative from each (right) coset of L, and having the property that if $t \in \mathscr{S}$ can be written as a reduced word $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, then any subword of t is also in $\mathscr{S}$, i.e., $1, x_1^{\epsilon_1}, x_1^{\epsilon_1}x_2^{\epsilon_2}, \dots, x_1^{\epsilon_1} \dots x_{n-1}^{\epsilon_{n-1}} \in \mathscr{S}$. It is not hard to show that a Schreier transversal can always be found.

For any $u \in F(X)$, let $\overline{u} \in \mathscr{S}$ such that $Lu = L\overline{u}$. Define the set $Y = \{(t,x) \in \mathscr{S} \times X \mid tx \neq \overline{tx}\}$, and the homomorphism $\phi : F(Y) \to L$ given by $\phi(t,x) = tx(\overline{tx})^{-1}$. In fact ϕ maps into $L \subset F(X)$, since

$$L\phi(t,x) = L(tx(\overline{tx})^{-1}) = L(\overline{tx})L\overline{(tx)^{-1}} = L(\overline{(tx)(tx)^{-1}}) = L.$$

Given $u = x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \in L$ written as a reduced word, we set up the following table

t_1	t_2	$\cdots$	t_n	t_{n+1}
$x_1^{\epsilon_1}$	$x_2^{\epsilon_2}$	$\cdots$	$x_n^{\epsilon_n}$	
v_1	v_2	$\cdots$	v_n	

The entries in the top row are computed in the following way:

$$t_1 = 1, \quad t_{i+1} = \overline{t_i x_i^{\epsilon_i}}, \quad 1 \leq i \leq n.$$

Since $Lt_{i+1} = L(\overline{t_i x_i^{\epsilon_i}}) = L(t_i x_i^{\epsilon_i}) = (Lt_i)x_i^{\epsilon_i} = \cdots = L(t_1)x_1^{\epsilon_1}\cdots x_i^{\epsilon_i} = L(x_1^{\epsilon_1}\cdots x_i^{\epsilon_i})$, We have that $t_{i+1} = \overline{x_1^{\epsilon_1}\cdots x_i^{\epsilon_i}}$, and in particular $t_{n+1} = \overline{x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}} = \overline{u} = 1$, because of $u \in L$.

The bottom row is computed as follows:

$$\binom{t}{x} = \begin{cases} (t,x) & \text{if} \quad tx \neq \overline{tx} \\ 1 & \text{otherwise,} \end{cases}$$

and for $1 \leq i \leq n$,

$$v_i = \begin{cases} \binom{t_i}{x_i} & \text{if} \quad \epsilon_i = +1 \\ \binom{t_{i+1}}{x_i}^{-1} & \text{if} \quad \epsilon_i = -1. \end{cases}$$

Note that $v_i \in F(Y)$. Now, if $\epsilon_i = +1$, we have

$$\phi(v_i) = t_i x_i^{\epsilon_i}(\overline{t_i x_i^{\epsilon_i}})^{-1} = t_i x_i^{\epsilon_i} t_{i+1}^{-1},$$

and if $\epsilon_i = -1$ we again have

$$\phi(v_i) = \overline{t_{i+1}x_i^{-\epsilon_i}}\, x_i^{-\epsilon_i} t_{i+1}^{-1} = t_i x_i^{\epsilon_i} t_{i+1}^{-1},$$

Therefore, if we let $v = v_1 \cdots v_n$, we have

$$\begin{aligned} \phi(v) &= \phi(v_1)\phi(v_2)\cdots\phi(v_n) \\ &= (t_1 x_1^{\epsilon_i} t_2^{-1})(t_2 x_2^{\epsilon_i} t_3^{-1})\cdots(t_n x_n^{\epsilon_i} t_{n+1}^{-1}) \\ &= t_1 x_1^{\epsilon_1}\cdots x_n^{\epsilon_n} t_{n+1}^{-1} \\ &= 1 x_1^{\epsilon_1}\cdots x_n^{\epsilon_n} 1 \\ &= u. \end{aligned}$$

So we define the rewriting map $\psi : L \to F(Y)$ by $\psi(u) = v$, and we have just seen that $\phi \circ \psi = \mathrm{id}_L$. In the definition of ψ we assumed that $x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}$ was a reduced word. It is easy to check that is not necessary.

Now, we want to show that $\psi \circ \phi = \mathrm{id}_{F(Y)}$. By construction, it's clear that ψ is a homomorphism, therefore it suffices to consider a generator $y = (t, x)$. Then $\phi(y) = tx(\overline{tx})^{-1}$, suppose $t = x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}$ and $\overline{tx} = \widetilde{x}_1^{\omega_1}\cdots\widetilde{x}_m^{\omega_m}$ as reduced words. Then because of the Schreier property of $\mathscr{S}$ we get the following table for $\phi(y)$

1	$x_1^{\epsilon_1}$	$x_1^{\epsilon_1}x_2^{\epsilon_2}$	$\cdots$	$x_1^{\epsilon_1}\cdots x_{n-1}^{\epsilon_{n-1}}$	t	$\widetilde{x}_1^{\omega_1}\cdots\widetilde{x}_m^{\omega_m}$	$\widetilde{x}_1^{\omega_1}\cdots\widetilde{x}_{m-1}^{\omega_{m-1}}$	$\cdots$	$\widetilde{x}_1^{\omega_1}$	1
$x_1^{\epsilon_1}$	$x_2^{\epsilon_2}$	$x_3^{\epsilon_3}$	$\cdots$	$x_n^{\epsilon_n}$	x	$\widetilde{x}_m^{\omega_m}$	$\widetilde{x}_{m-1}^{\omega_{m-1}}$	$\cdots$	$\widetilde{x}_1^{\omega_1}$	
1	1	1	$\cdots$	1	y	1	1	$\cdots$	1	

whereby $\psi(\phi(y)) = y$, which proves the following theorem.

Theorem A.1.2 (Nielsen-Schreier theorem)**.** *There is an isomorphism of groups* $L \cong F(Y)$.

We now turn to the general case of a subgroup H of a group G with presentations $\langle X \mid R \rangle$. This means that we have a short exact sequence

$$1 \longrightarrow \langle R \rangle \hookrightarrow F(X) \xrightarrow{\lambda} G \longrightarrow 1.$$

Let $L = \lambda^{-1}(H)$ be the subgroup of $F(X)$, and the preceding theorem gives us an isomorphism $\phi : F(Y) \to L$. Setting $\mu = \lambda \circ \phi : F(Y) \to H$, we then have the short exact sequence

$$1 \longrightarrow \ker \mu \hookrightarrow F(Y) \xrightarrow{\lambda} H \longrightarrow 1,$$

and all that remains is to find a set of relators $Q \subset F(Y)$, such that $\ker \mu = \langle Q \rangle$. So, let $v \in \ker \mu$, then $\phi(v) \in \ker \lambda = \langle R \rangle$, and hence

$$\phi(v) = (u_1 r_1^{\epsilon_1} u_1^{-1}) \cdots (u_n r_n^{\epsilon_n} u_n^{-1}),$$

for some $u_i \in F(X), r_i \in R$, and $\epsilon_i = \pm 1$. In order to be able to use ϕ^{-1}, we write u_i as $\widetilde{u}_i t_i$ with $\widetilde{u}_i \in L$, $t_i \in \mathscr{S}$, so that

$$\phi(v) \quad = \quad \widetilde{u}_1 (t_1 r_1 t_1^{-1})^{\epsilon_1} \widetilde{u}_1^{-1} \cdots \widetilde{u}_n (t_n r_n t_n^{-1})^{\epsilon_n} \widetilde{u}_n^{-1}, \qquad \text{(A.1.1)}$$

and since now both $\widetilde{u}_i \in L$ and $t_i r_i t_i^{-1} \in \ker \lambda \subset L$, applying ϕ^{-1} to A.1.1 and denoting by $v_i = \phi^{-1}(\widetilde{u}_i)$, we get

$$v = v_1 \phi^{-1}(t_1 r_1 t_1^{-1})^{\epsilon_1} v_1^{-1} \cdots v_n \phi^{-1}(t_n r_n t_n^{-1})^{\epsilon_n} v_n^{-1}.$$

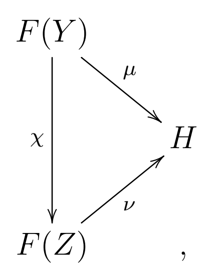
This shows that if we put $Q = \{\phi^{-1}(trt^{-1}) \mid t \in \mathscr{S}, r \in R\}$, we have $\ker \mu \subset \langle Q \rangle$. And the reverse inclusion is obvious, for $q \in Q$ we have $Q = \phi^{-1}(trt)$ for some $t \in \mathscr{S}$, $r \in R$, and then

$$\mu(q) = \lambda(trt^{-1}) = 1.$$

So $\ker \mu$ is a normal subgroup of $F(Y)$ containing the set Q, and $\langle Q \rangle$ is the smallest subgroup. This proves the following theorem.

Theorem A.1.3 (Reidemeister-Schreier)**.** *The subgroup H can be presented as $\langle Y | Q \rangle$.*

Sometimes it is possible to simplify this presentations by making use of the relators in R. We therefore imagine ourselves given a set Z, an epimorphism $\chi : F(Y) \to F(Z)$, and a homomorphism $\nu : F(Z) \to H$, such that the following diagram commutes,



,

i.e., $\nu \circ \chi = \mu = \lambda \circ \phi$. We see that ν is surjective, so that we have a short exact sequence

$$1 \longrightarrow \ker \nu \hookrightarrow F(Z) \overset{\nu}{\twoheadrightarrow} H \longrightarrow 1.$$

Let $w \in \ker \nu$, since χ is an epimorphism, $w = \chi(v)$ for some $v \in F(Y)$. As $1 = \nu(w) = \nu(\chi(v)) = \mu(v)$, we have $v \in \ker \mu = \langle Q \rangle$, so

$$v = (v_1 q_1^{\epsilon_1} v_1^{-1}) \cdots (v_n q_n^{\epsilon_n} v_n^{-1}),$$

with $v_i \in F(Y)$, $q_i \in Q$, $\epsilon = \pm 1$. If we denote by $w_i = \chi(v_i)$, we get

$$w = \chi(v) = w_1(\chi(q_1))^{\epsilon_1} w_1^{-1} \cdots w_n(\chi(q_n))^{\epsilon_n} w_n^{-1},$$

so if we put $P = \chi(Q) \setminus \{1\}$, we have $\ker \nu \subset \langle P \rangle$, and the reverse inculsion follows as before. We then have the next proposition.

Proposition A.1.4. *H can be presented as $\langle Z \,|\, P \rangle$.*

A.2 Presentation of $D_n^{(1)}$ y $D_n^{(2)}$

First recall that PB_n is given by the presentation $\langle \sigma_1, \ldots, \sigma_{n-1} | R_1 \cup R_2 \rangle$, where σ_i is the planar braid with only one double point like in figure 1.3, and relations

$$R_1 = \{(\sigma_j)^2 \quad | \quad 1 \leq j \leq n-1\} \tag{A.2.1}$$
$$R_2 = \{(\sigma_j \sigma_k)^2 \quad | \quad \text{if } |j-k| > 1, \text{for all } 1 \leq j, k \leq n-1\}. \tag{A.2.2}$$

Let $\varphi : PB_n \to \Sigma_n$ the homomorphism such that $\varphi(\sigma_j) = s_j$. We see Σ_{n-l} as the subgroup of Σ_n whose permutations map $n-i \mapsto n-i$ for $i = 0, 1, \ldots, l-1$. We set

$$D_n^{(l)} = \varphi^{-1}(\Sigma_{n-l}) \qquad l = 1, \ldots, n$$

Geometrically, $D_n^{(l)}$ consists of planar braids whose last l strands, do not change the order (see figure 2.1). We have that $D_n^{(n-1)} = D_n^{(n)} = PP_n$ and by convention $D_n^{(0)} = PB_n$. The chain of subgroups in Σ_n

$$\Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_2 \supset \Sigma_1 = \Sigma_0 = \{1\},$$

induce the chain of subgroups in PB_n

$$PB_n \supset D_n^{(1)} \supset \cdots \supset D_n^{(n-2)} \supset D_n^{(n-1)} = D_n^{(n)} = PP_n.$$

A Schreier transversal set of $D_n^{(l)}$ in $D_n^{(l-1)}$ is

$$\mathscr{S}_l = \{M_{n-l+1,i_l} \,|\, 0 \leq i \leq n-l\},$$

where $M_{n-l+1,i_l} = \sigma_{n-l}\sigma_{n-l-1}\cdots\sigma_{n-l-i_l}$ for $1 \le i \le n-l-1$ and $M_{n-l+1,0} = 1$.

Case $D_n^{(1)}$ in PB_n. Taking $l = 1$,

$$\mathscr{S}_1 = \{M_{n,i} \,|\, 0 \le i \le n-1\},$$

where $M_{n,i} = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_{n-i}$ for $1 \le i \le n-1$ and $M_{n,0} = 1$. We must now calculate $\overline{M_{n.i}\sigma_j}$ for all $0 \le i \le n-1$ and $1 \le j \le n-1$. It suffices to understand how the induced permutation acts on n. As

$$\begin{aligned} \varphi(M_{n,i}) &= (n-1,n)(n-2,n-1)\cdots(n-i,n-i+1) \\ &= (n-i,n-i+1,\ldots,n-1,n), \end{aligned}$$

and

$$n \overset{\varphi(M_{n,i})}{\longmapsto} n-i \overset{\varphi(M_{n,i})}{\longmapsto} \begin{cases} n-i & \text{if } j \ne n-i-1, n-i, \\ n-i-1 & \text{if } j = n-i-1, \\ n-i+1 & \text{if } j = n-i. \end{cases}$$

We must have

$$\overline{M_{n,i}\sigma_j} = \begin{cases} M_{n,i} & \text{if } i+j \ne n-1, n, \\ M_{n,i+1} & \text{if } i+j = n-1, \\ M_{n,i-1} & \text{if } i+j = n. \end{cases}$$

Now we can calculate $M_{n,i}\sigma_j(\overline{M_{n,i}\sigma_j})^{-1}$ for all $0 \le i \le n-1$, $1 \le j \le n-1$. It will be by cases.

- if $i + j \ne n-1, n$

$$\begin{aligned} M_{n,i}\sigma_j(\overline{M_{n,i}\sigma_j})^{-1} &= M_{n,i}\sigma_j(M_{n,i})^{-1} \\ &= (\sigma_{n-1}\cdots\sigma_{n-i})\sigma_j(\sigma_{n-i}^{-1}\cdots\sigma_{n-1}^{-1}) \end{aligned} \tag{A.2.3}$$

- if $i + j = n-1$

$$\begin{aligned} M_{n,i}\sigma_j(\overline{M_{n,i}\sigma_j})^{-1} &= M_{n,i}\sigma_j(M_{n,i+1})^{-1} \\ &= (\sigma_{n-1}\cdots\sigma_{j+1})\sigma_j(\sigma_j^{-1}\cdots\sigma_{n-1}^{-1}) \\ &= 1 \end{aligned}$$

- if $i + j = n$

$$\begin{aligned} M_{n,i}\sigma_j(\overline{M_{n,i}\sigma_j})^{-1} &= M_{n,i}\sigma_j(M_{n,i-1})^{-1} \\ &= (\sigma_{n-1}\cdots\sigma_j)\sigma_j(\sigma_{j+1}^{-1}\cdots\sigma_{n-1}^{-1}) \end{aligned} \tag{A.2.4}$$

Following the procedure described previously, we should therefore put $Y = \{(M_{n,i}, \sigma_j) | 0 \leq i \leq n-1, 1 \leq j \leq n-1, i+j \neq n-1\}$, and define the homomorphism $\phi : F(Y) \to D_n^{(1)}$ by

$$\phi(M_{n,i}, \sigma_j) = \begin{cases} \sigma_{n-1} \cdots \sigma_{n-i}\sigma_j\sigma_{n-i}^{-1} \cdots \sigma_{n-1}^{-1} & \text{if } i+j \neq n, \\ \sigma_{n-1} \cdots \sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1} \cdots \sigma_{n-1}^{-1} & \text{if } i+j = n, \end{cases}$$

However, as we mentioned, sometimes it is possible to reduce the generators using the relations of the bigger group, in this case PB_n. In this particular case, by the relation R_1 (see A.2.1) in the equation A.2.4, we have

$$\phi(M_{n,i}, \sigma_j) = 1 \quad \text{if } i+j = n$$

and in equation A.2.3, when $j+i < n-1$ we can use repeatedly times the relation R_2 (see A.2.2) to get

$$\begin{aligned} \phi(M_{n,i}, \sigma_j) &= \sigma_{n-1} \cdots \sigma_{n-i}\sigma_j\sigma_{n-i}^{-1} \cdots \sigma_{n-1}^{-1} \\ &= \sigma_{n-1} \cdots \sigma_{n-i+1}\sigma_j\sigma_{n-i}\sigma_{n-i}^{-1} \cdots \sigma_{n-1}^{-1} \\ &\vdots \\ &= \sigma_j, \end{aligned}$$

on the other hand, if $i+j > n$ and using also relation R_1

$$\begin{aligned} \phi(M_{n,i}, \sigma_j) &= \sigma_{n-1} \cdots \sigma_{n-i}\sigma_j\sigma_{n-i}^{-1} \cdots \sigma_{n-1}^{-1} \\ &= \sigma_{n-1} \cdots \sigma_{n-i+1}\sigma_j\sigma_{n-i}\sigma_{n-i}^{-1} \cdots \sigma_{n-1}^{-1} \\ &\vdots \\ &= \sigma_{n-1} \cdots \sigma_{j-1}\sigma_j\sigma_{j-1}^{-1} \cdots \sigma_{n-1}^{-1} \\ &= \sigma_{n-1} \cdots \sigma_{j-1}\sigma_j\sigma_{j-1} \cdots \sigma_{n-1} \end{aligned}$$

In conclusion we have that the generators can be reduced to

$$\begin{array}{cl} \sigma_j & \text{for all } 1 \leq j \leq n-2 \\ \sigma_{n-1} \cdots \sigma_{j-1}\sigma_j\sigma_{j-1} \cdots \sigma_{n-1} & \text{for all } 2 \leq j \leq n-1 \end{array} \tag{A.2.5}$$

Recalling that $\sigma_j = \sigma_j^{-1}$, the generators (A.2.5) are conjugation by $N_{n,j-1} = \sigma_{n-1} \cdots \sigma_{j-1}$. Then, if we write $N \cdot \sigma = N\sigma N^{-1}$, the generators (A.2.5) are rewritten as $N_{n,j-1} \cdot \sigma_j$ for all $2 \leq j \leq n-1$. We will therefore use the proposition A.1.4 to simplify the presentation. Let

$$Z = \{\sigma_1, \ldots, \sigma_{n-2}, N_{n,1} \cdot \sigma_2, \ldots, N_{n,n-2} \cdot \sigma_{n-1}\}$$

and define the homomorphism $\chi : F(Z) \to D_n^{(1)}$ by

$$\chi(M_{n,i},\sigma_j) = \begin{cases} \sigma_j & \text{if } i+j<n-1 \\ 1 & \text{if } i+j=n, \\ N_{n,j-1}\cdot\sigma_j & \text{if } i+j>n. \end{cases}$$

The final step is to compute the relators

$$P = \{\chi\circ\phi^{-1}(trt^{-1}) | t\in\mathscr{S}_1, r\in R = R_1\cup R_2\}\setminus\{1\}.$$

ϕ^{-1} is computed by means of tables as below. Note that

$$\chi\binom{M_{n,i}}{\sigma_j} = \begin{cases} \sigma_j & \text{if } i+j<n-1 \quad (1\le j\le n-2) \\ 1 & \text{if } i+j=n-1, \\ 1 & \text{if } i+j=n, \\ N_{n,j-1}\cdot\sigma_j & \text{if } i+j>n. \qquad (2\le j\le n-1) \end{cases} \tag{A.2.6}$$

For the relations, we compute $M_{n,i}(\sigma_j^2)M_{n,i}^{-1}$ for all $0\le i\le n-1, 1\le j\le n-1$ by cases as well as we did in the generators.

- If $i+j<n-1$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_j	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid \sigma_j \mid \sigma_j \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j=n-1$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i+1}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_j	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i+1}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid 1 \mid 1 \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j=n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i-1}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_j	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i-1}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid 1 \mid 1 \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j>n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_j	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid N_{n,j-1}\cdot\sigma_j \mid N_{n,j-1}\cdot\sigma_j \mid 1 \mid \dots \mid 1 \mid$$

Now we compute the relations $M_{n,i}(\sigma_j\sigma_k)M_{n,i}^{-1}$ for all $1 \le j,k \le n-1$ s.t. $|j-k|>1$, $0 \le i \le n-1$, and without lose of generality we can suppose that $k > j+1$.

- If $i+j<n-1$
  - ◇ if $i+k<n-1$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid \sigma_j \mid \sigma_k \mid \sigma_j \mid \sigma_j \mid 1 \mid \dots \mid 1 \mid$$

  - ◇ if $i+k=n-1$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i+1}$	$M_{n,i+1}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i+1}}{\sigma_j}$	$\binom{M_{n,i+1}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid \sigma_j \mid 1 \mid \sigma_j \mid 1 \mid 1 \mid \dots \mid 1 \mid$$

  - ◇ if $i+k=n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i-1}$	$M_{n,i-1}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i-1}}{\sigma_j}$	$\binom{M_{n,i-1}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid \sigma_j \mid 1 \mid \sigma_j \mid 1 \mid 1 \mid \dots \mid 1 \mid$$

◇ if $i+k>n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid \sigma_j \mid N_{n,k-1}\cdot\sigma_k \mid \sigma_j \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j=n-1$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i+1}$	$M_{n,i+1}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i+1}}{\sigma_k}$	$\binom{M_{n,i+1}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid 1 \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j=n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i-1}$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i-1}}{\sigma_k}$	$\binom{M_{n,i-1}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid 1 \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid \dots \mid 1 \mid$$

- If $i+j>n$

$M_{n,0}$	$M_{n,1}$	$\dots$	$M_{n,i-1}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$M_{n,i}$	$\dots$	$M_{n,1}$	$M_{n,0}$
σ_{n-1}	σ_{n-2}	$\dots$	σ_{n-i}	σ_j	σ_k	σ_j	σ_k	σ_{n-i}	$\dots$	σ_{n-1}	
$\binom{M_{n,0}}{\sigma_{n-1}}$	$\binom{M_{n,1}}{\sigma_{n-2}}$	$\dots$	$\binom{M_{n,i-1}}{\sigma_{n-i}}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_j}$	$\binom{M_{n,i}}{\sigma_k}$	$\binom{M_{n,i}}{\sigma_{n-i}}$	$\dots$	$\binom{M_{n,1}}{\sigma_{n-1}}$	

$$\chi \downarrow$$

$$1 \mid 1 \mid \dots \mid 1 \mid N_{n,j-1}\cdot\sigma_j \mid N_{n,k-1}\cdot\sigma_k \mid N_{n,j-1}\cdot\sigma_j \mid N_{n,k-1}\cdot\sigma_k \mid 1 \mid \dots \mid 1 \mid$$

Collecting the relations in the next tables:

$M_{n,i}(\sigma_j^2)M_{n,i}^{-1}$ for all $0 \leq i \leq n-1, 1 \leq j \leq n-1$		
$i+j < n-1$	$(\sigma_j)^2$	$1 \leq j \leq n-2$
$i+j = n-1$	1	
$i+j = n$	1	
$i+j > n$	$(N_{n,j-1} \cdot \sigma_j)^2$	$2 \leq j \leq n-1$

$M_{n,i}(\sigma_j\sigma_k)^2M_{n,i}^{-1}$ for all $1 \leq j,k \leq n-1$ s.t. $k > j+1$, $0 \leq i \leq n-1$			
$i+j < n-1$	$i+k < n-1$	$(\sigma_j\sigma_k)^2$	$1 \leq j,k \leq n-2$
	$i+k = n-1$	$(\sigma_j)^2$	$1 \leq j \leq n-2$
	$i+k = n$	$(\sigma_j)^2$	$1 \leq j \leq n-2$
	$i+k > n$	$[\sigma_j(N_{n,k-1} \cdot \sigma_k)]^2$	$1 \leq j \leq n-2$, $2 \leq k \leq n-1$, $k > j+2$
$i+j = n-1$		$(N_{n,k-1} \cdot \sigma_k)^2$	$2 \leq k \leq n-1$
$i+j = n$		$(N_{n,k-1} \cdot \sigma_k)^2$	$2 \leq k \leq n-1$
$i+j > n$		$[(N_{n,j-1} \cdot \sigma_j)(N_{n,k-1} \cdot \sigma_k)]^2$	$2 \leq j,k \leq n-1$, $k > j+1$

Re-indexing, $N_{n,j} \cdot \sigma_{j+1}$ for all $1 \leq j \leq n-2$, the presentation of $D_n^{(1)}$ looks like:

Proposition A.2.1. *The subgroup $D_n^{(1)}$ has a presentation with*

$$\begin{array}{ll} generators: & \{\sigma_j \quad | \quad j \in [1,n-2]\} \quad and \quad \{N_{n,j} \cdot \sigma_{j+1} \quad | \quad j \in [1,n-2]\} \\ relations: & (\sigma_j)^2 \\ & (N_{n,j} \cdot \sigma_{j+1})^2 \\ & (\sigma_j\sigma_k)^2 \quad |j-k| > 1 \\ & [(N_{n,j} \cdot \sigma_{j+1})(N_{n,k} \cdot \sigma_{k+1})]^2 \quad |j-k| > 1 \\ & [\sigma_j(N_{n,k} \cdot \sigma_{k+1})]^2 \quad k > j+1 \end{array}$$

Case $D_n^{(2)}$ in $D_n^{(1)}$. The computations are analogue to the laste case. We summarize them in the tables in the proof.

Proposition A.2.2. *The subgroup $D_n^{(2)}$ has a presentation with*

$$\begin{array}{ll} generators: & \{\sigma_j \quad | \quad j \in [1,n-3]\} \\ & \{N_{n-1,j} \cdot \sigma_{j+1} \quad | \quad j \in [1,n-3]\} \\ & \{M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1} \quad | \quad j \in [1,n-3], i_2 \in [0,n-4], i_2 < n-2-j\} \\ & \{N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3 \quad | \quad j \in [1,n-2]\} \\ & \{N_{n-1,j}N_{n,j+1} \cdot \sigma_{j+2} \quad | \quad j \in [1,n-3]\} \end{array} \tag{A.2.7}$$

$$
\begin{array}{lll}
relations: & (\sigma_j)^2 & \\
& (N_{n-1,j}\cdot\sigma_{j+1})^2 & \\
& (N_{n-1,j}N_{n,j+1}\cdot\sigma_{j+2})^2 & \\
& (M_{n-1,i_2}N_{n,j}\cdot\sigma_{j+1})^2 & \\
& (\sigma_j\sigma_k)^2 & |j-k|>1 \\
& [\sigma_j(N_{n-1,k}\cdot\sigma_{k+1})]^2 & k>j+1 \\
& [\sigma_j(N_{n-1,k}N_{n,k+1}\cdot\sigma_{k+2})]^2 & k>j+1 \\
& [\sigma_j(M_{n-1,i}N_{n,k}\cdot\sigma_{k+1})]^2 & k>j+1 \\
& [\sigma_j, N_{n-1,k+1}N_{n,k+2}\cdot(\sigma_{k+1}\sigma_k)^3] & k>j+1 \\
& [(N_{n-1,j}\cdot\sigma_{j+1})(N_{n-1,k}\cdot\sigma_{k+1})]^2 & |j-k|>1 \\
& [(N_{n-1,j}\cdot\sigma_{j+1})(N_{n-1,k}N_{n,k+1}\cdot\sigma_{k+2})]^2 & k>j+1 \\
& [(M_{n-1,i_2}N_{n,j}\cdot\sigma_{j+1})(M_{n-1,i_2}N_{n,k}\cdot\sigma_{k+1})]^2 & |j-k|>1 \\
& [(M_{n-1,i_2}N_{n,j}\cdot\sigma_{j+1})(N_{n-1,k}N_{n,k+1}\cdot\sigma_{k+2})]^2 & k>j+1 \\
& [(N_{n-1,j}N_{n,j+1}\cdot\sigma_{j+2})(N_{n-1,k}N_{n,k+1}\cdot\sigma_{k+2})]^2 & |j-k|>1 \\
& (N_{n-1,k+1}N_{n,j}\cdot\sigma_{j+1})(N_{n-1,k+1}N_{n,k+2}\cdot(\sigma_{k+1}\sigma_k)^3) & k>j+1 \\
& \quad = (N_{n-1,k+1}N_{n,k+2}\cdot(\sigma_{k+1}\sigma_k)^3)(N_{n-1,k}N_{n,j}\cdot\sigma_{j+1}) & \\
& [N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3, N_{n-1,k}N_{n,k+1}\cdot\sigma_{k+2}] & k>j.
\end{array}
$$

Proof. It follows by applying the Reidemeister-Schreier process to $D_n^{(1)}$ with Schreier transversal set $\mathscr{S}_2 = \{M_{n-1,i} | 0 \leq i \leq n-2\}$, where $M_{n-1,i} = \sigma_{n-2}\sigma_{n-3}\cdots\sigma_{n-1-i}$ for $1 \leq i \leq n-2$ and $M_{n-1,0} = 1$. We obtain the resulting morphisms:

For the first family of generators $\{\sigma_j\}_{j=1}^{n-2}$

$$
\chi\binom{M_{n-1,i}}{\sigma_j} = \begin{cases} \sigma_j & \text{if } i+j<n-2, \quad (1\leq j\leq n-3) \\ 1 & \text{if } i+j=n-2, \\ 1 & \text{if } i+j=n-1, \\ N_{n-1,j-1}\cdot\sigma_j & \text{if } i+j>n-1. \quad (2\leq j\leq n-2) \end{cases} \tag{A.2.8}
$$

For the second family of generators $\{N_{n,j}\cdot\sigma_{j+1}\}_{j=1}^{n-2}$

$$
\chi\binom{M_{n-1,i}}{N_{n,j}\cdot\sigma_{j+1}} = \begin{cases} M_{n-1,i_2}N_{n,j}\cdot\sigma_{j+1} & \text{if } i+j<n-2, \quad (1\leq j\leq n-3) \\ N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_{j+1}\sigma_j)^3 & \text{if } i+j=n-2, \quad (1\leq j\leq n-2) \\ N_{n-1,j+1}N_{n,j+2}\cdot(\sigma_j\sigma_{j+1})^3 & \text{if } i+j=n-1, \quad (1\leq j\leq n-2) \\ N_{n-1,j-1}N_{n,j}\cdot\sigma_{j+1} & \text{if } i+j>n-1. \quad (2\leq j\leq n-2) \end{cases} \tag{A.2.9}
$$

The relations are collected in the next tables

$M_{n-1,i}(\sigma_j^2)M_{n-1,i}^{-1}$ for all $0 \leq i \leq n-2, 1 \leq j \leq n-2$

$i+j<n-1$	$(\sigma_j)^2$	$1 \leq j \leq n-3$
$i+j=n-1$	1	
$i+j=n$	1	
$i+j>n$	$(N_{n-1,j-1} \cdot \sigma_j)^2$	$2 \leq j \leq n-2$

$M_{n-1,i}(N_{n,j} \cdot \sigma_{j+1})^2 M_{n-1,i}^{-1}$ for all $0 \leq i \leq n-2, 1 \leq j \leq n-3$

$i+j<n-1$	$(M_{n-1,i_2} N_{n,j} \cdot \sigma_{j+1})^2$	$1 \leq j \leq n-3$
$i+j=n-1$	$(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3)(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_j\sigma_{j+1})^3)$	$1 \leq j \leq n-2$
$i+j=n$	$(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_j\sigma_{j+1})^3)(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3)$	$1 \leq j \leq n-2$
$i+j>n$	$(N_{n-1,j-1}N_{n,j} \cdot \sigma_{j+1})^2$	$2 \leq j \leq n-2$

$M_{n-1,i}(\sigma_j\sigma_k)^2 M_{n-1,i}^{-1}$ for all $1 \leq j,k \leq n-2$ s.t. $k>j+1$, $0 \leq i \leq n-2$

$i+j<n-1$	$i+k<n-1$	$(\sigma_j\sigma_k)^2$	$1 \leq j,k \leq n-3$
	$i+k=n-1$	$(\sigma_j)^2$	$1 \leq j \leq n-3$
	$i+k=n$	$(\sigma_j)^2$	$1 \leq j \leq n-3$
	$i+k>n$	$[\sigma_j(N_{n,k-1} \cdot \sigma_k)]^2$	$1 \leq j \leq n-3$, $2 \leq k \leq n-2$, $k>j+2$
$i+j=n-1$		$(N_{n,k-1} \cdot \sigma_k)^2$	$2 \leq k \leq n-2$
$i+j=n$		$(N_{n,k-1} \cdot \sigma_k)^2$	$2 \leq k \leq n-2$
$i+j>n$		$[(N_{n,j-1} \cdot \sigma_j)(N_{n,k-1} \cdot \sigma_k)]^2$	$2 \leq j,k \leq n-2$, $k>j+1$

$M_{n-1,i}(\sigma_j(N_{n,k} \cdot \sigma_{k+1}))^2 M_{n-1,i}^{-1}$ for all $1 \leq j,k \leq n-2$ s.t. $k>j+1$, $0 \leq i \leq n-2$

$i+j<n-1$	$i+k<n-1$	$[\sigma_j(M_{n-1,i_2}N_{n,k} \cdot \sigma_{k+1})]^2$	$1 \leq j,k \leq n-3$
	$i+k=n-1$	$\sigma_j(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3)\sigma_j(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_k\sigma_{k+1})^3)$	$1 \leq j \leq n-3$
	$i+k=n$	$\sigma_j(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_k\sigma_{k+1})^3)\sigma_j(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3)$	$1 \leq j \leq n-3$
	$i+k>n$	$[\sigma_j(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})]^2$	$1 \leq j \leq n-3$, $2 \leq k \leq n-2$, $k>j+2$
$i+j=n-1$		$(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})^2$	$2 \leq k \leq n-2$
$i+j=n$		$(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})^2$	$2 \leq k \leq n-2$
$i+j>n$		$[(N_{n,j-1} \cdot \sigma_j)(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})]^2$	$2 \leq j,k \leq n-2$, $k>j+1$

$M_{n-1,i}((N_{n,j} \cdot \sigma_{j+1})(N_{n,k} \cdot \sigma_{k+1}))^2 M_{n-1,i}^{-1}$ for all $1 \leq j,k \leq n-2$ s.t. $k>j+1$, $0 \leq i \leq n-2$

$i+j<n-1$	$i+k<n-1$	$[(M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1})(M_{n-1,i_2}N_{n,k} \cdot \sigma_{k+1})]^2$	$1 \leq j,k \leq n-3$
	$i+k=n-1$	$(\underbrace{M_{n-1,i_2}}_{N_{n-1,k+1}} N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3)(\underbrace{M_{n-1,i_2+1}}_{N_{n-1,k}} N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_k\sigma_{k+1})^3)$	$1 \leq j \leq n-3$
	$i+k=n$	$(\underbrace{M_{n-1,i_2}}_{N_{n-1,k}} N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_k\sigma_{k+1})^3)(\underbrace{M_{n-1,i_2+1}}_{N_{n-1,k+1}} N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k+1}N_{n,k+2} \cdot (\sigma_{k+1}\sigma_k)^3)$	$1 \leq j \leq n-3$
	$i+k>n$	$[(M_{n-1,i_2}N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})]^2$	$1 \leq j \leq n-3$, $2 \leq k \leq n-2$, $k>j+2$
$i+j=n-1$		$(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3)(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_j\sigma_{j+1})^3)(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})$	$2 \leq k \leq n-2$
$i+j=n$		$(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_j\sigma_{j+1})^3)(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})(N_{n-1,j+1}N_{n,j+2} \cdot (\sigma_{j+1}\sigma_j)^3)(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})$	$2 \leq k \leq n-2$
$i+j>n$		$[(N_{n-1,j-1}N_{n,j} \cdot \sigma_{j+1})(N_{n-1,k-1}N_{n,k} \cdot \sigma_{k+1})]^2$	$2 \leq j,k \leq n-2$, $k>j+1$

□

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