CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
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Difusión Ultramétrica, Paisajes de Energía, Semigrupos de Feller y Operadores Pseudodiferenciales sobre p-ádicos

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Ultrametric Diffusion, Energy Landscapes, Feller Semigroups and Pseudodifferential Operators Over p-adics

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To my parents, brethren and nephews
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Resumen de la tesis

Durante los últimos treinta años ha existido un gran interés en el análisis p-ádico debido a sus conexiones con la física matemática, ver por ejemplo, [4]-[8], [17], [18], [41], [50]. Muchos de los modelos físicos propuestos utilizan ecuaciones pseudodiferenciales p-ádicas, como por ejemplo, modelos que describen la "difusión" sobre los p-ádicos o espacios ultramétricos. Más precisamente, Avetisov entre otros, proponen un nuevo tipo de modelos para ciertos sistemas complejos jerárquicos utilizando análisis p-ádico, ver por ejemplo, [3]-[8]. Por esta razón, entre otras, las ecuacione pseudodiferenciales se han estudiado intensamente en los últmos tiempos, vease por ejemplo [1], [32], [33], [35], [58] y sus referencias.

Durante los últimos diez años Zuñiga-Galindo y sus colaboradores han estudiado intensamente las ecuaciones pseudodiferenciales y los procesos estocásticos adjuntos relacionados con los modelos propuestos por Avetisov, entre otros, ver por ejemplo, [12], [15], [16], [26], [38], [39], [46], [47], [48], [54], [55], [56], [58], [59].
En esta disertación continuamos la investigación matemática de los modelos de Avetisov, entre otros. Esta disertación está organizada en dos partes. En la primera parte, estudiaremos ciertas ecuaciones pseudodiferenciales p-ádicas conectadas con paisajes de energía del tipo exponencial. Probaremos que la solución fundamental de estas ecuaciones son funciones de densidad de transición de procesos de Lévy con espacio de estado $\mathbb{Q}_p^n$, estudiaremos algunos aspectos de estos procesos, incluyendo el problema del tiempo del primer retorno. En la segunda parte, estudiaremos una nueva clase de operadores pseudodiferenciales no-Arquimedianos con coeficientes variables cuyos símbolos son funciones definidas negativas. Probaremos que estos operadores se extienden a generadores de semigrupos de Feller. También estudiaremos el problema de Cauchy para ciertas ecuaciones pseudodiferenciales naturalmente asociadas con estos operadores, mostrando que la solución fundamental de la

ecuación homogénea asociada está conectada con procesos de Lévy. Los resultados de esta disertación están destinados a ser publicados en dos artículos, escrito en cooperación con mi profesor asesor de doctorado Dr. W. A. Zúñiga-Galindo, see [48]-[47].

A lo largo de este trabajo, denotaremos el cuerpo de los números racionales por $\mathbb{Q}$, y por p un número primo fijado. El cuerpo de los números p-ádicos $\mathbb{Q}_p$ is definido como la completación del cuerpo de los números racionales $\mathbb{Q}$ con respecto a la norma p-ádica $|\cdot|_p$, la cual se define para $x \in \mathbb{Q}$ como

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

donde a y b son enteros coprimos con p. El entero $\gamma := ord(x)$, con $ord(0) := +\infty$, es llamado el orden p-ádico de x. Extendemos la norma $p-$ádica a $\mathbb{Q}_p^n$ tomando

$$||x||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

Puesto que $(\mathbb{Q}_p^n, +)$ es un grupo topológico localmente compacto, existe una medida de Haar aditiva $d^n x$, la cual es invariante bajo traslaciones, esto es, $d^n(a + x) = d^n(x)$, $a \in \mathbb{Q}_p^n$. Si la medida $d^n x$ es normalizada por la condición $\int_{\mathbb{Z}_p^n} d^n x = 1$, entonces $d^n x$ es ínica, donde $\mathbb{Z}_p^n$ denota la bola unitaria de $\mathbb{Q}_p^n$.

Este hecho nos permite usar el análisis armónico sobre $\mathbb{Q}_p^n$, ver por ejemplo, [40], [44], [51]. Denotaremos por $\mathcal{F}_{x\to\xi}$ la transformada de Fourier en $\mathbb{Q}_p^n$.

Un operador pseudodiferencial p-ádico tiene la forma $\mathcal{F}^{-1}_{\xi\to x}(a(\xi)\mathcal{F}_{x\to\xi}\varphi)$, donde la función $a(\xi)$, $\xi \in \mathbb{Q}_p^n$, es llamada el símbolo del operador, $\mathcal{F}^{-1}_{\xi\to x}$ es la transformada de Fourier inversa y φ es una función en un espacio adecuado.

La teoría de ecuaciones pseudodiferenciales p-ádicas está siendo motivada por el uso de modelos p-ádicos en física, ver por ejemplo, [1], [17], [32], [34], [51]. Muchos modelos propuestos implican ecuaciones pseudodiferenciales p-ádicas del tipo

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (Au)(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \geq 0, \\ \\ u\,(x, 0) = \varphi(x), \end{cases} \tag{1}$$

donde A es un operador pseudodiferencial, $\varphi(x)$ y $f(x,t)$, $x \in \mathbb{Q}_p^n$, $t \geq 0$, son funciones en un espacio adecuado. El caso más simple ocurre cuando $n = 1$ y A es el operador de Vladimirov: $(D^\alpha u)(x,t) = \mathcal{F}^{-1}_{\xi\to x}(|\xi|_p^\alpha \mathcal{F}_{x\to\xi} u(x,t))$, $\alpha > 0$. Vladimirov estudió ampliamente esta clase de operadores, mostrando también entre otras cosas, la existencia de soluciones fundamentales, ver [49], [51].

En [31], Kochubei estudió operadores pseudodiferenciales con símbolos de la forma $|f(\xi_1, \cdots, \xi_n)|_p^\alpha$, $\alpha > 0$, donde $f(\xi_1, \cdots, \xi_n)$ es una forma cuadrática tal que $f(\xi_1, \cdots, \xi_n) \neq 0$ cuando $|\xi_1|_p + \cdots + |\xi_n|_p \neq 0$. En [53], [57], Zúñiga-Galindo introduce operadores pseudodiferenciales con símbolos de la forma $|f(\xi_1, \cdots, \xi_n)|_p^\alpha$, $\alpha > 0$, donde $f(\xi_1, \cdots, \xi_n)$ es un polinomio no-constante, probando allí la existencia de soluciones fundamentales.

En [32], Kochubei estudió el problema de Cauchy (7), en el caso en el cual A es el operador de Vladimirov. El probó que la solución fundamental del problema de Cauchy es una densidad de transición de un proceso de Markov sin discontinuidades del segundo tipo. En adición, Kochubei construyó una gran clase de ecuaciones del tipo parabólico con coeficientes variables cuyas soluciones fundamentales tienen un significado probabilístico. Los resultados se han extendido en diferentes direcciones, ver por ejemplo [39], [56], [12], [15], [16]. En el libro [58], Zúñiga-Galindo presenta el estado del arte de la teoría de ecuaciones p-ádicas del tipo parabólico y sus procesos de Markov asociados. Ecuaciones del tipo (7), aparecen en modelos que describen la relajación en sistemas complejos tales como macromoléculas y proteínas. Un paradigma central en física de sistemas complejos afirma que la dinámica de una enorme clase de sistemas complejos pueden ser modelada como una caminata aleatoria en el paisaje de energía de los sistemas, ver por ejemplo [23]-[25], [33], [52], y sus referencias.

Un paisaje de energía (o simplemente un paisaje) es una función continua $\mathbb{U} : X \rightarrow \mathbb{R}$ que asigna a cada estado físico de un sistema su energía. En muchos casos podemos tomar X como un subconjunto de $\mathbb{R}^N$. El término paisaje complejo implica que la función $\mathbb{U}$ tiene muchos mínimos locales. En este caso el método de *cinénita entre cuencas* es aplicado, en este enfoque, el estudio de una caminata aleatoria en un paisaje complejo se basa en una descripción de la cinética generada por las transiciones entre grupos de estados (cuencas). Cuencas minimales corresponden a mínimos locales de energía, y cuencas grandes tienen una estructura jerárquica. Típicamente estos paisajes tienen un gran número de mínimos locales y la descripción de la dinámica en tales paisajes requieren una aproximación adecuada. La cinética entre cuencas ofrece una aceptable solución a este problema. En este sentido, la dinámica de un sistema complejo es codificada en una ecuación maestra la cual describe el comportamiento temporal de la probabilidad de salto entre dos estados del sistema, ver por ejemplo [33]. En [4]-[5] Avetisov entre otros, introducen una nueva clase de modelos para sistemas complejos basados en análisis p-ádico, estos modelos pueden ser aplicados, por ejemplo, a estudiar la relajación de sistemas complejos biológicos. La tasa de transición entre cuencas está deter-

minada por un factor de Arrhenius, el cual depende de la barrera de energía entre estas cuencas. Procedimientos para construir jerarquícamente cinéticas entre cuencas de cualquier paisaje de energía se han estudiado extensamente, ver por ejemplo, [9], [42], [43].
Usando estos métodos, un paisaje complejo es aproximado por un gráfico de desconexión (un árbol con raíz) y por una función en el árbol describiendo la distribución de las energías de activación. La dinámica del sistema es entonces codificada en un sistema de ecuaciones cinéticas de la forma:

$$\frac{\partial}{\partial t}u\left(i,t\right)=-\sum_{j}\left\{T(i,j)u\left(i,t\right)-T\left(j,i\right)u\left(j,t\right)\right\}v\left(j\right), \tag{2}$$

donde los índices i, j son los números de estados del sistema (que corresponden a los mínimos locales de energía), $T(i,j)\geq 0$ es la probabilidad por unidad de tiempo de una transición de i a j, y los $v(j)>0$ son los volúmenes de las cuencas. Para más detalles, el lector puede consultar [33] y sus referencias. Varios modelos de cinética entre cuencas y la dinámica jerárquica se han estudiado, ver por ejemplo [29], [33] y sus referencias, [37].
En [4]-[5] Avetisov, entre otros, desarrollaron una nueva clase de modelos de cinética entre cuencas utilizando difusión ultramétrica generada por operadores pseudodiferenciales p-ádicos. En estos modelos, el tiempo-evolución del sistema es controlado por una ecuación maestra de la forma

$$\frac{\partial u\left(x,t\right)}{\partial t}=\int_{\mathbb{Q}_p}\left\{j\left(x\mid y\right)u\left(y,t\right)-j\left(y\mid x\right)u\left(x,t\right)\right\}dy,\ x\in\mathbb{Q}_p,\ t\in\mathbb{R}_+, \tag{3}$$

donde la función $u\left(x,t\right):\mathbb{Q}_p\times\mathbb{R}_+\to\mathbb{R}_+$ es una distribución de densidad de probabilidad y la función $j\left(x\mid y\right):\mathbb{Q}_p\times\mathbb{Q}_p\to\mathbb{R}_+$ es la probabilidad de transición del estado y al estado x por unidad de tiempo. La ecuación maestra (9) es una versión continua de (8) obtenida de pasar el "límite continuo" en $\mathbb{Q}_p$ bajo las condiciones $v(j)=1$, $T(i,j)=j\left(|i-j|_p\right)$, para todo i,j, ver por ejemplo [33]. La transición de un estado y a un estado x puede percibirse como una superación de la barrera de energía que separa estos estados. En [4] una relación del tipo de Arrehenius fue utilizada, es decir,

$$j\left(x\mid y\right)\sim A(T)\exp\left\{-\frac{\mathbb{U}\left(x\mid y\right)}{kT}\right\},$$

donde $\mathbb{U}\left(x\mid y\right)$ es la altura de las barreras de activación para la transición del estado y al estado x, k es la constante de Boltzmann y T es la temperatura. Esta fórmula establece una relación entre la estructura del paisaje de energía

$\mathbb{U}(x \mid y)$ y la función de transición $j(x \mid y)$. El caso $j(x \mid y) = j(y \mid x)$ corresponde a un *paisaje de energía degenerado*. En este caso la ecuación maestra (9) toma la forma

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{Q}_p} j\left(|x-y|_p\right)\{u(y,t)-u(x,t)\}\,dy, \tag{4}$$

donde $j\left(|x-y|_p\right) = \frac{A(T)}{|x-y|_p}\exp\left\{-\frac{\mathbb{U}\left(|x-y|_p\right)}{kT}\right\}$. Escogiendo $\mathbb{U}$ convenientemente, varios paisajes de energía se pueden obtener. Siguiendo [4], existen tres paisajes básicos: (i) (logarítmico) $j\left(|x-y|_p\right) = \frac{1}{|x-y|_p \ln^{\alpha}\left(1+|x-y|_p\right)}$, $\alpha > 1$ (ii) (lineal) $j\left(|x-y|_p\right) = \frac{1}{|x-y|_p^{\alpha+1}}$, $\alpha > 0$, (iii) (exponencial) $j\left(|x-y|_p\right) = \frac{e^{-\alpha|x-y|_p}}{|x-y|_p}$, $\alpha > 0$.

En nuestra opinión, la novedad y la relevancia de los modelos idealistas de Avetisov, entre otros, provienen de dos hechos: primero, ellos codifican, en una lenguaje matemático, el paradigma central en física que afirma que la dinámica de varios sistemas complejos se puede describir como una caminata aleatoria sobre un árbol con raíz; segundo, estos modelos dan una descripción de los tipos característicos de relajación de sistemas complejos.

Los modelos originales de Avetisov, entre otros, fueron formulados en dimensión uno. Las ecuaciones maestras correspondientes fueron obtenidas estudiando una caminata aleatoria en un gráfico de desconexión, que proviene de "una" sección transversal del paisaje de energía de un sistema, usando el proceso de límite mencionado anteriormente, el árbol se convierte en $\mathbb{Q}_p$. En [23] Frauenfelder, entre otros, han señalado explícitamente que usar árboles con raíz (gráficos de desconexión) construidos a partir de "una" sección transversal de un paisaje de energía de un sistema complejo es engañoso en dos aspectos: "parece que una transición desde un estado inicial i a un estado final j debe seguir un camino único, y segundo, la entropía no juega un papel, " ver [23, p. 98 y las figuras 11.3 y 11.4]. Si consideramos varias secciones transversales de un paisaje de energía, cada uno de ellos da lugar a un gráfico de desconexión, y por lo tanto los índices i, j en la ecuación (9) son vectores que se ejecutan en el producto cartesiano de los gráficos de desconexión; en el modelo continuo, ver [10], las variables x, y se ejecutan a través de $\mathbb{Q}_p^n$. Por lo tanto, las ecuaciones maestras de los modelos de Avetisov, entre otros, debería estudiarse en una dimensión arbitraria por razones fśicas y por generalidad matemática. Este programa ha sido desarrollado por Zúñiga-Galindo y sus colaboradores en los últimos años, ver por ejemplo, [12], [15], [16], [39], [46], [56], [59].

En el capítulo 2, estudiamos algunos aspectos de la dinámica de "caminatas aleatorias" asociadas con los paisajes exponenciales introducidos en [4]. La correspondiente ecuación maestra toma la forma

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = (J * u(\cdot,t))(x) - u(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \geq 0 \quad (A) \\ \\ u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (B) \end{cases} \tag{5}$$

donde $J : \mathbb{Q}_p^n \to \mathbb{R}_+$ pertenece a una familia de funciones, dependiendo de varios parámetros, que codifican la estructura de un paisaje de energía. La familia de paisajes estudiados aquí son una generalización "integrable" de los paisajes exponenciales unidimensionales introducido en [4]. El primer paso es establecer que (11)-(A) es una ecuación de ultradifusión, es decir, que su solución fundamental es la densidad de transición de un proceso de Markov con espacio de estado $\mathbb{Q}_p^n$. Es importante mencionar aquí, que este hecho ha sido establecido rigurosamente solo para los paisajes lineales, en este caso (11)-(A) se convierte en una ecuación de calor p-ádica, y estas ecuaciones y sus procesos de Markov han sido estudiados intensamente últimamente, ver [12], [16], [15], [32], [51], [59], [56].

Para los paisajes exponenciales y logarítmicos en [4], la función j es solamente localmente integrable, y así el dominio natural del operador en el lado derecho de (10) no es evidente, y no se da en [4], por lo tanto el hecho que (10) es una ecuación de ultradifusión en el caso de paisajes exponenciales y logarítmicos no se estableció en [4]. Al imponer a la función J la condición de ser integrable, el operador en el lado derecho de (11)-(A) se convierte en un operador lineal acotado sobre L^ρ, $1 \leq \rho \leq \infty$, y en el caso de paisajes del tipo exponencial, mostramos que (11)-(A) es una ecuación de ultradifusión. En esta tesis, probaremos que la solución fundamental de (11)-(A) es la densidad de transición de un proceso de Lévy, ver Teorema 2.3.9.

Vale la pena mencionar que la contraparte real de la ecuación (11)-(A) se ha estudiado intensamente, en este contexto la ecuación se ha utilizado para modelar procesos de difusión, ver por ejemplo [2].

En el prefacio de [2], los autores muestran que para ciertos procesos de Markov sus funciones de densidad de transición satisfacen (11)-(A). En esta tesis, investigamos lo contrario de esta situación, en un sentido p-ádico.

También estudiamos el problema del tiempo del primer retorno para procesos estocásticos $\mathfrak{J}(t,\cdot)$ cada uno definido en un espacio de probabilidad dado $(\Omega, \mathcal{F}, P)$, cuya función de densidad de transición satisface (11)-(A)-(B), con $u_0(x)$ igual a la función característica de $\mathbb{Z}_p^n$. Más precisamente, estudiamos la variable aleatoria $\tau_{\mathbb{Z}_p^n}(\omega)$, $\omega \in \Omega$, definida como el tiempo más pequeño en el cual una trayectoria de $\mathfrak{J}(t,\omega)$, $\omega \in \Omega$, retorna a $\mathbb{Z}_p^n$. Mostrando que cualquier trayectoria de cualquiera de estos procesos retornan a $\mathbb{Z}_p^n$, casi seguro, ver Teorema 2.4.7.

Es importante mencionar que nuestros resultados no cubren todos los paisajes exponenciales introducidos aquí, esto requerirá el estudio de ecuaciones del tipo (11)-(A)-(B) en un sentido más general. Finalmente, los resultados en [16] presentan una generalización n-dimensional de los paisajes de energía lineales de [4], esta generalización se logró generalizando la ecuación del calor p-ádica. Hay varias diferencias importantes entre esta tesis y [16]. Primero, los operadores (Laplacianos) considerados in [16] son operadores no acotados y densamente definidos en subespacios adecuados de $L^2\left(\mathbb{Q}_p^n\right)$, mientras que los operadores estudiados aquí son operadores acotados definidos en $L^\rho\left(\mathbb{Q}_p^n\right)$, $1 \leq \rho \leq \infty$; segundo, las soluciones fundamentales en [16] son funciones integrables, mientras que aquí las soluciones fundamentales son distribuciones, lo cual hace que el estudio de los procesos estocásticos correspondientes sea más complicado.

En el Capítulo 3, estudiamos una gran clase de operadores pseudodiferenciales teniendo símbolos definidos negativos, los cuales tienen asociados semigrupos de Feller. Estos operadores tienen la forma

$$(A\,(x,t,\partial)\,\varphi)\,(x) = -\mathcal{F}^{-1}_{\xi\to x}\left(a\,(x,t,\xi)\,\mathcal{F}_{x\to\xi}\varphi\right)$$

donde el símbolo $\xi \to a\,(x,t,\xi)$ es una función definida negativa, para $x \in \mathbb{Q}_p^n$ y $t \in \mathbb{R}_+$. Un ejemplo típico de tales símbolos son funciones de la forma $\sum_{j=1}^m b_j(x,t)\boldsymbol{\psi}_j\,(\xi)$, donde $b_j : \mathbb{Q}_p^n \times \mathbb{R}_+ \to \mathbb{R}_+$ son funciones satisfaciendo

$$0 < \inf_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) \leq b_j(x,t) \leq \sup_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) < \infty,$$

y los $\boldsymbol{\psi}_j : \mathbb{Q}_p^n \to \mathbb{R}_+$ son funciones radiales, continuas y definidas negativas. En el sentido no-Arquimediano existen funciones definidas negativas "exóticas" tales como $\exp\left(\exp\left(\exp\sum_{i=0}^\infty a_j\,\|\xi\|_p^{\alpha_j}\right)\right)$ donde $\sum_{i=0}^\infty a_j y^{\alpha_j}$ es una serie real convergente con $a_j > 0$, $\alpha_j > 0$ and $\lim_{j\to\infty}\alpha_j = \infty$, ver Lema 3.1.8. Este tipo de funciones no se tienen en la contraparte Arquimediana.

Sea $P(x,t,\partial)$ denotando un operador pseudodiferencial cuyo símbolo es una función definida negativa $p\,(x,t,\xi)$ en la variable ξ. Nosotros introduciremos una nueva clase de espacios de funciones $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ asociados a funciones definidas negativas $\boldsymbol{\psi}$. Estos espacios son generalizaciones de los espacios $H_\infty\,(\mathbb{R})$ introducidos por Zúñiga-Galindo en [55]. Los espacios $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ son espacios nucleares Hilbert contables. Probaremos que $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ es el dominio natural para un operador del tipo $P(x,t,\partial)$ con un $\boldsymbol{\psi}$ adecuado. Bajo hipótesis débiles probaremos que $(P(x,t,\partial), B_{\boldsymbol{\psi},\infty}\,(\mathbb{R}))$ tienen una extensión cerrada a

$C_0\left(\mathbb{Q}_p^n,\mathbb{R}\right)$ (el $\mathbb{R}$-espacio vectorial de funciones continuas acotadas que se anulan en el infinito) el cual es el generador de un semigrupo de Feller, ver Teorema 3.3.12. En el caso en el cual el símbolo $p\left(x,t,\xi\right)$ no depende de t, probaremos que el problema de Cauchy

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = P(x,\partial)u(x,t) + f(x,t), \quad t \in [0,T],\, x \in \mathbb{Q}_p^n; \\ \\ u(x,0) = h(x) \in B_{\psi,\infty}\left(\mathbb{R}\right). \end{cases} \tag{6}$$

está bien definido y encontramos explícitamente el correspondiente semigrupo, el cual es un semigrupo de Feller, ver Teorema 3.4.4. Ecuaciones del tipo (12) aparecen como ecuaciones maestras en varios modelos que describen la dinámica de ciertos sistemas complejos jerárquicos, ver por ejemplo, [4]-[5], [16], [33] y sus referencias. Desde una perspectiva física, se espera que todas estas ecuaciones maestras deban describir la evolución de una densidad de probabilidad, nuestros resultados muestran que, de hecho, es el caso para una gran clase de símbolos. Como una consecuencia del trabajo de Kochubei, Zúñiga-Galindo y Chacón-Cortes, entre otros, una gran clase de ecuaciones pseudodiferenciales del tipo parabólico con coeficientes variables relacionados con procesos de Markov han sido estudiados, ver [32], [16], [15], [58] y sus referencias. Los resultados de la presente tesis complementan esta teoría. De hecho, la clase de ecuaciones consideradas aquí es en cierto sentido más grande que las clases considerada anteriormente. Sin embargo, necesitamos dos hipótesis que se mantienen para las ecuaciones estudiadas en [32], [58]. Primero, los operadores $P(x,t,\partial)$ preservan funciones real-valuadas, y segundo, los operadores $P(x,t,\partial)$ satisfacen el principio del máximo positivo. Estas hipótesis se cumplen para un clase grande de operadores pseudodiferenciales. En el sentido real Courrège estableció que todos los operadores pseudodiferenciales con símbolos definidos negativos satisfacen el principio del máximo positivo, ver [14]. Hoy en día, la contraparte no-Arquimediana de este resultado no es conocida.

Thesis Overview

During the last thirty years there has been a strong interest on p-adic analysis due to its connections with mathematical physics, see e.g. [4]-[8], [17], [18], [41], [50]. Many of the proposed physical models use p-adic pseudodifferential equations, for instance, models describing 'diffusion' on p-adic or ultrametric spaces. More precisely, Avetisov et al. proposed a new type of models for certain hiererarchical complex systems using p-adic analysis, see e.g. [3]-[8]. For this reason, among others, pseudodifferential equations have been studied intensively lately, see e.g. [1], [32], [33], [35], [58] and the references therein.
During the last ten years Zúñiga-Galindo and his collaborators have studied intensively the pseudodifferential equations and the attached stochastic processes related with the models proposed by Avetisov et al., see e.g. [12], [15], [16], [26], [38], [39], [46], [47], [48], [54], [55], [56], [58], [59]. In this dissertation we continue the mathematical investigation of Avetisov et al models. This dissertation is organized into two parts. In the first part, we study certain p-adic pseudodifferential equations connected with energy landscapes of exponential type. We show that the fundamental solutions of these equations are transition density functions of Lévy processes with state space $\mathbb{Q}_p^n$, we study some aspects of these processes, including the first passage time problem. In the second part, we study a new class of non-Archimedean pseudodifferential operators with variable coefficients whose symbols are negative definite functions. We prove that these operators extend to generators of Feller semigroups. We also study the Cauchy problem for certain pseudodifferential equations naturally associated with these operators, showing that the fundamental solution of the associated homogeneous equations are connected with Lévy processes. The results of this dissertation are intended to be published in two articles, written in cooperation with my doctoral advisor professor W. A. Zúñiga-Galindo, see [48]-[47].
Along this work, we denote the field of rational numbers by $\mathbb{Q}$, and by p a fixed prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of

the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined for $x \in \mathbb{Q}$ as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the p-adic order of x. We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

Since $(\mathbb{Q}_p^n, +)$ is a locally compact topological group, there exists an additive Haar measure $d^n x$, which is invariant under translations, i.e., $d^n(a + x) = d^n(x)$, $a \in \mathbb{Q}_p^n$. If the measure $d^n x$ is normalized by the condition $\int_{\mathbb{Z}_p^n} d^n x = 1$, then $d^n x$ is unique, where $\mathbb{Z}_p^n$ denotes the unit ball of $\mathbb{Q}_p^n$.
This fact allows us to use harmonic analysis on $\mathbb{Q}_p^n$, see e.g. [40], [44], [51]. We denote by $\mathcal{F}_{x\to\xi}$ the Fourier transform in $\mathbb{Q}_p^n$.
A p-adic pseudodifferential operators has the form $\mathcal{F}^{-1}_{\xi\to x}(a(\xi)\mathcal{F}_{x\to\xi}\varphi)$, where the function $a(\xi)$, $\xi \in \mathbb{Q}_p^n$, is called symbol of the operator, $\mathcal{F}^{-1}_{\xi\to x}$ is the inverse Fourier transform and φ is a function in a suitable space.
The theory of p-adic pseudodifferential equations is emerging motivated by the use of p-adic models in physic, see e.g. [1], [17], [32], [34], [51]. Many proposed models involve p-adic pseudodifferential equations of type

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (Au)(x,t) = f(x,t), \quad x \in \mathbb{Q}_p^n, \quad t \geq 0, \\ \\ u\,(x, 0) = \varphi(x), \end{cases} \tag{7}$$

where A is pseudodifferential operator, $\varphi(x)$ and $f(x,t)$, $x \in \mathbb{Q}_p^n$, $t \geq 0$, are functions in suitable spaces. The simplest case occurs when $n = 1$ and A is the Vladimirov operator: $(D^\alpha u)(x,t) = \mathcal{F}^{-1}_{\xi\to x}(|\xi|_p^\alpha \mathcal{F}_{x\to\xi} u(x,t))$, $\alpha > 0$. Vladimirov studied extensively this class of operators showing among other things, the existence of fundamental solutions, see [49], [51]. In [31], Kochubei studied pseudodifferential operators with symbols of the form $|f(\xi_1, \cdots, \xi_n)|_p^\alpha$, $\alpha > 0$, where $f(\xi_1, \cdots, \xi_n)$ is a quadratic form such that $f(\xi_1, \cdots, \xi_n) \neq 0$ when $|\xi_1|_p + \cdots + |\xi_n|_p \neq 0$. In [53], [57], Zúñiga-Galindo introduce pseudodifferential operators with symbols of the form $|f(\xi_1, \cdots, \xi_n)|_p^\alpha$, $\alpha > 0$, where $f(\xi_1, \cdots, \xi_n)$ is a non-constant polynomial and showed the existence of fundamental solutions.

In [32], Kochubei studied the Cauchy problem (7), in the case in which A is the Vladimirov operator. He showed that the fundamental solution of the Cauchy problem is a transition density of a Markov process without second kind discontinuities. In addition, Kochubei constructed a large class of parabolic-type equations with variable coefficients whose fundamental solutions have a probabilistic meaning. The results have been extended in different directions, see e.g. [39], [56], [12], [15], [16]. In the book [58], Zúñiga-Galindo presents state of art of the theory of p-adic parabolic-type equations and their associated Markov processes.
Equations of type (7), appeared in models describing the relaxation in complex systems such as macromolecules and proteins. A central paradigm in physics of complex systems asserts that the dynamics of a large class of complex systems can be modeled as a random walk in the energy landscape of the system, see e.g. [23]-[25], [33], [52], and the references therein.
An energy landscape (or simply a landscape) is a continuous function $\mathbb{U} : X \to \mathbb{R}$ that assigns to each physical state of a system its energy. In many cases we can take X to be a subset of $\mathbb{R}^N$. The term complex landscape means that function $\mathbb{U}$ has many local minima. In this case the method of *interbasin kinetics* is applied, in this approach, the study of a random walk on a complex landscape is based on a description of the kinetics generated by transitions between groups of states (basins). Minimal basins correspond to local minima of energy, and large basins have hierarchical structure. Typically these landscapes have a huge number of local minima and the description of the dynamics on such landscapes require an adequate approximation. The interbasin kinetics offers an acceptable solution to this problem. In this setting, the dynamics of a complex system is codified in a master equation which describes the temporal behavior of the jumping probability between two states of the system, see e.g. [33]. In [4]-[5] Avetisov et al. introduced a new class of models for complex systems based on p-adic analysis, these models can be applied, for instance, to study the relaxation of biological complex systems. The transition rate between basins is determined by the Arrhenius factor, which depends on the energy barrier between these basins. Procedures for constructing hierarchies of basins kinetics from any energy landscapes have been studied extensively, see e.g. [9], [42], [43]. By using these methods, a complex landscape is approximated by a *disconnectivity graph* (a rooted tree) and by a function on the tree describing the distribution of the activation energies. The dynamics of the system is then encoded in a system of kinetic

equations of the form:

$$\frac{\partial}{\partial t}u\left(i,t\right)=-\sum_{j}\left\{T(i,j)u\left(i,t\right)-T\left(j,i\right)u\left(j,t\right)\right\}v\left(j\right), \tag{8}$$

where the indices i, j number the states of the system (which correspond to local minima of energy), $T(i,j)\geq 0$ is the probability per unit time of a transition from i to j, and the $v(j)>0$ are the basin volumes. For further details the reader may consult [33, and the references therein]. Several models of interbasin kinetics and hierarchical dynamics have been studied, see e.g. [29], [33, and the references therein], [37].

In [4]-[5] Avetisov et al. developed new class of models of interbasin kinetics using ultrametric diffusion generated by p-adic pseudodifferential operators. In these models, the time-evolution of the system is controlled by a master equation of the form

$$\frac{\partial u\left(x,t\right)}{\partial t}=\int_{\mathbb{Q}_p}\left\{j\left(x\mid y\right)u\left(y,t\right)-j\left(y\mid x\right)u\left(x,t\right)\right\}dy,\ x\in\mathbb{Q}_p,\ t\in\mathbb{R}_+, \tag{9}$$

where the function $u\left(x,t\right):\mathbb{Q}_p\times\mathbb{R}_+\rightarrow\mathbb{R}_+$ is a probability density distribution, and the function $j\left(x\mid y\right):\mathbb{Q}_p\times\mathbb{Q}_p\rightarrow\mathbb{R}_+$ is the probability of transition from state y to the state x per unit time. Master equation (9) is a continuous version of (8) obtained from it by passing to a 'continuous limit' in $\mathbb{Q}_p$ under the conditions $v(j)=1$, $T(i,j)=j\left(\left|i-j\right|_p\right)$, for all i,j, see e.g. [33]. The transition from a state y to a state x can be perceived as overcoming the energy barrier separating these states. In [4] an Arrhenius type relation was used, that is,

$$j\left(x\mid y\right)\sim A(T)\exp\left\{-\frac{\mathbb{U}\left(x\mid y\right)}{kT}\right\},$$

where $\mathbb{U}\left(x\mid y\right)$ is the height of the activation barrier for the transition from the state y to state x, k is the Boltzmann constant and T is the temperature. This formula establishes a relation between the structure of the energy landscape $\mathbb{U}\left(x\mid y\right)$ and the transition function $j\left(x\mid y\right)$. The case $j\left(x\mid y\right)=j\left(y\mid x\right)$ corresponds to a *degenerate energy landscape*. In this case the master equation (9) takes the form

$$\frac{\partial u\left(x,t\right)}{\partial t}=\int_{\mathbb{Q}_p}j\left(\left|x-y\right|_p\right)\left\{u\left(y,t\right)-u\left(x,t\right)\right\}dy, \tag{10}$$

where $j\left(\left|x-y\right|_p\right)=\frac{A(T)}{\left|x-y\right|_p}\exp\left\{-\frac{\mathbb{U}\left(\left|x-y\right|_p\right)}{kT}\right\}$. By choosing $\mathbb{U}$ conveniently, several energy landscapes can be obtained. Following [4], there are three basic landscapes: (i) (logarithmic) $j\left(|x-y|_p\right) = \frac{1}{|x-y|_p \ln^{\alpha}\left(1+|x-y|_p\right)}$, $\alpha > 1$ (ii) (linear) $j\left(|x-y|_p\right) = \frac{1}{|x-y|_p^{\alpha+1}}$, $\alpha > 0$, (iii) (exponential) $j\left(|x-y|_p\right) = \frac{e^{-\alpha|x-y|_p}}{|x-y|_p}$, $\alpha > 0$.

In our opinion, the novelty and relevance of the idealistic models of Avetisov et al. come from two facts: first, they codify, in a mathematical language, the central physical paradigm asserting that the dynamics of several complex systems can be described as a random walk on a rooted tree; second, these models give a description of the characteristic types of relaxation of complex systems.

The original models of Avetisov et al. were formulated in dimension one. The corresponding master equations were obtained by studying a random walk on a disconnectivity graph, which comes from 'one' cross section of the energy landscape of a system, by using the above mentioned limit process, the tree becomes in $\mathbb{Q}_p$. In [23] Frauenfelder et al. have explicitly pointed out that using rooted trees (disconnectivity graphs) constructed from 'one' cross section of an energy landscape of a complex systems is misleading in two respects: "it appears that a transition from an initial state i to a final state j must follow a unique pathway, and second entropy does not play a role," see [23, p. 98 and figures 11.3 and 11.4]. If we consider several cross sections of an energy landscape, each of them gives rise to a disconnectivity graph, and hence the indices i, j in equation (9) are vectors running on the Cartesian product of the disconnectivity graphs; in the continuous model, see (10), the variables x, y run through $\mathbb{Q}_p^n$. Therefore, the master equations for the Avetisov et al. models should be studied in arbitrary dimension due to physical reasons and for mathematical generality. This program is being developed by Zúñiga-Galindo and his collaborators in recent years, see e.g. [12], [15], [16], [39], [46], [56], [59].

In Chapter 2, we study some aspects of the dynamics of 'random walks' associated with the exponential landscapes introduced in [4]. The corresponding master equation takes the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \left(J * u(\cdot,t)\right)(x) - u\left(x,t\right), \quad x \in \mathbb{Q}_p^n, \quad t \geq 0 \quad (A) \\ \\ u\left(x,0\right) = u_0\left(x\right), \qquad\qquad\qquad\qquad\qquad\qquad (B) \end{cases} \tag{11}$$

where $J : \mathbb{Q}_p^n \to \mathbb{R}_+$ belongs to a family of functions, depending on several parameters, which codify the structure of an energy landscape. The family of landscapes studied here is an 'integrable' generalization of the one-dimensional exponential landscapes introduced in [4]. The first step is to establish that

(11)-(A) is an ultradiffusion equation, i.e. that its fundamental solution is the transition density of a Markov process with space state $\mathbb{Q}_p^n$. It is important to mention here, that this fact has been established rigorously only for the linear landscapes, in this case (11)-(A) becomes a p-adic heat equation, and these equations and their Markov processes have been studied intensively lately, see [12], [16], [15], [32], [51], [59], [56]. For the exponential and logarithmic landscapes of [4], the function j is only locally integrable, and thus the natural domain of the operator in the right hand-side of (10) is not evident, and it is not given in [4], hence the fact that (10) is an ultradiffusion equation in the case of exponential and logarithm landscapes was not established in [4]. By imposing to the function J the condition of being integrable, the operator in the right hand-side of (11)-(A) becomes a linear bounded operator on L^ρ, $1 \leq \rho \leq \infty$, and in the case of exponential-type landscapes, we show that (11)-(A) is an ultradiffusion equation. In this thesis, we show that the fundamental solution of (11)-(A) is the transition density of a Lévy process, see Theorem 2.3.9. It is worth to mention that the real counterpart of equation (11)-(A) has been studied intensively, in this setting the equation has been used to model diffusion processes, see e.g. [2]. In the preface of [2], the authors show that for certain Markov processes their density transition functions satisfy (11)-(A). In this thesis, we investigate the converse of this situation, in a p-adic setting.

We also study the first passage time problem for stochastic processes $\mathfrak{J}(t, \cdot)$ each one defined on a given probability space $(\Omega, \mathcal{F}, P)$, whose transition density functions satisfy (11)-(A)-(B), with $u_0(x)$ equals to the characteristic function of $\mathbb{Z}_p^n$. More precisely, we study the random variable $\tau_{\mathbb{Z}_p^n}(\omega)$, $\omega \in \Omega$, defined as the smallest time in which a path of $\mathfrak{J}(t, \omega)$, $\omega \in \Omega$, returns to $\mathbb{Z}_p^n$. We show that every path of any of these processes returns to $\mathbb{Z}_p^n$, almost surely, see Theorem 2.4.7.

It is important to mention that our results do not cover all the exponential landscapes introduced here, this will require the study of equations of type (11)-(A)-(B) in a more general setting. Finally the results in [16] present an n-dimensional generalization of the linear energy landscapes of [4], this generalization was achieved by generalizing the p-adic heat equations. There are several important differences between this thesis and [16]. First, the operators (Laplacians) considered in [16] are unbounded operators densely defined in suitable subspaces the $L^2\left(\mathbb{Q}_p^n\right)$ while the operators studied here are bounded operators defined in $L^\rho\left(\mathbb{Q}_p^n\right)$, $1 \leq \rho \leq \infty$; second, the fundamental solutions in [16] are integrable functions of the position while here the fundamental solutions are distributions which makes the study of the corresponding stochastic processes more involved.

In Chapter 3, we study a large class of non-Archimedean pseudodifferential operators having negative definite symbols, which have attached Feller semigroups. These operators have the form

$$(A\,(x,t,\partial)\,\varphi)\,(x) = -\mathcal{F}^{-1}_{\xi\to x}\,(a\,(x,t,\xi)\,\mathcal{F}_{x\to\xi}\varphi)$$

where the symbol $\xi \to a\,(x,t,\xi)$ is a negative definite function, for $x \in \mathbb{Q}_p^n$ and $t \in \mathbb{R}_+$. A typical example of such symbols are functions of the form $\sum_{j=1}^m b_j(x,t)\boldsymbol{\psi}_j\,(\xi)$, where $b_j : \mathbb{Q}_p^n \times \mathbb{R}_+ \to \mathbb{R}_+$ are functions satisfying

$$0 < \inf_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) \le b_j(x,t) \le \sup_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) < \infty,$$

and the $\boldsymbol{\psi}_j : \mathbb{Q}_p^n \to \mathbb{R}_+$ are radial, continuous, negative definite functions. In the non-Archimedean setting there exist 'exotic' negative definite functions such as $\exp\left(\exp\left(\exp\sum_{i=0}^{\infty} a_j\,\|\xi\|_p^{\alpha_j}\right)\right)$ where $\sum_{i=0}^{\infty} a_j y^{\alpha_j}$ is a convergent real series with $a_j > 0$, $\alpha_j > 0$ and $\lim_{j\to\infty}\alpha_j = \infty$, see Lemma 3.1.8. This type of functions do not have Archimedean counterparts.

Let $P(x,t,\partial)$ denote a pseudodifferential operator whose symbol is a negative definite function $p\,(x,t,\xi)$ in the variable ξ. We introduce a new class of function spaces $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ attached to a negative definite function $\boldsymbol{\psi}$. These spaces are generalizations of the spaces $H_\infty\,(\mathbb{R})$ introduced by Zúñiga-Galindo in [55]. The spaces $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ are nuclear countably Hilbert spaces. We show that $B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})$ is the natural domain for an operator of type $P(x,t,\partial)$ for a suitable $\boldsymbol{\psi}$. Under mild hypotheses we show that $(P(x,t,\partial), B_{\boldsymbol{\psi},\infty}\,(\mathbb{R}))$ has a closed extension to $C_0\left(\mathbb{Q}_p^n,\mathbb{R}\right)$ (the $\mathbb{R}$-vector space of bounded continuous functions vanishing at infinity) which is the generator of a Feller semigroup, see Theorem 3.3.12. In the case in which the symbol $p\,(x,t,\xi)$ does not depend on t, we show that the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = P(x,\partial)u(x,t) + f(x,t), & t\in[0,T],\ x\in\mathbb{Q}_p^n; \\ \\ u(x,0) = h(x) \in B_{\boldsymbol{\psi},\infty}\,(\mathbb{R})\,. \end{cases} \tag{12}$$

is well-posed and find explicitly the corresponding semigroup, which is a Feller semigroup, see Theorem 3.4.4. Equations of type (12) appeared as master equations in several models that describe the dynamics of certain hierarchic complex systems, see e.g. [4]-[5], [16], [33] and the references therein. From a physical perspective, it is expected that all these master equations should describe the evolution of a probability density, our results show that this is, indeed, the case for a large class of symbols. As a consequence of the work of

Kochubei, Zúñiga-Galindo and Chacón-Cortes, among others, a large class of parabolic-type pseudodifferential equations with variable coefficients related with Markov processes have been studied, see [32], [16], [15], [58] and the references therein. The results of the present thesis complement this theory. Indeed, the class of equations considered here is in certain sense larger than the class considered before. However, we require two hypotheses that hold for the equations studied in [32], [58]. The first, operators $P(x,t,\partial)$ preserve real valued functions, and second, operators $P(x,t,\partial)$ satisfy the positive maximum principle. These hypotheses are satisfied for a large class of pseudodifferential operators. In the real setting Courrège established that all pseudodifferential operators with continuous definite negative symbols satisfy the positive maximum principle [14]. Nowadays, the non-Archimedean counterpart of this result is not known.

Contents

Chapter 1

p-adic Analysis: Essential Results

In this chapter, we will collect some basic results on the p-adic analysis and fix the notation that we will use through the thesis. All the results here are well known. For more details the reader can consult [1], [32], [51].

1.1 The field of p-adic numbers

Along this thesis p will denote a prime number. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0, & \text{if} \quad x = 0 \\ \\ p^{-\gamma}, & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of x*. We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [1], [51].

Any $p-$adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. So that, any $p-$adic number $x \neq 0$ can be represented uniquely as $x = p^{ord(x)} ac(x)$ where $ac(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the *angular component* of x. Notice that $\left|ac(x)\right|_p = 1$.
By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0 \\ \\ p^{ord(x)} \sum_{j=o}^{-ord_p(x)-1} x_j p^j, & \text{if} \quad ord(x) < 0. \end{cases}$$

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n ; ||x - a||_p \leq p^r\}$ *the ball of radius* p^r *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p ; |x - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of* $p-$*adic integers of* $\mathbb{Q}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n ; ||x - a||_p = p^r\}$ *the sphere of radius* p^r *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^n \subsetneq S_0^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [51, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

We will use $\Omega\left(p^{-r}||x - a||_p\right)$ to denote the characteristic function of the ball $B_r^n(a)$, $r \in \mathbb{Z}$. We will use the notation 1_A for the characteristic function of a set A. Along this thesis $d^n x$ will denote a Haar measure on $\mathbb{Q}_p^n$ normalized so that $\int_{\mathbb{Z}_p^n} d^n x = 1$.

1.2 The Bruhat-Schwartz space and distributions

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is *called locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^n. \tag{1.1}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n)$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, the largest number $l = l(\varphi)$ satisfying (1.1) is called *the exponent of local constancy (or the parameter of constancy) of* φ.

If U is an open subset of $\mathbb{Q}_p^n$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^{\rho}(U) = \left\{ \varphi : U \to \mathbb{C}; \|\varphi\|_{\rho} = \left\{ \int_{\mathbb{Q}_p^n} |\varphi(x)|^{\rho} \, d^n x \right\}^{\frac{1}{\rho}} < \infty \right\}.$$

Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \to \chi_p(\xi x)$, see e.g. [1, Section 2.3].

Let $\mathcal{D}'(\mathbb{Q}_p^n)$ denote the $\mathbb{C}$-vector space of all continuous functionals (distributions) on $\mathcal{D}(\mathbb{Q}_p^n)$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^n) \times \mathcal{D}(\mathbb{Q}_p^n) \to \mathbb{C}$ is denoted as (T, φ) for $T \in \mathcal{D}'(\mathbb{Q}_p^n)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, see e.g. [1, Section 4.4].

Every function $f \in L^1_{loc}$, defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ by the formula

$$(f, \varphi) := \int_{\mathbb{Q}_p^n} f(x)\, \varphi(x)\, d^n x, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

Such distributions are called *regular distributions.*

1.3 The Fourier transform

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^{n} \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

The Fourier transform is a linear (and continuous) isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [1, Section 4.8]. We will also use the notations $\mathcal{F}_{x\to\xi}\varphi$ or $\widehat{\varphi}$ for the Fourier transform of φ.

If $f \in L^1(\mathbb{Q}_p^n)$ then $(\mathcal{F}f)(\xi)$ is continuous on $\mathbb{Q}_p^n$ and

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) f(x) d^n x \to 0, \quad \xi \to \infty.$$

The Fourier transform $f \to \mathcal{F}f$ maps $L^2(\mathbb{Q}_p^n)$ onto $L^2(\mathbb{Q}_p^n)$ one-to-one and mutually continuous, where

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = lim_{\gamma \to \infty} \int_{B_\gamma^n} \chi_p(\xi \cdot x) f(x) d^n x \ \ in \ \ L^2(\mathbb{Q}_p^n);$$

$$(\mathcal{F}^{-1}\widehat{f})(x) = f(x) = lim_{\gamma \to \infty} \int_{B_\gamma^n} \chi_p(-x \cdot \xi) \widehat{f}(\xi) d^n \xi \ \ in \ \ L^2(\mathbb{Q}_p^n).$$

Moreover, the Parseval-Steklov equality holds:

$$(f, g) = (\mathcal{F}f, \mathcal{F}g), \qquad ||f||_{L^2} = ||\mathcal{F}f||_{L^2} \ \ f, g \in L^2(\mathbb{Q}_p^n),$$

where

$$(f, g) = \int_{\mathbb{Q}_p^n} f(x) \overline{g}(x) d^n x \qquad f, g \in L^2(\mathbb{Q}_p^n).$$

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

Chapter 2

Ultrametric Diffusion, Exponential Landscapes, and the First Passage Time Problem

In this chapter, we study certain ultradiffusion equations connected with energy landscapes of exponential type. These equations were introduced by Avetisov et al. in connection with certain p-adic models of complex systems, [4]-[3]. We show that the fundamental solutions of these equations are transition density functions of Lévy processes with state space $\mathbb{Q}_p^n$, we study some aspects of these processes including the first passage time problem.

2.1 Exponential landscapes

In this section we give several technical results for the functions J that codify the structure of the energy landscapes studied in this chapter.

Set $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$. We fix a continuous function $J : \mathbb{R}_+ \to \mathbb{R}_+$, and take $J(x) = J(||x||_p)$ for $x \in \mathbb{Q}_p^n$, then $J(x)$ is a *radial function* on $\mathbb{Q}_p^n$. In addition, we assume that $\int_{\mathbb{Q}_p^n} J(||x||_p) d^n x = 1$.

2.1.1 Definition. We say that function $J(\|x\|_p)$ is of exponential type if there exist positive real constants A, B, C_1, and a real constant $\gamma > -n$ such that

$$A \|x\|_p^\gamma e^{-C_1 \|x\|_p} \leq J(\|x\|_p) \leq B \|x\|_p^\gamma e^{-C_1 \|x\|_p}, \text{ for any } x \in \mathbb{Q}_p^n.$$

2.1.2 Remark. The condition $\gamma > -n$ is completely necessary to assure that $J \in L^1$. We notice that in dimension one the function $\frac{e^{-C|x|_p}}{|x|_p}$, $C > 0$, which was used in [4] as a j function, is not integrable. Indeed, assume that $\frac{e^{-C|x|_p}}{|x|_p} \in L^1$, then the following integral exists:

$$\begin{aligned} \int_{\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx &= \int_{p\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx + \int_{\mathbb{Z}_p^{\times}} \frac{e^{-C|x|_p}}{|x|_p} dx \\ &= \int_{\mathbb{Z}_p} \frac{e^{-Cp^{-1}|x|_p}}{|x|_p} dx + e^{-C}\left(1 - p^{-1}\right). \qquad (2.1) \end{aligned}$$

Now, since $C\,|x|_p \geq Cp^{-1}\,|x|_p$, we have $\int_{\mathbb{Z}_p} \frac{e^{-C|x|_p}}{|x|_p} dx - \int_{\mathbb{Z}_p} \frac{e^{-Cp^{-1}|x|_p}}{|x|_p} dx \leq 0$, which contradicts (2.1). This situation causes several mathematical problems that we will discuss later on.

2.1.3 Lemma. With the above notation, the following assertions hold:
(i) $\widehat{J}(\xi)$ is a real-valued, radial (i.e. $\widehat{J}(\xi) = \widehat{J}(\|\xi\|_p)$), and continuous function, satisfying $|\widehat{J}(\|\xi\|_p)| \leq 1$ and $\widehat{J}(0) = 1$;
(ii) for $\xi \in \mathbb{Q}_p^n \smallsetminus \{0\}$,

$$1 - \widehat{J}(\|\xi\|_p) = \|\xi\|_p^{-n} J(p\,\|\xi\|_p^{-1}) + p^n \|\xi\|_p^{-n} \sum_{l=0}^{\infty} p^{nl} J(p^{1+l}\,\|\xi\|_p^{-1});$$

(iii) if $-n < \gamma < 0$, then

$$1 - \widehat{J}(\|\xi\|_p) \leq B_1 \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + B_2 \|\xi\|_p^{-\gamma} e^{-C_1 p \|\xi\|_p^{-1}},$$

for $\xi \in \mathbb{Q}_p^n \smallsetminus \{0\}$, where B_1, B_2 are positive constants.

Proof. (i) The Fourier transform of an integrable radial function is a real-valued continuous radial function. Notice that $|\widehat{J}\left(\|\xi\|_p\right)| \leq \int_{\mathbb{Q}_p^n} |\chi_p(\xi \cdot x)|\, J(x) d^n x = \int_{\mathbb{Q}_p^n} J(\|x\|_p) d^n x = 1$. Now, since J is integrable $\widehat{J}(0) = \int_{\mathbb{Q}_p^n} J(\|x\|_p) d^n x = 1$.

(ii) Take $\xi = p^{ord(\xi)}\xi_0$, with $\xi_0 = \left(\xi_0^{(1)}, \ldots, \xi_0^{(n)}\right)$, $\|\xi_0\|_p = 1$, and $ord(\xi) \in \mathbb{Z}$, then

$$\widehat{J}\left(\|\xi\|_p\right) - 1 = \int_{\mathbb{Q}_p^n} J(||x||_p) \left\{\chi_p\left(p^{ord(\xi)}\xi_0 \cdot x\right) - 1\right\} d^n x.$$

By changing variables as $y_i = p^{ord(\xi)}\xi_0^{(i)}x_i$ for $i = 1, \ldots, n$, with $d^n x = p^{ord(\xi)n} d^n y$, we have

$$\widehat{J}\left(\|\xi\|_p\right) - 1 = p^{ord(\xi)n} \int_{\mathbb{Q}_p^n} J(p^{ord(\xi)}||y||_p) \left\{ \chi_p\left(\sum_{i=1}^{n} y_i\right) - 1 \right\} d^n y$$

$$= p^{ord(\xi)n} \int_{\mathbb{Q}_p^n \smallsetminus \mathbb{Z}_p^n} J(p^{ord(\xi)}||y||_p) \left\{ \chi_p\left(\sum_{i=1}^{n} y_i\right) - 1 \right\} d^n y$$

$$= p^{ord(\xi)n} \sum_{j=1}^{\infty} J(p^{j+ord(\xi)}) \int_{\|y\|_p = p^j} \left\{ \chi_p\left(\sum_{i=1}^{n} y_i\right) - 1 \right\} d^n y.$$

By changing variables as $y = p^{-j}z$, $d^n y = p^{nj} d^n z$, we get

$$\widehat{J}\left(\|\xi\|_p\right) - 1 = p^{ord(\xi)n} \sum_{j=1}^{\infty} p^{nj} J(p^{j+ord(\xi)}) \int_{S_0^n} \left\{ \chi_p\left(p^{-j}\sum_{i=1}^{n} z_i\right) - 1 \right\} d^n z$$

$$= p^{ord(\xi)n} \sum_{j=1}^{\infty} p^{nj} J(p^{j+ord(\xi)}) \begin{cases} -p^{-n} & \text{if } \ j \le 0 \\ -1 - p^{-n} & \text{if } \ j = 1 \\ -1 & \text{if } \ j \ge 2 \end{cases}$$

$$= -\left(1 + p^{-n}\right) p^{ord(\xi)n+n} J(p^{1+ord(\xi)}) - p^{ord(\xi)n} \sum_{j=2}^{\infty} p^{nj} J(p^{j+ord(\xi)})$$

$$= -p^{ord(\xi)n} J(p^{1+ord(\xi)}) - p^{ord(\xi)n+n} \sum_{l=0}^{\infty} p^{nl} J(p^{1+l+ord(\xi)}).$$

(iii) From (ii) and the fact that J is of exponential type, we get that

$$1 - \widehat{J}(\|\xi\|_p) \le Bp^{\gamma} \|\xi\|_p^{-n-\gamma} e^{-C_1 p \|\xi\|_p^{-1}} + Bp^{\gamma+n} \|\xi\|_p^{-n-\gamma} \sum_{l=0}^{\infty} p^{(n+\gamma)l} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}}. \tag{2.2}$$

By using that $-n < \gamma < 0$,

$$p^{\gamma} \sum_{l=0}^{\infty} p^{nl} p^{\gamma l} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}} \le \sum_{l=0}^{\infty} p^{nl} e^{-C_1 p^{1+l} \|\xi\|_p^{-1}}$$

$$= e^{-C_1 p \|\xi\|_p^{-1}} \sum_{l=0}^{\infty} p^{nl} e^{-C_1 p \|\xi\|_p^{-1} (p^l - 1)}$$

$$= e^{-C_1 p \|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p \|\xi\|_p^{-1} (p^l - 1)} \right\}. \tag{2.3}$$

By using that $p^l - 1 \geq p^{l-1}$ for any positive integer,

$$e^{-C_1 p\|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p\|\xi\|_p^{-1}(p^l-1)} \right\}$$

$$\leq e^{-C_1 p\|\xi\|_p^{-1}} \left\{ 1 + \sum_{l=1}^{\infty} p^{nl} e^{-C_1 p^l\|\xi\|_p^{-1}} \right\}$$

$$= e^{-C_1 p\|\xi\|_p^{-1}} \left\{ 1 + \frac{1}{1-p^{-n}} \int\limits_{\|y\|_p > 1} e^{-C_1 \|y\|_p \|\xi\|_p^{-1}} d^n y \right\}$$

$$\leq e^{-C_1 p\|\xi\|_p^{-1}} \left\{ 1 + \frac{1}{1-p^{-n}} \int\limits_{\mathbb{Q}_p^n} e^{-C_1 \|y\|_p \|\xi\|_p^{-1}} d^n y \right\} \leq e^{-C_1 p\|\xi\|_p^{-1}} \left\{ 1 + A_0 \|\xi\|_p^n \right\}, \tag{2.4}$$

where we used the well-known estimation

$$\int\limits_{\mathbb{Q}_p^n} e^{-\tau\|y\|_p} d^n y \leq C_0 \tau^{-n} \text{ for } \tau > 0.$$

Therefore from (2.2)-(2.4),

$$1 - \widehat{J}(\|\xi\|_p) \leq B_1 \|\xi\|_p^{-n-\gamma} e^{-C_1 p\|\xi\|_p^{-1}} + B_2 \|\xi\|_p^{-\gamma} e^{-C_1 p\|\xi\|_p^{-1}} \text{ for } \xi \neq 0.$$

2.1.4 Remark. We notice that the fact that J is of exponential type implies that supp $J\left(\|x\|_p\right) \subsetneqq \mathbb{Z}_p^n$. Then $1 - \widehat{J}(1) > 0$. Indeed, by Lemma 2.1.3-(ii),

$$\begin{aligned} 1 - \widehat{J}(1) &= J(p) + p^n \sum\nolimits_{l=0}^{\infty} p^{nl} J(p^{1+l}) = J(p) + \sum\nolimits_{k=1}^{\infty} p^{nk} J(p^k) \\ &= J(p) + \frac{1}{1-p^{-n}} \int_{\mathbb{Q}_p^n \smallsetminus \mathbb{Z}_p^n} J\left(\|x\|_p\right) d^n x. \end{aligned}$$

If $1 - \widehat{J}(1) = 0$, then $J(p) = 0$ and $J\left(\|x\|_p\right) \equiv 0$ for $x \in \mathbb{Q}_p^n \smallsetminus \mathbb{Z}_p^n$, i.e. supp $J\left(\|x\|_p\right) \subseteq \mathbb{Z}_p^n$.

2.1.5 Lemma. If $-n<\gamma<0$ and J is of exponential type, then

$$\frac{\Omega\left(\|\xi\|_p\right)}{1-\widehat{J}\left(\|\xi\|_p\right)}\notin L^1\left(\mathbb{Q}_p^n,d^n\xi\right).$$

Proof. By using Lemma 2.1.3-(iii),

$$\begin{aligned}\int_{\mathbb{Z}_p^n}\frac{d^n\xi}{1-\widehat{J}\left(||\xi||_p\right)}&\geq\int_{\mathbb{Z}_p^n}\frac{d^n\xi}{B_1\left\|\xi\right\|_p^{-n-\gamma}e^{-C_1p\|\xi\|_p^{-1}}+B_2\left\|\xi\right\|_p^{-\gamma}e^{-C_1p\|\xi\|_p^{-1}}}\\&=\int_{\mathbb{Z}_p^n}\frac{\|\xi\|_p^{\gamma}e^{C_1p\|\xi\|_p^{-1}}d^n\xi}{B_1\left\|\xi\right\|_p^{-n}+B_2}=\sum_{j=0}^{\infty}\frac{p^{-j\gamma}e^{C_1p^{j+1}}}{B_1p^{jn}+B_2}\int_{||\xi||_p=p^{-j}}d^n\xi\\&=\left(1-p^{-n}\right)\sum_{j=0}^{\infty}\frac{p^{-j(n+\gamma)}e^{C_1p^{j+1}}}{B_1p^{jn}+B_2}=\infty.\end{aligned}$$

2.1.6 Remark. (i) A function $f:\mathbb{Q}_p^n\to\mathbb{C}$ is called positive definite, if

$$\sum_{i,j=1}^{m}f(x_i-x_j)\lambda_i\overline{\lambda_j}\geq0$$

for all $m\in\mathbb{N}\backslash\{0\}$, $x_1,\ldots,x_m\in\mathbb{Q}_p^n$ and $\lambda_1,\ldots,\lambda_m\in\mathbb{C}$. By a direct calculation one verifies that $\widehat{J}(\|\xi\|_p)$ is a positive definite function.
(ii) A function $f:\mathbb{Q}_p^n\to\mathbb{C}$ is called negative definite, if

$$\sum\nolimits_{i,j=1}^{m}\left(f(x_i)+\overline{f(x_j)}-f(x_i-x_j)\right)\lambda_i\overline{\lambda_j}\geq0$$

for all $m\in\mathbb{N}\backslash\{0\}$, $x_1,\ldots,x_m\in\mathbb{Q}_p^n$, $\lambda_1,\ldots,\lambda_m\in\mathbb{C}$. A theorem due to Schoenberg asserts that a function $f:\mathbb{Q}_p^n\to\mathbb{C}$ is negative definite if and only if the following two conditions are satisfied:(1) $f(0)\geq0$ and (2) the function $x\longmapsto e^{-tf(x)}$ is positive definite for all $t>0$, cf. [10, Theorem 7.8]. By using Corollary 7.7 in [10, Theorem 7.8], the function $\widehat{J}(0)-\widehat{J}(\|\xi\|_p)=1-\widehat{J}(\|\xi\|_p)$ is negative definite, by applying Schoenberg Theorem, the function $e^{t(\widehat{J}(\|\xi\|_p)-1)}$ is positive definite for all $t>0$.

2.1.1 A class of nonlocal p-adic operators

We define the operator $Af = J * f - f$ with J as in Section 2.1. Then, for any $1 \leq \rho \leq \infty$, $A : L^\rho \longrightarrow L^\rho$ gives rise a well-defined linear bounded operator. Indeed, by the Young inequality

$$||Af||_{L^\rho} \leq ||J * f||_{L^\rho} + ||f||_{L^\rho} \leq ||J||_{L^1}||f||_{L^\rho} + ||f||_{L^\rho} \leq 2||f||_{L^\rho}.$$

2.1.7 Proposition. Consider $A : L^2\left(\mathbb{Q}_p^n\right) \to L^2\left(\mathbb{Q}_p^n\right)$ given by

$$Af\,(x) = \mathcal{F}^{-1}_{\xi\to x}\left((\widehat{J}(||\xi||_p) - 1)\mathcal{F}_{x\to\xi} f\right),$$

and the Cauchy problem :

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = Au(x,t), & t \in [0,\infty)\,,\, x \in \mathbb{Q}_p^n \\ \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n). \end{cases} \tag{2.5}$$

Then

$$u(x,t) = \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right) e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi) d^n\xi$$

is a classical solution of (2.5). In addition, $u(\cdot,t)$ is a continuous function for any $t \geq 0$.

Proof. The result follows from the following assertions.
Claim 1. $u(x,\cdot) \in C^1\left([0,\infty)\right)$ and

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right)(\widehat{J}(||\xi||_p) - 1)e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi) d^n\xi$$

for $t \geq 0$, $x \in \mathbb{Q}_p^n$.
The formula follows from the fact that

$$\left|\chi_p\left(-\xi\cdot x\right) e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi)\right| \leq |\widehat{u_0}(\xi)| \in L^1(\mathbb{Q}_p^n)$$

and that

$$\left|\chi_p\left(-\xi\cdot x\right)(\widehat{J}(||\xi||_p) - 1)e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi)\right| \leq 2\,|\widehat{u_0}(\xi)| \in L^1(\mathbb{Q}_p^n),$$

cf. Lemma 2.1.3-(i), by applying the Dominated Convergence Theorem.
Claim 2.

$$Au(x,t) = \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right)(\widehat{J}(||\xi||_p) - 1)e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi) d^n\xi$$

for $t \in [0,\infty)$, $x \in \mathbb{Q}_p^n$.

The formula follows from the fact that $u(x,t) = \mathcal{F}^{-1}_{\xi\to x}\left(e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi)\right) \in L^2\left(\mathbb{Q}_p^n\right)$ for any $t \geq 0$ and that $\left(\widehat{J}(||\xi||_p)-1\right)e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{u_0}(\xi) \in L^2\left(\mathbb{Q}_p^n\right)$ for any $t \geq 0$, cf. Lemma 2.1.3-(i).

2.2 Heat kernels

We define the *heat Kernel* attached to operator A as

$$Z(x,t) = \mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)t}) \in \mathcal{D}'(\mathbb{Q}_p^n) \text{ for } t \geq 0.$$

When considering $Z(x,t)$ as a function of x for t fixed we will write $Z_t(x)$.

We recall that a distribution $F \in \mathcal{D}'(\mathbb{Q}_p^n)$ is called *positive*, if $(F,\varphi) \geq 0$ for every *positive test function* φ, i.e. for $\varphi : \mathbb{Q}_p^n \longrightarrow \mathbb{R}_+$, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$.

A distribution F is *positive definite*, if for every test function φ, the inequality $(F, \overline{\varphi * \widetilde{\varphi}}) \geq 0$ holds, where $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$ and $\overline{\varphi(-x)}$ denotes the complex conjugate of $\varphi(-x)$.

2.2.1 Theorem. ($p-$adic Bochner-Schwartz) [59, Theorem 4.1] Every positive-definite distribution F on $\mathbb{Q}_p^n$ is the Fourier transform of a regular Borel measure μ on $\mathbb{Q}_p^n$, i.e.

$$(F,\varphi) = \int_{\mathbb{Q}_p^n} \widehat{\varphi}(\xi)d\mu(\xi), \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

2.2.2 Remark. If $f : \mathbb{Q}_p^n \longrightarrow \mathbb{C}$ is a continuous positive-definite function, then

$$(f, \overline{\varphi * \widetilde{\varphi}}) \geq 0 \quad \text{for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

which means that f generates a positive-definite distribution , see e.g. [10, Proposition 4.1].

2.2.3 Lemma. Let φ be a positive test function. Then

$$\int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right)e^{(\widehat{J}(\|\xi\|_p)-1)t}\widehat{\varphi}(\xi)d^n\xi = \left(Z_t * \varphi\right)(x) \geq 0 \quad \text{for } x \in \mathbb{Q}_p^n \text{ and } t \geq 0.$$

Proof. It is sufficient to show the lemma for $x \in \mathbb{Q}_p^n$ and $t > 0$. By Remark 2.1.6-(ii), the function $e^{t(\widehat{J}(\|\xi\|_p)-1)}$ is positive definite for all $t > 0$, by Remark

2.2.2 and Theorem 2.2.1, $Z_t(x) = \mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(\|\xi\|_p)-1)t})$ is a Borel measure on $\mathbb{Q}_p^n$ for $t > 0$, which is identified with a positive distribution. Then

$$\begin{aligned}(e^{(\widehat{J}(\|\xi\|_p)-1)t}, \chi_p(-\xi\cdot x)\,\widehat{\varphi}(\xi)) &= \left(\mathcal{F}^{-1}_{\xi\to y}(e^{(\widehat{J}(\|\xi\|_p)-1)t}), \mathcal{F}_{\xi\to y}(\chi_p(-\xi\cdot x)\,\widehat{\varphi}(\xi))\right)\\ &= (Z_t(y), \varphi(x-y)) \geq 0 \text{ for } t>0,\end{aligned}$$

since $\varphi(x-y)\geq 0$.

2.2.1 Decaying of the heat kernel at infinity

Let $h\left(\|\xi\|_p\right)\in L^1_{\mathrm{loc}}$, then

$$\sum_{j=-m}^{m} h\left(p^j\right) 1_{S_j^n}(\xi) \;\to h\left(\|\xi\|_p\right) \text{ in } \mathcal{D}'(\mathbb{Q}_p^n).$$

Now, by using [1, Theorem 4.9.3] and the fact that $\mathcal{F}^{-1}$ is continuous on $\mathcal{D}'(\mathbb{Q}_p^n)$,

$$\begin{aligned}\mathcal{F}^{-1}_{\xi\to x}\left(\sum_{j=-m}^{m} h\left(p^j\right) 1_{S_j^n}(\xi)\right) &= \sum_{j=-m}^{m} h\left(p^j\right)\mathcal{F}^{-1}_{\xi\to x}\left(1_{S_j^n}(\xi)\right)\\ &= \sum_{j=-m}^{m} h\left(p^j\right)\int_{S_j^n}\chi_p(-x\cdot\xi)\,d^n\xi \text{ in } \mathcal{D}'(\mathbb{Q}_p^n),\end{aligned}$$

therefore

$$\mathcal{F}^{-1}_{\xi\to x}\left(h\left(\|\xi\|_p\right)\right) = \sum_{j=-\infty}^{\infty} h\left(p^j\right)\int_{S_j^n}\chi_p(-x\cdot\xi)\,d^n\xi \text{ in } \mathcal{D}'(\mathbb{Q}_p^n). \tag{2.6}$$

Suppose now that $\sum_{k=0}^{\infty} p^{-kn} h\left(p^{-k}\|x\|_p^{-1}\right) < \infty$, then

$$\begin{aligned}\widetilde{h}(x) &:= \int_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\,h\left(\|\xi\|_p\right)d^n\xi := \lim_{m\to\infty}\sum_{j=-m}^{m}\int_{S_j^n}\chi_p(-x\cdot\xi)\,h\left(\|\xi\|_p\right)d^n\xi\\ &= \sum_{j=-\infty}^{\infty} h\left(p^j\right)\int_{S_j^n}\chi_p(-x\cdot\xi)\,d^n\xi\\ &= \left(1-p^{-n}\right)\|x\|_p^{-n}\sum_{k=0}^{\infty} p^{-kn} h\left(p^{-k}\|x\|_p^{-1}\right) - \|x\|_p^{-n}\, h\left(p\,\|x\|_p^{-1}\right) \text{ for } x\neq 0,\end{aligned}$$

and by comparing with (2.6), we get

$$\left(\mathcal{F}^{-1}_{\xi\to x}\left[h\left(\|\xi\|_p\right)\right],\phi\left(x\right)\right)=\left(\widetilde{h}\left(x\right),\phi\left(x\right)\right)$$

for $\phi\in\mathcal{D}(\mathbb{Q}_p^n)$ with supp $\phi\subset\mathbb{Q}_p^n\backslash\{0\}$. It is important to highlight that function $\widetilde{h}\left(x\right)$ is not the inverse Fourier transform of $h\left(\|\xi\|_p\right)$ in the classical sense, because we use an improper integral in its definition. This is the reason for the 'new notation'. We formally summarize the above reasoning in the following lemma:

2.2.4 Lemma. Let $h\left(\|\xi\|_p\right)\in L^1_{\text{loc}}$ satisfying $\sum_{k=0}^{\infty}p^{-kn}h\left(p^{-k}\|\xi\|_p^{-1}\right)<\infty$ for $\xi\neq 0$, then $\mathcal{F}^{-1}_{\xi\to x}\left[h\left(\|\xi\|_p\right)\right]=\widetilde{h}\left(x\right)$ as a distribution on $\mathcal{D}(\mathbb{Q}_p^n\backslash\{0\})$.

We now apply this lemma to the case $h\left(\|\xi\|_p\right)=e^{(\widehat{J}(\|\xi\|_p)-1)t}$, with $t\geq 0$:

$$\left(Z\left(x,t\right),\phi\left(x\right)\right)=\left(\widetilde{Z}\left(x,t\right),\phi\left(x\right)\right)$$

for $\phi\in\mathcal{D}(\mathbb{Q}_p^n)$ with supp $\phi\subset\mathbb{Q}_p^n\backslash\{0\}$, where

$$\widetilde{Z}\left(x,t\right)=||x||_p^{-n}\left[(1-p^{-n})\sum_{j=0}^{\infty}p^{-nj}e^{(\widehat{J}(p^{-j}\|x\|_p^{-1})-1)t}-e^{(\widehat{J}(p||x||_p^{-1})-1)t}\right]\tag{2.7}$$

for $t\geq 0$ and $x\neq 0$.

2.2.5 Proposition. Assume that J is of exponential type, then the following estimations hold:
(i) $\widetilde{Z}(x,t)\leq 2t||x||_p^{-n}$, for $x\in\mathbb{Q}_p^n\backslash\{0\}$ and $t>0$;
(ii) if $-n<\gamma<0$, then $\widetilde{Z}(x,t)\leq C_0 t\,\|x\|_p^{\gamma}\,e^{-C_1\|x\|_p}$, for $\|x\|_p>p^l$, $l\in\mathbb{Z}$, and $t>0$, where the positive constant C_0 depends on l.

Proof. The estimations follow from the following Claim:
Claim. $\widetilde{Z}(x,t)\leq t||x||_p^{-n}\left\{1-\widehat{J}(p||x||_p^{-1})\right\}$, for $x\in\mathbb{Q}_p^n\backslash\{0\}$ and $t>0$.

We notice that by Lemma 2.1.3-(ii) $1-\widehat{J}(p||x||_p^{-1})\geq 0$. The first estimation follows from Lemma 2.1.3-(i) and the Claim. The second estimation follows from the Claim and Lemma 2.1.3-(iii).

The proof of the Claim is as follows. By using that $e^{(\widehat{J}(p^{-j}\|x\|_p^{-1})-1)t} \leq 1$ for $j \in \mathbb{N}$, cf. Lemma 2.1.3-(i), we get that

$$\begin{aligned}
\widetilde{Z}(x,t) \leq ||x||_p^{-n} & \left[(1-p^{-n}) \sum_{j\geq 0} p^{-nj} - e^{(\widehat{J}(p||x||_p^{-1})-1)t} \right] \\
= ||x||_p^{-n} & \left[\sum_{j\geq 0} (p^{-nj} - p^{-n(j+1)}) - e^{(\widehat{J}(p||x||_p^{-1})-1)t} \right] \\
& = ||x||_p^{-n} \left\{ 1 - e^{(\widehat{J}(p||x||_p^{-1})-1)t} \right\}.
\end{aligned}$$

We now apply the Mean-Value Theorem to the real function $e^{(\widehat{J}(p||x||_p^{-1})-1)u}$ on $[0,t]$ with $t>0$,

$$e^{(\widehat{J}(p||x||_p^{-1})-1)t} - 1 = \left\{ \widehat{J}(p||x||_p^{-1}) - 1 \right\} t e^{(\widehat{J}(p||x||_p^{-1})-1)\tau}$$

for some $\tau \in (0,t)$, consequently $1 - e^{(\widehat{J}(p||x||_p^{-1})-1)t} \leq \left\{ 1 - \widehat{J}(p||x||_p^{-1}) \right\} t$. Hence,

$$\widetilde{Z}(x,t) \leq t||x||_p^{-n} \left\{ 1 - \widehat{J}(p||x||_p^{-1}) \right\}.$$

2.3 Lévy processes

For the basic results on Hunt, Lévy and Markov processes the reader may consult [19], [45], [11], [22].

2.3.1 Remark. We denote by $\mathcal{B}_0$ the family of subsets of $\mathbb{Q}_p^n$ formed by finite unions of disjoint balls and the empty set. This family has a natural structure of Boolean ring, i.e. if B_1, $B_2 \in \mathcal{B}_0$ then $B_1 \cup B_2 \in \mathcal{B}_0$ and $B_1 \backslash B_2 \in \mathcal{B}_0$. The Caratheodory Theorem asserts that if μ is a $\sigma-$finite measure on $\mathcal{B}_0$, then there is a unique measure also denoted by μ on $\mathcal{B}(\mathbb{Q}_p^n)$, the $\sigma-$ring generated by $\mathcal{B}_0$, which is $\sigma-$ring of Borel sets of $\mathbb{Q}_p^n$, see [28, Theorem A, p. 54]. Then every positive distribution can be identified with a Borel measure on $\mathbb{Q}_p^n$.

2.3.2 Definition. For $E \in \mathcal{B}_0(\mathbb{Q}_p^n)$, we define

$$p_t(x,E) = \begin{cases} Z_t(x) * 1_E(x), & \text{for } t>0,\ x \in \mathbb{Q}_p^n \\ 1_E(x), & \text{for } t=0,\ x \in \mathbb{Q}_p^n. \end{cases}$$

2.3.3 Lemma. $p_t(x,\cdot)$, $t \geq 0$, $x \in \mathbb{Q}_p^n$, is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$.

Proof. By Lemma 2.2.3 and the fact that

$$p_t(x,E) = \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right) e^{(\widehat{J}(\|\xi\|_p)-1)t}\ \widehat{1_E}(\xi) d^n\xi \text{ for } t>0,\ x \in \mathbb{Q}_p^n$$

$p_t(x,E)$ is a measure on $\mathcal{B}_0$, in addition, $p_t(x,\cdot)$ has a unique extension to a measure on the Borel $\sigma-$ring of $\mathbb{Q}_p^n$. We denote this extension also by $p_t(x,\cdot)$. Indeed, by the Caratheodory Theorem, see Remark 2.3.1, it is sufficient to show that $\mathbb{Q}_p^n$ is a countable disjoint union of balls B_i^n, $i \in \mathbb{N}$, satisfying $p_t(x,B_i^n) < \infty$ for any $i \in \mathbb{N}$. Indeed,

$$\mathbb{Q}_p^n = \bigsqcup_{\widetilde{x}_i \in (\mathbb{Q}_p/\mathbb{Z}_p)^n} B_0^n(\widetilde{x}_i),$$

where the elements of $\mathbb{Q}_p/\mathbb{Z}_p$ have the form $\widetilde{y} = a_{-m}p^{-m} + \cdots + a_{-1}p^{-1}$ with $a_i \in \{0,\cdots,p-1\}$. The correspondence $\widetilde{y} \mapsto a_m p^m + \cdots + a_1 p^1$ implies that $\mathbb{Q}_p/\mathbb{Z}_p$ is countable, and therefore $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ is also countable. Finally, the condition $p_t(x, B_0^n(\widetilde{x}_i)) < \infty$ follows from the fact that $\widehat{1_{B_0^n(\widetilde{x}_i)}}(\xi)$ has compact support.

2.3.4 Proposition. $p_t(x,E)$ for $t \geq 0$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$ is a Markov transition function on $\mathbb{Q}_p^n$.

Proof. Claim 1. $p_t(x,\cdot)$ is measure on $\mathcal{B}(\mathbb{Q}_p^n)$ and $p_t(x,\mathbb{Q}_p^n) = 1$ for all $t \geq 0$ and $x \in \mathbb{Q}_p^n$.

The first part of the assertion was established in Lemma 2.3.3. To show the second part of the assertion, we notice that B_k^n, $k \in \mathbb{N}$, is an increasing sequence of Borelian sets converging to $\mathbb{Q}_p^n$, i.e. $B_k^n \uparrow \mathbb{Q}_p^n$. Set $\Delta_k(x) := \Omega(p^{-k}||x||_p)$, $k \in \mathbb{N}$, hence $p_t(x,\mathbb{Q}_p^n) = \lim_{k\to\infty} p_t(x,\Delta_k)$. Now, since

$$\widehat{\Delta}_k(\xi) = \delta_k(\xi) = p^{kn} \begin{cases} 1 & \text{if} \quad ||\xi||_p \leq p^{-k} \\ \\ 0 & \text{if} \quad ||\xi||_p > p^{-k}, \end{cases}$$

$$\begin{aligned} p_t(x, \mathbb{Q}_p^n) &= \lim_{k\to\infty} \int_{\mathbb{Q}_p^n} \chi_p\left(-\xi\cdot x\right) e^{(\widehat{J}(||\xi||_p)-1)t}\delta_k(\xi)d^n\xi \\ &= \lim_{k\to\infty} p^{nk}\int_{||\xi||_p\le p^{-k}} \chi_p\left(-\xi\cdot x\right) e^{(\widehat{J}(||\xi||_p)-1)t}d^n\xi \quad (\text{taking } p^{-k}\xi = z) \\ &= \lim_{k\to\infty} \int_{||z||_p\le 1} \chi_p\left(-p^k z\cdot x\right) e^{(\widehat{J}(p^{-k}||z||_p)-1)t}d^n z \\ &= \lim_{k\to\infty} \int_{||z||_p\le 1} e^{(\widehat{J}(p^{-k}||z||_p)-1)t}d^n z, \text{ for } k \text{ big enough,} \end{aligned}$$

because for k big enough $p^k x \in \mathbb{Z}_p^n$ and thus $\chi_p\left(-p^k z\cdot x\right) \equiv 1$. By using that $e^{(\widehat{J}(p^{-k}||z||_p)-1)t} \le 1$ for any $t \ge 0$, and that $\lim_{k\to\infty} e^{(\widehat{J}(p^{-k}||z||_p)-1)t} = 1$ ($\widehat{J}$ is continuous at the origin and $\widehat{J}(0) = 1$), and by applying the Dominated Convergence Theorem,

$$p_t(x, \mathbb{Q}_p^n) = \int_{||z||_p\le 1} d^n z = 1.$$

Claim 2. $p_t(\cdot, E)$ is a Borel measurable function for all $t > 0$ and $E \in \mathcal{B}\left(\mathbb{Q}_p^n\right)$.

Define $E_k = \Delta_k E$, $k \in \mathbb{N}$, then $E_k \uparrow E$ with $E_k \in \mathcal{B}\left(\mathbb{Q}_p^n\right)$. By abuse of language, we use the notation $p_t(x, E_k)$ to mean a function of (t, x) with E_k fixed. Now, $p_t(x, E_k)$ is the solution of

$$\begin{cases} \frac{\partial}{\partial t}p_t(x, E_k) = J\left(x\right) * p_t(x, E_k) - p_t(x, E_k), & t\in[0,\infty),\ x\in\mathbb{Q}_p^n \\ \\ p_0(x, E_k) = 1_{E_k}, & 1_{E_k}\in L^1(\mathbb{Q}_p^n), \end{cases}$$

cf. Proposition 2.1.7. Then , $p_t(x, E_k)$ is a continuous function in x for any $t \ge 0$, which implies that $p_t(\cdot, E_k)$ is a measurable function of x for any $t \ge 0$. Now, by using that $E_k \uparrow E$ and the fact $p_t(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$, we get that $p_t(x, E_k) \to p_t(x, E)$ as $k \to \infty$, which implies that $p_t(\cdot, E)$ is the pointwise limit of a sequence of measurable functions $\{p_t(\cdot, E_k)\}_k$ and consequently $p_t(\cdot, E)$ is measurable.

Claim 3. $p_0(x, \{x\}) = 1$ for all $x \in \mathbb{Q}_p^n$.

This is a direct consequence of the definition of measure $p_t(x, E)$.

Claim 4. (The Chapman-Kolmogorov equation) For all $t, s \ge 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$,

$$p_{t+s}(x, E) = \int_{\mathbb{Q}_p^n} p_t(x, d^n y)p_s(y, E).$$

We consider the case $t,\ s>0$, since in the other cases the assertion is clear. We first note that

$$p_{t+s}(x,\cdot)=p_t(x,\cdot)*p_s(x,\cdot) \text{ in } \mathcal{D}'(\mathbb{Q}_p^n). \tag{2.8}$$

Indeed, for $E\in\mathcal{B}_0$, we have that $p_{t+s}(x,E)$ is equal to

$$\begin{aligned}\mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)(t+s)})*1_E &= \mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)t}\ e^{(\widehat{J}(||\xi||_p)-1)s})*1_E\\ &=\left[\mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)t})*\mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)s})\right]*1_E,\end{aligned}$$

since $e^{(\widehat{J}(||\xi||_p)-1)t}$, $e^{(\widehat{J}(||\xi||_p)-1)s}\in L^1_{loc}$. The Chapman-Kolmogorov equation is exactly (2.8). Indeed, by using the fact that the convolution of distributions is associative, we get from (2.8) that

$$\begin{aligned}p_{t+s}(x,E) &= \mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)t})*(\mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)s})*1_E)\\ &= \mathcal{F}^{-1}_{\xi\to x}(e^{(\widehat{J}(||\xi||_p)-1)t})*p_s(x,E),\end{aligned} \tag{2.9}$$

$E\in\mathcal{B}_0$. We now recall that the convolution of a distribution and a test function is a locally constant function, and hence its Fourier transform, as a distribution, is a function with compact support. By using this, from (2.9), we have

$$\begin{aligned}p_{t+s}(x,E) &= \mathcal{F}^{-1}_{\xi\to x}\left(e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{p_s}(\xi,E)\right)\\ &= \int_{\mathbb{Q}_p^n}\chi_p(-\xi\cdot x)\,e^{(\widehat{J}(||\xi||_p)-1)t}\widehat{p_s}(\xi,E)d^n\xi\end{aligned}$$

in $\mathcal{D}'(\mathbb{Q}_p^n)$. Let B_N^n be a ball containing the support of $\widehat{p_s}(\xi,E)$, with N depending on E and s, by using Fubini's Theorem,

$$p_{t+s}(x,E)=\int_{\mathbb{Q}_p^n}\chi_p(-\xi\cdot x)\left(e^{(\widehat{J}(||\xi||_p)-1)t}1_{B_N^n}(\xi)\right)\widehat{p_s}(\xi,E)d^n\xi$$

$$=\int_{\mathbb{Q}_p^n}\chi_p(-\xi\cdot x)\left(e^{(\widehat{J}(||\xi||_p)-1)t}1_{B_N^n}(\xi)\right)\left\{\int_{\mathbb{Q}_p^n}\chi_p(y\cdot\xi)\,p_s(y,E)d^ny\right\}d^n\xi$$

$$=\int_{\mathbb{Q}_p^n}\left(\int_{\mathbb{Q}_p^n}\chi_p(-(x-y)\cdot\xi)\,e^{(J(||\xi||_p)-1)t}1_{B_N^n}(\xi)d^n\xi\right)p_s(y,E)d^ny$$

$$=\int_{\mathbb{Q}_p^n}p_t(x-y,\widehat{1}_{B_N^n})p_s(y,E)d^ny \text{ for } E\in\mathcal{B}_0. \tag{2.10}$$

Formula (2.10) between positive distributions extends to a formula between Borel measures on $\mathbb{Q}_p^n$, by the Caratheodory Theorem.

2.3.5 Remark. (i) The transition function $p_t(x,\cdot)$ is normal, *i.e.* $\lim_{t\to 0^+} p_t(x,\mathbb{Q}_p^n) = 1$ for all $x \in \mathbb{Q}_p^n$. This follows from the fact that $p_t(x,\mathbb{Q}_p^n) = 1$, see proof of Claim 1.

(ii) From (2.8) we have $\{p_t(x,\cdot)\}_{t\geq 0}$ is a convolution semigroup in $\mathcal{D}'(\mathbb{Q}_p^n)$, and moreover $p_t(x,\cdot) \to \delta$ when $t \to 0^+$.

(iii) A function $p(x,E)$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$, is called a sub-Markovian transition function on $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$, if it satisfies (1) for every $x \in \mathbb{Q}_p^n$, $p(x,\cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$ such that $p(x,\mathbb{Q}_p^n) \leq 1$; (2) for every $E \in \mathcal{B}(\mathbb{Q}_p^n)$, $p(\cdot,E)$ is a Borel measurable function. Therefore, $p_t(x,E)$ for $t \geq 0$, $x \in \mathbb{Q}_p^n$, $E \in \mathcal{B}(\mathbb{Q}_p^n)$ is a sub-Markov transition function on $\mathbb{Q}_p^n$.

Let $C_b(\mathbb{Q}_p^n)$ be the space of real-valued, bounded, and continuous functions on $\mathbb{Q}_p^n$. This is a Banach space with the norm $\|f\|_\infty = \sup_{x\in\mathbb{Q}_p^n} |f(x)|$. We say that a function $f \in C_b(\mathbb{Q}_p^n)$ converges to zero as $x \to \infty$ if, for each $\epsilon > 0$, there exists a compact subset $E \subset \mathbb{Q}_p^n$ such that $|f(x)| < \epsilon$ for all $x \in \mathbb{Q}_p^n \smallsetminus E$. In such case we write $\lim_{x\to\infty} f(x) = 0$. We set

$$C_0(\mathbb{Q}_p^n) := \{f \in C_b(\mathbb{Q}_p^n);\ \lim_{x\to\infty} f(x) = 0\}.$$

The space $C_0(\mathbb{Q}_p^n)$ is a closed subspace of $C_b(\mathbb{Q}_p^n)$, and thus it is a Banach space.

2.3.6 Definition. Given a Markov transition function $p_t(x,\cdot)$, we attach to it the following operator:

$$T_t f(x) := \begin{cases} \int_{\mathbb{Q}_p^n} p_t(x,d^n y) f(y) & \text{if} \quad t > 0 \\ \\ f(x) & \text{if} \quad t = 0. \end{cases}$$

We say that $p_t(x,\cdot)$ is a C_0-function if the space $C_0(\mathbb{Q}_p^n)$ is an invariant subspace for the operators T_t, $t \geq 0$, i.e.

$$f \in C_0(\mathbb{Q}_p^n) \longrightarrow T_t f \in C_0(\mathbb{Q}_p^n).$$

2.3.7 Lemma. $p_t(x,\cdot)$ is a C_0-function. Furthermore, $T_t : C_0(\mathbb{Q}_p^n) \to C_0(\mathbb{Q}_p^n)$ is a bounded linear operator.

Proof. The result follows from the fact that $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $C_0(\mathbb{Q}_p^n)$, see e.g. [44, Proposition 1.3], by the following Claim:

Claim. $T_t : \left(\mathcal{D}(\mathbb{Q}_p^n), \|\cdot\|_\infty\right) \to C_0(\mathbb{Q}_p^n)$ is a bounded operator.

The proof of the Claim is as follows. Take $f\in\mathcal{D}(\mathbb{Q}_p^n)$ and $t>0$, then

$$\begin{aligned}\left|T_tf(x)\right| &= \left|\int_{\mathbb{Q}_p^n} p_t(x,d^ny)f(y)\right| \leq \int_{\mathbb{Q}_p^n} p_t(x,d^ny)\left|f(y)\right| \\ &= \int_{\mathbb{Q}_p^n\smallsetminus\{0\}} \widetilde{Z}_t(y)\left|f(x-y)\right|d^ny \leq \|f\|_\infty \int_{\mathbb{Q}_p^n\smallsetminus\{0\}}\widetilde{Z}_t(y)d^ny \\ &= \|f\|_\infty\, p_t(0,\mathbb{Q}_p^n)=\|f\|_\infty\,,\end{aligned}$$

cf. **Claim 1** in the proof of Proposition 2.3.4, this shows that T_t is a linear bounded operator from $\left(\mathcal{D}(\mathbb{Q}_p^n),\|\cdot\|_\infty\right)$ into $L^\infty(\mathbb{Q}_p^n)$. We now show that $\lim_{x\to\infty}T_tf(x)=0$. Take $f\in\mathcal{D}(\mathbb{Q}_p^n)$, with $\mathrm{supp}f=E$ and $t>0$, then

$$\begin{aligned}\left|T_tf(x)\right| = \left|\int_{\mathbb{Q}_p^n} p_t(x,d^ny)f(y)\right| &\leq \|f\|_\infty \int_E \widetilde{Z}_t(x-y)d^ny \\ &\leq Ct\,\|f\|_\infty\int_E ||x-y||_p^{-n}d^ny = Ct\,\|f\|_\infty\,||x||_p^{-n}vol(E),\end{aligned}$$

for $||x||_p$ big enough, cf. Proposition 2.2.5-(i). Finally, we show that $\lim_{x\to x_0}T_tf\,(x)=T_tf\,(x_0)$ for $t>0$. This fact follows by using the Dominated Convergence Theorem, since

$$T_tf\,(x)=\int_{\mathbb{Q}_p^n\smallsetminus\{0\}}\widetilde{Z}_t(y)f\,(x-y)\,d^ny$$

and $\left|\widetilde{Z}_t(y)f\,(x-y)\right|\leq\|f\|_\infty\,\widetilde{Z}_t(y)$ with $\int_{\mathbb{Q}_p^n\smallsetminus\{0\}}\widetilde{Z}_t(y)d^ny=p_t(0,\mathbb{Q}_p^n)=1$, cf. **Claim 1** in the proof of Proposition 2.3.4.

2.3.8 Remark. (i) We recall some results on Hunt, Lévy and Markov processes that we need to establish the main theorem of this section. All our processes have state space $(\mathbb{Q}_p^n,\mathcal{B}\left(\mathbb{Q}_p^n\right))$. A Hunt process is a Markov standard process which is quasi-left continuous on $[0,\infty)$, see [11, Definition 9.2 and the accompanying remarks].

(ii) Let $X=\{X_t\}_{t\geq0}$ be a Hunt process with state space $\mathbb{Q}_p^n$ and adjoined terminal state ∂ is a Lévy process on $\mathbb{Q}_p^n$ if (1) $P^x(X_t\in E)=P^0(X_t+x\in E)$ for $t\geq0$, 0, $x\in\mathbb{Q}_p^n$ and E a Borel subset of $\mathbb{Q}_p^n$; and (2) $P^0(X_t\in\mathbb{Q}_p^n)=1$ for $t\geq0$. Here $p_t(x,E)=P^x(X_t\in E)$.

(iii) The family of Borel probability measures $\{\mu_t,\,t\geq0\}$ given by

$$\mu_t(E)=P^0(X_t\in E) \tag{2.11}$$

is a convolution semigroup such that

$$\mu_t \to \delta \text{ as } t \to 0^+, \tag{2.12}$$

where δ denotes the Dirac distribution. Conversely, it can be shown that for any convolution semigroup $\{\mu_t,\ t \geq 0\}$ satisfying (2.12), it is possible to construct a Lévy process X_t with state space $\mathbb{Q}_p^n$ such that (2.11) is satisfied, see [22, Section 2] and [11, Exercise I-9-14].

2.3.9 Theorem. There exists a Lévy process $\mathfrak{X}(t,\omega)$, with state space $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$ and transition function $p_t(x,\cdot)$.

Proof. We first show that there exists a Hunt process $\mathfrak{X}(t,\omega)$ with state space $(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n))$ and transition function $p_t(x,\cdot)$. This result follow from [11, Theorem 9.4] by Proposition 2.3.4, Lemma 2.3.7 and Remark 2.3.5-(i), (iii). On the other hand, from Remarks 2.3.8 and 2.3.5-(ii), it follows that the Hunt process constructed is a Lévy process.

2.4 First passage time problem

Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = Au(x,t), & t \in [0,\infty),\ x \in \mathbb{Q}_p^n \\ \\ u(x,0) = \Omega(||x||_p). \end{cases} \tag{2.13}$$

By Proposition 2.1.7,

$$u(x,t) = Z_t(x) * \Omega(||x||_p), \tag{2.14}$$

is a classical solution of (2.13). We now define

$$q_t(x,E) = \begin{cases} (u(\cdot,t) * 1_E)(x) & \text{for } t > 0 \text{ and } E \in \mathcal{B}(\mathbb{Q}_p^n) \\ \\ 1_E(x) & \text{for } t = 0 \text{ and } E \in \mathcal{B}(\mathbb{Q}_p^n). \end{cases}$$

Since

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p^n) & \to & \mathbb{C} \\ \\ \phi(x) & \to & \phi(x) * \Omega(||x||_p) \end{array}$$

is linear continuous mapping, by the arguments given in the proof of Proposition 2.3.4, $q_t(x,E)$ is the transition function of a Markov process $\mathfrak{J}(t,\omega)$. Set

Υ to be the space of all paths $\mathfrak{J}(t,\omega)$. Then there exists a probability space $(\Upsilon,\mathcal{F},P)$, where P is a probability measure on Υ and $\mathfrak{J}(t,\cdot):(\Upsilon,\mathcal{F},P)\rightarrow\left(\mathbb{Q}_p^n,\mathcal{B}\left(\mathbb{Q}_p^n\right),d^nx\right)$ is random variable for each $t\geq 0$. The construction of this probability space follows from classical arguments, see, e.g., [20]. We notice that

$$\begin{aligned} P(\{\omega \;&\in\; \Upsilon:\ \mathfrak{J}(0,\omega)\in\mathbb{Z}_p^n\}) = q_0\left(0,\mathbb{Z}_p^n\right) = \Omega(||x||_p)*\Omega\left(||x||_p\right)|_{x=0} \\ &= \;\Omega\left(||x||_p\right)|_{x=0}=1. \end{aligned}$$

In this section we study the following random variable.

2.4.1 Definition. The random variable $\tau_{\mathbb{Z}_p^n}(\omega):\Upsilon\longrightarrow\mathbb{R}_+\cup\{+\infty\}$ defined by

$$\inf\{t>0;\mathfrak{J}(t,\omega)\in\mathbb{Z}_p^n \text{ and there exists } t' \text{ such that } 0<t'<t \text{ and } \mathfrak{J}(t',\omega)\notin\mathbb{Z}_p^n\}$$

is called the first passage time of a path of the random process $\mathfrak{J}(t,\omega)$ entering the domain $\mathbb{Z}_p^n$.

2.4.2 Remark. We notice that the condition

$$P(\{\omega\in\Upsilon:\tau_{\mathbb{Z}_p^n}(\omega)<\infty\})=1 \tag{2.15}$$

means that every path of $\mathfrak{J}(t,\omega)$ is sure to return to $\mathbb{Z}_p^n$. If (2.15) does not hold, then there exist paths of $\mathfrak{J}(t,\omega)$ that abandon $\mathbb{Z}_p^n$ and never go back.

2.4.3 Lemma. The function $u(x,t)=Z_t(x)*\Omega(||x||_p)$, $t\geq 0$, is pointwise differentiable in t and its derivative is given by the formula

$$\frac{\partial u}{\partial t}(x,t)=\int_{\mathbb{Z}_p^n}\chi_p(-x.\xi)e^{(\widehat{J}(\|\xi\|_p)-1)t}\left[\widehat{J}(\|\xi\|_p)-1\right]d^n\xi, \text{ for } t\geq 0.$$

Proof. The formula is obtained by applying the Dominated Convergence Theorem.

2.4.4 Lemma. The probability density function for a path of $\mathfrak{J}(t,\omega)$ to enter into $\mathbb{Z}_p^n$ at the instant of time t, with the condition that $\mathfrak{J}(0,\omega)\in\mathbb{Z}_p^n$ is given by

$$g(t)=\int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n}J(||y||_p)u(y,t)d^ny. \tag{2.16}$$

Proof. The survival probability, by definition

$$S(t) := S_{\mathbb{Z}_p^n}(t) = \int_{\mathbb{Z}_p^n} u(x,t)d^n x,$$

is the probability that a path of $\mathfrak{J}(t,\omega)$ remains in $\mathbb{Z}_p^n$ at the time t. Because there are no external or internal sources,

$$S'(t) = \begin{array}{l}\text{Probability that a path of } \mathfrak{J}(t,\omega)\\ \text{goes back to } \mathbb{Z}_p^n \text{ at the time } t\end{array} - \begin{array}{l}\text{Probability that a path of}\\ \mathfrak{J}(t,\omega) \text{ exists } \mathbb{Z}_p^n \text{ at the time } t\end{array}$$

$$= g(t) - CS(t), \text{ with } 0 < C \leq 1.$$

By using Lemma 2.4.3,

$$\begin{aligned}
S'(t) = \int_{\mathbb{Z}_p^n} \frac{\partial}{\partial t} u(x,t) d^n x &= \int_{\mathbb{Z}_p^n} \left\{ \int_{\mathbb{Q}_p^n} J(||y||_p) u(x-y,t) d^n y - u(x,t) \right\} d^n x \\
&= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n} J(||y||_p) \left\{ u(x-y,t) - u(x,t) \right\} d^n y d^n x \\
&= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Z}_p^n} J(||y||_p) \left\{ u(x-y,t) - u(x,t) \right\} d^n y d^n x \\
&+ \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(||y||_p) \left\{ u(x-y,t) - u(x,t) \right\} d^n y d^n x.
\end{aligned}$$

By Proposition 2.1.7, for $x, y \in \mathbb{Z}_p^n$,

$$\begin{aligned}
u(x-y,t) &= \int_{\mathbb{Z}_p^n} \chi_p\left(-(x-y)\cdot\xi\right) e^{(\widehat{J}(||\xi||_p)-1)t} d^n \xi = \int_{\mathbb{Z}_p^n} e^{(\widehat{J}(||\xi||_p)-1)t} d^n \xi \\
&= u(x,t),
\end{aligned}$$

i.e. $u(x-y,t) - u(x,t) \equiv 0$ for $x, y \in \mathbb{Z}_p^n$, consequently,

$$\begin{aligned}
S'(t) &= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(||y||_p)(u(x-y,t) - u(x,t)) d^n y d^n x \\
&= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(||y||_p) u(x-y,t) d^n y d^n x \\
&- \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(||y||_p) d^n y \int_{\mathbb{Z}_p^n} u(x,t) d^n x \\
&= \int_{\mathbb{Z}_p^n} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} J(||y||_p) u(x-y,t) d^n y d^n x - CS(t),
\end{aligned}$$

with $C := \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} J(||y||_p)d^ny \leq 1$, since J is of exponential type and $\int_{\mathbb{Q}_p^n} J(||y||_p)d^ny = 1$. We notice that if $x \in \mathbb{Z}_p^n$ and $y \in \mathbb{Q}_p^n\setminus\mathbb{Z}_p^n$, then

$$u(x-y,t) = \int_{\mathbb{Z}_p^n} \chi_p(-x\cdot\xi)\,\chi_p(y\cdot\xi)\,e^{(\widehat{J}(||\xi||_p)-1)t}d^n\xi$$

$$= \int_{\mathbb{Z}_p^n} \chi_p(y\cdot\xi)\,e^{(\widehat{J}(||\xi||_p)-1)t}d^n\xi = \int_{\mathbb{Z}_p^n} \chi_p(-y\cdot\xi)\,e^{(\widehat{J}(||\xi||_p)-1)t}d^n\xi = u(y,t),$$

and consequently

$$\begin{aligned} S'(t) &= \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} J(||y||_p)u(y,t)d^ny - CS(t) \\ &= g(t) - CS(t) \text{ with } 0 < C \leq 1. \end{aligned}$$

2.4.5 Proposition. The probability density function $f(t)$ of the random variable $\tau_{\mathbb{Z}_p^n}(\omega)$ satisfies the non-homogeneous Volterra equation of second kind

$$g(t) = \int_0^\infty g(t-\tau)f(\tau)d\tau + f(t). \tag{2.17}$$

Proof. The result follow from Lemma 2.4.4 by using the argument given in the proof of Theorem 1 in [3].

2.4.6 Lemma. The Laplace transform $G(s)$ of $g(t)$ is given by

$$G(s) = \int_{\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n} J(||y||_p)\int_{\mathbb{Z}_p^n} \frac{\chi_p(-y\cdot\xi)}{s+(1-\widehat{J}(\|\xi\|_p))}d^n\xi d^ny, \text{ for } Re(s)>0. \tag{2.18}$$

Proof. We first note that $e^{-st}J(||y||_p)e^{(\widehat{J}(||\xi||_p)-1)t}\Omega(||\xi||_p) \in L^1((0,\infty)\times\mathbb{Q}_p^n\setminus\mathbb{Z}_p^n\times\mathbb{Q}_p^n, dtd^n\xi d^ny)$ for $s \in \mathbb{C}$ with $Re(s) > 0$. The announced formula follows now from (2.16) and (2.14) by using Fubini's Theorem.

2.4.7 Theorem. If $-n < \gamma < 0$, then $P(\{\omega \in \Upsilon : \tau_{\mathbb{Z}_p^n}(\omega){<}\infty\}) = 1$.

Proof. By applying the Laplace transform to (2.17), we have

$$F(s) = \frac{G(s)}{1+G(s)} = 1 - \frac{1}{1+G(s)},$$

where $F(s)$ and $G(s)$ are the Laplace transforms of f and g, respectively. We understand $F(0) = \lim_{s\to 0} F(s)$ and $G(0) = \lim_{s\to 0} G(s)$. From $F(0) = \int_0^\infty f(t)dt = \frac{G(0)}{1+G(0)}$, it follows that if $G(0) = \infty$, then $\mathfrak{J}(t,\omega)$ is recurrent. Since $G(0) = \int_0^\infty g(t)dt$ is either a positive number or infinity, it is sufficient to show that $\lim_{s\to 0} G(s) = \infty$ for $s \in \mathbb{R}_+$. For $y \in \mathbb{Q}_p^n \backslash \mathbb{Z}_p^n$ with $||y||_p = p^i$, $i \in \mathbb{N}\backslash\{0\}$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p^n} \frac{\chi_p(-y\cdot\xi)}{s+1-\widehat{J}(\|\xi\|_p)} d^n\xi &= \sum_{j=0}^{\infty} \frac{1}{s+1-\widehat{J}(p^{-j})} \int_{||\xi||_p = p^{-j}} \chi_p(-y\cdot\xi)\, d^n\xi \\ &= \sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})},\end{aligned}$$

and by (2.18), we have that $G(s)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{\infty} J(p^i) \int_{||y||_p = p^i} \left[\sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})}\right] d^n y \\ &= \sum_{i=1}^{\infty} J(p^i) \left[\sum_{j=i}^{\infty} \frac{p^{-nj}(1-p^{-n})}{s+1-\widehat{J}(p^{-j})} - \frac{p^{-ni}}{s+1-\widehat{J}(p^{1-i})}\right] p^{ni}(1-p^{-n}) \\ &= (1-p^{-n}) \sum_{i=1}^{\infty} J(p^i) \left[(1-p^{-n}) \sum_{j=i}^{\infty} \frac{p^{n(i-j)}}{s+1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(p^{1-i})}\right].\end{aligned}$$

Now, since $\lim_{j\to\infty} 1 - \widehat{J}(p^{-j}) = 0$, given any $s > 0$, there exists $j_0(s) \in \mathbb{N}$ such that $1 - \widehat{J}(p^{-j}) < s$ for $j > j_0(s)$. In addition, $s \to 0^+$ implies that $j_0(s) \to \infty$. By using these observations, we have

$$\begin{aligned}G(s) &\geq (1-p^{-n})J(p) \left[(1-p^{-n}) \sum_{j=1}^{\infty} \frac{p^{n(1-j)}}{s+1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(1)}\right] \\ &\geq (1-p^{-n})J(p) \left[\frac{p^n(1-p^{-n})}{2} \sum_{j=1}^{j_0(s)} \frac{p^{-nj}}{1-\widehat{J}(p^{-j})} - \frac{1}{s+1-\widehat{J}(1)}\right],\end{aligned}$$

notice that by Remark 2.1.4, $1-\widehat{J}(1) > 0$. Therefore, $\lim_{s\to 0^+} G(s)$ exceeds the expression

$$(1-p^{-n})J(p)\left[\frac{p^n\left(1-p^{-n}\right)}{2}\sum_{j=0}^{\infty}\frac{p^{-nj}}{1-\widehat{J}(p^{-j})}-\frac{1+\frac{p^n\left(1-p^{-n}\right)}{2}}{1-\widehat{J}(1)}\right]$$

$$=(1-p^{-n})J(p)\left[\frac{p^n}{2}\int_{\mathbb{Z}_p^n}\frac{d^n\xi}{1-\widehat{J}(\|\xi\|_p)}-\frac{1+\frac{p^n\left(1-p^{-n}\right)}{2}}{1-\widehat{J}(1)}\right]=\infty,$$

cf. Lemma 2.1.5.

Chapter 3

Non-Archimedean Pseudodifferential Operators With Variable Coefficients and Feller Semigroups

In this chapter, we study a large class of non-Archimedean pseudodifferential operators whose symbols are negative definite functions. We prove that these operators extend to generators of Feller semigroups. In order to study these operators, we introduce a new class of anisotropic Sobolev spaces, which are the natural domains for the operators considered here. We also study the Cauchy problem for certain pseudodifferential equations and we show that the fundamental solution of the associated homogeneous equation is connected to Lévy process.

3.1 Positive Definite and Negative Definite Functions on $\mathbb{Q}_p^n$

In this section, we collect some results about positive definite and negative definite functions that we will use along the chapter, we refer the reader to [10] for further details.

We set $\mathbb{N}$ the set of nonnegative integers.

3.1.1 Definition. A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called positive definite, if

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\varphi(x_i - x_j)\lambda_i\overline{\lambda}_j \geq 0$$

for all $m \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_m \in \mathbb{Q}_p^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. Here, $\overline{\lambda}_j$ denotes the complex conjugate of λ_j.

The set of positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{P}(\mathbb{Q}_p^n)$ and the subset of $\mathcal{P}(\mathbb{Q}_p^n)$ consisting of the continuous positive definite functions on $\mathbb{Q}_p^n$ is denoted as $\mathcal{CP}(\mathbb{Q}_p^n)$. The following assertions hold: (i) $\mathcal{P}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$; (ii) if φ_1, $\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$, then $\varphi_1\varphi_2 \in \mathcal{P}(\mathbb{Q}_p^n)$; the non-negative constant functions belong to $\mathcal{P}(\mathbb{Q}_p^n)$; (iii) $\mathcal{CP}(\mathbb{Q}_p^n)$ is a convex cone which is a closed subset of the set of continuous complex-valued functions in the topology of compact convergence cf. [10, Proposition 3.6].

3.1.2 Example. (i) We set $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. Let $J : \mathbb{Q}_p^n \to \mathbb{R}_+$ be a radial (i.e. $J(x) = J(||x||_p)$) and continuous function. In addition, we assume that $\int_{\mathbb{Q}_p^n} J(||x||_p) d^n x = 1$. By a direct calculation one verifies that $\widehat{J}(\xi)$ is a radial, continuous and positive definite function on $\mathbb{Q}_p^n$ and moreover $|\widehat{J}(||\xi||_p)| \leq 1$.
(ii) The additive character $x \to \chi_p(x \cdot \alpha)$, for $\alpha \in \mathbb{Q}_p^n$, is a continuous, positive definite (complex-valued) function on $\mathbb{Q}_p^n$, see e.g. [10, p. 13].

3.1.3 Definition. A function $\psi : \mathbb{Q}_p^n \to \mathbb{C}$ is called negative definite, if

$$\sum_{i=1}^m \sum_{j=1}^m \left(\psi(x_i) + \overline{\psi(x_j)} - \psi(x_i - x_j)\right) \lambda_i \overline{\lambda}_j \geq 0 \tag{3.1}$$

for all $m \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_m \in \mathbb{Q}_p^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

We denote by $\mathcal{N}(\mathbb{Q}_p^n)$ the set of negative definite functions on $\mathbb{Q}_p^n$ and by $\mathcal{CN}(\mathbb{Q}_p^n)$ the set of continuous negative definite functions on $\mathbb{Q}_p^n$. The following assertions hold: (i) $\mathcal{N}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of pointwise convergence on $\mathbb{Q}_p^n$; (ii) The non-negative constant functions belong to $\mathcal{N}(\mathbb{Q}_p^n)$; (iii) $\mathcal{CN}(\mathbb{Q}_p^n)$ is a convex cone which is closed in the topology of compact convergence on $\mathbb{Q}_p^n$, cf. [10, Proposition 7.4].

Furthermore, if $\psi : \mathbb{Q}_p^n \to \mathbb{R}$ is negative definite function, then $\psi(-x) = \psi(x)$ and $\psi(x) \geq \psi(0) \geq 0$ for all $x \in \mathbb{Q}_p^n$, see e.g. [10, Proposition 7.5].

3.1.4 Example. Let $J : \mathbb{Q}_p^n \to \mathbb{R}_+$ the function given in Example 3.1.2-(i). By using Corollary 7.7 in [10, Theorem 7.8], the function $\widehat{J}(0) - \widehat{J}(||\xi||_p) = 1 - \widehat{J}(||\xi||_p)$ is negative definite. On the other hand, we have that $0 \leq 1 - \widehat{J}(||\xi||_p) \leq 2$, $\xi \in \mathbb{Q}_p^n$, see e.g. [48, Lemma 1-(i)].

3.1.5 Example. In [39], see also [32], Rodríguez-Vega and Zúñiga-Galindo considered the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + a(D_T^{\beta}u)(x,t) = f(x,t), \quad t \in (0,T_0],\, x \in \mathbb{Q}_p^n \\ \\ u(x,0) = \varphi(x), \end{cases} \tag{3.2}$$

where a, β, T_0 are positive real numbers, and $(D_T^{\beta}h)(x) = \mathcal{F}^{-1}_{\xi\to x}(||\xi||_p^{\beta}\mathcal{F}_{x\to\xi}h)$ is the Taibleson operator. They established that $e^{-at||\xi||_p^{\beta}} \in L^1(\mathbb{Q}_p^n)$ for $t > 0$, and that $Z(x,t) = \mathcal{F}^{-1}_{\xi\to x}(e^{-at||\xi||_p^{\beta}})$, for a, $t > 0$, is a transition function of a Markov process with space state $\mathbb{Q}_p^n$, cf. [39, Proposition 1 and Theorem 2]. By using a theorem due to Bochner, see [10, Theorem 3.12], the function $e^{-at||\xi||_p^{\beta}}$, for $t > 0$, is positive definite, and by a theorem due to Schoenberg, see [10, Theorem 7.8], $a||\xi||_p^{\beta}$ is a negative definite function, for any $\beta > 0$.

3.1.6 Example. Take $m \in \mathbb{N}\backslash\{0\}$ and let $b_j : \mathbb{Q}_p^n \to \mathbb{R}_+\backslash\{0\}$, $j = 1,\ldots,m$, be continuous positive functions. Let $0 < \alpha_1 \leq \ldots \leq \alpha_m$ be positive constants. Consider the function

$$q(x,\xi) = \sum\nolimits_{j=1}^{m} b_j(x)||\xi||_p^{\alpha_j}, \tag{3.3}$$

for x, $\xi \in \mathbb{Q}_p^n$. By Example 3.1.5, and the fact $\mathcal{N}(\mathbb{Q}_p^n)$ is a convex cone, $q(x,\xi)$ is radial, continuous, negative definite function in ξ for each x fixed.

3.1.7 Remark. (i) It is relevant to mention that the type of functions given in (3.3), with some $\alpha_j > 2$, occurs only in the non-Archimedean setting, since any function $\psi : \mathbb{R}^n \to \mathbb{R}$ locally bounded and negative definite, satisfies

$$|\psi(\xi)| \leq C_{\psi}(1 + ||\xi||_{\mathbb{R}}^2),$$

for some $C_\psi > 0$ and for all $\xi \in \mathbb{R}^n$, see e.g. [30, Lemma 3.6.22].
(ii) Let $h\,(y) = \sum_{j=0}^{\infty} a_j y^j$, $a_j \in \mathbb{R}_+$, be a convergent series in $\mathbb{R}_+$, which defines a non-constant function. By using Example 3.1.5 and the fact that $\mathcal{N}(\mathbb{Q}_p^n)$ is closed in the pointwise topology, it follows that $h\,(||x||_p) = \sum_{j=0}^{\infty} a_j ||x||_p^j$ is a negative definite function.

3.1.8 Lemma. (i) Let $\psi : \mathbb{Q}_p^n \to \mathbb{C}$ be a negative definite function such that $\xi \to [\psi\,(\xi)]^j$ is also a negative definite function for any $j \in \mathbb{N}$. Then $e^{\psi(\xi)}$ is a negative definite function. (ii) Set $\psi_0\,(\xi) := \sum_{j=1}^{\infty} c_j ||\xi||_p^{\alpha_j}$ with $c_j \geq 0$, $\alpha_j \in \mathbb{N}$

such that the real series $\sum_{j=1}^{\infty} c_j y^{\alpha_j}$ defines a non-constant real function. Then for any $j \in \mathbb{N}\backslash\{0\}$,

$$e^{e^{\cdot^{\cdot^{\cdot^{e^{\psi_0(\xi)}}}}}}, \; j - \text{powers} \tag{3.4}$$

is a continuous and negative definite function on $\mathbb{Q}_p^n$.

Proof. By the hypothesis, $\psi_m := \sum_{j=0}^{m} \frac{1}{j!} [\psi(\xi)]^j$, $m \in \mathbb{N}$, is negative definite, and since $\mathcal{N}(\mathbb{Q}_p^n)$ is closed in the pointwise topology, we have that $e^{\psi(\xi)}$ is negative definite. By using Remark 3.1.7-(ii), $\psi_0(\xi) = \sum_{j=1}^{\infty} c_j ||\xi||_p^{\alpha_j}$ is negative definite, and $[\psi_0(\xi)]^k = \left(\sum_{j=1}^{\infty} c_j ||\xi||_p^{\alpha_j}\right)^k = \sum_{j=1}^{\infty} d_j ||\xi||_p^{\beta_j}$, with $d_j = d_j(k) \geq 0$, $\beta_j = \beta_j(k) \in \mathbb{N}$, is also negative definite function. By the first part $e^{\psi_0(\xi)}$ is negative definite. By induction on j we obtain (3.4).

Negative definite functions of form (3.4) can only occur in the non-Archimedean setting.

From now on, $\boldsymbol{\psi} : \mathbb{Q}_p^n \to \mathbb{C}$ (or $\mathbb{R}$) denotes a radial, continuous and negative definite function. We consider three subclasses of negative definite functions, however, we do not expect that this classification be complete.

3.1.9 Definition. $\boldsymbol{\psi} : \mathbb{Q}_p^n \to \mathbb{C}$ (or $\mathbb{R}$) is called of type 0, if there exists a positive constant $C := C(\boldsymbol{\psi})$ such that

$$|\boldsymbol{\psi}(||\xi||_p)| \leq C, \text{ for all } \xi \in \mathbb{Q}_p^n.$$

3.1.10 Definition. $\boldsymbol{\psi} : \mathbb{Q}_p^n \to \mathbb{C}$ (or $\mathbb{R}$) is called of type 1, if there exist positive constants $C_0(\boldsymbol{\psi}) := C_0$, $C_1(\boldsymbol{\psi}) := C_1$, $\beta_0(\boldsymbol{\psi}) := \beta_0 \in \mathbb{R}_+\backslash\{0\}$ and $\beta_1(\boldsymbol{\psi}) := \beta_1 \in \mathbb{R}_+\backslash\{0\}$, with $\beta_1 \geq \beta_0$, such that

$$C_0 [\max\{1, ||\xi||_p\}]^{\beta_0} \leq \max\{1, |\boldsymbol{\psi}(||\xi||_p)|\} \leq C_1 [\max\{1, ||\xi||_p\}]^{\beta_1},$$

for all $\xi \in \mathbb{Q}_p^n$.

3.1.11 Definition. $\boldsymbol{\psi} : \mathbb{Q}_p^n \to \mathbb{C}$ (or $\mathbb{R}$) is called of type 2, if for all $\beta \geq 1$, there is a positive constant $C := C(\boldsymbol{\psi},\beta)$ such that

$$\max\{1, |\boldsymbol{\psi}(||\xi||_p)|\} > C [\max\{1, ||\xi||_p\}]^{\beta}, \text{ for all } \xi \in \mathbb{Q}_p^n.$$

3.2 Function Spaces Related to Negative Definite Functions

Along this section $\boldsymbol{\psi}:\mathbb{Q}_p^n\to\mathbb{C}$ denotes a negative definite, radial and continuous function of type 1 or 2, unless otherwise stated. In addition, we will assume that

$$0<\sup_{\xi\in\mathbb{Z}_p^n}|\boldsymbol{\psi}(||\xi||_p)|\leq 1. \tag{3.5}$$

This condition is achieved by multiplying $\boldsymbol{\psi}$ by a suitable positive constant. Condition (3.5) implies that

$$\varphi_l\,(x):=[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^l,\ l\in\mathbb{N},\text{ is a locally constant function,} \tag{3.6}$$

more precisely, $\varphi_l\,(x+x')=\varphi_l\,(x)$ for any $x'\in\mathbb{Z}_p^n$.

In this section, we introduce two classes of function spaces related to $\boldsymbol{\psi}$, namely $B_{\boldsymbol{\psi},l}\,(\mathbb{C})$, $l\in\mathbb{N}$, and $B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})$. These spaces are generalizations of the spaces $H_{\mathbb{C}}(l)$, $l\in\mathbb{N}$, and $H_{\mathbb{C}}(\infty)$ introduced by Zúñiga-Galindo in [55], see also [54]. The results presented in this section can be established by using the techniques presented in [55], [54].

For $\varphi,\gamma\in\mathcal{D}(\mathbb{Q}_p^n)$, and $l\in\mathbb{N}$, we define the following scalar product:

$$\langle\varphi,\gamma\rangle_{\boldsymbol{\psi},l}=\int_{\mathbb{Q}_p^n}[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^l\,\widehat{\varphi}(\xi)\overline{\widehat{\gamma}(\xi)}d^n\xi,$$

where the bar denotes the complex conjugate. We also set

$$||\varphi||^2_{\boldsymbol{\psi},l}:=\langle\varphi,\varphi\rangle_{\boldsymbol{\psi},l}\,.$$

Notice that $||\cdot||_{\boldsymbol{\psi},l}\leq||\cdot||_{\boldsymbol{\psi},m}$ for $l\leq m$. Let us denote by $B_{\boldsymbol{\psi},l}\,(\mathbb{C}):=B_{\boldsymbol{\psi},l}(\mathbb{Q}_p^n,\mathbb{C})$ the completion of $\mathcal{D}(\mathbb{Q}_p^n)$ with respect to $\langle\cdot,\cdot\rangle_{\boldsymbol{\psi},l}$. Then $B_{\boldsymbol{\psi},m}\,(\mathbb{C})\hookrightarrow B_{\boldsymbol{\psi},l}\,(\mathbb{C})$ (continuous embedding) for $l\leq m$.

We set

$$B_{\boldsymbol{\psi},\infty}\,(\mathbb{C}):=B_{\boldsymbol{\psi},\infty}(\mathbb{Q}_p^n,\mathbb{C})=\cap_{l\in\mathbb{N}}B_{\boldsymbol{\psi},l}\,(\mathbb{C})\,.$$

Notice that $B_{\boldsymbol{\psi},0}\,(\mathbb{C})=L^2(\mathbb{Q}_p^n)$ and $\mathcal{D}(\mathbb{Q}_p^n)\subset B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})\subset L^2(\mathbb{Q}_p^n)$. With the topology induced by the family of seminorms $\{||\cdot||_{\boldsymbol{\psi},l}\}_{l\in\mathbb{N}}$, $B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})$ becomes a locally convex topological space, which is metrizable. Indeed,

$$d_{\boldsymbol{\psi}}(f,g):=\max_{l\in\mathbb{N}}\left\{2^{-l}\frac{||f-g||_{\boldsymbol{\psi},l}}{1+||f-g||_{\boldsymbol{\psi},l}}\right\},\ \text{with } f,g\in B_{\boldsymbol{\psi},\infty},$$

is a metric for the topology of $B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})$ considered as a locally convex topological space. A sequence $\{f_l\}_{l\in\mathbb{N}}$ in $(B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})\,,d_{\boldsymbol{\psi}})$ converges to $f\in B_{\boldsymbol{\psi},\infty}\,(\mathbb{C})$

if and only if, $\{f_l\}_{l\in\mathbb{N}}$ converges to f in the norm $||\cdot||_{\psi,l}$ for all $l \in \mathbb{N}$. From this observation, it follows that the topology on $B_{\psi,\infty}(\mathbb{C})$ coincides with the projective limit topology τ_P. An open neighborhood base at zero of τ_P is given by the choice of $\epsilon > 0$ and $l \in \mathbb{N}$, and the set

$$U_{\epsilon,l} := \{f \in B_{\psi,\infty}; ||f||_{\psi,l} < \epsilon\}.$$

The space $B_{\psi,\infty}(\mathbb{C})$ endowed with the topology τ_P is a countable Hilbert space in the sense of Gel'fand and Vilenkin, see e.g. [27, Chapter I, Section 3.1] or [36, Section 1.2]. Furthermore $(B_{\psi,\infty}(\mathbb{C}), \tau_P)$ is metrizable and complete and hence a Fréchet space cf. [55, Lemma 3.3]. If $\psi = ||\cdot||_p$, then $B_{\psi,l}(\mathbb{C})$ coincides with the space $H_{\mathbb{C}}(l)$, respectively, $B_{\psi,\infty}(\mathbb{C})$ coincides with the space $H_{\mathbb{C}}(\infty)$, where $H_{\mathbb{C}}(l)$ and $H_{\mathbb{C}}(\infty)$ are the spaces introduced in [55], see also [54].

3.2.1 Remark. We will denote by $B_{\psi,l}(\mathbb{R}) := B_{\psi,l}(\mathbb{Q}_p^n, \mathbb{R})$, for all $l \in \mathbb{N}$, and by $B_{\psi,\infty}(\mathbb{R}) := B_{\psi,\infty}(\mathbb{Q}_p^n, \mathbb{R})$ the $\mathbb{R}$-vector spaces constructed from $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$. It is clear that $B_{\psi,l}(\mathbb{R}) \hookrightarrow B_{\psi,l}(\mathbb{C})$, $l \in \mathbb{N}$, and that $B_{\psi,\infty}(\mathbb{R}) \hookrightarrow B_{\psi,\infty}(\mathbb{C})$, where '$\hookrightarrow$' means continuous embedding.

3.2.2 Lemma. The following assertions hold:
(i) the completion of the metric space $(\mathcal{D}(\mathbb{Q}_p^n), d_\psi)$ is $(B_{\psi,\infty}(\mathbb{C}), d_\psi)$, which is a nuclear countably Hilbert space;
(ii) $B_{\psi,l}(\mathbb{C}) = \{f \in L^2; ||f||_{\psi,l} < \infty\} = \{T \in \mathcal{D}'(\mathbb{Q}_p^n); ||T||_{\psi,l} < \infty\}$;
(iii) $B_{\psi,\infty}(\mathbb{C}) = \{f \in L^2; ||f||_{\psi,l} < \infty$ for every $l \in \mathbb{N}\}$;
(iv) $B_{\psi,\infty}(\mathbb{C}) = \{T \in \mathcal{D}'(\mathbb{Q}_p^n); ||T||_{\psi,l} < \infty$ for every $l \in \mathbb{N}\}$;
(v) $B_{\psi,\infty}(\mathbb{C})$ is densely and continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{C})$;
(vi) $B_{\psi,\infty}(\mathbb{R})$ is densely and continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{R})$;
(vii) $B_{\psi,\infty}(\mathbb{C}) \subset L^1$. In particular, $\widehat{f} \in C_0(\mathbb{Q}_p^n, \mathbb{C})$ for $f \in B_{\psi,\infty}(\mathbb{C})$.

3.2.3 Remark. The condition $T \in \mathcal{D}'(\mathbb{Q}_p^n)$, $||T||_{\psi,l} < \infty$ assumes implicitly that $\widehat{T}$ is a regular distribution. The equalities (ii)-(iv) in Lemma 3.2.2 are in the sense of vector spaces. The statements (i)-(iv) are valid for the spaces $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$, $B_{\psi,l}(\mathbb{R})$ and $B_{\psi,\infty}(\mathbb{R})$.

Proof. (i) The proof is similar to [55, Lemma 3.4 and Theorem 3.6].

(ii) Take $f \in B_{\psi,l}(\mathbb{Q}_p^n)$, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{Q}_p^n)$ such that $f_n \overset{||\cdot||_{\psi,l}}{\to} f$ for any $l \in \mathbb{N}$, i.e.

$$[\max\{1, |\psi(||\xi||_p)|\}]^{\frac{l}{2}} \widehat{f_n} \overset{||\cdot||_{L^2}}{\to} [\max\{1, |\psi(||\xi||_p)|\}]^{\frac{l}{2}} \widehat{f}.$$

By taking $l=0$ and using that L^2 is complete, we have $\widehat{f}\in L^2$, i.e. $f\in L^2$. Conversely, take $f\in L^2$ such that $[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^{\frac{l}{2}}\widehat{f}\in L^2$. By using the fact that $\mathcal{D}\left(\mathbb{Q}_p^n\right)$ is dense in L^2, there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{D}\left(\mathbb{Q}_p^n\right)$ such that $f_m \underrightarrow{||.||_{L^2}} [\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^{\frac{l}{2}}\widehat{f}$. We now define $g_m(\xi):=\frac{f_m(-\xi)}{[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^{\frac{l}{2}}}\in\mathcal{D}\left(\mathbb{Q}_p^n\right)$, see (3.6). Then $\widehat{g}_m \underrightarrow{||.||_{\boldsymbol{\psi},l}} f$, for any $l\in\mathbb{N}$, i.e. $f\in B_{\boldsymbol{\psi},l}(\mathbb{Q}_p^n)$. The other equality follows from the fact that $T\in L^2$, $||T||_{\boldsymbol{\psi},l}<\infty \Leftrightarrow T\in\mathcal{D}'\left(\mathbb{Q}_p^n\right)$, $||T||_{\boldsymbol{\psi},l}<\infty$.

(iii) and (iv) are an immediate consequence of (ii).

(v) By using the fact that $\frac{1}{[\max\{1,||\xi||_p\}]^r}\in L^1$ for $r>n$, one verifies that

$$\frac{1}{[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^l}\in L^1 \text{ if}\begin{cases}\boldsymbol{\psi} \text{ is of type 1 and } l>\frac{n}{\beta_0}\\ \\ \boldsymbol{\psi} \text{ is of type 2 and } l\geq 1.\end{cases}\tag{3.7}$$

Take $f\in B_{\boldsymbol{\psi},\infty}(\mathbb{C})$ with $\boldsymbol{\psi}$ of type 1. By using the Cauchy-Schwarz inequality and (3.7), we have for all $l>\frac{n}{\beta_0}$,

$$\begin{aligned}\int_{\mathbb{Q}_p^n}\left|\widehat{f}(\xi)\right|d^n\xi &= \int_{\mathbb{Q}_p^n}\frac{1}{[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^{\frac{l}{2}}}[\max\{1,|\boldsymbol{\psi}(||\xi||_p)|\}]^{\frac{l}{2}}\left|\widehat{f}(\xi)\right|d^n\xi\\ &\leq C||f||_{\boldsymbol{\psi},l}.\end{aligned}$$

Thus $\widehat{f}\in L^1$, and since the Fourier transform of a function in L^1 is uniformly continuous, now by the Riemann-Lebesgue theorem, we obtain that $f\in C_0(\mathbb{Q}_p^n,\mathbb{C})$, i.e. $B_{\boldsymbol{\psi},\infty}\subset C_0(\mathbb{Q}_p^n,\mathbb{C})$. In addition,

$$||f||_{L^\infty}\leq||\widehat{f}||_{L^1}\leq C||f||_{\boldsymbol{\psi},l} \text{ for } l>\frac{n}{\beta_0}.\tag{3.8}$$

Since the topology of $B_{\boldsymbol{\psi},\infty}$ comes from the metric $d_{\boldsymbol{\psi}}$, the continuity of the embedding $B_{\boldsymbol{\psi},\infty}\to C_0(\mathbb{Q}_p^n,\mathbb{C})$ follows from (3.8), by using a standard argument based in sequences.

The proof is similar for functions $\boldsymbol{\psi}$ of type 2.

(vi) By (i) and Remark 3.2.3, and (v), we have

$$\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)\hookrightarrow B_{\boldsymbol{\psi},\infty}(\mathbb{R})\hookrightarrow B_{\boldsymbol{\psi},\infty}(\mathbb{C})\hookrightarrow C_0(\mathbb{Q}_p^n,\mathbb{C}),$$

which implies that $B_{\boldsymbol{\psi},\infty}(\mathbb{R})\hookrightarrow C_0(\mathbb{Q}_p^n,\mathbb{R})$. Finally, we recall that $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$ is dense in $C_0(\mathbb{Q}_p^n,\mathbb{R})$.

(vii) The proof of the fact $B_{\boldsymbol{\psi},\infty}(\mathbb{C})\subset L^1$ uses the same argument given [55] for Theorem 3.15-(ii). Then, by the Riemann-Lebesgue theorem, $\widehat{f}\in C_0(\mathbb{Q}_p^n,\mathbb{C})$ for $f\in B_{\boldsymbol{\psi},\infty}(\mathbb{C})$.

Recall that $H_{\mathbb{C}}(l) = B_{\psi,l}(\mathbb{C})$ and $H_{\mathbb{C}}(\infty) = B_{\psi,\infty}(\mathbb{C})$ when $\psi = ||\cdot||_p$, see [55], [54].

3.2.4 Lemma. $B_{\psi,\infty}(\mathbb{C}) \hookrightarrow H_{\mathbb{C}}(\infty)$.

Proof. By definition of the classes type 1 or 2,

$$||\varphi||_{\psi,l} \geq C(\psi, l)||\varphi||_l \text{ for } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^n\right) \text{ and } l \in \mathbb{N}. \tag{3.9}$$

By using the density of $\mathcal{D}\left(\mathbb{Q}_p^n\right)$ in $(B_{\psi,l}(\mathbb{C}), ||\cdot||_{\psi,l})$, we conclude that $B_{\psi,l}(\mathbb{C}) \subseteq H_{\mathbb{C}}(l)$ for any $l \in \mathbb{N}$. Consequently $B_{\psi,\infty}(\mathbb{C}) \subseteq H_{\mathbb{C}}(\infty)$. To check the continuity of the identity map, we use that if $f_n \overset{d_\psi}{\to} f$, i.e. $f_n \overset{||\cdot||_{\psi,l}}{\to} f$, for all $l \in \mathbb{N}$, then by (3.9) $f_n \overset{||\cdot||_l}{\to} f$ for any $l \in \mathbb{N}$, which means that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $H_{\mathbb{C}}(\infty)$.

3.3 Pseudodifferential operators and Feller semigroups

3.3.1 Yosida-Hille-Ray Theorem

We recall the Yosida-Hille-Ray Theorem in the setting of $(\mathbb{Q}_p^n, ||\cdot||_p)$. For a general discussion the reader may consult [21, Chapter 4].

A semigroup $\{T(t)\}_{t\geq 0}$ on $C_0(\mathbb{Q}_p^n, \mathbb{R})$ is said to be *positive* if $\{T(t)\}_{t\geq 0}$ is a positive operator for each $t \geq 0$, i.e. it maps non-negative functions to non-negative functions. An operator $(A, Dom(A))$ on $C_0(\mathbb{Q}_p^n, \mathbb{R})$ is said to satisfy the *positive maximum principle* if whenever $f \in Dom(A) \subseteq C_0(\mathbb{Q}_p^n, \mathbb{R})$, $x_0 \in \mathbb{Q}_p^n$, and $\sup_{x\in\mathbb{Q}_p^n} f(x) = f(x_0) \geq 0$ we have $Af(x_0) \leq 0$.

We recall that every linear operator on $C_0(\mathbb{Q}_p^n, \mathbb{R})$ satisfying the positive maximum principle is dissipative, see e.g. [21, Chapter 4, Lemma 2.1].

3.3.1 Theorem. (Hille-Yosida-Ray Theorem) [21, Chapter 4, Theorem 2.2] Assume that $(A, Dom(A))$ is a linear operator on $C_0(\mathbb{Q}_p^n, \mathbb{R})$. The closure $\overline{A}$ of A on $C_0(\mathbb{Q}_p^n, \mathbb{R})$ is single-valued and generates a strongly continuous, positive contraction semigroup $\{T(t)\}_{t\geq 0}$ on $C_0(\mathbb{Q}_p^n, \mathbb{R})$ if and only if:
(i) $Dom(A)$ is dense in $C_0(\mathbb{Q}_p^n, \mathbb{R})$;
(ii) A satisfies the positive maximum principle;
(iii) $\text{Rank}(\lambda I - A)$ is dense in $C_0(\mathbb{Q}_p^n, \mathbb{R})$ for some $\lambda > 0$.

3.3.2 Definition. A family of bounded linear operators $T_t : C_0(\mathbb{Q}_p^n, \mathbb{R}) \to C_0(\mathbb{Q}_p^n, \mathbb{R})$ is called a Feller semigroup if

(i) $T_{s+t} = T_s T_t$ and $T_0 = I$;
(ii) $\lim_{t\to 0} ||T_t f - f||_{L^\infty} = 0$ for any $f \in C_0(\mathbb{Q}_p^n, \mathbb{R})$;
(iii) $0 \le T_t f \le 1$ if $0 \le f \le 1$, with $f \in C_0(\mathbb{Q}_p^n, \mathbb{R})$ and for any $t \ge 0$.

Theorem 3.3.1 characterizes the Feller semigroups, more precisely, if $(A, Dom(A))$ satisfies Theorem 3.3.1, then A has a closed extension which is the generator of a Feller semigroup.

3.3.2 Pseudodifferential operators with variable coefficients attached to negative definite functions

3.3.3 Remark. Let $\boldsymbol{\psi}_j : \mathbb{Q}_p^n \to \mathbb{R}_+$ be radial, continuous, negative definite functions of types 0, 1 or 2, for $j = 1, \dots, m$, and let $b_j : \mathbb{Q}_p^n \times \mathbb{R}_+ \to \mathbb{R}_+ \backslash \{0\}$ be a function such that

$$0 < \inf_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) \text{ and } \sup_{x\in\mathbb{Q}_p^n,\ t\in\mathbb{R}_+} b_j(x,t) < \infty,$$

for $j = 1, \dots, m$. To these functions we attach the following symbol:

$$p(x,\xi,t) := \sum_{j=1}^{m} b_j(x,t)\boldsymbol{\psi}_j(||\xi||_p) \tag{3.10}$$

and the pseudodifferential operator

$$(P(x,t,\partial)\varphi)(x) = -\mathcal{F}^{-1}_{\xi\to x}(p(x,\xi,t)\mathcal{F}_{x\to\xi}\varphi)$$

for $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$. Notice that $(P(x,t,\partial)\varphi)(x) = -\sum_{j=1}^{m} b_j(x,t)(D_{\boldsymbol{\psi}_j}\varphi)(x)$, where $(D_{\boldsymbol{\psi}_j}\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\boldsymbol{\psi}_j(||\xi||_p)\mathcal{F}_{x\to\xi}\varphi)$ for $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$. Moreover, $P(x,t,\partial) : \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n) \cap C(\mathbb{Q}_p^n)$.

3.3.4 Remark. Notice that $p(x,\xi,t)$ defines a real-valued, negative definite function, and thus $\mathcal{F}(u_t) := e^{-tp(x,\xi,t)}$, $t > 0$, $\xi \in \mathbb{Q}_p^n$ and $x \in \mathbb{Q}_p^n$ fixed, is a convolution semigroup of measures, see [10, Theorem 8.3]. The condition $p(x,0,t) = 0$ implies that $(u_t)_{t>0}$ are probability measures, see [10, Corollary 8.6].

3.3.5 Remark. (i) If all the functions $\boldsymbol{\psi}_j$ appearing in (3.10) are of types 0 or 1 and there is at least one function $\boldsymbol{\psi}_j$ of type 1, then $p(x,\xi,t)$ is a negative definite, continuous and radial function in ξ of type 1.
(ii) If all the functions $\boldsymbol{\psi}_j$ appearing in (3.10) are of types 0, 1 or 2 and there is at least one function $\boldsymbol{\psi}_j$ of type 2, then $p(x,\xi,t)$ is a negative definite, continuous and radial function in ξ of type 2.

(iii) In the case (i), there are positive constants C, β, with $\beta \geq 1$, such that

$$p(x,\xi,t) \leq C[\max\{1,||\xi||_p\}]^{\beta}.$$

We set $\boldsymbol{\psi}(||\xi||_p) := ||\xi||_p^{\beta}$. Notice that $\max\{1,\boldsymbol{\psi}(||\xi||_p)\} = [\max\{1,||\xi||_p\}]^{\beta}$. We attach to $P(x,t,\partial)$ the space $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$. In the case (ii),

$$p(x,\xi,t) \leq C\sum_{j\in J}\boldsymbol{\psi}_j(||\xi||_p) =: \boldsymbol{\psi}(||\xi||_p),$$

where J is the set of indices $j \in \{1,2,\ldots,m\}$ for which $\boldsymbol{\psi}_j$ is of type 2. We attach to $P(x,t,\partial)$ the space $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$.
(iv) From now on we will assume that in (3.10) there is at least one function $\boldsymbol{\psi}_j$ of type 1 or type 2.

3.3.6 Remark. If $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$, then $(P(x,t,\partial)\varphi)(x)$ is a real-valued function, i.e. $(P(x,t,\partial)\varphi)(x) = \overline{(P(x,t,\partial)\varphi)(x)}$. Indeed, for $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$ we have that

$$\begin{aligned}\overline{(P(x,t,\partial)\varphi)(x)} &= -\int_{\mathbb{Q}_p^n}\chi_p(x\cdot\xi)p(x,\xi,t)\overline{(\mathcal{F}\varphi)(\xi)}d^n\xi \\ &= -\int_{\mathbb{Q}_p^n}\chi_p(x\cdot\xi)p(x,\xi,t)(\mathcal{F}^{-1}\varphi)(\xi)d^n\xi.\end{aligned}$$

Taking $\xi = -z$ and using $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$ we have that

$$\begin{aligned}\overline{(P(x,t,\partial)\varphi)(x)} &= -\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot z)p(x,z,t)(\mathcal{F}^{-1}\varphi)(-z)d^nz \\ &= -\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot z)p(x,z,t)(\mathcal{F}\varphi)(z)d^nz \\ &= (P(x,t,\partial)\varphi)(x).\end{aligned}$$

3.3.7 Lemma. With the conventions and notations introduced in Remark 3.3.5, the mapping $P(x,t,\partial): B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \to B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ is a well-defined continuous operator.

Proof. In the cases (i)-(ii) of Remark 3.3.5, for $\varphi \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$, we have

$$||P(x,t,\partial)\varphi||_{\boldsymbol{\psi},l} \leq C||\varphi||_{\boldsymbol{\psi},l+2}, \tag{3.11}$$

which implies (by using Remark 3.3.6 and the fact that $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n)$ is dense in $B_{\boldsymbol{\psi},l}(\mathbb{R})$ for any l) that $P(x,t,\partial): B_{\boldsymbol{\psi},l+2}(\mathbb{R}) \to B_{\boldsymbol{\psi},l}(\mathbb{R})$ is a well-defined continuous mapping for any $l \in \mathbb{N}$, and consequently $P(x,t,\partial) : B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \to B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ is a well-defined operator. The continuity is established by using an argument based on sequences and (3.11).

3.3.8 Example. Take $\beta > 0$ and set $\boldsymbol{\psi}_{\beta}\left(\|\xi\|_p\right) := \|\xi\|_p^{\beta}$. Then the operator $D_{\boldsymbol{\psi}_{\beta}}$ is the Taibleson operator D_T^{β}, see Example 3.1.5, which admits the following representation:

$$\left(D_{\boldsymbol{\psi}_{\beta}}\varphi\right)(x) = \frac{1-p^{\beta}}{1-p^{-\beta-n}} \int\limits_{\mathbb{Q}_p^n} \|y\|_p^{-\beta-n} \left(\varphi(x-y) - \varphi(x)\right) d^n y$$

for $\varphi \in \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$, see [39]. Notice that $-D_{\boldsymbol{\psi}_{\beta}}$ satisfies Remark 3.3.6 and the positive maximum principle.

3.3.9 Example. Set J as in Example 3.1.4. Then the function $\boldsymbol{\psi}_J\left(\|\xi\|_p\right) := 1 - \widehat{J}(||\xi||_p)$ is negative definite and it satisfies $0 \leq 1 - \widehat{J}(||\xi||_p) \leq 2$ for $\xi \in \mathbb{Q}_p^n$, which implies that $\boldsymbol{\psi}_J\left(\|\xi\|_p\right)$ is of type 0. Then

$$\begin{aligned}\left(D_{\boldsymbol{\psi}_J}\varphi\right)(x) &= \mathcal{F}^{-1}_{\xi\to x}\left(\left\{1 - \widehat{J}(||\xi||_p)\right\}\mathcal{F}_{x\to\xi}\varphi\right) \\ &= -\int\limits_{\mathbb{Q}_p^n} J\left(\|x-y\|_p\right)\left(\varphi(y) - \varphi(x)\right) d^n y\end{aligned}$$

for $\varphi \in \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$. Notice that $-D_{\boldsymbol{\psi}_{\beta}}$ satisfies Remark 3.3.6 and the positive maximum principle.

3.3.10 Example. Take $\boldsymbol{\psi}_{\beta_i}\left(\|\xi\|_p\right)$ for $i = 1,\ldots,l$ with $0 < \beta_1 < \cdots < \beta_l$ as in Example 3.3.8, and let J_i be functions as in Example 3.1.4 for $i = l+1,\ldots,m$, and set $\boldsymbol{\psi}_{J_i}\left(\|\xi\|_p\right) := 1 - \widehat{J}_i(||\xi||_p)$ as in Example 3.3.9. Then

$$\boldsymbol{\psi}(||\xi||_p) := \sum_{i=1}^{l} b_i(x,t)\boldsymbol{\psi}_{\beta_i}\left(\|\xi\|_p\right) + \sum_{i=l+1}^{m} b_i(x,t)\boldsymbol{\psi}_{J_i}\left(\|\xi\|_p\right),$$

where the $b_i(x,t)$'s are as before, is a negative definite function of type 1, and it satisfies $\boldsymbol{\psi}(||\xi||_p) \leq C \max\{1, ||\xi||_p\}^{\beta_l}$. Then the operator

$$P(x,t,\partial) = -\sum_{i=1}^{l} b_i(x,t) D_{\boldsymbol{\psi}_{\beta_i}} - \sum_{i=l+1}^{m} b_i(x,t) D_{\boldsymbol{\psi}_{J_i}}$$

satisfies Remark 3.3.6 and the positive maximum principle.

3.3.11 Lemma. For any fixed positive real number λ, the equation

$$(\lambda - P(x,t,\partial))u = f,\ \ f \in B_{\boldsymbol{\psi},\infty}(\mathbb{R}), \tag{3.12}$$

has a unique solution u in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$.

Proof. By using that $f \in L^2$, we have

$$\widehat{u}\,(\xi;t,\lambda) = \frac{\widehat{f}(\xi)}{\lambda + \sum_{j=1}^{m} b_j(x,t)\boldsymbol{\psi}_j(||\xi||_p)} := \frac{\widehat{f}(\xi)}{\lambda + H(x,t,||\xi||_p)}.$$

We now recall that $\boldsymbol{\psi}_j(||\xi||_p) \geq 0$ for any j, see Remark **??**, then

$$\inf_{x,t,\xi} H(x,t,||\xi||_p) \geq 0,$$

and consequently $|\widehat{u}\,(\xi;t,\lambda)| \leq \frac{|\widehat{f}(\xi)|}{\lambda}$. By Lemma 3.2.2 (ii)-(iii), $u\,(x;t,\lambda) \in B_{\boldsymbol{\psi},\infty}(\mathbb{C})$. The uniqueness follows from the fact that $(\lambda - P(x,t,\partial))u = 0$ has $u = 0$ as a unique solution since $B_{\boldsymbol{\psi},\infty}(\mathbb{C}) \subset C_0(\mathbb{Q}_p^n,\mathbb{C})$, see Lemma 3.2.2 (v). We now show that $u\,(x;t,\lambda)$ is a real valued function. Recall that

$$u\,(x;t,\lambda) = \lim_{k\to\infty} u_k\,(x;t,\lambda) \ \text{ in } L^2\text{-sense},$$

where

$$u_k\,(x;t,\lambda) = \int\limits_{||\xi||_p \leq p^k} \frac{\chi_p\,(-\xi\cdot x)\,\widehat{f}(\xi)}{\lambda + H(x,t,||\xi||_p)} d^n\xi.$$

Claim $u\,(x;t,\lambda) = \lim_{k\to\infty} u_k\,(x;t,\lambda)$ in L^2-sense, where the $u_k\,(x;t,\lambda)$'s are real valued functions.

Then there exists a subsequence of $\{u_k\,(x;t,\lambda)\}_{k\in\mathbb{N}}$ converging almost uniformly to the same limit, which implies that $u\,(x;t,\lambda)$ is a real valued function outside of a zero measure subset of $\mathbb{Q}_p^n$. Since $u\,(x;t,\lambda)$ is a continuous function in x, $B_{\boldsymbol{\psi},\infty}(\mathbb{C}) \subset C_0(\mathbb{Q}_p^n,\mathbb{C})$, necessarily $u\,(x;t,\lambda)$ is a continuous function at every $x \in \mathbb{Q}_p^n$.

Proof of the Claim. It is sufficient to show that $u_k\,(x;t,\lambda) = \overline{u_k\,(x;t,\lambda)}$ for any k. Indeed,

$$\begin{aligned}\overline{u_k\,(x;t,\lambda)} &= \int\limits_{||\xi||_p \leq p^k} \frac{\chi_p\,(\xi\cdot x)\,\overline{\widehat{f}(\xi)}}{\lambda + H(x,t,||\xi||_p)} d^n\xi = \int\limits_{||\xi||_p \leq p^k} \frac{\chi_p\,(\xi\cdot x)\left(\mathcal{F}^{-1}f\right)(\xi)}{\lambda + H(x,t,||\xi||_p)} d^n\xi \\ &= \int\limits_{||\xi||_p \leq p^k} \frac{\chi_p\,(-\xi\cdot x)\,\widehat{f}(\xi)}{\lambda + H(x,t,||\xi||_p)} d^n\xi,\end{aligned}$$

where we used that $\overline{\widehat{f}(\xi)} = \left(\mathcal{F}^{-1}f\right)(\xi)$ and $\left(\mathcal{F}^{-1}f\right)(-\xi) = \widehat{f}(\xi)$ for $f \in L^2$.

3.3.12 Theorem. Assume that $(P(x,t,\partial), B_{\boldsymbol{\psi},\infty}(\mathbb{R}))$ satisfies the positive maximum principle. Then the closure $(\overline{P(x,t,\partial)}, Dom(\overline{P(x,t,\partial)}))$ of $(P(x,t,\partial), B_{\boldsymbol{\psi},\infty}(\mathbb{R}))$ on $C_0(\mathbb{Q}_p^n,\mathbb{R})$ is the generator of a Feller semigroup.

Proof. We show that $(P(x,t,\partial), B_{\boldsymbol{\psi},\infty}(\mathbb{R}))$ satisfies conditions (i) and (iii) in the Hille-Yosida-Ray's theorem, see Theorem 3.3.1. By Lemma 3.3.7, $P(x,t,\partial) : B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \to B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ is a well-defined continuous operator, and by Lemma 3.2.2-(vi), $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ is densely and continuously embedded in $C_0(\mathbb{Q}_p^n,\mathbb{R})$. For the third condition, Lemma 3.3.11 implies that the rank of $\lambda - P(x,t,\partial)$ is $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ which is dense in $C_0(\mathbb{Q}_p^n,\mathbb{R})$.

3.4 Parabolic-Type Equations With Variable Coefficients

Let $T > 0$ and let $f(x,t) : \mathbb{Q}_p^n \times [0,T] \to \mathbb{R}$ such that $f(x,\cdot) : [0,T] \to C_0(\mathbb{Q}_p^n,\mathbb{R})$, let $b_j : \mathbb{Q}_p^n \to \mathbb{R}_+ \backslash \{0\}$ be a function such that

$$0 < \inf_{x\in\mathbb{Q}_p^n} b_j(x) \leq b_j(x) \leq \sup_{x\in\mathbb{Q}_p^n} b_j(x) < \infty,$$

for $j = 1,\ldots,m$, and consider

$$\begin{array}{rccc} P(x,\partial): & B_{\boldsymbol{\psi},\infty}(\mathbb{R}) & \to & B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \\ & g & \longrightarrow & -\sum_{j=1}^{m} b_j(x)(D_{\boldsymbol{\psi}_j} g)(x). \end{array}$$

The operators $P(x,\partial)$ are a particular case of the operators $P(x,t,\partial)$ considered before. Our aim is to study the following initial value problem:

$$\begin{cases} u(x,\cdot) \in C([0,T], B_{\boldsymbol{\psi},\infty}(\mathbb{R})) \cap C^1([0,T], C_0(\mathbb{Q}_p^n,\mathbb{R})); \\ \frac{\partial u}{\partial t}(x,t) = P(x,\partial)u(x,t) + f(x,t), \qquad t\in[0,T],\ x\in\mathbb{Q}_p^n; \\ u(x,0) = h(x) \in B_{\boldsymbol{\psi},\infty}(\mathbb{R}). \end{cases} \tag{3.13}$$

The proof of the following lemma is included for the sake of completeness.

3.4.1 Lemma. The operator $\overline{P(x,\partial)} : Dom(\overline{P(x,\partial)}) \to C_0(\mathbb{Q}_p^n,\mathbb{R})$ is $m-$dissipative.

Proof. We first verify that $\left(\overline{P(x,\partial)}, Dom(\overline{P(x,\partial)})\right)$ is dissipative, i.e.

$$\left\|\left(1-\lambda\overline{P(x,\partial)}\right)u\right\|_{L^\infty} \geq \|u\|_{L^\infty} \tag{3.14}$$

for $u \in Dom(\overline{P(x,\partial)})$. Take functions u, v such that $\overline{P(x,\partial)}u = v$, then there exist a sequences $\{u_m\}_{m\in\mathbb{N}}$ in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ and $\{v_m\}_{m\in\mathbb{N}}$ in $C_0(\mathbb{Q}_p^n,\mathbb{R})$ such that $u_m \to u$, $v_m \to v$, and $P(x,\partial)u_m = v_m$. Since $P(x,\partial)$ is dissipative, cf. [21, Chapter 4, Lemma 2.1], we have

$$\|(1-\lambda P(x,\partial))\,u_m\|_{L^\infty} \geq \|u_m\|_{L^\infty}\,. \tag{3.15}$$

Now (3.14) follows from (3.15) by taking the limit $m \to \infty$.

To show that $\left(\overline{P(x,\partial)}, Dom(\overline{P(x,\partial)})\right)$ is $m-$dissipative, we show that there exists $\lambda > 0$ such that for all $f \in C_0(\mathbb{Q}_p^n,\mathbb{R})$, there exists a solution $u \in Dom\left(\overline{P(x,\partial)}\right)$ of $\left(1-\lambda\overline{P(x,\partial)}\right)u = f$, cf. [13, Proposition 2.2.6]. Since $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ is dense in $C_0(\mathbb{Q}_p^n,\mathbb{R})$, there exists a sequence $\{f_m\}_{m\in\mathbb{N}}$ in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ such that $f_m \xrightarrow{||\cdot||_{\boldsymbol{\psi},l}} f$, for any $l \in \mathbb{N}$, now by using the density of $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$ in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$, there exists a sequence $\{g_m\}_{m\in\mathbb{N}}$ in $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$ such that $\|f_m - g_m\|_{\boldsymbol{\psi},l} \leq \frac{1}{m}$, thus $g_m \xrightarrow{||\cdot||_{\boldsymbol{\psi},l}} f$ and since $B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \hookrightarrow C_0(\mathbb{Q}_p^n,\mathbb{R})$ by Lemma 3.2.2 (vi), we get that $g_m \xrightarrow{||\cdot||_{L^\infty}} f$. For each g_m, there exists $u_m \in B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ such that for any $\lambda > 0$, $\left(1-\lambda^{-1}P(x,\partial)\right)\lambda u_m = g_m$, i.e. $P(x,\partial)u_m = \lambda u_m - g_m$, cf. Lemma 3.3.11.

Claim. The sequence $\{u_m\}_{m\in\mathbb{N}}$ is Cauchy in $B_{\boldsymbol{\psi},l}(\mathbb{R})$ for any $l \in \mathbb{N}$, i.e. is Cauchy in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$.

By the Claim, $u_m \to u \in B_{\boldsymbol{\psi},\infty}(\mathbb{R}) \hookrightarrow C_0(\mathbb{Q}_p^n,\mathbb{R})$ for some u. Therefore, $\overline{P(x,\partial)}u = \lambda u - f$, i.e. $\left(1-\lambda^{-1}\overline{P(x,\partial)}\right)\lambda u = f$.

Proof of the Claim.

By using that

$$\widehat{u_m}\,(\xi;t,\lambda) = \frac{\widehat{g_m}(\xi)}{\lambda + \sum_{j=1}^{m} b_j(x,t)\boldsymbol{\psi}_j(||\xi||_p)},$$

we have

$$|\widehat{u_m}\,(\xi;t,\lambda) - \widehat{u_n}\,(\xi;t,\lambda)| \leq \frac{|\widehat{g_m}(\xi) - \widehat{g_n}(\xi)|}{\lambda},$$

which implies that $\|u_m\,(\xi;t,\lambda) - u_n\,(\xi;t,\lambda)\|_{\boldsymbol{\psi},l} \leq \frac{1}{\lambda}||g_m(\xi) - g_n(\xi)||_{\boldsymbol{\psi},l}$ for any $l \in \mathbb{N}$, and since $g_m \xrightarrow{||\cdot||_{\boldsymbol{\psi},l}} f$, for any $l \in \mathbb{N}$, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is Cauchy in $B_{\boldsymbol{\psi},\infty}(\mathbb{R})$.

3.4.2 Lemma. Assume that at lest one of the following conditions hold:
(i) $f(x,\cdot) \in L^1((0,T), B_{\psi,\infty}(\mathbb{R}))$;
(ii) $f(x,\cdot) \in W^{1,1}((0,T), C_0(\mathbb{Q}_p^n, \mathbb{R}))$.
Then the initial value problem (3.13) has a unique solution of the form

$$u(x,t) = T(t)h(x) + \int_0^t T(t-s)f(x,s)ds, \text{ for all } t \in [0,T],$$

where $(T(t))_{t\geq 0}$ is the Feller semigroup associated to the operator $\overline{P(x,t,\partial)}$.

Proof. It follows from [13, Lemma 4.1.1], [13, Corollary 4.1.2] and [13, Proposition 4.1.6], by using the fact that $\overline{P(x,t,\partial)}$ is m-dissipative.

Consider the following Cauchy problem:

$$\begin{cases} u(x,\cdot) \in C([0,T], B_{\psi,\infty}(\mathbb{R})) \cap C^1([0,T], C_0(\mathbb{Q}_p^n, \mathbb{R})); \\ \frac{\partial u}{\partial t}(x,t) = P(x,\partial)u(x,t), \\ u(x,0) = u_0(x) \in \mathcal{D}\left(\mathbb{Q}_p^n, \mathbb{R}\right). \end{cases} \tag{3.16}$$

We define $p(x,\xi) = \sum_{j=1}^m b_j(x)(D_{\psi_j} g)(x)$,

$$u(x,t) = \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, e^{-tp(x,\xi)}\widehat{u_0}(\xi)d^n\xi, \text{ for } x \in \mathbb{Q}_p^n \text{ and } t \geq 0,$$

and

$$Z(x,t) = \mathcal{F}^{-1}_{\xi\to x}(e^{-tp(x,\xi)}) \text{ in } \mathcal{D}'\left(\mathbb{Q}_p^n\right), \text{ for } x \in \mathbb{Q}_p^n \text{ and } t \geq 0.$$

Notice that $u(x,t) = \mathcal{F}^{-1}_{\xi\to x}(e^{-tp(x,\xi)}) * u_0(x)$ in $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$, for $x \in \mathbb{Q}_p^n$ and $t \geq 0$.

3.4.3 Lemma. The function $u(x,t)$ defined above satisfies the following conditions:
(C1) $u(x,\cdot) \in C([0,T], B_{\psi,\infty}(\mathbb{R})) \cap C^1([0,T], C_0(\mathbb{Q}_p^n))$ and the derivative is given by

$$\frac{\partial u}{\partial t}(x,t) = -\int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, p(x,\xi)e^{-tp(x,\xi)}\widehat{u_0}(\xi)d^n\xi; \tag{3.17}$$

(C2) $u(x,\cdot) \in L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n)$ for any $t \geq 0$, and

$$P(x,\partial)u(x,t) = -\int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, p(x,\xi)e^{-tp(x,\xi)}\widehat{u_0}(\xi)d^n\xi. \tag{3.18}$$

Furthermore $u(x,\cdot)$ is a solution of the initial value problem (3.16).

Proof. The result is proved through the following claims:

Claim 1. $u(x,\cdot) \in C([0,T], B_{\boldsymbol{\psi},\infty}(\mathbb{R}))$

We first show that $u(\cdot,t) \in B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ for all $t \geq 0$. By using that $u(x,t) = \mathcal{F}^{-1}_{\xi\to x}(e^{-tp(x,\xi)}) * u_0(x)$ in $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$. This convolution exists because $u_0(x)$ has compact support, then

$$\widehat{u}(\xi,t) = e^{-tp(x,\xi)}\widehat{u_0}(\xi) \text{ in } \mathcal{D}'\left(\mathbb{Q}_p^n\right) \tag{3.19}$$

for $t \geq 0$. Now

$$\begin{aligned} ||u(\cdot,t)||^2_{\boldsymbol{\psi},l} &= \int_{\mathbb{Q}_p^n} [\max(1, \boldsymbol{\psi}(||\xi||_p))]^l |\widehat{u}(\xi,t)|^2 d^n\xi \\ &\leq \int_{\mathbb{Q}_p^n} [\max(1, \boldsymbol{\psi}(||\xi||_p))]^l |\widehat{u_0}(\xi)|^2 d^n\xi = ||u_0||^2_{\boldsymbol{\psi},l}, \end{aligned}$$

i.e. $||u(\cdot,t)||_{\boldsymbol{\psi},l} \leq ||u_0||_{\boldsymbol{\psi},l}$, for any $l \in \mathbb{N}$, which implies that $u(\cdot,t) \in B_{\boldsymbol{\psi},\infty}(\mathbb{R})$ for all $t \geq 0$.

We now verify that

$$\lim_{t\to t_0} ||u(x,t) - u(x,t_0)||^2_{\boldsymbol{\psi},l} = 0 \text{ for any } l \in \mathbb{N},$$

which implies the continuity of $u(\cdot,t)$. The verification of this fact is done by using (3.19) and the dominated convergence theorem.

Claim 2. $u(x,\cdot) \in C^1([0,T], C_0(\mathbb{Q}_p^n))$ and $\frac{\partial u}{\partial t}(x,t)$ is given by (3.17).

Set

$$h_t(x) := \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, p(x,\xi) e^{-tp(x,\xi)}\widehat{u_0}(\xi) d^n\xi, \text{ for } x \in \mathbb{Q}_p^n \text{ and } t \geq 0.$$

Notice that for any $t \geq 0$ fixed, and any $x \in \mathbb{Q}_p^n$ fixed, $p(x,\xi)e^{-tp(x,\xi)}\widehat{u_0}(\xi)$ is an integrable function in ξ, and thus by the Riemman-Lebesgue theorem $h_t(x) \in C_0(\mathbb{Q}_p^n)$ for any $t \geq 0$ fixed. Now, by applying the mean value theorem we have

$$\frac{e^{-tp(x,\xi)} - e^{-t_0 p(x,\xi)}}{t - t_0} = -p(x,\xi)e^{-\tau(x)p(x,\xi)},$$

for some $\tau(x)$ between t and t_0. So that

$$\begin{aligned}
&\lim_{t\to t_0}\left\|\frac{u(x,t)-u(x,t_0)}{t-t_0}+h_t(x)\right\|_{L^\infty}\\
&=\lim_{t\to t_0}\left\|\frac{\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\left\{e^{-tp(x,\xi)}-e^{-t_0p(x,\xi)}\right\}\widehat{u_0}(\xi)d^n\xi}{t-t_0}+h_t(x)\right\|_{L^\infty}\\
&=\lim_{t\to t_0}\left\|-\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\,p(x,\xi)e^{-\tau(x)p(x,\xi)}\widehat{u_0}(\xi)d^n\xi+h_t(x)\right\|_{L^\infty}\\
&=\lim_{t\to t_0}\left\|-\int_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\,p(x,\xi)\widehat{u_0}(\xi)\left[e^{-\tau(x)p(x,\xi)}-e^{-tp(x,\xi)}\right]d^n\xi\right\|_{L^\infty}
\end{aligned}$$

Now, when $t\to t_0$, $\tau(x)\to t$, so that

$$\lim_{t\to t_0}\left\|\frac{u(x,t)-u(x,t_0)}{t-t_0}+h_t(x)\right\|_{L^\infty}=0.$$

Claim 3. The assertion (C_2) holds.
Indeed, since u_0 has compact support

$$e^{-tp(x,\xi,t)}\widehat{u_0},\ p(x,\xi)e^{-tp(x,\xi,t)}\widehat{u_0}(\xi)\in L^1(\mathbb{Q}_p^n)\cap L^2(\mathbb{Q}_p^n),$$

and thus $P(x,\partial)u(x,t)=-\mathcal{F}^{-1}_{\xi\to x}(p(x,\xi)\mathcal{F}_{x\to\xi}u(x,t))$.

3.4.4 Theorem. Assuming that operator $P(x,\partial)$ satisfies the positive maximum principle. Then the Cauchy problem

$$\begin{cases}u(x,\cdot)\in C([0,T],B_{\boldsymbol{\psi},\infty}(\mathbb{R}))\cap C^1([0,T],C_0(\mathbb{Q}_p^n,\mathbb{R}));\\[1ex]
\frac{\partial u}{\partial t}(x,t)=P(x,\partial)u(x,t)+f(x,t)\\[1ex]
u(x,0)=u_0(x)\in B_{\boldsymbol{\psi},\infty}(\mathbb{R}),\end{cases}\tag{3.20}$$

where $f(x,t)$ is a function satisfying the assumptions of Lemma 3.4.2, has a unique solution given by

$$u(x,t)=Z_t(x)*u_0(x)+\int_0^t Z_{t-s}(x)f(x,s)ds$$

for all $t\in[0,T]$. Furthermore,

$$\begin{aligned}C_0(\mathbb{Q}_p^n,\mathbb{R})&\longrightarrow C_0(\mathbb{Q}_p^n,\mathbb{R})\\ h&\longmapsto Z_t(x)*h(x),\end{aligned}$$

for $t \geq 0$, gives rise to a Feller semigroup.

Proof. Set

$$(\digamma(t)u)(x) = Z_t(x) * u(x), \text{ for } t \geq 0,\ u \in \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right).$$

By using Lemmas 3.4.2, 3.4.3,

$$T(t) \mid_{\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)} = \digamma(t) \mid_{\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)} \text{ for } t \geq 0.$$

Now, since

$$\|\digamma(t)u\|_{L^\infty} = ||Z_t * u||_{L^\infty} \leq ||Z_t||_M ||u||_{L^\infty},$$

where $||Z_t||_M$ denotes the total variation of the finite Borel measure Z_t, and since $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^n\right)$ is dense in $C_0(\mathbb{Q}_p^n, \mathbb{R})$, we conclude $T(t) = \digamma(t)$ for $t \geq 0$. Finally, by Theorem 3.3.12, $\digamma(t)$ gives rise to a Feller semigroup.

3.4.5 Remark. With the hypotheses of Theorem 3.3.12 and assuming that $p(x, \xi, 0) = 0$ for any $x, \xi \in \mathbb{Q}_p^n$, we obtain the existence of a Lévy process $(X_t)_{t \geq 0}$ with state space $\mathbb{Q}_p^n$, such that $Z_t(x) = P_{X_t - X_0}(x)$, where $P_{X_t - X_0}$ denotes the distribution of the random variable $X_t - X_0$. This result follows from [22, Section 2], since Z_t is a convolution semigroup such that $Z_t \to \delta_0$ weakly as $t \to 0^+$.

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