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# Quaternionic Vekua Analysis in Domains in $\mathbb{R}^3$ with Application to Electromagnetic Systems of Equations

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**BRICEYDA BERENICE DELGADO LÓPEZ**

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Thesis advisor:

**Dr. R. Michael Porter Kamlin**

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# Análisis de Vekua cuaterniónico en dominios en $\mathbb{R}^3$ con aplicación a sistemas de ecuaciones electromagnéticas

TESIS

que presenta

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**Dr. R. Michael Porter Kamlin**

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# Contents

# Abstract

The first contribution of this thesis is an explicit general solution to the inhomogeneous div-curl system in fairly general bounded domains in $\mathbb{R}^3$. The construction of this solution is based on the fact that the div-curl system can be written as a $\overline{\partial}$-problem for the Moisil-Teodorescu operator in three-dimensional space, which permits applications of the Teodorescu operator.

This fundamental result is used to realize the second purpose of this work, which is the development of Vekua analysis. This refers to generalizing the theory of monogenic functions to a theory of solutions of the main Vekua equation in bounded domains in $\mathbb{R}^3$. A typical question is to construct the vector part of a solution when only the scalar part is known. Consideration of this question in terms of the boundary values of the solutions leads us to the construction of a Vekua-Hilbert transform for the main Vekua equation as well as a link with the quaternionic Dirichlet-to-Neumann map for the conductivity equation.

In addition to the information obtained for the conductivity equation, these results provide applications to a number of other equations of mathematical physics, including the double curl equation. Furthermore, we give an explicit solution for the case of static Maxwell's equations in a medium with a variable permeability.

# Resumen

La primera contribución de esta tesis es una solución general explícita al sistema div-rot inhomogéneo en ciertos dominios acotados de $\mathbb{R}^3$. La construcción de esta solución está basada en el hecho de que el sistema div-rot puede ser escrito como un $\overline{\partial}$-problema para el operador de Moisil-Teodorescu en el espacio tri-dimensional, lo cual permite aplicaciones del operador de Teodorescu.

Este resultado fundamental es utilizado para llevar a cabo el segundo propósito de este trabajo, es decir, el desarrollo del análisis de Vekua. Esto se refiere a generalizar la teoría de funciones monogénicas a la teoría de las soluciones de la ecuación de Vekua principal en dominios acotados de $\mathbb{R}^3$. Una pregunta natural es cómo construir la parte vectorial de una solución si solamente la parte escalar es conocida. Consideraciones de esta pregunta en términos de valores frontera de las soluciones nos conduce a la construcción de la transformada de Vekua-Hilbert para la ecuación de Vekua principal así como la conexión con el mapeo de Dirichlet-Neumann para la ecuación de conductividad.

Adicionalmente a la información obtenida de la ecuación de conductividad, estos resultados proporcionan aplicaciones a otras ecuaciones de la física matemática, incluyendo por ejemplo la ecuación doble rotacional. Además,

daremos una solución explícita para el caso de las ecuaciones de Maxwell en un medio con permeabilidad variable.

# Dedication

To my parents Luz and Federico and my sisters Lucero, Deysi and Cristal for their unconditional support and constant enthusiasm, and make me feel proud of the strongest values raised in our family.

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# Introduction

In this thesis we study the main quaternionic Vekua equation

$$DW = \frac{Df}{f}\overline{W}, \tag{1}$$

where $f$ is a nonvanishing scalar function defined in a domain in three dimensional space, $D = \sum_{i=1}^{3} e_i\partial_i$ is the Moisil-Teodorescu operator in $\mathbb{R}^3$ and $C_{\mathbb{H}}W = \overline{W}$ represents quaternionic conjugation. We give new results on methods of finding solutions to (1) as well as properties enjoyed by these solutions. Using these facts we produce further results on a number of equations of mathematical physics intimately closely related to (1).

Equation (1) is a particular case of the general Vekua equation $\overline{\partial}W = aW + b\overline{W}$, whose theory was introduced by Lipman Bers and Ilya Vekua [16, 99] for functions in $\mathbb{R}^2$ and plays an important role in the theory of pseudoanalytic functions (sometimes called generalized analytic functions), which has since been extended to wider contexts, including quaternionic analysis [14, 70, 71, 72, 77, 90]. We describe the theory of pseudoanalytic functions in more detail in Subsection 3.1.1.

In order to illustrate the analogies and applications of the Vekua equation, consider some factorizations of elliptic operators. If $(-\Delta+\nu)f=0$ where $\Delta$ is the Laplacian and $f$ is a nonvanishing solution, then

$$(-\Delta+\nu)\,u_0=\left(D+M^{\frac{Df}{f}}\right)\left(D-M^{\frac{Df}{f}}\right)u_0, \tag{2}$$

for every scalar function $u_0$, where $M^a$ denotes the operator of multiplication on the right by the function $a$. This quaternionic factorization (2) of the Schrödinger operator was obtained in [17, 18] in a form which requires a solution of an associated Riccati equation, which in [68] was shown to have the form $Df/f$. The operator $D-(Df/f)C_{\mathbb{H}}$ corresponding to (1) appears in factorizations of other operators [70]. For example, the elliptic operator representing the conductivity equation can be decomposed as

$$\nabla\cdot f^2\nabla u_0=-f\left(D+M^{\frac{Df}{f}}\right)\left(D-\frac{Df}{f}C_{\mathbb{H}}\right)fu_0, \tag{3}$$

where $\nabla\cdot$ is the divergence operator.

In [74] the difficulty of extending the concepts of pseudoanalytic function theory to the case of three or more dimensions was already noticed. Throughout this work we will use the term Vekua analysis to refer to generalizations of results concerning monogenic functions to solutions $W$ of the main Vekua equation (1). For instance, we are interested in the construction of the vector part of solutions of the main Vekua equation (construction of $f^2$-hyperconjugates), when only the scalar part is known. In the search for these $f^2$-hyperconjugates, or equivalently in the search for a solution of the homogeneous div-curl system $\operatorname{div}(f\vec{W})=0$, $\operatorname{curl}(f\vec{W})=-f^2\nabla(W_0/f)$, with $W=W_0+\vec{W}$, we were able to provide a general solution for the inhomogeneous div-curl system (Theorems 47, 48, 55). In the following we will give some information and historical advances in this problem of mathematical physics.

*Div-curl system.* We consider the general inhomogeneous div-curl system

$$\begin{aligned} \operatorname{div} \vec{w} &= g_0, \\ \operatorname{curl} \vec{w} &= \vec{g}, \end{aligned} \tag{4}$$

for appropriate assumptions on the scalar field $g_0$ and the vector field $\vec{g}$ and their domain of definition in three-dimensional space. This first order partial differential system governs, for example, static electromagnetic fields. In fact, Maxwell's equations consist of two simultaneous div-curl systems which describe how electrical and magnetic fields are generated by charges and currents together with their variations. Basic references to the theory of the classical Maxwell's equations are [20, 59]. Chapters 3 and 4 of [69] develop a quaternionic treatment for different systems of Maxwell's equations, and Chapter 2 of [67] does this for electrodynamical models.

Because of its fundamental importance in physics, the div-curl system has been studied from very many points of view. It was Hermann von Helmholtz who formulated the "Helmholtz Decomposition Theorem", which represents any vector field $\vec{w}$ in $\mathbb{R}^3$ as the sum of a divergence free vector field $\operatorname{curl} \vec{v}$ and an irrotational vector field $\operatorname{grad} v_0$:

$$\vec{w} = -\operatorname{grad} v_0 + \operatorname{curl} \vec{v},$$

where the *Helmholtz potentials* $v_0$ and $\vec{v}$ are given ([66, p. 166], [54, Th. 5.1.1]) by

$$v_0(\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\operatorname{div}\vec{w}}{|\vec{x}-\vec{y}|}\,d\vec{y}, \quad \vec{v}(\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\operatorname{curl}\vec{w}}{|\vec{x}-\vec{y}|}\,d\vec{y}. \tag{5}$$

Unfortunately, the solution is represented by integral operators over all of three-dimensional space, which is a serious limitation for many applications.

In [21] an existence result for a quaternionic solution of the related Moisil-Teodorescu equation $Dw = g$ was proved. The div-curl problem (4) consists of finding a purely vectorial solution. Explicit vector solutions have been found under diverse restrictive conditions, either on the data $g_0$ and $\vec{g}$ (beyond the evident requirement that $\vec{g}$ be solenoidal) or on the domain. For example, in [15, Section 4] a particular div-curl system with $g_0 = 0$ and $\vec{g} = 0$ is examined. On the other hand, for a solenoidal vector field, that is, for $g_0 = 0$, the Biot-Savart vector fields [41, 48] give a particular solution. In [62, Chapter 5] a numerical solution is given for the div-curl system under certain boundary conditions, based on the Least-Squares finite element method. A solution for star-shaped domains, based on a radial integral operator, was recently provided by Yu. M. Grigor'ev in [49, Th. 3.2], valid when the original data $g_0, \vec{g}$ in the system (4) are harmonic scalar and vectorial functions, respectively. Somewhat earlier, Colombo et. al. [30] produced a right inverse of curl under the condition that certain functions lie in the kernel of one of the components of the Teodorescu operator. This permits expressing the general solution for (4) under the assumption that a certain scalar field admits a hyperconjugate harmonic function. The condition given in [30] was echoed in [19] in the context of a one-parameter family of systems of which (4) is a particular case. In this thesis we give a complete solution to the inhomogeneous div-curl system with none of the abovementioned limitations.

*Hilbert transform.* Another purpose of this thesis is to formulate and solve certain boundary value problems associated to (1). For the case of constant $f$, the three-dimensional Hilbert transform takes scalar data on the boundary of a domain $\Omega \subseteq \mathbb{R}^3$ and produces the boundary value of the vector part of a quaternionic monogenic (hyperholomorphic) function of three real variables, for which the scalar part coincides with the original data. A series of articles of Brackx et al. [23, 24, 25] study the relationship between the Hilbert transforms and conjugate harmonic functions in the context of Clifford algebras on the unit sphere.

In the literature the Hilbert transform has sometimes been mistakenly identified with the vector part of the boundary value of the Cauchy integral, since they happen to coincide for half spaces in $\mathbb{R}^n$ [84, p. 758] and for the unit disk in the plane [8, Example 2.7(2)]. However, this does not hold for general domains, including higher dimensional balls [8, Example 2.7(3)]. Generalizing a representation of the Hilbert transform $\mathcal{H}$ given by T. Qian and others for bounded Lipschitz domains [8, 83, 84], we define the Hilbert transform $\mathcal{H}_f$ associated to the main Vekua equation (1). This leads to an investigation of the three-dimensional analogue of the Dirichlet-to-Neumann map for the conductivity equation.

In the 80's there were many successes in solving boundary value problems in Lipschitz domains. In particular, Dirichlet and Neumann problems for the Laplace equation were solved by means of boundary integral operators; more precisely the method of layer potentials; some references are [40, 60, 61, 100]. The main idea behind the above solution is to invert the operators $I \pm K_0$ or $I \pm K_0^*$ (depending on the problem), where $K_0$ is the scalar part of the singular Cauchy operator applied to scalar functions and $K_0^*$ its adjoint operator. Further, the solution of the div-curl system in Lipschitz domains (Theorem 55) as well as the generalization of the monogenic Hilbert transform $\mathcal{H}$ in the context of Sobolev spaces (Theorem 82) were possible thanks to the good properties of the operator $(I + K_0)^{-1}$.

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In Chapter 1 we will give some basic definitions and facts of quaternionic analysis. In addition, we study the properties of the operators and their component operators needed for the development of this thesis. With the purpose to make the work self-contained and to highlight the beauty of the interrelationships involved, we have included proofs of many facts which can be found elsewhere.

Chapter 2 may be considered as a completion of the analysis in [30]. We will show that in fact the required hyperconjugate harmonic function exists whenever $\vec{g}$ is solenoidal. As in [30] we rely heavily on the classical Teodorescu transform, more precisely in its component operators. The construction of the general solution of the div-curl system is carried out by, first constructing an explicit inverse to the curl, a result which is of independent interest. With this inverse we solve the homogeneous div-curl system (in which $g_0$ vanishes), and then follow [30] to show how to apply a correction to obtain the solution for the inhomogeneous system. Further, at the end of Chapter 2 we will give an improved generalized solution of the div-curl system in Lipschitz domains, removing the previous requirement of star-shapedness of the domain.

In Chapter 3 we apply this solution to several related problems, including some Dirichlet-type problems, the conductivity equation, the main Vekua equation, and the double curl-type equation, the latter of which is then used in a fundamental way for solving the static Maxwell's equations with variable permeability (system (3.12) below). All results obtained in the Chapters 2 and 3 are also valid for functions that take values in the the algebra $\mathbb{H}(\mathbb{C})$ of biquaternions (complex quaternions), but for simplicity we will work with the real quaternions $\mathbb{H}$.

In Chapter 4 some operator properties related to boundedness and invertibility of the Hilbert transform $\mathcal{H}$ are given, as well as an explicit form for its left inverse and adjoint. This is followed by the introduction of the scalar and vector Dirichlet-to-Neumann maps for the monogenic case. Furthermore, we construct the "Vekua-Hilbert transform" $\mathcal{H}_f$ associated to the main Vekua equation in bounded Lipschitz domains of $\mathbb{R}^3$, and establish some basic facts related to the elements of its construction.

In Chapter 5 we introduce the quaternionic Dirichlet-to-Neumann map (D-N) for the conductivity equation and connect their components (the scalar and vector D-N maps, $\Lambda_{0,f^2}$ and $\vec{\Lambda}_{f^2}$ respectively) with the Vekua-Hilbert transform $\mathcal{H}_f$. The vector component is a new concept. We further verify the continuous dependence on the boundary values of the conductivity $f^2$ for the Vekua-Hilbert transform $\mathcal{H}_f$ and the quaternionic D-N map, and analyze the properties of the Vekua-Hilbert transform when it is restricted to $\operatorname{Ker}\Lambda_{0,f^2}$ or $\operatorname{Ker}\vec{\Lambda}_{f^2}$.

Most of the content of this dissertation is also contained in [35, 36].

# Chapter 1

# Summary of results in quaternionic analysis

A fundamental concept of modern physics is the vector field. One way of greatly simplifying calculations involving derivatives of the components of vector fields is by using the notation of quaternions. Most of the material in this thesis will make use of this notation and terminology.

In this chapter we will summarize the facts we will need concerning monogenic (holomorphic) functions of a quaternionic variable, and properties of related integral operators.

## 1.1 Quaternions

Let $\mathbb{H}$ be the non-commutative algebra of quaternions over the real field $\mathbb{R}$. In this thesis we use a more modern notation $\{e_0, e_1, e_2, e_3\}$, which follows the multiplication rule

$$\begin{gathered} e_0 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1, \\ e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \end{gathered} \tag{1.1}$$

**Definition 1.** A *quaternion* is defined as a formal linear combination

$$x = x_0 + \sum_{i=1}^{3} e_i x_i \in \mathbb{H},$$

where $x_i \in \mathbb{R}$, $i = 0, 1, 2, 3$. The real number $x_0 = \operatorname{Sc} x$ is called the scalar part of $x$ and $\vec{x} = \sum_{i=1}^{3} e_i x_i = \operatorname{Vec} x$ is called the vector part of $x$. Thus we can write

$$x = x_0 + \vec{x}.$$

In this thesis we will identify the subspaces $\operatorname{Sc}\mathbb{H}$, $\operatorname{Vec}\mathbb{H}$ with the real numbers $\mathbb{R}$ and Euclidean space $\mathbb{R}^3$ respectively. Functions taking values in $\mathbb{R}$ will be called *scalar*; functions taking values in $\mathbb{R}^3$ will be called *vector fields*. The *multiplication* of two quaternions in this notation gives

$$xy = x_0 y_0 - \vec{x} \cdot \vec{y} + x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y}. \tag{1.2}$$

The *conjugate* of a quaternion $x$ is

$$\overline{x} = \operatorname{Sc} x - \operatorname{Vec} x = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3, \tag{1.3}$$

and its *norm* is given by $|x| = \sqrt{x\overline{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. For more details on the non-commutative algebra of quaternions see [42, 51, 52, 53, 54, 67, 94].

In mathematical physics in $\mathbb{R}^3$, two differential operators for vector fields of extremely importance are the *div* and *curl*, which are defined by

$$\operatorname{div}\vec{w} = \nabla\cdot\vec{w} = \sum_{i=1}^{3}\partial_i w_i, \tag{1.4}$$

$$\operatorname{curl}\vec{w} = \nabla\times\vec{w} = (\partial_2 w_3 - \partial_3 w_2)e_1 + (\partial_3 w_1 - \partial_1 w_3)e_2 + (\partial_1 w_2 - \partial_2 w_1)e_3.$$

Meanwhile, for scalar functions we define the *gradient* as follows:

$$\operatorname{grad} w_0 = \nabla w_0 = \partial_1 w_0 e_1 + \partial_2 w_0 e_2 + \partial_3 w_0 e_3. \tag{1.5}$$

For a domain $\Omega \subseteq \mathbb{R}^3 \subseteq \mathbb{H}$ and $r \geq 0$ we have function spaces such as $C^r(\Omega,\mathbb{H})$ which denotes the space of $r$-times continuously differentiable functions and $C^{r,\gamma}(\Omega,\mathbb{H})$ the space of $r$-times continuously differentiable Hölder functions with exponent $\gamma$ $(0<\gamma\leq 1)$. In particular we have the Sobolev spaces

$$W^{1,p}(\Omega,\mathbb{H}) = \left\{u \in L^p(\Omega,\mathbb{H})\colon \quad \operatorname{grad} u_i \in L^p(\Omega,\mathbb{R}^3)\right\},$$
$$W_0^{1,p}(\Omega,\mathbb{H}) = \overline{C_0^\infty(\Omega,\mathbb{H})} \subseteq W^{1,p}(\Omega,\mathbb{H}),$$

where $1 \leq p \leq \infty$ and $C_0^\infty$ denotes the linear space of smooth functions of compact support. Facts about Sobolev spaces are drawn from [1, 26, 50].

**Definition 2.** We will say that $\Omega \subseteq \mathbb{R}^3$ is a *Lipschitz domain* if every point $\vec{x}$ of $\partial\Omega$ lies in a ball neighborhood $\vec{x}+B$ such that $(\vec{x}+B)\cap\Omega$ can be obtained from

$$\{(x_1, x_2, x_3) \in B \colon x_3 > \psi(x_1, x_2)\},$$

by a rigid motion, where $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function, i.e.,

$$|\psi(x_1, x_2) - \psi(y_1, y_2)| \leq C|(x_1, x_2) - (y_1, y_2)|,$$

for some constant $C > 0$. Moreover, if the partial derivatives of $\psi$ are Hölder continuous with exponent $\gamma > 0$, then we will say that $\Omega$ is a $C^{1,\gamma}$ Lipschitz domain or simply a $C^{1,\gamma}$ domain.

Let $\Omega$ be a bounded Lipschitz domain and let $1 \leq p \leq \infty$. The space

$$W^{1-1/p,p}(\partial\Omega, \mathbb{H}) = \{\varphi \in L^p(\partial\Omega, \mathbb{H}) \colon \; u|_{\partial\Omega} = \varphi \text{ for some } u \in W^{1,p}(\Omega, \mathbb{H})\}$$

is the boundary values of functions in $W^{1,p}(\Omega, \mathbb{H})$. When $p = 2$, we will use the usual notation $H^{1/2}(\partial\Omega, \mathbb{H}) = W^{1-1/2,2}(\partial\Omega, \mathbb{H})$.

**Theorem 3.** *(Trace Theorem [50, Th. 1.5.1.10]) Let $\Omega$ be a bounded Lipschitz domain and $1 \leq p \leq \infty$. For every $u \in W^{1,p}(\Omega, \mathbb{H})$, the trace of $u$ exists. The trace operator*

$$\operatorname{tr} \colon W^{1,p}(\Omega, \mathbb{H}) \to L^p(\partial\Omega, \mathbb{H}), \quad \operatorname{tr} u = u|_{\partial\Omega},$$

*is bounded. Moreover, $\operatorname{tr} \colon W^{1,2}(\Omega, \mathbb{H}) \to H^{1/2}(\partial\Omega, \mathbb{H})$ is a surjective, bounded linear operator with a continuous right inverse.*

Whenever $\partial\Omega$ is mentioned we will specify the smoothness required for applying the basic facts about Sobolev spaces. The above applies as well to the Sobolev subspaces with $\mathbb{R}$ or $\mathbb{R}^3$ in place of $\mathbb{H}$. We gather in Theorem 23 below the facts we will need about certain integral operators on these spaces.

## 1.2 Monogenic functions

From now on $\vec{x} \in \mathbb{R}^3$. The Moisil-Teodorescu differential operator $D$ (also known as the Cauchy-Riemann or occasionally the Dirac operator) is defined by

$$D = e_1\partial_1 + e_2\partial_2 + e_3\partial_3, \tag{1.6}$$

where $\partial_i = \partial/\partial x_i$, $i = 1, 2, 3$. This is the three-dimensional extension of the classical Cauchy-Riemann operator $\partial_{\bar{z}} = (1/2)(\partial_x + i\partial_y)$, with $z = x + iy \in \mathbb{C}$.

The operator $D$ may be applied to differentiable functions $w\colon \Omega \to \mathbb{H}$, i.e. $w(\vec{x}) = w_0(\vec{x}) + \sum_{i=1}^{3} e_i w_i(\vec{x}) = \operatorname{Sc} w(\vec{x}) + \operatorname{Vec} w(\vec{x})$, where the coordinate functions $w_i$ are real-valued functions defined in $\Omega$ and $\operatorname{Sc} w = w_0$. Thus $D$ acts from both the left and right sides as follows:

$$\begin{aligned} Dw_0 &= w_0 D = \operatorname{grad} w_0, \\ D\vec{w} &= -\operatorname{div}\vec{w} + \operatorname{curl}\vec{w}, \quad \vec{w}D = -\operatorname{div}\vec{w} - \operatorname{curl}\vec{w}, \end{aligned} \tag{1.7}$$

expressing $D$ in terms of the gradient $\nabla$, the divergence $\nabla\cdot$ and the curl (or rotational) $\nabla\times$. Thus for $w = w_0 + \vec{w}$ the left and right operators are

$$
\begin{aligned}
Dw &= -\operatorname{div}\vec{w} + \operatorname{grad} w_0 + \operatorname{curl}\vec{w}, \\
wD &= -\operatorname{div}\vec{w} + \operatorname{grad} w_0 - \operatorname{curl}\vec{w}. \qquad (1.8)
\end{aligned}
$$

The following [51, 52] is a generalization of the Leibniz rule:

**Proposition 4.** *Let $w, v$ be functions in $C^1(\Omega, \mathbb{H})$. Then*

$$D[vw] = D[v]w + \overline{v}D[w] + 2(\operatorname{Sc}(vD))[w], \qquad (1.9)$$

*where we write*

$$(\operatorname{Sc}(vD))[w] = -\sum_{i=1}^{3} v_i \partial_i w.$$

Note that when $\operatorname{Vec} v = 0$, this simplifies to the classical formula $D[vw] = D[v]w + vD[w]$.

**Definition 5.** Let $\Omega \subseteq \mathbb{R}^3$ be an open subset. A function $w \in C^1(\Omega, \mathbb{H})$ is called *left-monogenic* (respectively *right-monogenic*) in $\Omega$ when $Dw = 0$ (respectively $wD = 0$) and we write $\mathfrak{M}(\Omega) = \mathfrak{M}(\Omega, \mathbb{H})$ and $\mathfrak{M}^r(\Omega) = \mathfrak{M}^r(\Omega, \mathbb{H})$ for the spaces of left-monogenic and right-monogenic functions.

In general, left-monogenic functions are not right-monogenic and vice versa. Simple examples are $w_\pm(\vec{x}) = x_1 \pm x_2 e_3$, where $w_+$ is only right-monogenic and with $w_-$ is only left-monogenic. The unqualified term "monogenic" will refer to left-monogenic functions; the terms "regular" or "hyperholomorphic" are also commonly used. By (1.8),

$$w \in \mathfrak{M}(\Omega) \Leftrightarrow \begin{cases} \operatorname{div} \vec{w} &= 0, \\ \operatorname{curl} \vec{w} &= -\operatorname{grad} w_0. \end{cases} \tag{1.10}$$

It is well known that if we attempt to define hyperholomorphic functions through the existence of the limit of the difference quotient, then we obtain a smaller set of functions; actually they will have a linear form [53, Th. 5.8].

**Definition 6.** When both $Dw = 0$ and $wD = 0$, $w$ is called a *monogenic constant*.

By (1.8), $w$ is a monogenic constant if and only if $w_0$ is constant and $\vec{w}$ satisfies $\operatorname{div} \vec{w} = 0$ and $\operatorname{curl} \vec{w} = 0$. If $w \in \mathfrak{M}(\Omega)$ with $\operatorname{Sc} w = 0$ or $\operatorname{Vec} w = 0$, then $w$ is a monogenic constant. Define

$$\operatorname{Sol}(\Omega, \mathbb{R}^3) = \{\vec{w}\colon \ \operatorname{div} \vec{w} = 0 \text{ in } \Omega\} \subseteq C^1(\Omega, \mathbb{R}^3),$$
$$\operatorname{Irr}(\Omega, \mathbb{R}^3) = \{\vec{w}\colon \ \operatorname{curl} \vec{w} = 0 \text{ in } \Omega\} \subseteq C^1(\Omega, \mathbb{R}^3).$$

Elements of $\operatorname{Sol}(\Omega, \mathbb{R}^3)$ are called solenoidal (or incompressible, or divergence free) fields, while elements of $\operatorname{Irr}(\Omega, \mathbb{R}^3)$ are called irrotational vector fields.

From the above analysis it can be seen that the space $\mathfrak{M}^c(\Omega) = \mathfrak{M}^c(\Omega, \mathbb{H}) = \mathfrak{M}(\Omega) \cap \mathfrak{M}^r(\Omega)$ of monogenic constants in $\Omega$ can be decomposed as

$$\mathfrak{M}^c(\Omega) = \mathbb{R} \oplus \operatorname{SI}(\Omega),$$

where

$$\mathrm{SI}(\Omega) = \mathrm{Sol}(\Omega, \mathbb{R}^3) \cap \mathrm{Irr}(\Omega, \mathbb{R}^3). \tag{1.11}$$

Elements of $\mathrm{SI}(\Omega)$ are called SI-vector fields and are studied in [47, 88, 92]. Locally they are gradients of real valued harmonic functions. See also the following work [7], in which holomorphic functions in domains of the 3D space (SI vector fields) are defined globally as gradients of harmonic functions.

Let $C^1(\overline{\Omega}, \mathbb{H}) = C^1(\Omega, \mathbb{H}) \cap C(\overline{\Omega}, \mathbb{H})$ and denote by $C_0^1(\overline{\Omega}, \mathbb{H}) \subseteq C^1(\overline{\Omega}, \mathbb{H})$ the subspace of functions that vanish on the boundary. Much of our work will be in the context of $L^p$ spaces or Sobolev spaces. For this reason we define the concept of generalized derivatives:

**Definition 7.** Let $w, v \in L^1(\Omega, \mathbb{H})$. We will say that $v$ is the *generalized derivative* of $w$ *with respect to the operator* $D$ if the relation

$$\int_\Omega \overline{w} Du \, d\vec{y} = -\int_\Omega \overline{v} u \, d\vec{y},$$

is satisfied for every $u \in C_0^1(\overline{\Omega}, \mathbb{H})$.

According to Definition 7, $\operatorname{div} \vec{w} = 0$ and $\operatorname{curl} \vec{w} = 0$ in the weak sense if

$$\int_\Omega \vec{w} \cdot \operatorname{grad} u_0 \, d\vec{y} = 0, \quad \text{and} \quad \int_\Omega \vec{w} \cdot \operatorname{curl} \vec{u} \, d\vec{y} = 0, \tag{1.12}$$

respectively, for every $u_0 + \vec{u} \in C_0^1(\overline{\Omega}, \mathbb{H})$.

## 1.3 Quaternionic analysis

We begin this section presenting some integral operators that will be fundamental in the development of this work. The following is a non-trivial example of a function that is left and right monogenic:

**Definition 8.** The *Cauchy kernel* is defined by

$$E(\vec{x}) = -\frac{\vec{x}}{4\pi|\vec{x}|^3}, \quad \vec{x} \in \mathbb{R}^3 - \{0\}. \tag{1.13}$$

Moreover, (1.13) is a fundamental solution of $D$.

Henceforth $\Omega \subseteq \mathbb{R}^3$ will always be a bounded domain and $\eta$ will denote the outward unit normal vector to $\partial\Omega$.

**Definition 9.** The *Cauchy operator* and the *Teodorescu transform* are defined by

$$F_{\partial\Omega}[\varphi](\vec{x}) = \int_{\partial\Omega} E(\vec{y}-\vec{x})\eta(\vec{y})\varphi(\vec{y})\, ds_{\vec{y}}, \quad \vec{x} \in \mathbb{R}^3 \setminus \partial\Omega, \tag{1.14}$$

$$T_\Omega[w](\vec{x}) = -\int_\Omega E(\vec{y}-\vec{x})w(\vec{y})\, d\vec{y}, \quad \vec{x} \in \mathbb{R}^3. \tag{1.15}$$

The following integral formula relates the operators (1.14) and (1.15).

**Proposition 10.** *(Formula of Borel-Pompeiu [53, Th. 7.8]) Let $\Omega$ be a bounded domain with sufficiently smooth boundary. Then for every $w \in C^1(\overline{\Omega}, \mathbb{H})$*

$$T_\Omega[Dw](\vec{x}) + F_{\partial\Omega}[\operatorname{tr} w](\vec{x}) = \begin{cases} w(\vec{x}), & \vec{x} \in \Omega, \\ 0, & \vec{x} \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{1.16}$$

**Remark 11.** A well-known result is that $T_\Omega[w]$ is differentiable in $\Omega$ for every $w \in C^1(\Omega)$ [53, Th. 8.2]. Moreover, the integral operator (1.15) makes sense as long as $w$ is integrable and $T_\Omega$ is the right inverse of $D$ in $L^p(\Omega, \mathbb{H})$ in the generalized sense of Definition 7:

**Proposition 12** ([51, Prop. 2.4.2])**.** *Let $w \in L^p(\Omega, \mathbb{H})$ and let $1 < p < \infty$. Then $T_\Omega[w] \in W^{1,p}(\Omega, \mathbb{H})$. Further,*

$$DT_\Omega[w] = w.$$

The following operator is obtained from the Cauchy operator (1.14)

**Definition 13.** Let $\varphi \in C^{0,\gamma}(\Omega, \mathbb{H})$ be a Hölder continuous function. The three-dimensional *singular Cauchy integral operator*

$$S_{\partial\Omega}[\varphi](\vec{x}) = 2\,\text{P.V.}\int_{\partial\Omega} E(\vec{y} - \vec{x})\eta(\vec{y})\varphi(\vec{y})ds_{\vec{y}}, \quad \vec{x} \in \partial\Omega. \tag{1.17}$$

Furthermore, $S_{\partial\Omega}$ is an involution [53, Cor. 7.20].

The formulas of following theorem are known as *the Plemelj-Sokhotski formulas* [89, 43]. For facility of notation, we will write $\text{tr}_+\, w(\vec{x})$ and $\text{tr}_-\, w(\vec{x})$ for the non-tangential limit of $w(\vec{y})$ as $\vec{y} \in \Omega^\pm$ tends to $\vec{x} \in \partial\Omega$, where $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

**Theorem 14.** *[53, Th. 7.17] Let $\Omega$ be a bounded domain with sufficiently smooth boundary and let $\varphi$ be a Hölder continuous function. Then*

$$\text{tr}_\pm\, F_{\partial\Omega}[\varphi](\vec{x}) = \frac{1}{2}[\pm\varphi(\vec{x}) + S_{\partial\Omega}[\varphi](\vec{x})]; \tag{1.18}$$

*from this it is seen that* $S_{\partial\Omega}[\varphi] = \varphi$ *is necessary and sufficient for* $\varphi$ *to represent the boundary values of a monogenic function defined in* $\Omega$*; i.e.* $\varphi = \operatorname{tr}_+ F_{\partial\Omega}[\varphi]$*; the condition* $S_{\partial\Omega}[\varphi] = -\varphi$ *is necessary and sufficient for* $\varphi$ *to have a monogenic continuation into the exterior domain* $\Omega^-$ *vanishing at* $\infty$*, i.e.* $\varphi = -\operatorname{tr}_- F_{\partial\Omega}[\varphi]$.

Proposition 10 and Theorem 14 can be enunciated for more general function spaces:

**Remark 15.** The surface integral operators (1.14) and (1.17) make sense when $\varphi \in L^p(\partial\Omega, \mathbb{H})$ [53]. Moreover, by [51, Cor. 2.5.4] the Borel-Pompeiu formula (1.16) is valid in $W^{1,p}(\Omega, \mathbb{H})$ for $p > 1$ and according to [51, Rmks. 2.5.11, 2.5.17], the Plemelj-Sokhotski formulas (1.18) are valid in the Sobolev spaces $W^{1-1/p,p}(\partial\Omega, \mathbb{H}) \subseteq L^p(\partial\Omega, \mathbb{H})$ for Lipschitz domains.

**Definition 16.** The *volume integral operator* and the *single-layer potential* [29, 31, 78] are defined by

$$L[w](\vec{x}) = -\int_\Omega \frac{w(\vec{y})}{4\pi|\vec{y}-\vec{x}|}\, d\vec{y}, \quad \vec{x} \in \Omega, \tag{1.19}$$

$$M[\varphi](\vec{x}) = \int_{\partial\Omega} \frac{\varphi(\vec{y})}{4\pi|\vec{y}-\vec{x}|}\, ds_{\vec{y}}, \quad \vec{x} \in \mathbb{R}^3 \setminus \partial\Omega, \tag{1.20}$$

when the integrals exist. The *boundary single-layer operator* $\operatorname{tr} M$ is obtained by evaluating the integral in (1.20) for $x \in \partial\Omega$, thus extending $M$ to all of $\mathbb{R}^3$.

In Propositions 18 and 19 we will show that $T_\Omega = -DL$ and $L$ is a right inverse of $\Delta$, respectively. Meanwhile, (1.26) illustrates the relationship between the operators $F_{\partial\Omega}$ and $M$.

**Theorem 17.** *Let $\Omega$ be a bounded domain and let $1 < p < \infty$. The Teodorescu transform [53, Th. 8.4] [54, Th. 4.1.7]*

$$T_\Omega \colon L^p(\Omega, \mathbb{H}) \to W^{1,p}(\Omega, \mathbb{H}),$$

*is a bounded operator.*

Some properties of the Teodorescu operator in the plane were studied in [99] as well as applications in gas dynamics and shell theory. Some other references are [56] and [46] for the Teodorescu operator in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. More recently, in [67, 101] variations and generalizations of this operator related to the $\alpha$-Dirac operator were analyzed.

## 1.4 Components of the Teodorescu operator

As a preliminary to providing the general solution to the div-curl system (2.1) in Chapter 2 below, we begin by analyzing the elements which form the Teodorescu operator (1.15). The following operators were introduced in [30] with different notation (see also [19]). After that, in [35] the notation

$$\begin{aligned} T_{0,\Omega}[\vec{w}](\vec{x}) &= \int_\Omega E(\vec{y}-\vec{x}) \cdot \vec{w}(\vec{y})\, d\vec{y}, \\ \overrightarrow{T}_{1,\Omega}[w_0](\vec{x}) &= -\int_\Omega w_0(\vec{y}) E(\vec{y}-\vec{x})\, d\vec{y}, \\ \overrightarrow{T}_{2,\Omega}[\vec{w}](\vec{x}) &= -\int_\Omega E(\vec{y}-\vec{x}) \times \vec{w}(\vec{y})\, d\vec{y}, \end{aligned} \tag{1.21}$$

was used, where $\cdot$ denotes the scalar (or inner) product of vectors and $\times$ denotes the cross product. Note that $\overrightarrow{T}_{1,\Omega}$ acts on $\mathbb{R}$-valued functions, while

$T_{0,\Omega}$, $\overrightarrow{T}_{2,\Omega}$ act on $\mathbb{R}^3$-valued functions, and $T_{0,\Omega}$ produces scalar-valued functions. Furthermore,

$$T_\Omega[w_0 + \vec{w}] = T_{0,\Omega}[\vec{w}] + \overrightarrow{T}_{1,\Omega}[w_0] + \overrightarrow{T}_{2,\Omega}[\vec{w}]. \tag{1.22}$$

This is an expression of the quaternionic multiplication formula $\vec{a}b = -\vec{a}\cdot\vec{b} + b_0\vec{a} + \vec{a}\times\vec{b}$.

The following result expresses the operators $T_{0,\Omega}$, $\overrightarrow{T}_{1,\Omega}$, $\overrightarrow{T}_{2,\Omega}$ in terms of the operator $L$ given in (1.19) acting on continuous functions and fields.

**Proposition 18.** *([30, Prop. 3.2]) For $w_0 \in C(\Omega,\mathbb{R})$, $\vec{w} \in C(\Omega,\mathbb{R}^3)$ integrable,*

$$\begin{aligned} T_{0,\Omega}[\vec{w}] &= \nabla\cdot L[\vec{w}] \\ \overrightarrow{T}_{1,\Omega}[w_0] &= -\nabla L[w_0], \\ \overrightarrow{T}_{2,\Omega}[\vec{w}] &= -\nabla\times L[\vec{w}]. \end{aligned}$$

*Consequently, $\overrightarrow{T}_{1,\Omega}[w_0] \in \mathrm{Irr}(\Omega,\mathbb{R}^3)$ and $\overrightarrow{T}_{2,\Omega}[\vec{w}] \in \mathrm{Sol}(\Omega,\mathbb{R}^3)$.*

**Proof.** The proof is a direct calculation, using $\nabla_{\vec{x}}(1/|\vec{x}-\vec{y}|) = -4\pi E(\vec{y}-\vec{x})$ and the product rules of vector analysis [51, Cor. 1.3.4]. For example for $T_{0,\Omega}$:

$$T_{0,\Omega}[\vec{w}](\vec{x}) = -\int_\Omega \mathrm{div}_{\vec{x}}\left(\frac{\vec{w}(\vec{y})}{4\pi|\vec{x}-\vec{y}|}\right)d\vec{y} = \mathrm{div}_{\vec{x}}\, L[\vec{w}](\vec{x}),$$

using the formula $\mathrm{div}(a\vec{A}) = \nabla a\cdot\vec{A} + a\,\mathrm{div}\,\vec{A}$. The conclusion regarding the images of $\overrightarrow{T}_{1,\Omega}$, $\overrightarrow{T}_{2,\Omega}$ is similar using the corresponding formulas for the operators grad and curl. □

**Proposition 19.** *The operator $L$ given by* (1.19) *is a right inverse of the Laplacian $\Delta$ on the space of integrable functions in $C(\Omega, \mathbb{H})$.*

**Proof.** Let $w = w_0 + \vec{w} \in C^1(\Omega, \mathbb{H})$. By the expressions given in Proposition 18 we have that $T_\Omega[w] = -DL[w]$, and using Proposition 12

$$w = DT_\Omega w = -DDL[w] = \Delta L[w]. \qquad \square$$

Proposition 19 may also be proved using the fact that $1/(4\pi|\vec{x} - \vec{y}|)$ is a fundamental solution for the Laplacian. By Proposition 18, we have

$$\operatorname{div} \overrightarrow{T}_{1,\Omega} = -\operatorname{div}\operatorname{grad} L = -\Delta L = -I, \tag{1.23}$$

where $I$ is the identity operator, which gives the following.

**Corollary 20.** *[30, Prop. 3.1] The operator $-\overrightarrow{T}_{1,\Omega}$ acting on integrable functions in $C(\Omega, \mathbb{R})$ is a right inverse for the divergence div.*

Another useful property is the following.

**Proposition 21.** *The relation* $\operatorname{grad} T_{0,\Omega} + \operatorname{curl} \overrightarrow{T}_{2,\Omega} = I$ *holds on the space of integrable functions $L^1(\Omega, \mathbb{R}^3)$.*

**Proof.** We have $T_\Omega[\vec{g}] = T_{0,\Omega}[\vec{g}] + \overrightarrow{T}_{2,\Omega}[\vec{g}]$, since $\vec{g}$ has no scalar part. The statement is obtained from the vectorial part of the relation of Proposition 12 according to (1.8). $\square$

The decomposition (1.22) as well as the intrinsic properties of every component operator (1.21) allow us to solve the div-curl system (2.1) under certain restrictions on the initial data in Chapter 2. Moreover, these same component operators intervene in the construction of the Vekua-Hilbert transform which will be introduced later in Chapter 4.

## 1.5 Components of the Cauchy and singular Cauchy integral operators

In this section we will examine the Cauchy operator (1.14) and the singular Cauchy operator (1.17) in order to analyze the component operators of the Hilbert transform (see Chapter 4) and the general solution of the div-curl system in the context of Lipschitz domains (see Subsection 2.3). In a similar way to (1.22) we give a decomposition of the Cauchy operator [51, Th. 2.5.5]

$$F_{\partial\Omega}\colon W^{1-1/p,p}(\partial\Omega,\mathbb{H}) \longrightarrow W^{1,p}(\Omega,\mathbb{H}) \cap \mathfrak{M}(\Omega). \tag{1.24}$$

For real-valued functions $\varphi_0 \in L^p(\partial\Omega,\mathbb{R})$ we decompose $F_{\partial\Omega}[\varphi_0] = F_{0,\partial\Omega}[\varphi_0] + \vec{F}_{1,\partial\Omega}[\varphi_0]$ into the normal and tangential components

$$\begin{aligned} F_{0,\partial\Omega}[\varphi_0](\vec{x}) &= -\int_{\partial\Omega} E(\vec{y}-\vec{x})\cdot\eta(\vec{y})\,\varphi_0(\vec{y})\,ds_{\vec{y}}, \\ \vec{F}_{1,\partial\Omega}[\varphi_0](\vec{x}) &= \int_{\partial\Omega} E(\vec{y}-\vec{x})\times\eta(\vec{y})\,\varphi_0(\vec{y})\,ds_{\vec{y}} \end{aligned} \tag{1.25}$$

for $\vec{x} \in \mathbb{R}^3 \setminus \partial\Omega$.

Analogous to Proposition 18 for $T_\Omega$, the components of $F_{\partial\Omega}$ can be expressed in terms of the single-layer potential $M$ of (1.20) as follows:

$$F_{0,\partial\Omega}[\varphi_0] = \nabla\cdot M[\varphi_0\eta], \quad \vec{F}_{1,\partial\Omega}[\varphi_0] = -\nabla\times M[\varphi_0\eta], \tag{1.26}$$

where $\overrightarrow{F}_{1,\partial\Omega}[\varphi_0] \in \mathrm{Sol}(\Omega,\mathbb{R}^3)$ and $\operatorname{curl} \overrightarrow{F}_{1,\partial\Omega}[\varphi_0] \in \mathrm{Irr}(\Omega,\mathbb{R}^3)$. More generally, for every $\varphi \in W^{1,p}(\Omega,\mathbb{H})$ we have that $F_{\partial\Omega}[\varphi] = -DM[\eta\varphi]$ ([51, Prop. 2.5.3], note the change of sign).

Similarly, we can decompose $S_{\partial\Omega} = K_0 + \overrightarrow{K}$, where the component operators are

$$\begin{aligned} K_0[\varphi](\vec{x}) &= 2\,\text{P.V.}\int_{\partial\Omega} -E(\vec{y}-\vec{x})\cdot\eta(\vec{y})\,\varphi(\vec{y})\,ds_{\vec{y}}, \\ \overrightarrow{K}[\varphi](\vec{x}) &= 2\,\text{P.V.}\int_{\partial\Omega} E(\vec{y}-\vec{x})\times\eta(\vec{y})\,\varphi(\vec{y})\,ds_{\vec{y}} = \sum_{i=1}^{3} e_i K_i[\varphi](\vec{x}), \end{aligned} \tag{1.27}$$

with $K_i[\varphi]$ $(i = 1, 2, 3)$ having as integration kernel the $i$th quaternionic component $[E(\vec{y}-\vec{x})\times\eta(\vec{y})]_i$. Note that $S_{\partial\Omega}$ is a right $\mathbb{H}$-linear operator, and in particular for real-valued functions $\varphi_0$, $\operatorname{Sc} S_{\partial\Omega}[\varphi_0] = K_0[\varphi_0]$ and $\operatorname{Vec} S_{\partial\Omega}[\varphi_0] = \overrightarrow{K}[\varphi_0]$. We will frequently use the fact that since a scalar constant $c_0 \in \mathbb{R}$ is monogenic, $S_{\partial\Omega}[c_0] = c_0$, so the scalar and vector parts give

$$K_0[c_0] = c_0, \quad \overrightarrow{K}[c_0] = 0. \tag{1.28}$$

The operators $K_0$ and $\overrightarrow{K}$ (1.27) acting on $L^p(\partial\Omega,\mathbb{H})$ and $L^p(\partial\Omega,\mathbb{R})$ respectively have as adjoints

$$\begin{aligned} K_0^*[\varphi](\vec{x}) &= 2\,\text{P.V.}\int_{\partial\Omega} E(\vec{y}-\vec{x})\cdot\eta(\vec{x})\,\varphi(\vec{y})\,ds_{\vec{y}}, \\ \overrightarrow{K}^*[\vec{\varphi}](\vec{x}) &= 2\,\text{P.V.}\int_{\partial\Omega} -E(\vec{y}-\vec{x})\times\eta(\vec{x})\cdot\vec{\varphi}(\vec{y})\,ds_{\vec{y}} = \sum_{i=1}^{3} K_i^*[\varphi_i](\vec{x}), \end{aligned}$$

on $L^q(\partial\Omega,\mathbb{H})$ and $L^q(\partial\Omega,\mathbb{R}^3)$, respectively, where the duality pairing of $\mathbb{H}$-valued functions is $\operatorname{Sc}\int_{\partial\Omega}\overline{\varphi(\vec{y})}\psi(\vec{y})\,ds_{\vec{y}}$ and $1/p+1/q=1$. The computations for the adjoint of $\overrightarrow{K}$ are

$$\begin{aligned}\langle \overrightarrow{K}[\varphi_0], \vec{\varphi}\rangle_{\partial\Omega} &= \mathrm{Sc}\, 2\ \text{P.V.}\int_{\partial\Omega} \overline{\int_{\partial\Omega} E(\vec{x}-\vec{y}) \times \eta(\vec{x})\varphi_0(\vec{x})\, ds_{\vec{x}}}\, \vec{\varphi}(\vec{y})\, ds_{\vec{y}} \\ &= 2\ \text{P.V.}\int_{\partial\Omega} \varphi_0(\vec{x}) \left( \int_{\partial\Omega} (E(\vec{x}-\vec{y}) \times \eta(\vec{x})) \cdot \vec{\varphi}(\vec{y})\, ds_{\vec{y}} \right) ds_{\vec{x}}.\end{aligned}$$

**Definition 22.** The *boundary averaging operator* is

$$A[\varphi] = \frac{1}{\sigma_\Omega} \int_{\partial\Omega} \varphi(\vec{y})\, ds_{\vec{y}} \tag{1.29}$$

with $\sigma_\Omega$ chosen so that $A[1] = 1$.

There is a natural induced mapping $I - A$ from $L^p(\partial\Omega, \mathbb{H})$ to $L^p_0(\partial\Omega, \mathbb{H})$, where $L^p_0(\cdot)$ is the subspace of functions in $L^p(\cdot)$ with mean 0.

Let $H^{-1/2}(\partial\Omega, \mathbb{H})$ be the dual of the Sobolev space $H^{1/2}(\partial\Omega, \mathbb{H})$. We summarize some results of a long series of research in the history of operator theory in quaternionic analysis.

**Theorem 23.** *Let $\Omega$ be a bounded Lipschitz domain. The following operators are continuous:*

*(a) Let $1 < p < \infty$. The singular Cauchy integral operator [79, p. 421]*

$$S_{\partial\Omega} \colon W^{1-1/p,p}(\partial\Omega, \mathbb{H}) \to W^{1-1/p,p}(\partial\Omega, \mathbb{H});$$

*(b) The single-layer potential [31, p. 38]*

$$M \colon H^{-1/2}(\partial\Omega, \mathbb{H}) \to W^{1,2}(\Omega, \mathbb{H});$$

(c) *The boundary single-layer operator [65, Prop. 2.4.7], [78, Th. 6.12]*

$$\operatorname{tr} M\colon H^{-1/2}(\partial\Omega,\mathbb{H}) \to H^{1/2}(\partial\Omega,\mathbb{H}).$$

The operator $K_0$ has been thoroughly studied due to its importance in solving the Dirichlet Problem, and has very good properties [32, 63]; for example on a $C^{1,\gamma}$ ($\gamma > 0$) domain

$$|E(\vec{y}-\vec{x})\cdot\eta(\vec{y})| \leq \frac{C}{|\vec{y}-\vec{x}|^{2-\gamma}},$$

and thus $K_0$ is a compact operator from $L^p(\partial\Omega)$ to itself ($1 < p < \infty$). Likewise $\overrightarrow{K}$ is bounded from $L^p(\partial\Omega,\mathbb{R})$ to $L^p(\partial\Omega,\mathbb{R}^3)$ and from $W^{1-1/p,p}(\partial\Omega,\mathbb{R})$ to $W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3)$, because $S_{\partial\Omega}$ is bounded in $L^p(\partial\Omega,\mathbb{H})$ [51, Th. 2.5.8] and in $W^{1-1/p,p}(\partial\Omega,\mathbb{H})$ (see Theorem 23(a)), respectively. When $\Omega$ is a bounded Lipschitz domain, although Fredholm theory is not applicable, it is possible to verify the invertibility of $I + K_0$. We summarize here the results of the operators $K_0$ and $K_0^*$ that we will need.

**Proposition 24.** *[32, 63] Let $\Omega$ be a bounded Lipschitz domain with connected complement. There is $\epsilon(\Omega)$, depending only on the Lipschitz character of $\partial\Omega$, such that*

(a) *If $\partial\Omega$ is Lipschitz, $2-\epsilon(\Omega) < p < \infty$, then $I + K_0$ is invertible on $L^p(\partial\Omega)$ with bounded inverse.*

(b) *If $\partial\Omega$ is Lipschitz, $1 < p < 2+\epsilon(\Omega)$, then $I + K_0$ is invertible on $W^{1-1/p,p}(\partial\Omega)$ with bounded inverse.*

(c) *If* $\Omega$ *is* $C^{1,\gamma}$ *Lipschitz for some* $\gamma > 0$, $1 < p < \infty$, *then* $I + K_0$ *is invertible both on* $L^p(\partial\Omega)$ *and* $W^{1-1/p,p}(\partial\Omega)$ *with bounded inverse.*

(d) *If* $\partial\Omega$ *is Lipschitz,* $1 < q < 2 + \epsilon(\Omega)$, *or* $C^{1,\gamma}$ *Lipschitz and* $1 < q < \infty$, *then* $I - K_0^*$ *is invertible on* $L_0^q(\partial\Omega)$ *with bounded inverse.*

**Proof.** Part (a) was established in [100, Theorem 3.1] for $p = 2$, and then in [32, Theorem 4.17] it was extended for $2 - \epsilon(\Omega) < p < \infty$.

To prove (b), let $1 < p < 2 + \epsilon(\Omega)$. In the proof of [100, Theorem 3.3] it is shown that $I + K_0 = M(I + K_0^*)M^{-1}$. We have noted previously that $I + K_0$ is bounded on $L^p(\partial\Omega)$ and has a bounded inverse. This fact is not sufficient for our purpose, but by the same reference [32, Theorems 4.17, 4.18], the single layer potential $M\colon L^p(\partial\Omega) \to W^{1-1/p,p}(\partial\Omega)$ and $I + K_0^*$ are bounded and have bounded inverses. From this the boundedness of $(I + K_0)^{-1}$ in $W^{1-1/p,p}(\partial\Omega)$ follows.

Parts (c) and (d) were stated in [63, p. 52] and [32, Theorem 4.17] (see also [100, Theorem 3.3] for $q = 2$), respectively. □

## 1.6 Div-curl spaces

In the quaternionic decomposition of the Moisil-Teodorescu operator (1.8) and in many electromagnetic problems there appear the differential operators grad, div and curl. One important interrelation between them is expressed by the linear homomorphisms

$$\mathbb{R} \to C^{r+3}(\Omega, \mathbb{R}) \overset{\text{grad}}{\to} C^{r+2}(\Omega, \mathbb{R}^3) \overset{\text{curl}}{\to} C^{r+1}(\Omega, \mathbb{R}^3) \overset{\text{div}}{\to} C^{r}(\Omega, \mathbb{R}) \to 0. \quad (1.30)$$

**Proposition 25.** *Let $r \geq 0$. The homomorphisms* (1.30) *form a sequence, that is,*

$$\mathbb{R} \subseteq \operatorname{Ker}\operatorname{grad}, \quad \operatorname{Im}\operatorname{grad} \subseteq \operatorname{Ker}\operatorname{curl}, \quad \operatorname{Im}\operatorname{curl} \subseteq \operatorname{Ker}\operatorname{div}, \quad \operatorname{Im}\operatorname{div} \subseteq C^r(\Omega, \mathbb{R}).$$

*When $\Omega$ is simply connected, the sequence is exact.*

**Proof.** The first containment is obvious, and the last two follow from $\operatorname{curl}\operatorname{grad} = 0$ and $\operatorname{div}\operatorname{curl} = 0$. The sequence is always exact at $C^{r+3}(\Omega, \mathbb{R})$ since the only functions with vanishing gradient are constants. It is also exact at $C^r(\Omega, \mathbb{R})$ by Corollary 20, that is, $\operatorname{Im}\operatorname{div} = C^r(\Omega, \mathbb{R})$. When $\Omega$ is simply connected, we may apply the Poincare's Lemma [34, Ch. 9, Lem. 3], to obtain

$$\begin{aligned} \operatorname{curl}\vec{w} = 0 &\Longleftrightarrow \vec{w} = \nabla u_0 \text{ for some } u_0 \text{ a scalar function } u_0, \\ \operatorname{div}\vec{w} = 0 &\Longleftrightarrow \vec{w} = \nabla \times \vec{u} \text{ for some } \vec{u} \text{ a vector field } \vec{u}. \end{aligned} \tag{1.31}$$

i.e., the corresponding inclusions are equalities. □

In (2.5) we will introduce the operator that produces the antigradient for $\vec{w}$ up to an arbitrary additive constant. The necessity of the second statement of (1.31) is related to finding a right inverse for curl, which is done in Theorem 43 for star-shaped domains and in Corollary 57 for Lipschitz domains.

Following [34, Chapter 9] and [44, Chapter 1], let us introduce the Hilbert spaces associated with the operators div and curl:

$$W^{2,\mathrm{div}}(\Omega,\mathbb{R}^3)=\left\{\vec{u}\in L^2(\Omega,\mathbb{R}^3)\colon\ \ \operatorname{div}\vec{u}\in L^2(\Omega,\mathbb{R})\right\},$$
$$W^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3)=\left\{\vec{u}\in L^2(\Omega,\mathbb{R}^3)\colon\ \ \operatorname{curl}\vec{u}\in L^2(\Omega,\mathbb{R}^3)\right\},$$

with norms $\|\vec{u}\|_{L^2}+\|\operatorname{div}\vec{u}\|_{L^2}$ and $\|\vec{u}\|_{L^2}+\|\operatorname{curl}\vec{u}\|_{L^2}$ respectively. Therefore, the intersection

$$W^{2,\,\text{div-curl}}(\Omega,\mathbb{R}^3)=W^{2,\mathrm{div}}(\Omega,\mathbb{R}^3)\cap W^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3)$$

is a Hilbert space with norm $\|\vec{u}\|_{L^2}+\|\operatorname{div}\vec{u}\|_{L^2}+\|\operatorname{curl}\vec{u}\|_{L^2}$. Observe that the conditions defining these spaces are weaker than requiring $\operatorname{grad} u$ to be in $L^2$, that is $W^{1,2}(\Omega,\mathbb{R}^3)\subset W^{2,\,\text{div-curl}}(\Omega,\mathbb{R}^3)$. For the opposite containment it is necessary to add certain boundary conditions; because taking $\vec{u}$ any harmonic function such that $\operatorname{tr}\vec{u}\in H^{1/2}(\partial\Omega,\mathbb{R}^3)\setminus H^{3/2}(\partial\Omega,\mathbb{R}^3)$, then $\nabla\vec{u}$ cannot be in $W^{1,2}(\Omega,\mathbb{R}^3)$. Thus $W^{2,\,\text{div-curl}}(\Omega,\mathbb{R}^3)\not\subset W^{1,2}(\Omega,\mathbb{R}^3)$. See Proposition 28 below for the required constraints. This result is sometimes enunciated as Friedrichs' inequality; references include [3, 75, 87].

**Definition 26.** The *normal and tangential trace operators* [34] are strongly defined by

$$\gamma_{\mathbf{n}}(\vec{u})=\vec{u}|_{\partial\Omega}\cdot\eta,\quad \gamma_{\mathbf{t}}(\vec{u})=\vec{u}|_{\partial\Omega}\times\eta. \tag{1.32}$$

on $W^{2,\mathrm{div}}(\Omega,\mathbb{R}^3)$ and $W^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3)$ respectively.

They are weakly defined as

$$\begin{aligned}\langle \gamma_{\mathbf{n}}(\vec{u}), \operatorname{tr} v_0 \rangle_{\partial\Omega} &= \int_\Omega \vec{u} \cdot \nabla v_0 \, d\vec{y} + \int_\Omega \operatorname{div} \vec{u} v_0 \, d\vec{y}, \\ \langle \gamma_{\mathbf{t}}(\vec{u}), \operatorname{tr} \vec{v} \rangle_{\partial\Omega} &= \int_\Omega \vec{u} \cdot \operatorname{curl} \vec{v} \, d\vec{y} - \int_\Omega \operatorname{curl} \vec{u} \cdot \vec{v} \, d\vec{y}, \end{aligned} \tag{1.33}$$

for every $v = v_0 + \vec{v} \in W^{1,2}(\Omega, \mathbb{H})$. In the literature, sometimes (1.33) are called *generalized Green's formulas*. Let $W_0^{2,\mathrm{div}}(\Omega, \mathbb{R}^3)$ and $W_0^{2,\mathrm{curl}}(\Omega, \mathbb{R}^3)$ be the kernels of the trace operators $\gamma_{\mathbf{n}}$ and $\gamma_{\mathbf{t}}$, respectively.

**Proposition 27.** *[34, Theorems 1, 2] Let $\Omega$ be a bounded Lipschitz domain. The normal trace operator $\gamma_{\mathbf{n}}$ and the tangential trace operator $\gamma_{\mathbf{t}}$ are bounded linear mappings from $W^{2,\mathrm{div}}(\Omega, \mathbb{R}^3)$ to $H^{-1/2}(\partial\Omega, \mathbb{R})$ and from $W^{2,\mathrm{curl}}(\Omega, \mathbb{R}^3)$ to $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$, respectively.*

**Proposition 28.** *(Friedrichs' inequalities [34, Chapter 9, Corollary 1], [3, Remark 2.14]) Let $\Omega$ be a $C^{1,1}$ bounded Lipschitz domain. If $\gamma_{\mathbf{n}}(\vec{u}) \in H^{1/2}(\partial\Omega, \mathbb{R})$ or $\gamma_{\mathbf{t}}(\vec{u}) \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, then*

$$\|\vec{u}\|_{W^{1,2}}^2 \leq C\Big( \|\vec{u}\|_{L^2}^2 + \|\operatorname{curl} \vec{u}\|_{L^2}^2 + \|\operatorname{div} \vec{u}\|_{L^2}^2 + \|\gamma_{\mathbf{n}}(\vec{u})\|_{H^{1/2}}^2 \Big)$$

*or*

$$\|\vec{u}\|_{W^{1,2}}^2 \leq C\Big( \|\vec{u}\|_{L^2}^2 + \|\operatorname{curl} \vec{u}\|_{L^2}^2 + \|\operatorname{div} \vec{u}\|_{L^2}^2 + \|\gamma_{\mathbf{t}}(\vec{u})\|_{H^{1/2}}^2 \Big),$$

*respectively, where $C > 0$ only depends on $\partial\Omega$.*

Let

$$\begin{aligned} W_{\mathbf{n}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3) &= W_0^{2,\mathrm{div}}(\Omega,\mathbb{R}^3)\cap W^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3), \\ W_{\mathbf{t}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3) &= W^{2,\mathrm{div}}(\Omega,\mathbb{R}^3)\cap W_0^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3), \end{aligned} \tag{1.34}$$

with the norm $\|\vec{u}\|_{L^2}^2 + \|\operatorname{div}\vec{u}\|_{L^2}^2 + \|\operatorname{curl}\vec{u}\|_{L^2}^2$.

The following statement (a) is a direct consequence of Proposition 28.

**Proposition 29.** [3, Theorems 2.8, 2.9, 2.12], [102].
*(a) Let $\Omega$ be a bounded $C^{1,1}$ Lipschitz domain. Then $W_{\mathbf{n}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3)$ and $W_{\mathbf{t}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3)$ are contained in $W^{1,2}(\Omega,\mathbb{R}^3)$.*
*(b) The inclusions of $W_{\mathbf{n}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3)$ and $W_{\mathbf{t}}^{2,\mathrm{div\,\text{-}\,curl}}(\Omega,\mathbb{R}^3)$ into $L^2(\Omega,\mathbb{R}^3)$ are compact operators.*

The next result is based on Proposition 29(a) and the Peetre-Tartar's Lemma [81, 97].

**Proposition 30.** [34, Chapter 9, Corollary 2]. *Let $\Omega$ be a bounded $C^{1,1}$ Lipschitz domain. The subspaces of "normally" and "tangentially" monogenic constants in the interior $\Omega = \Omega^+$ or exterior domain $\Omega^-$,*

$$\begin{aligned} \mathrm{SI}_{\mathbf{n}}(\Omega^{\pm}) &= \left\{\vec{u}\in L^2(\Omega^{\pm},\mathbb{R}^3):\ \operatorname{div}\vec{u}=0, \operatorname{curl}\vec{u}=0, \gamma_{\mathbf{n}}(\vec{u})=0\right\}, \\ \mathrm{SI}_{\mathbf{t}}(\Omega^{\pm}) &= \left\{\vec{u}\in L^2(\Omega^{\pm},\mathbb{R}^3):\ \operatorname{div}\vec{u}=0, \operatorname{curl}\vec{u}=0, \gamma_{\mathbf{t}}(\vec{u})=0\right\}, \end{aligned} \tag{1.35}$$

*have finite dimension.*

# Chapter 2

# Div-curl system

Consider the inhomogeneous div-curl system

$$\begin{aligned} \operatorname{div} \vec{w} &= g_0, \\ \operatorname{curl} \vec{w} &= \vec{g}. \end{aligned} \tag{2.1}$$

We will follow the custom of calling the system *homogeneous* when $g_0$ vanishes identically.

In this chapter we will give a complete solution to the reconstruction of a vector field from its divergence and curl under appropriate assumptions on the scalar field $g_0$ and the vector field $\vec{g}$ and their domain of definition in three-dimensional space.

An explicit general solution is given in terms of classical integral operators (components of the Teodorescu operator and the harmonic hyperconjugate operator), completing previously known results obtained under restrictive conditions.

## 2.1 Harmonic hyperconjugates

It is well known that every real valued harmonic function $u_0$ of a complex variable has locally a harmonic conjugate $u_1$ such that $u_0+iu_1$ is holomorphic, and $u_1$ is unique up to an additive constant. Similarly,

**Definition 31.** When $w = w_0 + \vec{w} \in \mathfrak{M}(\Omega)$ is monogenic, one says that $w_0, \vec{w}$ form a *hyperconjugate pair* or that $\vec{w}$ is the *harmonic hyperconjugate* of $w_0$.

Recall from (1.10) that $w \in \mathfrak{M}(\Omega)$ if and only if

$$\operatorname{div} \vec{w} = 0, \quad \operatorname{curl} \vec{w} = -\operatorname{grad} w_0.$$

The three-dimensional Laplacian is given by $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. Let $w \in \operatorname{Har}(\Omega, \mathbb{H}) = \{w\colon \Omega \to \mathbb{H},\ \Delta w = 0\}$, the set of harmonic functions defined in $\Omega$.

**Proposition 32.** *If $w$ is left- or right-monogenic, then $w \in \operatorname{Har}(\Omega, \mathbb{H})$.*

**Proof.** Since $\Delta w = -D^2 w$, we have the result. □

**Definition 33.** $\Omega$ is *star-shaped* with respect to the origin if $r\vec{x} \in \Omega$ whenever $\vec{x} \in \Omega$ and $0 \le r \le 1$.

The operators and results in this subsection are all well known [94, 52].

**Definition 34.** The *radial moment operator* of degree $\alpha$, applicable to $\mathbb{R}^n$-valued functions defined in star-shaped domains, is

$$I^\alpha[w](\vec{x}) = \int_0^1 t^\alpha w(t\vec{x})\, dt, \tag{2.2}$$

where usually $\alpha > -1$.

Via standard relations such as $\partial w_0(t\vec{x})/\partial t = \vec{x} \cdot \operatorname{grad} w_0(t\vec{x})$ one verifies the following.

**Lemma 35.** *[49]* $\operatorname{div} I^\alpha = I^{\alpha+1} \operatorname{div}$*;* $\operatorname{grad} I^\alpha = I^{\alpha+1} \operatorname{grad}$*;* $\operatorname{curl} I^\alpha = I^{\alpha+1} \operatorname{curl}$*;* $\Delta I^\alpha = I^{\alpha+2}\Delta$*;* $\vec{x}\cdot I^{\alpha+1}[\vec{w}] = I^\alpha[\vec{x}\cdot\vec{w}]$*;* $\vec{x}\times I^{\alpha+1}[\vec{w}] = I^\alpha[\vec{x}\times\vec{w}]$*;* $I^\alpha[(\vec{x}\cdot\operatorname{grad})w] = (\vec{x}\cdot\operatorname{grad})I^\alpha[w]$ *and*

$$I^\alpha[(\vec{x}\cdot\operatorname{grad})w] = w - (\alpha+1)I^\alpha[w].$$

A further property we will need is $I^\alpha[\vec{x}\cdot\operatorname{curl}\vec{w}] = \vec{x}\cdot\operatorname{curl} I^\alpha[\vec{w}]$, which yields $I^\alpha[\vec{x}\cdot\operatorname{Vec} D\vec{w}] = \vec{x}\cdot\operatorname{Vec} D I^\alpha[\vec{w}]$.

**Definition 36.** Let $\Omega$ be a star-shaped open set with respect to the origin. The *harmonic hyperconjugate operator* $\overrightarrow{U}_\Omega\colon \operatorname{Har}(\Omega,\mathbb{R}) \to \operatorname{Har}(\Omega,\mathbb{R}^3)$ is the composition

$$\overrightarrow{U}_\Omega = I^0[\operatorname{Vec}\vec{x}D]. \tag{2.3}$$

Recall that $Du$ is vectorial for scalar valued $u$; we have written $\vec{x}D$ for the operator $(\vec{x}D)[u](\vec{x}) = \vec{x}Du(\vec{x})$, which involves a quaternionic multiplication. Explicitly this is

$$\overrightarrow{U}_\Omega[w_0](\vec{x}) = \operatorname{Vec}\left(\int_0^1 t\vec{x}Dw_0(t\vec{x})\,dt\right) = \int_0^1 t\vec{x}\times\nabla w_0(t\vec{x})\,dt, \quad \vec{x}\in\Omega.$$

When $\Omega$ is star-shaped with respect to some other point, the definition of $\overrightarrow{U}_\Omega$ is adjusted by shifting the values of $\vec{x}$ accordingly. Versions of $\overrightarrow{U}_\Omega$ in $\mathbb{R}^n$

can be found in greater generality in [22] and [52, Sect. 2.1.5]; we give the proof of the following here for completeness, modifying slightly the argument which was given in [94] for functions in domains in $\mathbb{H}$.

**Proposition 37.** *Let $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set. The operator $\overrightarrow{U}_\Omega$ sends $\mathrm{Har}(\Omega, \mathbb{R})$ to $\mathrm{Har}(\Omega, \mathbb{R}^3)$. For every real-valued harmonic function $w_0 \in \mathrm{Har}(\Omega, \mathbb{R})$,*

$$w_0 + \overrightarrow{U}_\Omega[w_0] \in \mathfrak{M}(\Omega).$$

*Thus there is a monogenic function $w$ such that $\mathrm{Sc}\, w = w_0$.*

**Proof.** Let $w(\vec{x}) = w_0(\vec{x}) + \overrightarrow{U}_\Omega[w_0](\vec{x})$. Then since $\mathrm{Sc}\, \vec{x} D[w_0] = -\vec{x} \cdot \mathrm{grad}\, w_0$, by Lemma 35 we have $\mathrm{Sc}\, I^0[-\vec{x} D[w_0]] = w_0 - I^0[w_0]$, so (2.3) says

$$\begin{aligned} w &= -I^0[D[w_0]\vec{x}] + I^0[w_0] \\ &= \int_0^1 -t D w_0(t\vec{x})\vec{x}\, dt + \int_0^1 w_0(t\vec{x})\, dt, \end{aligned}$$

when $D[w_0]\vec{x}$ means the quaternionic multiplication $D[w_0](\vec{x})\vec{x}$. We apply $D$ and change the order of integration and derivation since $w_0$ and $Dw_0$ have continuous partial derivatives in $\Omega$:

$$(Dw)(\vec{x}) = \int_0^1 -t D_{\vec{x}}(D_{\vec{x}} w_0(t\vec{x})\vec{x})\, dt + \int_0^1 D_{\vec{x}}[w_0(t\vec{x})]\, dt. \tag{2.4}$$

The subscript in $D_{\vec{x}}$ is the variable with respect to which we apply the operator. Using the Leibniz formula (1.9),

$$
\begin{aligned}
D_{\vec{x}}\left(D_{\vec{x}} w_0(t\vec{x})\vec{x}\right) &= -\Delta_{\vec{x}}(w_0(t\vec{x}))\vec{x} + \overline{D_{\vec{x}} w_0(t\vec{x})} D\vec{x} - 2\sum_{i=1}^{3} \partial_i w_0(t\vec{x})\partial_i \vec{x} \\
&= t\Delta_{\vec{x}} w_0(t\vec{x}) + 3D_{\vec{x}} w_0(t\vec{x}) - 2D_{\vec{x}} w_0(t\vec{x}) \\
&= D_{\vec{x}} w_0(t\vec{x})
\end{aligned}
$$

since $w_0$ is harmonic. Finally, since the second integrand in (2.4) is equal to $tD_{\vec{x}} w_0(t\vec{x})$, we conclude $Dw = 0$ as required. □

**Definition 38.** The *antigradient operator* $\mathcal{A}$ is given by

$$
\begin{aligned}
\mathcal{A}[\vec{g}](x_1, x_2, x_3) = \int_{a_1}^{x_1} g_1(t, a_2, a_3)\, dt &+ \int_{a_2}^{x_2} g_2(x_1, t, a_3)\, dt \\
&+ \int_{a_3}^{x_3} g_3(x_1, x_2, t)\, dt,
\end{aligned} \tag{2.5}
$$

where $\vec{g}\colon \Omega \to \mathbb{R}^3$ is any vector field such that $\operatorname{curl} \vec{g} = 0$ and $(a_1, a_2, a_3) \in \Omega$.

**Proposition 39.** *[72] Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected domain. Then the scalar function $\psi = \mathcal{A}[\vec{g}]$ is a potential (or antigradient) for the irrotational field $\vec{g}$; i.e.* $\operatorname{grad} \psi = \vec{g}$.

Since potentials are defined up to an arbitrary additive constant, this local definition can be extended to give $\mathcal{A}\colon \operatorname{Irr}(\Omega, \mathbb{R}^3) \to C^2(\Omega, \mathbb{R})$.

By Proposition 37, every real-valued harmonic function is the scalar part of a monogenic function; conversely, the condition for completing a vector part to a hyperconjugate pair is for $\vec{w}$ to be harmonic and solenoidal:

**Proposition 40.** *Let* $\vec{w} \in \mathrm{Har}(\Omega, \mathbb{R}^3)$ *where* $\Omega$ *is simply connected. A necessary and sufficient condition for there to exist* $w \in \mathfrak{M}(\Omega)$ *such that* $\mathrm{Vec}\, w = \vec{w}$ *is that* $\mathrm{div}\, \vec{w} = 0$*. Moreover,* $w_0 = -\mathcal{A}[\mathrm{curl}\, \vec{w}]$ *up to additive constants.*

**Proof.** The necessity is given by (1.10). To prove the sufficiency, let $\vec{w}$ be solenoidal. Then $\mathrm{curl}\,\mathrm{curl}\, \vec{w} = \mathrm{grad}\,\mathrm{div}\, \vec{w} - \Delta\vec{w} = 0$, where $\Delta\vec{w}$ is the Laplacian applied to each component of the vector field. Thus we can define $w_0 = -\mathcal{A}[\mathrm{curl}\, \vec{w}]$ so that $\mathrm{curl}\, \vec{w} = -\,\mathrm{grad}\, w_0$ as required by (1.10). □

Part (i) of Proposition 41 below was noted in [30, Prop. 3.8]. This will be fundamental in the solutions of the div-curl system (2.1) in both star-shaped and Lipschitz domains, which will be given in Sections 2.2 and 2.3 respectively.

**Proposition 41.** *Suppose that* $\vec{w} \in L^1(\Omega, \mathbb{H})$*. (i)* $T_{0,\Omega}[\vec{w}] \in \mathrm{Har}(\Omega, \mathbb{R})$ *if and only if* $\vec{w} \in \mathrm{Sol}(\Omega, \mathbb{R}^3)$*; (ii)* $\overrightarrow{T}_{2,\Omega}[\vec{w}] \in \mathrm{Har}(\Omega, \mathbb{R}^3)$ *if and only if* $\vec{w} \in \mathrm{Irr}(\Omega, \mathbb{R}^3)$.

**Proof.** Using $\Delta = -D^2$ and the property $DT_\Omega = I$ in $L^p(\Omega, \mathbb{H})$ of Proposition 12 together with the decomposition of the operator $D$ given in (1.8) it follows that

$$\Delta T_\Omega[\vec{w}] = -D^2 T_\Omega[\vec{w}] = -D\vec{w} = \mathrm{div}\, \vec{w} - \mathrm{curl}\, \vec{w}.$$

The scalar and vector parts are $\Delta T_{0,\Omega}[\vec{w}] = \mathrm{div}\, \vec{w}$ and $\Delta \overrightarrow{T}_{2,\Omega}[\vec{w}] = -\,\mathrm{curl}\, \vec{w}$ by (1.22). □

**Remark 42.** Most of what we have done will go through equally well in the context of generalized derivatives. Thus Proposition 41 extends to the situation in which $\operatorname{div} \vec{w} = 0$ and $\operatorname{curl} \vec{w} = 0$ holds in the sense of distributions (1.12), because by Weyl's Lemma [37, 93], the weak solutions $T_{0,\Omega}[\vec{w}]$ and $\overrightarrow{T}_{2,\Omega}[\vec{w}]$ of the Laplace equation are smooth solutions.

## 2.2 Div-curl system in star-shaped domains

### 2.2.1 General solution

The first step in our solution of the div-curl system is to obtain an inverse for the curl operator, an object which is of independent interest. We will use the harmonic hyperconjugate operator $\overrightarrow{U}_\Omega$ of (2.3) and the component Teodorescu operators of (1.21). Note that the vanishing divergences $\operatorname{div} \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{w}]] = \operatorname{div} \overrightarrow{T}_{2,\Omega}[\vec{w}] = 0$ imply the a priori fact that

$$\overrightarrow{T}_{2,\Omega} - \overrightarrow{U}_\Omega T_{0,\Omega} \colon \operatorname{Sol}(\Omega, \mathbb{R}^3) \to \operatorname{Sol}(\Omega, \mathbb{R}^3).$$

**Theorem 43.** *Let $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set. The operator*

$$\overrightarrow{T}_{2,\Omega} - \overrightarrow{U}_\Omega T_{0,\Omega} \tag{2.6}$$

*is a right inverse for the curl acting on the class of integrable functions in* $\operatorname{Sol}(\Omega, \mathbb{R}^3)$.

**Proof.** Let $\vec{g} \in \operatorname{Sol}(\Omega, \mathbb{R}^3)$ and let $\vec{w} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[v_0]$ where $v_0 = T_{0,\Omega}[\vec{g}]$. By Proposition 41, $v_0 \in \operatorname{Har}(\Omega, \mathbb{R})$, so by Proposition 37, $v_0 + \overrightarrow{U}_\Omega[v_0]$ is a monogenic function whose equivalent system (1.10) is

$$\begin{aligned} \operatorname{div} \overrightarrow{U}_\Omega[v_0] &= 0, \\ \operatorname{curl} \overrightarrow{U}_\Omega[v_0] &= -\nabla v_0. \end{aligned} \tag{2.7}$$

Combining these equations with Proposition 21, we have that

$$\operatorname{curl} \vec{w} = -\nabla v_0 + \vec{g} + \nabla v_0 = \vec{g}. \qquad \square$$

**Remark 44.** In [30] it was shown that $\overrightarrow{T}_{2,\Omega}$ acts as a right inverse for curl *for elements of the kernel of* $T_{0,\Omega}$. We recover this fact here. Indeed, let $T_{0,\Omega}[\vec{g}] = 0$. By Proposition 41, the field $\vec{g}$ is indeed solenoidal, and since by (2.7) $\operatorname{div} \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]] = 0$, Theorem 43 says that $\operatorname{curl} \overrightarrow{T}_{2,\Omega}[\vec{g}] = \vec{g}$. It was recognized in [30] that to require $T_{0,\Omega}[\vec{g}]$ to vanish would be too strong a condition; now we see that the precise condition is for $T_{0,\Omega}[\vec{g}]$ to be harmonic.

**Corollary 45.** *Let $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set. Let $\vec{g} \in \operatorname{Sol}(\Omega, \mathbb{R}^3)$ be an integrable divergence free vector field. Then the general solution of the homogeneous system*

$$\operatorname{div} \vec{w} = 0, \quad \operatorname{curl} \vec{w} = \vec{g}$$

*in $\Omega$ has the form*

$$\vec{w} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]] + \nabla h, \tag{2.8}$$

*where $h \in \operatorname{Har}(\Omega, \mathbb{R})$ is an arbitrary real-valued harmonic function.*

**Proof.** Let $\vec{w}$ be admit the form (2.8). Then $\operatorname{div} \vec{w} = 0$ by the observations preceding the statement of the Corollary. A difference $\vec{v}$ of any two solutions of the div-curl system satisfies $\operatorname{div} v = 0$, $\operatorname{curl} v = 0$, i.e., $\vec{v}$ is a monogenic constant. Since $\Omega$ is star-shaped and therefore simply connected, $\vec{v}$ is the gradient of a harmonic function $h$. □

**Corollary 46.** *The operator*

$$-L - I^{-1}\left[\frac{|\vec{x}|^2}{2}\operatorname{grad} T_{0,\Omega}\right] \tag{2.9}$$

*where* $L$ *is defined in* (1.19)*, is a right inverse for the double curl operator* curl curl *acting on the class of integrable functions in* $\operatorname{Sol}(\Omega, \mathbb{R}^3)$*.*

(Note that $I^\alpha$ for the exponent $\alpha = -1$ can be applied here because of the factor $|\vec{x}|^2$ in the operand.)

**Proof.** It is enough to show that the curl applied after the operator (2.9) produces the right inverse of curl given in (2.6). But Lemma 35 with (2.3) gives

$$\begin{aligned}
\operatorname{curl} I^{-1}\left[\frac{|\vec{x}|^2}{2}\operatorname{grad} T_{0,\Omega}\right] &= I^0\left[\operatorname{curl}\left(\frac{|\vec{x}|^2}{2}\operatorname{grad} T_{0,\Omega}\right)\right] \\
&= I^0\left[\operatorname{grad}_{\vec{x}}\frac{|\vec{x}|^2}{2}\times\operatorname{grad} T_{0,\Omega}\right] \\
&= I^0\left[\vec{x}\times\operatorname{grad} T_{0,\Omega}\right] \\
&= \overrightarrow{U}_\Omega[T_{0,\Omega}],
\end{aligned}$$

while by Proposition 18,

$$\operatorname{curl} L = -\overrightarrow{T}_{2,\Omega}.$$

Adding these equalities we have the result. □

Now we can proceed to solve the inhomogeneous div-curl system with data $g_0, \vec{g}$. Assume again that $\vec{g}$ is solenoidal. By Proposition 12, the function $v = T_\Omega[-g_0+\vec{g}]$ satisfies $Dv = -g_0+\vec{g}$ and therefore is a *quaternionic* solution to (2.1). We seek to construct a vector solution $\vec{w}$ by subtracting a monogenic function whose scalar part is precisely the scalar part of $v$. Thus the key consists in taking the $T_{0,\Omega}$ component of $T_\Omega[\vec{g}]$, and using Proposition 37 to construct the monogenic conjugate of the $\mathbb{R}$-valued function $T_{0,\Omega}[\vec{g}]$. This is accomplished in the following result.

**Theorem 47.** *Let $\Omega$ be a star-shaped open set. Let $g_0 \in C(\Omega, \mathbb{R})$ and $\vec{g} \in \mathrm{Sol}(\Omega, \mathbb{R}^3)$ be integrable. The general solution of the inhomogeneous div-curl system* (2.1) *is given by*

$$\vec{w} = -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]] + \nabla h, \tag{2.10}$$

*where $h \in \mathrm{Har}(\Omega, \mathbb{R})$ is arbitrary.*

**Proof.** Since $\operatorname{div} \vec{g} = 0$, Proposition 41 says that $T_{0,\Omega}[\vec{g}]$ is an $\mathbb{R}$-valued harmonic function, so Proposition 37 permits us to complete it to the monogenic function $T_{0,\Omega}[\vec{g}] + \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]$. By (1.22), the difference

$$T_\Omega[-g_0 + \vec{g}] - (T_{0,\Omega}[\vec{g}] + \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]) = -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]$$

is purely vectorial. By Proposition 12 and the fact that $D\nabla h = 0$,

$$D\vec{w} = D(T_\Omega[-g_0 + \vec{g}]) = -g_0 + \vec{g}.$$

So (2.1) is satisfied because of (1.7). As in the proof of Corollary 45, we obtain the general solution adding to $\vec{w}$ the gradient of an arbitrary harmonic function. □

Note that (2.10) gives a Helmholtz decomposition [66, p. 166] of the solution $\vec{w}$ as the sum of an irrotational part $-\overrightarrow{T}_{1,\Omega}[g_0]$ and a solenoidal part $\overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]$. However the decomposition is not unique as in [54, Th. 5.1.1], because we are not requiring any boundary conditions. The decomposition becomes clearer by writing (2.10) as follows

$$\vec{w}(\vec{x}) = -\nabla L[g_0] - \operatorname{curl}\left(L[\vec{g}] + I^{-1}\left[\frac{|\vec{x}|^2}{2}\nabla T_{0,\Omega}[\vec{g}]\right]\right).$$

Proposition 41, Theorem 43, and Corollaries 45 and 46 are valid even in the distributional sense (weak solenoidal functions in $L^p$), because the properties of the Teodorescu transform $T_\Omega$ (see Remark 42) as well as Proposition 37 are also valid in Sobolev spaces. Recall the weak definition of solenoidal functions (1.12). Therefore we have the weak version of Theorem 47:

**Theorem 48.** *Let $1 < p < \infty$. Let $g_0 \in L^p(\Omega, \mathbb{R})$ and $\vec{g} \in L^p(\Omega, \mathbb{R}^3)$, and suppose* $\operatorname{div} \vec{g} = 0$ *in the weak sense. Then* (2.10) *is a weak solution of the div-curl system* (2.1)*.*

We give an explicit example of the above solution of the div-curl system.

**Example 49.** Let $\Omega$ be the unit ball in $\mathbb{R}^3$ minus any ray emanating from the origin. Take $g_0 = 0$ and $\vec{g} = \vec{x}/|\vec{x}|^3$ in the div-curl system (2.1). Since $\Omega$ is star-shaped with respect to any of its points $\vec{a}$ opposite to the ray, we shift the origin to $\vec{a}$ in the formula (2.3) for $\overrightarrow{U}_\Omega$ (cf. [94]) as follows:

$$\overrightarrow{U}_\Omega[w_0](\vec{x}) = \mathrm{Vec}\int_0^1 -tDw_0((1-t)\vec{a}+t\vec{x})(\vec{x}-\vec{a})dt. \tag{2.11}$$

Since the removed ray is of zero measure, we may use the explicit formula for the Teodorescu transform of $\vec{g}$ for the unit ball given in [52, p. 324, formula 28], namely $T_{0,\Omega}[\vec{g}] = 1 - 1/|\vec{x}|$ while $\overrightarrow{T}_{1,\Omega}[g_0] = \overrightarrow{T}_{2,\Omega}[\vec{g}] = 0$. Since $\vec{g}$ is solenoidal, the div-curl solution of Theorem 47 is

$$\begin{aligned} \vec{w}(\vec{x}) &= -\overrightarrow{U}_\Omega\left[-\frac{1}{|\vec{x}|}+1\right] \\ &= \frac{\vec{a}\times\vec{x}}{|\vec{x}|(\vec{a}\cdot\vec{x}-|\vec{a}||\vec{x}|)}. \end{aligned} \tag{2.12}$$

Since $\operatorname{div}\vec{w} = 0$ and $\operatorname{curl}\vec{w} = \vec{x}/|\vec{x}|^3$ are independent of $\vec{a}$, the difference of two such solutions is an SI field, as would be expected.

**Remark 50.** Suppose now that two given scalar and vectorial functions $g_0$ and $\vec{g}$ are harmonic. Under this additional hypothesis (and of course $\operatorname{div}\vec{g} = 0$), a solution

$$\vec{w} = -\vec{x}\times I^1[\vec{g}] + \operatorname{grad}\left(\frac{|\vec{x}|^2}{4}I^{1/2}[g_0 - \vec{x}\cdot I^2[\operatorname{curl}\vec{g}]]\right) \tag{2.13}$$

for the div-curl system (2.1) was given by Yu. M. Grigor'ev in [49, Th. 3.2], where the integrals $I^\alpha$ were defined in (2.2). We relate this to our solution (2.10).

By additivity we may consider $g_0$ and $\vec{g}$ independently. Suppose $g_0 = 0$, and let $\vec{w}$ be given by (2.13). Substitute $\vec{g} = \operatorname{curl}\overrightarrow{T}_{2,\Omega}[\vec{g}] + \operatorname{grad} T_{0,\Omega}[\vec{g}]$

(Proposition 21) to obtain $\vec{w} = \vec{u} - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]$ where

$$\vec{u}(x) = -\vec{x} \times I^1[\mathrm{curl}\,\overrightarrow{T}_{2,\Omega}\vec{g}] + \mathrm{grad}\left(\frac{|\vec{x}|^2}{4} I^{1/2}[-\vec{x}\cdot I^2[\mathrm{curl}\,\mathrm{curl}\,\overrightarrow{T}_{2,\Omega}[\vec{g}]]]\right).$$

Here we have used the fact that $\mathrm{curl}\,\mathrm{curl}\,\overrightarrow{T}_{2,\Omega}[\vec{g}] = \mathrm{curl}[\vec{g} - \mathrm{grad}\,T_{0,\Omega}[\vec{g}]] = \mathrm{curl}\,\vec{g}$.

The function $\vec{v} = \overrightarrow{T}_{2,\Omega}[\vec{g}]$ trivially satisfies

$$\begin{aligned} \mathrm{div}\,\vec{v} &= 0, \\ \mathrm{curl}\,\vec{v} &= \mathrm{curl}\,\overrightarrow{T}_{2,\Omega}[\vec{g}], \end{aligned}$$

while by (2.13), $\vec{u}$ satisfies the same system. Thus $\vec{u}$ differs from $\overrightarrow{T}_{2,\Omega}[\vec{g}]$ by an SI field. Since $\overrightarrow{T}_{1,\Omega}[g_0] = 0$, we have $\vec{w} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]] + \nabla h$ for some harmonic $h$, which agrees with our solution (2.10).

For $\vec{g} = 0$, we simply note that both (2.10) and (2.13) are left inverses of div applied to $g_0$.

It can be checked that $\vec{g}$ in Example 49 is harmonic, and that when one shifts appropriately the base point of integration in (2.13), the same solution (2.12) is obtained. We omit the detailed calculation.

### 2.2.2 Div-curl system with boundary data

We will rewrite the right inverse for the curl (2.6) in terms of boundary value integral operators. The following result tells us that $T_{0,\Omega}$ is in some sense a boundary integral operator, as it can be expressed in terms of the single-layer operator $M$ of (1.20) when the boundary values of $\vec{w}$ are known. Here

$\eta$ again denotes the outward normal vector to the boundary $\partial\Omega$, which from here on will be assumed to be smooth, and $\Omega$ will be bounded.

**Proposition 51.** *For every* $\vec{w} \in \mathrm{Sol}(\overline{\Omega}, \mathbb{R}^3) := \mathrm{Sol}(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3)$,

$$T_{0,\Omega}[\vec{w}] = M[\gamma_{\mathbf{n}}(\vec{w})],$$

*where* $\gamma_{\mathbf{n}}$ *is the normal trace defined in* (1.32).

**Proof.** Using the equality

$$\nabla_{\vec{x}}\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right) = -\frac{\vec{x}-\vec{y}}{4\pi|\vec{x}-\vec{y}|^3} = -\nabla_{\vec{y}}\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right),$$

we find that

$$\begin{aligned} T_{0,\Omega}[\vec{w}](\vec{x}) &= \int_\Omega \frac{\vec{x}-\vec{y}}{4\pi|\vec{x}-\vec{y}|^3}\cdot \vec{w}(\vec{y})\, d\vec{y} \\ &= \int_\Omega \nabla_{\vec{y}}\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right)\cdot \vec{w}(\vec{y})\, d\vec{y} \\ &= \int_\Omega \nabla_{\vec{y}}\cdot\left(\frac{\vec{w}(\vec{y})}{4\pi|\vec{x}-\vec{y}|}\right) d\vec{y}. \end{aligned}$$

since $\vec{w} \in \mathrm{Sol}(\Omega, \mathbb{R}^3)$. By the Divergence Theorem,

$$T_{0,\Omega}[\vec{w}](\vec{x}) = \int_{\partial\Omega} \frac{\vec{w}(\vec{y})}{4\pi|\vec{x}-\vec{y}|}\cdot \eta(\vec{y})\, ds_{\vec{y}}$$

as desired. □

Analogously to Proposition 51, the value of the expression $\overrightarrow{T}_{2,\Omega}[\vec{w}]$ can be recovered when we know the boundary values of functions $\vec{w}$ in $\mathrm{Irr}(\overline{\Omega}, \mathbb{R}^3) := \mathrm{Irr}(\Omega, \mathbb{R}^3) \cap C(\overline{\Omega}, \mathbb{R}^3)$:

**Proposition 52.** *For every* $\vec{w} \in \mathrm{Irr}(\overline{\Omega}, \mathbb{R}^3)$*,*

$$\overrightarrow{T}_{2,\Omega}[\vec{w}] = -M[\gamma_{\mathbf{t}}(\vec{w})],$$

*where* $\gamma_{\mathbf{t}}$ *is the tangential trace defined in* (1.32)*.*

**Proof.**

$$\begin{aligned}
\overrightarrow{T}_{2,\Omega}[\vec{w}](\vec{x}) &= \int_\Omega -\frac{\vec{x}-\vec{y}}{4\pi|\vec{x}-\vec{y}|^3} \times \vec{w}(\vec{y})\, d\vec{y} \\
&= -\int_\Omega \nabla_{\vec{y}} \left( \frac{1}{4\pi|\vec{x}-\vec{y}|} \right) \times \vec{w}(\vec{y})\, d\vec{y} \\
&= -\int_\Omega \nabla_{\vec{y}} \times \left( \frac{\vec{w}(\vec{y})}{4\pi|\vec{x}-\vec{y}|} \right) d\vec{y} + \int_\Omega \frac{\nabla_{\vec{y}} \times \vec{w}(\vec{y})}{4\pi|\vec{x}-\vec{y}|}\, d\vec{y}. \qquad (2.14)
\end{aligned}$$

Apply $\vec{w} \in \mathrm{Irr}(\overline{\Omega}, \mathbb{R}^3)$ and Stokes' theorem to obtain the desired result. □

**Remark 53.** If $\Omega$ is a bounded Lipschitz domain, then it is enough to consider $\vec{w}$ in $\mathrm{Sol}(\Omega, \mathbb{R}^3) \cap W^{1,p}(\Omega, \mathbb{R}^3)$ and $\mathrm{Irr}(\Omega, \mathbb{R}^3) \cap W^{1,p}(\Omega, \mathbb{R}^3)$ in Propositions 51 and 52, respectively.

We will write $\mathrm{SI}(\partial\Omega)$ for the space of boundary values of SI vector fields in $\overline{\Omega}$, which we recall from (1.11) are the purely vectorial monogenic constants. In other words, $\mathrm{SI}(\partial\Omega) = \{\vec{\varphi} \in C(\partial\Omega, \mathbb{R}^3) \colon S_{\partial\Omega}[\vec{\varphi}] = \vec{\varphi}\}$; recall the Plemelj-Sokhotski formulas of Theorem 14. By Proposition 32, we have that SI vector fields are harmonic, the extension of $\vec{\varphi} \in \mathrm{SI}(\partial\Omega)$ to the interior is unique by the Maximum Principle for harmonic functions [9]. We rewrite the right inverse of curl given in Theorem 43 as a boundary integral operator, under the condition that boundary data $\vec{\varphi}$ has an irrotational and solenoidal extension:

**Proposition 54.** *Let* $\vec{\varphi} \in \mathrm{SI}(\partial\Omega)$. *Define*

$$\vec{w} = -M[\vec{\varphi} \times \eta] - \overrightarrow{U}_{\Omega}[M[\vec{\varphi} \cdot \eta]] \tag{2.15}$$

*where again* $\eta$ *is the outward normal. Then*

$$\operatorname{div} \vec{w} = 0, \quad \operatorname{curl} \vec{w} = \vec{g},$$

*where* $\vec{g}$ *is the SI-vector field extension* $\vec{g}$ *of* $\vec{\varphi}$.

**Proof.** By Propositions 51 and 52, $T_{0,\Omega}[\vec{g}] = M[\vec{\varphi} \cdot \eta]$ and $\overrightarrow{T}_{2,\Omega}[\vec{g}] = -M[\vec{\varphi} \times \eta]$, respectively. The statement now follows from the Corollary 45. □

Let $\vec{\varphi} \in \mathrm{SI}(\partial\Omega)$. Since $\Delta = \operatorname{grad} \operatorname{div} - \operatorname{curl} \operatorname{curl}$, we have $\Delta\vec{w} = -\operatorname{curl} \vec{g}$, so the vector field (2.15) solves the following Dirichlet-type problem

$$\Delta\vec{w} = 0,$$
$$\operatorname{curl} \vec{w}|_{\partial\Omega} = \vec{\varphi}.$$

## 2.3 Div-curl system in Lipschitz domains

Note that Theorems 47 and 48 use the operator $\overrightarrow{U}_{\Omega}$, which can only be defined in star-shaped domains. Here we remove this restriction.

For this construction we will assume that $\mathbb{R}^3 \setminus \Omega$ is connected. Let $g_0 \in L^p(\Omega, \mathbb{R})$ and $\vec{g} \in L^p(\Omega, \mathbb{R}^3)$. The div-curl system is

$$\operatorname{div} \vec{w} = g_0, \quad \operatorname{curl} \vec{w} = \vec{g}. \tag{2.16}$$

Note that $\vec{g}$ is required to be weakly solenoidal,

$$\int_\Omega \vec{g}\cdot\nabla v_0\,d\vec{x} = 0$$

for all test functions $v_0 \in W_0^{1,q}(\Omega,\mathbb{R})$, in order for there to exist solutions to the second equation. Since $T_\Omega[\vec{g}] \in W^{1,p}(\Omega,\mathbb{H})$, the scalar function and the vector field

$$\alpha_0 = \operatorname{tr} T_{0,\Omega}[\vec{g}],\ \ \vec{\alpha} = \operatorname{tr}\,\overrightarrow{T}_{2,\Omega}[\vec{g}], \tag{2.17}$$

are well-defined and $\alpha = \alpha_0 + \vec{\alpha} = \operatorname{tr} T_\Omega[\vec{g}] \in W^{1-1/p,p}(\partial\Omega,\mathbb{H})$.

We now remove the restriction of starshapedness, presenting a solution of (2.16) for bounded Lipschitz domains with weaker topological constraints (for example, a solid torus will be admissible). This more general div-curl solution is expressed in terms of the inverse of $I + K_0$, as well as the Teodorescu transform $T_\Omega$ (1.21) and the Cauchy operator $F_{\partial\Omega}$ (1.25).

The hypothesis on $\partial\Omega$ in Theorem 55 is to guarantee that the operator $I + K_0$ is invertible in $L^p(\partial\Omega,\mathbb{R})$; it uses the value of $\epsilon(\Omega)$ which depends only of the Lipschitz character of $\partial\Omega$, as discussed in Proposition 24.

**Theorem 55.** *Let $\Omega$ be a bounded $C^{1,\gamma}$ Lipschitz domain with $\gamma > 0$ and $1 < p < \infty$, or a bounded Lipschitz domain for $2-\epsilon(\Omega) < p < \infty$. Suppose that $\mathbb{R}^3 \setminus \Omega$ is connected. Let $g_0 \in L^p(\Omega,\mathbb{R})$, $\vec{g} \in L^p(\Omega,\mathbb{R}^3)$ and suppose that* $\operatorname{div}\vec{g} = 0$ *in the weak sense. Then a weak solution $\vec{w}$ of the div-curl system*

(2.16) *is given by*

$$\begin{aligned}\vec{w} &= T_\Omega[-g_0 + \vec{g}] - F_{\partial\Omega}[2(I + K_0)^{-1}\alpha_0] \\ &= -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{F}_{1,\partial\Omega}[2(I + K_0)^{-1}\alpha_0],\end{aligned} \tag{2.18}$$

*where $\alpha_0$ was defined in* (2.17). *This solution is unique up to adding an arbitrary monogenic constant. Moreover, $\vec{w} \in W^{1,p}(\Omega, \mathbb{R}^3)$ when $1 < p < 2 + \epsilon(\Omega)$.*

**Proof.** Note that $h_0 = 2(I+K_0)^{-1}\alpha_0 \in L^p(\partial\Omega, \mathbb{R})$ and further $h_0 \in W^{1-1/p,p}(\partial\Omega, \mathbb{R})$ when $1 < p < 2 + \epsilon(\Omega)$ by Proposition 24(b). By the Plemelj-Sokhotski formula (1.18) and the decomposition $S_{\partial\Omega} = K_0 + \overrightarrow{K}$ (1.27), we have

$$\begin{aligned}\operatorname{tr}_+ F_{\partial\Omega}[h_0] &= \frac{1}{2}(h_0 + S_{\partial\Omega}[h_0]) \\ &= \frac{1}{2}(I + K_0)[h_0] + \frac{1}{2}\overrightarrow{K}[h_0] \\ &= \alpha_0 + \overrightarrow{K}(I + K_0)^{-1}[\alpha_0].\end{aligned}$$

By Proposition 41, $T_{0,\Omega}[\vec{g}]$ is harmonic, therefore $F_{\partial\Omega}[h_0] = T_{0,\Omega}[\vec{g}] + \overrightarrow{F}_{1,\partial\Omega}[h_0]$ is monogenic. Following the same argument used in the proof of Theorem 47, we see that

$$D\vec{w} = DT_\Omega[-g_0 + \vec{g}] - D(T_{0,\Omega}[\vec{g}] + \overrightarrow{F}_{1,\partial\Omega}[h_0]) = -g_0 + \vec{g}.$$

The fact that $\vec{w}$ belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^3)$ is a direct consequence of Theorem 17 and (1.24). □

The following result gives us an alternative way to complete a scalar-valued harmonic function to a monogenic function, similarly to the way the radial integration operator $\overrightarrow{U}_\Omega$ (2.3) did this for star-shaped domains in Proposition 37. It can be considered an "interior" version of the construction of the Hilbert transform $\mathcal{H}$ (see Section 4.1 below), in other words, a method to construct harmonic conjugates in Lipschitz domains of $\mathbb{R}^3$. See also the classical generalization of harmonic conjugates using SI-vector fields in the upper half space of $\mathbb{R}^n$ [91]. In this sense we can state the following

**Corollary 56.** *Let $\Omega$ be as in Theorem 55. Let $w_0 \in W^{1,p}(\Omega,\mathbb{R})$ be a scalar harmonic function. Let*

$$\vec{w} = \overrightarrow{F}_{1,\partial\Omega}[2(I+K_0)^{-1}\operatorname{tr} w_0].$$

*Then $w_0 + \vec{w}$ is monogenic in $\Omega$.*

**Corollary 57.** *Let $\Omega$ be as in Theorem 55. The following is a right inverse of* curl*:*

$$\vec{g} \mapsto \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{F}_{1,\partial\Omega}[2(I+K_0)^{-1}\alpha_0], \tag{2.19}$$

*acting on all $\vec{g} \in L^p(\Omega,\mathbb{R}^3)$ in the class of weakly divergence free vector fields.*

Since the right inverse of curl (2.19) acts as

$$\overrightarrow{T}_{2,\Omega} - 2\overrightarrow{F}_{1,\partial\Omega}(I+K_0)^{-1}\operatorname{tr} T_{0,\Omega}\colon \operatorname{Sol}(\Omega,\mathbb{R}^3) \to \operatorname{Sol}(\Omega,\mathbb{R}^3),$$

we have

**Corollary 58.** *Let $\Omega$ be as in Theorem 55. The following is a right inverse for the double curl operator:*

$$\vec{g} \mapsto -L[\vec{g}] + M[2(I + K_0)^{-1}[\alpha_0 \eta]],$$

*for every $\vec{g} \in L^p(\Omega, \mathbb{R}^3)$ in the class of weakly divergence free vector fields, where $\eta$ the outward pointing normal vector to $\partial\Omega$.*

**Proof.** This is a direct consequence of (1.26) and the fact that $\overrightarrow{T}_{2,\Omega}[\vec{g}] = -\operatorname{curl} L[\vec{g}]$ (see Proposition 18). □

The following result tell us that the right inverse operator of curl (2.19) can be expressed as a surface integral operator, when it acts over weak SI-vector fields whose trace is well defined:

**Corollary 59.** *Let $\Omega$ be as in Theorem 55. Let $\vec{\varphi}$ be the boundary values of the weakly SI-vector field $\vec{g} \in W^{1,p}(\Omega, \mathbb{R}^3)$. Define*

$$\vec{w} = -M[\vec{\varphi} \times \eta] - \overrightarrow{F}_{1,\partial\Omega}[2(I + K_0)^{-1} \operatorname{tr} M[\vec{\varphi} \cdot \eta]] \tag{2.20}$$

*where again $\eta$ is the outward normal. Then*

$$\operatorname{div} \vec{w} = 0, \quad \operatorname{curl} \vec{w} = \vec{g}.$$

**Proof.** The proof is similar to Proposition 54 with the corresponding right inverse of curl in the Lipschitz domain given by (2.19). □

**Remark 60.** It would be of great interest to remove the restriction that $\mathbb{R}^3 \backslash \Omega$ be connected in Theorem 55. This would require removing that restriction in Proposition 24, which is an active area of investigation sometimes referred to the technique of layer potentials [34].

# Chapter 3

# Application to diverse systems of differential equations

The general solution of the div-curl system (2.1) given in Chapter 2 allows us to answer questions related to the quaternionic main Vekua equation $DW = (Df/f)\overline{W}$ in $\mathbb{R}^3$, such as finding the vector part when the scalar part is known. In addition, using the general solution to the div-curl system and the known existence of the solution of the inhomogeneous conductivity equation, we prove the existence of solutions of the inhomogeneous double curl equation, and give an explicit solution for the case of static Maxwell's equations with a variable permeability.

## 3.1 Application to the three-dimensional main Vekua equation

We will study a special Vekua equation, which in [72] is called the main Vekua equation. We are interested in the natural generalization of this equation to the quaternionic case [72, Ch. 16], which possesses properties similar to those of the complex Vekua equation, including an intimate relation with the conductivity equation. The conductivity equation appears in many aspects of physics, and gives rise to inverse problems with applications to fields such as tomography. Here we apply the results obtained on the div-curl system to study solutions of these equations.

### 3.1.1 The main Vekua equation and equivalent formulations

**Definition 61.** The *main Vekua equation* is

$$DW = \frac{Df}{f}\overline{W}, \tag{3.1}$$

with $D$ the Moisil-Teodorescu operator given in (1.6), and $f$ a nonvanishing scalar valued function, sufficiently smooth to speak of $Df$ (made more precise in different contexts below).

The space of measurable weak solutions of the main Vekua equation (3.1) is denoted by

$$\mathfrak{M}_f(\Omega) = \left\{ W\colon \Omega \to \mathbb{H} \middle| \ \left( D - \frac{Df}{f} C_{\mathbb{H}} \right) W = 0 \right\}, \tag{3.2}$$

where $C_{\mathbb{H}} W = \overline{W}$ is the operator of quaternionic conjugation. This is a nontrivial linear subspace over $\mathbb{R}$.

The main Vekua equation is closely related to many other important differential equations. In [72, Chapter 16] we find results that relate solutions of the main Vekua equation to solutions of other differential equations and illustrate the analogies that exist with the theory for pseudoanalytic functions. For example,

$$F_0 = f, \quad F_1 = \frac{e_1}{f}, \quad F_2 = \frac{e_2}{f}, \quad F_3 = \frac{e_3}{f},$$

form a generating quartet for (3.1) in the terminology of [72]. That is, $F_i$ ($i = 0, 1, 2, 3$), are solutions of (3.1) and there exist scalar functions $G_i$ such that $W = \sum_{i=0}^{3} G_i F_i$. Moreover, $W$ satisfies (3.1) if and only if

$$\sum_{i=0}^{3} (DG_i) F_i = 0, \tag{3.3}$$

which is analogous to the Bers equation for pseudoanalytic functions of second kind [16, 72].

In particular, (3.1) is related to the $\mathbb{R}$-linear Beltrami equation (also called quaternionic Beltrami equation), as follows

$$DG = \mu D\overline{G}, \tag{3.4}$$

where $G = \sum_{i=0}^{3} G_i e_i = W_0/f + f\vec{W}$ and $\mu = (1-f^2)/(1+f^2)$. Recall that the above relation was essential to solve the Calderón problem in the complex case [5]. Other Bers generating functions for different first-order systems of mathematical physics were given in [73].

The main Vekua equation has an equivalent expression as a homogeneous div-curl system, which will be of interest to us throughout this work.

**Lemma 62.** *[72, Th. 161]* $W \in \mathfrak{M}_f(\Omega)$ *if and only if the scalar part* $W_0$ *and the vector part* $\vec{W}$ *satisfy the homogeneous div-curl system:*

$$\begin{aligned} \operatorname{div}(f\vec{W}) &= 0, \\ \operatorname{curl}(f\vec{W}) &= -f^2 \nabla \left( \frac{W_0}{f} \right). \end{aligned} \tag{3.5}$$

When $f$ is constant, this system reduces to (1.10) and $\mathfrak{M}_{f\equiv 1}(\Omega) = \mathfrak{M}(\Omega)$, the classical space of left-monogenic functions. Thus it is natural to wish to generalize results concerning monogenic functions to solutions of the main Vekua equation. This is one of the main goals of what can be called "Vekua analysis". We will give some results in this direction in this Section.

Suppose that $W \in C^2(\Omega, \mathbb{H})$. From the second equation of (3.5) we obtain by applying div, curl that [72, Th. 161]

$$\nabla \cdot f^2 \, \nabla \left( \frac{W_0}{f} \right) = 0, \tag{3.6}$$

$$\operatorname{curl}(f^{-2} \operatorname{curl}(f\vec{W})) = 0. \tag{3.7}$$

The first equation is the so-called *conductivity equation* and the second one is called the *double curl-type equation* for the conductivity $f^2$. These equations are satisfied separately by the scalar and vector parts of $W$ analogously to the way that two harmonic conjugates satisfy separately the Laplace equation; together they are not sufficient for $W_0 + \vec{W}$ to satisfy (3.5).

The conductivity equation (3.6) is equivalent to the Schrödinger equation

$$\Delta W_0 - \frac{\Delta f}{f} W_0 = 0.$$

Using the identity $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$ and the div-curl system (3.5), therefore an expression for the Laplacian of the vector part $\vec{W}$ is

$$\Delta \vec{W} + \frac{\Delta f}{f} \vec{W} = 2\nabla f \times \nabla W_0 + 2\left(\frac{\nabla f}{f} \cdot \nabla\right)\vec{W}.$$

Using (1.8), (3.5), and the fact that $f\vec{W}$ is vectorial, we have the following equivalence

$$DW = \frac{Df}{f}\overline{W} \;\Leftrightarrow\; D(f\vec{W}) = -f^2 \nabla\left(\frac{W_0}{f}\right). \tag{3.8}$$

**Definition 63.** For brevity we will say that $f^2$ is a *conductivity* when $f$ is a non-vanishing $\mathbb{R}$-valued function in the domain under consideration. The conductivity will be called *proper* when $f$ and $1/f$ are bounded. In other words, $f$ is proper when $\rho(f) := \sup(|f|, 1/|f|)$ is finite.

### 3.1.2 Completion of Vekua solutions from partial data

It is important to know what type of functions can be solutions to some main Vekua equation (i.e., for some $f$). Another question is how to complete an $f^2$-hyperconjugate pair, i.e. to recover the vector part $\vec{W}$ such that $W =$

$W_0 + \vec{W} \in \mathfrak{M}_f(\Omega)$ when the scalar part $W_0 \colon \Omega \to \mathbb{R}$ is known, or vice versa. We apply the results of Section 2.2 to these questions. First we treat the generalization of Proposition 37 for nonconstant conductivity.

**Definition 64.** $\vec{W}$ is an $f^2$-*hyperconjugate* for $W_0$ when $W_0+\vec{W}$ is a solution of (3.1).

**Theorem 65.** *Let $f^2$ be a conductivity of class $C^2$ in an open star-shaped set $\Omega \subseteq \mathbb{R}^3$. Suppose that $W_0 \in C^2(\Omega, \mathbb{R})$ satisfies the conductivity equation* (3.6) *in $\Omega$. Then the vector field $\vec{W}$ given by*

$$f\vec{W} = \vec{T}_{2,\Omega}\left[-f^2\nabla\left(\frac{W_0}{f}\right)\right] + \vec{U}_\Omega\left[T_{0,\Omega}\left[f^2\nabla\left(\frac{W_0}{f}\right)\right]\right] + \nabla h, \tag{3.9}$$

*is the most general $f^2$-hyperconjugate for $W_0$. Here $\nabla h$ is a purely vectorial additive monogenic constant, i.e., the gradient of a real harmonic function $h$.*

**Proof.** Observe that (3.5) is a homogeneous div-curl system (2.1) in the unknown $\vec{w} = f\vec{W}$, with $g_0 = 0$ and $\vec{g} = -f^2\nabla(W_0/f) \in \mathrm{Sol}(\Omega, \mathbb{R}^3)$, since by hypothesis $W_0$ satisfies (3.6). By Corollary 45, the general solution $\vec{w}$ is given by (3.9). By Lemma 62, $W = W_0 + \vec{W} \in \mathfrak{M}_f(\Omega)$. Moreover, $W \in \mathfrak{M}_f(\Omega) \cap C^2(\Omega, \mathbb{H})$. □

The differentiability assumptions of Theorem 65 can be relaxed as follows if one only requires a weak solution.

**Corollary 66.** *Let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity and let $W_0 \in W^{1,2}(\Omega, \mathbb{R})$ be a weak solution of* (3.6). *Then $\vec{W} \in W^{1,2}(\Omega, \mathbb{R}^3)$ given by* (3.9) *produces a weak solution $W_0 + \vec{W}$ of the main Vekua equation* (3.1).

Analogously to the well-known $\overline{\partial}$-problem in the complex case, the completion of the vector part of solutions of the main Vekua equation is given in terms of the integral operator $T_\Omega$. In [82] there is a generalization for the quaternionic case; however, the solution given there is not purely vectorial.

We now characterize the elements of the space $\operatorname{Vec}\mathfrak{M}_f(\Omega)$ of vector parts of solutions to the main Vekua equation, that is the $f^2$-hyperconjugate pair.

**Proposition 67.** *[74, Th. 10] Let $\vec{W} \in C^2(\Omega, \mathbb{R}^3)$ where $\Omega$ is a simply connected domain in $\mathbb{R}^3$. For the existence of $W \in \mathfrak{M}_f(\Omega)$ such that $\operatorname{Vec} W = \vec{W}$ it is necessary and sufficient that $\operatorname{div}(f\vec{W}) = 0$ together with the double curl-type equation* (3.7)*.*

**Proof.** The necessity is given by (3.5). For the sufficiency, the second condition implies that $f^{-2}\operatorname{curl}(f\vec{W})$ admits a potential $W_0$ obtained by applying $\mathcal{A}$ of (2.5). The function $W = W_0 + \vec{W}$ then satisfies (3.5) and hence also (3.1). □

### 3.1.3 Vekua boundary value problems

Now we consider boundary conditions on the Vekua equation. The following fact is essential to the solution of the Calderón problem in the plane [5]; see also [58, Th. 4.1] and the references therein, and a sketch of a proof in $\mathbb{R}^n$ in [6, p. 407]. Another reference for continuous proper conductivities is [80, p. 197, Th. 10]. The following conductivity problem reduces to the Dirichlet problem in the case when $f$ is constant, because $\nabla \cdot \nabla = \Delta$.

**Proposition 68.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with connected complement and $f^2$ a measurable proper conductivity in $\Omega$. Let $g_0 \in L^2(\Omega, \mathbb{R})$, given* *prescribed boundary values* $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$*, there exists a unique solution* $u_0 \in W^{1,2}(\Omega, \mathbb{R})$ *to the conductivity boundary value problem*

$$\begin{aligned} \nabla \cdot f^2 \, \nabla u_0 &= g_0, \\ u_0|_{\partial\Omega} &= \varphi_0. \end{aligned} \tag{3.10}$$

The methods of variational calculus applied in Section 3.3.2 can be used to obtain the existence of solutions of second-order elliptic equations such as this one.

**Theorem 69.** *Let* $f^2$ *be a proper conductivity in the bounded, star-shaped open set* $\Omega \subseteq \mathbb{R}^3$*,* $f \in W^{1,2}(\Omega, \mathbb{R})$ *and suppose that* $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$*. Then there exists a function* $W\colon \Omega \to \mathbb{H}$ *that satisfies the main Vekua equation* (3.1) *weakly and has boundary values* $\operatorname{Sc} W|_{\partial\Omega} = \varphi_0$*.*

**Proof.** Proposition 68 gives a solution $u_0 \in W^{1,2}(\Omega, \mathbb{R})$ of $\nabla \cdot f^2 \, \nabla u_0 = 0$ with boundary values $u_0|_{\partial\Omega} = \varphi_0/f \in H^{1/2}(\partial\Omega, \mathbb{R})$. The function $W_0 = f u_0$ satisfies the conditions of Theorem 65 and therefore has a completion $W_0 + \vec{W}$ satisfying the Vekua equation weakly. □

**Remark 70.** Theorem 69 provides a way to define a "Hilbert transform"

$$\mathcal{H}_f \colon H^{1/2}(\partial\Omega, \mathbb{R}) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$$

associated to the main Vekua equation (3.1) in star-shaped domains in $\mathbb{R}^3$, by

$$\mathcal{H}_f[\varphi_0] = \vec{W}|_{\partial\Omega},$$

where $\vec{W}$ is given by (3.9). However, we will introduce another construction in Subsection 4.2.1 for Lipschitz domains in $\mathbb{R}^3$, which generalizes a known Hilbert transform for monogenic functions in $\mathbb{R}^n$.

## 3.2 A result on Vekua-type operators

As was noted at the beginning of the Introduction, there are other operators quite similar to the Vekua operator which appear in factorizations of second order operators (2)–(3). For example, a particular case is the operator $D+\alpha$, with $\alpha$ constant, and its related $\alpha$-Teodorescu operator studied in [67] in $\mathbb{R}^3$ for different types of $\alpha$ and [101] in $\mathbb{R}^n$ with $\alpha$ a complex number.

We write $D_{\mathrm{r}}[w] = wD$ for the right-sided operator of (1.8). In [74], certain relations were established among the four operators

$$\begin{aligned} \mathbf{V} &= D - \frac{Df}{f} C_{\mathbb{H}}, \quad \overline{\mathbf{V}} = D_{\mathrm{r}} - M^{\frac{Df}{f}} C_{\mathbb{H}}, \\ \mathbf{V}_1 &= D_{\mathrm{r}} + \frac{Df}{f}, \quad \overline{\mathbf{V}}_1 = D + M^{\frac{Df}{f}}. \end{aligned} \tag{3.11}$$

where $M^{(\cdot)}$ denotes right multiplication.

**Definition 71.** [74] Let $W$ be a solution of the main Vekua equation (3.1), that is $\mathbf{V}W = 0$. Then the function $\overline{\mathbf{V}}W$ is called the "Bers derivative" of $W$.

In the following result we give a right inverse of the operator $\overline{\mathbf{V}}$ on a subspace analogous to the condition that $\vec{g} \in \mathrm{Sol}(\Omega, \mathbb{R}^3)$ for (2.1).

**Theorem 72.** *Let $f^2 \in C^1(\Omega, \mathbb{R})$ be a conductivity in the star-shaped domain $\Omega \subseteq \mathbb{R}^3$. Let $\vec{G} \in C^1(\Omega, \mathbb{R}^3)$ be a purely vectorial solution of the equation*

$\overline{\mathbf{V}}_1\vec{G} = 0$. *Then the general solution of the system* $\mathbf{V}W = 0$ *and* $\overline{\mathbf{V}}W = \vec{G}$ *in* $\Omega$ *is given by*

$$W = \frac{1}{2}\left( f\,\mathcal{A}\left[\frac{\vec{G}}{f}\right] - \frac{1}{f}\overrightarrow{T}_{2,\Omega}[f\vec{G}] + \frac{1}{f}\overrightarrow{U}_{\Omega}[T_{0,\Omega}[f\vec{G}]] + \frac{\nabla h}{f}\right),$$

*where* $h \in \mathrm{Har}(\Omega, \mathbb{R})$ *is arbitrary.*

**Proof.** The hypothesis $(D+M^{\frac{Df}{f}})\vec{G} = 0$ implies $\mathrm{div}(f\vec{G}) = 0$ and $\mathrm{curl}(\vec{G}/f) = 0$. Therefore we can apply the operator $\mathcal{A}$ of (2.5) to define $2W_0 = f\,\mathcal{A}[\vec{G}/f]$, which satisfies $\nabla(W_0/f) = \vec{G}/(2f)$. Thus

$$\mathrm{div}(f^2\nabla(W_0/f)) = \frac{1}{2}\,\mathrm{div}(f\vec{G}) = 0.$$

This justifies the application of Theorem 65, and we may define $\vec{W}$ via (3.9), so that $W = W_0 + \vec{W}$ satisfies (3.1).

It remains to verify that $\overline{\mathbf{V}}W = \vec{G}$. To do this we use the equivalent system (3.5) for the main Vekua equation:

$$\begin{aligned}
\overline{\mathbf{V}}W &= D_{\mathrm{r}}W - \overline{W}\frac{Df}{f} \\
&= -\frac{1}{f}\,\mathrm{div}(f\vec{W}) + f\nabla(W_0/f) - \frac{1}{f}\,\mathrm{curl}(f\vec{W}) \\
&= 2f\nabla(W_0/f),
\end{aligned}$$

as desired. □

A formula similar to that of Theorem 72 was given in [74, Theorem 9] but with another expression in place of $\overrightarrow{T}_{2,\Omega} - \overrightarrow{U}_{\Omega}T_{0,\Omega}$ for the inverse of curl, not applicable for bounded domains, since the Helmholtz potentials (5) given in the Introduction are valid only for $\Omega = \mathbb{R}^3$. Otherwise our proof is essentially the same.

## 3.3 Equation of double curl type

The following system of equations corresponds to the static Maxwell system, in a medium when just the permeability $f^2$ is variable ([69, Ch. 4] or [20, Ch. 2]):

$$\begin{aligned} \operatorname{div}(f^2\vec{H}) &= 0, \\ \operatorname{div}\vec{E} &= 0, \\ \operatorname{curl}\vec{H} &= \vec{g}, \\ \operatorname{curl}\vec{E} &= f^2\vec{H}. \end{aligned} \tag{3.12}$$

Here $\vec{E}$ and $\vec{H}$ represent electric and magnetic fields, respectively. We will apply our results to this system and to the double curl-type equation

$$\operatorname{curl}(f^{-2}\operatorname{curl}\vec{E}) = \vec{g}, \tag{3.13}$$

which is immediate from the last two equations of (3.12).

### 3.3.1 Generalized solutions of the Maxwell system

To obtain a general solution of (3.12) we will use the existence of solutions of the inhomogeneous conductivity problem (3.10).

The right inverse of the curl given by Theorem 43 permits us to invert the composed operator curl $f^{-2}$ curl, providing of course that this right inverse is applied to weakly solenoidal fields. The pair of fields $(\vec{E}, \vec{H})$ in the following result is constructed explicitly in terms of the operators defined in this paper.

**Theorem 73.** *Let the domain $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set, and assume that $f^2 \in W^{1,2}(\Omega, \mathbb{R})$ is a measurable proper conductivity in $\Omega$. Let $\vec{g} \in L^2(\Omega, \mathbb{R}^3)$ satisfy $\operatorname{div} \vec{g} = 0$. Then there exists a generalized solution $(\vec{E}, \vec{H})$ to the system* (3.12) *and its general form is given by*

$$\begin{aligned} \vec{E} &= \overrightarrow{T}_{2,\Omega}[f^2(\vec{B} + \nabla h)] - \overrightarrow{U}_{\Omega}[T_{0,\Omega}[f^2(\vec{B} + \nabla h)]] + \nabla h_1, \\ \vec{H} &= \vec{B} + \nabla h, \end{aligned} \tag{3.14}$$

*where $\vec{B} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_{\Omega}[T_{0,\Omega}[\vec{g}]]$, $h$ is solution of the conductivity equation $\operatorname{div}(f^2(\vec{B} + \nabla h)) = 0$ and $h_1$ is an arbitrary real valued harmonic function.*

**Proof.** Since $\operatorname{div} \vec{g} = 0$, by Corollary 45 the vector field

$$\vec{B} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{U}_{\Omega}[T_{0,\Omega}[\vec{g}]]$$

satisfies $\operatorname{curl} \vec{B} = \vec{g}$ and $\operatorname{div} \vec{B} = 0$ weakly. To solve

$$\operatorname{curl} \vec{E} = f^2(\vec{B} + \nabla h), \tag{3.15}$$

we must find an $\mathbb{R}-$valued function $h$ such that

$$\operatorname{div}(f^2(\vec{B} + \nabla h)) = 0.$$

Since $\operatorname{div}(f^2 \vec{B}) = \nabla f^2 \cdot \vec{B}$, we need to solve the inhomogeneous conductivity equation

$$\operatorname{div}(f^2\nabla h) = -\nabla f^2 \cdot \vec{B}, \tag{3.16}$$

It is no loss of generality to take the boundary condition $h|_{\partial\Omega} = 0$ in (3.6). By Proposition 68, this determines a unique generalized solution of (3.16) provided that $\nabla f^2 \cdot \vec{B} \in L^2(\Omega, \mathbb{R})$. But since $T_\Omega \colon L^2(\Omega, \mathbb{H}) \to W^{1,2}(\Omega, \mathbb{H})$ is bounded [53, Theorem 8.4], in fact $T_{0,\Omega}[\vec{g}] \in W^{1,2}(\Omega, \mathbb{R})$ and $\overrightarrow{T}_{2,\Omega}[\vec{g}] \in W^{1,2}(\Omega, \mathbb{R}^3)$. Combining with the fact that $\overrightarrow{U}_\Omega[T_{0,\Omega}[\vec{g}]]$ is harmonic by Proposition 37, we have $\vec{B} \in L^2(\Omega, \mathbb{R}^3)$. Thus, the hypothesis is fulfilled, and the desired $h$ exists. Applying the right inverse of curl to (3.15) we have the solution (3.14) where $h_1$ is an arbitrary harmonic function. Then $\operatorname{div}\vec{E} = 0$ by Corollary 45, and the remaining equations of (3.12) are then verified taking $\vec{H} = \vec{B} + \nabla h$. □

### 3.3.2 Variational methods for double curl boundary value problems

In this section we will prove that given $\vec{\varphi} \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ there exists an extension to the interior of $\Omega$ satisfying the double curl-type equation (3.7). Let $f^2 \in W^{1,2}(\Omega, \mathbb{R})$ be a measurable proper conductivity. We define the nonlinear functional $\varepsilon = \varepsilon_f \colon W^{1,2}(\Omega, \mathbb{R}^3) \to \mathbb{R}$ by

$$\varepsilon[\vec{W}] = \int_\Omega f^{-2} \operatorname{curl}\vec{W} \cdot \operatorname{curl}\vec{W}\, d\vec{y}. \tag{3.17}$$

We are interested in proving that for fixed $\vec{\varphi} \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, there exists at least one element that minimizes $\varepsilon$; we will use the results of the variational calculus which can be found, for example, in [33, Ch. 3].

Let $X$ be a reflexive Banach space and let $I\colon X \to \mathbb{R}$. We say that $I$ is weakly lower semicontinuous (w.l.s.) if $\liminf_{k\to\infty} I(u_k) \geq I(u)$ whenever $u_k \to u$ weakly in $X$. A functional $I$ is called coercive when there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $I(u) \geq \alpha \|u\|_X + \beta$ for all $u \in X$.

**Proposition 74.** *([33, Ch. 3, Th. 1.1]) Let $X$ be a reflexive Banach space and let $I\colon X \to \mathbb{R}$ be a w.l.s. and coercive functional. Then there exists at least one element $u_0 \in X$ such that*

$$I(u_0) = \inf \{I(u) \colon \ u \in X\}.$$

**Corollary 75.** *Under the hypotheses of Proposition 74, if $Y \subseteq X$ is a closed (in the norm of $X$) and convex subset, then exists $u_1 \in Y$ such that*

$$I(u_1) = \inf\{I(u) \colon \ u \in Y\}.$$

We apply these facts to the reflexive Banach space $X = W^{1,2}(\Omega, \mathbb{R}^3)$, and the functional $I = \varepsilon$ of (3.17), with $Y \subseteq X$ defined as follows:

$$Y = \{\vec{W} \in W^{1,2}(\Omega, \mathbb{R}^3) \colon \ \vec{W}|_{\partial\Omega} = \vec{\varphi}\}.$$

**Proposition 76.** *$Y \subseteq X$ and $\varepsilon$ satisfy the hypothesis of Corollary 75: (a) $Y$ is convex; (b) $Y$ is closed; (c) $\varepsilon$ is coercive; (d) $\varepsilon$ is w.l.s.*

**Proof.** (a) is immediate. To prove (b), let $\{\vec{W}_k\} \subseteq Y$ with $\vec{W}_k \to \vec{W}$; that is, $\|\vec{W}_k - \vec{W}\|_{H^1(\Omega)} \to 0$ as $k \to \infty$. By the Trace Theorem in Sobolev spaces [1] we have $C > 0$ such that

$$\|\vec{W}_k|_{\partial\Omega} - \vec{W}|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} \leq C\|\vec{W}_k - \vec{W}\|_{W^{1,2}(\Omega)}$$

for all $k$. And since $\vec{W}_k|_{\partial\Omega} = \vec{\varphi}$, then $\vec{W}|_{\partial\Omega} = \vec{\varphi}$ almost everywhere in $\partial\Omega$. By definition,

$$\varepsilon[\vec{W}] = \|f^{-1} \operatorname{curl} \vec{W}\|^2_{L^2(\Omega)}.$$

so we have (c). For (d), since the norm in any Banach space is w.l.s., we need to prove that if $\vec{W}_k \to \vec{W}$ weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$, then $\operatorname{curl} \vec{W}_k \to \operatorname{curl} \vec{W}$ weakly in $L^2(\Omega, \mathbb{R}^3)$. But this holds because $\partial\vec{W}_k/\partial x_i \to \partial\vec{W}/\partial x_i$ weakly in $L^2(\Omega, \mathbb{R}^3)$ $(i = 1, 2, 3)$, and because the curl is a combination of elements of $\partial/\partial x_i$. □

**Theorem 77.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary, and let $f^2$ be a measurable proper conductivity. Then given the boundary values $\vec{\varphi} \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ there exists an extension $\vec{W} \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that*

$$\begin{aligned} \operatorname{curl}\left(f^{-2} \operatorname{curl} \vec{W}\right) &= 0, \\ \vec{W}|_{\partial\Omega} &= \vec{\varphi}. \end{aligned} \tag{3.18}$$

**Proof.** By Corollary 75 and Proposition 76, the nonlinear functional (3.17) has a minimum $\vec{W}$ over $[\vec{\varphi}] + W_0^{1,2}(\Omega, \mathbb{R}^3)$. By definition, the second equation of the system (3.18) holds. To prove the first one, from the integration by parts formula for Sobolev spaces we have that

$$\langle \operatorname{curl} f^{-2} \operatorname{curl} \vec{W},\ \vec{v} \rangle = \int_\Omega f^{-2} \operatorname{curl} \vec{W} \cdot \operatorname{curl} \vec{v}\, d\vec{y} \tag{3.19}$$

when $\vec{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. The Gâteaux derivative of $\varepsilon$ at $\vec{w}$ in the direction $\vec{v}$ is

$$\begin{aligned} \varepsilon'_{\vec{v}}[\vec{W}] &= \lim_{t\to 0} \frac{\varepsilon[\vec{W} + t\vec{v}] - \varepsilon[\vec{W}]}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \int_\Omega (2tf^{-2} \operatorname{curl} \vec{v} \cdot \operatorname{curl} \vec{W} + t^2 f^{-2} \operatorname{curl} \vec{v} \cdot \operatorname{curl} \vec{v})\, d\vec{y} \\ &= 2 \int_\Omega f^{-2} \operatorname{curl} \vec{v} \cdot \operatorname{curl} \vec{W}\, d\vec{y}. \end{aligned}$$

Since $\vec{W}$ is an extreme point for $\varepsilon$, the integral vanishes, by (3.19) the first equation of (3.18) holds in the distributional sense. □

The vector field achieving the minimum in Theorem (77) is not unique, because $\varepsilon[\vec{W}] = \varepsilon[\vec{W} + \operatorname{grad} h]$ when $\operatorname{grad} h \in W_0^{1,2}(\Omega, \mathbb{R})$.

Similarly, we can find weak solutions for the inhomogeneous conductivity equation $\operatorname{div} f^2 \nabla W_0 = g_0$ (cf. Proposition 68). Now the functional to minimize is

$$\varepsilon[W_0] = \int_\Omega f^2\, \nabla W_0 \cdot \nabla W_0\, d\vec{y} + 2 \int_\Omega g_0 W_0\, d\vec{y},$$

given $g_0 \in L^2(\Omega, \mathbb{R})$.

# Chapter 4

# Hilbert transform for the Vekua equation

The aim of this chapter is to show the existence of a natural "Hilbert transform" $\mathcal{H}_f$ associated to the main Vekua equation (3.1) in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$, in the generality of solutions in the Sobolev space $H^{1/2}(\partial\Omega)$. This is a system of real equations in the dress of a quaternionic formula. The scalar part of a solution of the main Vekua equation satisfies a conductivity equation, while the vector part satisfies a double curl-type equation coupled with the condition of being divergence free (3.6)–(3.7). Our construction of $\mathcal{H}_f$ is inspired by the Hilbert transform given by T. Qian and others for the monogenic case of the Vekua equation, defined in terms of the component operators of the singular Cauchy integral operator and an inverse operator related to layer potentials [8, 83, 84].

## 4.1 Hilbert transform for monogenic functions

Before entering on the investigation of the Vekua equation in domains in $\mathbb{R}^3$, we begin our study of the Hilbert transform in the much simpler case of monogenic functions of three variables. This refers to a linear operator which produces the boundary values of the vector part of a monogenic function, given the boundary values of the scalar part, thus generalizing the classical operator defined by D. Hilbert for the unit disk or upper half plane in $\mathbb{C}$. This problem has been studied in the context of Clifford algebras for the unit sphere in $\mathbb{R}^n$ in [84, 24] and for $k$-forms in Lipschitz domains in [8]. Throughout this chapter we will always assume that the complement of $\Omega$ is connected.

### 4.1.1 Definition of $\mathcal{H}$

From now on $\Omega$ will be a bounded $C^{1,\gamma}$ Lipschitz domain with connected boundary, $\gamma > 0$ and $1 < p < \infty$, or $\Omega$ will be a bounded Lipschitz domain and $2 - \epsilon(\Omega) < p < \infty$ (unless another range of $p$ is specified). Then the operators $K_0$, $\overrightarrow{K}$ and $(I + K_0)^{-1}$ are all bounded from $L^p(\partial\Omega)$ to $L^p(\partial\Omega)$.

We recall the construction which was given in [83, 84] of the monogenic Hilbert transform for bounded Lipschitz domains and for the unit ball in $\mathbb{R}^n$.

**Definition 78.** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain, the *Hilbert transform*

$$\mathcal{H}\colon L^p(\partial\Omega, \mathbb{R}) \to L^p(\partial\Omega, \mathbb{R}^3)$$

is defined as

$$\mathcal{H}[\varphi_0] = \overrightarrow{K}(I+K_0)^{-1}\varphi_0 = \frac{1}{2}\overrightarrow{K}[h_0] \tag{4.1}$$

with $K_0$, $\overrightarrow{K}$ given in (1.27), and $h_0 = 2(I+K_0)^{-1}\varphi_0$.

By the Plemelj-Sokhotski formula (1.18), the non-tangential boundary limits of $F_{\partial\Omega}[h_0]$ exist, and since $h_0$ is $\mathbb{R}$-valued, for $\vec{x} \in \partial\Omega$ we have

$$\begin{aligned} \operatorname{tr}_+ F_{\partial\Omega}[h_0](\vec{x}) &= \frac{1}{2}\left(h_0(\vec{x}) + \operatorname{Sc}(S_{\partial\Omega}[h_0])(\vec{x})\right) + \frac{1}{2}\operatorname{Vec}(S_{\partial\Omega}[h_0])(\vec{x}) \\ &= \frac{1}{2}(I+K_0)h_0(\vec{x}) + \mathcal{H}[\varphi_0](\vec{x}) \\ &= \left(\varphi_0 + \mathcal{H}[\varphi_0]\right)(\vec{x}). \end{aligned} \tag{4.2}$$

Thus $\varphi_0 + \mathcal{H}[\varphi_0]$ is the boundary value of the monogenic function $F_{\partial\Omega}[h_0]$ in $\Omega$, which justifies calling $\mathcal{H}$ a Hilbert transform. The image of the Hilbert transform $\mathcal{H}$ belongs to the space of boundary functions whose harmonic extension is divergence free because from (1.10) and the construction (4.1), the vector part of the monogenic extension $W = F_{\partial\Omega}[h_0] = F_{\partial\Omega}[2(I+K_0)^{-1}\varphi_0]$ satisfies $\operatorname{div}\vec{W} = 0$.

**Proposition 79.** *On the Sobolev space $W^{1,2}(\Omega,\mathbb{R})$,*

$$2\operatorname{tr} T_{0,\Omega}\nabla = (I-K_0)\operatorname{tr}, \quad 2\operatorname{tr}\overrightarrow{T}_{2,\Omega}\nabla = -\overrightarrow{K}\operatorname{tr}.$$

**Proof.** Let $w_0 \in W^{1,2}(\Omega,\mathbb{R})$, $\varphi_0 = \operatorname{tr} w_0$. Apply (1.16) to $w_0$ and take the trace, and then apply (1.18):

$$\begin{aligned}\operatorname{tr} T_\Omega[\nabla w_0] =& \varphi_0 - \operatorname{tr} F_{\partial\Omega}[\varphi_0] = \varphi_0 - \frac{1}{2}(\varphi_0 + S_{\partial\Omega}[\varphi_0]) \\ =& \frac{1}{2}(I - K_0 - \overrightarrow{K})[\varphi_0].\end{aligned}$$

Now take the scalar and vector parts to obtain the desired formulas. $\square$

From Proposition 79 observe that the identity $2\,\mathcal{H}[\varphi_0] = \mathcal{H}[(I+K_0)\varphi_0] + \mathcal{H}[(I-K_0)\varphi_0]$ can now be expressed as

$$\mathcal{H}[\varphi_0] = -\operatorname{tr} \overrightarrow{T}_{2,\Omega}[\nabla w_0] + \mathcal{H}[\operatorname{tr} T_{0,\Omega}[\nabla w_0]]. \tag{4.3}$$

**Remark 80.** Consider the particular case $\Omega = \mathbb{B}^3 = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < 1\}$. The unit normal vector to $\partial\Omega = \mathbb{S}^2$ is $\eta(\vec{y}) = \vec{y}$, thus

$$E(\vec{y} - \vec{x})\eta(\vec{y}) = \frac{1}{4\pi}\left(\frac{1}{2|\vec{y} - \vec{x}|} + \frac{\vec{x} \times \vec{y}}{|\vec{y} - \vec{x}|^3}\right).$$

In this case, the operators $K_0$ and $\overrightarrow{K}$ of (1.27) are reduced to

$$K_{0,\mathbb{S}^2}[\varphi](\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\varphi(\vec{y})}{|\vec{y} - \vec{x}|}\, ds_{\vec{y}}, \quad \overrightarrow{K}_{\mathbb{S}^2}[\varphi](\vec{x}) = \mathrm{PV}\ \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{\vec{x} \times \vec{y}}{|\vec{y} - \vec{x}|^3}\varphi(\vec{y})\, ds_{\vec{y}}.$$

The following explicit representation [84] comes from the computation of the inner Poisson kernel and its Cauchy-type harmonic conjugates in the unit ball in $\mathbb{R}^n$. Recall $\partial\Omega = \mathbb{S}^2$.

**Theorem 81.** *[84, Th. 6] The Hilbert transform $\mathcal{H}$ given by the $L^p$-limits*

$$\mathcal{H}[\varphi_0](\vec{\xi}) = \lim_{r \to 1^-} \int_{\mathbb{S}^2} Q(r\vec{\xi}, \vec{y})\varphi_0(\vec{y}) ds_{\vec{y}} \tag{4.4}$$

*are bounded from $L^p(\mathbb{S}^2, \mathbb{R})$ to $L^p(\mathbb{S}^2, \mathbb{R}^3)$ $(1 < p < \infty)$. Where the kernel $Q$ is given by*

$$Q(\vec{x}, \vec{y}) = \frac{1}{4\pi} \left( \frac{2}{|\vec{x} - \vec{y}|^3} - \frac{1}{r^2} \int_0^r \frac{\rho}{|\rho\vec{\xi} - \vec{y}|^3} d\rho \right) \vec{x} \times \vec{y},$$

*with $\vec{x} = r\vec{\xi}$, $\vec{y}, \vec{\xi} \in \mathbb{S}^2$, $0 \leq r < 1$.*

### 4.1.2 Properties of $\mathcal{H}$ and its adjoint and inverse

We derive some basic facts of the Hilbert transform $\mathcal{H}$, as well as for the adjoint and a left inverse of $\mathcal{H}$. At the end of this subsection we will see that $\mathcal{H}$ belongs to the class of semi-Fredholm operators.

The Hilbert operator $\mathcal{H}$ is a bounded and non-compact operator in the $L^p$ norm. The boundedness was proved for the ball in [84, Th. 6] and for Lipschitz domains in [83, Theorem 3.2]. If $\mathcal{H}$ were compact, then $\overrightarrow{K}$ would also be compact, since $I + K_0$ is bounded on $L^p(\partial\Omega, \mathbb{R})$. But since $K_0$ is compact [63, Cor. 2.2.14] on $C^1$ domains, $S_{\partial\Omega}$ would then be compact by the decomposition (1.27), and then $S_{\partial\Omega}^2 = I$ would also be compact, which is absurd.

Now show that when we restrict the domain of the Hilbert transform $\mathcal{H}$ to Sobolev space, the property of boundedness is preserved. Recall the value $\epsilon(\Omega)$ discussed in Proposition 24.

**Theorem 82.** *Let $\Omega$ be a bounded Lipschitz domain. The restriction*

$$\mathcal{H}\colon W^{1-1/p,p}(\partial\Omega,\mathbb{R}) \to W^{1-1/p,p}(\partial\Omega,\mathbb{R}^3),$$

*of the Hilbert transform $\mathcal{H}$ is a bounded operator when $1 < p < 2+\epsilon(\Omega)$, and also when $1 < p < \infty$ and $\Omega$ is a $C^{1,\gamma}$ Lipschitz domain, $\gamma > 0$.*

**Proof.** We have noted that $\overrightarrow{K}$ is bounded, so the statement follows from (4.1) and Proposition 24, parts (b) and (c). □

From this it is straightforward to obtain the explicit form of the adjoint of $\mathcal{H}$. Write $\epsilon^{\pm}(\Omega) = (2 \pm \epsilon(\Omega))/(1 \pm \epsilon(\Omega))$.

**Proposition 83.** *Let $\Omega$ be a bounded Lipschitz domain. Then the adjoint $\mathcal{H}^*\colon L^q(\partial\Omega,\mathbb{R}^3) \to L^q(\partial\Omega,\mathbb{R})$*

$$\mathcal{H}^*[\vec{\varphi}] = (I + K_0^*)^{-1}\overrightarrow{K}^*[\vec{\varphi}] \tag{4.5}$$

*is bounded on $W^{1-1/q,q}(\partial\Omega)$ for $\epsilon^+(\Omega) < q < \infty$ and on $L^p(\partial\Omega)$ for $1 < q < \epsilon^-(\Omega)$. When $\Omega$ has $C^{1,\gamma}$ boundary, $\gamma > 0$, $\mathcal{H}^*\colon L^q(\partial\Omega,\mathbb{R}^3) \to L^q(\partial\Omega,\mathbb{R})$ is bounded for $1 < q < \infty$.*

We now discuss the invertibility of $\mathcal{H}$. The identity $S_{\partial\Omega}^2 = I$ combined with (1.27), when applied to real-valued functions, produces the identities

$$I - K_0^2 = -\sum_{i=1}^{3} K_i^2, \tag{4.6}$$

and $K_0K_i + K_iK_0 + K_jK_k - K_kK_j = 0$ for $(i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2)$. The equation (4.6) will be particularly useful below. The last three play a similar role to the commutative relations enjoyed by the Riesz transforms $R_i$ ($i = 1, 2, 3$) in a half space of $\mathbb{R}^3$ [67, p. 91].

In [83, 84] reference is made to the inverses of $I \pm K_0$ (see also [63]). However, we observe the following.

**Proposition 84.** *Let $\Omega$ be Lipschitz and $\epsilon^+(\Omega) < p < \infty$ or $C^{1,\gamma}$ Lipschitz and $1 < p < \infty$. Then $\mathrm{Ker}(I - K_0) = \mathbb{R}$ on $L^p(\partial\Omega)$.*

**Proof.** Let $c_0 \in \mathbb{R}$. Then (1.28) shows that $c_0 \in \mathrm{Ker}(I - K_0)$. We now verify that the only elements of $\mathrm{Ker}(I - K_0)$ are constants. Since the adjoint $I - K_0^*$ is invertible in $L_0^q(\partial\Omega)$ by Proposition 24(d), it follows from the Banach Closed Range Theorem that the image is $\mathrm{Im}\,(I - K_0) = L_0^p(\partial\Omega)$. Thus $\mathrm{Ker}(I - K_0)|_{L_0^p(\partial\Omega)} = \{0\}$. Finally, let $g \in L^p(\partial\Omega)$ such that $g \in \mathrm{Ker}(I - K_0)$. Let $f = (I - A)g$, where $A$ is the boundary averaging operator (1.29). Then by (1.28),

$$(I - K_0)f = f - (K_0[g] - K_0[Ag]) = f - (g - Ag) = 0.$$

Since $f \in L_0^p(\partial\Omega, \mathbb{R})$, we have $f = 0$; that is, $g = A[g] \in \mathbb{R}$. □

Note also that $K_0$ does not interfere with the averaging process:

$$AK_0[\varphi_0] = A[\varphi_0],$$

because $2\,\mathrm{P.V.}\!\int_{\partial\Omega} E(\vec{y} - \vec{x}) \cdot \eta(\vec{x})\, ds_{\vec{y}} = 1$. For this reason and by Proposition 84, the operator $I - K_0$ sends $L_0^p(\partial\Omega, \mathbb{R})$ to itself, and has an inverse

$$(I - K_0)^{-1} \colon L_0^p(\partial\Omega, \mathbb{R}) \to L_0^p(\partial\Omega, \mathbb{R})$$

with $\epsilon^+(\Omega) < p < \infty$ when $\Omega$ is Lipschitz and $1 < p < \infty$ when $\Omega$ is $C^{1,\gamma}$.

**Definition 85.** The *auxiliary transform* for the three-dimensional Hilbert transform is the operator $\mathcal{G}\colon L^p(\partial\Omega, \mathbb{R}^3) \to L^p_0(\partial\Omega, \mathbb{R})$

$$\mathcal{G}[\vec{\varphi}] = -(I - K_0)^{-1}(I - A)\overrightarrow{K} \cdot \vec{\varphi}. \tag{4.7}$$

In (4.7) we have used the notational convention

$$T \cdot \varphi = \sum_{i=0}^{3} T_i \varphi_i$$

which we will use whenever $T = \sum_{i=0}^{3} e_i T_i$ where $T_i$ are right $\mathbb{H}$-linear operators which send scalar-valued functions to scalar-valued functions, and $\varphi = \sum_{i=0}^{3} e_i \varphi_i$ with $\varphi_i$ scalar-valued.

**Proposition 86.** *Assume that $\Omega, p$ satisfy the hypotheses of Proposition 84. Then $\mathcal{G}$ is a left inverse for the Hilbert transform $\mathcal{H}$ on $L^p_0(\partial\Omega, \mathbb{R})$.*

**Proof.** Let $\varphi_0 \in L^p_0(\partial\Omega, \mathbb{R})$. By (4.1) and (4.6),

$$\begin{aligned}
\mathcal{G} \circ \mathcal{H}[\varphi_0] &= -(I - K_0)^{-1}(I - A)(\overrightarrow{K} \cdot \overrightarrow{K})(I + K_0)^{-1}\varphi_0 \\
&= (I - K_0)^{-1}(I - A)\left(-\sum_{i=1}^{3} K_i^2\right)(I + K_0)^{-1}\varphi_0 \\
&= (I - K_0)^{-1}(I - A)(I - K_0^2)(I + K_0)^{-1}\varphi_0 \\
&= (I - K_0)^{-1}(I - K_0 + AK_0 - A)\varphi_0 \\
&= \varphi_0,
\end{aligned}$$

where the last equality uses $AK_0 = A$. □

The proof of the non-compactness of $\mathcal{H}$ which we outlined at the beginning of this subsection fails in the case of bounded Lipschitz domains because $K_0$ does not need to be compact [39]. However, the existence of its left inverse automatically guarantees the non-compactness of $\mathcal{H}$. Other straightforward consequences are the following.

**Corollary 87.** *Under the same hypotheses,*

*(a) Restricted to the mean zero subspace $L_0^p(\partial\Omega,\mathbb{R})$, the Hilbert transform $\mathcal{H}$ is injective and its left inverse $\mathcal{G}\colon L^p(\partial\Omega,\mathbb{R}^3)\to L_0^p(\partial\Omega,\mathbb{R})$ is surjective.*

*(b) The left inverse $\mathcal{G}$ of the Hilbert transform is a bounded and non-compact operator.*

From (4.7) and $A^*=A$, the adjoint operator $\mathcal{G}^*\colon L_0^p(\partial\Omega,\mathbb{R})\to L^p(\partial\Omega,\mathbb{R}^3)$ is given by

$$\mathcal{G}^*[\varphi_0] = -\sum_{i=1}^{3} e_i K_i^*(I-A)(I-K_0^*)^{-1}[\varphi_0] = -\sum_{i=1}^{3} e_i K_i^*(I-K_0^*)^{-1}[\varphi_0].$$

We now look at the question of the images under $\mathcal{G}$ of the boundary values of SI vector fields. Recall that $\mathrm{SI}(\partial\Omega)$ is the space of boundary values of SI vector fields in $\Omega$ which extend to $\overline{\Omega}$, which we recall from (1.11) are the purely vectorial monogenic constants. Since SI vector fields are harmonic, the SI extension of $\vec{\varphi}\in\mathrm{SI}(\partial\Omega)$ to the interior is unique.

**Proposition 88.** *The elements of* $\mathrm{SI}(\partial\Omega)$ *are annihilated by* $\mathcal{G}$*; more precisely*

$$\mathrm{SI}(\partial\Omega)\cap L^p(\partial\Omega,\mathbb{R}^3)\subseteq \operatorname{Ker}\mathcal{G}\,.$$

**Proof.** Because for every $\vec{\varphi} \in \mathrm{SI}(\partial\Omega)$, $S_{\partial\Omega}[\vec{\varphi}] = \vec{\varphi}$, so $\overrightarrow{K} \cdot \vec{\varphi} = 0$. By (4.7), $\vec{\varphi} \in \operatorname{Ker} \mathcal{G}$. □

Clearly $\operatorname{Ker} \mathcal{H} = \mathbb{R}$ since the only scalar-valued monogenic functions are constants. One important fact about $\operatorname{Im} \mathcal{H}$ is $\mathrm{SI}(\partial\Omega) \cap L^p(\partial\Omega, \mathbb{R}^3) \cap \operatorname{Im} \mathcal{H} = \{\vec{0}\}$; moreover,

**Corollary 89.** *Under the same hypotheses of Proposition 86, the Hilbert transform $\mathcal{H}$ on $L^p(\partial\Omega, \mathbb{R})$ is a left semi-Fredholm operator.*

**Proof.** It is enough to prove that when the domain of $\mathcal{H}$ is restricted to $L^p_0(\partial\Omega, \mathbb{R})$, the image $\operatorname{Im} \mathcal{H}$ is closed in $L^p(\partial\Omega, \mathbb{R}^3)$ [76, Chapter 5]. By Proposition 86, $\mathcal{G}^* \circ \mathcal{H}^* = I$, so $\mathcal{H}^*$ is surjective. As a consequence of the Banach Closed Range theorem, $\mathcal{H}$ has closed range. □

Since $\mathbb{R} = \operatorname{Ker} \mathcal{H} = \operatorname{Ker} \overrightarrow{K}$ and $\operatorname{Im} \mathcal{H} = \operatorname{Im} \overrightarrow{K}$, the vector operator $\overrightarrow{K}$ is also left semi-Fredholm.

After enunciating some results of the monogenic Hilbert transform $\mathcal{H}$, we will recall the solution of the div-curl system in bounded Lipschitz domains presented in Section 2.3 and rewrite it in terms of $\mathcal{H}$ as follows

**Remark 90.** [36, Th. A.1] Under the same hypothesis of Theorem 55. Then a weak solution $\vec{w}$ of the div-curl system (2.16) is given by

$$\begin{aligned} \vec{w} &= T_\Omega[-g_0 + \vec{g}] + F_{\partial\Omega}[\vec{\alpha} - \mathcal{H}[\alpha_0]] \\ &= -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \overrightarrow{F}_{1,\partial\Omega}[2(I + K_0)^{-1}\alpha_0] \end{aligned} \tag{4.8}$$

where $\alpha_0$ and $\vec{\alpha}$ were defined in (2.17). Basically, the proof consists of showing that

$$F_{\partial\Omega}[\vec{\alpha} - \mathcal{H}[\alpha_0]] = -F_{\partial\Omega}[h_0],$$
$$\operatorname{Sc} F_{\partial\Omega}[\vec{\alpha} - \mathcal{H}[\alpha_0]] = -T_{0,\Omega}[\vec{g}] = -F_{0,\partial\Omega}[h_0].$$

Note that (1.16) applied to the function $T_\Omega[\vec{g}]$, and the fact that $DT_\Omega[\vec{g}] = \vec{g}$ yield $F_{\partial\Omega}[\alpha_0 + \vec{\alpha}] = 0$, so

$$\operatorname{tr} F_{\partial\Omega}[\vec{\alpha} - \mathcal{H}[\alpha_0]] = -\operatorname{tr} F_{\partial\Omega}[\alpha_0 + \mathcal{H}[\alpha_0]] = -\alpha_0 - \mathcal{H}[\alpha_0], \qquad (4.9)$$

which together with (4.2) proves the second equality in the solution (4.8).

### 4.1.3 Dirichlet-to-Neumann map

Intimately related to the Hilbert transform is the Dirichlet-to-Neumann (D-N) operator [10], which plays a fundamental role in the study of elliptic partial differential equations. In the rest of this chapter we restrict to the case $p = 2$ and work in domains $\Omega$ with Lipschitz boundary. Using the tools of the preliminary Section 1.6 we introduce

**Definition 91.** The "quaternionic Dirichlet-to-Neumann map" is given by

$$\begin{aligned} \Lambda\colon H^{1/2}(\partial\Omega, \mathbb{R}) &\to H^{-1/2}(\partial\Omega, \mathbb{H}) \\ \varphi_0 &\mapsto (Dw_0|_{\partial\Omega})\eta, \end{aligned} \qquad (4.10)$$

where $w_0 \in W^{1,2}(\Omega, \mathbb{R})$ is the unique harmonic extension of $\varphi_0$.

Note the essential use of quaternionic multiplication of vectors in (4.10).

Since $\mathrm{Sc}((Dw_0|_{\partial\Omega}\eta)\overline{v}) = -(\gamma_{\mathbf{n}}\nabla w_0)v_0 + (\gamma_{\mathbf{t}}\nabla w_0)\cdot\vec{v}$ for every $v = v_0 + \vec{v} \in W^{1,2}(\Omega,\mathbb{H})$, by the weak definitions (1.33) of $\gamma_{\mathbf{n}}$ and $\gamma_{\mathbf{t}}$, we have

$$\begin{aligned}\langle\Lambda[\varphi_0], \mathrm{tr}\, v\rangle_{\partial\Omega} &= \int_\Omega \nabla w_0\cdot(-\nabla v_0 + \mathrm{curl}\,\vec{v})\,d\vec{y}\\ &= \mathrm{Sc}\int_\Omega \nabla w_0 D_{\mathrm{r}} v\,d\vec{y},\end{aligned} \tag{4.11}$$

where we again write $D_{\mathrm{r}}$ for the right-sided operator

$$D_{\mathrm{r}}[v] = vD = \sum_{i=1}^{3}(\partial_i v)e_i = -\operatorname{div}\vec{v} + (\nabla v_0 - \mathrm{curl}\,\vec{v}).$$

The scalar and vector parts of the quaternionic product $(Dw_0|_{\partial\Omega})\eta$ give the decomposition

$$\Lambda[\varphi_0] = \Lambda_0[\varphi_0] + \vec{\Lambda}[\varphi_0] \tag{4.12}$$

with $\Lambda_0[\varphi_0] = -\gamma_{\mathbf{n}}\nabla w_0$, $\vec{\Lambda}[\varphi_0] = \gamma_{\mathbf{t}}\nabla w_0$. Thus the scalar part of $\Lambda[\varphi_0]$ coincides with the negative of the usual scalar D-N map for the Laplacian Dirichlet problem [32, 63]. We will verify in subsection 5.1 that $\Lambda[\varphi_0]$ does indeed lie in $H^{-1/2}(\partial\Omega,\mathbb{H})$ as implied by Definition 91.

(In the two-dimensional context, such as in [5], one has only a scalar D-N mapping, denoted commonly by "Λ".)

As usual $W^{2,2}(\Omega,\mathbb{R})$ is the notation for the Sobolev space of scalar functions whose gradient belongs to $W^{1,2}(\Omega,\mathbb{R}^3)$ and $H^{3/2}(\partial\Omega,\mathbb{R})$ is the space of boundary values of functions in $W^{2,2}(\Omega,\mathbb{R})$.

**Proposition 92.** $T_\Omega\nabla = -M\Lambda\,\mathrm{tr}$ *on* $\mathrm{Har}(\Omega,\mathbb{R})\cap W^{2,2}(\Omega,\mathbb{R})$.

**Proof.** In Propositions 51 and 52 it was seen that $T_{0,\Omega}[\vec{w}] = M[\gamma_{\mathbf{n}}(\vec{w})]$ for all $\vec{w} \in \mathrm{Sol}(\overline{\Omega}, \mathbb{R}^3)$ and $\overrightarrow{T}_{2,\Omega}[\vec{w}] = -M[\gamma_{\mathbf{t}}(\vec{w})]$ for all $\vec{w} \in \mathrm{Irr}(\overline{\Omega}, \mathbb{R}^3)$, where $M$ is the single-layer operator (1.20). See Remark 53 for the justification of the applicability to $W^{2,2}(\Omega, \mathbb{R})$ for the existence of boundary values. □

Note that by (1.20), $M$ is a scalar operator, so it respects the decomposition (4.12) of $\Lambda$:

$$T_{0,\Omega}\nabla = -M\Lambda_0\,\mathrm{tr}, \quad \overrightarrow{T}_{2,\Omega}\nabla = -M\vec{\Lambda}\,\mathrm{tr}\,. \tag{4.13}$$

We proved that the Hilbert transform $\mathcal{H}$ is a non-compact operator. However, when restricted to $\mathrm{Ker}\,\vec{\Lambda}$, by dimensional properties of $\mathrm{SI}_{\mathbf{t}}(\Omega)$, then $\mathcal{H}$ becomes compact. Recall that we are always assuming that $\partial\Omega$ is connected.

**Proposition 93.** $\mathrm{Ker}\,\Lambda_0 = \mathbb{R}$ *and* $\mathrm{Ker}\,\vec{\Lambda} = \mathbb{R}$.

**Proof.** Let $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$, and let $w_0$ be its harmonic extension. If $\varphi_0 \in \mathrm{Ker}\,\Lambda_0$, then $w_0$ satisfies a trivial Neumann condition and therefore is constant as claimed [32, Th. 4.18].

Now suppose instead that $\varphi_0 \in \mathrm{Ker}\,\vec{\Lambda}$. Since $\nabla w_0$ is a monogenic constant with vanishing tangential trace,

$$\mathcal{H}[\varphi_0] = -\,\mathrm{tr}\,\overrightarrow{T}_{2,\Omega}[\nabla w_0] + \mathcal{H}\,\mathrm{tr}\,T_{0,\Omega}[\nabla w_0]$$

lies in the image of the finite-dimensional space $\mathrm{SI}_{\mathbf{t}}(\Omega)$. Since $\partial\Omega$ is connected, $\mathrm{SI}_{\mathbf{t}}(\Omega) = 0$ because $\mathrm{SI}_{\mathbf{t}}(\Omega)$ is isomorphic to the second real cohomology space [13], so $\mathcal{H}(\mathrm{Ker}\,\vec{\Lambda}) = 0$. Thus $\mathrm{Ker}\,\vec{\Lambda} \subseteq \mathrm{Ker}\,\mathcal{H} = \mathbb{R}$. Clearly $\vec{\Lambda}$

annihilates constants (because the normal and tangential derivative of the constant extension must vanish), so the proof is finished. □

Some formulas expressing the topological characteristics (Betti numbers) of three-dimensional manifolds with connected boundary were derived in [11] through their Dirichlet-to-Neumann maps associated with scalar and vector harmonic fields. Moreover, in [12] a multidimensional generalization was shown.

## 4.2 Hilbert transform associated to the main Vekua equation

The definition of the Hilbert transform $\mathcal{H}$ for monogenic functions now permits us to define the analogous Hilbert transform $\mathcal{H}_f$ associated to the main Vekua equation (3.1).

Recall Definition 63. From now on $f \in W^{1,\infty}(\Omega, \mathbb{R})$. Note that $f$ and $(1/f)\vec{u}$ are simple examples of solutions of (3.1), where $\vec{u} \in \mathrm{SI}(\Omega)$ is a vectorial monogenic constant. We now extend some of our previous results, which are for the special case $f \equiv 1$, to the more general equation (3.1).

### 4.2.1 Construction of the Vekua-Hilbert transform

Results in [72, Ch. 16] relate solutions of the main Vekua equation to solutions of other differential equations. Recall that $W = W_0 + \vec{W}$ (see Lemma 62) satisfies (3.1) if and only if the scalar part $W_0$ and the vector part $\vec{W}$ satisfy the homogeneous div-curl system (3.5). Recall the conductivity equation satisfied by $W_0$

$$\nabla \cdot f^2 \nabla (W_0/f) = 0. \tag{4.14}$$

The following result is quite similar to Proposition 68. Here the hypothesis of Lipschitz boundary has been added, which permits using a basic estimate on elliptic boundary problems.

**Lemma 94.** ([58, Th. 4.1], [80, Th. 10] see also [38, 45, 50]) *Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary and let $f^2 \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Suppose that $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$ is known. Then there exists a unique extension $W_0 \in W^{1,2}(\Omega, \mathbb{R})$ satisfying* (4.14) *such that*

$$\mathrm{tr}(W_0/f) = \varphi_0 \tag{4.15}$$

*on $\partial\Omega$. Further,*

$$\|W_0/f\|_{W^{1,2}(\Omega,\mathbb{R})} \leq C_{\Omega,\rho(f)} \|\varphi_0\|_{H^{1/2}(\partial\Omega,\mathbb{R})} \tag{4.16}$$

*where $C_{\Omega,\rho(f)}$ only depends on $\Omega$ and $\rho(f)$.*

In Section 5 we will define a natural Neumann data for the conductivity equation (4.14). We will also prove a version of Lemma 94 for the vector part $\vec{W}$ of solutions of the Vekua equation.

To define the Hilbert transform for (3.1), let $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$ be a scalar boundary value function, and apply Lemma 94 to obtain $W_0$. The decomposition (1.22) of the Teodorescu operator applied to vector fields reduces to

$$T_\Omega\left[-f^2\nabla(W_0/f)\right] = T_{0,\Omega}\left[-f^2\nabla(W_0/f)\right] + \overrightarrow{T}_{2,\Omega}\left[-f^2\nabla(W_0/f)\right],$$

and by Theorem 17 both components lie in $W^{1,2}(\Omega)$.

**Definition 95.** Let $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. The *Vekua-Hilbert transform*

$$\mathcal{H}_f\colon H^{1/2}(\partial\Omega, \mathbb{R}) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$$

*associated to the main Vekua equation* (3.1) is defined as follows.

Let $\varphi_0, \vec{\varphi}$ be the *associated Teodorescu traces*

$$\alpha_0 = \operatorname{tr} T_{0,\Omega}\left[-f^2\nabla(W_0/f)\right],\ \vec{\alpha} = \operatorname{tr} \overrightarrow{T}_{2,\Omega}\left[-f^2\nabla(W_0/f)\right], \tag{4.17}$$

where $W_0$ is the solution of the conductivity equation (4.14) satisfying the boundary condition (4.15). Let us define

$$\mathcal{H}_f[\varphi_0] = \vec{\alpha} - \mathcal{H}[\alpha_0], \tag{4.18}$$

where $\mathcal{H}$ is the Hilbert transform defined in (4.1),

By the Trace Theorem 3 we have $\alpha = \alpha_0 + \vec{\alpha} \in H^{1/2}(\partial\Omega, \mathbb{H})$, and in fact by Proposition 18, $\vec{\alpha} \in \operatorname{Sol}(\partial\Omega)$.

Similarly to the Hilbert transform $\mathcal{H}$ for the monogenic case, $\mathcal{H}_f$ can be expressed as

$$\mathcal{H}_f[\varphi_0] = \vec{\alpha} - \frac{1}{2}\overrightarrow{K}[h_f] \tag{4.19}$$

with the real-valued function

$$h_f = 2(I + K_0)^{-1}\alpha_0 \in H^{1/2}(\partial\Omega, \mathbb{R}). \tag{4.20}$$

The term "Vekua-Hilbert transform" is justified by the following.

**Theorem 96.** *Let $\Omega$ be a bounded Lipschitz domain and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Suppose that $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. Then the quaternionic function*

$$f\varphi_0 + (1/f)\,\mathcal{H}_f[\varphi_0] \tag{4.21}$$

*is the trace of a solution of the main Vekua equation* (3.1)*.*

**Proof.** To produce $W = W_0 + \vec{W} \in W^{1,2}(\Omega, \mathbb{H})$ satisfying (3.1) such that

$$\operatorname{tr} W_0 = f\varphi_0, \ \operatorname{tr} f\vec{W} = \mathcal{H}_f[\varphi_0],$$

we take the extension $W_0$ of $f\varphi_0$ given by Lemma 94, and define the vector part $\vec{W}$ by

$$f\vec{W} = \overrightarrow{T}_{2,\Omega}[\vec{v}] - \overrightarrow{F}_{1,\partial\Omega}[h_f], \tag{4.22}$$

with $\vec{v} = -f^2\nabla(W_0/f)$ and $h_f$ given by (4.20); recall also (1.21)–(1.22) and (1.24)–(1.25). Since (3.1) is equivalent to $\operatorname{div}(f\vec{W}) = 0$, $\operatorname{curl}(f\vec{W}) = \vec{v}$, i.e. a div-curl system (2.16) with $g_0 = 0$, $\vec{g} = \vec{v}$, it follows from Theorem 55 that $W = W_0 + \vec{W}$ is a solution of (3.1). Further, by Remark 90 we have that

(4.22) can be rewritten as

$$f\vec{W} = T_\Omega[\vec{v}] + F_{\partial\Omega}[\mathcal{H}_f[\varphi_0]]. \tag{4.23}$$

Finally, taking trace to (4.23) and by (4.9) we have that

$$\operatorname{tr} f\vec{W} = \alpha_0 + \vec{\alpha} + \operatorname{tr} F_{\partial\Omega}[\vec{\alpha} - \mathcal{H}[\alpha_0]] = \vec{\alpha} - \mathcal{H}[\alpha_0] = \mathcal{H}_f[\varphi_0] \tag{4.24}$$

as required. □

**Remark 97.** When $f \equiv 1$, the transform $\mathcal{H}_f$ coincides with the Hilbert transform $\mathcal{H}$ of the monogenic case. To see this more clearly, first note that by (4.14), $W_0$ must be the harmonic extension of $\varphi_0$ to $\Omega$; similarly $\alpha_0 = -\operatorname{tr} T_{0,\Omega}[\nabla W_0]$ and $\vec{\alpha} = -\operatorname{tr} \overrightarrow{T}_{2,\Omega}[\nabla W_0]$ . Now (4.3) says that $\mathcal{H}[\varphi_0]$ is precisely the definition of $\mathcal{H}_{f\equiv 1}[\varphi_0]$.

**Remark 98.** In Remark 70 a slightly different definition was proposed for $\mathcal{H}_f$ in terms of the operators $T_{0,\Omega}$, $\overrightarrow{T}_{2,\Omega}$ and a certain radial integration operator, used in providing a general solution to the div-curl system valid in star-shaped domains. In that definition it is not possible to show the relationship with the monogenic Hilbert transform, because its construction is completely interior to domain $\Omega$.

**Remark 99.** We may consider $\mathcal{H}_f$ as a Hilbert transform for the quaternionic Beltrami equation (3.4) due to its relation to (3.1).

### 4.2.2 Properties of $\mathcal{H}_f$

**Proposition 100.** *Let $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. Then $\varphi_0 \in \operatorname{Ker} \mathcal{H}_f$ if and only if the associated Teodorescu traces $\alpha_0, \vec{\alpha}$ vanish identically.*

**Proof.** The Hodge decomposition [53, Th. 8.7] gives the orthogonal direct sum $L^2(\Omega, \mathbb{H}) = (\mathfrak{M}(\Omega) \cap L^2(\Omega, \mathbb{H})) \oplus D(W_0^{1,2}(\Omega, \mathbb{H}))$, where the subscript in $W_0^{1,2}$ indicates zero trace. Thus

$$\operatorname{tr} f\vec{W} = 0 \Leftrightarrow D(f\vec{W}) \in D(W_0^{1,2}(\Omega, \mathbb{H})) \Leftrightarrow \alpha_0 = 0,\ \vec{\alpha} = 0,$$

with $f\vec{W}$ as in (4.22) and where the last equivalence follows from the result [53, Prop. 8.9], which identifies orthogonality to all monogenic functions with the vanishing of the trace of the Teodorescu operator. By (4.24) we have the result. □

See [90] for the $n$-dimensional generalization of the above Hodge decomposition.

**Definition 101.** We will say that the vector part $\vec{W}$ of $W$ is *normalized* when it satisfies the boundary condition

$$\operatorname{tr} f\vec{W} = \mathcal{H}_f[\varphi_0]. \tag{4.25}$$

Let $W = W_0 + \vec{W}$ be an arbitrary solution of the main Vekua equation (3.1), and write $\varphi_0 = \operatorname{tr} W_0$, $\vec{\varphi} = \operatorname{tr} \vec{W}$. Consider

$$\vec{W}^* = \vec{W} - \frac{1}{f} F_{\partial\Omega}[f\vec{\varphi} + \mathcal{H}_f[\varphi_0]].$$

Then by (1.16), $f\vec{W}^*$ has the form (4.22) and hence satisfies the normalization condition (4.25), with $W_0 + \vec{W}^*$ a solution of (3.1).

On the other hand, let $W^1$ and $W^2$ be two solutions of (3.1) with the same scalar part and with normalized vector parts. If $\varphi^i = \operatorname{tr} W^i$, $i = 1, 2$, then

$$fW^1 - fW^2 = T_\Omega[D(f(W^1 - W^2))] + F_{\partial\Omega}[f\varphi^1 - f\varphi^2] = 0,$$

since $f(W^1 - W^2)$ is monogenic. Therefore $W^1 = W^2$; i.e. there is only one normalized vector part for a given scalar part of a solution of the main Vekua equation.

Some important facts about the solvability and regularity of the conductivity equation (4.14) permit us to prove the boundedness of $\mathcal{H}_f$:

**Theorem 102.** *Let $\Omega$ be a bounded Lipschitz domain and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Then the Vekua-Hilbert transform $\mathcal{H}_f\colon H^{1/2}(\partial\Omega, \mathbb{R}) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$ is a bounded operator, as are also the associated Teodorescu traces $\varphi_0 \longmapsto \alpha_0$ and $\varphi_0 \longmapsto \vec{\alpha}$ from $H^{1/2}(\partial\Omega, \mathbb{R})$ to $H^{1/2}(\partial\Omega, \mathbb{R})$ and $H^{1/2}(\partial\Omega, \mathbb{R}^3)$, respectively.*

**Proof.** Let $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. By Lemma 94, take $W_0 \in W^{1,2}(\Omega, \mathbb{R})$ satisfying (4.14)–(4.15). Since both $T_\Omega\colon L^2(\Omega) \to W^{1,2}(\Omega)$ and $\operatorname{tr}\colon W^{1,2}(\Omega) \to H^{1/2}(\partial\Omega)$ are continuous, by (4.16) we have

$$\begin{aligned}
\|\alpha_0 + \vec{\alpha}\|_{H^{1/2}(\partial\Omega)} &\le \|\operatorname{tr}\| \|T_\Omega\| \|f^2 \nabla(W_0/f)\|_{L^2(\Omega)} \\
&\le \|\operatorname{tr}\| \|T_\Omega\| \|f\|^2_{L^\infty(\Omega)} \|W_0/f\|_{W^{1,2}(\Omega)} \\
&\le C_{\Omega,\rho(f)} \|\operatorname{tr}\| \|T_\Omega\| \|f^2\|_{L^\infty(\Omega)} \|\varphi_0\|_{H^{1/2}(\partial\Omega)}. \qquad (4.26)
\end{aligned}$$

From this follows the continuity of $\alpha_0$ and $\vec{\alpha}$.

By the continuity of the Hilbert transform (Theorem 82),

$$\| \mathcal{H}[\alpha_0] \|_{H^{1/2}(\partial\Omega)} \leq \| \mathcal{H} \| \|\alpha_0\|_{H^{1/2}(\partial\Omega)}. \tag{4.27}$$

Using the inequalities (4.26)–(4.27), we have that

$$\begin{aligned}\| \mathcal{H}_f[\varphi_0] \|_{H^{1/2}(\partial\Omega)} &= \|\vec{\alpha} - \mathcal{H}[\alpha_0]\|_{H^{1/2}(\partial\Omega)} \\ &\leq \max(1, \| \mathcal{H} \|) \|\alpha_0 + \vec{\alpha}\|_{H^{1/2}(\partial\Omega)} \\ &\leq C_{\Omega,\rho(f)} \max(1, \| \mathcal{H} \|) \| \operatorname{tr} \| \|T_\Omega\| \|f^2\|_{L^\infty(\Omega)} \|\varphi_0\|_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

Therefore $\mathcal{H}_f$ is continuous. □

Analogous to the estimates for the solutions to the conductivity equation (4.16), we have

**Proposition 103.** *Let $\Omega$ be a $C^{1,1}$ bounded Lipschitz domain and let $f$ be a proper conductivity in $W^{1,\infty}(\Omega, \mathbb{R})$. Suppose that $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. Then the vector extension given by (4.22) satisfies*

$$\|f\vec{W}\|_{W^{1,2}(\Omega)} \leq C^*_{\Omega,\rho(f)} \|\varphi_0\|_{H^{1/2}(\partial\Omega)} \tag{4.28}$$

*where $C^*_{\Omega,\rho(f)}$ depends only on $\Omega$ and $\rho(f)$.*

**Proof.** Let $W_0 \in W^{1,2}(\Omega, \mathbb{R})$ be the unique solution of (4.14)–(4.15). Then as in (4.26),

$$\|T_\Omega[-f^2\nabla(W_0/f)]\|_{W^{1,2}(\Omega)} \leq C_{\Omega,\rho(f)} \|T_\Omega\| \|f^2\|_{W^{1,\infty}(\Omega)} \|\varphi_0\|_{H^{1/2}(\partial\Omega)}. \tag{4.29}$$

By (1.26) together with the fact that $\|\operatorname{curl}\vec{u}\|_{L^2}^2+\|\operatorname{div}\vec{u}\|_{L^2}^2 \leq 3\sum_{i=1}^3 \|\nabla(u_i)\|_{L^2}^2$ for every $\vec{u}\in W^{1,2}(\Omega)$ and Theorem 23(b),

$$\begin{aligned}\|F_{\partial\Omega}[h_f]\|_{L^2(\Omega)} &= \|\operatorname{div} M[\eta h_f] - \operatorname{curl} M[\eta h_f]\|_{L^2(\Omega)}\\ &\leq \sqrt{3}\|\nabla M[\eta h_f]\|_{L^2(\Omega)}\\ &\leq \sqrt{3}\|M[\eta h_f]\|_{W^{1,2}(\Omega)}\\ &\leq \sqrt{3}C_1\|M\|\|(I+K_0)^{-1}\|\|\eta\|_{W^{1,\infty}(\Omega)}\|\varphi_0\|_{H^{1/2}(\partial\Omega)}, \qquad (4.30)\end{aligned}$$

where the constant $C_1$ in the last inequality comes from the fact that both $(I+K_0)^{-1}\colon H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)$ and $\varphi_0\mapsto\alpha_0$ are bounded (see Proposition 24(c) and Theorem 102). By (4.29) and (4.30) and by the fact that $f\vec{W} = T_\Omega[-f^2\nabla(W_0/f)] - F_{\partial\Omega}[h_f]$ from (4.22),

$$\|f\vec{W}\|_{L^2(\Omega)} \leq C_2\|\varphi_0\|_{H^{1/2}(\partial\Omega)},$$

where $C_2 = C_{\Omega,\rho(f)}\|T_\Omega\|\|f^2\|_{W^{1,\infty}(\Omega)} + \sqrt{3}C_1\|M\|\|(I+K_0)^{-1}\|\|\eta\|_{W^{1,\infty}(\Omega)}$.

By the first Friedrichs inequality provided in Proposition 28, using the div-curl system (3.5) and the boundedness of the Vekua-Hilbert transform $\mathcal{H}_f$, we have

$$\begin{aligned}\|f\vec{W}\|_{W^{1,2}(\Omega)}^2 &\leq \|f\vec{W}\|_{L^2(\Omega)}^2 + \|\operatorname{curl}(f\vec{W})\|_{L^2(\Omega)}^2 + \|\mathcal{H}_f[\varphi_0]\cdot\eta\|_{H^{1/2}(\partial\Omega)}^2\\ &\leq C_2^2\|\varphi_0\|_{H^{1/2}(\partial\Omega)}^2 + \|f^2\nabla(W_0/f)\|_{L^2(\Omega)}^2\\ &\qquad + \|\mathcal{H}_f\|^2\|\eta\|_{W^{1,\infty}(\Omega)}^2\|\varphi_0\|_{H^{1/2}(\partial\Omega)}^2\\ &\leq C_{\Omega,\rho(f)}^{*2}\|\varphi_0\|_{H^{1/2}(\partial\Omega)}^2,\end{aligned}$$

where $C^{*2}_{\Omega,\rho(f)} = C_2^2 + C^2_{\Omega,\rho(f)} \|f^2\|^2_{L^\infty(\Omega)} + \| \mathcal{H}_f \|^2 \|\eta\|^2_{W^{1,\infty}(\Omega)}$. □

The construction of the Vekua-Hilbert transform $\mathcal{H}_f$ can be generalized to the context of Clifford algebras. More precisely, for $\Omega \subseteq \mathbb{R}^n$ a bounded Lipschitz domain with connected complement, almost all the results of Chapter 4 are valid in this $n$-dimensional framework except for Proposition 103, whose demonstration requires the boundedness of the operators div and curl.

# Chapter 5

# Dirichlet-to-Neumann map for the conductivity equation

The conductivity equation describes the behavior of an electric potential in a conductive medium. In 1980, A. P. Calderón [27] posed the question of whether it is possible to determine the electrical conductivity of a medium by making measurements at the boundary. Results obtained since then on the solvability, stability, uniqueness, and other properties of the Dirichlet problem associated to this kind of elliptic second order differential equation in $\mathbb{R}^n$ for $n \geq 3$ (e.g. Lemma 94 and [64, 95, 98]) will be essential in the development of the present work. The Calderón inverse problem is the subject of Electrical Impedance Tomography; for more about medical applications of the conductivity equation see [57]. This inverse problem is still open in three or more dimensions [86, 55, 28].

The goal of this chapter is to analyze in a direct way the relation between the Vekua-Hilbert transform proposed in Chapter 4 and the usual scalar

Dirichlet-to-Neuman (D-N) map, as well as with the quaternionic D-N map (which will be defined below). We hope that the development in this thesis may be useful in future investigation of such inverse problems.

## 5.1 Quaternionic Dirichlet-to-Neumann map

We now extend the concept of the D-N map given in Subsection 4.1.3 in the context of harmonic functions to the more general situation of solutions of the conductivity equation (4.14). There will be essential differences (see for example Proposition 110 below).

**Definition 104.** The scalar *Dirichlet-to-Neumann map* for the conductivity equation (4.14) is

$$\begin{aligned}\Lambda_{0,f^2}\colon H^{1/2}(\partial\Omega,\mathbb{R}) &\to H^{-1/2}(\partial\Omega,\mathbb{R}),\\ \varphi_0 &\mapsto -f^2\nabla(W_0/f)|_{\partial\Omega}\cdot\eta. \qquad (5.1)\end{aligned}$$

Here $\eta$ is again the unit outer normal vector to $\partial\Omega$ and $W_0\in W^{1,2}(\Omega,\mathbb{R})$ is the unique extension of $f\varphi_0$ as a solution of the conductivity equation (4.14) given in Lemma 94.

For $f^2$ smooth, $\Lambda_{0,f^2}[\varphi_0]$ is well-defined pointwise, but for general proper conductivities, the D-N map is only weakly defined by the relation

$$\langle\Lambda_{0,f^2}[\varphi_0],\operatorname{tr}v_0\rangle_{\partial\Omega} = -\int_\Omega f^2\nabla(W_0/f)\cdot\nabla v_0\,d\vec{y}, \qquad (5.2)$$

where $\nabla\cdot f^2\nabla(W_0/f)=0$, $\operatorname{tr}W_0=f\varphi_0$ and $v_0\in W^{1,2}(\Omega,\mathbb{R})$. One reference for the scalar D-N map is [85]. This map is an essential part of the solution of the Calderón problem [27], that is, to recover the pointwise conductivity $f^2$ interior to the domain $\Omega$ from electrical current measurements on the boundary $\partial\Omega$.

We will follow the definition given in [95], but we write $\Lambda_{0,f^2}$ rather than $\Lambda_{f^2}$ to emphasize the scalar nature of this quantity.

In analogy to (4.10) we introduce the following.

**Definition 105.** The *quaternionic Dirichlet-to-Neumann map* for the conductivity equation is defined strongly by

$$\Lambda_{f^2}[\varphi_0] = (f^2 D(W_0/f)|_{\partial\Omega})\eta \tag{5.3}$$

for functions $\varphi_0$ whose interior extension $W_0$ satisfy the conductivity equation (4.15).

**Theorem 106.** *The weak definition of* $\Lambda_{f^2}\colon H^{1/2}(\partial\Omega,\mathbb{R}) \to H^{-1/2}(\partial\Omega,\mathbb{H})$ *is given by*

$$\begin{aligned}\langle \Lambda_{f^2}[\varphi_0], \operatorname{tr} v\rangle_{\partial\Omega} = \operatorname{Sc}\int_\Omega f^2\nabla(W_0/f) f(D_{\mathrm{r}} + (Df/f))[v_0/f]\,d\vec{y}\\ + \operatorname{Sc}\int_\Omega f^2\nabla(W_0/f)\frac{1}{f}(D_{\mathrm{r}} - M^{Df/f}C_{\mathbb{H}})[f\vec{v}]\,d\vec{y},\end{aligned} \tag{5.4}$$

*for every* $v \in W^{1,2}(\Omega,\mathbb{H})$*, where* $W_0$ *is the solution of* (4.14) *with boundary values* (4.15)*,* $D_{\mathrm{r}}W = WD$*, and* $M^{(\cdot)}$ *denotes quaternionic multiplication from the right.*

With the notation $\Lambda_{f^2} = \Lambda_{0,f^2} + \vec{\Lambda}_{f^2}$ we can express (5.4) as

$$\langle \Lambda_{f^2}[\varphi_0], \operatorname{tr} v\rangle_{\partial\Omega} = \langle \Lambda_{0,f^2}[\varphi_0], \operatorname{tr} v_0\rangle_{\partial\Omega} + \langle \vec{\Lambda}_{f^2}[\varphi_0], \operatorname{tr}\vec{v}\rangle_{\partial\Omega},$$

where the scalar part is indeed the D-N map $\Lambda_{0,f^2}$ of (5.1), and the vector part (or *tangential D-N map*) is

$$\vec{\Lambda}_{f^2}[\varphi_0] = f^2\nabla(W_0/f)|_{\partial\Omega} \times \eta. \tag{5.5}$$

Recall Definition 71, where the operator $D_{\mathrm{r}} - M^{Df/f}C_{\mathbb{H}}$ appearing in the second integral of (5.4) is called [74] the "Bers derivative" of solutions of (3.1). When $f \equiv 1$, (5.4) reduces to (4.11). The proof of Theorem 106 is a long exercise in vector calculus, based on the ideas of (4.11), and the observation that by Green's formula,

$$\langle\vec{\Lambda}_{f^2}[\varphi_0], \mathrm{tr}\,\vec{v}\rangle_{\partial\Omega} = \int_\Omega f^2\nabla(W_0/f)\cdot \mathrm{curl}\,\vec{v}\,d\vec{y} - \int_\Omega \mathrm{curl}(f^2\nabla(W_0/f))\cdot\vec{v}\,d\vec{y}.$$

**Proposition 107.** *Let $\Omega$ be a bounded domain with Lipschitz boundary, and let $f^2 \in W^{1,\infty}(\Omega,\mathbb{R})$ be a proper conductivity. The quaternionic D-N map $\Lambda_{f^2}$ is continuous from $H^{1/2}(\partial\Omega,\mathbb{R})$ to the dual space $H^{-1/2}(\partial\Omega,\mathbb{H})$ of $H^{1/2}(\partial\Omega,\mathbb{H})$.*

**Proof.** Let $\varphi_0 \in H^{1/2}(\partial\Omega,\mathbb{R})$. Since $\Lambda_{0,f^2}[\varphi_0]$ and $\vec{\Lambda}_{f^2}[\varphi_0]$ are the normal and tangential traces respectively of $f^2\nabla(W_0/f) \in W^{2,\mathrm{div}}(\Omega,\mathbb{R}^3) \cap W^{2,\mathrm{curl}}(\Omega,\mathbb{R}^3)$, by (4.16), we have the estimates

$$\|\nabla(W_0/f)\|_{L^2} \le \|W_0/f\|_{W^{1,2}} \le C_{\Omega,\rho(f)}\|\varphi_0\|_{H^{1/2}}. \tag{5.6}$$

By Proposition 27,

$$\|\Lambda_{0,f^2}[\varphi_0]\|_{H^{-1/2}} \leq \|\gamma_{\mathbf{n}}\| \|f^2\nabla(W_0/f)\|_{L^2},$$
$$\|\vec{\Lambda}_{f^2}[\varphi_0]\|_{H^{-1/2}} \leq \|\gamma_{\mathbf{t}}\| \big(\|f^2\nabla(W_0/f)\|_{L^2} + \|\operatorname{curl}(f^2\nabla(W_0/f))\|_{L^2}\big).$$

Since

$$\|\operatorname{curl}(f^2\nabla(W_0/f))\|_{L^2} \leq \|\nabla f^2\|_{L^\infty} \|\nabla(W_0/f)\|_{L^2}, \tag{5.7}$$

by (5.6) we have

$$\|\vec{\Lambda}_{f^2}[\varphi_0]\|_{H^{-1/2}} \leq C_{\Omega,\rho(f)} \|\gamma_{\mathbf{t}}\| \|f^2\|_{W^{1,\infty}} \|\varphi_0\|_{H^{1/2}}. \tag{5.8}$$

Then

$$\|\Lambda_{f^2}[\varphi_0]\|_{H^{-1/2}(\partial\Omega,\mathbb{H})} \leq C_3 \|\varphi_0\|_{H^{1/2}(\partial\Omega,\mathbb{R})}$$

where $C_3 = C_{\Omega,\rho(f)}(\|\gamma_{\mathbf{n}}\| \|f\|_{L^\infty} + \|\gamma_{\mathbf{t}}\| \|f\|_{W^{1,\infty}})$. □

Proposition 107 justifies the claim made for the codomain of the D-N map for the monogenic case given in (4.10).

**Remark 108.** In the context of $\mathbb{R}^2$, the classical D-N map coincides with the tangential derivative of the Hilbert transform [4, Proposition 4.1]. In $\mathbb{R}^3$ the situation is intrinsically more complicated; some relations between the operators $\Lambda_{0,f^2}$, $\vec{\Lambda}_{f^2}$, and $\mathcal{H}_f$ will be developed in subsection 5.2. Here we only note that $\Lambda_{f^2}$ can be rewritten in various ways, as a consequence of

$\operatorname{tr} f\vec{W} = \mathcal{H}_f[\varphi_0]$ (Theorem 96):

$$\begin{aligned}\Lambda_{f^2}[\varphi_0] &= (f^2\nabla(W_0/f)|_{\partial\Omega})\eta = \operatorname{curl} f\vec{W}\Big|_{\partial\Omega}\cdot\eta - \operatorname{curl} f\vec{W}\Big|_{\partial\Omega}\times\eta \\ &= D(f\vec{W})\Big|_{\partial\Omega}\cdot\eta - D(f\vec{W})\Big|_{\partial\Omega}\times\eta = -(D(f\vec{W})|_{\partial\Omega})\eta \qquad (5.9)\end{aligned}$$

where $W_0 + \vec{W}$ is a solution of the main Vekua equation.

## 5.2 Norm properties of $\mathcal{H}_f$

Since the Vekua-Hilbert transform $\mathcal{H}_f$ is a generalization of the Hilbert transform $\mathcal{H}$, it is natural to expect that $\mathcal{H}_f$ preserves many of its properties; we will make use of the D-N mapping to investigate them. First we relate the Vekua-Hilbert transform $\mathcal{H}_f$ to the scalar D-N map $\Lambda_{0,f^2}$ and the vectorial D-N map $\vec{\Lambda}_{f^2}$ through the operator compositions (5.12) and (5.13).

**Proposition 109.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain and let $f^2 \in W^{1,\infty}(\Omega,\mathbb{R})$ be a proper conductivity. Then $\mathcal{H}_f$ can be written as*

$$\mathcal{H}_f[\varphi_0] = \operatorname{tr} M\vec{\Lambda}_{f^2}[\varphi_0] - \mathcal{H}\operatorname{tr} M\Lambda_{0,f^2}[\varphi_0] + \operatorname{tr} L[\operatorname{curl}(f^2\nabla(W_0/f))]. \quad (5.10)$$

**Proof.** First suppose that in fact $\varphi_0 \in H^{3/2}(\partial\Omega,\mathbb{R})$. Take $W_0 \in W^{1,2}(\Omega,\mathbb{R})$ satisfying (4.14)–(4.15). Since $\nabla(W_0/f) \in W^{1,2}(\Omega,\mathbb{R}^3)$, by the proof of Proposition 92 we have that

$$T_{0,\Omega}[f^2\nabla(W_0/f)] = -M[\Lambda_{0,f^2}[\varphi_0]]. \quad (5.11)$$

Thus we consider the associated trace $\alpha_0$ in the Vekua-Hilbert transform $\mathcal{H}_f$

(4.18) as constructed in the following way:

$$H^{1/2}(\partial\Omega,\mathbb{R}) \xrightarrow{\Lambda_{0,f^2}} H^{-1/2}(\partial\Omega,\mathbb{R}) \xrightarrow{\operatorname{tr} M} H^{1/2}(\partial\Omega,\mathbb{R}),$$

i.e.

$$\mathcal{H}[\alpha_0] = \mathcal{H}\operatorname{tr} M\Lambda_{0,f^2}[\varphi_0]. \tag{5.12}$$

By (2.14), the relation $\overrightarrow{T}_{2,\Omega}[\vec{w}] = -M[\gamma_{\mathbf{t}}(\vec{w})] - L[\operatorname{curl}\vec{w}]$ was proved for all $\vec{w} \in W^{1,2}(\Omega,\mathbb{R}^3)$, where $L$ is the right inverse of the Laplacian $\Delta$ given in (1.19). Therefore by (5.5)

$$\vec{\alpha} = \operatorname{tr} M\vec{\Lambda}_{f^2}[\varphi_0] + \operatorname{tr} L[\operatorname{curl}(f^2\nabla(W_0/f))]. \tag{5.13}$$

Thus (5.12)–(5.13) produce the expression (5.10) claimed for $\mathcal{H}_f$. This representation for $\mathcal{H}_f$ has been proved for functions $\varphi_0$ in the dense subspace $H^{3/2}(\partial\Omega,\mathbb{R})$, and by continuity is valid in the full space $H^{1/2}(\partial\Omega,\mathbb{R})$. □

**Proposition 110.** $\mathbb{R} \subseteq \operatorname{Ker}\mathcal{H}_f \cap H^{3/2}(\partial\Omega,\mathbb{R}) \subseteq \operatorname{Ker}\Lambda_{0,f^2} \subseteq H^{3/2}(\partial\Omega,\mathbb{R})$.

**Proof.** The first containment is straightforward from the uniqueness of the solutions of the conductivity equation. The proof of the second containment is a consequence of Proposition 100, equation (5.12) and the fact that $\operatorname{tr} M$ is an invertible operator from $L^2(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ [100, Th. 3.3]. Finally, the third containment follows from Proposition 29 (a). □

At the end of this section we will show that $\operatorname{Ker}\mathcal{H}_f$ in fact consists only of constants. We do not know whether the second containment of Proposition 110 is an equality for nonconstant $f$.

In [96, Th. 0.2] some estimates were presented to establish the continuous dependence of the scalar D-N map $\Lambda_{0,f^2}$ on the boundary values of the conductivity $f^2$. More specifically, as a consequence of [96, Th. 3.5] we have that $\|\Lambda_{0,f_n^2} - \Lambda_{0,f^2}\| \longrightarrow 0$ when $f_n \to f$ in $L^\infty$. We give a similar result for the quaternionic D-N map $\Lambda_{f^2}$ and for the Vekua-Hilbert transform $\mathcal{H}_f$:

**Theorem 111.** *Let $\Omega$ be a bounded Lipschitz domain. Let $\{f_n\} \subseteq W^{1,\infty}(\Omega,\mathbb{R})$ be a sequence of proper conductivities. Then*

*(a) if $f_n \to f$ in $L^\infty(\Omega,\mathbb{R})$, then $\|\,\mathcal{H}_{f_n} - \mathcal{H}_f\,\| \longrightarrow 0$;*

*(b) if $f_n \to f$ in $W^{1,\infty}(\Omega,\mathbb{R})$, then $\|\Lambda_{f_n^2} - \Lambda_{f^2}\| \longrightarrow 0$;*

*as operators on $H^{1/2}(\partial\Omega,\mathbb{R})$.*

**Proof.** Let $\varphi_0 \in H^{1/2}(\partial\Omega,\mathbb{R})$. Let $W_{0,n}, W_0 \in W^{1,2}(\Omega,\mathbb{R})$ be the respective extensions to solutions of the conductivity equations; that is,

$$\begin{aligned} \nabla\cdot f_n^2\,\nabla\,(W_{0,n}/f_n) = 0, &\quad \mathrm{tr}(W_{0,n}/f_n) = \varphi_0, \\ \nabla\cdot f^2\,\nabla\,(W_0/f) = 0, &\quad \mathrm{tr}(W_0/f) = \varphi_0. \end{aligned}$$

By (4.16), these unique solutions satisfy

$$\|\nabla(W_{0,n}/f_n)\|_{L^2} \le C_{\Omega,\rho(f_n)}\|\varphi_0\|_{H^{1/2}}, \quad \|\nabla(W_0/f)\|_{L^2} \le C_{\Omega,\rho(f)}\|\varphi_0\|_{H^{1/2}}.$$

It is a well-known fact about elliptic equations [96, Prop. 3.3] that

$$\|\nabla(W_{0,n}/f_n - W_0/f)\|_{L^2(\Omega)} \le c_n\|\varphi_0\|_{H^{1/2}(\partial\Omega)}, \tag{5.14}$$

where

$$c_n = \frac{\sup |f_n^2 - f^2|}{\inf f_n^2} \left( 1 + \left( \frac{\sup f^2}{\inf f^2} \right)^{1/2} \right).$$

Now consider the traces

$$\alpha_n = \alpha_{0,n} + \vec{\alpha}_n = \operatorname{tr} T_\Omega[-f_n^2 \nabla (W_{0,n}/f_n)],$$
$$\alpha = \alpha_0 + \vec{\alpha} = \operatorname{tr} T_\Omega[-f^2 \nabla (W_0/f)].$$

By (5.14), we have

$$\begin{aligned} &\|f_n^2 \nabla (W_{0,n}/f_n) - f^2 \nabla (W_0/f)\|_{L^2} \\ &\quad \leq \|f_n^2 - f^2\|_{L^\infty} \|\nabla (W_{0,n}/f_n)\|_{L^2} + \|f^2\|_{L^\infty} \|\nabla (W_{0,n}/f_n - W_0/f)\|_{L^2} \\ &\quad \leq \big( C_{\Omega,\rho(f_n)} \|f_n^2 - f^2\|_{L^\infty} + c_n \|f^2\|_{L^\infty} \big) \|\varphi_0\|_{H^{1/2}}. \end{aligned} \tag{5.15}$$

By (5.15) and the boundedness of the operators tr and $T_\Omega$ we have that

$$\begin{aligned} \|\vec{\alpha}_n - \vec{\alpha}\|_{H^{1/2}} &\leq \|\operatorname{tr}\| \|\overrightarrow{T}_{2,\Omega}\| \|f_n^2 \nabla (W_{0,n}/f_n) - f^2 \nabla (W_0/f)\|_{L^2} \\ &\leq \|\operatorname{tr}\| \|\overrightarrow{T}_{2,\Omega}\| \big( C_{\Omega,\rho(f_n)} \|f_n^2 - f^2\|_{L^\infty} + c_n \|f^2\|_{L^\infty} \big) \|\varphi_0\|_{H^{1/2}}. \end{aligned}$$

Analogously,

$$\begin{aligned} &\| \mathcal{H}[\alpha_{0,n}] - \mathcal{H}[\alpha_0] \|_{H^{1/2}} \\ &\quad \leq \| \mathcal{H} \| \|\operatorname{tr}\| \|T_{0,\Omega}\| \big( C_{\Omega,\rho(f_n)} \|f_n^2 - f^2\|_{L^\infty} + c_n \|f^2\|_{L^\infty} \big) \|\varphi_0\|_{H^{1/2}}. \end{aligned}$$

Since $c_n \to 0$, we obtain the limit of part (a). For part (b), by (5.8) and

(5.15) we have

$$
\begin{aligned}
&\|\vec{\Lambda}_{f_n^2}[\varphi_0] - \vec{\Lambda}_{f^2}[\varphi_0]\|_{H^{-1/2}} \\
&\qquad \le \|\gamma_{\mathbf{t}}\|\big(\|f_n^2\nabla(W_{0,n}/f_n) - f^2\nabla(W_0/f)\|_{L^2} \\
&\qquad\qquad + \|\operatorname{curl}(f_n^2\nabla(W_{0,n}/f_n) - f^2\nabla(W_0/f))\|_{L^2}\big) \\
&\qquad \le \|\gamma_{\mathbf{t}}\|\big(C_{\Omega,\rho(f_n)}\|f_n^2 - f^2\|_{W^{1,\infty}} + c_n\|f^2\|_{W^{1,\infty}}\big)\|\varphi_0\|_{H^{1/2}},
\end{aligned}
$$

as required. □

The stability question of the scalar D-N map asks whether two conductivities $f_1^2$, $f_2^2$ are close whenever $\Lambda_{0,f_1^2}$ is close to $\Lambda_{0,f_2^2}$. In [2, Th. 1] it was proved that there exists a continuous nondecreasing function $\omega\colon [0,\infty) \to [0,\infty)$ satisfying $\omega(t) \to 0$ as $t \to 0^+$ such that

$$\|f_1^2 - f_2^2\|_{L^\infty(\Omega)} \le \omega(\|\Lambda_{0,f_1^2} - \Lambda_{0,f_2^2}\|),$$

for a bounded open set $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, $f_i \in W^{s,2}(\Omega,\mathbb{R})$ ($L^2$ functions with derivatives up to order $s$ in $L^2$), $s > n/2$ and $n \ge 3$. However, the stability of the vector part $\vec{\Lambda}_{f^2}$ remains an open question.

In Theorem 102 it was established that $\varphi_0 \longmapsto \alpha_0$ and $\varphi_0 \longmapsto \vec{\alpha}$ are continuous. We will prove that these mappings are in fact compact when restricted to $\operatorname{Ker}\Lambda_{0,f^2}$ or $\operatorname{Ker}\vec{\Lambda}_{f^2}$.

**Proposition 112.** *Let $\Omega$ be a bounded $C^{1,1}$ Lipschitz domain and let $f \in W^{1,\infty}(\Omega,\mathbb{R})$ be a proper conductivity. The restrictions of $\mathcal{H}_f$ to $\operatorname{Ker}\Lambda_{0,f^2}$ and to $\operatorname{Ker}\vec{\Lambda}_{f^2}$ are compact mappings into $H^{1/2}(\partial\Omega,\mathbb{R}^3)$.*

**Proof.** Let $\varphi_0 \in \operatorname{Ker}\Lambda_{0,f^2}[\varphi_0]$, so the associated Teodorescu traces $\alpha_0, \vec{\alpha}$ are constructed as

$$\begin{array}{llll} \operatorname{Ker}\Lambda_{0,f^2} & \longrightarrow W^{2,\mathrm{div}\text{-}\mathrm{curl}}_{\mathbf{n}}(\Omega,\mathbb{R}^3) & \hookrightarrow L^2(\Omega,\mathbb{R}^3) & \longrightarrow H^{1/2}(\partial\Omega,\mathbb{H}), \\ \varphi_0 & \longmapsto \vec{v} & \longmapsto \vec{v} & \longmapsto \alpha_0+\vec{\alpha}, \end{array} \tag{5.16}$$

where $\vec{v}=f^2\nabla(W_0/f)$ and $\alpha_0+\vec{\alpha}=-\operatorname{tr}T_\Omega[\vec{v}]$. By (4.16), (5.6)–(5.7), the first mapping of (5.16) $\varphi_0\mapsto\vec{v}$ is a bounded operator from $H^{1/2}(\partial\Omega,\mathbb{R})$ to $W^{2,\mathrm{div}\text{-}\mathrm{curl}}_{\mathbf{n}}(\Omega,\mathbb{R}^3)$. Thus in fact all of the mappings shown are bounded. By Proposition 29, the inclusion mapping of $W^{2,\mathrm{div}\text{-}\mathrm{curl}}_{\mathbf{n}}(\Omega,\mathbb{R}^3)$ into $L^2(\Omega,\mathbb{R}^3)$ is compact. Therefore $\varphi_0\mapsto\alpha_0,\vec{\alpha}$ are compact, and in consequence $\mathcal{H}_f$ is also compact on $\operatorname{Ker}\Lambda_{0,f^2}$ as claimed. The proof for $\operatorname{Ker}\vec{\Lambda}_{f^2}$ is similar. □

In the following result we describe the Vekua-Hilbert transform $\mathcal{H}_f$ restricted to the kernel of the D-N operator $\Lambda_{0,f^2}$.

**Theorem 113.** *Let $\Omega$ be a $C^{1,1}$ bounded Lipschitz domain and let $f\in W^{1,\infty}(\Omega,\mathbb{R})$ be a proper conductivity. Then the Vekua-Hilbert transform $\mathcal{H}_f$ restricted to $\operatorname{Ker}\Lambda_{0,f^2}$ produces boundary values of monogenic constants in $\Omega^-$ which vanish at $\infty$.*

**Proof.** Let $\varphi_0\in\operatorname{Ker}\Lambda_{0,f^2}$. By Proposition 110, $f^2\nabla(W_0/f)\in W^{1,2}(\Omega,\mathbb{R}^3)$ . Taking the trace of (5.11) we have $\alpha_0=-\operatorname{tr}T_{0,\Omega}[f^2\nabla(W_0/f)]=0$, so

$$\mathcal{H}_f[\varphi_0]=\vec{\alpha}=\operatorname{tr}\overrightarrow{T}_{2,\Omega}[-f^2\nabla(W_0/f)]. \tag{5.17}$$

It is a classical fact [53, Prop. 8.1] that $T_\Omega[w](\vec{x})$ is always monogenic in $\Omega^-$ and tends to zero for $|\vec{x}|\to\infty$, so $T_{0,\Omega}[f^2\nabla(W_0/f)]$ vanishes in $\Omega^-$. By the conductivity equation, $f^2\nabla(W_0/f)$ is solenoidal and therefore [35, Prop. 3.1(i)] says that $T_{0,\Omega}[f^2\nabla(W_0/f)]$ is harmonic in $\Omega$, hence it vanishes in all of $\mathbb{R}^3$. Thus the vector field $\overrightarrow{T}_{2,\Omega}[f^2\nabla(W_0/f)] = T_\Omega[f^2\nabla(W_0/f)]$ is a monogenic constant in $\Omega^-$ vanishing at $\infty$, and the assertion follows from (5.17). $\square$

By Theorem 113 and Proposition 30, we know that

$$\dim(\operatorname{Ker}\mathcal{H}_f\,|_{\operatorname{Ker}\Lambda_{0,f^2}}) \leq \dim \mathrm{SI}_{\mathbf{t}}(\Omega^-) < \infty.$$

By Proposition 110, we have $\dim\operatorname{Ker}\mathcal{H}_f < \infty$. Therefore, since $\partial\Omega^- = \partial\Omega$ is connected we have $\operatorname{Ker}\mathcal{H}_f = \mathbb{R}$. We possess little information about the nature of $\operatorname{Ker}\Lambda_{0,f^2}$. It would be interesting, for example, to know whether all boundary values of exterior monogenic constants vanishing at $\infty$ are as Theorem 113.

# Conclusions and future work

We have given a general solution to the div-curl system, which is fundamental in mathematical physics.

This solution was then applied to several other related systems of differential equations, for example to find $f^2$-hyperconjugates of the main Vekua equation, to construct the pair of electromagnetic fields for solving the static Maxwell system with variable permeability, to give a right inverse of the operator representing the "Bers derivative" of solutions of the main Vekua equation, among others. Even though all the results in Chapter 3 were enunciated for star-shaped domains, we could remove this restriction and work in Lipschitz domains with connected complement (see Section 2.3).

We have shown the existence of a Hilbert transform associated to the main Vekua equation in bounded Lipschitz domains of $\mathbb{R}^3$. If we consider the $n$-dimensional main Vekua equation, that is for $\Omega \subseteq \mathbb{R}^n$ and $D = \sum_{i=1}^n e_i \partial_i$, we can generalize some results of Chapter 4 in the framework of Clifford algebras. For example, the construction of both the Vekua-Hilbert transform and the $f^2$-hyperconjugates of the main Vekua equation.

These results open the possibility of future research on the following questions:

- Div-curl system in multiply connected domains.
- Div-curl system in exterior domains.
- Development of Vekua analysis (formal powers, Cauchy Theorem, pseudoanalytic theory).
- Hardy and Bergman spaces for the three or $n$-dimensional Vekua equation.
- Inverse problems (Calderón problem, Gel'fand-Calderón problem).
- Transmutation operators related to the main Vekua equation.
- Numerical implementations of the above considerations.

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