CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE MATEMÁTICAS

“Solución de problemas con valores de frontera para ecuaciones diferenciales parciales parabólicas con coeficientes variables usando operadores de transmutación”

T E S I S

Que presenta

M. en C. Josafath Alfredo Otero Jiménez

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Ciudad de México **November 29, 2017**

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"Solution of boundary value problems for parabolic partial differential equations with variable coefficients using transmutation operators"

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Submitted by

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Advisors:

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Abstract

In the present work a complete system of solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t), \tag{1}$$

considered on a closed rectangle over the real plane is constructed. We assume that the coefficient q is a continuous complex valued function of an independent real variable x. The solutions represent the images of the heat polynomials under the action of a transmutation operator. The completeness of the system is with respect to the uniform norm in the closed rectangle. The system of solutions is shown to be useful for uniform approximation of solutions of initial boundary value problems for (2.1). The proposed numerical method is shown to reveal good accuracy.

For the case of a parabolic partial differential equation with time dependent potential of the form

$$\frac{\partial^2 u}{\partial x^2} - q(x)e^{i\omega t}u(x,t) = \frac{\partial u}{\partial t}(x,t), \tag{2}$$

where q is a continuously differentiable complex valued function, ω is a fixed real number and i is the imaginary unit; an explicit series representation of the images of the heat polynomials under the action of a transmutation operator is obtained. The construction of the series representation is based on a series representation of the transmutation kernel in terms of the positive integer powers of the exponential function involved in the potential of the equation. The truncated series representation of the transmuted heat polynomials shown to be useful for uniform approximation of solutions of initial boundary value problems for equation (2). Using the series representation of the transmutation kernel, a simple recursive integral procedure for the construction of the transmuted heat polynomials is presented.

Besides, an extension of the method of fundamental solutions for the equation (1) on a open rectangle using point sources outside the domain is presented. The method is based

on the construction of a system of functions being the images of the heat kernel under the action of a transmutation operator. A completeness result of the system of functions with respect to the uniform norm over the closed rectangle is obtained. A simple recursive procedure for calculation of the system is obtained. Then, using the collocation method an step by step method for approximation of solution of initial boundary value problems for the equation (1) is presented.

Finally, the use of the mapping property leads to an explicit solution of the noncharacteristic Cauchy problem for equation (1) with Cauchy data belonging to a Holmgren class of functions (see [3]). The solution is presented in terms of the formal powers arising in the spectral parameter power series (SPPS) method (see [14], [19]). On the other hand, an explicit formula for solution of the Cauchy problem for equation (1) with initial data of exponential growth order is presented. The solution is constructed with the aid of a transmutation operator defined over the whole real line for the space variable for which an adequate space is introduced. The Fourier-Legendre series representation of the transmutation kernel showed in [17] provide us a simple recursive procedure for constructions of the formula.

Resumen

En el trabajo se presenta un sistema completo de soluciones para la ecuación

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t), \tag{3}$$

considerada sobre un rectángulo del plano. Donde q es un coeficiente continuo complejo valuado de una variable real x. Las soluciones se construyen como las imágenes de los polinomios de calor bajo un operador de transmutación. La completes del sistema es con respecto a la norma uniforme en el rectángulo cerrado. El sistema muestra ser útil para la aproximación de soluciones a problemas con valores iniciales y de frontera para la ecuación (3). El método propuesto revela una buena aproximación.

En el caso de una ecuación diferencial parcial parabólica con un potencial dependiente del tiempo en la forma

$$\frac{\partial^2 u}{\partial x^2} - q(x)e^{i\omega t}u(x,t) = \frac{\partial u}{\partial t}(x,t), \tag{4}$$

donde q es una función complejo valuada continuamente diferenciable, ω es un número real positivo fijo e i es la unidad imaginaria; se muestra una representación explícita de las imágenes de los polinomios de calor bajo la acción de un operador de transmutación en forma de una serie. La representación se construye en base a una expresión en serie del núcleo integral del operador de transmutación en términos de las potencias enteras positivas de la función exponencial en el potencial. La serie truncada es útil para aproximar uniformemente los polinomios de calor transmutados y por tanto las soluciones de la ecuación (4). Usando la representación en series del núcleo de transmutación se presenta un procedimiento integral recursivo simple para la construcción de los polinomios de calor transmutados.

Además, se presenta una extensión del método de soluciones fundamentales para la ecuación (3) en un rectángulo cerrado usando fuentes puntuales en el exterior del dominio de la ecuación. El método esta basado en la construcción de las imágenes del núcleo de calor bajo la acción de un operador de transmutación. Un resultado de completes para

el sistema de funciones con respecto a la norma uniforme sobre el rectángulo cerrado es obtenido. Se presenta un procedimiento recursivo simple para el cálculo del sistema. En base a lo anterior se presenta un método para aproximar las soluciones de problemas con condiciones iniciales y de frontera para la ecuación (3) en un rectangulo del plano usando el método de colocación.

Finalmente, el uso de la propiedad de mapeo para los operadores de transmutación (ver [2]) lleva a una solución explicita del problema no caracteristico de Cauchy para la ecuacion (3) con datos iniciales en una clase de funciones de Holmgren (ver [3]). La solución es presentada en términos de las potencias formales que surgen en el método de series de potencias del parámetro espectral (el métod SPPS) (ver [14], [19]). Por otro lado, se presenta una fórmula explícita para la solución del problema de Cauchy para la ecuación (3) con datos iniciales de orden de crecimiento exponencial. La solución es construida con la ayuda de un operador de transmutación definido sobre toda la recta real para la variable espacial para el cual se introduce un espacio adecuado. El uso de la representación en series de Fourier-Legendre del núcleo de transmutación mostrado en [17] nos proporciona un procedimiento recursivo simple para la construcción de los coeficientes de la fórmula.

Contents

Overview

State of the art

The notion of a transmutation operator relating two linear differential operators was introduced in 1938 by J. Delsarte [7] and nowadays represents a widely used tool in the theory of linear differential equations (see, e.g., [1], [5], [26], [29]). For some classes of differential operators a transmutation operator can be realized in the form of a Volterra integral operator (see e.g., [6], [28]). The integral kernel of the transmutation operator can be obtained as a solution of a Goursat problem. In general the integral kernel can not be obtained explicitly and this restrict the application of the transmutation operator. There exist very few examples of the transmutation kernels available in a closed form (see [20]).

For $A := -\frac{d^2}{dx^2} + q(x)$, $B := -\frac{d^2}{dx^2}$ a transmutation operator can be realized in the form of a Volterra integral operator (see [29, Chapter 1]). In [2] the parametrized family of transmutation operators T_α, $\alpha \in \mathbb{C}$, was introduced. These family contains the transmutation operator for A and B introduced in [29]. In [2], [11], [17] and [20] some useful properties of the family T_α were proved as well as for the inverse transmutation operator.

In [2], [21], [22] a mapping property for transmutation operators in the family (1.14) was revealed making possible to apply the transmutation technique even when the integral kernel of the operator is unknown. The mapping property is very useful because of that fact that it is possible to know the result of application of the transmutation T_α to the non-negative integer powers of the independent variable, even not knowing the kernel of the transmutation operator. In particular, it was used to solve the Cauchy problem for the Klein-Gordon equation with a variable coefficient showing a remarkable performance in numerical applications (see [11]). Meanwhile, in [17] a Fourier-Legendre series representation of the integral kernel was obtained. The series representation was used for construction of a new representation of solutions of one dimensional Schrödinger equations.

On the other hand, the existence of a transmutation operator for the linear partial differential operators $D := \partial_x^2 - q(x,t) - \partial_t$, $C := \partial_x^2 - \partial_t$ and the possibility to construct complete systems of solutions by means of transmutation operators was proposed and explored by D. Colton (see [6]); in there, the approach developed requires the knowledge of the transmutation operators. In [25] and [10] a method of fundamental solutions for the heat operator C is proposed based on the completeness of the system of fundamental solutions restricted onto the parabolic boundary of the problem.

Contributions to the study of parabolic partial differential equations

In the present thesis, the solution of initial boundary value problem for parabolic partial differential equations with a variable coefficient of the form

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t), \tag{5}$$

where q is a continuous complex valued function of an independent real variable x, using the transmutation operator theory is studied.

First, in Chapter 2, a complete system of solutions of equation (5) considered on an open rectangle over the plane is obtained. The completeness of the system is with respect to the uniform norm in the closed rectangle. The system of solutions is shown to be useful for uniform approximation of solutions of initial boundary value problems for (5). The complete system of solutions is constructed with the aid of the transmutation operators relating (5) with the heat equation (see e.g.,[6], [11], [29]). The possibility to construct complete systems of solutions by means of transmutation operators was proposed and explored in [6], however the approach developed in [6] requires the knowledge of the transmutation operator. In the present work using a mapping property of the transmutation operators discovered in [2] we show that the construction of the complete system of solutions for equations of the form (5), representing transmuted heat polynomials, can be realized with no previous construction of the transmutation operator.

We illustrate the implementation of the complete system of the transmuted heat polynomials by a numerical solution of an initial boundary value problem for (5). The approximate solution is sought in the form of a linear combination of the transmuted heat polynomials and the initial and boundary conditions are satisfied by a collocation method. A remarkable accuracy is achieved in just few seconds using Matlab 2012 on a usual PC.

Following the approach in [6], for the parabolic partial differential equation with a time dependent coefficient of the form

$$\frac{\partial^2 u}{\partial x^2} - q(x)e^{i\omega t}u(x,t) = \frac{\partial u}{\partial t}(x,t) \tag{6}$$

where q is a continuously differentiable complex valued function, ω is a fixed positive real number and i is the imaginary unit; a system of functions for approximating solutions of equation (6) is obtained. The system of functions being an approximation of the transmuted heat polynomials is constructed in an explicit form using a transmutation operator in form of a second kind Volterra integral operator, relating (6) with the heat equation. In order to obtain the images of the heat polynomials under the action of the transmutation operator; a series representation of the transmutation kernel in terms of the positive integer powers of the complex exponential function involved in the potential of the equation is obtained. The coefficients of the series representation are time independent functions and some growth estimates on the uniform norm are obtained. Then, using the series representation of the transmutation kernel, we show that the construction of the system of functions can be realized in a recursive form. The system of functions is shown to be useful for uniform approximation of the transmuted heat polynomials hence it is useful for uniform approximation of solution of the initial boundary value problems for equation (6).

Next in Chapter 3, an extension of the method of fundamental solutions for equation (5) is presented. The method is based on the construction of a system of functions being the images of a system of fundamental solutions with external point sources for the heat equation under the action of a transmutation operator. Using the method of fundamental solutions for the heat equation with external point sources studied by Kupradze in [25] and by Johansson and Lesnic in [10], a completeness result of the system for approximating solutions of the equation (5) over a closed rectangle of the plane is proved. The completeness of the system is with respect to the uniform norm in the closed rectangle. The recently discovered Fourier-Legendre series representation for the transmutation kernel presented in [17] leads to a simple recursive procedure for calculating the system of functions in an explicit form. Then a step by step method for approximation of solution of initial boundary value problems for equation (5) is presented.

Finally in Chapter 4, explicit formulas for solution of the Cauchy problems are presented. The use of the mapping property leads to an explicit solution of the noncharacteristic Cauchy problem for equation (5) with Cauchy data belonging to a Holmgren class of functions [3]. The solution is presented in terms of the formal powers arising in the spectral parameter power series (SPPS) method (see [14], [19]). The solution of the Cauchy problem for equation (5) being the image of the solution of a Cauchy problem for heat equation under the action of a transmutation operator is presented. The use of the Fourier-Legendre series representation of the transmutation kernel leads to a system of functions as a convolution of Legendre polynomials with the heat kernel. The system provide us an explicit formula for the solution. Using the known recursive formulas of the Legendre polynomials a simple recursive procedure for the calculation of the system of functions is obtained. In oder to use the transmutation operator on the whole real line for the space variable an adequate functional space is introduced.

Approbation

The main contributions contained in this thesis were published in the following article.

- Vladislav V. Kravchenko, Josafath A. Otero, and Sergii M. Torba, " Analytic Approximation of Solutions of Parabolic Partial Differential Equations with Variable

Coefficients," Advances in Mathematical Physics, vol. 2017, Article ID 2947275, 5 pages, 2017. doi:10.1155/2017/2947275

The results contained in this thesis were accepted for presentation in the following congresses.

- XX International Symposium on Mathematical Methods Applied to the Sciences (SIMMAC), San José Costa Rica, 23-26 February 2016.

  Talk: Aproximación analítica a soluciones de ecuaciones parabólicas usando operadores de transmutación.

- International Conference Waves in Science and Engineering (WIS&E), Querétaro, México, August 22-26, 2016.

  Talk: Analytic approximation to solutions of parabolic differential equations and transmutation.

- Primeras Jornadas Matemáticas del CINVESTAV, D.F. México, Noviembre 22-25, 2016.

  Talk: Aproximación analítica a soluciones de ecuaciones parabólicas usando operadores de transmutación.

- 50 Congreso Nacional de la Sociedad Matemática Mexicana, Ciudad de México, Octubre 22-27, 2017.

  Talk: Una representación analítica de la solución al problema de Cauchy para ecuaciones diferenciales parciales parábolicas con coeficientes variables.

Chapter 1

Preliminaries

In this chapter some known results for the heat equation and the transmutation operator theory related to the present work are presented. These results are going to be used in the subsequent chapters.

The first section is dedicated to present some well known results of the heat equation about the solution of initial and boundary value problems which are the base of our study. The heat equation is a particular case of the parabolic partial differential equations with variable coefficients and this relation suggest us the use of the transmutation operator theory. In order to solve the initial and boundary value problems for parabolic partial differential equation with variable coefficients the section 2 present some necessary notation and definition concerning to the transmutation operator theory as well as a special system of functions called formal powers.

1.1 Parabolic partial differential equations with variable coefficients

In this section the parabolic partial differential equation in one space variable and one temporal variable is introduced.

Consider the general linear homogeneous parabolic partial differential equation of the second order in one space variable x and temporal variable t written in normal form as follows

$$\frac{\partial^2 u}{\partial x^2} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u(x,t) - c(x,t)\frac{\partial u}{\partial t}(x,t) = 0. \tag{1.1}$$

By making the change of the dependent variable

$$u(x,t) = v(x,t)\exp\left(-\frac{1}{2}\int_0^x a(\xi,t)d\xi\right)$$

we arrive at an equation for $v(x,t)$ of the same form as (1.1) but with $a \equiv 0$. On the other hand, for $c(x,t) > 0$ the change of variable

$$y = \int_0^x \sqrt{c(\xi,t)}d\xi$$

transforms equation (1.1) into an equation of the same form but with $c \equiv 1$. So, we can consider only parabolic partial differential equation in one space variable x and one temporal variable t in the canonical form

$$\frac{\partial^2 u}{\partial x^2} + Q(x,t)u(x,t) = \frac{\partial u}{\partial t}. \tag{1.2}$$

The function Q is called the potential of the equation (1.2). In what follows we shall assume that the potential Q is a continuous differentiable function with respect to the space variable x.

We are going to restrict our attention to classical solutions of (1.2) which are functions u, twice continuously differentiable with respect to x and continuously differentiable with respect to t and satisfying equation (1.2) in the domain of interest.

A well known particular case of (1.2) is the heat equation; it is obtained when the potential Q is identically zero. Then (1.2) takes the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}. \tag{1.3}$$

The heat equation is the base of our study so in the following section we present some results from the heat theory which are important to the present work (see e.g. [3], [30], [32] and [37]).

1.2 The heat equation

1.2.1 The fundamental solution

One of the most important solutions of the heat equation (1.3) is the so-called fundamental solution or heat kernel, defined by

$$F(x,t) := \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad x \in \mathbb{R}, t > 0. \tag{1.4}$$

It satisfies the heat equation (1.3) on the whole plane (x,t) with the exception of the origin $(0,0)$. The following theorem establishes some of its properties. These are useful in the proof of existence and uniqueness of the initial and boundary value problems for the heat equation.

Theorem 1 ([3]). *Properties of the fundamental solution.*

1. $F(x,t) > 0$ *for* $t > 0$.
2. *For fixed* $t > 0$, F *and its derivatives tend to zero exponentially fast as* $|x|$ *tends to infinity.*
3. *For any fixed* $\delta > 0$, $\lim_{t\downarrow 0} F(x,t) = 0$ *uniformly for all* $|x| \geq \delta$.
4. *For any fixed* $\delta > 0$, $\lim_{t\downarrow 0} \int_{|x|\geq\delta} F(x,t)dx = 0$.
5. *For all* $t > 0$, $\int_{-\infty}^{\infty} F(x,t)dx = 1$.
6. $\lim_{t\uparrow 0} - \int_0^t \frac{\partial F}{\partial x}(x, t-\eta)d\eta = -\frac{1}{2}$.
7. $\lim_{t\downarrow 0} - \int_0^t \frac{\partial F}{\partial x}(x, t-\eta)d\eta = \frac{1}{2}$.

As we see in the following sections, the fundamental solution is used to construct solutions to some specific problems for the heat equation; namely the Cauchy problems and the first boundary value problem among others.

1.2.2 The first boundary value problem

The first boundary value problem in the rectangle $\overline{\Omega}_1$ is one in which a solution of the equation (1.3), namely h, it is sought satisfying

$$h(-b,t) = f(t), \quad h(b,t) = g(t), \quad h(x,0) = \varphi(x) \tag{1.5}$$

for prescribed functions f, g and φ. Here Ω_1 is a bounded rectangle in $\mathbb{R}^2$ given by $\Omega_1 := (-b,b) \times (0,\tau)$. Consider Γ the so-called parabolic boundary of Ω_1 defined as

$$\Gamma := (\{-b\} \times [0,\tau]) \cup ([-b,b] \times \{0\}) \cup (\{b\} \times [0,\tau]) \,. \tag{1.6}$$

The following extreme-value theorem plays a crucial role in the uniqueness and in the continuous dependence of the solution to the first boundary value problem (1.3), (1.5) (see e.g., [3], [9] [13], [32], [37]) and it will be used in section 3.2.

Theorem 2 (Maximum Principle [32])**.** *If the function h, finite and continuous in the closed region $\overline{\Omega}_1$, satisfies the heat equation* (1.3) *in Ω_1, then the maximum and minimum values of the function h occur on the parabolic boundary Γ.*

Remark 1. *It is worth mentioning that the maximum principle for the heat equation in rectangles is valid using the L^2 norm as detailed in [30, Chapter 6].*

The existence and uniqueness of the solution of the first boundary value problem are well known results (see e.g., [3], [33] [37]).

Theorem 3 ([9])**.** *Let $f, g \in C^1([0,\tau])$ and $\varphi \in C^2([-b,b])$ satisfy $f(0) = \varphi(-b), \quad g(0) = \varphi(b)$. Then there exists a unique solution to* (1.3), (1.5) *and this solution depends continuously on the data.*

If in a certain problem, instead of the solution h of the heat equation (1.3) satisfying the initial and boundary conditions (1.5) we consider a solution $\tilde{h}$ of the same equation (1.3) but corresponding to another initial and boundary conditions

$$\tilde{h}(-b,t) = \tilde{f}(t), \quad \tilde{h}(b,t) = \tilde{g}(t), \quad \tilde{h}(x,0) = \tilde{\varphi}(x)$$

such that for all t in $[0,\tau]$ and x in $[-b,b]$

$$\left|f(t) - \tilde{f}(t)\right| \leq \varepsilon, \quad |g(t) - \tilde{g}(t)| \leq \varepsilon, \quad |\varphi(x) - \tilde{\varphi}(x)| \leq \varepsilon$$

for some given accuracy $\varepsilon > 0$ then

$$\left|h(x,t) - \tilde{h}(x,t)\right| \leq \varepsilon$$

for all (x,t) in $\overline{\Omega_1}$; i.e. the first boundary value problem (1.3),(1.5) is well posed according to Theorem 3 and the following theorem.

Theorem 4 ([32]). *If two solutions h_1 and h_2 of the heat equation* (1.3) *satisfy the inequality*

$$|h_1(x,t) - h_2(x,t)| \leq \varepsilon, \quad (x,t) \in \Gamma$$

for some $\varepsilon > 0$ then

$$|h_1(x,t) - h_2(x,t)| \leq \varepsilon, \quad (x,t) \in \overline{\Omega}_1.$$

1.2.3 The Cauchy problem

The Cauchy problem for the heat equation consists in the determination of a solution h of the equation (1.3) in the semiplane $\Omega_c := \mathbb{R} \times (0,\infty)$ such that on the line $t = 0$ it satisfies the condition

$$h(x,0) = \psi(x) \tag{1.7}$$

for a given function ψ.

The existence and uniqueness of the solution to the Cauchy problem (1.3), (1.7) is presented in terms of the following class of functions given by Mijailov [30].

Definition 1. *For $\sigma \geq 0$ let M_σ be the class of functions h for which there are positive constants C_1, C_2 such that*

$$|h(x,t)| \leq C_1 e^{C_2|x|^\sigma}, \quad (x,t) \in \Omega_c.$$

Theorem 5 (Uniqueness [30])**.** *The Cauchy problem (1.3), (1.7) in Ω_c cannot have more than one solution belonging to M_σ for any σ in $[0,2]$.*

Theorem 6 (Existence [3])**.** *For all piecewise continuous $\psi \in M_\sigma$, where $0 \leq \sigma < 2$, a solution of the Cauchy problem (1.3)-(1.7) in Ω_c is the function*

$$h(x,t) = \int_{\mathbb{R}} F(x-\xi,t)\psi(\xi)d\xi \tag{1.8}$$

where F is the fundamental solution given by (1.4).

The convolution (1.8) is called the Poisson transform.

1.2.4 The noncharacteristic Cauchy problem

The noncharacteristic Cauchy problem for the heat equation consists in finding a solution h of (1.3) in the rectangle of the first quarter plane $\Omega_{nc} := (0,b) \times (0,\tau)$ satisfying the following conditions

$$h(0,t) = F(t), \quad \frac{\partial h}{\partial x}(0,t) = G(t) \tag{1.9}$$

for given functions F, G.

The existence of the solution of the noncharacteristic Cauchy problem (1.3), (1.9) is presented in terms of the following class of functions.

Definition 2 (Holmgren's functions [3])**.** *For the positive constants γ_1, γ_2 and C_1, the Holmgren class $H(\gamma_1, \gamma_2, C_1, t_0)$ is the set of infinitely differentiable functions v defined on*

$|t - t_0| < \gamma_2$ *that satisfy*

$$\left|v^{(j)}(t)\right| \leq C_1 \gamma_1^{-2j} (2j)!, \quad j = 0, 1, \ldots$$

for all $t \in |t - t_0| < \gamma_2$.

Theorem 7 ([3])**.** *If* F *and* G *belong to* $H(b, \tau, C, 0)$*, then the power series*

$$h(x,t) = \sum_{k=0}^{\infty} \left(F^{(k)}(t) \frac{x^{2k}}{(2k)!} + G^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \right) \tag{1.10}$$

converges uniformly and absolutely for $|x| \leq r < b$ *and* h *is a solution of the noncharacteristic Cauchy problem*

$$\frac{\partial^2 h}{\partial x^2}(x,t) = \frac{\partial h}{\partial t}(x,t), \quad (x,t) \in \Omega_{nc}$$
$$h(0,t) = F(t), \quad \frac{\partial h}{\partial x}(0,t) = G(t), \quad t < \tau.$$

In order to obtain similar results for the first initial boundary value problem and Cauchy's problems for parabolic partial differential equations with variable coefficients in canonical form (1.2), in the following section the transmutation operator is introduced.

1.3 Transmutation operators

The notion of a transmutation operator relating two linear differential operators was introduced in 1938 by J. Delsarte [7] and nowadays represents a widely used tool in the theory of linear differential equations (see, e.g., [1], [5], [26], [29]). Very often in literature the transmutation operators are called the transformation operators. Here we keep the original term introduced by Delsarte and Lions [8].

We use a definition of a transmutation operator from [21] which is a modification of the definition given by Levitan [26], sufficient to the purpose of the present work. Let E

be a linear topological space and E_1 its linear subspace (not necessarily closed). Let A and B be linear operators: $E_1 \to E$.

Definition 3. *A linear invertible operator T defined on the whole E such that E_1 is invariant under the action of T is called a transmutation operator for the pair of operators A and B if it fulfills the following two conditions.*

1. *Both the operator T and its inverse T^{-1} are continuous in E;*

2. *The following operator equality is valid*

$$AT = TB$$

or which is the same

$$A = TBT^{-1}.$$

There are some cases of differential operators for which transmutation operators can be represented in a form of a Volterra integral operator (see e.g., [6], [28]). Whose integral kernel can be obtained as a solution of a Goursat problem. In general the integral kernel is unknown explicitly and this restrict the application of the transmutation operator. There exist very few examples of the transmutation kernels available in a closed form (see [20]).

One such particular case is the pair $A := -\frac{d^2}{dx^2} + q(x)$, $B := -\frac{d^2}{dx^2}$ for which an operator of transmutation can be realized in the form of a Volterra integral operator (see e.g. [29, Chapter 1])

$$Tv(x) = v(x) + \int_{-x}^{x} K(x,s)v(s)ds \tag{1.11}$$

where $K(x,s) = H(\frac{x+s}{2}, \frac{x-s}{2})$ and H is the unique solution of the Goursat problem

$$\frac{\partial^2 H}{\partial\xi\partial\eta}(\xi,\eta) = q(\xi+\eta)H(\xi,\eta),$$
$$H(\xi,0) = \frac{1}{2}\int_0^{\xi} q(r)dr, \quad H(0,\eta) = 0.$$

If the potential q is continuously differentiable, the kernel K itself is a classical solution of the Goursat problem

$$\frac{\partial^2 K}{\partial x^2} - q(x)K(x,s) = \frac{\partial^2 K}{\partial s^2}(x,s),$$
$$K(x,x) = \frac{1}{2}\int_0^x q(r)dr, \quad K(x,-x) = 0.$$

The transmutation operator (1.11) maps solutions of the equation

$$\frac{d^2 v}{dx^2} + \omega^2 v = 0$$

into solutions of the equation

$$\frac{d^2 u}{dx^2} - q(x)u + \omega^2 u = 0 \tag{1.12}$$

where ω is an arbitrary complex number. Following [29], denote by $e_0(i\omega, x)$ the solution of (1.12) satisfying the initial conditions

$$u(i\omega, 0) = 0, \quad \frac{du}{dx}(i\omega, 0) = i\omega,$$

then $e_0(i\omega, x) = T[e^{i\omega x}]$ (see [29], theorem 1.2.1). Moreover, the integral kernel K satisfies the following property of boundedness.

Theorem 8 ([28])**.** *If the operator A is defined on the whole axis $-\infty < x < \infty$, and the function q satisfies the condition*

$$\int_{-\infty}^{\infty} (1 + |x|)\,|q(x)|\,dx < \infty, \tag{1.13}$$

then the integral kernel K of the transmutation operator (1.11) *is uniformly bounded for $-\infty < x < \infty$ and the integral kernel of the inverse transmutation operator of* (1.11) *belongs to the function class M_1 in definition 1.*

This theorem is important in the solution of the Cauchy problem for parabolic partial differential equations with variable coefficients studied in section 4.1.

1.3.1 A parametrized family of transmutation operators

In [2] a parametrized family of operators T_α, $\alpha \in \mathbb{C}$, was introduced; it is given by

$$T_\alpha v(x) = v(x) + \int_{-x}^{x} \mathbf{K}(x, s; \alpha) v(s) ds \tag{1.14}$$

where

$$\mathbf{K}(x, s; \alpha) = \frac{\alpha}{2} + K(x, s) + \frac{\alpha}{2} \int_{s}^{x} K(x, r) - K(x, -r) dr. \tag{1.15}$$

Remark 2. *The integral kernel K of the transmutation operator* (1.11) *belongs to the family of integral kernels of the transmutation operators $\{T_\alpha\}_{\alpha\in\mathbb{C}}$, namely $\mathbf{K}(x, s; 0) = K(x, s)$ for any (x, s).*

In [2], [11], [17] and [20] some interesting properties of the family (1.14) were proved, here we write the most relevant of them to the present work.

Proposition 1 ([20])**.** *The operator T_α maps a solution v of an equation*

$$\frac{d^2 v}{dx^2} + \omega^2 v = 0,$$

where ω is a complex number, into the solution u of the equation

$$\frac{d^2 u}{dx^2} - q(x) u + \omega^2 u = 0$$

with the following correspondence of initial values

$$u(0) = v(0), \quad \frac{du}{dx}(0) = \frac{dv}{dx}(0) - \alpha v(0).$$

Theorem 9 ([20])**.** *In order for the function $\mathbf{K}$ to be the kernel of a transmutation operator* *acting as described in proposition 1, it is necessary and sufficient that*

$$\mathbf{H}(\xi,\eta;\alpha) := \mathbf{K}(\xi+\eta,\xi-\eta;\alpha)$$

be a solution of the Goursat problem

$$\frac{\partial^2 \mathbf{H}}{\partial\xi\partial\eta}(\xi,\eta;\alpha) = q(\xi+\eta)\mathbf{H}(\xi,\eta;\alpha),$$
$$\mathbf{H}(\xi,0) = \frac{\alpha}{2} + \frac{1}{2}\int_0^\xi q(r)dr, \quad \mathbf{H}(0,\eta) = \frac{\alpha}{2}.$$

If the potential q is continuously differentiable, the function $\mathbf{K}$ itself should be the solution of the Goursat problem

$$\frac{\partial^2 \mathbf{K}}{\partial x^2} - q(x)\mathbf{K}(x,s;\alpha) = \frac{\partial^2 \mathbf{K}}{\partial s^2}(x,s;\alpha),$$
$$\mathbf{K}(x,x;\alpha) = \frac{\alpha}{2} + \frac{1}{2}\int_0^x q(r)dr, \quad \mathbf{K}(x,-x;\alpha) = \frac{\alpha}{2}.$$

If the potential q is n times continuously differentiable then the kernel $\mathbf{K}$ is $n+1$ times continuously differentiable with respect to both independent variables.

Indeed $\{T_\alpha\}_{\alpha\in\mathbb{C}}$ is a family of transmutation operators according to the following theorem.

Theorem 10 ([24]). *Let $q \in C[-b,b]$. Then the operator T_α given by (1.14) satisfies the equality*

$$\left(-\frac{d^2}{dx^x} + q(x)\right) T_\alpha[v] = T_\alpha\left[-\frac{d^2 v}{dx^2}\right]$$

for any $v \in C^2[-b,b]$.

The inverse operator T_α^{-1} is calculated as a Volterra integral operator

$$T_\alpha^{-1}u(x) = u(x) + \int_{-x}^{x} \mathbf{L}(x,s;\alpha)u(s)ds.$$

Here the integral kernel $\mathbf{L}$ satisfies the Goursat problem

$$\frac{\partial^2 \mathbf{L}}{\partial x^2} - q(x)\mathbf{L}(x,s;\alpha) = \frac{\partial^2 \mathbf{L}}{\partial s^2}(x,s;\alpha),$$
$$\mathbf{L}(x,x;\alpha) = C_1 - \frac{1}{2}\int_0^x q(r)dr, \quad \mathbf{L}(x,-x;\alpha) = C_1.$$

The constant C_1 is calculated using Proposition 1 and is equal to $-\alpha/2$. Thus, $\mathbf{L}(x,s;\alpha) = -K(s,x;\alpha)$ (see [20]).

Proposition 2 ([11])**.** *Let q be a continuous complex valued function of an independent real variable $x \in [-b,b]$. Then the kernel $\mathbf{K}$ in the square $|x| \leq b$, $|t| \leq b$ satisfies the following estimate*

$$|\mathbf{K}(x,s;\alpha)| \leq \frac{|\alpha|}{2} I_0(\sqrt{c(x^2-t^2)}) + \frac{1}{2}\frac{\sqrt{c(x^2-t^2)}I_1(\sqrt{c(x^2-t^2)})}{|x-t|}$$

where $c = \max_{[-b,b]}|q(x)|$ and I_0 and I_1 are modified Bessel functions of the first kind.

Since the function $I_1(x)/x$ is monotone increasing for $x > 0$, it is obtained that

$$\frac{\sqrt{c(x^2-t^2)}I_1(\sqrt{c(x^2-t^2)})}{|x-t|} \leq 2\sqrt{c}I_1(b\sqrt{c})$$

for $|x| \leq b$ and $|t| \leq b$, and the following estimate for the norms of the transmutation operator and of its inverse immediately follows from Proposition 2.

Corollary 1 ([11])**.** *The following estimate holds*

$$\max\{\|T_\alpha\|, \|T_\alpha^{-1}\|\} \leq 1 + b(|h| I_0(b\sqrt{c}) + 2\sqrt{c}I_0(b\sqrt{c}))$$

where $c = \max_{[-b,b]}|q(x)|$ and I_0 and I_1 are modified Bessel functions of the first kind.

In [2] a mapping property for a certain transmutation operator in the class of the family (1.14) was revealed making possible to apply the transmutation technique even when the integral kernel of the operator is unknown, expanding the use of transmutations.

In what follows, such mapping property plays a crucial role; in order to explain it a system of recursive integrals is presented (see [2]).

1.3.2 System of recursive integrals

Let $f \in C^2(-b,b) \cap C^1[-b,b]$ be a complex valued function such that $f(x) \neq 0$ for any $x \in [-b,b]$. The interval $(-b,b)$ is supposed to be finite. Consider two sequences of recursive integrals (see [14], [19], [16])

$$X^{(0)} \equiv 1, \quad X^{(n)}(x) = n \int_{x_0}^{x} X^{(n-1)}(s)(f^2(s))^{(-1)^n} ds, \quad n \in \mathbb{N} \tag{1.16}$$

$$\tilde{X}^{(0)} \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_{x_0}^{x} \tilde{X}^{(n-1)}(s)(f^2(s))^{(-1)^{n-1}} ds, \quad n \in \mathbb{N} \tag{1.17}$$

where x_0 is an arbitrary fixed point in $[-b,b]$.

Definition 4 ([2]). *The family of functions $\{\varphi_k\}_{k=0}^{\infty}$ constructed according to the rule*

$$\varphi_k(x) = \begin{cases} f(x)X^{(k)}(x), & k \ \textit{impar} \\ f(x)\tilde{X}^{(k)}(x), & k \ \textit{par} \end{cases} \tag{1.18}$$

is called ***the system of formal powers associated with*** *f. As was shown in [22],* (1.18) *may be defined even when the condition of non-vanishing for f is removed.*

In [15] it was shown that the system $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $L^2(-b,b)$. In [16] its completeness in the space of continuous and piecewise continuously differentiable functions with respect to the maximum norm was obtained and the corresponding series expansions in terms of the functions φ_k were studied.

The formal powers arise in the spectral parameter power series (SPPS) representation for solutions of the one-dimensional Schrödinger equation ([14], [19]). The following theorem explains the SPPS method.

Theorem 11 (The SPPS representation [14], [19]). *Let q be a continuous complex valued function of an independent real variable $x \in [-b, b]$, λ be an arbitrary complex number. Suppose there exists a solution f of the equation*

$$\frac{d^2 f}{dx^2} - q(x) f(x) = 0, \tag{1.19}$$

on $[-b, b]$ such that $f \in C^2[-b, b]$ and $f(x) \neq 0$ on $[-b, b]$. Then the general solution of the equation

$$\frac{d^2 u}{dx^2} - q(x) u = \lambda u$$

on $(-b, b)$ has the form

$$u = c_1 u_1 + c_2 u_2$$

where c_1 and c_2 are arbitrary complex constants,

$$u_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \varphi_{2k}, \quad u_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \varphi_{2k+1}$$

and both series converge uniformly on $[-b, b]$.

The solutions u_1 and u_2 satisfy the initial conditions

$$\begin{aligned} u_1(x_0) = f(x_0), \quad & \tfrac{du_1}{dx}(x_0) = \tfrac{df}{dx}(x_0) \\ u_2(x_0) = 0, \quad & \tfrac{du_2}{dx}(x_0) = 1/f(x_0). \end{aligned}$$

The following theorem is very useful because due to it, in spite of not knowing the kernel of the transmutation operator, it becomes possible to know the result of application of the transmutation T_α to the non-negative integer powers of the independent variable.

Theorem 12 (Mapping property [2]). *Let q be a continuous complex valued function of an independent real variable $x \in [-b, b]$ for which there exists a particular solution f of (1.19) such that $f \in C^2[-b, b]$ and $f(x) \neq 0$ on $[-b, b]$ and normalized as $f(0) = 1$. Denote $\alpha := f'(0) \in \mathbb{C}$. Suppose T_α is the operator defined by (1.14) where the kernel K*

is a solution of the problem (1.16) *and* φ_k, $k \in \mathbb{N}_0$ *are functions defined by* (1.18) *with* $x_0 = 0$. *Then*

$$T_\alpha[x^k] = \varphi_k(x), \quad k \in \mathbb{N}_0. \tag{1.20}$$

In particular, it was used to solve the Cauchy problem for the Klein-Gordon equation with a variable coefficient showing a remarkable performance in numerical applications [11].

Finally another important theorem for the subsequent results is the following representation of the integral kernel K of (1.11) in terms of the Legendre-Fourier series using the formal powers $\{\varphi_k\}_{k\in\mathbb{N}_0}$ (1.18).

Let P_n denote the Legendre polynomial of order n, $l_{k,n}$ be the corresponding coefficient of x^k, that is $P_n(x) = \sum_{k=0}^{n} l_{k,n}x^k$.

Theorem 13 ([17])**.** *The transmutation kernel* $\mathbf{K}$ *of* (1.14) *has the form*

$$\mathbf{K}(x, s; \alpha) = \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} P_n\left(\frac{s}{x}\right) \tag{1.21}$$

where for every $x \in (0, b]$ *the series converges uniformly with respect to* $t \in [-x, x]$. *Here*

$$\beta_n(x) = \frac{2n+1}{2}\left(\sum_{k=0}^{n} \frac{l_{k,n}\varphi_k(x)}{x^k} - 1\right).$$

In [17] such representation was used to construct the solution of the one-dimensional Schrödinger equation expanding the use of transmutation even more.

With the aid of this last theorem an extension of the method of fundamental solutions for the first boundary value problem for parabolic partial differential equation with variable coefficients is presented in Chapter 3. Moreover Theorem 13 allows us to solve the Cauchy problem for parabolic partial differential equations with variable coefficients in subsection 4.1.2.

Chapter 2

Transmuted heat polynomials

In this chapter two systems of functions approximating the solutions of parabolic partial differential equations with variable coefficients are obtained. The variable coefficient can be a time independent or periodic time dependent potential. These systems of functions are shown to be useful for uniform approximation of solutions of boundary value problems.

The systems are constructed with the aid of the transmutation operators relating (1.2) with the heat equation (see e.g., [6], [11], [29]). The possibility to construct such systems by means of transmutation operators was proposed and explored by D. Colton (see [6]), however the approach presented in [6] requires the explicit knowledge of the transmutation operators.

The approach presented in this chapter is based on the mapping property of the transmutation operators exposed in Theorem 12 for the case of a time independent potential in one space variable (Section 2.1). Meanwhile for a periodic time dependent potential the construction is based on the knowledge of a series representation of the transmutation kernel in terms of exponential functions which we present in Theorem 16 (Section 2.2)

2.1 Transmuted heat polynomials for parabolic partial differential operators with time independent potential

In this section a complete system of solutions for the equation

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t) \tag{2.1}$$

considered on $\Omega := (-b,b) \times (0,\tau)$ is obtained. We assume that the potential $q \in C[-b,b]$ may be complex valued. The completeness of the system is with respect to the uniform norm in the closed rectangle $\bar{\Omega}$. The system of solutions may be applied, in particular, for approximation of solutions of initial boundary value problems for (2.1).

In Section 2.1.3 using the mapping property of the transmutation operators exposed in Theorem 12 we show that the construction of the complete systems of solutions for equations of the form (2.1), representing transmuted heat polynomials, can be realized with no previous construction of the transmutation operator.

We illustrate the implementation of the complete system of the transmuted heat polynomials by a numerical solution of an initial boundary value problem for (2.1). The approximate solution is sought in the form of a linear combination of the transmuted heat polynomials and the initial and boundary conditions are satisfied by a collocation method. A remarkable accuracy is achieved in few seconds using Matlab 2012 on a usual PC.

2.1.1 The heat polynomials

In this section the heat polynomials and some of their main properties are introduced. Let us consider the heat equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}. \tag{2.2}$$

The heat polynomials were introduced and widely studied in 1959 by Rosenbloom and Widder [31] and described further by Widder [34, 35, 36]. Heat polynomials are mainly used to construct an approximate solution of a given problem in a form of linear combination of the polynomials; it means that they can serve as a basis for expansion of other solutions of the heat equation.

A heat polynomial h_n of degree n is defined as the coefficient of $z^n/n!$ in the power series expansion

$$e^{zx+z^2t} = \sum_{n=0}^{\infty} h_n(x,t)\frac{z^n}{n!}, \quad -\infty < x < \infty, \quad t > 0. \tag{2.3}$$

They can be obtained from Cauchy's product of two power series, since $e^{zx+z^2t} = e^{zx}e^{z^2t}$. Setting

$$e^{zx} = \sum_{n=0}^{\infty} a_n z^n, \quad e^{z^2t} = \sum_{n=0}^{\infty} b_n z^n$$

where

$$a_n = \frac{x^n}{n!}, \quad b_n = \begin{cases} \frac{t^{n/2}}{(n/2)!}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases},$$

it follows that

$$e^{zx+z^2t} = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n = \sum_{j=0}^{n} b_j a_{n-j} = \sum_{k=0}^{[n/2]} \frac{t^k}{k!}\frac{x^{n-2k}}{(n-2k)!},$$

here $[\cdot]$ denotes the entire part of the number. From (2.3) it follows that

$$h_n(x,t) = n!c_n = \sum_{k=0}^{[n/2]} c_{nk}x^{n-2k}t^k, \quad c_{nk} = \frac{n!}{(n-2k)!k!}. \tag{2.4}$$

The first six heat polynomials are

$$h_0(x,t) = 1, \qquad h_1(x,t) = x, \qquad h_2(x,t) = x^2 + 2t,$$
$$h_3(x,t) = x^3 + 6xt, \quad h_4(x,t) = x^4 + 12x^2t + 12t^2, \quad h_5(x,t) = x^5 + 20x^3t + 60xt^2.$$

From simple derivation of (2.3) it is obtained that

$$\frac{\partial}{\partial x} h_n(x,t) = n h_{n-1}(x,t), \qquad \frac{\partial}{\partial t} h_n(x,t) = n(n-1) h_{n-2}(x,t).$$

Hence the following equality is valid

$$\frac{\partial^2}{\partial x^2} h_n(x,t) = \frac{\partial}{\partial t} h_n(x,t), \quad -\infty < x < \infty, \quad t > 0,$$

which means that the set of polynomials (2.4) is a family of solutions of (2.2).

The heat polynomials can be described in terms of Hermite polynomials of degree n, H_n [31]. Because of the fact that

$$e^{2zx - z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

it follows that

$$h_n(x,t) = (-t)^{n/2} H_n\left(\frac{x}{(-4t)^{1/2}}\right).$$

The heat polynomials can be used to approximate solutions of boundary value problems; for example of the first boundary value problem for heat equation. In such case the approximate solution is sought in the form of a linear combination of the polynomials (2.4); this solution satisfies the heat equation. In order to satisfy the initial and boundary conditions a discrepancy functional has to be minimized. In this way the coefficients of the linear combination of heat polynomials are calculated. The possibility of approximation is guaranted by the following theorem.

Theorem 14 ([6]). *Let h be a solution of the heat equation (2.2) in $(-b,b)\times(0,\tau)$ which is continuous in $[-b,b]\times[0,\tau]$. Then, for any given $\varepsilon>0$ there exists $N\in\mathbb{N}$ and the constants $a_0,a_1,\ldots,a_N$ such that*

$$\max_{[-b,b]\times[0,\tau]}\left|h(x,t)-\sum_{n=0}^{N}a_nh_n(x,t)\right|<\varepsilon.$$

This statement means that (2.4) is **a complete system of solutions of** (2.2) with respect to the uniform norm.

2.1.2 Transmutation operators

Fix α a complex constant and let f be the solution of (1.19) fulfilling the condition of Theorem 11 on $[-b,b]$ such that $f(0)=1,\ f'(0)=\alpha$.

As was shown in Section 1.3, for any $q\in C[-b,b]$ there exists a function $\mathbf{K}$ defined on the domain $0\le|s|\le x\le b$, continuously differentiable with respect to both arguments, such that the equality

$$ATv=TBv \tag{2.5}$$

is valid for all $v\in C^2[-b,b]$, where $A:=\frac{\partial^2}{\partial x^2}-q$, $B:=\frac{\partial^2}{\partial x^2}$ and T has the form of a second kind Volterra integral operator

$$Tv(x):=v(x)+\int_{-x}^{x}\mathbf{K}(x,s;\alpha)v(s)ds. \tag{2.6}$$

The function $\mathbf{K}$ is chosen so that $T[1]=f$ (see the mapping property Theorem 12). When $q\in C^1[-b,b]$ such function $\mathbf{K}$ is the unique solution of the Goursat problem

$$\frac{\partial^2\mathbf{K}}{\partial x^2}-q(x)\mathbf{K}(x,s;\alpha)=\frac{\partial^2\mathbf{K}}{\partial s^2}(x,s;\alpha),$$
$$\mathbf{K}(x,x;\alpha)=\frac{\alpha}{2}+\frac{1}{2}\int_0^x q(r)dr,\quad \mathbf{K}(x,-x;\alpha)=\frac{\alpha}{2}$$

For any $q\in C[-b,b]$ the kernel $\mathbf{K}$ can be defined as $\mathbf{K}(x,s;\alpha)=\mathbf{H}\big(\frac{x+s}{2},\frac{x-s}{2};\alpha\big)$, $|s|\le$

$|x| \leq b$, $\mathbf{H}$ being the unique solution of the Goursat problem

$$\frac{\partial^2 \mathbf{H}}{\partial \xi\, \partial \eta}(\xi, \eta; \alpha) = q(\xi + \eta)\mathbf{H}(\xi, \eta; \alpha),$$
$$\mathbf{H}(\xi, 0; \alpha) = \frac{\alpha}{2} + \frac{1}{2}\int_0^{\xi} q(s)\, ds, \qquad \mathbf{H}(0, \eta; \alpha) = \frac{\alpha}{2}.$$

The following results play a crucial role in what follows and they are an immediate consequence of the properties of the transmutation operators in Section 1.3.

Proposition 3. *The operator T given by* (2.6) *is of transmutation for the operators $D := \frac{\partial^2}{\partial x^2} - q - \frac{\partial}{\partial t}$, $C := \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$ in the functional space $C^2[-b, b] \times C^1[0, \tau]$.*

Proof. Because of the fact that the transmutation kernel $\mathbf{K}$ does not depend on the time variable t it is true that $\partial_t T w = T \partial_t w$ for every $w(x, t) \in C^2[-b, b] \times C^1[0, \tau]$. Hence, from the equation (2.5) it follows that

$$DTu = ATu - \partial_t Tu = TBu - T\partial_t u = TCu$$

for any $u \in C^2[-b, b] \times C^1[0, \tau]$. ∎

2.1.3 The transmuted heat polynomials

In this section a complete system of solutions of the equation (2.1) is presented.

Consider the heat polynomials $\{h_n\}_{n \in \mathbb{N}_0}$ defined in Section 2.1.1 by

$$h_n(x, t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{t^k x^{n-2k}}{k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Denote by $u_n = Th_n$ the images of the heat polynomials under the action of the transmutation operator T (2.6). From the mapping property of T (Theorem 12) we obtain that

$$u_n(x, t) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{t^k \varphi_{n-2k}(x)}{k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

Indeed, we have that $\{u_n\}_{n\in\mathbb{N}_0}$ are solutions of (2.1) for all $-b < x < b$ and $t > 0$

$$\frac{\partial^2 u_n}{\partial x^2} - q(x)u_n(x,t) - \frac{\partial u_n}{\partial t}(x,t) = T\left(\frac{\partial^2 h_n}{\partial x^2} - \frac{\partial h_n}{\partial t}(x,t)\right) = 0.$$

The completeness of the heat polynomials in Theorem 14 Section 2.1.1 and the uniform boundedness of T and T^{-1} in Corollary 1 imply the completeness of the transmuted heat polynomials (2.8) in the space of classical solutions of (2.1). Thus, the following statement is true.

Theorem 15. *Let u be continuous function in $\overline{\Omega}$ satsifying (2.1) in Ω. Then, given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and constants $a_0, a_1, \ldots, a_N$ such that*

$$\max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{n=0}^{N} a_n u_n(x,t) \right| < \varepsilon.$$

Proof. Choose $\varepsilon > 0$. Consider $h = T^{-1}u$. Due to the completeness of the heat polynomials (2.4) in Theorem 14, for any $\varepsilon_1 > 0$ there exists $N \in \mathbb{N}$ and constants $a_0, a_1, \ldots, a_N$ such that

$$\max_{(x,t)\in\overline{\Omega}} \left| h(x,t) - \sum_{n=0}^{N} a_n h_n(x,t) \right| < \varepsilon_1.$$

Then

$$\max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{n=0}^{N} a_n u_n(x,t) \right| = \max_{(x,t)\in\overline{\Omega}} \left| Th(x,t) - \sum_{n=0}^{N} a_n Th_n(x,t) \right| < C\varepsilon_1$$

where the constant C is the uniform norm of T. The choice of $\varepsilon_1 = \varepsilon/C$ finishes the proof. ■

2.1.4 Solution of the first boundary value problem

In this section we proceed to the step by step construction of a method to solve boundary value problems. The method is explained on the first boundary value problem; nevertheless other boundary value problems can be solved in analogous way. Thus, we consider the parabolic partial differential equation (2.1) subject to the Dirichlet boundary conditions

$$u(-b,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad t \in [0,\tau] \tag{2.9}$$

and the initial condition

$$u(x,0) = \varphi(x), \quad x \in [-b,b] \tag{2.10}$$

where ψ_1, ψ_2 and φ are continuously differentiable functions satisfying the compatibility conditions

$$\psi_1(0) = \varphi(-b), \quad \psi_2(0) = \varphi(b).$$

It is a well known result that the problem (2.1), (2.9), (2.10) is well posed, see Section 1.2.2.

The completeness of the transmuted heat polynomials given by (2.8) in Theorem 15 suggests the following simple method to approximate the solution of problem (2.1), (2.9), (2.10). The approximate solution $\tilde{u}$ is sought in the form of a linear combination

$$\tilde{u}(x,t) = \sum_{n=0}^{N} a_n u_n(x,t). \tag{2.11}$$

Since every u_n is a solution of (2.1), their linear combination satisfies (2.1) as well. The coefficients $\{a_n\}_{n=0}^{N}$ are sought in such way that $\tilde{u}$ satisfy the initial and the boundary conditions approximately. For this we used the collocation method where M points $\{(x_i,t_i)\}_{i=1}^{M}$ are chosen on the parabolic boundary Γ given by (1.6). Then, imposing the conditions (2.9) and (2.10) onto the approximate solution (2.11) the following $M \times (N+1)$

linear system of equations for the coefficients $\{a_n\}_{n=0}^N$ is obtained

$$\sum_{n=0}^{N} a_n u_n(x_i,t_i) = \begin{cases} \psi_1(t_i), & \text{if } x_i = -b \\ \varphi(x_i), & \text{if } t_i = 0 \\ \psi_2(t_i), & \text{if } x_i = b \end{cases}, \qquad i = 1,\dots,M. \tag{2.12}$$

The solution of the system (2.12) gives the approximate solution (2.11) on Ω.

Numerical illustration

We present a numerical example of the application of the method described in the previous section. It reveals a remarkable accuracy with very little computational efforts. The implementation was realized in Matlab 2012. In the appendix, the program is shown.

On the first step a nonvanishing solution f of (1.19) was computed using the SPPS method (see [19], [22]). The formal powers φ_k were constructed like in [11] using the `spapi` and `fnint` Matlab routines from the spline toolbox. Then the transmuted heat polynomials (2.8) were calculated. In order to obtain a unique solution of the system (2.12) $N+1$ equally spaced points on the parabolic boundary were chosen. Finally, the approximate solution (2.11) was computed on a mesh of 200×100 points in the interior of the rectangle Ω and compared with the corresponding exact solution.

Example 1. *Consider the initial Dirichlet problem*

$$u_{xx}(x,t) - x^2 u(x,t) = u_t(x,t), \qquad (x,t) \in (-1,1)\times(0,1), \tag{2.13}$$

$$u(x,0) = e^{-0.5x^2}, \qquad x \in [-1,1], \tag{2.14}$$

$$u(-1,t) = u(1,t) = e^{-0.5-t}, \qquad t \in [0,1]. \tag{2.15}$$

The exact solution of this problem has the form

$$u(x,t) = \exp\left(-\frac{1}{2}x^2 - t\right).$$

The distribution of the absolute error of the approximate solution for $N = 26$ *is presented on Figure 2.1. The maximum absolute error of the approximate solution is of order* 10^{-13}.

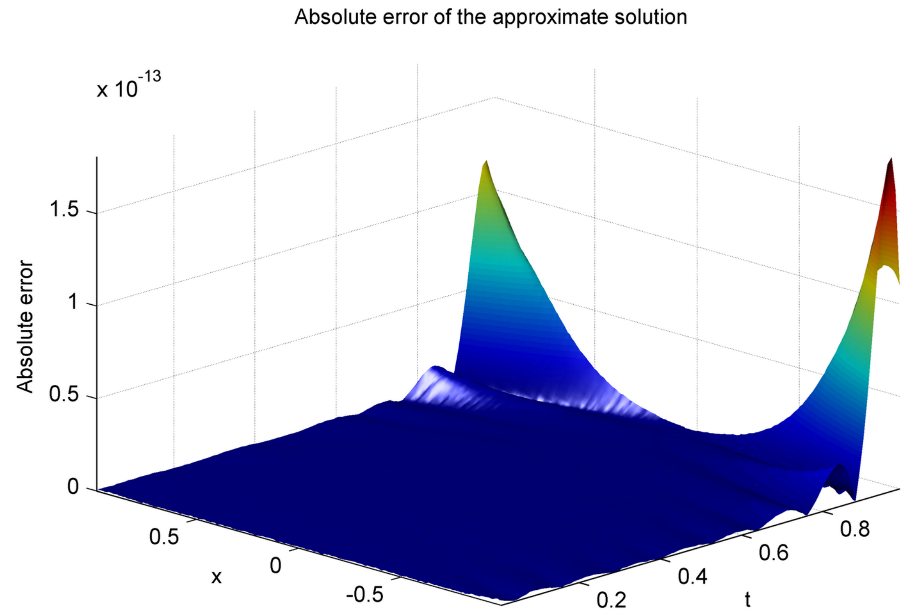

Figure 2.1: The absolute value of the difference $|u(x,t) - u_{26}(x,t)|$ between the exact and the approximate solutions for the problem (2.13)–(2.15).

It is often stated that boundary collocation methods (in particular, the heat polynomials method) lead to ill-conditioned systems of linear equations, see [4], [12], [27]. It is also the case for the proposed method. As is illustrated in Table 2.1, the condition number of the matrix in (2.12) *grows rather fast. Nevertheless, the straightforward implementation of the proposed method presented no numerical difficulties. The convergence and the robustness of the method are illustrated in Table 2.1 where the maximum absolute and the maximum relative error of the approximate solution for different values of* N *used for approximation* (2.11) *are presented. As one can appreciate, the convergence rate for small values of* N *is exponential. And even taking values of* N *much larger than the optimum one do not lead to any problem for collocation method nor to significant precision lost. Moreover, a simple test based on the accuracy of fulfilment of the initial and boundary conditions* (2.14)–(2.15) *can be utilized to estimate both the optimal* N *and the accuracy of the obtained approximate* *solution.*

N	Max. absolute error	Max. relative error	Cond. number
5	$2.3 \cdot 10^{-2}$	$6.2 \cdot 10^{-2}$	55.7
10	$1.4 \cdot 10^{-4}$	$5.5 \cdot 10^{-4}$	$1.82 \cdot 10^{5}$
15	$9.6 \cdot 10^{-8}$	$4.0 \cdot 10^{-7}$	$3.65 \cdot 10^{9}$
20	$2.0 \cdot 10^{-10}$	$8.6 \cdot 10^{-10}$	$1.76 \cdot 10^{14}$
23	$7.6 \cdot 10^{-13}$	$3.2 \cdot 10^{-12}$	$1.67 \cdot 10^{17}$
26	$1.8 \cdot 10^{-13}$	$7.8 \cdot 10^{-13}$	$2.59 \cdot 10^{23}$
29	$2.5 \cdot 10^{-12}$	$1.1 \cdot 10^{-11}$	$1.23 \cdot 10^{23}$
34	$1.7 \cdot 10^{-10}$	$7.3 \cdot 10^{-10}$	$1.37 \cdot 10^{25}$
39	$2.3 \cdot 10^{-9}$	$9.8 \cdot 10^{-9}$	$2.86 \cdot 10^{29}$
50	$4.7 \cdot 10^{-10}$	$2.1 \cdot 10^{-9}$	$1.25 \cdot 10^{41}$
75	$6.1 \cdot 10^{-11}$	$2.7 \cdot 10^{-10}$	$2.89 \cdot 10^{73}$
100	$2.8 \cdot 10^{-10}$	$1.2 \cdot 10^{-9}$	$5.63 \cdot 10^{105}$

Table 2.1: Maximal absolute and relative errors of the approximate solution and condition number of the matrix in (2.12) for the problem (2.13)–(2.15) obtained for different values of N in (2.11).

2.2 Transmuted heat polynomials for parabolic partial differential operators with time dependent potential

This section is dedicated to the construction of a system of functions for approximating solutions of the parabolic partial differential equations with a variable coefficient of the form

$$\frac{\partial^2 u}{\partial x^2} - q(x)e^{i\omega t}u(x,t) = \frac{\partial u}{\partial t}(x,t) \tag{2.16}$$

where ω is a positive real constant and i is the imaginary unit. We assume that q is a continuously differentiable complex valued function of an independent real variable $x \in [-b,b]$. The system of functions may be used, in particular, for uniform approximation of solutions of initial and boundary value problems for (2.16).

The system of functions being the approximate transmuted heat polynomials is constructed with the aid of a transmutation operator and a series representation of the transmutation kernel in terms of certain complex exponential functions. Using the successive approximation method the series representation is obtained as well as some estimates for the coefficients of the representation. The coefficients satisfying a sequence of Goursat problems does not depend on the temporal variable and define a second system of functions; being this the key to construct the approximate transmuted heat polynomials. Each function in the second system satisfies a recursive non homogeneous initial problem for the one-dimensional Schrödinger equation and can be calculated by a simple recursive procedure. Then, an analytic expression of the system of functions representing the approximate transmuted heat polynomials is obtained in terms of the second system of functions.

Based on the estimates of the coefficients and the completeness of the transmuted heat polynomials, a uniform approximation of the solutions of the equation (2.16) is proved.

Unlike the complete systems of solutions the new system presented here does not satisfy the equation. However, it approximates solutions of the equation like a complete system of solutions.

2.2.1 A transmutation operator

We use the transmutation operator in the form of a second kind Volterra integral operator

$$Th(x,t) = h(x,t) + \int_{-x}^{x} \mathcal{K}(x,s,t)h(s,t)ds \tag{2.17}$$

where the integral kernel $\mathcal{K}$ is the twice continuously differentiable solution of the Goursat problem

$$\frac{\partial^2 \mathcal{K}}{\partial x^2} - \frac{\partial \mathcal{K}}{\partial t} - q(x)e^{i\omega t}\mathcal{K}(x,s,t) = \frac{\partial^2 \mathcal{K}}{\partial s^2}, \tag{2.18}$$

$$\mathcal{K}(x,x,t) = \frac{e^{i\omega t}}{2}\int_0^x q(\sigma)d\sigma, \tag{2.19}$$

$$\mathcal{K}(x,-x,t) = 0 \tag{2.20}$$

for $0 < |s| \leq |x| < b$ and $0 < t < 2\pi/\omega$. This transmutation operator was studied by D. Colton in [6]. It maps classical solutions of the heat equation into classical solutions of equation (2.16). The transmutation operator (2.17) was used to construct a complete system of solutions of equation (2.16) (see [6, theorem 2.3.2]).

In order to understand the action of T given by (2.17) on the solutions of the heat equation the following results about the integral kernel $\mathcal{K}$ of T are presented.

Set

$$\xi = (x+s)/2, \quad \eta = (x-s)/2$$

and define $\tilde{K}(\xi,\eta,t) = \mathcal{K}(\xi+\eta,\xi-\eta,t)$. Then, the map $(x,s) \mapsto (\xi,\eta)$ transforms the Goursat problem (2.18)-(2.20) into the problem

$$\frac{\partial^2 \tilde{K}}{\partial \xi \partial \eta} - q(\xi+\eta)e^{i\omega t}\tilde{K}(\xi,\eta,t) = \frac{\partial \tilde{K}}{\partial t}, \tag{2.21}$$

$$\tilde{K}(\xi,0,t) = \frac{e^{i\omega t}}{2}\int_0^\xi q(\sigma)d\sigma, \tag{2.22}$$

$$\tilde{K}(0,\eta,t) = 0. \tag{2.23}$$

The solution of the Goursat problem (2.21)-(2.23) by the successive approximation method is given by

$$\tilde{K}(\xi,\eta,t) = \sum_{n=1}^{\infty} \tilde{K}_n(\xi,\eta,t), \tag{2.24}$$

where

$$\tilde{K}_1(\xi,\eta,t) = \frac{e^{i\omega t}}{2}\int_0^{\xi} q(\sigma)d\sigma \tag{2.25}$$

and for $n = 2, 3, \ldots$

$$\tilde{K}_n(\xi,\eta,t) = \int_0^{\eta}\int_0^{\xi}\left[\frac{\partial \tilde{K}_{n-1}}{\partial t}(\alpha,\beta,t) + q(\alpha+\beta)e^{i\omega t}\tilde{K}_{n-1}(\alpha,\beta,t)\right]d\alpha d\beta. \tag{2.26}$$

Proposition 4. *For every $n \in \mathbb{N}$ there exist functions $\{\kappa_{nj}\}_{j=1}^{n}$ such that the functions* (2.25), (2.26) *admit the form*

$$\tilde{K}_n(\xi,\eta,t) = \sum_{j=1}^{n}\kappa_{nj}(\xi,\eta)e^{ij\omega t} \tag{2.27}$$

for every $(\xi,\eta) \in (-b,b)\times(-b,b)$ and $t \in (0, 2\pi/\omega)$.

Proof. The proof is by induction. It is obvious for the case $n = 1$ with $\kappa_{11}(\xi,\eta) := (1/2)\int_0^{\xi} q(\sigma)d\sigma$. Then, from (2.25) it follows that

$$\tilde{K}_1(\xi,\eta,t) = \kappa_{11}(\xi,\eta)e^{i\omega t}.$$

Choose $n \in \mathbb{N}$ and $\kappa_{n1}(\xi,\eta), \kappa_{n2}(\xi,\eta), \ldots, \kappa_{nn}(\xi,\eta)$ such that

$$\tilde{K}_n(\xi,\eta,t) = \sum_{j=1}^{n}\kappa_{nj}(\xi,\eta)e^{ij\omega t}.$$

Thus, from (2.26) the following equality follows

$$
\begin{aligned}
\tilde{K}_{n+1}(\xi,\eta,t) &= \int_0^\eta \int_0^\xi \left[\frac{\partial \tilde{K}_n}{\partial t}(\alpha,\beta,t) + q(\alpha+\beta)e^{i\omega t}\tilde{K}_n(\alpha,\beta,t)\right] d\alpha d\beta \\
&= \int_0^\eta \int_0^\xi \left[\sum_{j=1}^{n} ij\omega\kappa_{nj}(\alpha,\beta)e^{ij\omega t} + q(\alpha+\beta)\sum_{j=2}^{n+1}\kappa_{n,j-1}(\alpha,\beta)e^{ij\omega t}\right] d\alpha d\beta \\
&= \int_0^\eta \int_0^\xi \left[i\omega e^{i\omega t}\kappa_{n1}(\alpha,\beta) + q(\alpha+\beta)e^{i\omega(n+1)t}\kappa_{nn}(\alpha,\beta)\right] d\alpha d\beta \\
&\quad + \sum_{j=2}^{n}\left(i\omega\int_0^\eta \int_0^\xi \left[\kappa_{nj}(\alpha,\beta) + q(\alpha+\beta)\kappa_{n,j-1}(\alpha,\beta)\right] d\alpha d\beta\right) e^{ij\omega t} \\
&= \sum_{j=1}^{n+1}\kappa_{n+1,j}(\xi,\eta)e^{ij\omega t},
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_{n+1,1}(\xi,\eta) &:= i\omega\int_0^\eta \int_0^\xi \kappa_{n1}(\alpha,\beta)d\alpha d\beta, \\
\kappa_{n+1,j}(\xi,\eta) &:= i\omega\sum_{j=2}^{n}\int_0^\eta \int_0^\xi \left[j\kappa_{nj}(\alpha,\beta) + q(\alpha+\beta)\kappa_{n,j-1}(\alpha,\beta)\right] d\alpha d\beta, \quad j = 2,\ldots,n \\
\kappa_{n+1,n+1}(\xi,\eta) &:= \int_0^\xi \int_0^\eta q(\alpha+\beta)\kappa_{nn}(\alpha,\beta)d\alpha d\beta.
\end{aligned}
$$

■

In what follows we consider additionally $\kappa_{n,j}$ for $j > n$ by setting $\kappa_{n,j} = 0$.

Proposition 5. *For every $n \in \mathbb{N}$ the functions $\{\kappa_{nj}\}_{j=1}^{\infty}$ admit the following form*

$$\kappa_{nj}(\xi,\eta) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \tilde{K}_n(\xi,\eta,t)e^{-ij\omega t}dt. \tag{2.28}$$

Proof. This equality is an immediate result of the orthogonality of the set $\{e^{ij\omega t}\}_{j\in\mathbb{Z}}$ in the functional space $L^2([0,2\pi/\omega])$ with the usual scalar product.

Choose $n \in \mathbb{N}$, then from (2.27) we have for every $j = 1, 2, 3, \ldots$ that

$$\int_0^{2\pi/\omega} \tilde{K}_n(\xi, \eta, t) e^{-ij\omega t} dt = \sum_{r=1}^{n} \kappa_{nr}(\xi, \eta) \int_0^{2\pi/\omega} e^{ir\omega t} e^{-ij\omega t} dt.$$

Due to the fact that

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{ir\omega t} e^{-ij\omega t} dt = \delta_{rj}$$

where δ_{rj} is the Kronecker delta the formula (2.28) follows. ■

The following results for the functions $\{\kappa_{nj}\}_{n,j \in \mathbb{N}}$ play an important role in the construction of a series representation for the integral kernel $\mathcal{K}$. Thus, consider the following notation. Choose a fixed constant T_0 such that $T_0 > 2\pi/\omega$ and denote $\tau_0 := 0$. We define

$$\tau_1 := -T_0 \ln \left| 1 - \frac{2\pi}{\omega T_0} \right|, \tag{2.29}$$

and for $n \geq 2$ set

$$\tau_n := \int_0^{2\pi/\omega} \left(1 - \frac{t}{T_0}\right)^{-n} dt = \frac{T_0}{n-1} \left[\left(1 - \frac{2\pi}{\omega T_0}\right)^{1-n} - 1 \right]. \tag{2.30}$$

Let C be a positive constant such that

$$|q(x)| \frac{\omega^k T_0^k}{k!} \leq C, \quad \forall k \in \mathbb{N}_0. \tag{2.31}$$

Then, the following estimates are valid.

Proposition 6. *For every $n \in \mathbb{N}$ and $j = 1, 2, \ldots, n$*

$$\begin{aligned} |\kappa_{nj}(\xi, \eta)| &\leq \frac{\omega}{2\pi} (2C)^n \frac{|\xi|^{n-1} |\eta|^{n-1}}{(n-1)!} \tau_n. && (2.32) \\ \left| \frac{\partial^2 \kappa_{nj}}{\partial \xi \partial \eta}(\xi, \eta) \right| &\leq \frac{\omega}{2\pi} (j\omega + |q(\xi + \eta)|)(2C)^{n-1} \frac{|\xi|^{n-2} |\eta|^{n-2}}{(n-2)!} \tau_{n-1}, \quad n \geq 2. && (2.33) \end{aligned}$$

Proof. From (2.26) and (2.31) by induction over n it follows that

$$\left|\tilde{K}_n(\xi,\eta,t)\right| \le (2C)^n \frac{|\xi|^{n-1}|\eta|^{n-1}}{(n-1)!}\left(1-\frac{t}{T_0}\right)^{-n}.$$

Hence and by the representation of the functions $\{\kappa_{nj}\}_{j=1}^n$ in formula (2.28) of Proposition 5 we obtain for $j=1,2\dots,n$ that

$$\begin{aligned} |\kappa_{nj}(\xi,\eta)| &\le \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left|\tilde{K}_n(\xi,\eta,t)\right|\left|e^{-ij\omega t}\right|dt \\ &\le \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(2C)^n\frac{|\xi|^{n-1}|\eta|^{n-1}}{(n-1)!}\left(1-\frac{t}{T_0}\right)^{-n}dt \\ &= \frac{\omega}{2\pi}(2C)^n\frac{|\xi|^{n-1}|\eta|^{n-1}}{(n-1)!}\tau_n. \end{aligned}$$

Thus, the inequality (2.32) is proved.

In an analogous way the second inequality of the proposition can be proved. From the representation of $\tilde{K}_n$ (2.26) and the integration by parts, if $n\ge 2$ then we have that

$$\begin{aligned} \frac{\partial^2\kappa_{nj}}{\partial\xi\partial\eta}(\xi,\eta) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{\partial^2\tilde{K}_n}{\partial\xi\partial\eta}(\xi,\eta,t)e^{-ij\omega t}dt \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{\partial\tilde{K}_{n-1}}{\partial t}(\xi,\eta,t)e^{-ij\omega t}+q(\xi+\eta)e^{i\omega t}\tilde{K}_{n-1}(\xi,\eta,t)e^{-ij\omega t}dt \\ &= \frac{\omega}{2\pi}\left[e^{-ij\omega t}\tilde{K}_{n-1}(\xi,\eta,t)\Big|_0^{2\pi/\omega}\right]+\frac{\omega}{2\pi}\int_0^{2\pi/\omega}ij\omega\tilde{K}_{n-1}(\xi,\eta,t)e^{-ij\omega t}dt \\ &\quad +\frac{\omega}{2\pi}\int_0^{2\pi/\omega}q(\xi+\eta)\tilde{K}_{n-1}(\xi,\eta,t)e^{-i(j-1)\omega t}dt \end{aligned}$$

Because of the fact that the exponential function in the first term is a $2\pi/\omega$ periodic function for every $j=1,2,\dots,n$ we have that

$$\frac{\partial^2\kappa_{nj}}{\partial\xi\partial\eta}(\xi,\eta)=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}[ij\omega e^{-ij\omega t}+q(\xi+\eta)e^{-i(j-1)\omega t}]\tilde{K}_{n-1}(\xi,\eta,t)dt.$$

Thus

$$\begin{aligned}
\left|\frac{\partial^2\kappa_{nj}}{\partial\xi\partial\eta}(\xi,\eta)\right| &\leq \frac{\omega}{2\pi}(j\omega+|q(\xi+\eta)|)\int_0^{2\pi/\omega}\left|\tilde{K}_{n-1}(\xi,\eta,t)\right|dt\\
&\leq \frac{\omega}{2\pi}(j\omega+|q(\xi+\eta)|)\int_0^{2\pi/\omega}(2C)^{n-1}\frac{|\xi|^{n-2}|\eta|^{n-2}}{(n-2)!}\left(1-\frac{t}{T_0}\right)^{-(n-1)}dt\\
&= \frac{\omega}{2\pi}(j\omega+|q(\xi+\eta)|)(2C)^{n-1}\frac{|\xi|^{n-2}|\eta|^{n-2}}{(n-2)!}\tau_{n-1}.
\end{aligned}$$

■

Remark 3. *Notice that* $\left|\frac{\partial^2\kappa_{11}}{\partial\xi\partial\eta}(\xi,\eta)\right|=0$ *for every* (ξ,η) *in* $(-b,b)\times(-b,b)$.

We note that due to the estimates (2.32) and (2.33) in Proposition 6 and the Weierstrass theorem we have that the series $\sum_{l=n}^{\infty}\kappa_{ln}(\xi,\eta)$ is uniformly convergent for every (ξ,η) in $(-b,b)\times(-b,b)$. Then, for $n\in\mathbb{N}$ we define the functions $\tilde{k}_n$ by

$$\tilde{k}_n(\xi,\eta):=\sum_{l=n}^{\infty}\kappa_{ln}(\xi,\eta),\quad (\xi,\eta)\in(-b,b)\times(-b,b). \tag{2.34}$$

Proposition 6 is used for obtaining the following estimates for functions $\{\tilde{k}_n\}_{n=1}^{\infty}$.

Corollary 2. *If* $n\in\mathbb{N}$ *then for every* $(\xi,\eta)\in(-b,b)\times(-b,b)$

$$\begin{aligned}
\left|\tilde{k}_n(\xi,\eta)\right| &\leq \tilde{A}\begin{cases}\exp(\tilde{C})+\ln|\omega T_0|-\ln|\omega T_0-2\pi|, & n=1\\ \tilde{C}^{n-1}\exp(\tilde{C})/(n-1)!, & n\geq 2\end{cases}\\
\left|\frac{\partial^2\tilde{k}_n}{\partial\xi\partial\eta}(\xi,\eta)\right| &\leq \tilde{A}(n\omega+|q(\xi+\eta)|)\begin{cases}\tilde{C}\exp(\tilde{C})+\ln|\omega T_0|-\ln|\omega T_0-2\pi|, & n\in\{1,2\}\\ \tilde{C}^{n-2}\exp(\tilde{C})/(n-2)!, & n\geq 3\end{cases}
\end{aligned}$$

where $\tilde{A}:=\omega CT_0/\pi$ *and* $\tilde{C}:=2b^2C\omega T_0/(\omega T_0-2\pi)$.

Proof. Choose $n\in\mathbb{N}$. Then, from definition of the $\tilde{k}_n$ functions in (2.34) and the estimates for the κ_{ln} functions in (2.32) it follows that

$$\left|\tilde{k}_n(\xi,\eta)\right|\leq\sum_{l=n}^{\infty}|\kappa_{ln}(\xi,\eta)|\leq\sum_{l=n}^{\infty}\frac{\omega}{2\pi}(2C)^l\frac{|\xi|^{l-1}|\eta|^{l-1}}{(l-1)!}\tau_l\leq\frac{\omega C}{\pi}\sum_{l=n}^{\infty}\frac{(2b^2C)^{l-1}}{(l-1)!}\tau_l.$$

Hence and from definition of τ_l in (2.29) and (2.30) it follows that, for $n = 1$

$$\begin{aligned}
\left|\tilde{k}_1(\xi,\eta)\right| &\leq \frac{\omega C}{\pi}\left(\tau_1 + \sum_{l=2}^{\infty}\frac{(2b^2C)^{l-1}}{(l-1)!}\tau_l\right) \\
&= \frac{\omega C}{\pi}\left(-T_0\ln\left|1-\frac{2\pi}{\omega T_0}\right| + \sum_{l=2}^{\infty}\frac{(2b^2C)^{l-1}}{(l-1)!}\frac{T_0}{l-1}\left[\left(1-\frac{2\pi}{\omega T_0}\right)^{1-l}-1\right]\right) \\
&\leq \tilde{A}\left(-\ln\left|1-\frac{2\pi}{\omega T_0}\right| + \sum_{l=1}^{\infty}\left(\frac{2b^2C}{1-\frac{2\pi}{\omega T_0}}\right)^{l}\frac{1}{l!}\right) \\
&\leq \tilde{A}\left(-\ln\left|1-\frac{2\pi}{\omega T_0}\right| + \sum_{l=0}^{\infty}\frac{\tilde{C}^l}{l!}\right)
\end{aligned}$$

where $\tilde{A} := \omega C T_0/\pi$ and $\tilde{C} = (2b^2C)/\left(1-\frac{2\pi}{\omega T_0}\right)$. Meanwhile for $n > 1$ we have that

$$\begin{aligned}
\left|\tilde{k}_n(\xi,\eta)\right| &\leq \frac{\omega C}{\pi}\sum_{l=n}^{\infty}\frac{(2b^2C)^{l-1}}{(l-1)!}\tau_l \\
&= \frac{\omega C}{\pi}\sum_{l=n}^{\infty}\frac{(2b^2C)^{l-1}}{(l-1)!}\frac{T_0}{l-1}\left[\left(1-\frac{2\pi}{\omega T_0}\right)^{1-l}-1\right] \\
&\leq \tilde{A}\sum_{l=n}^{\infty}\frac{\tilde{C}^{l-1}}{(l-1)!} \\
&= \tilde{A}\sum_{l=0}^{\infty}\frac{\tilde{C}^{n+l-1}}{(n+l-1)!} \\
&\leq \tilde{A}\frac{\tilde{C}^{n-1}}{(n-1)!}\exp\left(\tilde{C}\right).
\end{aligned}$$

In an analogous way from the estimates for the second derivative of κ_{ln} in (2.33) it follows, for $n \in \mathbb{N}$ that

$$\left|\frac{\partial^2\tilde{k}_n}{\partial\xi\partial\eta}(\xi,\eta)\right| \leq \sum_{l=n}^{\infty}\left|\frac{\partial^2\kappa_{ln}(\xi,\eta)}{\partial\xi\partial\eta}\right| \leq \frac{\omega}{2\pi}(n\omega + |q(\xi+\eta)|)\sum_{l=n}^{\infty}(2C)^{l-1}\frac{|\xi|^{l-2}|\eta|^{l-2}}{(l-2)!}\tau_{l-1}. \quad (2.35)$$

The last term in equation (2.35) is calculated as follows. From definition of τ_l in (2.29) and (2.30), for $n \in \{1,2\}$ the following inequality is valid

$$\begin{aligned}
\sum_{l=2}^{\infty}(2C)^{l-1}\frac{|\xi|^{l-2}|\eta|^{l-2}}{(l-2)!}\tau_{l-1} &= \sum_{l=1}^{\infty}(2C)^{l}\frac{|\xi|^{l-1}|\eta|^{l-1}}{(l-1)!}\tau_{l} \\
&= 2C\tau_1 + 2C\sum_{l=2}^{\infty}\frac{(2C|\xi||\eta|)^{l-1}}{(l-1)!}\frac{T_0}{l-1}\left[\left(\frac{1}{1-\frac{2\pi}{\omega T_0}}\right)^{l-1}-1\right] \\
&\leq 2C\left(\tau_1 + T_0\sum_{l=1}^{\infty}\frac{\tilde{C}^{l-1}}{(l-1)!}\right) \\
&= 2CT_0\left[-\ln\left|1-\frac{2\pi}{\omega T_0}\right| + \exp\tilde{C}\right]. \qquad (2.36)
\end{aligned}$$

Meanwhile for $n > 2$ we obtain that

$$\begin{aligned}
\sum_{l=n}^{\infty}(2C)^{l-1}\frac{|\xi|^{l-2}|\eta|^{l-2}}{(l-2)!}\tau_{l-1} &\leq 2C\sum_{l=n}^{\infty}\frac{(2b^2C)^{l-2}}{(l-2)!}\frac{T_0}{l-2}\left[\left(\frac{1}{1-\frac{2\pi}{\omega T_0}}\right)^{l-2}-1\right] \\
&\leq 2CT_0\sum_{l=n}^{\infty}\frac{\tilde{C}^{l-2}}{(l-2)!} \\
&\leq 2CT_0\exp\left(\tilde{C}\right)\frac{\tilde{C}^{l-2}}{(l-2)!}. \qquad (2.37)
\end{aligned}$$

Thus, substituting (2.36) and (2.37) into (2.35) we obtain the estimates for the second derivative of $\tilde{k}_n$. ■

Theorem 16. *The integral kernel $\tilde{K}$ of the transmutation operator* (2.17) *admits the following representation*

$$\tilde{K}(\xi,\eta,t) = \sum_{n=1}^{\infty}\tilde{k}_n(\xi,\eta)e^{in\omega t} \qquad (2.38)$$

for any $(\xi,\eta) \in (-b,b)\times(-b,b)$.

Proof. From the representation of the integral kernel $\tilde{K}$ by the succesive approximation method in (2.24) and the representation of functions $\tilde{k}_n$ in (2.34) in Proposition 4 we obtain that

$$\tilde{K}(\xi,\eta,t) = \sum_{n=1}^{\infty}\tilde{K}_n(\xi,\eta,t) = \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\kappa_{nj}(\xi,\eta)e^{ij\omega t}\right).$$

Hence and from the estimates of $\tilde{k}_n$ in Corollary 2, the result is obtained by changing the order of summation. Thus

$$\tilde{K}(\xi,\eta,t) = \sum_{n=1}^{\infty}\left(\sum_{l=n}^{\infty}\kappa_{ln}(\xi,\eta)\right)e^{in\omega t} = \sum_{n=1}^{\infty}\tilde{k}_n(\xi,\eta)e^{in\omega t}.$$

■

From this theorem we note that the knowledge of the functions $\{\tilde{k}_n\}_{n\in\mathbb{N}}$ leads to the construction of the transmutation T given by (2.17). Thus, the following lemma is important to us.

Lemma 1. *The functions* $\tilde{k}_n$ *given by* (2.34) *satisfy the Goursat problems*

$$\frac{\partial^2\tilde{k}_n}{\partial\xi\partial\eta} - in\omega\tilde{k}_n(\xi,\eta) = q(\xi+\eta)\tilde{k}_{n-1}(\xi,\eta), \tag{2.39}$$

$$\tilde{k}_n(\xi,0) = \begin{cases} \frac{1}{2}\int_0^{\xi} q(\sigma)d\sigma, & \text{if } n=1 \\ 0, & \text{if } n>1, \end{cases} \tag{2.40}$$

$$\tilde{k}_n(0,\eta) = 0, \quad n\geq 1 \tag{2.41}$$

considered on $(-b,b)\times(-b,b)$ *and* $\tilde{k}_0 := 0$.

Proof. It was shown in the proof of Proposition 6 that

$$\begin{aligned} \frac{\partial^2\kappa_{ln}}{\partial\xi\partial\eta}(\xi,\eta) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}[in\omega e^{-in\omega t} + q(\xi+\eta)e^{-i(n-1)\omega t}]\tilde{K}_{l-1}(\xi,\eta)dt \\ &= in\omega\kappa_{l-1,n}(\xi,\eta) + q(\xi+\eta)\kappa_{l-1,n-1}(\xi,\eta). \end{aligned}$$

Hence and from definition of $\tilde{k}_n$ in (2.34) the following equality follows.

If $n=1$ then $\tilde{k}_1(\xi,\eta)=\sum_{l=1}^{\infty}\kappa_{l1}(\xi,\eta)=\kappa_{11}(\xi,\eta)+\sum_{l=2}^{\infty}\kappa_{l1}(\xi,\eta)$. Thus,

$$\begin{aligned}\frac{\partial^2\tilde{k}_1}{\partial\xi\partial\eta}(\xi,\eta) &= \sum_{l=2}^{\infty}\frac{\partial^2\kappa_{l1}}{\partial\xi\partial\eta}(\xi,\eta)\\ &= \sum_{l=2}^{\infty} i\omega\kappa_{l-1,1}+q(\xi+\eta)\kappa_{l-1,0}.\end{aligned}$$

The last term in right-hand is zero because of (2.28), then $\frac{\partial^2\tilde{k}_1}{\partial\xi\partial\eta}(\xi,\eta)=i\omega\tilde{k}_1$

If $n\geq 2$ then

$$\begin{aligned}\frac{\partial^2\tilde{k}_n}{\partial\xi\partial\eta}(\xi,\eta) &= \sum_{l=n}^{\infty}\left[in\omega\kappa_{l-1,n}(\xi,\eta)+q(\xi+\eta)\kappa_{l-1,n-1}(\xi,\eta)\right]\\ &= in\omega\sum_{l=n}^{\infty}\kappa_{l-1,n}(\xi,\eta)+q(\xi+\eta)\sum_{l=n}^{\infty}\kappa_{l-1,n-1}(\xi,\eta)\\ &= in\omega\kappa_{n-1,n}(\xi,\eta)+in\omega\sum_{l=n}^{\infty}\kappa_{l,n}(\xi,\eta)+q(\xi+\eta)\sum_{l=n-1}^{\infty}\kappa_{l,n-1}(\xi,\eta).\end{aligned}$$

The first term in right-hand is zero by (2.28) then $\frac{\partial^2\tilde{k}_n}{\partial\xi\partial\eta}(\xi,\eta)=in\omega\tilde{k}_n(\xi,\eta)+q(\xi+\eta)\tilde{k}_{n-1}(\xi,\eta)$. Thus, (2.39) is proved.

On the other hand, due to the series representation for the integral kernel of Theorem 16

$$\tilde{K}(\xi,\eta,t)=\sum_{n=1}^{\infty}\tilde{k}_n(\xi,\eta)e^{in\omega t}$$

and the Goursat conditions of $\tilde{K}$ in (2.22) and (2.23)

$$\tilde{K}(\xi,0)=\frac{e^{i\omega t}}{2}\int_0^{\xi}q(\sigma)d\sigma,\quad \tilde{K}(0,\eta)=0.$$

Then, we have that

$$\sum_{n=1}^{\infty}\tilde{k}_n(\xi,0)e^{in\omega t}=\frac{e^{i\omega t}}{2}\int_0^{\xi}q(\sigma)d\sigma,\quad \sum_{n=1}^{\infty}\tilde{k}_n(0,\eta)e^{in\omega t}=0.$$

Thus, it follows that

$$\tilde{k}_1(\xi, 0) = \begin{cases} \frac{1}{2}\int_0^{\xi} q(\sigma)d\sigma, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

$$\tilde{k}_n(0, \eta) = 0, \quad n \geq 1.$$

■

Let us consider the inverse transformation of variables $(\xi, \eta) \mapsto (x, s)$ and define

$$k_n(x, s) = \tilde{k}\left(\frac{x+s}{2}, \frac{x-s}{2}\right).$$

Note that from (2.40) and (2.41) it follows that

$$k_n(x, x) = \begin{cases} \frac{1}{2}\int_0^{x} q(\sigma)d\sigma, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{2.42}$$

$$k_n(x, -x) = 0, \quad n \geq 1. \tag{2.43}$$

Then, consider the following definition.

Definition 5. *For $m \in \mathbb{N}_0, n \in \mathbb{N}$ we define*

$$u_{mn}(x) := \int_{-x}^{x} k_n(x, s)s^m ds, \quad x \in [-b, b]. \tag{2.44}$$

These functions play a crucial role in the following section and the following result is proved.

Theorem 17. *Let $q \in C^1[-b, b]$. Then, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ the functions u_{mn} satisfy the following Cauchy problems*

$$\frac{d^2 u_{mn}}{dx^2}(x) - in\omega u_{mn}(x) = q(x)u_{m,n-1}(x) + m(m-1)u_{m-2,n}(x) + q(x)x^m\delta_{n1}, \tag{2.45}$$

$$u_{mn}(0) = \frac{du_{mn}}{dx}(0) = 0,$$

where $\delta_{n1} = 1$ if $n = 1$ and 0 otherwise.

Proof. Choose $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. From definition of u_{mn} in (2.44) and the Goursat condition (2.43) it follows that

$$\frac{du_{mn}}{dx}(x) = \int_{-x}^{x} \frac{\partial k_n}{\partial x}(x,s)s^m ds + k_n(x,x)x^m \tag{2.46}$$

and

$$\begin{aligned}\frac{d^2u_{mn}}{dx^2}(x) &= \int_{-x}^{x} \frac{\partial^2 k_n}{\partial x^2}(x,s)s^m ds + x^m \frac{\partial k_n}{\partial x}(x,x) + (-x)^m \frac{\partial k_n}{\partial x}(x,-x) \\ &+ x^m \frac{d}{dx}(k_n(x,x)) + m x^{m-1} k_n(x,x).\end{aligned}$$

Hence and from the partial differential equation for $\tilde{k}_n$ (2.39) we obtain that

$$\begin{aligned}\frac{d^2u_{mn}}{dx^2}(x) &= \int_{-x}^{x} \left(\frac{\partial^2 k_n}{\partial s^2}(x,s) + q(x)k_{n-1}(x,s) + in\omega k_n(x,s) \right) s^m ds \\ &\quad + x^m \frac{\partial k_n}{\partial x}(x,x) + (-x)^m \frac{\partial k_n}{\partial x}(x,-x) + x^m \frac{d}{dx}(k_n(x,x)) \\ &\quad + m x^{m-1} k_n(x,x).\end{aligned}$$

Since the first integral in the right hand term can be written as follows (applying integration by parts twice)

$$\begin{aligned}\int_{-x}^{x} \frac{\partial^2 k_n}{\partial s^2}(x,s)s^m ds &= x^m \frac{\partial k_n}{\partial s}(x,x) - (-x)^m \frac{\partial k_n}{\partial s}(x,-x) - m x^{m-1} k_n(x,x) \\ &\quad + m(m-1) \int_{-x}^{x} k_n(x,s) s^{m-2} ds\end{aligned}$$

we obtain that

$$\begin{aligned}\frac{d^2u_{mn}}{dx^2}(x) &= x^m\frac{\partial k_n}{\partial s}(x,x)-(-x)^m\frac{\partial k_n}{\partial s}(x,-x)-mx^{m-1}k_n(x,x)\\ &+m(m-1)\int_{-x}^{x}k_n(x,s)s^{m-2}ds+q(x)\int_{-x}^{x}k_{n-1}(x,s)s^m ds\\ &+\int_{-x}^{x}in\omega k_n(x,s)s^m ds+x^m\frac{\partial k_n}{\partial x}(x,x)+(-x)^m\frac{\partial k_n}{\partial x}(x,-x)\\ &+x^m\frac{d}{dx}(k_n(x,x))+mx^{m-1}k_n(x,x).\end{aligned}$$

Note that

$$\frac{\partial k_n}{\partial s}(x,x)+\frac{\partial k_n}{\partial x}(x,x)=\frac{d}{dx}(k_n(x,x))=\frac{\delta_{n1}}{2}q(x)$$

(see (2.42)). Thus, from (2.44) the following equality is derived

$$\frac{d^2u_{mn}}{dx^2}(x)=x^mq(x)\delta_{n1}+m(m-1)u_{m-2,n}(x)+q(x)u_{m,n-1}(x)+in\omega u_{mn}(x).$$

The initial condition $u_{mn}(0)=0$ follows immediately from definition (2.44). The second initial condition is a consequence of equation (2.46) and the Goursat conditions (2.40), (2.41). ■

2.2.2 Approximate transmuted heat polynomials

In this section a family of functions approximating the solutions of (2.16) is obtained.

Denote by $u_m=Th_m$, $m\in\mathbb{N}_0$ the images of the heat polynomials in (2.4) under the action of the transmutation operator (2.17). Then, due to Theorem 16 the following statement is true. Denote $\Omega:=(-b,b)\times(0,2\pi/\omega)$.

Theorem 18. *The transmuted heat polynomials $\{u_m\}_{m\in\mathbb{N}_0}$ have the following form*

$$u_m(x,t)=h_m(x,t)+\sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\alpha_{mk}u_{m-2k,n}(x)t^ke^{in\omega t} \tag{2.47}$$

where the functions $\{u_{mn} : m \in \mathbb{N}_0, n \in \mathbb{N}\}$ *are defined by* (2.44) *and*

$$\alpha_{mk} := \frac{m!}{(m-2k)!k!}.$$

Proof. This representation is an immediate result of the representation for the integral kernel K in terms of the functions k_n (see Theorem 16). Indeed, we have that

$$\begin{aligned} Th_m(x,t) &= h_m(x,t) + \int_{-x}^{x} K(x,s,t)h_m(s,t) \\ &= h_m(x,t) + \int_{-x}^{x} \sum_{n=1}^{\infty} k_n(x,s)e^{in\omega t} \sum_{k=0}^{[m/2]} \alpha_{mk} s^{m-2k} t^k \\ &= h_m(x,t) + \sum_{n=1}^{\infty} \sum_{k=0}^{[m/2]} \left(\int_{-x}^{x} k_n(x,s) s^{m-2k} ds \right) \alpha_{mk} t^k e^{in\omega t}. \end{aligned}$$

■

Let us consider the functions

$$u_m^N(x,t) := h_m(x,t) + \sum_{n=1}^{N} \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} u_{m-2k,n}(x) t^k e^{in\omega t}, \quad m \in \mathbb{N}_0, \quad N \in \mathbb{N}. \tag{2.48}$$

Then the following estimate is valid.

Theorem 19. *For* $m \in \mathbb{N}_0$ *and* $\varepsilon > 0$ *there exists such* $N = N(\varepsilon)$ *that the following estimate is valid for any* $n \geq N$

$$\max_{(x,t)\in\overline{\Omega}} |u_m(x,t) - u_m^n(x,t)| < \varepsilon.$$

Proof. Set $m \in \mathbb{N}_0$. Choose $\varepsilon > 0$. Then, from the representation of the transmuted heat polynomial u_m in formula (2.47) and the truncated series in (2.48) we have the following inequality for every $n \in \mathbb{N}$

$$
\begin{aligned}
|u_m(x,t) - u_m^n(x,t)| &= \left| \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \sum_{r=n+1}^{\infty} u_{m-2k,r}(x) e^{ir\omega t} \right| \\
&\leq \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \sum_{r=n+1}^{\infty} |u_{m-2k,r}(x)| . \qquad (2.49)
\end{aligned}
$$

In order to obtain an estimate for $\sum_{r=n+1}^{\infty} |u_{m-2k,r}(x)|$ an estimate of $|u_{m-2k,r}(x)|$ is obtained. From definition of u_{mr} in equation (2.44) and the estimates for k_r in Corollary 2, after an integration it follows that

$$
\begin{aligned}
|u_{m-2k,r}(x)| &\leq \int_{-x}^{x} |k_r(x,s)| \, |s|^{m-2k} \, ds \\
&\leq 2\frac{\omega C T_0}{\pi} \int_0^x s^{m-2k} ds \frac{\tilde{C}^{r-1}}{(r-1)!} \exp(\tilde{C}) \\
&= 2\frac{\omega C T_0}{\pi} \frac{x^{m-2k+1}}{m-2k+1} \frac{\tilde{C}^{r-1}}{(r-1)!} \exp(\tilde{C}).
\end{aligned}
$$

Thus, we obtain

$$
|u_m(x,t) - u_m^n(x,t)| \leq \frac{2\omega C T_0}{\pi} \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \frac{x^{m-2k+1}}{m-2k+1} \sum_{r=n+1}^{\infty} \frac{\tilde{C}^{r-1}}{(r-1)!} \exp(\tilde{C}).
$$

Because of the fact that

$$
\sum_{r=n+1}^{\infty} \frac{\tilde{C}^{r-1}}{(r-1)!} = \sum_{r=0}^{\infty} \frac{\tilde{C}^{r+n}}{(r+n)!} \leq \exp(\tilde{C}) \frac{\tilde{C}^n}{n!}
$$

we have

$$
\begin{aligned}
|u_m(x,t) - u_m^n(x,t)| &\leq \frac{2\omega C T_0}{\pi} \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \frac{x^{m-2k+1}}{m-2k+1} \exp(2\tilde{C}) \frac{\tilde{C}^n}{n!} \\
&= \frac{2\omega C T_0}{\pi} \exp(2\tilde{C}) \frac{\tilde{C}^n}{n!} \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \frac{x^{m-2k+1}}{m-2k+1}.
\end{aligned}
$$

Thus, from (2.49) it follows that

$$
\max_{(x,t)\in\overline{\Omega}} |u_m(x,t) - u_m^n(x,t)| \leq \max_{(x,t)\in\overline{\Omega}} \frac{2\omega C T_0}{\pi} \exp(2\tilde{C}) \frac{\tilde{C}^n}{n!} \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \frac{x^{m-2k+1}}{m-2k+1}.
$$

The choice of N such that

$$
\frac{\tilde{C}^n}{n!} < \frac{\varepsilon\pi}{2\omega C T_0} \left(\max_{(x,t)\in\overline{\Omega}} \left| \sum_{k=0}^{\left[\frac{m}{2}\right]} \alpha_{mk} t^k \frac{x^{m-2k+1}}{m-2k+1} \right| \right)^{-1}, \quad \forall n \geq N
$$

finishes the proof. ■

Remark 4. *The explicit solution of the recursive system of ordinary differential equations* (2.45) *is given by*

$$
\begin{aligned}
u_{mn}(x) &= \frac{\sqrt{2}}{\sqrt{n\omega}(1+i)} \int_0^x h_{mn}(s) \sinh\left(\sqrt{\frac{n\omega}{2}}(x-s)\right) ds \\
&= \frac{\sqrt{2}}{\sqrt{n\omega}(1+i)} \sinh\left(\sqrt{\frac{n\omega}{2}} x\right) \int_0^x h_{mn}(s) \cosh\left(\sqrt{\frac{n\omega}{2}} s\right) ds \\
&- \frac{\sqrt{2}}{\sqrt{n\omega}(1+i)} \cosh\left(\sqrt{\frac{n\omega}{2}} x\right) \int_0^x h_{mn}(s) \sinh\left(\sqrt{\frac{n\omega}{2}} s\right) ds
\end{aligned}
$$

where

$$
h_{mn}(x) := m(m-1)u_{m-2,n} + q(x)u_{m,n-1}(x) + x^m q(x)\delta_{n1}
$$

for every $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Theorem 20. *Let u be a continuous function in $\overline{\Omega}$ satisfying the equation*

$$\frac{\partial^2 u}{\partial x^2} - q(x)e^{i\omega t}u(x,t) = \frac{\partial u}{\partial t}(x,t), \quad (x,t) \in \Omega. \tag{2.50}$$

Then, for every $\varepsilon > 0$ there exist $M, N \in \mathbb{N}$ and the coefficients $a_0, a_1, \dots, a_M$ such that

$$\max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{m=0}^{M} a_m u_m^N(x,t) \right| < \varepsilon.$$

Proof. Choose $\varepsilon > 0$. Let u be a solution of the equation (2.50). It is a well known result that $\{Th_m\}_{m\in\mathbb{N}_0}$ is a complete system of solutions of (2.50) (see [6, theorem 3.2.3]). Then, there exist $M \in \mathbb{N}$ and the coefficients $a_0, a_1, \dots, a_M$ such that

$$\max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{m=0}^{M} a_m u_m(x,t) \right| < \varepsilon/2.$$

From Theorem 19 for every $m = 0, 1, \dots, M$ there exists $N \in \mathbb{N}$ such that

$$\max_{(x,t)\overline{\Omega}} \left| u_m(x,t) - u_m^N(x,t) \right| < \frac{\varepsilon}{2\sum_{m=0}^{M} |a_m|}.$$

Hence and due to the triangle inequality the following inequality follows

$$\begin{aligned}
\max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{m=0}^{M} a_m u_m^N(x,t) \right| &\leq \max_{(x,t)\in\overline{\Omega}} \left| u(x,t) - \sum_{m=0}^{M} a_m u_m(x,t) \right| \\
&\quad + \max_{(x,t)\in\overline{\Omega}} \left| \sum_{m=0}^{M} u_m(x,t) - \sum_{m=0}^{M} a_m u_m^N(x,t) \right| \\
&\leq \frac{\varepsilon}{2} + \max_{(x,t)\in\overline{\Omega}} \sum_{m=0}^{M} |a_m| \left| u_m(x,t) - u_m^N(x,t) \right| < \varepsilon.
\end{aligned}$$

■

Remark 5. *The solution scheme is used to solve coupled systems of equations of the form*

$$\begin{aligned} f_{xx} - f_t + q(x)\cos(\omega t) f &= q(x)\sin(\omega t) g \\ g_{xx} - g_t + q(x)\sin(\omega t) g &= q(x)\cos(\omega t) f \end{aligned}$$

for $(x,t) \in (-b,b) \times (0,\tau)$*. Indeed, the function* $u(x,t) = f(x,t) + ig(x,t)$ *is solution of the equation* (2.16)*.*

Chapter 3

A method of fundamental solutions for parabolic partial differential equations with variable coefficients

In this chapter we return to the parabolic partial differential equation with a time independent potential. A system of functions being the images of the heat kernels under the action of the transmutation operator is constructed and a completeness result of the system for approximation of solutions of equation

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t) \tag{3.1}$$

on a rectangle of the plane (x,t) is proved. The completeness property is shown to be useful for uniform approximation of the solution of boundary value problems for (3.1), then an extension of the method of fundamental solutions is presented.

The system of functions is constructed with the aid of the transmutation operators presented in Section 1.3. Moreover, the use of the Fourier-Legendre series representation of the transmutation kernel given by Theorem 13 leads to a simple recursive procedure for computation of the system of functions.

The completeness property is a consequence of a similar result for the heat equation and the maximum principle for heat solutions over a rectangle in the plane.

The chapter contains the following sections. In Section 3.1 the known method of fundamental solutions for the heat equation is presented. Then in Section 3.2 the main theorem is presented. In Section 3.2.1 a recursive formula to construct the system of functions is obtained. Finally, a step by step method for approximate the solution of boundary value problems for (3.1) is proposed.

3.1 A method of fundamental solutions for the heat equation

Consider the problem of finding the solution h of the first boundary value problem

$$\frac{\partial^2 h}{\partial x^2}(x,t) = \frac{\partial h}{\partial t}(x,t), \tag{3.2}$$

$$h(x,0) = \Phi(x), \tag{3.3}$$

$$h(-b,t) = \Psi_1(t), \quad h(b,t) = \Psi_2(t) \tag{3.4}$$

considered on $\Omega := (-b,b) \times (0,\tau)$. Here Φ, Ψ_1 and Ψ_2 are continuously differentiable functions satisfying the compatibility conditions

$$\Psi_1(0) = \Phi(-b), \quad \Psi_2(0) = \Phi(b).$$

In order to apply the method of fundamental solutions the following notations and terminology are introduced.

Let $\{t_k \neq 0 : k = 1, 2, \ldots\}$ be a countable everywhere dense set of points in $(-\tau, \tau)$, $x_0 < -b$ and $x_1 > b$. Consider the functions

$$v_m(x,t) := \frac{H(t-t_n)}{2\sqrt{\pi(t-t_n)}} \exp\left(\frac{-(x-x_r)^2}{4(t-t_n)}\right), \quad m \in \mathbb{N} \tag{3.5}$$

for $(x,t) \in \Omega$ where n is the integer part of $m/2$, $r \in \{0,1\}$ and H is the Heaviside step function, whose value is zero for negative real numbers and one for positive real numbers.

Note that, $v_m(x,t) = H(t-t_n)F(x-x_r, t-t_n)$, where F is the fundamental solution of the heat operator defined by (1.4).

In order to approximate the solution of (3.2)-(3.4) by the method of fundamental solutions the following theorem is presented. It is the basis for the method.

Theorem 21 ([10]). *The restrictions $\{v_m(y,t) : y = -b, b\}_{m\in\mathbb{N}}$ constitute a linearly independent and complete set in $L^2((-\tau,\tau))$. The restrictions $\{v_m(x,0) : t_n < 0\}_{m\in\mathbb{N}}$ form a linearly independent and complete set in $L^2((-b,b))$.*

The first part of Theorem 21 was proved in 1964 by Kupradze in [25], meanwhile the second part is the principal result in [10].

Based on the previous Theorem 21 a method of fundamental solutions for approximating the solution of the problem (3.2)-(3.4) consists in the following.

Let M be a fixed positive integer number and consider M points $\{t_n\}_{n=1}^{M}$ in $(-\tau,\tau)\backslash\{0\}$. Choose $x_0 < -b$ and $x_1 > b$ and consider the $2M$ functions $\{v_m\}_{m=1}^{2M}$ constructed according to the rule (3.5). Then, due to Theorem 21 the approximate solution to (3.2), $\tilde{v}$ is sought in the form of a linear combination of the functions $\{v_m\}_{m=1}^{2M}$ and the coefficients are such that $\tilde{v}$ satisfy approximately the boundary conditions (3.3) and (3.4).

3.2 The transmuted heat kernel

In this section an extension of the method of fundamental solutions onto parabolic partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t} \tag{3.6}$$

considered on the bounded rectangle $\Omega := (-b,b)\times(0,\tau)$ is presented. We assume that the potential q is a continuous complex valued function of an independent real variable

$-b < x < b$. Denote by $\Gamma = \Gamma_1 \cup \Gamma_2$ the parabolic boundary of Ω such that

$$\Gamma_1 := \{(x,t) : x \in \{-b, b\}, \quad t \in (-\tau, \tau)\}, \quad \Gamma_2 := \{(x,t) : x \in (-b,b), \quad t = 0\}.$$

Let us consider the transmutation operator T defined in Section 1.3 as

$$Tv(x) = v(x) + \int_{-x}^{x} K(x,s)v(s)ds. \tag{3.7}$$

Then, consider the following definition.

Definition 6. *For every $m \in \mathbb{N}$ let u_m be the function*

$$u_m(x,t) := T[v_m](x,t), \quad (x,t) \in \Omega. \tag{3.8}$$

where v_m is the function defined by (3.5) *and the operator T is applied with respect to the variable x.*

Based on Theorem 21 and the maximum principle, a completeness result of the functions (3.8) for approximate solution of the parabolic partial differential equation (3.6) is proved in the following theorem.

Theorem 22. *Let u be a solution of equation (3.6) on Ω which is continuous in $\bar{\Omega}$. Then, for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ and the constants $\{c_m\}_{m=1}^{M}$ such that*

$$\left\| u - \sum_{m=1}^{M} c_m u_m \right\|_{L^2(\overline{\Omega})} \leq C\varepsilon_1.$$

Proof. Choose $\varepsilon > 0$. Consider $h = T^{-1}u$. Due to Theorem 21 we have that for any $\varepsilon_1 > 0$ there exist $M \in \mathbb{N}$ and constants $\{c_m\}_{m=1}^{M}$ such that

$$\left\| h(x,t) - \sum_{m=1}^{M} c_m v_m(x,t) \right\|_{L^2(\bar{\Omega})} < \varepsilon_1. \tag{3.9}$$

Then, using the boundedness of the transmutation operator and the maximum principle for the heat equation we obtain that

$$\begin{aligned}\left\| u - \sum_{m=1}^{M} c_m u_m \right\|_{L^2(\overline{\Omega})} &= \left\| Th - \sum_{m=1}^{M} c_m T v_m \right\|_{L^2(\overline{\Omega})} \\ &\leq C \left\| h - \sum_{m=1}^{M} c_m v_m \right\|_{L^2(\overline{\Omega})} \leq C \left\| h - \sum_{m=1}^{M} c_m v_m \right\|_{L^2(\Gamma)}\end{aligned}$$

where the constant C is the uniform norm of T guaranteed to exists in Corollary 1. Then, from (3.9) we obtain that

$$\left\| u - \sum_{m=1}^{M} c_m u_m \right\|_{L^2(\overline{\Omega})} \leq C\varepsilon_1.$$

The choice of $\varepsilon_1 = \varepsilon / C$ finishes the proof. ■

3.2.1 A recurrent procedure to compute the transmuted heat kernel

In this section we obtain a simple recursive procedure to compute the functions $\{u_m\}_{m\in\mathbb{N}}$ defined by (3.8). This fact is due to the Fourier-Legendre series representation of the integral kernel K which was mentioned in Theorem 13 given by

$$K(x,s) = \sum_{j=0}^{\infty} \frac{\beta_j(x)}{x} P_j\left(\frac{s}{x}\right) \tag{3.10}$$

where P_j denotes the Legendre polynomial of order j,

$$\beta_j(x) = \frac{2j+1}{2}\left(\sum_{k=0}^{j} \frac{l_{k,j}\varphi_k(x)}{x^k} - 1\right) \tag{3.11}$$

and $l_{k,j}$ are the corresponding coefficients of x^k in P_j, that is $P_j(x) = \sum_{k=0}^{j} l_{k,j} x^k$.

We substitute the Fourier-Legendre series (3.10) into equation (3.8). Then,

$$\begin{aligned} u_m(x,t) &= v_m(x,t) + \int_{-x}^{x} \sum_{j=0}^{\infty} \frac{\beta_j(x)}{x} P_j\left(\frac{s}{x}\right) \frac{1}{2\sqrt{\pi(t-t_n)}} \exp\left(-\frac{|s-x_r|^2}{4(t-t_n)}\right) ds \\ &= v_m(x,t) + \sum_{j=0}^{\infty} \frac{\beta_j(x)}{2x\sqrt{\pi(t-t_n)}} \int_{-x}^{x} P_j\left(\frac{s}{x}\right) \exp\left(-\frac{|s-x_r|^2}{4(t-t_n)}\right) ds \end{aligned}$$

Let us consider the functions

$$w_{m,j}(x,t) := \int_{-x}^{x} P_j\left(\frac{s}{x}\right) \exp\left(-\frac{|s-x_r|^2}{4(t-t_n)}\right) ds, \quad j \in \mathbb{N}_0, \quad m = 2n - r. \tag{3.12}$$

Thus,

$$u_m(x,t) = v_m(x,t) + \sum_{j=0}^{\infty} \frac{\beta_j(x) w_{m,j}(x,t)}{2x\sqrt{\pi(t-t_n)}}. \tag{3.13}$$

In order to obtain another way to compute the functions $\{u_m\}_{m\in\mathbb{N}}$ the following statement is proved.

Lemma 2. *Consider the functions*

$$\psi_j(x,t,r) := \int_{-x}^{x} P_j\left(\frac{s}{x}\right) \exp\left(\frac{-|s-r|^2}{4t}\right) ds, \quad j \in \mathbb{N}_0. \tag{3.14}$$

Then,

$$\psi_0(x,t,r) = \sqrt{\pi t}\left[Erf\left(\frac{x-r}{2\sqrt{t}}\right) + Erf\left(\frac{x+r}{2\sqrt{t}}\right)\right].$$

$$\psi_1(x,t,r) = \frac{2t}{x}\left[\exp\left(-\frac{|x+r|^2}{4t}\right) - \exp\left(-\frac{|x-r|^2}{4t}\right)\right] + \frac{r}{x}\psi_0(x,t,r).$$

$$\begin{aligned} \psi_2(x,t,r) = \frac{3rt}{x^2}&\left[\exp\left(-\frac{|x-r|^2}{4t}\right) - \exp\left(-\frac{|x+r|^2}{4t}\right)\right] + \frac{3r}{x}\psi_1(x,t,r) \\ &- \frac{3t}{x}\left[\exp\left(-\frac{|x-r|^2}{4t}\right) + \exp\left(-\frac{|x+r|^2}{4t}\right)\right] - \frac{3r^2+x^2-6t}{2x^2}\psi_0(x,t,r). \end{aligned}$$

$$\psi_3(x,t,r) = \left[\left(\frac{|x-r|^2}{4t}+1\right)\exp\left(-\frac{|x-r|^2}{4t}\right) - \left(\frac{|x+r|^2}{4t}+1\right)\exp\left(-\frac{|x+r|^2}{4t}\right)\right] + \frac{5r}{x}\psi_2(x,t,r) - \frac{3x^2+15}{2x^2}\psi_1(x,t,r) + \frac{15rx^2-5r^3}{6x^3}\psi_0(x,t,r).$$

Meanwhile, for $j \geq 4$

$$\psi_j(x,t,r) = \frac{[2(2j-1)(2j-5)t-x^2](2j-3)}{x^2 j(2j-5)}\psi_{j-2}(x,t,r) + \frac{(j-3)(2j-1)}{j(2j-5)}\psi_{j-4}(x,t,r) + \frac{r(2j-1)}{xj}\psi_{j-1}(x,t,r) - \frac{r(2j-1)}{xj}\psi_{j-3}(x,t,r) \quad (3.15)$$

Proof. Let us consider the change of variable $z = (s-r)/2\sqrt{t}$, then

$$\begin{aligned} \psi_0(x,t,r) &= \int_{-x}^{x} P_0\left(\frac{s}{x}\right)\exp\left(-\frac{|s-r|^2}{4t}\right)ds \\ &= \int_{-x}^{x} \exp\left(-\frac{|s-r|^2}{4t}\right)ds \\ &= 2\sqrt{t}\int_{z_1}^{z_2} e^{-z^2}dz \\ &= \sqrt{\pi t}\left[\mathrm{Erf}\left(\frac{x-r}{2\sqrt{t}}\right) + \mathrm{Erf}\left(\frac{x+r}{2\sqrt{t}}\right)\right]. \end{aligned}$$

where $z_1 := -(x+r)/2\sqrt{t}$, $z_2 := (x-r)/2\sqrt{t}$.

Note that before integration we have that

$$\psi_0(x,t,r) = 2\sqrt{t}\int_{z_1}^{z_2} e^{-z^2}dz. \quad (3.16)$$

From here it follows that

$$\begin{aligned}
\psi_1(x,t,r) &= 2\sqrt{t}\int_{z_1}^{z_2} P_1\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz \\
&= 2\sqrt{t}\int_{z_1}^{z_2} \left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz \\
&= \frac{4t}{x}\int_{z_1}^{z_2} ze^{-z^2}dz + \frac{2r\sqrt{t}}{x}\int_{z_1}^{z_2} e^{-z^2}dz \\
&= -\frac{2t}{x}\left[e^{-z^2}\right]_{z_1}^{z_2} + \frac{r}{x}\psi_0(x,t,r) \\
&= \frac{2t}{x}\left[\exp\left(-\frac{|x+r|^2}{4t}\right) - \exp\left(-\frac{|x-r|^2}{4t}\right)\right] + \frac{r}{x}\psi_0(x,t,r).
\end{aligned}$$

Again, we note that

$$\frac{4t}{x}\int_{z_1}^{z_2} ze^{-z^2}dz = \psi_1(x,t,r) - \frac{r}{x}\psi_0(x,t,r). \tag{3.17}$$

Moreover, by simple integration we have that

$$\int_{z_1}^{z_2} z^2e^{-z} = \frac{1}{4\sqrt{t}}\psi_0(x,t,r) - \frac{1}{2}\left[ze^{-z^2}\right]_{z_1}^{z_2} \tag{3.18}$$

then, using (3.16) and (3.17) we obtain that

$$\begin{aligned}
\psi_2(x,t,r) &= 2\sqrt{t}\int_{z_1}^{z_2} P_2\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz \\
&= 2\sqrt{t}\int_{z_1}^{z_2}\frac{3}{2}\left(\frac{2\sqrt{t}z+r}{x}\right)^2 e^{-z^2}dz - 2\sqrt{t}\int_{z_1}^{z_2}\frac{1}{2}e^{-z^2}dz \\
&= 3\sqrt{t}\int_{z_1}^{z_2}\left(\frac{2\sqrt{t}z+r}{x}\right)\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz - \frac{1}{2}\psi_0(x,t,r) \\
&= \frac{6t}{x}\int_{z_1}^{z_2} z\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz + \frac{3r}{2x}\psi_1(x,t,r) - \frac{1}{2}\psi_0(x,t,r) \\
&= \frac{12t^{3/2}}{x^2}\int_{z_1}^{z_2} z^2e^{-z^2}dz + \frac{6rt}{x^2}\int_{z_1}^{z_2} ze^{-z^2}dz + \frac{3r}{2x}\psi_1(x,t,r) - \frac{1}{2}\psi_0(x,t,r) \\
&= \frac{3t}{x^2}\psi_0(x,t,r) - \frac{6t^{3/2}}{x^2}\left[ze^{-z^2}\right]_{z_1}^{z_2} + \frac{3r}{2x}\left[\psi_1(x,t,r) - \frac{r}{x}\psi_0(x,t,r)\right] \\
&+ \frac{3r}{2x}\psi_1(x,t,r) - \frac{1}{2}\psi_0(x,t,r) \\
&= \frac{3rt}{x^2}\left[\exp\left(-\frac{|x-r|^2}{4t}\right) - \exp\left(-\frac{|x+r|^2}{4t}\right)\right] + \frac{3r}{x}\psi_1(x,t,r) \\
&- \frac{3t}{x}\left[\exp\left(-\frac{|x-r|^2}{4t}\right) + \exp\left(-\frac{|x+r|^2}{4t}\right)\right] - \frac{3r^2+x^2-6t}{2x^2}\psi_0(x,t,r).
\end{aligned}$$

In the same way, we note that

$$\frac{12t^{3/2}}{x^2}\int_{z_1}^{z_2} z^2e^{-z^2}dz = \psi_2(x,t,r) - \frac{3r}{x}\psi_1(x,t,r) + \frac{3r^2+x^2}{2x^2}\psi_0(x,t,r).$$

From here, (3.16), (3.17), (3.18) and due to the fact that

$$\int_{z_1}^{z_2} z^3e^{-z} = -\frac{1}{2}\left[(z^2+1)e^{-z^2}\right]_{z_1}^{z_2}$$

we obtain that

$$
\begin{aligned}
\psi_3(x,t,r) &= 2\sqrt{t}\int_{z_1}^{z_2} P_3\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz\\
&= 2\sqrt{t}\int_{z_1}^{z_2}\frac{5}{2}\left(\frac{2\sqrt{t}z+r}{x}\right)^3 e^{-z^2}dz - 2\sqrt{t}\int_{z_1}^{z_2}\frac{3}{2}\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz\\
&= 5\sqrt{t}\int_{z_1}^{z_2}\left(\frac{2\sqrt{t}z+r}{x}\right)\left(\frac{2\sqrt{t}z+r}{x}\right)^2 e^{-z^2}dz - 3\sqrt{t}\int_{z_1}^{z_2}\left(\frac{2\sqrt{t}z+r}{x}\right)e^{-z^2}dz\\
&= \frac{10t}{x}\int_{z_1}^{z_2} z\left(\frac{2\sqrt{t}z+r}{x}\right)^2 e^{-z^2}dz + \frac{5r\sqrt{t}}{x}\int_{z_1}^{z_2}\left(\frac{2\sqrt{t}z+r}{x}\right)^2 e^{-z^2}dz - \frac{3}{2}\psi_1\\
&= \frac{10t}{x^3}\int_{z_1}^{z_2}(4tz^3+4\sqrt{t}rz^2+r^2z)e^{-z^2}dz + \frac{5r}{3x}[\psi_2(x,t,r)+\frac{1}{2}\psi_0(x,t,r)] - \frac{3}{2}\psi_1\\
&= \frac{40t^2}{x^3}\int_{z_1}^{z_2} z^3e^{-z^2}dz + \frac{40t^{3/2}r}{x^3}\int_{z_1}^{z_2} z^2e^{-z^2}dz + \frac{10tr^2}{x^3}\int_{z_1}^{z_2} ze^{-z^2}dz\\
&+ \frac{5r}{3x}\psi_2(x,t,r) + \frac{5r}{6x}\psi_0(x,t,r) - \frac{3}{2}\psi_1(x,t,r)\\
&= \frac{40t^2}{x^3}\left[-\frac{1}{2}(z^2+1)e^{-z^2}\right]_{z_1}^{z_2} + \frac{10r}{3x}[\psi_2 - \frac{3r}{x}\psi_1 + \frac{3r^2+x^2}{2x^2}\psi_0] + \frac{5r^2}{2x^2}[\psi_1 - \frac{r}{x}\psi_0]\\
&+ \frac{5r}{3x}\psi_2(x,t,r) + \frac{5r}{6x}\psi_0(x,t,r) - \frac{3}{2}\psi_1(x,t,r)\\
&= -\frac{20t}{x^3}\left[(z^2+1)e^{-z^2}\right]_{z_1}^{z_2} + \frac{5r}{x}\psi_2(x,t,r) - \frac{3x^2+15}{2x^2}\psi_1(x,t,r) + \frac{15rx^2-5r^3}{6x^3}\psi_0\\
&= \left[\left(\frac{|x-r|^2}{4t}+1\right)\exp\left(-\frac{|x-r|^2}{4t}\right) - \left(\frac{|x+r|^2}{4t}+1\right)\exp\left(-\frac{|x+r|^2}{4t}\right)\right]\\
&+ \frac{5r}{x}\psi_2(x,t,r) - \frac{3x^2+15}{2x^2}\psi_1(x,t,r) + \frac{15rx^2-5r^3}{6x^3}\psi_0(x,t,r).
\end{aligned}
$$

On the other hand, using the known recursive formula for the Legendre polynomials

$$(2j+1)P_j(x) = P'_{j+1}(x) - P'_{j-1}(x)$$

and integrating by parts we have that

$$
\begin{aligned}
\psi_j(x,t,r) &= \int_{-x}^{x} P_j\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds \\
&= \frac{1}{2j+1}\int_{-x}^{x}\left[P'_{j+1}\left(\frac{s}{x}\right)-P'_{j-1}\left(\frac{s}{x}\right)\right]\exp\left(\frac{-|s-r|^2}{4t}\right)ds \\
&= \frac{x}{2j+1}\left[P_{j+1}\left(\frac{s}{x}\right)-P_{j-1}\left(\frac{s}{x}\right)\right]\exp\left(\frac{-|s-r|^2}{4t}\right)\Bigg|_{-x}^{x} \\
&+ \frac{x}{2(2j+1)t}\int_{-x}^{x}(s-r)\left[P_{j+1}\left(\frac{s}{x}\right)-P_{j-1}\left(\frac{s}{x}\right)\right]\exp\left(\frac{-|s-r|^2}{4t}\right)ds.
\end{aligned}
$$

Using the fact that $P_j(1)=1$ and $P_{j+1}(-1)=P_{j-1}(-1)$ for every j we have that

$$
\begin{aligned}
\psi_j(x,t,r) = \frac{x}{2(2j+1)t}\int_{-x}^{x} s\left[P_{j+1}\left(\frac{s}{x}\right)-P_{j-1}\left(\frac{s}{x}\right)\right]\exp\left(\frac{-|s-r|^2}{4t}\right)ds \\
-\frac{xr}{2(2j+1)t}\left[\psi_{j+1}(x,t,r)-\psi_{j-1}(x,t,r)\right]. \quad (3.19)
\end{aligned}
$$

Then, by the formula

$$(2j+1)xP_j(x) = (j+1)P_{j+1}(x) + jP_{j-1}(x)$$

we obtain that

$$
\begin{aligned}
\int_{-x}^{x} sP_{j+1}\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds = x\frac{j+2}{2j+3}\int_{-x}^{x} P_{j+2}\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds \\
+x\frac{j+1}{2j+3}\int_{-x}^{x} P_{j}\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds,
\end{aligned}
$$

$$\int_{-x}^{x} sP_{j-1}\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds = x\frac{j}{2j-1}\int_{-x}^{x} P_j\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds$$
$$+x\frac{j-1}{2j-1}\int_{-x}^{x} P_{j-1}\left(\frac{s}{x}\right)\exp\left(\frac{-|s-r|^2}{4t}\right)ds.$$

Then, replacing this last two equations in (3.19) we obtain that

$$\psi_j(x,t,r) = \frac{x^2}{2(2j+1)t}\left[\frac{j+2}{2j+3}\psi_{j+2}(x,t,r) + \frac{j+1}{2j+3}\psi_j(x,t,r) - \frac{j}{2j-1}\psi_j(x,t,r)\right.$$
$$\left.-\frac{j-1}{2j-1}\psi_{j-2}(x,t,r)\right] - \frac{xr}{2(2j+1)t}\psi_{j+1}(x,t,r) + \frac{xr}{2(2j+1)t}\psi_{j-1}(x,t,r),$$

Thus, we obtain that

$$\psi_{j+2}(x,t,r) = \frac{(2j+1)(2(2j+3)(2j-1)t - x^2)}{x^2(j+2)(2j-1)}\psi_j(x,t,r) + \frac{(2j+3)(j-1)}{(2j-1)(j+2)}\psi_{j-2}(x,t,r)$$
$$+\frac{(2j+3)r}{(j+2)x}\psi_{j+1}(x,t,r) - \frac{(2j+3)r}{(j+2)x}\psi_{j-1}(x,t,r).$$

then formula (3.15) is proved. ■

From the previous lemma, a simple recurrent procedure for computing (3.12) is obtained.

Theorem 23. *The functions $\{w_{m,j} : m \in \mathbb{N}, j \in \mathbb{N}_0\}$ given by (3.12) can be calculated by means of the following recurrent procedure*

$$w_{m,j}(x,t) = \frac{[2(2j-1)(2j-5)(t-t_n) - x^2](2j-3)}{x^2 j(2j-5)} w_{m,j-2}(x,t)$$
$$+\frac{(j-3)(2j-1)}{j(2j-5)}w_{m,j-4}(x,t) + \frac{x_r(2j-1)}{xj}w_{m,j-1}(x,t) - \frac{x_r(2j-1)}{xj}w_{m,j-3}(x,t)$$

with

$$w_{m,0}(x,t) = \sqrt{\pi(t-t_n)}\left[Erf\left(\frac{x-x_r}{2\sqrt{t-t_n}}\right) + Erf\left(\frac{x+x_r}{2\sqrt{t-t_n}}\right)\right],$$

$$w_{m,1}(x,t) = \frac{2(t-t_n)}{x}\left[\exp\left(\frac{-(x+x_r)^2}{4(t-t_n)}\right) - \exp\left(\frac{-(x-x_r)^2}{4(t-t_n)}\right)\right] + \frac{x_r}{x}w_0(x,t),$$

$$w_{m,2}(x,t) = \frac{3x_r(t-t_n)}{x^2}\left[\exp\left(-\frac{|x-x_r|^2}{4(t-t_n)}\right) - \exp\left(-\frac{|x+x_r|^2}{4(t-t_n)}\right)\right] + \frac{3x_r}{x} w_{m,1}(x,t)$$
$$-\frac{3(t-t_n)}{x}\left[\exp\left(-\frac{|x-x_r|^2}{4(t-t_n)}\right) + \exp\left(-\frac{|x+x_r|^2}{4(t-t_n)}\right)\right] - \frac{3x_r^2+x^2-6(t-t_n)}{2x^2} w_{m,0}(x,t),$$

$$w_{m,3}(x,t) = \left[\left(\frac{|x-x_r|^2}{4(t-t_n)}+1\right)\exp\left(-\frac{|x-x_r|^2}{4(t-t_n)}\right) - \left(\frac{|x+x_r|^2}{4(t-t_n)}+1\right)\exp\left(-\frac{|x+x_r|^2}{4(t-t_n)}\right)\right]$$
$$+\frac{5x_r}{x} w_{m,2}(x,t) - \frac{3x^2+15}{2x^2} w_{m,1}(x,t) + \frac{15x_r x^2 - 5x_r^3}{6x^3} w_{m,0}(x,t)$$

where $m = 2n - r$.

Proof. This equality is an immediate corollary of Lemma 2. Indeed, we have that $w_{m,j}(x,t) = \psi_j(x, t-t_n, x_r)$ where the Lemma 2 is used. ■

3.3 An extended method of fundamental solutions for boundary value problems for parabolic partial differential equations with variable coefficients

In this section a step by step method for approximation of the solution of boundary value problems for (3.6) is proposed. The basis of this method is Theorem 22 and the recursive formula in Theorem 23. The method is explained on the first boundary value problem but other boundary value problems can be solved in analogous way.

Let us consider equation (3.6) subject to the Dirichlet boundary conditions

$$u(-b,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad t \in [0,\tau], \tag{3.20}$$

and the initial condition

$$u(x,0) = \varphi(x), \quad x \in [-b,b]. \tag{3.21}$$

We assume that ψ_1, ψ_2, φ are continuously differentiable functions and satisfy the compatibility conditions

$$\psi_1(0) = \varphi(-b),\ \psi_2(0) = \varphi(b).$$

The problem (3.6), (3.20), (3.21) possesses a unique solution which depends continuously on the data (see e.g. [33]). Theorem 22 suggests the following simple method to approximate the solution of problem (3.6), (3.20), (3.21).

First of all let M be a positive integer and choose M points $\{t_k\}_{k=1}^{M}$ in the temporal interval $(-\tau, \tau)\backslash\{0\}$. After that, choose $x_0 < -b$, $x_1 > b$ and consider the $2M \times N$ functions $\{w_{m,j}(x,t)\}_{j=0}^{N-1}$ defined by (3.12) using the recursive formula given in Theorem 23. Then, consider the functions $\{\beta_j\}_{j=0}^{N-1}$ given by (3.11) and construct the functions $\{u_m\}_{m=1}^{2M}$ defined by formula (3.8).

The approximate solution $\tilde{u}$ of the equation (3.6) is sought as a linear combination of the functions (3.8) in the form

$$\tilde{u}(x,t) = \sum_{m=1}^{2M} c_m u_m(x,t). \tag{3.22}$$

The coefficients $\{c_m\}_{m=1}^{2M}$ are sought in such way that $\tilde{u}$ satisfy the initial and the boundary conditions approximately. For this we use the collocation method where L points $\{(x_i, t_i)\}_{i=1}^{L}$ are chosen on the parabolic boundary Γ. Then, imposing the boundary conditions (3.20), (3.21) onto the approximate solution (3.22) the following linear system of equations for the coefficients $\{c_m\}_{m=1}^{2M}$ is obtained

$$\sum_{m=1}^{2M} c_m u_m(x_i, t_i) = \begin{cases} \psi_1(t_i), & x_i = -b \\ \varphi(x_i), & t_i = 0 \\ \psi_2(t_i), & x_i = b \end{cases},\ i = 1, 2, \ldots, L. \tag{3.23}$$

According to the collocation method described in [10], [12], [27], the solution of the system (3.23) implies that function (3.22) is approximately equal to φ. ψ_1, ψ_2 on the parabolic boundary. By the maximum principle it is also an approximate solution on Ω.

Chapter 4

Cauchy problems for parabolic partial differential equations with variable coefficients in one space variable

This chapter is dedicated to obtain explicit solutions of the Cauchy and noncharacteristic Cauchy problems for parabolic partial differential equations with a variable coefficient of the form

$$\frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t) \tag{4.1}$$

where q is a continuous complex valued function of an independent real variable using the transmutation operators introduced in Section 1.3.

In Section 4.1 a system of functions for the representation of the solution of the Cauchy problem for (4.1) in terms of the $\{\beta_n\}_{n=0}^{\infty}$ functions given in Theorem 13 is obtained. The solution of the Cauchy problem for (4.1) being the image of certain Poisson transform under a transmutation operator is calculated using the Fourier-Legendre series representation of the transmutation kernel. Then, the system of functions is obtained as a convolution of the Legendre polynomials with the fundamental solution of the heat equation. Such integrals are calculated in a recursive way using the known recursive formula for the Legendre polynomials. In order to use the transmutation operator on the whole real line for the space variable an adequate functional space is introduced.

In Section 4.2 an explicit solution of the noncharacteristic Cauchy problem for equation (4.1) with Cauchy data belonging to a Holmgren class (see Section 1.2.4) is obtained. Using the mapping property the solution is presented in terms of the formal powers $\{\varphi_k\}_{k\in\mathbb{N}_0}$ arising in the spectral parameter power series (SPPS) method (see the mapping property, Theorem 12).

4.1 The Cauchy problem for a parabolic partial differential equation with variable coefficients

In this section we shall determine the solution u of the Cauchy problem for (4.1) considered on $-\infty < x < \infty$, $t > 0$ satisfying the initial condition on the line $t = 0$ given by

$$u(x,0) = \varphi(x) \quad \text{for } -\infty < x < \infty \tag{4.2}$$

where φ is a prescribed function. We assume that the potential q satisfies the condition

$$\int_{-\infty}^{\infty} (1 + |x|)\, |q(x)|\, dx < \infty. \tag{4.3}$$

In order to use a transmutation operator to solve the Cauchy problem, let us consider the following class of functions

Definition 7. *Let M be the class of functions f for which there are positive constants C_1, C_2 and $\sigma \in [0, 2)$ such that*

$$|f(x,t)| \le C_1 e^{C_2|x|^{\sigma}}, \quad x \in \mathbb{R}, t > 0.$$

It is worth mentioning that the functions of the class M satisfy the following properties. If $f, g \in M$ are such that

$$|f(x,t)| \leq Ae^{a|x|^{\alpha}}, \quad |g(x,t)| \leq Be^{b|x|^{\beta}},$$

then

$$|f(x,t)| \leq Ae^{a|x|^{\sigma}} \quad \forall\, \sigma \in [\alpha, 2), \qquad |(fg)(x,t)| \leq ABe^{(a+b)|x|^{\max\{\alpha,\beta\}}}.$$

So note that $f+g$ and fg belong to M. As we will see in the following section this class of functions is the adequate space for the definition of a transmutation operator and the Cauchy problem (4.1)-(4.2) to be studied.

4.1.1 A transmutation operator for parabolic operators on the whole real line

We use both transmutation operators T and its inverse T^{-1} introduced in Section 1.3 by

$$Th(x,t) = h(x,t) + \int_{-x}^{x} K(x,s)h(s,t)ds, \tag{4.4}$$

$$T^{-1}u(x,t) = u(x,t) + \int_{-x}^{x} L(x,s)u(s,t)ds. \tag{4.5}$$

where the integral kernel K is uniformly bounded, according to Theorem 8, for every $x \in \mathbb{R}$ and $|s| \leq x$ because of the condition (4.3). The integral kernel L belongs to the class of functions M with $\sigma = 1$ as we can see from Theorem 8. Thus, both K and L belongs to the class of functions M.

From the definition and the properties of class of functions M we obtain the following result for the domain of the transmutation operator T and its inverse T^{-1}.

Proposition 7. *The transmutation operator T (4.4) and its inverse T^{-1} (4.5) map the class of functions M into itself, i.e., T is a bijection on M.*

Proof. Set $f \in M$. Since the kernels K and L belong to M and due to the fact that the function $Ae^{a|x|^\sigma}$ is monotone increasing for $x > 0$ and for all A, a, $\sigma > 0$, we obtain that the integrals $\int_{-x}^{x} K(x,s)f(s,t)ds$, $\int_{-x}^{x} L(x,s)f(s,t)ds$ belong to M. Thus, Tf and $T^{-1}f \in M$. ■

4.1.2 Solution of the Cauchy problem

Let us now turn to the solution of Cauchy's problem for equation (4.1) with the initial condition (4.2) in the class of functions M.

Theorem 24. *The Cauchy problem*

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t) \quad -\infty < x < \infty,\ t > 0 \tag{4.6}$$

$$u(x,0) = \varphi(x), \quad -\infty < x < \infty \tag{4.7}$$

where $\varphi \in M$ and q is a continuous complex valued function which satisfies the condition (4.3)*, has a unique solution within the class of functions M.*

Moreover, the solution has the form

$$u(x,t) = h(x,t) + \sum_{n=0}^{\infty} \frac{\beta_n(x)}{2x\sqrt{\pi t}} \int_{-\infty}^{\infty} \psi_n(x,t,r)\psi(r)dr, \quad x \in \mathbb{R}, \quad t > 0 \tag{4.8}$$

where $\psi = T^{-1}\varphi$, h is the solution of (4.6) *with $q \equiv 0$ and initial condition ψ given by* (1.8) *and the functions $\{\psi_n\}_{n\in\mathbb{N}_0}$ are given by* (3.14)*. The functions $\{\beta_n\}_{n\in\mathbb{N}_0}$ are from Theorem 13.*

Proof. Denote $\psi := T^{-1}\varphi$. Let h be the unique solution of the Cauchy problem

$$\frac{\partial^2 h}{\partial x^2}(x,t) = \frac{\partial h}{\partial t}(x,t), \quad -\infty < x < \infty, \quad t > 0$$
$$h(x,0) = \psi(x), \qquad -\infty < x < \infty$$

defined by (1.8). Then, $u := Th$ is a solution of (4.6) because of Proposition 3. Also $T\psi = \varphi$. Thus, there exists a solution of the problem (4.6), (4.7). Due to the invertibility of T and the uniqueness of the Cauchy problem for the heat equation, u is the unique solution for (4.6), (4.7).

In order to construct the explicit formula (4.8) consider the integral kernel K in the form of the Fourier-Legendre series (1.21).

In order to compute u we note that h is given by the Poisson transform (1.8)

$$h(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4t}\right] \psi(r)dr. \tag{4.9}$$

Then, substituting (4.9) and (1.21) into the definition of T in (4.4) we obtain that

$$u(x,t) = h(x,t) + \int_{-x}^{x} \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} P_n\left(\frac{s}{x}\right) \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(s-r)^2}{4t}\right] \psi(r)drds.$$

By Fubini's theorem, changing the order of integration the following equality follows

$$\int_{-x}^{x} K(x,s)h(s,t)ds = \sum_{n=0}^{\infty} \frac{\beta_n(x)}{2x\sqrt{\pi t}} \int_{-\infty}^{\infty} \int_{-x}^{x} P_n\left(\frac{s}{x}\right) \exp\left(-\frac{|s-r|^2}{4t}\right) ds\psi(r)dr.$$

Thus,

$$u(x,t) = h(x,t) + \sum_{n=0}^{\infty} \frac{\beta_n(x)}{2x\sqrt{\pi t}} \int_{-\infty}^{\infty} \psi_n(x,t,r)\psi(r)dr$$

where ψ_n is defined by (3.14). ■

Remark 6. *Although the efficient construction of the inverse transmutation operator is a developing topic, the case when q is a compactly supported potential provides us an example of computation of ψ on $[-b,b]$. In this case, as was shown in [11], if φ admits a uniformly convergent series expansion in terms of the functions φ_k, then ψ is the uniformly convergent on $[-b,b]$ power series $\sum_{k=0}^{\infty} a_k x^k$ where the coefficients a_k are determined by the values of φ and its f-derivatives at* 0.

Remark 7. *Lemma 2 in Section 3.2.1 provides us with a recurrent procedure for computing the functions $\{\psi_n\}_{n\in\mathbb{N}_0}$. Then, a simple recurrent procedure to compute the transmuted Poisson transform* (4.8) *is obtained.*

4.2 Solution of the noncharacteristic Cauchy problem for parabolic partial differential equations with variable coefficients in one space variable

In this section an explicit solution of the noncharacteristic Cauchy problem for (4.1) in terms of the formal powers φ_k given by (1.18) is obtained. The boundary conditions on the line $x = 0$ are given by

$$u(0,t) = F(t), \quad \frac{\partial u}{\partial x}(0,t) = G(t) \tag{4.10}$$

where F and G belong to the Holmgren function class defined in Section 1.2.4

We use the transmutation operator T introduced in Section 2.1.2 by the second kind Volterra integral operator

$$Tv(x) := v(x) + \int_{-x}^{x} \mathbf{K}(x,s)v(s)ds. \tag{4.11}$$

The inverse operator T^{-1} also has the form of a second kind Volterra integral operator and satisfies the following correspondence of the initial values, see Section 1.3.1

$$v(0) = u(0), \qquad v'(0) = u'(0) - \alpha u(0), \tag{4.12}$$

where $v := T^{-1}u$.

As we have seen in Section 1.3.2, the transmutation operator T satisfies the mapping property $T[x^k] = \varphi_k(x)$ for every $k \in \mathbb{N}_0$, where the formal powers $\{\varphi_k\}_{k\in\mathbb{N}_0}$ are given by the rule (1.18). This makes it possible to prove the following theorem for the noncharacteristic Cauchy problem for equation (4.1) with initial conditions (4.10).

Theorem 25. *Let $q \in C[-b,b]$ and u be a solution of the noncharacteristic Cauchy* *problem*

$$\frac{\partial^2 u}{\partial x^2} - q(x)u(x,t) = \frac{\partial u}{\partial t}(x,t), \quad -b < x < b, \quad |t| < \tau \tag{4.13}$$

$$u(0,t) = F(t), \qquad |t| < \tau \tag{4.14}$$

$$\frac{\partial u}{\partial x}(0,t) = G(t), \qquad |t| < \tau \tag{4.15}$$

where $F, G \in H(b,\tau,C,0)$, $C > 0$. *Then the series*

$$\sum_{j=0}^{\infty} \left[\frac{F^{(j)}(t)}{(2j)!} \left(\varphi_{2j}(x) - \frac{\alpha}{2j+1}\varphi_{2j+1}(x) \right) + \frac{G^{(j)}(t)}{(2j+1)!}\varphi_{2j+1}(x) \right]$$

converges uniformly and absolutely for $|x| \le r < b$ *to the solution* $u(x,t)$ *where* φ_k *are the formal powers* (1.18).

Proof. Let u be a solution of (4.13). Consider the function $h := T^{-1}u$, where the operator T^{-1} is applied with respect to the variable x. According to Proposition 3, the function h is a solution of the heat equation. Due to (4.12) it satisfies the following noncharacteristic Cauchy problem

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}, \quad -b < x < b, \quad |t| < \tau$$

$$h(0,t) = F(t), \qquad |t| < \tau$$

$$\frac{\partial h}{\partial x}(0,t) = G(t) - \alpha F(t), \qquad |t| < \tau.$$

Since $G - \alpha F \in H(b,\tau,(1+\alpha)C,0)$, the solution of this problem is given by the absolutely and uniformly convergent series for $|x| \le r < b$ (see, e.g., [3], [37])

$$h(x,t) = \sum_{k=0}^{\infty} \left(\frac{F^{(k)}(t)}{(2k)!}x^{2k} + \frac{G^{(k)}(t) - \alpha F^{(k)}(t)}{(2k+1)!}x^{2k+1} \right).$$

Due to the mapping property of T we obtain that

$$u(x,t) = Th(x,t) = \sum_{k=0}^{\infty} \left[\frac{F^{(k)}(t)}{(2k)!} \left(\varphi_{2k}(x) - \frac{\alpha}{2k+1} \varphi_{2k+1}(x) \right) + \frac{G^{(k)}(t)}{(2k+1)!} \varphi_{2k+1}(x) \right].$$

This series converges uniformly and absolutely for $|x| \le r < b$ due to the uniform boundedness of T and T^{-1} guaranteed by Corollary 1. ∎

Appendix A

Numerical implementation

In this section, a program for the numerical implementation for approximate solution of initial boundary value problems for parabolic partial differential equations with time independent potential from Section 2.1.4 is presented. The program was realized in Matlab 2012.

The main code is presented in Listing A.1. Listings A.2-A.10 are auxiliar functions for the main code.

I would like to express my gratitude to Dr. R. Michael Porter for the code of the `ninteg` function in listing A.7 as well as to Dr. Sergii M. Torba for the code of the functions `ParticularSolution`, `ConstructFormalPowers` and `PowersX_n` in listingsA.8-A.10 which are very useful to the numerical implementation.

Listing A.1: Main code

```
% **********************************************************************
% Este codigo calcula la solucion numerica de la ecuacion
% u_{xx} - (x-1)^2 u(x,t) = u_t para  x en (0,2) y t en (0,1)
% con las siguientes condiciones iniciales y de frontera
% u(x,0) = uot, u(0,t) = uot, u(b,t) = ubt.

% Entrada : ax - limite espacial inferior
%           bx - limite espacial superior
```

```
%            at - limite temporal inferior
%            bt - limite temporal superior
%            N - Numero de potencias formales a usar
%            q - potencial de la ecuacion
%            uxo - condicion inicial
%            uot - condicion de frontera izquierda
%            ubt - condicion de frontera derecha

% Funciones externas : Este codigo usa las siguientes funciones externas
%            1) ptsparametrizados - parametrizacion de la frontera
%            2) ConstructFormalPowers
%            3) Particular solution
%            4) PowerX_n
%            5) ninteg

% Procesos :
%            a) Construccion de los polinomios de calor transmutados
%            evaluados en los puntos de colocacion
%            b) Calculo de los coeficientes de la aproximacion usando el
%            metodo de colocacion
%            c) Construccion de la solucion aproximada en el interior del
%            dominio de la ecauacion

%  Salida : Grafica el error absoluto de aproximacion
% ***********************************************************************

clc; clear; %limpiar pantalla y memoria

% Inicializacion del dominio de la solucion
ax = 0; % Limite espacial inferior
bx = 2; % Limite espacial superior
at = 0; % Limite temporal inferior
bt = 1; % Limite temporal superior
```

```
% Inicializacio de las potencias formales a usar
N = 27; % Cantidad de potencias formales N > 2

% Discretizacion de la frontera
[r,xo,to,indxo] = ptsparametrizados(ax,bx,at,bt,12001);

% Calculo de una solucion particular de f''-x^2f=0
[u1, u2, du1, du2] = ParticularSolution(xo, q(xo), N);
f = u1;%+i*u2; % asegurar la ausencia de ceros de f

% Construccion de las potencias formales phi y psi
[phi, psi] = ConstructFormalPowers(f, 1, 1, N, xo(end)-xo(1));

% Eleccion de la cantidad de puntos para colocacion
ind = floor(linspace(1,12001,N));

% Eleccion de los puntos en x
indx = find(indxo(1)<ind & ind<indxo(end) | indxo(1)==ind | ind==indxo(end))
    ;
X = r(ind(indx),1);

% Eleccion de los puntos en t sobre las fronteras laterales.
% Frontera lateral izquierda (x=0)
indt1 = indx(1)-1;
t1 = r(ind(1:indt1),2);

% Frontera lateral derecha (x=b)
indt2 = indx(end)+1;
t2 = r(ind(indt2:end),2);

% Evaluacion de las potencias formales en los puntos de colocacion sobre la
% frontera horizontal (t=0)
PHI = phi(:,ind(indx)-indxo(1)+1);
```

```
% Construccion de u_n(t,x) en los puntos de colocacion
for n = 1:N
    %agregar phi(0)
    if rem(n,2)==1;
        aux1 = (factorial(n-1)/factorial((n-1)/2))*t1.^((n-1)/2);
    else
        aux1 = zeros(length(t1),1);
    end

    aux2 = PHI(n,:)';%agregar phi_n(x);

    %agregar phi_n(b);
    aux3 = 0;
    for k = 0:floor((n-1)/2)
        aux3 = aux3 + (factorial(n-1)/factorial(k)/factorial(n-1-2*k))*phi(n
            -2*k,end)*t2.^k;
    end
    A(:,n) = [aux1;aux2;aux3];
end

B = [uot(t1);uxo(X);ubt(t2)];

a = A\B; % Solucion del sistema lineal para los coeficientes

% Construccion de la malla interna del dominio para aproximacion
auxx = linspace(ax,bx,102);
x = auxx(2:end-1);
p = spline(xo,phi,x);

auxt = linspace(at,bt,102);
t = auxt(2:end-1);

% Inicializacion de la solucion aproximada
U = zeros(length(t),length(x));
```

```
%Construccion de u_n(t,x)
for n = 1:N
    for k = 0:floor((n-1)/2)
        auxT(:,k+1) = (factorial(n-1)/factorial(n-1-2*k)/factorial(k))*t.^k;
        auxP(k+1,:) = p(n-2*k,:); %phi_0(x)->phi(1,:)==> phi_(n-1-2k)(x)->
            phi(n-1-2k+1,:)=phi(n-2k,:)
    end
    u(n,:,:) = auxT*auxP;
end

% Construccion de la solucion aproximada U(t,x) usando los polinomios de
% calor calculados u(n,t,x) y los coeficientes  a(n)
for k = 1:N
    U(:,:) = U(:,:) + squeeze(a(k)*u(k,:,:));
end

% Construccion de la solucion analitica conocida Ur(x,t)
for j = 1:length(t)
    Ur(:,j) = exp((-0.5)*(x-1).^2-t(j));
end

% Calculo del error absoluto y el error relativo en cada punto de la
% solucion aproximada
absoluteerror = abs(U'-Ur); % Ea(x,t)
relativeerror = abs((U'-Ur)./Ur); % Er(x,t)

% Encontrar el maximo de cada uno de los errores
ea = max(max(absoluteerror));
er = max(max(relativeerror));

% Grafica del error absoluto en el dominio de la ecuacion discretizado
surf(absoluteerror);
colormap jet
```

```
lighting phong
shading interp
axis tight
camlight left
xlabel t
ylabel x
zlabel 'Absolute error'
title 'The distribution of the absolute error'
grid on
```

Listing A.2: Auxiliar function

```
% *******************************************************************
% Este codigo parametriza la frontera parabolica de un rectangulo del
% plano
% *******************************************************************

% Entrada : ax, bx, at, bt - Limites del rectangulo
%           pts - cantidad de puntos deseados en la frontera

% Salida : r - Frontera parabolica parametrizada
%          x - Primera coordenada de r
%          t - Segunda coordenada de r
%          indx - cantidad de puntos en la base de la frontera parabolica

function [r,x,t,indx] = ptsparametrizados(ax,bx,at,bt,pts)

s = linspace(0,3,pts); % Discretizacion del dominio de la curva

r = zeros(pts,2); % Inicializacion de la curva
j = 0;
k = 0;

for i = 1:length(s)
    if 0<=s(i) && s(i)<1
```

```
        r(i,:) = [ax,s(i)*at+(1-s(i))*bt]; % Construccion izquierda de r
    elseif 1<=s(i) && s(i)<=2
        j = j+1;
        x(j) = (2-s(i))*ax+(s(i)-1)*bx; % Construccion de x
        r(i,:) = [x(j),at]; % Construccion de r en la parte inferior del
            rectangulo
        indx(j) = i; % Guardar los indices correspondientes a x
    elseif 2<s(i) && s(i)<=3
        k = k+1;
        t(k) = (3-s(i))*at+(s(i)-2)*bt; % Construccion de t
        r(i,:) = [bx,t(k)];  % Construccion derecha de r
    end
end
```

Listing A.3: Potential

```
%Potencial de la ecuacion q
function y = q(x)
    y = (x-1).^2;
end
```

Listing A.4: Initial condition

```
%Condicion inicial t=0
function y = uxo(x)
    y = exp((-1/2)*(x-1).^2);
end
```

Listing A.5: Left boundary condition

```
%Condicion de frontera en x=0
function y = uot(t)
    y = exp(-0.5-t);
end
```

Listing A.6: Right boundary condition

```
%Condicion de frontera en x=b
function y = ubt(t)
    y = exp(-0.5-t);
end
```

Listing A.7: ninteg function

```
function result = ninteg( dat, bminusa )

% Usage: ninteg(dat, intervallength ).
% Integrates list dat of n=5k+1 data points. Returns list of n points.
%  Uses 5-point integrated interpolating polynomial,
%    applied to points 1:6, 6:10, 11:16, etc.
%  Supposes points equally spaced on interval.
%
n = length(dat);                            % dat is to be integrated
if n < 6
    error( ['ninteg: requires at least 6 data points, received ' num2str(n)
       ] )
end
n2 = mod(n-1,5);               % number of points to process at the beginning
n1 = n - n2;                   % number of points to process afterwards

intmat = [   [475,1427,-798,482,-173,27]/1440     % Formula for
    [28,129,14,14,-6,1]/90                  % numerical
    3*[17,73,38,38,-7,1]/160                  % integration
    2*[7,32,12,32,7,0]/45                     % A 6 x 5 matrix
    5*[19,75,50,50,75,19]/288
    ] * bminusa/ (n-1);                         % Divide by interval length

inval = 0;
if n2 > 0
    mdat = reshape( dat(1:5), 5, 1 );
    row_n = mdat(1,:);                               % Create final row, so each
       column
```

```
    row_n = [ row_n(2:length(row_n)), dat(6) ];% ends with start of next
        column
    mdat = [ mdat', row_n' ]';                    % Annex the row
    m1 = intmat * mdat;         % numerical integration, gives 5 values
    incr = m1(5,:);             % prepare to calculate cumulative sums
    incr = cumsum( [ 0, incr(1:length(incr)-1) ] );
    incr = [ incr; incr; incr; incr; incr ];
    m1 = m1 + incr;             % matrix now contains the integrals
    result = reshape(m1,1,5);   % convert to a single list of length n
    tresult = result(1:n2);               % take the first n2 values
    inval = tresult(end);
end
% proceed the first part of the list
%if mod(n,5) == 1                          % Verify n is 5k+1
    mdat = reshape( dat(n2+1:n-1), 5, (n1-1)/5 ); % Break into 6 rows: first
         5 here
    row_n = mdat(1,:);                            % Create final row, so each
        column
    row_n = [ row_n(2:length(row_n)), dat(n) ];% ends with start of next
        column
    mdat = [ mdat', row_n' ]';                    % Annex the row
    m1 = intmat * mdat;         % numerical integration, gives 5 values
    incr = m1(5,:);             % prepare to calculate cumulative sums
    incr = cumsum( [ 0, incr(1:length(incr)-1) ] );
    incr = [ incr; incr; incr; incr; incr ];
    m1 = m1 + incr + inval;            % matrix now contains the integrals
    if n2 > 0
        result = [0, tresult, reshape(m1,1,n1-1) ];    % convert to a single
            list of length n
    else
        result = [0, reshape(m1,n2+1,n-1) ];    % convert to a single list of
             length n
    end
% else
```

```
%      error( ['ninteg: requires 5k+1 data points, received ' num2str(n) ] )
% end

end
```

Listing A.8: ParticularSolution function

```
% Compute a particular solution of the equation
% -y''+q y = 0 using the SPPS representation
% x - interval of interest
% q - coefficient q of the equation
% dq - derivative of the potential
% N - number of formal powers to use
function [u1, u2, du1, du2] = ParticularSolution(x, q, N)
f = ones(1, length(x));

[phi, psi] = ConstructFormalPowers(f, 1, q, N, x(end)-x(1));

u1 = ones(1, length(x));
u2 = zeros(1, length(x));
du1 = u2;
du2 = u2;
for i = 2:2:N
    u1 = u1 + phi(i+1,:) / factorial(i);
    du1 = du1 + psi(i, :) / factorial(i-1);
    u2 = u2 + phi(i, :) / factorial(i-1);
    du2 = du2 + psi(i-1, :) / factorial(i-2);
end
```

Listing A.9: ConstructFormalPowers function

```
% Construction of the formal powers
% Parameters: f - particular non-vanishing function,
% p and r - coefficients of the equation
% N - number of formal powers
```

```
% bmina - integration interval length
% The formal powers returned starting from the power 0,
% i.e., phi(1,:) coinside with phi_0.
function [phi, psi]=ConstructFormalPowers(f, p, r, N, bmina)
%*******************************************************************
% Iterative integrals
%*******************************************************************
q1 = f .^ 2 .* r;
q2 = (p .* f .^ 2).^(-1);
[X, Xtil] = PowersX_n(q1, q2, N, bmina);
%*******************************************************************
% Construction of the systems phi_k and psi_k
%*******************************************************************
phi = zeros(N+1, length(f));
psi = zeros(N+1, length(f));
% First function f1_0 coincides with f
finv = 1 ./ f;
phi(1,:) = f;
psi(1,:) = finv;

for n=1:N
    if rem(n,2) == 0;
        phi(n+1,:)=f.*Xtil(n,:);
        psi(n+1,:)=X(n,:).* finv;
    else
        phi(n+1,:)=f.*X(n,:);
        psi(n+1,:)=Xtil(n,:).* finv;
    end
end
```

Listing A.10: PowersX_n function

```
function [Xp,Xtilp]=PowersX_n(q1,q2,N1,bmina)
Xtilp = zeros(N1,length(q1));
Xp = zeros(N1,length(q1));
```

```
Xp(1,:) = ninteg(q2, bmina);
Xtilp(1,:) = ninteg(q1, bmina);
for n = 2:N1                                                %Cycle for
    calculating powers 2 to N of X and X~
    if rem(n,2) == 0                                         %Powers
        calculated for the even n
        Xtilp(n,:) = ninteg(n*Xtilp(n-1,:).*q2, bmina);
        Xp(n,:) = ninteg(n*Xp(n-1,:).*q1, bmina);
    else                                                     %Powers
        calculated for the odd n
        Xtilp(n,:) = ninteg(n*Xtilp(n-1,:).*q1, bmina);
        Xp(n,:) = ninteg(n*Xp(n-1,:).*q2, bmina);
    end
end
```

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