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Potential Differential Games: Stochastic and Deterministic Models

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Abstract

This thesis concerns noncooperative open-loop differential games with both finite and infinite horizon payoffs. Our main objective is to introduce the notion of a *potential differential game* (PDG) that, roughly put, is a differential game to which we can associate an optimal control problem (OCP) whose solutions are Nash equilibria for the original game.

A PDG has, therefore, two relevant features. Firstly, finding Nash equilibria for the game is greatly simplified, because it is easier to deal with a single-player problem, that is, an OCP, than with the original game itself. Secondly, the Nash equilibria obtained from the associated OCP are automatically "pure" (or deterministic) rather than "mixed" (or randomized).

The question then is how do we know when a given differential game is a PDG? Moreover, assuming that we have a PDG, how do we construct an associated OCP? And, what additional advantages can we obtain from associating an OCP to a PDG? Our main goal is to provide answers to these questions. Our approach is largely motivated by known results for static games [32], [39]. Finally, we extend to *stochastic* differential games some of our results on PDGs.

Resumen

En esta tesis estudiamos una clase de juegos diferenciales no cooperativos con horizonte finito e infinito. Nuestro objetivo principal es introducir la noción de un *juego diferencial potencial* que, en términos generales, es un juego diferencial al que podemos asociar un problema de control óptimo cuyas soluciones son equilibrios de Nash para el juego original. De aquí se sigue que un juego diferencial potencial tiene dos características relevantes. En primer lugar, encontrar equilibrios de Nash para el juego se simplifica enormemente, porque es más fácil tratar con un problema de control óptimo que con el juego original en sí mismo. En segundo lugar, los equilibrios de Nash obtenidos a partir del problema de control óptimo asociado son automáticamente "puros" (o deterministas) en lugar de "mixtos" (o aleatorizados). La pregunta obvia es, por supuesto, ¿cómo sabemos cuándo un juego diferencial dado es un juego diferencial potencial? Además, suponiendo que tenemos un juego diferencial potencial, ¿cómo construimos un problema de control óptimo asociado? ¿Qué más ventajas podemos obtener al encontrar un problema de control óptimo asociado a un juego diferencial potencial? Nuestro objetivo principal es proporcionar respuestas a estas preguntas. Nuestra investigación sigue una línea sugerida por algunos resultados sobre juegos estáticos [32], [39]. Por último, extendemos a juegos diferenciales *estocásticos* algunos de los resultados obtenidos para juegos diferenciales (deterministas).

Contents

Table of abbreviations and symbols

Abbreviations

OCP	Optimal control problem,
PDG	Potential differential game,
SDG	Stochastic differential game,
SDE	Stochastic differential equation,
a.e	Almost everywhere,
a.s	Almost sure.

Symbols

$\square$	End of a proof,
$\lozenge$	End of an example.

1 Introduction

This introduction consists of two sections. Section 1.1, is a brief introduction to the literature on potential games. We also explain the thesis' structure. Our presentation on potential differential games is based on papers by A. Fonseca-Morales and O. Hernández-Lerma [17, 18]. In Section 1.2, on static potential games we summarize the work of Monderer and Shapley [32] and Slade [39] to motivate some results on the differential case.

1.1 Potential games

Most known results on potential games are concentrated on *static* (or one-shot) games. These results come from two main lines of research. One of them can be traced back to the 1973 paper by Rosenthal [36], who identified a class of games with Nash equilibria in the family of *pure* strategies. A little over 20 years later, Monderer and Shapley [32] took Rosenthal's paper and extended it in several directions; they also coined the name "potential games" in analogy to the potential functions used in physics. The other line of research comes from Slade [39], who introduced the *optimization approach* to games. We will comment more of [32] and [39] in the next section.

For potential *dynamic* games there is just a handful of publications. For instance, for *discrete-time games*, see González-Sánchez and Hernández-Lerma [20, 22] and their references. For applications in communications engineering, see Zazo et al. [45, 46, 47].

For *differential* games, to the best of our knowledge, there are only the publications by Dragone et al. [13, 14, 15]. The relation between these last papers and our results here is not very close—for details, see Chapter 3, Remark 3.18, below.

The topic we are interested in is the notion of a *potential differential game* (PDG), that is, a differential game for which we can associate an optimal control problem whose solutions are Nash equilibria for this differential game. PDGs are introduced in Section 2.3.

Our main results are based on two approaches to identify PDGs: The *exact-potential approach* and the *fictitious-potential approach*. The first approach is based directly on particular features of the game's primitive data, namely, the payoff functions and the game dynamics. The second approach, on the other hand, depends on a given smooth auxiliary function, and it also imposes smoothness conditions on the game's primitive data. See Sections 3.1 and 3.2, below.

The names we use for our two approaches: "the exact-potential approach" and "the fictitious-potential approach" are analogous to the Monderer-Shapley's "exact potential games" [32] and Slade's "fictitious-objective function" [39], respectively.

We study numerous examples to illustrate our methods and results. A natural example of a PDG is a so-called team game in which all the players want to optimize the same payoff function. This naturally defines an associated optimal control problem. Other examples of PDGs are presented in Section 3.3, below.

For some PDGs, we can study the asymptotic behavior of Nash equilibria. Indeed, under suitable hypotheses, we apply a result by Trèlat and Zuazua [42] to the associated optimal control problem to obtain a turnpike property for Nash equilibria. See Section 4.1.

In Section 4.2, several classes of differential games that have Pareto-optimal Nash equilibria are identified. The results consist of two approaches, the *direct case* and the *potential case*. The former approach is based on particular features of the game's primitive data. In contrast, in the potential case, a key assumption is that we are dealing with *potential* games.

We also study PDGs in the *stochastic* case. On the one hand, in Section 5.4, we extend to stochastic games the *fictitious-potential approach* introduced in Section 3.2. We do not include the *exact-potential approach* proposed in Section 3.1 because it is quite similar. On the other hand, in Section 5.5, we establish some classes of stochastic PDGs that have Pareto-optimal Nash equilibria.

Summarizing, in Chapter 2, we introduce the differential games we are interested in. We include the definition of PDGs, and give some relevant remarks. In Chapter 3, we present two approaches to identify PDGs and illustrate the obtained results with numerous examples. In Chapter 4, we adapt a turnpike property to PDGs and, moreover, find several classes of PDGs that have Pareto-optimal Nash equilibria. In Chapter 5, we extend some results in previous sections to identify stochastic PDGs. Finally, in Chapter 6, we present some conclusions and open problems.

1.2 Potential games: The static case

To motivate our results we next present some of the pioneering concepts introduced by Monderer and Shapley [32] and Slade [39].

Consider a noncooperative N-player static game in *normal* (or *strategic*) form

$$\Gamma = (\bar{N}, A_1, \ldots, A_N, \pi^1, \ldots, \pi^N) \tag{1.1}$$

where $\bar{N} := \{1, \ldots, N\}$ is the set of players, and for every $i \in \bar{N}$,

- A_i denotes the action set for player i, and
- $\pi^i : A \to \mathbb{R}$ denotes the player i's payoff function, where

  $$A := A_1 \times \cdots \times A_N$$

  is the set of *multistrategies*, also known as *strategy profiles*.

A multistrategy $u^* = (u_1^*, \ldots, u_N^*) \in A$ is said to be a *Nash equilibrium* for the game Γ, assuming that the players want to maximize their payoff functions, if

$$\pi^i(u^*) \geq \pi^i(u_i, u_{-i}^*) \;\; \forall i \in \bar{N}, \tag{1.2}$$

where

$$(u_i, u_{-i}^*) := (u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*)$$

and $u_{-i}^* := (u_1^*, \ldots, u_{i-1}^*, u_{i+1}^*, \ldots, u_N^*)$ is a point in $A_{-i} := A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_N$.

Note from (1.2) that finding a Nash equilibrium is equivalent to solving N *coupled* maximization problems. A key feature of potential games, defined below, is that they yield Nash equilibria by means of a *single* optimization problem.

There are several classes of potential games, such as weighted, ordinal, best-reply, etc. (See González-Sánchez and Hernández-Lerma [23] or Monderer and Shapley [32].) The most basic class, however, is the following.

Definition 1.1. *(Monderer and Shapley [32].) The game Γ in (1.1) is called an exact potential game or simply a potential game if there is a function $P : A \to \mathbb{R}$ such that, for every $i \in \bar{N}$, and every $a_i, b_i \in A_i$, $u_{-i} \in A_{-i}$,*

$$\pi^i(a_i, u_{-i}) - \pi^i(b_i, u_{-i}) \;\; = \;\; P(a_i, u_{-i}) - P(b_i, u_{-i}).$$

In this case, P is called an exact potential function, or simply a potential function, for Γ.

From Definition 1.1 it is evident and easy to prove that if a multistrategy $u^* \in A$ maximizes P, then u^* is a Nash equibrium for Γ. In other words, solving a *single* optimization problem (rather than N, as in (1.2)) we obtain Nash equilibria for Γ. Hence, the simplest example of a potential game is the following.

Example 1.2. *The game* Γ *is called a team game (also known as a coordination game) if there is a function* $P : A \to \mathbb{R}$ *such that* $\pi^i = P$ *for all* $i = 1, \ldots, N$. *(In optimal control theory, a team game is known as a decentralized control problem. See [5].) Clearly, a team game is a potential game.* $\diamondsuit$

The following example is also well known.

Example 1.3. *The prisoner's dilemma is a well-known game in which the payoff functions are given as follows:*

	NC	*C*
NC	*(-1, -1)*	*(-9, 0)*
C	*(0, -9)*	*(-6, -6).*

We have a potential game in which, for every $k \in \mathbb{R}$, *the following defines a potential function for the game*

	NC	*C*
NC	*-4+k*	*-3+k*
C	*-3+k*	*k.*

This function attains its maximum value at (C, C), *with value* k; *hence,* (C, C) *is a Nash equilibrium for the prisoner's dilemma game.* $\diamondsuit$

The following theorem characterizes a potential game under smoothness conditions.

Theorem 1.4. *(Monderer and Shapley [32].) Let* Γ *be a static game for which the strategy sets are intervals of real numbers, and the payoff functions* $\pi^i : A \to \mathbb{R}$, $i \in \bar{N}$, *are continuously differentiable. A function* $P : A \to \mathbb{R}$ *is a potential for* Γ *if and only if* P *is continuously differentiable and*

$$\frac{\partial \pi^i}{\partial u_i} = \frac{\partial P}{\partial u_i} \quad \forall i \in \bar{N}. \tag{1.3}$$

Slade [39] introduced another line of research, the so-called *optimization approach*, which identifies games that can be solved by optimizing a function so-named *fictitious-objective function* that is the same as the potential function of Monderer and Shapley. One of Slade's main results is the following.

Proposition 1.5. *(Slade [39].) Let Γ be a static game in which the strategy sets A_i are compact intervals, and the functions π^i are in C^2. The following statements are equivalent.*

(a) The function $P : A \to \mathbb{R}$ is a potential function for the game Γ.

(b) For every $i = 1, \ldots, N$, there exists a function $c^i(u_{-i})$, such that

$$\pi^i(u) = P(u) + c^i(u_{-i}). \tag{1.4}$$

The following examples illustrate Proposition 1.5.

Example 1.6. *A team game, as in Example 1.2, satisfies (1.4) with $c^i(\cdot) = 0$ for every $i = 1, \ldots, N$.* $\Diamond$

Example 1.7. *Consider a market where N firms produce differentiated products. The payoff function of player i is, for $q = (q_1, \ldots, q_N)$,*

$$\pi^i(q) = R(q) + r^i(q_{-i}) - c_i(q_i), \;\; i = 1, \ldots, N.$$

By Proposition 1.5, a potential function for the game is

$$P(q) = R(q) - \sum_{i=1}^{N} c_i(q_i). \;\Diamond$$

The following example explains a relation between the potential function and the payoff functions' structure in a two-player zero-sum game.

Example 1.8. *Consider a two-player zero-sum game with payoff functions satisfying (1.4), i.e., $\pi^1 = -\pi^2$ and $\pi^i = P(u) + c^i(u_{-i})$. Then*

$$P = -\frac{1}{2}\sum_{k=1}^{2} c^k(u_{-k})$$

is a potential function for the game if and only if the game is separable, that is,

$$\pi^1(\cdot) = \frac{1}{2}(c^1(u_2) - c^2(u_1)).\Diamond$$

To conclude, note that Monderer and Shapley [32] and Slade [39] began from different viewpoints what we now call *potential games.*

For potential static games, there is a large number of related publications. See, for instance, González-Sánchez and Hernández-Lerma [23] for a small sample. In particular, for applications in engineering see Gopalakrishnan et al. [24], and La et al. [27].

2 Potential differential games (PDGs)

We introduce *potential differential games* (PDGs), which are the main subject of our study. To this end, the chapter is divided into three parts. In Section 2.1, we present the noncooperative differential games we are concerned with. In Section 2.2, we define some related optimal control problems (OCPs). Finally, in Section 2.3, we include the PDGs as well as some illustrative examples and remarks.

2.1 Differential games

Let $\bar{N} := \{1, \ldots, N\}$, $N \geq 2$, be the *set of players*, and $[0, T], T \leq \infty$, where T is the game's time *horizon.* Let $X \subset \mathbb{R}^l, l \geq 1$, be the *set of feasible states.* And, for each $i \in \bar{N}$ the *set of feasible controls* $U_i \subseteq \mathbb{R}^{m_i}$. Define

$$U := U_1 \times \cdots \times U_N \subseteq \mathbb{R}^m,$$

with $m := m_1 + \cdots + m_N$.

For each $i \in \bar{N}$, we introduce the *open-loop strategy space* for player i as

$$\mathbf{U}_i := \{\mathbf{u}_i : [0, T] \to U_i | \ \mathbf{u}_i \text{ is Borel-measurable}\}, \tag{2.1}$$

and $\mathbf{U} := \mathbf{U}_1 \times \cdots \times \mathbf{U}_N$ the space of *open-loop multistrategies.*

Given a multistrategy $\mathbf{u} \in \mathbf{U}$, a function $\mathbf{x} : [0, T] \to X$ is called the *admissible state path* for the game, corresponding to the multistrategy $\mathbf{u}$, if $\mathbf{x}$ is the unique solution to the system of ordinary differential equations

$$\begin{aligned} \dot{\mathbf{x}}(s) &= f(s, \mathbf{x}(s), \mathbf{u}(s)), \\ \mathbf{x}(0) &= x_0, \end{aligned} \tag{2.2}$$

where f is a given $\mathbb{R}^l$-valued function defined on $[0, T] \times X \times U$, and $x_0 := (x_{10}, \ldots, x_{N0}) \in X$ is a given *initial condition.* Sufficient conditions for existence and uniqueness of solutions to the system (2.2) are well known; see, for instance, [19] Chapter 1; [44], Chapter 3.

For each $i \in \bar{N}$, let $L^i : [0,T] \times X \times U \to \mathbb{R}$ be an *instantaneous* (or *current*) *payoff function* for player i, and $S^i : X \to \mathbb{R}$ a *terminal* (or *final*) *payoff function*, which is also known as a *salvage* or *bequest* function. The functions S^i vanish when $T = \infty$.

The *payoff function* for player i is defined for each $\mathbf{u} \in \mathbf{U}$ by

$$J_T^i(\mathbf{u}) := \begin{cases} \int_0^T L^i(s, \mathbf{x}(s), \mathbf{u}(s))ds + S^i(\mathbf{x}(T)) & \text{when } T < \infty, \\ \int_0^\infty e^{-\beta \cdot s} L^i(s, \mathbf{x}(s), \mathbf{u}(s))ds & \text{when } T = \infty, \end{cases} \tag{2.3}$$

where $\mathbf{x}$ is the admissible state path to the multistrategy $\mathbf{u}$, and $\beta > 0$ is the *intertemporal discount rate*, which is considered to be the same for every player.

For each $i \in \bar{N}$, consider the set of multistrategies

$$\mathbf{U}_{-i} := \mathbf{U}_1 \times \cdots \times \mathbf{U}_{i-1} \times \mathbf{U}_{i+1} \times \cdots \times \mathbf{U}_N.$$

For each $\mathbf{u}_i \in \mathbf{U}_i$ and each $\mathbf{u}^*_{-i} \in \mathbf{U}_{-i}$, we will write $(\mathbf{u}_i, \mathbf{u}^*_{-i})$ to denote the vector

$$(\mathbf{u}^*_1, \ldots, \mathbf{u}^*_{i-1}, \mathbf{u}_i, \mathbf{u}^*_{i+1}, \ldots, \mathbf{u}^*_N) \in \mathbf{U}.$$

Definition 2.1. *A multistrategy $\boldsymbol{u}^* = (\boldsymbol{u}^*_1, \ldots, \boldsymbol{u}^*_N) \in \boldsymbol{U}$ is called an open-loop Nash equilibrium for the differential game (2.2)-(2.3) if, for every $i = 1, \ldots, N$,*

$$J_T^i(\boldsymbol{u}_i, \boldsymbol{u}^*_{-i}) \leq J_T^i(\boldsymbol{u}^*) \ \ \forall \boldsymbol{u}_i \in \boldsymbol{U}_i.$$

Remark 2.2. *In many applications there are feasible states $X_i \subseteq \mathbb{R}^{l_i}$ for each player $i \in \bar{N}$ such that*

$$X := X_1 \times \cdots \times X_N \subseteq \mathbb{R}^l,$$

with $l := l_1 + \cdots + l_N$. Then we can write the system function f in (2.2) as a vector $(f^1, \ldots, f^N)$ where each coordinate f^i is an $\mathbb{R}^{l_i}-$valued function defined over $[0,T] \times X \times U$. Hence, the system (2.2) can be rewritten in terms of the $i-$th coordinate ($\boldsymbol{x}_i(s) \in \mathbb{R}^{l_i}; i = 1, \ldots, N$) as

$$\begin{aligned} \dot{\boldsymbol{x}}_i(s) &= f^i(s, \boldsymbol{x}(s), \boldsymbol{u}(s)), \\ \boldsymbol{x}_i(0) &= x_{i0}. \end{aligned} \tag{2.4}$$

As an example, some oligopolies [10] are expressed as in (2.4). If $j \in \bar{N}$ is such that $l_j = 0$, then we understand that player j has no state variable in the game. Despite this, the player is still affected by the admissible state path $\boldsymbol{x}$ for the game.

In a compact form, the class of differential games we are interested in can be described as

$$\Gamma_{x_0}^T := \left[\bar{N}, \{\mathbf{U}_i\}_{i\in\bar{N}}, \{J_T^i\}_{i\in\bar{N}}, f\right], \ T \leq \infty, \tag{2.5}$$

where $\bar{N} = \{1, \dots, N\}$ is the set of players and, for each $i \in \bar{N}, \mathbf{U}_i$ is the open-loop strategy space for player i in (2.1), J_T^i is the payoff function for player i as in (2.3), and f defines the system dynamics (2.2) (or (2.4)). In the infinite-horizon case, we write (2.5) as $\Gamma_{x_0}^\infty$.

2.2 Optimal control problems

From Definition 2.1 a Nash equilibrium for the game $\Gamma_{x_0}^T$, as in (2.5), induces N coupled OCPs. In contrast, using suitable functions and the spaces $U, X, \mathbf{U}$, etc, in Section 2.1, we can define an OCP as follows.

Definition 2.3. *Consider two functions $P : [0,T] \times X \times U \to \mathbb{R}$ and $S : X \to \mathbb{R}$. These functions define an OCP in which a single player (or controller) wants to maximize the payoff function defined, for each $\boldsymbol{u} \in \boldsymbol{U}$, by*

$$J_T(\boldsymbol{u}) := \begin{cases} \int_0^T P(s, \boldsymbol{x}(s), \boldsymbol{u}(s))ds + S(\boldsymbol{x}(T)) & \text{when } T < \infty, \\ \int_0^\infty e^{-\beta \cdot s} P(s, \boldsymbol{x}(s), \boldsymbol{u}(s))ds & \text{when } T = \infty, \end{cases}$$

subject to (2.2). A function $\boldsymbol{u}^* \in \boldsymbol{U}$ that solves this OCP is called an open-loop optimal control or optimal solution.

Note that the controller in the OCP in Definition 2.3 optimizes over the multistrategy space $\mathbf{U}$ for the game (2.5). To avoid confusions in our results, we clearly specify when we are referring to a control in $\mathbf{U}$ for the controller or to a multistrategy in $\mathbf{U}$ for the players.

2.3 PDGs

As we mentioned before, finding an open-loop Nash equilibrium for an N-player differential game is a difficult task due to the fact that a differential game is the coupling of N OCPs. In particular, note that all the state variables $x_1, \dots, x_N$ are included, in principle, in the constraint (2.4). These facts, among others, motivate the introduction of potential differential games (PDGs).

Definition 2.4. *A differential game $\Gamma_{x_0}^T, T \leq \infty$, as in (2.5), is called an open-loop PDG if there exists an OCP such that an open-loop optimal solution of this OCP is an open-loop Nash equilibrium for $\Gamma_{x_0}^T$.*

A special class of open-loop PDGs is the class of so-called team games. We introduced the static version of a team game in Examples 1.2 and 1.6. We now introduce the dynamic version in the next example.

Example 2.5. *Team games. The game (2.5), where T is either finite or infinite, is said to be a team game if there is a (payoff) function $J_T : \boldsymbol{U} \to \mathbb{R}$ such that*

$$J_T^i = J_T \ \ \forall i \in \bar{N}.$$

In others words, (2.5) is a team game if every player i has the same payoff function, say J_T.

It is easily seen that a team game is an open-loop PDG. Indeed, let $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_N^*) \in \boldsymbol{U}$ be a multistrategy that maximizes J_T subject to (2.2). Then, by definition of optimality, for every $i \in \bar{N}$, we have

$$J_T(\boldsymbol{u}^*) \geq J_T(\boldsymbol{u}_i, \boldsymbol{u}_{-i}^*) \ \ \forall \boldsymbol{u}_i \in \boldsymbol{U}_i.$$

Hence, by Definition 2.1, $\boldsymbol{u}^*$ is an open-loop Nash equilibrium. The converse is not true, however. That is, a team game can have Nash equilibria that do not maximize J_T. (A similar situation occurs for potential static games. For an example in which the optimizers of a potential game form a proper subset of the family of Nash equilibria of the game, see Remark 3.5 in [23].) $\diamondsuit$

Charalambous [5] has recently analyzed a class of stochastic team games from the viewpoint of *decentralized* stochastic optimal control. In his terminology, an optimal multistrategy $\mathbf{u}^*$ as above is called a *decentralized global optimal strategy* (GOS), which becomes a *decentralized person-by-person* (PbP) *optimal strategy* when seen as an open-loop Nash equilibrium. He also notes (as we did in the previous paragraph) that a GOS is necessarily a PbP optimal strategy, but not conversely.

The following Example 2.6 is a particular game suggested by one of the reviewers of a previous version of the paper [17]. It is important to mention that Slade [39] also makes a similar observation to that in Example 2.6 but for *static* games.

Example 2.6. *To fix ideas we consider the differential game (2.5) in the infinite horizon case. (The case $T < \infty$ is similar.) Let us suppose that* *we know in advance that the game has an open-loop Nash equilibrium,* $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{u}}_1, \dots, \hat{\boldsymbol{u}}_N)$*. Now consider an OCP, as in Definition 2.3, with payoff function* $J_\infty(\boldsymbol{u})$ *defined by*

$$P(s, \boldsymbol{x}(s), \boldsymbol{u}(s)) := -\sum_{i=1}^{N} \|\boldsymbol{u}_i(s) - \hat{\boldsymbol{u}}_i(s)\|^2 \quad \forall s \in [0, \infty).$$

Clearly, $J_\infty(\boldsymbol{u})$ *is maximized when* $\boldsymbol{u} = \hat{\boldsymbol{u}}$ *and, therefore, the original game is an open-loop PDG.*

In other words, a differential game that has an open-loop Nash equilibrium can be trivially transformed into an open-loop PDG. This means that the concept of potential differential game is really useful when we wish to show, by means of an OCP, that a given game has an open-loop Nash equilibrium. $\diamondsuit$

Example 2.6 also illustrates the fact (already noted in Example 2.5) that an OCP associated to an open-loop PDG might not identify all the Nash equilibria of the game. For instance, if the game in Example 2.6 has another Nash equilibrium $\bar{\mathbf{u}} \neq \hat{\mathbf{u}}$, then the OCP in that example will not identify $\bar{\mathbf{u}}$.

Going back to Definition 2.4, a serious limitation of it is that it does not specify how one can determine if a given differential game is an open-loop PDG. Or even if we know in advance that we have an open-loop PDG, the definition does not say how to find an associated OCP. To this end, we consider the following remark as a way to identify open-loop PDGs.

Remark 2.7. *A differential game* $\Gamma_{x_0}^{\infty}$*, as in (2.5) (with* $T = \infty$*), is an open-loop PDG if there exists a function* $P : [0, \infty) \times X \times U \to \mathbb{R}$ *such that the OCP given in Definition 2.3 with* $T = \infty$ *satisfies Definition 2.4. Analogously, a differential game* $\Gamma_{x_0}^{T}$*, as in (2.5) (with* $T < \infty$*), is an open-loop PDG if there exist functions* $P : [0, T] \times X \times U \to \mathbb{R}$ *and* $S : X \to \mathbb{R}$ *such that the OCP given in Definition 2.3 with* $T < \infty$ *satisfies Definition 2.4. Therefore, if* P *and* S *have these features, we call* P *a potential function and* S *a potential terminal payoff function for (2.5).*

2.4 Comments

We introduced some terminology and notation concerning the potential differential games we are interested in. In the next chapter, we will study two approaches to identify PDGs and find a corresponding OCP.

3 Identifying PDGs

From the definition of a PDG a key question arises: How can we identify PDGs and an associated OCP? We propose answers to this question by means of two approaches: The *exact-potential approach* and the *fictitious-potential approach*, which are described in Sections 3.1 and 3.2, respectively. In Section 3.3, we present several examples that illustrate our results. These results include both finite- and infinite-horizon (deterministic) differential games.

3.1 Exact-potential approach

The *exact-potential approach* is based on analyzing the particular features of a game's primitive data, namely, the payoff functions and the state dynamics of a differential game. It is so-named because of its similarity with the exact potential games for the static case in Section 1.2, above.

3.1.1 PDGs over an infinite horizon.

We begin with Theorem 3.1 on sufficient conditions for a differential game as in (2.5), with $T = \infty$, to be an open-loop PDG. To illustrate this theorem, see Examples 3.23 and 3.26 in Section 3.3, as well as Example 4.7 in Chapter 4.

Let us denote by X_{-i} the set $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$, and for functions $\mathbf{x}_i : [0, \infty) \to X_i$ and $\mathbf{x}^*_{-i} : [0, \infty) \to X_{-i}$, write

$$(\mathbf{x}_i, \mathbf{x}^*_{-i}) := (\mathbf{x}^*_1, \ldots, \mathbf{x}^*_{i-1}, \mathbf{x}_i, \mathbf{x}^*_{i+1}, \ldots, \mathbf{x}^*_N).$$

Theorem 3.1. *Let $\Gamma^{\infty}_{x_0}$ be a differential game as in (2.5), and $p : [0, \infty) \times X \times U \to \mathbb{R}$ a certain function. Assume that one of the following conditions holds for every $i \in \bar{N}$:*

(a) There exists a function $c^i : [0, \infty) \times U_{-i} \to \mathbb{R}$ such that

$$L^i(s, x, u) \quad = \quad p(s, x, u) + c^i(s, u_{-i}). \tag{3.1}$$

(b) There exist functions $c^i : [0, \infty) \times X \times U_{-i} \to \mathbb{R}$ and $g^i : [0, \infty) \times X \to X_i$

such that

$$\begin{aligned} L^i(s,x,u) &= p(s,x,u) + c^i(s,x,u_{-i}), \text{ and} \\ f^i(s,x,u) &= g^i(s,x). \end{aligned}$$

(c) There exist functions $c^i : [0,\infty) \times X_{-i} \times U_{-i} \to \mathbb{R}$ *and* $g^i : [0,\infty) \times X_i \times U_i \to X_i$ *such that*

$$\begin{aligned} L^i(s,x,u) &= p(s,x,u) + c^i(s,x_{-i},u_{-i}), \text{ and} \\ f^i(s,x,u) &= g^i(s,x_i,u_i). \end{aligned}$$

Then $\Gamma_{x_0}^{\infty}$ *is an open-loop PDG with potential function* p.

Proof. We prove the theorem for part (a) only. The proof for (b) or (c) is similar.

Consider the OCP in Definition 2.3 with $P := p$ and $T = \infty$. We wish to prove that this OCP satisfies Definition 2.4 (see Remark 2.7). To this end, let us assume that $\mathbf{u}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_N^*)$ is an open-loop optimal solution of this OCP, and $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*)$ is the corresponding feasible state path. Fix an arbitrary $i \in \bar{N}$, and let $\mathbf{u}_i \neq \mathbf{u}_i^*$ be an open-loop strategy for player i. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ be the new state trajectory given by (2.2) corresponding to $(\mathbf{u}_i, \mathbf{u}_{-i}^*)$. As $\mathbf{u}^*$ and $\mathbf{x}^*$ are optimal for the OCP, then

$$\int_0^{\infty} e^{-\beta\cdot s} p(s,\mathbf{x}(s),(\mathbf{u}_i(s),\mathbf{u}_{-i}^*(s)))ds \;\; \leq \;\; \int_0^{\infty} e^{-\beta\cdot s} p(s,\mathbf{x}^*(s),\mathbf{u}^*(s))ds.$$

Adding to both sides of this inequality the constant

$$\int_0^{\infty} e^{-\beta\cdot s} c^i(s,\mathbf{u}_{-i}^*(s))ds$$

we obtain from (3.1) that $J^i(\mathbf{u}_i, \mathbf{u}_{-i}^*) \leq J^i(\mathbf{u}^*)$ for all $\mathbf{u}_i \in \mathbf{U}_i$. Hence, since $i \in \bar{N}$ was arbitrary, by Definition 2.1, we conclude that $\mathbf{u}^*$ is an open-loop Nash equilibrium for $\Gamma_{x_0}^{\infty}$ when the condition (a) is satisfied. □

From Theorem 3.1, we again obtain that the team games in Example 2.5 are open-loop PDGs (take $c^i = 0$ in (3.1)). For static games, (3.1) becomes (1.4). Notice also that the particular game in part (b) of Theorem 3.1 satisfies that the state variable x_i for player i is fixed; it does not change when the players move their strategies. In this case, for discrete-time Markov games, it is said that the game has *action-independent transitions*. See, for instance, [23], Section 5, or [34], Section 4.

As in Theorem 3.1, it is easy to verify that the following special cases are open-loop PDGs.

Corollary 3.2. *Let $\Gamma_{x_0}^{\infty}$ be a differential game as in (2.5), and $p : [0, \infty) \times X \times U \to \mathbb{R}$ a given function. Let us assume that one of the following conditions holds for every $i \in \bar{N}$:*

(a) There exists a function $c^i : [0, \infty) \times U_i \to \mathbb{R}$ such that, instead of (3.1), we have

$$L^i(s, x, u) \;=\; p(s, x, u) + c^i(s, u_i).$$

(b) There exist functions $c^i : [0, \infty) \times X \times U_i \to \mathbb{R}$ and $g^i : [0, \infty) \times X \to X_i$ such that

$$\begin{aligned} L^i(s, x, u) &= p(s, x, u) + c^i(s, x, u_i), \text{ and} \\ f^i(s, x, u) &= g^i(s, x). \end{aligned}$$

(c) There exist functions $c^i : [0, \infty) \times X_i \times U_i \to \mathbb{R}$ and $g^i : [0, \infty) \times X_i \times U_i \to X_i$ such that

$$\begin{aligned} L^i(s, x, u) &= p(s, x, u) + c^i(s, x_i, u_i), \text{ and} \\ f^i(s, x, u) &= g^i(s, x_i, u_i). \end{aligned}$$

Then $\Gamma_{x_0}^{\infty}$ is an open-loop PDG with potential function $p + \sum_{j=1}^{N} c^j$.

Proof. Part (a) follows from Theorem 3.1(a) because, for any $i \in \bar{N}$, the instantaneous payoff function L^i can be rewritten as

$$L^i(s, x, u) \;=\; p(s, x, u) + \sum_{j=1}^{N} c^j(s, u_j) - \sum_{j \neq i} c^j(s, u_j). \tag{3.2}$$

Similar arguments give (b) and (c) in this corollary from (b) and (c) in Theorem 3.1, respectively. □

Notice that the potential function for the differential game in Theorem 3.1 is p, whereas the potential function in Corollary 3.2 is $p + \sum_{j=1}^{N} c^j$. Observe also that in Theorem 3.1 or Corollary 3.2 no assumption of regularity is required for the functions L^i and f^i. Dragone et al. [13] addressed the case in Corollary 3.2(c) with the function p being identically zero, and the functions c^i and f^i are assumed to be in $C^2([0, \infty) \times X \times U)$.

See Example 3.23, below, where using Corollary 3.2(c), it is shown that a game of extraction of exhaustible resources under common access is an open-loop PDG.

The following Corollaries 3.3, 3.4, and 3.5 specialize Theorem 3.1 to two-person zero-sum games ($L^1 + L^2 = 0$ which gives $J^1 + J^2 = 0$). Related results are presented by Potters et al. [34] for static games with finite action sets.

Corollary 3.3 establishes a relationship between Theorem 3.1(a) and Corollary 3.2(a) for two-person zero-sum games.

Corollary 3.3. *Suppose that $N = 2$, and let $\Gamma_{x_0}^{\infty}$ be a zero-sum differential game. Then the following statements are equivalent.*

(i) $\Gamma_{x_0}^{\infty}$ is an open-loop PDG that satisfies the features of part (a) in Theorem 3.1.

(ii) (Diagonal property.) For all $u_1, v_1 \in U_1$ and $u_2, v_2 \in U_2$ we have

$$L^1(s, u_1, u_2) + L^1(s, v_1, v_2) = L^1(s, u_1, v_2) + L^1(s, v_1, u_2).$$

(iii) (Separation property.) There are functions $\bar{g}^1 : [0, \infty) \times U_1 \to \mathbb{R}$ and $\bar{g}^2 : [0, \infty) \times U_2 \to \mathbb{R}$ such that

$$L^1(s, u_1, u_2) = \bar{g}^1(s, u_1) + \bar{g}^2(s, u_2).$$

(iv) $\Gamma_{x_0}^{\infty}$ has the characteristics of part (a) in Corollary 3.2.

Proof. (i) implies (ii). Given that $-L^1 = L^2$, it follows that

$$p(s, x, u_1, u_2) = -\frac{1}{2}[c^1(s, u_2) + c^2(s, u_1)].$$

Then, by (3.1), the diagonal property for L^1 is directly verified.

(ii) implies (iii). Consider fixed points $b_1 \in U_1$ and $b_2 \in U_2$, and functions

$$\begin{aligned} \bar{g}^1(s, u_1) &:= L^1(s, u_1, b_2) - \frac{1}{2}L^1(s, b_1, b_2) \text{ and} \\ \bar{g}^2(s, u_2) &:= L^1(s, b_1, u_2) - \frac{1}{2}L^1(s, b_1, b_2). \end{aligned}$$

Then $L^1(s, u_1, u_2) = \bar{g}^1(s, u_1) + \bar{g}^2(s, u_2)$.

(iii) implies (iv). Take

$$\begin{aligned} p(s, x, u_1, u_2) &= -[\bar{g}^1(s, u_1) - \bar{g}^2(s, u_2)], \\ c^1(s, u_1) &= 2\bar{g}^1(s, u_1), \text{ and} \\ c^2(s, u_2) &= -2\bar{g}^2(s, u_2). \end{aligned}$$

(iv) implies (i). Use (3.2) to obtain (3.1). □

Example 1.8, above, is a particular case of Corollary 3.3(i)-(iii) for a two-person zero-sum static game.

The following Corollary 3.4 establishes an equivalence between Theorem 3.1(b) and Corollary 3.2(b) for two-person zero-sum games.

Corollary 3.4. *Suppose that $N = 2$. For a zero-sum differential game $\Gamma_{x_0}^{\infty}$ the following statements are equivalent.*

(i) $\Gamma_{x_0}^{\infty}$ is an open-loop PDG that has the features of part (b) in Theorem 3.1. Let $\boldsymbol{x} : [0, \infty) \to X$ be the fixed state path, so the i-th coordinate $\boldsymbol{x}_i$ is the solution to

$$\begin{aligned} \dot{\boldsymbol{x}}_i(s) &= g^i(s, \boldsymbol{x}(s)) \ \ i = 1, \ldots, N, \\ \boldsymbol{x}_i(0) &= x_{i0}. \end{aligned}$$

(ii) For all $u_1, v_1 \in U_1$ and $u_2, v_2 \in U_2$ we have

$$L^1(s, \boldsymbol{x}(s), u_1, u_2) + L^1(s, \boldsymbol{x}(s), v_1, v_2) = L^1(s, \boldsymbol{x}(s), u_1, v_2) + L^1(s, \boldsymbol{x}(s), v_1, u_2).$$

(iii) There are functions $\bar{g}^1 : [0, \infty) \times X \times U_1 \to \mathbb{R}$ and $\bar{g}^2 : [0, \infty) \times X \times U_2 \to \mathbb{R}$ such that

$$L^1(s, \boldsymbol{x}(s), u_1, u_2) = \bar{g}^1(s, \boldsymbol{x}(s), u_1) + \bar{g}^2(s, \boldsymbol{x}(s), u_2).$$

(iv) $\Gamma_{x_0}^{\infty}$ has the characteristics of part (b) in Corollary 3.2.

We omit the proof of Corollary 3.4 (and also of Corollary 3.5 below) because it is similar to the proof of Corollary 3.3.

Corollary 3.5 establishes an equivalence between Theorem 3.1(c) and Corollary 3.2(c) for two-person zero-sum games.

Corollary 3.5. *Suppose $N = 2$. A zero-sum differential game satisfies that the following statements are equivalent.*

(i) $\Gamma_{x_0}^{\infty}$ is an open-loop PDG with the features of part (c) in Theorem 3.1.

(ii) For all $x_1, y_1 \in X_1, x_2, y_2 \in X_2$ and for all $u_1, v_1 \in U_1, u_2, v_2 \in U_2$ we have

$$\begin{aligned} L^1(s, x_1, x_2, u_1, u_2) \ &+ \ L^1(s, y_1, y_2, v_1, v_2) \\ &= \ L^1(s, x_1, y_2, u_1, v_2) \ + \ L^1(s, y_1, x_2, v_1, u_2). \end{aligned}$$

(iii) There are functions $\bar{g}^1 : [0, \infty) \times X_1 \times U_1 \to \mathbb{R}$ and $\bar{g}^2 : [0, \infty) \times X_2 \times U_2 \to \mathbb{R}$ such that

$$L^1(s, x_1, x_2, u_1, u_2) = \bar{g}^1(s, x_1, u_1) + \bar{g}^2(s, x_2, u_2).$$

(iv) $\Gamma_{x_0}^{\infty}$ has the characteristics of part (c) in Corollary 3.2.

3.1.2 PDGs over a finite horizon

In Theorem 3.6 below, we adapt Theorem 3.1 and Corollary 3.2 to games $\Gamma_{x_0}^T$ as in (2.5), with $T < \infty$, to be an open-loop PDG. By Remark 2.7, we need to show the existence of two functions $P : [0, T] \times X \times U \to \mathbb{R}$ and $S : X \to \mathbb{R}$ such that they define an OCP satisfying Definition 2.4.

Theorem 3.6. *Let $\Gamma_{x_0}^T$ be a finite-horizon differential game as in (2.5), and let $\bar{s} : X \to \mathbb{R}$ be a certain function. Let us assume that one of the following conditions holds for every $i \in \bar{N}$:*

(a') The functions f^i and L^i satisfy either part (a) in Theorem 3.1 or part (a) in Corollary 3.2 and, furthermore, the terminal payoff function is independent of i, that is,

$$S^i(x) = \bar{s}(x) \ \ \forall i \in \bar{N}.$$

(b') The functions f^i and L^i satisfy either (b) in Theorem 3.1 or (b) in Corollary 3.2. In either case, the final payoff function S^i for player i has no restrictions.

(c') The functions f^i and L^i satisfy (c) in Theorem 3.1 and, in addition, there exists a function $k^i : X_{-i} \to \mathbb{R}$ such that

$$S^i(x) \ = \ \bar{s}(x) + k^i(x_{-i}).$$

(d') The functions f^i and L^i satisfy (c) in Corollary 3.2, and there exists a function $k^i : X_i \to \mathbb{R}$ such that

$$S^i(x) \ = \ \bar{s}(x) + k^i(x_i).$$

Then the differential game $\Gamma_{x_0}^T$ is an open-loop PDG. The potential function is p or $p + \sum_{j=1}^N c^j$ when the assumptions considered are those of Theorem 3.1 or those of Corollary 3.2, respectively. Moreover, the potential terminal payoff function is $\bar{s}$ for (a'), identically zero for (b'), $\bar{s}$ for (c'), and $\bar{s} + \sum_{i=1}^N k^i$ for (d').

The proof of each part of Theorem 3.6 follows directly from the corresponding result in Theorem 3.1 or Corollary 3.2.

3.2 Fictitious-potential approach

We noted in Section 1.2, above, that the concept of a fictitious-objective function for a static game Γ as in (1.1), is a function $P : A \to \mathbb{R}$ that satisfies

$$\frac{\partial P}{\partial u_i} \ = \ \frac{\partial \pi^i}{\partial u_i} \ \ \forall i \in N. \tag{3.3}$$

See (1.3) to conclude that when P satisfies (3.3), then P is a potential function.

For a given differential game, the *fictitious-potential approach* consists of finding a smooth auxiliary function P (and another function S when $T < \infty$) such that certain characteristics similar to (3.3) are satisfied. The function P (and S when $T < \infty$) define the corresponding OCP. (See Remark 2.7.) Naturally, we can say that this approach is based on Slade's results [39].

3.2.1 Technical requirements

In the remainder of this section, we consider an arbitrary function $P : [0,T] \times X \times U \to \mathbb{R}$ and a differential game $\Gamma_{x_0}^T$, as in (2.5). We introduce regularity conditions and some notation to find requirements similar to (3.3). We use the term "Assumption" to describe certain features about differential games, and we use "Condition" to impose conditions on components from a "fictitious" optimal control problem.

Assumption 3.7. *Consider (2.5) with $T \leq \infty$. For each $i \in \bar{N}$,*
(a) the sets U_i and X_i are open and convex,
(b) the function L^i is in $C^2(X \times U)$,
(c) the function f^i is in $C^2(X \times U)$.

Denote, for each $i \in \bar{N}$, the gradient vector of the function L^i with respect to the vector u_i by

$$\nabla_{u_i} L^i := \left(\frac{\partial L^i}{\partial u_1^i}, \ldots, \frac{\partial L^i}{\partial u_{m_i}^i} \right).$$

Besides, for each fixed $(s, \bar{u}_{-i}) \in [0,T] \times U_{-i}$, the *Hessian matrix* of L^i with respect to the vector (x, u_i) is denoted by

$$Hess[L^i(s, x, (u_i, \bar{u}_{-i}))] \quad := \quad \begin{pmatrix} \frac{\partial^2 L^i}{\partial x_1^1 \partial x_1^1} & \cdots & \frac{\partial^2 L^i}{\partial u_{m_i}^i \partial x_1^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L^i}{\partial x_1^1 \partial u_{m_i}^i} & \cdots & \frac{\partial^2 L^i}{\partial u_{m_i}^i \partial u_{m_i}^i} \end{pmatrix}. \tag{3.4}$$

Analogously, $\nabla_{x_k} L^i, \nabla_{u_i} P$ and $\nabla_{x_k} P$ denote gradient vectors with their respective dimensions. Furthermore, for each fixed $(s, \bar{u}_{-i}) \in [0,T] \times U_{-i}$,

$$Hess[P(s, x, (u_i, \bar{u}_{-i}))]$$

denotes the Hessian matrix of the function P with respect to the vector $(x, u_i), i \in \bar{N}$, assuming that P has second order partial derivatives on $X \times U$.

Let $R \subseteq \bar{N}$ be the subset of indices k such that $l_k > 0$ (see Remark 2.2). Since $l \geq 1$ the set R is nonempty.

Assumption 3.8. *The functions* $L^1, \dots, L^N$ *satisfy that*

$$\nabla_{x_k} L^1 = \cdots = \nabla_{x_k} L^N \ \ \forall k \in R.$$

Assumption 3.9. *Let* r *be an index in* R.
(a) There is at least another index $k \in R \setminus \{r\}$.
(b) For every $j \in R$, $l_j = l_r$.

Assumption 3.10. *Let* $r \in R$ *be as in Assumption 3.9. For each* $i \in \bar{N}$, *the function* L^i *satisfies that*

$$\nabla_{x_r} L^i = \nabla_{x_j} L^i \ \ \forall j \in R.$$

Condition 3.11. *(Sufficient conditions.) The function* $P : [0,T] \times X \times U \to \mathbb{R}$ *is in* $C^2(X \times U)$, *is concave in* (x, u), *and for every* $i \in \bar{N}$ *satisfies*

$$\nabla_{u_i} P \;=\; \nabla_{u_i} L^i \tag{3.5}$$
$$\nabla_{x_i} P \;=\; \nabla_{x_i} L^i. \tag{3.6}$$

Note, first, that the condition (3.5) is similar to (3.3). Second, by (3.6), if an index $j \in \bar{N}$ is as in Remark 2.2 (so that $l_j = 0$), then the function P in Condition 3.11 and the functions $L^i, i \in \bar{N}$, depend only on state variables in a set with index in R. (The condition (3.6) can be modified, as we will see in Corollary 3.20.)

The following remark is a version of the maximum principle for OCPs with an infinite horizon. For more details, see [7], Chapter 22; [41]; [33] or [44], Chapter 3.

Remark 3.12. *Consider the OCP as in Definition 2.3 described by* P *with* $T = \infty$. *If* $\boldsymbol{u}^*$ *is an open-loop solution for this OCP, and* $\boldsymbol{x}^*$ *is the state path corresponding to* $\boldsymbol{u}^*$, *then there exists a vector of Lagrange multipliers* $\lambda^* : [0,\infty) \to \mathbb{R}^l$ *such that, using the notation* $(*) := (s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s)), s \in [0,\infty)$:
(i) for $k \in R$, *each coordinate* λ^{k*} *of* λ^* *is defined as the function* $\lambda^{k*} : [0,\infty) \to \mathbb{R}^{l_k}$ *that is the solution to the linear adjoint system*

$$\dot{\lambda}^{k*}(s) \;=\; \beta\lambda^{k*}(s) - \nabla_{x_k} P(*) - \nabla_{x_k}(f(*) \cdot \lambda^*(s)) \tag{3.7}$$

that satisfies the transversality conditions

$$\lim_{s\to\infty} e^{-\beta \cdot s} \lambda^{k*}(s) = 0; \quad \text{and} \tag{3.8}$$

(ii) for almost every $s \in [0,\infty)$, *the following maximality condition holds*

$$H(s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s), \lambda^*(s)) \;=\; \max_{u \in U} H(s, \boldsymbol{x}^*(s), u, \lambda^*(s)), \tag{3.9}$$

where $H : [0, \infty) \times X \times U \times \mathbb{R}^l \to \mathbb{R}$ *is the current value Hamiltonian function, or simply Hamiltonian function, defined by*

$$H(s, x, u, \lambda) := P(s, x, u) + f(s, x, u) \cdot \lambda. \tag{3.10}$$

Recall that, by (2.4), for each $k \in R$ *the vector* λ^{k*} *is of dimension* l_k*, that is,*

$$(\lambda_1^{k*}(s), \dots, \lambda_{l_k}^{k*}(s)) \in \mathbb{R}^{l_k} \;\; \forall s \in [0, \infty). \tag{3.11}$$

Condition 3.13. *(Sufficient conditions.) For each Lagrange multiplier* λ^* *as in Remark 3.12 the function*

$$(x, u) \mapsto H(s, x, u, \lambda^*(s))$$

is concave in (x, u)*.*

As a special case of Theorem 2 in [30], Chapter 6, Section 3, we present the following Lemma 3.14, which establishes, under certain assumptions, a relation between the concavity of the functions P and L^i on $X \times U_i, i \in \bar{N}$. Nevertheless, for completeness, we provide the proof.

Lemma 3.14. *Consider a game as in (2.5),* $T \leq \infty$*, under Assumption 3.7 and a function* P *satisfying (3.5)-(3.6) in Condition 3.11. Suppose that one of the following condition holds:*
(a) Assumption 3.8;
(b) Assumption 3.10 (and Assumption 3.9).
Then, for each $i \in \bar{N}$*, and each point* $(s, \bar{u}_{-i}) \in [0, T] \times U_{-i}$*,* P *is concave in* (x, u_i) *if and only if* L^i *is concave in* (x, u_i)*.*

Proof. For each $i \in \bar{N}$ and each fixed point $(s, \bar{u}_{-i})$ in $[0, T] \times U_{-i}$ we will show that

$$Hess[P(s, x, (u_i, \bar{u}_{-i}))] = Hess[L^i(s, x, (u_i, \bar{u}_{-i}))]. \tag{3.12}$$

First, note that by (3.5)-(3.6) we have for each $i, k \in \bar{N}$, and $q = 1, \dots, l_k$; $v, w = 1, \dots, m_i$, that:

$$\frac{\partial^2 P}{\partial x_q^k \partial u_v^i} = \frac{\partial^2 L^i}{\partial x_q^k \partial u_v^i}, \tag{3.13}$$

$$\frac{\partial^2 P}{\partial u_v^i \partial u_w^i} = \frac{\partial^2 L^i}{\partial u_v^i \partial u_w^i} \tag{3.14}$$

On the other hand, if we consider Assumption 3.8 and use equation (3.6), then for each $i, k, j \in \bar{N}$, and $q = 1, \dots, l_k$, $r = 1, \dots, l_j$,

$$\frac{\partial^2 P}{\partial x_q^k \partial x_r^j} = \frac{\partial^2 L^j}{\partial x_q^k \partial x_r^j} = \frac{\partial^2 L^i}{\partial x_q^k \partial x_r^j}. \tag{3.15}$$

Similarly, if we consider Assumptions 3.9, 3.10, and use equation (3.6), then for each $i, k, j \in \bar{N}$, and $q = 1, \dots, l_k$, $r = 1, \dots, l_j$,

$$\frac{\partial^2 P}{\partial x_q^k \partial x_r^j} = \frac{\partial^2 L^k}{\partial x_q^k \partial x_r^j} = \frac{\partial^2 L^k}{\partial x_q^k \partial x_r^i} = \frac{\partial^2 P}{\partial x_q^k \partial x_r^i} = \frac{\partial^2 L^i}{\partial x_q^k \partial x_r^i} = \frac{\partial^2 L^i}{\partial x_q^k \partial x_r^j}. \tag{3.16}$$

Hence, (3.13), (3.14), and (3.15) imply (3.12). And, (3.13), (3.14), and (3.16) imply (3.12). Therefore, for each $i \in \bar{N}$, P is concave in (x, u_i) if and only if (3.12) is negative semidefinite on $X \times U_i$ if and only if the function L^i is concave in (x, u_i). (See [30], Chapter 6, Section 3, Theorem 2.) □

3.2.2 PDGs over an infinite horizon

The following Theorems 3.15, 3.16, and 3.17 concern situations in which a differential game has a potential function P satisfying Conditions 3.11 and 3.13.

Theorem 3.15. *Suppose that a differential game $\Gamma_{x_0}^\infty$ as in (2.5) satisfies Assumptions 3.7 and 3.8. If there exists a function P satisfying Conditions 3.11 and 3.13, then $\Gamma_{x_0}^\infty$ is an open-loop PDG with potential function P.*

Proof. Consider the OCP in Definition 2.3 described by P, with $T = \infty$. Let $\mathbf{u}^*$ be an open-loop optimal solution for this OCP, and $\mathbf{x}^*$ the state path corresponding to $\mathbf{u}^*$. We will show that $\mathbf{u}^*$ is an open-loop Nash equilibrium. For notational ease, we will use again the notation $(*) := (s, \mathbf{x}^*(s), \mathbf{u}^*(s)), s \in [0, \infty)$. Then there exists a vector of Lagrange multipliers $\lambda^* : [0, \infty) \to \mathbb{R}^l$ that satisfies the features described in Remark 3.12. Note that the current hypotheses (3.5)-(3.6) on the functions P and $L^i, i \in \bar{N}$, imply that for each $i \in \bar{N}$ equation (3.9) holds, and

$$\nabla_{u_i} L^i(*) + \sum_{k \in R} \nabla_{u_i}(f^k(*) \cdot \lambda^{k*}(s)) = 0. \tag{3.17}$$

Moreover, by Assumption 3.8, equation (3.7) becomes

$$\dot{\lambda}^{k*}(s) \;=\; \beta \lambda^{k*}(s) - \nabla_{x_k} L^i(*) - \nabla_{x_k}(f(*) \cdot \lambda^*(s)), \;\; k \in R. \tag{3.18}$$

Now, for each $i \in \bar{N}$, define a Lagrange multiplier $p^{i*} : [0, \infty) \to \mathbb{R}^l$ where each coordinate $p_k^{i*} : [0, \infty) \to \mathbb{R}^{l_k}, k \in R$, is

$$p_k^{i*} := \lambda^{k*}, \tag{3.19}$$

i.e. $p^{i*} = \lambda^*$ for every $i \in \bar{N}$. Note that, by (2.4), (3.11) and (3.19), for each $i \in \bar{N}$ and each $s \in [0, \infty)$, the function $p_k^{i*}, k \in R$, takes l_k values, that is,

$$(p_{k1}^{i*}(s), \ldots, p_{kl_k}^{i*}(s)) \in \mathbb{R}^{l_k},$$

and for each $i \in \bar{N}$, $p_{kj}^{i*} = \lambda_j^{k*}, j = 1, \ldots, l_k, k \in R$.

By equation (3.18) we obtain that, for each $i \in \bar{N}$, the Lagrange multipliers $p_k^{i*}, k \in R$, solve the linear adjoint system

$$\dot{p}_k^{i*}(s) \;\; = \;\; \beta p_k^{i*}(s) - \nabla_{x_k} L^i(*) - \nabla_{x_k}(f(*) \cdot p^{i*}(s)) \tag{3.20}$$

under the transversality conditions

$$\lim_{s \to \infty} e^{-\beta \cdot s} p_k^{i*}(s) = 0. \tag{3.21}$$

Before completing the proof of Theorem 3.15 we note the following.

Remark: To obtain (3.25), below, we will use the following fact from [40], Chapter 7, Theorem 7.15. Let $\bar{g} : U_i \to \mathbb{R}$ be a concave and differentiable function on U_i, then u_i is an maximum of $\bar{g}$ if and only if $\nabla_{u_i} \bar{g}(u_i) = 0$.

Continuing the proof of Theorem 3.15, consider the *Hamiltonian function* $H^i : [0, \infty) \times X \times U \times \mathbb{R}^l \to \mathbb{R}$ for player i, which is defined as

$$H^i(s, x, u, p^i) := L^i(s, x, u) + f(s, x, u) \cdot p^i. \tag{3.22}$$

Hence, by Assumptions 3.7 and 3.8, Conditions 3.11 and 3.13, Lemma 3.14(a), and (3.19), for each $i \in \bar{N}$, the function

$$(x, u_i) \mapsto H^i(s, x, (u_i, \mathbf{u}_{-i}^*(s)), p^{i*}(s)) \tag{3.23}$$

is concave given that

$$Hess[H(s, x, (u_i, \mathbf{u}_{-i}^*(s)), \lambda^*(s))] = Hess[H^i(s, x, (u_i, \mathbf{u}_{-i}^*(s)), p^{i*}(s))]. \tag{3.24}$$

Then (3.17) can be rewritten, for every $i \in \bar{N}$, as

$$H^i(s, \mathbf{x}^*(s), \mathbf{u}^*(s), p^{i*}(s)) \;\; = \;\; \max_{u_i \in U_i} H^i(s, \mathbf{x}^*(s), (u_i, \mathbf{u}_{-i}^*(s)), p^{i*}(s)). \tag{3.25}$$

Moreover, since (3.23) is concave in (x, u_i) the function

$$x \mapsto \max_{u_i \in U_i} H^i(s, x, (u_i, \mathbf{u}^*_{-i}(s)), p^{i*}(s)) \tag{3.26}$$

is also concave.

Summarizing, for $\mathbf{u}^*$ and $\mathbf{x}^*$ we have Lagrange multipliers such that, for almost every $s \in [0, \infty)$, equations (3.20)-(3.25) hold. Therefore, by our hypotheses on $\mathbf{u}^*$ and $\mathbf{x}^*$, and the concavity of the equation (3.26), the multi-strategy $\mathbf{u}^*$ is an open-loop Nash equiibrium with state variable $\mathbf{x}^*$ for the game (2.5) with $T = \infty$ (see [26], Chapter 7; [33] or [44], Chapter 3). □

The Example 3.24, below, shows a two-player linear-quadratic differential game that is an open-loop PDG; that is, there is a function P that satisfies the hypotheses in Theorem 3.15.

Theorem 3.16. *Suppose that the differential game $\Gamma^{\infty}_{x_0}$ in (2.5), the set R, and the index $r \in R$ satisfy Assumptions 3.7 and 3.10. Moreover, suppose that for $j \in R$, there exist functions $g^j : [0, \infty) \times X_j \times U_j \to \mathbb{R}$ such that $f^j(s, x, u) = g^j(s, x_j, u_j)$ and*

$$\nabla_{x_r} g^r = \nabla_{x_j} g^j \;\; \forall j \in R.$$

If there exists a function P that satisfies the Conditions 3.11 and 3.13, then $\Gamma^{\infty}_{x_0}$ is an open-loop PDG with potential function P.

Proof. We use arguments similar to those in the proof of Theorem 3.15, but instead of (3.19) consider, for each $i \in \bar{N}$, the equality

$$p^{i*}_j := \lambda^{i*} \;\; \forall j \in R, \tag{3.27}$$

that is, for each $i \in \bar{N}$, we have $p^{i*}_{jk} = \lambda^{i*}_k$ for $k = 1, \ldots, l_i$ and $j \in R$.

To obtain equation (3.25), observe that for each $i \in \bar{N}$, the function (3.23) is concave in (x, u_i), because (3.24) holds by Assumptions 3.7 and 3.10, Conditions 3.11 and 3.13, Lemma 3.14(b), and (3.27). □

Note that in Theorem 3.16, p^{k*} is not necessarily equal to p^{r*} (in contrast to Theorem 3.15 where $p^{k*} = p^{r*}$). Another difference between Theorems 3.15 and 3.16 is that the latter requires that $l_k = l_r$, which is not needed in Theorem 3.15.

The following Theorem 3.17 characterizes open-loop PDGs in which we can remove, for each $i \in \bar{N}$, the Lagrange multipliers p^i_j for all $j \neq i$. It is important to notice that in general $p^i_j \equiv 0$ is not necessarily true in games satisfying Theorem 3.15 or 3.16, for instance.

Theorem 3.17. *Let $\Gamma_{x_0}^{\infty}$ be a differential game as in (2.5) where Assumption 3.7 holds. Assume that the set R satisfies only condition (a) in Assumption 3.9. Assume also that there exist functions $c^i : [0,\infty) \times X_i \times U \to \mathbb{R}$ for $i \in R$, $c^j : [0,\infty) \times U \to \mathbb{R}$ for $j \in \bar{N} \setminus R$, and $g^i : [0,\infty) \times X_i \times U_i \to X_i$ for $i \in R$ such that*

$$\begin{aligned} L^i(s,x,u) &:= c^i(s,x_i,u) \ \ i \in R, \\ L^i(s,x,u) &:= c^i(s,u) \ \ i \in \bar{N} \setminus R, \ \ \text{and} \\ f^i(s,x,u) &:= g^i(s,x_i,u_i) \ \ i \in R. \end{aligned}$$

If, in addition, a function P satisfies the Conditions 3.11 and 3.13, then $\Gamma_{x_0}^{\infty}$ is an open-loop PDG with potential function P.

Proof. The proof uses arguments similar to those in the proof of Theorem 3.15 except that, instead of (3.19), we now consider, for each $i \in R$,

$$p_i^{i*} := \lambda^{i*} \tag{3.28}$$

and for every $j \neq i, j \in R$,

$$p_j^{i*} := 0. \tag{3.29}$$

To justify equation (3.25), we use that P is concave in (x_i, u_i) if and only if $L^i, i \in \bar{N}$, is concave in (x_i, u_i). □

The condition (3.29) holds for the game (2.5); see [13], Proposition 3.

Lemma 3.14 and Theorems 3.15, 3.16 and 3.17 establish that for a differential game to have a smooth concave potential function, as in Conditions 3.11 and 3.13, it is necessary that each instantaneous payoff function L^i is concave on $X \times U_i$.

We mentioned earlier that a differential game satisfying Corollary 3.2(c) with $p \equiv 0$ was developed in Dragone et al. [13]. (The same situation was mentioned in [15].) This case is considered again in Theorem 3.17, above, but now the instantaneous payoff functions L^i can depend on the strategies of all players. The differentiability conditions in Assumption 3.7 is the price to pay in order to do this.

Besides the just mentioned, other differences between [17] (also the results in this thesis) and the works by Dragone et al. [13]-[15] are indicated in the following remark.

Remark 3.18. *The key difference between [17] and [13]-[15] is that we work directly with the primitive data (2.1)-(2.3) of a differential game to determine* *an associated OCP, whereas the approach in [13]-[15] is based on the Hamiltonian system of a differential game to determine whether there exists a so-called Hamiltonian potential function (HPF). It turns out that the Hamiltonians as in (3.10), when P exists, form a subset of the HPFs in [15]; see also [13], [14]. That is, a Hamiltonian function as in (3.10), associated to a certain OCP, is a HPF. The converse, however, is not true. Hence one can have a HPF that is not associated to an OCP. For examples of this situation see, for instance, Sections 4.3 and 4.6 in [13], and Section 4 in [15].*

Theorems 3.16 and 3.17 require $l \geq 2$; more specifically, require the existence of $r, k \in \bar{N}$, for which $l_r > 0$ and $l_k > 0$. The next Corollary 3.19 considers the case where there is a unique index $r \in R$ such that $l_r > 0$ and $l_j = 0$ for every $j \neq r$. Hence, the players only follow the state variable $\mathbf{x}_r$, that is, by (2.4), the state equation is $\dot{\mathbf{x}}_r(s) = f^r(s, \mathbf{x}_r(s), \mathbf{u}(s))$. Clearly, by (2.2), the case $l = 1$ is also considered when the state variable $\mathbf{x}(s)$ is in $\mathbb{R}$. These particular cases are included in Theorem 3.15. For a particular game with $l = 1$, see Example 3.23 below.

Corollary 3.19. *Let $\Gamma^{\infty}_{x_0}$ be a differential game as in (2.5) under Assumptions 3.7 and 3.8. Suppose that there is a single state for all players in the game, that is, the state variable $\boldsymbol{x}$ is independent of indices in $\bar{N}$. Suppose also that there exists a function P that satisfies Conditions 3.11 and 3.13. Then P is a potential function for $\Gamma^{\infty}_{x_0}$.*

Proof. Note that, in the context of Corollary 3.19, the condition (3.6) becomes $\nabla_x P = \nabla_x L^i$ for each $i \in \bar{N}$. Therefore, $\nabla_x L^1 = \nabla_x L^j$ for all $j \in \bar{N}$, which is as required in Theorem 3.15 for the function P. □

The Examples 3.23, 3.27, 3.28, 3.29, and 4.7, below, satisfy Corollary 3.19 (also Theorem 3.15).

The following Corollary 3.20, which is a consequence of Theorem 3.17, shows that equation (3.6) in Condition 3.11 can be modified.

Corollary 3.20. *Let $\Gamma^{\infty}_{x_0}$ be a differential game as in (2.5) where Assumption 3.7 holds. Let $k \neq j$ be two fixed indexes in R. Suppose that there exist functions*

$$\begin{aligned}
&c^i : [0,\infty) \times X_i \times U \to \mathbb{R} \;\; \textit{if } i \neq k, j;\\
&c^k : [0,\infty) \times X_j \times U \to \mathbb{R};\\
&c^j : [0,\infty) \times X_k \times U \to \mathbb{R}; \;\; \text{and}\\
&g^i : [0,\infty) \times X_i \times U_i \to X_i \;\; \textit{if } i \neq k, j;\\
&g^k : [0,\infty) \times X_j \times U_k \to X_k;\\
&g^j : [0,\infty) \times X_k \times U_j \to X_j,
\end{aligned}$$

such that

$$\begin{aligned}
L^i(s,x,u) &:= c^i(s,x_i,u) \text{ if } i \neq k,j;\\
L^k(s,x,u) &:= c^k(s,x_j,u);\\
L^j(s,x,u) &:= c^j(s,x_k,u); \text{ and}\\
f^i(s,x,u) &:= g^i(s,x_i,u_i) \text{ if } i \neq k,j;\\
f^k(s,x,u) &:= g^k(s,x_j,u_k);\\
f^j(s,x,u) &:= g^j(s,x_k,u_j).
\end{aligned}$$

Then a function $P : [0,\infty) \times X \times U \to \mathbb{R}$ *is a potential function for* $\Gamma^{\infty}_{x_0}$ *if for each* $i \neq j,k, P$ *satisfies equations (3.5)-(3.6), while for* j *and* k, P *satisfies the alternative equations*

$$\nabla_{x_k} P = \nabla_{x_j} L^k, \tag{3.30}$$
$$\nabla_{x_j} P = \nabla_{x_k} L^j. \tag{3.31}$$

Summarizing, Theorems 3.15, 3.16 and 3.17 describe open-loop PDGs whose potential functions, which clearly are not unique, satisfy Conditions 3.11 and 3.13. Moreover, the fact that (3.6) can be modified, as in Corollary 3.20, suggests that there may be other ways, in addition to Condition 3.11, to classify open-loop PDGs.

3.2.3 PDGs over a finite horizon

Theorem 3.21, below, identifies open-loop PDGs with a finite horizon satisfying some regularity conditions similar to those in the previous section. We consider new conditions for the potential terminal payoff function.

Theorem 3.21. *Consider a differential game* $\Gamma^T_{x_0}$ *as in (2.5), with* $T < \infty$. *Assume that one of the following conditions holds:*

(a) The set R *and the functions* L^i *and* f^i *satisfy the hypotheses in Theorem 3.15. Moreover, the terminal payoff functions* $S^i, i \in \bar{N}$, *satisfy that*

$$\nabla_{x_k} S^1 = \cdots = \nabla_{x_k} S^N \;\; \forall k \in R. \tag{3.32}$$

(b) The set R *and the functions* L^i *and* f^i *satisfy the hypotheses of Theorem 3.16, whereas the terminal payoff function* S^i *satisfies that*

$$\nabla_{x_1} S^i = \cdots = \nabla_{x_N} S^i \;\; \forall i \in \bar{N}.$$

(c) The set R and the functions L^i and f^i satisfy the hypotheses of Theorem 3.17, and there exists a function $k^i : X_i \to \mathbb{R}$ such that the terminal payoff function for player i can be written as

$$S^i(x) = k^i(x_i) \ \forall i \in \bar{N}.$$

In addition, suppose that there are functions $P : [0,T] \times X \times U \to \mathbb{R}$ and $S : X \to \mathbb{R}$ such that P satisfies Conditions 3.11 and 3.13, and S is concave in x and satisfies that

$$\nabla_{x_i} S = \nabla_{x_i} S^i \ \forall i \in R. \tag{3.33}$$

Then the differential game $\Gamma_{x_0}^T$ is an open-loop PDG with potential function P and potential terminal payoff function S.

Proof. Consider the OCP described in Definition 2.3 by P and S, with $T < \infty$. Let $\mathbf{u}^*$ be an open-loop optimal solution to this OCP and $\mathbf{x}^*$ the corresponding admissible path. By arguments similar to those in the proof of Theorem 3.15, and adapting Remark 3.12 to $T < \infty$, there exists a vector of Lagrange multipliers $\lambda^* : [0,T] \to \mathbb{R}^l$ such that, with $\beta = 0$, for each index $k \in R$ we have the equation (3.7) and the final condition

$$\lambda^{k*}(T) = \nabla_{x_k} S(\mathbf{x}^*(T)), \tag{3.34}$$

that is, the transversality condition (3.8) is replaced by (3.34). Moreover, for almost every $s \in [0,T]$, the maximality condition (3.9) holds, too.

We only prove part (a). Considering the current hypotheses (3.5)-(3.6) on the functions P and L^i, together with (3.32), (3.33) on the functions S^i and S, we have that for each i and k in R, the Lagrange multiplier $p_k^{i*} : [0,T] \to \mathbb{R}^{l_k}$ defined as in (3.19) satisfies the linear differential equation (3.20) with $\beta = 0$ and the final condition

$$p_k^{i*}(T) = \nabla_{x_k} S^i(\mathbf{x}^*(T));$$

see also (3.34). Since (3.32) and (3.33) hold, S is concave in x if and only if $S^i, i \in \bar{N}$, is concave in x. Thus, using Lemma 3.14(a) and the concavity of the function (3.23), the maximality condition (3.25) holds, which combined with the concavity of the function in (3.26), the multistrategy $\mathbf{u}^*$ is an open-loop Nash equilibrium with state variable $\mathbf{x}^*$ for the game (2.5) under constraints of (a).

The proof of parts (b) and (c) follows as in the proof of Theorems 3.16 and 3.17, respectively. □

The condition (3.29) holds for the game (2.5) in part (c) of Theorem 3.21; see [9], Proposition 3.2.

We illustrate Theorem 3.21(a) in Example 3.31, below.

The following remark describes a change done to Theorem 6 in [17].

Remark 3.22. *We assumed in Theorem 6 in [17], which corresponds to our present Theorem 3.21, above, the existence of a function S convex in x instead of concave in x. We fix the mistake and the corresponding proof in Theorem 3.21.*

3.3 PDGs: Examples

In this section, we show numerous examples to illustrate our results in the previous sections.

The following example on the extraction of exhaustible resources under common access is an open-loop PDG, by Corollary 3.2.

Example 3.23. *(Amir and Nannerup [1], Long [29].) Extraction of exhaustible resources under common access.*

As in (2.5), let $\bar{N} = \{1, \ldots, N\}$. Player i decides its quantity q_i to extract. The utility function for each player $i \in \bar{N}$, is $L^i(q_1, \ldots, q_N) := q_i^{r_i}$, where $0 < r_i < 1$, and a common stock of exhaustible resource x is considered for the players.

The payoff function for player i to maximize is

$$\int_0^\infty e^{-\beta t}[q_i^{r_i}(t)]dt$$

subject to

$$\dot{x}(t) = -q_i(t) - \sum_{j \neq i} q_j(t), \tag{3.35}$$

with $q_i(t) \geq 0, \lim_{t\to\infty} x(t) \geq 0, x(0) = x_0 > 0$, and β is the discount rate.

By Corollary 3.2(a), we have an open-loop PDG with potential function

$$P(q_1, \ldots, q_N) := \sum_{i=1}^N q_i^{r_i}.$$

To calculate an open-loop Nash equilibrium, we can consider the Hamiltonian system of the OCP defined by the potential function P and the game dynamics

(3.35), that is,

$$\begin{aligned} H(t,x(t),q(t),\lambda(t)) &:= \sum_{i=1}^{N} q_i^{r_i}(t) - \lambda(t)\sum_{i\in\bar{N}} q_i(t), \\ r_i q^{r_i-1} - \lambda &= 0 \;\; \forall i \in \bar{N}, \\ \dot{\lambda}(t) &= \beta\lambda(t). \end{aligned}$$

Hence

$$q_i^*(t) = q_i^0 e^{\frac{\beta t}{r_i-1}}, \;\; i \in \bar{N},$$

is the $i-th$ *coordinate of the open-loop Nash equilibrium and the corresponding state path is*

$$x^*(t) = x_0 - \sum_i q_i^0 \frac{r_i-1}{\beta}\left[e^{\frac{\beta t}{r_i-1}} - 1\right].$$

Then player i *chooses his/her restriction as*

$$q_i^0 \le \frac{\beta x_0}{1-r_i} - \sum_{j\neq i} q_j^0 \frac{1-r_j}{1-r_i}. \;\; \Diamond$$

By Theorem 3.15, the following two-player linear-quadratic differential game is an open-loop PDG.

Example 3.24. *Consider, for* $i = 1, 2,$ *the matrices*

$$A_i := \begin{pmatrix} a & a_i \\ q-a_i & b \end{pmatrix}, \qquad R_i := \begin{pmatrix} r_{i1} & 0 \\ 0 & r_{i2} \end{pmatrix}, \tag{3.36}$$

with constants $a, b, q, r_{i1}, r_{i2} < 0, a_i \in \mathbb{R},$ *and* $|q| \le 2\sqrt{ab}.$

Denote by x_i *the state variable and by* u_i *the strategy of player* i*. Here* $x := (x_1, x_2)', u := (u_1, u_2)'$*, and prime* $(')$ *denotes the transpose of a vector or a matrix.*

Player i *wants to maximize*

$$J^i = \frac{1}{2}\int_0^\infty e^{-\beta t}\left\{x'(t)A_i x(t) + u'(t)R_i u(t)\right\} dt$$

where the i*-th coordinate of* x *is subject to*

$$\dot{x}_i(t) = c_{i1}x_1(t) + c_{i2}x_2(t) + c_{i3}u_1(t) + c_{i4}u_2(t),$$

with constants $c_{ij} \in \mathbb{R}$ *for each* $i = 1, 2$ *and* $j = 1, \ldots, 4.$

Calculating, for $i = 1, 2$, *the gradient vectors* $\nabla_{x_i} L^i$ *and* $\nabla_{u_i} L^i$, *and using equations (3.5)-(3.6) in Condition* 3.11, *we see from Theorem 3.15 that the function*

$$P := \frac{1}{2} x' A x + u' R u,$$

with

$$A := \begin{pmatrix} a & q/2 \\ q/2 & b \end{pmatrix} \text{ and } R := \begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix},$$

is a potential function for the game. Thus, the associated OCP is to maximize

$$J = \frac{1}{2} \int_0^\infty e^{-\beta t} \{x' A x + u' R u\} \, dt$$

subject to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Now, define the matrices

$$Q := \begin{pmatrix} a & q/4 \\ q/4 & b \end{pmatrix}, \quad M := \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}, \quad C := \begin{pmatrix} c_{13} & c_{23} \\ c_{14} & c_{24} \end{pmatrix}.$$

The Hamiltonian function for the OCP is therefore given by

$$H := P + \lambda^1 \dot{x}_1 + \lambda^2 \dot{x}_2,$$

where λ_1, λ_2 *satisfy*

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \beta \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} - Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{3.37}$$

with first order condition

$$R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + C \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.38}$$

Assuming that C *has an inverse matrix* C^{-1}, *we obtain that (3.37) and (3.38) can be expressed as*

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}, \tag{3.39}$$

where

$$E := \begin{pmatrix} \beta I - C^{-1}RMR^{-1}C & C^{-1}RQ \\ C' & M' \end{pmatrix}.$$

Here, I is the identity matrix.

We can obtain an open-loop Nash equilibrium for the game, if we solve the linear system (3.39) and obtain an analytical solution according to the signs of the eigenvalues of E. $\diamondsuit$

The following Example 3.25 illustrates Theorem 3.16.

Example 3.25. *(See [9].) We consider N players and $[0,T]$, $T < \infty$. For each $i = 1,\dots,N$, let $M^i : [0,T] \times U_i \to \mathbb{R}$ and $R^i : [0,T] \times U_{-i} \to \mathbb{R}$ be differentiable functions, with M^i a concave function on U_i. Player i has the instantaneous payoff function*

$$L^i(t,x,u) = a_i \sum_{j=1}^{N} x_j + M^i(t,u_i) + R^i(t,u_{-i}).$$

Consider $S^i = 0, i \in \bar{N}$. The state variable x_i satisfies

$$\dot{x}_i = G(t)x_i + F^i(t,u_i), \;\; x_i(0) = x_{i0},$$

with $G : [0,T] \to \mathbb{R}$, $F^i : [0,T] \times U_i \to \mathbb{R}$, and F^i is concave function in u_i.

By Theorem 3.16, computing $\nabla_{x_i} L^i$ and $\nabla_{u_i} L^i$, and considering Condition 3.11, we have that

$$P(t,x,u) = \sum_{j=1}^{N} \left[a_j x_j + M^j(t,u_j)\right]$$

is a potential function, and $S = 0$ is the potential terminal payoff function for the game. Thus the OCP corresponding to this game is to maximize

$$\int_0^T e^{-\beta t} \sum_{j=1}^{N} \left[a_j x_j(t) + M^j(t,u_j(t))\right] dt$$

subject to

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} G(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & G(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} F^1(t,u_1) \\ \vdots \\ F^N(t,u_N) \end{pmatrix}.$$

From (3.10) the Hamiltonian function for this OCP is

$$H(t,x,u) = \sum_{j=1}^{N}\left[a_j x_j + M^j(t,u_j)\right] + \sum_{j=1}^{N}\lambda^j\left[G(t)x_j + F^j(t,u_j)\right].$$

Therefore, the Hamiltonian system to analyze the Nash equilibrium for the game is, for $i = 1,\dots,N$,

$$\begin{aligned}\dot{\lambda}^i &= \beta\lambda^i - \left[a_i + G(t)\lambda^i\right], \lambda^i(T) = 0,\\ 0 &= \frac{\partial}{\partial u_i}M^i(t,u_i) + \lambda^i\frac{\partial}{\partial u_i}F^i(t,u_i),\\ \dot{x}_i &= G(t)x_i + F^i(t,u_i).\ \Diamond\end{aligned}$$

We now give a list of open-loop PDGs with applications in economics and management science that can be found in [10], where it is not mentioned, of course, that they are PDGs. We assume that the variable x denotes the common state and u_i the strategy of player i.

By Theorem 3.1, the Example 3.26 is an open-loop PDG.

Example 3.26. (See [10], p. 88.) Consider $\bar{N} = \{1,2\}$.

$$\begin{aligned}L^1 &:= -x - \frac{\alpha}{2}u_1^2 + u_2,\ \ \alpha > 0,\\ L^2 &:= -x + u_2,\\ f &:= -u_1\sqrt{x} + u_2 + 1.\ \Diamond\end{aligned}$$

On the other hand, by Theorem 3.15 (or Corollary 3.19), each one of the following games is an open-loop PDG.

Example 3.27. (See [10], p. 107.)

$$\begin{aligned}L^i &:= -x - \frac{1}{2}u_i^2,\ \ i \in \bar{N},\\ f &:= -\sqrt{x}\left(\sum_{i\in\bar{N}} u_i\right).\ \Diamond\end{aligned}$$

Example 3.28. (See [10], p. 108.)

$$\begin{aligned}L^i &:= \sqrt{u_i},\ \ i \in \bar{N},\\ f &:= x - \sum_{i\in\bar{N}} u_i.\ \Diamond\end{aligned}$$

In the following Examples 3.29, 3.30, and 3.31 we have $\bar{N} = \{1, 2\}$.

Example 3.29. (See [10], p. 108.)

$$\begin{aligned} L^i &:= x - \alpha u_i, \;\; \alpha > 0, \\ f &:= u_1 u_2. \;\; \Diamond \end{aligned}$$

Example 3.30. A two-player affine-quadratic differential game.

$$L^i = -\frac{1}{2}\left[(1-x)^2 + (1-u_i)^2\right],$$

The state variable $x(t)$ is the solution of the nonlinear differential equation

$$\dot{x}(t) \;\; = \;\; -\left[x^2(t)u_1(t) + u_2(t)\right], \;\; x(0) = x_0. \;\; \Diamond$$

Finally, the following finite-horizon game is an open-loop PDG by Theorem 3.21.

Example 3.31. (See [10], p. 189.)

$$\begin{aligned} L^i &:= x - K_i(u_i), \;\; S^i = F_i(x), \\ f &:= u_1 + u_2 - \alpha x, \;\; \alpha > 0, \end{aligned}$$

where, for each $i = 1, 2$, K_i is a convex function, and F_i a concave function. $\Diamond$

3.4 Comments

In this chapter, we propose two different approaches to identify PDGs and an associated OCP: The *exact-potential approach* and the *fictitious-potential approach*. They yield several alternative ways to identify PDGs; See for instance, Theorems 3.1, 3.6, and 3.15–3.17, 3.21, and their corresponding corollaries. These results are illustrated by the examples in Section 3.3. Chapter 4 presents some applications of PDGs, whereas Chapter 5 considers *stochastic* PDGs.

4 Some properties of PDGs

In this chapter, we present two applications for PDGs. We first apply a turnpike theorem to some PDGs. Having a PDG, a turnpike theorem by Trèlat and Zuazua [42] helps us to study the asymptotic behavior of optimal solutions of the associated OCPs and, therefore, the behavior of Nash equilibria for the PDG. A second topic is the clasification of PDGs that have Pareto-optimal Nash equilibria. These equilibria are strategies that give the best benefit to the society and, at the same time, the players receive fair payments. Players do not have incentives to change the accorded strategies to play the game, obtaining a reduction of risks and costs caused by a state of anarchy among the players [2].

4.1 A turnpike property for PDGs

In this section we consider time-homogeneous PDGs, which are particular cases of (2.5) with $T = \infty$, and satisfy the hypotheses of any of Theorems 3.15 or 3.16 or 3.17. We denote by $\bar{\Gamma}$ a game with such a description, and by OCP($\bar{\Gamma}$) the associated OCP for $\bar{\Gamma}$. Therefore, we assume that the OCP($\bar{\Gamma}$) has a payoff function

$$J(\mathbf{u}) \;:=\; \int_0^\infty e^{-\beta t} P(\mathbf{x}(t), \mathbf{u}(t))dt,$$

which is to be maximized subject to

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f(\mathbf{x}(t), \mathbf{u}(t)) \;\; \forall t \geq 0, \\ \mathbf{x}(0) &= x_0, \end{aligned}$$

where P is a potential function for $\bar{\Gamma}$. We also assume that P satisfies the Conditions 3.11 and 3.13.

The main objective is to study the asymptotic behavior of Nash equilibria of these games $\bar{\Gamma}$ via a turnpike theorem in Trèlat and Zuazua [42]. To this end, we require suitable assumptions and some terminology that we introduce as follows.

For each $0 < \tau < \infty$, we denote by $\mathbf{u}_i^\tau \in \mathbf{U}_i$ the strategy of player i when the time set is $[0, \tau]$, and by $\mathbf{u}_\tau = (\mathbf{u}_1^\tau, \ldots, \mathbf{u}_N^\tau)$ the corresponding multistrategy in $\mathbf{U}$. Remember $\mathbf{U}_i$ and $\mathbf{U}$ defined by (2.1).

Assumption 4.1. *For each $\tau > 0$, the open-loop multistrategies $\boldsymbol{u}_\tau$ are in $L^\infty([0, \tau])$, the space of essentially bounded functions on $[0, \tau]$.*

For each $\tau > 0$, the *τ-truncated-optimal control problem* $\mathrm{OCP}(\bar{\Gamma})_\tau$ of $\mathrm{OCP}(\bar{\Gamma})$ is defined by the payoff function

$$\int_0^\tau e^{-\beta t} P(\mathbf{x}_\tau(t), \mathbf{u}_\tau(t)) dt$$

subject to

$$\begin{aligned} \dot{\mathbf{x}}_\tau(t) &= f(\mathbf{x}_\tau(t), \mathbf{u}_\tau(t)), \;\; 0 \leq t \leq \tau, \\ \mathbf{x}_\tau(0) &= x_{\tau 0}. \end{aligned}$$

Remark 4.2. *Note that, obviously, Theorem 3.21 holds when $S^i := 0$ for every $i \in \bar{N}$. Thus the OCP($\bar{\Gamma}$)$_\tau$ can be associated to the τ-truncated-potential differential game $\bar{\Gamma}_\tau$: it is the game $\bar{\Gamma}$ where, for each $i \in \bar{N}$, the terminal payoff function S^i for player i is zero and the time horizon is τ instead of $T = \infty$.*

Assumption 4.3. *For each finite $\tau > 0$, OCP($\bar{\Gamma}$)$_\tau$ has at least one optimal solution $(\boldsymbol{x}_\tau^*, \boldsymbol{u}_\tau^*)$.*

For each solution $(\mathbf{x}_\tau^*, \mathbf{u}_\tau^*)$ of $\mathrm{OCP}(\bar{\Gamma})_\tau$, let λ_τ^* be the corresponding Lagrange multiplier given by the maximum principle. (See Remark 3.12 and the proof of Theorem 3.21.)

A *static optimization problem* associated to the $\mathrm{OCP}(\bar{\Gamma})$ consists in finding a pair $(\bar{x}, \bar{u}) \in \mathbb{R}^l \times \mathbb{R}^m$ that solves

$$\max_{(x,u) \in X \times U} P(x, u),$$

subject to $f(x, u) = 0$.

Assumption 4.4. *The static optimization problem has at least one optimal solution $(\bar{x}, \bar{u})$.*

For each point $(\bar{x}, \bar{u})$ as in Assumption 4.4, let $\bar{\lambda}$ be the corresponding Lagrange multiplier given by the maximum principle in $(\bar{x}, \bar{u})$, that is, $\bar{\lambda}$ satisfies

$$\begin{aligned} \nabla_x H(\bar{\Delta}) &= 0, \\ \nabla_u H(\bar{\Delta}) &= 0, \\ \nabla_\lambda H(\bar{\Delta}) &= 0, \end{aligned}$$

where $\bar{\Delta} := (\bar{x}, \bar{u}, \bar{\lambda})$ and H is the current value Hamiltonian as in (3.10). The triplet $\bar{\Delta}$ is called an *equilibrium point* (or *steady-state*) for OCP($\bar{\Gamma}$).

The matrix H_{xx} is defined, for each $i, j \in \bar{N}$, given that $x_i = (x_1^i, \dots, x_{l_i}^i)$ and $x_j = (x_1^j, \dots, x_{l_j}^j)$, by

$$[H_{xx}]_{\{ij\}_{kr}} := \frac{\partial}{\partial x_k^i} \frac{\partial}{\partial x_r^j} H,$$

for each $k = 1, \dots, l_i$ and $r = 1, \dots, l_j$. The matrices $H_{x\lambda}$, $H_{u\lambda}$, H_{uu}, and H_{xu} are similarly defined. Let

$$\begin{aligned} A &:= \left[H_{x\lambda} - H_{u\lambda} H_{uu}^{-1} H_{xu}\right](\bar{\Delta}), \\ B &:= H_{u\lambda}(\bar{\Delta}), \\ F &:= \left[-H_{xx} + H_{ux} H_{uu}^{-1} H_{xu}\right](\bar{\Delta}). \end{aligned}$$

The matrices A, B, and F are well defined when we assume that H_{uu} is symmetric negative definite, see Theorem 4.5, below.

We have shown in Chapter 3 that the existence of a potential function P for the game $\bar{\Gamma}$ induces a refinement of the set of Nash equilibria. Therefore, if the OCP($\bar{\Gamma}$) defined by P satisfies the hypotheses in the following Theorem 4.5, then $\bar{\Gamma}$ has at least one Nash equilibrium satisfying the turnpike property (4.1)-(4.5), below. An explicit statement is given in Corollary 4.6.

Theorem 4.5. *(Trèlat and Zuazua [42], Theorem 1) Suppose that the Assumptions 4.1, 4.3, 4.4 hold and, moreover, the OCP($\bar{\Gamma}$) satisfies that H_{uu} is symmetric negative definite, F is symmetric positive definite, and the Kalman controllability condition*

$$rank(B, AB, \dots, A^{N-1}B) = N$$

holds. Then we have the following: There exist constants $\epsilon > 0, c_1 > 0, c_2 > 0$, and a time $T_0 > 0$ such that, if

$$|\bar{x} - x_0| < \epsilon, \tag{4.1}$$

then for every $\tau > T_0$, the τ-truncated optimal control problem OCP($\bar{\Gamma}$)$_\tau$ has at least one open-loop optimal solution $(\boldsymbol{x}_\tau^*, \boldsymbol{u}_\tau^*)$ that satisfies, for every $t \in [0, \tau]$,

$$\|\boldsymbol{x}_\tau^*(t) - \bar{x}\| + \|\boldsymbol{u}_\tau^*(t) - \bar{u}\| + \left\|\lambda_\tau^*(t) - \bar{\lambda}\right\| \leq c_1(e^{-c_2 t} + e^{-c_2(\tau - t)}), \tag{4.2}$$

where λ^*_τ *is the Lagrange multiplier corresponding to* $(\boldsymbol{x}^*_\tau, \boldsymbol{u}^*_\tau)$. *Therefore*

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \boldsymbol{x}^*_\tau(t)dt = \bar{x}, \tag{4.3}$$

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \boldsymbol{u}^*_\tau(t)dt = \bar{u}, \tag{4.4}$$

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \lambda^*_\tau(t)dt = \bar{\lambda}. \tag{4.5}$$

As a direct consequence of Theorem 4.5 we obtain the following.

Corollary 4.6. *Let* $\bar{\Gamma}$ *be a time-homogeneous open-loop PDG that satisfies the hypotheses of any of the Theorems 3.15 or 3.16 or 3.17. Moreover, suppose that its corresponding OCP*($\bar{\Gamma}$) *satisfies the assumptions in Theorem 4.5. Then there exist at least one open-loop Nash equilibrium* $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ *such that if* x_0 *is as in* (4.1), *the pair* $(\boldsymbol{x}^*_\tau, \boldsymbol{u}^*_\tau)$ *satisfies (4.2) for every* $t \in [0, \tau]$. *Hence, (4.3)-(4.5) also hold.*

The following example gives an application of Theorem 3.17 and Corollary 4.6.

Example 4.7. *Consider an affine-quadratic differential game described as follows: the instantaneous payoff function for player* i *is*

$$L^i(x_i, u) = -\frac{1}{2}\left[(1+x_i)^2 + \sum_{i=1}^{N}\sum_{j=1}^{m_i}(1+u^i_j)^2\right],$$

with state variable x_i *that is the solution of the nonlinear differential equation*

$$\dot{x}_i(t) = -\sum_{j=1}^{m_i} x_i^2(t)u^i_j(t), \ \ x_i(0) = x_{i0}. \tag{4.6}$$

By Theorem 3.17 (or Theorem 3.1) this differential game is an open-loop PDG. The potential function is

$$P(x,u) := -\frac{1}{2}\sum_{i=1}^{N}\left[(1+x_i)^2 + \sum_{j=1}^{m_i}(1+u^i_j)^2\right]$$

and it defines the OCP with payoff function

$$\int_0^\infty -\frac{1}{2}e^{-\beta t}\sum_{i=1}^{N}\left[(1+x_i(t))^2 + \sum_{j=1}^{m_i}(1+u^i_j(t))^2\right]$$

subject to (4.6) considering every index $i = 1, \ldots, N$.

The corresponding Hamiltonian function is

$$H(x,u,\lambda) = -\frac{1}{2}\sum_{i=1}^{N}\left[(1+x_i)^2 + \sum_{j=1}^{m_i}(1+u_j^i)^2\right] - \sum_{i=1}^{N}\sum_{j=1}^{m_i}\lambda^i x_i^2 u_j^i.$$

Thus, the Hamiltonian system for this OCP is, for every $i = 1, \ldots, N$,

$$\begin{aligned}
\dot{x}_i(t) &= -\sum_{j=1}^{m_i} x_i^2(t)u_j^i(t), \;\; x_i(0) = x_{i0}, \\
0 &= -(1+u_j^i(t)) - \lambda^i(t)x_i^2(t), \;\; j = 1, \ldots, m_i, \\
\dot{\lambda}^i(t) &= \beta\lambda^i(t) + 1 + x_i(t) + 2\lambda^i(t)x_i(t)\sum_{j=1}^{m_i} u_j^i(t), \\
0 &= \lim_{t\to\infty} e^{-\beta t}\lambda^i(t).
\end{aligned}$$

Corollary 4.6 states that the Nash equilibrium corresponding to the optimal solution to the τ*-truncated control problem is close to the steady-state point*

$$\bar{x}_i = -1, \;\; \bar{u}_j^i = 0, \;\; \bar{\lambda}^i = -1,$$

for every $i = 1, \ldots, N$ *and* $j = 1, \ldots, m_i$*, which is the solution to the system*

$$\begin{aligned}
0 &= -\sum_{j=1}^{m_i} x_i^2 u_j^i, \\
0 &= -(1+u_j^i) - \lambda^i x_i^2, \;\; j = 1, \ldots, m_i, \\
0 &= 1 + x_i + 2\lambda^i x_i \sum_{j=1}^{m_i} u_j^i.
\end{aligned}$$

Therefore, for each initial condition x_0 *in a neighborhood of radius* $\epsilon > 0$ *of the point* $\bar{x} = (-1, \ldots, -1)$*, the corresponding Nash equilibrium* u_τ^* *and its admissible path* x_τ^* *are close to the points* $\bar{u} = (0, \ldots, 0)$ *and* $\bar{x}$*, respectively, in the sense of* (4.2). $\diamondsuit$

4.2 PDGs with Pareto-optimal Nash equilibria

4.2.1 Cooperative differential games

In this section, we consider a game as in (2.5) with $T = \infty$, but in the *cooperative* case. To this end, we first introduce some notation and terminology.

Notation. For $u = (u_1, \ldots, u_N)$ and $v = (v_1, \ldots, v_N)$ in $\mathbb{R}^N$,

$$u \geq v \text{ means: } u_i \geq v_i \ \ \forall i \in \bar{N};$$
$$u > v \text{ means: } u \geq v \text{ and } u \neq v;$$
$$u \gg v \text{ means: } u_i > v_i \ \ \forall i \in \bar{N}.$$

Let $r(\mathbf{u}) := (J_1(\mathbf{u}), \dots, J_N(\mathbf{u}))$ be the reward vector for each $\mathbf{u} \in \mathbf{U}$. A multistrategy $\mathbf{u}^* \in \mathbf{U}$ is called *Pareto optimal* (or *nonsuperior* or *unimprovable*) for the game (2.5) if there is no $\mathbf{u} \in \mathbf{U}$ such that

$$r(\mathbf{u}) > r(\mathbf{u}^*). \tag{4.7}$$

The corresponding reward vector $r(\mathbf{u}^*)$ is said to be a *Pareto point.* Moreover, if instead of (4.7), $r(\mathbf{u}) \gg r(\mathbf{u}^*)$ holds, then $\mathbf{u}^*$ is called *weakly Pareto optimal.*

We recall the following known facts for cooperative games; see, for instance, [16], [35], [38].

Lemma 4.8. *(a) Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ be such that $\lambda_i > 0$ for all $i \in \bar{N}$, and $\lambda_1 + \cdots + \lambda_N = 1$. If $\boldsymbol{u}^* \in \boldsymbol{U}$ maximizes the scalar product $\lambda \cdot r(\boldsymbol{u}) = \sum_{i=1}^N \lambda_i J_i(\boldsymbol{u})$, that is,*

$$\lambda \cdot r(\boldsymbol{u}^*) = \max_{\boldsymbol{u}} \lambda \cdot r(\boldsymbol{u}),$$

then $\boldsymbol{u}^*$ is Pareto optimal.

(b) The converse of (a) is true provided that $\boldsymbol{U}$ is convex and $J_1, \dots, J_N$ are all concave.

Remark 4.9. *The converse of Lemma 4.8(a) does **not** hold, in general. See examples in [16], [35].*

Lemma 4.10. *[38] A multistrategy $\boldsymbol{u}^* \in \boldsymbol{U}$ is Pareto optimal if and only if, for every $i \in \bar{N}, \boldsymbol{u}^*$ maximizes J_i on the set*

$$\mathscr{U}_i := \{\boldsymbol{u} \in \boldsymbol{U} \mid J_j(\boldsymbol{u}) \geq J_j(\boldsymbol{u}^*) \ \ \forall j \neq i\}.$$

In view of Lemmas 4.8 and 4.10, finding a Pareto solution is essentially the same as solving an OCP. Hence, for the existence of such solutions it suffices to give conditions for the existence of optimal controls. See [3], [16], [28], [35], [38], [44], for instance.

4.2.2 The direct case

The simplest example of a PDG with Pareto-optimal Nash equilibria is a *team game.* We saw in Example 2.5, above, that if $\mathbf{u}^* \in \mathbf{U}$ optimizes the corresponding OCP defined by the potential function, then $\mathbf{u}^*$ is a *Nash equilibrium.* On the other hand, it is obvious that $\mathbf{u}^*$ is also *Pareto optimal*, by Lemma 4.8. Therefore, a team game lies in the class of games we are interested in.

The following theorem provides a direct way to identify classes of PDGs that have Pareto-optimal Nash equilibria.

Theorem 4.11. *Consider a differential game as in (2.5) with* $f = (f_1, \ldots, f_N)$. *Suppose that there are functions* $\hat{g}_i, \hat{f}_i$, *such that one of the following conditions holds for every* $i \in \bar{N}$:
(a) $L_i(t, x, u) = \hat{g}_i(t, u_i)$.
(b) $L_i(t, x, u) = \hat{g}_i(t, x, u_i)$, $f_i(t, x, u) = \hat{f}_i(t, x)$.
(c) $L_i(t, x, u) = \hat{g}_i(t, x_i, u_i)$, $f_i(t, x, u) = \hat{f}_i(t, x_i, u_i)$.
Then the differential game (2.5) is a PDG and has a potential function

$$P = \hat{g}_1 + \cdots + \hat{g}_N. \tag{4.8}$$

Hence, if $\boldsymbol{u}^* = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N) \in \boldsymbol{U}$ *maximizes* J, *then* $\boldsymbol{u}^*$ *is an open-loop Nash equilibrium. In addition, if* $\boldsymbol{U}$ *is convex and* J_i *is concave on* $\boldsymbol{U}_i$ *for every* $i \in \bar{N}$, *then* $\boldsymbol{u}^*$ *is also Pareto optimal.*

Proof. This result follows from Corollary 3.2 and Lemma 4.8 above. □

The Example 3.23 above, about extraction of exhaustible resources under common access, has a Pareto-optimal Nash equilibrium by Theorem 4.11.

4.2.3 The potential case

In the remainder of this section we use the following condition **G** to simplify our presentation.
G: Let (2.5) be a PDG where the associated OCP is as in Definition 2.3.

Recall the notation in Lemma 4.10, above.

Theorem 4.12. *Assume* $\boldsymbol{G}$. *If* $\boldsymbol{u}^*$ *is a multistrategy such that, for every* $i \in \bar{N}$, $\boldsymbol{u}^*$ *maximizes* J_i *on* $\mathscr{U}_i$ *and, in addition,* $\boldsymbol{u}^*$ *maximizes the payoff function in Definition 2.3, then* $\boldsymbol{u}^*$ *is a Pareto-optimal Nash equilibrium.*

Proof. The theorem follows directly from Lemma 4.10 and the definition of a PDG. □

The following Corollaries 4.13 and 4.14 follow from Theorem 4.12 and Lemma 4.10, respectively.

Corollary 4.13. *Assume* $\boldsymbol{G}$*. If a multistrategy* $\boldsymbol{u}^*$ *is the unique maximizer of* J_k *for some* $k \in \bar{N}$*, and if* $\boldsymbol{u}^*$ *is also a maximizer for the payoff function in Definition 2.3; that is, there is an index* $k \in \bar{N}$ *such that, for every* $\mathbf{u} \in \boldsymbol{U}$,

$$J_k(u^*) \geq J_k(u) \text{ and } \tag{4.9}$$
$$J(u^*) \geq J(u), \tag{4.10}$$

then $\boldsymbol{u}^*$ *is a Pareto-optimal Nash equilibrium. Furthermore, if* $\boldsymbol{u}^*$ *is not the unique multistrategy satisfying (4.9), then* $\boldsymbol{u}^*$ *is a Nash equilibrium and it is also weakly Pareto optimal.*

Corollary 4.14. *Consider the game (2.5). If a multistrategy* $\boldsymbol{u}^*$ *is such that, for each* $i \in \bar{N}$,

$$J_i(u^*) \geq J_i(u) \ \forall u \in \boldsymbol{U}, \tag{4.11}$$

then $\boldsymbol{u}^*$ *is a Nash equilibrium that is also Pareto optimal.*

In the following example, the conditions (4.9)-(4.10) in Corollary 4.13 are satisfied.

Example 4.15. *(For a version of this model, see [10], p. 87.) Consider two players. Let* $X = [0, \infty)$ *be the state space of the game, and fix an initial state* $x_0 \in X$*. The feasible control set for player 1 is* $U^1 = [0, \infty)$ *and for player 2 is* $U^2 = [0, 1]$.

The payoff function for player 1 is

$$J^1(\boldsymbol{u}) = \int_0^\infty e^{-\beta t}[u_2(t) - x(t) - \frac{\alpha}{2} u_1^2(t)]dt,$$

with $\alpha, \beta > 0$*, and for player 2 is*

$$J^2(\boldsymbol{u}) = \int_0^\infty e^{-\beta t}[u_2(t) - x(t)]dt,$$

which are subject to the system equation

$$\dot{x}(t) = 1 + u_2(t) - u_1(t)\sqrt{x(t)}, \ \ x(0) = x_0. \tag{4.12}$$

This model is a PDG with potential function

$$P(t, x, u) = u_2 - x - \frac{\alpha}{2} u_1^2.$$

The potential function P and (4.12) define the associated OCP to this game. To obtain the optimal solution for the associated OCP, we have the following Hamiltonian system:

$$\begin{aligned}
H(\cdot) &= u_2 - x - \frac{\alpha}{2}u_1^2 + \lambda[1 + u_2 - u_1\sqrt{x}], \\
u_1 &= -\frac{\lambda}{\alpha}\sqrt{x}, \\
\lambda(t) + 1 &\geq 0, \;\; 0 \leq u_2 \leq 1, \\
\dot{\lambda} &= 1 + \beta\lambda - \frac{1}{2\alpha}\lambda^2, \\
\dot{x} &= 1 + u_2 - u_1\sqrt{x}, \;\; x(0) = x_0.
\end{aligned}$$

Solving this Hamiltonian system, we have that the optimal solution $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \boldsymbol{u}_2^*)$ *is*

$$\begin{aligned}
\boldsymbol{u}_1^*(t) &= -\frac{\lambda^*(t)}{\alpha}\sqrt{\boldsymbol{x}^*(t)}, \\
\boldsymbol{u}_2^*(t) &= \begin{cases} 0 \;\; if \;\; \lambda^*(t) < -1 \\ \\ 1 \;\; if \;\; \lambda^*(t) \geq -1, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\lambda^*(t) &= \frac{\alpha[(\beta + C) + k_0 e^{-Ct}(\beta - C)]}{1 + k_0 e^{-Ct}}, \\
C &= \sqrt{\beta^2 + \frac{2}{\alpha}}, \;\; \lambda_0 = \alpha\beta + \frac{2\alpha C}{\sqrt{2\alpha} + 2}, \\
k_0 &= \frac{(\frac{\sqrt{2\alpha}}{2} - 1)C - [\beta - \frac{\lambda_0}{\alpha}]}{(\frac{\sqrt{2\alpha}}{2} + 1)C + [\beta - \frac{\lambda_0}{\alpha}]}.
\end{aligned}$$

The corresponding state path is given by

$$\boldsymbol{x}^*(t) = \exp\left(\int_0^t \frac{\lambda^*(\tau)}{\alpha}d\tau\right)\left[\int_0^t (1 + \boldsymbol{u}_2^*(s))\exp\left(\int_0^s \frac{\lambda^*(\tau)}{\alpha}d\tau\right)ds + x_0\right].$$

In addition, (4.9)-(4.10) hold. Therefore, the multistrategy $\boldsymbol{u}^*$ *is a Nash equilibrium and Pareto optimal for the game.* ◊

4.2.4 Remarks

The following remarks relate our results in this Section 4.2 and some known facts in the literature.

1. All the cases considered in Theorem 4.11 satisfy (4.11).
2. A game as in Corollary 4.14 is not necessarily a PDG.
3. Remark 1 in [37] presents a particular case of Corollary 4.14 but requiring differentiability and convexity conditions for (2.5).
4. Example 4.15 has payoff functions depending on the state variable of the game and, in addition, player 1's payoff function also depends on the player 2's strategy variable. This is in contrast to Theorem 4.11, where, for each i, the (instantaneous) payoff function $\hat{g}_i$ depends only on the strategies of player i. Thus, Theorem 4.11 and Theorem 4.12 give different ways to identify differential games with Pareto-optimal Nash equilibria.
5. A result similar to Theorem 4.11(a) was obtained in [11] for a dynamic resource management game that considers overtaking multistrategies.
6. Another result similar to Theorem 4.11(a) appears in [1].
7. Using PDGs to obtain Pareto-optimal Nash equilibria, it is possible to relax concavity and differentiability conditions on the game's primitive data. See [17] and [37].

4.3 Comments

In this chapter, we conclude our findings related to deterministic potential differential games. We have studied some properties of PDGs, such as the asymptotic behavior of Nash equilibria, and how we can use the potential function to ensure a Pareto-optimal Nash equilibrium. In the next chapter, we present some results for a stochastic differential game to be a potential game.

5 Stochastic PDGs

Our aim in this chapter is to identify potential games within the class of stochastic differential games. To this end, we follow a procedure similar to the fictitious-potential approach presented in Section 3.2. We also include a section about stochastic PDGs that have Pareto-optimal Nash equilibria.

5.1 Stochastic differential games

We next introduce the stochastic differential games (SDGs) we are concerned with.

Consider a *complete filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P})$ on which a *d-dimensional standard Brownian motion* $W(\cdot)$ is defined. Let us assume that $\{\mathscr{F}_t\}_{t\geq 0}$ is its *natural filtration* and $\mathscr{F}_0$ includes all $\mathbf{P}$-null sets of $\mathscr{F}$.

Let $\bar{N} := \{1, \ldots, N\}$ be the *set of players*, with $N \geq 2$. For each $i \in \bar{N}$, define the space of the *open-loop control processes* for player i as

$$\bar{\mathbf{U}}_i := \{\mathbf{u}_i : \Omega \times [0,T] \to U_i | \ \mathbf{u}_i \text{ is } \{\mathscr{F}_t\}_{t\geq 0} - \text{adapted}, \ \|\mathbf{u}_i\|_2 < \infty\}, \quad (5.1)$$

where $0 < T < \infty$ is the *game's time horizon*, $U_i \subset \mathbb{R}^{m_i}$ is a nonempty Borel set, and $\|\mathbf{u}_i\|_2 := (E\int_0^T |\mathbf{u}_i(s)|^2 ds)^{1/2}$.

The space of *multistrategy processes* for the players is

$$\bar{\mathbf{U}} := \prod_{i=1}^{N} \bar{\mathbf{U}}_i. \quad (5.2)$$

A multistrategy $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$ in $\bar{\mathbf{U}}$ takes values in the set $U = U_1 \times \cdots \times U_N \subset \mathbb{R}^m$, with $m = m_1 + \cdots + m_N$.

Now, for each $\mathbf{u} \in \bar{\mathbf{U}}$, we consider the controlled stochastic differential equation (SDE) given by

$$\begin{aligned} d\mathbf{x}(s) &= f(s, \mathbf{x}(s), \mathbf{u}(s))ds + \sigma(s, \mathbf{x}(s), \mathbf{u}(s))dW(s), \\ \mathbf{x}(0) &= x_0, \end{aligned} \quad (5.3)$$

where $f : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}^l$ and $\sigma : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}^{l\times d}$ are given functions, and $x_0 \in \mathbb{R}^l$ is a given deterministic initial condition. We call

$\mathbf{x} : \Omega \times [0,T] \to \mathbb{R}^l$ the admissible state process of the game corresponding to the multistrategy $\mathbf{u}$.

As in Remark 2.2, we can note the following. The SDE (5.3) can be rewritten as the coupled system

$$\begin{aligned} d\mathbf{x}_i(s) &= f^i(s,\mathbf{x}(s),\mathbf{u}(s))ds + \sum_{k=1}^{d} \sigma^{ik}(s,\mathbf{x}(s),\mathbf{u}(s))dW_k(s), \qquad (5.4) \\ \mathbf{x}_i(0) &= x_{i0}, \end{aligned}$$

for each $i \in \bar{N}$, with $f = (f^1, \ldots, f^N)'$, and σ^{ik} is the ik-entry of the matrix σ. The apostrophe $(')$ represents the transpose of a vector or a matrix.

We will assume that $\mathbf{x}_i \in \mathbb{R}^{l_i}$ is the state process for the i-th player and that it solves the equation (5.4). Furthermore, for $j \in \bar{N}$, we understand that player j has no state variable in the game if $l_j = 0$. (Remember that $l = l_1 + \cdots + l_N$.) Let $R \subseteq \bar{N}$ be the subset of indices k, such that $l_k > 0$. Clearly, since $l \geq 1$, the set R is nonempty.

Sufficient conditions for the existence and uniqueness of solutions to the system (5.3) (or (5.4)) are well known; see, for instance, [44], Chapters 2 and 3.

For each $i \in \bar{N}$ and $\mathbf{u} \in \bar{\mathbf{U}}$, we define the payoff functional for player i by

$$\bar{J}^i(\mathbf{u}) := E[\int_0^T L^i(s,\mathbf{x}(s),\mathbf{u}(s))ds + h^i(\mathbf{x}(T))], \qquad (5.5)$$

where $\mathbf{x}$ is the admissible state process to the multistrategy process $\mathbf{u}$, and $L^i : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}$ and $h^i : \mathbb{R}^l \to \mathbb{R}$ are the *instantaneous* (or *current*) *payoff function* and the *terminal* (or *final*) *payoff function* for player i, respectively.

Notation. For each $i \in \bar{N}$, we define the set of multistrategies $\bar{\mathbf{U}}_{-i} := \bar{\mathbf{U}}_1 \times \cdots \times \bar{\mathbf{U}}_{i-1} \times \bar{\mathbf{U}}_{i+1} \times \cdots \times \bar{\mathbf{U}}_N$. For each $\mathbf{u}_i \in \bar{\mathbf{U}}_i$ and each $\mathbf{u}^*_{-i} \in \bar{\mathbf{U}}_{-i}$, we write $(\mathbf{u}_i, \mathbf{u}^*_{-i})$ to denote the vector

$$(\mathbf{u}^*_1, \ldots, \mathbf{u}^*_{i-1}, \mathbf{u}_i, \mathbf{u}^*_{i+1}, \ldots, \mathbf{u}^*_N) \in \bar{\mathbf{U}}.$$

Definition 5.1. *A multistrategy process $\boldsymbol{u}^* = (\boldsymbol{u}^*_1, \ldots, \boldsymbol{u}^*_N) \in \bar{\boldsymbol{U}}$ is called an open-loop Nash equilibrium for the SDG (5.3)-(5.5) if, for every $i \in \bar{N}$,*

$$\bar{J}^i(\boldsymbol{u}_i, \boldsymbol{u}^*_{-i}) \leq \bar{J}^i(\boldsymbol{u}^*) \ \ \forall \boldsymbol{u}_i \in \bar{\boldsymbol{U}}_i.$$

In a compact form, the class of SDGs we are interested in is described as

$$\bar{\Gamma}_{x_0} \ := \ \left[\bar{N}, \bar{\mathbf{U}}, \{\bar{J}^i\}_{i\in\bar{N}}, f, \sigma\right], \ \ T < \infty, \qquad (5.6)$$

where $\bar{N} = \{1, \ldots, N\}$ is the set of players, $\bar{\mathbf{U}}$ is the open-loop multistrategy process space defined in (5.2), $\bar{J}^i, i \in \bar{N}$, is the payoff functional for player i, as in (5.5), and f, σ define the system dynamics (5.3) (or (5.4)).

5.2 Stochastic optimal control problems

We define some stochastic OCPs using suitable functions and the same spaces $U, \bar{\mathbf{U}}$, etc, described in the previous Section 5.1. This is similar to what we did for the deterministic case in Section 2.2.

Definition 5.2. *Consider two functions $P : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}$ and $h : \mathbb{R}^l \to \mathbb{R}$. The stochastic OCP defined by P and h consists of a single player (or controller) who wants to maximize the payoff functional*

$$\bar{J}(\boldsymbol{u}) := E[\int_0^T P(s, \boldsymbol{x}(s), \boldsymbol{u}(s))ds + h(\boldsymbol{x}(T))] \tag{5.7}$$

subject to (5.3) (or (5.4)). A function $\boldsymbol{u}^* \in \bar{\boldsymbol{U}}$ that solves a stochastic OCP is called an open-loop optimal control or optimal solution.

By simplifying, an stochastic OCP can be expressed in compact form as

$$\bar{\Lambda}_{x_0} = [\bar{\mathbf{U}}, \bar{J}, f, \sigma] \tag{5.8}$$

where $\bar{\mathbf{U}}$ is given by (5.2), $\bar{J}$ is described in (5.7) as the payoff function of a controller, and f, σ define (5.3) (or (5.4)).

As in Section 2.2, to avoid confusions in our following results, we clearly specify when a point in $\bar{\mathbf{U}}$ is a control or a multistrategy.

5.3 Technical requirements

As a direct extension of Definition 2.4 on PDGs, we say that a stochastic differential game $\bar{\Gamma}_{x_0}$, as in (5.6), is called an open-loop stochastic PDG if there exists a stochastic OCP such that an open-loop optimal solution of this stochastic OCP is an open-loop Nash equilibrium for $\bar{\Gamma}_{x_0}$.

Clearly, stochastic team games are open-loop stochastic PDGs. (See Example 2.5.)

We specifically give in Remarks 5.4 and 5.5, below, two versions of the maximum principle: the necessary conditions for an optimal solution of $\bar{\Lambda}_{x_0}$, an stochastic OCP, as in (5.8), and sufficient conditions for a Nash equilibrium of an SDG $\bar{\Gamma}_{x_0}$, as in (5.6). In both cases, we consider the following assumption. (See [33] and [44], Chapter 3, for more details.)

Assumption 5.3. *Let Γ_{x_0} be an SDG, as in (5.6), and $\bar{\Lambda}_{x_0}$ a stochastic OCP, as in (5.8). For each $i \in \bar{N}$,*

(a) *The sets* U_i *are open and convex.*
(b) *The functions* L^i, P, f, *and* σ *are in* $C^2(\mathbb{R}^l \times U)$.
(c) *The functions* h, h^i *are in* $C^2(\mathbb{R}^l)$.

Remark 5.4. *(Necessary conditions for optimal solutions. See Theorem 3.2 in [44], p. 118.) Consider a stochastic OCP* $\bar{\Lambda}_{x_0}$ *as in (5.8). We define the associated Hamiltonian as*

$$H(s,x,u,p,q) \;:=\; P(s,x,u) + p \cdot f(s,x,u) + \mathrm{tr}[q'\sigma(s,x,u)], \qquad (5.9)$$

for $(s,x,u,p,q) \in [0,T] \times \mathbb{R}^l \times U \times \mathbb{R}^l \times \mathbb{R}^{l\times d}$. *If* $\boldsymbol{u}^* \in \bar{\boldsymbol{U}}$ *is an open-loop optimal solution for this stochastic OCP, and* $\boldsymbol{x}^*$ *is the state process corresponding to* $\boldsymbol{u}^*$, *then there exist processes* $(\bar{p}, \bar{q})$ *and* $(\bar{P}, \bar{Q})$ *such that, using the notation* $(*) := (s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s)), s \in [0,T]$:

$$\begin{aligned}
d\bar{p}(s) &= -H_x((*), \bar{p}(s), \bar{q}(s))ds + \bar{q}(s)dW(s),\\
\bar{p}(T) &= h_x(\boldsymbol{x}^*(T)),\\
d\bar{P}(s) &= -[H_{xx}((*), \bar{p}(s), \bar{q}(s)) + \bar{P}(s)f_x(*) + f_x(*)'\bar{P}(s)]ds\\
&\quad + \sum_{k=1}^{d}[\sigma_x^k(*)'\bar{P}(s)\sigma_x^k(*) + \bar{Q}_k(*)\sigma_x^k(*) + \sigma_x^k(*)'\bar{Q}_k(s)]ds\\
&\quad + \sum_{k=1}^{d}\bar{Q}_k(s)dW_k(s),\\
\bar{P}(T) &= h_{xx}(\boldsymbol{x}^*(T)),
\end{aligned}$$

where $\sigma^k := (\sigma^{1k}, \ldots, \sigma^{Nk})', 1 \le k \le d$, *is the* k*-th column of* σ. *Moreover, we have that*

$$H_{u_i}((*), \bar{p}(s), \bar{q}(s)) = 0 \;\; \forall \;\; i \in \bar{N}, \;\; a.e. \;\; s \in [0,T], \;\; \boldsymbol{P} - a.s. \qquad (5.10)$$

Remark 5.5. *(Sufficient conditions for Nash equilibria. See Theorem 6.2 in [33], p. 564.) Consider an SDG* $\bar{\Gamma}_{x_0}$ *as in (5.6). We define the associated Hamiltonian for player* i *as follows:*

$$H^i(s,x,u,p^i,q^i) \;:=\; L^i(s,x,u) + p^i \cdot f(s,x,u) + \mathrm{tr}[q^{i\prime}\sigma(s,x,u)], \qquad (5.11)$$

for $(s,x,u,p^i,q^i) \in [0,T] \times \mathbb{R}^l \times U \times \mathbb{R}^l \times \mathbb{R}^{l\times d}$. *Let* $\boldsymbol{u}^* \in \bar{\boldsymbol{U}}$ *be an open-loop multistrategy for the players, and* $\boldsymbol{x}^*$ *be the state process corresponding to* $\boldsymbol{u}^*$. *Suppose for each* $i \in \bar{N}$, *there exist processes* $(\bar{p}^i, \bar{q}^i)$ *and* $(\bar{P}^i, \bar{Q}^i)$ *such that,* *using the notation* $(*) := (s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s)), s \in [0,T]$*:*

$$
\begin{aligned}
d\bar{p}^i(s) &= -H^i_x((*), \bar{p}^i(s), \bar{q}^i(s))ds + q^i(s)dW(s) && (5.12)\\
\bar{p}^i(T) &= h^i_x(\boldsymbol{x}^*(T)) && (5.13)\\
d\bar{P}^i(s) &= -[H^i_{xx}((*), \bar{p}^i(s), \bar{q}^i(s)) + \bar{P}^i(s)f_x(*) + f_x(*)'\bar{P}^i(s)]ds && (5.14)\\
&\quad + \sum_{k=1}^{d}[\sigma^k_x(*)'\bar{P}^i(s)\sigma^k_x(*) + \bar{Q}^i_k(*)\sigma^k_x(*) + \sigma^k_x(*)'\bar{Q}^i_k(s)]ds\\
&\quad + \sum_{k=1}^{d}\bar{Q}^i_k(s)dW_k(s),\\
\bar{P}^i(T) &= h^i_{xx}(\boldsymbol{x}^*(T)) && (5.15)
\end{aligned}
$$

are satisfied. Moreover, suppose that, for each $i \in \bar{N}$,

(a) $h^i(x)$ *is concave in* x,

(b) $H^i(s, x, u_i, \boldsymbol{u}^*_{-i}(s), \bar{p}^i(s), \bar{q}^i(s))$ *is concave in* (x, u_i) *for* $s \in [0,T]$, *and*

(c) $H^i_{u_i}((*), \bar{p}^i(s), \bar{q}^i(s)) = 0$, *a.e.* $s \in [0,T]$, $\boldsymbol{P}$*-a.s.*

Then $\boldsymbol{u}^*$ *is Nash equilibrium of the game* $\bar{\Gamma}_{x_0}$.

We denote the Hamiltonian function for the controller of a stochastic OCP by H in (5.9), and for the player $i \in \bar{N}$ of an SDG by H^i in (5.11). (This is in fact a slight abuse of notation, because we use the same letters H and H^i that we used in the deterministic context.)

5.4 Fictitious-potential approach: The stochastic version

We next extend to stochastic PDGs some results in Section 3.2. In particular, we use the terms "Assumption" and "Condition" with the same meaning as in Section 3.2.1.

Consider a stochastic differential games, as in (5.6), under Assumption 5.3. To identify stochastic PDGs we require the next condition, which is as Condition 3.11 to differential games.

Condition 5.6. *There are functions* $P : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}$ *and* $h : \mathbb{R}^l \to \mathbb{R}$ *that satisfy for each* $(\cdot) = (s, x, u) \in [0,T] \times \mathbb{R}^l \times U$,

$$
\begin{aligned}
P_{u_i}(\cdot) &= L^i_{u_i}(\cdot), \;\; i \in \bar{N}, && (5.16)\\
P_{x_i}(\cdot) &= L^i_{x_i}(\cdot), \;\; i \in R, && (5.17)\\
h_{x_i}(x) &= h^i_{x_i}(x), \;\; i \in R. && (5.18)
\end{aligned}
$$

Note that if an index $j \in \bar{N}$ is such that $l_j = 0$, i.e., $j \notin R$, then in Condition 5.6, the functions P and $L^i, i \in \bar{N}$, depend only on state variables with index in R.

The next condition is a stochastic version of Condition 3.13.

Condition 5.7. *(a) For each $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ an optimal solution with their respective Lagrange multiplier $(\bar{p}, \bar{q})$ and $(\bar{P}, \bar{Q})$, the function*

$$(x, u_i) \mapsto H(s, x, u_i, \boldsymbol{u}^*_{-i}(s), \bar{p}(s), \bar{q}(s))$$

is concave in (x, u_i) for each $i \in \bar{N}$.

(b) The function h is concave in x.

The following Assumption 5.8 is an analogue of Assumption 3.8.

Assumption 5.8. *Let $\bar{\Gamma}_{x_0}$ be as in (5.6). The functions $L^1, \ldots, L^N$ satisfy that for each $(\cdot) = (s, x, u) \in [0, T] \times \mathbb{R}^l \times U$, and $k \in R$,*

$$L^1_{x_k}(\cdot) = \ldots = L^N_{x_k}(\cdot), \tag{5.19}$$

$$h^1_{x_k}(\cdot) = \ldots = h^N_{x_k}(\cdot). \tag{5.20}$$

Following similar arguments to those in the proofs of Lemma 3.14 and Theorem 3.15 yield the next lemma.

Lemma 5.9. *Under Condition 5.6 and Assumption 5.8:*

(a) The pairs $(\bar{p}, \bar{q}), (\bar{P}, \bar{Q})$ satisfy (5.12)-(5.15) in Remark 5.5 with $(\bar{p}^i, \bar{q}^i) = (\bar{p}, \bar{q})$ and $(\bar{P}^i, \bar{Q}^i) = (\bar{P}, \bar{Q})$ for every $i \in \bar{N}$, and a.e. $s \in [0, T]$, $\boldsymbol{P}$-a.s.

(b) For each $i \in \bar{N}$, and a.e. $s \in [0, T]$, $\boldsymbol{P}$-a.s.,

$$H_{u_i}(s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s), \bar{p}(s), \bar{q}(s)) = H^i_{u_i}(s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s), \bar{p}(s), \bar{q}(s)) = 0.$$

The stochastic version of Theorem 3.15 is the following.

Theorem 5.10. *Let $\bar{\Gamma}_{x_0}$ be as in (5.6). If Conditions 5.6 and 5.7, and Assumption 5.8 hold, then $\bar{\Gamma}_{x_0}$ is an open-loop stochastic PDG with potential function P and potential final function h.*

Proof. Each open-loop optimal solution u^* of the stochastic OCP defined by P and h is a Nash equilibrium for the SDG because for each $i \in \bar{N}$, (5.16), (5.17), and (5.19) imply that the functions $H(s, \cdot, \cdot, \mathbf{u}^*_{-i}, \bar{p}, \bar{q})$ and $H^i(s, \cdot, \cdot, \mathbf{u}^*_{-i}, \bar{p}, \bar{q})$ have the same Hessian matrix. (See Lemma 3.14, above.) Thus, under Condition 5.7(a), a relation of concavity in (x, u_i) between them is established. Note also that combining (5.18) and (5.20), the function $h^i, i \in \bar{N}$ has the same Hessian matrix that the function h. Thus, if Condition 5.7(b) holds, a relation of concavity in x between them is obtained. Moreover, in view of Lemma 5.9, we have sufficient conditions that satisfy Remark 5.5. □

Note that Theorem 5.10 does not require conditions on the dynamics (5.3).

Example 5.11. *Consider the game (5.6) and let* $p : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}$, $c^i : [0,T] \times U_{-i} \to \mathbb{R}$, *and* $h : \mathbb{R}^l \to \mathbb{R}$ *be functions such that*

$$\begin{aligned} L^i(s,x,u) &= p(s,x,u) + c^i(s,u_{-i}), \\ h^i(x) &= h(x) + k_i, \end{aligned}$$

with $k_i \in \mathbb{R}$. *By Theorem 5.10,* p *is a potential function and* h *is the potential final function for the game.* $\Diamond$

Example 5.12. *Consider the game (5.6), and let* $p : [0,T] \times \mathbb{R}^l \times U \to \mathbb{R}$, $c^i : [0,T] \times U_i \to \mathbb{R}$, *and* $h : \mathbb{R}^l \to \mathbb{R}$ *be functions such that*

$$\begin{aligned} L^i(s,x,u) &= p(s,x,u) + \sum_{r \in R} x_r + c^i(s,u_i), \\ h^i(x) &= h(x) + k_i, \end{aligned}$$

with $k_i \in \mathbb{R}$. *By Theorem 5.10,* $p + \sum_{r \in R} x_r + \sum_{i=1}^{N} c^i$ *is a potential function and* h *is the potential final function for the game.* $\Diamond$

The following Assumption 5.13 is almost similar to Assumption 3.9.

Assumption 5.13. *(a)* $N \leq l$.
(b) For every $j \in R$, $l_j = l_1$.

Assumption 5.14 means the state process can be rewritten as a uncoupled system of SDEs, we assumed this in Theorems 3.16, and 3.17.

Assumption 5.14. *Suppose that, for each* $i \in R$, *the functions* $f^i, \sigma^{ik}, 1 \leq k \leq d$, *depend only on the variables* (s, x_i, u_i); *that is, (5.4) can be written as*

$$\begin{aligned} d\boldsymbol{x}_i(s) &= f^i(s, \boldsymbol{x}_i(s), \boldsymbol{u}_i(s))ds + \sum_{k=1}^{d} \sigma^{ik}(s, \boldsymbol{x}_i(s), \boldsymbol{u}_i(s))dW_k(s), \\ \boldsymbol{x}_i(0) &= x_{i0}. \end{aligned}$$

We now consider Assumption 5.15 instead of Assumption 3.10.

Assumption 5.15. *Let* $\bar{\Gamma}_{x_0}$ *be as in (5.6). The functions* $L^i, h^i, i \in \bar{N}, f$, *and* σ *satisfy, for each* $k \in R$, *and each* $(\cdot) = (s,x,u) \in [0,T] \times \mathbb{R}^l \times U$,

$$\begin{aligned} &L^i_{x_1}(\cdot) = L^i_{x_k}(\cdot), \quad \text{and} \quad f^1_{x_1}(\cdot) = f^k_{x_k}(\cdot), \\ &h^i_{x_1}(x) = h^i_{x_k}(x), \qquad\qquad \sigma^{1j}_{x_1}(\cdot) = \sigma^{kj}_{x_k}(\cdot), \;\; 1 \leq j \leq d. \end{aligned}$$

Notation. We write p_i to denote the i-th coordinate of a vector $p \in \mathbb{R}^l$ and q_{kj} as the kj-th entry of a matrix $q \in \mathbb{R}^{l\times d}$.

We get the next lemma, following the same ideas as in Lemma 5.9.

Lemma 5.16. *Under Conditions 5.6 and 5.7, and Assumptions 5.13, 5.14, and 5.15, the Lagrange multipliers $(\bar{p}^i, \bar{q}^i), (\bar{P}^i, \bar{Q}^i)$ defined for each $k \in R$ and $1 \leq j \leq d$ by*

$$\bar{p}^i_k := \bar{p}_i \qquad \bar{P}^i_k := \bar{P}_i, \tag{5.21}$$

$$\bar{q}^i_{kj} := \bar{q}_{ij} \qquad \bar{Q}^i_{kj} := \bar{Q}_{ij}, \tag{5.22}$$

satisfy that:

(a) The functions $H(s, \cdot, \cdot, \boldsymbol{u}^*_{-i}, \bar{p}, \bar{q})$ and $H^i(s, \cdot, \cdot, \boldsymbol{u}^*_{-i}, \bar{p}^i, \bar{q}^i), i \in \bar{N}$ are concave in (x, u_i). The functions h and $h^i, i \in N$, are concave in x.

(b) The equations (5.12)-(5.15) in Remark 5.5 hold, a.e. $s \in [0, T]$, $\boldsymbol{P}$-a.s.

(c) For each $i \in \bar{N}$, and a.e. $s \in [0, T]$, $\boldsymbol{P}$-a.s.,

$$H_{u_i}(s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s), \bar{p}(s), \bar{q}(s)) = H^i_{u_i}(s, \boldsymbol{x}^*(s), \boldsymbol{u}^*(s), \bar{p}^i(s), \bar{q}^i(s)) = 0.$$

Therefore, Theorem 5.17 provides a criterion for identifying stochastic PDGs similar to Theorem 3.16 for deterministic models.

Theorem 5.17. *A stochastic differential game $\bar{\Gamma}_{x_0}$, as in (5.6), that satisfies Conditions 5.6 and 5.7, and Assumptions 5.13, 5.14, and 5.15, is an open-loop stochastic PDG with potential function P and potential final function h.*

Proof. Let $(\mathbf{x}^*, \mathbf{u}^*)$ be an optimal control pair for the stochastic OCP defined by P and h. By Lemma 5.16, the processes in (5.21) and (5.22) satisfy the requirements in Remark 5.5. Therefore, $\mathbf{u}^*$ is a Nash equilibrium. □

Example 5.18. *For each $i = 1, \ldots, N$, let $K^i : [0, T] \times U_i \to \mathbb{R}$ and $R^i : [0, T] \times U_{-i} \to \mathbb{R}$ be differentiable functions, with K^i concave on U_i. Moreover, player i has the instantaneous payoff function*

$$L^i(t, x, u) = a_i(\sum_{j=1}^{N} x_j) + K^i(t, u_i) + R^i(t, u_{-i})$$

with state variable x_i that satisfies

$$dx_i = [G(t)x_i + M^i(t, u_i)]dt + S(t)(u_i + x_i)dW_i(t), \;\; x_i(0) = x_{i0}.$$

where $M^i : [0, T] \times U_i \to \mathbb{R}$ is a concave function in u_i. Note that the final payoff function of each player is zero.

By Theorem 5.17, we have that

$$P(t,x,u) = \sum_{j=1}^{N} \left[a_j x_j + K^j(t,u_j)\right]$$

is a potential function and $h = 0$ *is the potential final function for the game.* $\diamondsuit$

A particular case of Example 5.18 is the Example 5.23, bellow.

Note that in Theorem 5.17, p^{k*} is not necessarily equal to p^{r*} (in contrast to Theorem 5.10 where $p^{k*} = p^{r*}$). Another difference between Theorems 5.10 and 5.17 is that Theorem 5.17 requires Condition 5.13, which is not needed in Theorem 5.10.

Finally, the next assumption and theorem correspond to Theorem 3.17.

Assumption 5.19. *Assume Condition 5.13(a). Each function* L^i *depends only on* (t, x_i, u)*, and* h_i *only on* x_i*.*

Theorem 5.20. *Let* $\bar{\Gamma}_{x_0}$ *be a differential game, as in (5.6), where Conditions 5.6 and 5.7, and Assumptions 5.14, and 5.19 hold. Then* $\bar{\Gamma}_{x_0}$ *is an open-loop PDG with potential function* P*, and potential final function* h*.*

Proof. The proof uses arguments similar to those in the proof of Theorem 5.10 and Theorem 5.17, except that the Lagrange processes are now as follows: for each $i \in R$ and $1 \le j \le d$,

$$\bar{p}^i_i := \bar{p}_i, \qquad \bar{p}^i_k := 0, k \neq i, \tag{5.23}$$

$$\bar{q}^i_{ij} := \bar{q}_{ij}, \qquad \bar{q}^i_{kj} := 0, k \neq i, \tag{5.24}$$

and

$$\bar{P}^i_i := \bar{P}_i, \qquad \bar{P}^i_k := 0, k \neq i, \tag{5.25}$$

$$\bar{Q}^i_{ij} := \bar{Q}_{ij}, \qquad \bar{Q}^i_{kj} := 0, k \neq i. \tag{5.26}$$

We use Condition 5.7 so that the function $H^i(s, x_i, u_i, u_{-i}, \bar{p}^i, \bar{q}^i)$ is concave in (x_i, u_i). $\square$

Example 5.21. *Consider an affine-quadratic differential game described as follows: the instantaneous payoff function for player* i *is*

$$g^i(x_i,u) = -\frac{1}{2}\left[(1+x_i)^2 + \sum_{i=1}^{N}\sum_{j=1}^{m_i}(1+u^i_j)^2\right],$$

with state variable x_i that is the solution of the nonlinear SDE

$$dx_i(t) \quad = \quad -\sum_{j=1}^{m_i} x_i^2(t)u_j^i(t) + dW(t), \;\; x_i(0) = x_{i0}. \tag{5.27}$$

By Theorem 5.20 this differential game is an open-loop PDG. The potential function

$$P(x,u) := -\frac{1}{2}\sum_{i=1}^{N}\left[(1+x_i)^2 + \sum_{j=1}^{m_i}(1+u_j^i)^2\right]$$

defines the associated stochastic OCP.

Theorem 5.20 characterizes open-loop PDGs in which we can remove, for each $i \in N$, the Lagrange multipliers $\bar{p}_k^i, \bar{q}_{kj}^i, \bar{P}_k^i, \bar{Q}_{kj}^i$ for all $k \neq i$ and $1 \leq j \leq d$, as an alternative to solve the corresponding equation system described in Remark 5.5. It is important to notice that in general this assignment is not necessarily true in games satisfying Theorems 5.10 or 5.17, for instance.

Remark. We comment why we are using the Hamiltonian function H in (5.9), above, instead of the function $\mathcal{H}$ defined in [44], p. 118, equation (3.16). To this end, consider

- Assumption 5.3, and
- $\bar{\Lambda}_{x_0}$, an stochastic OCP, as in (5.8).

Note that by the necessary conditions of optimal solution for $\bar{\Lambda}_{x_0}$ described by the maximum principle in Theorem 3.2 of [44], p. 118, we have that, if $\mathbf{u}^*$ is an optimal solution, then the so-called maximality condition (3.20), [44], p. 119, is satisfied in $\mathbf{u}^*$. Therefore, by Lemma 2.3 in [44], p. 106, for each $i \in \bar{N}$,

$$\mathcal{H}_{u_i}(s, \mathbf{x}^*(s), \mathbf{u}^*(s), \bar{p}(s), \bar{q}(s)) = 0.$$

Moreover, by Lemma 5.1 [44], p. 138, our condition (5.10), above, also holds. Hence, for the purposes of our present context, it suffices to consider the function H instead of $\mathcal{H}$.

5.5 Stochastic PDGs with Pareto-optimal Nash equilibria

We now consider the infinite-horizon stochastic differential game described by

$$d\mathbf{x}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t))dt + \sigma(t, \mathbf{x}(t))dW(t), \;\; \mathbf{x}(0) = x_0, \;\; t \geq 0, \tag{5.28}$$

and

$$J_i(\mathbf{u}) := E\left[\int_0^\infty e^{-\beta t} L^i(t, \mathbf{x}(t), \mathbf{u}(t))dt\right]. \tag{5.29}$$

Let $\mathbf{u}(\cdot)$ be an open-loop multistrategy in $\bar{\mathbf{U}}$ for which (5.28) and (5.29) are well-defined; see [44], for instance. Note that σ does not depend on $\mathbf{u}$.

Notation. For each $i \in \bar{N}$, let $\sigma_i := (\sigma_{i1}, \ldots, \sigma_{id})$ be row i of the $N \times d$ matrix σ in (5.28).

In the present stochastic case, Theorem 4.11, above, becomes as follows.

Theorem 5.22. *[18]. Consider a stochastic differential game as in (5.28)-(5.29), with $f = (f_1, \ldots, f_N)$. Suppose that there are functions $\hat{g}_i$, $\hat{f}_i$ and $\hat{\sigma}_i$, such that one of the following conditions holds for every $i \in \bar{N}$:*
(a) $L^i(t, x, u) = \hat{g}^i(t, u_i)$.
(b) $L^i(t, x, u) = \hat{g}^i(t, x, u_i)$, $f^i(t, x, u) = \hat{f}^i(t, x)$.
(c) $L^i(t, x, u) = \hat{g}^i(t, x_i, u_i)$, $f^i(t, x, u) = \hat{f}^i(t, x_i, u_i)$, $\sigma_i(t, x) = \hat{\sigma}_i(t, x_i)$.
Then the game (5.28)-(5.29) is a stochastic PDG where the associated OCP has objective function J as in Definition 5.2 (with $T = \infty$ and $h = 0$) and potential function

$$P = \hat{g}^1 + \cdots + \hat{g}^N. \tag{5.30}$$

Hence, if $\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_N^*) \in \bar{\boldsymbol{U}}$ maximizes J, then $\boldsymbol{u}^*$ is an open-loop Nash equilibrium. If, in addition, $\bar{\boldsymbol{U}}$ is convex and J_i is concave on $\bar{\boldsymbol{U}}_i$ for every $i \in \bar{N}$, then the reward vector $r(\boldsymbol{u}^*)$ is a Pareto point.

Proof. The proof follows the arguments in the proof of Theorem 4.11, and therefore, is omitted □

The following example illustrates Theorem 5.22.

Example 5.23. *Competition for consumption of a productive asset [25].* Assume that there are N players. The control sets are $U_i := [0, \infty)$ for all $i \in \bar{N}$. The players wish to maximize the expected discounted utility of *consumption*

$$J_i(\mathbf{u}) := E\left[\int_0^\infty e^{-\beta t} L^i(u_i(t))dt\right]$$

with $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$, subject to the stock dynamics

$$d\mathbf{x}(t) = [F(\mathbf{x}(t)) - \sum_{i=1}^N \mathbf{u}_i(t)]dt + \sigma(\mathbf{x}(t))dW(t), \;\; x(0) = x,$$

where F and σ are given functions that usually depend on the resource being exploited [25]. This game is as in Theorem 5.22(a). Hence it is a potential stochastic differential game with potential function

$$P(\mathbf{u}) := \sum_{i=1}^{N} L^i(\mathbf{u}_i).$$

Assuming that the instantaneous utility functions L^i are strictly concave, the optimal solutions of the associated OCP are both Nash equilibria and Pareto optimal. $\Diamond$

5.6 Comments

In this chapter we presented results on stochastic PDGs. These results were then used to identify SDGs with Pareto-optimal Nash equilibria. All of these facts are motivated, of course, by our findings in Chapter 3 and 4.

6 Conclusions

This work concerns deterministic and stochastic differential games, with finite- and infinite-horizon objective functions. The main aim is to introduce the concept of a *potential differential game* (PDG), which, roughly put, is a non- cooperative differential game to which we can associate an optimal control problem (OCP) whose optimal solutions are Nash equilibria for the given differential game. PDGs have two significant features. 1) It is greatly simplified finding Nash equilibria because it is easier to analyze an OCP than a differential game. And, 2) the obtained Nash equilibria—being solutions of an OCP—are necessarily pure or deterministic, rather than mixed or randomized. After an introduction to static (or one-shot) potential games in Section 1.2, we study deterministic PDGs in Chapters 3 and 4, and stochastic PDGs in Chapter 5. Most of the material on PDGs comes from the papers [17, 18], except for some parts of Chapter 5. We work in the context of *open-loop* (OL) strategies so that, in fact, we are dealing with OL-PDGs.

Some of our main findings are in Chapter 3. We propose two approaches to identify PDGs: The exact-potential approach and the fictitious-potential approach. The former approach is based on the games primitive data, which are mainly the game dynamics (2.2) and the payoff functions (2.3). We establish Theorem 3.1 for infinite-horizon differential games and Theorem 3.6 for the finite horizon case.

The fictitious-potential approach, on the other hand, assumes the existence of a smooth concave function that together with the game's primitive data and some suitable assumptions (see Condition 3.11, for instance) allow us to classify potential differential games. We establish Theorems 3.15, 3.16, and 3.17 for infinite-horizon differential games and Theorem 3.21 for finite horizon models.

We illustrate these approaches with numerous examples. See Section 3.3.

In Chapter 4, as an application of our results obtained in Chapter 3, we show that some time-homogeneous differential games satisfy a turnpike property; see Corollary 4.6. Thus, for some PDGs, it is possible to analyze the asymptotic behavior of Nash equilibria. To this end, we use a result by Trèlat and Zuazua [42] from control theory, which we state in Theorem 4.5.

Another topic we consider is the classification of PDGs with Pareto-optimal

Nash equilibria. This is established in Theorem 4.11 and Theorem 4.12. See also [18].

In Chapter 5, we study stochastic differential games as in (5.6), obtaining Theorems 5.10, 5.17, and 5.20. We develop the fictitious-potential approach for the stochastic case. Moreover, we obtain Theorem 5.22, which identifies classes of stochastic PDGs having Pareto-optimal Nash equilibria.

There are, of course, many open problems such as the following.

1. Slade [39] noted that Nash equilibria determined from the OCP associated with a class of discrete-time deterministic potential games have some (Lyapunov-like) stability properties. For the case of PDGs, it would be interesting to identify classes, if any, whose Nash equilibria are stable in some sense.

2. Is it possible to study PDGs in the class of closed-loop strategies?

3. By definition of PDG, an optimal solution of the associated OCP is a Nash equilibrium for the given differential game. Is it possible to provide conditions under which the converse is true? (Note that the answer is affirmative in some trivial cases, for instance, under some concavity conditions ensuring the existence of a unique Nash equilibrium. See also Example 2.6.)

4. For a class of discrete-time stochastic games, [22] uses inverse control problems (as in [21]) to analyze potential games. Is it possible to use inverse control problems to study PDGs?

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