Centro de Investigación y de Estudios Avanzados
del Instituto Politécnico Nacional
Unidad Zacatenco
Departamento de Matemáticas

Asintóticas para los valores propios de matrices de Toeplitz con símbolos que tienen un cero de orden 4

Tesis que presenta

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para obtener el grado de

Doctor en Ciencias

en la Especialidad de Matemáticas

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Ciudad de México Octubre, 2018

Center for Research and Advanced Studies
of the National Polytechnic Institute
Campus Zacatenco
Department of Mathematics

Asymptotics of eigenvalues for Toeplitz matrices with symbols that have a zero of order 4

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TO OBTAIN THE DEGREE OF
DOCTOR OF SCIENCE
IN THE SPECIALITY OF MATHEMATICS

Thesis Advisor: Dr. Sergey Grudskiy

Mexico City October, 2018

Agradecimientos

Agradezco al Dr. Sergey Grudskiy por sus enseñanzas, dedicación y todo el apoyo brindado en la elaboración de este trabajo.

Mi profundo agradecimiento al Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional y al Consejo Nacional de Ciencia y Tecnología (CONACYT) por el apoyo económico proporcionado que me permitió realizar este trabajo.

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Resumen

El propósito de esta tesis es explorar y describir el comportamiento asintótico para los valores propios de las matrices de Toeplitz con símbolos que tienen un único mínimo de orden 4. Para este propósito, primero decidimos considerar las matrices de Toeplitz $T_n(a)$ generadas por el símbolo (3.1.1), con el objetivo de obtener fórmulas asintóticas no solo para los valores propios extremos, sino también para los valores propios internos. Usando las fórmulas de Elouafi [20] derivamos ecuaciones exactas para todos los valores propios de $T_n(a)$, es decir, las ecuaciones (3.2.1) y (3.2.2). Estas ecuaciones nos permitieron realizar el análisis asintótico y, por lo tanto, deducimos fórmulas para sus valores propios que son uniformes en j. Teniendo en cuenta los resultados obtenidos para las matrices de Toeplitz generadas por el símbolo (3.1.1), que no satisface la condición simple-loop, proporcionamos un contraejemplo a una conjetura presentada en [19] por Ekström, Garoni y Serra-Capizzano y, al mismo tiempo, revelamos que la condición simple-loop es esencial para la existencia de la expansión asintótica regular. En la etapa final de esta tesis, exploramos la clase de todas las funciones racionales a que satisfacen las condiciones del símbolo (3.1.1). En lugar de recurrir al cálculo de los determinantes de las matrices de Toeplitz, consideramos la ecuación para los vectores propios, $T_n(a-\lambda)X = 0$, y representamos $a - \lambda = a - \psi(s)$ como un producto $p(\cdot, s)p_\varepsilon(\cdot, s)b_\varepsilon(\cdot, s)$, donde $p(\cdot, s)p_\varepsilon(\cdot, s)$ es un polinomio de Laurent que tiene los mismos ceros de $a - \psi(s)$ y $b_\varepsilon(\cdot, s)$ es positiva con $\inf b_\varepsilon > 0$. Finalmente, obtenemos un sistema homogéneo de dos ecuaciones lineales con dos incógnitas y, como el determinante de ese sistema tiene que ser cero, deducimos una ecuación exacta para los valores propios. Después de aplicar las herramientas de análisis asintótico, convertimos la ecuación para los valores propios en dos ecuaciones asintóticas más simples, que nos permiten proporcionar las fórmulas asintóticas para todos los valores propios que son uniformes en j.

Summary

The purpose of this thesis is to explore and describe the asymptotic behavior for the eigenvalues of Toeplitz matrices with symbols that have one minimum of order 4. For this purpose, we first decided to consider the Toeplitz matrices $T_n(a)$ generated by the symbol (3.1.1), with the aim of obtaining asymptotic formulas not only for the extreme eigenvalues, but also for the inner eigenvalues. Using Elouafi's formulas [20] we derived exact equations for all eigenvalues of $T_n(a)$, that is, equations (3.2.1) and (3.2.2). These equations allowed us to carry out the asymptotic analysis and, therefore, we deduced formulas for its eigenvalues that are uniform in j. Taking into account the obtained results for Toeplitz matrices generated by the symbol (3.1.1), which does not satisfy the simple-loop condition, we provide a counterexample to a conjecture presented in [19] by Ekström, Garoni, and Serra-Capizzano and, at the same time, we reveal that the simple-loop condition is essential for the existence of the regular asymptotic expansion. In the final stage of this thesis, we explore the class of all rational functions a satisfying the conditions of symbol (3.1.1). Instead of having recourse to Toeplitz determinants, we considered the equation for the eigenvectors, $T_n(a-\lambda)X = 0$, and represented $a - \lambda = a - \psi(s)$ as a product $p(\cdot, s)p_\varepsilon(\cdot, s)b_\varepsilon(\cdot, s)$, where $p(\cdot, s)p_\varepsilon(\cdot, s)$ is a certain Laurent polynomial which inherits the zeros of $a - \psi(s)$ and $b_\varepsilon(\cdot, s)$ is positive and separated from zero. Eventually, we get a homogeneous system of two linear equations with two unknowns and, as the determinant of that system has to be zero, we obtain an exact equation for the eigenvalues. After applying the asymptotic analysis tools, we converted this equation for the eigenvalues in two simpler asymptotic equations, which allow us to provide the asymptotic formulas for all eigenvalues that are uniform in j.

Introduction

In operator theory, a Toeplitz operator is the compression of a multiplication operator acting on $L^2(\mathbb{T})$. The matrix representation of a Toeplitz operator (in $\ell^2(\mathbb{Z}_+)$), with respect to the orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$ of the Hardy space H^2, is an infinite complex matrix having constant diagonals (parallel to the main one). This infinite matrix is known as a Toeplitz matrix.

An infinite Toeplitz matrix is completely determined by the (complex) numbers that constitute its first row and its first column. The function in $L^\infty(\mathbb{T})$ (the equivalence class of $L^\infty(\mathbb{T})$ containing it) whose Fourier coefficients are just these numbers is referred to as the symbol of the matrix.

Toeplitz matrices arise in plenty of applications and have been extensively studied for a long time. This does not imply that there is nothing left to say on the topic. To the contrary, Toeplitz matrices are an active field of research with many facets, and the amount of material gathered only in the last decade would easily fill several volumes.

The asymptotic behavior of their spectral characteristics, as n goes to infinity, is always at the heart of the matter. Moreover, this topic is extensively studied [9, 10, 23] and has applications in various areas, including stochastic processes and time series analysis [13, 34], signal processing [25, 40, 41], numerical solutions of differential and integral equations [27, 38, 44], image processing [14, 24], and quantum mechanics [17, 18].

In the early twenties, Szegő studied in detail the distribution of the eigenvalues of the finite sections of infinite Toeplitz matrices associated with a function defined in $[-\pi, \pi]$ and integrable in the sense of Lebesgue. Thus, the most famous and arguably the most important result describing the collective asymptotic behavior of the eigenvalues of Hermitian Toeplitz matrices, as n goes to infinity, is Szegő's first limit theorem.

For a banded Toeplitz matrix, the corresponding symbol is a Laurent polynomial. In such a case the work of Schimidt and Spitzer [39] in 1960, gives a full description of its asymptotic spectral. Years later in 1975 Day [15] extended the results of Schimidt and Spitzer to the rational case. However, despite the efforts of many mathematicians, in the general case (i.e., when the symbol belongs to $L^\infty(\mathbb{T})$) almost nothing is known about the asymptotic spectral behavior of a Toeplitz matrix.

In 1963 Brown and Halmos [12] showed that for a symbol a in $L^\infty(\mathbb{T})$, the eigenvalues of the corresponding Toeplitz matrices lie in the convex hull of the essential range of a. The first Szegő limit theorem and its many generalizations, see [10, 11] for details, show that under certain circumstances, up to $o(n)$ possible outliers, the eigenvalues of the corresponding Toeplitz matrices cluster along the essential range of the symbol, in such a case we say that the eigenvalues have the *canonical distribution*. In 1994 Widom conjectured that in "general", see [46] for details, the eigenvalues of a Toeplitz matrix have the canonical distribution.

On the other hand, the literature is very short for results on the individual asymptotic behavior of the eigenvalues, we mention the works of Böttcher, Bogoya, Grudsky and Maximenko [1, 7, 2, 6] and Deift, Its, and Krasovsky [16], which study the Hermitian case. In [1] the authors describe the individual asymptotic behavior of the eigenvalues of a class of large Hessenberg Toeplitz matrices, i.e., the class of symbols considered is characterized by having one power singularity over the unit circle $\mathbb{T}$. In [7, 2, 6] the authors explored the behavior of the *jth* eigenvalue of an $n \times n$ Hermitian Toeplitz matrix as n tends to infinity and provided asymptotic formulas that are uniform in j for $1 \leq j \leq n$. Moreover, we emphasize that the Toeplitz matrices are generated by symbols obey the *simple-loop* condition, that is, the real-valued generating function defined on $[0, 2\pi]$ has only two intervals of monotonicity and the second derivative at the maximum and minimum points is non-zero. As well, it was shown that the eigenvalues of Toeplitz matrices generated by a simple-loop symbols admit certain *regular asymptotic expansion* into negative powers of $n+1$, that is, given any natural number p and g satisfying the simple-loop condition, the eigenvalues $\lambda_{n,1} < \cdots < \lambda_{n,n}$ of $T_n(g)$ admit an asymptotic expansion

$$\lambda_{n,j} = \sum_{k=0}^{p} \frac{f_k(\frac{j\pi}{n+1})}{(n+1)^k} + O\left(\frac{1}{(n+1)^{p+1}}\right) \quad (n \to \infty), \tag{0.0.1}$$

with the error term being uniform in $1 \leq j \leq n$ and with continuous functions $f_0, \ldots, f_p : [0, \pi] \to \mathbb{R}$.

The main goal of this investigation is to consider the Toeplitz matrix $T_n(a)$ such that its symbol a has one minimum of order 4. We notice that the simplest case of Hermitian Toeplitz matrices generated by simple-loop symbols is tridiagonal Toeplitz matrices, i.e., the real-valued generating function is $g(\varphi) = 2 - 2\cos\varphi = \left(2\sin\frac{\varphi}{2}\right)^2$ for $\varphi \in [0, 2\pi]$. In this case are known the exact formulas for the eigenvalues, that is,

$$\lambda_{n,j} = 2 - 2\cos\frac{j\pi}{n+1} = \left(2\sin\frac{j\pi}{2n+2}\right)^2,$$

and hence they satisfy the regular asymptotic expansion (0.0.1) with $f_0 = g$ and $f_k = 0$ for $k \geq 1$. The simplest case for Toeplitz matrices with a symbol that has one minimum of order 4 is the pentadiagonal Toeplitz matrices, which frequently arise from discretization by finite differences on boundary value problems involving fourth-order derivatives. Therefore, the real-valued generating function defined by $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$ for $\varphi \in [0, 2\pi]$, allows us to consider pentadiagonal Toeplitz matrices with a symbol that has one minimum of order 4. In this concrete case, the problem of finding asymptotic formulas for all eigenvalues is nontrivial and, therefore, the first part of our investigation is devoted to this problem. Moreover, the asymptotic formulas have essentially other structure compared to the equation (0.0.1). This fact is proved in the second part of our investigation. In the third part of this work, we consider the general class of symbols with one minimum of order 4 and, therefore, it should be noted that the method to carry out the investigation, in this stage, is essentially other compared to the firs part.

This work is organized as follows.

Chapter 1 is devoted to familiarize the reader with the basic concepts and definitions needed to understand our work. This ideas comes from functional analysis, C^*-algebras, and asymptotic analysis. The reader need not study this chapter very carefully; it suffices to glance through it, pick up some notations, and backtrack whenever the necessity arises.

In Chapter 2 we give the basic theory of Toeplitz operators. Toeplitz operators with continuous symbols are explained in Section 2.1 together with the first spectral results of Toeplitz. In Section 2.2 we present the stability theory that was introduced by Gohberg and Hirschman in the sixties, and the enlarged and simplified by Silbermann in the early eighties. With that theory we can use all the C^*-algebras theory as an useful tool in analyzing the spectral characteristics of Toeplitz operators. In Section 2.3 we resume the well-known results about the asymptotic behavior on the extreme eigenvalues of the Hermitian Toeplitz matrices. Finally, in Section 2.4 we formulate some fresh results that are used when it is studied the asymptotic behavior on the extreme and inner eigenvalues of Hermitian Toeplitz matrices with simple-loop symbols.

In Chapter 3 we study the asymptotic behavior of the $n \times n$ sections of certain pentadiagonal symmetric Toeplitz matrices generated by the symbol $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$, as n goes to infinity. Moreover, we establish asymptotic formulas for all the eigenvalues of these finite sections. The entries of the matrices are real and the real-valued generating function g has a minimum and a maximum such that its fourth derivative at the minimum and its second derivative at the maximum are nonzero. This is not the simple-loop case considered in [2], [6], and [7]. We apply the main result of [20] and obtain nonlinear equations for the eigenvalues. It should be noted that our equations have a more complicated structure than the equations in [2], [6], and [7]. Therefore, we required a more delicate method for its asymptotic analysis.

In Chapter 4 we use the results obtained in the Chapter 3 to show that the eigenvalues of the pentadiagonal Toeplitz matrices generated by the symbol $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$, which does not satisfy the simple-loop condition, do not admit a regular asymptotic expansion. Thus, we give a counterexample to a conjecture presented in [19] by Ekström, Garoni, and Serra-Capizzano, and at the same time we reveal that the simple-loop condition is essential for the existence of the regular asymptotic expansion.

In Chapter 5 we continue the line of investigation of Chapter 3, i.e., it is studied the asymptotic behavior of the eigenvalues of $T_n(a)$, as n goes to infinity, assuming that the symbol a is a rational function and the generating function $g(\sigma) := a(e^{i\sigma})$ has one minimum and one maximum with its fourth derivative at the minimum and its second derivative at the maximum are nonzero.

Chapter 1

Preliminaries

For understanding the results of this work, the reader must be familiar with some basic knowledges of real and complex analysis, linear algebra, and functional analysis. For this purpose, in the present chapter we explain the preliminaries that we need.

1.1 Banach and Hilbert Space

We start by fixing notation for a number of standard sets that will be in constant use throughout this work. We denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the standard entire, real, and complex sets, respectively. Additionally, let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and we denote by $\mathbb{T}$ the unit circle in the complex plane, i.e., $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

A *normed space* is a pair $(\mathscr{X}, \|\cdot\|)$, where $\mathscr{X}$ is a vector space (or linear space) over an infinite scalar field $\mathbb{F}$ and $\|\cdot\|$ is a norm on $\mathscr{X}$. A *Banach space* is a normed space that is complete with respect to the metric defined by the norm, i.e., $d(x, y) := \|x - y\|$ for $x, y \in \mathscr{X}$.

A *Hilbert space* is a vector space $\mathscr{X}$ over $\mathbb{F}$ with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathscr{X}$ is a complete metric space with the metric given by

$$d(x, y) := \|x - y\| := (\langle x - y, x - y \rangle)^{\frac{1}{2}}.$$

Therefore, every Hilbert space is also a Banach space.

As well, we introduce the spaces that will play an important role throughout this work. The space $L^2(\mathbb{T})$ is the set of all functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty.$$

Similarly, $L^\infty(\mathbb{T})$ is the set of all essentially bounded functions over the unit circle, that is, the set of all functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbb{T}} |f(t)| < \infty,$$

where ess sup is the supremum but without taking into account the zero measure sets.

As is customary, we often abuse the language and view $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ as spaces of functions rather than as spaces of equivalence classes of functions.

The space $\ell^2(\mathbb{Z})$ is the set of all complex sequences $x = \{x_n\}_{n=-\infty}^{\infty}$ such that

$$\|x\|_2 := \left(\sum_{n=-\infty}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} < \infty.$$

The space $\ell^2(\mathbb{Z}_+)$ is the subset of $\ell^2(\mathbb{Z})$ of all the complex sequences with null negative components, i.e., the set of all complex sequences $x = \{x_n\}_{n=-\infty}^{\infty}$ such that $x_n = 0$ for every $n < 0$.

The *Hardy Space*, denoted by H^2, consists of all functions having power series representations with square-summable complex coefficients. That is,

$$H^2 := \left\{ f : \mathbb{T} \to \mathbb{C} : f(t) = \sum_{k=0}^{\infty} f_k t^k \text{ and } \|f\| := \left(\sum_{k=0}^{\infty} |f_k|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

For example, it is well known that $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T})$, and H^2 are Hilbert spaces under the inner products

$$\langle x, y \rangle := \sum_{n=-\infty}^{\infty} x_n \overline{y_n} \quad \text{for} \quad x, y \in \ell^2(\mathbb{Z}),$$
$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta \quad \text{for} \quad f, g \in L^2(\mathbb{T}),$$

and

$$\langle f, g \rangle := \sum_{k=0}^{\infty} f_k \overline{g_k} \quad \text{for} \quad f, g \in H^2,$$

respectively.

The following theorem allow us to consider the elements of the space $L^\infty(\mathbb{T})$ like elements of $L^2(\mathbb{T})$.

Theorem 1.1.1. $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$.

Proof. If f is an arbitrary element of $L^\infty(\mathbb{T})$, then

$$\|f\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \|f\|_\infty^2 d\theta = \|f\|_\infty^2 < \infty.$$

Therefore, it follows that $f \in L^2(\mathbb{T})$. □

1.2 Operators and algebras

A *linear operator* from a Banach space $\mathscr{X}$ to a Banach space $\mathscr{Y}$ is a mapping $L : \mathscr{X} \to \mathscr{Y}$ such that $L(\alpha x + \beta y) = \alpha Lx + \beta Ly$ for all $x,\ y \in \mathscr{X}$ and all scalars $\alpha,\ \beta \in \mathbb{F}$. We often write Lx instead of $L(x)$, this simplification is standard in functional analysis.

We say that the operator L is *bounded* if there exists a number $\alpha > 0$ such that $\|Lx\| \leq \alpha \|x\|$ for all $x \in \mathscr{X}$. Additionally, we can define the *operator norm* as $\|L\| := \sup\limits_{\|x\|=1} \|Lx\|$. It follows that $\|Lx\| \leq \|L\| \|x\|$ for all $x \in \mathscr{X}$, i.e., $\|L\|$ is the least number α such that $\|Lx\| \leq \alpha \|x\|$ for all $x \in \mathscr{X}$.

The set of all bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$ is denoted by $\mathcal{B}(\mathscr{X}, \mathscr{Y})$, and it is well known that equipped with point-wise operations and the operator norm, it is a Banach space. In the case $\mathscr{X} = \mathscr{Y}$ we write $\mathcal{B}(\mathscr{X})$ instead of $\mathcal{B}(\mathscr{X}, \mathscr{X})$.

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, for every $A \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ there is a unique $B \in \mathcal{B}(\mathscr{K}, \mathscr{H})$ such that

$$\langle Ah, k\rangle = \langle h, Bk\rangle,$$

for every $h \in \mathscr{H}$ and every $k \in \mathscr{K}$. That unique operator B is called the *adjoint* of A and is denoted by $B = A^*$. Moreover, we have the following properties

$$\begin{aligned} (L+S)^* &= L^* + S^*, \\ (\alpha L)^* &= \overline{\alpha} L^*, \\ (L^*)^* &= L, \\ (LS)^* &= S^* L^*, \\ \|L^*\| &= \|L\|, \\ \|L^* L\| &= \|L\|^2, \end{aligned}$$

for arbitrary operators L and S and scalar α.

A linear operator $L : \mathscr{X} \to \mathscr{Y}$ is called a *compact operator (or completely continuous operator)* if, for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathscr{X}$, the sequence $\{Lx_n\}_{n=1}^{\infty}$ contains a convergent subsequence in $\mathscr{Y}$. The set of all compact operators from $\mathscr{X}$ into $\mathscr{Y}$ is denoted by $\mathcal{K}(\mathscr{X}, \mathscr{Y})$, and we write $\mathcal{K}(\mathscr{X})$ instead of $\mathcal{K}(\mathscr{X}, \mathscr{X})$. If $\dim \mathscr{X} = \infty$, the identity operator $\mathcal{I} : \mathscr{X} \to \mathscr{X}$ is not compact, although it is bounded.

An *algebra* over a field $\mathbb{F}$ is a vector space $\mathcal{A}$ over $\mathbb{F}$ with an associative and distributive multiplication such that if $\alpha \in \mathbb{F}$ and $a, b \in \mathcal{A}$, then $\alpha(ab) = (\alpha a)b = a(\alpha b)$. A *Banach algebra* is an algebra $\mathcal{A}$ over $\mathbb{F}$ that has a norm $\|\cdot\|$ relative to which $\mathcal{A}$ is a Banach space and such that for all $a, b \in \mathcal{A}$, $\|ab\| \le \|a\|\|b\|$. If $\mathcal{A}$ has a multiplicative identity e we say that $\mathcal{A}$ is a *unital algebra* and it is assumed that $\|e\| = 1$. The map $\alpha \mapsto \alpha e$ is an isomorphism of $\mathbb{F}$ into $\mathcal{A}$ and $\|\alpha e\| = |\alpha|$. So it will be assumed that $\mathbb{F} \subset \mathcal{A}$ by this identification.

There are several examples of Banach algebras, the most common are the following:

1. $\mathbb{C}$ is the simplest example of an abelian Banach algebra with identity $e = 1$.

2. If K is a compact topological space, then the space of all continuous functions over K, denoted by $C(K)$, is an abelian Banach algebra with identity $e = 1$, point-wise algebraic operations, and the uniform norm.

3. If K is a locally compact topological space, then the space of all continuous functions over K which vanish at infinity, denoted by $C_0(K)$, is an abelian Banach algebra with point-wise algebraic operations and the uniform norm. Furthermore, $C_0(K)$ does not have an identity.

4. If (X, Ω, μ) is a σ-finite measure space, then $L^{\infty}(X, \Omega, \mu)$ is an abelian Banach algebra with identity $e = 1$, point-wise algebraic operations, and the norm $\|\cdot\|_{\infty}$.

5. If $\mathscr{X}$ is a Banach space, then $\mathcal{B}(\mathscr{X})$ is a Banach algebra with identity $e = \mathcal{I}$ (the identity operator on $\mathscr{X}$) and the multiplication is defined by composition of operators. $\mathcal{B}(\mathscr{X})$ is not abelian, unless $\dim \mathscr{X} = 1$.

6. If $\mathscr{X}$ is a Banach space, then $\mathcal{K}(\mathscr{X})$ is a Banach algebra without identity, unless $\dim \mathscr{X} < \infty$. When $\dim \mathscr{X} < \infty$ it follows that $\mathcal{K}(\mathscr{X}) = \mathcal{B}(\mathscr{X})$.

A conjugate linear map $a \mapsto a^*$ of a Banach algebra $\mathcal{A}$ into itself is called an *involution* if $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. Finally, a C^*-algebra is a Banach algebra $\mathcal{A}$ with an involution such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$. A unital C^*-algebra is a C^*-algebra $\mathcal{A}$ which has an element e such that $ae = ea = a$ for all $a \in \mathcal{A}$. A C^*-algebra $\mathcal{A}$ is said to be commutative (or abelian) if $ab = ba$ for all $a, b \in \mathcal{A}$.

If $\mathscr{H}$ is a Hilbert space, then $\mathcal{B}(\mathscr{H})$ and $\mathcal{K}(\mathscr{H})$ are C^*-algebra under the operator norm, the usual point-wise operations, and the adjoint as involution. The set $L^\infty(\mathbb{T})$ is also a C^*-algebra under point-wise operations, the norm $\|\cdot\|_\infty$, and the passage to complex conjugate as involution. The C^*-algebra $L^\infty(\mathbb{T})$ is commutative. The C^*-algebras $\mathcal{B}(\mathscr{H})$ and $\mathcal{K}(\mathscr{H})$ are not commutative for $\dim \mathscr{H} \geq 2$. The C^*-algebra $\mathcal{K}(\mathscr{H})$ is unital if and only if $\dim \mathscr{H} < \infty$, in which case $\mathcal{K}(\mathscr{H}) = \mathcal{B}(\mathscr{H})$.

An element a of a unital C^*-algebra $\mathcal{A}$ is said to be *invertible* if there is a $b \in \mathcal{A}$ such that $ab = ba = e$. If it exists, this element b is unique; it is denoted by a^{-1} and is called the *inverse* of a. The *spectrum* of an element a of a unital C^*-algebra $\mathcal{A}$ is the compact and non-empty set given by

$$sp_{\mathcal{A}} a := \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible in } \mathcal{A}\}.$$

A subset $\mathcal{B}$ of a C^*-algebra $\mathcal{A}$ is called a C^*-subalgebra of $\mathcal{A}$ if $\mathcal{B}$ itself is a C^*-algebra with the norm and the operations of $\mathcal{A}$. The following theorem says that C^*-algebras are “inverse closed”.

Theorem 1.2.1. *If $\mathcal{B}$ is a unital C^*-algebra with unit element e and if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$ which contains e, then $sp_{\mathcal{A}} a = sp_{\mathcal{B}} a$, for every $a \in \mathcal{A}$.*

Proof. See [29, p. 41]. □

By virtue of this theorem, we will abbreviate $sp_{\mathcal{A}} a$ to $sp\, a$. A C^*-subalgebra $\mathcal{J}$ of C^*-algebra $\mathcal{A}$ is called a *closed ideal* of $\mathcal{A}$ if $aj \in \mathcal{J}$ and $ja \in \mathcal{J}$ for all $a \in \mathcal{A}$ and all $j \in \mathcal{J}$.

Theorem 1.2.2. *If $\mathcal{A}$ is a C^*-algebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$, then the quotient algebra $\mathcal{A}/\mathcal{J}$ is a C^*-algebra with the usual quotient operations,*

$$\begin{aligned} \lambda(a + \mathcal{J}) &:= \lambda a + \mathcal{J}, \\ (a + \mathcal{J}) + (b + \mathcal{J}) &:= (a + b) + \mathcal{J}, \\ (a + \mathcal{J})(b + \mathcal{J}) &:= ab + \mathcal{J}, \\ (a + \mathcal{J})^* &:= a^* + \mathcal{J}, \end{aligned}$$

and the usual quotient norm $\|a + \mathcal{J}\| := \inf_{j \in \mathcal{J}} \|a + j\|$.

Proof. See [29, p. 80]. □

A $*$-*homomorphism* is a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ of a C^*-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ which satisfies $\varphi(a)^* = \varphi(a^*)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. When $\mathcal{A}$ and $\mathcal{B}$ are unital, we also require that $*$-homomorphism map the unit of $\mathcal{A}$ to the unit element of $\mathcal{B}$. Bijective $*$-homomorphisms are referred to as $*$-*isomorphisms*.

Theorem 1.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebra and suppose that $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism. Then, the following hold.*

(i) *The map φ is contractive, $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.*

(ii) *The image $\varphi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$.*

(iii) *If φ is injective, then φ preserves spectra and norms, $sp(\varphi(a)) = sp\, a$ and $\|\varphi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.*

Proof. See [10, p. 68]. □

1.3 Fredholm operators

Let $\mathscr{H}$ be a Hilbert space. An operator $A \in \mathcal{B}(\mathscr{H})$ is said to be *Fredholm* if it is invertible modulo compact operators, that is, if the coset $A + \mathcal{K}(\mathscr{H})$ is invertible in the quotient algebra $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$. It is well known that an operator $A \in \mathcal{B}(\mathscr{H})$ is Fredholm if and only if it is *normally solvable* (which means that its range $\operatorname{Im} A$ is a closed subspace of $\mathscr{H}$) and both the *kernel*

$$\ker A := \{x \in \mathscr{H} : Ax = 0\}.$$

and the *cokernel*

$$\operatorname{coker} A := \mathscr{H} / \operatorname{Im} A,$$

have finite dimensions. Therefore, for a Fredholm operator A the *index* $\operatorname{Ind} A := \dim\ker A - \dim\operatorname{coker} A$, is a well defined integer.

Example 1.3.1. For $n \in \mathbb{Z}$ and $t \in \mathbb{T}$, let χ_n be the function given by $\chi_n(t) := t^n$. It is readily seen that $T(\chi_n)$, the Toeplitz operator with symbol χ_n over the space ℓ^2, acts by the rule

$$T(\chi_n) : (x_j)_{j=0}^{\infty} \mapsto \begin{cases} (\underbrace{0,\dots,0}_{n}, x_0, x_1, \dots) & \text{if } n \geq 0, \\ (x_{|n|}, x_{|n|+1}, \dots) & \text{if } n < 0. \end{cases}$$

Consequently,

$$\operatorname{Im} T(\chi_n) = \operatorname{Im} \overline{T(\chi_n)},$$
$$\dim\ker T(\chi_n) = \begin{cases} 0 & \text{if } n \geq 0, \\ |n| & \text{if } n < 0, \end{cases}$$
$$\dim\operatorname{coker} T(\chi_n) = \begin{cases} n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

whence $\operatorname{Ind} T(\chi_n) = -n$ for all $n \in \mathbb{Z}$.

The following theorem summarizes some well-known properties of the index.

Theorem 1.3.1. *Let $A, B \in \mathcal{B}(\mathscr{H})$ be Fredholm operators.*

(*i*) *If $K \in \mathcal{K}(\mathscr{H})$, then $A + K$ is Fredholm and* $\operatorname{Ind}(A + K) = \operatorname{Ind} A$.

(*ii*) *There is an $\varepsilon = \varepsilon(A) > 0$ such that $A + C$ is Fredholm and* $\operatorname{Ind}(A + C) = \operatorname{Ind} A$ *whenever $C \in \mathcal{B}(\mathscr{H})$ and $\|C\| < \varepsilon$.*

(*iii*) *The product AB is Fredholm and* $\operatorname{Ind}(AB) = \operatorname{Ind} A + \operatorname{Ind} B$.

(*iv*) *The adjoint operator A^* is Fredholm and* $\operatorname{Ind} A^* = -\operatorname{Ind} A$.

The *spectrum of an operator* $A \in \mathcal{B}(\mathscr{H})$ is its $sp\, A$ as an element of the C^*-algebra $\mathcal{B}(\mathscr{H})$, that is,

$$sp\, A := \{\lambda \in \mathbb{C} : A - \lambda\mathcal{I} \text{ is not invertible}\}.$$

By Theorem 1.2.2 the quotient algebra $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$ is also a C^*-algebra. For $A \in \mathcal{B}(\mathscr{H})$ the *essential spectrum*, denoted by $sp_{ess} A$, is defined as the spectrum of $A + \mathcal{K}(\mathscr{H})$ in $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$, i.e.,

$$sp_{ess} A := sp(A + \mathcal{K}(\mathscr{H})) = \{\lambda \in \mathbb{C} : A - \lambda\mathcal{I} \text{ is not Fredholm}\}.$$

Additionally, the *essential norm* $\|A\|_{ess}$ is defined as the norm of $A+\mathcal{K}(\mathscr{H})$ in $\mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$,

$$\|A\|_{ess} := \|A+\mathcal{K}(\mathscr{H})\| = \inf_{K\in\mathcal{K}(\mathscr{H})} \|A+K\|.$$

From the definitions it is easy to verify that $sp_{ess}\, A \subset sp\, A$ and $\|A\|_{ess} \leq \|A\|$.

1.4 Big O and little o

Big O and *little o*, also called Landau's symbols, are symbolisms used in complexity theory, computer science, and mathematics to describe the asymptotic behavior of functions. Basically, it tells us how fast a function grows or declines.

Consider a variable x in a Hausdorff space $\mathscr{X}$. The variable x ranges over a subset R of $\mathscr{X}$ and x_0 is a limit point of R (which may or may not belong to R). Let ϕ, ψ, and similar symbols denote real or complex-valued numerical functions of x defined when x is in R.

We define $\phi \in O(\psi)$ as $x \to x_0$, which is usually written as $\phi(x) = O(\psi(x))$, if there is a constant $C > 0$ and a neighborhood U of x_0 such that $|\phi(x)| \leq C|\psi(x)|$ for all x common to U and R; and we define $\phi \in o(\psi)$ as $x \to x_0$ if for each $\varepsilon > 0$ there is a neighborhood U_ε of x_0 such that $|\phi(x)| < \varepsilon|\psi(x)|$ for all x common to U_ε and R. As with big O notation, $\phi \in o(\psi)$ is usually written as $\phi(x) = o(\psi(x))$, which is a slight abuse of notation. If $\psi(x) \neq 0$, the relation $\phi(x) = o(\psi(x))$ is equivalent to $\lim\limits_{x\to x_0} \frac{\phi(x)}{\psi(x)} = 0$.

If the functions involved in an order relation depend on parameters, in general also the constant C, and the neighborhoods U, U_ε involved in the definitions will depend on the parameters. If C, U, U_ε may be chosen to be independent of the parameters, the order relation is said to hold *uniformly* in the parameters.

Operations with order relations are governed by a number of simple rules. We shall set out the most frequently used rules for the big O notation, and given that $o(\psi) \subset O(\psi)$, the corresponding rules hold for the little o notation. In the following rules R and x_0 are fixed, and the qualifying phrase "as $x \to x_0$" is omitted throughout.

If $\phi = O(\psi)$ and $\alpha > 0$, then

$$|\phi|^\alpha = O(|\psi|^\alpha). \tag{1.4.1}$$

If $\phi_\ell = O(\psi_\ell)$ and the α_ℓ are constants for $\ell = 1, \ldots, k$, then

$$\sum_\ell \alpha_\ell \phi_\ell = O\left(\sum_\ell |\alpha_\ell||\psi_\ell|\right). \tag{1.4.2}$$

This relation holds also for infinite series provided that $\phi_\ell = O(\psi_\ell)$ uniformly in ℓ. In the case of infinite series, (1.4.2) and similar statements will be interpreted in the following manner. If $\sum\limits_\ell |\alpha_\ell \psi_\ell|$ converges, then so does $\sum\limits_\ell \alpha_\ell \phi_\ell$ and (1.4.2) is true, and if $\sum\limits_\ell |\alpha_\ell \psi_\ell|$ diverges, then there is nothing to state.

If $\phi_\ell = O(\psi_\ell)$ and the α_ℓ are constants for $\ell = 1, \ldots, k$, with $|\psi_\ell| \leq |\psi|$ for $\ell = 1, \ldots, k$ and for all x common to R and to some neighborhood U_0 of x_0, then

$$\sum_\ell \alpha_\ell \phi_\ell = O\left(\psi\right). \tag{1.4.3}$$

Analogously, this relation holds for infinite series provided that $\phi_\ell = O(\psi_\ell)$ uniformly in ℓ, and $\sum_\ell |\alpha_\ell| < \infty$.

If $\phi_\ell = O(\psi_\ell)$ for $\ell = 1, \dots, k$, then

$$\prod_\ell \phi_\ell = O\left(\prod_\ell \psi_\ell\right). \tag{1.4.4}$$

The proof of (1.4.1) is immediate. To prove (1.4.2), we remark that by assumption there are numbers C_ℓ and neighborhoods U_ℓ of x_0 associated with the ϕ_ℓ. If the number of the ϕ_ℓ is finite, there is a C larger than all the C_ℓ, and a neighborhood U contained in all the U_ℓ such that

$$\left|\sum_\ell \alpha_\ell \phi_\ell\right| \le \sum_\ell |\alpha_\ell| C_\ell |\psi_\ell| \le C \sum_\ell |\alpha_\ell||\psi_\ell|,$$

when x is common to R and U, and this proves (1.4.2). If there is an infinite number of ψ_ℓ, then the existence of C and U follows from the uniformity in ℓ, of the order relation. Rule (1.4.3) can be deduced from (1.4.2) since under the circumstances envisaged we may take U above to be contained in U_0 and therefore,

$$C \sum_\ell |\alpha_\ell||\psi_\ell| \le \left(C \sum_\ell |\alpha_\ell|\right) |\psi| = C_1 \psi,$$

where $C_1 := C \sum_\ell |\alpha_\ell|$ is a finite constant. The proof of rule (1.4.4) is similar to that of (1.4.2).

Order relations may be *integrated* either with respect to the independent variable or with respect to parameters. For simplicity we shall restrict ourselves to integrals with respect to real variables. Extensions to complex and abstract variables are possible.

Let x be a real variable and $R = (a, b)$. If $\phi = O(\psi)$ as $x \to b$, where ϕ and ψ are measurable functions in R, then

$$\int_x^b \phi(t)dt = O\left(\int_x^b |\psi(t)|dt\right) \quad \text{as } x \to b. \tag{1.4.5}$$

The proof of this rule is as follows: If $\int_x^b |\psi(t)|dt = \infty$, there is nothing to prove. If $\int_x^b |\psi(t)|dt < \infty$ for some x, then we can choose C and x_1 such that $\int_x^b |\psi(t)|dt < \infty$ and $|\phi(x)| \le C|\psi(x)|$ for $x_1 < x < b$, and hence

$$\left|\int_x^b \phi(t)dt\right| \le \int_x^b |\phi(t)|dt \le C \int_x^b |\psi(t)|dt \quad \text{for } x_1 < x < b.$$

Let x be a variable element of the set R in a Hausdorff space and let y be real parameter such that $\alpha < y < \beta$. If $\phi(x, y) = O(\psi(x, y))$ uniformly in y, as $x \to x_0$, where ϕ and ψ are measurable functions in (α, β) for each fixed x in R, then

$$\int_\alpha^\beta \phi(x, y)dy = O\left(\int_\alpha^\beta |\psi(x, y)|dy\right) \quad \text{as } x \to x_0. \tag{1.4.6}$$

The proof of rule (1.4.6) is similar to that of rule (1.4.5).

In general, it is not always permissible to *differentiate* order relations either with respect to the independent variable or with respect to parameters. For example, if $\phi(x) = x^2 \sin\frac{1}{x}$, then $\phi(x) = o(x)$ but $\phi'(x) \ne o(1)$ as $x \to 0$. However, some general results on the differentiation of order relations exist in the case of analytic functions of a complex variable.

1.5 Asymptotic expansions

In this Section $R,\ x,\ x_0,\ \phi$ have the same meaning as in the previous Section. A finite or infinite sequence of functions $\phi_1, \phi_2, \ldots,$ will be abbreviated as $\{\phi_n\}$.

The sequence of functions $\{\phi_n\}$ is called an *asymptotic sequence uniformly in* n as $x \to x_0$, if for each $n = 1, 2, \ldots,$ ϕ_n is defined in R and $\phi_{n+1} = o(\phi_n)$ uniformly in n. Furthermore, if the ϕ_n depend on parameters and $\phi_{n+1} = o(\phi_n)$ uniformly in the parameters, then $\{\phi_n\}$ is said to be an *asymptotic sequence uniformly in the parameters.*

Let $\{\phi_n\}$ be an asymptotic sequence as $x \to x_0$, ϕ be a numerical function of x defined in R, and $a_1, a_2, \ldots,$ be constants (i.e., independent of x). The (formal) series $\sum_n a_n \phi_n(x)$ is said to be an *asymptotic expansion to* N terms of ϕ as $x \to x_0$ if

$$\phi(x) = \sum_{n=1}^{N} a_n \phi_n(x) + o(\phi_N) \quad \text{as } x \to x_0.$$

An asymptotic expansion to N terms will often be indicated as

$$\phi(x) \approx \sum_n a_n \phi_n(x) \quad \text{to } N \text{ terms as } x \to x_0 \text{ in } R.$$

An asymptotic expansion to 1 term will be written as

$$\phi(x) \approx a_1 \phi_1(x) \quad \text{as } x \to x_0$$

and will be called an *asymptotic representation*; and an asymptotic expansion to any number of terms (i.e., with $N = \infty$) will be written

$$\phi(x) \approx \sum_n a_n \phi_n(x) \quad \text{as } x \to x_0$$

and will be called *asymptotic expansion.* An asymptotic expansion may be convergent or divergent.

If an asymptotic expansion to N terms, with N finite, involves certain parameters, we shall say that it holds *uniformly* in these parameters if the remainder is uniform in the parameters. An asymptotic expansion ($N = \infty$) involving certain parameters will be said to hold uniformly in these parameters if $\phi(x) - \sum_{n=1}^{M} a_n \phi_n(x) = o(\phi_M)$ uniformly in the parameters for each sufficiently large M (but not necessarily uniformly in M).

The formal (finite or infinite) series $\sum_n a_n \phi_n(x)$ will be called an *asymptotic series.* Asymptotic expansions with respect to the sequence $\phi_n(x) = x^{\pm n}$ will be called *asymptotic power series.*

By definition, the coefficients in an asymptotic expansion to N terms may be computed by means of the recurrence formula

$$a_m = \lim_{x \to x_0} \frac{\phi(x) - \sum_{n=1}^{m-1} a_n \phi_n(x)}{\phi_m(x)}, \quad m = 1, \ldots, N.$$

If $\sum_n a_n \phi_n(x)$ is an asymptotic expansion to N terms of ϕ, then the same formal series will also provide an asymptotic expansion to any lesser number of terms of the same function. We also have the somewhat sharper result

$$\phi(x) = \sum_{n=1}^{M} a_n \phi_n(x) + O(\phi_{M+1}) \quad \text{as } x \to x_0, \ M = 1, \ldots, N-1.$$

With x_0 and R fixed, the asymptotic expansion to a given number of terms of a given function is *unique* if the asymptotic sequence is known. On the other hand, one and the same function may have asymptotic expansions involving two different asymptotic sequences, and the two sequences are not necessarily equivalent. For instance,

$$\frac{1}{1+x} \approx \sum_{n=1}^{\infty}(-1)^{n-1}x^{-n} \quad \text{and} \quad \frac{1}{1+x} \approx \sum_{n=1}^{\infty}(x-1)x^{-2n} \quad \text{as } x \to \infty.$$

In this example both asymptotic expansions are convergent series when $|x| > 1$. It often happens that some asymptotic expansions of a function diverge while others converge. The transformation of divergent asymptotic expansions into converges ones is of great analytical, although of very little computational interest.

An asymptotic expansion does not determine its "sum" uniquely. For instance, the functions $(1+x)^{-1}$, $(1+e^{-x})/(1+x)$, $(1+e^{-\sqrt{x}}+x)^{-1}$, all possess the asymptotic expansion $\sum_{n=0}^{\infty}(-1)^n x^{-n-1}$ as $x \to \infty$. A given (finite or infinite) asymptotic sequence $\{\phi_n\}$, for $x \to x_0$ in R establishes an *equivalence relation* among functions defined in R, that is, ϕ and ψ are *asymptotically equal* with respect to $\{\phi_n\}$ if $\phi(x) - \psi(x) = o(\phi_n)$ as $x \to x_0$ in R, for all n occurring in the sequence. An asymptotic series represents a class of asymptotically equal functions rather than a single function.

The theory of asymptotic series is very large, many operations like sum, multiplication, integration, differentiation, and reversion, can be defined. We refer the reader to the book [21] for more details.

Chapter 2

On Toeplitz operators and matrices

Toeplitz operators or equivalent, infinite Toeplitz matrices are operators on ℓ^p. When studying large finite Toeplitz matrices, it is natural look also at their infinite counterparts. The spectral phenomena of the latter are sometimes easier to understand than those of the former. The question whether properties of infinite Toeplitz matrices mimic the corresponding properties of their large finite sections is very delicate and is, in a sense, the topic of this chapter.

2.1 Toeplitz operators with continuous symbols

In this section we will see that the problem of determining when a Toeplitz operator is invertible has been replaced by that of determining when it is a Fredholm operator and what is its index. If the symbol of the operator is continuous, then it is readily done. The result is due to a number of authors including Gohberg, Widom, and Simonenko.

2.1.1 Infinite Toeplitz matrices as bounded operators

Let $f(t) = \sum_{k=-\infty}^{\infty} f_k t^k$ be an element of $L^2 := L^2(\mathbb{T})$. Consider the projection operator $P : L^2 \to L^2$ given by

$$P\left(\sum_{k=-\infty}^{\infty} f_k t^k\right) := \sum_{k=0}^{\infty} f_k t^k,$$

the space $P(L^2)$ is referred to as the Hardy space H^2.

By Theorem 1.1.1, for each $a \in L^\infty := L^\infty(\mathbb{T})$ it follows that a is also an element of L^2, thus a has convergent Fourier expansions like $\sum_{k=-\infty}^{\infty} a_k t^k$. Consider the multiplication operator

$$M_a : \begin{array}{ccc} L^2 & \to & L^2 \\ f & \mapsto & af \end{array}$$ and the *Toeplitz operator* PM_aP given by

$$PM_aP(f) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} f_k a_{n-k}\right) t^n. \tag{2.1.1}$$

It is possible to consider the operator (2.1.1) as an operator in the space $\ell^2 := \ell^2(\mathbb{Z}_+)$ as follows: The set $\{e_n\}_{n=0}^{\infty}$ with $e_n := t^n$ for $t \in \mathbb{T}$, is an orthonormal basis for the closed subspace H^2 of L^2 under the usual inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta. \tag{2.1.2}$$

With the norm induced by (2.1.2) we obtain

$$\|f\|_2^2 = \langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |f_k|^2,$$

where the last expression is the norm of $(f_0, f_1, f_2, \dots)$ in the space ℓ^2. Thus, for each element of H^2 we can associate the infinite vector of its Fourier coefficients with an element in ℓ^2 in a natural way. Hence, the operator (2.1.1) act over ℓ^2 with the following matrix representation

$$T(a) := \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1.3}$$

The matrix (2.1.3) is known as a *Toeplitz matrix* and, clearly, it is characterized by the property of being constant along the parallels to the main diagonal. The matrix (2.1.3) is completely determined by its entries in the first row and first column, that is, by the sequence

$$\{a_k\}_{k=-\infty}^{\infty} = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}.$$

More precisely, in the bounded case, for $x, y \in \ell^2$ the operator (2.1.1) acts by the rule

$$y = T(a)x \quad \text{with} \quad \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

It is also possible to get the adjoint of the operator (2.1.1) defining

$$T^*(a) := \begin{pmatrix} \overline{a_0} & \overline{a_1} & \overline{a_2} & \cdots \\ \overline{a_{-1}} & \overline{a_0} & \overline{a_1} & \cdots \\ \overline{a_{-2}} & \overline{a_{-1}} & \overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that is, $T^*(a)$ is the conjugate transpose of $T(a)$. Instead of the projection P we can consider a finite projection like $P_n : L^2 \to L^2,\ n \in \mathbb{N}$, given by

$$P_n\left(\sum_{k=-\infty}^{\infty} f_k t^k\right) := \sum_{k=0}^{n-1} f_k t^k.$$

Similarly, the Toeplitz operator $P_n M_a P_n|_{\mathrm{Im}\, P_n}$ acts over $\ell_n^2 := \ell_n^2(\mathbb{Z}_+) = \{\{x_k\}_{k=0}^{\infty} : x_k = 0 \text{ for } k \geq n\}$ and have the following matrix representation

$$T_n(a) := \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & a_0 & \cdots & a_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}. \tag{2.1.4}$$

Naturally $T_n^*(a)$ is the conjugate transpose of $T_n(a)$. A natural question emerges here, given a sequence $\{a_n\}_{n=-\infty}^{\infty}$ of complex numbers, when does the matrix (2.1.3) induce a bounded operator on ℓ^2?. The answer is a classical result by Otto Toeplitz (see [42]).

Theorem 2.1.1 (Toeplitz 1911)**.** *The matrix* (2.1.3) *defines a bounded operator on ℓ^2 if and only if the numbers a_n are the Fourier Coefficients of some function $a \in L^\infty$,*

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}d\theta, \quad n \in \mathbb{Z}. \tag{2.1.5}$$

Moreover, the norm of the operator (2.1.3) *satisfies* $\|T(a)\| = \|a\|_\infty := \operatorname{ess}\sup_{t\in\mathbb{T}} |a(t)|$.

Proof. The multiplication operator $M_a : \begin{array}{ccc} L^2 & \to & L^2 \\ f & \mapsto & af \end{array}$ is bounded if and only if $a \in L^\infty$, in which case $\|M_a\| = \|a\|_\infty$, (see [28, p. 44]).

An orthonormal basis of L^2 under the inner product (2.1.2) is given by $\{e_n\}_{n=-\infty}^{\infty}$, where $e_n(t) := t^n$ for $t \in \mathbb{T}$. The matrix representation of M_a with respect to the basis $\{e_n\}$ is easily seen to be

$$L(a) := \left(\begin{array}{ccc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \hline \cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right), \tag{2.1.6}$$

where the numbers a_n are defined by (2.1.5). Thus, we arrive at the conclusion that $L(a)$ defines a bounded operator on $\ell^2(\mathbb{Z})$ if and only if $a \in L^\infty$ and $\|L(a)\| = \|a\|_\infty$ in this case.

The matrix (2.1.3) is the lower right quarter of (2.1.6), that is, we may think of $T(a)$ as the compression of $L(a)$ to the space ℓ^2. This implies that if $a \in L^\infty$, then

$$\|T(a)\| \leq \|L(a)\| = \|a\|_\infty. \tag{2.1.7}$$

For a natural number n, let S_n be the projection on $\ell^2(\mathbb{Z})$ given by

$$S_n : \{x_k\}_{k=-\infty}^{\infty} \mapsto \{\ldots, 0, 0, x_{-n}, \ldots, x_{-1}, x_0, x_1, x_2\}.$$

The matrix representation of the operator $S_n L(a) S_n|_{\mathrm{Im}\, S_n}$ results from (2.1.6) by deleting all rows and columns indexed by a number in $\{\ldots, -(n+2), -(n+1)\}$. Hence, the matrix representation of $S_n L(a) S_n|_{\mathrm{Im}\, S_n}$ is just (2.1.3). This shows that

$$\|T(a)\| = \|S_n L(a) S_n\|. \tag{2.1.8}$$

The operator S_n converges strongly to the identity operator on $\ell^2(\mathbb{Z})$, thus $S_nL(a)S_n$ converges strongly to $L(a)$, whence

$$\|L(a)\| \leq \liminf_{n\to\infty} \|S_nL(a)S_n\|. \tag{2.1.9}$$

From (2.1.8) and (2.1.9) we see that $L(a)$ and thus M_a must be bounded whenever $T(a)$ is bounded and that

$$\|L(a)\| \leq \|T(a)\|. \tag{2.1.10}$$

Consequently, $T(a)$ is bounded if and only if $a \in L^\infty$, in which case (2.1.7) and (2.1.10) give the equality $\|T(a)\| = \|a\|_\infty$. □

Clearly, if there is a function $a \in L^\infty$ satisfying (2.1.5), then this function (the equivalence class of L^∞ containing it) is unique. Therefore, we denote both the matrix (2.1.3) and the operator it induces on ℓ^2 by $T(a)$ and similarly, we denote both the matrix (2.1.4) and the operator it induces on ℓ_n^2 by $T_n(a)$. The function a is in this context referred to as the *symbol of the Toeplitz matrix/operator* $T(a)$ or *symbol of the Toeplitz matrix/operator* $T_n(a)$, respectively.

Theorem 2.1.2. *We have*

$$\limsup_{n\to\infty} \|T_n(a)\| = \|T(a)\| = \|a\|_\infty.$$

Proof. Observe that $T_n(a) = P_nT(a)P_n$ implies that $\|T_n(a)\| \leq \|P_n\| \cdot \|T(a)\| \cdot \|P_n\| = \|T(a)\|$, we thus get

$$\limsup_{n\to\infty} \|T_n(a)\| \leq \|T(a)\|.$$

Let ε be a small positive constant, then there is a unit vector $x_\varepsilon \in \ell^2$ such that $\|T(a)x_\varepsilon\|_2 > \|T(a)\| - \varepsilon$. Note that

$$T_n(a)P_nx_\varepsilon = P_nT(a)P_nx_\varepsilon \underset{n}{\to} T(a)x_\varepsilon.$$

Thus, there is a natural number $N = N(\varepsilon)$ such that for every $n > N$ we obtain $\|T_n(a)P_nx\varepsilon\|_2 > \|T(a)\| - \varepsilon$. That is, $\|T_n(a)\| \geq \|T_n(a)P_nx\varepsilon\|_2 > \|T(a)\| - \varepsilon$, getting the remaining inequality

$$\limsup_{n\to\infty} \|T_n(a)\| \geq \|T(a)\|.$$

Finally, the equality $\|T(a)\| = \|a\|_\infty$ comes from Theorem 2.1.1. □

2.1.2 Compactness and self-adjointness

In this section we cite two simple results that are illustrative and useful.

Theorem 2.1.3 (Gohberg 1952)**.** *The only compact Toeplitz operator with essentially bounded symbol is the zero operator.*

Proof. Let $a \in L^\infty$ and suppose $T(a)$ is compact. Let Q_n be the projection

$$Q_n : \ell^2 \to \ell^2, \quad (x_0, x_1, x_2, \dots) \mapsto (0, \dots, 0, x_n, x_{n+1}, \dots).$$

Given that $Q_n \to 0$ strongly and $T(a)$ is compact, it follows that $\|Q_nT(a)Q_n\|$ converges to 0. The compression $Q_nT(a)Q_n|_{\mathrm{Im}\,Q_n}$ has the same matrix as $T(a)$ whence $\|T(a)\| = \|Q_nT(a)Q_n\|$. Consequently, $\|T(a)\| = 0$. □

Because of $T(a) - \lambda\mathcal{I} = T(a-\lambda)$ for every $\lambda \in \mathbb{C}$, we learn from Theorem 2.1.3 that $T(a)$ is never of the form $\lambda\mathcal{I}$ plus a compact operator, unless $T(a) = \lambda\mathcal{I}$.

Theorem 2.1.4. *The Toeplitz operator $T(a)$ is self-adjoint if and only if a is real-valued.*

Proof. We know that $T(a) = T^*(a)$ if and only if $a_n = \overline{a_{-n}}$ for all n, which happens if and only if $a(t) = \overline{a(t)}$ for each $t \in \mathbb{T}$. □

2.1.3 Spectrum

We are mainly concerned with Toeplitz operators with continuous symbols. Let $C := C(\mathbb{T})$ be the set of all complex-valued continuous functions on the unit circle $\mathbb{T}$. It is well known that C is a C^*-subalgebra of L^∞. We give $\mathbb{T}$ the counterclockwise orientation. For a function $a \in C$, the image $\mathcal{R}(a)$ is a closed continuous and naturally oriented curve in the complex plane. If a point $\lambda \in \mathbb{C}$ is not located on $\mathcal{R}(a)$, we denoted by $\operatorname{wind}_\lambda(a)$ the winding number of the curve $\mathcal{R}(a)$ about λ. To describe the spectrum of $T(a)$ we need two classical results from Gohberg, (see [22, p. 165-174]).

Theorem 2.1.5 (Gohberg 1952)**.** *Let $a \in C$. The operator $T(a)$ is Fredholm if and only if $0 \notin \mathcal{R}(a)$. In this case* $\operatorname{Ind} T(a) = -\operatorname{wind}_0(a)$.

The proof of this theorem is based on two auxiliary results. For $a \in L^\infty$ and $t \in \mathbb{T}$, we define the function $\tilde{a}(t) := a\left(\frac{1}{t}\right)$. In terms of Fourier series

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n \mapsto \tilde{a}(t) = \sum_{n=-\infty}^{\infty} a_{-n} t^n.$$

Clearly,

$$T(a) := \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad T(\tilde{a}) := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_{-1} & a_0 & a_1 & \cdots \\ a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, $T(\tilde{a})$ is the transpose of $T(a)$. The *Hankel operator* $H(a)$ generated by a is given by the matrix $(a_{j+k+1})_{j,k=0}^\infty$ and similarly, $\tilde{a}$ generates the Hankel operator $H(\tilde{a})$ given by the matrix $(a_{-j-k-1})_{j,k=0}^\infty$, that is,

$$H(a) := \begin{pmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H(\tilde{a}) := \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Because $H(a)$ may be identified with the matrix in the lower left quarter of (2.1.6), we see that if $a \in L^\infty$, then $H(a)$ induces a bounded operator on ℓ^2 and

$$\|H(a)\| \leq \|a\|_\infty. \tag{2.1.11}$$

Since $\|a\|_\infty = \|\tilde{a}\|_\infty$, we also have

$$\|H(\tilde{a})\| \leq \|a\|_\infty. \tag{2.1.12}$$

Theorem 2.1.6. *If $a, b \in L^\infty$, then*

$$T(a)T(b) = T(ab) - H(a)H(\tilde{b}). \tag{2.1.13}$$

Proof. The (l, j) entry in the right side of (2.1.13) is

$$\begin{aligned}\sum_{k=-\infty}^{\infty} a_k b_{l-j-k} - \sum_{s=0}^{\infty} a_{l+s} b_{-s-j} &= \sum_{k=-\infty}^{\infty} a_k b_{l-j-k} - \sum_{k=l}^{\infty} a_k b_{l-j-k} \\ &= \sum_{k=-\infty}^{l-1} a_k b_{l-j-k} = \sum_{s=0}^{\infty} a_{l-1-s} b_{s-j+1},\end{aligned}$$

which is the same entry in the left side. □

Theorem 2.1.7. *If $c \in C$, then $H(c)$ and $H(\tilde{c})$ are compact operators on ℓ^2.*

Proof. Let $\{f_n\}_n$ be a sequence of trigonometric polynomials such that the norm $\|c - f_n\|_\infty \to 0$ as $n \to \infty$, (for example, let f_n be the nth Fejér-Césaro mean of c, [47, p. 94]). From (2.1.11) and (2.1.12) we deduce that

$$\begin{aligned}\|H(c) - H(f_n)\| \le& \|H(c - f_n)\| \le \|c - f_n\|_\infty = o(1), \\ \|H(\tilde{c}) - H(\tilde{f}_n)\| \le& \|H(\tilde{c} - \tilde{f}_n)\| \le \|c - f_n\|_\infty = o(1),\end{aligned}$$

and as $H(f_n)$ and $H(\tilde{f}_n)$ are finite-rank operators, it follows that $H(c)$ and $H(\tilde{c})$ are compact operators. □

Proof of the Theorem 2.1.5. Consider the map

$$\varphi : C \to \mathcal{B}(\ell^2)/\mathcal{K}(\ell^2), \quad a \mapsto T(a) + \mathcal{K}(\ell^2).$$

(The quotient algebra $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$ is widely known as the Calkin algebra of ℓ^2). This map is obviously linear, we have

$$\varphi(a)^* = (T(a) + \mathcal{K}(\ell^2))^* = T(\overline{a}) + \mathcal{K}(\ell^2) = \varphi(\overline{a}),$$

and Theorems 2.1.6 and 2.1.7 imply that

$$\varphi(a)\varphi(b) =(T(a) + \mathcal{K}(\ell^2))(T(b) + \mathcal{K}(\ell^2)) = T(ab) + \mathcal{K}(\ell^2) = \varphi(ab).$$

Thus, φ is a $*$-homomorphism. From Theorem 2.1.3 we know that φ is injective. Consequently, by Theorem 1.2.3 φ is a $*$-isomorphism of C onto the C^*-subalgebra $\varphi(C)$ of $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$. From Theorems 1.2.1 and 1.2.3 we deduce that $T(a)$ is Fredholm if and only if a is invertible in C, that is, if and only if $0 \notin \mathcal{R}(a)$. Theorem 1.2.3 also gives the equality

$$\|T(a)\|_{ess} = \|a\|_\infty. \tag{2.1.14}$$

The index formula follows from a simple homotopy argument. Let $\Phi(\ell^2)$ be the set of Fredholm operators on ℓ^2 and let GC be the set of all $a \in C$ for which $0 \notin \mathcal{R}(a)$. If $a \in GC$ and $\mathrm{wind}_0(a) = n$, then there is a continuous function

$$[0,1] \to GC, \quad \mu \mapsto a_\mu,$$

such that $a_0 = a$ and $a_1 = \chi_n$ (recall Example 1.3.1). The function

$$[0,1] \to \Phi(\ell^2), \quad \mu \mapsto T(a_\mu),$$

is also continuous, and Theorem 1.3.1 shows that the map

$$[0,1] \to \mathbb{Z}, \quad \mu \mapsto \operatorname{Ind} T(a_\mu),$$

is continuous and locally constant, that is, constant. This implies that

$$\operatorname{Ind} T(a) = \operatorname{Ind} T(a_0) = \operatorname{Ind} T(a_1) = \operatorname{Ind} T(\chi_n).$$

Finally, Example 1.3.1 tell us that $\operatorname{Ind} T(\chi_n) = -n$.

Theorem 2.1.8 (Gohberg 1952)**.** *Let $a \in C$. The operator $T(a)$ is invertible if and only if it is Fredholm of index zero.*

Proof. If $T(a)$ is invertible in $\mathcal{B}(\ell^2)$, then it will be obviously invertible in the Calkin algebra $\mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$, by which $T(a)$ is Fredholm. Moreover, $\ker T(a) = \{0\}$ and $\operatorname{Im} T(a) = \ell^2$, and therefore $\operatorname{coker} T(a) = \{0\}$ and $\operatorname{Ind} T(a) = \dim\ker T(a) - \dim\operatorname{coker} T(a) = 0$.

On the other hand, suppose $T(a)$ is Fredholm of index zero and, contrary to what we want, let us assume that $T(a)$ is not invertible. Then,

$$\dim\ker T(a) = \dim\operatorname{coker} T(a) > 0,$$

and since

$$\dim\operatorname{coker} T(a) = \dim\ker T^*(a) = \dim\ker T(\overline{a}),$$

there are non-zero $x_+, y_+ \in \ell^2$ such that $T(a)x_+ = 0$ and $T(\overline{a})y_+ = 0$. Extend x_+ and y_+ by zero to all $\ell^2(\mathbb{Z})$ and let $L(a)$ be the operator (2.1.6). Then,

$$L(a)x_+ = x_-, \text{ where } x_- \in \ell^2(\mathbb{Z}) \text{ and } (x_-)_n = 0 \text{ for } n \geq 0,$$

$$L(\overline{a})y_+ = y_-, \text{ where } y_- \in \ell^2(\mathbb{Z}) \text{ and } (y_-)_n = 0 \text{ for } n \geq 0.$$

The convolution $u * v$ of two sequences $u, v \in \ell^2(\mathbb{Z})$ is the sequence $\{(u*v)_n\}_{n=-\infty}^{\infty}$ given by

$$(u*v)_n = \sum_{n=-\infty}^{\infty} u_k v_{n-k}.$$

Note that $u * v$ is a well defined sequence in $\ell^\infty(\mathbb{Z})$, because $|(u*v)_n| \leq \|u\|_2\|v\|_2 < \infty$, where $\|\cdot\|_2$ denotes the norm in $\ell^2(\mathbb{Z})$. Let $b \in \ell^2(\mathbb{Z})$ be the sequence of the Fourier coefficients of $a \in C \subset L^2$. Given a sequence $f = \{f_n\}_{n=-\infty}^{\infty}$, we define the sequence $(f^\sharp)_n := \overline{f_{-n}}$. It is easy to see that $(u*v)^\sharp = u^\sharp * v^\sharp$ for $u, v \in \ell^2(\mathbb{Z})$. We have

$$L(a)x_+ = b * x_+ = x_- \text{ and } L(\overline{a})y_+ = b^\sharp * y_+ = y_-.$$

Hence,

$$y_-^\sharp * x_+ = (b^\sharp * y_+)^\sharp * x_+ = (b * y_+^\sharp) * x_+ = (y_+^\sharp * b) * x_+. \tag{2.1.15}$$

We claim that

$$(y_+^\sharp * b) * x_+ = y_+^\sharp * (b * x_+). \tag{2.1.16}$$

This is easily verified if y_+ and x_+ have finite supports. Because

$$|((y_+^\sharp * b) * x_+)_n| \leq \|y_+^\sharp * b\|_2 \|x_+\|_2 \leq \|y_+\|_2 \|a\|_\infty \|x_+\|_2,$$

$$|(y_+^\sharp * (b * x_+))_n| \leq \|y_+^\sharp\|_2 \|b * x_+\|_2 \leq \|y_+\|_2 \|a\|_\infty \|x_+\|_2,$$

it follows that (2.1.16) is true for arbitrary $y_+, x_+ \in \ell^2(\mathbb{Z})$. From (2.1.15) and (2.1.16) we obtain

$$y_-^\sharp * x_+ = y_+^\sharp * (b * x_+) = y_+^\sharp * x_-. \tag{2.1.17}$$

Given that $(y_-^\sharp * x_+)_n = 0$ for $n \leq 0$, we see that (2.1.17) is the zero sequence. In particular $(y_+^\sharp * x_-)_n = 0$ for all $n \geq 0$, which means that

$$\begin{aligned}
&\overline{(y_+)_0}(x_-)_{-1} = 0, \\
&\overline{(y_+)_0}(x_-)_{-2} + \overline{(y_+)_1}(x_-)_{-1} = 0, \\
&\overline{(y_+)_0}(x_-)_{-3} + \overline{(y_+)_1}(x_-)_{-2} + \overline{(y_+)_2}(x_-)_{-1} = 0, \\
&\cdots \quad \cdots \quad \cdots
\end{aligned}$$

As $y_+ \neq 0$, it results that $(x_-)_{-1} = (x_-)_{-2} = (x_-)_{-3} = \cdots = 0$. Hence, $x_- = 0$. This implies that $L(a)x_+ = 0$. The Fredholmness of $T(a)$ in conjunction with Theorem 2.1.5 shows that a has no zeros on $\mathbb{T}$. Consequently, $a^{-1} \in L^\infty$ and as $L(a^{-1})$ and $L(a)$ are unitarily equivalent to $M_{a^{-1}}$ and M_a, respectively; we obtain that $L(a^{-1})$ is the inverse of $L(a)$. It follows that $x_+ = 0$, which is a contradiction. □

To end this section we summarize all we have seen here in the following theorem.

Theorem 2.1.9. *If $a \in C$, then*

$$sp_{ess}T(a) = \mathcal{R}(a) \quad \text{and} \quad sp\,T(a) = \mathcal{R}(a) \cup \{\lambda \in \mathbb{C} : \text{wind}_\lambda(a) \neq 0\}.$$

Proof. Using that $T(a) - \lambda\mathcal{I} = T(a - \lambda)$ and Theorem 2.1.5 and 2.1.8 we can see that

$$\begin{aligned}
sp_{ess}T(a) =& \{\lambda \in \mathbb{C} : T(a) - \lambda\mathcal{I} \text{ is not Fredholm}\} \\
=& \{\lambda \in \mathbb{C} : 0 \in \mathcal{R}(a - \lambda)\} \\
=& \{\lambda \in \mathbb{C} : \lambda = a(t) \text{ for some } t \in \mathbb{T}\} \\
=& \mathcal{R}(a),
\end{aligned}$$

and

$$\begin{aligned}
sp\,T(a) =& \{\lambda \in \mathbb{C} : T(a) - \lambda\mathcal{I} \text{ is not invertible}\} \\
=& \{\lambda \in \mathbb{C} : T(a - \lambda) \text{ is not Fredholm}\} \cup \{\lambda \in \mathbb{C} : \text{Ind}(T(a - \lambda)) \neq 0\} \\
=& \mathcal{R}(a) \cup \{\lambda \in \mathbb{C} : \text{wind}_0(a - \lambda) \neq 0\} \\
=& \mathcal{R}(a) \cup \{\lambda \in \mathbb{C} : \text{wind}_\lambda(a) \neq 0\}.
\end{aligned}$$

□

2.2 Stability

In this section, we study when and how, the Toeplitz matrices $T_n(a)$ mimic the behavior of the operator $T(a)$. We know that $T(a)$ is invertible if and only if $0 \notin \mathcal{R}(a)$ and $\text{wind}_0(a) = 0$, in the case $a \in C$. Whence the finite sections of an invertible Toeplitz operator $T(a)$, can be used to approximate the solution of the equation $T(a)x = y$ in ℓ^2 and for every sufficiently large n, all this finite sections $T_n(a)$ will be invertible as well.

2.2.1 Finite section method

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of $n \times n$ matrices. This sequence is said to be *stable* if there is a natural number n_0 such that the matrices A_n are invertible for all $n \geq n_0$ and $\sup\limits_{n \geq n_0} \|A_n^{-1}\| < \infty$. Using the convention to put $\|A^{-1}\| = \infty$ if A is not invertible, we can say that $\{A_n\}_{n=1}^{\infty}$ is a stable sequence if and only if $\limsup\limits_{n\to\infty} \|A_n^{-1}\| < \infty$.

Let $A \in \mathcal{B}(\ell^2)$ be a given operator and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of $n \times n$ matrices. In order to solve the equation

$$Ax = y, \tag{2.2.1}$$

we can have recourse to the finite systems

$$A_n x^{(n)} = P_n y, \quad x^{(n)} \in \operatorname{Im} P_n, \tag{2.2.2}$$

where here and throughout what follows, P_n is the same projection of Subsection 2.1.1, but this time acting over ℓ^2, i.e.,

$$P_n : \ell^2 \to \ell^2, \quad (x_0, x_1, x_2, \dots,) \mapsto (x_0, x_1, \dots, x_{n-1}, 0, 0, \dots). \tag{2.2.3}$$

The image of P_n is a subspace of ℓ^2, but we freely identify it with $\mathbb{C}^n$. This fact allows us to think of A_n and A_n^{-1} as operators on ℓ^2. We can make the identifications $A_n = A_n P_n$ and $A_n^{-1} = A_n^{-1} P_n$.

Suppose A is invertible. We say that the *method* $\{A_n\}$ *is applicable to* A if there is a natural number n_0 such that the equation (2.2.2) is uniquely solvable for every $y \in \ell^2$ and every $n \geq n_0$, and if their solutions $x^{(n)}$ converge in ℓ^2 to the solution x of (2.2.1) for every $y \in \ell^2$. Equivalently, the method $\{A_n\}$ is applicable to A if and only if the matrices A_n are invertible for all sufficiently large n and $A_n^{-1} \to A^{-1}$ strongly. In the case where $A_n = P_n A P_n|_{\operatorname{Im} P_n}$, we speak of the *finite section method.*

Theorem 2.2.1. *Let $A \in \mathcal{B}(\ell^2)$ be invertible and suppose $\{A_n\}$ is a sequence of $n \times n$ matrices such that $A_n \to A$ strongly. Then, the method $\{A_n\}$ is applicable to A if and only if the sequence $\{A_n\}$ is stable.*

Proof. If $A_n^{-1} \to A^{-1}$ strongly, then $\limsup\limits_{n\to\infty} \|A_n^{-1}\| < \infty$ due to the *principle of uniform boundedness* (see [10, p. 41]). Hence, $\{A_n\}$ is stable. Conversely, suppose $\{A_n\}$ is stable, then for each $y \in \ell^2$,

$$\|A_n^{-1} P_n y - A^{-1} y\| \leq \|A_n^{-1} P_n y - P_n A^{-1} y\| + \|P_n A^{-1} y - A^{-1} y\|,$$

the second term on the right goes to zero because $P_n \to \mathcal{I}$ strongly, and the first term on the right is

$$\|A_n^{-1}(P_n y - A_n P_n A^{-1}) y\| \leq M \|P_n y - A_n P_n A^{-1} y\| = o(1),$$

since $A_n P_n A^{-1} \to A A^{-1} = \mathcal{I}$ strongly. $\square$

Theorem 2.2.2. *Let $\{A_n\}$ be a sequence of $n\times n$ matrices and suppose there is an operator $A \in \mathcal{B}(\ell^2)$ such that $A_n \to A$ and $A_n^* \to A^*$ strongly. If $\{A_n\}$ is stable, then A is necessarily invertible and*

$$\|A^{-1}\| \le \liminf_{n\to\infty} \|A_n^{-1}\|.$$

Proof. Suppose $\|A_n^{-1}\| \le M$ for infinitely many n. For $x \in \ell^2$ and these n,

$$\|P_n x\| = \|A_n^{-1} A_n P_n x\| \le M\|A_n P_n x\|, \quad \|P_n x\| = \|(A_n^*)^{-1} A_n^* P_n x\| \le M\|A_n^* P_n x\|.$$

Passing to the limit $n \to \infty$, we obtain $\|x\| \le M\|Ax\|$, $\|x\| \le M\|A^* x\|$, which implies that A is invertible and $\|A^{-1}\| \le M$. □

2.2.2 Perturbed Toeplitz matrices

For $a \in L^\infty$, let $T_n(a)$ be the matrix in (2.1.4). Obviously

$$T_n(a) \to T(a), \quad T_n^*(a) = T_n(\overline{a}) \to T(\overline{a}) = T^*(a),$$

strongly. In particular, Theorem 2.2.2 tells us that the finite section method $\{T_n(a)\}$ is applicable to an invertible Toeplitz operator $T(a)$ if and only if $\{T_n(a)\}$ is stable.

Instead of the pure Toeplitz matrices $\{T_n(a)\}$, we consider more general matrices, namely, matrices of the form

$$A_n = T_n(a) + P_n K P_n + W_n L W_n + C_n, \tag{2.2.4}$$

where $a \in L^\infty$, $K \in \mathcal{K}(\ell^2)$, $L \in \mathcal{K}(\ell^2)$, $\{C_n\}$ is a sequence of $n \times n$ matrices such that $\|C_n\| \to 0$, P_n is given by (2.2.3), and W_n is defined by

$$W_n : \ell^2 \to \ell^2, \quad (x_0, x_1, x_2, \dots) \mapsto (x_{n-1}, x_{n-2}, \dots, x_0, 0, 0, \dots).$$

Once again, we freely identify $\operatorname{Im} W_n \subset \ell^2$ and $\mathbb{C}^n$, and frequently we think of W_n as being the matrix

$$W_n := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $W_n \cdot W_n$ is the $n \times n$ identity matrix and $W_n^2 = P_n$. Consideration of matrices of the form (2.2.4) is motivated by the following result, which is the $n \times n$ analogue of Theorem 2.1.6.

Theorem 2.2.3 (Widom 1976)**.** *If $a, b \in L^\infty$, then*

$$T_n(a)T_n(b) = T_n(ab) - P_n H(a) H(\tilde{b}) P_n - W_n H(\tilde{a}) H(b) W_n. \tag{2.2.5}$$

Proof. The (l, j) entry in the right side of (2.2.5) is

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} a_k b_{l-j-k} - \sum_{k=l}^{\infty} a_k b_{l-j-k} - \sum_{k=n-l+1}^{\infty} a_{-k} b_{l-j+k} \\ &= \sum_{k=-\infty}^{l-1} a_k b_{l-j-k} - \sum_{k=-\infty}^{l-n-1} a_k b_{l-j-k} = \sum_{k=l-n}^{l-1} a_k b_{l-j-k} \\ &= \sum_{s=0}^{n-1} a_{l-1-s} b_{s-j+1}, \end{aligned}$$

which is just the same entry in the left side. □

When $a, b \in C$, Theorems 2.2.3 and 2.1.7 imply that

$$T_n(a)T_n(b) = T_n(ab) - P_nKP_n - W_nLW_n,$$

where $K, L \in \mathcal{K}(\ell^2)$.

2.2.3 Algebraization of stability

In this section we will see that the sequence $\{A_n\}$ defined by matrices of the form (2.2.4) with $a \in C$ constitute a C^*-algebra.

Let $\mathcal{S}$ be the set of all sequences $\{A_n\} := \{A_n\}_{n=1}^{\infty}$ of $n \times n$ matrices A_n such that $\sup\limits_{n\geq 1} \|A_n\| < \infty$, and let $\mathcal{N}$ be the set of all sequences $\{A_n\}$ in $\mathcal{S}$ for which $\lim\limits_{n\to\infty} \|A_n\| = 0$. It is easily see that $\mathcal{S}$ is a C^*-algebra with the operations

$$\begin{aligned} \lambda\{A_n\} &:= \{\lambda A_n\}, \\ \{A_n\} + \{B_n\} &:= \{A_n + B_n\}, \\ \{A_n\}\{B_n\} &:= \{A_nB_n\}, \\ \{A_n\}^* &:= \{A_n^*\}, \end{aligned}$$

and the norm $\|\{A_n\}\| := \sup\limits_{n\geq 1} \|A_n\|$, and that $\mathcal{N}$ is a closed ideal of $\mathcal{S}$. Thus, by Theorem 1.2.2, the quotient algebra $\mathcal{S}/\mathcal{N}$ is also C^*-algebra. For $\{A_n\} \in \mathcal{S}$, we abbreviate the coset $\{A_n\} + \mathcal{N}$ by $\{A_n\}^\nu$. Note that,

$$\|\{A_n\}^\nu\| = \inf_{\{B_n\}\in\mathcal{N}} \|\{A_n + B_n\}\| = \inf_{\{B_n\}\in\mathcal{N}} \sup_{n\geq 1} \|\{A_n + B_n\}\| = \limsup_{n\to\infty} \|A_n\|.$$

Theorem 2.2.4. *A sequence $\{A_n\} \in \mathcal{S}$ is stable if and only if $\{A_n\}^\nu$ is invertible in $\mathcal{S}/\mathcal{N}$.*

Proof. If $\{A_n\}$ is stable, then the matrices A_n are invertible from some natural number n_0, thus there is a sequence $\{B_n\} \in \mathcal{S}$ such that

$$B_nA_n = P_n + C_n', \quad A_nB_n = P_n + C_n'', \tag{2.2.6}$$

where $C_n' = C_n'' = 0$ for all $n \geq n_0$. This implies that $\{B_n\}^\nu$ is the inverse of $\{A_n\}^\nu$. On the other hand, if $\{A_n\}^\nu$ has the inverse $\{B_n\}^\nu$ in $\mathcal{S}/\mathcal{N}$, then (2.2.6) holds with certain $\{C_n'\} \in \mathcal{N}$ and $\{C_n''\} \in \mathcal{N}$. Clearly, $\|C_n'\| \leq 1/2$ for all sufficiently large n. For these n, the matrix $(P_n + C_n')|_{\operatorname{Im} P_n} = I + C_n'$ is invertible, whence

$$\|A_n^{-1}\| = \|(I + C_n')^{-1}B_n\| \leq 2\|B_n\|,$$

which shows that $\{A_n\}$ is stable. ☐

The C^*-algebra $\mathcal{S}/\mathcal{N}$ is very large and therefore difficult to understand. In order to study Toeplitz operators with continuous symbols, we can restrict ourselves to a much smaller algebra. We define $\mathcal{S}(C)$ as the subset of $\mathcal{S}$ which consists of all elements $\{A_n\}$ with

$$A_n = T_n(a) + P_nKP_n + W_nLW_n + C_n,$$

where $a \in C$; $K, L \in \mathcal{K}(\ell^2)$, $\{C_n\} \in \mathcal{N}$, and we let $\mathcal{S}(C)/\mathcal{N}$ stands for the subset of $\mathcal{S}/\mathcal{N}$ consisting of the coset $\{A_n\}^\nu$ with $\{A_n\} \in \mathcal{S}(C)$.

Theorem 2.2.5. *If* $\{A_n\} = \{T_n(a) + P_nKP_n + W_nLW_n + C_n\}$ *is a sequence in* $\mathcal{S}(C)$, *then*

$$A_n \underset{str}{\longrightarrow} A := T(a) + K, \tag{2.2.7}$$

$$W_nA_nW_n \underset{str}{\longrightarrow} \widetilde{A} := T(\tilde{a}) + L. \tag{2.2.8}$$

Here the subindex "str" means strong convergence.

Proof. Let $\mathbf{x} = \{x_k\}_{k=0}^{\infty}$, $\mathbf{y} = \{y_k\}_{k=0}^{\infty} \in \ell^2$ be arbitrary. Thus, for every $\varepsilon > 0$ there is a natural number n_0 such that

$$\sum_{k=n_0+1}^{\infty} |x_k|^2 < \varepsilon^2, \quad \sum_{k=n_0+1}^{\infty} |y_k|^2 < \varepsilon^2.$$

Fix a positive ε, take $\mathbf{x}_{n_0} := (x_0, x_1, \ldots, x_{n_0}, 0, 0, \ldots)$ and $\mathbf{r}_{n_0} := \mathbf{x} - \mathbf{x}_{n_0}$, thus we have $\mathbf{x} = \mathbf{x}_{n_0} + \mathbf{r}_{n_0}$ with $\|\mathbf{r}_{n_0}\|_2 < \varepsilon$. For each $n \geq 2n_0 + 1$ we obtain $\langle W_n\mathbf{x}, \mathbf{y}\rangle = \langle W_n\mathbf{x}_{n_0}, \mathbf{y}\rangle + \langle W_n\mathbf{r}_{n_0}, \mathbf{y}\rangle$ and

$$\begin{aligned}
|\langle W_n\mathbf{x}, \mathbf{y}\rangle| &\leq |\langle W_n\mathbf{x}_{n_0}, \mathbf{y}\rangle| + |\langle W_n\mathbf{r}_{n_0}, \mathbf{y}\rangle| \\
&\leq \sum_{k=n-n_0-1}^{n-1} |x_{n-k-1}\overline{y_k}| + \sum_{k=0}^{n-n_0-2} |x_{n-k-1}\overline{y_k}| \\
&\leq \left(\sum_{k=n-n_0-1}^{n-1} |x_{n-k-1}|^2\right)^{1/2} \left(\sum_{k=n-n_0-1}^{n-1} |y_k|^2\right)^{1/2} \\
&\quad + \left(\sum_{k=0}^{n-n_0-2} |x_{n-k-1}|^2\right)^{1/2} \left(\sum_{k=0}^{n-n_0-2} |y_k|^2\right)^{1/2} \\
&\leq \|\mathbf{x}_{n_0}\|_2 \cdot \varepsilon + \varepsilon \cdot \|\mathbf{y}\|_2 = \varepsilon\left(\|\mathbf{x}_{n_0}\|_2 + \|\mathbf{y}\|_2\right).
\end{aligned}$$

Thus, $W_n \to 0$ weakly. Since L is compact, it follows that $LW_n \to 0$ strongly. As $\|W_n\| = 1$, we see that $W_nLW_n \to 0$ strongly (see [9, p. 60]). This implies (2.2.7). Using the fact that $W_nT_n(a)W_n = T_n(\widetilde{a})$ we obtain

$$W_nA_nW_n = T_n(\tilde{a}) + W_nKW_n + P_nLP_n + W_nC_nW_n,$$

and since $\|W_n\| = 1$, (2.2.8) is a consequence of (2.2.7). □

Theorem 2.2.6 (Silbermann 1981)**.** *The spaces* $\mathcal{S}(C)$ *and* $\mathcal{S}(C)/\mathcal{N}$ *are* C^**-subalgebras of* $\mathcal{S}$ *and* $\mathcal{S}/\mathcal{N}$, *respectively.*

Proof. As the quotient map $\mathcal{S} \to \mathcal{S}/\mathcal{N}$ is clearly a continuous $*$-homomorphism and $\mathcal{S}(C)$ is the pre-image of $\mathcal{S}(C)/\mathcal{N}$, it suffices to show that $\mathcal{S}(C)/\mathcal{N}$ is a C^*-subalgebra of $\mathcal{S}/\mathcal{N}$.

We first show that $\mathcal{S}(C)/\mathcal{N}$ is closed. So let

$$\{A_n^i\}^{\nu} = \{T_n(a_i) + P_nK_iP_n + W_nL_iW_n\}^{\nu}, \quad i = 1, 2, \ldots,$$

be a Cauchy sequence in $\mathcal{S}(C)/\mathcal{N}$. Then, given $\varepsilon > 0$, there is an $I = I(\varepsilon) > 0$ such that $\|\{A_n^i\}^{\nu} - \{A_n^j\}^{\nu}\| < \varepsilon$ for all $i, j \geq I$. From (2.2.7) we obtain that

$$\begin{aligned}
\|T(a_i) + K_i - T(a_j) - K_j\| &\leq \liminf_{n\to\infty} \|A_n^i - A_n^j\| \leq \limsup_{n\to\infty} \|A_n^i - A_n^j\| \\
&= \|\{A_n^i\}^{\nu} - \{A_n^j\}^{\nu}\| < \varepsilon.
\end{aligned}$$

This shows that $\{T(a_i)+K_i\}_{i=1}^{\infty}$ is a Cauchy sequence. By (2.1.14) we obtain

$$\|a_i - a_j\|_\infty = \|T(a_i - a_j)\|_{ess} \le \|T(a_i) - T(a_j) + K_i - K_j\|$$

and hence $\{a_i\}_{i=1}^{\infty}$ is a Cauchy sequence. It follows that $\{a_i\}_{i=1}^{\infty}$ converges in L^∞ to some $a \in L^\infty$. Since $\{T(a_i)+K_i\}_{i=1}^{\infty}$ is a Cauchy sequence, we now see that $\{K_i\}_{i=1}^{\infty}$ is also a Cauchy sequence. Hence, there is a $K \in \mathcal{K}(\ell^2)$ such that $K_i \to K$ uniformly.

Now consider $\{W_n A_n^i W_n\}$. Since $\|W_n\| = 1$, we have

$$\|\{W_n A_n^i W_n\}^\nu - \{W_n A_n^j W_n\}^\nu\| \le \|\{A_n^i\}^\nu - \{A_n^j\}^\nu\| < \varepsilon,$$

for all $i, j \ge I$. This show that $\{W_n A_n^i W_n\}^\nu$ is a Cauchy sequence. Using (2.2.8) instead of (2.2.7), we obtain as above that $\{L_i\}_{i=1}^{\infty}$ converges to some $L \in \mathcal{K}(\ell^2)$. In summary, $\{A_n^i\}^\nu \to \{T_n(a) + P_n K P_n + W_n L W_n\}^\nu$ as $n \to \infty$, by which it is completed the proof of the closedness of $\mathcal{S}(C)/\mathcal{N}$.

It is clear that $\mathcal{S}(C)/\mathcal{N}$ is invariant under the two linear operations and the involution. It remains to show that the product of two elements of $\mathcal{S}(C)/\mathcal{N}$ is again in $\mathcal{S}(C)/\mathcal{N}$. From Theorems 2.2.3 and 2.1.7 we infer that if $a, b \in C$, then

$$\{T_n(a)\}^\nu \{T_n(b)\}^\nu \in \mathcal{S}(C)/\mathcal{N}.$$

Now let $a \in C$ and $K \in \mathcal{K}(\ell^2)$. Then,

$$\{T_n(a)\}^\nu \{P_n K P_n\}^\nu = \{P_n T(a) P_n K P_n\}^\nu = \{P_n T(a) K P_n\}^\nu - \{P_n T(a) Q_n K P_n\}^\nu,$$

where $Q_n := \mathcal{I} - P_n$. Obviously, $T(a)K \in \mathcal{K}(\ell^2)$. Since $Q_n = Q_n^* \to 0$ strongly, it follows that $Q_n K \to 0$ uniformly (see [9, p. 60]), whence $\{P_n T(a) Q_n K P_n\}^\nu \in \mathcal{N}$. Thus,

$$\{T_n(a)\}^\nu \{P_n K P_n\}^\nu = \{P_n T(a) K P_n\}^\nu \in \mathcal{S}(C)/\mathcal{N}.$$

If $a \in C$ and $L \in \mathcal{K}(\ell^2)$, we have

$$\begin{aligned}\{T_n(a)\}^\nu \{W_n L W_n\}^\nu &= \{P_n T(a) W_n L W_n\}^\nu = \{W_n W_n T(a) W_n L W_n\}^\nu \\ &= \{W_n T(\tilde{a}) P_n L W_n\}^\nu = \{W_n T(\tilde{a}) L W_n\}^\nu - \{W_n T(\tilde{a}) Q_n L W_n\}^\nu\end{aligned}$$

and it results as above that $\{W_n T(\tilde{a}) Q_n L W_n\}^\nu \in \mathcal{N}$, whence

$$\{T_n(a)\}^\nu \{W_n L W_n\}^\nu = \{W_n T(\tilde{a}) L W_n\}^\nu \in \mathcal{S}(C)/\mathcal{N}.$$

The remaining cases can be checked similarly. □

Corollary 2.2.7. *A sequence $\{A_n\} \in \mathcal{S}(C)$ is stable if and only if $\{A_n\}^\nu$ is invertible in $\mathcal{S}(C)/\mathcal{N}$.*

Proof. By Theorems 2.2.4 and 2.2.6, $\{A_n\}$ is stable if and only if $\{A_n\}^\nu$ is invertible in $\mathcal{S}/\mathcal{N}$. By Theorem 1.2.1, $\{A_n\}^\nu$ is invertible in $\mathcal{S}/\mathcal{N}$ if and only if $\{A_n\}^\nu$ is invertible in $\mathcal{S}(C)/\mathcal{N}$. □

Thus, for the sequences $\{A_n\}$ that we are interested, we have reduced the stability problem to an invertibility problem in the C^*-algebra $\mathcal{S}(C)/\mathcal{N}$.

2.2.4 Silbermann theory

We now begin with the harvest from Theorem 2.2.6 and its Corollary 2.2.7.

For $\{A_n\} \in \mathcal{S}(C)$, let A and $\tilde{A}$ be as in Theorem 2.2.5. It is clear that the maps

$$\psi_0 : \mathcal{S}(C)/\mathcal{N} \to \mathcal{B}(\ell^2),\ \{A_n\}^\nu \mapsto A,\ \psi_1 : \mathcal{S}(C)/\mathcal{N} \to \mathcal{B}(\ell^2),\ \{A_n\}^\nu \mapsto \tilde{A},$$

are well defined $*$-homomorphisms.

Theorem 2.2.8 (Silbermann 1981)**.** *A sequence $\{A_n\}$ in the algebra $\mathcal{S}(C)$ is stable if and only if the two operators A and $\tilde{A}$ are invertible.*

Proof. Consider the $*$-homomorphism given by

$$\psi = \psi_0 \oplus \psi_1 : \mathcal{S}(C)/\mathcal{N} \to \mathcal{B}(\ell^2) \oplus \mathcal{B}(\ell^2),\ \{A_n\}^\nu \mapsto (A, \tilde{A}).$$

Note that $\mathcal{B}(\ell^2) \oplus \mathcal{B}(\ell^2)$ stands for the C^*-algebra of all ordered pairs (A, B), $A, B \in \mathcal{B}(\ell^2)$ with componentwise operations and the norm $\|(A, B)\| := \max\{\|A\|, \|B\|\}$.

We claim that ψ is injective. Indeed, if

$$A = T(a) + K = 0, \quad \tilde{A} = T(\tilde{a}) + L = 0,$$

then, $a = 0$ by Theorem 2.1.3 and hence $K = L = 0$, which implies that $\{A_n\}^\nu = 0$. From Theorem 1.2.3 we now deduce that ψ preserves spectra: $\{A_n\}^\nu$ is invertible if and only if A and $\tilde{A}$ are invertible. As the invertibility of $\{A_n\}^\nu$ is equivalent to the stability of $\{A_n\}$ (Corollary 2.2.7), we arrive at the assertion. □

Corollary 2.2.9 (Baxter 1963)**.** *Let $a \in C$. The sequence $\{T_n(a)\}$ is stable if and only if $T(a)$ is invertible.*

Proof. Since $\tilde{A} = T(\tilde{a})$ is the transpose of $A = T(a)$ and thus invertible if and only if $T(a)$ is invertible, this corollary is an immediate consequence of Theorem 2.2.8. □

Corollary 2.2.10. *The finite section method is applicable to every invertible Toeplitz operator with a continuous symbol.*

Proof. Theorem 2.2.1 and Corollary 2.2.9. □

2.2.5 Asymptotic inverses

The following result reveals the structure of the inverse of matrices of the form (2.2.4) for large n.

Theorem 2.2.11 (Widom 1976 and Silbermann 1981)**.** *Let $\{A_n\} = \{T_n(a) + P_nKP_n + W_nLW_n + C_n\} \in \mathcal{S}(C)$. If $T(a) + K$ and $T(\tilde{a}) + L$ are invertibles, then for all sufficiently large n,*

$$A_n^{-1} = T_n(a^{-1}) + P_nXP_n + W_nYW_n + D_n,$$

where $\|D_n\| \to 0$ as $n \to \infty$ and the compact operators X and Y are given by

$$X = (T(a) + K)^{-1} - T(a^{-1}), \quad Y = (T(\tilde{a}) + L)^{-1} - T(\tilde{a}^{-1}).$$

Proof. If $T(a)+K$ and $T(\tilde{a})+L$ are invertible, then $\{A_n\}^\nu$ is invertible in $\mathcal{S}(C)/\mathcal{N}$ by virtue of Corollary 2.2.7 and Theorem 2.2.8. Hence, A_n^{-1} is of the form

$$A_n^{-1} = T_n(b) + P_nXP_n + W_nYW_n + D_n, \tag{2.2.9}$$

with $b \in C$, $X, Y \in \mathcal{K}(\ell^2)$, and $\{D_n\} \in \mathcal{N}$. Rewriting (2.2.9) in the form

$$P_n = A_n(T_n(b) + P_nXP_n + W_nYW_n + D_n)$$
$$P_n = W_nA_nW_n(T_n(\tilde{b}) + W_nXW_n + P_nYP_n + W_nD_nW_n)$$

and taking into account Theorem 2.2.5 we obtain

$$\mathcal{I} = (T(a)+K)(T(b)+X), \quad \mathcal{I} = (T(\tilde{a})+L)(T(\tilde{b})+Y),$$

whence

$$X = (T(a)+K)^{-1} - T(b), \quad Y = (T(\tilde{a})+L)^{-1} - T(\tilde{b}).$$

Finally, from Theorems 2.1.6 and 2.1.7 we deduce that

$$\mathcal{I} = T(a)T(b) + KT(b) + T(a)X + KX = T(ab) + \textit{compact operator},$$

and Theorem 2.1.3 implies that $ab = 1$. □

2.3 Spectral theory for the extreme eigenvalues of Hermitian Toeplitz matrices

The purpose of this section is to formulate some results that are well known when it is studied the asymptotic behavior on the extreme eigenvalues of Hermitian Toeplitz matrices, i.e., we present the results of [31] and [45].

In [31], S. V. Parter extended the results of Kac, Murdock, and Szegő (see [26]) considering the class of functions g with the following properties:

(1) g is a real, continuous, and periodic function with period 2π, where $\min g = g(0) = m$ and $\varphi = 0$ is the only value of φ (mod 2π) for which this minimum is attained.

(2) g has continuous derivatives of order 2α ($\alpha \in \mathbb{N}$) in some neighborhood of $\varphi = 0$ and $g^{(2\alpha)}(0) = \sigma^2 > 0$ is the first non-vanishing derivative of g at $\varphi = 0$.

The assumption that the minimum of g is achieved at $\varphi = 0$ is unimportant, i.e., the point 0 may be replaced by any φ_0. Since the Toeplitz matrix corresponding to any translate $g(\varphi + \varphi_0)$ of $g(\varphi)$ has exactly the same eigenvalues as $g(\varphi)$.

Theorem 2.3.1 ([31] Theorem 3)**.** *Let g be a function which satisfies the conditions* (1) *and* (2) *with $\alpha = 2$. Let $\lambda_{j,n}$ be the eigenvalues of $T_n(a)$ ordered in non-decreasing order. Then, for fixed $j = 1, 2, \dots,$ as $n \to \infty$, we have*

$$\lambda_{j,n} = m + \frac{1}{4!}\left(\frac{(2j+1)\pi + E_j}{2(n+3)}\right)^4 g^{(4)}(0) + o(n^{-4}), \quad n \to \infty, \tag{2.3.1}$$

where the E_j is determined by the equation

$$\tan\left(\frac{(2j+1)\pi + E_j}{4}\right) = (-1)^j \tanh\left(\frac{(2j+1)\pi + E_j}{4}\right).$$

Similarly, In [45], H. Widom re-obtained and extended the results of Kac, Murdock, and Szegő considering the class of functions g satisfying:

(3) g is continuous and periodic with period 2π; $\max g = g(0) = M$ and $\varphi = 0$ is the only value of φ (mod 2π) for which this maximum is reached. Assume also that g is even, has continuous derivatives up to the fourth order in some neighborhood of $\varphi = 0$, and $\sigma^2 = -g^{(2)}(0) \neq 0$.

Once again, the assumption that the maximum of g is achieved at $\varphi = 0$ is unimportant, i.e., the point 0 may be replaced by any φ_0.

Theorem 2.3.2 ([45] Theorem 2.1)**.** *Let g be a function which satisfies the condition* (3)*. Let $\lambda_{j,n}$ be the eigenvalues of $T_n(a)$ ordered in decreasing order. Then, as $n \to \infty$, for fixed $j = 1, 2, \ldots,$ we have*

$$\lambda_{j,n} = M - \frac{\sigma^2\pi^2 j^2}{2(n+1)^2}\left(1 + \frac{\rho}{n+1}\right) + o(n^{-3}), \tag{2.3.2}$$

where

$$\rho = 2 + \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{g'(\varphi)}{g(\varphi) - M} - 2\csc\varphi\right)\cot\frac{\varphi}{2}d\varphi.$$

Remark 2.3.3. In [45], there is a misprint when calculating ρ, i.e., the constant 2 wasn't included. Hence, we make the respective correction here.

2.4 Spectral theory for the eigenvalues of Hermitian Toeplitz matrices with simple-loop symbols

The purpose of the present section is to formulate some fresh results that are used when it is studied the asymptotic behavior on the extreme and inner eigenvalues of Hermitian Toeplitz matrices with simple-loop symbols, i.e., we present the results of [6].

In [6], simple-loop means that if t moves along $\mathbb{T}$, then $a(t)$ moves strictly monotonically from its minimum to its maximum and then strictly monotonically back to its minimum, without any rests in the minimum and the maximum (which includes that the second derivative at these points is nonzero). The authors obtained asymptotic formulas not only for the extreme eigenvalues, but also for the inner eigenvalues.

For $\alpha \geq 0$, we denote by W^α the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ that can be represented in the form

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j \quad (t \in \mathbb{T}),$$

with Fourier coefficients satisfying

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty.$$

If $\alpha \geq 1$, then we let SL^α denote the set of all real-valued functions $a \in W^\alpha$ such that the function g defined by $g(\sigma) := a(e^{i\sigma})$ has the following properties:

i) The range of g is the segment $[0, M]$ with $M > 0$.

ii) $g(0) = g(2\pi) = 0$ and $g''(0) = g''(2\pi) > 0$.

iii) There is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = M$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

The condition a is real-valued function is equivalent to the condition that all matrices $T_n(a)$ ($n \in \mathbb{Z}_+$) are Hermitian (self-adjoint). If $a \in W^\alpha$, then $g \in C^{\lfloor \alpha \rfloor}[0, 2\pi]$ where $\lfloor \alpha \rfloor$ is the integer part of α. Therefore, the condition $a \in SL^\alpha$ with $\alpha \geq 1$ implies, in particular, that g belongs to $C^1[0, 2\pi]$.

Now, for each $\alpha \geq 1$ we introduce a new class of symbols MSL^α (the modified simple-loop class). Namely, $a \in MSL^\alpha$ if $a \in SL^\alpha$ and

iv) There exist functions $q_1, q_2 \in W^\alpha$ satisfying

$$a(t) = (t-1)q_1(t) \quad \text{and} \quad a(t) - a(e^{i\varphi_0}) = (t - e^{i\varphi_0})q_2(t). \tag{2.4.1}$$

Moreover, for every $\alpha \geq 1$ the class $SL^{\alpha+1}$ is contained in MSL^α. So every statement in Theorems 2.4.1, 2.4.2, and 2.4.3 formulated for $a \in MSL^\alpha$ is true for all $a \in SL^{\alpha+1}$.

Then for each $\lambda \in [0, M]$, there are exactly one $\varphi_1(\lambda) \in [0, \varphi_0]$ such that $g(\varphi_1(\lambda)) = \lambda$, and exactly one $\varphi_2(\lambda) \in [\varphi_0, 2\pi]$ such that $g(\varphi_2(\lambda)) = \lambda$. Denote by $\varphi(\lambda)$ the arithmetic mean of the lengths of the segments $[0, \varphi_1(\lambda)]$ and $[\varphi_2(\lambda), 2\pi]$,

$$\varphi(\lambda) := \frac{1}{2}\left(\varphi_1(\lambda) - \varphi_2(\lambda)\right) + \pi = \frac{1}{2}\left|\{\sigma \in [0, 2\pi] : g(\sigma) \leq \lambda\}\right|,$$

where $|\cdot|$ is the Lebesgue measure on $[0, 2\pi]$. The function $\varphi : [0, M] \to [0, \pi]$ is continuous and bijective. We let $\psi : [0, \pi] \to [0, M]$ stand for the inverse function.

As well, we define two functions $\beta : [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ and $\eta : [0, \pi] \to \mathbb{R}$ by

$$\begin{aligned}
\beta(\sigma, s) &:= \frac{(g(\sigma) - \psi(s))e^{is}}{(e^{i\sigma} - e^{i\varphi_1(\psi(s))})(e^{-i\sigma} - e^{-i\varphi_2(\psi(s))})}, \\
\eta(s) &:= \frac{1}{4\pi}\int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \varphi_2(\psi(s))}{2}} d\sigma - \frac{1}{4\pi}\int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \varphi_1(\psi(s))}{2}} d\sigma,
\end{aligned}$$

the integrals taken in the sense of the principal value.

Theorem 2.4.1 ([6] Theorem 2.1)**.** *Let $\alpha \geq 1$ and $a \in MSL^\alpha$. Then for every $n \geq 1$:*

(i) The eigenvalues of $T_n(a)$ are all distinct, i.e., $\lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$.

(ii) The numbers $s_j^{(n)} := \varphi(\lambda_j^{(n)})$ $(j = 1, \ldots, n)$ satisfy

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j), \tag{2.4.2}$$

where $\Delta_1^{(n)}(j) = o(1/n^{\alpha-1})$ as $n \to \infty$.

(iii) For every sufficiently large n, (2.4.2) *has exactly one solution $s_j^{(n)} \in [0, \pi]$ for each $j = 1, \ldots, n$.*

To write the individual asymptotics of the eigenvalues, we introduce the parameter

$$d_j^{(n)} := \frac{\pi j}{n+1}.$$

Theorem 2.4.2 ([6] Theorem 2.2)**.** *Under the conditions of Theorem 2.4.1,*

$$s_j^{(n)} = d_j^{(n)} + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{p_k\left(d_j^{(n)}\right)}{(n+1)^k} + \Delta_2^{(n)}(j),$$

where $\Delta_2^{(n)}(j) = o(1/n^\alpha)$ *as* $n \to \infty$*. The coefficients* p_k *can be calculated explicitly; in particular,*

$$p_1(s) = -\eta(s) \quad \textit{and} \quad p_2(s) = \eta(s)\eta'(s).$$

Using $\lambda = \psi(s)$ and Theorem 2.4.2, we can formulate the result that allow us to describe completely the asymptotic behavior for the inner and extreme eigenvalues.

Theorem 2.4.3 ([6] Theorem 2.3)**.** *Under the conditions of Theorem 2.4.1,*

$$\lambda_j^{(n)} = \psi\left(d_j^{(n)}\right) + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{f_k\left(d_j^{(n)}\right)}{(n+1)^k} + \Delta_3^{(n)}(j),$$

where $\Delta_3^{(n)}(j)$ *is* $o\left(\frac{1}{n^\alpha}\left(d_j^{(n)}(\pi - d_j^{(n)})\right)^{\alpha-1}\right)$ *if* $1 \le \alpha \le 2$ *and* $o\left(\frac{d_j^{(n)}}{n^\alpha}(\pi - d_j^{(n)})\right)$ *if* $\alpha \ge 2$ *as* $n \to \infty$*. The coefficients* f_k *can be calculated explicitly; in particular,*

$$f_1(s) = -\psi'(s)\eta(s) \quad \textit{and} \quad f_2(s) = \frac{1}{2}\psi''(s)\eta^2(s) + \psi'(s)\eta(s)\eta'(s).$$

Chapter 3

Asymptotics of eigenvalues for pentadiagonal symmetric Toeplitz matrices

In this chapter we study the asymptotic behavior of the $n \times n$ sections of certain pentadiagonal symmetric Toeplitz matrices as n goes to infinity. Moreover, we establish asymptotic formulas for all the eigenvalues of these finite sections. The entries of the matrices are real and we consider the case where the real-valued generating function has a minimum and a maximum such that its fourth derivative at the minimum and its second derivative at the maximum are nonzero. This is not the simple-loop case considered in [2], [6], and [7]. We apply the main result of [20] and obtain nonlinear equations for the eigenvalues. It should be noted that our equations have a more complicated structure than the equations in [2], [6], and [7]. Therefore, we required a more delicate method for its asymptotic analysis.

3.1 Introduction

The asymptotic behavior of the eigenvalues of $T_n(a)$ as n goes to infinity has been thoroughly studied by mathematicians and physicists for a long time. Toeplitz matrices arise in particular in many problems of statistical physics and there are questions about the asymptotics of their spectrum, principally their determinants, eigenvalues, and eigenvectors are always at the heart of the matter.

In the Hermitian case, i.e., when a is real-valued, there is a lot of known work about the extreme eigenvalues of the matrices $T_n(a)$. For example, In [45], H. Widom considered the class of functions $g(\varphi) := a(e^{i\varphi})$ such that g is real-valued, smooth, even, $M := g(0) = \max g$, and $g''(0) \neq 0$. There, he established an asymptotic formula for the extreme eigenvalues near M. Similarly, in [31], S. V. Parter considered the class of functions $g(\varphi) := a(e^{i\varphi})$ such that g is real-valued, smooth, $m := g(0) = \min g$, and $g^{(2k)}(0) \neq 0$ for $k \in \mathbb{N}$. Once again, there he established an asymptotic formula for the extreme eigenvalues near m. See [8], [26], [30], [32], [35], and [36] for more information about the behavior of extreme eigenvalues.

Results on the individual asymptotic formulas for all (extreme and inner) eigenvalues of Hermitian Toeplitz matrices were obtained only recently in [2], [6], [7], and [16]. In [7] it is assumed that the symbol a is a Laurent polynomial such that $a(e^{i\varphi})$ is strictly increasing from its minimum to its maximum and strictly decreasing from its maximum back to its minimum. Furthermore, $a(e^{i\varphi})$ has nonzero second derivatives at the minimum and the maximum. These requirements were relaxed in

[2] where it was considered a class of smooth symbols with 4-th derivative in the Wiener algebra W^{α} ($\alpha \geq 4$). Similarly, in [6] these requirements were relaxed again when it was considered a class of symbols in the Wiener algebra W^{α} ($\alpha \geq 1$) with some additional condition of smoothness at the points where its minimum and maximum are reached (see (2.4.1)). Finally, the results of [16] are stated for C^{∞} symbols.

In this chapter, we present a model case such that the second derivative of the symbol a at the minimum is equal to 0. Therefore, throughout the Chapter we suppose that the symbol is

$$a(t) = \left(\frac{1}{t} + t - 2\right)^2 \quad (t = e^{i\varphi} \in \mathbb{T}), \tag{3.1.1}$$

and the generating function

$$g(\varphi) := a(e^{i\varphi}) = (2 - 2\cos\varphi)^2 = M \sin^4\left(\frac{\varphi}{2}\right), \tag{3.1.2}$$

where $\varphi \in [-\pi, \pi]$ and $M := 16$. Notice that the Fourier coefficients of the pentadiagonal Toeplitz matrix with symbol a are $a_0 = 6$, $a_{-1} = a_1 = -4$, $a_{-2} = a_2 = 1$, and $a_k = 0$ for $|k| > 2$.

Moreover, pentadiagonal Toeplitz matrices frequently arise from boundary value problems involving fourth order derivatives, i.e., the Toeplitz matrix with symbol a comes from discretization, by equi-spaced Finite Differences (FD) of precision order two, of the unidimensional fourth derivative with homogeneous boundary conditions both on the function and on the first derivative. See, for example [37].

The function g has the following properties:

i) The range of g is the segment $[0, M]$ with $M := g(\pi) > 0$.

ii) $g(0) = 0$, $g(-\pi) = g(\pi) = M$, $g'(-\pi) = g'(\pi) = 0$, $g''(-\pi) = g''(\pi) < 0$, $g'(0) = g''(0) = g'''(0) = 0$, and $g^{(4)}(0) > 0$.

The function g is even. Furthermore, $g : [0, \pi] \to [0, M]$ is bijective and increasing. Let $g^{-1} : [0, M] \to [0, \pi]$ be the inverse function of g restricted to $[0, \pi]$. Thus, we get $\lambda = g(\varphi)$ if and only if $\varphi = g^{-1}(\lambda)$.

We will express all the main objects in terms of φ rather than λ itself, because this approach will simplify the proof of our main results. Using the main result of [20], we will find formulas for the determinant of $T_n(a - \lambda) = T_n(a) - \lambda I_n$ that will allow us to obtain asymptotic formulas in terms of φ. Given that $\lambda = g(\varphi)$ if and only if $\varphi = g^{-1}(\lambda)$, we will get the asymptotic formulas for the eigenvalues of $T_n(a)$.

It should be noted that our case is essentially different to the simple-loop case presented in [2], [6], [7], and [16]; because we have a fourth order zero at the point $\varphi = 0$. Therefore, the asymptotic formulas on a neighborhood of the minimum, have a different form of the formulas presented in [2], [6], and [7]. Furthermore, the nonlinear equations for the eigenvalues of $T_n(a)$ (obtained with the help of the main result of [20]) have a more complicated structure than in [2], [6], [7], and [16]. In particular, since all the terms in these equations depend on n, we require a more delicate method for its asymptotic analysis.

Once again, it is emphasized that we get asymptotic formulas for all eigenvalues of $T_n(a)$. We simplify the asymptotic formulas for inner and extreme eigenvalues and compare our results for extreme eigenvalues with the well known articles H. Widom [45] and S. V. Parter [31].

As well, our model case (3.1.1) allow us to consider more general smooth symbols satisfying *i)* and *ii)*. Moreover, it should be noted that our results can easily be generalized to symbols

$$A(t) = c_1 a(t) + c_2 \quad (t = e^{i\varphi} \in \mathbb{T}),$$

where $c_1, c_2 \in \mathbb{R}$. In fact, for A we obtain that $T_n(A) = c_1 T_n(a) + c_2$.

3.2 Main results

In this section, we begin by introducing the functions $\beta : [0,\pi] \to \mathbb{R}$ and $f : [0,\pi] \to \mathbb{R}$, where

$$\beta(\varphi) := 2\log\left(\sin\frac{\varphi}{2} + \sqrt{1+\sin^2\frac{\varphi}{2}}\right) \quad \text{and} \quad f(\varphi) := \frac{\cos\frac{\varphi}{2}}{\sqrt{1+\sin^2\frac{\varphi}{2}}}.$$

By a simple calculation $f(\varphi) = \beta'(\varphi)$. These functions will be in constant use throughout this Chapter.

Now, we formulate our main results starting with the results that allow us to analyze the asymptotic behavior of the eigenvalues.

Theorem 3.2.1. *A number λ is an eigenvalue of $T_n(a)$ if and only if there exists a number $\varphi \in (0,\pi)$ such that $\lambda = g(\varphi)$ and*

$$\tan\left(\frac{n+2}{2}\varphi\right) = -f(\varphi)\tanh\left(\frac{n+2}{2}\beta(\varphi)\right), \tag{3.2.1}$$

or

$$\tan\left(\frac{n+2}{2}\varphi\right) = \frac{1}{f(\varphi)}\tanh\left(\frac{n+2}{2}\beta(\varphi)\right). \tag{3.2.2}$$

Theorem 3.2.2. *If n is sufficiently large, then*

(i) the equation

$$\frac{n+2}{2}\varphi + \arctan\left(f(\varphi)\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)\right) = \pi j \tag{3.2.3}$$

has exactly one solution $\varphi_{2j-1}^{(n)} \in \left(\frac{(2j-1)\pi}{n+2}, \frac{2\pi j}{n+2}\right)$ for $j = 1,\ldots,[\frac{n+1}{2}]$, with $\lambda_{2j-1}^{(n)} = g(\varphi_{2j-1}^{(n)})$.

(ii) the equation

$$\frac{n+2}{2}\varphi - \arctan\left(\frac{1}{f(\varphi)}\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)\right) = \pi j \tag{3.2.4}$$

has exactly one solution $\varphi_{2j}^{(n)} \in \left(\frac{2\pi j}{n+2}, \frac{(2j+1)\pi}{n+2}\right)$ for $j = 1,\ldots,[\frac{n}{2}]$, with $\lambda_{2j}^{(n)} = g(\varphi_{2j}^{(n)})$.

(iii) the eigenvalues of $T_n(a)$ are all distinct and we can write $\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}$.

Remark 3.2.3. Theorem 3.2.2 shows that, for sufficiently large n, the equations (3.2.3) and (3.2.4) have a unique solution for each $j \in \{1,\ldots,[\frac{n+1}{2}]\}$ and $j \in \{1,\ldots,[\frac{n}{2}]\}$, respectively. Furthermore, if n is an even number, then the equation (3.2.3) gives us $\frac{n}{2}$ solutions on $(0,\pi)$ and the equation (3.2.4) gives us $\frac{n}{2}$ solutions on $(0,\pi)$, as well. Thus, we have exactly n solutions. But, if n is an odd number, then the equations (3.2.3) and (3.2.4) give us $\frac{n+1}{2}$ solutions and $[\frac{n}{2}]$ solutions on $(0,\pi)$, respectively. Again, we have exactly n solutions. As well, we have to notice that the problem of finding the eigenvalues of $T_n(a)$ is reduced to solve the nonlinear equations (3.2.3) and (3.2.4) on the intervals $\left(\frac{(2j-1)\pi}{n+2}, \frac{2\pi j}{n+2}\right)$ and $\left(\frac{2\pi j}{n+2}, \frac{(2j+1)\pi}{n+2}\right)$, respectively. Each one of these equations can be solved numerically in many ways.

The following theorems give us the asymptotic approach. For this goal we introduce the parameters $d_- := d_{2j}^{(n)} = \frac{2\pi j}{n+2}$ and $d_+ := d_{2j+1}^{(n)} = \frac{(2j+1)\pi}{n+2}$. We need to consider the equations

$$u = -\arctan(f(d_-)\tanh\Psi_j^-(u)) \quad u \in \left[-\frac{\pi}{2}, 0\right] \tag{3.2.5}$$

and

$$w = -\operatorname{arccot}\left(\frac{1}{f(d_+)}\tanh\Psi_j^+(w)\right) \quad w \in \left[-\frac{\pi}{2}, 0\right], \tag{3.2.6}$$

where $\Psi_j^-(u) := \frac{n+2}{2}\beta(d_-) + \beta'(d_-)u$ and $\Psi_j^+(w) := \frac{n+2}{2}\beta(d_+) + \beta'(d_+)w$.

Theorem 3.2.4. *If $n \to \infty$, then*

(i)

$$\varphi_{2j-1}^{(n)} = d_- + \frac{2u_{1,j}}{n+2} + \frac{4u_{2,j}}{(n+2)^2} + \Delta_1(n,j), \tag{3.2.7}$$

where $|\Delta_1(n,j)| \le \frac{b_1}{n^3}$ and the constant b_1 does not depend on n and j, for $j = 1, \dots, [\frac{n+1}{2}]$. The coefficient $u_{1,j}$ is the unique solution of (3.2.5) in $[-\frac{\pi}{2}, 0]$. The coefficient $u_{2,j}$ is defined in terms of $u_{1,j}$ (see (3.4.11)).

(ii)

$$\varphi_{2j}^{(n)} = d_+ + \frac{2w_{1,j}}{n+2} + \frac{4w_{2,j}}{(n+2)^2} + \Delta_2(n,j), \tag{3.2.8}$$

where $|\Delta_2(n,j)| \le \frac{b_2}{n^3}$ and the constant b_2 does not depend on n and j, for $j = 1, \dots, [\frac{n}{2}]$. The coefficient $w_{1,j}$ is the unique solution of (3.2.6) in $[-\frac{\pi}{2}, 0]$. The coefficient $w_{2,j}$ is defined in terms of $w_{1,j}$ (see (3.4.19)).

Using $\lambda = g(\varphi)$ and Theorem 3.2.4, we can formulate the results that allow us to describe completely the asymptotic behavior for the inner and extreme eigenvalues. For the inner eigenvalues we have the following asymptotic expressions.

Theorem 3.2.5. *Let $\varepsilon > 0$ be a small number such that $\varepsilon \ge \gamma_1 \frac{\log n}{n}$ for some positive constant γ_1.*

(i) If $\pi\varepsilon < d_- < (1-\varepsilon)\pi$, then

$$\lambda_{2j-1}^{(n)} = g(d_-) + \frac{2u_{1,j}^* g'(d_-)}{n+2} + \frac{2(u_{1,j}^*)^2 g''(d_-) + 4u_{2,j}^* g'(d_-)}{(n+2)^2} + \Delta_3(n,j), \tag{3.2.9}$$

where $|\Delta_3(n,j)| \le \frac{b_3}{n^3}$ and the constant b_3 does not depend on n and j. The coefficients $u_{1,j}^*$ and $u_{2,j}^*$ are defined by

$$u_{1,j}^* = -\arctan f(d_-) \quad \textit{and} \quad u_{2,j}^* = \frac{f'(d_-)\arctan f(d_-)}{1 + f^2(d_-)}.$$

(ii) If $\pi\varepsilon < d_+ < (1-\varepsilon)\pi$, then

$$\lambda_{2j}^{(n)} = g(d_+) + \frac{2w_{1,j}^* g'(d_+)}{n+2} + \frac{2(w_{1,j}^*)^2 g''(d_+) + 4w_{2,j}^* g'(d_+)}{2(n+2)^2} + \Delta_4(n,j), \tag{3.2.10}$$

where $|\Delta_4(n,j)| \le \frac{b_4}{n^3}$ and the constant b_4 does not depend on n and j. The coefficients $w_{1,j}^*$ and $w_{2,j}^*$ are defined by

$$w_{1,j}^* = -\operatorname{arccot}\frac{1}{f(d_+)} \quad \textit{and} \quad w_{2,j}^* = \frac{f'(d_+)\operatorname{arccot}\frac{1}{f(d_+)}}{1 + f^2(d_+)}.$$

Now, it is formulated the result for the extreme eigenvalues near zero.

Theorem 3.2.6. *(i) If $d_- \to 0$ as $n \to \infty$, then*

$$\lambda_{2j-1}^{(n)} = \frac{(2\pi j + 2u_{1,j})^4}{(n+2)^4} + \frac{16u_{2,j}(2\pi j + 2u_{1,j})^3}{(n+2)^5} + \Delta_5(n,j), \tag{3.2.11}$$

where $|\Delta_5(n,j)| \le b_5 \left(\frac{d_-^3}{n^3} + d_-^8\right)$ and the constant b_5 does not depend on n and j. Moreover, $u_{1,j}$ is the unique solution on $[-\frac{\pi}{2}, 0]$ of the equation (3.2.5), and the coefficient $u_{2,j}$ is defined in terms of $u_{1,j}$ (see (3.4.11)).

(ii) If $d_+ \to 0$ as $n \to \infty$, then

$$\lambda_{2j}^{(n)} = \frac{((2j+1)\pi + 2w_{1,j})^4}{(n+2)^4} + \frac{16w_{2,j}((2j+1)\pi + 2w_{1,j})^3}{(n+2)^5} + \Delta_6(n,j), \tag{3.2.12}$$

where $|\Delta_6(n,j)| \le b_6 \left(\frac{d_+^3}{n^3} + d_+^8\right)$ and the constant b_6 does not depend on n and j. The coefficient $w_{1,j}$ is the unique solution of (3.2.6) in $[-\frac{\pi}{2}, 0]$ and the coefficient $w_{2,j}$ is defined in terms of $w_{1,j}$ (see (3.4.19)).

Finally, it is presented the result for the extreme eigenvalues near M.

Theorem 3.2.7. *(i) If $\pi - d_{2j}^{(n)} \to 0$ as $n \to \infty$, then*

$$\lambda_{2j-1}^{(n)} = M - \frac{M(d_{2j}^{(n)} - \pi)^2}{2} + \frac{2Mf'(\pi)(d_{2j}^{(n)} - \pi)^2}{n+2} + \Delta_7(n,j), \tag{3.2.13}$$

where $|\Delta_7(n,j)| \le b_7 \left((d_{2j}^{(n)} - \pi)^4 + \frac{1}{n^3}\right)$ and the constant b_7 does not depend on n and j.

(ii) If $\pi - d_{2j+1}^{(n)} \to 0$ as $n \to \infty$, then

$$\lambda_{2j}^{(n)} = M - \frac{M(d_{2j+1}^{(n)} - \pi)^2}{2} + \frac{2Mf'(\pi)(d_{2j+1}^{(n)} - \pi)^2}{n+2} + \Delta_8(n,j), \tag{3.2.14}$$

where $|\Delta_8(n,j)| \le b_8 \left((d_{2j+1}^{(n)} - \pi)^4 + \frac{1}{n^3}\right)$ and the constant b_8 does not depend on n and j.

Remark 3.2.8. The formulas in the Theorems 3.2.4-3.2.7 have 3 asymptotic terms. However, our method allows us to obtain as many terms as we desire.

3.3 Determinants

The purpose of this section is to obtain formulas for the determinant of $T_n(a - \lambda)$ that will allow us to analyze the asymptotic behavior of the eigenvalues. We will derive the equations (3.2.1) and (3.2.2) using the main result of [20].

We start by introducing the Chebyshev polynomials $\{\mathcal{T}_n\}$, $\{\mathcal{U}_n\}$, $\{\mathcal{V}_n\}$, and $\{\mathcal{W}_n\}$. They all satisfy the same recurrence relation

$$\mathcal{T}_{n+1}(x) = 2x\mathcal{T}_n(x) - \mathcal{T}_{n-1}(x) \quad \text{for} \quad n = 1, 2, \ldots$$

and the different initial conditions are

$$\mathcal{T}_0(x) = \mathcal{U}_0(x) = 1, \quad 2\mathcal{T}_1(x) = \mathcal{U}_1(x) = 2x,$$

$$\mathcal{W}_0(x) = \mathcal{V}_0(x) = 1, \quad \text{and} \quad \mathcal{W}_1(x) = \mathcal{V}_1(x) + 2 = 2x + 1.$$

Furthermore, we use the following useful properties of them:

$$\mathcal{T}_n(\cos\varphi) = \cos n\varphi, \quad \mathcal{U}_n(\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi},$$

$$\mathcal{V}_n(\cos\varphi) = \frac{\cos(n+\frac{1}{2})\varphi}{\cos\frac{\varphi}{2}}, \quad \text{and} \quad \mathcal{W}_n(\cos\varphi) = \frac{\sin(n+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}}.$$

Now, we formulate the main results of [20] where the symbol $a(t) = \sum_{k=-r}^{r} a_k t^k$, with $a_r \neq 0$ and $a_k = a_{-k}$, was considered.

Theorem 3.3.1 (Elouafi 2014)**.** *Let ξ_j and $\frac{1}{\xi_j}$ $(j = 1, \ldots, r)$ be the (distinct) zeros of the polynomial $g_1(t) = t^r a(t)$. Then, for all $p \geq 1$, $\det T_{2p}$ equals*

$$\frac{a_r^{2p}}{2^{r(r-1)}} \times \frac{\begin{vmatrix} \mathcal{V}_p(\alpha_1) & \cdots & \mathcal{V}_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ \mathcal{V}_{p+r-1}(\alpha_1) & \cdots & \mathcal{V}_{p+r-1(\alpha_r)} \end{vmatrix}}{\prod_{1\leq i<j\leq r} (\alpha_j - \alpha_i)} \times \frac{\begin{vmatrix} \mathcal{W}_p(\alpha_1) & \cdots & \mathcal{W}_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ \mathcal{W}_{p+r-1}(\alpha_1) & \cdots & \mathcal{W}_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i<j\leq r} (\alpha_j - \alpha_i)}$$

and $\det T_{2p+1}$ equals

$$\frac{(-1)^r a_r^{2p+1}}{2^{r(r-2)}} \times \frac{\begin{vmatrix} \mathcal{U}_p(\alpha_1) & \cdots & \mathcal{U}_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ \mathcal{U}_{p+r-1}(\alpha_1) & \cdots & \mathcal{U}_{p+r-1(\alpha_r)} \end{vmatrix}}{\prod_{1\leq i<j\leq r} (\alpha_j - \alpha_i)} \times \frac{\begin{vmatrix} \mathcal{T}_{p+1}(\alpha_1) & \cdots & \mathcal{T}_{p+1}(\alpha_r) \\ \vdots & \ddots & \vdots \\ \mathcal{T}_{p+r}(\alpha_1) & \cdots & \mathcal{T}_{p+r}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i<j\leq r} (\alpha_j - \alpha_i)},$$

where $\alpha_k = \frac{1}{2}\left(\xi_k + \frac{1}{\xi_k}\right)$ $(k = 1, \ldots, r)$ are the zeros of the polynomial $h_1(x) = a_0 + 2\sum_{k=1}^{r} a_k \mathcal{T}_k(x)$.

Proof. See [20, p. 27-31]. □

In our case, the coefficients of the polynomial h_1 are $a_0 = 6 - \lambda$, $a_1 = -4$, and $a_2 = 1$. Since $\mathcal{T}_1(x) = x$ and $\mathcal{T}_2(x) = 2x\mathcal{T}_1(x) - \mathcal{T}_0(x) = 2x^2 - 1$, it follows that $h_1(x) = 4x^2 - 8x + (4 - \lambda)$. Notice that h_1 has the zeros $\alpha_1 = 1 - \frac{\sqrt{\lambda}}{2}$ and $\alpha_2 = 1 + \frac{\sqrt{\lambda}}{2}$. Given that $t \in \mathbb{T}$ and $\lambda = g(\varphi) = M\sin^4\left(\frac{\varphi}{2}\right)$, we obtain that $\alpha_1 = \cos(\varphi)$ and $\alpha_2 = \cos(\xi)$, where $\xi := \xi(\varphi) = -i\beta(\varphi)$.

Lemma 3.3.2. *Let a be as in* (3.1.1)*, $g(\varphi) = a(e^{i\varphi})$ and $\lambda = g(\varphi)$ for $\varphi \in (0, \pi)$. Then, for all $p \geq 1$, $\det T_{2p}(a - g(\varphi))$ equals*

$$\begin{aligned}&(\cos((p+\frac{1}{2})\varphi)\cos((p+\frac{3}{2})\xi(\varphi)) - \cos((p+\frac{1}{2})\xi(\varphi))\cos((p+\frac{3}{2})\varphi)) \\ &\times \frac{\sin((p+\frac{1}{2})\varphi)\sin((p+\frac{3}{2})\xi(\varphi)) - \sin((p+\frac{1}{2})\xi(\varphi))\sin((p+\frac{3}{2})\varphi)}{\lambda\sin\varphi\sin\xi(\varphi)}\end{aligned}$$

and $\det T_{2p+1}(a-g(\varphi))$ *equals*

$$(\cos((p+1)\varphi)\cos((p+2)\xi(\varphi))-\cos((p+1)\xi(\varphi))\cos((p+2)\varphi))$$
$$\times\frac{\sin((p+1)\varphi)\sin((p+2)\xi(\varphi))-\sin((p+1)\xi(\varphi))\sin((p+2)\varphi)}{\lambda\sin\varphi\sin\xi(\varphi)}.$$

Proof. By Theorem 3.3.1 $\det T_{2p}$ equals

$$\frac{(\mathcal{V}_p(\alpha_1)\mathcal{V}_{p+1}(\alpha_2)-\mathcal{V}_p(\alpha_2)\mathcal{V}_{p+1}(\alpha_1))(\mathcal{W}_p(\alpha_1)\mathcal{W}_{p+1}(\alpha_2)-\mathcal{W}_p(\alpha_2)\mathcal{W}_{p+1}(\alpha_1))}{4(\alpha_2-\alpha_1)^2}$$
$$=(\cos((p+\frac{1}{2})\varphi)\cos((p+\frac{3}{2})\xi(\varphi))-\cos((p+\frac{1}{2})\xi(\varphi))\cos((p+\frac{3}{2})\varphi))$$
$$\times\frac{\sin((p+\frac{1}{2})\varphi)\sin((p+\frac{3}{2})\xi(\varphi))-\sin((p+\frac{1}{2})\xi(\varphi))\sin((p+\frac{3}{2})\varphi)}{\lambda\sin\varphi\sin\xi(\varphi)}$$

and $\det T_{2p+1}$ equals

$$\frac{(\mathcal{U}_p(\alpha_1)\mathcal{U}_{p+1}(\alpha_2)-\mathcal{U}_p(\alpha_2)\mathcal{U}_{p+1}(\alpha_1))(\mathcal{T}_{p+1}(\alpha_1)\mathcal{T}_{p+2}(\alpha_2)-\mathcal{T}_{p+1}(\alpha_2)\mathcal{T}_{p+2}(\alpha_1))}{(\alpha_2-\alpha_1)^2}$$
$$=\frac{\sin((p+1)\varphi)\sin((p+2)\xi(\varphi))-\sin((p+1)\xi(\varphi))\sin((p+2)\varphi)}{\lambda\sin\varphi\sin\xi(\varphi)}$$
$$\times(\cos((p+1)\varphi)\cos((p+2)\xi(\varphi))-\cos((p+1)\xi(\varphi))\cos((p+2)\varphi)).$$

□

Lemma 3.3.3. *For all* $p\geq 1$ *and* $\varphi\in(0,\pi)$, *we have* $\det T_{2p}(a-g(\varphi))=0$ *and* $\det T_{2p+1}(a-g(\varphi))=0$ *are equivalent to the equations* (3.2.1) *and* (3.2.2).

Proof. From Lemma 3.3.2 we obtain that $\det T_{2p}(a-g(\varphi))=0$ is equivalent to

$$\cos((p+\frac{1}{2})\varphi)\cos((p+\frac{3}{2})\xi(\varphi))-\cos((p+\frac{1}{2})\xi(\varphi))\cos((p+\frac{3}{2})\varphi)=0 \tag{3.3.1}$$

or

$$\sin((p+\frac{1}{2})\varphi)\sin((p+\frac{3}{2})\xi(\varphi))-\sin((p+\frac{1}{2})\xi(\varphi))\sin((p+\frac{3}{2})\varphi)=0. \tag{3.3.2}$$

Taking into account that $\cos(-ix)=\cosh(x)$ and $\xi(\varphi)=-i\beta(\varphi)$ in (3.3.1), we arrive at the equation

$$\frac{\cos((p+\frac{3}{2})\varphi)}{\cos((p+\frac{1}{2})\varphi)}=\frac{\cosh((p+\frac{3}{2})\beta(\varphi))}{\cosh((p+\frac{1}{2})\beta(\varphi))}. \tag{3.3.3}$$

Notice that (3.3.1) and (3.3.3) are equivalent. Furthermore, $\cos((p+\frac{3}{2})\varphi)=\cos((p+\frac{1}{2})\varphi)\cos\varphi-\sin((p+\frac{1}{2})\varphi)\sin\varphi$. Therefore, if $\cos((p+\frac{1}{2})\varphi)=0$, then $\varphi=\frac{(2j+1)\pi}{2p+1}$ with $j\in\mathbb{Z}$. Thus, $\cos((p+\frac{3}{2})\varphi)=(-1)^{j+1}\sin\frac{(2j+1)\pi}{2p+1}$. As $j\neq p$, it follows that $\cos((p+\frac{3}{2})\varphi)\neq 0$.

The equation (3.3.3) can be rewritten as follows

$$\frac{\cos((p+1)\varphi+\frac{\varphi}{2})}{\cos((p+1)\varphi-\frac{\varphi}{2})}=\frac{\cosh((p+1)\beta(\varphi)+\frac{\beta(\varphi)}{2})}{\cosh((p+1)\beta(\varphi)-\frac{\beta(\varphi)}{2})}.$$

Taking $p = \frac{n}{2}$ a simple calculation shows that

$$\tan\left(\frac{n+2}{2}\varphi\right) = -\cot\frac{\varphi}{2}\tanh\frac{\beta(\varphi)}{2}\tanh\left(\frac{n+2}{2}\beta(\varphi)\right). \tag{3.3.4}$$

Given that $\beta(\varphi) = 2\log\left(\sin\frac{\varphi}{2} + \sqrt{1+\sin^2\frac{\varphi}{2}}\right)$ and $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we easily see that

$$\cot\frac{\varphi}{2}\tanh\frac{\beta(\varphi)}{2} = \frac{\cos\frac{\varphi}{2}}{\sqrt{1+\sin^2\frac{\varphi}{2}}} = f(\varphi). \tag{3.3.5}$$

Substituting (3.3.5) in (3.3.4) we get the equation (3.2.1), i.e.,

$$\tan\left(\frac{n+2}{2}\varphi\right) = -f(\varphi)\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)$$

Similarly, substituting $\xi(\varphi) = -i\beta(\varphi)$ in (3.3.2) and using $\sin(-ix) = -i\sinh x$, we arrive at the equation

$$\frac{\sin((p+\frac{3}{2})\varphi)}{\sin((p+\frac{1}{2})\varphi)} = \frac{\sinh((p+\frac{3}{2})\beta(\varphi))}{\sinh((p+\frac{1}{2})\beta(\varphi))}. \tag{3.3.6}$$

Again, notice that (3.3.2) and (3.3.6) are equivalent. As well, we have $\sin((p+\frac{3}{2})\varphi) = \sin((p+\frac{1}{2})\varphi)\cos\varphi + \cos((p+\frac{1}{2})\varphi)\sin\varphi$. Therefore, if $\sin((p+\frac{1}{2})\varphi) = 0$, then $\varphi = \frac{2j\pi}{2p+1}$ with $j \in \mathbb{Z}$. Thus, $\sin((p+\frac{3}{2})\varphi) = (-1)^j \sin\frac{2j\pi}{2p+1}$. As $j \neq 0$, it follows that $\sin((p+\frac{3}{2})\varphi) \neq 0$.

The equation (3.3.6) can be rewritten as follows

$$\frac{\sin((p+1)\varphi + \frac{\varphi}{2})}{\sin((p+1)\varphi - \frac{\varphi}{2})} = \frac{\sinh((p+1)\beta(\varphi) + \frac{\beta(\varphi)}{2})}{\sinh((p+1)\beta(\varphi) - \frac{\beta(\varphi)}{2})}.$$

Simplifying this equation and using $p = \frac{n}{2}$, we obtain (3.2.2), i.e.,

$$\tan\left(\frac{n+2}{2}\varphi\right) = \frac{1}{f(\varphi)}\tanh\left(\frac{n+2}{2}\beta(\varphi)\right).$$

Once again, from Lemma 3.3.2 we find that $\det T_{2p+1}(a-\lambda) = 0$ is equivalent to

$$\cos((p+1)\varphi)\cos((p+2)\xi(\varphi)) - \cos((p+1)\xi(\varphi))\cos((p+2)\varphi) = 0 \tag{3.3.7}$$

or

$$\sin((p+1)\varphi)\sin((p+2)\xi(\varphi)) - \sin((p+1)\xi(\varphi))\sin((p+2)\varphi) = 0. \tag{3.3.8}$$

Substituting $\xi(\varphi) = -i\beta(\varphi)$ in (3.3.7) and since $\cos(-ix) = \cosh x$, we obtain the equation

$$\frac{\cos((p+2)\varphi)}{\cos((p+1)\varphi)} = \frac{\cosh((p+2)\beta(\varphi))}{\cosh((p+1)\beta(\varphi))}.$$

Again, if $\cos((p+1)\varphi) = 0$ then, $\cos((p+2)\varphi) \neq 0$. Therefore, the last equation can be rewritten as follows

$$\frac{\cos((p+\frac{3}{2})\varphi + \frac{\varphi}{2})}{\cos((p+\frac{3}{2})\varphi - \frac{\varphi}{2})} = \frac{\cosh((p+\frac{3}{2})\beta(\varphi) + \frac{\beta(\varphi)}{2})}{\cosh((p+\frac{3}{2})\beta(\varphi) - \frac{\beta(\varphi)}{2})}.$$

Simplifying this equation and using $p = \frac{n}{2} - \frac{1}{2}$, we obtain (3.2.1).

Analogously, substituting $\xi(\varphi) = -i\beta(\varphi)$ in (3.3.8) with $\sin(-ix) = -i\sinh x$, we arrive at the equation

$$\frac{\sin((p+2)\varphi)}{\sin((p+1)\varphi)} = \frac{\sinh((p+2)\beta(\varphi))}{\sinh((p+1)\beta(\varphi))}.$$

Once again, if $\sin((p+1)\varphi) = 0$, then $\sin((p+2)\varphi) \neq 0$. Then, the last equation can be rewritten as follows

$$\frac{\sin((p+\frac{3}{2})\varphi + \frac{\varphi}{2})}{\sin((p+\frac{3}{2})\varphi - \frac{\varphi}{2})} = \frac{\sinh((p+\frac{3}{2})\beta(\varphi) + \frac{\beta(\varphi)}{2})}{\sinh((p+\frac{3}{2})\beta(\varphi) - \frac{\beta(\varphi)}{2})}.$$

Finally, simplifying this equation and using $p = \frac{n}{2} - \frac{1}{2}$, we get (3.2.2). □

3.4 Proof of the main results

Proof of Theorem 3.2.1. This theorem follows directly from Lemma 3.3.3. □

Now, we introduce two functions $F : [0,\pi] \times \mathbb{N} \to \mathbb{R}$ and $G : [0,\pi] \times \mathbb{N} \to \mathbb{R}$ given by

$$F(\varphi, n) := \frac{n+2}{2}\varphi + \arctan\left(f(\varphi)\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)\right)$$

and

$$G(\varphi, n) := \frac{n+2}{2}\varphi - \arctan\left(\frac{1}{f(\varphi)}\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)\right).$$

The functions F and G will have an important role in this work, the solutions of

$$F(\varphi, n) = \pi j \quad \text{and} \quad G(\varphi, n) = \pi j \quad (j \in \mathbb{N}),$$

will allow us to find asymptotic formulas for the eigenvalues of $T_n(a)$.

Lemma 3.4.1. *If n is sufficiently large, then*

(*i*) *F is increasing on $[0,\pi]$.*

(*ii*) *G is increasing on $[\frac{\pi}{2(n+2)}, \pi]$.*

Proof. (*i*) For $\varphi \in [0,\pi]$, we have

$$F'(\varphi, n) = \frac{n+2}{2} + \frac{\frac{n+2}{2} f^2(\varphi)\operatorname{sech}^2(\frac{n+2}{2}\beta(\varphi))}{1 + f^2(\varphi)\tanh^2(\frac{n+2}{2}\beta(\varphi))} + \frac{f'(\varphi)\tanh(\frac{n+2}{2}\beta(\varphi))}{1 + f^2(\varphi)\tanh^2(\frac{n+2}{2}\beta(\varphi))},$$

where $f'(\varphi) = -\frac{\sin\frac{\varphi}{2}}{2\sqrt{1+\sin^2\frac{\varphi}{2}}}\,(1 + f^2(\varphi))$.

Since $\frac{\frac{n+2}{2} f^2(\varphi)\operatorname{sech}^2(\frac{n+2}{2}\beta(\varphi))}{1+f^2(\varphi)\tanh^2(\frac{n+2}{2}\beta(\varphi))} \geq 0$ for $\varphi \in [0,\pi]$, it follows that

$$F'(\varphi, n) \geq \frac{n+2}{2} + \frac{f'(\varphi)\tanh(\frac{n+2}{2}\beta(\varphi))}{1 + f^2(\varphi)\tanh^2(\frac{n+2}{2}\beta(\varphi))}. \tag{3.4.1}$$

On the other hand, taking into account that f decreases on $[0,\pi]$ and using the formula of f, we see that $0 \leq f(\varphi) \leq 1$. Accordingly, combining this estimate and the formula of f', we show that

$$|f'(\varphi)| \leq 1, \quad \varphi \in [0,\pi].$$

Additionally, the function β satisfies the estimate $0 \le \beta(\varphi) \le 2\log\left(1+\sqrt{2}\right)$, since β increases on $[0,\pi]$. Hence, if n is large enough, then we deduce

$$\left|\frac{f'(\varphi)\tanh\left(\frac{n+2}{2}\beta(\varphi)\right)}{1+f^2(\varphi)\tanh^2\left(\frac{n+2}{2}\beta(\varphi)\right)}\right| < \tanh\left(\frac{n+2}{2}\beta(\varphi)\right) \le \tanh\left(\log\left(1+\sqrt{2}\right)(n+2)\right)$$
$$= 1 + O\left(e^{-2\log\left(1+\sqrt{2}\right)(n+2)}\right) < 1.2.$$

Substituting this inequality in (3.4.1) we arrive at the estimate

$$F'(\varphi,n) > \frac{n}{2} + 1 - 1.2 = \frac{n}{2} - \frac{1}{5}.$$

Therefore, if n satisfies $\frac{n}{2} > \frac{1}{5}$, then $F'(\varphi,n) > 0$.

(ii) Similarly, for $\varphi \in [\frac{\pi}{2(n+2)}, \pi]$ we obtain that

$$G'(\varphi,n) = \frac{n+2}{2} - \frac{\frac{n+2}{2}\operatorname{sech}^2(\frac{n+2}{2}\beta(\varphi))}{1+\frac{1}{f^2(\varphi)}\tanh^2(\frac{n+2}{2}\beta(\varphi))} + \frac{f'(\varphi)\tanh(\frac{n+2}{2}\beta(\varphi))}{f^2(\varphi)+\tanh^2(\frac{n+2}{2}\beta(\varphi))}.$$

Since $|f'(\varphi)| \le 1$ for $\varphi \in [0,\pi]$, we arrive at the estimate

$$G'(\varphi,n) > \frac{n+2}{2} - \frac{n+2}{2\cosh^2(\frac{n+2}{2}\beta(\varphi))} - \coth\left(\frac{n+2}{2}\beta(\varphi)\right). \tag{3.4.2}$$

Given that β increases on $[0,\pi]$ and $\frac{\pi}{2(n+2)} \le \varphi \le \pi$, it follows that $\frac{n+2}{2}\beta\left(\frac{\pi}{2(n+2)}\right) \le \frac{n+2}{2}\beta(\varphi)$. Therefore, if n is large enough and taking into account the asymptotic expansion $\beta(\varphi) = \varphi + O(\varphi^3)$ as $\varphi \to 0^+$, then we get

$$\frac{1}{\cosh^2(\frac{n+2}{2}\beta(\varphi))} \le \left(\frac{2}{e^{\frac{1}{2}(n+2)(\frac{\pi}{2(n+2)}+O((\frac{\pi}{2(n+2)})^3))} + e^{-\frac{1}{2}(n+2)(\frac{\pi}{2(n+2)}+O((\frac{\pi}{2(n+2)})^3))}}\right)^2$$
$$= \left(\frac{2}{e^{\frac{\pi}{4}+O\left(\frac{1}{(n+2)^2}\right)} + e^{-\frac{\pi}{4}+O\left(\frac{1}{(n+2)^2}\right)}}\right)^2 \le \left(\frac{2.3}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)^2 < 0.8$$

and

$$\coth\left(\frac{n+2}{2}\beta(\varphi)\right) \le \coth\left(\frac{n+2}{2}\beta\left(\frac{\pi}{2(n+2)}\right)\right) = \coth\left(\frac{\pi}{4} + O\left(\frac{1}{(n+2)^2}\right)\right)$$
$$= \frac{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}{e^{\frac{\pi}{4}}-e^{-\frac{\pi}{4}}} + \frac{c_3}{(n+2)^2} < 1.6,$$

where c_3 does not depend on n.

Using these inequalities in (3.4.2) we conclude that

$$G'(\varphi,n) > \frac{n}{10} - \frac{7}{5}.$$

Finally, if n satisfies $n > 14$, then $G'(\varphi,n) > 0$. $\square$

Proof of Theorem 3.2.2. Let n be an even number.

(i) Taking into account that for each j the function F is continuous on the segment $\left[\frac{(2j-1)\pi}{n+2}, \frac{2\pi j}{n+2}\right]$ and as

$$\arctan\left(f\left(\frac{(2j-1)\pi}{n+2}\right)\tanh\left(\frac{n+2}{2}\beta\left(\frac{(2j-1)\pi}{n+2}\right)\right)\right)-\frac{\pi}{2}<0,$$
$$\arctan\left(f\left(\frac{2\pi j}{n+2}\right)\tanh\left(\frac{n+2}{2}\beta\left(\frac{2\pi j}{n+2}\right)\right)\right)>0,$$

it is easy to verify that

$$F\left(\frac{(2j-1)\pi}{n+2},n\right)<\pi j, \quad F\left(\frac{2\pi j}{n+2},n\right)>\pi j.$$

Accordingly, the Intermediate Value Theorem tells us that (3.2.3) have solutions, i.e., for $1\le j\le\frac{n}{2}$ there is at least one $\varphi_{2j-1}^{(n)}$ with $F(\varphi_{2j-1}^{(n)},n)=\pi j$. By Lemma 3.4.1, F is a strictly increasing function on $[0,\pi]$ and we conclude that these solutions are unique.

(ii) Similarly, for each j the function G is continuous on the segment $\left[\frac{2\pi j}{n+2}, \frac{(2j+1)\pi}{n+2}\right]$ and given that

$$\arctan\left(\frac{1}{f\left(\frac{2\pi j}{n+2}\right)}\tanh\left(\frac{n+2}{2}\beta\left(\frac{2\pi j}{n+2}\right)\right)\right)>0$$

and

$$\frac{\pi}{2}-\arctan\left(\frac{1}{f\left(\frac{(2j+1)\pi}{n+2}\right)}\tanh\left(\frac{n+2}{2}\beta\left(\frac{(2j+1)\pi}{n+2}\right)\right)\right)>0,$$

we easily see that

$$G\left(\frac{2\pi j}{n+2},n\right)<\pi j, \quad G\left(\frac{(2j+1)\pi}{n+2},n\right)>\pi j.$$

Once again, the Intermediate Value Theorem tells us that (3.2.4) have solutions, i.e., for $1\le j\le\frac{n}{2}$ there is at least one $\varphi_{2j}^{(n)}$ with $G(\varphi_{2j}^{(n)},n)=\pi j$. By Lemma 3.4.1, G is a strictly increasing function on $[\frac{\pi}{2(n+2)},\pi]$ and we conclude that these solutions are unique.

(iii) For example, when n is an even number, from (i) and (ii) it follows that

$$\varphi_1^{(n)}<\varphi_2^{(n)}<\cdots<\varphi_{2j-1}^{(n)}<\varphi_{2j}^{(n)}<\cdots<\varphi_{n-1}^{(n)}<\varphi_n^{(n)}.$$

Furthermore, given that g is a strictly increasing function on $[0,\pi]$, we obtain

$$g(\varphi_1^{(n)})<g(\varphi_2^{(n)})<\cdots<g(\varphi_{2j-1}^{(n)})<g(\varphi_{2j}^{(n)})<\cdots<g(\varphi_{n-1}^{(n)})<g(\varphi_n^{(n)}).$$

Thus, using these inequalities and taking into account that $\lambda=g(\varphi)$, we show that

$$\lambda_1^{(n)}<\lambda_2^{(n)}<\cdots<\lambda_{n-1}^{(n)}<\lambda_n^{(n)}.$$

□

To simplify the notation in the following proofs, we will denote $\frac{n+2}{2}$ by the variable q.

Proof of Theorem 3.2.4. (i) Making the variable change $\frac{u}{q} = \varphi - d_-$ in (3.2.3), we arrive at the equation

$$u = -\arctan\left(f\left(d_- + \frac{u}{q}\right)\tanh\left(q\beta\left(d_- + \frac{u}{q}\right)\right)\right), \tag{3.4.3}$$

where $u \in (-\frac{\pi}{2}, 0)$.

By Taylor's theorem, we have

$$f\left(d_- + \frac{u}{q}\right) = f(d_-) + \frac{f'(d_-)u}{q} + O\left(\frac{u^2}{q^2}\right) \quad (q \to \infty) \tag{3.4.4}$$

and

$$q\beta\left(d_- + \frac{u}{q}\right) = q\beta(d_-) + \beta'(d_-)u + \frac{\beta''(d_-)u^2}{2q} + O\left(\frac{u^3}{q^2}\right) \quad (q \to \infty). \tag{3.4.5}$$

Replacing (3.4.4) and (3.4.5) in (3.4.3), we obtain

$$u = -\arctan\left[\left(f(d_-) + \frac{f'(d_-)u}{q} + O\left(\frac{u^2}{q^2}\right)\right)\tanh\left(q\beta(d_-) + \beta'(d_-)u + \frac{\beta''(d_-)u^2}{2q} + O\left(\frac{u^3}{q^2}\right)\right)\right]. \tag{3.4.6}$$

We will seek an asymptotic formula for the solution $u_{*,j}$ of (3.4.3) with the form

$$u_{*,j} := u_{1,j} + \frac{u_{2,j}}{q} + O\left(\frac{1}{q^2}\right) \quad (q \to \infty) \tag{3.4.7}$$

where the coefficients $u_{1,j}$ and $u_{2,j}$ will be properly defined.

Substituting (3.4.7) in (3.4.4) and (3.4.5), as $q \to \infty$, we arrive at

$$f(d_-) + \frac{f'(d_-)u_{*,j}}{q} + O\left(\frac{u_{*,j}^2}{q^2}\right) = f(d_-) + \frac{f'(d_-)u_{1,j}}{q} + O\left(\frac{1}{q^2}\right) \tag{3.4.8}$$

and

$$q\beta(d_-) + \beta'(d_-)u_{*,j} + \frac{\beta''(d_-)u_{*,j}^2}{2q} + O\left(\frac{u_{*,j}^3}{q^2}\right) = \Psi_j^- + \frac{\beta''(d_-)u_{1,j}^2 + 2\beta'(d_-)u_{2,j}}{2q} + O\left(\frac{1}{q^2}\right), \tag{3.4.9}$$

where $\Psi_j^- := q\beta(d_-) + \beta'(d_-)u_{1,j}$.

Combining (3.4.7), (3.4.8), and (3.4.9) with (3.4.6), it follows that

$$\begin{aligned} u_{1,j} + \frac{u_{2,j}}{q} = &-\left(\frac{f(d_-)[\beta''(d_-)u_{1,j}^2 + 2\beta'(d_-)u_{2,j}] + f'(d_-)u_{1,j}\sinh(2\Psi_j^-)}{2\cosh^2\Psi_j^-[1 + f^2(d_-)\tanh^2\Psi_j^-]}\right)\frac{1}{q} \\ &- \arctan\left(f(d_-)\tanh\Psi_j^-\right) + O\left(\frac{1}{q^2}\right) \quad (q \to \infty). \end{aligned}$$

Notice that $u_{1,j}$ can be defined as the solution of the equation

$$u = -\arctan(f(d_-)\tanh\Psi_j^-(u)),$$

where $\Psi_j^-(u) := q\beta(d_-) + \beta'(d_-)u$. Indeed, if we define $F_1(u) := u + \arctan(f(d_-)\tanh\Psi_j^-(u))$ for $u \in [-\frac{\pi}{2}, 0]$, then

$$F_1'(u) = 1 + \frac{f^2(d_-)}{\cosh^2\Psi_j^-(u) + f^2(d_-)\sinh^2\Psi_j^-(u)} > 0.$$

This implies that F_1 is a strictly increasing function on $(-\frac{\pi}{2}, 0)$. Since F_1 is continuous on $[-\frac{\pi}{2}, 0]$ with $F_1(-\frac{\pi}{2}) < 0$ and $F_1(0) > 0$, there is a unique $u_{1,j}$ such that

$$u_{1,j} = -\arctan(f(d_-)\tanh\Psi_j^-(u_{1,j})). \tag{3.4.10}$$

Additionally, $u_{2,j}$ should satisfy

$$u_{2,j} = -\frac{f(d_-)[\beta''(d_-)u_{1,j}^2 + 2\beta'(d_-)u_{2,j}] + f'(d_-)u_{1,j}\sinh(2\Psi_j^-)}{2\left[\cosh^2\Psi_j^- + f^2(d_-)\sinh^2\Psi_j^-\right]}.$$

A simple calculation reveals that

$$u_{2,j} = -\frac{f(d_-)\beta''(d_-)u_{1,j}^2 + f'(d_-)u_{1,j}\sinh(2\Psi_j^-)}{2\left[1 + f^2(d_-)\right]\cosh^2\Psi_j^-}. \tag{3.4.11}$$

Taking into account that $u_{*,j} = q\left(\varphi_{2j-1}^{(n)} - d_-\right)$ and $q = \frac{n+2}{2}$, we get the asymptotic formula (3.2.7), i.e.,

$$\varphi_{2j-1}^{(n)} = d_- + \frac{2u_{1,j}}{n+2} + \frac{4u_{2,j}}{(n+2)^2} + O\left(\frac{1}{n^3}\right) \quad (n \to \infty),$$

for $j = 1, \ldots, [\frac{n+1}{2}]$.

(ii) Let $h(\varphi) := \frac{1}{f(\varphi)}$. The variable change $\frac{w}{q} = \varphi - d_+$ in (3.2.4) leads us to

$$w = -\operatorname{arccot}\left(h\left(d_+ + \frac{w}{q}\right)\tanh\left(q\beta\left(d_+ + \frac{w}{q}\right)\right)\right), \tag{3.4.12}$$

where $w \in (-\frac{\pi}{2}, 0)$.

Substituting (3.4.4) and (3.4.5) in (3.4.12), with d_+ and h instead of d_- and f, it follows that

$$w = -\operatorname{arccot}\left[\left(h(d_+) + \frac{h'(d_+)w}{q} + O\left(\frac{w^2}{q^2}\right)\right)\tanh\left(q\beta(d_+) + \beta'(d_+)w + \frac{\beta''(d_+)w^2}{2q} + O\left(\frac{w^3}{q^2}\right)\right)\right]. \tag{3.4.13}$$

We also seek an asymptotic formula for the solution $w_{*,j}$ of (3.4.12) with the form

$$w_{*,j} := w_{1,j} + \frac{w_{2,j}}{q} + O\left(\frac{1}{q^2}\right) \quad (q \to \infty), \tag{3.4.14}$$

where the coefficients $w_{1,j}$ and $w_{2,j}$ will be properly defined.

Now, as $q \to \infty$, using (3.4.14) we arrive at

$$h(d_+) + \frac{h'(d_+)w_{*,j}}{q} + O\left(\frac{w_{*,j}^2}{q^2}\right) = h(d_+) + \frac{h'(d_+)w_{1,j}}{q} + O\left(\frac{1}{q^2}\right) \tag{3.4.15}$$

and

$$q\beta(d_+) + \beta'(d_+)w_{*,j} + \frac{\beta''(d_+)w_{*,j}^2}{2q} + O\left(\frac{w_{*,j}^3}{q^2}\right) = \Psi_j^+ + \frac{\beta''(d_+)w_{1,j}^2 + 2\beta'(d_+)w_{2,j}}{2q} + O\left(\frac{1}{q^2}\right), \tag{3.4.16}$$

where $\Psi_j^+ := q\beta(d_+) + \beta'(d_+)w_{1,j}$.

Substituting (3.4.14), (3.4.15), and (3.4.16) in (3.4.13), it follows that

$$w_{1,j} + \frac{w_{2,j}}{q} = \left(\frac{h(d_+)[\beta''(d_+)w_{1,j}^2 + 2\beta'(d_+)w_{2,j}] + h'(d_+)w_{1,j}\sinh(2\Psi_j^+)}{2\cosh^2\Psi_j^+[1 + h^2(d_+)\tanh^2\Psi_j^+]} \right) \frac{1}{q}$$
$$- \operatorname{arccot}\left(h(d_+)\tanh\Psi_j^+\right) + O\left(\frac{1}{q^2}\right) \quad (q \to \infty).$$

Similarly, notice that $w_{1,j}$ can be defined as the solution of the equation

$$w = -\operatorname{arccot}(h(d_+)\tanh\Psi_j^+(w)),$$

where $\Psi_j^+(w) := q\beta(d_+) + \beta'(d_+)w$. Indeed, if we define $F_2(w) := w + \operatorname{arccot}(h(d_+)\tanh\Psi_j^+(w))$ for $w \in [-\frac{\pi}{2}, 0]$, then

$$F_2'(w) = 1 - \frac{1}{\cosh^2\Psi_j^+(w)[1 + h^2(d_+)\tanh^2\Psi_j^+(w)]}.$$

Additionally, we have

$$q\beta(d_+) - \frac{\beta'(d_+)\pi}{2} < \Psi_j^+(w) < q\beta(d_+),$$

for every $w \in (-\frac{\pi}{2}, 0)$. Since $d_+ > \frac{\pi}{q}$ and $\beta'(d_+) < 1$, we see that

$$q\beta\left(\frac{\pi}{q}\right) - \frac{\pi}{2} < q\beta(d_+) - \frac{\beta'(d_+)\pi}{2} < \Psi_j^+(w). \tag{3.4.17}$$

Using the inequality (3.4.17) we obtain

$$\frac{1}{\cosh^2\Psi_j^+(w)[1 + h^2(d_+)\tanh^2\Psi_j^+(w)]} < \frac{1}{\cosh^2\Psi_j^+(w)} < \frac{1}{\cosh^2(q\beta(\frac{\pi}{q}) - \frac{\pi}{2})}$$
$$= \left(\frac{2}{e^{q\beta(\frac{\pi}{q}) - \frac{\pi}{2}} + e^{-q\beta(\frac{\pi}{q}) + \frac{\pi}{2}}}\right)^2 < \left(\frac{4}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}\right)^2 < 0.8.$$

This implies that F_2 is a strictly increasing function on $(-\frac{\pi}{2}, 0)$. Given that F_2 is continuous on $[-\frac{\pi}{2}, 0]$ with $F_2(-\frac{\pi}{2}) < 0$ and $F_2(0) > 0$, there is a unique $w_{1,j}$ such that

$$w_{1,j} = -\operatorname{arccot}(h(d_+)\tanh\Psi_j^+(w_{1,j})). \tag{3.4.18}$$

On the other hand, the coefficient $w_{2,j}$ satisfies the equation

$$w_{2,j} = \frac{h'(d_+)w_{1,j}\sinh(2\Psi_j^+) + h(d_+)[\beta''(d_+)w_{1,j}^2 + 2\beta'(d_+)w_{2,j}]}{2\left[\cosh^2\Psi_j^+ + h^2(d_+)\sinh^2\Psi_j^+\right]}.$$

A simple calculation reveals that

$$w_{2,j} = \frac{f(d_+)\beta''(d_+)w_{1,j}^2 - f'(d_+)w_{1,j}\sinh(2\Psi_j^+)}{2\,[1 + f^2(d_+)]\sinh^2\Psi_j^+}. \tag{3.4.19}$$

Taking into account that $w_{*,j} = q\left(\varphi_{2j}^{(n)} - d_+\right)$ and $q = \frac{n+2}{2}$, we get the asymptotic formula (3.2.8), i.e.,

$$\varphi_{2j}^{(n)} = d_+ + \frac{2w_{1,j}}{n+2} + \frac{4w_{2,j}}{(n+2)^2} + O\left(\frac{1}{n^3}\right) \quad (n \to \infty),$$

for $j = 1, \dots, [\frac{n}{2}]$. □

To deduce the asymptotic formulas of the eigenvalues, the idea is to use the formulas $\lambda_{2j-1}^{(n)} = g(\varphi_{2j-1}^{(n)})$ and $\lambda_{2j}^{(n)} = g(\varphi_{2j}^{(n)})$. By Taylor's theorem applied to (3.2.7) and (3.2.8), as $q = \frac{n+2}{2} \to \infty$, there are points $\zeta_1 \in (d_- - \frac{\pi}{2q}, d_-)$ and $\zeta_2 \in (d_+ - \frac{\pi}{2q}, d_+)$ such that

$$\begin{aligned}\lambda_{2j-1}^{(n)} &= g(d_-) + g'(d_-)\left(\frac{u_{1,j}}{q} + \frac{u_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right) + \frac{g''(d_-)}{2}\left(\frac{u_{1,j}}{q} + \frac{u_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right)^2 \\ &\quad + \frac{g'''(\zeta_1)}{6}\left(\frac{u_{1,j}}{q} + \frac{u_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right)^3\end{aligned} \tag{3.4.20}$$

and

$$\begin{aligned}\lambda_{2j}^{(n)} &= g(d_+) + g'(d_+)\left(\frac{w_{1,j}}{q} + \frac{w_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right) + \frac{g''(d_+)}{2}\left(\frac{w_{1,j}}{q} + \frac{w_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right)^2 \\ &\quad + \frac{g'''(\zeta_2)}{6}\left(\frac{w_{1,j}}{q} + \frac{w_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)\right)^3.\end{aligned} \tag{3.4.21}$$

Proof of Theorem 3.2.5. (i) Notice that g', g'', and g''' are bounded. The relation $q = \frac{n+2}{2}$ in (3.4.20) shows that

$$\lambda_{2j-1}^{(n)} = g(d_-) + \frac{2u_{1,j}g'(d_-)}{n+2} + \frac{2(u_{1,j})^2 g''(d_-) + 4u_{2,j}g'(d_-)}{(n+2)^2} + O\left(\frac{1}{n^3}\right) \quad (n \to \infty).$$

Now, we are going to analyze $u_{1,j}$ and $u_{2,j}$. Let $c_4 := \beta(\pi\varepsilon)$. From (3.4.10) we know that

$$u_{1,j} = -\arctan(f(d_-)\tanh \Psi_j^-(u_{1,j})),$$

where $\Psi_j^-(u_{1,j}) = q\beta(d_-) + \beta'(d_-)u_{1,j}$. Given that $\pi\varepsilon < d_- < (1-\varepsilon)\pi$ and β is an increasing function, we deduce that $-2q\beta(d_-) < -2q\beta(\pi\varepsilon)$. Furthermore, if q is sufficiently large, then we see that

$$f(d_-)\tanh \Psi_j^-(u_{1,j}) = f(d_-)(1 + O(e^{(-2\beta(\pi\varepsilon)q)})) = f(d_-) + O(e^{-2c_4 q}) \quad (q \to \infty),$$

which implies that

$$u_{1,j} = -\arctan f(d_-) + O\left(e^{-c_4(n+2)}\right) \quad (n \to \infty).$$

Similarly, using (3.4.11) we get

$$u_{2,j} = \frac{f'(d_-)\arctan f(d_-)}{1 + f^2(d_-)} + O\left(e^{-c_4(n+2)}\right) \quad (n \to \infty).$$

Therefore, if $u_{1,j}^* := -\arctan f(d_-)$ and $u_{2,j}^* := \frac{f'(d_-)\arctan f(d_-)}{1+f^2(d_-)}$, then it follows that

$$|u_{1,j} - u_{1,j}^*| = O\left(e^{-c_4(n+2)}\right), \quad |u_{2,j} - u_{2,j}^*| = O\left(e^{-c_4(n+2)}\right) \quad (n \to \infty).$$

(ii) Analogously, using (3.4.21) we arrive at the formula (3.2.10). On the other hand, we can analyze $w_{1,j}$ and $w_{2,j}$ as we did with $u_{1,j}$ and $u_{2,j}$ in the first part of the proof. Thus, from (3.4.18) and (3.4.19) we verify that $w_{1,j}$ and $w_{2,j}$ satisfy, as $n \to \infty$,

$$w_{1,j} = -\operatorname{arccot}\frac{1}{f(d_+)} + O\left(e^{-c_4(n+2)}\right)$$

and

$$w_{2,j} = \frac{f'(d_+)\operatorname{arccot}\frac{1}{f(d_+)}}{1+f^2(d_+)} + O\left(e^{-c_4(n+2)}\right).$$

Finally, if $w^*_{1,j} := -\operatorname{arccot}\frac{1}{f(d_+)}$ and $w^*_{2,j} := \frac{f'(d_+)\operatorname{arccot}\frac{1}{f(d_+)}}{1+f^2(d_+)}$, then it follows that

$$|w_{1,j} - w^*_{1,j}| = O\left(e^{-c_4(n+2)}\right), \quad |w_{2,j} - w^*_{2,j}| = O\left(e^{-c_4(n+2)}\right) \quad (n \to \infty).$$

□

Proof of Theorem 3.2.6. Recall that $g(\varphi) := M\sin^4(\varphi/2)$, for $\varphi \in [0,\pi]$.

(*i*) Given that $\lambda^{(n)}_{2j-1} = g(\varphi^{(n)}_{2j-1})$, where $\varphi^{(n)}_{2j-1}$ is given by (3.2.7), we obtain

$$\lambda^{(n)}_{2j-1} = M\sin^4\left(\frac{d_-}{2} + \frac{u_{1,j}}{2q} + \frac{u_{2,j}}{2q^2} + O\left(\frac{1}{q^3}\right)\right) \quad (q \to \infty).$$

Let $C_j := 1 + \frac{u_{1,j}}{j\pi}$. Taking into account that $\sin x = x - \frac{1}{6}x^3 + O(x^5)$, as $x \to 0$, a simple calculation shows that

$$\begin{aligned}
\lambda^{(n)}_{2j-1} &= C_j^4 d_-^4 \left[1 + \frac{u_{2,j}}{d_-q^2C_j} + O\left(\frac{1}{d_-q^3}\right) - \frac{d_-^2C_j^2}{24}\left(1 + \frac{u_{2,j}}{d_-q^2C_j} + O\left(\frac{1}{d_-q^3}\right)\right)^3 + O\left(d_-^4\right)\right]^4 \\
&= C_j^4 d_-^4 \left[1 + \frac{u_{2,j}}{d_-q^2C_j} + O\left(\frac{1}{d_-q^3}\right) + O\left(d_-^4\right)\right]^4 \\
&= C_j^4 d_-^4 + \frac{4u_{2,j}C_j^3d_-^3}{q^2} + O\left(\frac{d_-^3}{q^3}\right) + O\left(d_-^8\right) \quad (q \to \infty).
\end{aligned}$$

Substituting the expressions for C_j, d_-, and q we arrive at (3.2.11), i.e.,

$$\lambda^{(n)}_{2j-1} = \frac{(2\pi j + 2u_{1,j})^4}{(n+2)^4} + \frac{16u_{2,j}(2\pi j + 2u_{1,j})^3}{(n+2)^5} + O\left(\frac{d_-^3}{n^3}\right) + O\left(d_-^8\right) \quad (d_- \to 0),$$

where $u_{1,j}$ and $u_{2,j}$ are given by (3.4.10) and (3.4.11), respectively.

(*ii*) To deduce the formula (3.2.12), we can proceed as we did with (3.2.11). In this case $w_{1,j}$ and $w_{2,j}$ are given by (3.4.18) and (3.4.19), respectively. □

In Theorem 3.2.6, the constants $u_{1,j}$ and $w_{1,j}$ are obtained as the unique solution of the equations (3.4.10) and (3.4.18), respectively. In order to simplify the calculations, we present the following result.

Corollary 3.4.2. *Let* $a(t) = \left(\frac{1}{t} + t - 2\right)^2$ *and suppose that* $d_- \to 0$ *and* $d_+ \to 0$ *as* $n \to \infty$. *Then,*

(*i*) *If* $\pi j \gg 1$, *as* $j \to \infty$, *then*

$$|u_{1,j} - u^*_{1,j}| = O\left(e^{-\pi j}\right),$$

where $u^*_{1,j} := -\arctan f(d_-)$.

(*ii*) *If* $(j + \frac{1}{2})\pi \gg 1$, *as* $j \to \infty$, *then*

$$|w_{1,j} - w^*_{1,j}| = O\left(e^{-\pi j}\right),$$

where $w^*_{1,j} := -\operatorname{arccot}\frac{1}{f(d_+)}$.

(iii) If j is bounded, as $n \to \infty$, then

$$u_{1,j} = -\arctan(\tanh(\pi j + u_{1,j})) + O\left(\frac{1}{n^2}\right),$$
$$w_{1,j} = -\arctan\left(\coth\left(\left(j + \frac{1}{2}\right)\pi + w_{1,j}\right)\right) + O\left(\frac{1}{n^2}\right).$$

Proof. (i) From (3.4.10) we know that

$$u_{1,j} = -\arctan\left(f(d_-)\tanh\left(\frac{n+2}{2}\beta(d_-) + \beta'(d_-)u_{1,j}\right)\right). \tag{3.4.22}$$

Given that $d_- \to 0$, we get the asymptotic expansions $\beta(d_-) = d_- + O(d_-^3)$ and $\beta'(d_-) = 1 + O(d_-^2)$. Replacing these asymptotic expansions in (3.4.22) we deduce that

$$u_{1,j} = -\arctan\left(f(d_-)\tanh\left(\pi j + u_{1,j} + O\left(\frac{j^3}{n^2}\right)\right)\right) \quad (n \to \infty). \tag{3.4.23}$$

Now, if $\pi j \gg 1$ as $j \to \infty$, then we show that

$$\tanh\left(\pi j + u_{1,j} + O\left(\frac{j^3}{n^2}\right)\right) = 1 + O\left(e^{-\pi j}\right). \tag{3.4.24}$$

Combining (3.4.24) with (3.4.23) we see that

$$u_{1,j} = -\arctan f(d_-) + O\left(e^{-\pi j}\right) \quad (j \to \infty).$$

Once again, if $u_{1,j}^* := -\arctan f(d_-)$, then we obtain

$$|u_{1,j} - u_{1,j}^*| = O\left(e^{-\pi j}\right) \quad (j \to \infty).$$

(ii) Similarly, from (3.4.18) and given that $d_+ \to 0$ we arrive at

$$w_{1,j} = -\operatorname{arccot}\left(\frac{1}{f(d_+)}\tanh\left(\left(j + \frac{1}{2}\right)\pi + w_{1,j} + O\left(\frac{j^3}{n^2}\right)\right)\right).$$

On the other hand, if $\left(j + \frac{1}{2}\right)\pi \gg 1$ as $j \to \infty$, then it is possible, in a similar fashion, to show that

$$|w_{1,j} - w_{1,j}^*| = O\left(e^{-\pi j}\right),$$

where $w_{1,j}^* := -\operatorname{arccot}\frac{1}{f(d_+)}$.

(iii) Given that j is bounded and $d_- \to 0$ as $n \to \infty$, we have

$$f(d_-) = 1 + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

Substituting this asymptotic expansion in (3.4.22) it follows that

$$u_{1,j} = -\arctan(\tanh(\pi j + u_{1,j})) + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

Analogously, from (3.4.18) and taking into account that j is bounded and $d_+ \to 0$, as $n \to \infty$, we deduce that

$$w_{1,j} = -\arctan\left(\coth\left(\left(j + \frac{1}{2}\right)\pi + w_{1,j}\right)\right) + O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

□

Proof of Theorem 3.2.7. (i) Notice that, as $n \to \infty$,

$$g\left(d_{2j}^{(n)}\right) = M - \frac{M}{2}(d_{2j}^{(n)} - \pi)^2 + O((d_{2j}^{(n)} - \pi)^4),$$

which implies that $g'\left(d_{2j}^{(n)}\right) = -M(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3)$ and $g''\left(d_{2j}^{(n)}\right) = -M + O((d_{2j}^{(n)} - \pi)^2)$. Using these equalities in (3.2.9) we get

$$\lambda_{2j-1}^{(n)} = M - \frac{M(d_{2j}^{(n)} - \pi)^2}{2} - \frac{Mu_{1,j}^*(d_{2j}^{(n)} - \pi)}{q} - \frac{Mu_{2,j}^*(d_{2j}^{(n)} - \pi)}{q^2} - \frac{M(u_{1,j}^*)^2}{2q^2} + O\left((d_{2j}^{(n)} - \pi)^4\right) + O\left(\frac{1}{n^3}\right) \quad (n \to \infty). \tag{3.4.25}$$

From Theorem 3.2.5 we know that $u_{1,j}^* = -\arctan f(d_{2j}^{(n)})$. Furthermore, as $n \to \infty$, $f\left(d_{2j}^{(n)}\right) = f'(\pi)(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3)$ and $f'\left(d_{2j}^{(n)}\right) = f'(\pi) + O((d_{2j}^{(n)} - \pi)^2)$. Thus, we see that

$$u_{1,j}^* = -\arctan(f'(\pi)(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3)) = -f'(\pi)(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3),$$

i.e., $u_{1,j}^*$ is given by

$$u_{1,j}^* = -f'(\pi)(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3) \quad (n \to \infty). \tag{3.4.26}$$

Once again, by Theorem 3.2.5 we have

$$u_{2,j}^* = \frac{f'(d_{2j}^{(n)}) \arctan f(d_{2j}^{(n)})}{1 + f^2(d_{2j}^{(n)})}.$$

Similarly, taking into account the asymptotic expansions of $f\left(d_{2j}^{(n)}\right)$ and $f'\left(d_{2j}^{(n)}\right)$, it follows that $u_{2,j}^*$ is given by

$$u_{2,j}^* = (f'(\pi))^2(d_{2j}^{(n)} - \pi) + O((d_{2j}^{(n)} - \pi)^3) \quad (n \to \infty). \tag{3.4.27}$$

Substituting (3.4.26) and (3.4.27) in (3.4.25) with $q = \frac{n+2}{2}$, we arrive at (3.2.13), i.e.,

$$\lambda_{2j-1}^{(n)} = M - \frac{M(d_{2j}^{(n)} - \pi)^2}{2} + \frac{2Mf'(\pi)(d_{2j}^{(n)} - \pi)^2}{n+2} + O\left((d_{2j}^{(n)} - \pi)^4\right) + O\left(\frac{1}{n^3}\right) \quad (n \to \infty).$$

(ii) (3.2.14) can be obtained as we did with (3.2.13). □

3.5 Extreme eigenvalues

In this section, the well-known results of [31] and [45] are compared with our Theorems 3.2.6 and 3.2.7.

3.5.1 Smallest eigenvalues

For the eigenvalues near zero, we compare the first asymptotic term in (3.2.11) and (3.2.12) with (2.3.1).

Given that $a(t) = \left(\frac{1}{t} + t - 2\right)^2$ it follows that $g(\varphi) = a(e^{i\varphi}) = 4(\cos\varphi - 1)^2$ satisfies the conditions (1) and (2). Moreover, as $n \to \infty$, using $m = g(0) = 0$ and $g^{(4)}(0) = 4!$ we can rewrite (2.3.1) as

$$\lambda_{2k-1,n} = \left(\frac{2k\pi + \frac{E_{2k-1}-\pi}{2}}{n+3}\right)^4 + o(n^{-4}) \tag{3.5.1}$$

and

$$\lambda_{2k,n} = \left(\frac{(2k+1)\pi + \frac{E_{2k}-\pi}{2}}{n+3}\right)^4 + o(n^{-4}), \tag{3.5.2}$$

where $k = 1, 2, \ldots$. On the other hand, the constants E_{2k} and E_{2k-1} satisfy the following equations

$$\tan\left(\frac{E_{2k-1} - \pi}{4}\right) = -\tanh\left(\frac{(4k-1)\pi + E_{2k-1}}{4}\right) \tag{3.5.3}$$

and

$$\tan\left(\frac{E_{2k} + \pi}{4}\right) = \tanh\left(\frac{(4k+1)\pi + E_{2k}}{4}\right). \tag{3.5.4}$$

By Theorem 3.2.6, for every fixed k and $n \to \infty$, we get

$$\lambda_{2k-1}^{(n)} = \frac{(2k\pi + 2u_{1,k})^4}{(n+2)^4} + O\left(\frac{1}{n^5}\right) \tag{3.5.5}$$

and

$$\lambda_{2k}^{(n)} = \frac{((2k+1)\pi + 2w_{1,k})^4}{(n+2)^4} + O\left(\frac{1}{n^5}\right), \tag{3.5.6}$$

where $u_{1,k}$ and $w_{1,k}$ satisfy

$$u_{1,k} = -\arctan\left(f(d_-)\tanh\left(\frac{n+2}{2}\beta(d_-) + \beta'(d_-)u_{1,k}\right)\right),$$
$$w_{1,k} = -\arctan\left(f(d_+)\coth\left(\frac{n+2}{2}\beta(d_+) + \beta'(d_+)w_{1,k}\right)\right),$$

where $d_- = \frac{2k\pi}{n+2}$ and $d_+ = \frac{(2k+1)\pi}{n+2}$.

Notice that the first term in (3.5.1) and (3.5.2) coincides with the first term in (3.5.5) and (3.5.6), respectively, if and only if $u_{1,k} = \frac{E_{2k-1}-\pi}{4}$ and $w_{1,k} = \frac{E_{2k}-\pi}{4}$. Hence, if $u_{1,k} = \frac{E_{2k-1}-\pi}{4}$ and $w_{1,k} = \frac{E_{2k}-\pi}{4}$, (3.5.3) and (3.5.4) can be rewritten as

$$\tan u_{1,k} = -\tanh(k\pi + u_{1,k}) \quad \text{and} \quad \tan w_{1,k} = -\coth\left(\left(k + \frac{1}{2}\right)\pi + w_{1,k}\right),$$

respectively.

Since $d_- \to 0$, $d_+ \to 0$, and k is fixed, in view of Corollary 3.4.2, as $n \to \infty$, we know that

$$\tan(u_{1,k}) = -\tanh(\pi k + u_{1,k}) + O\left(\frac{1}{n^2}\right),$$
$$\tan(w_{1,k}) = -\coth\left(\left(k + \frac{1}{2}\right)\pi + w_{1,k}\right) + O\left(\frac{1}{n^2}\right),$$

which implies that Theorems 3.2.6 and 2.3.1 coincide, when k is fixed.

Remark 3.5.1. The formulas obtained in [31] work for fixed eigenvalues (j is fixed). Our formulas work for fixed j as well, but we can consider new cases, for example $j \ll n$.

3.5.2 Largest eigenvalues

For the eigenvalues near M, we compare now the asymptotic terms of (3.2.13) and (3.2.14) with (2.3.2).

In order to satisfy the condition (3), we need to modify the symbol a. Therefore, we consider $a_1(t) := a(-t) = \left(\frac{1}{t} + t + 2\right)^2$ and $g(\varphi) := a_1(e^{i\varphi})$ for $\varphi \in [-\pi, \pi]$. Notice that $T_n(a)$ and $T_n(a_1)$ are similar matrices with same eigenvalues.

Taking into account that $t = e^{i\varphi}$ and $g'(\varphi) = 2i(e^{-i\varphi} + e^{i\varphi} + 2)(e^{i\varphi} - e^{-i\varphi})$ for $\varphi \in [-\pi, \pi]$, we can rewrite $\rho - 2$ as

$$\rho - 2 = \frac{1}{\pi} \int_{\mathbb{T}} \left(\frac{\frac{2i(t-1)(t+1)^3}{t^2}}{(t^{-1} + t + 2)^2 - 16} - \frac{4i}{t - t^{-1}} \right) \frac{t+1}{t(t-1)} dt = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{-4(t^2 + 4t + 1)}{t(t^2 + 6t + 1)} dt.$$

Therefore, if $h_2(t) := \frac{-4(t^2+4t+1)}{t(t^2+6t+1)}$ for $t \in \mathbb{T}$, then using the Residue Theorem we get

$$\rho = 2 + \mathrm{Res}(h_2, 0) + \mathrm{Res}(h_2, -3 + 2\sqrt{2}) = -2 + \sqrt{2}.$$

Accordingly, for every $k = 1, 2, \ldots$ with $n \to \infty$, (2.3.2) can be written as

$$\lambda_{2k,n} = 16 - \frac{8(2k\pi)^2}{(n+1)^2} + \frac{8(2 - \sqrt{2})(2k\pi)^2}{(n+1)^3} + o(n^{-3}) \tag{3.5.7}$$

and

$$\lambda_{2k-1,n} = 16 - \frac{8((2k-1)\pi)^2}{(n+1)^2} + \frac{8(2 - \sqrt{2})((2k-1)\pi)^2}{(n+1)^3} + o(n^{-3}). \tag{3.5.8}$$

From Theorem 3.2.7, as $n \to \infty$, we have

$$\lambda_{2j-1}^{(n)} = 16 - 8(d_{2j}^{(n)} - \pi)^2 + \frac{32 f'(\pi)(d_{2j}^{(n)} - \pi)^2}{n+2} + O\left((d_{2j}^{(n)} - \pi)^4\right) \tag{3.5.9}$$

and

$$\lambda_{2j}^{(n)} = 16 - 8(d_{2j+1}^{(n)} - \pi)^2 + \frac{32 f'(\pi)(d_{2j+1}^{(n)} - \pi)^2}{n+2} + O\left((d_{2j+1}^{(n)} - \pi)^4\right), \tag{3.5.10}$$

where $d_{2j}^{(n)} = \frac{2\pi j}{n+2}$, $d_{2j+1}^{(n)} = \frac{(2j+1)\pi}{n+2}$ and $f'(\pi) = -\frac{\sqrt{2}}{4}$.

On the other hand, if n is an even number, then $\pi - d_{2j}^{(n)} = \frac{(n-2j+2)\pi}{n+2}$ for $j = \frac{n}{2}, \frac{n}{2} - 1, \ldots$. Thus, making the variable change $2k = n - 2j + 2$ in (3.5.9), for $k = 1, 2, \ldots$, we arrive at the equation

$$\lambda_{n-2k+1}^{(n)} = 16 - \frac{8(2k\pi)^2}{(n+2)^2} + \frac{32 f'(\pi)(2k\pi)^2}{(n+2)^3} + O\left(\frac{k^4}{n^4}\right).$$

Since $(1 + \frac{1}{n+1})^{-p} = 1 - \frac{p}{n+1} + O\left(\frac{1}{n^2}\right)$ ($p \in \mathbb{R}$), we see that

$$\lambda_{n-2k+1}^{(n)} = 16 - \frac{8(2k\pi)^2}{(n+1)^2} + \frac{(16 + 32 f'(\pi))(2k\pi)^2}{(n+1)^3} + O\left(\frac{k^4}{n^4}\right).$$

Given that $f'(\pi) = -\frac{\sqrt{2}}{4}$ it follows that $16 + 32f'(\pi) = 8(2-\sqrt{2})$. Therefore, we show that

$$\lambda_{n-2k+1}^{(n)} = 16 - \frac{8(2k\pi)^2}{(n+1)^2} + \frac{8(2-\sqrt{2})(2k\pi)^2}{(n+1)^3} + O\left(\frac{k^4}{n^4}\right),$$

which coincides with (3.5.7), when k is fixed.

Similarly, from (3.5.10) we can deduce that

$$\lambda_{n-2k+2}^{(2)} = 16 - \frac{8((2k-1)\pi)^2}{(n+1)^2} + \frac{8(2-\sqrt{2})((2k-1)\pi)^2}{(n+1)^3} + O\left(\frac{k^4}{n^4}\right),$$

which coincides with (3.5.8), when k is fixed. Now, if n is an odd number, then we can proceed as we did for an even number.

3.6 Numerical tests

The main idea of the theorems in Section 3.2 is to provide us asymptotic expansions which reveal the fine structure of the eigenvalues bulk and show that this structure is essentially independent of the matrix dimension n. Furthermore, our asymptotic formulas also have the potential to be of use for design of numerical algorithms for the computation of the eigenvalues. For instance, we make all numerical tests of the eigenvalues of $T_n(a)$ for several moderately sized n by *Matlab R*2015*a*. Here, all exact eigenvalues are determined by means of the Matlab function *eig*.

The numerical tests in this section allow us to check that we calculated the coefficients in the asymptotic formulas correctly and we want to demonstrate that the asymptotic formulas deliver acceptable approximations not only for astronomically large n, but already for n in the early hundreds.

Denote by $\lambda_{2j-1}^{(n,k)}$ and $\lambda_{2j}^{(n,k)}$ the approximations of $\lambda_{2j-1}^{(n)}$ and $\lambda_{2j}^{(n)}$ with k terms, respectively. These approximations are obtained from our formulas (3.2.9), (3.2.10), (3.2.11), (3.2.12), (3.2.13), and (3.2.14). For example, from (3.2.9), we have

$$\lambda_{2j-1}^{(n,2)} = g(d_-) + \frac{2u_{1,j}^* g'(d_-)}{n+2}, \quad \lambda_{2j-1}^{(n,3)} = g(d_-) + \frac{2u_{1,j}^* g'(d_-)}{n+2} + \frac{2(u_{1,j}^*)^2 g''(d_-) + 4u_{2,j}^* g'(d_-)}{(n+2)^2}.$$

For each j, we put $\epsilon_{2j-1}^{(n,k)} := |\lambda_{2j-1}^{(n)} - \lambda_{2j-1}^{(n,k)}|$, $\epsilon_{2j}^{(n,k)} := |\lambda_{2j}^{(n)} - \lambda_{2j}^{(n,k)}|$.

To use the formulas (3.2.7) and (3.2.8), we denote by $\tilde{\lambda}_{2j-1}^{(n,k)} := g(\varphi_{2j-1,*}^{(n,k)})$ and $\tilde{\lambda}_{2j}^{(n,k)} := g(\varphi_{2j,*}^{(n,k)})$ the approximations of $\lambda_{2j-1}^{(n)}$ and $\lambda_{2j}^{(n)}$, where $\varphi_{2j-1,*}^{(n,k)}$ and $\varphi_{2j,*}^{(n,k)}$ contain k terms. For example, when taking (3.2.7) and (3.2.8), we have

$$\tilde{\lambda}_{2j-1}^{(n,2)} = g\left(d_- + \frac{2u_{1,j}}{n+2}\right), \quad \tilde{\lambda}_{2j-1}^{(n,3)} = g\left(d_- + \frac{2u_{1,j}}{n+2} + \frac{4u_{2,j}}{(n+2)^2}\right), \tag{3.6.1}$$

$$\tilde{\lambda}_{2j}^{(n,2)} = g\left(d_+ + \frac{2w_{1,j}}{n+2}\right), \quad \tilde{\lambda}_{2j}^{(n,3)} = g\left(d_+ + \frac{2w_{1,j}}{n+2} + \frac{4w_{2,j}}{(n+2)^2}\right). \tag{3.6.2}$$

Furthermore, if n is an even number, then we denote by $\epsilon^{(n,k)}$ the corresponding maximal error, i.e.,

$$\epsilon^{(n,k)} := \max_{j \in \{1,2,\ldots,\frac{n}{2}\}} \{\epsilon_{2j-1}^{(n,k)}, \epsilon_{2j}^{(n,k)}\}.$$

n	64	128	512	1024
$\epsilon^{(n,2)}$	$1.23 \cdot 10^{-3}$	$3.18 \cdot 10^{-4}$	$2.03 \cdot 10^{-5}$	$5.11 \cdot 10^{-6}$
$\epsilon^{(n,3)}$	$1.03 \cdot 10^{-5}$	$1.35 \cdot 10^{-6}$	$1.77 \cdot 10^{-8}$	$3.25 \cdot 10^{-10}$

Table 3.1: Maximal absolute errors for the eigenvalues of $T_n(a)$ obtained with our formulas (3.6.1) and (3.6.2).

n	64	128	512	1024
$\epsilon_1^{(n,1)}$	$4.38 \cdot 10^{-8}$	$7.49 \cdot 10^{-10}$	$1.96 \cdot 10^{-13}$	$3.19 \cdot 10^{-15}$
$\epsilon_1^{(n,2)}$	$1.96 \cdot 10^{-8}$	$3.35 \cdot 10^{-10}$	$8.81 \cdot 10^{-14}$	$1.48 \cdot 10^{-15}$

Table 3.2: Maximal absolute errors for the eigenvalues of $T_n(a)$ obtained with our formula (3.2.11), i.e., $j = 1$.

n	64	128	512	1024
$\epsilon_2^{(n,1)}$	$6.32 \cdot 10^{-7}$	$1.08 \cdot 10^{-8}$	$2.83 \cdot 10^{-12}$	$4.45 \cdot 10^{-14}$
$\epsilon_2^{(n,2)}$	$4.59 \cdot 10^{-7}$	$7.85 \cdot 10^{-9}$	$2.05 \cdot 10^{-12}$	$3.22 \cdot 10^{-14}$

Table 3.3: Maximal absolute errors for the eigenvalues of $T_n(a)$ obtained with our formula (3.2.12), i.e., $j = 1$.

n	64	128	512	1024
$\epsilon^{(n,1)}$	$7.47 \cdot 10^{-4}$	$7.76 \cdot 10^{-5}$	$1.13 \cdot 10^{-6}$	$1.62 \cdot 10^{-7}$
$\epsilon^{(n,2)}$	$7.28 \cdot 10^{-4}$	$7.65 \cdot 10^{-5}$	$1.12 \cdot 10^{-6}$	$1.61 \cdot 10^{-7}$

Table 3.4: Maximal absolute errors for the eigenvalues of $T_n(a)$ near zero, obtained with our formulas (3.2.11) and (3.2.12), where $\epsilon^{(n,k)} := \max\{\epsilon_{2j-1}^{(n,k)}, \epsilon_{2j}^{(n,k)} : j \in \{1, \ldots, [\sqrt{n}]\}\}$. These errors show that we can consider the case $j \ll n$.

n	64	128	512	1024	2048
$\epsilon^{(n,2)}$	$1.0 \cdot 10^{-3}$	$3.06 \cdot 10^{-4}$	$1.95 \cdot 10^{-5}$	$4.91 \cdot 10^{-6}$	$1.23 \cdot 10^{-6}$
$\epsilon^{(n,3)}$	$2.01 \cdot 10^{-5}$	$2.63 \cdot 10^{-6}$	$4.24 \cdot 10^{-8}$	$5.33 \cdot 10^{-9}$	$6.68 \cdot 10^{-10}$

Table 3.5: Maximal absolute errors for the inner eigenvalues of $T_n(a)$ obtained with our formulas (3.2.9) and (3.2.10), where $\epsilon^{(n,k)} := \max\{\epsilon_{2j-1}^{(n,k)}, \epsilon_{2j}^{(n,k)} : j \in \{[\frac{n}{6}] + 1, \ldots, n - [\frac{n}{6}]\}\}$.

n	64	128	512	1024	2048
$\epsilon_{n-1}^{(n,2)}$	$1.43 \cdot 10^{-3}$	$1.95 \cdot 10^{-4}$	$3.25 \cdot 10^{-6}$	$4.11 \cdot 10^{-7}$	$5.17 \cdot 10^{-8}$
$\epsilon_{n-1}^{(n,3)}$	$1.17 \cdot 10^{-4}$	$7.63 \cdot 10^{-6}$	$3.06 \cdot 10^{-8}$	$1.92 \cdot 10^{-9}$	$1.2 \cdot 10^{-10}$

Table 3.6: Maximal absolute errors for the penultimate eigenvalue of $T_n(a)$ obtained with our formula (3.2.13), i.e., $j = \frac{n}{2}$.

n	64	128	512	1024	2048
$\epsilon_{n}^{(n,2)}$	$3.85 \cdot 10^{-4}$	$5.06 \cdot 10^{-5}$	$8.21 \cdot 10^{-7}$	$1.03 \cdot 10^{-7}$	$1.29 \cdot 10^{-8}$
$\epsilon_{n}^{(n,3)}$	$2.61 \cdot 10^{-6}$	$1.63 \cdot 10^{-7}$	$6.39 \cdot 10^{-10}$	$3.99 \cdot 10^{-11}$	$2.49 \cdot 10^{-12}$

Table 3.7: Maximal absolute errors for the very last eigenvalue of $T_n(a)$ obtained with our formula (3.2.14), i.e., $j = \frac{n}{2}$.

Chapter 4

Eigenvalues of even very nice Toeplitz matrices can be unexpectedly erratic

In this chapter we use the results obtained in the Chapter 3 to show that the eigenvalues of the pentadiagonal Toeplitz matrices generated by the symbol $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$, which does not satisfy the simple-loop condition, do not admit a regular asymptotic expansion. Thus, we give a counterexample to a conjecture presented in [19] by Ekström, Garoni, and Serra-Capizzano, and at the same time we reveal that the simple-loop condition is essential for the existence of the regular asymptotic expansion.

4.1 Introduction and main results

It was shown in a series of recent publications [7, 16, 2, 6] that the eigenvalues of $n \times n$ Toeplitz matrices generated by a simple-loop symbols admit certain regular asymptotic expansions into negative powers of $n+1$. We recall that, in a more general context, the starting point is a 2π-periodic bounded function $g : \mathbb{R} \to \mathbb{R}$ with Fourier series $g(\varphi) \sim \sum_{k=-\infty}^{\infty} \hat{g}_k e^{ik\varphi}$. The $n \times n$ Toeplitz matrix generated by g is the matrix $T_n(g) = (\hat{g}_{j-k})_{j,k=1}^n$. The function g is referred to as the symbol of the matrix sequence $\{T_n(g)\}_{n=1}^\infty$. Examples of simple-loop symbols are even 2π-periodic C^∞ functions $g : \mathbb{R} \to \mathbb{R}$ satisfying $g'(\varphi) > 0$ for every φ in $(0,\pi)$, $g'(0) = 0$, $g''(0) > 0$, $g'(\pi) = 0$, $g''(\pi) < 0$. The requirement that g be a real-valued and even function implies that the matrices $T_n(g)$ are real and symmetric.

In the beginning of Section 7 of [2], The authors noted that the mere existence of such regular asymptotic expansions already helps to approximate the eigenvalues of large matrices by using the eigenvalues of small matrices and some sort of extrapolation.

In [19], Ekström, Garoni, and Serra-Capizzano worked out the idea of such extrapolation in detail. They also emphasized that the symbols of interest in connection with the discretization of differential equations are of the form

$$g_m(\varphi) = (2 - 2\cos\varphi)^m = \left(2\sin\frac{\varphi}{2}\right)^{2m}. \tag{4.1.1}$$

In the simplest case $m = 1$, the $n \times n$ Toeplitz matrices $T_n(g_1)$ generated by $g_1(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^2$ are tridiagonal matrices. For example,

$$T_4(g_1) = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}.$$

The eigenvalues of these matrices are known exactly,

$$\lambda_{n,j} = 2 - 2\cos\frac{j\pi}{n+1} = \left(2\sin\frac{j\pi}{2n+2}\right)^2.$$

A crucial observation of [19] is that the symbols g_m are no longer simple-loop symbols for $m \geq 2$, because then the second derivative at 0 vanishes. Our concrete symbol (3.1.2) is just g_2 and hence not a simple-loop symbol. As well, Ekström, Garoni, and Serra-Capizzano conjectured that the regular asymptotic expansions stay true for smooth even real-valued symbols that are monotone on $[0, \pi]$ and that may have a minimum or a maximum of higher-order. They verified this conjecture numerically for some examples and for small values of p. This conjecture has attracted a lot of attention.

Independently and at the same time, we considered just the symbol (3.1.2) and derived exact equations and asymptotic expansions for the eigenvalues of $T_n(g)$. Later, when paper [19] came to our attention, we realized to our surprise that the results of Chapter 3 imply that for $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$ the eigenvalues do not admit a regular asymptotic expansion.

In [19] the conjecture was formulated and presented in the following form: given any natural number p, the eigenvalues $\lambda_{n,1} < \cdots < \lambda_{n,n}$ of $T_n(g)$ admit an asymptotic expansion

$$\lambda_{n,j} = \sum_{k=0}^{p} \frac{f_k(\frac{j\pi}{n+1})}{(n+1)^k} + O\left(\frac{1}{(n+1)^{p+1}}\right) \quad (n \to \infty), \tag{4.1.2}$$

with the error term being uniform in $1 \leq j \leq n$ and with continuous functions $f_0, \ldots, f_p : [0, \pi] \to \mathbb{R}$. Notice that the eigenvalues of $T_n(g_1)$ obey a regular asymptotic expansion with $f_0 = g_1$ and $f_k = 0$ for $k \geq 1$.

The following theorem, which is the main result of the present chapter, shows that this is surprisingly false for $p = 4$.

Theorem 4.1.1. *Let g and $T_n(g)$ be as above. There do not exist continuous functions $f_0, \ldots, f_4 : [0, \pi] \to \mathbb{R}$ and numbers $C > 0$, $N \in \mathbb{N}$ such that*

$$\left|\lambda_{n,j} - \sum_{k=0}^{4} \frac{f_k\left(\frac{j\pi}{n+1}\right)}{(n+1)^k}\right| \leq \frac{C}{(n+1)^5} \tag{4.1.3}$$

for every $n \geq N$ and every $j \in \{1, \ldots, n\}$.

Unfortunately, there is an unpleasant complication. We call it the n, $n+1$, $n+2$ problem. In (4.1.2) and (4.1.3) we used the denominator $n+1$. This denominator is very convenient when tackling simple-loop symbols. However, when dealing with the symbol (3.1.2), the denominator $n+2$ is naturally emerging. Therefore, we decided to work mostly with $n+2$. We will denote the coefficient functions by f_k if the denominator is $n+1$ and by d_k in case it is $n+2$. To avoid any confusion, let us state the $n+2$.

Theorem 4.1.2. *Let g and $T_n(g)$ be as above and let $p \geq 0$ be an integer.*

(*i*) *There exist continuous functions $d_0, \ldots, d_p : [0, \pi] \to \mathbb{R}$ and a number $D_p > 0$ such that*

$$\left| \lambda_{n,j} - \sum_{k=0}^{p} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} \right| \leq \frac{D_p}{(n+2)^{p+1}} \tag{4.1.4}$$

whenever $n \geq 1$ and $\frac{p}{2}\log(n+2) \leq j \leq n$. These functions $d_0, \ldots, d_p$ are uniquely determined.

(*ii*) *There is a constant $C > 0$ such that*

$$\left| \lambda_{n,j} - \sum_{k=0}^{3} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} \right| \leq \frac{C}{(n+2)^4} \tag{4.1.5}$$

for all $n \geq 1$ and all $j \in \{1, \ldots, n\}$.

(*iii*) *However, there do not exist numbers $C > 0$ and $N \in \mathbb{N}$ such that*

$$\left| \lambda_{n,j} - \sum_{k=0}^{4} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} \right| \leq \frac{C}{(n+2)^5} \tag{4.1.6}$$

for all $n \geq N$ and all $j \in \{1, \ldots, n\}$.

Part (*ii*) of the Theorem 4.1.2 might suggest that all eigenvalues $\lambda_{n,j}$ are *moderately well* approximated by the sums $\sum_{k=0}^{3} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k}$. In fact, as we will show this approximation is *extremely bad* for the first eigenvalues, in the sense that the corresponding relative errors do not converge to zero. However, as Theorem 4.1.2 part (*i*) shows, asymptotic expansions of the form (4.1.2) for $p = 2, 3, 4, \ldots$ can be used outside a small neighborhood of the point at which the symbol has a zero of order greater than 2.

4.2 Regular expansions of the eigenvalues

In this and the following sections, we work in abstract settings and use the denominator $n+s$, where s is an arbitrary positive constant ("shift"). This allows us to unify the situations with $n+1$ and $n+2$ and to simplify the subsequent references in the last sections of the chapter. We first introduce some notation and recall some facts. Given a 2π-periodic bounded real-valued function g on the real line, we denote by $\lambda_{n,1}, \ldots, \lambda_{n,n}$ the eigenvalues of the corresponding Toeplitz matrices $T_n(g)$, ordered in the ascending order: $\lambda_{n,1} \leq \cdots \leq \lambda_{n,n}$. Using the first Szegő limit theorem and criteria for weak convergence of probability measures, the authors proved in [5, 3] that if the essential range of g is a segment of the real line, then $\lambda_{n,j}$ can be uniformly approximated by the values of the quantile function Q (associated to g) at the points $\frac{j}{n+s}$:

$$\max_{1 \leq j \leq n} \left| \lambda_{n,j} - Q\left(\frac{j}{n+s}\right) \right| = o(1) \quad (n \to \infty). \tag{4.2.1}$$

If g is continuous, even, and strictly increasing on $[0, \pi]$, then $Q(\varphi)$ is just $g(\pi\varphi)$. Denote by $u_{n,j}$ the points of the uniform mesh $\frac{j\pi}{n+s}$, $j \in \{1, \ldots, n\}$. Then (4.2.1) can be rewritten in the form

$$\max_{1 \leq j \leq n} |\lambda_{n,j} - g(u_{n,j})| = o(1) \quad (n \to \infty). \tag{4.2.2}$$

In [43], Trench proved that for this class of symbols the eigenvalues are all distinct:

$$g(0) < \lambda_{n,1} < \cdots < \lambda_{n,n} < g(\pi).$$

Thus, there exist real numbers $\varphi_{n,1}, \ldots, \varphi_{n,n}$ such that $0 < \varphi_{n,1} < \cdots < \varphi_{n,n} < \pi$ and $\lambda_{n,j} = g(\varphi_{n,j})$. Taking into account (4.2.2), we can try to use $u_{n,j}$ as an initial approximation for $\varphi_{n,j}$. This approximation can be very inaccurate, but it is better than nothing.

Now let J be an arbitrary set of integer pairs (n,j) such that $1 \le j \le n$ for every (n,j) in J. Suppose that for each (n,j) in J the number $\varphi_{n,j}$ is the unique solution of the equation

$$\varphi = u_{n,j} + \frac{\eta(\varphi)}{n+s} + \rho_{n,j}(\varphi), \tag{4.2.3}$$

where η is an infinitely smooth real-valued function on $[0,\pi]$ and $\{\rho_{n,j}\}_{(n,j)\in J}$ is a family of infinitely smooth real-valued functions on $[0,\pi]$ such that

$$\sup_{0\le\varphi\le\pi} \sup_{j:(n,j)\in J} |\rho_{n,j}(\varphi)| = O\left(\frac{1}{(n+s)^p}\right) \tag{4.2.4}$$

for some p in $\mathbb{N}$.

In the simple-loop case, the function ρ_n did not depend on j, and J was of the form $\{(n,j) : n \ge N,\ 1 \le j \le n\}$ for some N.

Let us show how to derive asymptotic expansions of $\varphi_{n,j}$ and $\lambda_{n,j}$ from equation (4.2.3).

Theorem 4.2.1. *Let η be an infinitely smooth real-valued function on $[0,\pi]$, and $\{\rho_{n,j}\}_{(n,j)\in J}$ be a family of real-valued functions on $[0,\pi]$ satisfying (4.2.4) for some natural number p. Suppose that for all $(n,j) \in J$ equation (4.2.3) has a unique solution $\varphi_{n,j}$. Then there exists a sequence of real-valued infinitely smooth functions $c_0, c_1, c_2, \ldots$ defined on $[0,\pi]$ such that there is a number $r_p > 0$ ensuring that, for all (n,j) in J,*

$$\left|\varphi_{n,j} - \sum_{k=0}^{p} \frac{c_k(u_{n,j})}{(n+s)^k}\right| \le \frac{r_p}{(n+s)^{p+1}}. \tag{4.2.5}$$

Furthermore, if g is an infinitely smooth 2π-periodic real-valued even function on $\mathbb{R}$, strictly increasing on $[0,\pi]$, then there exists a sequence of real-valued infinitely smooth functions $d_0, d_1, d_2, \ldots$ defined on $[0,\pi]$ such that the numbers $\lambda_{n,j} := g(\varphi_{n,j})$ can be approximated as follows: there exists an R_p such that, for all (n,j) in J,

$$\left|\lambda_{n,j} - \sum_{k=0}^{p} \frac{d_k(u_{n,j})}{(n+s)^k}\right| \le \frac{R_p}{(n+s)^{p+1}}. \tag{4.2.6}$$

Proof. This proposition was essentially proved in [2, 6], with a slightly different notation and reasoning, including a justification of the fixed-point method. Here we propose a simpler proof. Our goal is to show that (4.2.5) and (4.2.6) are direct and trivial consequences of the main equation (4.2.3).

In order to simplify notation, we denote by $O\left(\frac{1}{(n+s)^p}\right)$ any expression that may depend on n and j but can be estimated from above by $\frac{C}{(n+s)^p}$ with C independent of n or j. Then (4.2.3) implies that

$$\varphi_{n,j} = u_{n,j} + O\left(\frac{1}{n+s}\right).$$

Substitute this expression into (4.2.3) and expand η by Taylor's formula around the point $u_{n,j}$:

$$\begin{aligned}\varphi_{n,j} &= u_{n,j} + \frac{\eta\left(u_{n,j} + O\left(\frac{1}{n+s}\right)\right)}{n+s} + O\left(\frac{1}{(n+s)^2}\right)\\ &= u_{n,j} + \frac{\eta(u_{n,j})}{n+s} + O\left(\frac{1}{(n+s)^2}\right).\end{aligned}$$

Substituting the last expression into (4.2.3) and expanding η by Taylor formula around $u_{n,j}$ we get

$$\begin{aligned}\varphi_{n,j} &= u_{n,j} + \frac{\eta\left(u_{n,j} + \frac{\eta(u_{n,j})}{n+s} + O\left(\frac{1}{(n+s)^2}\right)\right)}{n+s} + O\left(\frac{1}{(n+s)^3}\right)\\ &= u_{n,j} + \frac{\eta(u_{n,j})}{n+s} + \frac{\eta(u_{n,j})\eta'(u_{n,j})}{(n+s)^2} + O\left(\frac{1}{(n+s)^3}\right).\end{aligned}$$

This "Münchhausen trick" can be applied again and again (we refer to the story when Baron von Münchhausen saved himself from being drowned in a swamp by pulling on his own hair), yielding an asymptotic expansion of the form (4.2.5) of any desired order p.

The first of the functions c_k are

$$\begin{aligned}&c_0(\varphi) = \varphi, \quad c_1(\varphi) = \eta(\varphi), \quad c_2(\varphi) = \eta(\varphi)\eta'(\varphi),\\ &c_3 = \eta(\eta')^2 + \frac{1}{2}\eta^2\eta'', \quad c_4 = \eta(\eta')^3 + \frac{3}{2}\eta^2\eta'\eta'' + \frac{1}{6}\eta^3\eta''',\\ &c_5 = \eta(\eta')^4 + 3\eta^2(\eta'^2)\eta'' + \frac{1}{2}\eta^3(\eta'')^2 + \frac{2}{3}\eta^3\eta'\eta''' + \frac{1}{24}\eta^4\eta^{(4)}.\end{aligned} \tag{4.2.7}$$

By induction on p it is straightforward to show that c_k is a uniquely determined polynomial in $\eta, \eta', \ldots, \eta^{(k-1)}$ also for $k \geq 6$.

Once we have the asymptotic formulas for $\varphi_{n,j}$, we can use the formula $\lambda_{n,j} = g(\varphi_{n,j})$ and expand the function g by Taylor's formula around the point $u_{n,j}$ to obtain:

$$\begin{aligned}\lambda_{n,j} &= g\left(u_{n,j} + \sum_{k=1}^{p} \frac{c_k(u_{n,j})}{(n+s)^k} + O\left(\frac{1}{(n+s)^{p+1}}\right)\right)\\ &= \sum_{m=0}^{p} \frac{g^{(m)}(u_{n,j})}{m!}\left(\sum_{k=1}^{p} \frac{c_k(u_{n,j})}{(n+s)^k} + O\left(\frac{1}{(n+s)^{p+1}}\right)\right)^m + O((\varphi_{n,j} - u_{n,j})^{p+1}).\end{aligned}$$

Expanding the powers, regrouping the summands, and writing $\varphi_{n,j} - u_{n,j}$ as $O\left(\frac{1}{n+s}\right)$, we arrive at the regular asymptotic formula for $\lambda_{n,j}$:

$$\lambda_{n,j} = \sum_{k=0}^{p} \frac{d_k(u_{n,j})}{(n+s)^k} + O\left(\frac{1}{(n+s)^{p+1}}\right). \tag{4.2.8}$$

The first of the functions $d_0, d_1, d_2, \ldots$ can be computed by the formulas

$$\begin{aligned}&d_0 = g, \quad d_1 = g'c_1, \quad d_2 = g'c_2 + \frac{1}{2}g''c_1^2, \quad d_3 = g'c_3 + g''c_1c_2 + \frac{1}{6}g'''c_1^3,\\ &d_4 = g'c_4 + g''\left(c_1c_3 + \frac{1}{2}c_2^2\right) + \frac{1}{2}g'''c_1^2c_2 + \frac{1}{24}g^{(4)}c_1^4,\\ &d_5 = g'c_5 + g''(c_2c_3 + c_1c_4) + \frac{1}{2}g'''(c_1^2c_3 + c_1c_2^2) + \frac{1}{6}g^{(4)}c_1^3c_2 + \frac{1}{120}g^{(5)}c_1^5.\end{aligned} \tag{4.2.9}$$

It can again be proved by induction on p that the functions $c_0, c_1, c_2, \ldots$ are polynomials in $\eta, \eta', \eta'', \ldots$ and that the functions $d_0, d_1, d_2, \ldots$ are polynomials in $c_0, c_1, c_2, \ldots$ and $g, g', g'', \ldots$. As a consequence, all the functions c_k and d_k are infinitely smooth. □

Remark 4.2.2. If the functions $d_0, d_1, \ldots$ are infinitely smooth, then one can easily transform an asymptotic expansion into negative powers of $n + s_1$ into an asymptotic expansion in negative powers of $n + s_2$. For example, we suppose that

$$\lambda_{n,j} = \sum_{k=0}^{p} \frac{d_k(u_{n,j})}{(n+2)^k} + O\left(\frac{1}{(n+2)^{p+1}}\right)$$

and we want

$$\lambda_{n,j} = \sum_{k=0}^{p} \frac{f_k(u_{n,j})}{(n+1)^k} + O\left(\frac{1}{(n+1)^{p+1}}\right).$$

For $k = 0, 1$, we have

$$\begin{aligned} d_k\left(\frac{j\pi}{n+2}\right) &= d_k\left(\frac{j\pi}{(n+1)\left(1+\frac{1}{n+1}\right)}\right) \\ &= d_k\left(\frac{j\pi}{n+1} - \frac{j\pi}{(n+1)^2} + O\left(\frac{1}{(n+1)^2}\right)\right) \\ &= d_k\left(\frac{j\pi}{n+1}\right) - d_k'\left(\frac{j\pi}{n+1}\right)\frac{j\pi}{n+1}\frac{1}{n+1} + O\left(\frac{1}{(n+1)^2}\right), \end{aligned}$$

and thus

$$\begin{aligned} d_0\left(\frac{j\pi}{n+2}\right) &+ d_1\left(\frac{j\pi}{n+2}\right)\frac{1}{n+2} + O\left(\frac{1}{(n+2)^2}\right) \\ &= d_0\left(\frac{j\pi}{n+1}\right) - d_0'\left(\frac{j\pi}{n+1}\right)\frac{j\pi}{n+1}\frac{1}{n+1} \\ &+ d_1\left(\frac{j\pi}{n+1}\right)\frac{1}{n+1} + O\left(\frac{1}{(n+1)^2}\right), \end{aligned}$$

resulting in the equalities $f_0(\varphi) = d_0(\varphi)$ and $f_1(\varphi) = d_1(\varphi) - \varphi d_0'(\varphi)$.

Remark 4.2.3. The hard part of the work in [2, 6] was to derive equation (4.2.3) and an explicit formula for η, to verify that η is sufficiently smooth, to establish upper bounds for the functions ρ_n, and to prove that (4.2.3) has a unique solution for every n large enough and for every j. Moreover, all this work was done under the assumption that g has some sort of smoothness of a finite order. In Theorem 4.2.1 we just require all these properties.

4.3 Uniqueness of the regular asymptotic expansion

As in the previous section, we fix some $s > 0$.

If there exists an asymptotic expansion of the form (4.2.8), then the functions $d_0, d_1, d_2, \ldots$ are uniquely determined. Let us state and prove this fact formally. Instead of requiring (4.2.8) for all n and j, we assume it holds for a set of pairs (n, j) such that the quotients $u_{n,j} := \frac{j\pi}{n+s}$ "asymptotically fill" $[0, \pi]$. Here is the corresponding technical definition.

Definition 4.3.1. Let J be a subset of $\mathbb{N}^2$. We say that J *asymptotically fills* $[0,\pi]$ *by quotients* if for every φ in $[0,\pi]$, every N in $\mathbb{N}$ and every $\delta>0$ there is a pair of numbers (n,j) in J such that $n\ge N$, $1\le j\le n$, and $|u_{n,j}-\varphi|\le\delta$.

It is easy to see that J asymptotically fills $[0,\pi]$ by quotients if and only if the set $\{u_{n,j}:(n,j)\in J\}$ is dense in $[0,\pi]$.

Theorem 4.3.2. *Let $p\ge 0$ be an integer, let $d_0,d_1,\ldots,d_p$ and $\widetilde{d}_0,\widetilde{d}_1,\ldots,\widetilde{d}_p$ be continuous functions on $[0,\pi]$, let $C>0$, and let J be a subset of $\mathbb{N}^2$ such that J asymptotically fills $[0,\pi]$ by quotients. Suppose that for every pair (n,j) in J the inequalities*

$$\left|\lambda_{n,j}-\sum_{k=0}^{p}\frac{d_k(u_{n,j})}{(n+s)^k}\right|\le\frac{C}{(n+s)^{p+1}},\qquad\left|\lambda_{n,j}-\sum_{k=0}^{p}\frac{\widetilde{d}_k(u_{n,j})}{(n+s)^k}\right|\le\frac{C}{(n+s)^{p+1}}$$

hold. Then $d_k(\varphi)=\widetilde{d}_k(\varphi)$ for every $k\in\{0,\ldots,p\}$ and every φ in $[0,\pi]$.

Proof. Denote the function $d_p-\widetilde{d}_p$ by h_p. It is clear that $h_0=0$. Proceeding by mathematical induction over p, we assume that h_k is the zero constant for every k with $k<p$, and we have to show that h_p is the zero constant.

Let $\varphi\in[0,\pi]$ and $\varepsilon>0$. Using the continuity of h_p at the point φ, choose $\delta>0$ such that $|\psi-\varphi|\le\delta$ implies $|h_p(\psi)-h_p(\varphi)|\le\varepsilon/2$. Take N such that $2C/(N+s)\le\varepsilon/2$. After that, pick n and j such that $(n,j)\in J$, $n\ge N$, and $|u_{n,j}-\varphi|\le\delta$. Then

$$\left|\frac{d_p(u_{n,j})}{(n+s)^p}-\frac{\widetilde{d}_p(u_{n,j})}{(n+s)^p}\right|\le\frac{2C}{(n+s)^{p+1}},$$

which implies

$$|h_p(u_{n,j})|\le\frac{2C}{n+s}\le\frac{2C}{N+s}\le\frac{\varepsilon}{2}.$$

Finally,

$$|h_p(\varphi)|\le|h_p(\varphi)-h_p(u_{n,j})|+|h_p(u_{n,j})|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

As $\varepsilon>0$ can be chosen arbitrarily, it follows that h_p is identically zero. □

4.4 An example with a minimum of the fourth order

The purpose of this section is to recall some results of Chapter 3 and to derive some new corollaries. We first consider the pentadiagonal Toeplitz matrices generated by the trigonometric polynomial (3.1.2), i.e., $g(\varphi)=\left(2\sin\frac{\varphi}{2}\right)^4$. From Chapter 3 we know that the function g takes real values, is even, and strictly increases on $[0,\pi]$. Moreover, g does not belong to simple-loop class, because g has a minimum of the fourth order: $g(0)=g'(0)=g''(0)=g'''(0)=0$, $g^{(4)}(0)>0$.

We begin by recalling and introducing some functions, which arise in Section 3.2 :

$$\beta(\varphi) = 2\log\left(\sin\frac{\varphi}{2} + \sqrt{1+\left(\sin\frac{\varphi}{2}\right)^2}\right),$$
$$f(\varphi) = \frac{\cos\frac{\varphi}{2}}{\sqrt{1+(\sin\frac{\varphi}{2})^2}},$$
$$\eta_n^{odd}(\varphi) := 2\arctan\left(\frac{1}{f(\varphi)}\coth\frac{(n+2)\beta(\varphi)}{2}\right),$$
$$\eta_n^{even}(\varphi) := 2\arctan\left(\frac{1}{f(\varphi)}\tanh\frac{(n+2)\beta(\varphi)}{2}\right),$$
$$\eta_{n,j}(\varphi) := \begin{cases}\eta_n^{odd}(\varphi), & \text{if } j \text{ is odd,}\\ \eta_n^{even}(\varphi), & \text{if } j \text{ is even.}\end{cases}$$

As previously, we denote by $\varphi_{n,j}$ the points in $(0,\pi)$ such that $\lambda_{n,j} = g(\varphi_{n,j})$. In this example, we let $u_{n,j}$ stand for $j\pi/(n+2)$.

In Theorems 3.2.1 and 3.2.2 we used Elouafi's formulas [20] for the determinants of Toeplitz matrices and derived exact equations for the eigenvalues of $T_n(g)$. Namely, it was proved that there exists an N_0 such that if $n \geq N_0$ and $j \in \{1,\ldots,n\}$, then $\varphi_{n,j}$ is the unique solution in the interval $(u_{n,j}, u_{n,j+1})$ of the equation

$$\varphi = u_{n,j} + \frac{\eta_{n,j}(\varphi)}{n+2}. \tag{4.4.1}$$

The equations (3.2.3) and (3.2.4) correspond wiht (4.4.1). Moreover, this equations are written in a slightly different (but equivalent) form, without joining the cases of odd and even values of j.

Equation (4.4.1) is more complicated than (4.2.3), in the sense that now instead of one function η we have a family of functions, depending on n and on the parity of j.

Notice that if φ is not too close to zero, then $\beta(\varphi)$ is not too close to zero. Thus, the product $\frac{n+2}{2}\beta(\varphi)$ is large and the expressions $\tanh\frac{(n+2)\beta(\varphi)}{2}$ and $\coth\frac{(n+2)\beta(\varphi)}{2}$ are very close to 1, when n is large enough. Denote by η the function obtained from η_n^{odd} and η_n^{even} by neglecting these expressions, that is,

$$\eta(\varphi) := 2\arctan\left(\frac{1}{f(\varphi)}\right). \tag{4.4.2}$$

and put

$$\rho_{n,j}(\varphi) := \frac{\eta_{n,j}(\varphi) - \eta(\varphi)}{n+2}.$$

Therefore, the main equation (4.4.1) takes the form (4.2.3) with $s = 2$:

$$\varphi = u_{n,j} + \frac{\eta(\varphi)}{n+2} + \rho_{n,j}(\varphi). \tag{4.4.3}$$

Figure 4.1 shows that the functions η_{64}^{odd}, η_{64}^{even} and η almost coincide outside of a certain neighborhood of zero.

The following lemma provides us with upper estimates for $\rho_{n,j}(\varphi)$.

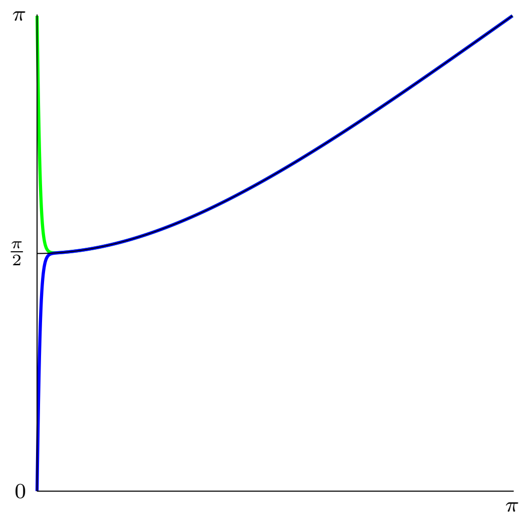

Figure 4.1: Plots of η_{64}^{odd} (green), η_{64}^{even} (blue) and η (black)

Lemma 4.4.1. *Let $n, j \in \mathbb{N}$. If $1 \le j \le n/2$, then*

$$\sup_{u_{n,j} \le \varphi \le u_{n,j+1}} |\rho_{n,j}(\varphi)| \le \frac{6e^{-2j}}{n+2}. \tag{4.4.4}$$

If $n/2 \le j \le n$, then

$$\sup_{u_{n,j} \le \varphi \le u_{n,j+1}} |\rho_{n,j}(\varphi)| \le 6e^{-(n+2)\pi/2}. \tag{4.4.5}$$

Proof. First suppose that $1 \le j \le n/2$ and $u_{n,j} \le \varphi \le u_{n,j+1}$. Then

$$\frac{j\pi}{n+2} \le \varphi \le \frac{(j+1)\pi}{n+2} \le \frac{\pi}{2}.$$

It is readily verified that $\beta(\varphi) \ge 2\varphi/\pi$ for every φ in $[0, \pi/2]$. Consequently,

$$\frac{(n+2)\beta(\varphi)}{2} \ge j.$$

It is also easy to see that $0 \le 1 - \tanh(y) \le 2e^{-2y}$ and $0 \le \coth(y) - 1 \le 3e^{-2y}$ for $y \ge 1$, $f(\varphi) > 1/2$ for φ in $[0, \pi/2]$, and that arctan is Lipschitz continuous with coefficient 1. Thus

$$|\eta_{n,j}(\varphi) - \eta(\varphi)| \le 6e^{-2j},$$

which yields (4.4.4).

Now, If $n/2 \le j \le n$, then we use the estimate $\beta(\varphi) \ge \varphi/2$ and $f(\varphi) > 1/(n+2)$ to obtain

$$\frac{(n+2)\beta(\varphi)}{2} \ge \frac{(n+2)\pi}{4},$$
$$|\eta_{n,j}(\varphi) - \eta(\varphi)| \le 6(n+2)e^{-(n+2)\pi/2},$$

which implies (4.4.5) □

The next proposition gives asymptotic formulas for the eigenvalues $\lambda_{n,j}$ provided j is "not too small". It mimics Theorem 3.2.5 from Chapter 3, the novelty being that we here join the cases of odd and even values of j and state the result for an arbitrary order p.

Theorem 4.4.2. *For every $p \in \mathbb{N}$, the functions $\rho_{n,j}$ admit the asymptotic upper estimate*

$$\max_{(p/2)\log(n+2)\le j\le n} \sup_{\varphi\in[u_{n,j},u_{n,j+1}]} |\rho_{n,j}(\varphi)| = O\left(\frac{1}{n^{p+1}}\right). \tag{4.4.6}$$

Moreover, for every $p \in \mathbb{N}$, every $n \in \mathbb{N}$, and every j satisfying

$$\frac{p}{2}\log(n+2) \le j \le n, \tag{4.4.7}$$

the numbers $\varphi_{n,j}$ and $\lambda_{n,j}$ have asymptotic expansions of the form

$$\begin{aligned}\varphi_{n,j} &= \sum_{k=0}^{p} \frac{c_k(u_{n,j})}{(n+2)^k} + O\left(\frac{1}{(n+2)^{p+1}}\right),\\ \lambda_{n,j} &= \sum_{k=0}^{p} \frac{d_k(u_{n,j})}{(n+2)^k} + O\left(\frac{1}{(n+2)^{p+1}}\right),\end{aligned} \tag{4.4.8}$$

where the upper estimates of the residue terms are uniform in j, the functions c_k and d_k are infinitely smooth and can be expressed in terms of η and g by the formulas shown in the proof of Theorem 4.2.1.

Proof. We have to verify the upper bound (4.4.6). The other statements the follow from Theorem 4.2.1. Let $p, n \in \mathbb{N}$ and j satisfy (4.4.7). If $j \le n/2$, then (4.4.4) gives

$$\frac{6e^{-2j}}{n+2} \le \frac{6e^{-p\log(n+2)}}{n+2} = \frac{6}{(n+2)^{p+1}}.$$

On the other hand, if $j > n/2$, then we obtain from (4.4.5) that

$$e^{-(n+2)\pi/2} = O\left(\frac{1}{n^{p+1}}\right).$$

Joining these two cases we arrive at (4.4.6). □

In Theorem 4.2.1 we expressed the first of the coefficients c_k and d_k in terms of the first derivatives of g and η. Here are explicit formulas for $g', \ldots, g^{(4)}$:

$$\begin{aligned} g'(\varphi) &= 23\cos\frac{\varphi}{2}\left(\sin\frac{\varphi}{2}\right)^3, \qquad g''(\varphi) = 16(1+2\cos(\varphi))\left(\sin\frac{\varphi}{2}\right)^2,\\ g'''(\varphi) &= -8\sin(\varphi) + 16\sin(2\varphi), \quad g^{(4)}(\varphi) = -8\cos(\varphi) + 32\cos(2\varphi),\\ g^{(5)}(\varphi) &= 8\sin(\varphi) - 64\sin(2\varphi).\end{aligned} \tag{4.4.9}$$

For $\eta', \ldots, \eta^{(4)}$ we have

$$\begin{aligned} \eta'(\varphi) &= \frac{\sin\frac{\varphi}{2}}{\left(1+(\sin\frac{\varphi}{2})^2\right)^{1/2}}, \qquad \eta''(\varphi) = \frac{\sqrt{2}\cos\frac{\varphi}{2}}{(3-\cos(\varphi))^{3/2}},\\ \eta'''(\varphi) &= -\frac{5\sin\frac{\varphi}{2}+\sin\frac{3\varphi}{2}}{\sqrt{2}(3-\cos(\varphi))^{5/2}}, \quad \eta^{(4)}(\varphi) = \frac{-4\cos\frac{\varphi}{2}+19\cos\frac{3\varphi}{2}+\cos\frac{5\varphi}{2}}{2\sqrt{2}(3-\cos(\varphi))^{7/2}}.\end{aligned} \tag{4.4.10}$$

Numerical test 4.4.3. In order to test (4.4.8) numerically for $p = 4$, we computed $g', \ldots, g^{(4)}$ by (4.4.9), $\eta, \eta', \ldots, \eta^{(3)}$ by (4.4.2) and (4.4.10), $c_0, c_1, \ldots, c_4$ by (4.2.7) and $d_0, d_1, \ldots, d_4$ by (4.2.9). The exact eigenvalues were computed by the simple iteration in equation (4.4.3) and independently by general eigenvalue algorithms (for $n \leq 1024$). All computations were made in high-precision arithmetic with 100 decimal digits after the floating point, in SageMath and independently in Wolfram Mathematica. Denote by $E_{n,4}$ the maximal error in (4.4.8), with $p = 4$:

$$E_{n,4} := \max_{2\log(n+2) \leq j \leq n} \left| \lambda_{n,j} - \sum_{k=0}^{4} \frac{d_k(u_{n,j})}{(n+2)^k} \right|.$$

The following table shows $E_{n,4}$ and $(n+2)^5 E_{n,4}$ for various values of n.

	$n = 64$	$n = 256$	$n = 1024$	$n = 4096$	$n = 16384$
$E_{n,4}$	$2.4 \cdot 10^{-7}$	$3.1 \cdot 10^{-10}$	$3.2 \cdot 10^{-13}$	$3.2 \cdot 10^{-16}$	$3.1 \cdot 10^{-19}$
$(n+2)^5 E_{n,4}$	306.72	354.87	366.61	369.52	370.25

We see that the numbers $E_{n,4}$ really behave like $O(1/(n+2)^5)$.

4.5 An asymptotic formula for the first eigenvalues in the example

In this section we study the asymptotic behavior of $\lambda_{n,j}$ as n tends to ∞, considering j as a fixed parameter.

Using the definition of arctan and the formula for $\tan(\varphi + j\pi/2)$, we rewrite equation (4.4.1) in the following equivalent form:

$$f(\varphi)^{(-1)^{j+1}} \tanh \frac{(n+2)\beta(\varphi)}{2} = (-1)^j \tan \frac{(n+2)\varphi}{2}. \tag{4.5.1}$$

The first factor in the left-hand side of (4.5.1) is just $f(\varphi)$ for odd values of j and $1/f(\varphi)$ for even values of j. We know that

$$\frac{j\pi}{n+2} \leq \varphi_{n,j} \leq \frac{(j+1)\pi}{n+2},$$

and it is natural to expect that the product $(n+2)\varphi_{n,j}$ has a certain finite limit α_j as n tends to infinity and j is fixed. Assuming this and taking into account that

$$f(\varphi) \to 1, \quad \beta(\varphi) \sim \varphi, \quad \text{as} \quad \varphi \to 0,$$

we can pass to the limit in (4.5.1) to obtain a simple transcendental equation for α_j. This is an informal motivation of the following formal reasoning.

For each j in $\mathbb{N}$, denote by α_j the unique real number that belongs to the interval $(j\pi, (j+1)\pi)$ and satisfies

$$\tanh \frac{\alpha_j}{2} = (-1)^j \tan \frac{\alpha_j}{2}. \tag{4.5.2}$$

Equation (4.5.2) is easy to solve by numerical methods for each j. Approximately,

$$\alpha_1 \approx 4.73004, \qquad \alpha_2 \approx 7.85320, \qquad \alpha_3 \approx 10.99560.$$

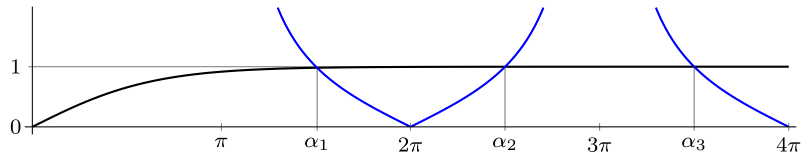

Figure 4.2: The left-hand side (black) and the right-hand side (blue) of (4.5.2), for $j=1$ on $(\pi,2\pi)$, for $j=2$ on $(2\pi,3\pi)$, and for $j=3$ on $(3\pi,4\pi)$.

It follows from (4.5.2) that $\alpha_j > \frac{(2j+1)\pi}{2}$ if j is odd and $\alpha_j < \frac{(2j+1)\pi}{2}$ if j is even. In particular,

$$\alpha_1 > \frac{3\pi}{2}, \qquad \alpha_2 < \frac{5\pi}{2}, \qquad \alpha_3 > \frac{7\pi}{2}.$$

We remark that differences between α_j and $(2j+1)\pi/2$ are extremely small:

$$\alpha_1 - \frac{3\pi}{2} \approx 1.8\cdot 10^{-2}, \quad \alpha_2 - \frac{5\pi}{2} \approx -7.8\cdot 10^{-4}, \quad \alpha_3 - \frac{7\pi}{2} \approx 3.3\cdot 10^{-5}.$$

Contrary to the general agreement of this chapter, the upper estimates of the residual terms in the following theorem (and in its proof) are not uniform in j. Thus we use the notation O_j instead of O.

Theorem 4.5.1. *Let g be the function defined by* (3.1.2) *and define $\varphi_{n,j}\in(0,\pi)$ by $\lambda_{n,j}=g(\varphi_{n,j})$. Then for each fixed j in $\mathbb{N}$, $\varphi_{n,j}$ and $\lambda_{n,j}$ satisfy the asymptotic formulas*

$$\varphi_{n,j} = \frac{\alpha_j}{n+2} + O_j\left(\frac{1}{(n+2)^3}\right), \tag{4.5.3}$$

$$\lambda_{n,j} = \left(\frac{\alpha_j}{n+2}\right)^4 + O_j\left(\frac{1}{(n+2)^6}\right). \tag{4.5.4}$$

Proof. Fix j in $\mathbb{N}$. We are going to analyze (4.5.1) by asymptotic methods, as n tends to ∞. Put

$$\delta_{n,j} := (n+2)\varphi_{n,j} - \alpha_j,$$

i.e., represent the product $(n+2)\varphi_{n,j}$ in the form

$$(n+2)\varphi_{n,j} = \alpha_j + \delta_{n,j}.$$

It is easy to verify that, as $\varphi\to 0$,

$$f(\varphi) = 1 + O(\varphi^2), \qquad \beta(\varphi) = \varphi + O(\varphi^3).$$

Moreover, we know that $\frac{j\pi}{n+2} \le \varphi_{n,j} \le \frac{(j+1)\pi}{n+2}$, thus $\varphi_{n,j} = O_j(1/(n+2))$. Therefore

$$f(\varphi_{n,j}) = 1 + O_j\left(\frac{1}{(n+2)^2}\right), \qquad \frac{1}{f(\varphi_{n,j})} = 1 + O_j\left(\frac{1}{(n+2)^2}\right),$$

$$\frac{(n+2)}{2}\beta(\varphi_{n,j}) = \frac{\alpha_j+\delta_{n,j}}{2} + O_j\left(\frac{1}{(n+2)^2}\right),$$

$$\tanh\frac{(n+2)}{2}\beta(\varphi_{n,j}) = \tanh\frac{\alpha_j+\delta_{n,j}}{2} + O_j\left(\frac{1}{(n+2)^2}\right).$$

By the Mean Value Theorem, there exist some numbers $\xi_{1,n,j}$ and $\xi_{2,n,j}$ between $\alpha_j/2$ and $(\alpha_j+\delta_{n,j})/2$ such that

$$\tanh\frac{\alpha_j+\delta_{n,j}}{2}-\tanh\frac{\alpha_j}{2}=\tanh'(\xi_{1,n,j})\frac{\delta_{n,j}}{2}$$

and

$$\tan\frac{\alpha_j+\delta_{n,j}}{2}-\tan\frac{\alpha_j}{2}=\tan'(\xi_{2,n,j})\frac{\delta_{n,j}}{2}.$$

After replacing φ by $\varphi_{n,j}$, equation (4.5.1) takes the form

$$\tanh\frac{\alpha_j}{2}+\tanh'(\xi_{1,n,j})\frac{\delta_{n,j}}{2}+O_j\left(\frac{1}{(n+2)^2}\right)=(-1)^j\left(\tan\frac{\alpha_j}{2}+\tan'(\xi_{2,n,j})\frac{\delta_{n,j}}{2}+O_j\left(\frac{1}{(n+2)^2}\right)\right).$$

Using the definition of α_j, this can be simplified to

$$\left(\tan'(\xi_{2,n,j})+(-1)^{j-1}\tanh'(\xi_{1,n,j})\right)\delta_{n,j}=O_j\left(\frac{1}{(n+2)^2}\right).$$

The coefficient before $\delta_{n,j}$ is strictly positive and bounded away from zero. Indeed, for all φ from the considered domain $(j\pi/2,(j+1)\pi/2)$ we have $\tan'(\varphi)>1$ and

$$\tanh'(\varphi)=\frac{1}{1+\varphi^2}<\frac{1}{1+\frac{\pi^2}{4}}<\frac{1}{2},$$

thus

$$\tan'(\xi_{2,n,j})+(-1)^{j-1}\tanh'(\xi_{1,n,j})>\frac{1}{2}.$$

Therefore $\delta_{n,j}=O_j(1/(n+2)^2)$, which is equivalent to (4.5.3). The function g has the following asymptotic expansion near the point 0:

$$g(\varphi)=\varphi^4+O(\varphi^6). \tag{4.5.5}$$

Using the formula $\lambda_{n,j}=g(\varphi_{n,j})$ and combining (4.5.3) with (4.5.5), we arrive at (4.5.4). □

Numerical test 4.5.2. Denote by $\varepsilon_{n,j}$ the absolute value of the residue in (4.5.4):

$$\varepsilon_{n,j}:=\left|\lambda_{n,j}-\left(\frac{\alpha_j}{n+2}\right)^4\right|.$$

Similarly to Numerical test 4.4.3, the exact eigenvalues $\lambda_{n,j}$ and the coefficients α_j are computed in the high-precision arithmetic with 100 decimal digits after the floating point. The next table shows $\varepsilon_{n,j}$ and $(n+2)^6\varepsilon_{n,j}$ corresponding to $j=1,2$ and to various values of n.

	$n=64$	$n=256$	$n=1024$	$n=4096$	$n=16384$
$\varepsilon_{n,1}$	$6.3\cdot10^{-9}$	$1.8\cdot10^{-11}$	$4.5\cdot10^{-16}$	$1.1\cdot10^{-19}$	$2.7\cdot10^{-23}$
$(n+2)^6\varepsilon_{n,1}$	523.37	524.39	524.46	524.46	524.46
$\varepsilon_{n,2}$	$1.1\cdot10^{-7}$	$3.1\cdot10^{-11}$	$7.9\cdot10^{-15}$	$2.0\cdot10^{-18}$	$4.9\cdot10^{-22}$
$(n+2)^6\varepsilon_{n,2}$	9315.7	9266.9	9263.7	9263.5	9263.4

Moreover, numerical experiments show that

$$\max_{1\le n\le 100000}((n+2)^6\varepsilon_{n,1})<524.47.$$

Remark 4.5.3. Theorem 4.5.1 has trivial corollaries about the norm of the inverse matrix and the condition number:

$$\|T_n^{-1}(g)\|_2 \sim \left(\frac{n+2}{\alpha_j}\right)^4, \quad \mathrm{cond}_2(T_n(g)) \sim 16\left(\frac{n+2}{\alpha_j}\right)^4, \quad \text{as} \quad n \to \infty.$$

Remark 4.5.4. Theorem 4.5.1 is not really new. Parter [31, 33] showed that if g_m is given by (4.1.1), then the corresponding eigenvalues satisfy

$$\lambda_{n,j} = \frac{\gamma_j(m)}{(n+2)^{2m}} + o\left(\frac{1}{(n+2)^{2m}}\right), \quad \text{as} \quad n \to \infty,$$

with some constant $\gamma_j(m)$ for each fixed j. Our theorem identifies $\gamma_j(2)$ as α_j^4 and improves the $o(1/(n+2)^4)$ to $O(1/(n+2)^6)$. Parter also had explicit formulas for $\gamma_j(2)$ in terms of the solutions of certain transcendental equations.

Remark 4.5.5. If we pass to the denominator $n+1$ in formula (4.5.4), then it become more complicated:

$$\lambda_{n,j} = \frac{\alpha_j^4}{(n+1)^4} - \frac{4\alpha_j^4}{(n+1)^5} + O_j\left(\frac{1}{(n+2)^6}\right).$$

This reveals that the denominator $n+2$ is more convenient when studying the asymptotic behavior of the first eigenvalues in the example (3.1.2).

4.6 The regular asymptotic expansion with four terms for the example

Lemma 4.6.1. *Let* $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$ *and let* $d_0, \ldots, d_4$ *be the same functions as in Theorem 4.4.2. Then, as* $n \to \infty$*, we have the asymptotic expansions*

$$\sum_{k=0}^{3} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} = \frac{(j\pi + \eta(0))^4 - \eta^4(0)}{(n+2)^4} + O\left(\frac{j^4}{(n+2)^5}\right), \tag{4.6.1}$$

$$\sum_{k=0}^{4} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} = \frac{(j\pi + \eta(0))^4}{(n+2)^4} + O\left(\frac{j^4}{(n+2)^5}\right), \tag{4.6.2}$$

uniformly in j.

Proof. By (4.4.9), the function and its derivative admit the following asymptotic expansions near the point 0:

$$g(\varphi) = \varphi^4 + O(\varphi^6), \quad g'(\varphi) = 4\varphi^3 + O(\varphi^5), \quad g''(\varphi) = 12\varphi^2 + O(\varphi^4), \tag{4.6.3}$$
$$g'''(\varphi) = 24\varphi + O(\varphi^3), \quad g^{(4)}(\varphi) = 24 + O(\varphi^2) \quad (\varphi \to 0).$$

Applying (4.4.10) and taking into account that η is smooth, we see that

$$c_0(\varphi) = \varphi, \quad c_1(\varphi) = \eta(\varphi) = \eta(0) + O(\varphi) \quad (\varphi \to 0) \tag{4.6.4}$$

and that the functions c_2, c_3, and c_4 are bounded. Substituting (4.6.3) and (4.6.4) into the formulas (4.2.9), we get the following expansions of $d_0(\varphi), \ldots, d_4(\varphi)$, as $\varphi \to 0$:

$$\begin{aligned} &d_0(\varphi) = \varphi^4 + O(\varphi^6), \quad d_1(\varphi) = 4\varphi^3\eta(0) + O(\varphi^4), \quad d_2(\varphi) = 6\varphi^2\eta^2(0) + O(\varphi^3), \\ &d_3(\varphi) = 4\varphi\eta^3(0) + O(\varphi^2), \quad d_4(\varphi) = \eta^4(0) + O(\varphi). \end{aligned}$$

Using these formulas and the binomial theorem, we arrive at (4.6.2). Moving in (4.6.2) the summand with $k = 4$ to the right-hand side we obtain (4.6.1). □

The following theorem proves Theorem 4.1.2 (*ii*).

Theorem 4.6.2. *Let $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$ and $d_0, \ldots, d_3 : [0, \pi] \to \mathbb{R}$ be the functions from the proof of Theorem 4.4.2. Then, there exists a $C > 0$ such that*

$$\left| \lambda_{n,j} - \sum_{k=0}^{3} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} \right| \leq \frac{C}{(n+2)^4} \tag{4.6.5}$$

for all $n \geq 1$ and all $j \in \{1, \ldots, n\}$.

Proof. By Theorem 4.4.2 we are left with the case $j < 2\log(n+2)$. Using (4.4.3), the upper estimate (4.4.4), and the smoothness of η, we conclude that

$$\varphi_{n,j} = \frac{j\pi + \eta(0)}{n+2} + O\left(\frac{j}{(n+2)^2}\right) + O\left(\frac{e^{-2j}}{n+2}\right).$$

Therefore, taking into account (4.5.5) we obtain that

$$\lambda_{n,j} = g(\varphi_{n,j}) = \varphi_{n,j}^4 + O\left(\frac{(\log(n+2))^6}{(n+2)^6}\right) = \varphi_{n,j}^4 + O\left(\frac{1}{(n+2)^4}\right).$$

Expanding $\varphi_{n,j}^4$ by the multinomial theorem and separating the main term, we get

$$\varphi_{n,j}^4 = \left(\frac{j\pi + \eta(0)}{n+2}\right)^4 + \sum_{\substack{p,q,r \leq 0 \\ p+q+r=4 \\ p<4}} O\left(\frac{(j\pi + \eta(0))^p j^q e^{-2jr}}{(n+2)^{p+2q+r}}\right).$$

The sum over p, q, r can be divided into the part with $q > 0$ and the part with $q = 0$ and estimated by

$$\sum_{\substack{p,q,r \leq 0 \\ p+q+r=4 \\ q<0}} O\left(\frac{(j\pi + \eta(0))^p j^q}{(n+2)^{4+q}}\right) + \sum_{\substack{p,r \leq 0 \\ p+r=4 \\ r<0}} O\left(\frac{(j\pi + \eta(0))^p e^{-2jr}}{(n+2)^4}\right) = O\left(\frac{1}{(n+2)^4}\right).$$

Thus, the true asymptotic expansion of $\lambda_{n,j}$ under the condition $j < 2\log(n+2)$ is

$$\lambda_{n,j} = \left(\frac{j\pi + \eta(0)}{n+2}\right)^4 + O\left(\frac{1}{(n+2)^4}\right). \tag{4.6.6}$$

On the other hand, using (4.6.1) and the fact $j^4 = O(n+2)$, we get

$$\sum_{k=0}^{3} \frac{d_k(u_{n,j})}{(n+2)^k} = \left(\frac{j\pi + \eta(0)}{n+2}\right)^4 + O\left(\frac{1}{(n+2)^4}\right). \tag{4.6.7}$$

Comparing (4.6.6) and (4.6.7), we obtain the required result. □

Numerical test 4.6.3. Denote by Δ_n the maximal error in (4.6.5):

$$\Delta_n := \max_{1\le j\le n}\left|\lambda_{n,j} - \sum_{k=0}^{3}\frac{d_k(u_{n,j})}{(n+2)^k}\right|.$$

The following table shows Δ_n and $(n+2)^4\Delta_n$ for various values of n.

	$n=64$	$n=256$	$n=1024$	$n=4096$	$n=16384$
Δ_n	$7.6\cdot 10^{-6}$	$3.2\cdot 10^{-8}$	$1.3\cdot 10^{-10}$	$5.1\cdot 10^{-13}$	$2.0\cdot 10^{-15}$
$(n+2)^4\Delta_n$	143.97	143.05	142.81	142.75	142.74

According to this table, the numbers Δ_n really behave like $O(1/(n+2)^4)$.

Remark 4.6.4. Let us again embark on the case $p=3$ and thus on Theorem 4.1.2 (ii) and the previous Numerical test 4.6.3. This test suggests that we could be satisfied by an error of 10^{-15} for $n=16384$. However, as the first eigenvalues are also of order 10^{-15} we obtain nothing but an upper bound for them. In other words, the approximation of the first eigenvalues $\lambda_{n,j}$ by $\sum\limits_{k=0}^{3}\frac{d_k(u_{n,j})}{(n+2)^k}$ is bad in the sense that the absolute error of this approximation is of the same order $O(1/(n+2)^4)$ as the eigenvalue $\lambda_{n,j}$ which we want to approximate. To state it in different terms, for each fixed j, the residues

$$\omega_{n,j} := \lambda_{n,j} - \sum_{k=0}^{3}\frac{d_k(\frac{j\pi}{n+2})}{(n+2)^k}$$

decay at the same rate $O(1/(n+2)^4)$ as the eigenvalues $\lambda_{n,j}$ and the distances between them, and the corresponding relative errors do not tend to zero:

$$\frac{\omega_{n,j}}{\lambda_{n,j}} \to \frac{\alpha_j^4+\eta^4(0)-(j\pi+\eta(0))^4}{\alpha_j^4} \neq 0$$

$$\frac{\omega_{n,j}}{\lambda_{n,j+1}-\lambda_{n,j}} \to \frac{\alpha_j^4+\eta^4(0)-(j\pi+\eta(0))^4}{\alpha_{j+1}^4-\alpha_j^4} \neq 0.$$

Compared to this, the residues of the asymptotic expansions for simple-loop symbols (see [2, 6]) can be bounded by $o\left(\frac{j(n+1-j)}{n^2}\frac{1}{n^p}\right)$, where p is related with the smoothness of the symbols, and the expression $\frac{j(n+1-j)}{n^2}$ is in the simple-loop case always comparable with the distance $\lambda_{n,j+1}-\lambda_{n,j}$ between the consecutive eigenvalues, i.e., there exist $C_1>0$ and $C_2>0$ such that

$$C_1\frac{j(n+1-j)}{n^2} \le \lambda_{n,j+1}-\lambda_{n,j} \le C_2\frac{j(n+1-j)}{n^2}.$$

Clearly, the quotient $\frac{|\omega_{n,j}|}{\lambda_{n,j+1}-\lambda_{n,j}}$ is a more adequate measure of the quality of the approximation that just the absolute error $|\omega_{n,j}|$.

4.7 There is no regular five terms asymptotic expansion for the example

As said, Ekström, Garoni, and Serra-Capizzano [19] conjectured that for every infinitely smooth 2π-periodic real-valued even function g, strictly increasing on $[0,\pi]$, the eigenvalues $\lambda_{n,j}$ of the corresponding Toeplitz matrices admit an asymptotic expansion of the regular form (4.1.2) for every order p.

We now show that for the symbol $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$ an asymptotic expansion of the form (4.1.2) cannot be true for $p = 4$. This disproves Conjecture 1 from [19].

We remark that the following theorem is actually stronger than the third part of Theorem 4.1.2. Namely, Theorem 4.1.2 (*iii*) states that (4.1.6) cannot hold with functions $d_1, \ldots, d_4$ appearing in (4.1.4). The following theorem tells us that (4.1.6) is also impossible for any other choice of continuous functions $d_1, \ldots, d_4$. The reason is of course Theorem 4.3.2.

Theorem 4.7.1. *Let* $g(\varphi) = \left(2\sin\frac{\varphi}{2}\right)^4$. *Denote by* $\lambda_{n,1}, \ldots, \lambda_{n,n}$ *the eigenvalues of the Toeplitz matrices* $T_n(g)$, *written in the ascending order. Then, there do not exist continuous functions* $d_0, \ldots, d_4 : [0,\pi] \to \mathbb{R}$ *and numbers* $C > 0$, $N \in \mathbb{N}$, *such that for every* $n \geq N$ *and every* j *in* $\{1, \ldots, n\}$

$$\left| \lambda_{n,j} - \sum_{k=1}^{4} \frac{d_k\left(\frac{j\pi}{n+2}\right)}{(n+2)^k} \right| \leq \frac{C}{(n+2)^5}. \tag{4.7.1}$$

Proof. Reasoning by contradiction, assume that there exist functions $d_0, \ldots, d_4$ and numbers C and N with the required properties. Put

$$J = \{(n,j) \in \mathbb{N}^2 : \ n \geq N, \ 2\log(n+2) \leq j \leq n\}.$$

Clearly, this set J asymptotically fills $[0,\pi]$ by quotients.

So, by Theorem 4.3.2, the functions $d_0, \ldots, d_4$ from (4.7.1) must be the same as the functions $d_0, \ldots, d_4$ from Theorem 4.4.2. In other words, the asymptotic expansion (4.4.8) from Theorem 4.4.2 holds for every pair (n,j) with n large enough and j in $\{1, \ldots, n\}$, that is, without the restriction $j \geq 2\log(n+2)$.

Combining (4.7.1) with (4.6.2), we see that for each fixed j the eigenvalue $\lambda_{n,j}$ must have the asymptotic behavior

$$\lambda_{n,j} = \left(\frac{j\pi + \eta(0)}{n+2}\right)^4 + O_j\left(\frac{1}{(n+2)^5}\right).$$

Since $\eta(0) = 2\arctan(1) = \frac{\pi}{2}$, we obtain for $j = 1$ that

$$\lambda_{n,1} = \left(\frac{3\pi/2}{n+2}\right)^4 + O\left(\frac{1}{(n+2)^5}\right).$$

which contradicts Theorem 4.5.1 because $3\pi/2 \neq \alpha_1$. □

Proof of Theorem 4.1.2. The existence of the asymptotic expansions (4.1.4) follows from Theorem 4.4.2, its uniqueness is a consequence of Theorem 4.3.2, Formula (4.1.5) was established in Theorem 4.6.2, and the impossibility of (4.1.6) is just Theorem 4.7.1. □

Proof of Theorem 4.1.1. The functions $d_0, d_1, \ldots$ from Theorem 4.4.2 are infinitely smooth on $[0,\pi]$, and thus, by Remark 4.2.2, the expansion (4.4.8) can be rewritten in the form (4.1.3) with some infinitely smooth functions $f_0, \ldots, f_4$. So, (4.1.3) is true for all (n,j) satisfying that $2\log(n+2) \leq j \leq n$.

Contrary to what we want, assume that there are $f_0, \ldots, f_4$, C, and N as in the statement of Theorem 4.1.1. Then, by Theorem 4.3.2, the functions $f_0, \ldots, f_4$ are the same as those in the previous paragraph. In particular, $f_0, \ldots, f_4$ must be infinitely smooth. In this case, the asymptotic expansion (4.1.3) can be rewritten in powers of $1/(n+2)$ and is true for all n and j with $n \geq N$ and $1 \leq j \leq n$. This contradicts Theorem 4.7.1. $\square$

Chapter 5

Asymptotics of eigenvalues for Toeplitz matrices with rational symbols that have a minimum of order 4

In this chapter we continue the line of investigation of Chapter 3. We considered there, the Toeplitz matrix $T_n(a)$ generated by symbol (3.1.1). In the present chapter we study the asymptotic behavior of the eigenvalues of $T_n(a)$, as n goes to infinity, assuming that the symbol a is a rational function and the generating function $g(\sigma) := a(e^{i\sigma})$ has one minimum and one maximum with its fourth derivative at the minimum and its second derivative at the maximum are nonzero. Furthermore, if $\varepsilon > 0$ is fixed and enough small, then we get (5.2.13), which is an exact equation for the eigenvalues of $T_n(a)$. Taking into account the cases $0 < s < \varepsilon$ and $\varepsilon \leq s \leq \pi$, and the asymptotic expansions of the elements that appear in (5.2.13), we simplify this equation.

5.1 Introduction and main results

We consider the class of all symbols a satisfying the following conditions:

i) a is a real-valued rational function.

ii) The function $g : [0, 2\pi] \to \mathbb{R}$, given by $g(\sigma) := a(e^{i\sigma})$, has range $[0, M]$ with $M > 0$.

iii) $g(0) = g(2\pi) = 0$, $g'(0) = g'(2\pi) = g''(0) = g''(2\pi) = g'''(0) = g'''(2\pi) = 0$, and $g^{(4)}(0) = g^{(4)}(2\pi) > 0$.

iv) There is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = M$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

For each $\lambda \in [0, M]$, there are exactly one $\varphi_1(\lambda) \in [0, \varphi_0]$ such that $g(\varphi_1(\lambda)) = \lambda$, and exactly one $\varphi_2(\lambda) \in [\varphi_0, 2\pi]$ such that $g(\varphi_2(\lambda)) = \lambda$. Note that $\varphi_{1,2}$ are the inverse function of g restricted to the intervals $[0, \varphi_0]$ and $[\varphi_0, 2\pi]$, respectively. For each $\lambda \in [0, M]$, the function g takes values less than or equal to λ on the segments $[0, \varphi_1(\lambda)]$ and $[\varphi_2(\lambda), 2\pi]$. Denote by $\varphi(\lambda)$ the arithmetic mean of the lengths of these two segments,

$$\varphi(\lambda) := \frac{1}{2}\left(\varphi_1(\lambda) - \varphi_2(\lambda)\right) + \pi = \frac{1}{2}\left|\{\sigma \in [0, 2\pi] : g(\sigma) \le \lambda\}\right|,$$

where $|\cdot|$ is the Lebesgue measure on $[0, 2\pi]$. The function $\varphi : [0, M] \to [0, \pi]$ is continuous and bijective. We let $\psi : [0, \pi] \to [0, M]$ stand for the inverse function.

Similarly, in this chapter we will express all the main objects in terms of $s = \varphi(\lambda)$ rather than λ itself. This approach will simplify the statement of our main results. In particular we put

$$\sigma_1(s) = \varphi_1(\psi(s)) = \varphi_1(\lambda) \quad \text{and} \quad \sigma_2(s) = \varphi_2(\psi(s)) = \varphi_2(\lambda).$$

Additionally, we have

$$g(\sigma_1(s)) = g(\sigma_2(s)) = \psi(s) = \lambda.$$

For each $s \in [0, \pi]$ the symbol $a - \psi(s)$ has 2 zeros: one at $t = e^{i\sigma_1(s)}$ and one at $t = e^{i\sigma_2(s)}$. Consequently $T(a - \psi(s))$ is not invertible. Furthermore, for $s = 0$ the symbol $a - \psi(s)$ has a zero of order 4.

In this and the following sections, we suppose that $\varepsilon > 0$ is enough small. Moreover, we will consider the cases $0 < s < \varepsilon$ and $\varepsilon \le s \le \pi$.

In order to get an invertible Toeplitz operator, we introduce the function $b : \mathbb{T} \times [0, \pi] \to \mathbb{R}$ defined by

$$b(t, s) := b_\varepsilon(t, s) := \frac{(a(t) - \psi(s))\, e^{is} e^{s}}{(t - e^{i\sigma_1(s)})(t^{-1} - e^{-i\sigma_2(s)})(t - e^{s})(t^{-1} - e^{s})}. \tag{5.1.1}$$

A simple calculation shows that

$$b_\varepsilon(e^{i\sigma}, s) = \frac{\psi(s) - g(\sigma)}{\left[4 \sin\frac{\sigma - \sigma_1(s)}{2} \sin\frac{\sigma - \sigma_2(s)}{2}\right]\left[2\cosh s - 2\cos\sigma\right]}. \tag{5.1.2}$$

The equation (5.1.2) reveals that b_ε is positive and, therefore, b_ε generates invertible Toeplitz matrices for all n. The subscript ε of b_ε has essentially sense in the case $0 < s < \varepsilon$, but it is convenient to preserve the notation $b_\varepsilon(\cdot, s)$ for all s. For each $s \in [0, \pi]$, Lemma 5.2.1 guarantees the existence of a Wiener-Hopf factorization $b_{\varepsilon,\pm}(\cdot, s)$ of $b_\varepsilon(\cdot, s)$ and the invertibility of $T(b_\varepsilon(\cdot, s))$. Therefore, the finite section method is applicable and the related Toeplitz matrices $T_n(b_\varepsilon(\cdot, s))$ are also invertible. Moreover, their inverses are uniformly bounded with respect to $n \in \mathbb{N}$ and $s \in [0, \pi]$, which is crucial in our forthcoming calculations.

Furthermore, given that a is a rational function, it follows that $b_\varepsilon(\cdot, s)$ is also rational. Thus, both functions a and $b_\varepsilon(\cdot, s)$ belong to W^α for each $\alpha \ge 0$, where W^α was defined in Section 2.4. The function $b_\varepsilon(\cdot, s)$ can be written as $b_\varepsilon(t, s) = \frac{q_1(t,s)}{q_2(t,s)}$, where q_1 and q_2 are polynomials in the variable t. It is well known that a Wiener-Hopf factorization for $b_\varepsilon(\cdot, s)$ is $b_\varepsilon(t, s) = b_{\varepsilon,+}(t, s) b_{\varepsilon,-}(t, s)$ with

$$b_{\varepsilon,-}(t, s) = \frac{\prod_{j=1}^{m}\left(1 - \frac{1}{\overline{v}_j(s)t}\right)}{\prod_{j=1}^{k}\left(1 - \frac{1}{\overline{u}_j(s)t}\right)}, \quad b_{\varepsilon,+}(t, s) = \kappa_0(s)\frac{\prod_{j=1}^{m}\left(1 - \frac{t}{v_j(s)}\right)}{\prod_{j=1}^{k}\left(1 - \frac{t}{u_j(s)}\right)},$$

where $v_j(s)$ and $\frac{1}{\overline{v}_j(s)}$ are the $2m$ zeros of $q_1(\cdot, s)$, $u_j(s)$ and $\frac{1}{\overline{u}_j(s)}$ are the $2k$ zeros of $q_2(\cdot, s)$, and $\kappa_0(s)$ is a constant.

Let $d_j^{(n)} = \frac{\pi j}{n+1}$ and $\gamma : [\varepsilon, \pi] \to \mathbb{R}$ be a function defined by

$$\gamma(s) := \eta(s) + \arctan\left(\frac{\sin\sigma_2(s)}{e^s - \cos\sigma_2(s)}\right) - \arctan\left(\frac{\sin\sigma_1(s)}{e^s - \cos\sigma_1(s)}\right),$$

where

$$\eta(s) := \frac{1}{4\pi}\int_0^{2\pi} \frac{\log b_\varepsilon(e^{i\sigma}, s)}{\tan\frac{\sigma-\sigma_2(s)}{2}} d\sigma - \frac{1}{4\pi}\int_0^{2\pi} \frac{\log b_\varepsilon(e^{i\sigma}, s)}{\tan\frac{\sigma-\sigma_1(s)}{2}} d\sigma.$$

The singular integrals above are understood in the Cauchy principal-value sense. Moreover, η satisfies Lemma 5.2.6 and $\arctan(\cdot) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now we are ready to formulate our main results for the case $\varepsilon \le s \le \pi$.

Theorem 5.1.1. *Let $\varepsilon \le s \le \pi$ and a be a function satisfying the conditions i), ii), iii), and iv). Then*

$$s_j^{(n)} = d_j^{(n)} + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{p_k\left(d_j^{(n)}\right)}{(n+1)^k} + \Delta_1^{(n)}(j), \tag{5.1.3}$$

where $\Delta_1^{(n)}(j) = O\left(e^{-\delta n}\right)$ as $n \to \infty$, uniformly in j. The coefficients p_k can be calculated explicitly; in particular,

$$p_1(s) = -\gamma(s) \quad \text{and} \quad p_2(s) = \gamma(s)\gamma'(s).$$

Using $\lambda = \psi(s)$ and Theorem 5.1.1, we can formulate the results that allow us to describe the asymptotic behavior for the eigenvalues $\lambda_j^{(n)}$ in the case $\varepsilon \le s \le \pi$.

Theorem 5.1.2. *Under the conditions of Theorem 5.1.1,*

$$\lambda_j^{(n)} = \psi\left(d_j^{(n)}\right) + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{f_k\left(d_j^{(n)}\right)}{(n+1)^k} + \Delta_2^{(n)}(j), \tag{5.1.4}$$

where $\Delta_2^{(n)}(j) = O\left(e^{-\delta n}\right)$ as $n \to \infty$, uniformly in j. The coefficients f_k can be calculated explicitly; in particular,

$$f_1(s) = -\psi'(s)\gamma(s) \quad \text{and} \quad f_2(s) = \frac{1}{2}\psi''(s)\gamma^2(s) + \psi'(s)\gamma(s)\gamma'(s).$$

We now present the main result for the case $0 < s < \varepsilon$.

Theorem 5.1.3. *Let $0 < s < \varepsilon$ and a be a function satisfying the conditions i), ii), iii), and iv). Then for each fixed j in $\mathbb{N}$, the eigenvalues $\lambda_j^{(n)}$ satisfy*

$$\lambda_j^{(n)} = \frac{g^{(4)}(0)}{4!}\left(\frac{\beta_j}{n+3}\right)^4 + O\left(\frac{1}{n^5}\right) \quad \text{as} \quad n \to \infty, \tag{5.1.5}$$

where β_j is the unique solution of the equation

$$\tan\frac{\beta}{2} = (-1)^j\left(\frac{\sinh\beta - \sin\beta}{\cosh\beta - \cos\beta}\right)^{(-1)^j}.$$

5.2 Derivation of the equation for the eigenvalues

Let $f(t) = \sum_{j=-\infty}^{\infty} f_j t^j$ $(t \in \mathbb{T})$ be a function in the Hilbert space $L_2(\mathbb{T})$. Recall the function $\chi_n(t) := t^n$ and the familiar operators

$$[P_n f](t) := \sum_{j=0}^{n-1} f_j \chi_j(t), \quad [Q_n f](t) := \sum_{j=n}^{\infty} f_j \chi_j(t), \quad [Pf](t) := \sum_{j=0}^{\infty} f_j \chi_j(t),$$
$$[Qf](t) := \sum_{j=-\infty}^{-1} f_j \chi_j(t), \text{ and } [W_n f](t) := \sum_{j=0}^{n-1} f_{n-1-j} \chi_j(t).$$

We denote by $L_2^{(n)}(\mathbb{T})$ the image of the operator P_n. In particular, we have

$$L_2^{(n)}(\mathbb{T}) = \operatorname{span}\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}.$$

For a symbol $a \in L_\infty(\mathbb{T})$, the Toeplitz matrix $T_n(a)$ may be identified with the operator $T_n(a) : L_2^{(n)}(\mathbb{T}) \to L_2^{(n)}(\mathbb{T})$ given by $T_n(a)f := P_n(af)$.

The following result allows us to show that the function b_ε is positive and bounded away from zero.

Lemma 5.2.1. *Let a be a function satisfying the conditions i), ii), iii), and iv). Then the function $b_\varepsilon : \mathbb{T} \times [0, \pi] \to \mathbb{R}$, defined by* (5.1.1), *is positive and bounded away from zero*

Proof. Starting with (5.1.1) rewrite b_ε in the form

$$b_\varepsilon(t, s) := \Xi(t, s) \frac{t}{(t - e^s)(e^{-s} - t)}, \tag{5.2.1}$$

where $\Xi(t, s) := \frac{(a(t)-\psi(s))e^{is}}{(t-e^{i\sigma_1(s)})(t^{-1}-e^{-i\sigma_2(s)})}$. An elementary computation shows that

$$\Xi(t, s) = e^{\frac{i}{2}(\sigma_1(s)+\sigma_2(s))} t \frac{a(t) - \psi(s)}{(t - e^{i\sigma_1(s)})(t - e^{i\sigma_2(s)})}.$$

Using the relation $\psi(s) = a\left(e^{i\sigma_1(s)}\right) = a\left(e^{i\sigma_2(s)}\right)$, for $s \in (0, \pi)$ we obtain the following "partial fraction decomposition" of Ξ:

$$\Xi(t, s) = \frac{e^{\frac{i}{2}(\sigma_1(s)+\sigma_2(s))} t}{e^{i\sigma_1(s)} - e^{i\sigma_2(s)}} \left(\frac{a(t) - a\left(e^{i\sigma_1(s)}\right)}{t - e^{i\sigma_1(s)}} - \frac{a(t) - a\left(e^{i\sigma_2(s)}\right)}{t - e^{i\sigma_2(s)}} \right).$$

Consider the function q_3 defined by $q_3(t) := \frac{a(t)}{t-1}$. A simple calculation reveals that Ξ can be rewritten as

$$\Xi(t, s) = t \left(c_1(s) b_1(t, s) + c_2(s) b_2(t, s) \right), \tag{5.2.2}$$

where

$$b_1(t, s) := \frac{q_3(t) - q_3\left(e^{i\sigma_1(s)}\right)}{t - e^{i\sigma_1(s)}}, \quad c_1(s) := -\frac{\sin \frac{\sigma_1(s)}{2}}{\sin \frac{s}{2}} e^{\frac{i}{2}\sigma_1(s)}, \tag{5.2.3}$$
$$b_2(t, s) := \frac{q_3(t) - q_3\left(e^{i\sigma_2(s)}\right)}{t - e^{i\sigma_2(s)}}, \quad c_2(s) := \frac{\sin \frac{\sigma_2(s)}{2}}{\sin \frac{s}{2}} e^{\frac{i}{2}\sigma_2(s)}.$$

In order to analize the situation near the point 0, we consider the case $(\sigma, s) \in [0, \varepsilon] \times [0, \varepsilon]$, where $\varepsilon > 0$ is fixed and enough small. Taking into account that $(\sigma, s) \in [0, \varepsilon] \times [0, \varepsilon]$, we can easily obtain the following asymptotic expansions:

$$\begin{aligned} \sigma_1(s) = s + O(|s|^2), \quad \sigma_2(s) = 2\pi - s + O(|s|^2), \quad e^{i\sigma_{1,2}(s)} = 1 \pm is + O(|s|^2), \\ e^{i\sigma} = 1 + i\sigma + O(|\sigma|^2), \quad \sin s = s + O(|s|^3), \quad e^{\pm s} = 1 \pm s + O(|s|^2). \end{aligned} \tag{5.2.4}$$

Then, substituting the asymptotic formulas from (5.2.4) in (5.2.3) we arrive at the representations

$$c_1(s) = -1 + O(|s|), \quad c_2(s) = -1 + O(|s|). \tag{5.2.5}$$

On the other hand, using the condition *iii)* we see that q_3, q_3', q_3'', and q_3''' have the following expansions:

$$\begin{aligned} &q_3(t) = \xi(t-1)^3 + O(|t-1|^4), \quad q_3'(t) = 3\xi(t-1)^2 + O(|t-1|^3), \quad q_3''(t) = 6\xi(t-1) + O(|t-1|^2), \\ &q_3'''(t) = 6\xi + O(|t-1|), \end{aligned} \tag{5.2.6}$$

where $\xi := \frac{g^{(4)}(0)}{4!} = \frac{a^{(4)}(1)}{4!}$. Once again, a simple calculation shows that $b_1(e^{i\sigma}, s)$ can rewrite as

$$b_1(e^{i\sigma}, s) = q_3'\left(e^{i\sigma_1(s)}\right) + \frac{q_3''\left(e^{i\sigma_1(s)}\right)}{2}\left(e^{i\sigma} - e^{i\sigma_1(s)}\right) + \frac{q_3'''\left(e^{i\sigma_1(s)}\right)}{3!}\left(e^{i\sigma} - e^{i\sigma_1(s)}\right)^2 + O\left(|e^{i\sigma} - e^{i\sigma_1(s)}|^3\right).$$

Combining this with the asymptotic formulas from (5.2.4) and (5.2.6) we get

$$b_1(e^{i\sigma}, s) = -\xi\left(\sigma^2 + s^2 + \sigma s\right) + O_3(\sigma, s), \tag{5.2.7}$$

with $O_3(\sigma, s) := O\left(|\sigma|^3 + |\sigma|^2|s| + |\sigma||s|^2 + |s|^3\right)$. In a similar fashion, it is possible to show that

$$b_2(e^{i\sigma}, s) = -\xi\left(\sigma^2 + s^2 - \sigma s\right) + O_3(\sigma, s). \tag{5.2.8}$$

Thus (5.2.5), (5.2.7), and (5.2.8) mixed with (5.2.2), gives us

$$\Xi(e^{i\sigma}, s) = 2\xi\left(\sigma^2 + s^2\right) + O_3(\sigma, s). \tag{5.2.9}$$

Similarly, using the asymptotic formulas from (5.2.4) it follows that

$$(e^{i\sigma} - e^{s})(e^{-s} - e^{i\sigma}) = \sigma^2 + s^2 + O_3(\sigma, s). \tag{5.2.10}$$

Finally, combining (5.2.9) and (5.2.10) with (5.2.1) we obtain

$$b_\varepsilon(e^{i\sigma}, s) = 2\xi + O(|\sigma| + |s|),$$

which implies that b_ε is positive and bounded away from zero, in the case $(\sigma, s) \in [0, \varepsilon] \times [0, \varepsilon]$.

In the case $(\sigma, s) \in K := [0, 2\pi] \times [0, \pi] \setminus [0, \varepsilon] \times [0, \varepsilon]$, from (5.1.2) we see that b_ε is continuous and positive on the compact set K and, therefore,

$$\inf\{b_\varepsilon(e^{i\sigma}, s) : (\sigma, s) \in K\} > 0.$$

□

Our first objective is to derive an exact equation for the eigenvalues of $T_n(a)$. By virtue of (5.1.1), we arrive at the representation

$$a(t) - \psi(s) = p(t,s)p_\varepsilon(t,s)b_\varepsilon(t,s), \tag{5.2.11}$$

where $p(t,s) := -e^{-is}(t - e^{i\sigma_1(s)})(t^{-1} - e^{-i\sigma_2(s)})$ and $p_\varepsilon(t,s) := (t - e^s)(1 - \frac{e^{-s}}{t})$. Since the function $p_\varepsilon(\cdot,s)b_\varepsilon(\cdot,s)$ is negative, for s is not too close to zero, then the operator $T(p_\varepsilon(\cdot,s)b_\varepsilon(\cdot,s))$ is invertible. Moreover, this guarantees that the related Toeplitz matrices $T_n(p_\varepsilon(\cdot,s)b_\varepsilon(\cdot,s))$ are invertible and that their inverses are uniform bounded with respect to $n \in \mathbb{N}$ and $s \in [\varepsilon, \pi]$. Consider the polynomials Θ_k and $\widehat{\Theta}_k$ defined for $k \geq 1$ by

$$\Theta_k(t,s) = [T_k^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0](t) \quad \text{and} \quad \widehat{\Theta}_k(t,s) = [T_{n+2}^{-1}(\widetilde{b}_\varepsilon(\cdot,s)\widetilde{p}_\varepsilon(\cdot,s))\chi_0](1/t), \tag{5.2.12}$$

where $\widetilde{b}_\varepsilon(t,s) := b_\varepsilon(t^{-1},s)$ and $\widetilde{p}_\varepsilon(t,s) := p_\varepsilon(t^{-1},s)$. The following result establishes a relation between Θ_k and $\widehat{\Theta}_k$, and this lemma can be proved analogously to Lemma 3.1 in [4]. Therefore, we state this result without a proof.

Lemma 5.2.2. *We have*

$$\widehat{\Theta}_k(t,s) = \overline{\Theta_k(t,s)}.$$

The following result allows us to derive a tractable equation for the eigenvalues of $T_n(a)$. Moreover, Equation (5.2.13) will be the starting point to analyze the asymptotic behavior of the eigenvalues.

Lemma 5.2.3. *Let $n \geq 1$ and a be a function satisfying the conditions i), ii), iii), and iv). A number $\lambda = \psi(s)$ is an eigenvalue of $T_n(a)$ if and only if*

$$e^{i(n+1)\sigma_2(s)}\Theta_{n+2}(e^{i\sigma_1(s)},s)\overline{\Theta_{n+2}(e^{i\sigma_2(s)},s)} = e^{i(n+1)\sigma_1(s)}\Theta_{n+2}(e^{i\sigma_2(s)},s)\overline{\Theta_{n+2}(e^{i\sigma_1(s)},s)}. \tag{5.2.13}$$

Proof. We are searching for all values of λ belonging to $[0,M]$ such that the equation $T_n(a)X = \lambda X$ has nonzero solutions X in $L_2^{(n)}(\mathbb{T})$. By the change of variable $\lambda = \psi(s)$, the latter equation can be rewritten as

$$T_n(a - \psi(s))X = 0. \tag{5.2.14}$$

By virtue of (5.2.11), Equation (5.2.14) is equivalent to

$$P_n b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)p(\cdot,s)X = 0. \tag{5.2.15}$$

Multiply (5.2.15) by the function χ_1 to get

$$(P_{n+1} - P_1)b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)\chi_1 p(\cdot,s)X = 0. \tag{5.2.16}$$

Note that $\chi_1 p(\cdot,s)X \in L_2^{(n+2)}(\mathbb{T})$ and $(P_{n+1} - P_1)P_{n+2} = P_{n+1} - P_1$. Then (5.2.16) can be rewritten as $(P_{n+1} - P_1)Y = 0$, where $Y := P_{n+2}b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)\chi_1 p(\cdot,s)X = T_{n+2}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_1 p(\cdot,s)X$. From $(P_{n+1} - P_1)Y = 0$, it follows that Y has the form

$$Y = y_0\chi_o + y_{n+1}\chi_{n+1}.$$

Given that $T_{n+2}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$ is invertible, we obtain $T_{n+2}^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))Y = \chi_1 p(\cdot,s)X$, that is,

$$y_0[T_{n+2}^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0](t) + y_{n+1}[T_{n+2}^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_{n+1}](t) = tp(t,s)X(t). \tag{5.2.17}$$

Taking into account the identity

$$W_{n+2}T_{n+2}(b)W_{n+2} = T_{n+2}(\widetilde{b}),$$

it is easy to verify that

$$[T_{n+2}^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_{n+1}](t) = t^{n+1}[T_{n+2}^{-1}(\widetilde{b}_\varepsilon(\cdot,s)\widetilde{p}_\varepsilon(\cdot,s))\chi_0](1/t) = t^{n+1}\widehat{\Theta}_{n+2}(t,s).$$

Therefore (5.2.17) can be written as

$$y_0\Theta_{n+2}(t,s) + y_{n+1}t^{n+1}\widehat{\Theta}_{n+2}(t,s) = tp(t,s)X(t).$$

Since $p(t,s)$ is zero at both $t = e^{i\sigma_1(s)}$ and $t = e^{i\sigma_2(s)}$, it follows that y_0 and y_{n+1} must satisfy the homogeneous linear system

$$\begin{aligned} \Theta_{n+2}(e^{i\sigma_1(s)},s)y_0 + e^{i(n+1)\sigma_1(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)},s)y_{n+1} &= 0, \\ \Theta_{n+2}(e^{i\sigma_2(s)},s)y_0 + e^{i(n+1)\sigma_2(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)},s)y_{n+1} &= 0. \end{aligned} \tag{5.2.18}$$

Therefore, the initial equation (5.2.14) has a non-trivial solution X if and only if the determinant of system (5.2.18) is zero. Thus we arrive at the equation

$$e^{i(n+1)\sigma_2(s)}\Theta_{n+2}(e^{i\sigma_1(s)},s)\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)},s) - e^{i(n+1)\sigma_1(s)}\Theta_{n+2}(e^{i\sigma_2(s)},s)\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)},s) = 0. \tag{5.2.19}$$

Finally, using Lemma 5.2.2 in (5.2.19) we get (5.2.13). □

At this point asymptotic analysis makes his entrance. Relation (5.2.13) is an exact equation for the eigenvalues of $T_n(a)$, but it is too complicated to be solved for s. In order to simplify (5.2.13), we need information about the asymptotic behavior of Θ_n as $n \to \infty$. The Toeplitz operator $T(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$ related to the infinite matrix $((b_\varepsilon p_\varepsilon)_{j-k}(s))_{j,k=0}^{\infty}$ is the bounded linear operator on $PL_2(\mathbb{T}) = H^2$ acting by the rule

$$T(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))f := Pb_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)f.$$

Recall that $b_{\varepsilon,\pm}(\cdot,s)$ are the Wiener-Hopf factors of $b_\varepsilon(\cdot,s)$. Furthermore, they have expansions of the form

$$b_{\varepsilon,+}(t,s) = \sum_{j=0}^{\infty} u_j(s)t^j \quad\text{and}\quad b_{\varepsilon,-}(t,s) = \sum_{j=0}^{\infty} v_j(s)t^{-j},$$

and are analytic functions of t inside and outside the complex unit circle, respectively. Then $b_\varepsilon(t,s)p_\varepsilon(t,s)$ can be represented as $(t-e^s)b_{\varepsilon,+}(t,s)(1-\frac{e^{-s}}{t})b_{\varepsilon,-}(t,s)$. Therefore, it follows that

$$[T^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0](t) = (t-e^s)^{-1}\, b_{\varepsilon,+}^{-1}(t,s)P\left[\left(1-\frac{e^{-s}}{\cdot}\right)^{-1} b_{\varepsilon,-}^{-1}(\cdot,s)\chi_0\right](t).$$

On the other hand, note that $P\left[\left(1-\frac{e^{-s}}{\cdot}\right)^{-1} b_{\varepsilon,-}^{-1}(\cdot,s)\chi_0\right](t) = 1$ and thus we get

$$[T^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0](t) = (t-e^s)^{-1}\, b_{\varepsilon,+}^{-1}(t,s) \tag{5.2.20}$$

The following result is well known and allows us to obtain an estimate for the Fourier coefficients, when f is a complex-valued integrable function. We formulate it and present a proof.

Lemma 5.2.4. *If $f:\mathbb{C}\to\mathbb{C}$ is a rational function, then its Fourier coefficients satisfy $|f_j|\le D_1 e^{-\delta|j|}$ for some constants $D_1>0$ and $\delta>0$.*

Proof. Let $z_1,\dots,z_m,y_1,\dots,y_k$ be the poles of the function f and let $\delta_1:=\max\{|z_1|,\dots,|z_m|\}<1$ and $\delta_2:=\min\{|y_1|,\dots,|y_k|\}>1$. Given that f is analytic for $\delta_1<|t|<\delta_2$ and assuming that $j\in\mathbb{N}$ and $\varrho>0$, we get

$$f_j=\frac{1}{2\pi i}\int_{|t|=1}\frac{a(t)}{t^{j+1}}dt=\frac{1}{2\pi i}\int_{|t|=\delta_2-\varrho}\frac{a(t)}{t^{j+1}}dt\quad\text{and}\quad f_{-j}=\frac{1}{2\pi i}\int_{|t|=1}\frac{a(t)}{t^{-j+1}}dt=\frac{1}{2\pi i}\int_{|t|=\delta_1+\varrho}\frac{a(t)}{t^{-j+1}}dt,$$

and hence we arrive at the estimates

$$|f_j|\le C_1e^{-\ln(\delta_2-\varrho)j}\quad\text{and}\quad |f_{-j}|\le C_2e^{-\ln(\delta_1+\varrho)(-j)}.$$

Put $\delta:=\min\left\{\ln(\delta_2-\varrho),\ln(\delta_1+\varrho)^{-1}\right\}$. □

The following lemma establishes an asymptotic expansion for Θ_n as $n\to\infty$, and this lemma can be proved analogously to Lemma 5.4 in [6].

Lemma 5.2.5. *Let a be a function satisfying the conditions i), ii), iii), and iv). There is a constant $\delta>0$ such that we have the asymptotic expansion*

$$\Theta_n(t,s)=(t-e^s)^{-1}\,b_{\varepsilon,+}^{-1}(t,s)+R_1^{(n)}(t,s),$$

where $\sup\{|R_1^{(n)}(t,s)|:(t,s)\in\mathbb{T}\times[\varepsilon,\pi]\}=O\left(e^{-\delta n}\right)$ *as $n\to\infty$, uniformly in s, t, and n.*

Proof. It follows from the definitions in the beginning of this section that P_n converges strongly to P, and $T_n(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$ converges strongly to $T(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$ as $n\to\infty$. From (5.2.12) and (5.2.20) we get $\Theta_n(t,s)=(t-e^s)^{-1}\,b_{\varepsilon,+}^{-1}(t,s)+R_1^{(n)}(t,s)$, where

$$\begin{aligned}R_1^{(n)}(t,s)&=T_n^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0-T^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0\\&=T_n^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\{T(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))-T_n(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\}T^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0\\&-Q_nT^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))\chi_0\end{aligned}$$

Using the identities $T(b_\varepsilon p_\varepsilon)=Pb_\varepsilon p_\varepsilon P=P_nb_\varepsilon p_\varepsilon P_n+P_nb_\varepsilon p_\varepsilon Q_n+Q_nb_\varepsilon p_\varepsilon P$, $T_n(b_\varepsilon p_\varepsilon)=P_nb_\varepsilon p_\varepsilon P_n$, $Q_nb_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)(\cdot-e^s)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)=Q_n\left(1-\frac{e^s}{\cdot}\right)b_{\varepsilon,-}(\cdot,s)=0$, and (5.2.20), we obtain

$$R_1^{(n)}(\cdot,s)=T_n^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))P_nb_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)-Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s).\tag{5.2.21}$$

Given that the finite section method is applicable to $T(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$, it follows that the matrices $T_n^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))$ are uniformly bounded with respect to n and s, i.e., there exists $C>0$ satisfying

$$\|T_n^{-1}(b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s))y\|_{\ell^1}\le C\|y\|_{\ell^1}\quad(y\in\mathbb{C}^n,\ n\in\mathbb{Z}_+,\ s\in[\varepsilon,\pi]).$$

Note that $\|\cdot\|_{\ell^1}=\|\cdot\|_0$ and $\|\cdot\|_\infty\le\|\cdot\|_0$. Thus (5.2.21) give us

$$\begin{aligned}\|R_1^{(n)}(\cdot,s)\|_\infty&\le C\|P_nb_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)\|_0+\|Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)\|_0\\&\le C\|P_nb_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)\|_0\|Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)\|_0+\|Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)\|_0\\&\le\left(C\|b_\varepsilon(\cdot,s)p_\varepsilon(\cdot,s)\|_0+1\right)\|Q_n\left(\cdot-e^s\right)^{-1}b_{\varepsilon,+}^{-1}(\cdot,s)\|_0.\end{aligned}\tag{5.2.22}$$

Finally, applying Lemma 5.2.4 to the estimate (5.2.22) with $f(t)=(t-e^s)^{-1}\,b_{\varepsilon,+}^{-1}(t,s)$, we conclude that $\|R_1^{(n)}(\cdot,s)\|_\infty=O\left(e^{-\delta n}\right)$ as $n\to\infty$. □

The next lemma allows us to relate the factors $b_{\varepsilon,\pm}$ and obtain an equivalent writing for them. This result can be proved analogously to Lemma 3.6 in [2]. Therefore, we state this result without a proof.

Lemma 5.2.6. *For every $s \in [\varepsilon, \pi]$ we have*

$$\frac{b_{\varepsilon,+}(e^{i\sigma_1(s)}, s)b_{\varepsilon,-}(e^{i\sigma_2(s)}, s)}{b_{\varepsilon,+}(e^{i\sigma_2(s)}, s)b_{\varepsilon,-}(e^{i\sigma_1(s)}, s)} = e^{2i\eta(s)},$$

where the real-valued function η is a continuous argument of $\frac{b_{\varepsilon,+}(e^{i\sigma_1(s)},s)}{b_{\varepsilon,+}(e^{i\sigma_2(s)},s)}$.

The following lemma gives us an implicit equation for the eigenvalues of $T_n(a)$ when s is not too close to zero. We will use this equation in the following section to find the corresponding asymptotic expansion.

Lemma 5.2.7. *Let $\varepsilon \le s \le \pi$ and a be a function satisfying the conditions i), ii), iii), and iv). Then for every sufficiently large natural number n there exists a real-valued function $R_2^{(n)} \in C[\varepsilon, \pi]$ with the following property:*

A number $\lambda = \psi(s)$ is an eigenvalue of $T_n(a)$ if and only if there is a $j \in \mathbb{Z}$ such that

$$(n+1)s + \eta(s) + \arctan\left(\frac{\sin\sigma_2(s)}{e^s - \cos\sigma_2(s)}\right) - \arctan\left(\frac{\sin\sigma_1(s)}{e^s - \cos\sigma_1(s)}\right) - R_2^{(n)}(s) = j\pi, \tag{5.2.23}$$

where $\|R_2^{(n)}\|_\infty = O\left(e^{-\delta n}\right)$ as $n \to \infty$, uniformly in $s \in [\varepsilon, \pi]$.

Proof. We will deduce (5.2.23) from (5.2.13). First rewrite equation (5.2.13) in the form

$$e^{(n+1)i(\sigma_1(s)-\sigma_2(s))} = \frac{\Theta_{n+2}(e^{i\sigma_1(s)}, s)\overline{\Theta_{n+2}(e^{i\sigma_2(s)}, s)}}{\Theta_{n+2}(e^{i\sigma_2(s)}, s)\overline{\Theta_{n+2}(e^{i\sigma_1(s)}, s)}}. \tag{5.2.24}$$

Taking into account that $\sigma_1(s) - \sigma_2(s) = 2s - 2\pi$ the left-hand side of (5.2.24) is just

$$e^{(n+1)i(\sigma_1(s)-\sigma_2(s))} = e^{i(n+1)(2s-2\pi)} = e^{2i(n+1)s}. \tag{5.2.25}$$

On the other hand, Lemma 5.2.5 and the equality $b_{\varepsilon,-}^{-1}(t,s) = e^{[\log b_\varepsilon(\cdot,s)]_0}\overline{b_{\varepsilon,+}^{-1}(t,s)}$ allow us to rewrite the right-hand side of (5.2.24) as

$$\begin{aligned}\frac{\Theta_{n+2}(e^{i\sigma_1(s)}, s)\overline{\Theta_{n+2}(e^{i\sigma_2(s)}, s)}}{\Theta_{n+2}(e^{i\sigma_2(s)}, s)\overline{\Theta_{n+2}(e^{i\sigma_1(s)}, s)}} &= \left[\frac{e^s - e^{i\sigma_2(s)}}{e^s - e^{-i\sigma_2(s)}}\right]\left[\frac{e^s - e^{-i\sigma_1(s)}}{e^s - e^{i\sigma_1(s)}}\right] \\ &\times \left[\frac{b_{\varepsilon,+}(e^{i\sigma_2(s)}, s)b_{\varepsilon,-}(e^{i\sigma_1(s)}, s)}{b_{\varepsilon,+}(e^{i\sigma_1(s)}, s)b_{\varepsilon,-}(e^{i\sigma_2(s)}, s)}\right] \\ &\times \left[1 + \frac{R_3^{(n)}(s) - R_4^{(n)}(s)}{1 + R_4^{(n)}(s)}\right],\end{aligned} \tag{5.2.26}$$

where

$$\begin{aligned}R_3^{(n)}(s) :=& \left(e^{-i\sigma_2(s)} - e^s\right)\overline{b_{\varepsilon,+}(e^{i\sigma_2(s)}, s)R_1^{(n)}(e^{i\sigma_2(s)}, s)} + \left(e^{i\sigma_1(s)} - e^s\right)b_{\varepsilon,+}(e^{i\sigma_1(s)}, s)R_1^{(n)}(e^{i\sigma_1(s)}, s) \\ &+ \left(e^{i\sigma_1(s)} - e^s\right)\left(e^{-i\sigma_2(s)} - e^s\right)b_{\varepsilon,+}(e^{i\sigma_1(s)}, s)\overline{b_{\varepsilon,+}(e^{i\sigma_2(s)}, s)}R_1^{(n)}(e^{i\sigma_1(s)}, s)\overline{R_1^{(n)}(e^{i\sigma_2(s)}, s)},\end{aligned}$$

and

$$R_4^{(n)}(s) := \left(e^{-i\sigma_1(s)} - e^s\right) \overline{b_{\varepsilon,+}(e^{i\sigma_1(s)}, s) R_1^{(n)}(e^{i\sigma_1(s)}, s)} + \left(e^{i\sigma_2(s)} - e^s\right) b_{\varepsilon,+}(e^{i\sigma_2(s)}, s) R_1^{(n)}(e^{i\sigma_2(s)}, s)$$
$$+ \left(e^{i\sigma_2(s)} - e^s\right)\left(e^{-i\sigma_1(s)} - e^s\right) b_{\varepsilon,+}(e^{i\sigma_2(s)}, s)\overline{b_{\varepsilon,+}(e^{i\sigma_1(s)}, s)} R_1^{(n)}(e^{i\sigma_2(s)}, s)\overline{R_1^{(n)}(e^{i\sigma_1(s)}, s)}.$$

From Lemma 5.2.5 we easily see that $\|R_{3,4}^{(n)}\|_\infty = O\left(e^{-\delta n}\right)$ as $n \to \infty$. Using Lemma 5.2.6 in (5.2.26) and taking into account (5.2.25) it follows that Equation (5.2.24) can be rewritten as

$$e^{2i(n+1)s} = e^{2i\left(\arg\left(e^s - e^{i\sigma_2(s)}\right) + \arg\left(e^s - e^{-i\sigma_1(s)}\right) - \eta(s) + R_2^{(n)}(s)\right)},$$

where $R_2^{(n)}(s) := \frac{1}{2}\arg\left(1 + \frac{R_3^{(n)}(s) - R_4^{(n)}(s)}{1 + R_4^{(n)}(s)}\right)$ is a real function satisfying $\|R_2^{(n)}\|_\infty = O\left(e^{-\delta n}\right)$ as $n \to \infty$.

Note also that

$$\arg\left(e^s - e^{i\sigma_2(s)}\right) = -\arctan\left(\frac{\sin\sigma_2(s)}{e^s - \cos\sigma_2(s)}\right)$$

and

$$\arg\left(e^s - e^{-i\sigma_1(s)}\right) = \arctan\left(\frac{\sin\sigma_1(s)}{e^s - \cos\sigma_1(s)}\right).$$

Therefore, it follows that

$$e^{2i(n+1)s} = e^{2i\left(-\arctan\left(\frac{\sin\sigma_2(s)}{e^s - \cos\sigma_2(s)}\right) + \arctan\left(\frac{\sin\sigma_1(s)}{e^s - \cos\sigma_1(s)}\right) - \eta(s) + R_2^{(n)}(s)\right)},$$

which is equivalent to (5.2.23) with some $j \in \mathbb{Z}$. □

The following lemma gives us an implicit equation for the eigenvalues of $T_n(a)$ when s is too close to zero. We will use this equation in the following section to find the corresponding asymptotic expansion.

Lemma 5.2.8. *Let $0 < s < \varepsilon$ and a be a function satisfying the conditions i), ii), iii), and iv). Then for every sufficiently large natural number n there exists a real-valued function $R_5^{(n)} \in C[0, \varepsilon]$ with the following property:*

A number $\lambda = \psi(s)$ is an eigenvalue of $T_n(a)$ if and only if there is a $j \in \mathbb{Z}$ such that

$$(n+3)s - 2\arctan\left(\frac{\sinh(n+3)s - \sin(n+3)s}{\cosh(n+3)s - \cos(n+3)s}\right) - R_5^{(n)}(s) = j\pi. \tag{5.2.27}$$

where $R_5^{(n)}(s) = O(s) + O\left(s^{-1}e^{-\delta n}\right)$ as $n \to \infty$.

Proof. We suppose that $[T_{n+2}^{-1}(b_\varepsilon(\cdot, s)p_\varepsilon(\cdot, s))\chi_0](t) = X_{n+2}(t)$ with $X_{n+2}(t) \in L_2^{(n+2)}(\mathbb{T})$. Given that $P_{n+2} = I - Q - Q_{n+2}$ we have

$$b_\varepsilon(t, s)p_\varepsilon(t, s)X_{n+2}(t) = 1 + X_-(t) + t^{n+2}X_+(t), \tag{5.2.28}$$

where $X_-(t) := [Qb_\varepsilon(\cdot, s)p_\varepsilon(\cdot, s)X_{n+2}](t)$ and $X_+(t) := t^{-(n+2)}[Q_{n+2}b_\varepsilon(\cdot, s)p_\varepsilon(\cdot, s)X_{n+2}](t)$. Multiply (5.2.28) by the function χ_1 to get

$$b_\varepsilon(t, s)\chi_1 p_\varepsilon(t, s)X_{n+2}(t) = t + tX_-(t) + t^{n+3}X_+(t). \tag{5.2.29}$$

Note that $\chi_1 p_\varepsilon(\cdot,s)X_{n+2} \in L_2^{(n+4)}(\mathbb{T})$ and applying the operator P_{n+4} to (5.2.29) we obtain

$$T_{n+4}(b_\varepsilon(\cdot,s))\chi_1 p_\varepsilon(t,s)X_{n+2}(t) = P_{n+4}b_\varepsilon(t,s)\chi_1 p_\varepsilon(t,s)X_{n+2}(t) = t + (X_-)_{-1}(s) + (X_+)_0(s)t^{n+3}.$$

Since $T_{n+4}(b_\varepsilon(\cdot,s))$ is invertible and $[T_{n+4}^{-1}(b_\varepsilon(\cdot,s))\chi_{n+3}](t) = t^{n+3}[T_{n+4}^{-1}(\widetilde{b}_\varepsilon(\cdot,s))\chi_0](t^{-1})$, we can write $X_{n+2}(t)$ as

$$X_{n+2}(t) = \frac{[T_{n+4}^{-1}(b_\varepsilon(\cdot,s))\chi_1](t) + (X_-)_{-1}(s)[T_{n+4}^{-1}(b_\varepsilon(\cdot,s))\chi_0](t) + (X_+)_0(s)t^{n+3}[T_{n+4}^{-1}(\widetilde{b}_\varepsilon(\cdot,s))\chi_0](t^{-1})}{\chi_1 p_\varepsilon(t,s)}. \tag{5.2.30}$$

We denote the polynomials $[T_{n+4}^{-1}(b_\varepsilon(\cdot,s))\chi_0](t)$ and $[T_{n+4}^{-1}(\widetilde{b}_\varepsilon(\cdot,s))\chi_0](t^{-1})$ by $\Phi_{n+4}(t,s)$ and $\widehat{\Phi}_{n+4}(t,s)$, respectively. Moreover, Φ_{n+4} and $\widehat{\Phi}_{n+4}$ satisfy the relation $\widehat{\Phi}_{n+4}(t,s) = \overline{\Phi_{n+4}(t,s)}$. The polynomial $[T_{n+4}^{-1}(b_\varepsilon(\cdot,s))\chi_1](t)$ is denoted by $\Psi_{n+4}(t,s)$.

It should be noted that the numerator in (5.2.30) must be zero at both $t = e^s$ and $t = e^{-s}$. Consequently, the coefficients $(X_-)_{-1}$ and $(X_+)_0$ must satisfy the non-homogeneous linear system

$$\begin{aligned}
\Phi_{n+4}(e^s,s)(X_-)_{-1}(s) + e^{(n+3)s}\overline{\Phi_{n+4}(e^s,s)}(X_+)_0(s) &= -\,\Psi_{n+4}(e^s,s),\\
\Phi_{n+4}(e^{-s},s)(X_-)_{-1}(s) + e^{-(n+3)s}\overline{\Phi_{n+4}(e^{-s},s)}(X_+)_0(s) &= -\,\Psi_{n+4}(e^{-s},s).
\end{aligned}$$

By Cramer's rule applied to the latter linear system, it follows that $(X_-)_{-1}(s)$ and $(X_+)_0(s)$ can be written as

$$(X_-)_{-1}(s) = \frac{e^{(n+3)s}\overline{\Phi_{n+4}(e^s,s)}\Psi_{n+4}(e^{-s},s) - e^{-(n+3)s}\overline{\Phi_{n+4}(e^{-s},s)}\Psi_{n+4}(e^s,s)}{e^{-(n+3)s}\overline{\Phi_{n+4}(e^{-s},s)}\Phi_{n+4}(e^s,s) - e^{(n+3)s}\overline{\Phi_{n+4}(e^s,s)}\Phi_{n+4}(e^{-s},s)},$$

and

$$(X_+)_0(s) = \frac{\Phi_{n+4}(e^{-s},s)\Psi_{n+4}(e^s,s) - \Phi_{n+4}(e^s,s)\Psi_{n+4}(e^{-s},s)}{e^{-(n+3)s}\overline{\Phi_{n+4}(e^{-s},s)}\Phi_{n+4}(e^s,s) - e^{(n+3)s}\overline{\Phi_{n+4}(e^s,s)}\Phi_{n+4}(e^{-s},s)}.$$

Similarly, we need information about the asymptotic behavior of $\Phi_n(e^{\pm s},s)$ and $\Psi_n(e^{\pm s},s)$ as $n \to \infty$. Given that the inverse of the Toeplitz operator $T(b_\varepsilon(\cdot,s))$ is

$$T^{-1}(b_\varepsilon(\cdot,s)) = b_{\varepsilon,+}^{-1}(\cdot,s)Pb_{\varepsilon,-}^{-1}(\cdot,s),$$

and $Pb_{\varepsilon,-}^{-1}(t,s) = b_{\varepsilon,-}^{-1}(\infty,s) = 1$, we get

$$[T^{-1}(b_\varepsilon(\cdot,s))\chi_0](t) = b_{\epsilon,+}^{-1}(t,s).$$

Analogously, it follows that

$$[T^{-1}(b_\varepsilon(\cdot,s))\chi_1](t) = [b_{\varepsilon,+}^{-1}(\cdot,s)Pb_{\varepsilon,-}^{-1}(\cdot,s)\chi_1](t) = b_{\varepsilon,+}^{-1}(t,s)(t + (b_{\varepsilon,-}^{-1})_{-1}(s)),$$

where

$$(b_{\varepsilon,-}^{-1})_{-1}(s) = \frac{1}{2\pi i}\int_{\mathbb{T}} b_{\varepsilon,-}^{-1}(\tau,s)d\tau.$$

In a similar way to Lemma 5.2.5, we can suppose that

$$\Phi_n(t,s) = b_{\varepsilon,+}^{-1}(t,s) + R_6^{(n)}(t,s) \tag{5.2.31}$$

and

$$\Psi_n(t,s) = \left(t + (b^{-1}_{\varepsilon,-})_{-1}(s)\right) b^{-1}_{\varepsilon,+}(t,s) + R^{(n)}_7(t,s), \tag{5.2.32}$$

with $\|R^{(n)}_6\|_\infty = \|R^{(n)}_7\|_\infty = O\left(e^{-\delta n}\right)$ as $n \to \infty$, uniformly in s, t, and n.

On the other hand, it is natural to expect that the product $(n+3)s$ has a finite limit as n tends to infinity and using (5.2.31) and (5.2.32) we can rewrite $(X_+)_0(s)$ as

$$(X_+)_0(s) = \left[\frac{2\,\text{sinh}s + R^{(n)}_8(s)}{\eta_{1,n+3}(s) + R^{(n)}_9(s)}\right] \tag{5.2.33}$$

where

$$\begin{aligned}\eta_{1,n+3}(s) :=& \frac{b_{\varepsilon,+}(e^{-s},s)}{\overline{b_{\varepsilon,+}(e^{-s},s)}} e^{-(n+3)s} - \frac{b_{\varepsilon,+}(e^{s},s)}{\overline{b_{\varepsilon,+}(e^{s},s)}} e^{(n+3)s},\\ \|R^{(n)}_8\|_\infty =& \|R^{(n)}_9\|_\infty = O\left(e^{-\delta n}\right) \quad \text{as} \quad n \to \infty.\end{aligned}$$

Once again, by virtue of (5.2.31) and (5.2.32) we can rewrite $(X_-)_{-1}(s)$ as

$$(X_-)_{-1}(s) = \left[\frac{\eta_{2,n+3}(s) + R^{(n)}_{10}(s)}{\eta_{1,n+3}(s) + R^{(n)}_9(s)}\right], \tag{5.2.34}$$

where

$$\begin{aligned}\eta_{2,n+3}(s) :=& \frac{b_{\varepsilon,+}(e^{s},s)}{\overline{b_{\varepsilon,+}(e^{s},s)}} \left(e^{-s} + (b^{-1}_{\varepsilon,-})_{-1}(s)\right) e^{(n+3)s} - \frac{b_{\varepsilon,+}(e^{-s},s)}{\overline{b_{\varepsilon,+}(e^{-s},s)}} \left(e^{s} + (b^{-1}_{\varepsilon,-})_{-1}(s)\right) e^{-(n+3)s},\\ \|R^{(n)}_{10}\|_\infty =& O\left(e^{-\delta n}\right) \quad \text{as} \quad n \to \infty.\end{aligned}$$

Substituting (5.2.31), (5.2.32), (5.2.33), and (5.2.34) in (5.2.30) we obtain the following asymptotic expansion

$$X_{n+2}(t) = \frac{\left[\left(t + (b^{-1}_{\varepsilon,-})_{-1}(s)\right) \eta_{1,n+3}(s) + \eta_{2,n+3}(s)\right] b^{-1}_{\varepsilon,+}(t,s) + 2\,\text{sinh}s\ \overline{b^{-1}_{\varepsilon,+}(t,s)}\ t^{n+3} + R^{(n)}_{11}(t,s)}{\eta_{1,n+3}(s)\chi_1 p_\varepsilon(t,s)} \tag{5.2.35}$$

and $\|R^{(n)}_{11}\|_\infty = O\left(e^{-\delta n}\right)$ as $n \to \infty$.

Notice that $\eta_{2,n+3}(s) = \eta_{3,n+2}(s) - (b^{-1}_{\varepsilon,-})_{-1}(s)\eta_{1,n+3}(s)$ with

$$\eta_{3,n+2}(s) := \frac{b_{\varepsilon,+}(e^{s},s)}{\overline{b_{\varepsilon,+}(e^{s},s)}} e^{(n+2)s} - \frac{b_{\varepsilon,+}(e^{-s},s)}{\overline{b_{\varepsilon,+}(e^{-s},s)}} e^{-(n+2)s}.$$

Thus, (5.2.35) can be simplified as

$$X_{n+2}(t) = \frac{\left[t\eta_{1,n+3}(s) + \eta_{3,n+2}(s)\right] b^{-1}_{\varepsilon,+}(t,s) + 2\,\text{sinh}s\ \overline{b^{-1}_{\varepsilon,+}(t,s)}\ t^{n+3} + R^{(n)}_{11}(t,s)}{\eta_{1,n+3}(s)\chi_1 p_\varepsilon(t,s)}. \tag{5.2.36}$$

We now rewrite equation (5.2.13) in the form

$$e^{(n+1)i(\sigma_1(s)-\sigma_2(s))} = \frac{\Theta_{n+2}(e^{i\sigma_1(s)},s)\overline{\Theta_{n+2}(e^{i\sigma_2(s)},s)}}{\Theta_{n+2}(e^{i\sigma_2(s)},s)\overline{\Theta_{n+2}(e^{i\sigma_1(s)},s)}}. \tag{5.2.37}$$

Using (5.2.36) in (5.2.37) and taking into account that $\frac{p_\varepsilon(e^{i\sigma_2(s)},s)\overline{p_\varepsilon(e^{i\sigma_1(s)},s)}}{p_\varepsilon(e^{i\sigma_1(s)},s)\overline{p_\varepsilon(e^{i\sigma_2(s)},s)}} = 1$ and $\sigma_1(s)-\sigma_2(s) = 2s-2\pi$, we get

$$\begin{aligned}&\left[\frac{\left[e^{i\sigma_1(s)}\eta_{1,n+3}(s)+\eta_{3,n+2}(s)\right] b^{-1}_{\varepsilon,+}(e^{i\sigma_1(s)},s) + 2\sinh s\,\overline{b^{-1}_{\varepsilon,+}(e^{i\sigma_1(s)},s)}\; e^{i(n+3)\sigma_1(s)} + R^{(n)}_{11}(e^{i\sigma_1(s)},s)}{\left[e^{-i\sigma_1(s)}\overline{\eta_{1,n+3}(s)}+\overline{\eta_{3,n+2}(s)}\right] \overline{b^{-1}_{\varepsilon,+}(e^{i\sigma_1(s)},s)} + 2\sinh s\; b^{-1}_{\varepsilon,+}(e^{i\sigma_1(s)},s)\; e^{-i(n+3)\sigma_1(s)} + \overline{R^{(n)}_{11}(e^{i\sigma_1(s)},s)}}\right]\\
&\left[\frac{\left[e^{-i\sigma_2(s)}\overline{\eta_{1,n+3}(s)}+\overline{\eta_{3,n+2}(s)}\right] \overline{b^{-1}_{\varepsilon,+}(e^{i\sigma_2(s)},s)} + 2\sinh s\; b^{-1}_{\varepsilon,+}(e^{i\sigma_2(s)},s)\; e^{-i(n+3)\sigma_2(s)} + \overline{R^{(n)}_{11}(e^{i\sigma_2(s)},s)}}{\left[e^{i\sigma_2(s)}\eta_{1,n+3}(s)+\eta_{3,n+2}(s)\right] b^{-1}_{\varepsilon,+}(e^{i\sigma_2(s)},s) + 2\sinh s\,\overline{b^{-1}_{\varepsilon,+}(e^{i\sigma_2(s)},s)}\; e^{i(n+3)\sigma_2(s)} + R^{(n)}_{11}(e^{i\sigma_2(s)},s)}\right]\\
&= e^{2i(n+3)s}.\end{aligned} \tag{5.2.38}$$

Given that $0 < s < \varepsilon$ we can easily obtain the following asymptotic expansions

$$\begin{aligned}&\sigma_1(s) = s+O(s^2),\quad \sigma_2(s) = 2\pi-s+O(s^2),\quad e^{i\sigma_1(s)} = 1+is+O(s^2),\quad e^{i\sigma_2(s)} = 1-is+O(s^2),\\
&\sinh s = s+O(s^3),\quad b^{-1}_{\varepsilon,+}(e^{i\sigma_1(s)},s) = b^{-1}_{\varepsilon,+}(1,0)+\left(i\frac{\partial b^{-1}_{\varepsilon,+}(1,0)}{\partial t}+\frac{\partial b^{-1}_{\varepsilon,+}(1,0)}{\partial s}\right)s+\cdots = b^{-1}_{\varepsilon,+}(1,0)+O(s),\\
&b^{-1}_{\varepsilon,+}(e^{i\sigma_2(s)},s) = b^{-1}_{\varepsilon,+}(1,0)+\left(-i\frac{\partial b^{-1}_{\varepsilon,+}(1,0)}{\partial t}+\frac{\partial b^{-1}_{\varepsilon,+}(1,0)}{\partial s}\right)s+\cdots = b^{-1}_{\varepsilon,+}(1,0)+O(s)\end{aligned} \tag{5.2.39}$$

In a similar way to Lemma 5.2.6 we can suppose that $\frac{b_{\varepsilon,+}(e^{-s},s)}{\overline{b_{\varepsilon,+}(e^{-s},s)}} = e^{2i\eta_1(s)}$ and $\frac{b_{\varepsilon,+}(e^{s},s)}{\overline{b_{\varepsilon,+}(e^{s},s)}} = e^{2i\eta_2(s)}$, where $\eta_1(0) = \eta_2(0)$. Thus, substituting the asymptotic formulas from (5.2.39) and taking into account $\eta_1(s)-\eta_2(s) = O(s)$ for $s \to 0^+$, we see that

$$\begin{aligned}e^{i\sigma_1(s)}\eta_{1,n+3}(s)+\eta_{3,n+2}(s) &= -2e^{2i\eta_1(0)}\left[\cosh(n+3)s + i\sinh(n+3)s\right]s + O(s^2),\\
e^{i\sigma_2(s)}\eta_{1,n+3}(s)+\eta_{3,n+2}(s) &= -2e^{2i\eta_1(0)}\left[\cosh(n+3)s - i\sinh(n+3)s\right]s + O(s^2).\end{aligned} \tag{5.2.40}$$

Now, combining (5.2.39) and (5.2.40) with (5.2.38), it follows that

$$e^{2i(n+3)s} = \left[\frac{(\cosh(n+3)s-\cos(n+3)s)+i(\sinh(n+3)s-\sin(n+3)s)}{(\cosh(n+3)s-\cos(n+3)s)-i(\sinh(n+3)s-\sin(n+3)s)}\right]^2\left[1+O(s)+O\left(s^{-1}e^{-\delta n}\right)\right]$$

Finally, since

$$\left[\frac{(\cosh(n+3)s-\cos(n+3)s)+i(\sinh(n+3)s-\sin(n+3)s)}{(\cosh(n+3)s-\cos(n+3)s)-i(\sinh(n+3)s-\sin(n+3)s)}\right]^2 = e^{4i\arctan\left(\frac{\sinh(n+3)s-\sin(n+3)s}{\cosh(n+3)s-\cos(n+3)s}\right)},$$

we get Equation (5.2.27) with some $j \in \mathbb{Z}$. $\square$

5.3 Solution of the equations for the eigenvalues

This section is devoted to the solution of (5.2.23) and (5.2.27) with its "truncated versions" (without the error term). We first begin with (5.2.27) and its truncated version, the latter being

$$(n+3)s - 2\arctan\left(\frac{\sinh(n+3)s - \sin(n+3)s}{\cosh(n+3)s - \cos(n+3)s}\right) = j\pi. \tag{5.3.1}$$

Using the definition of arctan and the formula $\tan(s - \pi j/2) = \begin{cases} -\cot s & \text{if } j = 2k-1, \\ \tan s & \text{if } j = 2k, \end{cases}$ we can rewrite equation (5.3.1) in the equivalent form

$$\tan\left(\frac{n+3}{2}s\right) = (-1)^j \left(\frac{\sinh(n+3)s - \sin(n+3)s}{\cosh(n+3)s - \cos(n+3)s}\right)^{(-1)^j}. \tag{5.3.2}$$

It is easy to verify that the solutions $\widehat{s}_j^{(n)}$ of (5.3.2) belong to $\left(\frac{(2j-1)\pi}{n+3}, \frac{(2j+1)\pi}{n+3}\right)$. Moreover, we know that $\widehat{s}_{2j-1}^{(n)} \in \left(\frac{(2j-1)\pi}{n+3}, \frac{2j\pi}{n+3}\right)$ and $\widehat{s}_{2j}^{(n)} \in \left(\frac{2j\pi}{n+3}, \frac{(2j+1)\pi}{n+3}\right)$, which allow us to infer that

$$\frac{j\pi}{n+3} < \widehat{s}_j^{(n)} < \frac{(j+1)\pi}{n+3}.$$

Therefore, taking into account this, we can suppose that $(n+3)\widehat{s}_j^{(n)}$ has a finite limit as n tends to infinity and j is fixed.

For each j in $\mathbb{N}$, denote by β_j the unique real number that belongs to the interval $(j\pi, (j+1)\pi)$ and satisfies the transcendental equation

$$\tan\frac{\beta_j}{2} = (-1)^j \left(\frac{\sinh\beta_j - \sin\beta_j}{\cosh\beta_j - \cos\beta_j}\right)^{(-1)^j}. \tag{5.3.3}$$

Furthermore, for each j, the transcendental equation (5.3.3) is easy to solve by numerical methods. Approximately,

$$\beta_1 \approx 4.730040, \quad \beta_2 \approx 7.853204, \quad \beta_3 \approx 10.995607.$$

Note that β_j is a solution of (5.3.3) if and only if $\widehat{s}_j^{(n)} = \frac{\beta_j}{n+3}$ is a solution of (5.3.1).

The solution of the equation (5.2.27) is denoted by $s_j^{(n)}$. The following result allows us to derive an asymptotic expansion of $s_j^{(n)}$.

Lemma 5.3.1. *Let $\mu_0 > 0$ be a small number such that $\frac{\mu_0}{n} < s < \varepsilon$ and a be a function satisfying the conditions i), ii), iii), and iv). Then for each fixed j in $\mathbb{N}$, the solutions of the equations* (5.2.27) *and* (5.3.1) *satisfy*

$$s_j^{(n)} = \frac{\beta_j}{n+3} + O\left(\frac{1}{n^2}\right) \quad \textit{and} \quad \left|s_j^{(n)} - \widehat{s}_j^{(n)}\right| = O\left(\frac{1}{n^2}\right) \quad \textit{as} \quad n \to \infty.$$

Proof. Fix j in $\mathbb{N}$. Given that $R_5^{(n)}(s) = O(s) + O(s^{-1}e^{-\delta n})$ and as $\frac{\mu_0}{n} < s$, it follows that

$$R_5^{(n)}(s) = O(s) + O(ne^{-\delta n}) \text{ as } n \to \infty.$$

Making the variable change $z = (n+3)s$ in (5.2.27), we get

$$z - 2\arctan\left(\frac{\sinh z - \sin z}{\cosh z - \cos z}\right) - R_5^{(n)}\left(\frac{z}{n+3}\right) = j\pi. \tag{5.3.4}$$

We now define the function $F(z) := z - 2\arctan\left(\frac{\sinh z - \sin z}{\cosh z - \cos z}\right)$. Using the definition of β_j we can deduce that $F(\beta_j) = j\pi$. Therefore, if z_j is a solution of (5.3.4) and as $F(\beta_j) = j\pi$, then we obtain

$$|F(z_j) - F(\beta_j)| = |F(z_j) - j\pi| = \left|R_5^{(n)}\left(\frac{z_j}{n+3}\right)\right|. \tag{5.3.5}$$

On the other hand, by Lagrange's theorem, there exists a point $\tilde{z}$ between z_j and β_j such that

$$|F(z_j) - F(\beta_j)| = |F'(\tilde{z})|\,|z_j - \beta_j|. \tag{5.3.6}$$

Combining (5.3.5) and (5.3.6), and noting that F' is a bounded function on $[z_j, \beta_j]$ or $[\beta_j, z_j]$, we arrive at the estimate

$$|z_j - \beta_j| = O\left(\frac{1}{n}\right) + O(ne^{-\delta n}).$$

Finally, taking into account that $z_j = (n+3)s_j^{(n)}$, we see that

$$\left|s_j^{(n)} - \frac{\beta_j}{n+3}\right| = O\left(\frac{1}{n^2}\right) + O(e^{-\delta n}),$$

which implies that $s_j^{(n)} = \frac{\beta_j}{n+3} + O\left(\frac{1}{n^2}\right)$ and $\left|s_j^{(n)} - \widehat{s}_j^{(n)}\right| = O\left(\frac{1}{n^2}\right)$ as $n \to \infty$.

□

Proof of Theorem 5.1.3. Fix j in $\mathbb{N}$. The idea is to use the relation $\lambda_j^{(n)} = \psi\left(s_j^{(n)}\right)$. We first have to deduce an asymptotic expansion for the function ψ as $s \to 0^+$. Indeed, from *iii*) we know that $g(0) = g'(0) = g''(0) = g'''(0) = 0$ and $g^{(4)}(0) > 0$. Thus g has the following asymptotic expansion

$$g(\sigma) = \zeta\sigma^4 + O(\sigma^5) \quad (\sigma \to 0^+), \tag{5.3.7}$$

where $\zeta := \frac{g^{(4)}(0)}{4!}$.

Using the change of variable $\sigma = \varphi_1(\lambda)$ and taking into account that $\lambda = g(\varphi_1(\lambda))$ and $\varphi_1(\lambda) \to 0$ as $\lambda \to 0^+$, we obtain from (5.3.7) that

$$\varphi_1(\lambda) = \zeta^{-1/4}\lambda^{1/4} + O(\lambda^{1/2}) \quad (\lambda \to 0^+). \tag{5.3.8}$$

Once again, from *iii*) we know that $g(2\pi) = g'(2\pi) = g''(2\pi) = g'''(2\pi) = 0$ and $g^{(4)}(2\pi) = g^{(4)}(0) > 0$. Thus g has the following asymptotic expansion

$$g(\sigma) = \zeta(\sigma - 2\pi)^4 + O(|\sigma - 2\pi|^5) \quad (\sigma \to 2\pi^-). \tag{5.3.9}$$

In a similar fashion, given that $\lambda = g(\varphi_2(\lambda))$ and $\varphi_2(\lambda) \to 2\pi$ as $\lambda \to 0^+$, it is possible to deduce from (5.3.9) that

$$\varphi_2(\lambda) = 2\pi - \zeta^{-1/4}\lambda^{1/4} + O(\lambda^{1/2}), \quad \lambda \to 0^+. \tag{5.3.10}$$

Subtracting (5.3.10) from (5.3.8) and applying the formula $\varphi(\lambda) = \frac{1}{2}(\varphi_1(\lambda) - \varphi_2(\lambda) + 2\pi)$ we get an asymptotic expansion of φ:

$$\varphi(\lambda) = \zeta^{-1/4}\lambda^{1/4} + O(\lambda^{1/2}) \quad (\lambda \to 0^+). \tag{5.3.11}$$

Therefore, making the variable changes $s = \varphi(\lambda)$ and $\lambda = \psi(s)$, it follows from (5.3.11) that

$$\psi(s) = \zeta s^4 + O(s^5) \quad (s \to 0^+). \tag{5.3.12}$$

By virtue of Lemma 5.3.1 we obtain that $s_j^{(n)} = \frac{\beta_j}{n+3} + O\left(\frac{1}{n^2}\right)$. Finally, combining this with (5.3.12) and the formula $\lambda_j^{(n)} = \psi\left(s_j^{(n)}\right)$, we arrive at (5.1.5). □

We now study the solution of (5.2.23) and its truncated version, the latter being

$$(n+1)s + \gamma(s) = j\pi, \tag{5.3.13}$$

where $\gamma : [\varepsilon, \pi] \to \mathbb{R}$ is defined by

$$\gamma(s) := \eta(s) + \arctan\left(\frac{\sin\sigma_2(s)}{e^s - \cos\sigma_2(s)}\right) - \arctan\left(\frac{\sin\sigma_1(s)}{e^s - \cos\sigma_1(s)}\right)$$

and $\gamma \in C^1[\varepsilon, \pi]$.

We now introduce the following notation:

$$F_n(s) := (n+1)s + \gamma(s), \quad d_j^{(n)} := \frac{\pi j}{n+1}, \quad e_j^{(n)} := d_j^{(n)} - \frac{\gamma\left(d_j^{(n)}\right)}{n+1}, \quad \text{and}$$
$$I_j^{(n)} := \left[e_j^{(n)} - \frac{r(n)}{n+1}, e_j^{(n)} + \frac{r(n)}{n+1}\right],$$

with $r(n) := 2\|R_2^{(n)}\|_\infty + \frac{4\|\gamma\|_\infty\|\gamma'\|_\infty}{n+1}$. Thus there exists $C_3 > 0$ such that for every sufficiently large n

$$e_{j+1}^{(n)} - e_j^{(n)} \geq \frac{C_3}{n} \quad \text{and} \quad \operatorname{diam} I_j^{(n)} = O\left(\frac{1}{n^2}\right). \tag{5.3.14}$$

Therefore for every sufficiently large n, relations in (5.3.14) show that the intervals $I_j^{(n)}$ are disjoint.

The next lemma shows that, for every sufficiently large n, equation (5.2.23) has one and only one solution denoted by $s_j^{(n)}$. By virtue of Lemma 5.3.1, we can suppose that the equation (5.2.27) gives us the first solutions on $(0, \pi)$, i.e., we have j_1 solutions and these solutions are unique.

Lemma 5.3.2. *Let $\varepsilon \leq s \leq \pi$ and a be a function satisfying the conditions i), ii), iii), and iv). Then for every sufficiently large natural number n the equation* (5.2.23) *has a unique solution $s_j^{(n)} \in I_j^{(n)}$, for every $j = j_1 + 1, j_1 + 2, \ldots, n$.*

Proof. The function $F_n - R_2^{(n)}$ is continuous on the interval $I_j^{(n)}$. On the other hand, a simple calculation reveals that

$$\begin{aligned} F_n\left(e_j^{(n)} + \frac{r(n)}{n+1}\right) &= (n+1)\left(e_j^{(n)} + \frac{r(n)}{n+1}\right) + \gamma\left(e_j^{(n)} + \frac{r(n)}{n+1}\right) \\ &= \pi j - \gamma\left(d_j^{(n)}\right) + r(n) + \gamma\left(e_j^{(n)} + \frac{r(n)}{n+1}\right). \end{aligned} \tag{5.3.15}$$

Additionally, notice that $|r(n)| \leq \|\gamma\|_\infty$ for every sufficiently large n, which implies the following estimate

$$\left|e_j^{(n)} + \frac{r(n)}{n+1} - d_j^{(n)}\right| = \frac{\left|r(n) - \gamma\left(d_j^{(n)}\right)\right|}{n+1} \leq \frac{2\|\gamma\|_\infty}{n+1}. \tag{5.3.16}$$

Thus combining (5.3.15) and (5.3.16) we obtain

$$\begin{aligned} F_n\left(e_j^{(n)}+\frac{r(n)}{n+1}\right)-\pi j-R_2^{(n)}\left(e_j^{(n)}+\frac{r(n)}{n+1}\right) &= r(n)+\left[\gamma\left(e_j^{(n)}+\frac{r(n)}{n+1}\right)-\gamma\left(d_j^{(n)}\right)\right]-R_2^{(n)}\left(e_j^{(n)}+\frac{r(n)}{n+1}\right) \\ &\geq 2\|R_2^{(n)}\|_\infty + \frac{4\|\gamma\|_\infty\|\gamma'\|_\infty}{n+1} - \frac{2\|\gamma\|_\infty\|\gamma'\|_\infty}{n+1} - \|R_2^{(n)}\|_\infty \\ &= \|R_2^{(n)}\|_\infty + \frac{2\|\gamma\|_\infty\|\gamma'\|_\infty}{n+1} \geq 0. \end{aligned}$$

That is,

$$F_n\left(e_j^{(n)}+\frac{r(n)}{n+1}\right)-R_2^{(n)}\left(e_j^{(n)}+\frac{r(n)}{n+1}\right) \geq \pi j. \tag{5.3.17}$$

In a similar fashion, it is possible to show that

$$F_n\left(e_j^{(n)}-\frac{r(n)}{n+1}\right)-R_2^{(n)}\left(e_j^{(n)}-\frac{r(n)}{n+1}\right) \leq \pi j. \tag{5.3.18}$$

Relations (5.3.17) and (5.3.18) together the Intermediate Value Theorem show that in the interval $I_j^{(n)}$ there exists a solution of the equation (5.2.23).

At this moment we do not know whether this solution is unique. Contrary to what we want, assume that for some $j \in \{j_1+1, j_1+2, \ldots, n\}$ equation (5.2.23) has another solution s belonging to $I_j^{(n)}$. The $n+1$ numbers $s_1^{(n)}, \ldots, s_{j_1}^{(n)}, s_{j_1+1}^{(n)}, s_{j_1+2}^{(n)}, \ldots, s_n^{(n)}, s$ are different. Since ψ is injective on $[0,\pi]$, the corresponding eigenvalues $\psi\left(s_1^{(n)}\right), \ldots, \psi\left(s_{j_1}^{(n)}\right), \psi\left(s_{j_1+1}^{(n)}\right), \psi\left(s_{j_1+2}^{(n)}\right), \ldots, \psi\left(s_n^{(n)}\right), \psi(s)$ are different, too. This contradicts the fact that the matrix $T_n(a)$ has only n eigenvalues. ☐

Now (5.3.13) is equivalent to the equation

$$s = d_j^{(n)} - \frac{\gamma(s)}{n+1}. \tag{5.3.19}$$

The following result justifies the application of the fixed-point iteration method to (5.3.19). As previously, we denote by j_1 the number of solutions on $(0,\pi)$ of the equation (5.2.27).

Lemma 5.3.3. *Let $\varepsilon \leq s \leq \pi$ and a be a function satisfying the conditions i), ii), iii), and iv). For every sufficiently large natural number n, we have:*

(*i*) *The equation* (5.3.13) *has a unique solution $\widehat{s}_j^{(n)} \in I_j^{(n)}$, for every $j = j_1+1, j_1+2, \ldots, n$.*

(*ii*) *The function $f_j^{(n)} : I_j^{(n)} \to \mathbb{R}$ defined by*

$$f_j^{(n)}(s) := d_j^{(n)} - \frac{\gamma(s)}{n+1}$$

is contractive on $I_j^{(n)}$ with a Lipschitz constant $\|f_j^{(n)\prime}\|_\infty \leq \frac{\|\gamma'\|_\infty}{n+1}$, for every $j = j_1+1, j_1+2, \ldots, n$.

(iii) $\left|s_j^{(n)} - \widehat{s}_j^{(n)}\right| = O(e^{-\delta n})$.

(iv) The sequence defined by

$$\widehat{s}_{j,0}^{(n)} := d_j^{(n)} \quad \text{and} \quad \widehat{s}_{j,\ell}^{(n)} := f_j^{(n)}\left(\widehat{s}_{j,\ell-1}^{(n)}\right) \quad (\ell \geq 1)$$

satisfies $\left|\widehat{s}_{j,\ell}^{(n)} - \widehat{s}_j^{(n)}\right| = O\left(\frac{1}{n^{\ell+1}}\right)$.

(v) $\left|s_j^{(n)} - \widehat{s}_{j,\ell}^{(n)}\right| = O(e^{-\delta n}) + O\left(\frac{1}{n^{\ell+1}}\right)$.

Proof. (i) The existence and uniqueness for the solutions of (5.3.13) in $I_j^{(n)}$ can be proved as in the previous lemma, just take 0 instead of $R_2^{(n)}(s)$.

(ii) The estimate for $\|f_j^{(n)\prime}\|_\infty$ is immediate. We now have to show that $f_j^{(n)}$ maps $I_j^{(n)}$ to $I_j^{(n)}$. Indeed, if $\left|s - e_j^{(n)}\right| \leq \frac{r(n)}{n+1}$, then for every sufficiently large n, we get

$$\begin{aligned}\left|f_j^{(n)}(s) - e_j^{(n)}\right| &= \frac{\left|\gamma\left(d_j^{(n)}\right) - \gamma(s)\right|}{n+1} \leq \frac{\|\gamma'\|_\infty}{n+1}\left|d_j^{(n)} - s\right| \leq \frac{\|\gamma'\|_\infty}{n+1}\left|e_j^{(n)} - s\right| + \frac{\|\gamma\|_\infty\|\gamma'\|_\infty}{(n+1)^2} \\ &\leq \frac{\|\gamma'\|_\infty r(n)}{(n+1)^2} + \frac{\|\gamma\|_\infty\|\gamma'\|_\infty}{(n+1)^2} \leq \frac{r(n)}{n+1}.\end{aligned}$$

Therefore, it follows that the function $f_j^{(n)}$ is contractive on $I_j^{(n)}$ and has *exactly one* fixed point, which we denote by $\widehat{s}_j^{(n)}$.

(iii) Since $s_j^{(n)}$ and $\widehat{s}_j^{(n)}$ are the solutions of (5.2.23) and (5.3.13), respectively, then we obtain

$$\left|F_n\left(s_j^{(n)}\right) - F_n\left(\widehat{s}_j^{(n)}\right)\right| = \left|F_n\left(s_j^{(n)}\right) - \pi j\right| = \left|R_2^{(n)}\left(s_j^{(n)}\right)\right| \leq \|R_2^{(n)}\|_\infty = O\left(e^{-\delta n}\right). \tag{5.3.20}$$

On the other hand, it is easy to see that $F_n'(s) \geq \frac{n+1}{2}$ for every sufficiently large n. Combining this with Lagrange's theorem we get

$$\left|F_n\left(s_j^{(n)}\right) - F_n\left(\widehat{s}_j^{(n)}\right)\right| \geq \frac{n+1}{2}\left|s_j^{(n)} - \widehat{s}_j^{(n)}\right|. \tag{5.3.21}$$

Therefore, from (5.3.20) and (5.3.21) we arrive at the estimate

$$\left|s_j^{(n)} - \widehat{s}_j^{(n)}\right| = O(e^{-\delta n}).$$

(iv) Note that

$$\left|\widehat{s}_{j,1}^{(n)} - \widehat{s}_{j,0}^{(n)}\right| = \left|e_j^{(n)} - d_j^{(n)}\right| = \frac{\left|\gamma\left(d_j^{(n)}\right)\right|}{n+1} \leq \frac{\|\gamma\|_\infty}{n+1}. \tag{5.3.22}$$

By the well-known result to estimate the speed of convergence for the fixed point iterations, it follows that

$$\left|\widehat{s}_{j,\ell}^{(n)} - \widehat{s}_j^{(n)}\right| \leq \frac{\|f_j^{(n)\prime}\|_\infty^\ell}{1 - \|f_j^{(n)\prime}\|_\infty}\left|\widehat{s}_{j,1}^{(n)} - \widehat{s}_{j,0}^{(n)}\right|.$$

Combining this with (5.3.22) and as $\|f_j^{(n)'}\|_\infty \leq \frac{\|\gamma'\|_\infty}{n+1}$, we get

$$\left|\widehat{s}_{j,\ell}^{(n)} - \widehat{s}_j^{(n)}\right| \leq \frac{n+1}{n+1-\|\gamma'\|_\infty} \frac{\|\gamma\|_\infty}{\|\gamma'\|_\infty} \left(\frac{\|\gamma'\|_\infty}{n+1}\right)^{\ell+1} = O\left(\frac{1}{n^{\ell+1}}\right).$$

(*v*) Applying (*iii*) and (*iv*) we arrive at the estimate $\left|s_j^{(n)} - \widehat{s}_{j,\ell}^{(n)}\right| = O(e^{-\delta n}) + O\left(\frac{1}{n^{\ell+1}}\right)$. ☐

Proof of Theorem 5.1.1. This theorem follows directly from Theorem 2.4.2, Lemma 5.3.2, and Lemma 5.3.3. ☐

Proof of Theorem 5.1.2. This theorem follows directly from Theorem 2.4.3 and Theorem 5.1.1. ☐

5.4 Numerical tests

This section allows to check that we calculated the coefficients in the asymptotic formulas correctly, i.e., we want to demonstrate that the asymptotic formulas deliver acceptable approximations not only for astronomically large n, but already for n in the early hundreds. For instance, we make all numerical tests of the eigenvalues of $T_n(a)$ for several moderately sized n by *Matlab R*2015*a*. Here, all exact eigenvalues are determined by means of the Matlab function *eig*.

Denote by $\lambda_j^{(n,k)}$ the approximation of $\lambda_j^{(n)}$ with k terms. These approximations are obtained from our formulas (5.1.4) and (5.1.5). For example, from (5.1.4), we have

$$\lambda_j^{(n,3)} = \psi\left(d_j^{(n)}\right) - \frac{\psi'\left(d_j^{(n)}\right)\gamma\left(d_j^{(n)}\right)}{n+1} + \frac{\psi''\left(d_j^{(n)}\right)\left(\gamma\left(d_j^{(n)}\right)\right)^2 + 2\psi'\left(d_j^{(n)}\right)\gamma\left(d_j^{(n)}\right)\gamma'\left(d_j^{(n)}\right)}{2(n+2)^2},$$

$$\lambda_j^{(n,2)} = \psi\left(d_j^{(n)}\right) - \frac{\psi'\left(d_j^{(n)}\right)\gamma\left(d_j^{(n)}\right)}{n+1}, \quad \lambda_j^{(n,1)} = \psi\left(d_j^{(n)}\right).$$

For each j, we put $\epsilon_j^{(n,k)} := |\lambda_j^{(n)} - \lambda_j^{(n,k)}|$. To use the formulas (5.1.3) we denote by $\tilde{\lambda}_j^{(n,k)} := \psi\left(s_{j,*}^{(n,k)}\right)$ the approximations of $\lambda_j^{(n)}$, where $s_{j,*}^{(n,k)}$ contains k terms. For example, from (5.1.3), we have

$$\tilde{\lambda}_j^{(n,2)} = \psi\left(d_j^{(n)} - \frac{\gamma\left(d_j^{(n)}\right)}{n+1}\right), \quad \tilde{\lambda}_j^{(n,3)} = \psi\left(d_j^{(n)} - \frac{\gamma\left(d_j^{(n)}\right)}{n+1} + \frac{\gamma\left(d_j^{(n)}\right)\gamma'\left(d_j^{(n)}\right)}{(n+1)^2}\right). \tag{5.4.1}$$

In order to test numerically the asymptotic formulas (5.1.3), (5.1.4), and (5.1.5) we consider the symbol $a(t) = \frac{1}{t^3} + \frac{1}{t^2} - \frac{13}{t} + 22 - 13t + t^2 + t^3$, which has the generating function $g(\sigma) = 2\cos 3\sigma + 2\cos 2\sigma - 26\cos\sigma + 22$. Obviously, the related Toeplitz matrix $T_n(a)$ will be symmetric and heptadiagonal. Moreover, we have $\psi = g$, $\sigma_1(s) = s$, $\sigma_2(s) = 2\pi - s$, and $b_\varepsilon(e^{i\sigma}, s) = \frac{g(\sigma)-g(s)}{4(\cos s - \cos\sigma)(\cosh s - \cos\sigma)}$.

n	64	128	512	1024
$\epsilon_1^{(n,1)}$	$1.5 \cdot 10^{-6}$	$5.3 \cdot 10^{-8}$	$5.5 \cdot 10^{-11}$	$1.7 \cdot 10^{-12}$
$\epsilon_2^{(n,1)}$	$1.1 \cdot 10^{-5}$	$4.0 \cdot 10^{-7}$	$4.2 \cdot 10^{-10}$	$1.3 \cdot 10^{-11}$

Table 5.1: Maximal absolute errors for the eigenvalues of $T_n(a)$ obtained with our formula (5.1.5), i.e., $j = 1, 2$.

n	64	128	512	1024
$\epsilon^{(n,2)}$	$4.2 \cdot 10^{-4}$	$1.08 \cdot 10^{-4}$	$6.08 \cdot 10^{-6}$	$1.7 \cdot 10^{-6}$
$\epsilon^{(n,3)}$	$4.2 \cdot 10^{-6}$	$5.4 \cdot 10^{-7}$	$8.6 \cdot 10^{-9}$	$1.08 \cdot 10^{-9}$

Table 5.2: Maximal absolute errors for the eigenvalues of $T_n(a)$ obtained with our formulas (5.4.1), where $\epsilon^{(n,k)} := \max\{\epsilon_j^{(n,k)} : j \in \{[\frac{n}{6}] + 1, \ldots, n\}\}$.

n	64	128	512	1024
$\epsilon^{(n,2)}$	$3.5 \cdot 10^{-4}$	$9.08 \cdot 10^{-5}$	$5.7 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$
$\epsilon^{(n,3)}$	$3.6 \cdot 10^{-6}$	$4.6 \cdot 10^{-7}$	$7.47 \cdot 10^{-9}$	$9.3 \cdot 10^{-10}$

Table 5.3: Maximal absolute errors for the inner eigenvalues of $T_n(a)$ obtained with our formula (5.1.4), where $\epsilon^{(n,k)} := \max\{\epsilon_j^{(n,k)} : j \in \{[\frac{n}{6}] + 1, \ldots, n\}\}$.

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