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Contents

Abstract

This thesis consists basically of two chapters, the first chapter is devoted to the higher topological complexity of real projective spaces $\mathbb{R}\mathrm{P}^m$. We give a thorough analysis of the gap between the upper and lower bounds of the inequality $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) \leq \mathbf{TC}_s(\mathbb{R}\mathrm{P}^m) \leq sm$, which allows us to give an estimation for $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^m)$. Further, we explain how such estimation seems to be closely related to the determination of the Euclidean immersion dimension of $\mathbb{R}\mathrm{P}^m$. The second chapter is devoted to the effective topological complexity of the orientable surfaces of genus Σ_g, $g \geq 2$. There, by finding effective-zero-divisors of dimension 1 (in the 256 systems of local coefficients having as group $\mathbb{Z}$) we present some indirect evidence that suggests that the effective topological complexity of Σ_g, $g \geq 2$, would be 3 instead of 4.

Resumen

Esta tesis consiste básicamente de dos capítulos, el primer capítulo está dedicado a la complejidad topológica superior de los espacios proyectivos reales $\mathbb{R}\mathrm{P}^m$. Hacemos un análisis exhaustivo de la diferencia entre los límites superior e inferior de la desigualdad $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) \leq \mathbf{TC}_s(\mathbb{R}\mathrm{P}^m) \leq sm$, el cual nos permite dar una estimación para $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^m)$. Además explicamos cómo tal estimación parece estar relacionada con la determinación de la dimensión de inmersión euclideana de $\mathbb{R}\mathrm{P}^m$. El segundo capítulo está dedicado a la complejidad topológica efectiva de las superficies orientables de género $g \geq 2$, Σ_g. Encontramos divisores del cero efectivos de dimensión 1 (en los 256 sistemas de coeficientes locales que tienen como grupo a $\mathbb{Z}$) que dan cierta evidencia indirecta que sugiere que la complejidad efectiva de Σ_g, $g \geq 2$, es 3 en lugar de 4.

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Introduction

The concept of topological complexity of a topological space X, $\mathbf{TC}(X)$, was introduced in [13] by M. Farber, and it can be thought of as one less than the minimal number of rules, motion planning rules, required to tell how to move between any two points of X. Later this notion was generalized to higher topological complexity of a topological space X, $\mathbf{TC}_s(X)$, by Y. Rudyak in [25]. The latter can be interpreted as one less than the number of rules required to tell how to move consecutively between any s specified points of the space X. Both notions are homotopic invariants and $\mathbf{TC}_2(X) = \mathbf{TC}(X)$. In [14], M. Farber, S. Tabachnikov and S. Yuzvinsky showed that $\mathbf{TC}(\mathbb{R}P^m) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^m)$ provided $m \neq 1, 3, 7$. Research on the immersion problem for projective spaces has yielded evidence that suggests a relation of the form $\mathbf{TC}(\mathbb{R}P^m) = 2m - \delta(m)$, where $\delta(m) = \mathcal{O}(\alpha(m))$ and $\alpha(m)$ denotes the number of ones in the binary expansion of m. Since $\mathbf{TC}_s(\mathbb{R}P^m) = sm - \delta_s(m)$ for some non-negative integer $\delta_s(m)$ and $\mathbf{TC}_2(\mathbb{R}P^m)$ is usually equal to the immersion dimension of $\mathbb{R}P^m$, the first part of this thesis is intended to be an initial step to understand the smallest dimension of Euclidean spaces where $\mathbb{R}P^m$ can be immersed, at least for $m \not\equiv 3 \mod 4$, through the study of δ_s.

Standard techniques in algebraic topology can be used to estimate the topological complexity of interesting spaces such as the closed orientable surface of genus g. However, the case of the closed non-orientable surface of genus $g \in \{2, 3\}$ turned out to be a particularly challenging task, a partial solution, case $g = 2$, was indicated by D. Davis in [10], and a complete solution was accomplished by D. Cohen and L. Vandembroucq in [7]. The second part of this thesis arose from a desire to recover and generalize these results from Błaszczyk-Kaluba's version of Farber's topological complexity for mechanical systems whose configuration spaces exhibit symmetries, namely effective topological complexity, see [3]. Our main result says that there is a monotonic sequence of effective topological complexities for orientable surfaces as follows

$$3 \leq \mathbf{TC}^{\sigma}(\Sigma_2) \leq \mathbf{TC}^{\sigma}(\Sigma_3) \leq \mathbf{TC}^{\sigma}(\Sigma_4) \leq \cdots \leq 4,$$

for more details see section 2.4 and chapter 4.

2

Preliminaries

2.1 Higher topological complexity

In this chapter we review some of the concepts and fundamental results from topological complexity and cohomology of groups that will be needed in the sequel.

Definition 2.1. The *reduced topological complexity* of X, $\mathbf{TC}(X)$, is defined as the smallest integer k such that there exists an open cover $\{U_0, \ldots, U_k\}$ of $X \times X$ such that the restriction of the end points evaluation map $e_{0,1} : PX \to X \times X$ to each U_i admits a continuous section.

Definition 2.2. The *reduced sth topological complexity* $\mathbf{TC}_s(X)$ of a space X is the smallest integer k such that there exists an open cover $\{U_0, \ldots, U_k\}$ of X^s such that the restriction of the path fibration

$$\begin{array}{rccl} f: & PX & \longrightarrow & X^s \\ & \gamma & \mapsto & \left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \ldots, \gamma\left(\frac{s-2}{s-1}\right), \gamma\left(\frac{s-1}{s-1}\right)\right) \end{array}$$

to each U_i admits a continuous section.

Remark 2.1. Notice that $\mathbf{TC}_2(X) = \mathbf{TC}(X)$.

Definition 2.3. Given a commutative ring R, the *s-th zero-divisor cup-length* of X, $\mathrm{zcl}_s(X)$, is the maximal number of elements in $\ker(\Delta_s^* \colon H^*(X^s; R) \to H^*(X; R))$ having a non-trivial product, where $\Delta_s \colon X \to X^s$ is the s-fold iterated diagonal.

Proposition 2.1. *For a c-connected space X having the homotopy type of a CW complex,*

$$\mathrm{zcl}_s(X) \le \mathbf{TC}_s(X) \le s\,\mathrm{hdim}(X)/(c+1).$$

The notation $\mathrm{hdim}(X)$ stands for the (cellular) homotopy dimension of X, i.e. the minimal dimension of CW complexes having the homotopy type of X. The lower bound in proposition 2.1 comes from noticing that $\mathbf{TC}_s(X)$ is equal to the Schwarz genus of the diagonal map $d_s : X \to X^s$ and the upper bound comes from obstruction theory, for details see [1, theorem 3.9].

Lemma 2.2 (Lucas's theorem 1878)**.** *Let m and n non-negative integers, and p a prime, the following congruence relation holds*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} (\mathrm{mod}\, p)$$

where

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$$

and

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

are the base p expansions of m and n respectively.

The following result is a particular case of Kummer's theorem.

Corollary 2.2.1. $\binom{m}{n}$ *is divisible by a prime p if and only if at least one of the base p digits of n is greater than the corresponding digit of m.*

2.2 *Diagonal approximation and rewriting systems*

For a detailed treatment of the following topics, we refer the reader to [4], [19], [15], and [20]. We must also clarify that in this thesis we only consider left $\mathbb{Z}G$-modules (and their corresponding morphisms).

Definition 2.4. Let G be a group. The *group ring* $\mathbb{Z}G$ is a ring associated to G. Additively it is the free abelian group on G, i.e., an element of $\mathbb{Z}G$ is a finite linear combination of the group elements

$$n_1 g_1 + \cdots + n_k g_k, \qquad n_i \in \mathbb{Z},\ g_i \in G.$$

The sum and product of $\mathbb{Z}G$ are defined by

$$\sum_i m_i g_i + \sum_i n_i g_i = \sum_i (m_i + n_i) g_i,$$

and

$$\left(\sum_i m_i g_i\right)\left(\sum_j m_j g_j\right) = \sum_{i,j} m_i n_j g_i g_j.$$

Definition 2.5. For any group G the *augmentation map* is the ring homomorphism $\epsilon\colon \mathbb{Z}G \to \mathbb{Z}$ such that $\epsilon(g) = 1$ for all $g \in G$. The kernel of ϵ is called the *augmentation ideal* of $\mathbb{Z}G$ and is denoted $I(G)$.

Remark 2.2. A basis for $I(G)$ as $\mathbb{Z}$-module consists of the elements $g - 1 \in \mathbb{Z}G, g \in G \setminus \{1\}$.

Definition 2.6. For a group G and a left $\mathbb{Z}G$-module M, we define its nth cohomology group with coefficients in M to be

$$H^n(G, M) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M),$$

for $n \geq 0$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, i.e. $H^n(G, M)$ is the nth right derived functor of the left exact functor $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$.

The above definition provides us with a method of calculating $H^n(G, M)$. Namely, we first find a projective resolution $\mathbf{F}$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over $\mathbb{Z}G$, then we consider the complex $\mathrm{Hom}_{\mathbb{Z}G}(\mathbf{F}, M)$, and the cohomology groups $H^n(G, M)$ are the cohomology groups of this last complex.

Among many possible topics within techniques from group cohomology we are interested in the cup product.

Theorem 2.3. *Let* $\mathbf{F}, \mathbf{F}'$ *be two free (or projective) resolutions of a module M, then there is a chain map* $\mathbf{F} \to \mathbf{F}'$ *over the identity on M. Further, any two such maps are chain homotopy equivalent. In particular, any such chain map* $\mathbf{F} \to \mathbf{F}'$ *over the identity on M is a chain homotopy equivalence.*

Proposition 2.4. *Let* G, G' *be groups. If* $\mathbf{F}$ *and* $\mathbf{F}'$ *are projective resolutions of* $\mathbb{Z}$ *over* $\mathbb{Z}G$ *and* $\mathbb{Z}G$*, respectively, then* $\mathbf{F} \otimes \mathbf{F}'$ *is a projective resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}[G \times G']$*, and* $G \times G'$ *acts diagonally over* $\mathbf{F} \otimes \mathbf{F}'$.

Recall that

$$\left(\mathbf{F}\otimes\mathbf{F}'\right)_n=\bigoplus_{i=0}^{n}F_i\otimes F_{n-i},$$

and

$$\partial_n\left(f_i\otimes f'_{n-i}\right)=d_i(f_i)\otimes f'_{n-i}+(-1)^i f_i\otimes d'_{n-i}(f'_{n-i}).$$

Remark 2.3. If $\mathbf{F}$ is a projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, then $\mathbf{F}\otimes\mathbf{F}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G\times G]$. By theorem 2.3 there is a $\mathbb{Z}G$-linear map Ψ, called a *diagonal approximation*, that extends $\mathrm{id}_{\mathbb{Z}}:\mathbb{Z}\to\mathbb{Z}$.

Definition 2.7. Let $\mathbf{F}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and let M and N be $\mathbb{Z}G$-modules. Given the cocyles $u\in\mathrm{Hom}_{\mathbb{Z}G}(F_p,M)$ and $v\in\mathrm{Hom}_{\mathbb{Z}G}(F_q,M)$, representatives of $[u]$ and $[v]$, we define the *cup product* of $[u]$ and $[v]$ to be the cohomology class

$$[(u\otimes v)\circ\Psi_{pq}]\in H^{p+q}(G,M\otimes N),$$

where $\Psi_{pq}\colon F_{p+q}\to F_p\otimes F_q$ is the composition of $\Psi_{p+q}\colon F_{p+q}\to(\mathbf{F}\otimes\mathbf{F})_{p+q}$ with the projection $\pi_{pq}\colon(\mathbf{F}\otimes\mathbf{F})_{p+q}\to F_p\otimes F_q$.

Definition 2.8. Let G be a group and $\mathbf{F}$ a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over $\mathbb{Z}G$. A *contracting homotopy* $\mathbf{T}$ for $\mathbf{F}$ consists of a sequence of $\mathbb{Z}$-homomorphism $T_q\colon X_q\to X_{q+1}$, $q\geq-1$, such that $d_{q+1}T_q+T_{q-1}d_q=\mathrm{id}_{X_q}$ for each $q\geq0$. Here we write $X_{-1}=\mathbb{Z}$ and $d_0=\epsilon$.

Since in order to calculate the cup product of two cohomology classes it is necessary a diagonal approximation, the propositions below will allow us to achive such task in our cases of interest.

Proposition 2.5. *Let G be a group, $\mathbf{X}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and $\mathbf{T}$ a contracting homotopy for $\mathbf{X}$. Extend $T_{-1}\epsilon\colon X_0\to X_0$ to a chain map $T_{-1}\epsilon\colon\mathbf{X}\to\mathbf{X}$ over $\mathbb{Z}$ by defining $(T_{-1}\epsilon)_i=0$ if $i\neq0$. Let $U_q\colon(\mathbf{X}\otimes\mathbf{X})_q\to(\mathbf{X}\otimes\mathbf{X})_{q+1}$ for $q\geq-1$ be the $\mathbb{Z}$-homomorphisms given by $U_{-1}=T_{-1}\otimes T_{-1}\colon\mathbb{Z}\otimes\mathbb{Z}\to X_0\otimes X_0$, and $U_q(u\otimes v)=T_i(u)\otimes v+(T_{-1}\epsilon)_i(u)\otimes T_{q-i}(v)$ for $u\in X_i$, $v\in X_{q-i}$, $0\leq i\leq q$. Then the U_q constitute a contracting homotopy for $\mathbf{X}\otimes\mathbf{X}$.*

Proposition 2.6. *Let G be a group, $\mathbf{X}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and $\mathbf{U}$ a contracting homotopy for $\mathbf{X}\otimes\mathbf{X}$. Suppose that for each $q\geq0$, B_q is a $\mathbb{Z}G$-basis for X_q such that $\epsilon(b)=1$ for each $b\in B_0$. Let $\psi_0\colon X_0\to X_0\otimes X_0$ be the left $\mathbb{Z}G$-module homomorphism determined by $\psi_0(b)=b\otimes b$ for $b\in B_0$. For $q>0$ let $\psi_q\colon X_q\to(\mathbf{X}\otimes\mathbf{X})_q$ be the left $\mathbb{Z}G$-module homomorphism determined inductively by $\psi_q(b)=U_{q-1}\psi_{q-1}\partial_q(b)$ for $b\in B_q$. Then ψ is a diagonal approximation for $\mathbf{X}$.*

Basically, the previous propositions state that given a finitely generated free resolution $\mathbf{F}$ of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ over $\mathbb{Z}G$, we can calculate a diagonal approximation $\psi\colon\mathbf{F}\to\mathbf{F}\otimes\mathbf{F}$ whenever we have a contracting homotopy for $\mathbf{F}$. Furthermore, the next lemma and its proof are quite useful, essentially it says that if you have defined $s_k:F_k\to F_{k+1}$ such that $d_{k+1}s_k+s_{k-1}d_k=\mathrm{id}_{F_k}$ then you can construct $s_{k+1}:F_{k+1}\to F_{k+2}$ satisfying $d_{k+2}s_{k+1}+s_kd_{k+1}=\mathrm{id}_{F_{k+1}}$ by defining $s_{k+1}(u)=v$ provided $d_{k+2}(v)+s_kd_{k+1}(u)=u$, where u is an element of a basis of F_{k+1}.

Lemma 2.7. *Let G be a group and $\mathbf{F}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Suppose that, for a given $n\geq0$, there are $\mathbb{Z}$-homomorphisms $s_k\colon F_k\to F_{k+1}$ such that $d_{k+1}s_k+s_{k-1}d_k=\mathrm{id}_{F_k}$ for $-1\leq k\leq n$ $(F_{-1}=\mathbb{Z})$. Then there is a map of abelian groups $s_{n+1}\colon F_{n+1}\to F_{n+2}$ such that $d_{n+2}s_{n+1}+s_nd_{n+1}=\mathrm{id}_{F_{n+1}}$.*

Next we introduce concepts and results that we need to construct contracting homotopies in chapter 4.

Definition 2.9. Let S be a set (alphabet) and let S^* be the free monoid on S. A *rewriting system on* S^* is a subset $R\subseteq S^*\times S^*$. An element $(u,v)\in R$, also written $u\to v$, is called a *rule* of R.

Given a rewriting system R, we write:

- $x \to y$ for $x, y \in S^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$,
- $x \overset{*}{\to} y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of arrows.

Definition 2.10. $x \in S^*$ is called *irreducible* with respect to R if there is no possible reduction $x \to y$, otherwise x is called *reducible*.

Definition 2.11. A *rewriting system for a monoid* M is a tuple (S, R) such that $\langle S | v_1 = v_2 \text{ if } (v_1, v_2) \in R \rangle$ is a presentation for M. A *rewriting system for a group* G is a rewriting system for G as monoid.

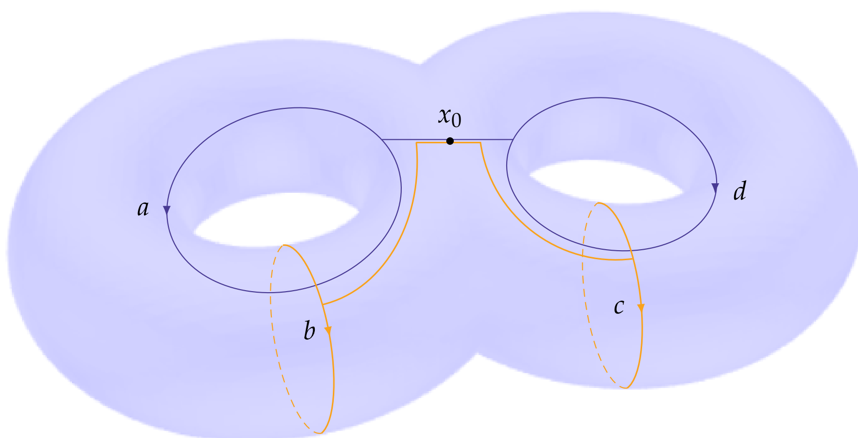
Definition 2.12. Given a rewriting system, we say that

1. (R, S) is *noetherian* if there is no infinite chain $x \to x_1 \to x_2 \to \cdots$ for any $x \in S^*$.
2. (R, S) is *confluent* if wherever $x \overset{*}{\to} y_1$ and $x \overset{*}{\to} y_2$, there is a z such that $y_1 \overset{*}{\to} z$ and $y_2 \overset{*}{\to} z$.
3. (R, S) is *complete* if it is Noetherian and confluent.
4. (R, S) is *finite* if both R and S are finite.

A group with a complete rewriting system has the property that there is exactly one irreducible word representing each of the group elements. So a finite complete rewriting system gives a solution to the word problem for the group.

Proposition 2.8. *There is a finite complete rewriting system for the fundamental group of a closed orientable surface of genus g, using the alphabet S of the usual generators and their inverses.*

In fact, the rewriting system we use here for the fundamental group of the genus 2 orientable surface Σ_2, $\pi_1(\Sigma_2, x_0)$, with generators a, b, c and d as shown in the figure (see lemma 4.5) has the following twelve rules:



$x\bar{x} \to 1$ for $x \in \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}$, $cd \to ba\bar{b}\bar{a}dc$, $c\bar{d} \to \bar{d}ab\bar{a}\bar{b}c$, $\bar{c}\bar{d} \to \bar{d}\bar{c}ba\bar{b}\bar{a}$, $\bar{c}ba\bar{b}\bar{a}d \to d\bar{c}$.

2.3 Local coefficient systems

In this section we provide the main definitions and properties of cohomology with local systems of coefficients. This kind of cohomology is use widely here, our main reference is [27, chapter 6].

Definition 2.13. Let R be a ring. A *local system* of R-modules L over a topological space X is defined as a function which makes the assignations $x \mapsto L(x)$, and $\gamma \mapsto L(\gamma)$, where $x \in X$, $L(x)$ is a left R-module, $\gamma\colon [0,1] \to X$ is a continuous path, $L(\gamma)\colon L(\gamma(1)) \to L(\gamma(0))$ is an R-homomorphism, and the following conditions are satisfied:

- If two paths $\gamma_1, \gamma_2 \colon [0,1] \to X$ have the same end points, and are homotopic with respect to the end points, then $L(\gamma_1) = L(\gamma_2)$.
- If γ is the constant path at $x \in X$, then $L(\gamma) \colon L(x) \to L(x)$ is the identity map.
- If $\gamma_1, \gamma_2 \colon [0,1] \to X$ are two paths such that $\gamma_1(1) = \gamma_2(0)$, we have $L(\gamma_1\gamma_2) = L(\gamma_1) \circ L(\gamma_2)$.

Thus for any two points $x, y \in X$ in the same path connected component of X, $L(x)$ and $L(y)$ are isomorphic R-modules, since any path γ connecting x to y defines an isomorphism $L(\gamma) \colon L(y) \to L(x)$. In particular, any loop $\gamma \colon [0,1] \to X$ at $x \in X$, defines an automorphism that depens only on the homotopy class of the loop. An action of the fundamental group $\pi = \pi_1(X, x)$ on $L(x)$ can be defined by the correspondence $[\gamma] \mapsto L(\gamma)$, with this action $L(x)$ is a left module over the group ring $R\pi$.

If X is path connected, then $L(x)$ viewed as left $R\pi$-module, determines the local system L. In fact, given an arbitrary left R-module M with an action of $\pi_1(X, x)$ by R-automorphisms, there is a local coefficient system L over X such that $L(x) \cong M$ as $R\pi$-modules. Furthermore, L is unique up to functorial isomorphism.

An important result on cohomology with local coefficients that we will use in this work states that for a path connected space X,

$$H^1(X; L) \cong Q\left(\pi, L(x)\right) / P\left(\pi, L(x)\right),$$

where the last quotient denotes crossed homomorphisms over principal homomorphisms, see [27, theorem 3.3 on p. 276].

2.3.1 *The setting for cohomology with local coefficients*

For a more detailed treatment of what follows we recommend to see [27, chapter 6, section 2].

Definition 2.14. Let G and H local coefficient systems over X and Y respectively. A *map* $\Theta = (\theta_1, \theta_2) \colon (X; G) \to (Y; H)$ consists of a continuous map $\theta_1 \colon X \to Y$ and a homomorphism $\theta_2 \colon \theta_1^* H \to G$ of systems of local coefficients over X.
Two maps $\Theta, \Psi \colon (X; G) \to (Y; B)$ are *homotopic* if there is a map $\Lambda = (\lambda_1, \lambda_2) \colon (X \times I, p^*G) \to (Y; H)$ such that

$$\begin{aligned} \lambda_1 i_0 &= \theta_1, & \text{and} \qquad i_0^* \lambda_2 &= \theta_2, \\ \lambda_1 i_1 &= \psi_1, & i_1^* \lambda_2 &= \psi_2, \end{aligned}$$

where

$$\begin{array}{rccclcrccc} i_t : & X & \longrightarrow & X \times I & & \text{and} & p : & X \times I & \longrightarrow & X \\ & x & \mapsto & (x,t),\ t \in \{0,1\} & & & & (x,t) & \mapsto & x. \end{array}$$

Lemma 2.9. *Let $K \colon X \times I \to Y$ be a homotopy from $f := K(-,0)$ to $g := K(-,1)$ and H a system of local coefficients in Y. Then f^*H and g^*H are isomorphic.*

Proof. Let us write $G = g^*H$ and $R = f^*H$. We define $\Phi \colon G \to R$ such that $\Phi(b) \colon G(b) \to R(b)$ is given by $H(K(b, -))$ for each $b \in X$. If α is a path in X from b_1 to b_2, then

$$\begin{aligned} \Phi(b_1) \circ G(\alpha) &= H(K(b_1, -) * (g \circ \alpha)) \\ R(\alpha) \circ \Phi(b_2) &= H((f \circ \alpha) * K(b_2, -)). \end{aligned}$$

Now $\Phi(b_1) \circ G(\alpha) = R(\alpha) \circ \Phi(b_2)$ provided $(f \circ \alpha) * K(b_2, -) \cong K(b_1, -) * (g \circ \alpha)$, to see the last homotopy, it is enough to show the existence of a map $L' \colon I \times I \to Y$, such that

$$\begin{aligned} L'(s, 0) &= (f \circ \alpha)(s), \\ L'(s, 1) &= (g \circ \alpha)(s), \\ L'(0, t) &= K(b_1, t), \\ L'(1, t) &= K(b_2, t). \end{aligned}$$

Take $L' = K \circ (\alpha \times \mathrm{id}_I)$. ∎

Remark 2.4. Suppose that $f, g\colon X \to Y$ are homotopic through K and H is a system of local coefficients in Y. Consider the maps $\Theta, \Psi\colon (X, f^*H) \to (Y, H)$ given by $\Theta = (f, \mathrm{id})$ and $\Psi = (g, g')$, where g' is the isomorphism given by lemma 2.9. Define $\Lambda = (K, \lambda')$ where $\lambda' = H_{(K(x,s)|[0,t])}$. Then $\Theta \overset{\Lambda}{\simeq} \Psi$ and therefore they induce the same map in cohomology, which means there is the commutative diagram

$$\begin{array}{ccc} H^*(X, f^*H) & \xleftarrow{f^*} & H^*(Y, H) \\ \cong\uparrow & \swarrow_{g^*} & \\ H^*(X, g^*H), & & \end{array}$$

where the vertical isomorphism is induced by g'. In particular $\ker f^* = \ker g^*$.

2.4 Effective topological complexity

Definition 2.15. The *Schwarz genus* or *sectional category* of a fibration $p : E \to B$, $\mathbf{secat}(p)$, is the smallest $k \in \mathbb{N}$ such that B can be covered by open sets $U_0, \ldots, U_k$ in such a way that for every $i = 0, \ldots, k$, there exists a map $s_i : U_i \to E$ such that $p \circ s_i$ is the inclusion map $U_i \hookrightarrow B$.

Remark 2.5. Notice that $\mathbf{TC}(X)$ and $\mathbf{TC}_s(X)$ are particular cases of the Schwarz genus of a fibration.

Let X be a space with a left principal action of a topological group G. For $i \in \{0, 1\}$ let $e_i : PX \to X$ be the fibration defined by $e_i(\gamma) = \gamma(i)$. Consider the pullback of the morphisms $\epsilon_0 : PX \xrightarrow{e_0} X \xrightarrow{q} X/G$ and $\epsilon_1 : PX \xrightarrow{e_1} X \xrightarrow{q} X/G$, where q denotes the canonical quotient map,

$$\begin{array}{ccccc} PX \times_{X/G} PX & & \longrightarrow & & PX \\ & & & & \downarrow e_0 \\ \downarrow & & & & X \\ & & & & \downarrow q \\ PX & \xrightarrow{e_1} & X & \xrightarrow{q} & X/G. \end{array}$$

For brevity let us write $P_2(X)$ for the fibered product

$$PX \times_{X/G} PX = \{(\alpha, \beta) \in PX \times PX \mid \alpha(1) \equiv \beta(0) \bmod G\}.$$

The next definition is due to Z. Błaszczyk and M. Kaluba, and arises from the idea that in the problem of motion planning, the symmetries present in configuration spaces can be used, see [3].

Definition 2.16. The *reduced effective topological complexity* $\mathbf{TC}^{G,2}(X)$ is the sectional category of the fibration $p_2 : P_2(X) \to X \times X$ given by $p_2(\alpha, \beta) = (\alpha(0), \beta(1))$.

For a topological group G acting on a space X, higher versions $\mathbf{TC}^{G,k}(X)$, $k \geq 1$, are central in [3], however if G acts freely on X, $\mathbf{TC}^{G,k}(X) = \mathbf{TC}^{G,2}(X)$ for all $k \geq 2$. Since we are interested in the case of a free action, we will only deal with the case $k = 2$.

Whenever we want to study in detail a fibration, an important ingredient is knowing the monodromy, that is, the action of the fundamental group of the base space on the fiber. In order to do this, as first step we give a reformulation of the fibration p_2, and as second, and final step, we describe such action in terms of the previous step. From now on we will assume that G is a discrete and finite group.

First, for $g \in G$ let us define

$$P_{2,g}(X) := \{(\alpha, \beta) \in P(X) \times P(X) \mid \alpha(1) = g \cdot \beta(0)\},$$

then for the fibration p_2, since G is discrete, finite and acts freely on X, $P_2(X)$ is the topological disjoint union

$$P_2(X) = \coprod_{g \in G} P_{2,g}(X).$$

Furthermore, notice that $P_{2,g}(X) \cong P_{2,e}(X) \cong P(X)$, $g \in G \setminus \{e\}$ [1].

In these terms, the fibration p_2 takes the form

$$p_2(\gamma) = (\gamma(0), \overline{g} \cdot \gamma(1)),$$

for γ in the copy of $P_2(X)$ corresponding to $P_{2,g}(X)$, and

$$\begin{aligned} p_2^{-1}(x_0, x_0) &= \coprod_{g \in G} \mathrm{Maps}(I, 0, 1; X, x_0, g \cdot x_0) \\ &\simeq G \times \Omega X. \end{aligned}$$

[1] The homomorphisms are given by the maps

$$\begin{array}{rccc} f_g : & P_{2,g}(X) & \longrightarrow & P_{2,e}(X) \\ & (\alpha, \beta) & \mapsto & (\alpha, g \cdot \beta), \end{array}$$

$$\begin{array}{rccc} h_g : & P_{2,e}(X) & \longrightarrow & P_{2,g}(X) \\ & (\alpha, \beta) & \mapsto & (\alpha, \overline{g} \cdot \beta), \end{array}$$

$$\begin{array}{rccc} f : & P_{2,e}(X) & \longrightarrow & P(X) \\ & (\alpha, \beta) & \mapsto & \alpha * \beta, \end{array}$$

and

$$\begin{array}{rccc} h : & P(X) & \longrightarrow & P_{2,e}(X) \\ & \gamma & \mapsto & (\alpha_\gamma, \beta_\gamma), \end{array}$$

where

$$\alpha_\gamma(s) = \gamma\left(\frac{s}{2}\right) \text{ and } \beta_\gamma(s) = \left(\frac{1+s}{2}\right).$$

The last homotopy equivalence is given as follows. For each $g \in G$, fix $\phi_g \in \mathrm{Maps}(I, 0, 1; X, x_0, g \cdot x_0)$, we choose ϕ_e as the constant curve at x_0, const_{x_0} . For a continuous curve c from x_0 to $g \cdot x_0$, $c * \overline{\phi_g}$ is a loop at x_0, and for a loop ℓ_{x_0} at x_0, $\ell_{x_0} * \phi_g$ is a continuous curve from x_0 to $g \cdot x_0$. Then

$$\begin{array}{rccc} f'_g : & \mathrm{Maps}(I, 0, 1; X, x_0, g \cdot x_0) & \longrightarrow & \{g\} \times \Omega X \\ & c & \mapsto & (g, c * \overline{\phi_g}) \end{array}$$

and

$$\begin{array}{rccc} h'_g : & \{g\} \times \Omega X & \longrightarrow & \mathrm{Maps}(I, 0, 1; X, x_0, g \cdot x_0) \\ & (g, \ell_{x_0}) & \mapsto & \ell_{x_0} * \phi_g \end{array}$$

are homotopy equivalences, one inverse of the other.

Second, we describe the action of $\pi_1(X \times X, (x_0, x_0))$ on $p_2^{-1}(x_0, x_0)$ in terms of the homotopy equivalence $p_2^{-1}(x_0, x_0) \simeq G \times \Omega X$ noted above. Given a loop

$$\begin{array}{rccc} \sigma : & I & \longrightarrow & X \times X \\ & \tau & \mapsto & (\sigma_1(\tau), \sigma_2(\tau)) \end{array}$$

at (x_0, x_0), the HLP for p_2 says that the following diagram has (a non-unique) solution H_σ

$$\begin{array}{ccc} (G \times \Omega X) \times \{1\} & \hookrightarrow & P_2(X) \\ \big\downarrow & \overset{H_\sigma}{\nearrow} & \big\downarrow p_2 \\ (G \times \Omega X) \times I & \xrightarrow[\sigma \circ \mathrm{proj}_2]{} & X \times X. \end{array}$$

Under the above considerations the top map in the diagram is given by

$$\begin{array}{ccc} (G \times \Omega X) \times \{1\} & \longrightarrow & P_2(X) \\ ((g, \ell_{x_0}), 1) & \mapsto & (\ell_{x_0}, \overline{g} \cdot \phi_g), \end{array}$$

and an explicit solution $H_\sigma : (G \times \Omega X) \times I \longrightarrow P_2(X)$ is given by

$$H_\sigma((g, \ell_{x_0}), \tau) = \left(\alpha_{g,\ell_{x_0},\tau} \, , \, \overline{g} \cdot \beta_{g,\ell_{x_0},\tau}\right),$$

where $\alpha_{g,\ell_{x_0},\tau}(s) = \gamma_{g,\ell_{x_0},\tau}\left(\frac{1}{2}s\right)$ and $\beta_{g,\ell_{x_0},\tau}(s) = \gamma_{g,\ell_{x_0},\tau}\left(\frac{1}{2} + \frac{1}{2}s\right)$ for

$$\begin{array}{rccl} \gamma_{g,\ell_{x_0},\tau} : & I & \longrightarrow & X \\ & s & \mapsto & \begin{cases} \sigma_1(3s+\tau), & 0 \leq s \leq \frac{1-\tau}{3}, \\ (\ell_{x_0} * \phi_g)\left(\frac{3s+\tau-1}{1+2\tau}\right), & \frac{1-\tau}{3} \leq s \leq \frac{2+\tau}{3}, \\ g \cdot \sigma_2(\tau + 3 - 3s), & \frac{2+\tau}{3} \leq s \leq 1. \end{cases} \end{array}$$

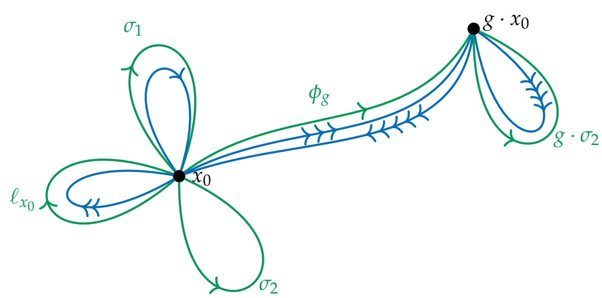
In these terms, the monodromy action of $[\sigma] \in \pi_1\left(X \times X, (x_0, x_0)\right)$ on the homotopy fiber of p_2 corresponds to the (homotopy class of the) map

$$\begin{array}{ccc} G \times \Omega X & \longrightarrow & p_2^{-1}(x_0, x_0) \\ (g, \ell_{x_0}) & \mapsto & H_\sigma(g \, , \ell_{x_0} \, , 0). \end{array}$$

In terms of the discussion in the first step, the latter map takes the form

$$\begin{array}{rccl} f_\sigma : & G \times \Omega X & \longrightarrow & G \times \Omega X \\ & (\, g \, , \, \ell_{x_0}) & \mapsto & \left(g \, , \sigma_1 * \ell_{x_0} * \phi_g * g \cdot \overline{\sigma_2} * \overline{\phi_g}\,\right). \end{array}$$

Where a bar on top of a path stands for the path transversed in the opposite direction. Graphically,



To close this section, we interpret the action of $\pi_1(X \times X, (x_0, x_0))$ on $G \times \Omega X$ in the setting for 1-dimensional obstruction theory when the fiber is not path connected i.e., on reduced 0-dimensional homology. This is, of course, the "effective" analogue of the interpretations done in [8] for regular **TC**.
On the one hand, the induced map

$$(f_\sigma)_* : \quad \overline{H_0}(G \times \Omega X) \quad \longrightarrow \quad \overline{H_0}(G \times \Omega X)$$

is an isomorphism and the above construction gives a monodromy on $\overline{H_0}(G \times \Omega X)$. On the other hand, as abelian group, $H_0(G \times \Omega X)$ is free on the elements of $\pi_0(G \times \Omega X)$. Since there is a canonical bijection of pointed sets: $\pi_1(Y, y_0) \cong \pi_0(\Omega(Y, y_0))$, it follows that $H_0(G \times \Omega X)$ is free on the elements of $G \times \pi_1(X, x_0)$ and $\overline{H_0}(G \times \Omega X)$ coincides with the augmentation ideal

$$I(G \times \pi_1(X, x_0)) = \ker(\mathbb{Z}[G \times \pi_1(X, x_0)] \to \mathbb{Z}).$$

From the considerations in the second step above, we see that $\pi_1(X \times X, (x_0, x_0))$ acts on a basis element of $I(G \times \pi_1(X, x_0))$ by

$$\begin{aligned}([\sigma_1], [\sigma_2]) \cdot [(g, [\ell_{x_0}]) - (e, 1)] &= \left(g, [\sigma_1]\,[\ell_{x_0}]\,[\phi_g]\,[\psi_g \circ \overline{\sigma_2}]\,[\overline{\phi_g}]\right) - (e, [\sigma_1]\,[\overline{\sigma_2}]) \\ &= \left(g, [\sigma_1]\,[\ell_{x_0}]\,[\phi_g]\,\overline{\psi_{g*}(\sigma_2)}\,[\overline{\phi_g}]\right) - (e, [\sigma_1]\,[\overline{\sigma_2}]),\end{aligned}$$

with

$$\begin{array}{rccc} \psi_g: & (X, x_0) & \longrightarrow & (X, g \cdot x_0) \\ & x & \mapsto & g \cdot x. \end{array}$$

An important property of the effective topological complexity is given by the inequality

$$\mathbf{TC}^{G,2}(X) \leq \mathbf{TC}(X/G),$$

if X is a free G-space. This inequality is not elementary, it follows from two facts. First, easily from the definitions one can see that $\mathbf{TC}^{G,2}(X) \leq \mathbf{TC}^{G}(X)$, where $\mathbf{TC}^{G}(X)$ denotes the invariant topological complexity of the space X, see [23]. Second, from the properties of the invariant topological complexity for free actions, $\mathbf{TC}^{G}(X) = \mathbf{TC}(X/G)$.

Remark 2.6. In unpublished work, Z. Błaszczyk, J. González and M. Kaluba showed that if G is a group acting principally on X, the map

$$\begin{array}{rccc} h: & X \times G & \to & P_2(X) = P(X) \times_{X/G} P(X) \\ & (x, g) & \mapsto & (\text{const}_x, \text{const}_{\overline{g} \cdot x}) \end{array}$$

is a homotopy equivalence. Furthermore, the "fattened diagonal"

$$\begin{array}{rccc} j: & G \times X & \hookrightarrow & X \times X \\ & (g, x) & \mapsto & (x, g \cdot x) \end{array}$$

has $p_2 : P_2(X) \to X \times X$ as fibrational replacement.

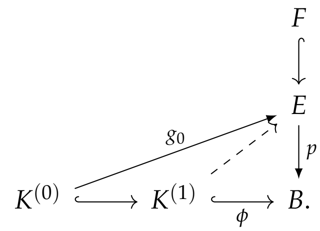
2.5 *The obstruction theory setting*

2.5.1 *Sectioning a fibration over the* 1*-skeleton*

In this subsection we review some of the fine (and not so standard) points in the obstruction-theory viewpoint for the problem of constructing a section of a fibration.

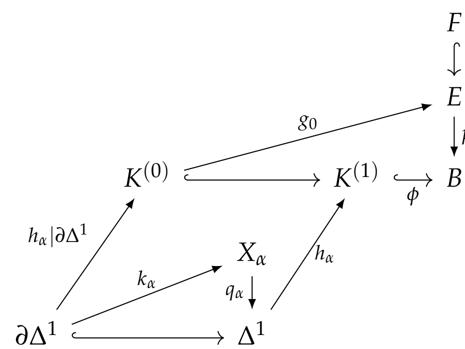
Let $p : E \to B$ be a fibration with B a path connected CW complex and whose fiber F is $(k-1)$-connected, $k \geq 1$ (if $k = 1$, it is also assumed that $\pi_1(F)$ is abelian, so that F is 1-simple). The primary obstruction to the existence of a section of p is a cohomology class $o(p) \in H^{k+1}(B; \pi_k(F))$. Here the coefficients may be twisted by $\pi_1(B)$. The definition involves choosing a section on the k-skeleton and analyzing, withing a cohomological setting, the obvious homotopy obstructions to extend the chosen fibration on each $(k+1)$-cell. The resulting primary obstruction $o(p)$ is canonical, as it does not depend on the chosen section on the k-skeleton. A clarification has to be made here about finding a section over the 1-skeleton of B. A section over the 0-skeleton always can be extended to the 1-skeleton if F is path connected, the interesting case occurs when F is not. It turns out that in this case, the primary obstruction to the existence of a section of p over the 1-skeleton of B is defined as well, it is a cohomology class $o(p) \in H^1(B; \overline{H_0}(F))$, and can be defined directly in terms of the monodromy action of the fundamental group of the base space on the reduced 0-dimensional homology of the fiber space. We review below the technical details in the latter situation (see for instance [8]).

Let $p \colon E \to B$ be a fibration with fiber F and 0-connected base space B. Denote for $K^{(0)}$ and $K^{(1)}$ the skeletons of dimension 0 and 1 of B respectively. Let $\phi \colon K^{(1)} \hookrightarrow B$ be the inclusion map and $g_0 \colon K^{(0)} \to E$ such that $p \circ g_0 = \phi | K^{(0)}$. We want to study the problem of extending the section g_0 of p on $K^{(0)}$ to a section of p on $K^{(1)}$:



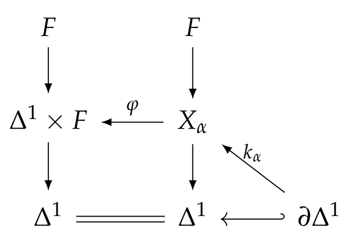
Let e_α be a 1-cell of B with characteristic map $h_\alpha\colon (\Delta^1, \partial\Delta^1) \to (e_\alpha, \partial e_\alpha)$. Consider the fibration $q_\alpha\colon X_\alpha \to \Delta^1$ induced by the map $\phi \circ h_\alpha$, and consider the partial cross-section k_α of q_α determined by the maps $g_0 \circ h_\alpha|\partial\Delta^1$ and $\partial\Delta^1 \hookrightarrow \Delta^1$.

$$\begin{array}{rcll} k_\alpha : & \partial\Delta^1 & \to & X_\alpha \\ & 0 & \mapsto & (0, g_0 h_\alpha(0)) \\ & 1 & \mapsto & (1, g_0 h_\alpha(1)). \end{array}$$

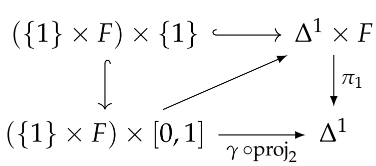


Clearly, the study of extending g_0 form $K^{(0)}$ to $K^{(1)}$ is equivalent to studying whether or not $k_\alpha(0)$ and $k_\alpha(1)$ lie in the same path component of X_α for any α. Note that any extension of k_α to Δ^1 can be fixed to be a section for q_α due to the HLP of the latter fibration.

Since Δ^1 is contractible, the fibration $q_\alpha\colon X_\alpha \to \Delta^1$ is fiber homotopically trivial and the space X_α has the same homotopy type as $\Delta^1 \times F$, and therefore as F. Let $\varphi\colon X_\alpha \to \Delta^1 \times F$ be a fiber homotopy equivalence. Deciding whether or not $k_\alpha(0)$ and $k_\alpha(1)$ lie in the same path component of X_α can be posed in terms of φ: Are $\varphi k_\alpha(0)$ and $\varphi k_\alpha(1)$ in the same path component?



In turn, this last problem can be posed in monodromy terms as follows: Since Δ^1 is 0-connected, the path components of $\Delta^1 \times F$ are in one to one correspondence (via the projection $\pi_2 : \Delta^1 \times F \to F$ onto the second factor) with the path components of F. Writing $\varphi k_\alpha(i) = (i, f_i) \in \Delta^1 \times F$ for $i = 0, 1$, we then see that $\varphi k_\alpha(0)$ and $\varphi k_\alpha(1)$ lie in the same path component of $\Delta^1 \times F$ if and only if $(0, f_0)$ and $(0, f_1)$ lie in the same path component of $\{0\} \times F$, which is a fiber of the projection $\pi_1 : \Delta^1 \times F \to \Delta^1$. The point to note is that π_1 (is a trivial fibration and, as such) has trivial monodromy, so that $(0, f_1)$ is the monodromy action on $(1, f_1) = \varphi k_\alpha(1)$ of the identity path $\gamma : [0, 1] \to \Delta^1$. Summarizing, the obstruction for $\varphi k_\alpha(0) = (0, f_0)$ and $\varphi k_\alpha(1) = (1, f_1)$ to be in the same path component is given by



$$[\gamma] \cdot [\varphi k_\alpha(1)] - [\varphi k_\alpha(0)] \in \overline{H_0}(F),$$

where brackets around γ stand for homotopy (rel 0, 1) class, and brackets around $\varphi k_\alpha(0)$ and $\varphi k_\alpha(1)$ stand for 0-dimensional class.

Since φ is a fiber homotopy equivalence, and since monodromy actions are functorial with respect to pull backs of fibrations, we deduce that the obstruction cochain $c = c(g_0) \in C^1\left(B; \underline{\widetilde{H}_0}(F)\right)$ for extending g_0 to a section of p over the 1-skeleton assigns to each 1-cell e_α with characteristic map $h_\alpha : (\Delta^1, \partial\Delta^1) \to \left(K^{(1)}, K^{(0)}\right)$ the element

$$[h_\alpha] \cdot [g_0(h_\alpha(1))] - [g_0(h_\alpha(0))] \in \widetilde{H}_0\left(F_{h_\alpha(0)}\right)$$

where $F_{h_{\alpha(0)}}$ is the fiber of p over $\varphi(h_\alpha(0))$.

Obstruction theory will be used in conjunction with the fiberwise join product, which we review below.

Definition 2.17. Given two fibrations $p_i : E_i \to B$, $i = 1, 2$, the *fiberwise join* of p_1 and p_2 is the fibration $p_1 \underset{B}{*} p_2 : E_1 \underset{B}{*} E_2 \to B$, where $E_1 *_B E_2$ consists of the formal sums

$$\{te_1 + (1-t)e_2 \in E_1 * E_2 \mid p_1(e_1) = p_2(e_2), t \in \{0, 1\}\}$$

and

$$\left(p_1 \underset{B}{*} p_2\right)(te_1 + (1-t)e_2) = p_1(e_1) = p_2(e_2).$$

The join construction can be iterated. In particular, given a fibration $p : E \to B$ we can form the $(k+1)$-fold iterated self-join of p, $\underset{k}{*}\, p$. Thus

$$\underset{0}{*}\, p = p \qquad \text{and} \qquad \underset{k+1}{*}\, p = \left(\underset{k}{*}\, p\right) \underset{B}{*}\, p.$$

Note that if p has fiber F, then $\underset{k}{*}\, p$ has fiber $F^{*(k+1)}$, the $(k+1)$-iterated regular join of F with itself.

The following two theorems were proved by Schwarz, who connected the genus of a fibration with the existence of a section in a certain join of that fibration. See [26, theorem 2 on p. 66] and [26, theorem 4 on p. 73]. In [26] all the fiber spaces are supposed to be locally-trivial and all topological spaces normal and paracompact.

Theorem 2.10. *For a fibration $p : E \to B$, $\mathbf{secat}(p) \le k$ if and only if $\underset{k}{*}\, p$ admits a global section.*

Theorem 2.11 ([26]). *Let $\theta \in H^1\left(B; \underline{\widetilde{H}_0(F)}\right)$ be the obstruction for sectioning a fibration $F \hookrightarrow E \xrightarrow{p} B$ over the 1-dimensional skeleton of the CW complex B. Note that $\overset{k+1}{*} F$ is $(k-1)$-connected, so that $\underset{k}{*}\, p$ admits a section over the kth skeleton of B. If $k \ge 2$, then the primary obstruction*

$$\theta_{k+1} \in H^{k+1}\left(B; \underline{\pi_k\left(\overset{k+1}{*} F\right)}\right)$$

for sectioning $\underset{k}{*}\, p$ on the $(k+1)$-skeleton corresponds with the $(k+1)$th power

$$\theta^{k+1} \in H^{k+1}\left(B; \underline{\widetilde{H}_0(F)}^{\otimes(k+1)}\right)$$

under the isomorphism of coefficients given as the composition

$$\widetilde{H}_0(F)^{\otimes(k+1)} \cong \widetilde{H}_0(F)\left(F^{\wedge(k+1)}\right) \cong H_k\left(\overset{k+1}{*} F\right) \cong \pi_k\left(\overset{k+1}{*} F\right).$$

Here the first isomorphism comes by the Künneth formula, the second isomorphism is the suspension isomorphism in homology $\left(\text{since } \overset{k+1}{*} F \cong \Sigma^k \wedge_{k+1} F\right)$ and the third isomorphism comes from the Hurewicz theorem (since $k \ge 2$).

Now suppose that X is a path connected CW complex, $\dim(X) = n \geq 2$, and you are interested in knowing if the inequalities $\mathbf{TC}(X) \leq 2n-1$ and $\mathbf{TC}^{G,2}(X) \leq 2n-1$ hold. In terms of theorem 2.10, you are asking if the fibrations

$$\underset{2n-1}{*} e_{0,1} \qquad \text{and} \qquad \underset{2n-1}{*} p_2$$

admit a global section. There is an obvious commutative diagram of fibrations

$$\begin{array}{ccc} \Omega X = \{e\} \times \Omega X & \hookrightarrow & G \times \Omega X \\ \downarrow & & \downarrow \\ PX = P_{2,e}(X) & \hookrightarrow & P_2(X) \\ {\scriptstyle e_{0,1}}\downarrow & & \downarrow{\scriptstyle p_2} \\ X \times X & = & X \times X \end{array}$$

which yields the fiberwise analogue

$$\begin{array}{ccc} (\Omega X)^{*2n} & \hookrightarrow & (G \times \Omega X)^{*2n} \\ \downarrow & & \downarrow \\ \overset{2n}{\underset{X\times X}{*}}(PX) & \hookrightarrow & \overset{2n}{\underset{X\times X}{*}}(P_2(X)) \\ {\scriptstyle \underset{2n-1}{*} e_{0,1}}\downarrow & & \downarrow{\scriptstyle \underset{2n-1}{*} p_2} \\ X \times X & = & X \times X, \end{array}$$

since both fibers are $(2n-2)$-connected, and since $\dim(X \times X) = 2n$, there is only one primary obstruction

$$\mathcal{O} \in H^{2n}\left(X \times X; \underline{\pi_{2n-1}\left((\Omega X)^{*2n}\right)}\right)$$

for the inequality $\mathbf{TC}(X) \leq 2n-1$, and only one primary obstruction

$$\mathcal{O}_G \in H^{2n}\left(X \times X; \underline{\pi_{2n-1}\left((G \times \Omega X)^{*2n}\right)}\right)$$

for the inequality $\mathbf{TC}^{G,2}(X) \leq 2n-1$, where underlined groups indicate twisted coefficients. Furthermore, the functioriality of primary obstructions implies that $\mathcal{O}$ hits $\mathcal{O}_G$ under the morphism

$$\rho_{2n} : H^{2n}\left(X \times X; \underline{\pi_{2n-1}\left((\Omega X)^{*2n}\right)}\right) \to H^{2n}\left(X \times X; \underline{\pi_{2n-1}\left((G \times \Omega X)^{*2n}\right)}\right) \tag{2.1}$$

corresponding to the map of twisted coefficients induced by the inclusion of fibers $(\Omega X)^{*2n} \hookrightarrow (G \times \Omega X)^{*2n}$. Note that the primary obstructions for sectioning $e_{0,1}$ and p_2 on the 1-dimensional skeleton of $X \times X$ are cohomology classes $o \in H^1\left(X \times X; \underline{\overline{H_0}\,(\Omega X)}\right)$ and $o_G \in H^1\left(X \times X; \underline{\overline{H_0}\,(G \times \Omega X)}\right)$. Using again the functoriality [2] of primary obstructions, we see that $\rho_1(o) = o_G$, where

$$\rho_1 : H^1\left(X \times X; \underline{\overline{H_0}\left(\Omega X\right)}\right) \to H^1\left(X \times X; \underline{\overline{H_0}\left(G \times \Omega X\right)}\right)$$

is the 1-dimensional analogue of (2.1). Further, in terms of theorem 2.11, $o^{2n} = \mathcal{O}$ and $o_G^{2n} = \mathcal{O}_G$.

[2] The functoriality asserted here is easily seen from the fact, (reviewed in this section) that 1-dimensional obstructions are given in terms of monodromy actions.

3

The stability of the higher topological complexity of real projective spaces: an approach to their immersion dimension

This chapter is an exposition of joint work with Jesús González and Aldo Guzmán-Sáenz [5] [3]. Here we describe a number $r(m)$, which depends on the structure of zeros and ones in the binary expansion of m, and with the property that $\mathbf{TC}_s(\mathbb{R}P^m)$ is given by sm with an error of at most one provided $s \geq r(m)$ and $m \not\equiv 3 \mod 4$ (the error vanishes for even m). Recently, D. Davis, based on the results that we describe in this chapter, has found the best lower bound for $\mathbf{TC}_s(\mathbb{R}P^m)$ coming from mod 2 singular cohomology considerations, see [9].

[3] [5] was published as part of [6].

Throughout this chapter, we will only be concerned with simple coefficients in $R = \mathbb{Z}_2$, and will omit reference of coefficients in writing a cohomology group $H^*(X)$. In these terms,

$$\Delta_s^*\colon H^*(X^s) = H^*(X)^{\otimes s} \to H^*(X)$$

is given by the s-fold iterated cup-multiplication, which explains the notation "zcl" (zero-divisors cup-length) for elements in the kernel of Δ_s^*.

3.1 *Cohomology input*

Recall from [1] the inequalities that relate $\text{cat}(-)$ and $\mathbf{TC}(-)$:

$$\text{cat}(X^{s-1}) \leq \mathbf{TC}_s(X) \leq \text{cat}(X^s).$$

Since $\text{cat}((\mathbb{R}P^m)^s) = sm$ for any s, the monotonic sequence

$$\mathbf{TC}_2(\mathbb{R}P^m) \leq \mathbf{TC}_3(\mathbb{R}P^m) \leq \cdots \leq \mathbf{TC}_s(\mathbb{R}P^m) \leq \cdots$$

has an average linear growth. This chapter's goal is to study the actual deviation of the above growth from the linear function sm.

The inequalities $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) \leq \mathbf{TC}_s(\mathbb{R}\mathrm{P}^m) \leq sm$, provided by propositon 2.1, suggest the following definition.

Definition 3.1. For $m \geq 1$ and $s \geq 2$, set

$$G_s(m) = sm - \mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m)$$

and

$$\delta_s(m) = sm - \mathbf{TC}_s(\mathbb{R}P^m).$$

Remark 3.1. (i) We have the inequalities

$$0 \leq \delta_s(m) \leq G_s(m). \tag{3.1}$$

(ii) Before stating and proving theorems involving δ_s and G_s, we inform the reader that the proofs of such theorems use the following notation. For $s \geq 2$, let $x_i \in H^1((\mathbb{R}\mathrm{P}^m)^s)$ be the pull back of the non-trivial class in $H^1(\mathbb{R}\mathrm{P}^m)$ under the i-th projection map $(\mathbb{R}\mathrm{P}^m)^s \to \mathbb{R}\mathrm{P}^m$. Note that we do not stress the dependence of x_i on s. This is because, if $s' > s$ and $\pi_{s,s'}\colon (\mathbb{R}\mathrm{P}^m)^{s'} \to (\mathbb{R}\mathrm{P}^m)^s$ is the projection onto the first s coordinates, then we think of the map induced in cohomology by $\pi_{s,s'}$ as a honest inclusion. Note that, in these conditions, the standard (graded) basis of $H^*((\mathbb{R}\mathrm{P}^m)^s)$ consists of the monomials $x_1^{e_1}x_2^{e_2}\cdots x_s^{e_s}$ where $0 \leq e_i \leq m$ —recall that $x_i^{m+1} = 0$.

To make this chapter self-contained, we include some theorems that are due to J. González, D. Gutiérrez and A. Lara, see [17] [4].

[4] [17] was published as part of [6].

Theorem 3.1 ([17]). *For $m \geq 1$, the sequence of non-negative integers $\{G_s(m)\}_s$ is non-increasing,*

$$G_2(m) \geq G_3(m) \geq \cdots \geq 0. \tag{3.2}$$

Proof. The statement follows once the inequality

$$\mathrm{zcl}_{s+1}(\mathbb{R}\mathrm{P}^m) \geq \mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) + m, \quad s \geq 2$$

is verified. If $z \in H^*((\mathbb{R}\mathrm{P}^m)^s)$ is a non-zero product of s-th zero-divisors, then

$$z \cdot (x_1 + x_{s+1})^m = z \cdot (x_{s+1}^m + \cdots) \neq 0.$$

□

The non-increasing sequence (3.2) stabilizes to some non-negative integer $G(m)$ which is bounded from above by $2^{e(m)} - 1$, a fact that follows from theorem 3.2. Here $e(m)$ stands for the length of the block of consecutive ones ending the binary expansion of m. Theorem 3.3 shows that in fact the equality $G(m) = 2^{e(m)} - 1$ holds.

Theorem 3.2 ([17])**.** *The inequalities $0 \leq \delta_s(m) \leq 2^{e(m)} - 1$ hold provided $s \geq \ell(m)$. Here $\ell(m) = \max\{(m+1)/2^{e(m)}, 2\}$ and $e(m)$ stands for the length of the block of consecutive ones ending the binary expansion of m (e.g., $e(m) = 0$ if m is even [5]). In particular, if m is even and $s \geq \ell(m)$, $\delta_s(m) = 0$, so that $\mathbf{TC}_s(\mathbb{R}P^m) = sm$.*

Proof. From equation (3.1), it is enough to show that for $s \geq \ell(m)$, $G_s(m) \leq 2^{e(m)} - 1$. Set $e = e(m)$.
For $e \geq 1$ and $s \geq 2$, the inequality $G_s(2^e - 1) \leq 2^e - 1$ holds due to the non-triviality of the s-th zero-divisor [6]

$$(x_1 + x_2)^{2^e-1}(x_1 + x_3)^{2^e-1} \cdots (x_1 + x_s)^{2^e-1} \in H^*((\mathbb{R}\mathrm{P}^{2^e-1})^s).$$

If $m > 2^e - 1$ and η stands for $(m+1)/2^e$ ($\eta \geq 3$), then the product of η-th zero-divisors

$$(x_1 + x_\eta)^{m+2^e} \cdots (x_{\eta-1} + x_\eta)^{m+2^e} \neq 0 \in H^*((\mathbb{R}\mathrm{P}^m)^\eta).$$

This claim follows from lemma 2.2: the hypothesis on m and e implies that the binomial coefficient $\binom{m+2^e}{2^e}$ is odd, so

$$(x_i + x_\eta)^{m+2^e} = x_i^m x_\eta^{2^e} + \text{terms involving powers } x_i^j \text{ with } j < m$$

for $1 \leq i \leq \eta - 1$. Thus, ignoring basis elements $x_1^{a_1} \cdots x_\eta^{a_\eta}$ having $a_i < m$ for some $i \in \{1, \ldots, \eta - 1\}$, the product of η-th zero-divisors under consideration becomes

$$(x_1^m x_\eta^{2^e})(x_2^m x_\eta^{2^e}) \cdots (x_{\eta-1}^m x_\eta^{2^e}) = x_1^m x_2^m \cdots x_{\eta-1}^m x_\eta^{(\eta-1)2^e},$$

which is a basis element. This yields $G_\eta(m) \leq 2^e - 1$. □

Definition 3.2. Consider the finite sequence of numbers

$$\{\delta_{\ell(m)}(m),\ \delta_{\ell(m)-1}(m),\ \ldots,\ \delta_2(m)\},^7 \tag{3.4}$$

we say that an element in this sequence is *well controlled* if it is less than or equal to $2^{e(m)} - 1$.

Definition 3.3. Let $\lambda : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ be the function whose value in m, $\lambda(m)$, is the smallest integer s such that for $t \geq s$, $\delta_t(m)$ is well controlled.

[5] In fact $e(m)$ is defined by the formula

$$m \equiv 2^{e(m)} - 1 \bmod 2^{e(m)+1} \tag{3.3}$$

[6] It was shown in [17] that for a fix $j \in \{1, \ldots, s\}$, the ideal of s zero-divisors in $H^*(\mathbb{R}\mathrm{P}^{2^e-1})^{\otimes s}$ is geneated by the elements $x_i + x_j$ with $1 \leq i \leq s$ and $i \neq j$. A sketch of the proof is given below. For $j = s$, let

$$z = \sum_{(a_1,\ldots,a_s)} x_1^{a_1} \cdots x_s^{a_s}$$

be the expression, in terms of the standard basis, of a homogeneous s-th zero-divisor. Note that the number of summands must be even if $\deg(z) \leq m$. Then, it is suffices to prove that the following elements lie in the ideal I_s generated by the binomials $x_i + x_s$:

(i) The sum of any two basis elements in degree at most m.

(ii) A basis element in degree grater than m.

Items (i) and (ii) are dealt with inductive arguments. For $j \neq s$, the argument is identical to the previous one.

[7] where $\ell(m)$ is as in theorem 3.2.

Remark 3.2.

(i) We refer to the tail elements

$$\{\delta_{\lambda(m)}(m),\ \delta_{\lambda(m)-1}(m),\ \ldots,\ \delta_2(m)\}, \tag{3.5}$$

as the critical sequence.

(ii) Notice that $e(m) = 0$ for $m \equiv (0,2) \bmod 4$, and $e(m) = 1$ for $m \equiv 1 \bmod 4$. Thus the above theorem says that, provided $s \geq \ell(m)$,

$$\delta_s(m) \in \{0,1\} \tag{3.6}$$

and

$$\delta_s(m) = 0, \text{ if } m \text{ is even}, \tag{3.7}$$

where

$$\ell(m) = \begin{cases} m+1, & \text{if } m \text{ is even;} \\ \frac{m+1}{2}, & \text{if } m \equiv 1 \bmod 4. \end{cases}$$

Therefore in our case of interest, $m \not\equiv 3 \mod 4$, an element in the sequence (3.4) is well controlled if satisfies (3.6) and (3.7).

The functions δ_s are notably difficult to deal with as they reflect the intrinsic homotopy phenomenology of the multi-sectioning problem for the fibrations in definition 2.2. A more accessible task is to deal with the functions G_s since, by construction, these objects depend only on the mod 2 cohomology ring of $\mathbb{R}\mathrm{P}^m$. However, in a large portion of the cases, we have $\delta_s(m) = G_s(m)$, which justifies a careful analysis of the functions G_s. Two central tasks in such a direction are (i) the computation of the stabilized $G(m)$, and (ii) the estimation of the smallest integer $s(m) \geq 2$ satisfying

$$G_s(m) = G(m) \quad \text{for} \quad s \geq s(m). \tag{3.8}$$

Theorem 3.3 below, addressed task (i) above. Furthermore, its proof serves as preparation for one of the main results of this chapter, theorem 3.4, which addresses task (ii) above.

Theorem 3.3. $G(m) = 2^{e(m)} - 1$.

Proof. As before, set $e = e(m)$. The proof of theorem 3.2 gives $G(m) \leq 2^e - 1$. Since the ideal of s-th zero-divisors of $\mathbb{R}\mathrm{P}^m$ is generated by the classes $x_1 + x_i$ with $2 \leq i \leq s$, it suffices to show that no non-zero product

$$(x_1+x_2)^{m+i_2}(x_1+x_3)^{m+i_3}\cdots(x_1+x_s)^{m+i_s}, \tag{3.9}$$

where $s \geq 2$ and $m + i_j \geq 0$, can yield a gap $G_s(m)$ smaller than $2^e - 1$. In view of (3.3)

$$m + 1 = 2^e q \tag{3.10}$$

for some positive odd integer q. Assume, for a contradiction, that there is a non-zero product (3.9) with $sm - \sum_{j=2}^{s}(m+i_j) < 2^e - 1$ or, equivalently, with

$$m < 2^e - 1 + \sum_{j=2}^{s} i_j. \tag{3.11}$$

It can be assumed in addition that each i_j is positive, for otherwise we just remove the corresponding factor $(x_1+x_j)^{m+i_j}$ from (3.9) without altering (3.11). In this setting, we have that

$$x_1^{(u+1)2^e} \text{ divides } (x_1+x_j)^{m+i_j}, \text{ if } i_j > 2^e u \text{ for some } u \geq 0, \tag{3.12}$$

for in fact $(x_1+x_j)^{m+i_j} = (x_1+x_j)^{m+1}(x_1+x_j)^{2^e u}(x_1+x_j)^{i_j-2^e u-1}$, where (3.10) gives

$$(x_1+x_j)^{m+1}(x_1+x_j)^{2^e u} = (x_1^{2^e} + x_j^{2^e})^{q+u},$$

which is divisible by $x_1^{(u+1)2^e}$ as $x_j^{2^e q} = x_j^{m+1} = 0$.

Now, for $c \geq 1$, let p_c be the number of integers $i_2, \ldots, i_s$ in (3.9) that lie in the interval

$$\{2^e(c-1)+1, 2^e(c-1)+2, \ldots, 2^e c\}.$$

Then (3.10) and (3.11) yield

$$2^e q = m+1 < 2^e + \sum_{c\geq 1} 2^e c\, p_c \quad \text{i.e.} \quad q \leq \sum_c c\, p_c.$$

The punch line is that (3.12) implies that (3.9) is divisible by x_1^η where

$$\eta = \sum_c 2^e c\, p_c \geq 2^e q = m+1$$

which, in view of the relation $x_1^{m+1} = 0$, contradicts the non-triviality of (3.9). □

Theorem 3.4 below estimates $s(m)$ by a function whose value in m depends strongly on the number and distribution on ones in the dyadic expansion of m. The formulation of this theorem requires the following definition.

Definition 3.4. Let m be a positive integer such that $m+1$ is not a 2-power, and set $e = e(m)$. Let k be the first positive integer with $2^k > m$ (so $k > e$), and set $d_0 = 2^k - m - 1$ (so d_0 is a positive integer divisible by 2^e). Consider the non-negative integer $t = (d_0 - 2^e)/2^e$ and, for $1 \leq \ell \leq t$, set $d_\ell = d_0 - 2^e \ell$ (so $d_0 > d_1 > \cdots > d_t = 2^e$). Define non-negative integers r_ℓ $(0 \leq \ell \leq t)$ by the recursive equations

$$r_0 = \begin{cases} \left\lfloor \frac{m-(2^e-1)}{d_0} \right\rfloor, & \text{if } \binom{m+d_0}{d_0} \equiv 1 \bmod 2; \\ 0, & \text{otherwise,} \end{cases}$$

$$r_1 = \begin{cases} \left\lfloor \frac{m-(2^e-1)-d_0 r_0}{d_1} \right\rfloor, & \text{if } \binom{m+d_1}{d_1} \equiv 1 \bmod 2; \\ 0, & \text{otherwise,} \end{cases}$$

$$r_2 = \begin{cases} \left\lfloor \frac{m-(2^e-1)-d_0 r_0 - d_1 r_1}{d_2} \right\rfloor, & \text{if } \binom{m+d_2}{d_2} \equiv 1 \bmod 2; \\ 0, & \text{otherwise,} \end{cases}$$

$$\cdots$$

$$r_t = \begin{cases} \left\lfloor \frac{m-(2^e-1)-d_0 r_0 - d_1 r_1 - \cdots - d_{t-1} r_{t-1}}{d_t} \right\rfloor, & \text{if } \binom{m+d_t}{d_t} \equiv 1 \bmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, set $r(m) = 1 + \sum_{\ell=0}^{t} r_\ell$.

In definition 3.4, the dyadic expansion of d_0 is the "complement" of that for m. So $\binom{m+d_0}{d_0}$ is odd (and thus $r_0 = \lfloor \frac{m-(2^e-1)}{d_0} \rfloor$). Further, since $d_t = 2^e$, the binomial coefficient $\binom{m+d_t}{d_t}$ is odd too (recall that $m \equiv 2^e - 1$ (mod 2^{e+1})). In addition, since 2^e divides $m - (2^e - 1)$ as well as each d_ℓ, we actually have

$$r_t = \frac{m - (2^e - 1) - d_0 r_0 - d_1 r_1 - \cdots - d_{t-1} r_{t-1}}{d_t}. \tag{3.13}$$

Theorem 3.4. *With the notation in (3.8) and definition 3.4, $s(m) \leq r(m)$.*

Proof. Let $s = r(m)$ and $s_\ell = 1 + \sum_{i=0}^{\ell-1} r_i$ for $0 \leq \ell \leq t$. Consider the product of s-th zero-divisors

$$\prod_{\ell=0}^{t} \left(\prod_{i_\ell=1}^{r_\ell} (x_1 + x_{s_\ell+i_\ell})^{m+d_\ell} \right). \tag{3.14}$$

By Proposition 2.1, it suffices to check that the expansion of (3.14) in terms of the standard basis of $H^*((\mathbb{R}\mathrm{P}^m)^s)$ contains the basis element $x_1^{m-(2^e-1)} x_2^m \cdots x_s^m$.

Note that the ℓ-th factor in (3.14) is to be neglected if $r_\ell = 0$ and, by construction, this happens whenever $\binom{m+d_\ell}{d_\ell}$ is even. On the other hand, if $r_\ell > 0$ (so that $\binom{m+d_\ell}{d_\ell}$ is odd), then each of the factors $(x_1 + x_{s_\ell+i_\ell})^{m+d_\ell}$ in (3.14) takes the form

$$(x_1 + x_{s_\ell+i_\ell})^{m+d_\ell} = x_1^{d_\ell} x_{s_\ell+i_\ell}^m + \text{monomials involving powers } x_{s_\ell+i_\ell}^p,$$

with $p < m$. Therefore, for the purpose of keeping track of basis elements of the form $x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$ with $a_i = m$ for $2 \leq i \leq s$, equation (3.14) becomes

$$\prod_{\ell=0}^{t} \left(\prod_{i_\ell=1}^{r_\ell} \left(x_1^{d_\ell} \cdot x_{s_\ell+i_\ell}^m \right) \right) = x_1^{d_0 r_0 + d_1 r_1 + \cdots + d_t r_t} x_2^m \cdots x_s^m.$$

The result then follows from (3.13). □

Remark 3.3. (i) Theorems 3.3 and 3.4 give

$$\lambda(m) \leq s(m) \leq r(m).$$

(ii) As mentioned at the beginning of the chapter, D. Davis found the best lower bound for $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^m)$ coming from the techniques in this work, in [9] he gives a descriptions (both recursive and direct) of the function $s(m)$.

3.2 *Binary expansions*

In this section we illustrate the way in which the values of the function $r(m)$ depend on the number and distribution of ones in the binary expansion of m. With this in mind, it is convenient to set a suitably flexible notation.

Definition 3.5. Let m be a positive integer. Write $m = \sum_{i=0}^{\mu} b_i 2^i$ with $b_i \in \{0, 1\}$ and $b_\mu = 1$. The binary expansion of m, that is, the string of zeros and ones $b_\mu b_{\mu-1} \cdots b_0$, starts (from the left) with a block of ones, say n_1 of them; then it has a block of zeros, say z_1 of them; then it has a second block of ones, say n_2 of them, and so forth. The *codified binary expansion* of m, $\mathrm{cbe}(m)$, is the (finite) sequence of positive integers $\mathrm{cbe}(m) = (n_1, z_1, n_2, \ldots)$.

Remark 3.4. (i) The length of the sequence $\mathrm{cbe}(m)$ agrees mod 2 with m, and μ is the integral part of $\log_2(n)$.

(ii) It is standard to set $\alpha := \sum n_i$ (the number of ones in the binary expansion of m) and $\nu := \min\{i \colon b_i \neq 0\}$ (the exponent in the highest 2-power dividing m). For instance, $\nu = z_\omega$ when $\mathrm{cbe}(m) = (n_1, z_1, \ldots, n_\omega, z_\omega)$. If we need to stress the dependence of the parameters e, α, μ, ν, b_i, n_i, or z_i on m, we use the notation $e(m)$, $\alpha(m)$, $\mu(m)$, $\nu(m)$, $b_i(m)$, $n_i(m)$, or $z_i(m)$, accordingly.

(iii) The relation $\mathrm{cbe}(m) = (n_1, z_1, n_2, \ldots)$ sets a bijective correspondence from the set of positive integers m to the set of finite sequences of positive integers $(n_1, z_1, n_2, \ldots)$, and we use $p_2(n_1, z_1, n_2, \cdots) = m$ for the inverse function. In fact, it will be convenient to replace the notation $p_2(n_1, z_1, n_2, \cdots)$ by the corresponding binary expansion $1^{n_1} 0^{z_1} 1^{n_2} \cdots$, where exponents indicate the number of times that a zero or a one is to be repeated.

Proposition 3.5. *Let m be even with* $\mathrm{cbe}(m) = (n_1, z_1, \ldots, n_\omega, z_\omega)$ *and* $n_1 < n_2 < \cdots < n_\omega$. *Assume* $n_u < z_u$ *for* $1 \leq u \leq \omega$ *(this condition can be thought of as saying that the blocks of ones in the binary expansion of m are "suitably" spaced). Then* $r(m) = 1 + 2^{n_\omega}$. *More explicitly, the non-zero numbers* r_ℓ $(0 \leq \ell \leq t)$ *in definition 3.4 hold for* $\ell \in \{\kappa_u, \ell_u : 1 \leq u \leq \omega\}$ *where*

$$\kappa_u = 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_{u-1}} 1^{z_{u-1}} 0^{n_u + z_u + \cdots + n_\omega + z_\omega};$$
$$\ell_u = 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_{u-1}} 1^{z_{u-1}} 0^{n_u} 1^{z_u - n_u} 0^{n_u + n_{u+1} + z_{u+1} + \cdots + n_\omega + z_\omega}.$$

(Just as the sum $\sum_{i=u+1}^{\omega} (n_i + z_i)$ *is ignored for* $u = \omega$, *the initial segment* $1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_{u-1}} 1^{z_{u-1}}$ *in the two binary expansions above should be ignored for* $u = 1$. *For instance* $\kappa_1 = 0$.*) Furthermore*

$$r_{\kappa_1} = 2^{n_1} - 1 \ \text{ with } \ d_{\kappa_1} = 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega},$$
$$r_{\ell_1} = 1 \ \text{ with } \ d_{\ell_1} = 1^{n_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega},$$

and, for $u \geq 2$,

$$r_{\kappa_u} = 2^{n_u} - 2^{n_{u-1}} - 1 \ \text{ with } \ d_{\kappa_u} = 1^{z_u} 0^{n_{u+1}} 1^{z_{u+1}} \cdots 0^{n_\omega} 1^{z_\omega},$$
$$r_{\ell_u} = 1 \ \text{ with } \ d_{\ell_u} = 1^{n_u} 0^{n_{u+1}} 1^{z_{u+1}} \cdots 0^{n_\omega} 1^{z_\omega}.$$

Proof. The assertion following definition 3.4 obviously generalizes to the observation that, for any $u \in \{1, \ldots, \omega\}$, the binary expansions of d_{κ_u} and d_{ℓ_u} are complementary to that of m. In particular all binomial coefficients $\binom{m + d_{\kappa_u}}{d_{\kappa_u}}$ and $\binom{m + d_{\ell_u}}{d_{\ell_u}}$ with $u \in \{1, \ldots, \omega\}$ are odd.

We start by considering in detail the (slightly special) case $u = 1$. The equality $r_0 = 2^{n_1} - 1$ follows from the fact that $(2^{n_1} - 1)d_0 \leq m < 2^{n_1} d_0$, which in turn holds since

$$\begin{aligned} m - (2^{n_1} - 1)d_0 &= 1^{n_1} 0^{z_1} 1^{n_2} 0^{z_2} \cdots 1^{n_\omega} 0^{z_\omega} - 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} 0^{n_1} + \\ &\quad 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} \\ &= 1^{n_1 + z_1 + n_2 + z_2 + \cdots + n_\omega + z_\omega} - 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} 0^{n_1} \\ &= 1^{n_2} 0^{z_2} \cdots 1^{n_\omega} 0^{z_\omega} 1^{n_1} \geq 0 \end{aligned} \tag{3.15}$$

and $m - 2^{n_1} d_0 = 1^{n_2} 0^{z_2} \cdots 1^{n_\omega} 0^{z_\omega} 1^{n_1} - 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} < 0$, due to the assumption $n_1 < z_1$.

Next we show that

$$r_\ell = 0 \ \text{ for } \ 0 < \ell < \ell_1. \tag{3.16}$$

For such a value of ℓ we have

$$0^{n_1} 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} = d_0 > d_\ell = d_0 - \ell > d_0 - \ell_1 = 0^{z_1} 1^{n_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega}, \tag{3.17}$$

so that the binary expansion of d_ℓ must have at least one of the zeros on the right-hand side of (3.17) changed to a 1. If such a 1 appears in one of the blocks 0^{n_i} with $2 \leq i \leq \omega$, then the binomial coefficient $\binom{m + d_\ell}{d_\ell}$ is obviously even, and so $r_\ell = 0$. Otherwise, the 1 must appear in the block 0^{z_1}, so that

$$d_\ell \geq 2^{n_1 + n_2 + z_2 + \cdots + n_\omega + z_\omega}.$$

In such a situation the vanishing of r_ℓ follows from the easy-to-check fact that

$$2^{n_1 + n_2 + z_2 + \cdots + n_\omega + z_\omega} > m - d_0 r_0.$$

For the case $\ell = \ell_1$, note that

$$d_{\ell_1} \leq m - d_0 r_0 < 2 d_{\ell_1}, \tag{3.18}$$

which yields $r_{\ell_1} = 1$. The first inequality in (3.18) holds since

$$\begin{aligned} d_{\ell_1} + 2^{n_1} d_0 &= 0^{z_1} 1^{n_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} + 1^{z_1} 0^{n_2} 1^{z_2} \cdots 0^{n_\omega} 1^{z_\omega} 0^{n_1} \\ &\leq 1^{n_1 + z_1 + \cdots + n_\omega + z_\omega} = m + d_0, \end{aligned}$$

where the inequality comes from the fact that both

$$0^{z_1}1^{n_1}0^{n_2}\cdots \quad \text{and} \quad 1^{z_1}0^{n_2}\cdots$$

have zeros in their (n_1+z_1+1)-st position counted from the left, so that any previous carry in the binary sum disappears at that spot, while no further carries appear from that point on. The second inequality in (3.18) holds since

$$\begin{aligned} m+d_0 &= 1^{n_1+z_1+\cdots+n_\omega+z_\omega} \\ &< 0^{z_1-1}1^{n_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}0 + 1^{z_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}0^{n_1} \\ &= 2d_{\ell_1}+2^{n_1}d_0 \end{aligned}$$

where the inequality is due to the fact that a carry is forced at the end of the binary sum of

$$0^{z_1-1}1^{n_1}\cdots \quad \text{and} \quad 1^{z_1}0^{n_2}\cdots.$$

It is convenient to note at this point that the numerator in the quotient defining the next non-trivial r_ℓ ($\ell>\ell_1$) is

$$\begin{aligned} m-(2^{n_1}-1)d_0-d_{\ell_1} &= m-1^{z_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}0^{n_1}+1^{z_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega} \\ &\quad -1^{n_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega} \\ &= m-1^{z_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}0^{n_1} \\ &\quad +1^{z_1-n_1}0^{n_1+n_2+z_2+\cdots+n_\omega+z_\omega} \\ &= 1^{n_1}0^{z_1}1^{n_2}0^{z_2}\cdots 1^{n_\omega}0^{z_\omega} \\ &\quad -1^{n_1}0^{z_1-n_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}0^{n_1}. \end{aligned} \tag{3.19}$$

The case $u=1$ will be complete once we show that $r_\ell=0$ for $\ell_1<\ell<k_2$. Such a value of ℓ has $d_0-\ell_1>d_\ell=d_0-\ell>d_0-\kappa_2$, i.e.

$$0^{z_1}1^{n_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}=d_0-\ell_1>d_\ell>d_0-\kappa_2=0^{n_1+z_1}0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}, \tag{3.20}$$

so that the binary expansion of d_ℓ must have at least one of the zeros on the right-hand side of (3.20) changed to a 1. As above, if such a 1 appears in one of the blocks 0^{n_i} with $2\le i\le\omega$, then the binomial coefficient $\binom{m+d_\ell}{d_\ell}$ is obviously even, and so $r_\ell=0$. Otherwise, the 1 must appear in the block $0^{n_1+z_1}$, so that $d_\ell\ge 2^{n_2+z_2+\cdots+n_\omega+z_\omega}$. In such a situation the vanishing of r_ℓ follows from the fact that $2^{n_2+z_2+\cdots+n_\omega+z_\omega}$ is strictly larger than (3.19), which in turn is observed from the binary-sum setup below.

$$\begin{array}{rl} & \overbrace{1^{n_1}0^{z_1-n_1}}^{z_1}\,\overbrace{0^{n_2}1^{z_2}\cdots 0^{n_\omega}1^{z_\omega}}^{n_2+z_2+\cdots+n_\omega+z_\omega}\,\overbrace{0^{n_1}}^{n_1} \\ + & \underbrace{\qquad}_{n_1}\;\underbrace{\qquad\qquad\underbrace{\quad}_{n_1}}_{z_1}1\,\underbrace{0^{n_2}0^{z_2}\cdots 0^{n_\omega}0^{z_\omega}}_{n_2+z_2+\cdots+n_\omega+z_\omega} \\ \hline 1^{n_1}0^{z_1}\cdots & < \quad 1^{n_1}\underbrace{000\cdots0}_{z_1-1}1\cdots \end{array}$$

The cases $u\ge 2$ can now be dealt with recursively, using part of the previous analysis. For the start of the recursion we have to check that $r_{\kappa_2}=2^{n_2}-2^{n_1}-1$ or, equivalently, that

$$(2^{n_2}-2^{n_1}-1)d_{\kappa_2}\le m-(2^{n_1}-1)d_0-d_{\ell_1}<(2^{n_2}-2^{n_1})d_{\kappa_2}.$$

Again, both inequalities are verifiable from the corresponding binary expansions. Indeed, (3.19) yields

$$\begin{aligned}
m-&(2^{n_1}-1)d_0-d_{\ell_1}-(2^{n_2}-2^{n_1}-1)d_{\kappa_2}\\
&=1^{n_1}0^{z_1}1^{n_2}0^{z_2}\cdots1^{n_\omega}0^{z_\omega}-1^{n_1}0^{z_1-n_1}0^{n_2}1^{z_2}\cdots0^{n_\omega}1^{z_\omega}0^{n_1}-\\
&\quad(2^{n_2}-2^{n_1}-1)d_{\kappa_2}\\
&=1^{n_1}0^{z_1}1^{n_2}0^{z_2}\cdots1^{n_\omega}0^{z_\omega}+0^{n_2}1^{z_2}\cdots0^{n_\omega}1^{z_\omega}-\\
&\quad1^{n_1}0^{z_1-n_1}0^{n_2}1^{z_2}\cdots0^{n_\omega}1^{z_\omega}0^{n_1}-(2^{n_2}-2^{n_1})d_{\kappa_2}\\
&=1^{n_1}0^{z_1}1^{n_2+z_2+\cdots+n_\omega+z_\omega}-1^{n_1}0^{z_1-n_1}0^{n_2}1^{z_2}\cdots0^{n_\omega}1^{z_\omega}0^{n_1}+\\
&\quad0^{n_2}1^{z_2}0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}0^{n_1}-2^{n_2}d_{\kappa_2}\\
&=1^{n_1}0^{z_1}1^{n_2+z_2+\cdots+n_\omega+z_\omega}-1^{n_1}0^{z_1-n_1}0^{n_2+z_2+\cdots+n_\omega+z_\omega+n_1}-2^{n_2}d_{\kappa_2}\\
&=1^{n_1}0^{z_1}1^{n_2+z_2+\cdots+n_\omega+z_\omega}-1^{n_1}0^{z_1+n_2+z_2+\cdots+n_\omega+z_\omega}-\\
&\quad1^{z_2}0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}0^{n_2}\\
&=1^{n_1}0^{z_1}1^{n_2+z_2+\cdots+n_\omega+z_\omega}-1^{n_1}0^{z_1}1^{z_2}0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}0^{n_2}\\
&=1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}1^{n_2}\geq0,
\end{aligned}\tag{3.21}$$

and so

$$\begin{aligned}
m-(2^{n_1}-1)d_0-d_{\ell_1}-(2^{n_2}-2^{n_1})d_{\kappa_2}=&1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}1^{n_2}-\\
&1^{z_2}0^{n_3}1^{n_3}\cdots0^{n_\omega}1^{z_\omega}<0.
\end{aligned}$$

From this point on, the proof enters a recursive loop which starts (for $u=2$) with the fact that the two relations

$$r_\ell=0\quad\text{for}\quad\kappa_u<\ell<\ell_u\ \textit{ and }\ r_{\ell_u}=1$$

are shown for $2\leq u\leq w$ following the arguments proving (3.16) and (3.18), respectively. In fact, the situation is formally identical as the reader will note by comparing (3.21) with (3.15), as well as by comparing the easily verified fact that (3.19) takes the form

$$1^{n_2}0^{z_2}1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}-0^{n_2-n_1}1^{z_2}0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}0^{n_1}$$

with

$$\begin{aligned}
m-(2^{n_1}-1)d_0-&d_{\ell_1}-(2^{n_2}-2^{n_1}-1)d_{\kappa_2}-d_{\ell_2}\\
&=1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}1^{n_2}-d_{\ell_2}\qquad\qquad\text{(by (3.21))}\\
&=1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}1^{n_2}-1^{n_2}0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}\\
&=1^{n_3-n_2}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}1^{n_2}-0^{n_3}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}\\
&=1^{n_3}0^{z_3}\cdots1^{n_\omega}0^{z_\omega}-0^{n_3-n_2}1^{z_3}\cdots0^{n_\omega}1^{z_\omega}0^{n_2},
\end{aligned}$$

where the last equality is obtained by complementing with respect to $1^{n_3+z_3+\cdots+n_\omega+z_\omega}$. □

Remark 3.5. Proposition 3.5 can be generalized slightly. Specifically, the hypothesis $n_u<z_u$ for $1\leq u\leq\omega$ can be weakened to requiring only $n_u\leq z_u$ for $1\leq u\leq\omega$ without altering the main conclusion $r(m)=1+2^{n_\omega}$. The explicit description of the non-trivial r_ℓ's changes only slightly since $\kappa_u=\ell_u$ (and of course $d_{\kappa_u}=d_{\ell_u}$) whenever $n_u=z_u$, in which case the two values of r_{κ_u} and r_{ℓ_u} will merge into the single

$$r_{\kappa_u}=r_{\ell_u}=2^{n_u}-2^{n_{u-1}}$$

(interpreting 2^{n_0} as zero).

Remark 3.6. More generally, for m as in proposition 3.5 (and with the hypothesis $n_u < z_u$ replaced by the more general requirement $n_u \leq z_u$), assume that the binomial coefficient

$$\binom{m+j}{m+i} \tag{3.22}$$

is odd for some integer $i \in \{0, 1, \ldots, 2^{z_\omega} - 1\}$ and $j = (i+1)(2^{n_\omega} + 1) - 2$. Then the proof of proposition 3.5 can be adapted to give $r(m+i) = 1 + 2^{n_\omega}$. Note that the hypothesis $i < 2^{z_\omega}$ implies that the first $2\omega - 2$ terms in $\mathrm{cbe}(m+i)$ agree with those of $\mathrm{cbe}(m)$, though the number and form of the subsequent terms in the two codified binary sequences might bear no relationship to each other. Thus, this example allows us to identify instances of m' where the suitable spacing conditions in proposition 3.5 hold only partially, and yet $r(m')$ has a simple description (see proposition 3.8).

Remark 3.7. The binomial coefficient (3.22) is odd in a many instances. A simple way to see this is by observing that the mod 2 value of (3.22) agrees with that of the binomial coefficient $\binom{j}{i}$ whenever the condition $i < 2^{z_\omega}$ is strengthened to $j < 2^{z_\omega}$. (The latter hypothesis can be thought of as requiring that the dyadic "tail" i in $m+i$ is "far enough" from the last block of ones in the binary expansion of m). It is then worth noticing that the mod 2 values of $\binom{j}{i}$ (as i varies) have an interesting arithmetical behavior. Consider, for simplicity, the case $\omega = 1 = n_1$, where a nice Fibonacci-type fractal pattern arises for the parity properties of the resulting binomial coefficient $\binom{j}{i} = \binom{3i+1}{i}$. Indeed, if we list the mod 2 values of $\binom{3i+1}{i}$ in the range $2^\ell \leq i \leq 2^{\ell+1} - 2$ with i even, we get the first $2^{\ell-2}$ numbers in the series

$$1^3, 0, 1^2, 0^2, 1^3, 0^5, 1^3, 0, 1^2, 0^{10}, 1^3, 0, 1^2, 0^2, 1^3, 0^{21}, 1^3, 0, 1^2, 0^2, 1^3, 0^5, 1^3, 0, 1^2, 0^{42}, \ldots \tag{3.23}$$

followed by $2^{\ell-2}$ zeros. Here the notation "a^b" stands for "$a, a, \ldots, a$" where a is repeated b times. The Fibonacci-type behavior enters as follows: Let f_c denote the sequence of the first 2^c digits in (3.23). For instance $f_0 = (1)$ and $f_1 = (1, 1)$. Then, for $c \geq 2$, the sequence f_c is the concatenation of f_{c-1} followed by f_{c-2}, and followed finally by 2^{c-2} zeros.

Remark 3.8. The case $i = 0$ in remark 3.6 specializes to proposition 3.5. In turn, the case $n_1 = \omega = 1$ in proposition 3.5 was obtained in [18, theorem 4.3] as part of a series of sharp results for the higher topological complexity of certain families of flag manifolds. In fact, it was the stabilization phenomenon "$\mathbf{TC}_s \to s\,\mathrm{hdim}$" noted in [18] for flag manifolds that motivated us to take a closer look at the situation for real projective spaces.

Other instances in the explicit description of the function $r(m)$ are described in the remainder of this section (for m even), with attention restricted to cases where the binary expansion of m has at most two blocks of consecutive ones. For instance, proposition 3.5 (as generalized in remark 3.5) and proposition 3.6 below account for a full description of the function $r(m)$ when m is even and has a single block of consecutive ones. The proofs of propositions 3.6 and 3.7–3.10 below follow the same strategy as that used in the proof of proposition 3.5; however the actual arguments are much easier and, consequently, will be left as an exercise for the diligent reader. We will only focus on cases where m is even, in particular $e(m) = 0$ and $d_\ell = d_0 - \ell$. Accordingly, we will specify (the binary expansion of) d_ℓ, but will omit explicit reference to ℓ. (Both ℓ and d_ℓ were described explicitly in the statement of proposition 3.5 for proof-referencing purposes.)

Proposition 3.6. *Let m be even with $\mathrm{cbe}(m) = (n, z)$ and $n > z$. Then*

$$r(m) = 1 + \sum_{i=0}^{\eta} 2^{n-iz},$$

where η stands for the largest integer which is strictly smaller than n/z. Explicitly, the non-zero numbers r_ℓ $(0 \leq \ell \leq t)$ in definition 3.4 hold for $\ell \in \{\ell_1, \ell_2\}$ with

$$r_{\ell_1} = \sum_{i=0}^{\eta} 2^{n-iz} - 1$$

and $r_{\ell_2} = 1$, where $d_{\ell_1} = 1^z$ and $d_{\ell_2} = 1^{n-\eta z}$.

Just as in remark 3.5, when $n - \eta z = z$, so that $\ell_1 = \ell_2$, the two values r_{ℓ_1} and r_{ℓ_2} should be interpreted as merging into the single

$$r_{\ell_1} = r_{\ell_2} = \sum_{i=0}^{\eta} 2^{n-iz}.$$

A similar phenomenon applies with propositions 3.7–3.10, but we will make no further comments on such a direction.

Remark 3.9. The weakest instance in Proposition 3.6 holds with $z_1 = 1$ (that is, when $m+2$ is a 2-power), for then $r(m)$ agrees with the linear function $\ell(m)$ in section3.1.

When m is even and $\text{cbe}(m) = (n_1, z_1, n_2, z_2)$, the hypotheses in proposition 3.5 (as generalized in remark 3.5) become

$$n_1 < n_2, \; n_1 \leq z_1, \text{ and } n_2 \leq z_2. \tag{3.24}$$

The following results describe the value of $r(m)$ when all but one of these inequalities hold. Note that proposition 3.7 below is analogous to Proposition 3.6, while Proposition 3.8 below fits in the setting of Remark 3.6 (as simplified in remark 3.7).

Proposition 3.7. *Let m be even with $\text{cbe}(m) = (n_1, z_1, n_2, z_2)$, $n_1 \leq z_1$ and $\max\{n_1, z_2\} < n_2$. Then*

$$r(m) = 1 + \sum_{i=0}^{\eta} 2^{n_2 - iz_2}$$

where η stands for the largest integer which is strictly smaller than n_2/z_2. Explicitly, the non-zero numbers r_ℓ $(0 \leq \ell \leq t)$ in definition 3.4 hold for $\ell \in \{\kappa_1, \ell_1, \kappa_2, \ell_2\}$ with $r_{\kappa_1} = 2^{n_1} - 1$, $r_{\kappa_2} = \sum_{i=0}^{\eta} 2^{n_2 - iz_2} - 2^{n_1} - 1$, and $r_{\ell_1} = r_{\ell_2} = 1$, where $d_{\kappa_1} = 1^{z_1}0^{n_2}1^{z_2}$, $d_{\kappa_2} = 1^{z_2}$, $d_{\ell_1} = 1^{n_1}0^{n_2}1^{z_2}$, and $d_{\ell_2} = 1^{n_2 - \eta z_2}$.

Proposition 3.8. *Let m be even with $\text{cbe}(m) = (n_1, z_1, n_2, z_2)$ and $n_2 \leq n_1 \leq \min\{z_1, z_2\}$. Then $r(m) = 1 + 2^{n_1}$. Explicitly, the non-zero numbers r_ℓ $(0 \leq \ell \leq t)$ in definition 3.4 hold for $\ell \in \{\kappa_1, \ell_1\}$ with $r_{\kappa_1} = 2^{n_1} - 1$ and $r_{\ell_1} = 1$, where $d_{\kappa_1} = 1^{z_1}0^{n_2}1^{z_2}$ and $d_{\ell_1} = 1^{n_2}0^{z_2}1^{n_1}$.*

Proposition 3.9. *Let m be even with $\text{cbe}(m) = (n_1, z_1, n_2, z_2)$ and $n_2 \leq z_2 < n_1 \leq z_1$. Then*

$$r(m) = 1 + 2^{n_1} + 2^{\min\{n_2, n_1 - z_2\}}.$$

Explicitly, the non-zero numbers r_ℓ $(0 \leq \ell \leq t)$ in definition 3.4 hold for $\ell \in \{\kappa_1, \ell_1, \kappa_2, \ell_2\}$ with $r_{\kappa_1} = 2^{n_1} - 1$, $r_{\kappa_2} = 2^{\min\{n_2, n_1 - z_2\}} - 1$ and $r_{\ell_1} = r_{\ell_2} = 1$, where $d_{\kappa_1} = 1^{z_1}0^{n_2}1^{z_2}$, $d_{\kappa_2} = 1^{z_2}$, and

$$\begin{array}{lll} d_{\ell_1} = 1^{n_2}0^{n_1}1^{z_2} & \text{and } d_{\ell_2} = 1^{n_1 - z_2}, & \text{provided } n_2 \geq n_1 - z_2, \\ d_{\ell_1} = 1^{n_2}0^{z_2}1^{n_1 - n_2 - z_2}0^{n_2}1^{z_2} & \text{and } d_{\ell_2} = 1^{n_2}, & \text{provided } n_2 \leq n_1 - z_2. \end{array}$$

Proposition 3.10. *Let m be even with $\text{cbe}(m) = (n_1, z_1, n_2, z_2)$ and $z_1 < n_1 < n_2 \leq z_2$. Then $r(m) = 1 + 2^{n_2}$. Explicitly, the non-zero numbers r_ℓ $(0 \leq \ell \leq t)$ in definition 3.4 hold for $\ell \in \{\kappa_1, \ell_1, \kappa_2, \ell_2\}$ with*

$$r_{\kappa_1} = 2^{n_2} - r_{\kappa_2} - 2,$$

$$r_{\kappa_2} = 2^{n_1+1}(2^{n_2 - n_1 - 1} - 1) + 2^{\rho}(2^{z_1 - 1} - 1)\sum_{i=0}^{q} 2^{iz_1 + 2} + 2^{\rho+1} - 1,$$

$$r_{\ell_1} = r_{\ell_2} = 1,$$

where $d_{\kappa_1} = 1^{z_1}0^{n_2}1^{z_2}$, $d_{\kappa_2} = 1^{z_2}$, $d_{\ell_1} = 1^{\rho+1}0^{n_2}1^{z_2}$, and $d_{\ell_2} = 1^{n_2}$. Here q and ρ stand, respectively, for the quotient and remainder in the division of $n_1 - z_1 - 1$ by z_1.

3.3 Immersion dimension via higher **TC***: An example*

The idea behind this section is to exemplify not only that $G_s(m)$ and $\delta_s(m)$ are closely related, but that the monotonous behavior of the sequence $\{G_s(m)\}_s$ seems to be particularly attractive in the critical sequence (3.5).

Let us start by considering small-dimensional examples, and the way they become part of larger families sharing similar properties.

There are three singular situations: $\mathbb{R}\mathrm{P}^1$ is a circle, and it certainly fits in the well known description of the higher topological complexity of spheres, where the dimension of the sphere plays the decisive role: $\mathbf{TC}_s(S^{2k}) = s$, while $\mathbf{TC}_s(S^{2k+1}) = s-1$, for any $s \geq 2$, see [1]. Closely related is the case of the H-spaces $\mathbb{R}\mathrm{P}^3$ and $\mathbb{R}\mathrm{P}^7$, so that [24, theorem 1] gives $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^m) = \mathrm{cat}((\mathbb{R}\mathrm{P}^m)^{s-1}) = m(s-1)$ for any s if $m \in \{1,3,7\}$.

The first truly interesting case is that of the projective plane, which immerses optimally in three-dimensional Euclidean space as the Boy Surface, so $\mathbf{TC}_2(\mathbb{R}\mathrm{P}^2) = 3$. Note that this is just one below the dimensional bound in Proposition 2.1, which contrasts with the fact (from theorem 3.4) that

$$\mathbf{TC}_s(\mathbb{R}\mathrm{P}^2) = 2s \text{ for any } s \geq 3. \tag{3.25}$$

It is worth remarking that (3.25) is part of a more general phenomenon: Any closed (orientable or not) surface S, other than the sphere, and the torus, has $\mathbf{TC}_s(S) = 2s$ whenever $s \geq 3$ (c.f. [18, theorem 5.1]); this should also be compared to the fact that $\mathbf{TC}_2(S) = 4$ for any (orientable or not) closed surface S other than sphere, the torus and the projective plane. For our purposes, a much more interesting observation to make at this point is that (3.25) generalizes (again in view of theorem 3.4) to the fact that, for $a \geq 1$,

$$\mathbf{TC}_s(\mathbb{R}\mathrm{P}^{2^a}) = 2^a s \text{ for any } s \geq 3,$$

while

$$\mathbf{TC}_2(\mathbb{R}\mathrm{P}^{2^a}) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^{2^a}) = 2^{a+1} - 1$$

In terms of the δ_s functions, such a situation translates into the equalities

$$\delta_3(2^a) = 0 \text{ and } \delta_2(2^a) = 1. \tag{3.26}$$

Since $r(2^a) = 3$ (recall that $a \geq 1$), this yields a nicely regular increasing behavior for the critical sequence (3.5) when $m = 2^a$. Admittedly, the length of the sequence (3.26) is ridiculously short but, as discussed next, a similar regularity phenomenon could actually be holding in the next obvious example, namely $m = 2^a + 2^{a+1}$ with $a \geq 1$ (the special case $m = 3$ has been considered above), which we discuss next.

At first sight, the situation is slightly special for $m = 2^a + 2^{a+1}$ if $a = 1$, so we consider it first. The immersion dimension of $\mathbb{R}\mathrm{P}^6$ is known to be $\mathbf{TC}_2(\mathbb{R}\mathrm{P}^6) = 7$, while proposition 3.6 gives $r(6) = 7$. Thus, the critical sequence (3.5) now becomes

$$\delta_7(6) = 0, \ \delta_6(6) = ?, \ \delta_5(6) = ?, \ \delta_4(6) = ?, \ \delta_3(6) = ?, \ \delta_2(6) = 5. \tag{3.27}$$

Furthermore, the proof of theorem 3.4 yields

$$(x_1 + x_2)^7 (x_1 + x_3)^7 \cdots (x_1 + x_7)^7 \neq 0.$$

In particular, if we only consider the first $j-1$ factors ($2 \leq j \leq 7$), we obtain the last instance in the chain of inequalities $\delta_j(6) \leq G_j(6) \leq 7 - j$. Consequently, (3.27) becomes

$$\delta_7(6) = 0, \ \delta_6(6) \leq 1, \ \delta_5(6) \leq 2, \ \delta_4(6) \leq 3, \ \delta_3(6) \leq 4, \ \delta_2(6) = 5. \tag{3.28}$$

It would be interesting to know whether (3.28) really has the nice steady increasing behavior suggested by (3.26), namely if

$$\delta_j(6) = 7 - j \text{ for } 2 \leq j \leq 7. \tag{3.29}$$

For instance, (3.29) would hold provided one could prove that the inequalities in (3.28) held in the stronger form $\delta_i(6) \leq \delta_{i+1}(6) + 1$. At any rate, the following considerations are intended to give numerical evidence toward the possibility that the $(a \geq 2)$-analogue of (3.29) holds in a suitably generalized way.

For $a \geq 2$, proposition 3.5 gives $r(2^a + 2^{a+1}) = 5$, thus the critical sequence (3.5) now takes the slightly shorter form

$$\delta_5(2^a + 2^{a+1}) = 0, \quad \delta_4(2^a + 2^{a+1}) = ?, \quad \delta_3(2^a + 2^{a+1}) = ?,$$
$$\delta_2(2^a + 2^{a+1}) = ?$$

The currently known information about $\mathrm{Imm}(\mathbb{R}\mathrm{P}^{2^a+2^{a+1}})$ for $a \leq 5$ yields

Case $a = 2$: $\delta_2(12) = 6$.

Case $a = 3$: $\delta_2(24) \in \{9, 10\}$.[8]

Case $a = 4$: $\delta_2(48) \in \{9, 10, 11\}$.

Case $a = 5$: $\delta_2(96) \in \{13, 14, \ldots, 18\}$.

[8] Note that $\mathbb{R}\mathrm{P}^{24}$ is the smallest-dimensional projective space whose immersion dimension is not fully known; yet our purely homological methods suffice to get the exact value of $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^{24})$ for $s \geq 5$.

The punch line is that the above facts provide some evidence for potentially extending the estimates in (3.28) by the following:

Conjecture 3.11. *For $a \geq 1$ and $2 \leq j \leq r(2^a + 2^{a+1})$,*

$$\delta_j(2^a + 2^{a+1}) \leq (r(2^a + 2^{a+1}) - j)a. \tag{3.30}$$

Remark 3.10. Kitchloo-Wilson's non-immersion result $\mathbb{R}\mathrm{P}^{48} \not\subseteq \mathbb{R}^{84}$ (the lowest-dimensional new result in [22] giving $\delta_2(48) < 12$) implies that we should not expect equality to hold in (3.30) —which seems to be compatible with the fact that the potential new non-immersion result in conjecture 3.12 below is still far from the expected optimal Euclidean immersion of $\mathbb{R}\mathrm{P}^m$, namely $2m - 2\alpha(m) + o(\alpha(m))$.

Our interest in the above discussion comes from the fact that conjecture 3.11 obviously contains (with $j = 2$) what would be the new (as far as we are aware of) non-immersion result $\delta_2(2^a + 2^{a+1}) \leq 3a$ for $a \geq 2$, i.e.:

Conjecture 3.12. *For $a \geq 2$, $\mathrm{Imm}(\mathbb{R}\mathrm{P}^{2^a+2^{a+1}}) \geq 2^{a+1} + 2^{a+2} - 3a$.*

For example, with $a = 3$, conjecture 3.12 would settle the value of the currently open immersion dimension of $\mathbb{R}\mathrm{P}^{24}$ (one of the iconic cases back in the decade of the 1970's) to be $\mathbf{TC}_2(\mathbb{R}\mathrm{P}^{24}) = 39$.

Of course, one could try to apply the $\mathbf{TC}_s$ approach to $\mathrm{Imm}(\mathbb{R}\mathrm{P}^m)$ for other families of projective spaces $\mathbb{R}\mathrm{P}^m$. For instance, some of the phenomena described above seem to hold for spaces of the form $\mathbb{R}\mathrm{P}^{2^a+2^{a+1}+2^{a+2}}$ with $a \geq 2$ and, more generally, for spaces $\mathbb{R}\mathrm{P}^m$ with $\mathrm{cbe}(m) = (n_1, z_1)$ and

$z_1 \geq n_1 - 1$. One could even try to use the same strategy in order to prove (positive) immersion results. Indeed, just as (3.30) is a statement about the possibility that the increasing behavior of the critical sequence (3.5) is bounded from above by some linear function, it is natural to try to prove a general statement asserting that, for some fixed integer $\phi(m)$, $\delta_j(m) \geq \delta_{j+1}(m) + \phi(m)$ in the range of the critical sequence (3.5). Such a possibility will most likely need to use stronger homotopy methods (e.g. the Hopf-type obstruction methods recently developed in [16]), rather than the homological methods in this paper. For instance, the homotopy obstruction methods in [8, Section 2] seem to lead to a proof of equality in (3.30) for $j = r(2^a + 2^{a+1})$.

4

On the effective topological complexity of Σ_g, $g \geq 2$

Let Σ_g be embedded in $\mathbb{R}^3$ so that Σ_g is invariant under reflections in the xy- ,yz- , and xz-planes. Let σ stand for the "antipodal" (orientation-reversing) involution on Σ_g given by $\sigma(x,y,z) = (-x,-y,-z)$. If we consider Σ_g with a left action of $\mathbb{Z}_2 = \{e, \sigma\}$, then according with definition 2.16,

$$\mathbf{TC}^{\sigma}(\Sigma_g) := \mathbf{TC}^{\mathbb{Z}_2,2}(\Sigma_g) = \mathbf{secat}\left(p_2 : P_2(\Sigma_g) \to \Sigma_g \times \Sigma_g\right).$$

For the fibration $p_2 : P_2(\Sigma_g) \to \Sigma_g \times \Sigma_g$, note that $P_2(\Sigma_g)$ is the topological disjoint union of two copies of $P(\Sigma_g)$, and we write

$$P_2(\Sigma_g) = P_e(\Sigma_g) \coprod P_\sigma(\Sigma_g),$$

with $P_e(\Sigma_g)$ corresponding to the condition $\alpha(1) = \beta(0)$ and $P_\sigma(\Sigma_g)$ to the condition $\alpha(1) = \sigma \cdot \beta(0)$. In these terms,

$$p_2(\gamma) = \begin{cases} e_{0,1}(\gamma), & \text{for } \gamma \in P_e(\Sigma_g) \\ \epsilon_{0,1}(\gamma), & \text{for } \gamma \in P_\sigma(\Sigma_g), \end{cases}$$

where $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$ and $\epsilon_{0,1}(\gamma) = (\gamma(0), \sigma \cdot \gamma(1))$ is the "twisted" evaluation map. In particular, if $x_0 \in \Sigma_g$ is the base point of Σ_g, then

$$\begin{aligned} F &:= p_2^{-1}(x_0, x_0) \\ &= \Omega\Sigma_g \coprod \mathrm{Maps}(I, 0, 1; \Sigma_g, x_0, \sigma \cdot x_0) \\ &\simeq \mathbb{Z}_2 \times \Omega\Sigma_g. \end{aligned}$$

As mentioned in the preliminaries, asking if the inequalities $\mathbf{TC}(\Sigma_g) \leq 3$ and $\mathbf{TC}^{\sigma}(\Sigma_g) \leq 3$ hold is equivalent to asking if the fibrations

$$\underset{3}{*}\, e_{0,1} \qquad \text{and} \qquad \underset{3}{*}\, p_2$$

admit a global section. There are commutative diagrams of fibrations

$$\begin{array}{ccc} \Omega\Sigma_g & \hookrightarrow & \mathbb{Z}_2 \times \Omega\Sigma_g \\ \downarrow & & \downarrow \\ P\Sigma_g & \hookrightarrow & P_2(\Sigma_g) \\ {\scriptstyle e_{0,1}}\downarrow & & \downarrow{\scriptstyle p_2} \\ \Sigma_g \times \Sigma_g & = & \Sigma_g \times \Sigma_g \end{array} \tag{4.1}$$

and

$$\begin{array}{ccc}
(\Omega\Sigma_g)^{*4} & \hookrightarrow & (\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4} \\
\downarrow & & \downarrow \\
\overset{4}{\underset{\Sigma_g\times\Sigma_g}{*}}(P\Sigma_g) & \hookrightarrow & \overset{4}{\underset{\Sigma_g\times\Sigma_g}{*}}(P_G(\Sigma_g)) \\
\downarrow {\scriptstyle \underset{3}{*} e_{0,1}} & & \downarrow {\scriptstyle \underset{3}{*} p_2} \\
\Sigma_g \times \Sigma_g & = & \Sigma_g \times \Sigma_g,
\end{array} \tag{4.2}$$

both fibers in the last diagram are 2-connected, and since $\dim(\Sigma_g \times \Sigma_g) = 4$, there is only one primary obstruction

$$\mathcal{O}_g \in H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\Omega\Sigma_g)^{*4}\right)}\right)$$

for the inequality $\mathbf{TC}(\Sigma_g) \leq 3$, and only one primary obstruction

$$\mathcal{O}_{g,\mathbb{Z}_2} \in H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right)}\right)$$

for the inequality $\mathbf{TC}^{\sigma}(\Sigma_g) \leq 3$. Functoriality of primary obstructions implies that $\mathcal{O}_g$ hits $\mathcal{O}_{g,\mathbb{Z}_2}$ under the morphism

$$\rho_4 : H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\Omega\Sigma_g)^{*4}\right)}\right) \to H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\mathbb{Z}_2 \times \Omega X)^{*4}\right)}\right) \tag{4.3}$$

corresponding to the map of twisted coefficients induced by the inclusion of fibers $(\Omega\Sigma_g)^{*4} \hookrightarrow (\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}$. The primary obstructions for sectioning $e_{0,1}$ and p_2 on the 1-dimensional skeleton of $\Sigma_g \times \Sigma_g$ are cohomology classes

$$o_g \in H^1\left(\Sigma_g \times \Sigma_g; \underline{\overline{H_0}\left(\Omega\Sigma_g\right)}\right) \quad \text{and} \quad o_{g,\mathbb{Z}_2} \in H^1\left(\Sigma_g \times \Sigma_g; \underline{\overline{H_0}\left(\mathbb{Z}_2 \times \Omega\Sigma_g\right)}\right).$$

Again the functoriality of primary obstructions yields $\rho_1(o_g) = o_{g,\mathbb{Z}_2}$, with

$$\rho_1 : H^1\left(\Sigma_g \times \Sigma_g; \underline{\overline{H_0}\left(\Omega\Sigma_g\right)}\right) \to H^1\left(\Sigma_g \times \Sigma_g; \underline{\overline{H_0}\left(\mathbb{Z}_2 \times \Omega\Sigma_g\right)}\right).$$

The structural details in the domains of both ρ_1 and ρ_4 are well understood from [8]. Write π_g for $\pi_1(\Sigma_g, x_0)$, then:

- $\left(\overline{H_0}(\Omega\Sigma_g)\right)^{\otimes 4} \cong \overline{H_0}\left((\Omega\Sigma_g)^{\wedge 4}\right) \cong \overline{H_3}\left(\Sigma^3\left(\Omega\Sigma_g\right)^{\wedge 4}\right) \cong \overline{H_3}\left((\Omega\Sigma_g)^{*4}\right) \cong \pi_3\left((\Omega\Sigma_g)^{*4}\right)$ with $\overline{H_0}(\Omega\Sigma_g) \cong I(\pi_g)$, the kernel of the augmentation morphism $\mathbb{Z}[\pi_g] \to \mathbb{Z}$.
- The monodromy in the domain of ρ_4 (coming from the fibration on the left of (4.2)) is the fourth tensor-power of the monodromy in the domain of ρ_1, the latter one being the action (by homomorphisms) $(\pi_g \times \pi_g) \times I(\pi_g) \to I(\pi_g)$ given by

  $$(x, y) \cdot \sum n_i c_i = \sum n_i x c_i \overline{y},$$

  where $\overline{y}$ denotes the inverse of the element $y \in \pi_g$.

- $\mathcal{O}_g$ is the 4-th power of $o_g \in H^1\left(\Sigma_g \times \Sigma_g; \underline{I(\pi_g)}\right)$, where the latter class corresponds to the crossed morphism given by
$$\begin{array}{rccc} \phi_g : & \pi_g \times \pi_g & \longrightarrow & I(\pi_g) \\ & (x,y) & \mapsto & x\overline{y} - 1. \end{array} \tag{4.4}$$

The structural details on the range of ρ_1 and ρ_4 are spelled out next.

Lemma 4.1. *1. $\pi_3((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}) \cong \left(\overline{H_0}(\mathbb{Z}_2 \times \Omega\Sigma_g)\right)^{\otimes 4}$ with $\overline{H_0}(\mathbb{Z}_2 \times \Omega\Sigma_g) \cong I(\mathbb{Z}_2 \times \pi_g)$.*

2. *The monodromy in the range of ρ_4 (coming from the fibration on the right of (4.2)) is the 4-th tensor-power of the monodromy in the range of ρ_1, the latter one being the restriction of the action (by homomorphisms) $(\pi_g \times \pi_g) \times \mathbb{Z}[\mathbb{Z}_2 \times \pi_g] \to \mathbb{Z}[\mathbb{Z}_2 \times \pi_g]$ determined on basis elements by*
$$(x,y) \cdot (e,c) = (e, xc\overline{y}) \quad \text{and} \quad (x,y) \cdot (\sigma, c) = \left(\sigma, xc\overline{\widetilde{\sigma}(y)}\right). \tag{4.5}$$

   Here
$$\begin{array}{rccc} \widetilde{\sigma} : & \pi_g & \longrightarrow & \pi_g \\ & x & \mapsto & [\phi_0] \cdot \sigma_*(x) \cdot [\,\overline{\phi_0}\,], \end{array} \tag{4.6}$$

   where σ_* is the isomorphism $\pi_g = \pi_1(\Sigma_g, x_0) \to \pi_1(\Sigma_g, \sigma \cdot x_0)$ induced by the based map $\sigma : (\Sigma_g, x_0) \to (\Sigma_g, \sigma \cdot x_0)$, $\phi_0 \in \mathrm{Maps}(I, 0, 1; \Sigma_g, x_0, \sigma \cdot x_0)$ is a fixed path, and $\overline{\phi_0}$ is its inverse path.

3. *$\mathcal{O}_{g,\mathbb{Z}_2}$ is the fourth power of $o_{g,\mathbb{Z}_2} \in H^1\left(\Sigma_g \times \sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_g)}\right)$, where the latter class corresponds to the crossed morphism*
$$\begin{array}{rccc} \phi_{g,\mathbb{Z}_2} : & \pi_g \times \pi_g & \longrightarrow & I(\mathbb{Z}_2 \times \pi_g) \\ & (x,y) & \mapsto & (e, x\overline{y}) - (e,1). \end{array} \tag{4.7}$$

Before proceeding with the proof, we would like to note that both ϕ_g and $\phi_{g,\mathbb{Z}_2}$ are determined up to principal homomorphisms, and the expressions given in (4.4) and (4.7) come from taking an explicit and obvious lifting over the 0-skeleton. For instance, (4.7) comes from taking a lifting over the 0-skeleton into the component $P_e(\Sigma_g)$.

Proof. Regarding the first assertion,
$$\begin{aligned} \pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right) &\cong \overline{H_3}\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right), \text{ by connectivity} \\ &\cong \overline{H_3}\left(\Sigma^3(\mathbb{Z}_2 \times \Omega\Sigma_g)^{\wedge 4}\right) \\ &\cong \overline{H_0}\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{\wedge 4}\right), \text{ by suspension isomorphism} \\ &\cong \left(\overline{H_0}(\mathbb{Z}_2 \times \Omega\Sigma_g)\right)^{\otimes 4}, \end{aligned}$$
and the isomorphism $\overline{H_0}(\mathbb{Z}_2 \times \Omega\Sigma_g) \cong I(\mathbb{Z}_2 \times \pi_g)$ is indicated in the preliminaries.
Regarding the second assertion, the fact that the monodromy in the range of ρ_4 is the fourth tensor power of that in the range of ρ_1 follows from [26], and the monodromy in the range of ρ_1 is described, in a general case, in the preliminaries.
In view of (4.1) and as in the observation containing (4.3), the third assertion follows from (4.4) and the functoriality of primary obstructions, recalling that o_g (resp. $o_{g,\mathbb{Z}_2}$) is the primary obstruction for sectioning $e_{0,1}$ (resp. p_2) on the 1-dimensional skeleton of $\Sigma_g \times \Sigma_g$. In other words, if the same section is taken on the 0-skeleton for the fibration in (4.1), then $\phi_{g,\mathbb{Z}_2}$ is forced to be the composition of ϕ_g followed by the obvious inclusion $I(\pi_g) \to I(\mathbb{Z}_2 \times \pi_g)$.

□

Lemma 4.2. *Let $[\Sigma_g \times \Sigma_g]$ denote the fundamental class of $\Sigma_g \times \Sigma_g$. The map*

$$\rho : \left(\pi_3\left((\Omega\Sigma_g)^{*4}\right)\right)_{\pi_g\times\pi_g} \to \left(\pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right)\right)_{\pi_g\times\pi_g}$$

induced in coinvariants (i.e. in zero dimensional twisted homology) by the fourth join-power of the inclusion of fibers $\Omega\Sigma_g \hookrightarrow \mathbb{Z}_2 \times \Omega\Sigma_g$ in (4.1) sends the Poincaré image obstruction $\mathcal{O}_g \cap [\Sigma_g \times \Sigma_g]$ into the Poincaré image obstruction $\mathcal{O}_{g,\mathbb{Z}_2} \cap [\Sigma_g \times \Sigma_g]$, the latter of which vanishes if and only if $\mathbf{TC}^\sigma(\Sigma_g) \leq 3$.

Proof. This is just a reinterpretation, in terms of Poincaré duality. The key point is that the morphism (4.3), induced by the map of coefficients, fits in the commutative diagram

$$\begin{array}{ccc}
H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\Omega\Sigma_g)^{*4}\right)}\right) & \xrightarrow{\rho_4} & H^4\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right)}\right) \\
\cong\downarrow & & \downarrow\cong \\
H_0\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\Omega\Sigma_g)^{*4}\right)}\right) & \longrightarrow & H_0\left(\Sigma_g \times \Sigma_g; \underline{\pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right)}\right) \\
\| & & \| \\
\left(\pi_3\left((\Omega\Sigma_g)^{*4}\right)\right)_{\pi_g\times\pi_g} & \xrightarrow{\rho} & \left(\pi_3\left((\mathbb{Z}_2 \times \Omega\Sigma_g)^{*4}\right)\right)_{\pi_g\times\pi_g},
\end{array}$$

where the top two vertical isomorphisms are given by Poincaré duality, i.e. by capping with the fundamental class $[\Sigma_g \times \Sigma_g] \in H_4(\Sigma_g \times \Sigma_g; \mathbb{Z})$. □

Remark 4.1. In his early work on topological complexity, M. Farber realized (through methods much simpler than obstruction theory) that $\mathbf{TC}(\Sigma_g) = 4$ for $g \geq 2$. Thus, $\mathcal{O}_g$ and its Poincaré-isomorphic image $\mathcal{O}_g \cap [\Sigma_g \times \Sigma_g]$ are non-zero. The problem addressed in the present chapter of the thesis is to decide whether $\mathcal{O}_g \cap [\Sigma_g \times \Sigma_g]$ lies in the kernel of ρ. Although we do not yet know the answer of the latter question, the following considerations are intented to give a partial answer, namely, that the posibility that $\mathcal{O}_g \cap [\Sigma_g \times \Sigma_g] \in \ker(\rho)$ actually has a "monotonic" behavior on g. Details follow.

We use the following presentation of the fundamental group of Σ_g

$$\pi_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle,$$

and we consider the map

$$\phi : \Sigma_g \to \Sigma_{g-1}, \quad g - 1 \geq 2 \tag{4.8}$$

whose homotopy class corresponds to the homomorphism

$$\begin{array}{rccll}
\varphi : & \pi_g & \longrightarrow & \pi_{g-1} & \\
& a_i & \mapsto & a_i, & 1 \leq i \leq g-1 \\
& b_i & \mapsto & b_i, & 1 \leq i \leq g-1 \\
& a_g, b_g & \mapsto & 1. &
\end{array} \tag{4.9}$$

Recall that oriented closed surfaces we deal with are Eilenberg-MacLane spaces of type $(\pi_g, 1)$. Specifically, for an Eilenberg-MacLane spaces of type (G, n) and for any $(n-1)$-connected CW complex X there is a bijection $[X, Y] \to \mathrm{Hom}(\pi_n(X), \pi_n(Y))$.

Lemma 4.3. *The map $H_2(\pi_g; \mathbb{Z}) \to H_2(\pi_{g-1}; \mathbb{Z})$ induced by φ (or, equivalently, by ϕ) is an isomorphism.*

Proof. A free resolution for the trivial $\mathbb{Z}[\pi_g]$-module $\mathbb{Z}$ is given by [9]

[9] See [21], page 86.

$$0 \longrightarrow M_2^g \xrightarrow{d_2} M_1^g \xrightarrow{d_1} M_0^g \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{4.10}$$

where $M_0^g = \mathbb{Z}[\pi_g]\langle x_g\rangle$, $M_1^g = \mathbb{Z}[\pi_g]\langle d_{1,g}, e_{1,g}, \dots, d_{g,g}, e_{g,g}\rangle$, $M_2^g = \mathbb{Z}[\pi_g]\langle \omega_g\rangle$, and

$$\begin{aligned}
\epsilon(x_g) &= 1,\\
d_1(d_{i,g}) &= a_i - 1,\\
d_1(e_{i,g}) &= b_i - 1,\\
d_2(\omega_g) &= \sum_{i=1}^{g} ([a_1,b_1]\cdots[a_{i-1},b_{i-1}] - [a_1,b_1]\cdots[a_i,b_i]b_i)\, d_{i,g} + \\
&\quad \sum_{i=1}^{g} ([a_1,b_1]\cdots[a_{i-1},b_{i-1}]a_i - [a_1,b_1]\cdots[a_i,b_i])\, e_{i,g},\\
&= \sum_{i=1}^{g} \left[\left(\prod_{k=0}^{i-1}[a_k,b_k]\right)\left(d_{i,g} - a_i b_i a_i^{-1} d_{i,g}\right)\right] + \qquad (4.11)\\
&\quad \sum_{i=1}^{g} \left[\left(\prod_{k=0}^{i-1}[a_k,b_k]\right) a_i \left(e_{i,g} - b_i a_i^{-1} b_i^{-1} e_{i,g}\right)\right]
\end{aligned}$$

where $[a_0, b_0]$ is understood as 1.
Equation (4.11) implies that the class $\overline{\omega_g \otimes t} \in M_2^g \otimes_{\pi_g} \mathbb{Z}$ generates the kernel of $d_2 \otimes \mathrm{id}$. Furthermore, the map

$$\begin{aligned}
c_g &\mapsto c_{g-1}\\
d_{i,g} &\mapsto d_{i,g-1}, \quad 1 \leq i \leq g-1\\
e_{i,g} &\mapsto e_{i,g-1}, \quad 1 \leq i \leq g-1\\
d_g, e_g &\mapsto 0\\
f_g &\mapsto f_{g-1}
\end{aligned}$$

induces a chain map $f : M_*^g \to M_*^{g-1}$ [10], which in turns induces an isomorphism $H_2(\phi, \varphi) : H_2(\pi_g; \mathbb{Z}) \longrightarrow H_2(\pi_{g-1}; \mathbb{Z})$. □

[10] f is compatible with $\varphi : \pi_g \to \pi_{g-1}$.

Corollary 4.3.1. *The map $H_4(\pi_g \times \pi_g; \mathbb{Z}) \to H_4(\pi_{g-1} \times \pi_{g-1}; \mathbb{Z})$ induced by $\varphi \times \varphi$ (or, equivalently, by $\phi \times \phi$ is an isomorphism.*

Proof. It follows from the general Künneth formula. □

The expression in (4.7) yields a commutative diagram

$$\begin{array}{ccc} \pi_g \times \pi_g & \xrightarrow{\phi_{g,\mathbb{Z}_2}} & I(\mathbb{Z}_2 \times \pi_g) \\ \downarrow{\scriptstyle \varphi\times\varphi} & & \downarrow{\scriptstyle I(\mathrm{id}_{\mathbb{Z}_2}\times\varphi)} \\ \pi_{g-1} \times \pi_{g-1} & \xrightarrow[\phi_{g-1,\mathbb{Z}_2}]{} & I(\mathbb{Z}_2 \times \pi_{g-1}). \end{array}$$

In cohomological terms, this means that the classes $o_{g,\mathbb{Z}_2}$ and $o_{g-1,\mathbb{Z}_2}$ hit a common class $o \in H^1\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})}\right)$ under morphisms

$$\begin{aligned} H^1\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_g)}\right) &\to H^1\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})}\right) \\ H^1\left(\Sigma_{g-1} \times \Sigma_{g-1}; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})}\right) &\to H^1\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})}\right). \end{aligned}$$

The former map is induced at coefficient level, by the morphism $I(\mathrm{id}_{\mathbb{Z}_2} \times \varphi)$, whereas the latter map is induced, at topological space level, by $\phi \times \phi$. Take fourth powers to get the corresponding mapping situation $\mathcal{O}_{g-1,\mathbb{Z}_2}, \mathcal{O}_{g,\mathbb{Z}_2} \mapsto o^4$ on the two vertical maps on the left of the diagram

$$\begin{array}{ccc} H^4(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_g)^{\otimes 4}}) & & \\ \downarrow{\scriptstyle I(\mathrm{id}_{\mathbb{Z}_2}\times\varphi)^{\otimes 4}} & & \\ H^4\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})^{\otimes 4}}\right) & \xrightarrow{\cdot \cap [\Sigma_g \times \Sigma_g]} & H_0\left(\Sigma_g \times \Sigma_g; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})^{\otimes 4}}\right) \\ \uparrow{\scriptstyle (\phi\times\phi)^*} & & \downarrow{\scriptstyle (\phi\times\phi)_*} \\ H^4\left(\Sigma_{g-1} \times \Sigma_{g-1}; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})^{\otimes 4}}\right) & \xrightarrow{\cdot \cap [\Sigma_{g-1} \times \Sigma_{g-1}]} & H_0\left(\Sigma_{g-1} \times \Sigma_{g-1}; \underline{I(\mathbb{Z}_2 \times \pi_{g-1})^{\otimes 4}}\right). \end{array}$$

The commutativity of the previous diagram comes from the usual naturality behavior of cap products and corollary 4.3.1. Since the bottom horizontal map in the diagram is an isomorphism, a chase of elements yields that the non-triviality of $\mathcal{O}_{g-1,\mathbb{Z}_2}$ implies the non-triviality of $\mathcal{O}_{g,\mathbb{Z}_2}$. In other words, the inequality $\mathbf{TC}^{\sigma}(\Sigma_{g-1}) \geq 4$ implies the inequality $\mathbf{TC}^{\sigma}(\Sigma_g) \geq 4$. Since $\mathbf{TC}^{\sigma}(\Sigma_g)$ is bounded from above by $\mathbf{TC}(\Sigma_g/\mathbb{Z}_2)$, which is known to be 4, the conclusion above becomes that whenever we have $\mathbf{TC}^{\sigma}(\Sigma_{g-1}) = 4$, we must also have $\mathbf{TC}^{\sigma}(\Sigma_g) = 4$. Further, in unpublished work, Z. Błaszczyk, J. González and M. Kaluba proved that $\mathbf{TC}^{\sigma}(\Sigma_g) \geq 3$ for $g \geq 2$. We thus obtain:

Theorem 4.4. $3 \leq \mathbf{TC}^{\sigma}(\Sigma_2) \leq \mathbf{TC}^{\sigma}(\Sigma_3) \leq \cdots \leq 4$.

The monotonic behavior on the previous theorem extends to small genera: Błaszczyk, J. González and M. Kaluba also showed that $\mathbf{TC}^{\sigma}(S^2) = 1$ and $\mathbf{TC}^{\sigma}(S^1 \times S^1) = 2$.

In the rest of the chapter we focus attention on the case of Σ_2. In particular, we recover, from a purely group-cohomology viewpoint, Błaszczyk - González-Kaluba's inequality $\mathbf{TC}^{\sigma}(\Sigma_2) \geq 3$. In fact, the results in the following sections seem to suggest that the latter inequality could actually be a strict equality.

4.1 Orientable surface of genus 2

We change a little bit the notation used before. Consider the following presentation of the fundamental group of Σ_2

$$\pi_2 = \pi_1(\Sigma_2) = \langle a, b, c, d \mid [a,b][c,d] = 1 \rangle,$$

and the free resolution $\mathbf{M}$ for the trivial $\mathbb{Z}[\pi_2]$-module $\mathbb{Z}$ given by

$$0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where M_0, M_1 and M_2 are free $\mathbb{Z}[\pi_2]$-modules with basis $\{x\}$, $\{\alpha,\beta,\gamma,\delta\}$ and $\{\omega\}$ respectively, and,

$$\begin{aligned} \epsilon(x) &= 1, & d_1(\alpha) &= ax - x, \\ d_1(\beta) &= bx - x, & d_1(\gamma) &= cx - x, \\ d_1(\delta) &= dx - x, & d_2(\omega) &= (1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + (ab\bar{a}\bar{b} - d)\gamma + (ab\bar{a}\bar{b}c - 1)\delta. \end{aligned}$$

We proceed to construct contracting homotopies $\mathbf{s}$, $\mathbf{U}$ and $\mathbf{T}$ for the resolutions $\mathbf{M}$, $\mathbf{M}^{\otimes 2}$ and $\mathbf{M}^{\otimes 4}$ respectively.

Since the definitions of $\mathbf{U}$ and $\mathbf{T}$ are direct once we have $\mathbf{s}$, suppose for the moment that the latter one has already been defined, then:

- A contracting homotopy $\mathbf{U}$ for $\mathbf{M}^{\otimes 2}$ is given by

$$\begin{aligned} U_{-1}(1) &= x \otimes x \\ U_k(u \otimes v) &= \mathbf{s}(u) \otimes v + s_{-1}\epsilon(u) \otimes \mathbf{s}(v), \quad k \geq 0. \end{aligned}$$

- A contracting homotopy $\mathbf{T}$ for $\mathbf{M}^{\otimes 4}$ is given by

$$\begin{aligned} T_{-1}(1) &= x \otimes x \otimes x \otimes x \\ T_k(u \otimes v \otimes w \otimes z) &= \mathbf{s}(u) \otimes v \otimes w \otimes z + s_{-1}\epsilon(u) \otimes \mathbf{s}(v) \otimes w \otimes z \\ &\quad + s_{-1}\epsilon(u) \otimes s_{-1}\epsilon(v) \otimes \mathbf{s}(w) \otimes z \\ &\quad + s_{-1}\epsilon(u) \otimes s_{-1}\epsilon(v) \otimes s_{-1}\epsilon(w) \otimes \mathbf{s}(z), \quad k \geq 0. \end{aligned}$$

The construction of a contracting homotopy for $\mathbf{M}$ is a much more elaborate task, which we now work out. It should be said that when we refer to normal forms, these will be taken with respect to the rewriting system for π_2 described in section 2.2.

A contracting homotopy $\mathbf{s}$ for $\mathbf{M}$ is given by $\mathbb{Z}$-homomorphisms $s_{-1}\colon \mathbb{Z} \to M_0$, $s_0\colon M_0 \to M_1$, and $s_1\colon M_1 \to M_2$ such that $\epsilon s_{-1} = \mathrm{id}_{\mathbb{Z}}$ and $d_{k+1}s_k + s_{k-1}d_k = \mathrm{id}_{M_k}$ for $0 \leq k \leq 2$, recall that $d_0 = \epsilon$.

1. Define $s_{-1}(1) = x$, notice that the condition $\epsilon s_{-1} = \mathrm{id}_{\mathbb{Z}}$ is satisfied.
2. For s_0 we begin by defining

$$\begin{aligned} &s_0(x) = 0, \\ &s_0(ax) = \alpha, && s_0(bx) = \beta, && s_0(cx) = \gamma, && s_0(dx) = \delta, \\ &s_0(\bar{a}x) = -\bar{a}\alpha, && s_0(\bar{b}x) = -\bar{b}\beta, && s_0(\bar{c}x) = -\bar{c}\gamma, && s_0(\bar{d}x) = -\bar{d}\delta. \end{aligned}$$

Now if $\ell_1 \cdots \ell_k$ is in normal form, define

$$s_0(\ell_1\ell_2 \cdots \ell_k x) = s_0(\ell_1 x) + \ell_1 s_o(\ell_2 x) + \cdots + \ell_1 \cdots \ell_{k-1} s_0(\ell_k x).$$

Notice that $s_0(\ell_1 \cdots \ell_{m+1} \cdots \ell_k x) = s_0(\ell_1 \cdots \ell_m x) + \ell_1 \cdots \ell_m s_0(\ell_{m+1} \cdots \ell_k x)$, since

$$\begin{aligned} s_0(\ell_1 \cdots \ell_m \cdots \ell_k x) &= s_0(\ell_1 x) + \ell_1 s_o(\ell_2 x) + \cdots + \ell_1 \cdots \ell_m s_0(\ell_{m+1} x) + \cdots + \ell_1 \cdots \ell_{k-1} s_0(\ell_k x) \\ &= s_0(\ell_1 x) + \cdots + \ell_1 \cdots \ell_m \left(s_0(\ell_{m+1} x) + \cdots + \ell_{m+1} \cdots \ell_{m-1} s_0(\ell_k x)\right) \\ &= s_0(\ell_1 \cdots \ell_m x) + \ell_1 \cdots \ell_m s_0(\ell_{m+1} \cdots \ell_k x). \end{aligned}$$

Let us check that

$$d_1 s_0 + s_{-1}\epsilon = \mathrm{id}_{M_0}.$$

Notice that the relation holds when we evaluate on x. Now for $(\lambda, \ell) \in \{(\alpha, a), (\beta, b), (\gamma, c), (\delta, d)\}$

$$d_1 s_0(\ell x) = d_1\lambda = \ell x - x = \ell x - s_{-1}\epsilon(\ell x),$$

and

$$d_1 s_0\left(\overline{\ell} x\right) = d_1\left(-\overline{\ell}\lambda\right) = -\overline{\ell} d_1(\lambda) = -\overline{\ell}(\ell x - x) = -x + \overline{\ell} x = \overline{\ell} x - s_{-1}\epsilon\left(\overline{\ell} x\right).$$

Thus,

$$\begin{aligned} d_1 s_0(\ell_1 \ell_2 \cdots \ell_k x) &= d_1 s_0(\ell_1 x) + d_1\left(\ell_1 s_0(\ell_2 x)\right) + \cdots + d_1\left(\ell_1 \cdots \ell_{k-1} s_0(\ell_k x)\right) \\ &= (\ell_1 x - x) + \ell_1(\ell_2 x - x) + \cdots + \ell_1 \cdots \ell_{k-1}(\ell_k x - x) \\ &= \ell_1 x - x + \ell_1\ell_2 x - \ell_1 x + \cdots + \ell_1 \cdots \ell_{k-1}\ell_k x - \ell_1 \cdots \ell_{k-1} x \\ &= \ell_1 \cdots \ell_k x - x \\ &= \ell_1 \cdots \ell_k x - s_{-1}\epsilon\left(\ell_1 \cdots \ell_k x\right). \end{aligned}$$

3. In order to define s_1 we begin by putting

$$T_i = c\left(ab\overline{a}\overline{b}c\right)^i, \quad i \geq 0$$

$$U_i = \left(\overline{c}ba\overline{b}\overline{a}\right)^i, \quad i \geq 0.$$

For $y = \ell_1\ell_2\cdots\ell_k \in \pi_2$ in normal form, we define

$$s_1(y\lambda) = 0, \text{ for } \lambda \in \{\alpha, \beta, \gamma\},$$

$$s_1(y\delta) = \begin{cases} \left(y\sum_{i=1}^{n+1} U_i\right)\omega, & \text{if } y \text{ ends as } T_n \text{ but not in } T_{n+1}, n \geq 0 \\ -\left(y\sum_{i=0}^{n} \overline{U_i}\right)\omega, & \text{if } y \text{ ends as } U_{n+1} \text{ but not in } U_{n+2}, n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Next the relation $d_2 s_1 + s_0 d_1 = \mathrm{id}_{M_1}$ is verified, starting by evaluating it at $y\alpha$. We have:

$$s_0(d_1(y\alpha)) = s_0(\ell_1 \cdots \ell_k a x) - s_0(\ell_1 \cdots \ell_k x).$$

(a) If $\ell_k \neq \overline{a}$, then $\ell_1 \cdots \ell_k a$ is in normal form, then

$$s_0(\ell_1 \cdots \ell_k a x) = s_0(\ell_1 \cdots \ell_k x) + \ell_1 \cdots \ell_k \alpha,$$

thus

$$s_0 d_1(y\alpha) = \ell_1 \cdots \ell_k \alpha = y\alpha = y\alpha - d_2 s_1(y\alpha),$$

since $s_1(y\alpha) = 0$.

(b) If $\ell_k = \overline{a}$, then $\ell_1 \cdots \ell_k a = \ell_1 \cdots \ell_{k-1}$, which is in normal form, thus

$$\begin{aligned} s_0(d_1(y\alpha)) &= s_0(\ell_1 \cdots \ell_{k-1} x) - s_0(\ell_1 \cdots \ell_{k-1} x) - \ell_1 \cdots \ell_{k-1} s_0(\ell_k x) \\ &= -\ell_1 \cdots \ell_{k-1} s_0(\overline{a} x) \\ &= -\ell_1 \cdots \ell_{k-1}(-\overline{a}\alpha) \\ &= y\alpha \\ &= y\alpha - d_2 s_1(y\alpha), \end{aligned}$$

since $s_1(y\alpha) = 0$.

The verification of the relation $s_0d_1 + d_2s_1 = \mathrm{id}_{M_1}$ at $y\beta$ and $y\gamma$ is similar and omitted. In order to check the relation at $y\delta$, where there are three cases to consider, we must clarify the terminology: we will understand by "w_1 ends as w_2" that the spelling of w_1 ends the same as the spelling of the word w_2. We now check the relation at $y\delta$.

Case 1: y ends as T_n but not in T_{n+1}, $n \geq 0$.

We begin with the verification for $n = 0$. If $y = \ell_1 \cdots \ell_k c$ is in normal form, then

$$s_0d_1(y\delta) = s_0(\ell_1 \cdots \ell_k ba\bar{b}\bar{a}dcx) - s_0(\ell_1 \cdots \ell_k cx).$$

Notice that $\ell_k \neq \bar{c}$ since x is in normal form. Furthermore, by hypothesis $\ell_{k-4}\ell_{k-3}\ell_{k-2}\ell_{k-1}\ell_k c \neq cab\bar{a}\bar{b}c = T_1$.
In order to check the desired relation, we have to consider whether $\ell_1 \cdots \ell_k ba\bar{b}\bar{a}dc$ is in normal form or not.

(a) If $\ell_1 \cdots \ell_k ba\bar{b}\bar{a}dc$ is in normal form, then

$$\begin{aligned} y\delta - s_0d_1(y\delta) &= \ell_1 \cdots \ell_k \left[(ba\bar{b}\bar{a} - b)\alpha + (ba\bar{b} - 1)\beta + (1 - ba\bar{b}\bar{a}d)\gamma + (c - ba\bar{b}\bar{a})\delta\right] \\ &= d_2(\ell_1 \cdots \ell_k ba\bar{b}\bar{a}\omega) \\ &= d_2(s_1(y\delta)). \end{aligned}$$

(b) If $\ell_1 \cdots \ell_k ba\bar{b}\bar{a}dc$ is not in normal form:

i. $y = \ell_1 \cdots \ell_{k-1}\bar{b}c$, $\ell_k = \bar{b}$, $\ell_{k-1} \neq \bar{a}$. In this case

$$s_0d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-1}a\bar{b}\bar{a}dcx) - s_0(\ell_1 \cdots \ell_{k-1}\bar{b}cx),$$

notice that $\ell_1 \cdots \ell_{k-1}a\bar{b}\bar{a}dc$ is in normal form since $\ell_{k-1} \neq b$, then

$$\begin{aligned} y\delta - s_0d_1(y\delta) &= \ell_1 \cdots \ell_{k-1} \left[(a\bar{b}\bar{a} - 1)\alpha + (a\bar{b} - \bar{b})\beta + (\bar{b} - a\bar{b}\bar{a}d)\gamma + (\bar{b}c - a\bar{b}\bar{a})\delta\right] \\ &= d_2(\ell_1 \cdots \ell_{k-1}a\bar{b}\bar{a}\omega) \\ &= d_2s_1(y\delta), \end{aligned}$$

since

$$s_1(y\delta) = \ell_1 \cdots \ell_{k-1}a\bar{b}\bar{a}\omega.$$

ii. $y = \ell_1 \cdots \ell_{k-2}\bar{a}\bar{b}c$, $\ell_k = \bar{b}$, $\ell_{k-1} = \bar{a}$, $\ell_{k-2} \neq b$. In this case

$$s_0d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-2}\bar{b}\bar{a}dcx) - s_0(\ell_1 \cdots \ell_{k-2}\bar{a}\bar{b}cx),$$

notice that $\ell_1 \cdots \ell_{k-2}\bar{b}\bar{a}dc$ is in normal form since $\ell_{k-2} \neq a$, then

$$\begin{aligned} y\delta - s_0d_1(y\delta) &= \ell_1 \cdots \ell_{k-2} \left[(\bar{b}\bar{a} - \bar{a})\alpha + (\bar{b} - \bar{a}\bar{b})\beta + (\bar{a}\bar{b} - \bar{b}\bar{a}d)\gamma + (\bar{a}\bar{b}c - \bar{b}\bar{a})\delta\right] \\ &= d_2(\ell_1 \cdots \ell_{k-2}\bar{b}\bar{a}\omega) \\ &= d_2s_1(y\delta), \end{aligned}$$

since

$$s_1(y\delta) = \ell_1 \cdots \ell_{k-2}\bar{b}\bar{a}\omega.$$

iii. $y = \ell_1 \cdots \ell_{k-3} b\overline{a}\overline{b}c$, $\ell_k = \overline{b}$, $\ell_{k-1} = \overline{a}$, $\ell_{k-2} = b$, $\ell_{k-3} \neq a$. In this case

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-3}\overline{a}dcx) - s_0(\ell_1 \cdots \ell_{k-3} b\overline{a}\overline{b}cx),$$

notice that $\ell_1 \cdots \ell_{k-3}\overline{a}dc$ is in normal form since $\ell_{k-3} \neq \overline{b}$, then

$$\begin{aligned} y\delta - s_0 d_1(y\delta) &= \ell_1 \cdots \ell_{k-3} \left[(\overline{a} - b\overline{a})\alpha + (1 - b\overline{a}\overline{b})\beta + (b\overline{a}\overline{b} - \overline{a}d)\gamma + (b\overline{a}\overline{b}c - \overline{a})\delta\right] \\ &= d_2(\ell_1 \cdots \ell_{k-3}\overline{a}\omega) \\ &= d_2 s_1(y\delta), \end{aligned}$$

since

$$s_1(y\delta) = \ell_1 \cdots \ell_{k-3}\overline{a}\omega.$$

iv. $y = \ell_1 \cdots \ell_{k-4} ab\overline{a}\overline{b}c$, $\ell_k = \overline{b}$, $\ell_{k-1} = \overline{a}$, $\ell_{k-2} = b$, $\ell_{k-3} = a$, $\ell_{k-4} \neq \overline{d}$. In this case

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-4}dcx) - s_0(\ell_1 \cdots \ell_{k-4} ab\overline{a}\overline{b}cx),$$

notice that $\ell_1 \cdots \ell_{k-4}dc$ is in normal form since $\ell_{k-4} \neq \overline{a}, c$, then

$$\begin{aligned} y\delta - s_0 d_1(y\delta) &= \ell_1 \cdots \ell_{k-4} \left[(1 - ab\overline{a})\alpha + (a - ab\overline{a}\overline{b})\beta + (ab\overline{a}\overline{b} - d)\gamma + (ab\overline{a}\overline{b}c - 1)\delta\right] \\ &= d_2(\ell_1 \cdots \ell_{k-4}\omega) \\ &= d_2 s_1(y\delta), \end{aligned}$$

since

$$s_1(y\delta) = \ell_1 \cdots \ell_{k-4}\omega.$$

v. $y = \ell_1 \cdots \ell_{k-5}\overline{d}ab\overline{a}\overline{b}c$, $\ell_k = \overline{b}$, $\ell_{k-1} = \overline{a}$, $\ell_{k-2} = b$, $\ell_{k-3} = a$, $\ell_{k-4} = \overline{d}$. In this case

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-5}cx) - s_0(\ell_1 \cdots \ell_{k-5}\overline{d}ab\overline{a}\overline{b}cx),$$

notice that $\ell_1 \cdots \ell_{k-5}c$ is in normal form since $\ell_{k-5} \neq c$, then

$$\begin{aligned} y\delta - s_0 d_1(y\delta) &= \ell_1 \cdots \ell_{k-5} \left[(\overline{d} - \overline{d}ab\overline{a})\alpha + (\overline{d}a - \overline{d}ab\overline{a}\overline{b})\beta + (\overline{d}ab\overline{a}\overline{b} - 1)\gamma + (\overline{d}ab\overline{a}\overline{b}c - \overline{d})\delta\right] \\ &= d_2(\ell_1 \cdots \ell_{k-5}\overline{d}\omega) \\ &= d_2 s_1(y\delta), \end{aligned}$$

since

$$s_1(y\delta) = \ell_1 \cdots \ell_{k-5}\overline{d}\omega.$$

This completes the verification of case 1 when $n = 0$, now we proceed with the corresponding verification for $n \geq 1$.
If $y = \ell_1 \cdots \ell_k c(ab\overline{a}\overline{b}c)^n$ is in normal form, then

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_k ba\overline{b}\overline{a}dc^{n+1}x) - s_0(\ell_1 \cdots \ell_k(cab\overline{a}\overline{b})^n cx).$$

Notice that $\ell_k \neq \overline{c}$ since x is in normal form, and $\ell_{k-4}\ell_{k-3}\ell_{k-2}\ell_{k-1}\ell_k \neq cab\overline{a}\overline{b}$ since y does not end as T_{n+1}.
In order to check the desired relation, we have to consider whether $\ell_1 \cdots \ell_k ba\overline{b}\overline{a}dc^{n+1}$ is in normal form or not.

(a) If $\ell_1 \cdots \ell_k ba\overline{b}\overline{a}dc^{n+1}$ is in normal form, then

$$\begin{aligned} y\delta - s_0 d_1(y\delta) = \ell_1 \cdots \ell_k \Bigg\{ & \left[\sum_{i=0}^{n-1} \left((cab\overline{a}\overline{b})^i c - (cab\overline{a}\overline{b})^i cab\overline{a} \right) - b + ba\overline{b}\overline{a} \right] \alpha + \\ & \left[\sum_{i=1}^{n} \left((cab\overline{a}\overline{b})^{i-1} ca - (cab\overline{a}\overline{b})^i \right) - 1 + ba\overline{b} \right] \beta + \\ & \left[\sum_{i=0}^{n} \left((cab\overline{a}\overline{b})^i - (ba\overline{b}\overline{a}dc^i) \right) \right] \gamma + \left[c(ab\overline{a}\overline{b}c)^n - ba\overline{b}\overline{a} \right] \delta \Bigg\} \\ = d_2 & \left(\ell_1 \cdots \ell_k \left[\sum_{i=0}^{n-1} (cab\overline{a}\overline{b})^i c + ba\overline{b}\overline{a} \right] \omega \right), \end{aligned}$$

which agrees with $d_2 s_1(y\delta)$ since

$$\begin{aligned} \ell_1 \cdots \ell_k \left(\sum_{i=0}^{n-1} \left((cab\overline{a}\overline{b})^i c \right) + ba\overline{b}\overline{a} \right) \omega &= y \left(\sum_{i=0}^{n-1} \left((ab\overline{a}\overline{b}c)^{-n+i} \right) + (\overline{c}ba\overline{b}\overline{a})^{n+1} \right) \omega \\ &= y \left(\sum_{i=1}^{n} \left((\overline{c}ba\overline{b}\overline{a})^i \right) + (\overline{c}ba\overline{b}\overline{a})^{n+1} \right) \omega \\ &= y \left(\sum_{i=1}^{n+1} U_i \right) \omega \\ &= s_1(y\delta). \end{aligned}$$

(b) If $\ell_1 \cdots \ell_k ba\overline{b}\overline{a}dc^{n+1}$ is not in normal form:

i. $y = \ell_1 \cdots \ell_{k-1} \overline{b}c(ab\overline{a}\overline{b}c)^n$, $\ell_k = \overline{b}$, $\ell_{k-1} \neq \overline{a}$. In this case

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-1} a\overline{b}\overline{a}dc^{n+1}x) - s_0(\ell_1 \cdots \ell_{k-1} \overline{b}(cab\overline{a}\overline{b})^n cx),$$

notice that $\ell_1 \cdots \ell_{k-1} a\overline{b}\overline{a}dc^{n+1}$ is in normal form since $\ell_{k-1} \neq b$, then

$$\begin{aligned} y\delta - s_0 d_1(y\delta) = \ell_1 \cdots \ell_{k-1} \Bigg\{ & \left[\sum_{i=0}^{n-1} \left((\overline{b}cab\overline{a})^i \overline{b}c - (\overline{b}cab\overline{a})^i \right) + a\overline{b}\overline{a} - (\overline{b}cab\overline{a})^n \right] \alpha + \\ & \left[\sum_{i=0}^{n-1} \left((\overline{b}cab\overline{a})^i \overline{b}ca - (\overline{b}cab\overline{a})^i \overline{b} \right) + a\overline{b} - (\overline{b}cab\overline{a})^n \overline{b} \right] \beta + \\ & \left[\sum_{i=0}^{n} \left((\overline{b}cab\overline{a})^i \overline{b} - (a\overline{b}\overline{a}dc^i) \right) \right] \gamma + \left[\overline{b}c(ab\overline{a}\overline{b}c)^n - a\overline{b}\overline{a} \right] \delta \Bigg\} \\ = d_2 & \left(\ell_1 \cdots \ell_{k-1} \left[\sum_{i=0}^{n-1} \overline{b}c(ab\overline{a}\overline{b}c)^i + a\overline{b}\overline{a} \right] \omega \right) \\ = d_2 & s_1(y\delta), \end{aligned}$$

since

$$\begin{aligned}\ell_1\cdots\ell_{k-1}\left(\sum_{i=0}^{n-1}\left(\overline{b}c(ab\overline{a}\overline{b}c)^i\right)+a\overline{b}\overline{a}\right)\omega &= y\left(\sum_{i=0}^{n-1}\left((ab\overline{a}\overline{b}c)^{i-n}\right)+(ab\overline{a}\overline{b}c)^{-n}\overline{c}ba\overline{b}\overline{a}\right)\omega\\ &= y\left(\sum_{i=1}^{n}\left((\overline{c}ba\overline{b}\overline{a})^i\right)+(\overline{c}ba\overline{b}\overline{a})^{n+1}\right)\omega\\ &= y\left(\sum_{i=1}^{n+1}U_i\right)\omega\\ &= s_1(y\delta).\end{aligned}$$

ii. $y=\ell_1\cdots\ell_{k-2}\overline{a}\overline{b}c(ab\overline{a}\overline{b}c)^n$, $\ell_k=\overline{b}$, $\ell_{k-1}=\overline{a}$, $\ell_{k-2}\neq b$. In this case

$$s_0d_1(y\delta)=s_0(\ell_1\cdots\ell_{k-2}\overline{b}\overline{a}dc^{n+1}x)-s_0(\ell_1\cdots\ell_{k-2}\overline{a}\overline{b}(cab\overline{a}\overline{b})^ncx),$$

notice that $\ell_1\cdots\ell_{k-2}\overline{b}\overline{a}dc^{n+1}$ is in normal form since $\ell_{k-2}\neq a$, then

$$\begin{aligned}y\delta-s_0d_1(y\delta)=\ell_1\cdots\ell_{k-2}\Bigg\{&\left[\sum_{i=0}^{n-1}\left((\overline{a}\overline{b}cab)^i\overline{a}\overline{b}c-(\overline{a}\overline{b}cab)^i\overline{a}\right)+\overline{b}\overline{a}-(\overline{a}\overline{b}cab)^n\overline{a}\right]\alpha+\\ &\left[\sum_{i=0}^{n-1}\left((\overline{a}\overline{b}cab)^i\overline{a}\overline{b}ca-(\overline{a}\overline{b}cab)^i\overline{a}\overline{b}\right)+\overline{b}-(\overline{a}\overline{b}cab)^n\overline{a}\overline{b}\right]\beta+\\ &\left[\sum_{i=0}^{n}\left((\overline{a}\overline{b}cab)^i\overline{a}\overline{b}-\overline{b}\overline{a}dc^i\right)\right]\gamma+\left[\overline{a}\overline{b}c(ab\overline{a}\overline{b}c)^n-\overline{b}\overline{a}\right]\delta\Bigg\}\\ &= d_2\left(\ell_1\cdots\ell_{k-2}\left[\sum_{i=0}^{n-1}\overline{a}\overline{b}c(ab\overline{a}\overline{b}c)^i+\overline{b}\overline{a}\right]\omega\right)\\ &= d_2s_1(y\delta),\end{aligned}$$

since

$$\begin{aligned}\ell_1\cdots\ell_{k-2}\left(\sum_{i=0}^{n-1}\left(\overline{a}\overline{b}c(ab\overline{a}\overline{b}c)^i\right)+\overline{b}\overline{a}\right)\omega &= y\left(\sum_{i=0}^{n-1}\left((ab\overline{a}\overline{b}c)^{-n}(ab\overline{a}\overline{b}c)^i\right)+(ab\overline{a}\overline{b}c)^{-n}\overline{c}ba\overline{b}\overline{a}\right)\omega\\ &= y\left(\sum_{i=0}^{n-1}\left((ab\overline{a}\overline{b}c)^{-n+i}\right)+(\overline{c}ba\overline{b}\overline{a})^{n+1}\right)\omega\\ &= y\left(\sum_{i=1}^{n+1}U_i\right)\omega\\ &= s_1(y\delta).\end{aligned}$$

iii. $y=\ell_1\cdots\ell_{k-3}b\overline{a}\overline{b}c(ab\overline{a}\overline{b}c)^n$, $\ell_k=\overline{b}$, $\ell_{k-1}=\overline{a}$, $\ell_{k-2}=b$, $\ell_{k-3}\neq a$. In this case we have

$$s_0d_1(y\delta)=s_0(\ell_1\cdots\ell_{k-3}\overline{a}dc^{n+1}x)-s_0(\ell_1\cdots\ell_{k-3}b\overline{a}\overline{b}(cab\overline{a}\overline{b})^ncx),$$

notice that $\ell_1 \cdots \ell_{k-3}\bar{a}dc^{n+1}$ is in normal form since $\ell_{k-3} \neq \bar{b}$, then

$$\begin{aligned}
y\delta - s_0 d_1(y\delta) = \ell_1 \cdots \ell_{k-3} \Bigg\{ & \left[\sum_{i=0}^{n-1} \Big((b\bar{a}\bar{b}ca)^i b\bar{a}\bar{b}c - (b\bar{a}\bar{b}ca)^i b\bar{a} \Big) + \bar{a} - (b\bar{a}\bar{b}ca)^n b\bar{a} \right] \alpha + \\
& \left[\sum_{i=0}^{n} \Big((b\bar{a}\bar{b}ca)^i - (b\bar{a}\bar{b}ca)^i b\bar{a}\bar{b} \Big) \right] \beta + \left[\sum_{i=0}^{n} \Big((b\bar{a}\bar{b}ca)^i b\bar{a}\bar{b} - (\bar{a}dc^i) \Big) \right] \gamma + \\
& \Big[b\bar{a}\bar{b}c(ab\bar{a}\bar{b}c)^n - \bar{a} \Big] \delta \Bigg\} \\
= {} & d_2 \left(\ell_1 \cdots \ell_{k-3} \left[\sum_{i=0}^{n-1} \Big(b\bar{a}\bar{b}c(ab\bar{a}\bar{b}c)^i \Big) + \bar{a} \right] \omega \right) \\
= {} & d_2 s_1(y\delta),
\end{aligned}$$

since

$$\begin{aligned}
\ell_1 \cdots \ell_{k-3} \left(\sum_{i=0}^{n-1} \Big(b\bar{a}\bar{b}c(ab\bar{a}\bar{b}c)^i \Big) + \bar{a} \right) \omega &= y \left(\sum_{i=0}^{n-1} \Big((ab\bar{a}\bar{b}c)^{-n} (ab\bar{a}\bar{b}c)^i \Big) + (ab\bar{a}\bar{b}c)^{-n} \bar{c}ba\bar{b}\bar{a} \right) \omega \\
&= y \left(\sum_{i=1}^{n} \Big((\bar{c}ba\bar{b}\bar{a})^i \Big) + (\bar{c}ba\bar{b}\bar{a})^{n+1} \right) \omega \\
&= y \left(\sum_{i=1}^{n+1} U_i \right) \omega \\
&= s_1(y\delta).
\end{aligned}$$

iv. $y = \ell_1 \cdots \ell_{k-4}(ab\bar{a}\bar{b}c)^{n+1}$, $\ell_k = \bar{b}$, $\ell_{k-1} = \bar{a}$, $\ell_{k-2} = b$, $\ell_{k-3} = a$, $\ell_{k-4} \neq \bar{d}$. In this case we have

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-4} dc^{n+1} x) - s_0(\ell_1 \cdots \ell_{k-4} ab\bar{a}\bar{b}(cab\bar{a}\bar{b})^n cx),$$

notice that $\ell_1 \cdots \ell_{k-4}dc^{n+1}$ is in normal form since $\ell_{k-4} \neq \bar{a}, c$, then

$$\begin{aligned}
y\delta - s_0 d_1(y\delta) = \ell_1 \cdots \ell_{k-4} \Bigg\{ & \left[\sum_{i=0}^{n} \Big((ab\bar{a}\bar{b}c)^i - (ab\bar{a}\bar{b}c)^i ab\bar{a} \Big) \right] \alpha + \\
& \left[\sum_{i=0}^{n} \Big((ab\bar{a}\bar{b}c)^i a - (ab\bar{a}\bar{b}c)^i ab\bar{a}\bar{b} \Big) \right] \beta + \\
& \left[\sum_{i=0}^{n} \Big((ab\bar{a}\bar{b}c)^i ab\bar{a}\bar{b} - dc^i \Big) \right] \gamma + \Big[(ab\bar{a}\bar{b}c)^{n+1} - 1 \Big] \delta \Bigg\} \\
= {} & d_2 \left(\ell_1 \cdots \ell_{k-4} \left[\sum_{i=0}^{n} (ab\bar{a}\bar{b}c)^i \right] \omega \right) \\
= {} & d_2 s_1(y\delta),
\end{aligned}$$

since

$$\begin{aligned}
\ell_1 \cdots \ell_{k-4}\left(\sum_{i=0}^{n}(ab\overline{a}\overline{b}c)^i\omega\right) &= \ell_1 \cdots \ell_{k-4}(ab\overline{a}\overline{b}c)^{n+1}\left(\sum_{i=0}^{n}(ab\overline{a}\overline{b}c)^{-n-1}(ab\overline{a}\overline{b}c)^i\right)\omega \\
&= y\left(\sum_{i=0}^{n}(ab\overline{a}\overline{b}c)^{-n-1+i}\right)\omega \\
&= y\left(\sum_{i=1}^{n+1}(\overline{c}ba\overline{b}\overline{a})^i\right)\omega \\
&= y\left(\sum_{i=1}^{n+1}U_i\right)\omega \\
&= s_1(y\delta).
\end{aligned}$$

v. $y = \ell_1 \cdots \ell_{k-5}\overline{d}(ab\overline{a}\overline{b}c)^{n+1}$, $\ell_k = \overline{b}$, $\ell_{k-1} = \overline{a}$, $\ell_{k-2} = b$, $\ell_{k-3} = a$, $\ell_{k-4} = \overline{d}$. In this case we have

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_{k-5}c^{n+1}x) - s_0(\ell_1 \cdots \ell_{k-5}\overline{d}ab\overline{a}\overline{b}(cab\overline{a}\overline{b})^n cx),$$

notice that $\ell_1 \cdots \ell_{k-5}c^{n+1}$ is in normal form since $\ell_{k-5} \neq \overline{c}$, then

$$\begin{aligned}
y\delta - s_0 d_1(y\delta) = \ell_1 \cdots \ell_{k-5}\Bigg\{ &\left[\sum_{i=0}^{n}\left(\overline{d}(ab\overline{a}\overline{b}c)^i - \overline{d}(ab\overline{a}\overline{b}c)^i ab\overline{a}\right)\right]\alpha + \\
&\left[\sum_{i=0}^{n}\left(\overline{d}(ab\overline{a}\overline{b}c)^i a - \overline{d}(ab\overline{a}\overline{b}c)^i ab\overline{a}\overline{b}\right)\right]\beta + \\
&\left[\sum_{i=0}^{n}\left(\overline{d}(ab\overline{a}\overline{b}c)^i ab\overline{a}\overline{b} - c^i\right)\right]\gamma + \left[\overline{d}(ab\overline{a}\overline{b}c)^{n+1} - \overline{d}\right]\delta\Bigg\} \\
= d_2&\left(\ell_1 \cdots \ell_{k-5}\left[\sum_{i=0}^{n}\overline{d}(ab\overline{a}\overline{b}c)^i\right]\omega\right) \\
= d_2&s_1(y\delta),
\end{aligned}$$

since

$$\begin{aligned}
\ell_1 \cdots \ell_{k-5}\left(\sum_{i=0}^{n}\overline{d}(ab\overline{a}\overline{b}c)^i\right)\omega &= \ell_1 \cdots \ell_{k-5}\overline{d}(ab\overline{a}\overline{b}c)^{n+1}\left(\sum_{i=0}^{n}(ab\overline{a}\overline{b}c)^{-n-1}(ab\overline{a}\overline{b}c)^i\right)\omega \\
&= y\left(\sum_{i=0}^{n}(ab\overline{a}\overline{b}c)^{-n-1+i}\right)\omega \\
&= y\left(\sum_{i=1}^{n+1}(\overline{c}ba\overline{b}\overline{a})^i\right)\omega \\
&= y\left(\sum_{i=1}^{n+1}U_i\right)\omega \\
&= s_1(y\delta).
\end{aligned}$$

Case 2: y ends as U_{n+1} but not in U_{n+2}, $n \geq 0$. In this case the details are similar but less elaborate than those in case 1, therefore we just indicate the essential steps to reconstruct the whole argument.

Suppose $y = \ell_1 \cdots \ell_k(\overline{c}ba\overline{b}\overline{a})^{n+1}$ is in normal form and $\ell_1 \cdots \ell_k$ does not end as $\overline{c}ba\overline{b}\overline{a}$.

(a) If $\ell_1 \cdots \ell_k$ does not end as $\overline{d}$:

$$\begin{aligned}
y\delta - s_0 d_1(y\delta) &= \ell_1 \cdots \ell_k \Bigg\{ \left[\sum_{i=0}^{n} \left((\bar{c}ba\overline{b}\overline{a})^i \bar{c}b - (\bar{c}ba\overline{b}\overline{a})^{i+1} \right) \right] \alpha + \\
&\left[\sum_{i=0}^{n} (\bar{c}ba\overline{b}\overline{a})^i \bar{c} - (\bar{c}ba\overline{b}\overline{a})^i \bar{c}ba\overline{b} \right] \beta + \\
&\left[\sum_{i=1}^{n+1} dc^{-i} - \sum_{i=0}^{n} (\bar{c}ba\overline{b}\overline{a})^i \bar{c} \right] \gamma + \left[(\bar{c}ba\overline{b}\overline{a})^{n+1} - 1 \right] \delta \Bigg\} \\
&= d_2 \left(-\ell_1 \cdots \ell_k \sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^i \omega \right) \\
&= d_2 s_1(y\delta),
\end{aligned}$$

since

$$\begin{aligned}
-\ell_1 \cdots \ell_k \left(\sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^i \right) \omega &= -\ell_1 \cdots \ell_k (\bar{c}ba\overline{b}\overline{a})^{n+1} \left(\sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^{-n-1+i} \right) \omega \\
&= -y \left(\sum_{i=0}^{n} (ab\overline{a}\overline{b}c)^i \right) \omega \\
&= -y \left(\sum_{i=0}^{n} \overline{U_i} \right) \omega \\
&= s_1(y\delta).
\end{aligned}$$

(b) If $\ell_1 \cdots \ell_k$ ends as $\overline{d}$:

$$\begin{aligned}
y\delta - s_0 d_1(y\delta) &= \ell_1 \cdots \ell_{k-1} \Bigg\{ \left[\sum_{i=0}^{n} \overline{d} (\bar{c}ba\overline{b}\overline{a})^i \bar{c}b - \sum_{i=1}^{n+1} \overline{d} (\bar{c}ba\overline{b}\overline{a})^i \right] \alpha + \\
&\left[\sum_{i=0}^{n} \left(\overline{d} (\bar{c}ba\overline{b}\overline{a})^i \bar{c} - \overline{d} (\bar{c}ba\overline{b}\overline{a})^i \bar{c}ba\overline{b} \right) \right] \beta + \\
&\left[\sum_{i=1}^{n+1} c^{-i} - \sum_{i=0}^{n} \overline{d} (\bar{c}ba\overline{b}\overline{a})^i \bar{c} \right] \gamma + \left[\overline{d} (\bar{c}ba\overline{b}\overline{a})^{n+1} - \overline{d} \right] \delta \Bigg\} \\
&= d_2 \left(-\ell_1 \cdots \ell_{k-1} \overline{d} \sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^i \omega \right) \\
&= d_2 s_1(y\delta),
\end{aligned}$$

since

$$\begin{aligned}
-\ell_1 \cdots \ell_{k-1} \overline{d} \left(\sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^i \right) \omega &= -\ell_1 \cdots \ell_k (\bar{c}ba\overline{b}\overline{a})^{n+1} \left(\sum_{i=1}^{n+1} (\bar{c}ba\overline{b}\overline{a})^{-n-1+i} \right) \omega \\
&= -y \left(\sum_{i=0}^{n} (ab\overline{a}\overline{b}c)^i \right) \omega \\
&= -y \left(\sum_{i=0}^{n} \overline{U_i} \right) \omega \\
&= s_1(y\delta).
\end{aligned}$$

Case 3: $y = \ell_1 \cdots \ell_k$ is in normal form and does not end as the words mentioned in the previous cases. Then

$$s_0 d_1(y\delta) = s_0(\ell_1 \cdots \ell_k dx) - s_0(\ell_1 \cdots \ell_k x).$$

If $\ell_1 \cdots \ell_k d$ is in normal form, we have

$$s_0(d_1(y\delta)) = s_0(\ell_1 \cdots \ell_k dx) - s_0(\ell_1 \cdots \ell_k x) = \ell_1 \cdots \ell_k \delta = y\delta = y\delta - d_2 s_1(y\delta),$$

since $s_1(y\delta) = 0$.

Otherwise, $\ell_k = \overline{d}$, and so $\ell_1 \cdots \ell_k d = \ell_1 \cdots \ell_{k-1}$, which is in normal form. We thus have

$$\begin{aligned} s_0(d_1(y\delta)) &= -\ell_1 \cdots \ell_{k-1} s_0(\overline{d}x) \\ &= \ell_1 \cdots \ell_{k-1} \overline{d}\delta \\ &= y\delta \\ &= y\delta - d_2 s_1(y\delta), \end{aligned}$$

since $s_1(y\delta) = 0$.

Finally, notice that for $y \in \pi_2$, $s_1 d_2(y\omega) = \omega$ by the above, i.e. $s_1 d_2 = \mathrm{id}_{M_2}$. In fact,

$$d_2(s_1 d_2 - \mathrm{id}_{M_2}) = (d_2 s_1) d_2 - d_2 = (\mathrm{id}_{M_2} - s_0 d_1) d_2 - d_2 = d_2 - d_2 = 0,$$

so that $s_1 d_2 = \mathrm{id}_{M_2}$, as d_2 is injective.

Next we describe diagonal approximations $\psi : \mathbf{M} \to \mathbf{M}^{\otimes 2}$ and $\varphi : \mathbf{M}^{\otimes 2} \to \mathbf{M}^{\otimes 4}$. The formulas we present below are obtained by virtue of propositions 2.5 and 2.6, and since ψ and φ are meant to be, respectively, π_2 and $(\pi_2 \times \pi_2)$-homomorphisms, it is enough to define them on the corresponding basic elements (all the formulas were computed by hand and verified on the computer, except for the formula for $\varphi_4(\omega \otimes \omega)$, which was obtained with the help of a computer). In fact,

$$\psi_0(x) \quad = x \otimes x.$$

$$\psi_1(\lambda) \quad = \lambda \otimes \ell x + x \otimes \lambda, \text{ for } (\lambda, \ell) \in \{(\alpha, a), (\beta, b), (\gamma, c), (\delta, d)\}.$$

$$\begin{aligned} \psi_2(\omega) \quad = &\ ((-1 + ab\overline{a})\alpha + (-a + ab\overline{a}\overline{b})\beta) \otimes ab\overline{a}\overline{b}\beta + \alpha \otimes a\beta + \omega \otimes dcx \\ &+ ((1 - ab\overline{a})\alpha + (a - ab\overline{a}\overline{b})\beta + (ab\overline{a}\overline{b}\gamma) \otimes ab\overline{a}\overline{b}c\delta + x \otimes \omega \\ &+ ((-1 + ab\overline{a})\alpha - a\beta) \otimes ab\overline{a}\alpha + ((1 - ab\overline{a})\alpha + (a - ab\overline{a}\overline{b})\beta) \otimes ab\overline{a}\overline{b}\gamma \\ &- \delta \otimes d\gamma. \end{aligned}$$

$$\varphi_0(x \otimes x) = x \otimes x \otimes x \otimes x.$$

$$\varphi_1(\lambda \otimes x) = \lambda \otimes x \otimes \ell x \otimes x + x \otimes x \otimes \lambda \otimes x, \text{ for } (\lambda, \ell) \in \{(\alpha, a), (\beta, b), (\gamma, c), (\delta, d)\}.$$

$$\varphi_1(x \otimes \lambda) = x \otimes \lambda \otimes x \otimes \ell x + x \otimes x \otimes x \otimes \lambda, \text{ for } (\lambda, \ell) \in \{(\alpha, a), (\beta, b), (\gamma, c), (\delta, d)\}.$$

$$\begin{aligned}
\varphi_2(\omega \otimes x) = {} & ((-1+ab\overline{a})\alpha - a\beta) \otimes x \otimes ab\overline{a}\alpha \otimes x + \alpha \otimes x \otimes a\beta \otimes x \\
& + ((-1+ab\overline{a})\alpha + (-a+ab\overline{a}\overline{b})\beta) \otimes x \otimes ab\overline{a}\overline{b}\beta \otimes x \\
& + ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes x \otimes ab\overline{a}\overline{b}\gamma \otimes x \\
& - \delta \otimes x \otimes d\gamma \otimes x + \omega \otimes x \otimes dcx \otimes x \\
& + ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma) \otimes x \otimes ab\overline{a}\overline{b}c\delta \otimes x \\
& + x \otimes x \otimes \omega \otimes x.
\end{aligned}$$

$$\begin{aligned}
\varphi_2(x \otimes \omega) = {} & x \otimes ((-1+ab\overline{a})\alpha - a\beta) \otimes x \otimes ab\overline{a}\alpha + x \otimes \alpha \otimes x \otimes a\beta \\
& + x \otimes ((-1+ab\overline{a})\alpha + (-a+ab\overline{a}\overline{b})\beta) \otimes x \otimes ab\overline{a}\overline{b}\beta \\
& + x \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes x \otimes ab\overline{a}\overline{b}\gamma \\
& - x \otimes \delta \otimes x \otimes d\gamma \quad + x \otimes \omega \otimes x \otimes dcx \\
& + x \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma) \otimes x \otimes ab\overline{a}\overline{b}c\delta \\
& + x \otimes x \otimes x \otimes \omega.
\end{aligned}$$

$$\begin{aligned}
\varphi_2(\lambda_1 \otimes \lambda_2) = {} & \lambda_1 \otimes \lambda_2 \otimes \ell_1 x \otimes \ell_2 x + \lambda_1 \otimes x \otimes \ell_1 x \otimes \lambda_2 \\
& + x \otimes x \otimes \lambda_1 \otimes \lambda_2 - x \otimes \lambda_2 \otimes \lambda_1 \otimes \ell_2 x, \text{ for } (\lambda_i, \ell_i) \text{ in } \{(\alpha,a),(\beta,b),(\gamma,c),(\delta,d)\}, i \in \{0,1\}.
\end{aligned}$$

$$\begin{aligned}
\varphi_3(\lambda \otimes \omega) = {} & \lambda \otimes ((-1+ab\overline{a})\alpha - a\beta) \otimes \ell x \otimes ab\overline{a}\alpha + \lambda \otimes \alpha \otimes \ell x \otimes a\beta \\
& + \lambda \otimes ((-1+ab\overline{a})\alpha + (-a+ab\overline{a}\overline{b})\beta \otimes \ell x \otimes ab\overline{a}\overline{b}\beta \\
& + \lambda \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes \ell x \otimes ab\overline{a}\overline{b}\gamma \\
& - \lambda \otimes \delta \otimes \ell x \otimes d\gamma + \lambda \otimes \omega \otimes \ell x \otimes dcx \\
& + \lambda \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma) \otimes \ell x \otimes ab\overline{a}\overline{b}c\delta \\
& + \lambda \otimes x \otimes \ell x \otimes \omega + x \otimes x \otimes \lambda \otimes \omega - x \otimes ((-1+ab\overline{a})\alpha - a\beta) \otimes \lambda \otimes ab\overline{a}\alpha \\
& - x \otimes \alpha \otimes \lambda \otimes a\beta - x \otimes ((-1+ab\overline{a})\alpha + (-a+ab\overline{a}\overline{b})\beta) \otimes \lambda \otimes ab\overline{a}\overline{b}\beta \\
& - x \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes \lambda \otimes ab\overline{a}\overline{b}\gamma + x \otimes \delta \otimes \lambda \otimes d\gamma \\
& - x \otimes ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma) \otimes \lambda \otimes ab\overline{a}\overline{b}c\delta \\
& + x \otimes \omega \otimes \lambda \otimes dcx, \text{ for } (\lambda,\ell) \in \{(\alpha,a),(\beta,b),(\gamma,c),(\delta,d)\}.
\end{aligned}$$

$$\begin{aligned}
\varphi_3(\omega \otimes \lambda) = {} & ((ab\overline{a}-1)\alpha - a\beta) \otimes x \otimes ab\overline{a}\alpha \otimes \lambda \\
& + ((1-ab\overline{a})\alpha + a\beta) \otimes \lambda \otimes ab\overline{a}\alpha \otimes \ell x \\
& + \alpha \otimes x \otimes a\beta \otimes \lambda - \alpha \otimes \lambda \otimes a\beta \otimes \ell x \\
& + ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta) \otimes x \otimes ab\overline{a}\overline{b}\beta \otimes \lambda \\
& + ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes \lambda \otimes ab\overline{a}\overline{b}\beta \otimes \ell x \\
& + ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta) \otimes x \otimes ab\overline{a}\overline{b}\gamma \otimes \lambda \\
& + ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta) \otimes \lambda \otimes ab\overline{a}\overline{b}\gamma \otimes \ell x \\
& - \delta \otimes x \otimes d\gamma \otimes \lambda + \delta \otimes \lambda \otimes d\gamma \otimes \ell x \\
& + \omega \otimes \lambda \otimes dcx \otimes \ell x + \omega \otimes x \otimes dcx \otimes \lambda \\
& + ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma) \otimes x \otimes ab\overline{a}\overline{b}c\delta \otimes \lambda \\
& + x \otimes x \otimes \omega \otimes \lambda + ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta - ab\overline{a}\overline{b}\gamma) \otimes \lambda \otimes ab\overline{a}\overline{b}c\delta \otimes \ell x \\
& + x \otimes \lambda \otimes \omega \otimes \ell x, \text{ for } (\lambda,\ell) \in \{(\alpha,a),(\beta,b),(\gamma,c),(\delta,d)\}.
\end{aligned}$$

$$
\begin{aligned}
\varphi_4(\omega\otimes\omega) =& ((ab\overline{a}-1)\alpha - a\beta)\otimes x\otimes ab\overline{a}\alpha\otimes\omega + ((1-ab\overline{a})\alpha + a\beta)\otimes((ab\overline{a}-1)\alpha - a\beta)\otimes ab\overline{a}\alpha\otimes ab\overline{a}\alpha \\
&+ ((1-ab\overline{a})\alpha + a\beta)\otimes\alpha\otimes ab\overline{a}\alpha\otimes a\beta \\
&+ ((1-ab\overline{a})\alpha + a\beta)\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes ab\overline{a}\alpha\otimes ab\overline{a}\overline{b}\beta \\
&+ ((1-ab\overline{a})\alpha + a\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes ab\overline{a}\alpha\otimes ab\overline{a}\overline{b}\gamma \\
&- ((1-ab\overline{a})\alpha + a\beta)\otimes\delta\otimes ab\overline{a}\alpha\otimes d\gamma \\
&+ ((1-ab\overline{a})\alpha + a\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma)\otimes ab\overline{a}\alpha\otimes ab\overline{a}\overline{b}c\delta \\
&+ ((ab\overline{a}-1)\alpha - a\beta)\otimes\omega\otimes ab\overline{a}\alpha\otimes dcx \\
&+ \alpha\otimes x\otimes a\beta\otimes\omega - \alpha\otimes((ab\overline{a}-1)\alpha - a\beta)\otimes a\beta\otimes ab\overline{a}\alpha \\
&- \alpha\otimes\alpha\otimes a\beta\otimes a\beta - \alpha\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes a\beta\otimes ab\overline{a}\overline{b}\beta \\
&- \alpha\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes a\beta\otimes ab\overline{a}\overline{b}\gamma \\
&+ \alpha\otimes\delta\otimes a\beta\otimes d\gamma - \alpha\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + (ab\overline{a}\overline{b})\gamma)\otimes a\beta\otimes ab\overline{a}\overline{b}c\delta \\
&+ \alpha\otimes\omega\otimes a\beta\otimes dcx + ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes x\otimes ab\overline{a}\overline{b}\beta\otimes\omega \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes((ab\overline{a}-1)\alpha - a\beta)\otimes ab\overline{a}\overline{b}\beta\otimes ab\overline{a}\alpha \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes\alpha\otimes ab\overline{a}\overline{b}\beta\otimes a\beta \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes ab\overline{a}\overline{b}\beta\otimes ab\overline{a}\overline{b}\beta \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes ab\overline{a}\overline{b}\beta\otimes ab\overline{a}\overline{b}\gamma \\
&- ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes\delta\otimes ab\overline{a}\overline{b}\beta\otimes d\gamma \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + (ab\overline{a}\overline{b})\gamma)\otimes ab\overline{a}\overline{b}\beta\otimes ab\overline{a}\overline{b}c\delta \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes\omega\otimes ab\overline{a}\overline{b}\beta\otimes dcx \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes x\otimes ab\overline{a}\overline{b}\gamma\otimes\omega \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes(ab\overline{a}-1)\alpha - a\beta)\otimes ab\overline{a}\overline{b}\gamma\otimes ab\overline{a}\alpha \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes\alpha\otimes ab\overline{a}\overline{b}\gamma\otimes a\beta \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes ab\overline{a}\overline{b}\gamma\otimes ab\overline{a}\overline{b}\beta \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes ab\overline{a}\overline{b}\gamma\otimes ab\overline{a}\overline{b}\gamma \\
&- ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes\delta\otimes ab\overline{a}\overline{b}\gamma\otimes d\gamma \\
&+ ((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma)\otimes ab\overline{a}\overline{b}\gamma\otimes ab\overline{a}\overline{b}c\delta \\
&+ ((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes\omega\otimes ab\overline{a}\overline{b}\gamma\otimes dcx \\
&- \delta\otimes x\otimes d\gamma\otimes\omega + \delta\otimes((ab\overline{a}-1)\alpha - a\beta)\otimes d\gamma\otimes ab\overline{a}\alpha \\
&+ \delta\otimes\alpha\otimes d\gamma\otimes a\beta + \delta\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes d\gamma\otimes ab\overline{a}\overline{b}\beta \\
&+ \delta\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes d\gamma\otimes ab\overline{a}\overline{b}\gamma - \delta\otimes\delta\otimes d\gamma\otimes d\gamma \\
&+ \delta\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta + ab\overline{a}\overline{b}\gamma)\otimes d\gamma\otimes ab\overline{a}\overline{b}c\delta \\
&- \delta\otimes\omega\otimes d\gamma\otimes dcx + \omega\otimes((ab\overline{a}-1)\alpha - a\beta)\otimes dcx\otimes ab\overline{a}\alpha \\
&+ \omega\otimes\alpha\otimes dcx\otimes a\beta + \omega\otimes((ab\overline{a}-1)\alpha + (ab\overline{a}\overline{b}-a)\beta)\otimes dcx\otimes ab\overline{a}\overline{b}\beta \\
&+ \omega\otimes((1-ab\overline{a})\alpha + (a-ab\overline{a}\overline{b})\beta)\otimes dcx\otimes ab\overline{a}\overline{b}\gamma \\
&- \omega\otimes\delta\otimes dcx\otimes d\gamma + \omega\otimes\omega\otimes dcx\otimes dcx
\end{aligned}
$$

$$
\begin{aligned}
&+ \omega \otimes ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + ab\bar{a}\bar{b}\gamma) \otimes dcx \otimes ab\bar{a}\bar{b}c\delta \\
&+ \omega \otimes x \otimes dcx \otimes \omega + ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + ab\bar{a}\bar{b}\gamma) \otimes x \otimes ab\bar{a}\bar{b}c\delta \otimes \omega \\
&+ x \otimes x \otimes \omega \otimes \omega + ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes \alpha \otimes ab\bar{a}\bar{b}c\delta \otimes a\beta \\
&+ ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes ((ab\bar{a} - 1)\alpha - a\beta) \otimes ab\bar{a}\bar{b}c\delta \otimes ab\bar{a}\alpha \\
&+ ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta) \otimes ab\bar{a}\bar{b}c\delta \otimes ab\bar{a}\bar{b}\beta \\
&+ ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta) \otimes ab\bar{a}\bar{b}c\delta \otimes ab\bar{a}\bar{b}\gamma \\
&- ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes \delta \otimes ab\bar{a}\bar{b}c\delta \otimes d\gamma \\
&+ ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta - ab\bar{a}\bar{b}\gamma) \otimes ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + ab\bar{a}\bar{b}\gamma) \otimes ab\bar{a}\bar{b}c\delta \otimes ab\bar{a}\bar{b}c\delta \\
&+ ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + ab\bar{a}\bar{b}\gamma) \otimes \omega \otimes ab\bar{a}\bar{b}c\delta \otimes dcx \\
&+ x \otimes ((ab\bar{a} - 1)\alpha - a\beta) \otimes \omega \otimes ab\bar{a}\alpha + x \otimes \alpha \otimes \omega \otimes a\beta \\
&+ x \otimes ((ab\bar{a} - 1)\alpha + (ab\bar{a}\bar{b} - a)\beta) \otimes \omega \otimes ab\bar{a}\bar{b}\beta \\
&+ x \otimes ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta) \otimes \omega \otimes ab\bar{a}\bar{b}\gamma \\
&- x \otimes \delta \otimes \omega \otimes d\gamma + x \otimes \omega \otimes \omega \otimes dcx \\
&+ x \otimes ((1 - ab\bar{a})\alpha + (a - ab\bar{a}\bar{b})\beta + ab\bar{a}\bar{b}\gamma) \otimes \omega \otimes ab\bar{a}\bar{b}c\delta.
\end{aligned}
$$

4.1.1 *The cohomology of $\Sigma_2 \times \Sigma_2$ with coefficients in $\widetilde{\mathbb{Z}}$*

There are 256 systems of local coefficients on $\Sigma_2 \times \Sigma_2$ having $\mathbb{Z}$ as underlying group. Let s_x and s'_x be in $\{-1, 1\}$, for $x \in \{a, b, c, d\}$. We write $\widetilde{\mathbb{Z}} := \mathbb{Z}[s_a, s_b, s_c, s_d, s'_a, s'_b, s'_c, s'_d]$ to mean that $\mathbb{Z}$ is the $(\pi_2 \times \pi_2)$-module where $(x, 1)$ (resp. $(1, x)$) acts non-trivially on $\mathbb{Z}$ if and only if $s_x = -1$ (resp. $s'_x = -1$), otherwise it acts trivially. As evidenced by calculations, if just one generator of the group $\pi_2 \times \pi_2$ acts by change of sign on $\mathbb{Z}$, then all the groups $H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right)$ are isomorphic, as well as their multiplicative structure $H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right) \otimes H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right) \xrightarrow{\cup} H^*(\pi_2 \times \pi_2; \mathbb{Z})$.
By routine calculations we obtain that the groups $H^*(\pi_2 \times \pi_2; \mathbb{Z})$ and $H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right)$ are given by

$$
\begin{aligned}
H^0(\pi_2 \times \pi_2; \mathbb{Z}) &= \mathbb{Z}, & H^0(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) &= 0, \\
H^1(\pi_2 \times \pi_2; \mathbb{Z}) &= \mathbb{Z}^8, & H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}^2, \\
H^2(\pi_2 \times \pi_2; \mathbb{Z}) &= \mathbb{Z}^{18} & H^2(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) &\cong \mathbb{Z}_2^5 \oplus \mathbb{Z}^8, \\
H^3(\pi_2 \times \pi_2; \mathbb{Z}) &= \mathbb{Z}^8, & H^3(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) &\cong \mathbb{Z}_2^5 \oplus \mathbb{Z}^2, \\
H^4(\pi_2 \times \pi_2; \mathbb{Z}) &= \mathbb{Z}, & H^4(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) &\cong \mathbb{Z}_2.
\end{aligned}
$$

In fact, we have

$$\begin{aligned}
H^0(\pi_2\times\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle x^*\otimes x^*\rangle,\\
H^1(\pi_2\times\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle x^*\otimes\alpha^*, x^*\otimes\beta^*, x^*\otimes\gamma^*, x^*\otimes\delta^*, \alpha^*\otimes x^*,\\
&\qquad \beta^*\otimes x^*, \gamma^*\otimes x^*, \delta^*\otimes x^*\rangle,\\
H^2(\pi_2\times\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle x^*\otimes\omega^*, \omega^*\otimes x^*, \alpha^*\otimes\alpha^*, \alpha^*\otimes\beta^*, \alpha^*\otimes\gamma^*,\\
&\qquad \alpha^*\otimes\delta^*, \beta^*\otimes\alpha^*, \beta^*\otimes\beta^*, \beta^*\otimes\gamma^*, \beta^*\otimes\delta^*,\\
&\qquad \gamma^*\otimes\alpha^*, \gamma^*\otimes\beta^*, \gamma^*\otimes\gamma^*, \gamma^*\otimes\delta^*, \delta^*\otimes\alpha^*,\\
&\qquad \delta^*\otimes\beta^*, \delta^*\otimes\gamma^*, \delta^*\otimes\delta^*\rangle,\\
H^3(\pi_2\times\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle \omega^*\otimes\alpha^*, \omega^*\otimes\beta^*, \omega^*\otimes\gamma^*, \omega^*\otimes\delta^*, \alpha^*\otimes\omega^*,\\
&\qquad \beta^*\otimes\omega^*, \gamma^*\otimes\omega^*, \delta^*\otimes\omega^*\rangle,\\
H^4(\pi_2\times\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle\omega^*\otimes\omega^*\rangle,
\end{aligned}$$

notice that these expressions are just a clear manifestation of Künneth formula for cohomology.

As mentioned above, if just one generator of $\pi_2\times\pi_2$ acts non-trivially on $\mathbb{Z}$, then all the groups $H^*\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right)$ are isomorphic. We present below the results when $(a,1)$ is such a generator.

$$\begin{aligned}
H^0\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right) &= 0,\\
H^1\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right) &= \mathbb{Z}_2\langle(\alpha^*\otimes x^*)_\theta\rangle\oplus\mathbb{Z}\langle(\gamma^*\otimes x^*)_\theta, (\delta^*\otimes x^*)_\theta\rangle,\\
H^2\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right) &= \mathbb{Z}_2\langle(\omega^*\otimes x^*)_\theta, (\alpha^*\otimes\alpha^*)_\theta, (\alpha^*\otimes\beta^*)_\theta, (\alpha^*\otimes\gamma^*)_\theta, (\alpha^*\otimes\delta^*)_\theta\rangle\oplus\\
&\quad \mathbb{Z}\langle(\gamma^*\otimes\alpha^*)_\theta, (\gamma^*\otimes\beta^*)_\theta, (\gamma^*\otimes\gamma^*)_\theta, (\gamma^*\otimes\delta^*)_\theta, (\delta^*\otimes\alpha^*)_\theta, (\delta^*\otimes\beta^*)_\theta, (\delta^*\otimes\gamma^*)_\theta, (\delta^*\otimes\delta^*)_\theta\rangle,\\
H^3\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right) &= \mathbb{Z}_2\langle(\omega^*\otimes\alpha^*)_\theta, (\omega^*\otimes\beta^*)_\theta, (\omega^*\otimes\gamma^*)_\theta, (\omega^*\otimes\delta^*)_\theta, (\alpha^*\otimes\omega^*)_\theta\rangle\oplus\mathbb{Z}\langle(\gamma^*\otimes\omega^*)_\theta, (\delta^*\otimes\omega^*)_\theta\rangle,\\
H^4\left(\pi_2\times\pi_2;\widetilde{\mathbb{Z}}\right) &= \mathbb{Z}_2\langle(\omega^*\otimes\omega^*)_\theta\rangle,
\end{aligned}$$

where we put the subscript θ to distinguish these generators from those above. Notice that Künneth formula for cohomology does not make obvious these expressions immediately previous.

Up to anti-commutativity, the only non-trivial products

$$H^*\left(\pi_2\times\pi_2;\mathbb{Z}\right)\otimes H^*\left(\pi_2\times\pi_2;\mathbb{Z}\right)\xrightarrow{\cup}H^*\left(\pi_2\times\pi_2;\mathbb{Z}\right)$$

are listed below.

$$\begin{array}{lcll}
(x^*\otimes x^*)\cup(x^*\otimes\lambda^*) &=& x^*\otimes\lambda^*, & \text{for } \lambda\in\{\alpha,\beta,\gamma,\delta\},\\
(x^*\otimes x^*)\cup(\lambda^*\otimes x^*) &=& \lambda^*\otimes x^*, & \text{for } \lambda\in\{\alpha,\beta,\gamma,\delta\},\\
(x^*\otimes x^*)\cup(x^*\otimes\omega^*) &=& x^*\otimes\omega^*, &\\
(x^*\otimes x^*)\cup(\omega^*\otimes x^*) &=& \omega^*\otimes x^*, &\\
(x^*\otimes x^*)\cup(\lambda_1^*\otimes\lambda_2^*) &=& \lambda_1^*\otimes\lambda_2^*, & \text{for } \lambda_1,\lambda_2\in\{\alpha,\beta,\gamma,\delta\},\\
(x^*\otimes x^*)\cup(\omega^*\otimes\lambda^*) &=& \omega^*\otimes\lambda^*, & \text{for } \lambda\in\{\alpha,\beta,\gamma,\delta\},\\
(x^*\otimes x^*)\cup(\lambda^*\otimes\omega^*) &=& \lambda^*\otimes\omega^*, & \text{for } \lambda\in\{\alpha,\beta,\gamma,\delta\},\\
(x^*\otimes x^*)\cup(\omega^*\otimes\omega^*) &=& \omega^*\otimes\omega^*, &
\end{array}$$

$$\begin{array}{lcr}
(x^*\otimes\alpha^*)\cup(x^*\otimes\beta^*) &=& x^*\otimes\omega^*,\\
(x^*\otimes\alpha^*)\cup(\alpha^*\otimes x^*) &=& -\alpha^*\otimes\alpha^*,\\
(x^*\otimes\alpha^*)\cup(\beta^*\otimes x^*) &=& -\beta^*\otimes\alpha^*,\\
(x^*\otimes\alpha^*)\cup(\gamma^*\otimes x^*) &=& -\gamma^*\otimes\alpha^*,\\
(x^*\otimes\alpha^*)\cup(\delta^*\otimes x^*) &=& -\delta^*\otimes\alpha^*,
\end{array}$$

$$\begin{aligned}
(x^* \otimes \beta^*) \cup (\alpha^* \otimes x^*) &= -\alpha^* \otimes \beta^*, \\
(x^* \otimes \beta^*) \cup (\beta^* \otimes x^*) &= -\beta^* \otimes \beta^*, \\
(x^* \otimes \beta^*) \cup (\gamma^* \otimes x^*) &= -\gamma^* \otimes \beta^*, \\
(x^* \otimes \beta^*) \cup (\delta^* \otimes x^*) &= -\delta^* \otimes \beta^*,
\end{aligned}$$

$$\begin{aligned}
(x^* \otimes \gamma^*) \cup (x^* \otimes \delta^*) &= x^* \otimes \omega^*, \\
(x^* \otimes \gamma^*) \cup (\alpha^* \otimes x^*) &= -\alpha^* \otimes \gamma^*, \\
(x^* \otimes \gamma^*) \cup (\beta^* \otimes x^*) &= -\beta^* \otimes \gamma^*, \\
(x^* \otimes \gamma^*) \cup (\gamma^* \otimes x^*) &= -\gamma^* \otimes \gamma^*, \\
(x^* \otimes \gamma^*) \cup (\delta^* \otimes x^*) &= -\delta^* \otimes \gamma^*,
\end{aligned}$$

$$\begin{aligned}
(x^* \otimes \delta^*) \cup (\alpha^* \otimes x^*) &= -\alpha^* \otimes \delta^*, \\
(x^* \otimes \delta^*) \cup (\beta^* \otimes x^*) &= -\beta^* \otimes \delta^*, \\
(x^* \otimes \delta^*) \cup (\gamma^* \otimes x^*) &= -\gamma^* \otimes \delta^*, \\
(x^* \otimes \delta^*) \cup (\delta^* \otimes x^*) &= -\delta^* \otimes \delta^*,
\end{aligned}$$

$$\begin{aligned}
(\alpha^* \otimes x^*) \cup (\beta^* \otimes x^*) &= \omega^* \otimes x^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes x^*) &= \omega^* \otimes x^*,
\end{aligned}$$

$$\begin{aligned}
(x^* \otimes \alpha^*) \cup (\omega^* \otimes x^*) &= \omega^* \otimes \alpha^*, \\
(x^* \otimes \alpha^*) \cup (\alpha^* \otimes \beta^*) &= -\alpha^* \otimes \omega^*, \\
(x^* \otimes \alpha^*) \cup (\beta^* \otimes \beta^*) &= -\beta^* \otimes \omega^*, \\
(x^* \otimes \alpha^*) \cup (\gamma^* \otimes \beta^*) &= -\gamma^* \otimes \omega^*, \\
(x^* \otimes \alpha^*) \cup (\delta^* \otimes \beta^*) &= -\delta^* \otimes \omega^*,
\end{aligned}$$

$$\begin{aligned}
(\alpha^* \otimes x^*) \cup (x^* \otimes \omega^*) &= \alpha^* \otimes \omega^*, \\
(\alpha^* \otimes x^*) \cup (\beta^* \otimes \alpha^*) &= \omega^* \otimes \alpha^*, \\
(\alpha^* \otimes x^*) \cup (\beta^* \otimes \beta^*) &= \omega^* \otimes \beta^*, \\
(\alpha^* \otimes x^*) \cup (\beta^* \otimes \gamma^*) &= \omega^* \otimes \gamma^*, \\
(\alpha^* \otimes x^*) \cup (\beta^* \otimes \delta^*) &= \omega^* \otimes \delta^*,
\end{aligned}$$

$$\begin{aligned}
(x^* \otimes \beta^*) \cup (\omega^* \otimes x^*) &= \omega^* \otimes \beta^*, \\
(x^* \otimes \beta^*) \cup (\alpha^* \otimes \alpha^*) &= \alpha^* \otimes \omega^*, \\
(x^* \otimes \beta^*) \cup (\beta^* \otimes \alpha^*) &= \beta^* \otimes \omega^*, \\
(x^* \otimes \beta^*) \cup (\gamma^* \otimes \alpha^*) &= \gamma^* \otimes \omega^*, \\
(x^* \otimes \beta^*) \cup (\delta^* \otimes \alpha^*) &= \delta^* \otimes \omega^*,
\end{aligned}$$

$$\begin{aligned}
(\beta^* \otimes x^*) \cup (x^* \otimes \omega^*) &= \beta^* \otimes \omega^*, \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes \alpha^*) &= -\omega^* \otimes \alpha^*, \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes \beta^*) &= -\omega^* \otimes \beta^*, \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes \gamma^*) &= -\omega^* \otimes \gamma^*, \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes \delta^*) &= -\omega^* \otimes \delta^*,
\end{aligned}$$

$$\begin{aligned}
(x^* \otimes \gamma^*) \cup (\omega^* \otimes x^*) &= \omega^* \otimes \gamma^*, \\
(x^* \otimes \gamma^*) \cup (\alpha^* \otimes \delta^*) &= -\alpha^* \otimes \omega^*, \\
(x^* \otimes \gamma^*) \cup (\beta^* \otimes \delta^*) &= -\beta^* \otimes \omega^*, \\
(x^* \otimes \gamma^*) \cup (\gamma^* \otimes \delta^*) &= -\gamma^* \otimes \omega^*, \\
(x^* \otimes \gamma^*) \cup (\delta^* \otimes \delta^*) &= -\delta^* \otimes \omega^*,
\end{aligned}$$

$$\begin{aligned}
(\gamma^* \otimes x^*) \cup (x^* \otimes \omega^*) &= \gamma^* \otimes \omega^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \alpha^*) &= \omega^* \otimes \alpha^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \beta^*) &= \omega^* \otimes \beta^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \gamma^*) &= \omega^* \otimes \gamma^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \delta^*) &= \omega^* \otimes \delta^*,
\end{aligned}$$

$$\begin{array}{rcl}
(x^* \otimes \delta^*) \cup (\omega^* \otimes x^*) & = & \omega^* \otimes \delta^*, \\
(x^* \otimes \delta^*) \cup (\alpha^* \otimes \gamma^*) & = & \alpha^* \otimes \omega^*, \\
(x^* \otimes \delta^*) \cup (\beta^* \otimes \gamma^*) & = & \beta^* \otimes \omega^*, \\
(x^* \otimes \delta^*) \cup (\gamma^* \otimes \gamma^*) & = & \gamma^* \otimes \omega^*, \\
(x^* \otimes \delta^*) \cup (\delta^* \otimes \gamma^*) & = & \delta^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(\delta^* \otimes x^*) \cup (x^* \otimes \omega^*) & = & \delta^* \otimes \omega^*, \\
(\delta^* \otimes x^*) \cup (\gamma^* \otimes \alpha^*) & = & -\omega^* \otimes \alpha^*, \\
(\delta^* \otimes x^*) \cup (\gamma^* \otimes \beta^*) & = & -\omega^* \otimes \beta^*, \\
(\delta^* \otimes x^*) \cup (\gamma^* \otimes \gamma^*) & = & -\omega^* \otimes \gamma^*, \\
(\delta^* \otimes x^*) \cup (\gamma^* \otimes \delta^*) & = & -\omega^* \otimes \delta^*,
\end{array}$$

$$\begin{array}{rcl}
(x^* \otimes \alpha^*) \cup (\omega^* \otimes \beta^*) & = & \omega^* \otimes \omega^*, \\
(x^* \otimes \beta^*) \cup (\omega^* \otimes \alpha^*) & = & -\omega^* \otimes \omega^*, \\
(x^* \otimes \gamma^*) \cup (\omega^* \otimes \delta^*) & = & \omega^* \otimes \omega^*, \\
(x^* \otimes \delta^*) \cup (\omega^* \otimes \gamma^*) & = & -\omega^* \otimes \omega^*, \\
(\alpha^* \otimes x^*) \cup (\beta^* \otimes \omega^*) & = & \omega^* \otimes \omega^*, \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes \omega^*) & = & -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \omega^*) & = & \omega^* \otimes \omega^*, \\
(\delta^* \otimes x^*) \cup (\gamma^* \otimes \omega^*) & = & -\omega^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(x^* \otimes \omega^*) \cup (\omega^* \otimes x^*) & = & \omega^* \otimes \omega^*, \\
(\omega^* \otimes x^*) \cup (x^* \otimes \omega^*) & = & \omega^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(\alpha^* \otimes \alpha^*) \cup (\beta^* \otimes \beta^*) & = & -\omega^* \otimes \omega^*, \\
(\alpha^* \otimes \beta^*) \cup (\beta^* \otimes \alpha^*) & = & \omega^* \otimes \omega^*, \\
(\alpha^* \otimes \gamma^*) \cup (\beta^* \otimes \delta^*) & = & -\omega^* \otimes \omega^*, \\
(\alpha^* \otimes \delta^*) \cup (\beta^* \otimes \gamma^*) & = & \omega^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(\beta^* \otimes \alpha^*) \cup (\alpha^* \otimes \beta^*) & = & \omega^* \otimes \omega^*, \\
(\beta^* \otimes \beta^*) \cup (\alpha^* \otimes \alpha^*) & = & -\omega^* \otimes \omega^*, \\
(\beta^* \otimes \gamma^*) \cup (\alpha^* \otimes \delta^*) & = & \omega^* \otimes \omega^*, \\
(\beta^* \otimes \delta^*) \cup (\alpha^* \otimes \gamma^*) & = & -\omega^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(\gamma^* \otimes \alpha^*) \cup (\delta^* \otimes \beta^*) & = & -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes \beta^*) \cup (\delta^* \otimes \alpha^*) & = & \omega^* \otimes \omega^*, \\
(\gamma^* \otimes \gamma^*) \cup (\delta^* \otimes \delta^*) & = & -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes \delta^*) \cup (\delta^* \otimes \gamma^*) & = & \omega^* \otimes \omega^*,
\end{array}$$

$$\begin{array}{rcl}
(\delta^* \otimes \alpha^*) \cup (\gamma^* \otimes \beta^*) & = & \omega^* \otimes \omega^*, \\
(\delta^* \otimes \beta^*) \cup (\gamma^* \otimes \alpha^*) & = & -\omega^* \otimes \omega^*, \\
(\delta^* \otimes \gamma^*) \cup (\gamma^* \otimes \delta^*) & = & \omega^* \otimes \omega^*, \\
(\delta^* \otimes \delta^*) \cup (\gamma^* \otimes \gamma^*) & = & -\omega^* \otimes \omega^*.
\end{array}$$

Similarly, up to anti-commutativity, the only non-zero products

$$H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right) \otimes H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right) \xrightarrow{\cup} H^*\left(\pi_2 \times \pi_2; \mathbb{Z}\right)$$

are the following:

$$\begin{array}{rcl}
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes x^*)_\theta & = & \omega^* \otimes x^*,
\end{array}$$

$$\begin{array}{rcl}
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes \alpha^*)_\theta & = & \omega^* \otimes \alpha^*, \\
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes \beta^*)_\theta & = & \omega^* \otimes \beta^*, \\
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes \gamma^*)_\theta & = & \omega^* \otimes \gamma^*, \\
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes \delta^*)_\theta & = & \omega^* \otimes \delta^*,
\end{array}$$

$$
\begin{aligned}
(\delta^* \otimes x^*)_\theta \cup (\gamma^* \otimes \alpha^*)_\theta &= -\omega^* \otimes \alpha^*, \\
(\delta^* \otimes x^*)_\theta \cup (\gamma^* \otimes \beta^*)_\theta &= -\omega^* \otimes \beta^*, \\
(\delta^* \otimes x^*)_\theta \cup (\gamma^* \otimes \gamma^*)_\theta &= -\omega^* \otimes \gamma^*, \\
(\delta^* \otimes x^*)_\theta \cup (\gamma^* \otimes \delta^*)_\theta &= -\omega^* \otimes \delta^*, \\
(\gamma^* \otimes x^*)_\theta \cup (\delta^* \otimes \omega^*)_\theta &= \omega^* \otimes \omega^*, \\
(\delta^* \otimes x^*)_\theta \cup (\gamma^* \otimes \omega^*)_\theta &= -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes \alpha^*)_\theta \cup (\delta^* \otimes \beta^*)_\theta &= -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes \beta^*)_\theta \cup (\delta^* \otimes \alpha^*)_\theta &= \omega^* \otimes \omega^*, \\
(\gamma^* \otimes \gamma^*)_\theta \cup (\delta^* \otimes \delta^*)_\theta &= -\omega^* \otimes \omega^*, \\
(\gamma^* \otimes \delta^*)_\theta \cup (\delta^* \otimes \gamma^*)_\theta &= \omega^* \otimes \omega^*, \\
(\delta^* \otimes \alpha^*)_\theta \cup (\gamma^* \otimes \beta^*)_\theta &= \omega^* \otimes \omega^*, \\
(\delta^* \otimes \beta^*)_\theta \cup (\gamma^* \otimes \alpha^*)_\theta &= -\omega^* \otimes \omega^*, \\
(\delta^* \otimes \gamma^*)_\theta \cup (\gamma^* \otimes \delta^*)_\theta &= \omega^* \otimes \omega^*, \\
(\delta^* \otimes \delta^*)_\theta \cup (\gamma^* \otimes \gamma^*)_\theta &= -\omega^* \otimes \omega^*.
\end{aligned}
$$

Finally, up to anti-commutativity, the only non-zero products

$$
H^*(\pi_2 \times \pi_2; \mathbb{Z}) \otimes H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right) \xrightarrow{\cup} H^*\left(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}\right)
$$

are given below.

$$
\begin{aligned}
(x^* \otimes x^*) \cup (\lambda^* \otimes x^*)_\theta &= (\lambda^* \otimes x^*)_\theta, \text{ for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(x^* \otimes x^*) \cup (\lambda_1^* \otimes \lambda_2^*)_\theta &= (\lambda_1^* \otimes \lambda_2^*)_\theta, \text{ for } \lambda_1 \in \{\alpha, \gamma, \delta\} \text{ and } \lambda_2 \in \{\alpha, \beta, \gamma, \delta\}, \\
(x^* \otimes x^*) \cup (\omega^* \otimes x^*)_\theta &= (\omega^* \otimes x^*)_\theta, \\
(x^* \otimes x^*) \cup (\omega^* \otimes \lambda^*)_\theta &= (\omega^* \otimes \lambda^*)_\theta, \text{ for } \lambda \in \{\alpha, \beta, \gamma, \delta\}, \\
(x^* \otimes x^*) \cup (\lambda^* \otimes \omega^*)_\theta &= (\lambda^* \otimes \omega^*)_\theta, \text{ for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(x^* \otimes x^*) \cup (\omega^* \otimes \omega^*)_\theta &= (\omega^* \otimes \omega^*)_\theta,
\end{aligned}
$$

$$
\begin{aligned}
(x^* \otimes \lambda_1^*) \cup (\lambda_2^* \otimes x^*)_\theta &= -(\lambda_2^* \otimes \lambda_1^*)_\theta, \text{ for } \lambda_1 \in \{\alpha, \beta, \gamma, \delta\} \text{ and } \lambda_2 \in \{\alpha, \gamma, \delta\} \\
(\beta^* \otimes x^*) \cup (\alpha^* \otimes x^*)_\theta &= -(\omega^* \otimes x^*)_\theta, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes x^*)_\theta &= (\omega^* \otimes x^*)_\theta,
\end{aligned}
$$

$$
\begin{aligned}
(x^* \otimes \lambda^*) \cup (\omega^* \otimes x^*)_\theta &= (\omega^* \otimes \lambda^*)_\theta, && \text{for } \lambda \in \{\alpha, \beta, \gamma, \delta\}, \\
(x^* \otimes \alpha^*) \cup (\lambda^* \otimes \beta^*)_\theta &= -(\lambda^* \otimes \omega^*)_\theta, && \text{for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(x^* \otimes \beta^*) \cup (\lambda^* \otimes \alpha^*)_\theta &= (\lambda^* \otimes \omega^*)_\theta, && \text{for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(x^* \otimes \gamma^*) \cup (\lambda^* \otimes \delta^*)_\theta &= -(\lambda^* \otimes \omega^*)_\theta, && \text{for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(x^* \otimes \delta^*) \cup (\lambda^* \otimes \gamma^*)_\theta &= (\lambda^* \otimes \omega^*)_\theta, && \text{for } \lambda \in \{\alpha, \gamma, \delta\}, \\
(\lambda_1^* \otimes x^*) \cup (\lambda_2^* \otimes \lambda_3^*)_\theta &= -(\omega^* \otimes \lambda_3^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\beta, \alpha), (\delta, \gamma)\} \text{ and } \lambda_3 \in \{\alpha, \beta, \gamma, \delta\}, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \lambda^*)_\theta &= (\omega^* \otimes \lambda^*)_\theta, && \text{for } \lambda \in \{\alpha, \beta, \gamma, \delta\},
\end{aligned}
$$

$$
\begin{aligned}
(x^* \otimes \lambda_1^*) \cup (\omega^* \otimes \lambda_2^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\alpha, \beta), (\gamma, \delta)\}, \\
(x^* \otimes \lambda_1^*) \cup (\omega^* \otimes \lambda_2^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\beta, \alpha), (\delta, \gamma)\}, \\
(\lambda_1^* \otimes x^*) \cup (\lambda_2^* \otimes \omega^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\beta, \alpha), (\delta, \gamma)\}, \\
(\gamma^* \otimes x^*) \cup (\delta^* \otimes \omega^*)_\theta &= (\omega^* \otimes \omega^*)_\theta,
\end{aligned}
$$

$$
\begin{aligned}
(x^* \otimes \omega^*) \cup (\omega^* \otimes x^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, \\
(\lambda_1^* \otimes \lambda_2^*) \cup (\lambda_2^* \otimes \lambda_1^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\beta, \alpha), (\delta, \gamma), (\gamma, \delta)\}, \\
(\beta^* \otimes \gamma^*) \cup (\alpha^* \otimes \delta^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, \\
(\delta^* \otimes \alpha^*) \cup (\gamma^* \otimes \beta^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, \\
(\gamma^* \otimes \beta^*) \cup (\delta^* \otimes \alpha^*)_\theta &= (\omega^* \otimes \omega^*)_\theta, \\
(\lambda_1^* \otimes \lambda_1^*) \cup (\lambda_2^* \otimes \lambda_2^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta, && \text{for } (\lambda_1, \lambda_2) \in \{(\beta, \alpha), (\delta, \gamma), (\gamma, \delta)\}, \\
(\beta^* \otimes \delta^*) \cup (\alpha^* \otimes \gamma^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta, \\
(\delta^* \otimes \beta^*) \cup (\gamma^* \otimes \alpha^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta, \\
(\gamma^* \otimes \alpha^*) \cup (\delta^* \otimes \beta^*)_\theta &= -(\omega^* \otimes \omega^*)_\theta.
\end{aligned}
$$

4.1.2 *Effective-zero–divisors*

Definition 4.1. Let $\mathcal{M}$ be a local system of coefficients on $X \times X$. A cohomology class $u \in H^*(X \times X; \mathcal{M})$ is called a zero-divisor if its restriction to the diagonal is trivial, i.e. $0 = \Phi^*(u) \in H^*(X; \mathcal{M}|X)$, where $\mathcal{M}|X$ denotes the local system induced by the diagonal map $\Phi : X \to X \times X$.

Remark 4.2. The importance of zero–divisors stems from the following fact: suppose that there are zero–divisors $u_i \in H^*(X \times X; \mathcal{A}_i)$, $i = 1, \dots, k$, such that their cup-product $u_1 \cup \cdots \cup u_k \neq 0$ in $H^*(X \times X; \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k)$, then $k \leq \mathbf{TC}(X)$. See [12, corollary 4.40].

Definition 4.2. Let $\mathcal{M}$ be a local coefficient system on $X \times X$ and let $j^*(\mathcal{M})$ be the local coefficient system on $G \times X$ induced by the "fattened diagonal". A cohomology class $\overline{f} \in H^1(X \times X; \mathcal{M})$ is called an effective-zero–divisor if $j^*(\overline{f}) = 0$ in $H^*(G \times X; j^*(\mathcal{M}))$.

Remark 4.3. Notice that for a discrete group G, $G \times X = \coprod_{g \in G} X$, then

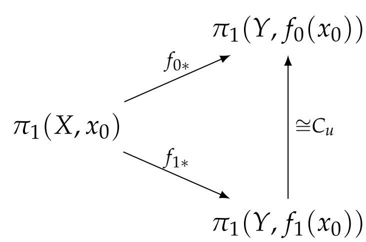
$$H^*(G \times X; j^*(\mathcal{M})) \cong \prod_{g \in G} H^*\left(X; j^*(\mathcal{M})_g\right),$$

where $j^*(\mathcal{M})_g$ denotes the restriction of $j^*(\mathcal{M})$ to the copy of X corresponding to $g \in G$.

Definition 4.3. Let $f_0, f_1 \colon X \to Y$ and $u \colon I \to Y$ a path. Suppose there is a homotopy $F \colon X \times I \to Y$ from f_0 to f_1 such that $F(x_0, t) = u(t)$. Then we say f_0 is *freely homotopic* to f_1 along u.

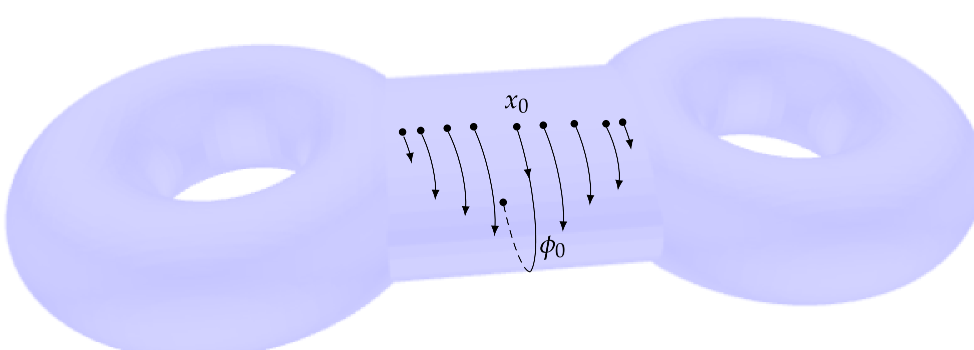
Remark 4.4. If $f_0, f_1 \colon (X, x_0) \to (Y, y_0)$, then u is a loop.

1. The diagram



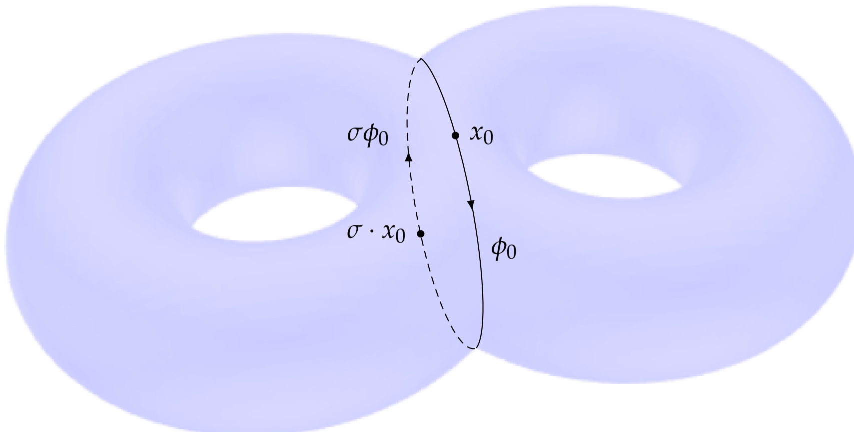
   commutes, where C_u denotes conjugation by u.

2. $\mathrm{id}_{\Sigma_2} \colon (\Sigma_2, x_0) \to (\Sigma_2, x_0)$ is freely homotopic to a map $\iota \colon (\Sigma_2, x_0) \to (\Sigma_2, \sigma \cdot x_0)$ throught a homotopy $H \colon \Sigma_2 \times I \to \Sigma_2$ such that $H(x_0, t) = \phi_0$. An example of such a homotopy is depicted in the picture below.



Let $G := \sigma \circ H \colon \Sigma_2 \times I \to \Sigma_2$, then

$$\begin{aligned} G(-,0) &= \sigma \\ G(-,1) &= \sigma \circ \iota, \quad \text{and} \quad G(x_0,1) = \sigma \circ \iota(x_0) = x_0 \\ G(x_0,-) &= \sigma \circ \phi_0 \colon I \to \Sigma_2. \end{aligned}$$



Then

$$L := (\mathrm{proj}_1, G)\colon \quad \begin{array}{ccc} \Sigma_2 \times I & \to & \Sigma_2 \times \Sigma_2 \\ (s,t) & \mapsto & (s, G(s,t)) \end{array}$$

satisfies

$$\begin{aligned} L(-,0) &= (\mathrm{id}_{\Sigma_2}, \sigma) =: \Phi', \\ L(-,1) &= (-, \sigma \circ \iota) = (\mathrm{id}_{\Sigma_2}, \sigma \circ \iota), \\ L(x_0,-) &= (x_0, \sigma \circ \phi_0). \end{aligned}$$

Thus, $\Phi'' := (\mathrm{id}_{\Sigma_2}, \sigma \circ \iota)\colon (\Sigma_2, x_0) \to (\Sigma_2 \times \Sigma_2, (x_0,x_0))$ is homotopic to Φ' through L, and they induce, up to an isomorphism, the same map in cohomology according to remark 2.4. What will be relevant for us is that the morphism induced by Φ' and Φ'' have the same kernel, since we are interested in finding effective-zero–divisors and these are precisely the elements of $\ker(\Phi'')^* \cap \ker \Phi^*$.

Given a $(\pi_2 \times \pi_2)$-module M, let us denote by $\mathcal{M}$ the local system over $\Sigma_2 \times \Sigma_2$ determined by M. In what follows $Q(\pi_2 \times \pi_2, M)$ (resp. $P(\pi_2 \times \pi_2, M)$) stands for the group of all crossed homomorphisms from $\pi_2 \times \pi_2$ into M (the group of all principal homomorphisms of $\pi_2 \times \pi_2$ into M).

Our goal is to understand the kernel of the map

$$(\Phi'')^*\colon H^1(\Sigma_2 \times \Sigma_2; \mathcal{M}) \to H^1(\Sigma_2; (\Phi'')^*(\mathcal{M})),$$

which can be understood in terms of crossed morphims,

$$\begin{array}{ccc} H^1\left(\Sigma_2 \times \Sigma_2; \mathcal{M}\right) & \xrightarrow{(\Phi'')^*} & H^1\left(\Sigma_2; (\Phi'')^*(\mathcal{M})\right) \\ \downarrow \cong & & \downarrow \cong \\ Q\left(\pi_2 \times \pi_2, M\right) / P\left(\pi_2 \times \pi_2, M\right) & \xrightarrow{(\Phi'')_*} & Q\left(\pi_2, (\Phi'')_*(M)\right) / P\left(\pi_2, (\Phi'')_*(M)\right), \end{array}$$

where $(\Phi'')_*$ is given by composition with $\Phi''_* : \pi_2 \to \pi_2 \times \pi_2$ and the latter fits in the following commutative diagram

$$\begin{array}{ccc} & & \pi_2 \times \pi_2 \\ & \overset{\Phi''_*}{\nearrow} & \uparrow {\scriptstyle \mathrm{id} \times C_{\sigma\phi_0}} \\ \pi_2 & & \\ & \underset{\Phi'_*}{\searrow} & \\ & & \pi_2 \times \pi_1(\Sigma_2, \sigma \cdot x_0). \end{array}$$

Then $\ker((\Phi')^*)$ can be calculated in terms of

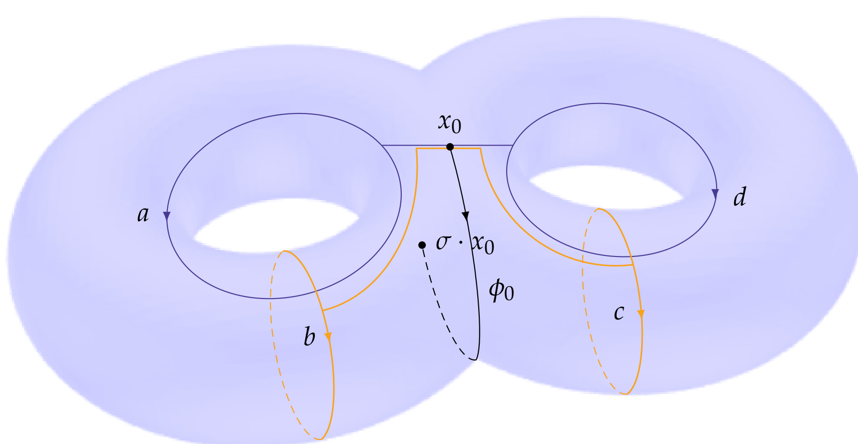
$$\begin{array}{ccccc} \pi_2 & \xrightarrow{\Phi'_*} & \pi_2 \times \pi_1(\Sigma_2, \sigma \cdot x_0) & \xrightarrow{\mathrm{id} \times C_{\sigma \circ \phi_0}} & \pi_2 \times \pi_2 \\ [\alpha] & \longmapsto & ([\alpha], [\sigma\alpha]) & \longmapsto & \left([\alpha], [\sigma\overline{\phi_0}][\sigma\alpha][\sigma\phi_0]\right). \end{array}$$

Let $\widehat{\sigma} \colon \pi_2 \to \pi_2$ be given by $\widehat{\sigma}([\alpha]) = [\sigma\overline{\phi_0}][\sigma\alpha][\sigma\phi_0]$ and $\widetilde{\sigma}$ as in lemma 4.1. Then

$$\begin{aligned} \widehat{\sigma}\left(\widetilde{\sigma}([\alpha])\right) &= \left[\sigma\overline{\phi_0}\right] \left[\sigma\left(\phi_0 \cdot \sigma\alpha \cdot \overline{\phi_0}\right)\right] \left[\sigma\phi_0\right] \\ &= \left[\sigma\overline{\phi_0}\right] \left[\sigma\phi_0 \cdot \alpha \cdot \sigma\overline{\phi_0}\right] \left[\sigma\phi_0\right] \\ &= \left[\sigma\overline{\phi_0}\right] \left[\sigma\phi_0\right] [\alpha] \left[\sigma\overline{\phi_0}\right] \left[\sigma\phi_0\right] \\ &= [\alpha], \end{aligned}$$

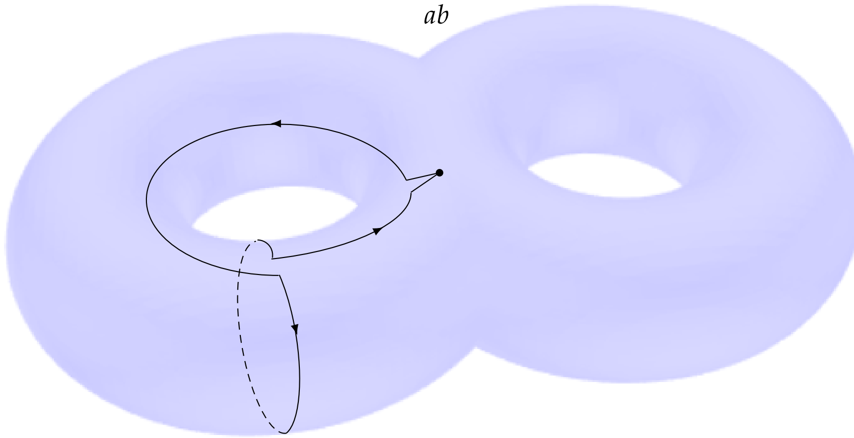
i.e., $\widehat{\sigma} = (\widetilde{\sigma})^{-1}$.

Lemma 4.5. *Loops representing the generators a, b, c and d of π_2 can be chosen as depicted in the following picture:*

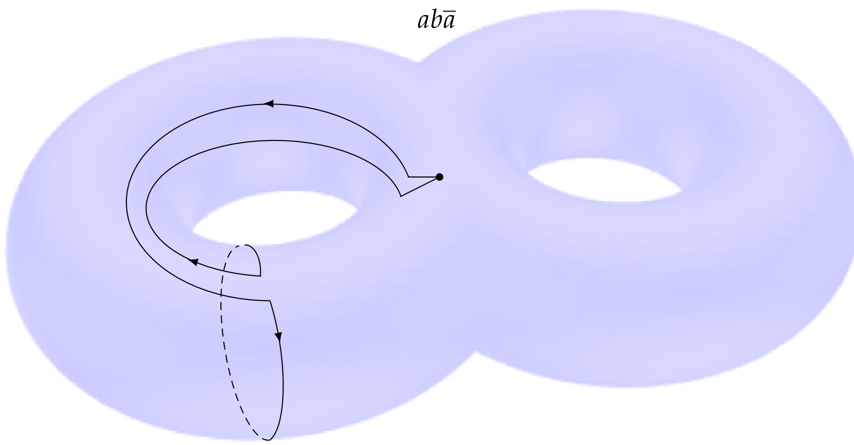


Proof. Take the loops chosen in the statement of lemma 4.5 for the homotopy classes a, b, c and d. We get, by direct inspection, the representative loops shown below of the indicated elements:

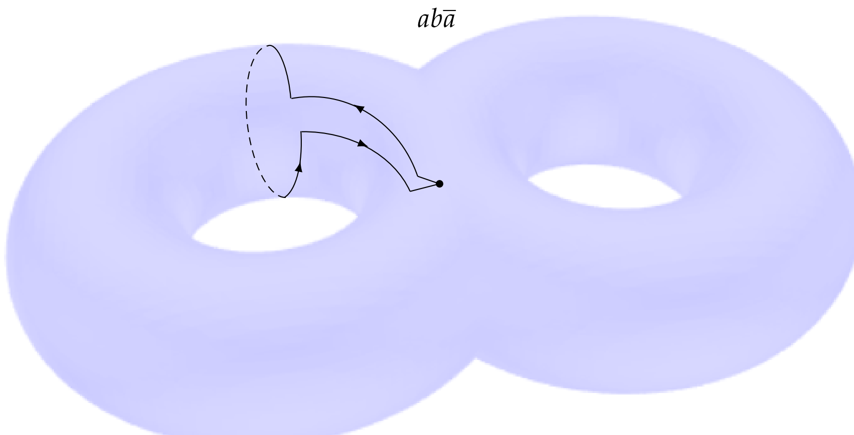
ab



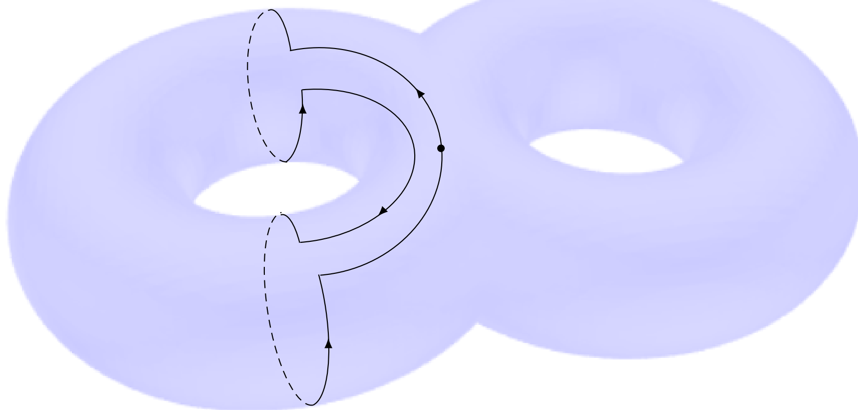
$ab\overline{a}$



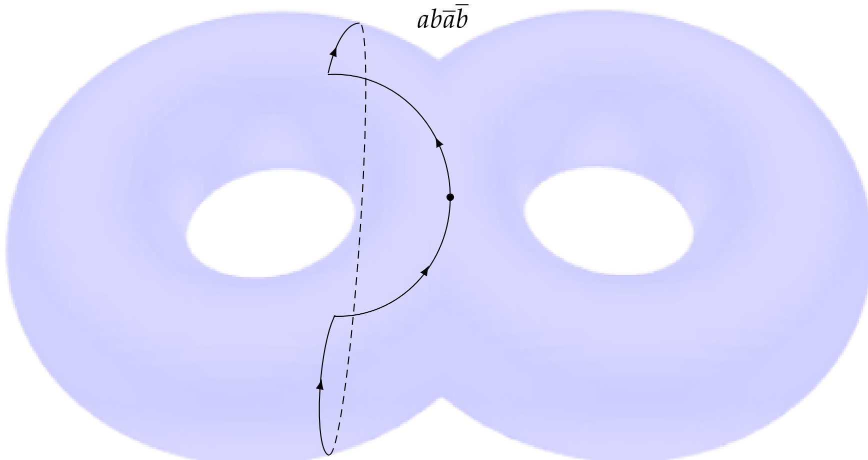
$ab\overline{a}$



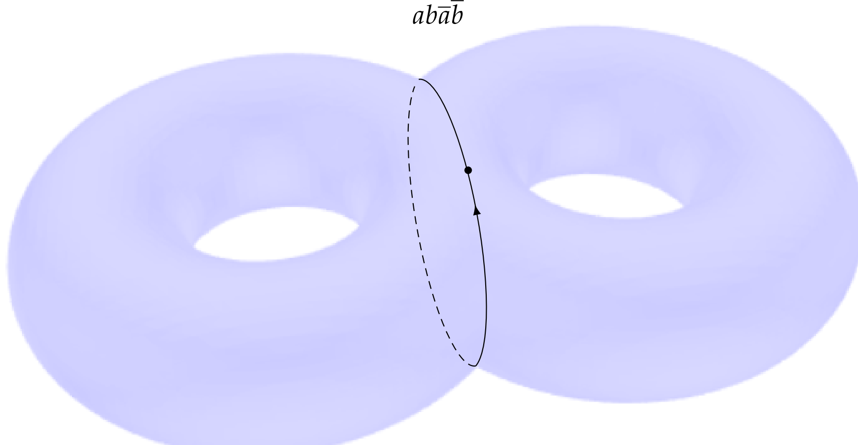
$ab\overline{a}\overline{b}$



$ab\overline{a}\overline{b}$



$ab\overline{a}\overline{b}$



In a similar fashion, the latter loop can be seen to be a representative for $dc\overline{d}\overline{c}$ too. Thus $ab\overline{a}\overline{b} = dc\overline{d}\overline{c}$ which is equivalent to the standard relation $ab\overline{a}\overline{b}cd\overline{c}\overline{d} = e$ defining π_2. □

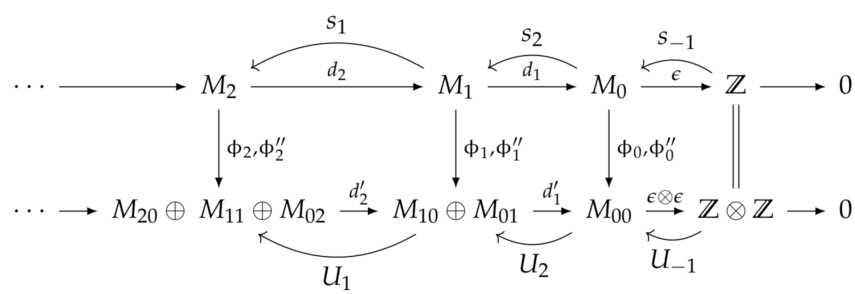
Just as lemma 4.5, lemma 4.6 below can be proved by making use of explicit pictures.

Lemma 4.6. *The isomorphisms $\widetilde{\sigma}$ and $\widetilde{\sigma}^{-1}$ are determined by the relations*

$$\begin{aligned} \widetilde{\sigma}(a) &= \overline{d}ab\overline{a}\overline{b}, & \widetilde{\sigma}^{-1}(a) &= ab\overline{a}\overline{b}\,\overline{d}, \\ \widetilde{\sigma}(b) &= ba\overline{b}\overline{a}\,\overline{c}, & \widetilde{\sigma}^{-1}(b) &= \overline{c}ba\overline{b}\overline{a}, \\ \widetilde{\sigma}(c) &= ba\overline{b}\overline{a}\overline{b}, & \widetilde{\sigma}^{-1}(c) &= a\overline{b}\overline{a}, \\ \widetilde{\sigma}(d) &= b\overline{a}\overline{b}, & \widetilde{\sigma}^{-1}(d) &= ab\overline{a}\overline{b}\overline{a}, \end{aligned} \tag{4.12}$$

where the expression on the right of each of these equalities is in normal form with respect to the finite complete rewriting system described right after proposition 2.8.

The above considerations allow us to identify, in the setting of resolutions, the "twisted" and "usual" diagonals Φ'' and Φ. Consider the diagram below



where

1. M_{ij} denotes $M_i \otimes M_j$, for $i, j \in \{0, 1, 2\}$.
2. ϕ_i'' is defined on $\mathbb{Z}[\pi_2]$-basis elements by

   $$\phi_0''(x) = x \otimes x,$$

   and

   $$\phi_i''(\rho) = U_{i-1}\phi_{i-1}''d_i(\rho), \text{ for } i = 1, 2.$$

   Furthermore, if $z \in \pi_2$, then $\phi_i''(z \cdot \rho) = (z, \widetilde{\sigma}^{-1}(z)) \cdot \phi_i''(\rho)$. This corresponds to the "twisted" diagonal

   $$\begin{array}{rccc} \Phi'': & \pi_2 & \to & \pi_2 \times \pi_2 \\ & z & \mapsto & \left(z, \widetilde{\sigma}^{-1}(z)\right). \end{array}$$

3. ϕ_i is defined on $\mathbb{Z}[\pi_2]$-basis elements by

   $$\phi_0(x) = x \otimes x,$$

   and

   $$\phi_i(\rho) = U_{i-1}\phi_{i-1}d_i(\rho) \text{ for } i = 1, 2.$$

   Furthermore, if $z \in \pi_2$, then $\phi_i(z \cdot \rho) = (z, z) \cdot \phi_i(\rho)$. This corresponds to the "usual" diagonal

   $$\begin{array}{rccc} \Phi: & \pi_2 & \to & \pi_2 \times \pi_2 \\ & z & \mapsto & (z,\, z). \end{array}$$

If $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, s_a', s_b', s_c', s_d']$ is a local system over $\Sigma_2 \times \Sigma_2$, then the systems of local coefficients induced by Φ and Φ'' over Σ_2 are, respectively, $\mathbb{Z}_\Phi = \mathbb{Z}[s_a s_a', s_b s_b', s_c s_c', s_d s_d']$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}[s_a s_d', s_b s_c', s_c s_b', s_d s_a']$.

Sometimes we write $\mathbb{Z}_x$ to mean the local coefficient system on Σ_2 such that only $x \in \{a,b,c,d\}$ acts non-trivially on $\mathbb{Z}$, $\mathbb{Z}_{xy}$ to mean the local coefficient system where only $x, y \in \{a,b,c,d\}$ act non-trivially on $\mathbb{Z}$, and so on.

Before proceeding to find effective zero–divisors in dimension 1, we begin by writing down, in a much more explicit way than the one that appears in [15], the additive 1-dimensional cohomology of π_2 for any local system having as underlying group the integers $\mathbb{Z}$.

$$\begin{aligned}
H^1(\pi_2;\mathbb{Z}) &= \mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_a) &= \mathbb{Z}_2\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_b) &= \mathbb{Z}_2\langle\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_c) &= \mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}_2\langle\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_d) &= \mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}_2\langle\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{ab}) &= \mathbb{Z}_2\langle\alpha^*+\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{ac}) &= \left(\mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\gamma^*\rangle \oplus \mathbb{Z}\langle\beta^*-\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{ad}) &= \left(\mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\delta^*\rangle \oplus \mathbb{Z}\langle\beta^*+\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{bc}) &= \left(\mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle\right)/2\mathbb{Z}\langle\beta^*+\gamma^*\rangle \oplus \mathbb{Z}\langle\alpha^*+\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{bd}) &= \left(\mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle\right)/2\mathbb{Z}\langle\beta^*+\delta^*\rangle \oplus \mathbb{Z}\langle\alpha^*-\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{cd}) &= \mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}_2\langle\gamma^*+\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{abc}) &= \left(\mathbb{Z}\langle\alpha^*+\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\beta^*+\gamma^*\rangle \oplus \mathbb{Z}\langle\alpha^*+\delta^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{abd}) &= \left(\mathbb{Z}\langle\alpha^*+\beta^*\rangle \oplus \mathbb{Z}\langle\delta^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\beta^*+\delta^*\rangle \oplus \mathbb{Z}\langle\alpha^*-\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{acd}) &= \left(\mathbb{Z}\langle\alpha^*\rangle \oplus \mathbb{Z}\langle\gamma^*+\delta^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\gamma^*+\delta^*\rangle \oplus \mathbb{Z}\langle\beta^*+\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{bcd}) &= \left(\mathbb{Z}\langle\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*+\delta^*\rangle\right)/2\mathbb{Z}\langle\beta^*+\gamma^*+\delta^*\rangle \oplus \mathbb{Z}\langle\alpha^*-\gamma^*\rangle \\
H^1(\pi_2;\mathbb{Z}_{abcd}) &= \left(\mathbb{Z}\langle\alpha^*+\beta^*\rangle \oplus \mathbb{Z}\langle\gamma^*+\delta^*\rangle\right)/2\mathbb{Z}\langle\alpha^*+\beta^*+\gamma^*+\delta^*\rangle \oplus \mathbb{Z}\langle\alpha^*-\gamma^*\rangle
\end{aligned}$$

In order to look for 1-dimensional effective zero–divisors for the 256 systems of local coefficients over $\Sigma_2 \times \Sigma_2$, we consider sixteen cases. In turn, each of the first fifteen cases has two sub-cases, namely when the 1-cohomology in question is additively $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$.

1. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, -1, -1, -1]$, we have

$$\begin{aligned}
\mathbb{Z}_\Phi &= \mathbb{Z}[-s_a, -s_b, -s_c, -s_d], \\
\mathbb{Z}_{\Phi''} &= \mathbb{Z}[-s_a, -s_b, -s_c, -s_d].
\end{aligned}$$

   By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\,\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}\langle r_3\rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\beta^* + x^*\otimes\gamma^* + x^*\otimes\delta^*,$$

   and

$$\begin{cases}
r_1 = x^*\otimes\alpha^* + x^*\otimes\beta^* + x^*\otimes\gamma^* + x^*\otimes\delta^*, \\
r_2 = x^*\otimes\gamma^* + x^*\otimes\delta^*, \\
r_3 = x^*\otimes\alpha^* - x^*\otimes\gamma^*.
\end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{ccccccc} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha-\alpha\otimes x & \overset{r}{\mapsto} & 1-\frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x\otimes\beta-\beta\otimes x & \mapsto & 1-\frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\ \gamma & \mapsto & x\otimes\gamma-\gamma\otimes x & \mapsto & 1-\frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x\otimes\delta-\delta\otimes x & \mapsto & 1-\frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = s_c = s_d = -1$. In such a case

$$\begin{array}{rcl} r & = & \alpha^*\otimes x^* + \beta^*\otimes x^* + \gamma^*\otimes x^* + \delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\beta^* + x^*\otimes\gamma^* + x^*\otimes\delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a,-s_b,-s_c,-s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a,-s_b,-s_c,-s_d] = \mathbb{Z}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{ccccc} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha-2x\otimes\beta+x\otimes\delta-\alpha\otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & 2x\otimes\alpha-2x\otimes\beta+x\otimes\gamma-\beta\otimes x & \mapsto & 0, \\ \gamma & \mapsto & 2x\otimes\alpha-x\otimes\beta-\gamma\otimes x & \mapsto & 0, \\ \delta & \mapsto & 3x\otimes\alpha-2x\otimes\beta-\delta\otimes x & \mapsto & 0. \end{array}$$

It follows that in this case r is an effective-zero–divisor.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{\Phi''} = \mathbb{Z}_{abcd}$, and if we take a look at the "usual" diagonal we see that

$$\begin{array}{ccccc} \alpha & \overset{\Phi}{\mapsto} & x^*\otimes\alpha^*-\alpha^*\otimes x^* & \overset{r_1,r_2,r_3}{\mapsto} & 1,0,1, \\ \beta & \mapsto & x^*\otimes\beta^*-\beta^*\otimes x^* & \mapsto & 1,0,0, \\ \gamma & \mapsto & x^*\otimes\gamma^*-\gamma^*\otimes x^* & \mapsto & 1,1,-1, \\ \delta & \mapsto & x^*\otimes\delta^*-\delta^*\otimes x^* & \mapsto & 1,1,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ, then for the second case there are not effective-zero–divisors.

2. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, -1, -1, 1]$, we have

$$\begin{array}{rcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a,-s_b,-s_c,s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a,-s_b,-s_c,-s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2\times\pi_2;\widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}\langle r_3\rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\beta^* + x^*\otimes\gamma^*,$$

and

$$\left\{\begin{array}{rcl} r_1 & = & x^*\otimes\alpha^* + x^*\otimes\beta^* + x^*\otimes\gamma^*, \\ r_2 & = & x^*\otimes\beta^* + x^*\otimes\gamma^* - x^*\otimes\delta^*, \\ r_3 & = & x^*\otimes\gamma^*. \end{array}\right.$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{ccccccc} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha-\alpha\otimes x & \overset{r}{\mapsto} & 1-\frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x\otimes\beta-\beta\otimes x & \mapsto & 1-\frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\ \gamma & \mapsto & x\otimes\gamma-\gamma\otimes x & \mapsto & 1-\frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x\otimes\delta+\delta\otimes x & \mapsto & 0+\frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = s_c = -1$ and $s_d = 1$. In such a case

$$\begin{array}{lll} r & = & \alpha^* \otimes x^* + \beta^* \otimes x^* + \gamma^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \beta^* + x^* \otimes \gamma^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, -s_b, -s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, -s_b, -s_c, -s_d] = \mathbb{Z}_{ad}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lllll} \alpha & \overset{\Phi''}{\mapsto} & 2x \otimes \alpha - 2x \otimes \beta - x \otimes \gamma + \alpha \otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & 2x \otimes \alpha - 2x \otimes \beta + x \otimes \gamma - \beta \otimes x & \mapsto & 0, \\ \gamma & \mapsto & 2x \otimes \alpha - x \otimes \beta - \gamma \otimes x & \mapsto & 0, \\ \delta & \mapsto & 3x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{abc}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{bcd}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lllll} \alpha & \overset{\Phi''}{\mapsto} & 2x \otimes \alpha - 2x \otimes \beta - x \otimes \gamma + \alpha \otimes x & \overset{r_1, r_2, r_3}{\mapsto} & -1, -3, -1, \\ \beta & \mapsto & 2x \otimes \alpha - 2x \otimes \beta + x \otimes \gamma - \beta \otimes x & \mapsto & 1, -1, 1, \\ \gamma & \mapsto & 2x \otimes \alpha - x \otimes \beta - \gamma \otimes x & \mapsto & 1, -1, 0, \\ \delta & \mapsto & 3x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1, -2, 0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', and therefore there are not effective-zero–divisors in this case.

3. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, -1, 1, -1]$, we have

$$\begin{array}{lll} \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, -s_b, s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, -s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2 \langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}\langle r_3 \rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \beta^* + x^* \otimes \delta^*,$$

and

$$\left\{ \begin{array}{lll} r_1 & = & x^* \otimes \alpha^* + x^* \otimes \beta^* + x^* \otimes \delta^*, \\ r_2 & = & x^* \otimes \beta^* + x^* \otimes \gamma^* + x^* \otimes \delta^*, \\ r_3 & = & x^* \otimes \delta^*. \end{array} \right.$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lllllll} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha - \alpha \otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x \otimes \beta - \beta \otimes x & \mapsto & 1 - \frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x \otimes \delta - \delta \otimes x & \mapsto & 1 - \frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = s_d = -1$ and $s_c = 1$. In such a case

$$\begin{array}{lll} r & = & \alpha^* \otimes x^* + \beta^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \beta^* + x^* \otimes \delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, -s_b, s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, -s_c, -s_d] = \mathbb{Z}_{bc}. \end{array}$$

Now analyzing the “twisted” diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - 2x\otimes\beta + x\otimes\delta - \alpha\otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & -2x\otimes\alpha + 2x\otimes\beta - x\otimes\gamma + \beta\otimes x & \mapsto & 1, \\ \gamma & \mapsto & 2x\otimes\alpha - x\otimes\beta - \gamma\otimes x & \mapsto & 1, \\ \delta & \mapsto & 3x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 0. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{abd}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{acd}$, and if we take a look at the “twisted” diagonal we see that

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - 2x\otimes\beta - x\otimes\gamma + \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,-3,0, \\ \beta & \mapsto & 2x\otimes\alpha - 2x\otimes\beta + x\otimes\gamma - \beta\otimes x & \mapsto & 0,-1,0, \\ \gamma & \mapsto & 2x\otimes\alpha - x\otimes\beta - \gamma\otimes x & \mapsto & 1,-1,0, \\ \delta & \mapsto & 3x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 1,-2,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', and therefore there are not effective-zero–divisors in this case.

4. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, -1, 1, 1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, -s_b, s_c, s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, -s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}\langle r_3\rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\beta^*,$$

and

$$\begin{cases} r_1 = x^*\otimes\alpha^* + x^*\otimes\beta^*, \\ r_2 = x^*\otimes\gamma^*, \\ r_3 = x^*\otimes\delta^*. \end{cases}$$

Let us analyze in the first case the “usual” diagonal:

$$\begin{array}{lclclcl} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha - \alpha\otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x\otimes\beta - \beta\otimes x & \mapsto & 1 - \frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\ \gamma & \mapsto & x\otimes\gamma + \gamma\otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x\otimes\delta + \delta\otimes x & \mapsto & 0 + \frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = -1$ and $s_c = s_d = 1$. In such a case

$$\begin{array}{lcl} r & = & \alpha^*\otimes x^* + \beta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\beta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, -s_b, s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, -s_c, -s_d] = \mathbb{Z}_{abcd}. \end{array}$$

Now analyzing the “twisted” diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - 2x\otimes\beta - x\otimes\delta + \alpha\otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & -2x\otimes\alpha + 2x\otimes\beta - x\otimes\gamma + \beta\otimes x & \mapsto & 1, \\ \gamma & \mapsto & 2x\otimes\alpha - x\otimes\beta - \gamma\otimes x & \mapsto & 1, \\ \delta & \mapsto & 3x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{ab}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{cd}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - 2x\otimes\beta - x\otimes\delta + \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,0,-1, \\ \beta & \mapsto & -2x\otimes\alpha + 2x\otimes\beta - x\otimes\gamma + \beta\otimes x & \mapsto & 0,-1,0, \\ \gamma & \mapsto & 2x\otimes\alpha - x\otimes\beta - \gamma\otimes x & \mapsto & 1,0,0, \\ \delta & \mapsto & 3x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 1,0,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', and therefore there are not effective-zero–divisors in this case.

5. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, 1, -1, -1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, s_b, -s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, -s_b, s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2\times\pi_2;\widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}\langle r_3\rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\gamma^* + x^*\otimes\delta^*,$$

and

$$\begin{cases} r_1 & = & x^*\otimes\alpha^*, \\ r_2 & = & x^*\otimes\beta^* - x^*\otimes\delta^*, \\ r_3 & = & x^*\otimes\alpha^* + x^*\otimes\gamma^* + x^*\otimes\delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{cclclcl} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha - \alpha\otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x\otimes\beta + \beta\otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x\otimes\gamma - \gamma\otimes x & \mapsto & 1 - \frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x\otimes\delta - \delta\otimes x & \mapsto & 1 - \frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_c = s_d = -1$ and $s_b = 1$. In such a case

$$\begin{array}{lcl} r & = & \alpha^*\otimes x^* + \gamma^*\otimes x^* + \delta^*\otimes x^* + x^*\otimes\alpha^* + x^*\otimes\gamma^* + x^*\otimes\delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, s_b, -s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, -s_b, s_c, -s_d] = \mathbb{Z}_{bc}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x\otimes\beta + x\otimes\delta - \alpha\otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & -2x\otimes\beta + x\otimes\gamma - \beta\otimes x & \mapsto & 1, \\ \gamma & \mapsto & x\otimes\beta + \gamma\otimes x & \mapsto & 1, \\ \delta & \mapsto & x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 0. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{acd}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{abd}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x\otimes\beta + x\otimes\delta - \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,-3,1, \\ \beta & \mapsto & -2x\otimes\beta + x\otimes\gamma - \beta\otimes x & \mapsto & 0,-2,1, \\ \gamma & \mapsto & x\otimes\beta + \gamma\otimes x & \mapsto & 0,1,0, \\ \delta & \mapsto & x\otimes\alpha - 2x\otimes\beta - \delta\otimes x & \mapsto & 1,-2,1. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', and therefore there are not effective-zero–divisors in this case.

6. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, 1, -1, 1]$, we have

$$\begin{array}{rcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, s_b, -s_c, s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, -s_b, s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}\langle r_3 \rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \gamma^*,$$

and

$$\begin{cases} r_1 & = & x^* \otimes \alpha^*, \\ r_2 & = & x^* \otimes \beta^* - x^* \otimes \delta^*, \\ r_3 & = & x^* \otimes \alpha^* + x^* \otimes \gamma^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lclclcl} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha - \alpha \otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x \otimes \beta + \beta \otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x \otimes \gamma - \gamma \otimes x & \mapsto & 1 - \frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x \otimes \delta + \delta \otimes x & \mapsto & 0 + \frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_c = -1$ and $s_b = s_d = 1$. In such a case

$$\begin{array}{lcl} r & = & \alpha^* \otimes x^* + \gamma^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \gamma^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, s_b, -s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, -s_b, s_c, -s_d] = \mathbb{Z}_{abcd}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta - x \otimes \delta + \alpha \otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & -2x \otimes \beta + x \otimes \gamma - \beta \otimes x & \mapsto & 1, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 1, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{ac}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{bd}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta - x \otimes \delta + \alpha \otimes x & \overset{r_1, r_2, r_3}{\mapsto} & 0, -1, 0, \\ \beta & \mapsto & -2x \otimes \beta + x \otimes \gamma - \beta \otimes x & \mapsto & 0, -2, 1, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 0, 1, 0, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1, -2, 1. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', and therefore there are not effective-zero–divisors in this case.

7. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, 1, 1, -1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_{\Phi} & = & \mathbb{Z}[-s_a, s_b, s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2 \langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}_2\langle r_3 \rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \delta^*,$$

and

$$\begin{cases} r_1 & = & x^* \otimes \alpha^*, \\ r_2 & = & x^* \otimes \beta^* + x^* \otimes \gamma^*, \\ r_3 & = & x^* \otimes \alpha^* + x^* \otimes \delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lclclcl} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha - \alpha \otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x \otimes \beta + \beta \otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x \otimes \delta - \delta \otimes x & \mapsto & 1 - \frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_d = -1$ and $s_b = s_c = 1$. In such a case

$$\begin{array}{lcl} r & = & \alpha^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \delta^*, \\ \mathbb{Z}_{\Phi} & = & \mathbb{Z}[-s_a, s_b, s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, s_c, -s_d] = \mathbb{Z}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta + x \otimes \delta - \alpha \otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & 2x \otimes \beta - x \otimes \gamma + \beta \otimes x & \mapsto & 0, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 0, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 0. \end{array}$$

It follows that in this case r is an effective-zero–divisor.

For the second case, it turns out that $\mathbb{Z}_{\Phi} = \mathbb{Z}_{ad}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{ad}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta + x \otimes \delta - \alpha \otimes x & \overset{r_1, r_2, r_3}{\mapsto} & 0, -2, 1, \\ \beta & \mapsto & 2x \otimes \beta - x \otimes \gamma + \beta \otimes x & \mapsto & 0, 1, 0, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 0, 1, 0, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1, -2, 1. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

8. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, -1, 1, 1, 1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_{\Phi} & = & \mathbb{Z}[-s_a, s_b, s_c, s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, s_c, -s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2 \langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}\langle r_3 \rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \alpha^*,$$

and

$$\begin{cases} r_1 &= x^* \otimes \alpha^*, \\ r_2 &= x^* \otimes \gamma^*, \\ r_3 &= x^* \otimes \delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lclclcl} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha - \alpha \otimes x & \overset{r}{\mapsto} & 1 - \frac{1-s_a}{2} & = & \frac{s_a+1}{2}, \\ \beta & \mapsto & x \otimes \beta + \beta \otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x \otimes \delta + \delta \otimes x & \mapsto & 0 + \frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = -1$ and $s_b = s_c = s_d = 1$. In such a case

$$\begin{array}{lcl} r & = & \alpha^* \otimes x^* + x^* \otimes \alpha^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[-s_a, s_b, s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, s_c, -s_d] = \mathbb{Z}_{ad}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta - x \otimes \delta + \alpha \otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & 2x \otimes \beta - x \otimes \gamma + \beta \otimes x & \mapsto & 0, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 0, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_a$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_d$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & -2x \otimes \beta - x \otimes \delta + \alpha \otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,0,1, \\ \beta & \mapsto & 2x \otimes \beta - x \otimes \gamma + \beta \otimes x & \mapsto & 0,-1,0, \\ \gamma & \mapsto & x \otimes \beta + \gamma \otimes x & \mapsto & 0,0,0, \\ \delta & \mapsto & x \otimes \alpha - 2x \otimes \beta - \delta \otimes x & \mapsto & 1,0,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

9. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, -1, -1, -1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, -s_b, -s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, -s_b, -s_c, s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2 \langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}\langle r_3 \rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \beta^* + x^* \otimes \gamma^* + x^* \otimes \delta^*,$$

and

$$\begin{cases} r_1 &= x^*\otimes\beta^* + x^*\otimes\gamma^* + x^*\otimes\delta^*,\\ r_2 &= x^*\otimes\gamma^* + x^*\otimes\delta^*,\\ r_3 &= x^*\otimes\alpha^* + x^*\otimes\delta^*.\end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lclclcl} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha+\alpha\otimes x & \overset{r}{\mapsto} & 0+\frac{1-s_a}{2} & = & \frac{1-s_a}{2},\\ \beta & \mapsto & x\otimes\beta-\beta\otimes x & \mapsto & 1-\frac{1-s_b}{2} & = & \frac{s_b+1}{2},\\ \gamma & \mapsto & x\otimes\gamma-\gamma\otimes x & \mapsto & 1-\frac{1-s_c}{2} & = & \frac{s_c+1}{2},\\ \delta & \mapsto & x\otimes\delta-\delta\otimes x & \mapsto & 1-\frac{1-s_d}{2} & = & \frac{s_d+1}{2},\end{array}$$

thus $\Phi^*(r)=0$ only for $s_a=1$ and $s_b=s_c=s_d=-1$. In such a case

$$\begin{array}{lcl} r & = & \beta^*\otimes x^*+\gamma^*\otimes x^*+\delta^*\otimes x^*+x^*\otimes\beta^*+x^*\otimes\gamma^*+x^*\otimes\delta^*,\\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a,-s_b,-s_c,-s_d]=\mathbb{Z},\\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a,-s_b,-s_c,s_d]=\mathbb{Z}_{ad}.\end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha+x\otimes\delta-\alpha\otimes x & \overset{r}{\mapsto} & 1,\\ \beta & \mapsto & 2x\otimes\alpha+x\otimes\gamma-\beta\otimes x & \mapsto & 0,\\ \gamma & \mapsto & 2x\otimes\alpha+x\otimes\beta-\gamma\otimes x & \mapsto & 0,\\ \delta & \mapsto & x\otimes\alpha+\delta\otimes x & \mapsto & 1.\end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi=\mathbb{Z}_{bcd}$ and $\mathbb{Z}_{\Phi''}=\mathbb{Z}_{abc}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha+x\otimes\delta-\alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 1,1,3,\\ \beta & \mapsto & 2x\otimes\alpha+x\otimes\gamma-\beta\otimes x & \mapsto & 1,1,2,\\ \gamma & \mapsto & 2x\otimes\alpha+x\otimes\beta-\gamma\otimes x & \mapsto & 1,0,2,\\ \delta & \mapsto & x\otimes\alpha+\delta\otimes x & \mapsto & 0,0,1.\end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

10. $\widetilde{\mathbb{Z}}=\mathbb{Z}[s_a,s_b,s_c,s_d,1,-1,-1,1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a,-s_b,-s_c,s_d],\\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a,-s_b,-s_c,s_d],\end{array}$$

By direct calculation we obtain that $H^1(\pi_2\times\pi_2;\widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1\rangle\oplus\mathbb{Z}_2\langle r_2\rangle\oplus\mathbb{Z}\langle r_3\rangle$ (if all of the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\beta^* + x^*\otimes\gamma^*,$$

and

$$\begin{cases} r_1 &= x^*\otimes\alpha^* + x^*\otimes\delta^*,\\ r_2 &= x^*\otimes\beta^* + x^*\otimes\gamma^*,\\ r_3 &= x^*\otimes\gamma^*.\end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lllllll}
\alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha+\alpha\otimes x & \overset{r}{\mapsto} & 0+\frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\
\beta & \mapsto & x\otimes\beta-\beta\otimes x & \mapsto & 1-\frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\
\gamma & \mapsto & x\otimes\gamma-\gamma\otimes x & \mapsto & 1-\frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\
\delta & \mapsto & x\otimes\delta+\delta\otimes x & \mapsto & 0+\frac{1-s_d}{2} & = & \frac{1-s_d}{2},
\end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_d = 1$ and $s_b = s_c = -1$. In such a case

$$\begin{array}{lll}
r & = & \beta^*\otimes x^*+\gamma^*\otimes x^*+x^*\otimes\beta^*+x^*\otimes\gamma^*, \\
\mathbb{Z}_\Phi & = & \mathbb{Z}[s_a,-s_b,-s_c,s_d]=\mathbb{Z}, \\
\mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a,-s_b,-s_c,s_d]=\mathbb{Z}.
\end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lllll}
\alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha-x\otimes\delta+\alpha\otimes x & \overset{r}{\mapsto} & 0, \\
\beta & \mapsto & 2x\otimes\alpha+x\otimes\gamma-\beta\otimes x & \mapsto & 0, \\
\gamma & \mapsto & 2x\otimes\alpha+x\otimes\beta-\gamma\otimes x & \mapsto & 0, \\
\delta & \mapsto & x\otimes\alpha+\delta\otimes x & \mapsto & 0.
\end{array}$$

It follows that in this case r is an effective-zero–divisor.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{bc}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{bc}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lllll}
\alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha-x\otimes\delta+\alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 1,0,0, \\
\beta & \mapsto & 2x\otimes\alpha+x\otimes\gamma-\beta\otimes x & \mapsto & 2,1,1, \\
\gamma & \mapsto & 2x\otimes\alpha+x\otimes\beta-\gamma\otimes x & \mapsto & 2,1,0, \\
\delta & \mapsto & x\otimes\alpha+\delta\otimes x & \mapsto & 1,0,0.
\end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

11. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a,s_b,s_c,s_d,1,-1,1,-1]$, we have

$$\begin{array}{lll}
\mathbb{Z}_\Phi & = & \mathbb{Z}[s_a,-s_b,s_c,-s_d], \\
\mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a,s_b,-s_c,s_d].
\end{array}$$

By direct calculation we obtain that $H^1(\pi_2\times\pi_2;\widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1\rangle\oplus\mathbb{Z}_2\langle r_2\rangle\oplus\mathbb{Z}\langle r_3\rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\beta^* + x^*\otimes\delta^*,$$

and

$$\begin{cases}
r_1 = x^*\otimes\alpha^*-x^*\otimes\gamma^*, \\
r_2 = x^*\otimes\beta^*+x^*\otimes\delta^*, \\
r_3 = x^*\otimes\delta^*.
\end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lllllll}
\alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha+\alpha\otimes x & \overset{r}{\mapsto} & 0+\frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\
\beta & \mapsto & x\otimes\beta-\beta\otimes x & \mapsto & 1-\frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\
\gamma & \mapsto & x\otimes\gamma+\gamma\otimes x & \mapsto & 0+\frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\
\delta & \mapsto & x\otimes\delta-\delta\otimes x & \mapsto & 1-\frac{1-s_d}{2} & = & \frac{s_d+1}{2},
\end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_c = 1$ and $s_b = s_d = -1$. In such a case

$$\begin{array}{rcl} r & = & \beta^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \beta^* + x^* \otimes \delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, -s_b, s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, -s_c, s_d] = \mathbb{Z}_{abcd}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{rclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x \otimes \alpha + x \otimes \delta - \alpha \otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & -2x \otimes \alpha - x \otimes \gamma + \beta \otimes x & \mapsto & 1, \\ \gamma & \mapsto & 2x \otimes \alpha + x \otimes \beta - \gamma \otimes x & \mapsto & 1, \\ \delta & \mapsto & x \otimes \alpha + \delta \otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{bd}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{ac}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{rclcl} \alpha & \overset{\Phi''}{\mapsto} & 2x \otimes \alpha + x \otimes \delta - \alpha \otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 2, 1, 1, \\ \beta & \mapsto & -2x \otimes \alpha - x \otimes \gamma + \beta \otimes x & \mapsto & -1, 0, 0, \\ \gamma & \mapsto & 2x \otimes \alpha + x \otimes \beta - \gamma \otimes x & \mapsto & 2, 1, 0, \\ \delta & \mapsto & x \otimes \alpha + \delta \otimes x & \mapsto & 1, 0, 0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

12. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, -1, 1, 1]$, we have

$$\begin{array}{rcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, -s_b, s_c, s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, -s_c, s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2 \langle r \rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}_2\langle r_1 \rangle \oplus \mathbb{Z}\langle r_2 \rangle \oplus \mathbb{Z}\langle r_3 \rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \beta^*,$$

and

$$\begin{cases} r_1 & = & x^* \otimes \beta^*, \\ r_2 & = & x^* \otimes \gamma^*, \\ r_3 & = & x^* \otimes \delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{rclclcl} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha + \alpha \otimes x & \overset{r}{\mapsto} & 0 + \frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\ \beta & \mapsto & x \otimes \beta - \beta \otimes x & \mapsto & 1 - \frac{1-s_b}{2} & = & \frac{s_b+1}{2}, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x \otimes \delta + \delta \otimes x & \mapsto & 0 + \frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_c = s_d = 1$ and $s_b = -1$. In such a case

$$\begin{array}{rcl} r & = & \beta^* \otimes x^* + x^* \otimes \beta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, -s_b, s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, s_b, -s_c, s_d] = \mathbb{Z}_{bc}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lllll} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - x\otimes\delta + \alpha\otimes x & \overset{r}{\mapsto} & 0, \\ \beta & \mapsto & -2x\otimes\alpha - x\otimes\gamma + \beta\otimes x & \mapsto & 1, \\ \gamma & \mapsto & 2x\otimes\alpha + x\otimes\beta - \gamma\otimes x & \mapsto & 1, \\ \delta & \mapsto & x\otimes\alpha + \delta\otimes x & \mapsto & 0. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_b$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_c$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{lllll} \alpha & \overset{\Phi''}{\mapsto} & 2x\otimes\alpha - x\otimes\delta + \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,0,-1, \\ \beta & \mapsto & -2x\otimes\alpha - x\otimes\gamma + \beta\otimes x & \mapsto & 0,-1,0, \\ \gamma & \mapsto & 2x\otimes\alpha + x\otimes\beta - \gamma\otimes x & \mapsto & 1,0,0, \\ \delta & \mapsto & x\otimes\alpha + \delta\otimes x & \mapsto & 0,0,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

13. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, 1, -1, -1]$, we have

$$\begin{array}{lll} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, -s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, -s_b, s_c, s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}_2\langle r_3\rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \gamma^* + x^* \otimes \delta^*,$$

and

$$\begin{cases} r_1 = x^* \otimes \alpha^*, \\ r_2 = x^* \otimes \beta^*, \\ r_3 = x^* \otimes \gamma^* + x^* \otimes \delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{lllllll} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha + \alpha\otimes x & \overset{r}{\mapsto} & 0 + \frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\ \beta & \mapsto & x\otimes\beta + \beta\otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x\otimes\gamma - \gamma\otimes x & \mapsto & 1 - \frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x\otimes\delta - \delta\otimes x & \mapsto & 1 - \frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = 1$ and $s_c = s_d = -1$. In such a case

$$\begin{array}{lll} r & = & \gamma^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \gamma^* + x^* \otimes \delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, -s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, -s_b, s_c, s_d] = \mathbb{Z}_{abcd}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{lllll} \alpha & \overset{\Phi''}{\mapsto} & x\otimes\delta - \alpha\otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & x\otimes\gamma - \beta\otimes x & \mapsto & 1, \\ \gamma & \mapsto & -x\otimes\beta + \gamma\otimes x & \mapsto & 1, \\ \delta & \mapsto & -x\otimes\alpha + \delta\otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_{cd}$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_{ab}$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & x\otimes\delta - \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,0,1, \\ \beta & \mapsto & x\otimes\gamma - \beta\otimes x & \mapsto & 0,0,1, \\ \gamma & \mapsto & -x\otimes\beta + \gamma\otimes x & \mapsto & 0,-1,0, \\ \delta & \mapsto & -x\otimes\alpha + \delta\otimes x & \mapsto & 1,0,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

14. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, 1, -1, 1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, -s_c, s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, -s_b, s_c, s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}_2\langle r_3\rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^*\otimes x^* + \frac{1-s_b}{2}\beta^*\otimes x^* + \frac{1-s_c}{2}\gamma^*\otimes x^* + \frac{1-s_d}{2}\delta^*\otimes x^* + x^*\otimes\gamma^*,$$

and

$$\left\{ \begin{array}{lcl} r_1 & = & x^*\otimes\alpha^*, \\ r_2 & = & x^*\otimes\beta^*, \\ r_3 & = & x^*\otimes\gamma^*. \end{array} \right.$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{cclclcl} \alpha & \overset{\Phi}{\mapsto} & x\otimes\alpha + \alpha\otimes x & \overset{r}{\mapsto} & 0 + \frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\ \beta & \mapsto & x\otimes\beta + \beta\otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x\otimes\gamma - \gamma\otimes x & \mapsto & 1 - \frac{1-s_c}{2} & = & \frac{s_c+1}{2}, \\ \delta & \mapsto & x\otimes\delta + \delta\otimes x & \mapsto & 0 + \frac{1-s_d}{2} & = & \frac{1-s_d}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = s_d = 1$ and $s_c = -1$. In such a case

$$\begin{array}{lcl} r & = & \gamma^*\otimes x^* + x^*\otimes\gamma^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, -s_c, s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[s_a, -s_b, s_c, s_d] = \mathbb{Z}_{bc}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & -x\otimes\delta + \alpha\otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & x\otimes\gamma - \beta\otimes x & \mapsto & 0, \\ \gamma & \mapsto & -x\otimes\beta + \gamma\otimes x & \mapsto & 0, \\ \delta & \mapsto & -x\otimes\alpha + \delta\otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_c$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_b$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{cclcl} \alpha & \overset{\Phi''}{\mapsto} & -x\otimes\delta + \alpha\otimes x & \overset{r_1,r_2,r_3}{\mapsto} & 0,0,0, \\ \beta & \mapsto & x\otimes\gamma - \beta\otimes x & \mapsto & 0,0,1, \\ \gamma & \mapsto & -x\otimes\beta + \gamma\otimes x & \mapsto & 0,-1,0, \\ \delta & \mapsto & -x\otimes\alpha + \delta\otimes x & \mapsto & -1,0,0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

15. $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, 1, 1, -1]$, we have

$$\begin{array}{lcl} \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, s_c, -s_d], \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, s_c, s_d]. \end{array}$$

By direct calculation we obtain that $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}})$ is either $\mathbb{Z}_2\langle r\rangle$ (if at least one of the s_x is -1) or $\mathbb{Z}\langle r_1\rangle \oplus \mathbb{Z}\langle r_2\rangle \oplus \mathbb{Z}_2\langle r_3\rangle$ (if all the s_x are 1), where

$$r = \frac{1-s_a}{2}\alpha^* \otimes x^* + \frac{1-s_b}{2}\beta^* \otimes x^* + \frac{1-s_c}{2}\gamma^* \otimes x^* + \frac{1-s_d}{2}\delta^* \otimes x^* + x^* \otimes \delta^*,$$

and

$$\begin{cases} r_1 & = & x^* \otimes \alpha^*, \\ r_2 & = & x^* \otimes \beta^*, \\ r_3 & = & x^* \otimes \delta^*. \end{cases}$$

Let us analyze in the first case the "usual" diagonal:

$$\begin{array}{ccccccc} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha + \alpha \otimes x & \overset{r}{\mapsto} & 0 + \frac{1-s_a}{2} & = & \frac{1-s_a}{2}, \\ \beta & \mapsto & x \otimes \beta + \beta \otimes x & \mapsto & 0 + \frac{1-s_b}{2} & = & \frac{1-s_b}{2}, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x & \mapsto & 0 + \frac{1-s_c}{2} & = & \frac{1-s_c}{2}, \\ \delta & \mapsto & x \otimes \delta - \delta \otimes x & \mapsto & 1 - \frac{1-s_d}{2} & = & \frac{s_d+1}{2}, \end{array}$$

thus $\Phi^*(r) = 0$ only for $s_a = s_b = s_c = 1$ and $s_d = -1$. In such a case

$$\begin{array}{lcl} r & = & \delta^* \otimes x^* + x^* \otimes \delta^*, \\ \mathbb{Z}_\Phi & = & \mathbb{Z}[s_a, s_b, s_c, -s_d] = \mathbb{Z}, \\ \mathbb{Z}_{\Phi''} & = & \mathbb{Z}[-s_a, s_b, s_c, s_d] = \mathbb{Z}_{ad}. \end{array}$$

Now analyzing the "twisted" diagonal we obtain

$$\begin{array}{ccccc} \alpha & \overset{\Phi''}{\mapsto} & x \otimes \delta - \alpha \otimes x & \overset{r}{\mapsto} & 1, \\ \beta & \mapsto & -x \otimes \gamma + \beta \otimes x & \mapsto & 0, \\ \gamma & \mapsto & -x \otimes \beta + \gamma \otimes x & \mapsto & 0, \\ \delta & \mapsto & -x \otimes \alpha + \delta \otimes x & \mapsto & 1. \end{array}$$

It follows that in this case there are not effective-zero–divisors.

For the second case, it turns out that $\mathbb{Z}_\Phi = \mathbb{Z}_d$ and $\mathbb{Z}_{\Phi''} = \mathbb{Z}_a$, and if we take a look at the "twisted" diagonal we see that

$$\begin{array}{ccccc} \alpha & \overset{\Phi''}{\mapsto} & x \otimes \delta - \alpha \otimes x & \overset{r_1, r_2, r_3}{\mapsto} & 0, 0, 1, \\ \beta & \mapsto & -x \otimes \gamma + \beta \otimes x & \mapsto & 0, 0, 0, \\ \gamma & \mapsto & -x \otimes \beta + \gamma \otimes x & \mapsto & 0, -1, 0, \\ \delta & \mapsto & -x \otimes \alpha + \delta \otimes x & \mapsto & -1, 0, 0. \end{array}$$

From where it follows that there is no linear combination of r_1, r_2 and r_3 that vanishes over Φ'', then for the second case the are not effective-zero–divisors.

16. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[s_a, s_b, s_c, s_d, 1, 1, 1, 1]$, we have sixteen cases.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,-1,-1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* + \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* - \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{abcd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,-1,-1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* + \gamma^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* + \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{abc})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,-1,1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* - \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{abd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,-1,1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \beta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{ab})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,1,-1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \beta^* \otimes x^* + \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{acd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,1,-1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \gamma^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \beta^* \otimes x^* - \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{ac})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,1,1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \beta^* \otimes x^* + \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{ad})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,1,1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \alpha^* \otimes x^* \rangle}{2\mathbb{Z}\langle \alpha^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{a})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,-1,-1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \beta^* \otimes x^* + \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* - \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{bcd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,-1,-1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle}{2\mathbb{Z}\langle \beta^* \otimes x^* + \gamma^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* + \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{bc})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,-1,1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \beta^* \otimes x^* + \delta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \alpha^* \otimes x^* - \gamma^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{bd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,-1,1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \frac{\mathbb{Z}\langle \beta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \beta^* \otimes x^* \rangle} \oplus \mathbb{Z}\langle \gamma^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \delta^* \otimes x^* \rangle \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{b})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,1,-1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \frac{\mathbb{Z}\langle \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \gamma^* \otimes x^* + \delta^* \otimes x^* \rangle} \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_{cd})$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,1,-1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \frac{\mathbb{Z}\langle \gamma^* \otimes x^* \rangle}{2\mathbb{Z}\langle \gamma^* \otimes x^* \rangle} \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_c)$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,1,1,-1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \mathbb{Z}\langle \alpha^* \otimes x^* \rangle \oplus \mathbb{Z}\langle \beta^* \otimes x^* \rangle \oplus \frac{\mathbb{Z}\langle \delta^* \otimes x^* \rangle}{2\mathbb{Z}\langle \delta^* \otimes x^* \rangle} \overset{\Phi^*}{\cong} H^1(\pi_2; \mathbb{Z}_d)$.

- For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,1,1,1,1,1,1,1]$,
  $H^1(\pi_2 \times \pi_2; \widetilde{\mathbb{Z}}) = \mathbb{Z}\langle x^* \otimes \alpha^*, x^* \otimes \beta^*, x^* \otimes \gamma^*, x^* \otimes \delta^*, \alpha^* \otimes x^*, \beta^* \otimes x^*, \gamma^* \otimes x^*, \delta^* \otimes x^* \rangle$.
  The "usual" diagonal takes the following values

$$\begin{array}{ccl} \alpha & \overset{\Phi}{\mapsto} & x \otimes \alpha + \alpha \otimes x, \\ \beta & \mapsto & x \otimes \beta + \beta \otimes x, \\ \gamma & \mapsto & x \otimes \gamma + \gamma \otimes x, \\ \delta & \mapsto & x \otimes \delta + \delta \otimes x, \end{array}$$

  and the "twisted" diagonal takes the following values

$$\begin{array}{ccl} \alpha & \overset{\Phi''}{\mapsto} & -x \otimes \delta + \alpha \otimes x, \\ \beta & \mapsto & -x \otimes \gamma + \beta \otimes x, \\ \gamma & \mapsto & -x \otimes \beta + \gamma \otimes x, \\ \delta & \mapsto & -x \otimes \alpha + \delta \otimes x. \end{array}$$

  It follows that in this case that

$$\alpha^* \otimes x^* - \delta^* \otimes x^* - x^* \otimes \alpha^* + x^* \otimes \delta^*$$

  and

$$\beta^* \otimes x^* - \gamma^* \otimes x^* - x^* \otimes \beta^* + x^* \otimes \gamma^*$$

  are effective-zero–divisors.

Summarizing, we have found five 1-dimensional effective zero–divisors, namely

1. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,-1,-1,-1,-1,-1,-1,-1]$

$$d_1 = \alpha^* \otimes x^* + \beta^* \otimes x^* + \gamma^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \beta^* + x^* \otimes \gamma^* + x^* \otimes \delta^*.$$

2. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[-1,1,1,-1,-1,1,1,-1]$

$$d_2 = \alpha^* \otimes x^* + \delta^* \otimes x^* + x^* \otimes \alpha^* + x^* \otimes \delta^*.$$

3. For $\widetilde{\mathbb{Z}} = \mathbb{Z}[1,-1,-1,1,1,-1,-1,1]$

$$d_3 = \beta^* \otimes x^* + \gamma^* \otimes x^* + x^* \otimes \beta^* + x^* \otimes \gamma^*.$$

4. For $\mathbb{Z}$,

$$d_4 = \alpha^* \otimes x^* - \delta^* \otimes x^* - x^* \otimes \alpha^* + x^* \otimes \delta^*.$$

   and

$$d_5 = \beta^* \otimes x^* - \gamma \otimes x^* - x^* \otimes \beta^* + x^* \otimes \gamma^*.$$

Now if we look for products of length four of these zero–divisors, we will find that all of them are zero. Nevertheless, with coefficients $\mathbb{Z}$ trivial,

$$d_4 \cup d_5 \;=\; -\alpha^* \otimes \beta^* + \alpha^* \otimes \gamma^* + \delta^* \otimes \beta^* - \delta^* \otimes \gamma^* + \beta^* \otimes \alpha^* - \gamma^* \otimes \alpha^* - \beta^* \otimes \delta^* + \gamma^* \otimes \delta^*.$$

Furthermore, with coefficients $\mathbb{Z}$ trivial there are sixteen effective zero–divisors in dimension 2, of which

$$\alpha^* \otimes \alpha^*, \alpha^* \otimes \delta^*, \beta^* \otimes \beta^*, \beta^* \otimes \gamma^*, \gamma^* \otimes \beta^*, \gamma^* \otimes \gamma^*, \delta^* \otimes \alpha^*, \delta^* \otimes \delta^*,$$

have trivial product with $d_4 \cup d_5$, and the remaining,

$$\begin{array}{ll} w^* \otimes x^* + 2\delta^* \otimes \gamma^* + x^* \otimes w^*, & \alpha^* \otimes \beta^* + \delta^* \otimes \gamma^*, \\ \alpha^* \otimes \gamma^* - \delta^* \otimes \gamma^* - x^* \otimes w^*, & \beta^* \otimes \alpha^* - \delta^* \otimes \gamma^*, \\ \beta^* \otimes \delta^* + \delta^* \otimes \gamma^* + x^* \otimes w^*, & \gamma^* \otimes \alpha^* - \delta^* \otimes \gamma^* - x^* \otimes w^*, \\ \gamma^* \otimes \delta^* + \delta^* \otimes \gamma^*, & \delta^* \otimes \beta^* + \delta^* \otimes \gamma^* + x^* \otimes w^*, \end{array}$$

each multiply with $d_4 \cup d_5$ to give $\pm 2\omega^* \otimes \omega^*$. We have therefore established:

Proposition 4.7. $3 \leq \mathbf{TC}^{\sigma}(\Sigma_2) \leq 4$.

4.2 *The Bernstein-Costa-Farber class*

In [8] A. Costa and M. Farber explicitly described, for a connected CW complex X, a crossed homomorphism representing the primary obstruction to the existence of a continuous section of $e_{0,1}$. Here we discuss the universality of such a class and its analogous class $o_{g,\mathbb{Z}_2}$. This universal property is suggested by the corresponding property for the Bernstein class in the Lusternik–Schnirelmann category, see [2], [26], and [11].

Let $x_0 \in X$ the base point, $\pi = \pi_1(X, x_0)$ a discrete group, $I(\pi) = \ker(\epsilon : \mathbb{Z}[\pi] \to \mathbb{Z})$, and M an abelian group on which $\pi \times \pi$ operates. $I(\pi)$ is a $\mathbb{Z}[\pi \times \pi]$-module via the restriction of the action of $\mathbb{Z}[\pi \times \pi]$ over $\mathbb{Z}[\pi]$,

$$(x, y) \cdot \sum n_i c_i = \sum n_i x c_i \overline{y}, \quad x, y, c_i \in \pi \text{ and } n_i \in \mathbb{Z}.$$

Fix the crossed homomorphism (introduced by A. Costa and M. Farber)

$$\begin{array}{rccc} \mathrm{v}: & \pi \times \pi & \longrightarrow & I(\pi) \\ & (x, y) & \mapsto & x\overline{y} - 1 \end{array} \tag{4.13}$$

and denote the corresponding one-dimensional cohomology class by $\overline{\mathrm{v}} \in H^1\left(X \times X; \underline{I(\pi)}\right)$.

Remark 4.5. In fact v represents a zero-divisor. See [8].

Lemma 4.8. *For $f \in Q(\pi \times \pi, M)$ being trivial on the diagonal $\Delta_{\pi\times\pi} \subset \pi \times \pi$ is equivalent to the condition $f(x, 1) = f(1, \overline{x})$ for all $x \in \pi$.*

Proof. Given $x \in \pi$,

$$\begin{aligned} f(x, x) &= f((1, x), (x, 1)) \\ &= f(1, x) + (1, x) f(x, 1) \end{aligned}$$

since f is a crossed morphism. Then for all $x \in \pi$,

$$(1, \overline{x}) f(x, x) - f(x, 1) = (1, \overline{x}) f(1, x).$$

Furthermore, $f(1,1) = 0$, and for all $x \in \pi$,

$$\begin{aligned} f(1,1) =& f((1,\overline{x})(1,x)) \\ =& f(1,\overline{x}) + (1,\overline{x})f(1,x) \\ =& f(1,\overline{x}) + (1,\overline{x})f(x,x) - f(x,1) \end{aligned}$$

i.e.

$$f(x,1) = f(1,\overline{x}) + (1,\overline{x})f(x,x), \quad x \in \pi.$$

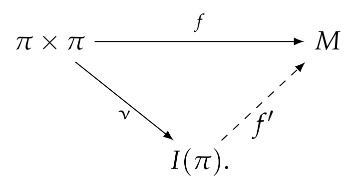
The result follows. □

Remark 4.6. If $f \in Q(\pi \times \pi, M)$ is trivial on $\Delta_{\pi\times\pi}$, then $f(x,y) = f(xz,yz)$ for all $x,y,z \in \pi$. Since

$$\begin{aligned} f(xz,yz) &= f\left((x,y)(z,z)\right) \\ &= f(x,y) + (x,y)\cdot f(z,z) \\ &= f(x,y) + (x,y)\cdot 0 \\ &= f(x,y), \end{aligned}$$

where the last equality is because the action is by homomorphisms.

Lemma 4.9. *Let $f \in Q(\pi \times \pi, M)$ such that it is trivial when restricted to the diagonal $\Delta_{\pi\times\pi} \subset \pi \times \pi$. Then there exists a $(\pi \times \pi)$-morphism $f' : I(\pi) \to M$ making commutative the diagram*



Proof. Suppose f as in the statement. Consider the additive morphism $f' : I(\pi) \to M$ determined on basis elements by setting

$$f'(x-1) = f(x,1) \text{ for } x \in \pi,$$

and extending it by linearity. The diagram commutes since

$$\begin{aligned} f' \circ \nu(x,y) =& f'(x\overline{y} - 1) \\ =& f(x\overline{y}, 1) \\ =& f(x,y), \text{ by remark 4.6}. \end{aligned}$$

In order to show that f' is a $(\pi \times \pi)$-morphism notice that

$$\begin{aligned} f'\left((y,1)\cdot(x-1)\right) - (y,1)\cdot f'(x-1) =& f'(yx - y) - (y,1)\cdot f'(x-1) \\ =& f(yx,1) - f(y,1) - (y,1)\cdot f(x,1) \\ =& f(y,1) + (y,1)\cdot f(x,1) - f(y,1) - (y,1)\cdot f(x,1) \\ =& 0 \end{aligned}$$

and

$$\begin{aligned} f'\left((1,y)\cdot(x-1)\right) - (1,y)\cdot f'(x-1) =& f'(x\overline{y} - \overline{y}) - (1,y)\cdot f'(x-1) \\ =& f(x\overline{y},1) - f(\overline{y},1) - (1,y)\cdot f(x,1) \\ =& f(x,1) + (x,1)\cdot f(\overline{y},1) - f(\overline{y},1) - (1,y)\cdot f(x,1) \\ =& f(x,1) + (x,1)\cdot f(1,y) - f(1,y) - (1,y)\cdot f(x,1) \\ =& f((x,1)(1,y)) - f((1,y)(x,1)) \\ =& 0 \end{aligned}$$

i.e.

$$f'((y,1)\cdot(x-1)) = (y,1)\cdot f'(x-1)$$

and

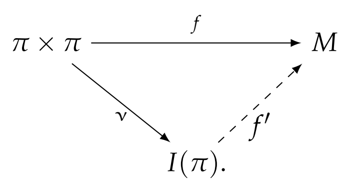
$$f'((1,y)\cdot(x-1)) = (1,y)\cdot f'(x-1).$$

Then

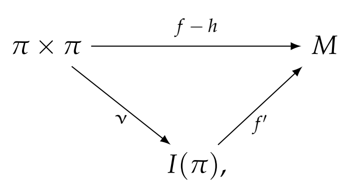
$$\begin{aligned} f'((y,z)\cdot(x-1)) =& f'((y,1)\cdot[(1,z)\cdot(x-1)]) \\ =&(y,1)\cdot f'([(1,z)\cdot(x-1)]) \\ =&(y,1)\cdot(1,z)\cdot f'(x-1) \\ =&(y,z)\cdot f'(x-1). \end{aligned}$$

□

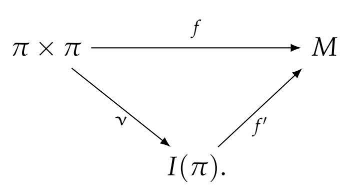
Corollary 4.9.1. *Let $f \in Q(\pi\times\pi, M)$ such that $f(x,x) = (x,x)\cdot m - m$ for some $m \in M$. Then there exists a $(\pi\times\pi)$-morphism $f' : I(\pi) \to M$ making commutative the diagram up to a principal morphism*



Proof. Let $h \in P(\pi\times\pi, M)$ such that $h(x,y) = (x,y)\cdot m - m$. Then $f-h$ is a crossed morphism strictly trivial on the diagonal, therefore there is a $(\pi\times\pi)$-morphism $f' : I(\pi) \to M$ making commutative the diagram



i.e. the diagram below commutes up to a principal morphism



□

The theorem 4.10 is an immediate consequence of the discussion above, lemma 4.9 and corollary 4.9.1.

Theorem 4.10. *For any zero divisor $\overline{f} \in H^1(X\times X;\mathcal{M})$ there exists a $(\pi\times\pi)$-morphism $I(\pi)\to M$ such that the induced homomorphism for cohomology takes $\overline{\nu}$ to $\overline{f}$ (where $\mathcal{M}$ denotes the local coefficient system over $X\times X$ determined by M).*

An analogous result holds for 1-dimensional effective-zero–divisors.

Theorem 4.11. *Let M be a $\mathbb{Z}[\pi_2 \times \pi_2]$-module and $\varphi : \pi_2 \times \pi_2 \to M$ a crossed homomorphism which corresponds to an effective–zero–divisor via the isomorphism $H^1(\Sigma_2 \times \Sigma_2; \mathcal{M}) \cong Q(\pi_2 \times \pi_2, M)/P(\pi_2 \times \pi_2, M)$, i.e.*

$$\varphi(x,x) = (x,x) \cdot m_1 - m_1,$$
$$\varphi(x, \widetilde{\sigma}^{-1}(x)) = (x, \widetilde{\sigma}^{-1}(x)) \cdot m_2 - m_2,$$

for elements $m_1, m_2 \in M$. Then there exists a $(\pi_2 \times \pi_2)$-morphism $\psi : I(\mathbb{Z}_2 \times \pi_2) \to M$ such that the induced homomorphism for cohomology takes $\mathfrak{o}_{2,\mathbb{Z}_2}$ to $\overline{\varphi}$.

Proof. We define $\psi \colon I(\mathbb{Z}_2 \times \pi_2) \to M$ by

$$\psi((e,x) - (e,1)) = (\varphi - \varphi_{m_1})(x,1), \tag{4.14}$$
$$\psi((\sigma,x) - (e,1)) = (1 - \widetilde{\sigma}^{-1}) \cdot (m_1 - m_2) + (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(x),1), \tag{4.15}$$

where $\varphi_{m_1}(x,y) = (x,y) \cdot m_1 - m_1$.
For (4.15)

$$\begin{aligned}\psi((a,1) \cdot [(\sigma,x) - (e,1)]) =&(1, \widetilde{\sigma}^{-1}(\overline{x})\widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2) \\ &+ (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(ax),1) - (\varphi - \varphi_{m_1})(a,1),\end{aligned} \tag{4.16}$$
$$\begin{aligned}(a,1) \cdot \psi((\sigma,x) - (e,1)) =&(a, \widetilde{\sigma}^{-1}(\overline{x})) \cdot (m_1 - m_2) \\ &+ (a,1) \cdot (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(x),1).\end{aligned} \tag{4.17}$$

Let us check that (4.16) and (4.17) are equal. In fact, notice that

$$\begin{aligned}(\varphi - \varphi_{m_1})(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) &= \varphi(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) - \varphi_{m_1}(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \\ &= \varphi(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) - [(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot m_1 - m_1] \\ &= [(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot m_2 - m_2] \cdot [(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot m_1 - m_1] \qquad (4.18) \\ &= (m_1 - m_2) - (\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2). \qquad (4.19)\end{aligned}$$

From (4.19):

$$(\varphi - \varphi_{m_1})(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) + (\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2) = m_1 - m_2. \tag{4.20}$$

Substracting $(\varphi - \varphi_{m_1})(1, \widetilde{\sigma}^{-1}(x))$ in (4.20) we obtain the equality

$$\begin{aligned}&(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2) + (\varphi - \varphi_{m_1})(\overline{a}\widetilde{\sigma}^{-1}(\overline{a}), \widetilde{\sigma}^{-1}(x)) + \\ &(\overline{a}\widetilde{\sigma}^{-1}(\overline{a}), \widetilde{\sigma}^{-1}(x)) \cdot (\varphi - \varphi_{m_1})(1, \widetilde{\sigma}^{-1}(\overline{x})) - ((\varphi - \varphi_{m_1})(1, \widetilde{\sigma}^{-1}(\overline{x})) \\ &= (m_1 - m_2) + (1, \widetilde{\sigma}^{-1}(\overline{x})) \cdot (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(\overline{x}),1).\end{aligned} \tag{4.21}$$

Where the left hand side of (4.21) turns to be equal to

$$(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2) + (\overline{a}, \widetilde{\sigma}^{-1}(x)) \cdot (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(ax),1) - (\overline{a}, \widetilde{\sigma}^{-1}(x))(\varphi - \varphi_{m_1})(a,1).$$

Then we have

$$\begin{aligned}&(\overline{a}, \widetilde{\sigma}^{-1}(\overline{a})) \cdot (m_1 - m_2) + (\overline{a}, \widetilde{\sigma}^{-1}(x)) \cdot (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(ax),1) \\ &- (\overline{a}, \widetilde{\sigma}^{-1}(x))(\varphi - \varphi_{m_1})(a,1) = (m_1 - m_2) \\ &+ (1, \widetilde{\sigma}^{-1}(x)) \cdot (\varphi - \varphi_{m_1})(\widetilde{\sigma}^{-1}(x),1).\end{aligned} \tag{4.22}$$

Acting on both sides of (4.22) by $(\overline{a}, \widetilde{\sigma}^{-1}(\overline{x}))$, we get that

$$\psi((a,1) \cdot [(\sigma,x) - (e,1)]) = (a,1) \cdot \psi((\sigma,x) - (e,1)).$$

∎

5

Conclusions and future work

First, we gave a thorough analysis of the gap between the upper and lower bounds of the inequalities $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) \leq \mathbf{TC}_s(\mathbb{R}\mathrm{P}^m) \leq sm$, which allowed us to give an estimation for $\mathbf{TC}_s(\mathbb{R}\mathrm{P}^m)$. Further, we explained how such estimation seems to be closely related to the determination of the Euclidean immersion dimension of $\mathbb{R}\mathrm{P}^m$.

Second, by finding effective-zero–divisors of dimension 1 (in the 256 systems of local coefficients having as group $\mathbb{Z}$) we presented some indirect evidence suggesting that the effective topological complexity of the orientable surfaces of genus $g \geq 2$ would be 3 instead of 4.

Finally, as an immediate future work we have to try to produce motion planners that prove that in effect the indirect evidence mentioned above is true.

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