CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE MATEMÁTICAS

"Propiedades y aplicaciones de operadores de transmutación para sistemas de Dirac unidimensionales"

T E S I S
Que presenta

M. en C. Nelson Jades Gutiérrez Jiménez

Para obtener el grado de
DOCTOR EN CIENCIAS
EN LA ESPECIALIDAD DE MATEMÁTICAS

Directores de la Tesis:
Dr. Vladislav V. Kravchenko
Dr. Sergii M. Torba

Ciudad de México — Octubre, 2018

CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

CAMPUS ZACATENCO
DEPARTMENT OF MATHEMATICS

"Properties and applications of transmutation operators for one-dimensional Dirac systems"

A thesis submitted by

M.S. Nelson Jades Gutiérrez Jiménez

To obtain the degree of
DOCTOR IN SCIENCE
IN THE SPECIALITY OF MATHEMATICS

Thesis advisors:
Dr. Vladislav V. Kravchenko
Dr. Sergii M. Torba

Ciudad de México | Octubre, 2018

To my parents,
Sonia and Ramiro

Acknowledgments

I would like to express my most sincere gratitude to my advisors, Professor Sergii M. Torba, and Professor Vladislav V. Kravchenko for the guidance and help during this process. I really appreciate them for the beautiful mathematics they practice, build and share.

I would like to thank Professor R. Michael Porter for his support, teaching and useful talks during this process. Furthermore, I would like to thank the reviewing committee for agreeing to examine this work I appreciate your support and suggestions.

Last but not least, I would like to thank the CINVESTAV for providing all the facilities for carrying out this PhD research, and my fellow students, in particular, Josafath for his great friendship. In addition, I express my gratitude to the Consejo Nacional de Ciencia y Tecnología, CONACYT, for the financial support in the form of a PhD scholarship.

Contents

Resumen

En este trabajo se desarrolla la aproximación analítica de operadores de transmutación (AATO) para operadores Dirac unidimensionales. Se trata de la construcción analítica de un sistema completo de soluciones para un sistema diferencial parcial hiperbólico en las variables reales x y t. Se sabe que un operador de transmutación para operadores de Dirac unidimensionales puede realizarse en la forma de un operador integral Volterra y su núcleo integral satisface el sistema hiperbólico que se menciona, condicionado en las curvas características $x = t$ y $x = -t$. El sistema de soluciones que se construye para dicho sistema, se presenta como un producto de funciones matriciales 2×2 en la variable real x por potencias de t, lo que conduce a una aproximación del núcleo integral del operador de transmutación en la forma $\sum_{n=0}^{N} t^n \mathcal{K}_n(x)$. Esta forma conveniente proporciona una manera simple de construir aproximaciones con cotas de error uniforme a las soluciones del sistema Dirac unidimensional estacionario y resolver problemas espectrales relacionados. Se presenta una explicación exhaustiva del método anterior indicando su uso desde un punto de vista práctico. Como puente en la construcción anterior, también se desarrolla la representación SPPS para el sistema Dirac unidimensional. Esta última representación también puede ser utilizada para resolver problemas espectrales para el mismo sistema.

Abstract

In this work we develop an analytic approximation of transmutation operators (AATO) for one-dimensional Dirac operators. It is based on an analytical construction of a complete system of solutions for a hyperbolic partial differential system in the real variables x and t. It is known that a transmutation operator for one-dimensional Dirac operators can be realized in the form of a Volterra integral operator and its integral kernel satisfies the mentioned hyperbolic system with conditions on the characteristics curves $x = t$ and $x = -t$. The system of constructed solutions is presented as a product of 2×2 matrix functions in the real variable x by powers of t, which leads to an approximation of the integral kernel of the transmutation operator in the form $\sum_{n=0}^{N} t^n \mathcal{K}_n(x)$. This convenient form provides a simple way to construct approximations with uniform error bounds to the solutions of one-dimensional stationary Dirac system and to solve related spectral problems. We present an exhaustive explanation of the previous method and discuss its use from a practical point of view. As a bridge in the previous construction, we also present the SPPS representation for one-dimensional Dirac system. This last representation also works to solve spectral problems for the one-dimensional Dirac system.

Notation

1. $C([a,b];\mathbb{C})$: space of continuous complex function on $[a,b]$. Let $f \in C([a,b];\mathbb{C})$.

$$\|f\| := \|f(\cdot_t)\|_{\max\limits_{t\in[a,b]}} = \max_{t\in[a,b]} |f(t)| .$$

2. Space of continuously differentiable functions on $[a,b]$.

$$C^1([a,b];\mathbb{C}) := \{f \mid f' \in C([a,b];\mathbb{C})\} .$$

3. $AC[a,b]$: space of absolutely continuous functions on $[a,b]$.

4. $L^p(a,b)$ space, $1 \le p < \infty$.

$$L^p(a,b) := \left\{ f : (a,b) \to \mathbb{C} \,\middle|\, f \text{ is measurable and } \|f\|_{L^p} := \left(\int_a^b |f|^p \, dx \right)^{1/p} < \infty \right\} .$$

5. $W_p^r(a,b)$ Sovolev space, $1 \le p < \infty$.

$$W_p^r(a,b) := \left\{ f \in L^p \,\middle|\, D^k f \in L^p, k = 1, \dots, r \right\} .$$

   If there is no ambiguity, we may omit the interval (a,b) and write simply L^p and W_p^r.

6. We will write $(y_1, \dots, y_n)^T$ to denote a vector-valued function on $[a,b]$. Let $Y = (y_1, \dots, y_n)^T$. If $y_i \in C([a,b];\mathbb{C})$ for each i, $i = 1, \dots, n$, then

$$\|Y\| := \max \{\|y_1\|, \dots, \|y_n\|\} .$$

7. $AC(-b,b)^n$ space.

$$AC(-b,b)^n = \left\{ Y = (y_1, \dots, y_n)^T \mid y_i \in AC(-b,b), i = 1 \dots n \right\} .$$

8. $L^p(-b,b)^n$ space.

$$L^p(-b,b)^n = \left\{ Y = (y_1, \dots, y_n)^T \mid y_i \in L^p(-b,b), i = 1 \dots n \right\} .$$

   Let $\boldsymbol{f} = (f_1, f_2)^T$ and $\boldsymbol{g} = (g_1, g_2)^T$. The inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle := \int_a^b \boldsymbol{f}(x) \cdot \overline{\boldsymbol{g}(x)} \, dx = \int_a^b f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)} \, dx$$

   and the norm

$$\|\boldsymbol{f}\| := \langle \boldsymbol{f}, \boldsymbol{f} \rangle^{1/2} .$$

   on $L^2(-b,b)^2$.

9. $W\,(-b,b)^n$ space.

$$W^r_p(a,b)^n = \left\{Y = (y_1,\ldots,y_n)^T \,|\, y_i \in W^r_p(a,b), i = 1\ldots n\right\}.$$

10. We denote by $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$ the space of $n\times n$ matrices with complex entries. Let $A \in \mathcal{M}_n$.

$$\|\,A\,\|_\infty = \max\left\{\sum_{i=1}^n |a_{i1}|,\ldots,\sum_{i=1}^n |a_{in}|\right\},\quad A = (a_{ij}),\quad i,j = 1,\ldots,n.$$

11. Similar to [30], we will write $\mathcal{M}_n$-valued continuous function to indicate an element of the space

$$C([a,b];\mathcal{M}_n) := \left\{(f_{ij})\,\middle|\, f_{ij} \in C\,[a,b]\,,\quad i,j = 1,\ldots,n\right\}.$$

Let $F \in C([a,b];\mathcal{M}_n)$. Then

$$\|F\| := \max\left\{\sum_{i=1}^n \|f_{i1}\|\,,\ldots,\sum_{i=1}^n \|f_{in}\|\right\}.$$

12. $L^p((a,b)\,,\mathcal{M}_n)$: space of $\mathcal{M}_n$-valued functions whose components belong to L^p. Let $F \in L^p((a,b)\,,\mathcal{M}_n)$. Then

$$\|F\|_{L^p} := \max\left\{\sum_{i=1}^n \|f_{i1}\|_{L^p}\,,\ldots,\sum_{i=1}^n \|f_{in}\|_{L^p}\right\}.$$

13. The notation $g(\cdot_x)$ indicates function of variable x: $x \mapsto g(x)$. Similar by, $M(\cdot_x,\cdot_t)$ indicates function of variables x and t: $(x,t) \mapsto M(x,t)$ and $w(\cdot_x,t)$ indicates function of variable x: $x \mapsto w(x,t)$.

Overview

Introduction

This dissertation is oriented to the development of a recently discovered method focused on the one-dimensional stationary Dirac system. The method was introduced in [43] and it is called an analytic approximation of transmutation operators (AATO). As a fundamental part in the development of this method for the treated system, the SPPS method for the same system is also presented. Both methods are analytic and of easy numerical implementation that allows one to approximate the solution of initial value problems as well as the solution of spectral problems for the indicated system, each of these with its own characteristics.

The document is structured in five chapters. In the preliminary chapter we present the one-dimensional Dirac system, we also present a brief historical summary of the methods implemented in the previous papers with the purpose of motivating and showing the type of results obtained in the following chapters. In the second chapter, we present the SPPS representation for the solutions and introduce the main ingredient, the formal powers for the one-dimensional Dirac system. This representation is not only presented in the context of continuous solutions but also in the context of weak solutions. In addition, as a particular case of the previous representation, we present a very interesting example that relates the formal powers constructed in [41] for the Sturm-Liouville equation. Consequently, our representation extends the result established in [41]. Chapter 3 is devoted to the study of transmutation operators for one-dimensional Dirac operators, it is a very important chapter since it is the basis of the method used. In Chapter 4 we present a detailed explanation of the AATO method. It is worth noting that the analytic approximation method developed here provides a different way of obtaining the results from [43]. Finally, Chapter 5 presents a practical point of view on the AATO method and how it is implemented to solve associated problems for the one-dimensional Dirac system. Of course, the results obtained in this dissertation are in collaboration with professors Vladislav V. Kravchenko and Sergii Torba, I am very grateful for suggesting me the beautiful thematic for this dissertation and I deeply appreciate the guidance and support during this process.

State of the Art

Due to its importance, the one-dimensional stationary Dirac system has been the object of study in various areas of mathematics and mathematical physics ([2] [14] [29] [54]) and currently provides applications to graphene [31]. For an introduction to the Dirac equation see for example [4] [5] [25] [60].

In the recent paper [43] a new method for solving spectral problems for Sturm-Liouville equations is introduced, see also [37]. The method is called the analytic approximation of transmutation operators (AATO). Roughly speaking, the analytic approximation method is based on the notion of transmutation operator introduced by Delsate [16] and on the results from [33] where a new complete system of solution for the Klein-Gordon equation was constructed. This method presents a highly accurate approximation to the solution of spectral problems because the approximations do not depend on the size of the spectral parameter, moreover the approximate solutions are obtained in analytical form and offer an easy numerical implementation. Added to all the above, the approach proposed in [43] allows one the use of transmutation operators as a practical and easy-to-use tool, rather than a theoretical tool as is normally used. All these characteristics are the initial motivation to extend this method to other equations of interest in mathematical physics, in particular due to its importance to the one-dimensional Dirac system.

In this dissertation we examine the linear differential operators $\mathcal{A}_Q = B\frac{d}{dx} + Q(x)$ and $\mathcal{A}_0 := B\frac{d}{dx}$, where $B^2 = -\mathcal{I}$, $BQ \equiv -QB$, $\mathcal{I}$ being the the identity matrix. Similar in spirit to [49], a transmutation operator $\boldsymbol{T}$ for the above operators (in the sense of Definition 1.3.1) is a Volterra integral operator and its integral kernel $\boldsymbol{K}(\cdot_x, \cdot_t)$ satisfies certain boundary conditions on characteristic curves $t = x$ and $t = -x$ attached to the non-homogeneous matrix transport equation type $BK_x + K_tB = -Q(x)K$, see Theorem 3.2.10. One of the main characteristics of these transmutation operators is derived from the definition, the operator $\boldsymbol{T}$ maps a vector-valued solution $y(x)$ of the simple Dirac system $\mathcal{A}_0 y = \lambda y$ (with potential $Q \equiv 0$) into a solution $\boldsymbol{T}[y](x)$ of the complicated Dirac system $\mathcal{A}_Q y = \lambda y$. The important point to emphasize here is that, the integral kernel $\boldsymbol{K}$ in general is unknown and only known in closed form for a few particular potentials, see for example [42].

Based on the ideas of [43], the main result of this dissertation states that

Theorem 4.4.1. *Let $\{a_n, c_n\}_{n=0}^N$ and $\{b_n, d_n\}_{n=0}^N$ be complex numbers such that*

$$\left\| \tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right\| < \epsilon_1$$

and

$$\left\| \tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right\| < \epsilon_2.$$

for every $x \in [0, b]$. Then the kernel $\mathbf{K}(x,t)$ is approximated by the linear combination

$$K_N(x,t) = \sum_{n=0}^{N} \Big[a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big],$$

in such a way that for every $(x,t) \in \Omega^+$ the following inequality holds

$$\big\| \mathbf{K}(x,t) - K_N(x,t) \big\| < C_{\epsilon_1, \epsilon_2}.$$

In this theorem, $\{\mathcal{O}_m^i,\ i = 1,2,3,4\}_{m=0}^{\infty}$ is the matrix system indicated in Definition 4.2.7, p and q are the components of the potential Q and the 2×2 matrix-valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ are restrictions of K_N to the characteristic curves $t = x$ and $t = -x$. The result established in Theorem 4.4.1 not only extends the approximation obtained in [43] but also establishes it in a different way.

The main contribution of this dissertation is oriented to the study of the one-dimensional stationary Dirac system linked to the theory of transmutation operators.

- A complete system of solutions for the hyperbolic system $BK_x + K_t B = -Q(x)K$ is constructed.
- As a result of the convenient approximation K_N for the integral kernel, a highly competitive method for approximate solution of spectral problems for the one-dimensional Dirac system $\mathcal{A}_Q y = \lambda y$ is proposed.
- Convergence rate estimates depending on the smoothness of the potential are presented.

Approbation

The results presented in this dissertation are written in two articles in collaboration with professors Vladislav V. Kravchenko and Sergii Torba [26], [27].

The results contained in this thesis were accepted for presentation in the following congresses.

- International Conference Waves in Science and Engineering (WIS&E), Querétaro, México, August 22-26, 2016. Talk: Analytic approximation of a transmutation operator related to one-dimensional Dirac operators
- XLIX Congreso Nacional de la Sociedad Matemática Mexicana, Aguascalientes, México, Octubre 23-28, 2016. Talk: Aproximación analítica de un operador de transmutación para el operador de Dirac unidimensional.
- Primeras Jornadas Matemáticas del CINVESTAV, D.F. México, Noviembre 22-25, 2016. Talk: Aproximación analítica de un operador de transmutación entre operadores de Dirac unidimensionales.

Finally, we mention that as a closely related project the result established in Theorem 4.4.1 can be extended to the case when the potential is an $n \times n$ matrix-valued function satisfying certain commutativity relation, under more general conditions these operators have been considered by Marchenko [51]. In addition, now that we have the new representations that have emerged recently for integral kernels of transmutation operators [45]-[46], we are going to examine this representation in the case of one-dimensional Dirac operators.

Chapter 1

Preliminaries

The aim of this chapter is to motivate our investigation and establish the terminology used. We give a brief exposition of the two methods developed here focused on the system in question. In Section 1.1 we make a brief presentation of the one-dimensional stationary Dirac system. Section 1.2 contains a brief historical summary of the SPPS representation. In Section 1.3 we introduce the notion of transmutation operator. Section 1.4 deals with the AATO (analytic approximation of transmutation operators) method and we briefly outline its main characteristics.

1.1 The one-dimensional stationary Dirac system

As its name implies, this system is derived from the fascinating equation that Paul Dirac introduced in 1928 [19] [20] to describe the relativistic motion of spin-1/2 particles. Mainly in this work the system in question is considered as follows:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} y = \lambda y, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}. \tag{1.1}$$

From the physical point of view, system (1.1) corresponds to stationary state of a particle under the action of external forces, whose characteristics are included in the entries $p(x)$ and $q(x)$ of the potential matrix which we will denote by Q ([4], [5], [25], [57], [60]). For example, following [60], an argument close to that provided by Dirac is presented which leads to his famous equation in the form

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = H_0\psi(t,\boldsymbol{x}), \quad t\in\mathbb{R}, \quad \boldsymbol{x}\in\mathbb{R}^3,$$

where $\hbar$ is the Planck constant and H_0 is the Dirac Hamiltonian. In the one-dimensional case, adding to the Hamiltonian H_0 the potential matrix Q, the sum can be written as

$$H_0 + Q(x) = -i\hbar c\sigma_2\frac{d}{dx} + \sigma_3 mc^2 + p(x)\sigma_3 + q(x)\sigma_1, \tag{1.2}$$

where σ_1, σ_2, σ_3 are known as the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So that, writing out the expression for $H_0 + Q$ in (1.2) it is easily seen that the differential operator on the left-hand side of (1.1) is nothing but the previous sum except by the constants. It should be noted that currently the system (1.1) is used in particular, in graphene applications [31].

Due to its importance, the system (1.1) has been widely studied in many articles from a mathematical point of view and, although at first glance this system seems somewhat restrictive, in reality it corresponds to a canonical form of a more general system, namely,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix} y = \lambda y, \; p_{12}(x) \equiv p_{21}(x). \tag{1.3}$$

Following Levitan and Sargsjan [49], the above system can be transformed into any of the following systems via an orthogonal transformation in the two-dimensional space,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dz}{dx} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} z = \lambda z, \tag{1.4}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dz}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} z = \lambda z, \tag{1.5}$$

which are known as canonical forms of the system (1.3), which includes the Dirac radial equation, see [57].

For simplicity of notation, we shall write the matrix equation (1.1) as

$$B\frac{dY}{dx} + Q(x)Y = \lambda Y, \qquad Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{1.6}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

This form will be used for convenience in most of the work.

Since the methods developed in Chapters 2 and 4 allow us to solve initial value problems, as well as boundary value problems associated with the system (1.1), we briefly mention that the type of boundary conditions associated with the system (1.1) can be considered in the form

$$U(y(a), y(b)) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} y_1(a) \\ y_2(a) \end{pmatrix} + \begin{pmatrix} u_{13} & u_{14} \\ u_{23} & u_{24} \end{pmatrix} \begin{pmatrix} y_1(b) \\ y_2(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To obtain specialized information regarding the one-dimensional Dirac operators we suggest the work [54].

1.2 Spectral parameter power series

Based on pseudoanalytic functions theory [7], V.V. Kravchenko [35] introduced a new representation for solutions of a Sturm-Liouville equation, it has the form of a powers series with respect to the spectral parameter, whose coefficients are obtained by simple recursive integration procedure involving a non-trivial solution for the associated homogeneous equation. Later in [38], V. V. Kravchenko and R. M. Porter extended this representation to the equation

$$(pu')' + qu = \lambda ru, \tag{1.7}$$

separating it from the theory of pseudo analytic functions and called a spectral parameter power series (SPPS) repesentation, showing that the SPPS representation provides a simple and powerful method for numerical solution of initial value, boundary value and spectral problems.

Under the assumptions, u_0 being a non-trivial solution of equation

$$(pu_0')' + qu_0 = 0, \tag{1.8}$$

p, q, r, u_0 are complex-valued functions of the real variable x and λ is an arbitrary complex constant.

Theorem 1.2.1 ([38]). *Assume that on a finite interval $[a, b]$, equation (1.8) possesses a particular solution u_0 such that the functions $u_0^2 r$ and $1/(u_0^2 p)$ are continuous on $[a, b]$. Then the general solution of (1.7) on (a, b) has the form*

$$u = c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are arbitrary complex constants,

$$u_1(x) = u_0(x) \sum_{k=0}^{\infty} \lambda^k \widetilde{X}^{(2k)}(x), \quad \text{and} \quad u_2(x) = u_0(x) \sum_{k=0}^{\infty} \lambda^k X^{(2k+1)}(x), \tag{1.9}$$

with $\widetilde{X}^{(n)}$ and $X^{(n)}$ being defined by the recursive relations

$$\widetilde{X}^{(n)} \equiv 1, \quad \widetilde{X}^{(n)} \equiv 1,$$

$$\widetilde{X}^{(n)}(x) = \begin{cases} \displaystyle\int_{x_0}^{x} \widetilde{X}^{(n-1)}(s) u_0^2(s) r(s)\, ds, & n \text{ odd}, \\[2ex] \displaystyle\int_{x_0}^{x} \widetilde{X}^{(n-1)}(s) \frac{1}{u_0^2(s) p(s)}\, ds, & n \text{ even}, \end{cases} \tag{1.10}$$

$$X^{(n)}(x) = \begin{cases} \displaystyle\int_{x_0}^{x} X^{(n-1)}(s) \frac{1}{u_0^2(s) p(s)} ds, & n \text{ odd}, \\[2ex] \displaystyle\int_{x_0}^{x} X^{(n-1)}(s) u_0^2(s) r(s) ds, & n \text{ even}, \end{cases} \tag{1.11}$$

where x_0 is an arbitrary point in $[a, b]$ such that p is continuous at x_0 and $p(x_0) \neq 0$. Further, both series in (1.9) converge uniformly on $[a, b]$.

Subsequently, in joint works of V. V. Kravchenko and his colleague S. M. Torba, the SPPS method was improved and expanded considerably, in particular in [40] the condition of being non-vanishing imposed on u_0 is removed. Currently, there are already many works that involve this representation, see [33] and references therein, and within its main characteristics the SPPS representation allows one to solve approximately spectral problems by means the roots calculation of a polynomial. The aim of Chapter 2 is to obtain the analogous of the above theorem for the solutions of the one-dimensional stationary Dirac system (1.1).

1.3 Transmutation operators

A crucial fact in the theory of linear differential equations is the concept of the transmutation operator. This idea goes back to Delsarte, Lions, Povzner, Marchenko ([16], [17], [51]) and consists in relating two linear differential operators to examine a more complicated equation in terms of a simpler one. Very often in the literature this concept is also known as transformation operators. Mainly of a theoretical nature, it is a tool involved in dozens of works associated with direct and inverse problems for lineal differential equations. It should be noted that, as part of the state of the art on the use of transmutation operators, the approach proposed by the authors Kravchenko and Torba is purely practical.

We give the definition of transmutation operator from [41] which is a modification of the definition proposed by Levitan [48] adapted to the purposes of the present work. Let E be a linear topological space and E_1 its linear subspace (not necessarily closed). Let $\mathcal{A}_1, \mathcal{A}_2 : E_1 \to E$ be linear operators.

Definition 1.3.1. A linear invertible operator T defined on the whole E such that E_1 is invariant under the action of T is called a transmutation operator for the pair of operators $\mathcal{A}_1$ and $\mathcal{A}_2$ if it fulfills the following two conditions.

1. Both the operator T and its inverse T^{-1} are continuous in E;

2. The following operator equality is valid

$$\mathcal{A}_1 T = T \mathcal{A}_2$$

or which is the same

$$\mathcal{A}_1 = T \mathcal{A}_2 T^{-1}.$$

The relation $\mathcal{A}_1 T = T \mathcal{A}_2$ is known as transmutation property and perhaps is the most important property of the previous definition, for example in the finite dimensional case this property corresponds to the definition of two similar matrices and is well-known that similar matrices share all desirable properties. The basic idea of transmutation is to relate two linear differential operators, one simpler than the other, which allows us to transform a more complicated equation into a simpler one, as later indicated.

1.4 Analytic approximation of transmutation operators

The analytic approximation of transmutation operators (AATO) is a recent method introduced in [43] to approximate solution of initial value and spectral problems for Sturm-Liouville equations, see also [37]. Roughly speaking, this method is based on the concept of transmutation operator introduced in 1938 by J. Dersarte [16], and on the results from [33] where a new complete system of solution for the Klein-Gordon equation was constructed.

Since one of the objectives established in the present dissertation is to introduce this method oriented to one-dimensional Dirac operators and with the purpose to give briefly sketch about this method, we have compiled basic information from [43] that motivated the extension of this attractive method to one-dimensional Dirac operators. In addition, we shall write one of the main results obtained in [43] since in Section 5.4 it will be referenced with the purpose of connecting results and drawing some conclusions. In what follows, we will transcribe only some aspects from [43].

In the sense of Definition 1.3.1, a parametrized family of transmutation operators for the one-dimensional Schrödinger operators $A_1 = -\frac{d^2}{dx^2} + q_1(x)$ and $A_2 = -\frac{d^2}{dx^2}$ can be realized in the form of the Volterra integral operators

$$T_h u(x) = u(x) + \int_{-x}^{x} \boldsymbol{K}_h(x,t)u(t)\,dt, \tag{1.12}$$

where h is a complex parameter, see [9], [39], and the integral kernel $\boldsymbol{K}_h(x,t)$ can be obtained as a solution of a Klein-Gordon type equation with boundary conditions on the characteristics curves $t = x$ and $t = -x$. These last type of problems are called Goursart problems associated to the integral kernel.

The interest of the operator T_h is that $v := T_h u$ satisfies $-v'' + q_1(x)v = \omega^2 v$, as long as, u being a solution of the equation $-u'' = \omega^2 u$. This last property follows from the transmutation property $\mathcal{A}_1 T_h = T_h \mathcal{A}_2$. Note also that the latter equation is simpler than the first. We would like to emphasize here that, in practice, this attractive property of the operator T_h is only available under the condition that the integral kernel $\boldsymbol{K}_h$ is known, and we already know that them are only known in exact form for some particular potentials, see [39]. In addition to the above, the classical approach of successive approximations can be used to approximate the kernels, however, the numerical implementation requires large computational resources.

In essence, the AATO method consists of the analytical approximation of the integral kernel $\boldsymbol{K}_h$, this approximation is obtained by means of approximating the one-dimensional data on the characteristic curves $t = x$ and $t = -x$.

Under certain normalization of the constant h, in [43] it is established that

Theorem 1.4.1 ([43])**.** *Let the complex numbers $a_0, \ldots, a_N$ and $b_1, \ldots, b_N$ be such that*

$$\left| \frac{h}{2} + \frac{1}{4}\int_0^x q_1(s)\,ds - \sum_{n=0}^{N} a_n \boldsymbol{c}_n(x) \right| < \epsilon_1$$

and

$$\left|\frac{1}{4}\int_0^x q_1(s)\,ds - \sum_{n=0}^{N} b_n \boldsymbol{s}_n(x)\right| < \epsilon_2$$

for every $x \in [-b, b]$. Then the kernel is approximated by the function

$$K_{h,N}(x,t) = a_0 u_0(x,t) + \sum_{n=1}^{N} a_n u_{2n-1}(x,t) + \sum_{n=1}^{N} b_n u_{2n}(x,t)$$

in such a way that for every $(x, t) \in [-b, b] \times [-b, b]$ the inequality holds

$$|\boldsymbol{K}_h(x,t) - K_{h,N}(x,t)| < C_{\epsilon_1,\epsilon_2}.$$

In the previous theorem, the families of functions $\boldsymbol{c}_n$ and $\boldsymbol{s}_n$ are the restrictions of the generalized wave polynomials $u_{2n-1}(x,t)$ and $u_{2n}(x,t)$ on the characteristic curves $t = x$ and $y = -x$ respectively, see [33]. As a result of combining transmutation operators with the convenient representation for the approximate kernel $K_{h,N}$, the AATO method is a high performance method to approximate solutions. Within its main characteristics we point out that the constructed approximations have uniformly bounded error and do not depend on the size of the spectral parameter and it is also easy to implement numerically.

Our main objective is to establish an analogue of the previous theorem for one-dimensional Dirac operators. For this purpose, a rigorous justification is required, which is presented in Chapter 4, based on the results of Chapter 3 of this dissertation.

Chapter 2

The SPPS method for the Dirac system

In this chapter we develop the SPPS method for one-dimensional stationary Dirac systems. We show how the general solution for the first order system

$$y_2' + p(x)y_1 + q(x)y_2 = -\lambda y_1, \tag{2.1}$$
$$-y_1' + q(x)y_1 - p(x)y_2 = -\lambda y_2, \tag{2.2}$$

can be represented in the form of Spectral Parameter Power Series. We observe that the previous system corresponds to the matrix equation indicated in (1.1). In Section 2.1 we introduce the systems of generalized formal powers and give estimates in terms of the uniform norm that guarantees the uniform convergence of related series. In Section 2.2 the SPPS representation for the system (2.1)-(2.2) is stated and proved, which is the main result of the chapter established by Theorem 2.2.4. Section 2.3 presents as an example a very important particular case of Theorem 2.2.4. This leads to an interesting relation between the formal powers involved in the SPPS representation for the one-dimensional Schrödinger equation and our recursive formulas. In Section 2.4 we give a brief exposition of some of the standard facts on the SPPS representation. Section 2.5 presents the SPPS representation in a more general setting of L^1 coefficients. This idea comes from [8] and the proof was adapted under our assumptions.

2.1 Systems of generalized formal powers

Let $p, q \in C[a, b]$ be complex-valued functions of the real variable x. Henceforth we assume that $(f, g)^T$ is a solution of the homogeneous Dirac system corresponding to equation (1.6), i.e.

$$g' + p(x)f + q(x)g = 0, \tag{2.3}$$
$$-f' + q(x)f - p(x)g = 0, \tag{2.4}$$

such that both functions f and g are non-vanishing on $[a,b]$. Let x_0 be a point from the segment $[a,b]$ and

$$\kappa := (f(x_0)g(x_0))^{-1}. \tag{2.5}$$

Consider the following systems of functions defined by recursive relations:

$$X^{(0)}(x) = -\int_{x_0}^{x} \frac{p(s)}{f^2(s)} ds, \tag{2.6}$$

$$Y^{(0)}(x) = \kappa + \int_{x_0}^{x} \frac{p(s)}{g^2(s)} ds, \tag{2.7}$$

$$Z^{(n)}(x) = \int_{x_0}^{x} \Big(f^2(s)X^{(n-1)}(s) + g^2(s)Y^{(n-1)}(s)\Big)ds, \qquad n = 1,2,\dots \tag{2.8}$$

$$X^{(n)}(x) = \int_{x_0}^{x} \Big(\frac{p(s)}{f^2(s)}Z^{(n)}(s) + \frac{g(s)}{f(s)}Y^{(n-1)}(s)\Big)ds, \qquad n = 1,2,\dots \tag{2.9}$$

$$Y^{(n)}(x) = -\int_{x_0}^{x} \Big(\frac{p(s)}{g^2(s)}Z^{(n)}(s) + \frac{f(s)}{g(s)}X^{(n-1)}(s)\Big)ds. \qquad n = 1,2,\dots \tag{2.10}$$

Similarly we use as the initial functions

$$\widetilde{X}^{(0)}(x) = \kappa + \int_{x_0}^{x} \frac{p(s)}{f^2(s)}\, ds, \tag{2.11}$$

$$\widetilde{Y}^{(0)}(x) = -\int_{x_0}^{x} \frac{p(s)}{g^2(s)}\, ds \tag{2.12}$$

and define functions $\widetilde{Z}^{(n)}$, $\widetilde{X}^{(n)}$ and $\widetilde{Y}^{(n)}$, $n \geq 0$ by formulas (2.8)–(2.10) replacing $X^{(n)}$, $Y^{(n)}$ and $Z^{(n)}$ by $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$ and $\widetilde{Z}^{(n)}$.

The following lemmas give some estimates in terms of the uniform norm.

Lemma 2.1.1. *Under the above conditions for f and g, the following estimates hold for the functions $X^{(n)}$, $Y^{(n)}$, $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$, $Z^{(n)}$, $\widetilde{Z}^{(n)}$, $n \geq 0$:*

$$|X^{(n)}(x)|, |Y^{(n)}(x)|, |\widetilde{X}^{(n)}(x)|, |\widetilde{Y}^{(n)}(x)| \leq \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{c_2}{c_1}\Big)^k \Big(\frac{c_3}{c_1}\Big)^{n-k} \frac{(|\kappa| + c_1|x-x_0|)^{n+k+1}}{(n+k+1)!}, \tag{2.13}$$

$$|Z^{(n)}(x)|, |\widetilde{Z}^{(n)}(x)| \leq \sum_{k=0}^{n-1} \binom{n-1}{k} \Big(\frac{c_2}{c_1}\Big)^{k+1} \Big(\frac{c_3}{c_1}\Big)^{n-k-1} \frac{(|\kappa| + c_1|x-x_0|)^{n+k+1}}{(n+k+1)!}, \tag{2.14}$$

where the constants c_i, $i = 1,2,3$, are given by

$$c_1 = \max\left\{\left\|\frac{p}{f^2}\right\|, \left\|\frac{p}{g^2}\right\|, 1\right\},\ c_2 = \left\|f^2\right\| + \left\|g^2\right\|,\ c_3 = \max\left\{\left\|\frac{f}{g}\right\|, \left\|\frac{g}{f}\right\|\right\}, \tag{2.15}$$

and $\|\cdot\|$ denotes the uniform norm on $[a,b]$.

Proof. The proof is by induction. Let us suppose $x > x_0$; for $x < x_0$ it is similar. The estimate (2.13) is trivial when $n = 0$. It follows that

$$\left|Z^{(1)}(x)\right| \leq c_2 \int_{x_0}^{x} \big(|\kappa| + c_1(s-x_0)\big)\, ds \leq \frac{c_2}{c_1}\frac{1}{2}\big(|\kappa| + c_1(s-x_0)\big)^2. \tag{2.16}$$

Therefore the base step has been shown. Suppose (2.13) and (2.14) are true for $n = m$. Then for $n = m+1$ we have

$$\begin{aligned}\left|Z^{(m+1)}(x)\right| &\leq c_2 \sum_{k=0}^{m} \binom{m}{k} \left(\frac{c_2}{c_1}\right)^k \left(\frac{c_3}{c_1}\right)^{m-k} \int_{x_0}^{x} \frac{\big(|\kappa| + c_1(s-x_0)\big)^{m+k+1}}{(m+k+1)!}\, ds \\ &\leq \frac{c_2}{c_1} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{c_2}{c_1}\right)^k \left(\frac{c_3}{c_1}\right)^{m-k} \frac{\big(|\kappa| + c_1(x-x_0)\big)^{m+k+2}}{(m+k+2)!} \\ &\leq \sum_{k=0}^{m} \binom{m}{k} \left(\frac{c_2}{c_1}\right)^{k+1} \left(\frac{c_3}{c_1}\right)^{m-k} \frac{\big(|\kappa| + c_1(x-x_0)\big)^{m+k+2}}{(m+k+2)!}.\end{aligned} \tag{2.17}$$

From (2.9), substituting the estimates (2.13) and (2.17), we obtain

$$\begin{aligned}\left|X^{(m+1)}(x)\right| &\leq \int_{x_0}^{x} c_1\left|Z^{(m+1)}(s)\right| + c_3\left|Y^{(m)}(s)\right| ds \\ &\leq \int_{x_0}^{x} c_1\left(\sum_{k=0}^{m} \binom{m}{k} \left(\frac{c_2}{c_1}\right)^{k+1} \left(\frac{c_3}{c_1}\right)^{m-k} \frac{\big(|\kappa| + c_1(s-x_0)\big)^{m+k+2}}{(m+k+2)!}\right) \\ &\qquad + c_3\left(\sum_{k=0}^{m} \binom{m}{k} \left(\frac{c_2}{c_1}\right)^{k} \left(\frac{c_3}{c_1}\right)^{m-k} \frac{\big(|\kappa| + c_1(s-x_0)\big)^{m+k+1}}{(m+k+1)!}\right) ds.\end{aligned} \tag{2.18}$$

From (2.18), after separating the terms with indexes $k = m$ and $k = 0$ for the first sum and the second sum respectively, and using the well-known relation

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k},$$

the integrand on the right-hand side of the above inequality can be rewritten as

$$\begin{aligned}&c_1\left(\frac{c_2}{c_1}\right)^{m+1} \frac{\big(|\kappa| + c_1(s-x_0)\big)^{2m+2}}{(2m+2)!} \\ &\quad + \sum_{k=1}^{m} \binom{m+1}{k} \left(\frac{c_2}{c_1}\right)^{k} \left(\frac{c_3^{m-k+1}}{c_1^{m-k}}\right) \frac{\big(|\kappa| + c_1(s-x_0)\big)^{m+k+1}}{(m+k+1)!} \\ &\qquad + c_3\left(\frac{c_3}{c_1}\right)^{m} \frac{\big(|\kappa| + c_1(s-x_0)\big)^{m+1}}{(m+1)!}.\end{aligned} \tag{2.19}$$

Integrating from x_0 to x each of the terms in this last expression, we obtain

$$\left(\frac{c_2}{c_1}\right)^{m+1}\frac{(|\kappa|+c_1(s-x_0))^{2m+3}}{(2m+3)!} + \sum_{k=1}^{m}\binom{m+1}{k}\left(\frac{c_2}{c_1}\right)^{k}\left(\frac{c_3}{c_1}\right)^{m-k+1}\frac{(|\kappa|+c_1(s-x_0))^{m+k+2}}{(m+k+2)!} + \left(\frac{c_3}{c_1}\right)^{m+1}\frac{(|\kappa|+c_1(s-x_0))^{m+2}}{(m+2)!}. \quad (2.20)$$

From (2.18), following the chain of inequalities and adding the terms in (2.20), we have

$$\left|X^{(m+1)}(x)\right| \le \sum_{k=0}^{m+1}\binom{m+1}{k}\left(\frac{c_2}{c_1}\right)^{k}\left(\frac{c_3}{c_1}\right)^{m-k+1}\frac{(|\kappa|+c_1(s-x_0))^{m+k+2}}{(m+k+2)!}.$$

The estimates (2.13) and (2.14) have been shown for the function $X^{(n)}$ and $Z^{(n)}$; the proof for the functions $Y^{(n)}$, $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$, $\widetilde{Z}^{(n)}$ are similar. □

Corollary 2.1.2. *Under the conditions of Lemma 2.1.1, setting $c(x) = |\kappa| + c_1|x - x_0|$, the right-hand sides of the estimates* (2.13) *and* (2.14) *can be replaced respectively by*

$$A_n(x) = \frac{(c(x))^{n+1}}{(n+1)!}\left(\frac{c_3}{c_1} + \frac{c_2}{c_1}c(x)\right)^{n}, \quad (2.21)$$

$$B_n(x) = \frac{(c(x))^{n+1}}{(n+1)!}\frac{c_2}{c_1}\left(\frac{c_3}{c_1} + \frac{c_2}{c_1}c(x)\right)^{n-1}. \quad (2.22)$$

Definition 2.1.3. The systems of vector-valued functions $\{Y_1^{(n)}\}_{n=0}^{\infty}$ and $\{Y_2^{(n)}\}_{n=0}^{\infty}$ in terms of generalized formal powers (2.6)-(2.12) are defined as follows:

$$Y_1^{(n)}(x) := \begin{pmatrix} f(x)\widetilde{X}^{(n)}(x) \\ g(x)\widetilde{Y}^{(n)}(x)\end{pmatrix} \quad \text{and} \quad Y_2^{(n)}(x) := \begin{pmatrix} f(x)X^{(n)}(x) \\ g(x)Y^{(n)}(x)\end{pmatrix}. \quad (2.23)$$

The systems in (2.23) play the main role in the SPPS representation for the one-dimensional Dirac system.

2.2 The SPPS representation

In this section we obtain the SPPS representation for the one-dimensional Dirac system. It is given in terms of the vector-valued functions presented in Definition 2.1.3. The proof is based on the following two propositions which show that the infinite sequences in (2.23) are solutions of certain Dirac systems. The first proposition establishes the general solution of the homogeneous Dirac system as a linear combination of the functions in (2.23) with $n = 0$ while the second one shows that the functions from the infinite system (2.23) with $n \ge 1$ are solutions of non-homogeneous systems where the non-homogeneous parts are the previous functions of the same infinite sequence.

Proposition 2.2.1. *The general solution of the homogeneous Dirac system has the form*

$$c_1 \begin{pmatrix} f(x)\left(\kappa + \displaystyle\int_{x_0}^{x} \frac{p_(s)}{f^2(s)}\,ds\right) \\ -g(x)\displaystyle\int_{x_0}^{x} \frac{p(s)}{g^2(s)}\,ds \end{pmatrix} + c_2 \begin{pmatrix} -f(x)\displaystyle\int_{x_0}^{x} \frac{p_2(s)}{f^2(s)}\,ds \\ g(x)\left(\kappa + \displaystyle\int_{x_0}^{x} \frac{p(s)}{g^2(s)}\,ds\right) \end{pmatrix}, \tag{2.24}$$

where c_1 and c_2 are arbitrary complex constants.

Proof. Note that the first vector-valued function in the above linear combination corresponds to the vector-valued function $Y_1^{(0)}$ in (2.23) and the second one to $Y_2^{(0)}$. Let us first consider the vector-valued function $Y_1^{(0)}$, for $Y_2^{(0)}$ is similar. Since f and g satisfy the homogeneous system (2.3) and (2.4), i.e.

$$\begin{aligned} g' &= -pf + qg, \\ -f' &= -qf + pg, \end{aligned} \tag{2.25}$$

one can easily check the following equality:

$$\frac{p(x)}{f^2(x)} + \frac{p(x)}{g^2(x)} = \frac{d}{dx}\Big(\frac{1}{f(x)g(x)}\Big). \tag{2.26}$$

Taking into account equality (2.26) and by direct verification we have that

$$\Big(B\frac{d}{dx} + Q(x)\Big)Y_1^{(0)}(x) = \begin{pmatrix} pf\displaystyle\int_{x_0}^{x} \frac{d}{ds}\left(\frac{1}{f(s)g(s)}\right)ds + \kappa pf - \frac{p}{g} \\ pg\displaystyle\int_{x_0}^{x} \frac{d}{ds}\left(\frac{1}{f(s)g(s)}\right)ds + \kappa pg - \frac{p}{f} \end{pmatrix}.$$

After a simple integration, it immediately follows that the right hand side of the last expression is equal to zero, i.e., the vector-valued function $Y_1^{(0)}$ satisfies the homogeneous Dirac system. Finally note that the values of $Y_1^{(0)}$ and $Y_2^{(0)}$ at $x = x_0$ are given by

$$Y_1^{(0)}(x_0) = \frac{1}{g(x_0)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad Y_2^{(0)}(x_0) = \frac{1}{f(x_0)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.27}$$

therefore $W(Y_1^{(0)}, Y_2^{(0)})(x_0) = \kappa$ and linear independence is obtained. □

Proposition 2.2.2. *The vector-valued functions in* (2.23) *satisfy the following recursive nonhomogeneous Dirac systems*

$$B\frac{d}{dx}Y_i^{(n)} + Q(x)Y_i^{(n)} = -Y_i^{(n-1)}, \quad i = 1, 2, \quad n \geq 0, \tag{2.28}$$

where $Y_i^{(-1)} \equiv 0$.

Proof. Let us prove the statement for $i = 2$; for $i = 1$ similar reasoning can be applied. By induction, Proposition 2.2.1 gives the base step. Suppose that (2.28) is valid for $n = m-1$. Then for $n = m$, taking into account the expressions (2.9) and (2.10) we have

$$B\frac{d}{dx}Y_2^{(m)} + Q(x)Y_2^{(m)} = \begin{pmatrix} g'Y^{(m)} - \frac{p}{g}Z^{(m)} - fX^{(m-1)} + pfX^{(m)} + qgY^{(m)} \\ -f'X^{(m)} - \frac{p}{f}Z^{(m)} - gY^{(m-1)} + qfX^{(m)} - pgX^{(m)} \end{pmatrix}. \tag{2.29}$$

Now substituting the corresponding terms for f' and g' in the right hand side of (2.29) respectively we find that

$$B\frac{d}{dx}Y_2^{(m)} + Q(x)Y_2^{(m)} = \begin{pmatrix} pf(X^{(m)} - Y^{(m)}) - \frac{p}{g}Z^{(m)} - fX^{(m-1)} \\ pg(X^{(m)} - Y^{(m)}) - \frac{p}{f}Z^{(m)} - gX^{(m-1)} \end{pmatrix}. \tag{2.30}$$

Again from (2.9), (2.10) and (2.26) after integrating by parts the following relationship is obtained:

$$X^{(m)}(x) - Y^{(m)}(x) = \frac{1}{f(x)g(x)}Z^{(m)}(x),$$

and substituting this last expression in (2.30) we obtain the validity of the expression (2.28) for $n = m$. □

Remark 2.2.3. In summary infinite sequences of vector-valued functions associated with the differential operator $L_Q = B\frac{d}{dx} + Q$ have been constructed which allow us to write the general solution of the Dirac system. Under a proper normalization of the formal powers in (2.8)-(2.10), these types of system are known as L-bases related to an operator, see [23].

Theorem 2.2.4. *Assume that the homogeneous Dirac system* (2.3)-(2.4) *possesses a particular solution* $(f, g)^T$ *such that both functions* f *and* g *are non-vanishing on* $[a, b]$. *Then the general solution of the Dirac system* (2.1)-(2.2) *has the form*

$$Y = c_1Y_1 + c_2Y_2$$

where c_1 *and* c_2 *are arbitrary complex constants and*

$$Y_1(x) = \sum_{n=0}^{\infty} \lambda^n Y_1^{(n)}(x), \quad Y_2(x) = \sum_{n=0}^{\infty} \lambda^n Y_2^{(n)}(x). \tag{2.31}$$

The solutions Y_1 and Y_2 satisfy the following initial conditions:

$$Y_1(x_0) = \frac{1}{g(x_0)}\begin{pmatrix}1\\0\end{pmatrix}, \quad Y_2(x_0) = \frac{1}{f(x_0)}\begin{pmatrix}0\\1\end{pmatrix}.$$

Proof. Consider the monotone increasing successions (2.21) and (2.22), see Corollary 2.1.2. Let us also note that the series $\sum \lambda^n A_n(b)$, $\sum \lambda^n B_n(b)$ are convergent. Since for every $x \in [a, b]$,

$$\begin{aligned} |Y_i^{(n)}(x)| &\leq \left\|(f, g)^T\right\| A_n(b), \\ \left|\frac{d}{dx}Y_i^{(n)}(x)\right| &\leq \left\|(f', g')^T\right\| A_n(b) + \left\|(f, g)^T\right\| A_{n-1}(b) + \left\|(f, g)^T\right\| c_1B_n(b), \end{aligned}$$

by the Weierstrass M-test, both series in (2.31) converge absolutely and uniformly in the interval $[a,b]$, as well as the series of termwise derivatives. Hence applying the Dirac operator $B\frac{d}{dx}+Q(x)$ to both series in (2.31), and taking into account Propositions 2.2.1 and 2.2.2 it follows that

$$\begin{aligned}\left(B\frac{d}{dx}+Q(x)\right)Y(x)&=-c_1\sum_{n=1}^{\infty}\lambda^n Y_1^{(n-1)}(x)-c_2\sum_{n=1}^{\infty}\lambda^n Y_2^{(n-1)}(x)\\&=-c_1\lambda\sum_{n=1}^{\infty}\lambda^{n-1}Y_1^{(n-1)}(x)-c_2\lambda\sum_{n=1}^{\infty}\lambda^{n-1}Y_2^{(n-1)}(x)\\&=-c_1\lambda\sum_{n=0}^{\infty}\lambda^{n}Y_1^{(n)}(x)-c_2\lambda\sum_{n=0}^{\infty}\lambda^{n}Y_2^{(n)}(x)\\&=-\lambda Y(x).\end{aligned}$$

The linear independence follows from (2.27) and the simple observation that the functions in (2.23) satisfy the initial conditions $Y_i^{(n\geq 1)}(x_0)=(0,0)^T$, $i=1,2$. Therefore $W(Y_1,Y_2)(x_0)=\kappa\neq 0$. □

2.3 One-dimensional Dirac equations with Lorentz scalar potential

In this section the most relevant example of the chapter is presented. It deals with the SPPS representation given by Theorem 2.2.4 when the entry p of the potential Q equals zero, namely

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\frac{dY}{dx}+\begin{pmatrix}0&q(x)\\q(x)&0\end{pmatrix}Y=-\lambda Y;\quad Y=\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}. \tag{2.32}$$

The importance lies not only in the physical context but also in the mathematical context, since it establishes connections with formal powers involved in the SPPS representation for the one-dimensional Schrödinger equation and its Darboux-transformed equation, see [38], [39], [41], [43]. It should be noted that the main relationship obtained in this section, Proposition 2.3.2, will be referenced later in order to show examples of transmutation operator kernels in relation to the ATTO method that is developed for the Dirac system in the following chapters.

Following [41] by setting $q(x)=-(m+S(x))$, where m $(m>0)$ is the mass of a particle and $S(x)$ is a Lorentz scalar, (2.32) is called one-dimensional Dirac system with Lorentz scalar potential. In this case, formulas for the general solution in terms of infinite sequences of functions $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ are obtained. We emphasize that the same formulas are verified by the representation given by Theorem 2.2.4.

According to Theorem 2.2.4, the general solution of the system (2.32) has the form

$$Y(x)=c_1\sum_{n=0}^{\infty}\lambda^k\begin{pmatrix}f(x)\widetilde{X}^{(n)}(x)\\g(x)\widetilde{Y}^{(n)}(x)\end{pmatrix}+c_2\sum_{n=0}^{\infty}\lambda^k\begin{pmatrix}f(x)X^{(n)}(x)\\g(x)Y^{(n)}(x)\end{pmatrix} \tag{2.33}$$

where both functions f and g are non-vanishing solutions of the homogeneous system

$$g' + qg = 0, \tag{2.34}$$
$$-f' + qf = 0, \tag{2.35}$$

which can always be chosen as

$$f(x) = \exp\left(\int_{x_0}^{x} q(s)\, ds\right) \quad \text{and} \quad g(x) = \exp\left(-\int_{x_0}^{x} q(s)\, ds\right). \tag{2.36}$$

Note that recursive formulas (2.6)-(2.10) are simplified substantially by setting p equal to zero, moreover the recursive formulas only depend on the square of the function $f(x)$ and its multiplicative inverse since

$$\frac{f(x)}{g(x)} = f^2(x) \quad \text{and} \quad \frac{g(x)}{f(x)} = \frac{1}{f^2(x)}. \tag{2.37}$$

We present a brief illustration of the previous fact in the following table.

n	0	1	2	3	4	...
$\widetilde{X}^{(n)}$	1	0	$-\int \frac{1}{f^2} \int f^2$	0	$\int \frac{1}{f^2} \int f^2 \int \frac{1}{f^2} \int f^2$	...
$\widetilde{Y}^{(n)}$	0	$-\int f^2$	0	$\int f^2 \int \frac{1}{f^2} \int f^2$	0	...
$X^{(n)}$	0	$\int \frac{1}{f^2}$	0	$-\int \frac{1}{f^2} \int f^2 \int \frac{1}{f^2}$	0	...
$Y^{(n)}$	1	0	$-\int f^2 \int \frac{1}{f^2}$	0	$\int f^2 \int \frac{1}{f^2} \int f^2 \int \frac{1}{f^2}$	...

Given the above, it is possible to rewrite the systems of formal powers $X^{(n)}$, $Y^{(n)}$, $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$ in terms of only two systems of recursive integrals which alternate the multiplication and division by f^2. Consider the systems of recursive integrals

$$\mathcal{X}^{(0)}(x) = 1, \qquad \mathcal{X}^{(n)}(x) = n\int_{x_0}^{x} \mathcal{X}^{(n-1)}(s)(f^2(s))^{(-1)^n} ds, \qquad n = 1, 2, \ldots \tag{2.38}$$

$$\widetilde{\mathcal{X}}^{(0)}(x) = 1, \qquad \widetilde{\mathcal{X}}^{(n)}(x) = n\int_{x_0}^{x} \widetilde{\mathcal{X}}^{(n-1)}(s)(f^2(s))^{(-1)^{n-1}} ds, \qquad n = 1, 2, \ldots. \tag{2.39}$$

Taking into account the signs at each step it is straightforward to obtain the relations

$$\widetilde{X}^{(k)} = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{k!}\widetilde{\mathcal{X}}^{(k)}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \qquad X^{(k)} = \begin{cases} 0, & k \text{ even}, \\ \frac{(-1)^{\frac{k-1}{2}}}{k!}\mathcal{X}^{(k)}, & k \text{ odd}. \end{cases} \tag{2.40}$$

$$\widetilde{Y}^{(k)} = \begin{cases} 0, & k \text{ even}, \\ \frac{(-1)^{\frac{k+1}{2}}}{k!}\widetilde{\mathcal{X}}^{(k)}, & k \text{ odd}. \end{cases} \qquad Y^{(k)} = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{k!}\mathcal{X}^{(k)}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \tag{2.41}$$

After writing out the expression (2.33) in odd and even series and using the expressions (2.40)-(2.41), the general solution of the system (2.32) has the form

$$\begin{aligned} Y =& c_1 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k}}{(2k)!} f\widetilde{\mathcal{X}}^{(2k)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\lambda^{2k+1}}{(2k+1)!} \frac{1}{f}\widetilde{\mathcal{X}}^{(2k+1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &+ c_2 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k}}{(2k)!} \frac{1}{f}\mathcal{X}^{(2k)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1}}{(2k+1)!} f\mathcal{X}^{(2k+1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

or which is the same

$$y_1(x) = c_1 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k}}{(2k)!} \varphi_{2k}(x) + c_2 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1}}{(2k+1)!} \varphi_{2k+1}(x), \tag{2.42}$$

$$y_2(x) = c_2 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k}}{(2k)!} \psi_{2k}(x) - c_1 \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1}}{(2k+1)!} \psi_{2k+1}(x), \tag{2.43}$$

where the infinite sequences of functions $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ are given by

$$\varphi_k(x) = \begin{cases} f(x)\widetilde{\mathcal{X}}^{(k)}(x), & k \text{ even}, \\ f(x)\mathcal{X}^{(k)}(x), & k \text{ odd}, \end{cases} \quad \text{and} \quad \psi_k(x) = \begin{cases} \frac{\mathcal{X}^{(k)}(x)}{f(x)}, & k \text{ even}, \\ \frac{\widetilde{\mathcal{X}}^{(k)}(x)}{f(x)}, & k \text{ odd}, \end{cases} \tag{2.44}$$

respectively. The families of functions in (2.44) are known as systems of formal powers and are related to the one-dimensional Schrödinger equation $u'' - q(x)u = -\omega u$. These systems have been used in many articles (see [39], [41], [43]) and play the main role in order to approximate solution of spectral problems. The above motivates the following definition.

Definition 2.3.1. The infinite sequences of vector-valued functions $\{\Phi_k\}_{k=0}^{\infty}$ and $\{\Psi_k\}_{k=0}^{\infty}$ are given respectively by

$$\Phi_k(x) = \begin{cases} \frac{k!}{(-1)^{\frac{k-1}{2}}} \frac{1}{g(x_0)} \begin{pmatrix} f(x)X^{(k)}(x) \\ g(x)Y^{(k)}(x) \end{pmatrix}, & k \text{ odd}, \\ \frac{k!}{(-1)^{\frac{k}{2}}} \frac{1}{f(x_0)} \begin{pmatrix} f(x)\widetilde{X}^{(k)}(x) \\ g(x)\widetilde{Y}^{(k)}(x) \end{pmatrix}, & k \text{ even} \end{cases} \tag{2.45}$$

and

$$\Psi_k(x) = \begin{cases} \frac{k!}{(-1)^{\frac{k+1}{2}}} \frac{1}{f(x_0)} \begin{pmatrix} f(x)\widetilde{X}^{(k)}(x) \\ g(x)\widetilde{Y}^{(k)}(x) \end{pmatrix}, & k \text{ odd}, \\ \frac{k!}{(-1)^{\frac{k}{2}}} \frac{1}{g(x_0)} \begin{pmatrix} f(x)X^{(k)}(x) \\ g(x)Y^{(k)}(x) \end{pmatrix}, & k \text{ even}. \end{cases} \tag{2.46}$$

As a consequence of Definition 2.3.1 we present the following proposition.

Proposition 2.3.2. *Under the conditions stated above, if moreover $p(x) = 0$ for all x, the system of vector-valued functions $\{\Phi_k\}_{k=0}^{\infty}$ and $\{\Psi_k\}_{k=0}^{\infty}$ has the form*

$$\Phi_k(x) = \begin{pmatrix} \varphi_k(x) \\ 0 \end{pmatrix}, \quad \textit{and} \quad \Psi_k(x) = \begin{pmatrix} 0 \\ \psi_k(x) \end{pmatrix}, \quad n = 0, 1, 2, \ldots$$

Proof. The proof follows from relations (2.40) and (2.41). □

Example 2.3.3. Setting $f(x) = 1$, $g(x) = 1$ and $p(x) = 0$, we find that

$$\Phi_k(x) = \begin{pmatrix} x^k \\ 0 \end{pmatrix}, \quad \text{and} \quad \Psi_k(x) = \begin{pmatrix} 0 \\ x^k \end{pmatrix}, \quad n = 0, 1, 2, \ldots$$

2.4 Necessary facts on the SPPS representation

2.4.1 Construction of a non-vanishing particular solution

Simpler recursive formulas than the formulas (2.6)-(2.12) enable us to construct solutions of the homogeneous Dirac system $B\frac{dY}{dx} + Q(x)Y = 0$. We observe that this system can be written as

$$B\frac{dY}{dx} + \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix} Y = -\begin{pmatrix} p(x) & 0 \\ 0 & -p(x) \end{pmatrix} Y \ \equiv \ B\frac{dY}{dx} + Q_0(x)Y = -P_0(x)Y.$$

In addition, the vector-valued function $Y_0 = (f_0, g_0)^T$ with entries given by

$$f_0(x) = \exp\left(\int_{x_0}^{x} q(s)\,ds\right) \quad \text{and} \quad g_0(x) = \exp\left(-\int_{x_0}^{x} q(s)\,ds\right), \tag{2.47}$$

is a non-vanishing solution of the homogeneous Dirac system with $p(x) \equiv 0$, i.e.

$$\begin{array}{r} g_0' + q(x)g_0 = 0, \\ -f_0' + q(x)f_0 = 0, \end{array} \quad \equiv \quad B\frac{dY_0}{dx} + Q_0(x)Y_0 = 0. \tag{2.48}$$

With minor modifications to the recursive formulas (2.6)-(2.12), we present the following proposition from which solutions of the homogeneous Dirac system are obtained in the particular case $\lambda = 1$.

Proposition 2.4.1. *Let λ be an arbitrary complex spectral parameter. Then the general solution of the system*

$$B\frac{dY}{dx} + Q_0(x)Y = -\lambda P_0(x)Y$$

has the form

$$Y(x) = c_1 \sum_{n=0}^{\infty} \lambda^n \begin{pmatrix} f_0(x)X^{(n)}(x) \\ 0 \end{pmatrix} + c_2 \sum_{n=0}^{\infty} \lambda^n \begin{pmatrix} 0 \\ g_0(x)Y^{(n)}(x) \end{pmatrix}, \quad c_1, c_2 \in \mathbb{C}$$

where $X^{(n)}$ and $Y^{(n)}$ are defined by recursive relations

$$X^{(0)}(x) = 1, \qquad X^{(n)}(x) = -\int_{x_0}^{x} \frac{g_0(s)}{f_0(s)} p(s) Y^{(n-1)}(s)ds, \qquad n = 1, 2, \ldots \tag{2.49}$$

$$Y^{(0)}(x) = 1, \qquad Y^{(n)}(x) = -\int_{x_0}^{x} \frac{f_0(s)}{g_0(s)} p(s) X^{(n-1)}(s)ds. \qquad n = 1, 2, \ldots. \tag{2.50}$$

As shown in the above proposition, the SPPS method allows one to construct solutions of the homogeneous Dirac system. However, the question is how one can obtain non-vanishing solutions (including the case of complex-valued potentials). We are not aware of a general deterministic method for constructing them. The following proposition is well known and can be used when both functions in the potential are real valued.

Proposition 2.4.2 ([26]). *Let the coefficients p and q of the system* (2.3)-(2.4) *are real-valued functions and $(u_1, v_1)^T$ and $(u_2, v_2)^T$ are two linearly independent real-valued solutions of the system* (2.3)-(2.4)*. Then the linear combination*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + i \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

is a non-vanishing solution of the system (2.3)-(2.4)*, i.e., both functions u and v do not have zeros on $[a, b]$.*

In the general case, a non-vanishing linear combination of the solutions $(u_1, v_1)^T$ and $(u_2, v_2)^T$ always exists (and, in some sense, almost all linear combinations with complex coefficients are non-vanishing).

Proposition 2.4.3 ([26]). *Let $(u_1, v_1)^T$ and $(u_2, v_2)^T$ be two linearly independent solutions of* (2.3)-(2.4)*. Then there exists a linear combination*

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + c_2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

such that both functions u and v are non-vanishing on $[a, b]$.

2.4.2 The spectral shift technique

Based on [38], the spectral shift technique allows one changing the centre of the power series in the parameter λ indicated in Theorem 2.2.4, to obtain more precise approximations for larger λ . Note that the series in (2.31) are centered at $\lambda_0 = 0$. Under the assumption that for some $\lambda_0 \neq 0$, the vector-valued function $Y = (f, g)^T$ is a non-vanishing solution of

$$B\frac{dY}{dx} + Q(x)Y = -\lambda_0 Y, \tag{2.51}$$

which is known, and let us consider the system

$$B\frac{dY}{dx} + \left(Q(x) + \lambda_0 \mathcal{I}\right) Y = -(\lambda - \lambda_0)Y. \tag{2.52}$$

According to the recursive formulas (2.6)-(2.12) we obtain that the solutions in (2.31) have the form

$$Y_1(x) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n \, Y_1^{(n)}(x), \quad Y_2(x) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n \, Y_2^{(n)}(x).$$

and satisfy (2.52).

2.5 Discontinuous coefficients

The aim of this section is to extend Theorem 2.2.4 to integrable coefficients. This fact allows us to consider a wider range of applications than those established by Theorem 2.2.4, for example, certain discontinuous potentials. The results here are generalize from [8].

Proposition 2.5.1. *Let f and g be functions in $L^2(a,b)$ such that $\frac{p}{f^2}$, $\frac{p}{g^2}$, $\frac{f}{g}$ and $\frac{g}{f}$ are integrable functions of the independent real variable $x \in [a,b]$. Let us consider the function*

$$h(x) = \max\left\{ \left|\frac{p(x)}{f^2(x)}\right|, \left|\frac{p(x)}{g^2(x)}\right|, \left|\frac{f(x)}{g(x)}\right|, \left|\frac{g(x)}{f(x)}\right| \right\} \tag{2.53}$$

and define

$$R(x) := |\kappa| + \int_{x_0}^{x} h(s)\,ds \qquad \text{and} \qquad P(x) := \int_{x_0}^{x} |f(s)|^2 + |g(s)|^2\,ds. \tag{2.54}$$

Then the functions X^n, Y^n, Z^n, $\widetilde{X}^n$, $\widetilde{Y}^n$, $\widetilde{Z}^n$ are absolutely continuous functions and satisfy the following estimates

$$|X^{(n)}(x)|, |Y^{(n)}(x)|, |\widetilde{X}^{(n)}(x)|, |\widetilde{Y}^{(n)}(x)| \leq \frac{(R(x))^{n+1}}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k}}{(n-k)!}, \tag{2.55}$$

$$|Z^{(n)}(x)|, |\widetilde{Z}^{(n)}(x)| \leq \frac{(R(x))^{n}}{n!} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(P(x))^{n-k}}{(n-k)!}. \tag{2.56}$$

Proof. It is well known that the maximum function of a finite number of integrable functions is an integrable function too (see e.g. [10]) therefore the function h defined in (2.53) is integrable and consequently a well-defined function a.e., moreover all formal powers involved are absolutely continuous functions since their derivatives belong to the space L^1 . The proof is by induction. For $n = 0$, the estimates (2.55)-(2.56) follows directly from (2.54). Suppose that (2.55) and (2.56) are true for some n. Then assuming $x \geq x_0$ we obtain

$$\begin{aligned} \left|Z^{(n+1)}(x)\right| &\leq \frac{(R(x))^{n+1}}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} \int_{x_0}^{x} (|f(s)|^2 + |g(s)|^2) \frac{(P(s))^{n-k}}{(n-k)!}\,ds \\ &= \frac{(R(x))^{n+1}}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k+1}}{(n-k+1)!}. \end{aligned} \tag{2.57}$$

Therefore from the last inequality and by the inductive hypothesis it follows that

$$\left|X^{(n+1)}(x)\right| \leq \int_{x_0}^{x} \left[\left| \frac{p(s)}{f^2(s)} \right| \left|Z^{(n+1)}(s)\right| + \left| \frac{g(s)}{f(s)} \right| \left|Y^{(n)}(s)\right| \right] ds \tag{2.58}$$

$$\leq \sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k+1}}{(n-k+1)!} \int_{x_0}^{x} h(s) \frac{(R(s))^{n+1}}{(n+1)!} ds \tag{2.59}$$

$$+ \sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k}}{(n-k)!} \int_{x_0}^{x} h(s) \frac{(R(s))^{n+1}}{(n+1)!} ds \tag{2.60}$$

$$\leq \frac{(R(x))^{n+2}}{(n+2)!} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k+1}}{(n-k+1)!} + \sum_{k=0}^{n} \binom{n}{k} \frac{(P(x))^{n-k}}{(n-k)!} \right). \tag{2.61}$$

Making obvious shifts in the indices of the summations on the right-hand side of the previous inequality and using the well-known relation

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

we obtain that the estimate (2.55) is true for $n+1$, i.e.

$$\left|X^{(n+1)}(x)\right| \leq \frac{(R(s))^{n+2}}{(n+2)!} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(P(x))^{n-k+1}}{(n-k+1)!}. \tag{2.62}$$

The proof for the functions $Y^{(n)}$, $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$, $\widetilde{Z}^{(n)}$ is similar. □

Corollary 2.5.2. *Under the conditions of Proposition 2.5.1, the spectral parameter power series $\sum_{n=0}^{\infty} \lambda^n X^{(n)}$, $\sum_{n=0}^{\infty} \lambda^n Y^{(n)}$ and $\sum_{n=0}^{\infty} \lambda^n Z^{(n)}$ converge uniformly on $[a, b]$, as well as the series obtained by replacing $X^{(n)}$, $Y^{(n)}$ and $Z^{(n)}$ by $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$ and $\widetilde{Z}^{(n)}$.*

Proof. Setting $c_1 = R(b)$ and $c_2 = P(b) + 1$, it follows directly that the estimates on right-hand sides of (2.55) and (2.56) can be replaced respectively by

$$\frac{c_1 (c_1 c_2)^n}{(n+1)!} \quad \text{and} \quad \frac{(c_1 c_2)^n}{n!}. \tag{2.63}$$

Consequently it follows that

$$\sum_{n=0}^{\infty} \left|\lambda^n X^{(n)}(x)\right|, \sum_{n=0}^{\infty} \left|\lambda^n Y^{(n)}(x)\right| < C_1 := \sum_{n=0}^{\infty} \frac{c_1 (\lambda c_1 c_2)^n}{(n+1)!} < \infty, \tag{2.64}$$

$$\sum_{n=0}^{\infty} \left|\lambda^n Z^{(n)}(x)\right| < C_2 := \sum_{n=0}^{\infty} \frac{(\lambda c_1 c_2)^n}{n!} < \infty. \tag{2.65}$$

Therefore by the Weierstrass M-test the result is obtained. □

Proposition 2.5.3 ([8]). *Let $\{V_n\}_{n=0}^{\infty}$ be a secuence of absolutely continuous functions on $[a,b]$. If the series $\sum_{n=0}^{\infty} V_n(x_0)$ converges at some $x_0 \in [a,b]$ and the series $\sum_{n=0}^{\infty} V_n'$ converges to $v \in L^1[a,b]$ in the norm of $L^1[a,b]$, then $\sum_{n=0}^{\infty} V_n$ converges uniformly to $V \in AC[a,b]$ and $V' = v$ a.e.*

Proposition 2.5.4. *Under the conditions of Proposition 2.5.1, the spectral parameter power series $\sum_{n=0}^{\infty} \lambda^n X^{(n)}$, $\sum_{n=0}^{\infty} \lambda^n Y^{(n)}$ and $\sum_{n=0}^{\infty} \lambda^n Z^{(n)}$ converge uniformly to functions u, v, and w which belong to the space $AC[a,b]$, respectively for the series obtained by replacing $X^{(n)}$, $Y^{(n)}$ and $Z^{(n)}$ by $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$ and $\widetilde{Z}^{(n)}$.*

Proof. First we consider the series $\sum_{n=0}^{\infty} \lambda^n X^{(n)}$. By Corollary 2.5.2 let u, v and w be functions such that

$$\sum_{n=0}^{N} \lambda^n X^{(n)} \to u, \sum_{n=0}^{N} \lambda^n Y^{(n)} \to v, \sum_{n=0}^{N} \lambda^n Z^{(n)} \to w,$$

uniformly as N goes to infinity. Let us estimate the derivative of the partial sum $\sum_{n=0}^{N} \lambda^n X^{(n)}$

$$\begin{aligned} \left|\sum_{n=0}^{N} \lambda^n \frac{d}{dx} X^{(n)}(x)\right| &= \left|\sum_{n=0}^{N} \Big(\frac{p(x)}{f^2(x)} \lambda^n Z^{(n)}(x) + \frac{g(x)}{f(x)} \lambda^n Y^{(n-1)}(x) \Big)\right| && (2.66) \\ &\leq |h(x)| \sum_{n=0}^{N} \big(|\lambda^n Z^{(n)}(x)| + |\lambda^n Y^{(n-1)}(x)| \big) && (2.67) \\ &\leq |h(x)| \left(\max_{x\in[a,b]} |v(x)| + \max_{x\in[a,b]} |w(x)| \right). && (2.68) \end{aligned}$$

Since $h(x)$ belongs to the space $L^1(a,b)$ by the dominated convergence theorem we have that the series obtained by term-wise differentiation from the series of $\sum_{n=0}^{\infty} \lambda^n X^{(n)}$ belongs to the space $L^1(a,b)$, moreover this series converges to the function $u' = \frac{p}{f^2} w + \frac{g}{f} v$. From the above and bearing in mind Proposition 2.5.3 the result follows for the series $\sum_{n=0}^{\infty} \lambda^n X^{(n)}$, and similarly for the others. □

Theorem 2.5.5. *Suppose that both functions f and g are absolutely continuous and non-vanishing functions on $[a,b]$ that satisfy the homogeneous Dirac system (2.3)-(2.4) a.e. on $[a,b]$. Assume that $\frac{p}{f^2}$, $\frac{p}{g^2}$, $\frac{f}{g}$ and $\frac{g}{f}$ are integrable functions and λ be an arbitrary complex parameter. Then the general solution of the Dirac system $B\frac{dY}{dx} + Q(x)Y = -\lambda Y$ has the form*

$$Y(x) = c_1 \sum_{n=0}^{\infty} \lambda^n Y_1^{(n)}(x) + c_2 \sum_{n=0}^{\infty} \lambda^n Y_2^{(n)}(x)$$

where c_1 and c_2 are arbitrary complex constants and both series converge uniformly to vector-valued functions $(u_i, v_i)^T$, $u_i, v_i \in AC[a,b]$, $i = 1, 2$.

Proof. From Proposition 2.5.4 we conclude that for some vector-valued functions $\boldsymbol{y}_i, i = 1, 2$, belonging to the space $W_1^1 \times W_1^1$, $\sum_{n=0}^{\infty} \lambda^n Y_i^{(n)} = \boldsymbol{y}_i$ a.e. on $[a, b]$. Therefore, as in the proof of Theorem 2.2.4 we obtain that

$$\begin{aligned}\left(B\frac{d}{dx} + Q(x)\right) Y(x) &= -c_1 \sum_{n=1}^{\infty} \lambda^n Y_1^{(n-1)}(x) - c_2 \sum_{n=1}^{\infty} \lambda^n Y_2^{(n-1)}(x) \\ &= -c_1\lambda \sum_{n=1}^{\infty} \lambda^{n-1} Y_1^{(n-1)}(x) - c_2\lambda \sum_{n=1}^{\infty} \lambda^{n-1} Y_2^{(n-1)}(x) \\ &= -c_1\lambda \sum_{n=0}^{\infty} \lambda^n Y_1^{(n)}(x) - c_2\lambda \sum_{n=0}^{\infty} \lambda^n Y_2^{(n)}(x) \\ &= -\lambda Y(x), \quad \text{a.e. on } [a, b]\end{aligned}$$

Finally, since $W(\boldsymbol{y}_1, \boldsymbol{y}_2)(x_0) = \kappa \neq 0$, it follows that Y is the general solution. □

2.6 General scheme for numerical implementation

Here we present a brief sketch for numerical implementation. Since the main objective of this work is the analytic approximation of integral kernels we do not present here examples in this direction. Based on the main results of this chapter, Theorem 2.2.4 and Theorem 2.5.5 we provide the following algorithm.

Consider a one-dimensional Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} y = -\lambda y, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{2.69}$$

with some initial condition

$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

or a boundary condition

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} u_{13} & u_{14} \\ u_{23} & u_{24} \end{pmatrix} \begin{pmatrix} y_1(b) \\ y_2(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.70}$$

1. Find a non-vanishing solution $y = (f, g)^T$ of the homogeneous Dirac system associated to equation (2.69), see Proposition 2.4.1.

2. Compute the functions $X^{(k)}, Y^{(k)}, Z^{(k)}, \widetilde{X}^{(k)}, \widetilde{Y}^{(k)}$ and $\widetilde{Z}^{(k)}$, $k = 0, \ldots, N$ using (2.6)-(2.10).

3. According to Definition 2.1.3 compute the vector-valued functions $Y_1^{(n)}$ and $Y_2^{(n)}$, $k = 0, \ldots, N$ using (2.23).

4. The approximation to the solution of the initial value problem is given by $Y_N(x) = a \cdot g(0) \sum_{n=0}^{N} \lambda^n Y_1^{(n)}(x) + b \cdot f(0) \sum_{n=0}^{N} \lambda^n Y_2^{(n)}(x)$.

5. To approximate solutions with boundary conditions given by (2.70). Consider an approximation in the form $Y_N(x) = c_1 \sum_{n=0}^{N} \lambda^n Y_1^{(n)}(x) + c_2 \sum_{n=0}^{N} \lambda^n Y_2^{(n)}(x)$. Substituting this into (2.70) we obtain the homogeneous system $M_N(\lambda)(c_1, c_2)^T = 0$, where $M_N(\lambda)$ is a 2×2 matrix which depends on the spectral parameter λ whose entries are given by

$$m_{11}(\lambda) = \frac{u_{11}}{g(0)} + \sum_{n=0}^{N} \lambda^n \left(u_{13} f(b) \widetilde{X}^{(n)}(b) + u_{14} g(b) \widetilde{Y}^{(n)}(b) \right),$$
$$m_{12}(\lambda) = \frac{u_{12}}{f(0)} + \sum_{n=0}^{N} \lambda^n \left(u_{13} f(b) X^{(n)}(b) + u_{14} g(b) Y^{(n)}(b) \right),$$
$$m_{21}(\lambda) = \frac{u_{21}}{g(0)} + \sum_{n=0}^{N} \lambda^n \left(u_{23} f(b) \widetilde{X}^{(n)}(b) + u_{24} g(b) \widetilde{Y}^{(n)}(b) \right),$$
$$m_{22}(\lambda) = \frac{u_{22}}{f(0)} + \sum_{n=0}^{N} \lambda^n \left(u_{23} f(b) X^{(n)}(b) + u_{24} g(b) Y^{(n)}(b) \right).$$

Since non-trivial solutions are sought, an approximation to the characteristic equation of the boundary problem (2.69)-(2.70) has the form

$$\det \left(M_N(\lambda) \right) = 0, \tag{2.71}$$

Observe that equation (2.71) is a polynomial in λ.

6. Find roots of the equation (2.71). These roots are approximations to the spectral problem.

Chapter 3

Transmutation operators

We discuss in this Chapter some of the standard facts on transmutation operators for one-dimensional Dirac operators on a finite interval. The results presented here are of primary use in the next chapter and can be considered as a bridge between the SPPS representation and the AATO method.

We shall write the Dirac operators under consideration as differential expressions of the form

$$\mathcal{A}_Q := B\frac{d}{dx} + Q(x), \tag{3.1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

and $\mathcal{A}_0$ denotes the expression (3.1), having Q as the null matrix-valued function. Although the references are so broad on the subject, in order to compare the results presented here, the most pertinent references are [49] and [51]. It is known that a transmutation operator for the pair of operators $\mathcal{A}_0$ and $\mathcal{A}_Q$ can be realized as a Volterra integral operator, where one can obtain its integral kernel from a Goursat problem for a certain hyperbolic system, similar to [49]. In addition, problems of this type can be solved by the classical method of successive approximations. We will restrict our attention to guaranteeing existence and uniqueness of a solution of Goursat problem with values given on the characteristic curves $x = t$ and $x = -t$. It is worth mentioning that our approach is different from [2] and [49], it does not require additional assumptions on the potential. Although similar to [51], it was obtained independently after having thoroughly manipulated the integral representations given in [49].

In Section 3.1 we will look more closely at notion of transmutation operator between one-dimensional Dirac operators. In particular, according to Definition 1.3.1, we briefly review the approach given in [49] and we establish the requirements that we seek to satisfy in this area. In Section 3.2 the existence of transmutation operators in the form of Volterra integral operator on a symmetric segment for the one-dimensional Dirac operators is proved. Finally with the aim to provide analytical tools that we use in the next chapter, Section 3.3 is devoted to the study of stability for the Goursat problem associated to the integral kernel of the transmutation operator for one-dimensional Dirac operators. In Section 3.4 an important operator is constructed that allows to establish completeness properties for a certain type of matrix-valued functions.

3.1 Transmutation operator for the Dirac system

In order to illustrate Definition 1.3.1 for a pair of operators $\mathcal{A}_Q$ and $\mathcal{A}_R$ under the assumption that Q and R are continuous matrix-valued functions, we consider the spaces $E = C[-b,b]\times C[-b,b]$ and $E_1 = C^1[-b,b]\times C^1[-b,b]$ equipped with the maximum norm. We begin with a general result on such operators under these last assumptions.

Proposition 3.1.1. *Suppose that Q and R are continuous matrix-valued functions on $[-b,b]$. Then a transmutation operator for $\mathcal{A}_Q$ and $\mathcal{A}_R$ can be realized in the form of a Volterra integral operator*

$$Ty(x) = y(x) + \int_{-x}^{x} K(x,t)y(t)\,dt, \tag{3.2}$$

where $K(x,t)$ is a 2×2 matrix-valued function satisfying the partial differential equation

$$BK_x(x,t) + K_t(x,t)B = K(x,t)R(t) - Q(x)K(x,t) \tag{3.3}$$

with the Goursat conditions

$$BK(x,x) - K(x,x)B = R(x) - Q(x), \tag{3.4}$$

$$BK(x,-x) + K(x,-x)B = 0. \tag{3.5}$$

Conversely, if $K(x,t)$ is the solution of the problem (3.3), (3.4), (3.5), *then the operator T determined by the formula* (3.2) *is a transmutation operator for the pair of operators $\mathcal{A}_Q$ and $\mathcal{A}_R$.*

The proof involves looking at ([49], Th. 10.3.1) and can be handled in much the same way, the only difference being in the form of the Volterra integral operator (3.2). The results from [49] require boundary condition at $x = 0$, while we will not use any additional restriction at $x = 0$. The absence of any initial condition requires extending the integration in (3.2) onto the symmetric segment $(-x,x)$ which leads to different conditions for the Goursat problem (3.4)-(3.5). Another difference of the present work from the results of [49] is that we consider the potentials Q and R on the finite segment only, while to apply the proof from [49] one needs the potentials to be defined on the whole semi-axes.

The method of proof given by [49] is based on the construction of the Cauchy problem for an equivalent non-homogeneous matrix equation. It should be noted that equation (3.3) corresponds to a second-order hyperbolic system and the resulting equation is a wave type equation. Although it was possible to completely adapt this scheme, it turned out that the integral equations obtained were difficult to manipulate for purposes of this work.

The important point to note here is that we are interested in studying the Goursat problem (3.3)-(3.5) in the domains

$$\Omega^+ = \{(x,t) \,|\, 0 \le x < b,\, |t| \le x\},$$
$$\Omega^- = \{(x,t) \,|\, -b < x \le 0,\, |t| \le |x|\},$$

in order to obtain additional information about the smoothness of the integral kernel $K(x,t)$ and also obtain results about continuous dependence on the Goursat data mainly in the case $R \equiv 0$.

Remark 3.1.2. Note that the Goursat problem (3.3)-(3.5) can be solved independently on the domains Ω^+ and Ω^-.

In addition to [49], there are different approaches to prove the existence and uniqueness of solutions for Goursat problems similar to (3.3)-(3.5), see [15], [61], [30], and in more general setting in [30]. It should be noted that for the integral kernel K to be the classical solution of the Goursat problem, Q must be continuously differentiable.

Before moving on to another section where we respond positively to the problems raised in the last paragraph, we would like to point out that we follow a different approach than those mentioned.

3.2 Existence and uniqueness of solutions for Goursat problems

In the remainder of this chapter, more generality, we can assume that B is a constant matrix of order $n \times n$ such that $B^2 = -\mathcal{I}$, where $\mathcal{I}$ is the identity matrix of order $n \times n$, and the matrix-valued function Q satisfies $BQ \equiv -QB$.

Let us first outline how the Goursat problem (3.3)-(3.5) is transformed into terms of new variables

$$\xi = \frac{x+t}{2}, \quad \text{and} \quad \eta = \frac{x-t}{2}. \tag{3.6}$$

To do this, it is convenient to introduce the auxiliary symbol $\mathcal{H}$ to denote the space of all matrix-valued functions. For simplicity of presentation we omit the independent variable in the notations. Define

$$\mathcal{P}^+[A] := \frac{1}{2}(A + BAB) \quad \text{and} \quad \mathcal{P}^-[A] := \frac{1}{2}(A - BAB), \tag{3.7}$$

where $A \in \mathcal{H}$. Roughly speaking, the proposition below says that $\mathcal{P}^+$ and $\mathcal{P}^-$ are projectors and that $\mathcal{H}$ is the direct sum of the subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$.

Proposition 3.2.1. *The following properties are fulfilled*

1. $\mathcal{I} = \mathcal{P}^+ + \mathcal{P}^-$.
2. $(\mathcal{P}^+)^2 = \mathcal{P}^+$, $(\mathcal{P}^-)^2 = \mathcal{P}^-$.

3. $\mathcal{H}^- := \operatorname{Ker}\mathcal{P}^- = \{\Phi \in \mathcal{H} \mid \Phi B = -B\Phi\}$; $\operatorname{Im}\mathcal{P}^+ = \mathcal{H}^-$.

4. $\mathcal{H}^+ := \operatorname{Ker}\mathcal{P}^+ = \{\Psi \in \mathcal{H} \mid \Psi B = B\Psi\}$; $\operatorname{Im}\mathcal{P}^- = \mathcal{H}^+$.

5. $\mathcal{H}^- \cap \mathcal{H}^+ = \{0\}$.

6. $\mathcal{P}^+[AC] = \mathcal{P}^+[A]\,\mathcal{P}^-[C] + \mathcal{P}^-[A]\,\mathcal{P}^+[C]$.

7. $\mathcal{P}^-[AC] = \mathcal{P}^+[A]\,\mathcal{P}^+[C] + \mathcal{P}^-[A]\,\mathcal{P}^-[C]$.

Proof. The proof follows from (3.7) by direct computation. □

The task is now to establish a change of coordinates from the xy-plane into the $\xi\eta$-plane. Indeed, choose $\xi = \frac{1}{2}(x+t)$ and $\eta = \frac{1}{2}(x-t)$, and set $K(x,t) = H(\xi(x,t), \eta(x,t))$. We see at once that

$$K_x = \frac{1}{2}(H_\xi + H_\eta) \quad \text{and} \quad K_t = \frac{1}{2}(H_\xi - H_\eta), \tag{3.8}$$

which is clear by the chain rule. Substituting (3.8) into (3.3), after multiplying on the left by $-B$ and keeping in mind (3.7) we get

$$\mathcal{P}^-[H_\xi(\xi,\eta)] + \mathcal{P}^+[H_\eta(\xi,\eta)] = BQ(\xi+\eta)H(\xi,\eta) - BH(\xi,\eta)R(\xi-\eta).$$

Also note that the left hand sides in (3.4)-(3.5) have the form

$$\begin{aligned} BH(\xi,0) - H(\xi,0)B &= 2B\mathcal{P}^+[H(\xi,0)], \\ BH(0,\eta) + H(0,\eta)B &= 2B\mathcal{P}^-[H(0,\eta)]. \end{aligned}$$

From the above, one can see that the problem (3.3)-(3.5) becomes

$$\mathcal{P}^-[H_\xi(\xi,\eta)] + \mathcal{P}^+[H_\eta(\xi,\eta)] = BQ(\xi+\eta)H(\xi,\eta) - BH(\xi,\eta)R(\xi-\eta), \tag{3.9}$$

$$\mathcal{P}^+[H(\xi,0)] = -\frac{1}{2}B\mathcal{E}_1(\xi), \tag{3.10}$$

$$\mathcal{P}^-[H(0,\eta)] = -\frac{1}{2}B\mathcal{E}_2(\eta), \tag{3.11}$$

with the compatibility condition $\mathcal{E}_1 \in \mathcal{H}^-$ and $\mathcal{E}_2 \in \mathcal{H}^+$, in the domains

$$\begin{aligned} \Xi^+ &= \{(\xi,\eta) \mid 0 \le \xi < b,\ 0 \le \eta < b - \xi\}, \\ \Xi^- &= \{(\xi,\eta) \mid -b < \xi \le 0,\ -b + \xi < \eta \le 0\}. \end{aligned}$$

Remark 3.2.2. Similar to Remark 3.1.2, the problem (3.9)-(3.11) can be solved independently on the domains Ξ^+ and Ξ^-.

Remark 3.2.3. It is clear that in the case of Proposition 3.1.1, $\mathcal{E}_1(\xi) = R(\xi) - Q(\xi)$ and $\mathcal{E}_2(\eta) = 0$. Moreover, we do not need additional assumptions on the potential in order to transform problem (3.3)-(3.5) into (3.9)-(3.11), see [15], [30], [49], [61]. Thus, $K(x,t)$ is a solution of the problem (3.3)-(3.5) if and only if $H(\xi,\eta)$ is a solution of (3.9)-(3.11)

Our next goal is to state the existence and uniqueness for solution of the problem (3.9)-(3.11). We shall get this statement using the standard method, establishing the same statement for an equivalent integral equation. To obtain an integral equation equivalent to the problem (3.9)-(3.11), we proceed as follows, applying $\mathcal{P}^-$ on both sides of (3.9), and integrating with respect to ξ yields

$$\mathcal{P}^-\left[H(\xi,\eta)\right] = \mathcal{P}^-\left[H(0,\eta)\right] + \int_0^\xi \mathcal{P}^-\left[BQ(u+\eta)H(u,\eta) - BH(u,\eta)R(u-\eta)\right] du.$$

In the same manner, applying $\mathcal{P}^+$ and integrating respect to η we have

$$\mathcal{P}^+\left[H(\xi,\eta)\right] = \mathcal{P}^+\left[H(\xi,0)\right] + \int_0^\eta \mathcal{P}^-\left[BQ(\xi+v)H(\xi,v) - BH(\xi,v)R(\xi-v)\right] dv.$$

Thus, on account of Proposition 3.2.1, we conclude that

$$\begin{aligned} H(\xi,\eta) = {} & \mathcal{P}^-\left[H(0,\eta)\right] + \mathcal{P}^+\left[H(\xi,0)\right] \\ & + \int_0^\xi BQ(u+\eta)\mathcal{P}^+\left[H(u,\eta)\right] + \mathcal{P}^+\left[H(u,\eta)\right]BR(u-\eta)\,du \\ & + \int_0^\eta BQ(\xi+v)\mathcal{P}^-\left[H(\xi,v)\right] - \mathcal{P}^-\left[H(\xi,v)\right]BR(\xi-v)\,dv. \end{aligned} \tag{3.12}$$

The existence of a solution to the integral equation (3.12) is established by the method of successive approximations.

Remark 3.2.4. For our purposes, and to simplify the formulas here and subsequently we assume $R(x) = 0$. This assumption does not represent an essential drawback in the proof of the theorem below. We would like to mention that our proof does not need to extend the potential Q onto a ray or whole line, which was necessary for the approach given in [49]. Although if we need to extend the potential R, without loss of generality it can be extended by zero to the interval $[-b, 0)$.

Theorem 3.2.5. *Let Q be a matrix-valued function belonging to $L^2(0,b)$. Then the integral equation*

$$H(\xi,\eta) = \frac{1}{2}BQ(\xi) + \int_0^\xi BQ(u+\eta)\mathcal{P}^+\left[H(u,\eta)\right] du + \int_0^\eta BQ(\xi+v)\mathcal{P}^-\left[H(\xi,v)\right] dv \tag{3.13}$$

has a unique solution, this solution belongs to $L^2(\Xi^+, \mathcal{M}_n)$. Moreover, if the the matrix-valued function Q is continuous, then the kernel $H(\xi,\eta)$ is continuous and satisfies the inequality

$$|H(\xi,\eta)| \leq \frac{1}{2}\|Q\|\exp(b\|Q\|), \quad (\xi,\eta) \in \Xi^+.$$

Proof. The proof is standard by the method of successive approximations. Let $\{H_n\}_{n=0}^\infty$ be a sequence of matrix-valued functions given by

$$H_n(\xi,\eta) = \int_0^\xi BQ(u+\eta)\mathcal{P}^+\left[H_{n-1}(u,\eta)\right] du + \int_0^\eta BQ(\xi+v)\mathcal{P}^-\left[H_{n-1}(\xi,v)\right] dv, \tag{3.14}$$

where $H_0(\xi,\eta) = \frac{1}{2}BQ(\xi)$. Let us first proceed by induction in order to get the following estimate

$$|H_n(\xi,\eta)| \leq \left(\frac{\xi^{\lfloor\frac{n-1}{2}\rfloor}}{\lfloor\frac{n-1}{2}\rfloor!}\right)^{1/2}\left(\frac{\eta^{\lfloor\frac{n}{2}\rfloor}}{\lfloor\frac{n}{2}\rfloor!}\right)^{1/2}\left(\frac{\|Q\|^{n+1}_{L^2(0,\xi+\eta)}}{(n+1)!}\right)^{1/2}, \quad n = 1,2\ldots \tag{3.15}$$

Indeed, we have $H_1(\xi,\eta) = \frac{1}{2}\int_0^\xi BQ(u+\eta)BQ(u)\,du$, because H_0 belongs to $\mathcal{H}^-$. It follows that

$$|H_1(\xi,\eta)| \leq \frac{1}{2}\left(\int_0^\xi |Q(u+\eta)|^2\,du\right)^{1/2}\left(\int_0^\xi |Q(u)|^2\,du\right)^{1/2} \leq \frac{1}{2}\left(\int_0^{\xi+\eta}|Q(\theta)|^2\,d\theta\right).$$

Choose $\sigma(\xi+\eta) = \int_0^{\xi+\eta}|Q(\theta)|^2\,d\theta$ and note that $\sigma'(\xi+\eta) = |Q(\xi+\eta)|^2$. Similarly, as H_1 belongs to $\mathcal{H}^+$, we see that $H_2(\xi,\eta) = \int_0^\eta BQ(\xi+v)H_1(\xi,v)\,dv$. Thus,

$$\begin{aligned}|H_2(\xi,\eta)| \leq \frac{1}{2}\int_0^\eta |Q(\xi+v)|\sigma(\xi+v)dv &\leq \frac{1}{2}\left(\int_0^\eta dv\right)^{1/2}\left(\int_0^\eta |Q(\xi+v)|^2\sigma^2(\xi+v)dv\right)^{1/2}\\ &= \frac{1}{\sqrt{2}\sqrt{2}}\eta^{1/2}\left(\frac{w^3}{3}\Big|_{\sigma(\xi)}^{\sigma(\xi+\eta)}\right)^{1/2}\\ &\leq \eta^{1/2}\frac{\sigma^{3/2}(\xi+\eta)}{\sqrt{2}\sqrt{3}}.\end{aligned}$$

Notice that in the last inequality we have used the Holder's inequality and neglected the term $\sigma(\xi)$. If we continue in this fashion there is no loss of generality in assuming $H_n \in \mathcal{H}^-$, and supposing that (3.15) holds for n, we will prove it for $n+1$. It follows that

$$\begin{aligned}|H_{n+1}(\xi,\eta)| &\leq \left(\frac{\eta^{\lfloor\frac{n}{2}\rfloor}}{\lfloor\frac{n}{2}\rfloor!}\right)^{1/2}\int_0^\xi |Q(u+\eta)|\left(\frac{u^{\lfloor\frac{n-1}{2}\rfloor}}{\lfloor\frac{n-1}{2}\rfloor!}\right)^{1/2}\left(\frac{\sigma^{n+1}(u+\eta)}{(n+1)!}\right)^{1/2}du,\\ &\leq \left(\frac{\eta^{\lfloor\frac{n}{2}\rfloor}}{\lfloor\frac{n}{2}\rfloor!}\right)^{1/2}\left(\int_0^\xi \frac{u^{\lfloor\frac{n-1}{2}\rfloor}}{\lfloor\frac{n-1}{2}\rfloor!}du\right)^{1/2}\left(\int_0^\xi |Q(u+\eta)|^2\frac{\sigma^{n+1}(u+\eta)}{(n+1)!}du\right)^{1/2},\\ &\leq \left(\frac{\eta^{\lfloor\frac{n}{2}\rfloor}}{\lfloor\frac{n}{2}\rfloor!}\right)^{1/2}\left(\frac{\xi^{\lfloor\frac{n}{2}\rfloor}}{\lfloor\frac{n}{2}\rfloor!}\right)^{1/2}\left(\frac{\sigma^{n+2}(\xi+\eta)}{(n+2)!}\right)^{1/2},\end{aligned}$$

which gives the claim (3.15). Having disposed of this preliminary step, we can now return to (3.14) and consider the series of these terms. Since the series $\sum_{n=0}^\infty H_n(\xi,\eta)$ is majorized over Ξ^+ by the converging numerical series

$$|H_n(\xi,\eta)| \leq \frac{b^{\lfloor(n-1)/2\rfloor}}{\lfloor(n-1)/2\rfloor!}\frac{\|Q\|^{n+1}_{L^2(0,b)}}{\sqrt{(n+1)!}},$$

we conclude that the series $\sum_{n=0}^\infty H_n(\xi,\eta)$ converges for each $(\xi,\eta) \in \Xi^+$. Moreover, this series converges to a function belonging to the space L^2, being a consequence of the estimation (3.15). As usual, from (3.14) we obtain that $H(\xi, \eta) := \sum_{n=0}^{\infty} H_n(\xi, \eta)$ satisfies the integral equation (3.13) as well as the uniqueness of the solution. Finally, a slight change in the way of obtaining the estimate (3.15) yields the corresponding result under the assumption of continuity of Q. Namely, (3.15) becomes

$$|H(\xi + \eta)| \leq \frac{1}{2} \frac{\|Q\|^{n+1} (\xi + \eta)^n}{n!}, \quad n = 1, 2 \dots \tag{3.16}$$

As H_n is continuous for each n, combining (3.16) with the Weierstrass M-test the statement follows. □

The proposition below establishes the relation between the smoothness of the potential Q and the integral kernel H.

Proposition 3.2.6. *Under the assumptions of Theorem 3.2.5, if moreover the matrix-valud function Q has $r \geq 0$ continuous derivatives then $H(\xi, \eta)$ has continuous derivative of all orders up to r with respect to both variables.*

Proof. The proof is by induction on r. In fact, we need only consider $r \geq 1$. If Q is continuously differentiable we see at once that $H(\xi, \eta)$ can be derived with respect to both variables, which is clear from the right hand side of (3.13). Hence, differentiating and using integration by parts leads to

$$H_\xi(\xi, \eta) = \frac{1}{2} BQ'(\xi) + \frac{1}{2} Q^2(\xi) + \int_0^\eta \big(BQ'(\xi + v) + Q^2(\xi + v)\big) H(\xi, v) \, dv, \tag{3.17}$$

and

$$H_\eta(\xi, \eta) = \int_0^\xi \big(BQ'(u + \eta) + Q^2(u + \eta)\big) H(u, \eta) \, du. \tag{3.18}$$

Since the right hand sides of these last two equalities are continuous, (3.17)-(3.18) shows that $H(\xi, \eta)$ has continuous derivatives with respect to both variables, which is our assertion for $r = 1$. Assume the statement holds for degree r; we will prove it for $r + 1$. We now apply the previous argument again, with $r = 1$ replaced by $r + 1$, and see from (3.13) that $H(\xi, \eta)$ has $r + 1$ derivatives with respect to both variables. Given that

$$\partial_\xi^{r+1} H(\xi, \eta) = \partial_\xi^r H_\xi(\xi, \eta) \quad \text{and} \quad \partial_\eta^{r+1} H(\xi, \eta) = \partial_\eta^r H_\eta(\xi, \eta), \tag{3.19}$$

on account of (3.17) and (3.18), we obtain the continuity for the above expressions on the right hand side of (3.19), which is the desired conclusion. □

Remark 3.2.7. From the proof of the last proposition we can assert that in order for $K(\cdot_x, \cdot_t)$ to be a classical solution for the Goursat problem (3.3)-(3.5), it is necessary and sufficient that Q be continuously differentiable. This last result was announced without proof in [51] (see, Chap.1, Sec.2, Prob. 5).

The only point remaining concerns to the suitable spaces for considering a transmutation operator of the form (3.2), relating the operators $\mathcal{A}_R$ and $\mathcal{A}_Q$ in the sense to hold the equality

$$T\mathcal{A}_R Y = \mathcal{A}_Q T Y, \tag{3.20}$$

on some vector space. For the sake of simplicity suppose that Q and R are continuously differentiable. It is sufficient to consider the set $AC(-b,b)^n$.

Proposition 3.2.8. *The transmutation operator T satisfies the equality*

$$T\mathcal{A}_R Y = \mathcal{A}_Q T Y,$$

for any $Y \in AC(-b,b)^n$.

Proof. Let $Y \in AC(-b,b)^n$. It follows that

$$\begin{aligned}\mathcal{A}_Q TY(x) &= B\frac{d}{dx}\left(Y(x) + \int_{-x}^{x} K(x,t)Y(t)\,dt\right) + Q(x)\left(Y(x) + \int_{-x}^{x} K(x,t)Y(t)\,dt\right), \\ &= BY'(x) + \int_{-x}^{x} \big(B\partial_x K(x,t) + Q(x)K(x,t)\big)Y(t)\,dt \\ &\qquad + \big(BK(x,x) + Q(x)\big)Y(x) + BK(x,-x)Y(-x), \end{aligned} \tag{3.21}$$

where we have used the formula

$$\partial_x \int_{-x}^{x} K(x,t)Y(t)dt = K(x,x)Y(x) + K(x,-x)Y(-x) + \int_{-x}^{x} \partial_x K(x,t)Y(t)dt. \tag{3.22}$$

On the other hand, we get

$$\begin{aligned}T\mathcal{A}_R Y(x) &= BY'(x) + R(x)Y(x) + \int_{-x}^{x} K(x,t)\big(BY'(t) + R(t)Y(t)\big)\,dt, \\ &= BY'(x) + \int_{-x}^{x} \big(K(x,t)R(t) - \partial_t K(x,t)B\big)Y(t)\,dt \\ &\qquad + \big(K(x,x)B + R(x)\big)Y(x) - K(x,-x)BY(-x), \end{aligned} \tag{3.23}$$

where we have used the integration by parts, which leads to the formula

$$\int_{-x}^{x} K(x,t)BY'(t)dt = K(x,x)BY(x) - K(x,-x)BY(-x) - \int_{-x}^{x} \partial_t K(x,t)BY(t)dt. \tag{3.24}$$

Note that the right hand sides of the formulas (3.21) and (3.23) are well-defined as vector-valued functions in L^1. Finally, if K satisfies the Goursat problem (3.3)-(3.5), equality (3.20) is obtained, conversely, if we have the property (3.20) for all $Y \in AC(-b,b)^n$, equating (3.21) and (3.23) leads to the Goursat problem (3.3)-(3.5). □

Remark 3.2.9. Following Hryniv and Pronska [30], the transmutation property can be established for larger class of potentials Q in the distributions sense, the part of the differentiability is justified from de fact that the formulas (3.22) and (3.24) are valid in the same context. Consequently, the differentiability of K should be understood in the distributional sense and it is only required that K being a mild solution of (3.3)

Summarizing, we can now state an analogue of Proposition 3.1.1 in a more general setting, for this purpose let us consider the functional space $W_2^1(-b,b)^n$.

Theorem 3.2.10. *Let $Q \in L_2((-b,b),\mathcal{M}_n)$. Then a transmutation operator T, relating the operators $\mathcal{A}_0$ and $\mathcal{A}_Q$ in the sense that*

$$T\mathcal{A}_0 Y = \mathcal{A}_Q TY, \tag{3.25}$$

for all $Y \in W_2^1(-b,b)^n$ can be realized in the form of a Volterra integral operator

$$TY(x) = Y(x) + \int_{-x}^{x} K(x,t)Y(t)\,dt. \tag{3.26}$$

Its integral kernel $K(x,t)$ is a mild solution of the Goursat problem

$$BK_x(x,t) + K_t(x,t)B = -Q(x)K(x,t) \tag{3.27}$$

$$BK(x,x) - K(x,x)B = -Q(x), \tag{3.28}$$

$$BK(x,-x) + K(x,-x)B = 0. \tag{3.29}$$

Conversely, if $K(x,t)$ is the solution of the problem (3.27), (3.28), (3.29), *then the operator T determined by the formula is a transmutation operator for the pair of operators $\mathcal{A}_0$ and $\mathcal{A}_Q$.*

3.3 Continuous dependence on the Goursat data

The purpose of this section is to present some estimates for the difference of two solutions of problems (3.27) having different Goursat data. These estimates serve in implementing methods to approximate integral kernels of transmutation operators, in particular, for the AATO method established in the next chapter.

The difference of two solutions satisfies the Goursat problem

$$BK_x(x,t) + K_t(x,t)B = -Q(x)K(x,t), \tag{3.30}$$

$$BK(x,x) - K(x,x)B = \mathcal{E}_1(x), \tag{3.31}$$

$$BK(x,-x) + K(x,-x)B = \mathcal{E}_2(x), \tag{3.32}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are differences of the corresponding Goursat data, satisfying compatibility conditions $\mathcal{E}_1 \in \mathcal{H}^-$ and $\mathcal{E}_2 \in \mathcal{H}^+$. The equivalent integral equation reads

$$\begin{aligned} H(\xi,\eta) = -\frac{1}{2}B\mathcal{E}_1(\xi) - \frac{1}{2}B\mathcal{E}_2(\eta) + \int_0^{\xi} BQ(u+\eta)\mathcal{P}^+\left[H(u,\eta)\right]du \\ + \int_0^{\eta} BQ(\xi+v)\mathcal{P}^-\left[H(\xi,v)\right]dv. \end{aligned} \tag{3.33}$$

The latter is deduced from (3.12) by taking $R \equiv 0$.

In order to motivate the results of this section let us explain the main reasons to treat with the Goursat problem (3.30)-(3.32). Guaranteeing the existence and uniqueness of solutions for this last problem not only establishes the stability but also allows us to approximate the solution of the problem (3.27)-(3.29) by solutions of (3.30) such that their restriction at $t = x$ and $t = -x$ provide an approximation of the Goursat data. This is one of the most important aspects of the AATO approach, since it reduces a problem of approximate solution of the Goursat problem into approximation on the characteristics.

Remark 3.3.1. Note that any solution of Eq. (3.30) satisfies the conditions (3.31)-(3.32) as well as the difference between the solution of (3.27)-(3.29) and an arbitrary solution of (3.30).

Proposition 3.3.2. *Let $Q \in C([0, b], \mathcal{M}_n)$. Then the Goursat problem* (3.30)-(3.32) *in the domain Ω^+ is well-posed for $\mathcal{E}_1$ and $\mathcal{E}_2$ being $\mathcal{M}_n$-valued continuous functions. Its solution is continuous and satisfies the inequality*

$$|K(x,t)| \leq \frac{1}{2}\left(\|\mathcal{E}_1\| + \|\mathcal{E}_2\|\right) \exp\left(2\int_0^x |Q(t)|\, dt\right). \tag{3.34}$$

where $\|\cdot\|$ is the maximum norm.

Proof. The proof is similar to the proof of Theorem 3.2.5. From (3.33) by the successive approximations method we get the estimate. □

Remark 3.3.3. The estimate established in the previous proposition improves the estimate that would have been obtained using the Levitan [49] approach, namely, using the latter we get

$$|K(x,t)| \leq \left(\|\mathcal{E}_1\| + \|\mathcal{E}_2\|\right) e^{2b\|Q\|\left(e^{2b\|Q\|}-1\right)}.$$

In view of this fact our main aim in the next chapter is to approximate the integral kernel. We would like to point out that Proposition 3.3.2 gives a direct estimate of the difference between the solution of the Goursat problem (3.27)-(3.29) and any solution of (3.30)-(3.32) in terms of uniform norm. However, as mentioned previously in Section 3.1, the use of a transmutation operator in the form (3.26) only involves the values $K(x, \cdot_t)$, for $-x \leq t \leq x$. Hence the following results are convenient.

Proposition 3.3.4. *Let $\mathcal{E}_1, \mathcal{E}_2 \in L^1((0,b), \mathcal{M}_n)$, $Q \in L^\infty((0,b), \mathcal{M}_n)$. Then the matrix-valued function $K(x,\cdot) = H(\xi(x,\cdot), \eta(x,\cdot))$ belongs to $L^1(-x,x)$; moreover, the following estimate holds*

$$\int_{-x}^x |K(x,t)|\, dt \leq \sqrt{2}\left(\int_0^x |\mathcal{E}_1(t)| + |\mathcal{E}_2(t)|\, dt\right) \exp\left(2\sqrt{2}\int_0^x |Q(t)|\, dt\right), \tag{3.35}$$

for all $x \in [0, b]$.

Proof. To prove the estimate (3.35) it is sufficient to use (3.33) together with the observation that for fixed x the vertical segment $-x \le t \le x$, corresponds to the path $\eta = x - \xi$ for $\xi \in [0, x]$ in the $\xi\eta$-plane. Fix $x \in [0, b]$. Let us consider the line integral

$$\int_\gamma |H(\xi, \eta)| \, dl, \quad \text{where} \quad \gamma(s) = (x - s, s), \quad 0 \le s \le x.$$

From the above, by definition of line integral, we get

$$\begin{aligned} \|H(x - \cdot, \cdot)\|_{L^1(0,x)} &\le \sqrt{2} \int_0^x \bigg(\frac{1}{2} |\mathcal{E}_1(x - s) + \mathcal{E}_2(s)| + \int_0^{x-s} |Q(u + s) H(u, s)| \, du \\ &\qquad + \int_0^s |Q(x - s + v) H(x - s, v)| \, dv \bigg) ds \\ &= \frac{\sqrt{2}}{2} \Big(\|\mathcal{E}_1\|_{L^1(0,x)} + \|\mathcal{E}_2\|_{L^1(0,x)} \Big) \\ &\qquad + \sqrt{2} \int_0^x \int_0^{x-s} |Q(u + s)| \big(|H(u, s)| + |H(s, u)| \big) \, du \, ds \\ &= \frac{\sqrt{2}}{2} \Big(\|\mathcal{E}_1\|_{L^1(0,x)} + \|\mathcal{E}_2\|_{L^1(0,x)} \Big) \\ &\qquad + \sqrt{2} \int_0^x \int_s^x |Q(u)| \big(|H(u - s, s)| + |H(s, u - s)| \big) \, du \, ds. \end{aligned}$$

By changing integration order, we obtain the inequality

$$\begin{aligned} \|H(x - \cdot, \cdot)\|_{L^1(0,x)} &\le \frac{\sqrt{2}}{2} \Big(\|\mathcal{E}_1\|_{L^1(0,x)} + \|\mathcal{E}_2\|_{L^1(0,x)} \Big) \\ &\qquad + \sqrt{2} \int_0^x |Q(u)| \big(\|H(u - \cdot, \cdot)\|_{L^1(0,u)} + \|H(\cdot, u - \cdot)\|_{L^1(0,u)} \big) \, du. \end{aligned} \tag{3.36}$$

Repeating the previous argument and using the opposite parametrization to γ leads to

$$\begin{aligned} \|H(\cdot, x - \cdot)\|_{L^1(0,x)} &\le \frac{\sqrt{2}}{2} \Big(\|\mathcal{E}_1\|_{L^1(0,x)} + \|\mathcal{E}_2\|_{L^1(0,x)} \Big) \\ &\qquad + \sqrt{2} \int_0^x |Q(u)| \big(\|H(u - \cdot, \cdot)\|_{L^1(0,u)} + \|H(\cdot, u - \cdot)\|_{L^1(0,u)} \big) \, du. \end{aligned} \tag{3.37}$$

Summing (3.36) and (3.37), and applying Grönwall's inequality we conclude that

$$\int_0^x |H(x - s, s)| + |H(s, x - s)| \, ds \le \sqrt{2} \Big(\int_0^x |\mathcal{E}_1(s)| + |\mathcal{E}_2(s)| \, ds \Big) \exp\Big(2\sqrt{2} \int_0^x |Q(s)| \, ds \Big).$$

From this and the simple observation that $\int_{-x}^x |K(x, t)| \, dt = \int_0^x |H(s, x - s)| \, ds$ we obtain estimate (3.35), and the proposition follows. $\square$

The proposition below is similar to the latter under the assumption that the Goursat data belongs to $L^2(\mathcal{M}_n)$, the proof can be handled in much the same way, however we present the main ideas of the proof.

Proposition 3.3.5. *Let $\mathcal{E}_1, \mathcal{E}_2, Q \in L^2((0,b), \mathcal{M}_n)$. Then the matrix-valued function $K(x,\cdot) = H(\xi(x,\cdot), \eta(x,\cdot))$ belongs to $L^2(-x,x)$; moreover, the following estimate holds*

$$\int_{-x}^{x} |K(x,t)|^2 \, dt \leq 2\sqrt{2} \left(\int_0^x |\mathcal{E}_1(t)|^2 + |\mathcal{E}_2(t)|^2 \, dt \right) \exp\left(8x \int_0^x |Q(t)|^2 \, dt \right), \tag{3.38}$$

for all $x \in [0,b]$.

Proof. Let γ be as before. Using Minkowski's inequality we find that

$$\begin{aligned} \|H(x-\cdot,\cdot)\|_{L^2(0,x)} \leq \frac{1}{2}\Big(\|\mathcal{E}_1\|_{L^2(0,x)} + \|\mathcal{E}_2\|_{L^2(0,x)} \Big) \\ + \left(\int_0^x \Big(\int_0^{x-s} |Q(u+s)H(u,s)| \, du \right. \\ \left. + \int_0^s |Q(x-s+v)H(x-s,v)| \, dv \Big)^2 \, ds \right)^{1/2}. \end{aligned} \tag{3.39}$$

By simple changes of variables and using Hölder's inequality, the sum of the last two terms on the right-hand side of (3.39), yields

$$\left(2x \int_0^x \int_s^x |Q(\theta)|^2 \left(|H(\theta-s,s)|^2 + |H(s,\theta-s)|^2 \right) d\theta \, ds \right)^{1/2},$$

and by changing integration order we have

$$\left(2x \int_0^x |Q(\theta)|^2 \int_0^\theta \left(|H(\theta-s,s)|^2 + |H(s,\theta-s)|^2 \right) ds \Big) \, d\theta \right)^{1/2}.$$

Taking into account these recent changes in inequality (3.39), it follows that

$$\begin{aligned} \|H(x-\cdot,\cdot)\|^2_{L^2(0,x)} \leq \|\mathcal{E}_1\|^2_{L^2(0,x)} + \|\mathcal{E}_2\|^2_{L^2(0,x)} \\ + 4x \int_0^x |Q(\theta)|^2 \left(\|H(\theta-\cdot,\cdot)\|^2_{L^2(0,\theta)} + \|H(\cdot,\theta-\cdot)\|^2_{L^2(0,\theta)} \right) d\theta. \end{aligned} \tag{3.40}$$

Similarly, using the opposite parametrization to γ the same estimate as in (3.40) is obtained for $\|H(\cdot,x-\cdot)\|^2_{L^2(0,x)}$. Hence

$$\begin{aligned} \|H(\cdot,x-\cdot)\|^2_{L^2(0,x)} \leq \|\mathcal{E}_1\|^2_{L^2(0,x)} + \|\mathcal{E}_2\|^2_{L^2(0,x)} \\ + 4x \int_0^x |Q(\theta)|^2 \left(\|H(\theta-\cdot,\cdot)\|^2_{L^2(0,\theta)} + \|H(\cdot,\theta-\cdot)\|^2_{L^2(0,\theta)} \right) d\theta. \end{aligned} \tag{3.41}$$

Combining (3.40) with (3.41) and applying the Grönwall's inequality, we conclude that

$$\int_0^x |H(x-\theta,\theta)|^2+|H(\theta,x-\theta)|^2\,d\theta \leq 2\Big(\int_0^x |\mathcal{E}_1(\theta)|^2+|\mathcal{E}_2(\theta)|^2\,d\theta\Big)\exp\Big(8x\int_0^x |Q(\theta)|^2\,d\theta\Big),$$

and the proposition follows. ☐

3.4 Goursat-to-Goursat transmutation operator

The aim of this section is to perform a direct analysis of the Goursat problem in the preceding section via an operator transforming the boundary data of the Goursat problem for a Matrix Transport Equation into the boundary data of the Goursat problem (3.30)-(3.32). This idea goes back to [42] and once again indicates the use of transmutation operators as an analytical tool to examine the Goursat problem (3.30)-(3.32).

Let us start introducing a simpler problem than the main problem of the previous section, namely, the Goursat problem (3.30), (3.31) and (3.32), by setting $Q \equiv 0$, $K(x,t)$ being defined on $\Omega^+ \cup \Omega^-$. We recall B as being defined on Section 3.2. For convenience we change the notation of this new problem as follows

$$B\boldsymbol{k}_x(x,t) + \boldsymbol{k}_t(x,t)B = 0, \tag{3.42}$$

$$B\boldsymbol{k}(x,x) - \boldsymbol{k}(x,x)B = \Phi(x), \tag{3.43}$$

$$B\boldsymbol{k}(x,-x) + \boldsymbol{k}(x,-x)B = \Psi(x), \tag{3.44}$$

where $x \in [-b,b]$ and Φ and Ψ belong to the spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively.

Lemma 3.4.1. *The general solution of the equation*

$$B\boldsymbol{k}_x + \boldsymbol{k}_tB = 0, \tag{3.45}$$

has the following form

$$\boldsymbol{k}(x,t) = \mathcal{P}^+[H_1]\left(\frac{x+t}{2}\right) + \mathcal{P}^-[H_2]\left(\frac{x-t}{2}\right). \tag{3.46}$$

where H_1 and H_2 are arbitrary continuously differentiable $\mathcal{M}_n$-valued functions.

Proof. An easy computation shows that the right hand side of (3.46) satisfy (3.45). On the other hand, let $\boldsymbol{k}$ be a solution of equation (3.45). Define $h(\xi(x,t),\eta(x,t)) = \boldsymbol{k}(x,t)$ via (3.6). Then $2\boldsymbol{k}_x = h_\xi + h_\eta$ and $2\boldsymbol{k}_t = h_\xi - h_\eta$. Substituting these into (3.45) yields $\mathcal{P}^-[h_\xi](\xi,\eta)+\mathcal{P}^+[h_\eta](\xi,\eta) = 0$. Applying $\mathcal{P}^+$ and $\mathcal{P}^-$ we conclude that $\mathcal{P}^+[h_\eta](\xi,\eta) = 0$ and $\mathcal{P}^-[h_\xi](\xi,\eta) = 0$. From this, if we now integrate respect to the variables η and ξ we obtain that $\mathcal{P}^+[h](\xi,\eta) = C_1(\xi)$ and $\mathcal{P}^-[h](\xi,\eta) = C_2(\eta)$ for some $C_1 \in \mathcal{H}^-$ and $C_2 \in \mathcal{H}^+$. Thus,

$$\boldsymbol{k}(x,t) = \mathcal{P}^+[h](\xi,\eta) + \mathcal{P}^-[h](\xi,\eta) = C_1\left(\frac{x+t}{2}\right) + C_2\left(\frac{x-t}{2}\right).$$

Finally, we have $C_1 = \mathcal{P}^+[H_1]$ and $C_2 = \mathcal{P}^-[H_2]$, for some $H_1, H_2 \in \mathcal{H}$, because the image of $\mathcal{P}^+$ is $\mathcal{H}^-$ and the image of $\mathcal{P}^-$ is $\mathcal{H}^-$, which completes the proof. ☐

Remark 3.4.2. Actually (3.46) can be put in the form

$$\boldsymbol{k}(x,t) = \mathcal{P}^{+}[H]\,(x+t) + \mathcal{P}^{-}[H]\,(x-t)\,, \tag{3.47}$$

as well as (3.47) becomes (3.46). We recall that $\mathcal{H}^{+} \cap \mathcal{H}^{-} = \{0\}$, therefore if we replace $H(x) = \mathcal{P}^{+}H_1(x/2) + \mathcal{P}^{-}H_2(x/2)$ into (3.47) we recover (3.46).

The proof of the proposition below is based on the result of the preceding lemma, which establishes the general solution of equation (3.42). Having disposed of this preliminary step, by an easy computation, the proposition follows.

Proposition 3.4.3. *The general solution of the Goursat problem* (3.42)-(3.44) *is given by*

$$\boldsymbol{k}(x,t) = \frac{1}{2}\left(\Phi\left(\frac{x+t}{2}\right) - \Psi\left(\frac{x-t}{2}\right)\right)B. \tag{3.48}$$

With the purpose to establish the following proposition and to illustrate how the transmutation property works, we consider a transmutation operator T being as in Theorem 3.2.10, and $\boldsymbol{k}(x,t)$ being a solution of equation (3.42). But $T\mathcal{A}_0 = \mathcal{A}_Q T$, as was described in (3.20), and it follows that

$$\mathcal{A}_Q T\boldsymbol{k}(x,t) = T\mathcal{A}_0\boldsymbol{k}(x,t) = -T\partial_t\boldsymbol{k}(x,t)B = -\partial_t T\boldsymbol{k}(x,t).$$

In the other words, the image of $\boldsymbol{k}$ under the transmutation operator T satisfies (3.30). The same holds for K being a solution of equation (3.30). Taking into account that T induces a transmutation operator acting by columns on the space $\mathcal{H}$, by abuse of notation we use the same letter T.

Proposition 3.4.4. *Under the conditions stated above, there exist a bounded operator T_G mapping the Goursat data corresponding to the Goursat problem* (3.42)-(3.44) *into the Goursat data* (3.31)-(3.32).

Proof. Let Φ and Ψ be continuously differentiable $\mathcal{M}_n$-valued functions defined on $[-b,b]$. Setting $\boldsymbol{k}$ by the formula (3.48) and applying the transmutation operator to this expression we get

$$\begin{aligned}\mathcal{U}(x,t) = T\boldsymbol{k}(x,t) = \Bigg(\Phi\left(\frac{x+t}{2}\right) + \int_{-x}^{x} K(x,\tau)\Phi\left(\frac{\tau+t}{2}\right)\,d\tau \\ - \Psi\left(\frac{x+t}{2}\right) + \int_{-x}^{x} K(x,\tau)\Psi\left(\frac{\tau+t}{2}\right)\,d\tau\Bigg)B.\end{aligned} \tag{3.49}$$

Considering the values of these expressions when $t = x$ and $t = -x$, and making some changes of variables yields

$$\begin{aligned}\mathcal{U}(x,x) = \frac{1}{2}\Bigg(\Phi\,(x) + 2\int_{0}^{x} K(x,2t-x)\Phi\,(t)\;dt \\ - 2\int_{-x}^{0} K(x,2t+x)\Psi\,(t)\;dt - \Psi\,(0)\Bigg)B\end{aligned} \tag{3.50}$$

and

$$\mathcal{U}(x,-x) = \frac{1}{2}\Bigg(-\Psi(x) + 2\int_{-x}^{0} K(x,2t+x)\Phi(t)\,dt \\ - 2\int_{0}^{x} K(x,2t-x)\Psi(t)\,dt + \Phi(0)\Bigg)B. \tag{3.51}$$

Of course the expressions $\mathcal{U}(x,x)$ and $\mathcal{U}(x,-x)$ may not satisfy the compatibility conditions for Goursat data (3.30)-(3.32), nevertheless applying $2B\mathcal{P}^+$ and $2B\mathcal{P}^-$ respectively to (3.50) and (3.51) we obtain a mapping

$$\begin{pmatrix}\Phi(x)\\ \Psi(x)\end{pmatrix} \xmapsto{T_G} \begin{pmatrix}\mathcal{E}_1(x)\\ \mathcal{E}_2(x)\end{pmatrix}, \quad x\in[-b,b] \tag{3.52}$$

where $\mathcal{E}_1 \in \mathcal{H}^+$ and $\mathcal{E}_2 \in \mathcal{H}^-$. Note that $2B\mathcal{P}^+$ and $2B\mathcal{P}^-$ lead to the Goursat conditions (3.31)-(3.32). These last are given by

$$\mathcal{E}_1(x) = \Phi(x) + \int_0^x (K^*+K)(x,2t-x)\Phi(t)\,dt + \int_{-x}^0 (K^*-K)(x,2t+x)\Psi(t)\,dt \tag{3.53}$$

and

$$\mathcal{E}_2(x) = \Psi(x) + \int_0^x (K^*+K)(x,2t-x)\Psi(t)\,dt + \int_{-x}^0 (K^*-K)(x,2t+x)\Phi(t)\,dt \tag{3.54}$$

where K^* denotes the term $K^* = -BKB$. We consider the operator T_G defined by (3.52) acting on the space

$$\widehat{\mathcal{H}} = \left\{ \begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \,\middle|\, \Phi\in\mathcal{H}^+, \Psi\in\mathcal{H}^- \right\} \subseteq C^1((-b,b),\mathcal{M}_n)\times C^1((-b,b),\mathcal{M}_n),$$

equipped by the norm of the sum of two $\mathcal{M}_n$-valued functions. What is left is to show that T_G is a bounded operator. In fact, by definition,

$$\begin{aligned}\left\| T_G(\Phi,\Psi)^T(\cdot_x)\right\|_\infty &= \|\mathcal{E}_1\|_\infty + \|\mathcal{E}_2\|_\infty \\ &= \max_{x\in[-b,b]} \left|2B\mathcal{P}^+[\mathcal{U}](x,x)\right| + \max_{x\in[-b,b]}\left|2B\mathcal{P}^-[\mathcal{U}](x,-x)\right| \\ &\le 4\max_{(x,t)\in[-b,b]} |\mathcal{U}(x,t)| \\ &= 4\|\mathcal{U}(\cdot_x,\cdot_t)\|_\infty .\end{aligned}$$

Since $\mathcal{U} = T\boldsymbol{k}(\cdot_x,\cdot_t)$ satisfies (3.30)-(3.32), using the estimate (3.34) of Proposition 3.3.2 it follows that

$$\left\|T_G(\Phi,\Psi)^T\right\|_\infty \le 2\Big(\|\mathcal{E}_1\|_\infty + \|\mathcal{E}_2\|_\infty\Big)\exp\left(2\int_{-b}^{b}|Q(t)|\,dt\right).$$

Finally, an easy computation shows that

$$|\mathcal{E}_1(x)| + |\mathcal{E}_2(x)| \leq 2 \left\|(\Phi, \Psi)^T\right\|_\infty \left(1 + 2b \left\|K(x,t)\right\|_\infty\right),$$

which follows from (3.53) and (3.54). According to these latter inequalities , we can assert that

$$\|T_G\| \leq 4\Big(1 + 2b \left\|K(x,t)\right\|_\infty\Big) \exp\left(2 \int_{-b}^{b} |Q(t)| \; dt\right),$$

and the proof is complete. □

Remark 3.4.5. Actually the Goursat-to-Goursat operator T_G is also a bounded operator on the linear space $\widehat{\mathcal{H}}$ as subspace of $W^{1,2}(\mathcal{M}_n) \times W^{1,2}(\mathcal{M}_n)$, which follows from Proposition 3.3.5 and the fact that the transmutation operator T is well defined on the Sobolev space $W^{1,2}(\mathcal{M}_n)$, see Theorem 3.2.10.

Moreover, T_G is a diagonalizable operator. We observe that the $\mathcal{M}_n$-valued functions Φ and Ψ appear in both (3.53) and (3.54). Similar to [43] the Goursat operator can be decoupled in a pair of operators acting separately on each $\mathcal{M}_n$-valued function belonging to $\mathcal{H}^+$ and $\mathcal{H}^-$.

To do this, we define $\mathcal{G}$ to be UT_GU , where U is given by the block matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & -\mathcal{I} \end{pmatrix},$$

and $\mathcal{I}$ is the identity matrix of size n. Let $(\Phi, \Psi)^T \in \widehat{\mathcal{H}}$. Computing UT_GU we get a new mapping from $\widehat{\mathcal{H}}$ into itself in the form

$$\begin{pmatrix} \mathcal{E}_1(x) \\ \mathcal{E}_2(x) \end{pmatrix} = \begin{pmatrix} \Phi(x) + \int_0^x \left(K^* + K\right)(x, 2t - x)\Phi(t)\, dt + \int_{-x}^0 \left(K^* - K\right)(x, 2t + x)\Phi(t)\, dt \\ \Psi(x) + \int_0^x \left(K^* + K\right)(x, 2t - x)\Psi(t)\, dt - \int_{-x}^0 \left(K^* - K\right)(x, 2t + x)\Psi(t)\, dt \end{pmatrix}$$

which can be rewritten as follows

$$\mathcal{G}\begin{pmatrix} \Phi(x) \\ \Psi(x) \end{pmatrix} = \begin{pmatrix} \Phi(x) \\ \Psi(x) \end{pmatrix} + \int_{-x}^{x} \begin{pmatrix} K^{(1)}(x,t) & 0 \\ 0 & K^{(2)}(x,t) \end{pmatrix} \begin{pmatrix} \Phi(t) \\ \Psi(t) \end{pmatrix} dt, \tag{3.55}$$

where

$$K^{(1)}(x,t) := \begin{cases} (K^* + K)(x, 2t - x), & 0 < t \leq x, \\ (K^* - K)(x, 2t + x), & -x \leq t < 0. \end{cases} \tag{3.56}$$

$$K^{(2)}(x,t) := \begin{cases} (K^* + K)(x, 2t - x), & 0 < t \leq x, \\ (K - K^*)(x, 2t + x), & -x \leq t < 0. \end{cases} \tag{3.57}$$

or which is the same $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$,

$$\mathcal{G}_1\Phi(x) = \Phi(x) + \int_{-x}^{x} K^{(1)}(x,t)\Phi(t)\, dt \tag{3.58}$$

and

$$\mathcal{G}_2\Psi(x) = \Psi(x) + \int_{-x}^{x} K^{(2)}(x,t)\Psi(t)\, dt. \tag{3.59}$$

Remark 3.4.6. In general neither $K^{(1)}(x,t)$ nor $K^{(2)}(x,t)$ are continuous integral kernels. A necessary but not sufficient condition for $K^{(1)}$ and $K^{(2)}$ to be continuous integral kernels is that the potential Q satisfies the condition $Q(0)=0$. For example, using the Goursat conditions (3.28)-(3.29) one can verify that $K^{(1)}(x,t)$ tends to 0 along the line $t=x$, $x>0$, whereas $K^{(1)}(x,t)$ tends to $-BQ(0)$ along the line $t=-x$, $x>0$. Nevertheless, it is known that operators (3.58) and (3.59) are invertible operators.

Taking advantage of the projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ we observe that the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ can be written as

$$\mathcal{G}_1\Phi(x)=\Phi(x)+2\int_0^x \mathcal{P}^-[K](x,2t-x)\Phi(t)\,dt-2\int_{-x}^0 \mathcal{P}^+[K](x,2t+x)\Phi(t)\,dt \tag{3.60}$$

and

$$\mathcal{G}_2\Psi(x)=\Psi(x)+2\int_0^x \mathcal{P}^-[K](x,2t-x)\Psi(t)\,dt+2\int_{-x}^0 \mathcal{P}^+[K](x,2t+x)\Psi(t)\,dt. \tag{3.61}$$

In addition it is worth noting that the linear operators $\mathcal{G}_1$ and $\mathcal{G}_2$ may be extended from the spaces $\mathcal{H}^-$ and $\mathcal{H}^+$ respectively to the collection of all $\mathcal{M}_n$-valued functions. They are quite symmetric and satisfy the following relations

$$B\mathcal{G}_1[\Phi]=-\mathcal{G}_2[\Phi]B,\quad \Phi\in\mathcal{H}^-, \tag{3.62}$$
$$B\mathcal{G}_1[\Psi]=\mathcal{G}_2[\Psi]B,\quad \Psi\in\mathcal{H}^+, \tag{3.63}$$

Combining these we can apply the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ for any $\mathcal{M}_n$-valued function $\mathcal{F}$ as follows by decomposing $\mathcal{F}=\Phi+\Psi$, $\Phi\in\mathcal{H}^-$, $\Psi\in\mathcal{H}^+$,

$$\mathcal{G}_1[\mathcal{F}]=\mathcal{G}_1[\Phi+\Psi]=\mathcal{G}_1[\Phi]-B\mathcal{G}_2[\Psi]B, \tag{3.64}$$
$$\mathcal{G}_2[\mathcal{F}]=\mathcal{G}_2[\Phi+\Psi]=B\mathcal{G}_1[\Phi]B+\mathcal{G}_2[\Psi]. \tag{3.65}$$

The natural question arises how a Volterra integral operator with integral kernel K^* works? The proposition below solve this question.

Corollary 3.4.7. *Let K be as in Theorem 3.2.10. Set $K^*=-BKB$. Then the Volterra integral operator*

$$T^*Y(x)=Y(x)+\int_{-x}^x K^*(x,t)Y(t)\,dt, \tag{3.66}$$

is a transmutation operator for the pair of differentiable operators $\mathcal{A}_0$ and $\mathcal{A}_{-Q}$, and its integral kernel satisfies

$$BK_x^*(x,t)+K_t^*(x,t)B=Q(x)K^*(x,t), \tag{3.67}$$
$$BK^*(x,x)-K^*(x,x)B=Q(x), \tag{3.68}$$
$$BK^*(x,-x)+K^*(x,-x)B=0. \tag{3.69}$$

Proof. It is sufficient to show the transmutation property for the operator T^*. We first recall that $B^2 = -\mathcal{I}$, $BQ(x) = -Q(x)B$. Since $T^*[Y](x) = -BT[BY](x)$, which follows easily from (3.66), we see that

$$T^*[\mathcal{A}_0 Y](x) = -BT[B\mathcal{A}_0 Y](x) = -BT[\mathcal{A}_0 BY](x). \tag{3.70}$$

Applying the transmutation property (3.25) of the operator T to BY leads to

$$T[\mathcal{A}_0 BY](x) = \mathcal{A}_Q T[BY](x) = B\frac{d}{dx}T[BY](x) + Q(x)T[BY](x).$$

Combining (3.70) with this latter equality gives

$$T^*\mathcal{A}_0 Y(x) = \mathcal{A}_{-Q}T^*Y(x),$$

and the transmutation property for T^* is proved. $\square$

Chapter 4

The AATO method for the Dirac system

The aim of this chapter is to present an analytic approximation of the transmutation operators (AATO) for one-dimensional Dirac operators, which is the main result of this work. This method allows one to obtain an approximation of the solutions of the one-dimensional Dirac system by means of transmutation operators.

Unless otherwise stated throughout this chapter we assume that the potential Q belongs to $C([0,b],\mathcal{M}_2)$ and is extended, if needed, onto $[-b,0)$ as a continuous function. Therefore in accordance with Theorem 3.2.10 it is sufficient to use the transmutation property on $C^1(-b,b)\times C^1(-b,b)$.

In the next section we establish the mapping property which is an important property that connects the solutions in the SPPS representation for the one-dimensional Dirac system with solutions of the same system through transmutation operators. Section 4.2 is devoted to construction of the complete system of solutions for the integral kernel equation. In Section 4.3 we develop the approximate construction of the integral kernel based on the conditions that the integral kernel of transmutation operator satisfies on the characteristics curves $x=t$ and $x=-t$. In Section 4.4 we obtain the main result of the work.

4.1 Mapping property

We begin by recalling the notion of transmutation operators for operators of interest $\mathcal{A}_0$ and $\mathcal{A}_Q$ indicated by Theorem 3.2.10 in the previous chapter. Here and subsequently, B and Q denote 2×2 matrix-valued functions in the standard form

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad \text{and} \quad Q(x)=\begin{pmatrix}p(x) & q(x)\\ q(x) & -p(x)\end{pmatrix}.$$

A transmutation operator in the sense of Definition 1.3.1 can be realized in the form of a Volterra integral operator

$$Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)dt, \tag{4.1}$$

and its integral kernel satisfies a Goursat problem. Furthermore, the transmutation property makes it possible for T to map a solution $v = (v_1, v_2)^T$ of the equation

$$B\frac{dv}{dx} + \lambda v = 0, \tag{4.2}$$

where λ is a complex number, into a solution $u = (u_1, u_2)^T$ of the equation

$$B\frac{du}{dx} + Q(x)u + \lambda u = 0, \tag{4.3}$$

with the correspondence of the initial values $u(0) = v(0)$.

Unfortunately the integral kernel of the operator T can be found in closed form only for a few particular potentials, in general it is unknown. So there is no way to determine the result of T acting on an arbitrary vector-valued function. However, it is possible to determine the result of T acting on an arbitrary vector function of the form $(x^k, x^m)^T$, and, hence, on arbitrary vector-function $(p_1, p_2)^T$, where p_1 and p_2 are polynomials.

For this purpose the following assumptions will be needed throughout the chapter. We restrict the use of Theorem 2.2.4 to segment $(-b, b)$ and take $x_0 = 0$. In addition, the non-vanishing solution $(f, g)^T$ is normalized according to $f(0)g(0) = 1$. From these last assumptions, combining Theorem 2.2.4 and Theorem 3.2.10 we conclude that

$$Ty(x) = c_1 \sum_{k=0}^{\infty} \lambda^k \begin{pmatrix} f\widetilde{X}^{(k)}(x) \\ g\widetilde{Y}^{(k)}(x) \end{pmatrix} + c_2 \sum_{k=0}^{\infty} \lambda^k \begin{pmatrix} fX^{(k)}(x) \\ gY^{(k)}(x) \end{pmatrix}, \tag{4.4}$$

where $c_1$and c_2 are arbitrary complex constants and y is a solution of (4.2).

Remark 4.1.1. By Theorem 2.5.5, which is a generalization of Theorem 2.2.4 the equality (4.4) is valid on the linear space $W^{1,1}(-b, b) \times W^{1,1}(-b, b)$.

On the other hand it is easy to check that the general solution of equation (4.2) is given by the linear combination of vector-valued functions

$$\begin{pmatrix} \cos(\lambda x) \\ -\sin(\lambda x) \end{pmatrix}, \quad \begin{pmatrix} \sin(\lambda x) \\ \cos(\lambda x) \end{pmatrix}. \tag{4.5}$$

Thus from (4.4) taking into account the initial conditions it follows that

$$T\begin{pmatrix} \cos(\lambda x) \\ -\sin(\lambda x) \end{pmatrix} = \frac{1}{f(0)} \sum_{k=0}^{\infty} \lambda^k \begin{pmatrix} f(x)\widetilde{X}^{(k)}(x) \\ g(x)\widetilde{Y}^{(k)}(x) \end{pmatrix}, \tag{4.6}$$

and,

$$T\begin{pmatrix} \sin(\lambda x) \\ \cos(\lambda x) \end{pmatrix} = \frac{1}{g(0)} \sum_{k=0}^{\infty} \lambda^k \begin{pmatrix} f(x)X^{(k)}(x) \\ g(x)Y^{(k)}(x) \end{pmatrix}. \tag{4.7}$$

Finally, by representing the functions $\sin(\lambda x)$ and $\cos(\lambda x)$ in their respective power series, and since the solutions are analytical with respect to the parameter λ, by comparing coefficients respectively in (4.6) and (4.7) we obtain

$$T\begin{pmatrix} x^{2k} \\ 0 \end{pmatrix} = \tfrac{(2k)!}{(-1)^k}\tfrac{1}{f(0)}\begin{pmatrix} f(x)\widetilde{X}^{(2k)}(x) \\ g(x)\widetilde{Y}^{(2k)}(x) \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ x^{2k+1} \end{pmatrix} = \tfrac{(2k+1)!}{(-1)^{k+1}}\tfrac{1}{f(0)}\begin{pmatrix} f(x)\widetilde{X}^{(2k+1)}(x) \\ g(x)\widetilde{Y}^{(2k+1)}(x) \end{pmatrix},$$
$$T\begin{pmatrix} 0 \\ x^{2k} \end{pmatrix} = \tfrac{(2k)!}{(-1)^k}\tfrac{1}{g(0)}\begin{pmatrix} f(x)X^{(2k)}(x) \\ g(x)Y^{(2k)}(x) \end{pmatrix}, \quad T\begin{pmatrix} x^{2k+1} \\ 0 \end{pmatrix} = \tfrac{(2k+1)!}{(-1)^k}\tfrac{1}{g(0)}\begin{pmatrix} f(x)X^{(2k+1)}(x) \\ g(x)Y^{(2k+1)}(x) \end{pmatrix}.$$

If we now utilize the infinite sequences of vector-valued functions $\{\Phi_k\}_{k=0}^{\infty}$ and $\{\Psi_k\}_{k=0}^{\infty}$ introduced in Definition 2.3.1, we obtain the following

Theorem 4.1.2 (Mapping theorem)**.** *Let $p, q \in L^2[-b, b]$ be complex valued functions. Let f and g be as in Theorem 2.2.4 normalized according to the condition $f(0)g(0) = 1$. Let T be the transmutation operator for $\mathcal{A}_0$ and $\mathcal{A}_Q$, and let Φ_k and Ψ_k be vector-valued functions defined by* (2.45) *and* (2.46) *respectively. Then*

$$T\begin{pmatrix} x^k \\ 0 \end{pmatrix} = \Phi_k(x) \qquad \text{and} \qquad T\begin{pmatrix} 0 \\ x^k \end{pmatrix} = \Psi_k(x), \quad k = 0, 1, 2, \dots. \tag{4.8}$$

Corollary 4.1.3. *Under the assumptions of Theorem 4.1.2 with T replaced by T^* the following mapping property holds*

$$T^*\begin{pmatrix} x^k \\ 0 \end{pmatrix} = B\Psi_k(x) \quad \text{and} \quad T^*\begin{pmatrix} 0 \\ x^k \end{pmatrix} = -B\Phi_k(x), \quad k = 0, 1, 2, \dots. \tag{4.9}$$

4.2 Construction of complete systems of solutions

Our aim in this section is to introduce a complete system of solutions for the integral kernel equation. As explained in Section 3.4 this problem is naturally connected with pre-images of solutions of the integral kernel equation.

4.2.1 Equation $B\boldsymbol{k}_x + \boldsymbol{k}_t B = 0$.

In order to approximate solutions of the above equation we introduce a new system of matrix-valued functions. It is Lemma 3.4.1 that makes this definition allowable.

Definition 4.2.1 (Wave Matrices)**.** Let us apply the formula (3.47) replacing H by each of these four $\mathcal{M}_2$-valued functions

$$\begin{pmatrix} x^m & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x^m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x^m & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x^m \end{pmatrix}, \quad m = 0, 1, 2, \dots$$

which leads to the following four $\mathcal{M}_2$-valued functions

$$P_m^1(x,t) = \begin{pmatrix} p_{2m-1}(x,t) & 0 \\ 0 & -p_{2m}(x,t) \end{pmatrix}, \qquad P_m^2(x,t) = \begin{pmatrix} 0 & p_{2m-1}(x,t) \\ p_{2m}(x,t) & 0 \end{pmatrix},$$
$$P_m^3(x,t) = \begin{pmatrix} 0 & p_{2m}(x,t) \\ p_{2m-1}(x,t) & 0 \end{pmatrix}, \qquad P_m^4(x,t) = \begin{pmatrix} -p_{2m}(x,t) & 0 \\ 0 & p_{2m-1}(x,t) \end{pmatrix},$$

where, $m \geq 0$, $p_0(x,t) = 1$,

$$p_{2m-1}(x,t) = \frac{1}{2}\left((x+t)^m + (x-t)^m\right), \tag{4.10}$$
$$p_{2m}(x,t) = \frac{1}{2}\left((x+t)^m - (x-t)^m\right). \tag{4.11}$$

In this $P_m^1, \ldots, P_m^4$, are called wave matrices.

Remark 4.2.2. Of course the $\mathcal{M}_2$-valued functions P_m^i, $i = 1,2,3,4$ satisfy the wave-matrix equation $\partial_x^2 K(x,t) = \partial_t^2 K(x,t)$. In addition, the functions in (4.10)-(4.11) are known as wave polynomials and these can be rewritten as

$$p_0(x,t) = 1,\ p_{2m-1}(x,t) = \sum_{even\, k=0}^{m} \binom{m}{k} x^{m-k} t^k,\ p_{2m}(x,t) = \sum_{odd\, k=1}^{m} \binom{m}{k} x^{m-k} t^k \tag{4.12}$$

The fact that they arise here does not cause us any surprise since the wave polynomials are complete in the linear space of regular solutions of the wave equation with respect to the maximum norm, see [33] for more details.

In order to motivate our results, let us take a look at the Goursat problem (3.42), (3.43) and (3.44) in Section 3.4 . We offer a similar version to Proposition 1 of [33].

Proposition 4.2.3. *Let φ and ψ be $\mathcal{M}_2$-valued functions corresponding to the Goursat problem* (3.42)-(3.44)*, which entries being uniformly convergent series on* $[-b,b]$

$$\varphi(x) = \sum_{n=0}^{\infty} \begin{pmatrix} a_n x^n & b_n x^n \\ b_n x^n & -a_n x^n \end{pmatrix}, \quad \psi(x) = \sum_{n=0}^{\infty} \begin{pmatrix} c_n x^n & d_n x^n \\ -d_n x^n & c_n x^n \end{pmatrix}. \tag{4.13}$$

Then the unique solution of the Goursat problem (3.42), (3.43), (3.44) *is equal to*

$$\boldsymbol{k}(x,t) = \sum_{n=0}^{\infty} \frac{d_n - b_n}{2^{n+1}} P_n^1(x,t) + \frac{a_n - c_n}{2^{n+1}} P_n^2(x,t) + \frac{a_n + c_n}{2^{n+1}} P_n^3(x,t) + \frac{b_n + d_n}{2^{n+1}} P_n^4(x,t). \tag{4.14}$$

Proof. It is sufficient to use (3.48) in Proposition 3.4.3 together with the Goursat data (4.13), the proposition follows by direct computation. □

Proposition 4.2.4. *Let* $\boldsymbol{k}(\cdot_x, \cdot_t)$ *be an* $\mathcal{M}_2$*-valued solution of equation* (3.45). *Given any* $\epsilon > 0$*, there exists a linear combination of wave matrices in the form*

$$\boldsymbol{k}_m(x,t) = \sum_{n=0}^{m} \Big(a_n P_n^1(x,t) + b_n P_n^2(x,t) + c_n P_n^3(x,t) + d_n P_n^4(x,t) \Big) \tag{4.15}$$

such that for every $(x,t) \in [-b,b] \times [-b,b]$

$$|\boldsymbol{k}(x,t) - \boldsymbol{k}_m(x,t)| < \epsilon. \tag{4.16}$$

Proof. Let $\boldsymbol{k}$ be given. In accordance with Lemma 3.4.1 and (3.47) there exists $H \in C([-b,b], \mathcal{M}_2)$, so that $\boldsymbol{k}(x,t) = \mathcal{P}^+[H(x+t)] + \mathcal{P}^-[H(x-t)]$. Let

$$H(x) = \begin{pmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{pmatrix}.$$

Given $\epsilon > 0$, by the Weierstrass Approximation Theorem, there exist polynomials p_{ij}, $i,j \in \{1,2\}$, of the same degree such that

$$|h_{ij}(x) - p_{ij}(x)| < \frac{\epsilon}{2},$$

for every $x \in [-b,b]$. To construct an approximation in (4.16) let us first consider

$$\begin{aligned} P(x) &= \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix} \\ &= \sum_{n=0}^{m} \left[a_n \begin{pmatrix} x^n & 0 \\ 0 & 0 \end{pmatrix} + b_n \begin{pmatrix} 0 & x^n \\ 0 & 0 \end{pmatrix} + c_n \begin{pmatrix} 0 & 0 \\ x^n & 0 \end{pmatrix} + d_n \begin{pmatrix} 0 & 0 \\ 0 & x^n \end{pmatrix} \right]. \end{aligned} \tag{4.17}$$

After applying the formula on the right hand side of (3.47) we get

$$\begin{aligned} \boldsymbol{k}_m(x,t) &= \mathcal{P}^+ \left[P(x+t) \right] + \mathcal{P}^- \left[P(x-t) \right] \\ &= \sum_{n=0}^{m} a_n P_n^1(x,t) + b_n P_n^2(x,t) + c_n P_n^3(x,t) + d_n P_n^4(x,t). \end{aligned} \tag{4.18}$$

We conclude that

$$\begin{aligned} |\boldsymbol{k}(x,t) - \boldsymbol{k}_m(x,t)| &= \left| \mathcal{P}^+ \left[H(x+t) - P(x+t) \right] + \mathcal{P}^- \left[H(x-t) - P(x-t) \right] \right| \\ &\le |H(x+t) - P(x+t)| + |H(x-t) - P(x-t)| \\ &< \epsilon. \end{aligned}$$

□

Remark 4.2.5. Propositions 4.2.3 and 4.2.4 may be summarized by saying that the wave matrices provide a complete system of solutions for the matrix equation $B\boldsymbol{k}_x + \boldsymbol{k}_t B = 0$, in such a way that any solution of the equation (3.45) can be approximated by a linear combination (4.15), furthermore if the solution's values at $x = t$ and $t = -x$ admit power series representation the linear combination (4.15) converge when m tends to ∞.

Remark 4.2.6. Under the assumption that $\boldsymbol{k}(x,t) \in W^{1,1}([-b,b]\times[-b,b],\mathcal{M}_2)$ the previous proof does not require any essential modification in the sense that the above result is also valid in the norm of $W^{1,1}$. This follows from the fact that any function belonging to the space $W^{1,1}$ can be approximated by polynomials, see [18, Ch.7],

4.2.2 Equation $BK_x + K_tB = -Q(x)K$.

Having disposed of this preliminary step, we are in position to introduce a complete system of solutions for the integral kernel equation (3.27) via the mapping property (4.8) indicated by Theorem 4.1.2. According to the expression (4.12) in Remark 4.2.2 we consider the wave matrices from Definition 4.2.1. As we mention before the transmutation operator T in (4.1) acts on vector-valued functions of one real variable, however in a natural way T induces a transmutation operator acting on the space of $\mathcal{M}_2$-valued functions, in particular on the wave matrices.

Definition 4.2.7. The generalized wave matrices $\mathcal{O}_m^i(x,t) = TP_m^i$ are the images of the wave matrices P_m^i of Definition 4.2.1 under the transmutation operator T of (4.1) with respect to the variable x for each fixed t. Specifically,

$$\begin{aligned} \mathcal{O}_m^1(x,t) &= \begin{bmatrix}\mathcal{U}_{2m-1}(x,t) & -\mathcal{V}_{2m}(x,t)\end{bmatrix}, & \mathcal{O}_m^2(x,t) &= \begin{bmatrix}\mathcal{V}_{2m}(x,t) & \mathcal{U}_{2m-1}(x,t)\end{bmatrix}, \\ \mathcal{O}_m^3(x,t) &= \begin{bmatrix}\mathcal{V}_{2m-1}(x,t) & \mathcal{U}_{2m}(x,t)\end{bmatrix}, & \mathcal{O}_m^4(x,t) &= \begin{bmatrix}-\mathcal{U}_{2m}(x,t) & \mathcal{V}_{2m-1}(x,t)\end{bmatrix}, \end{aligned} \tag{4.19}$$

where $m \geq 0$ and the vector-valued functions $\mathcal{U}_{2m-1}$, $\mathcal{U}_{2m}$, $\mathcal{V}_{2m-1}$ and $\mathcal{V}_{2m}$ are given by

$$\mathcal{U}_{2m-1}(x,t) = \sum_{\text{even } k=0}^{m} \binom{m}{k} \Phi_{m-k}(x)t^k \qquad \mathcal{U}_{2m}(x,t) = \sum_{\text{odd } k=1}^{m} \binom{m}{k} \Phi_{m-k}(x)t^k, \tag{4.20}$$

$$\mathcal{V}_{2m-1}(x,t) = \sum_{\text{even } k=0}^{m} \binom{m}{k} \Psi_{m-k}(x)t^k, \qquad \mathcal{V}_{2m}(x,t) = \sum_{\text{odd } k=1}^{m} \binom{m}{k} \Psi_{m-k}(x)t^k. \tag{4.21}$$

Remark 4.2.8. In the following theorem, the complete term is similar to that established in Proposition 4.2.4, see Remark 4.2.5.

Theorem 4.2.9. *The system of $\mathcal{M}_2$-valued functions $\{\mathcal{O}_m^i,\ i=1,2,3,4\}_{m=0}^{\infty}$ is a complete system of solutions of the equation*

$$BK_x(x,t) + K_t(x,t)B = -Q(x)K(x,t) \tag{4.22}$$

in Ω^+.

Proof. Let K be a solution of (4.22). Define $\boldsymbol{k}(\cdot_x,t) = T^{-1}K(\cdot_x,t)$. Then as a consequence of the transmutation property (3.25), $\boldsymbol{k}$ satisfies (3.45). Choose $\epsilon > 0$, by Proposition 4.2.4 there exists a linear combination of wave matrices in the form (4.15) such that

$$|\boldsymbol{k}(x,t) - \boldsymbol{k}_m(x,t)| < \frac{\epsilon}{\|T\|},$$

for every $(x,t) \in [-b,b] \times [-b,b]$. We next set $K_m(\cdot_x, t) = T\boldsymbol{k}_m(\cdot_x, t)$. We thus get

$$K_m(x,t) = T\boldsymbol{k}_m(x,t) = \sum_{n=0}^{m} \Big[a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big], \quad (4.23)$$

since $\mathcal{O}_m^i = TP_m^i$, for each $i = 1, \dots, 4$. Then due to the boundedness of the operators T and T^{-1}, we conclude that

$$|K(x,t) - K_m(x,t)| \leq \|T\| \, |\boldsymbol{k}(x,t) - \boldsymbol{k}_m(x,t)| < \epsilon,$$

which completes the proof. □

Remark 4.2.10. As we can notice the previous proof is based on the properties of operators T, T^{-1}, and the completeness of the wave matrices. Therefore the completeness in Theorem 4.2.9 is established in the sense that one can approximate solutions of (4.22) uniformly by linear combinations (4.23) in the space of classical solutions of equation (4.22).

Remark 4.2.11. We recall that the systems of vector-valued functions $\{\Phi_k\}_{k=0}^{\infty}$ and $\{\Psi_k\}_{k=0}^{\infty}$ belong to $W^{1,1}([-b,b], \mathcal{M}_2)$, see Proposition 2.5.1 and Definition 2.3.1. So that, by construction the systems of generalized wave matrices $\{\mathcal{O}_m^i,\ i = 1,2,3,4\}_{m=0}^{\infty}$ belong to $W^{1,1}(\Omega^+, \mathcal{M}_2)$, see (4.20)-(4.21). Combining these with the fact that $\boldsymbol{k}(\cdot_x, t) = T^{-1}K(\cdot_x, t) \in L^1$, as long as $Q(x)$ belongs to L^2, and taking into account Remark 4.2.6 the result of the completeness is also valid on the space $W^{1,1}(\Omega^+, \mathcal{M}_2)$.

4.3 Approximate construction of integral kernels

Roughly speaking, Theorem 4.2.9 says that any solution of the equation (4.22) satisfied, in particular, by the integral kernel K can be approximated by linear combinations of generalized wave matrices. What is really desired is an algorithm to determine approximate coefficients based on the Goursat data only. This issue needs handling with great care because it represents one of the most important aspects of the proposed approach for the approximation of integral kernels of transmutation operators. The importance is due to the fact that the problem of finding the integral kernel is reduced to solution of a much easier problem, namely, an approximation problem on a segment.

In allusion to the above, the aim is to perform an approximation of the integral kernel in the form

$$K_N(x,t) = \sum_{n=0}^{N} \Big(a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big), \quad (4.24)$$

where the coefficients $\{a_n, b_n, c_n, d_n\}_{n=0}^{N} \subseteq \mathbb{C}$ are to be obtained from the Goursat conditions

$$B\mathbf{K}(x,x) - \mathbf{K}(x,x)B = -Q(x), \quad (4.25)$$

$$B\mathbf{K}(x,-x) + \mathbf{K}(x,-x)B = 0. \quad (4.26)$$

We shall show that having sufficiently good approximation of the data on the characteristics curves $t = x$ and $t = -x$, a good approximation of the integral kernel K in form (4.24) is guaranteed on the whole Ω^+.

In order to examine the values of $K_N(\cdot_x, \cdot_t)$ on the characteristic curves $t = x$ and $t = -x$, we start with the following example that shows how the Goursat-to-Goursat operator T_G works on the solutions base of equation $B\boldsymbol{k}_x + \boldsymbol{k}_t B = 0$.

Example 4.3.1. Consider the problem (3.42)-(3.44) with the Goursat data $\Phi(x) = 2P_n^1(x,x)$ and $\Psi(x) = 2P_n^1(x,-x)$. Then, by Proposition 3.4.3 we have that the solution of this problem is the wave matrix $P_n^3(x,t)$. From this, if we now apply the result of Proposition 3.4.4, we obtain that

$$\begin{aligned}\mathcal{E}_1(x) &= 2B\mathcal{P}^+\left[\mathcal{O}_n^3\right](x,x)\\ &= B\mathcal{O}_n^3(x,x) - \mathcal{O}_n^3(x,x)B\\ &= B\left[\mathcal{V}_{2n-1}(x,x)\quad \mathcal{U}_{2n}(x,x)\right] - \left[\mathcal{V}_{2n-1}(x,x)\quad \mathcal{U}_{2n}(x,x)\right]B\\ &= \left[\mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x)\quad -\mathcal{V}_{2n-1}(x,x) + B\mathcal{U}_{2n}(x,x)\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_2(x) &= 2B\mathcal{P}^+\left[\mathcal{O}_n^3\right](x,-x)\\ &= B\mathcal{O}_n^3(x,-x) + \mathcal{O}_n^3(x,-x)B\\ &= B\left[\mathcal{V}_{2n-1}(x,-x)\quad \mathcal{U}_{2n}(x,-x)\right] + \left[\mathcal{V}_{2n-1}(x,-x)\quad \mathcal{U}_{2n}(x,-x)\right]B\\ &= \left[\mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x)\quad \mathcal{V}_{2n-1}(x,x) - B\mathcal{U}_{2n}(x,x)\right],\end{aligned}$$

where we have used the relations

$$\mathcal{U}_{2n}(x,-x) = -\mathcal{U}_{2n}(x,x) \quad\text{and}\quad \mathcal{V}_{2n-1}(x,-x) = \mathcal{V}_{2n-1}(x,x),$$

and therefore T_G is given by

$$2^n\begin{pmatrix} x^n & 0\\ 0 & -x^n\\ x^n & 0\\ 0 & x^n\end{pmatrix} \xmapsto{T_G} \begin{pmatrix}\mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x) & -\mathcal{V}_{2n-1}(x,x) + B\mathcal{U}_{2n}(x,x)\\ \mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x) & \mathcal{V}_{2n-1}(x,x) - B\mathcal{U}_{2n}(x,x)\end{pmatrix}. \tag{4.27}$$

Similarly, using the Goursat data $\Phi(x) = 2P_n^2(x,x)$ and $\Psi(x) = 2P_n^2(x,-x)$ the solution of (3.42)-(3.44) is the wave matrix $P_n^4(x,t)$, hence

$$T_G(\Phi(x),\Psi(x))^T = \left(2B\mathcal{P}^+\left[\mathcal{O}_n^4\right](x,x), 2B\mathcal{P}^-\left[\mathcal{O}_n^4\right](x,-x)\right)^T,$$

which yields

$$2^n\begin{pmatrix} 0 & x^n\\ x^n & 0\\ 0 & x^n\\ -x^n & 0\end{pmatrix} \xmapsto{T_G} \begin{pmatrix}\mathcal{V}_{2n-1}(x,x) - B\mathcal{U}_{2n}(x,x) & \mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x)\\ -\mathcal{V}_{2n-1}(x,x) + B\mathcal{U}_{2n}(x,x) & \mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x)\end{pmatrix}. \tag{4.28}$$

If we continue in this fashion with $\Phi(x) = -2P_n^i(x,x)$ and $\Psi(x) = -2P_n^i(x,-x)$ for each $i \in \{3,4\}$ the maps

$$2^n \begin{pmatrix} 0 & -x^n \\ -x^n & 0 \\ 0 & x^n \\ -x^n & 0 \end{pmatrix} \xmapsto{T_G} \begin{pmatrix} -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) & -\mathcal{U}_{2n-1}(x,x) - B\mathcal{V}_{2n}(x,x) \\ -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) & \mathcal{U}_{2n-1}(x,x) + B\mathcal{V}_{2n}(x,x) \end{pmatrix} \quad (4.29)$$

and

$$2^n \begin{pmatrix} x^n & 0 \\ 0 & -x^n \\ -x^n & 0 \\ 0 & -x^n \end{pmatrix} \xmapsto{T_G} \begin{pmatrix} \mathcal{U}_{2n-1}(x,x) + B\mathcal{V}_{2n}(x,x) & -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) \\ -\mathcal{U}_{2n-1}(x,x) - B\mathcal{V}_{2n}(x,x) & -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) \end{pmatrix} \quad (4.30)$$

are similarly obtained from

$$T_G \begin{pmatrix} \Phi(x) \\ \Psi(x) \end{pmatrix} = \begin{pmatrix} 2B\mathcal{P}^+ \left[\mathcal{O}_n^j\right](x,x) \\ 2B\mathcal{P}^- \left[\mathcal{O}_n^j\right](x,-x) \end{pmatrix},$$

respectively for each $j \in \{1,2\}$.

The Goursat conditions involve all coefficients in the desired approximation. By considering the half-sum and half-difference of the Goursat conditions we may partially separate the approximation problems.

Lemma 4.3.2. *The half-sum and half-difference of the Goursat data corresponding to K_N have the following form:*

$$B\mathcal{P}^+ \left[K_N(x,x)\right] + B\mathcal{P}^- \left[K_N(x,-x)\right] = \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad (4.31)$$

$$B\mathcal{P}^+ \left[K_N(x,x)\right] - B\mathcal{P}^- \left[K_N(x,-x)\right] = \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix}, \quad (4.32)$$

where the matrix valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ are given by

$$\mathcal{N}_n(x) = \begin{pmatrix} -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) & \mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x) \end{pmatrix}, \quad (4.33)$$

$$\mathcal{M}_n(x) = \begin{pmatrix} \mathcal{U}_{2n-1}(x,x) + B\mathcal{V}_{2n}(x,x) & \mathcal{V}_{2n-1}(x,x) - B\mathcal{U}_{2n}(x,x) \end{pmatrix}. \quad (4.34)$$

Moreover, the following relations hold:

$$B\mathcal{N}_n(x) = -\mathcal{M}_n(x) \quad \textit{and} \quad B\mathcal{M}_n(x) = \mathcal{N}_n(x). \quad (4.35)$$

Proof. The proof is by direct verification. According to the above example, we see at once that the difference of $2B\mathcal{P}^+ \left[\mathcal{O}_n^i\right](x,x)$ and $2B\mathcal{P}^- \left[\mathcal{O}_n^i\right](x,-x)$ as well as the sum produce matrices with at least one null column, which is clear from the right hand side of (4.27)-(4.30), so that

$$B\mathcal{P}^{+}\left[K_N\right]+B\mathcal{P}^{-}\left[K_N\right]=\sum_{n=0}^{N}\Big(a_n\left[-\mathcal{V}_{2n}+B\mathcal{U}_{2n-1}\quad 0\right]+b_n\left[0\quad -\mathcal{V}_{2n}+B\mathcal{U}_{2n-1}\right]$$
$$+c_n\left[\mathcal{U}_{2n}+B\mathcal{V}_{2n-1}\quad 0\right]+d_n\left[0\quad \mathcal{U}_{2n}+B\mathcal{V}_{2n-1}\right]\Big)=$$
$$\sum_{n=0}^{N}\left[\left[-\mathcal{V}_{2n}+B\mathcal{U}_{2n-1}\quad \mathcal{U}_{2n}+B\mathcal{V}_{2n-1}\right]\begin{pmatrix}a_n\\ c_n\end{pmatrix}\quad \left[-\mathcal{V}_{2n}+B\mathcal{U}_{2n-1}\quad \mathcal{U}_{2n}+B\mathcal{V}_{2n-1}\right]\begin{pmatrix}b_n\\ d_n\end{pmatrix}\right]$$
$$=\sum_{n=0}^{N}\left[-\mathcal{V}_{2n}+B\mathcal{U}_{2n-1}\quad \mathcal{U}_{2n}+B\mathcal{V}_{2n-1}\right]\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix},$$

which gives (4.34). To obtain (4.33)

$$B\mathcal{P}^{+}\left[K_N\right]-B\mathcal{P}^{-}\left[K_N\right]=\sum_{n=0}^{N}\Big(a_n\left[0\quad -\mathcal{U}_{2n-1}-B\mathcal{V}_{2n}\right]+b_n\left[\mathcal{U}_{2n-1}+B\mathcal{V}_{2n}\quad 0\right]$$
$$+c_n\left[0\quad -\mathcal{V}_{2n-1}+B\mathcal{U}_{2n}\right]+d_n\left[\mathcal{V}_{2n-1}-B\mathcal{U}_{2n}\quad 0\right]\Big)=$$
$$\sum_{n=0}^{N}\left[\left[\mathcal{U}_{2n-1}+B\mathcal{V}_{2n}\quad \mathcal{V}_{2n-1}-B\mathcal{U}_{2n}\right]\begin{pmatrix}b_n\\ d_n\end{pmatrix}\quad \left[\mathcal{U}_{2n-1}+B\mathcal{V}_{2n}\quad \mathcal{V}_{2n-1}-B\mathcal{U}_{2n}\right]\begin{pmatrix}-a_n\\ -c_n\end{pmatrix}\right]$$
$$=\sum_{n=0}^{N}\left[\mathcal{U}_{2n-1}+B\mathcal{V}_{2n}\quad \mathcal{V}_{2n-1}-B\mathcal{U}_{2n}\right]\begin{pmatrix}b_n & -a_n\\ d_n & -c_n\end{pmatrix},$$

and the lemma follows. □

Proposition 4.3.3. *Let* $\{a_n, c_n\}_{n=0}^{N}$ *and* $\{b_n, d_n\}_{n=0}^{N}$ *be complex numbers such that for every* $x \in [0, b]$,

$$\left\|\tfrac{1}{2}\begin{pmatrix}-p(x)\\ -q(x)\end{pmatrix}-\sum_{n=0}^{N}\mathcal{N}_n(x)\begin{pmatrix}a_n\\ c_n\end{pmatrix}\right\|<\epsilon_1$$

and

$$\left\|\tfrac{1}{2}\begin{pmatrix}-p(x)\\ -q(x)\end{pmatrix}-\sum_{n=0}^{N}\mathcal{M}_n(x)\begin{pmatrix}b_n\\ d_n\end{pmatrix}\right\|<\epsilon_2.$$

Then for every $x \in [0, b]$

$$\left\|\tfrac{1}{2}\begin{pmatrix}-p(x) & -q(x)\\ -q(x) & p(x)\end{pmatrix}-\sum_{n=0}^{N}\mathcal{N}_n(x)\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}\right\|<\epsilon_1+\epsilon_2$$

and

$$\left\|\tfrac{1}{2}\begin{pmatrix}-p(x) & -q(x)\\ -q(x) & p(x)\end{pmatrix}-\sum_{n=0}^{N}\mathcal{M}_n(x)\begin{pmatrix}b_n & -a_n\\ d_n & -c_n\end{pmatrix}\right\|<\epsilon_1+\epsilon_2.$$

Proof. The proof is based on the relations satisfied by the matrix-valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ and Proposition 4.3.8. Using (4.35) it can be easily seen that

$$\frac{1}{2}\begin{pmatrix} -p(x) & -q(x) \\ -q(x) & p(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} =$$
$$\left(\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right] \ \left[\tfrac{1}{2}\begin{pmatrix} -q(x) \\ p(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right] \right) =$$
$$\left(\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right] \ B\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right] \right) \tag{4.36}$$

and

$$\frac{1}{2}\begin{pmatrix} -p(x) & -q(x) \\ -q(x) & p(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix} =$$
$$\left(\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right] \ \left[\tfrac{1}{2}\begin{pmatrix} -q(x) \\ p(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} -a_n \\ -c_n \end{pmatrix} \right] \right) =$$
$$\left(\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right] \ B\left[\tfrac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right] \right). \tag{4.37}$$

We now take the maximum of the norms by columns. From (4.36)-(4.37) the assertion of the lemma is obtained. □

The principal significance of Proposition 4.3.3 is that it allows one to obtain independently the coefficients involved in the desired approximation for the integral kernel in the form (4.24). Moreover, since the half-sum and half-difference of the Goursat data corresponding to (4.25)-(4.26) are equal to $-Q(x)/2 \in \mathcal{H}^-$ the above lemma also states that the sum of the linear combinations indicated in (4.36)-(4.37) approaches the matrix-valued function $-Q(x)$ and the corresponding subtraction approaches zero.

Remark 4.3.4. Both $\mathcal{M}_2$-valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ (as a consequence of Theorem 4.1.2 and Corollary 4.1.3) can be introduced, also, as the result of applying the operator $T+T^*$ to the $\mathcal{M}_n$-valued functions

$$\begin{pmatrix} 0 & (x+t)^n \\ -(x+t)^n & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (x+t)^n & 0 \\ 0 & (x+t)^n \end{pmatrix},$$

and then restrict to the characteristics. Of course, these last matrices are a linear combination of wave matrices indicated by Definition 4.2.1, and as (4.35) they also differ by multiplication to the left by B. The point to emphasize here is that the completeness of $\mathcal{N}_n(x)$ and $\mathcal{M}_n(x)$ on the characteristics does not follow directly from the completeness of the generalized wave matrices $\mathcal{O}_n^i(\cdot_x, \cdot_t)$. The fact that any data on the characteristics can be approximated is made possible by the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ introduced in Section 3.4.

In order to know the pre-image for both matrix-valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ under operators $\mathcal{G}_1$ and $\mathcal{G}_2$ it is convenient to consider the representations below

$$\mathcal{G}_1\Phi(x) = \Phi(x) + \int_{-x}^{x} K^*(x,\tau)\frac{1}{2}\left(\Phi\left(\frac{\tau+x}{2}\right) + \Phi\left(\frac{\tau-x}{2}\right)\right) d\tau$$
$$+ \int_{-x}^{x} K(x,\tau)\frac{1}{2}\left(\Phi\left(\frac{\tau+x}{2}\right) - \Phi\left(\frac{\tau-x}{2}\right)\right) d\tau \tag{4.38}$$

and

$$\mathcal{G}_2\Psi(x) = \Psi(x) + \int_{-x}^{x} K^*(x,\tau)\frac{1}{2}\left(\Psi\left(\frac{\tau+x}{2}\right) - \Psi\left(\frac{\tau-x}{2}\right)\right) d\tau$$
$$+ \int_{-x}^{x} K(x,\tau)\frac{1}{2}\left(\Psi\left(\frac{\tau+x}{2}\right) + \Psi\left(\frac{\tau-x}{2}\right)\right) d\tau \tag{4.39}$$

which follows from (3.60)-(3.61) by a simple change of variables. If we now replace Φ and $\Psi$by the matrix-valued functions

$$\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix},$$

respectively in the expressions (4.38)-(4.39) according to the assignments (4.8) and (4.9) given by the mapping theorem and its corollary we find that

$$2^n\mathcal{G}_1\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix} = \mathcal{N}_n(x) \tag{4.40}$$

and

$$2^n\mathcal{G}_2\begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix} = \mathcal{M}_n(x). \tag{4.41}$$

Remark 4.3.5. It is possible to find the above assignments directly from (4.27)-(4.30), since by construction the operator G maps the half-sum and the half-difference of data Φ and Ψ for the problem (3.42)-(3.44) into the same part for data $\mathcal{E}_1$ and $\mathcal{E}_2$ for the problem (3.30)-(3.32). From (4.28) and (4.29) we get

$$\frac{2^n}{2}\begin{pmatrix} 0 & x^n \\ 0 & 0 \end{pmatrix} \xmapsto{\mathcal{G}_1} \frac{1}{2}\begin{bmatrix} 0 & \mathcal{U}_{2n}(x,x) + B\mathcal{V}_{2n-1}(x,x) \end{bmatrix}$$

and

$$\frac{2^n}{2}\begin{pmatrix} 0 & 0 \\ -x^n & 0 \end{pmatrix} \xmapsto{\mathcal{G}_1} \frac{1}{2}\begin{bmatrix} -\mathcal{V}_{2n}(x,x) + B\mathcal{U}_{2n-1}(x,x) & 0 \end{bmatrix}$$

respectively, thus (4.40) is obtained by linearity of the operator $\mathcal{G}_1$. For (4.41),

$$2^n\mathcal{G}_2\begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix} = 2^n\left[\mathcal{G}_2\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix}\right](-B) = -2^nB\mathcal{G}_1\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix}$$
$$= -B\mathcal{N}_n(x) = \mathcal{M}_n(x),$$

where we have used the relations (3.63) and (4.35).

Lemma 4.3.6. *Each of the systems of matrix-valued functions* $\{\mathcal{N}_n\}_{n=0}^{\infty}$ *and* $\{\mathcal{M}_n\}_{n=0}^{\infty}$ *is linearly independent on* $[-b,b]$.

Proof. Let $a_0,\ldots,a_N$ be a non-trivial sequences of complex numbers such that the linear combination

$$a_0\mathcal{N}_0(x)+\ldots+a_N\mathcal{N}_N(x)=0.$$

Applying $\mathcal{G}_1^{-1}$ to the latter equality we get

$$a_0\begin{pmatrix}0 & x^0\\ -x^0 & 0\end{pmatrix}+\ldots+a_N\begin{pmatrix}0 & x^N\\ -x^N & 0\end{pmatrix}=0,$$

which contradicts the linear independence of the set

$$\left\{\begin{pmatrix}0 & x^0\\ -x^0 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & x^N\\ -x^N & 0\end{pmatrix}\right\}.$$

The proof for $\{\mathcal{M}_n\}_{n=0}^{\infty}$ is similar. □

Let us suppose for the moment that $\boldsymbol{g}$ is a 2×2 continuous matrix-valued function. As in the proof of Proposition 4.2.4, the Weierstrass Approximation Theorem yields a 2×2 polynomial matrix $\boldsymbol{p}$ which approximates $\boldsymbol{g}$ in the uniform norm with any desired precision. Consider the part of $\boldsymbol{p}$ belonging to $\mathcal{H}^+$ and the part of $\boldsymbol{p}$ belonging to $\mathcal{H}^-$, which can be written in standard form as follows:

$$\boldsymbol{p}^+(x)=\sum_{n=0}^{N}\begin{pmatrix}\widetilde{a}_nx^n & \widetilde{b}_nx^n\\ \widetilde{b}_nx^n & -\widetilde{a}_nx^n\end{pmatrix}=\sum_{n=0}^{N}\begin{pmatrix}0 & x^n\\ -x^n & 0\end{pmatrix}\begin{pmatrix}-\widetilde{b}_n & \widetilde{a}_n\\ \widetilde{a}_n & \widetilde{b}_n\end{pmatrix}\tag{4.42}$$

and

$$\boldsymbol{p}^-(x)=\sum_{n=0}^{N}\begin{pmatrix}\widetilde{c}_nx^n & \widetilde{d}_nx^n\\ -\widetilde{d}_nx^n & \widetilde{c}_nx^n\end{pmatrix}=\sum_{n=0}^{N}\begin{pmatrix}0 & x^n\\ -x^n & 0\end{pmatrix}\begin{pmatrix}\widetilde{d}_n & -\widetilde{c}_n\\ \widetilde{c}_n & \widetilde{d}_n\end{pmatrix}.\tag{4.43}$$

The operator $\mathcal{G}_1$ satisfies an associative property type with respect to multiplication on the right by a constant matrix, i.e. $\mathcal{G}_1[\Phi(x)C]=\mathcal{G}_1[\Phi(x)]C$ for all constant matrix C. Using (4.40) we conclude from (4.42)-(4.43) that

$$\mathcal{G}_1[\boldsymbol{p}^+]=\sum_{n=0}^{N}\mathcal{N}_n(x)\begin{pmatrix}-\widetilde{b}_n & \widetilde{a}_n\\ \widetilde{a}_n & \widetilde{b}_n\end{pmatrix}\quad\text{and}\quad\mathcal{G}_1[\boldsymbol{p}^-]=\sum_{n=0}^{N}\mathcal{N}_n(x)\begin{pmatrix}\widetilde{d}_n & -\widetilde{c}_n\\ \widetilde{c}_n & \widetilde{d}_n\end{pmatrix},\tag{4.44}$$

hence that

$$\mathcal{G}_1[\boldsymbol{p}]=\mathcal{G}_1[\boldsymbol{p}^++\boldsymbol{p}^-]=\sum_{n=0}^{N}\mathcal{N}_n(x)\begin{pmatrix}\widetilde{d}_n-\widetilde{b}_n & \widetilde{a}_n-\widetilde{c}_n\\ \widetilde{a}_n+\widetilde{c}_n & \widetilde{b}_n+\widetilde{d}_n\end{pmatrix}.\tag{4.45}$$

Setting

$$a_n=\widetilde{d}_n-\widetilde{b}_n,\qquad b_n=\widetilde{a}_n-\widetilde{c}_n,$$
$$c_n=\widetilde{a}_n+\widetilde{c}_n,\qquad d_n=\widetilde{b}_n+\widetilde{d}_n,$$

it follows that

$$\mathcal{G}_1[\boldsymbol{p}] = \mathcal{G}_1[\boldsymbol{p}^+ + \boldsymbol{p}^-] = \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}. \tag{4.46}$$

Finally, since $\mathcal{G}_1$ is a bounded operator on $C((-b,b),\mathcal{M}_2)$ we can estimate the image of $\boldsymbol{g}-\boldsymbol{p}$ under the operator $\mathcal{G}_1$ by the expression $\|\mathcal{G}_1\|\,\|\boldsymbol{g}-\boldsymbol{p}\|$. In the same manner we can obtain the preceding statement for $\mathcal{M}_n$ instead of $\mathcal{N}_n$ which involves the operator $\mathcal{G}_2$.

Remark 4.3.7. It should be noted that the product on the right of the matrix $\mathcal{N}_n$ by a constant matrix is actually a linear combination of $\mathcal{N}_n$, $\mathcal{N}_nB$, $\mathcal{N}_nBC$, $\mathcal{N}_nC$ with scalar coefficients, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

namely,

$$\begin{aligned} \mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} &= \mathcal{N}_n(x)\left[\mathcal{P}^-\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + \mathcal{P}^+\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right] \\ &= \frac{a_n+d_n}{2}\mathcal{N}_n(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{b_n-c_n}{2}\mathcal{N}_n(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &\quad + \frac{a_n-d_n}{2}\mathcal{N}_n(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{b_n+c_n}{2}\mathcal{N}_n(x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \widetilde{d}_n\mathcal{N}_n(x) - \widetilde{c}_n\mathcal{N}_n(x)B - \widetilde{b}_n\mathcal{N}_n(x)BC + \widetilde{a}_n\mathcal{N}_n(x)C, \\ &= \widetilde{d}_n\mathcal{G}_1\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix} + \widetilde{c}_n\mathcal{G}_1\begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix} \\ &\qquad + \widetilde{b}_n\mathcal{G}_1\begin{pmatrix} 0 & x^n \\ x^n & 0 \end{pmatrix} + \widetilde{a}_n\mathcal{G}_1\begin{pmatrix} x^n & 0 \\ 0 & -x^n \end{pmatrix}, \end{aligned} \tag{4.47}$$

and similarly for the product of the matrix-valued function $\mathcal{M}_n$ by an arbitrary matrix

$$\mathcal{M}_n(x)\begin{pmatrix} e_n & f_n \\ g_n & h_n \end{pmatrix} = -\widetilde{h}_n\mathcal{M}_n(x) + \widetilde{g}_n\mathcal{M}_n(x)B + \widetilde{f}_n\mathcal{M}_n(x)BC - \widetilde{e}_n\mathcal{M}_n(x)C. \tag{4.48}$$

In spirit, the effect of multiplying the matrix-valued functions $\mathcal{N}_n$ and $\mathcal{M}_n$ by the above matrices is to interchange the columns of the matrices $\mathcal{N}_n$ and $\mathcal{M}_n$. Combining this with the properties of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ the matrix-valued functions indicated in the linear combination (4.47) and (4.48) are linearly independent.

Proposition 4.3.8. *Let F be a 2×2 continuous matrix-valued function. Given any $\epsilon > 0$, there exists a sequence of matrices $\{C_n\}_{n=0}^{N}$ such that*

$$\left\| F(\cdot_x) - \sum_{n=0}^{N} \mathcal{N}_n(\cdot_x) C_n \right\|_\infty < \epsilon.$$

The same is true for $\mathcal{M}_n$ in place of $\mathcal{N}_n$.

Proof. Let F be given. Define $\boldsymbol{f} = \mathcal{G}_1^{-1}[F]$ and consider $\boldsymbol{f} = \mathcal{P}^+[\boldsymbol{f}] + \mathcal{P}^-[\boldsymbol{f}]$. Let

$$\mathcal{P}^+[\boldsymbol{f}] = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} \quad \text{and} \quad \mathcal{P}^-[\boldsymbol{f}] = \begin{pmatrix} r(x) & s(x) \\ -s(x) & r(x) \end{pmatrix}.$$

Given $\epsilon > 0$, according to (4.42)-(4.43) by Weierstrass Approximation Theorem we can find matrix polynomials $\boldsymbol{p}^+$ and $\boldsymbol{p}^-$ such that

$$\left\| \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} - \boldsymbol{p}^+(x) \right\| < \frac{\epsilon}{2\left\|\mathcal{G}_1\right\|} \quad \text{and} \quad \left\| \begin{pmatrix} r(x) & s(x) \\ -s(x) & r(x) \end{pmatrix} - \boldsymbol{p}^-(x) \right\| < \frac{\epsilon}{2\left\|\mathcal{G}_1\right\|}.$$

Hence

$$\begin{aligned} \left\| F - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\| &\leq \left\|\mathcal{G}_1\right\| \left\| (\mathcal{P}^+[\boldsymbol{f}] + \mathcal{P}^-[\boldsymbol{f}]) - (\boldsymbol{p}^+ + \boldsymbol{p}^-) \right\| \\ &\leq \left\|\mathcal{G}_1\right\| \Big(\left\|\mathcal{P}^+[\boldsymbol{f}] - \boldsymbol{p}^+\right\| + \left\|\mathcal{P}^-[\boldsymbol{f}] - \boldsymbol{p}^-\right\| \Big) \\ &< \epsilon. \end{aligned}$$

□

Remark 4.3.9. In the following theorem it is convenient recall Definition 4.2.7.

Theorem 4.3.10. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\mathcal{M}_2$-valued continuous functions corresponding to the Goursat problem* (3.30)-(3.32)*. Suppose that the half-sum and the half-difference of the Goursat data admit representations in the form*

$$\frac{1}{2}\left(\mathcal{E}_1(x) + \mathcal{E}_2(x)\right) = \sum_{n=0}^{\infty} \mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \tag{4.49}$$

and,

$$\frac{1}{2}\left(\mathcal{E}_1(x) - \mathcal{E}_2(x)\right) = \sum_{n=0}^{\infty} \mathcal{M}_n(x) \begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix}, \tag{4.50}$$

for every $x \in [-b, b]$ where $\mathcal{M}_n$, $\mathcal{N}_n$ are given by (4.33)-(4.34)*. Then the solution of the equation $B\partial_x K + \partial_t K B = -Q(x)K$ with the data $\mathcal{E}_1$, $\mathcal{E}_2$ is equal to*

$$K(x,t) = \sum_{n=0}^{\infty} \Big(a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big),$$

for every $(x,t) \in [-b, b] \times [-b, b]$.

Proof. We see at once that

$$\mathcal{E}_1(x) = BK(x,x) - K(x,x)B = \sum_{n=0}^{\infty} \left[\mathcal{N}_n(x) \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + \mathcal{M}_n(x) \begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix} \right],$$

and

$$\mathcal{E}_2(x) = BK(x,-x) + K(x,-x)B = \sum_{n=0}^{\infty}\left[\mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} - \mathcal{M}_n(x)\begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix}\right].$$

which is clear from (4.49) and (4.50). Note that $\mathcal{E}_1(x)$ and $\mathcal{E}_2(x)$ belong to spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively. To see this, use the relations (4.35) and recall that $B^2 = -\mathcal{I}_2$. Combining the above and applying T_G^{-1} to $(\mathcal{E}_1, \mathcal{E}_2)^T$ we obtain

$$\varphi(x) = \sum_{n=0}^{\infty}\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix}\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + \begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix}\begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix},$$

and

$$\psi(x) = \sum_{n=0}^{\infty}\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix}\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} - \begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix}\begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix},$$

or which is the same,

$$\varphi(x) = \sum_{n=0}^{\infty} 2^{n+1}\left(d_n - a_n\right)\begin{pmatrix} 0 & x^n \\ x^n & 0 \end{pmatrix} + 2^{n+1}\left(b_n + c_n\right)\begin{pmatrix} x^n & 0 \\ 0 & -x^n \end{pmatrix}, \tag{4.51}$$

and

$$\psi(x) = \sum_{n=0}^{\infty} 2^{n+1}\left(d_n + a_n\right)\begin{pmatrix} 0 & x^n \\ -x^n & 0 \end{pmatrix} - 2^{n+1}\left(b_n - c_n\right)\begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix}. \tag{4.52}$$

Since φ and ψ are admissible Goursat data for the problem (3.42)-(3.44), we are now in a position to apply Proposition 4.2.3, which leads to

$$\boldsymbol{k}(x,t) = \sum_{n=0}^{\infty}\Big(a_n P_n^1(x,t) + b_n P_n^2(x,t) + c_n P_n^3(x,t) + d_n P_n^4(x,t)\Big).$$

From this, after applying the transformation operator T we obtain the desired conclusion. $\square$

It remains to be said that the above lemma conforms to Remark 4.3.7, i.e. in order to approximate the integral kernel from the Goursart conditions (4.25)-(4.26) it is sufficient to use the part of (4.47) belonging to $\mathcal{H}^-$ to approximate $-Q(x)$ and the part corresponding to $\mathcal{H}^+$ to approximate $0 \in \mathcal{H}^+$. More precisely, from (4.47) we find that

$$\begin{aligned}
\mathcal{S}_n(x) &:= \mathcal{P}^+\left[\mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right] \\
&= \widetilde{d}_n\mathcal{P}^+\left[\mathcal{N}_n\right] - \widetilde{c}_n\mathcal{P}^+\left[\mathcal{N}_n B\right] - \widetilde{b}_n\mathcal{P}^+\left[\mathcal{N}_n BC\right] + \widetilde{a}_n\mathcal{P}^+\left[\mathcal{N}_n C\right] \\
&= \widetilde{d}_n\mathcal{P}^+\left[\mathcal{N}_n\right] - \widetilde{c}_n\mathcal{P}^+\left[\mathcal{N}_n\right]B - \widetilde{b}_n\mathcal{P}^-\left[\mathcal{N}_n\right]BC + \widetilde{a}_n\mathcal{P}^-\left[\mathcal{N}_n\right]C \\
&= \mathcal{P}^+\left[\mathcal{N}_n\right]\left(\widetilde{d}_n\mathcal{I} - \widetilde{c}_n B\right) + \mathcal{P}^-\left[\mathcal{N}_n\right]\left(\widetilde{a}_n C - \widetilde{b}_n BC\right) \\
&= \mathcal{P}^+\left[\mathcal{N}_n\right]\begin{pmatrix} \widetilde{d}_n & -\widetilde{c}_n \\ \widetilde{c}_n & \widetilde{d}_n \end{pmatrix} + \mathcal{P}^-\left[\mathcal{N}_n(x)\right]\begin{pmatrix} -\widetilde{b}_n & \widetilde{a}_n \\ \widetilde{a}_n & \widetilde{b}_n \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_n(x) &:= \mathcal{P}^-\left[\mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right] \\
&= \widetilde{d}_n\mathcal{P}^-\left[\mathcal{N}_n\right] - \widetilde{c}_n\mathcal{P}^-\left[\mathcal{N}_nB\right] - \widetilde{b}_n\mathcal{P}^-\left[\mathcal{N}_nBC\right] + \widetilde{a}_n\mathcal{P}^-\left[\mathcal{N}_nC\right] \\
&= \widetilde{d}_n\mathcal{P}^-\left[\mathcal{N}_n\right] - \widetilde{c}_n\mathcal{P}^-\left[\mathcal{N}_n\right]B - \widetilde{b}_n\mathcal{P}^+\left[\mathcal{N}_n\right]BC + \widetilde{a}_n\mathcal{P}^+\left[\mathcal{N}_n\right]C \\
&= \mathcal{P}^-\left[\mathcal{N}_n\right]\left(\widetilde{d}_n\mathcal{I} - \widetilde{c}_nB\right) + \mathcal{P}^+\left[\mathcal{N}_n\right]\left(\widetilde{a}_nC - \widetilde{b}_nBC\right) \\
&= \mathcal{P}^-\left[\mathcal{N}_n\right]\begin{pmatrix} \widetilde{d}_n & -\widetilde{c}_n \\ \widetilde{c}_n & \widetilde{d}_n \end{pmatrix} + \mathcal{P}^+\left[\mathcal{N}_n(x)\right]\begin{pmatrix} -\widetilde{b}_n & \widetilde{a}_n \\ \widetilde{a}_n & \widetilde{b}_n \end{pmatrix}.
\end{aligned}$$

If we now write out the corresponding projections of the matrix-valued function $\mathcal{N}_n$ the use of relations (4.35) leads to

$$\mathcal{S}_n(x) = \mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} + \mathcal{M}_n(x)\begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix} = \mathcal{N}_n(x)C_n - \mathcal{M}_n(x)C_nB, \tag{4.53}$$

$$\mathcal{C}_n(x) = \mathcal{N}_n(x)\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} - \mathcal{M}_n(x)\begin{pmatrix} b_n & -a_n \\ d_n & -c_n \end{pmatrix} = \mathcal{N}_n(x)C_n + \mathcal{M}_n(x)C_nB. \tag{4.54}$$

where C_n is the coefficient matrix.

4.4 Analytic approximation of the integral kernel

Before presenting and demonstrating the main result of this chapter, it is convenient to briefly review the relevant ideas on the analytical approximation of the integral kernel of the transmutation operator that relates the one-dimensional operators

$$\mathcal{A}_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dx} \quad \text{and} \quad \mathcal{A}_Q := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}. \tag{4.55}$$

Firstly Theorem 3.2.10 establishes a transmutation operator T for the operators $\mathcal{A}_0$ and $\mathcal{A}_Q$ in the sense that $T\mathcal{A}_0 = \mathcal{A}_QT$ as a Volterra integral operator and its integral kernel satisfies the Goursart problem below,

$$BK_x(x,t) + K_t(x,t)B = -Q(x)K(x,t) \tag{4.56}$$

$$BK(x,x) - K(x,x)B = -Q(x), \tag{4.57}$$

$$BK(x,-x) + K(x,-x)B = 0, \tag{4.58}$$

and on account of Theorem 3.2.5 the existence and uniqueness of the solution of the previous problem is guaranteed. Let us denote this solution by $\boldsymbol{K}$.

Secondly, since the approximation K_N of the integral kernel $\boldsymbol{K}$ is obtained from approximation of the Goursat data $\boldsymbol{K}(x,\pm x)$ in (4.57)-(4.58) and the difference of two solutions of (4.56) in general satisfy, the conditions

$$BK(x,x) - K(x,x)B = \mathcal{E}_1(x), \tag{4.59}$$

$$BK(x,-x) + K(x,-x)B = \mathcal{E}_2(x), \tag{4.60}$$

it is necessary to examine the Goursat problem (4.56), (4.59)-(4.60), in particular, continuous dependence of the solution on $\mathcal{E}_1$ and $\mathcal{E}_2$. The results obtained in Section 3.3 not only provide the existence of solutions for this problem but also establish the stability, explicitly Propositions 3.3.4 and 3.3.5.

Thirdly combining the use of the transmutation property $T\mathcal{A}_0 = \mathcal{A}_Q T$ with the mapping property indicated by Theorem 4.1.2 makes it possible to introduce a complete system of solutions $\{K_N(\cdot_x, \cdot_t)\}$ for equation (4.56) via images of the wave matrices involved in Proposition 4.2.4. See also Definitions 4.2.1 and 4.2.7.

Finally, the task of being able to approximate any Goursat data for $Q \in C([-b,b], \mathcal{M}_2)$ by the matrix-valued functions that result from restricting K_N to the characteristic curves $t = x$ and $t = -x$ is guaranteed by the operator T_G provided in Proposition 3.4.4. We can now formulate the main result of this chapter.

Theorem 4.4.1. *Let $\{a_n, c_n\}_{n=0}^N$ and $\{b_n, d_n\}_{n=0}^N$ be complex numbers such that*

$$\left\| \frac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right\|_\infty < \epsilon_1 \tag{4.61}$$

and

$$\left\| \frac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right\|_\infty < \epsilon_2 \tag{4.62}$$

for every $x \in [-b,b]$. Then the kernel $\boldsymbol{K}(x,t)$ is approximated by the linear combination

$$K_N(x,t) = \sum_{n=0}^{N} \Big(a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big), \tag{4.63}$$

in such a way that for every $(x,t) \in \Omega^+ \cup \Omega^-$ the following inequality holds

$$\big\| \boldsymbol{K}(x,t) - K_N(x,t) \big\|_\infty < C_{\epsilon_1, \epsilon_2, T, T^{-1}}. \tag{4.64}$$

Proof. Suppose that complex numbers $\{a_n, c_n\}_{n=0}^N$ and $\{b_n, d_n\}_{n=0}^N$ are set according to (4.61) and (4.62) respectively. Setting

$$\mathcal{E}_1(x) = \sum_{n=0}^{N} \mathcal{S}_n(x) \quad \text{and} \quad \mathcal{E}_2(x) = \sum_{n=0}^{N} \mathcal{C}_n(x),$$

where $\mathcal{S}_n$ and $\mathcal{C}_n$ being as in (4.53) and (4.54) respectively, (4.63) is a solution of the same equation (4.56) that the integral kernel $\boldsymbol{K}$ satisfies, this last being a consequence of

Theorem 4.3.10. Define $\widetilde{K}_N(x,t) = \boldsymbol{K}(x,t) - K_N(x,t)$. Since $\widetilde{K}_N$ satisfies (4.56), (4.59)-(4.60) due to Proposition 3.4.4 we get a solution $\widetilde{k}_N(x,t)$ of the problem (3.42)-(3.44) where the Goursat data are given by

$$\begin{pmatrix} \Phi_N(x) \\ \Psi_N(x) \end{pmatrix} = T_G^{-1} \begin{pmatrix} Q(x) - \mathcal{E}_1(x) \\ -\mathcal{E}_2(x) \end{pmatrix},$$

and $K(x,t) = T\widetilde{k}_N(x,t)$. Since T_G^{-1} is a bounded operator and

$$\begin{aligned} Q(x) - \mathcal{E}_1(x) &= \left(\frac{1}{2}Q(x) - \sum_{n=0}^{N} \mathcal{N}_n(x) C_n\right) + \left(\frac{1}{2}Q(x) - \sum_{n=0}^{N} -\mathcal{M}_n(x) C_n B\right), \\ -\mathcal{E}_2(x) &= \left(\frac{1}{2}Q(x) - \sum_{n=0}^{N} \mathcal{N}_n(x) C_n\right) - \left(\frac{1}{2}Q(x) - \sum_{n=0}^{N} -\mathcal{M}_n(x) C_n B\right), \end{aligned}$$

on account of Proposition 4.3.3, it follows that

$$\left\|(\Phi_N, \Psi_N)^T\right\| \leq \left\|T_G^{-1}\right\| \left\|(Q - \mathcal{E}_1, -\mathcal{E}_2)^T\right\| \leq 2\left\|T_G^{-1}\right\| (\epsilon_1 + \epsilon_2).$$

Finally,

$$\begin{aligned} \|\boldsymbol{K}(x,t) - K_N(x,t)\| = \left\|T\widetilde{k}_N(x,t)\right\| \leq \|T\| \left\|\widetilde{k}_N(x,t)\right\| &\leq \frac{1}{2}\|T\| \left(\|\Phi_N(x)\| + \|\Psi_N(x)\|\right) \\ &\leq 2\|T\| \left\|T_G^{-1}\right\| (\epsilon_1 + \epsilon_2). \end{aligned}$$

□

The above theorem is an analogue of [43] in the context of integral kernels for transmutation operators between one-dimensional Dirac operators and corresponds to one of the main objectives of this work. Although the constant C does not represent any inconvenience since we have the estimate for the integral kernel $\boldsymbol{K}$, see Theorem 3.2.5, there exists a disadvantage from the practical point of view since finding coefficients in a uniform norm is not a simple task, even more in the context of vector-valued functions, see [18]. The remainder of this work will be devoted to establishing a practical and simple way of solving the approximation problems (4.61) and (4.62) and finding estimates for resulting approximation error for the integral kernel.

Chapter 5

Practical point of view on the AATO method

This chapter is devoted to show the practical use of the analytic approximation for the integral kernels developed in the previous chapter in order to approximate solutions of one-dimensional Dirac system. Here we emphasize that the approximation method uses the transmutation operators as a practical tool instead of a theoretical tool. In addition, it is easy to implement and the constructed approximations have uniform error bounds.

Using Jackson type approximation theorems Section 5.1 deals with the convergence rate estimates for the AATO method in the uniform norm. According to the analytic approximation theorem which Theorem 4.4.1 establishes in the uniform norm and based on the preceding results, in Section 5.2 it is shown that if we find coefficients in L^2-norm the convergence rate estimates do not get worse. In Section 5.3 we derive an interesting formula for the approximate solution of one-dimensional Dirac system. It is an important section because it corresponds to main motivation of this work. In Section 5.4 we apply the developed theory to one-dimensional Dirac equations with scalar Lorentz potential, as a result we obtain a different possibility to approximate integral kernels than the one offered in [43]. We also provide a brief description of an algorithm for the numerical implementation and based on a few integral kernels that are known [42] we present an example which show the validity of constructed formulas.

5.1 Convergence rate estimates for the analytic approximation of transmutation operators

In what follows we establish relations between the smoothness of the potential Q and the decrease rate of $\epsilon_{1,2}$ from Theorem 4.4.1 as functions of N. As a result, we establish convergence rate estimates for the approximations K_N of the integral kernel $\boldsymbol{K}$. In what follows, $\boldsymbol{K}$ is the integral kernel of the transmutation operator given by (3.2).

Note that we do not need to consider any continuation of the potential Q onto $[-b, 0)$ in order to establish this result.

Theorem 5.1.1. *Let $Q \in C^r([0,b],\mathcal{M}_2)$. Then for every $N > r$ there exist coefficients $\{a_n, b_n, c_n, d_n\}_{n=0}^{N}$ such that*

$$|\boldsymbol{K}(x,t) - K_N(x,t)| < \frac{C_r}{N^r} \tag{5.1}$$

for every (x,t) in the domain $0 \le |t| \le x \le b$, here the constant C_r depends on Q and does not depend on N.

The proof of Theorem 5.1.1 is based on two lemmas. The first one shows an already mentioned statement, namely that K_N is a $\mathcal{M}_2$-valued polynomial function in the variable t. The second demonstrates that the following Cauchy problem is well posed in the domain $0 \le x \le b$, $-b \le t \le b$.

$$\begin{cases} BK_x(x,t) + K_t(x,t)B = -Q(x)K(x,t), \\ K(b,t) = \mathcal{F}(t). \end{cases} \tag{5.2}$$

Note by simple inspection that each generalized wave matrix $\mathcal{O}_n^i$, $i = 1 \dots 4$, contains terms with powers of t whose degree is less than or equal to n, and from a long but simple procedure we obtain

Lemma 5.1.2. *For each fixed $x \in [0,b]$, the linear combination in* (4.63)*, is a $\mathcal{M}_2$-valued polynomial function in the variable t whose degree is less than or equal to N.*

$$K_N(x,t) = \sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \left(\sum_{k=0}^{N-2n} \binom{2n+k}{2n} \begin{bmatrix} a_{2n+k}\Phi_k + c_{2n+k}\Psi_k & b_{2n+k}\Phi_k + d_{2n+k}\Psi_k \end{bmatrix} \right) t^{2n} +$$
$$\sum_{n=0}^{\lceil \frac{N}{2} \rceil - 1} \left(\sum_{k=0}^{N-2n-1} \binom{2n+1+k}{2n+1} \begin{bmatrix} b_{2n+1+k}\Psi_k - d_{2n+1+k}\Phi_k & c_{2n+1+k}\Phi_k - a_{2n+1+k}\Psi_k \end{bmatrix} \right) t^{2n+1}, \tag{5.3}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denotes the floor and the ceiling functions.

Lemma 5.1.3. *The Cauchy problem* (5.2) *with initial data $\mathcal{F}$ that belongs to $C^1[-b,b]$ is well posed in the domain Ω^+, moreover if $|\mathcal{F}(t) - \mathcal{F}_1(t)| < \delta$ then*

$$|K(x,t)| < \delta e^{\int_0^x |Q(\tau)|\, d\tau}. \tag{5.4}$$

Proof. The existence and uniqueness as well as the continuous dependence of the initial data follows from applying in a usual way the method of successive approximations to the following integral equation.

$$\begin{aligned} K(x,t) &= \mathcal{P}^+[\mathcal{F}(t+x)] + \mathcal{P}^-[\mathcal{F}(t-x)] \\ &\quad + \int_0^x BQ(\tau)\Big(\mathcal{P}^+[K(\tau,\tau+t-x)] + \mathcal{P}^-[K(\tau,t+x-\tau)]\Big)d\tau. \end{aligned}$$

□

Now we present the proof of Theorem 5.1.1. It uses a Jackson type approximation theorem, see Theorem 6.2 in ([18], Chap. 7, Sec. 6.)

Definition 5.1.4 ([18])**.** The r-th modulus of smoothness of $f \in L^p([a,b])$, $1 \le p \le \infty$, $r = 1, \dots, h \in \mathbb{R}$, is defined by

$$\omega_r(f,t)_p := \sup_{0 \le h \le t} \|\Delta_h^r(f,\cdot)\|_{L^p[a,b-rh]}, \quad t \ge 0,$$

where,

$$\Delta_h^r(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

In the case $r = 1$ it is called modulus of continuity and $\omega_0(f,t)_p := \|f\|_{L_p}$. If $p = \infty$ and $f \in C[a,b]$,

$$\omega(f,t) = \omega_1(f,t) = \sup_{\substack{|x-y| \le t \\ x,y \in [a,b]}} |f(x) - f(y)|, \quad t \ge 0.$$

We also note that in the latter case $\omega(f,t) \le 2\|f\|_\infty$.

Theorem 5.1.5 ([18])**.** *For a function $f \in W_p^r[a,b]$, $n > r$, $r = 0, 1, \dots$, $1 \le p \le \infty$, the error of the best polynomial approximation satisfies*

$$E_n(f)_p \le \frac{C}{n^r} \omega(f^{(r)}, 1/n)_p. \tag{5.5}$$

Here C is a constant which does not depend on f.

Proof of Theorem 5.1.1. According to the hypothesis, let $N > r$. Proposition 3.2.6 now shows that $\boldsymbol{K}(b,\cdot_t) \in C^r([-b,b],\mathcal{M}_2)$, and on account of Theorem 5.1.5, without loss of generality there exist an $\mathcal{M}_2$-valued polynomial of degree N in the variable t

$$P_N(t) = \sum_{n=0}^{N} \begin{pmatrix} \widetilde{a}_n & \widetilde{b}_n \\ \widetilde{c}_n & \widetilde{d}_n \end{pmatrix} t^n \tag{5.6}$$

such that

$$\|\boldsymbol{K}(b,t) - P_N(t)\|_\infty < \frac{C_r}{N^r}. \tag{5.7}$$

We are now in a position to find an approximation for $\boldsymbol{K}(b,\cdot_t)$ in the form $K_N(b,\cdot_t)$ as follows. Using Lemma 5.1.2 and equating the coefficients in even and odd powers of variable t from the expressions (5.3) and (5.6), we obtain the following equations respectively for $n = \lfloor \frac{N}{2} \rfloor, \dots, 0$ and $n = \lceil \frac{N}{2} \rceil - 1, \dots, 0$.

$$\begin{pmatrix} \widetilde{a}_{2n} & \widetilde{b}_{2n} \\ \widetilde{c}_{2n} & \widetilde{d}_{2n} \end{pmatrix} = \sum_{k=0}^{N-2n} \binom{2n+k}{2n} \left[a_{2n+k}\Phi_k(b) + c_{2n+k}\Psi_k(b) \quad b_{2n+k}\Phi_k(b) + d_{2n+k}\Psi_k(b) \right], \tag{5.8}$$

and

$$\begin{pmatrix} \widetilde{a}_{2n+1} & \widetilde{b}_{2n+1} \\ \widetilde{c}_{2n+1} & \widetilde{d}_{2n+1} \end{pmatrix} = \sum_{k=0}^{N-2n-1} \binom{2n+1+k}{2n+1} \Big(\\ \big[b_{2n+1+k}\Psi_k(b) - d_{2n+1+k}\Phi_k(b) \quad c_{2n+1+k}\Phi_k(b) - a_{2n+1+k}\Psi_k(b) \big] \Big). \tag{5.9}$$

Since the vectors $\Phi_0(b)$, $\Psi_0(b)$ are linearly independent, the following recursive formulas uniquely determine the coefficients $\{a_n, b_n, c_n, d_n\}_{n=0}^N$ as follows.

$$\begin{pmatrix} a_j \\ c_j \end{pmatrix} = \begin{cases} \big[\Phi_0(b) \quad \Psi_0(b)\big]^{-1} \left(\begin{pmatrix} \widetilde{a}_j \\ \widetilde{c}_j \end{pmatrix} - \sum\limits_{k=1}^{N-j} \binom{j+k}{j} \left(a_{j+k}\Phi_k(b) + c_{j+k}\Psi_k(b) \right) \right), & j \text{ even}, \\ \big[-\Psi_0(b) \quad \Phi_0(b)\big]^{-1} \left(\begin{pmatrix} \widetilde{b}_j \\ \widetilde{d}_j \end{pmatrix} - \sum\limits_{k=1}^{N-j} \binom{j+k}{j} \left(c_{j+k}\Phi_k(b) - a_{j+k}\Psi_k(b) \right) \right), & j \text{ odd}. \end{cases} \tag{5.10}$$

$$\begin{pmatrix} b_j \\ d_j \end{pmatrix} = \begin{cases} \big[\Phi_0(b) \quad \Psi_0(b)\big]^{-1} \left(\begin{pmatrix} \widetilde{b}_j \\ \widetilde{d}_j \end{pmatrix} - \sum\limits_{k=1}^{N-j} \binom{j+k}{j} \left(b_{j+k}\Phi_k(b) + d_{j+k}\Psi_k(b) \right) \right), & j \text{ even}, \\ \big[\Psi_0(b) \quad -\Phi_0(b)\big]^{-1} \left(\begin{pmatrix} \widetilde{a}_j \\ \widetilde{c}_j \end{pmatrix} - \sum\limits_{k=1}^{N-j} \binom{j+k}{j} \left(b_{j+k}\Psi_k(b) - d_{j+k}\Phi_k(b) \right) \right), & j \text{ odd}. \end{cases} \tag{5.11}$$

Consequently the inequality (5.7) is valid for $K_N(b,t)$. Now consider the Cauchy problem with initial data $\mathcal{F}(t) := \boldsymbol{K}(b,t) - K_N(b,t)$. By Lemma 5.1.3 there exists a unique continuous extension $K(x,t)$ in the domain Ω^+ that satisfies the Cauchy problem. $\square$

5.2 To find the approximations of the integral kernel

Since the task of finding the coefficients $\{a_n, c_n\}_{n=0}^N$ and $\{b_n, d_n\}_{n=0}^N$ indicated in Theorem 4.4.1 in the uniform norm requires some effort, see for example [18], these coefficients can be easily found using the least squares method, although this latter does not provide the best approximation in the uniform norm.

Based on the results of Section 3.3 and Proposition 4.3.3, we present a version of Theorem 4.4.1, whose proof does not involve the inverse operator T^{-1} and does not require any continuation of Q onto $[-b, 0)$.

Proposition 5.2.1. *Let $\{a_n, c_n\}_{n=0}^N$ and $\{b_n, d_n\}_{n=0}^N$ be complex numbers such that*

$$\left\| \frac{1}{2} \begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{N}_n(x) \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right\|_{L^2[0,b]} < \epsilon_1, \tag{5.12}$$

and

$$\left\| \frac{1}{2}\begin{pmatrix} -p(x) \\ -q(x) \end{pmatrix} - \sum_{n=0}^{N} \mathcal{M}_n(x) \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right\|_{L^2[0,b]} < \epsilon_2. \tag{5.13}$$

Then the kernel $\boldsymbol{K}(x,t)$ *is approximated by the linear combination*

$$K_N(x,t) = \sum_{n=0}^{N} \Big(a_n \mathcal{O}_n^1(x,t) + b_n \mathcal{O}_n^2(x,t) + c_n \mathcal{O}_n^3(x,t) + d_n \mathcal{O}_n^4(x,t) \Big), \tag{5.14}$$

in such a way that for every $x \in [-b,b]$ *the following inequality holds*

$$\|\boldsymbol{K}(x,\cdot_t) - K_N(x,\cdot_t)\|_{L^2[-x,x]} < C_{\epsilon_1,\epsilon_2,Q}. \tag{5.15}$$

Proof. Let $\boldsymbol{K}(x,t)$ be the solution of the Goursat problem (4.56)-(4.58). Define

$$\mathcal{E}_1(x) = -Q(x) - \sum_{n=0}^{N} \mathcal{S}_n(x) \quad \text{and} \quad \mathcal{E}_2(x) = -\sum_{n=0}^{N} \mathcal{C}_n(x).$$

From Proposition 4.3.3 we obtain that

$$\|\mathcal{E}_1(\cdot_x)\|_{L^2[0,b]} < \epsilon_1 + \epsilon_2 \quad \text{and} \quad \|\mathcal{E}_2(\cdot_x)\|_{L^2[0,b]} < \epsilon_1 + \epsilon_2$$

Since the diference $\boldsymbol{K}(x,t) - K_N(x,t)$ satisfy the Goursat problem (3.30)-(3.32) it follows from Proposition 3.3.5 that

$$\begin{aligned} \|\boldsymbol{K}(x,\cdot_t) - K_N(x,\cdot_t)\|_{L^2[-x,x]} &= \left(\int_{-x}^{x} |\boldsymbol{K}(x,t) - K_N(x,t)|^2 \, dt \right)^{1/2} \\ &\le \left(2\sqrt{2} \left(\|\mathcal{E}_1(\cdot)\|^2_{L^2[0,x]} + \|\mathcal{E}_2(\cdot)\|^2_{L^2[0,x]} \right) \exp\left(8x \, \|Q\|^2_{L^2[0,x]} \right) \right)^{1/2} \\ &\le 2^{5/4}(\epsilon_1 + \epsilon_2) \exp\left(8b \, \|Q\|^2_{L^2[0,b]} \right) \end{aligned} \tag{5.16}$$

□

Let K_N denote the approximation of the integral kernel $\boldsymbol{K}$ of the form (5.14) with the coefficients $\{a_n, b_n, c_n, d_n\}_{n=0}^N$ being obtained by applying the least squares method to minimize (5.12) and (5.13). Then the following estimate can be proved for fixed x, and $t \in [-x,x]$.

Proposition 5.2.2. *For a matrix-valued function* $Q \in C^r([0,b], \mathcal{M}_2)$, $N > r$, *the error of approximation by* K_N *satisfies*

$$\|\boldsymbol{K}(x,\cdot_t) - K_N(x,\cdot_t)\|_{L^2[-x,x]} \le \frac{C_r}{N^r}\sqrt{b}.$$

Proof. According to Theorem 5.1.1, there exists coefficients $\left\{\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n, \widetilde{d}_n\right\}_{n=0}^N$ and the matrix function $\widetilde{K}_N$ such that

$$|\boldsymbol{K}(x,t) - \widetilde{K}_N(x,t)| < \frac{\widetilde{C}_r}{N^r} \tag{5.17}$$

for every (x,t) in the domain $0 \le |t| \le x \le b$. In particular, (5.17) holds for $t = x$ and $t = -x$, from which one may deduce that for these coefficients $\left\{\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n, \widetilde{d}_n\right\}_{n=0}^N$ the inequalities (5.12) and (5.13) are satisfied with $\epsilon_{1,2} = \frac{2\widetilde{C}_r}{N^r}\sqrt{b}$. Indeed, from (5.17) and the equality

$$B\mathcal{P}^+[\boldsymbol{K} - \widetilde{K}_N](x,x) + B\mathcal{P}^-[\boldsymbol{K} - \widetilde{K}_N](x,-x) = \frac{1}{2}Q(x) - \sum_{n=0}^N \widetilde{\mathcal{N}}_n(x)\widetilde{C}_n,$$

it follows that

$$\begin{aligned}\left\| \frac{1}{2}\begin{pmatrix} -p(\cdot_x) \\ -q(\cdot_x)\end{pmatrix} - \sum_{n=0}^N \widetilde{\mathcal{N}}_n(\cdot_x)\begin{pmatrix}\widetilde{a}_n \\ \widetilde{c}_n\end{pmatrix}\right\|_{L^2[0,b]} &\le \left(\int_0^x \left|\frac{1}{2}Q(x) - \sum_{n=0}^N \widetilde{\mathcal{N}}_n(x)\widetilde{C}_n\right|^2 dt\right)^{1/2} \\ &< \left(\int_0^x \left(\frac{2\widetilde{C}_r}{N^r}\right)^2 dt\right)^{1/2} = \frac{2\widetilde{C}_r}{N^r}\sqrt{b}.\end{aligned}$$

For the coefficients $\{a_n, b_n, c_n, d_n\}_{n=0}^N$ obtained by applying the least squares method the errors $\epsilon_{1,2}$ in (5.12) and (5.13) can not be larger. Now the statement follows by application of (5.16), i.e.

$$\|\boldsymbol{K}(x,\cdot_t) - K_N(x,\cdot_t)\|_{L^2(-x,x)} \le 2^{13/4}\frac{\widetilde{C}_r}{N^r}\sqrt{b}\exp\left(8b\,\|Q\|^2_{L^2[0,b]}\right).$$

□

5.3 Approximate solution of the one-dimensional Dirac system

An important fact of the approximation found in the form (5.14) is that it provides a simple manner to construct approximations with uniform error bounds to the solutions of one-dimensional stationary Dirac system $\mathcal{A}_Q y = -\lambda y$.

According to Lemma 5.1.2, we will denote by $\mathcal{K}_{2n}$ the $\mathcal{M}_2$-valued function which is accompanied by the even powers of t, and by $\mathcal{K}_{2n+1}$ the $\mathcal{M}_2$-valued function which is accompanied by the odd powers of t respectively.

$$\begin{aligned}\mathcal{K}_{2n}(x) &:= \sum_{k=0}^{N-2n}\binom{2n+k}{2n}\begin{bmatrix}a_{2n+k}\Phi_k + c_{2n+k}\Psi_k & b_{2n+k}\Phi_k + d_{2n+k}\Psi_k\end{bmatrix}\\ &= \sum_{k=0}^{N-2n}\binom{2n+k}{2n}\begin{bmatrix}\Phi_k(x) & \Psi_k(x)\end{bmatrix} C_{2n+k}.\end{aligned} \tag{5.18}$$

$$\begin{aligned}\mathcal{K}_{2n+1}(x) &:= \sum_{k=0}^{N-2n-1}\binom{2n+1+k}{2n+1}\begin{bmatrix}b_{2n+1+k}\Psi_k - d_{2n+1+k}\Phi_k & c_{2n+1+k}\Phi_k - a_{2n+1+k}\Psi_k\end{bmatrix}\\ &= \sum_{k=0}^{N-2n-1}\binom{2n+1+k}{2n+1}\begin{bmatrix}\Phi_k(x) & \Psi_k(x)\end{bmatrix} BC_{2n+1+k}B.\end{aligned} \tag{5.19}$$

Therefore, the approximation K_N can be written as

$$K_N(x,t) = \sum_{n=0}^{\lfloor \frac{N}{2}\rfloor}\mathcal{K}_{2n}(x)t^{2n} + \sum_{n=0}^{\lceil \frac{N}{2}\rceil -1}\mathcal{K}_{2n+1}(x)t^{2n+1}. \tag{5.20}$$

Since the general solution of the equation $\mathcal{A}_0\boldsymbol{y} = -\lambda\boldsymbol{y}$ is given by the linear combination below

$$\boldsymbol{y} = \begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix} = c_1\begin{pmatrix}\cos(\lambda x)\\ -\sin(\lambda x)\end{pmatrix} + c_2\begin{pmatrix}\sin(\lambda x)\\ \cos(\lambda x)\end{pmatrix}, \tag{5.21}$$

it follows from the transmutation property $T\mathcal{A}_0 = \mathcal{A}_Q T$ that the general solution of one-dimensional Dirac system $\mathcal{A}_Q\boldsymbol{v} = -\lambda\boldsymbol{v}$ admits the representation

$$\boldsymbol{v} = T\boldsymbol{y} = \begin{pmatrix} v_1\\ v_2\end{pmatrix} = c_1 T\begin{pmatrix}\cos(\lambda x)\\ -\sin(\lambda x)\end{pmatrix} + c_2 T\begin{pmatrix}\sin(\lambda x)\\ \cos(\lambda x)\end{pmatrix} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}.$$

Let us now consider an approximation of the integral kernel $\boldsymbol{K}$ given by Theorem 4.4.1. And let us consider the approximations corresponding to the solutions of satisfying the initial conditions $(1,0)^T$ and $(0,1)^T$ as follows,

$$\widetilde{\boldsymbol{v_1}}_N = \begin{pmatrix}\cos(\lambda x)\\ -\sin(\lambda x)\end{pmatrix} + \int_{-x}^{x}\sum_{n=0}^{N}\mathcal{K}_n(x)t^n\begin{pmatrix}\cos(\lambda t)\\ -\sin(\lambda t)\end{pmatrix}dt, \tag{5.22}$$

and

$$\widetilde{\boldsymbol{v_2}}_N = \begin{pmatrix}\sin(\lambda x)\\ \cos(\lambda x)\end{pmatrix} + \int_{-x}^{x}\sum_{n=0}^{N}\mathcal{K}_n(x)t^n\begin{pmatrix}\sin(\lambda t)\\ \cos(\lambda t)\end{pmatrix}dt, \tag{5.23}$$

respectively. We have two facts to emphasize here, first we observe that both integrals in (5.22) and (5.23) can be calculated in a closed form. Thus, the approximations (5.22)-(5.23) are simple linear combinations which only involves the vector-valued functions

$\Phi_k(x)$ and $\Psi_k(x)$ indicated by the mapping theorem. Using the formulas

$$\int t^k \sin(\lambda t)\,dt = -\sum_{j=0}^{k} j!\binom{k}{j}\frac{t^{k-j}}{\lambda^{j+1}}\cos\left(\lambda t + \frac{j\pi}{2}\right),$$

$$\int t^k \cos(\lambda t)\,dt = \sum_{j=0}^{k} j!\binom{k}{j}\frac{t^{k-j}}{\lambda^{j+1}}\sin\left(\lambda t + \frac{j\pi}{2}\right),$$

see [24], it follows from (5.22) and (5.23) that

$$\begin{aligned}\widetilde{\boldsymbol{v_1}}_N = \begin{pmatrix}\cos(\lambda x)\\ -\sin(\lambda x)\end{pmatrix} &+ \sum_{n=0}^{\lfloor \frac{N}{2}\rfloor}\mathcal{K}_{2n}(x)\sum_{j=0}^{2n} j!\binom{2n}{j}\frac{t^{2n-j}}{\lambda^{j+1}}\begin{pmatrix}\sin\left(\lambda t+\frac{j\pi}{2}\right)\\ \cos\left(\lambda t+\frac{j\pi}{2}\right)\end{pmatrix}\Bigg|_{-x}^{x}\\ &+\sum_{n=0}^{\lceil \frac{N}{2}\rceil -1}\mathcal{K}_{2n+1}(x)\sum_{j=0}^{2n+1} j!\binom{2n+1}{j}\frac{t^{2n+1-j}}{\lambda^{j+1}}\begin{pmatrix}\sin\left(\lambda t+\frac{j\pi}{2}\right)\\ \cos\left(\lambda t+\frac{j\pi}{2}\right)\end{pmatrix}\Bigg|_{-x}^{x},\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\widetilde{\boldsymbol{v_2}}_N = \begin{pmatrix}\sin(\lambda x)\\ \cos(\lambda x)\end{pmatrix} &+ \sum_{n=0}^{\lfloor \frac{N}{2}\rfloor}\mathcal{K}_{2n}(x)\sum_{j=0}^{2n} j!\binom{2n}{j}\frac{t^{2n-j}}{\lambda^{j+1}}\begin{pmatrix}-\cos\left(\lambda t+\frac{j\pi}{2}\right)\\ \sin\left(\lambda t+\frac{j\pi}{2}\right)\end{pmatrix}\Bigg|_{-x}^{x}\\ &+\sum_{n=0}^{\lceil \frac{N}{2}\rceil -1}\mathcal{K}_{2n+1}(x)\sum_{j=0}^{2n+1} j!\binom{2n+1}{j}\frac{t^{j+1}}{\lambda^{j+1}}\begin{pmatrix}-\cos\left(\lambda t+\frac{j\pi}{2}\right)\\ \sin\left(\lambda t+\frac{j\pi}{2}\right)\end{pmatrix}\Bigg|_{-x}^{x},\end{aligned} \tag{5.25}$$

respectively.

Second, the errors of approximations to the solutions of the one-dimensional Dirac system can be bounded independently on the size of the spectral parameter

Proposition 5.3.1. *Let λ be a real parameter. Assume that*

$$\|\boldsymbol{K}(x,\cdot_t) - K_N(x,\cdot_t)\|_{L^2[-x,x]} \le \epsilon(x).$$

Then

$$|\boldsymbol{v_1}(\lambda,x) - \widetilde{\boldsymbol{v_1}}_N(\lambda,x)| \le \epsilon\sqrt{2x}, \tag{5.26}$$

and similarly for $\boldsymbol{v_2} - \widetilde{\boldsymbol{v_2}}_N$.

Proof. Let us first observe that

$$\begin{aligned}\boldsymbol{v_1}(\lambda,x) - \widetilde{\boldsymbol{v_1}}_N(\lambda,x) &= \int_{-x}^{x}\left(\boldsymbol{K} - K_N\right)(x,t)(\cos(\lambda t), -\sin(\lambda t))^T\,dt\\ &= \int_{-x}^{x}\begin{pmatrix}(K_{11},K_{12})\cdot(\cos(\lambda t),-\sin(\lambda t))^T\\ (K_{21},K_{22})\cdot(\cos(\lambda t),-\sin(\lambda t))^T\end{pmatrix}dt,\end{aligned}$$

where $K_{ij},\ i,j = 1,2$ denote the entries for the difference $\boldsymbol{K} - K_N$, and $(\cdot)$ being the scalar product in $\mathbb{R}^2$, hence

$$\begin{aligned}
|\boldsymbol{v_1}(\lambda,x) - \widetilde{\boldsymbol{v_1}}_N(\lambda,x)| &\leq \|(\boldsymbol{K} - K_N)(x,\cdot_t)\|^2_{L^2[-x,x]} \left\|(\cos(\lambda(\cdot_t)), -\sin(\lambda(\cdot_t)))^T\right\|^2_{L^2[-x,x]} \\
&\leq \epsilon\left(\int_{-x}^{x} \cos^2(\lambda t) + \sin^2(\lambda t)\,dt\right)^{1/2} \\
&\leq \epsilon\sqrt{2x}.
\end{aligned}$$

□

5.4 The AATO method applied to Dirac system with Lorentz scalar potential

As in Section 2.3 here we put into practice the theory developed in Chapter 4 applied to the Dirac system with Lorentz scalar potential

Consider the one-dimensional Dirac operators $\mathcal{A}_0$ and $\mathcal{A}_Q$ as in (4.55) in the case $p \equiv 0$. For convenience we will rewrite the differential expression $\mathcal{A}_Q$ as

$$\mathcal{A}_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + q(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv B\frac{d}{dx} + q(x)C.$$

Recall that $B^2 = -\mathcal{I}$, $C^2 = \mathcal{I}$ and $BC = -CB$. So that, the integral kernel $\boldsymbol{K}$ for the transmutation operator T relating the operators $\mathcal{A}_0$ and $\mathcal{A}_q$ satisfy

$$BK_x(x,t) + K_t(x,t)B = -q(x)CK(x,t) \tag{5.27}$$

with the Goursat data

$$BK(x,x) - K(x,x)B = -q(x)C, \tag{5.28}$$
$$BK(x,-x) + K(x,-x)B = 0. \tag{5.29}$$

With the purpose of relating the results obtained in this chapter with respect to [43]-[44], in the following, we will briefly summarize facts established by the authors V.V. Kravchenko and S. M. Torba for transmutation operators for one-dimensional Schrödinger operators, some sentences will be written verbatim and the same notation will be used.

Let f be a solution of the equation $f'' - qf = 0$ such that $f(x) \neq 0$, for any $x \in [-b,b]$ and satisfying the initial conditions $f(0) = 1$ and $h := f'(0)$. Then there exists a unique operator of the form

$$T_f u(x) = u(x) + \int_{-x}^{x} \boldsymbol{K}_f(x,t)u(t)\,dt, \tag{5.30}$$

in such a way that $T_f[1] = f$ and

$$\left(-\frac{d^2}{dx^2} - q_1(x)\right)T_f[u] = T_f\left[-\frac{d^2}{dx^2}(u)\right] \tag{5.31}$$

for any $u \in C^2[-b,b]$. The preceding statement states that the operator (5.30) is actually a transmutation operator in the sense of Definition 1.3.1 for the Schrödinger operators $A_1 := -\frac{d^2}{dx^2} + q_1(x)$ and $B := -\frac{d^2}{dx^2}$. See [42]. The transmutation operator T_f maps a solution v of the equation $v'' + \omega^2 v = 0$, where ω is a complex number, into a solution u of the equation

$$u'' - q_1(x)u + \omega^2 u = 0 \tag{5.32}$$

with the following correspondence of the initial values $u(0) = v(0)$, $u'(0) = v'(0) + hv(0)$. In addition the integral kernel of the transmutation operator in (5.30) satisfies the following Goursat problem

$$\left(\frac{d^2}{dx^2} - q_1(x)\right)K(x,t) = \frac{d^2}{dx^2}K(x,t), \tag{5.33}$$

$$K(x,x) = \frac{h}{2} + \frac{1}{2}\int_0^x q_1(s)\,ds, \quad K(x,-x) = \frac{h}{2}. \tag{5.34}$$

Under the assumption stated for the function f, let us suppose that u is a solution of equation (5.32). Set

$$v = \left(\partial_x - \frac{f'}{f}\right)u = \left(f\partial_x \frac{1}{f}\right)u, \tag{5.35}$$

then the function v is a solution of the equation $v'' - q_2(x)v + \omega^2 v = 0$, where

$$q_2(x) = 2\left(\frac{f'}{f}\right)^2 - q_1(x).$$

We shall write this last expression as $A_2 v = \omega v$, where

$$A_2 := -\frac{d^2}{dx^2} + q_2(x) \quad \text{and} \quad q_2(x) = 2\left(\frac{f'}{f}\right)^2 - q_1(x).$$

The operator A_2 is known as the Darboux transformation of the operator A_1. Following [39] the transmutation operator for the Schrödinger operator with Darboux potential has been constructed. It has the same form of the operator in (5.30) and since $1/f$ is a non-vanishing solution of the equation $A_2 v = 0$, the transmutation operator for the operators A_2 and B will be denoted by

$$T_{1/f}u(x) = u(x) + \int_{-x}^{x} \boldsymbol{K}_{1/f}(x,t)u(t)\,dt, \tag{5.36}$$

the integral kernel $\boldsymbol{K}_{1/f}$ also satisfy a Goursat problem similar to (5.33)-(5.34), where q_1 is replaced by q_2, and h is replaced by $-h$.

To conclude with this brief introduction of the operators T_f and $T_{1/f}$, recall that as a particular case of the formal powers introduced in (2.6)-(2.10) we obtain the families of functions $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ which are involved in the SPPS representation for the Sturm-Liouville equation (5.32) and its associated Darbox equation, see (2.44). Consider the following four function systems

$$u_{2m-1}(x,t) = \sum_{\text{even } k=0}^{m} \binom{m}{k} \varphi_{m-k}(x) t^k \qquad u_{2m}(x,t) = \sum_{\text{odd } k=1}^{m} \binom{m}{k} \varphi_{m-k}(x) t^k, \tag{5.37}$$

$$v_{2m-1}(x,t) = \sum_{\text{even } k=0}^{m} \binom{m}{k} \psi_{m-k}(x) t^k, \qquad v_{2m}(x,t) = \sum_{\text{odd } k=1}^{m} \binom{m}{k} \psi_{m-k}(x) t^k. \tag{5.38}$$

and their respective values on the characteristics $x = t$ and $t = -x$.

$$\begin{aligned}
\boldsymbol{c}_{2m-1}(x) &= u_{2m-1}(x,x), & m &= 1,2,\ldots & &\text{and} & \boldsymbol{c}_0(x) &= f(x),\\
\boldsymbol{s}_{2m}(x) &= u_{2m}(x,x), & m &= 1,2,\ldots & &\text{and} & \boldsymbol{s}_0(x) &\equiv 0.\\
\widetilde{\boldsymbol{c}}_{2m-1}(x) &= v_{2m-1}(x,x), & m &= 1,2,\ldots & &\text{and} & \widetilde{\boldsymbol{c}}_0(x) &= 1/f(x),\\
\widetilde{\boldsymbol{s}}_{2m}(x) &= v_{2m}(x,x), & m &= 1,2,\ldots & &\text{and} & \widetilde{\boldsymbol{s}}_0(x) &\equiv 0.
\end{aligned}$$

As shown in [43] the systems of functions $\{\boldsymbol{c}_n\}_{n=0}^{\infty}$ and $\{\boldsymbol{s}_n\}_{n=0}^{\infty}$ are linearly independent and complete in $C^1[-b,b]$ and $C_0^1[-b,b]$ respectively, and are closely related to the analytic approximation method for the integral kernel $\boldsymbol{K}_f$.

The first result to be indicated as a direct consequence of Theorem 4.1.2 and due to Proposition 2.3.2 is presented below.

Proposition 5.4.1. *Under the assumptions of Theorem 4.1.2 with* $p(x) \equiv 0$*. The integral kernel* $\boldsymbol{K}$ *for the transmutation operator related to operators* $\mathcal{A}_0$ *and* $\mathcal{A}_q$ *has the form*

$$\boldsymbol{K}(x,t) = \begin{pmatrix} \boldsymbol{K}_f(x,t) & 0 \\ 0 & \boldsymbol{K}_{1/f}(x,t) \end{pmatrix} \tag{5.39}$$

Proof. To this end, we set the operator $\mathcal{T}$ acting on $\mathbb{H} := L^2(-b,b) \times L^2(-b,b)$ as follows

$$\mathcal{T}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \int_{-x}^{x} \begin{pmatrix} \boldsymbol{K}_f(x,t) & 0 \\ 0 & \boldsymbol{K}_{1/f}(x,t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} dt,$$

and to shorten notation, we write $\mathcal{T}y(x) = y(x) + \int_{-x}^{x} \mathcal{K}(x,t) y(t)\,dt$, $y = (y_1, y_2)^T$. Therefore, according to Theorem 4.1.2 and due to Proposition 2.3.2 it follows that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \Phi_m(x) + \Psi_n(x) - \begin{pmatrix} \varphi_m(x) \\ \psi_n(x) \end{pmatrix} = (T - \mathcal{T}) \begin{pmatrix} x^m \\ x^n \end{pmatrix} = \int_{-x}^{x} (\boldsymbol{K}(x,t) - \mathcal{K}(x,t)) \begin{pmatrix} x^m \\ x^n \end{pmatrix} dt.$$

Since each row of matrix-valued function $\boldsymbol{K}(x,t) - \mathcal{K}(x,t)$ is orthogonal to all vector-valued functions in the form $(x^m, x^n)^T$ on the space $\mathbb{H}$, we conclude that $\boldsymbol{K}(x,t) \equiv \mathcal{K}(x,t)$. On the other hand, it is easy to check that an integral kernel in form (5.27) satisfy the Goursat data (5.28)-(5.29), i.e.,

$$B\boldsymbol{K}(x,x) - \boldsymbol{K}(x,x)B = -\left(\frac{2h}{2} + \frac{1}{2}\int_0^x q_1(s) - q_2(s)\,ds\right)C = -q(x)C, \tag{5.40}$$

$$B\boldsymbol{K}(x,-x) + \boldsymbol{K}(x,-x)B = \left(\frac{h}{2} - \frac{h}{2}\right)B = 0, \tag{5.41}$$

which completes the proof. □

Remark 5.4.2. In other words Proposition 5.4.1 asserts that combining the integral kernels of the operators T_f and $T_{1/f}$ in the form (5.39) satisfy (5.27)-(5.29). It follows immediately from [43] that it is possible to implement the analytic approximation method for the integral kernels $\boldsymbol{K}_f$ and $\boldsymbol{K}_{1/f}$ in the particular case discussed here. As a result of the above a question arises naturally, what is the appearance of the functions $\mathcal{N}_n$ and $\mathcal{M}_n$ involved in Theorem 4.4.1. While it is true that the construction of Theorem 4.4.1 is based on the ideas in [43], it is worth noting that Theorem 4.4.1 does really not match with Theorem 5.1 from [43]. In fact, Theorem 4.4.1 offers a different possibility to approximate the integral kernel in (5.30), namely, the approximation of the data at $x = t$ and $x = -t$ is obtained by generalized derivatives of the systems $\{\boldsymbol{c}_n\}_{n=0}^{\infty}$ and $\{\boldsymbol{s}_n\}_{n=0}^{\infty}$.

For convenience and not lose to sight of this last fact, the proof of the following lemma is straightforward by induction and using the relations already known.

$$\partial_x \varphi_k = \frac{f'}{f}\varphi_k + k\psi_{k-1}, \qquad k = 0, 1, \dots$$

$$f\partial_x\left(\frac{1}{f}(x^l\varphi_k)\right) = lx^{l-1}\varphi_k + kx^l\psi_{k-1}, \qquad k = 0, 1, \dots \text{ and } l \geq 0.$$

Lemma 5.4.3. *Under the assumption stated for the function f, the relations*

$$\boldsymbol{c}_n(x) + \widetilde{\boldsymbol{s}}_n(x) = \frac{1}{n+1} f\partial\frac{1}{f}\boldsymbol{s}_{n+1}(x) \quad \textit{and} \quad \boldsymbol{s}_n(x) + \widetilde{\boldsymbol{c}}_n(x) = \frac{1}{n+1} f\partial\frac{1}{f}\boldsymbol{c}_{n+1}(x) \tag{5.42}$$

hold for each $n = 0, 1 \dots$

Consequently, on account of the definitions in (4.20)-(4.21) and Proposition 2.3.2, it follows easily that

$$\mathcal{N}_n(x) = \begin{pmatrix} 0 & \widetilde{\boldsymbol{c}}_n(x) + \boldsymbol{s}_n(x) \\ -\widetilde{\boldsymbol{c}}_n(x) - \boldsymbol{s}_n(x) & 0 \end{pmatrix} = \frac{1}{n+1} f\partial\frac{1}{f}\begin{pmatrix} 0 & \boldsymbol{c}_{n+1} \\ -\boldsymbol{s}_{n+1} & 0 \end{pmatrix}, \tag{5.43}$$

$$\mathcal{M}_n(x) = \begin{pmatrix} \boldsymbol{c}_n(x) + \widetilde{\boldsymbol{s}}_n(x) & 0 \\ 0 & \widetilde{\boldsymbol{c}}_n(x) + \boldsymbol{s}_n(x) \end{pmatrix} = \frac{1}{n+1} f\partial\frac{1}{f}\begin{pmatrix} \boldsymbol{s}_{n+1} & 0 \\ 0 & \boldsymbol{c}_{n+1} \end{pmatrix}. \tag{5.44}$$

Corollary 5.4.4. *Under the assumptions of Theorem 4.1.2 with $p(x) \equiv 0$. Then the integral kernel $\boldsymbol{K}(x,t)$ in (5.39) is approximated by the linear combination*

$$K_N(x,t) = \sum_{n=0}^{N} \Big[a_n \mathcal{O}_n^1(x,t) + d_n \mathcal{O}_n^4(x,t) \Big]$$
$$= \mathcal{I}_2 \begin{pmatrix} a_0 u(x,t) \\ -d_0 v(x,t) \end{pmatrix} + \mathcal{I}_2 \sum_{n=1}^{N} \begin{pmatrix} a_n u_{2n-1}(x,t) - d_n u_{2n}(x,t) \\ -a_n v_{2n-1}(x,t) + d_n v_{2n}(x,t) \end{pmatrix}, \tag{5.45}$$

in the domain Ω^+. Here $f(x) = \exp\left(\int_0^x q(s)\,ds\right)$ and the complex numbers $\{a_n\}_{n=0}^N$ and $\{d_n\}_{n=0}^N$ are taken in such a way that

$$\left\| \left(-\frac{q}{2}\right) - \sum_{n=0}^{N} \frac{a_n}{n+1} f \partial \frac{1}{f} \boldsymbol{s}_{n+1} \right\| < \epsilon_1 \quad \textit{and} \quad \left\| \left(-\frac{q}{2}\right) - \sum_{n=0}^{N} \frac{d_n}{n+1} f \partial \frac{1}{f} \boldsymbol{c}_{n+1} \right\| < \epsilon_2. \tag{5.46}$$

Proof. Combining (5.43)-(5.44) with Proposition 4.3.3 we can see that the coefficients c_n and b_n may really be assumed as zero. Thus, to obtain (5.45) from definition of the matrix-valued functions $\mathcal{O}_n^1$ and $\mathcal{O}_n^4$, and the proof is complete. □

5.4.1 General scheme for numerical implementation

Based on the practical point of view of Theorem 4.4.1 provided in this chapter, we present a brief sketch for numerical implementation. We provide the following algorithm.

Consider a one-dimensional Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} y = -\lambda y, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \tag{5.47}$$

with some initial condition

$$y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

or a boundary condition

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} u_{13} & u_{14} \\ u_{23} & u_{24} \end{pmatrix} \begin{pmatrix} y_1(b) \\ y_2(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.48}$$

1. Find a non-vanishing solution $y = (f, g)^T$ of the equation $B\frac{dy}{dx} + Q(x)y = 0$, see Proposition 2.4.1. This solution is normalized in such a way that $f(0)g(0) = 1$.

2. Compute the functions $X^{(k)}, Y^{(k)}, Z^{(k)}, \widetilde{X}^{(k)}, \widetilde{Y}^{(k)}$ and $\widetilde{Z}^{(k)}$, $k = 0, \ldots, N$ using (2.6)-(2.10).

3. Compute the vector functions Φ_k and Ψ_k, $k = 0, \ldots, N$ using (2.45) and (2.46), see Definition 2.3.1.

4. According to (4.20) and (4.21) compute the matrix functions $\mathcal{N}_k$ and $\mathcal{M}_k$, $k = 0,\dots,N$, using (4.33) and (4.34).

5. Find coefficients $\{a_n, c_n\}_{n=0}^{N}$ and $\{b_n, d_n\}_{n=0}^{N}$ using (5.12) and (5.13) respectively.

6. Compute the matrix functions $\mathcal{K}_{2n}$ and $\mathcal{K}_{2n+1}$ using (5.18) and (5.19) respectively.

7. Compute the vector functions $\widetilde{\boldsymbol{v_1}}_N$ and $\widetilde{\boldsymbol{v_2}}_N$ using (5.24) and (5.25). The approximation to the solution of the initial value problem is given by $y_N = a\widetilde{\boldsymbol{v_1}}_N + b\widetilde{\boldsymbol{v_2}}_N$.

8. To solve spectral problem the boundary condition (5.48) gives an approximation for the characteristic function of the problem in terms of Y_N. Namely, this approximation has the form
$$\det\left(M_N(\lambda)\right) = 0, \tag{5.49}$$
where
$$M_N(\lambda) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} u_{13} & u_{14} \\ u_{23} & u_{24} \end{pmatrix} \left[\widetilde{\boldsymbol{v_1}}_N(b) \quad \widetilde{\boldsymbol{v_2}}_N(b)\right].$$

9. Find roots of the equation in the last step.

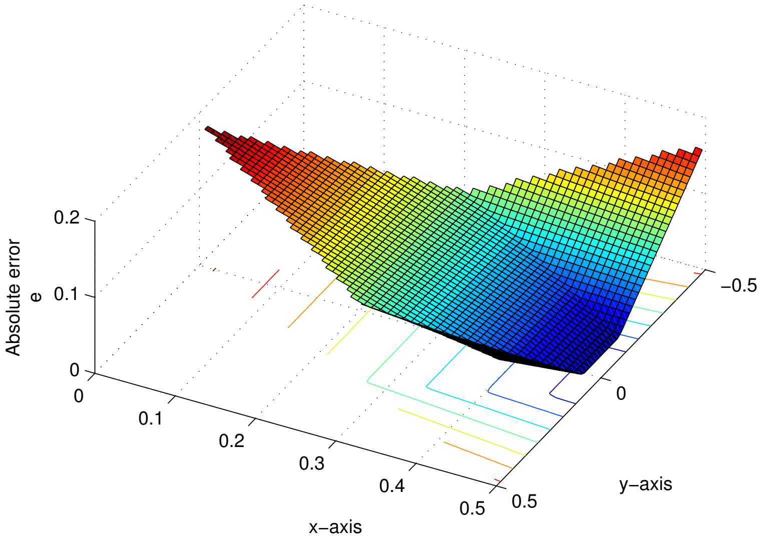
5.4.2 Examples

The integral kernels $\boldsymbol{K}$ for the problems below are known explicitly [42]. We use formulas (5.18), (5.19) and (5.20) to obtain in a closed form the approximations K_N. All calculations performed with the aid of symbolic tool from Matlab 2013.

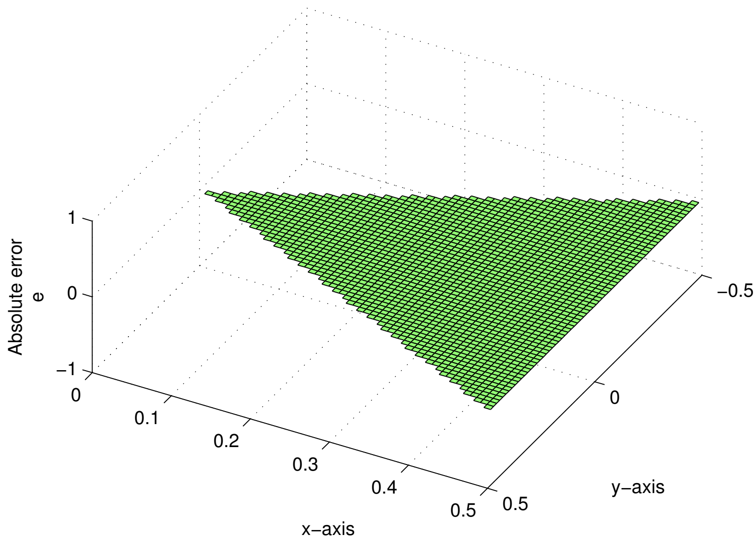
Example 5.4.5. Consider the Dirac system with Lorentz scalar potential $q(x) = \frac{1}{x+1}$, i.e.
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} 0 & \frac{1}{x+1} \\ \frac{1}{x+1} & 0 \end{pmatrix} y = -\lambda y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{5.50}$$

The non-vanishing solution of the equation $B\frac{dy}{dx} + Q(x)y = 0$ are given by $y = (f,g)^T$, where $f(x) = 1 + x$, $g(x) = \frac{1}{1+x}$. By Proposition 5.4.1 the integral kernel $\boldsymbol{K}$ for the transmutation operator related to operators $\mathcal{A}_0$ and $\mathcal{A}_q$ has the form (5.39). From [42] we have that
$$\boldsymbol{K}(x,t) = \begin{pmatrix} K_f(x,t) & 0 \\ 0 & K_{1/f}(x,t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{t-1}{2(x+1)} \end{pmatrix}.$$

The graphs in Figure 5.1 correspond to the error of the approximation of the integral kernel $\boldsymbol{K}$ by K_N. In this case the approximation is exact for $N = 1$. From (5.24) and (5.25) we obtain the two linear by independent solution of (5.50),
$$Y_1(\lambda, x) = \widetilde{\boldsymbol{v_1}}_2(\lambda, x) = \begin{pmatrix} \cos(\lambda x) + \dfrac{\sin(\lambda x)}{\lambda} \\ -\sin(\lambda x) + \dfrac{\lambda x \cos(\lambda x) - \sin(\lambda x)}{\lambda^2 (x+1)} \end{pmatrix} \tag{5.51}$$



(a) $N = 0$, $b = 0.5$.



(b) $N = 1$, $b = 0.5$.

Figure 5.1: The absolute error $e = |K(x,t) - K_N(x,t)|$ on Ω^+ for (5.50).

and

$$Y_2(\lambda, x) = \widetilde{\boldsymbol{v_2}}_2(\lambda, x) = \begin{pmatrix} \sin(\lambda x) \\ \cos(\lambda x) + \dfrac{-\sin(\lambda x)}{\lambda(x+1)} \end{pmatrix}. \tag{5.52}$$

Example 5.4.6. Consider the Dirac system with Lorentz scalar potential $q(x) = \tanh(x)$, i.e.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} 0 & \tanh(x) \\ \tanh(x) & 0 \end{pmatrix} y = -\lambda y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{5.53}$$

The non-vanishing solution of the equation $B\frac{dy}{dx} + Q(x)y = 0$ is given by $y = (f, g)^T$, where $f(x) = \cosh(x)$, $g(x) = 1/\cosh(x)$. Combining (5.39) with [42], the integral kernel $\boldsymbol{K}$ for this example is known in the following form

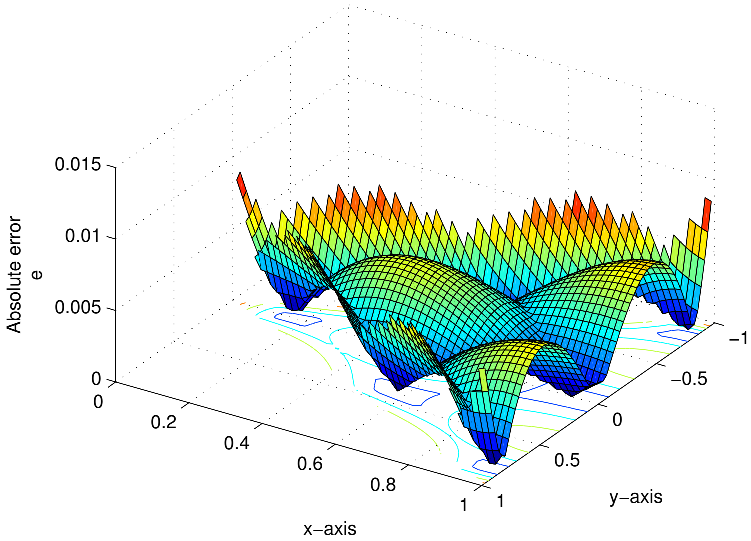
$$\boldsymbol{K}(x,t) = \begin{pmatrix} K_f(x,t) & 0 \\ 0 & K_{1/f}(x,t) \end{pmatrix},$$

where

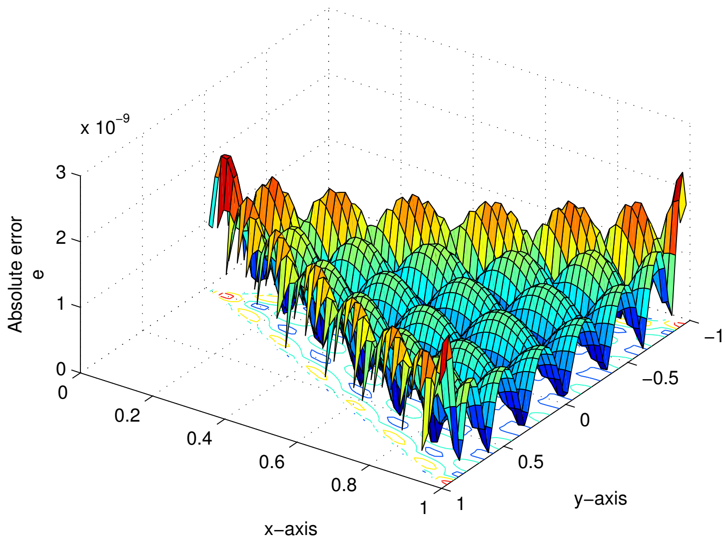
$$K_f(x,t) = -\frac{1}{2}\frac{\sqrt{x^2-t^2}\, I_1\left(\sqrt{x^2-t^2}\right)}{x-t},$$

$$K_{1/f}(x,t) = -\frac{1}{2\cosh(x)} \int_{-t}^{x} \left(\frac{I_0\left(\sqrt{x^2-t^2}\right) t}{x-t} + \frac{\sqrt{x^2-t^2}\, I_1\left(\sqrt{x^2-t^2}\right)}{x-t} \right) \cosh(s)\, ds,$$

and I_0, I_1 are the modified Bessel functions of the first kind. Even though it is not a closed form, it can be used for comparison. The graphs in Figure 5.2 show the error of the approximation of the integral kernel $\boldsymbol{K}$ by K_N.



(a) $N = 2$, $b = 1$.



(b) $N = 8$, $b = 1$.

Figure 5.2: The absolute error $e = |K(x,t) - K_N(x,t)|$ on Ω^+ for (5.53).

Conclusions and future work

Two analytical methods SPPS and AATO focused on the solutions of the one-dimensional Dirac system are developed. Each of these methods admits numerical implementation and works to approximate the solution of initial value problems as well as solutions of boundary value problems for the indicated system.

In the SPPS method, the solutions of the one-dimensional Dirac system are represented in the form of a power series in the spectral parameter. This representation is valid for both continuous potentials and discontinuous potentials. In addition, the representation obtained generalize the corresponding result in [41].

The AATO method considers representation of the solutions via a transmutation operator and provides an analytical way to approximate its integral kernel. For the hyperbolic system of equations associated with the integral kernel: We provide estimates and continuous dependence results on the Goursat data, a complete system of solutions is constructed, we show how to obtain the coefficients involved in the approximations. As a result, the AATO method provides analytic approximations to the solutions of the one-dimensional Dirac system.

Finally, we show how to apply in practice the analytic approximation. Approximations are obtained for the solutions of the one-dimensional Dirac system, whose approximation permits estimate independent on the spectral parameter. In addition, the convergence rate estimates are presented for the method.

These results open the possibility of future research on the following questions:

- Since the results in Chapter 3 do not depend on the size of the matrix-valued functions, the result established in Theorem 4.4.1 and the AATO method can be extended to the case when the potential is an $n \times n$ matrix-valued function satisfying certain commutativity relation. Under more general conditions these operators have been considered by Marchenko [51].

- Due to the results of new emerging representations for solutions associated with the one-dimensional Schrödinger equation and perturbed Bessel equations [45], [46], it is appropriate to examine representation of solutions to the one-dimensional Dirac system in terms of Neumann series of Bessel functions.

- Due to the relevance of symmetric potentials in mathematical physics, we consider studying the radial Dirac equations and will try to establish analytical methods for the representation of solutions.

Bibliography

[1] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, Philadelphia: SIAM, 1981.

[2] S. Albeverio, R. O. Hryniv, and Y. Mikytyuk. *Inverse spectral problems for Dirac operators with summable potentials.* Russian J. Math. Phys. 12 (4), (2005). 406-423.

[3] M. H. Annaby, M. M Tharwat *On the computation of the eigenvalues of Dirac systems* Calcolo 49 (2012). 221-240.

[4] V. G. Bagrov, D. M. Gitman. *Exact Solutions of Relativistic Wave Equations.* Springer. (2012). 324pgs.

[5] V. G. Bagrov, D. M. Gitman. *The Dirac Equation and its Solutions.* De Gruyter. (2014). 430pgs.

[6] H. Begehr, R. Gilbert, *Transformations, Transmutations and Kernel Functions*, vol. 12. Longman Scientific & Technical, Harlow (1992)

[7] L. Bers, *Theory of Pseudo-Analytic Functions.* New York University. New York. (1952).

[8] H. Blancarte, H. M. Campos, K. V. Khmelnytskaya, *Spectral parameter power series method for discontinuous coefficients*, Math. Meth. Appl. Sci. 38 (2015), 2000-2011.

[9] H. M. Campos, V. V. Kravchenko, S. M. Torba. *Transmutation, L-bases and complete families of solutions of stationary Schrödinger equation in the plane*, J. Math. Anal. Appl. 389 (2) (2012), 1222-1238.

[10] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* Springer. (2011). 599pgs.

[11] R. Camporesi, A. J. Di Scala, A generalization of a theorem of Mammana, Colloq. Math. 122 (2011), 215–223.

[12] R. W. Carroll, *Transmutation Theory and Applications.Mathematics Studies,* vol. 117. North-Holland, Amsterdam (1985)

[13] R. Castillo-Perez, V. V. Kravchenko, S. M. Torba. *Spectral parameter power series for perturbed Bessel equations*, Appl. Math. Comput. 220 (2013) 676–694.

[14] S. Clark, F. Gesztesy. *Weyl-Titchmarsh M-Function Asymptotics, Local Uniqueness Results, Trace Formulas, and Borg-type Theorems for Dirac Operators*, Trans. Amer. Math. Soc. 354, 3475–3534 (2002).

[15] S. Cox and R. Knobel. *An inverse spectral problem for a nonnormal first order differential operator*, Integral Equations Operator Theory, 25(2): 147-162, 1996.

[16] J. Delsarte, *Sur certaines transformations fonctionnelles relatives aux équations linéaires aux dérivé es partielles du second ordre*, C. R. Acad. Sc. 206 (1938), 178-182.

[17] J. Delsarte, J. L. Lions, *Transmutations d'opérateurs différentiels dans le domaine complexe*, Comment. Math. Helv. 32 (1956), 113-128.

[18] R. Devore, G. Lorentz. *Constructive approximation*, Springer-Verlag, Berlin, 1993.

[19] P. A. M., Dirac. *The quantum theory of the electron.* Proc. Roy. Soc. (London). A111, 610-624 (1928) - Notes 1.1.

[20] P. A. M., Dirac. *The quantum theory of the electron, II.* Proc. Roy. Soc. (London). A118, 351-361 (1928) - Notes 1.1.

[21] P. A. M., Dirac. *Principles of Quantum Mechanics.* Oxford University Press, New York, (1935).

[22] J. Eckhardt, A. Kostenko and G. Teschl *Inverse uniqueness results for one-dimensional weighted Dirac operators.* Spectral Theory and Differential Equations: V.A. Marchenko 90th Anniversary Collection, E. Khruslov, L. Pastur, and D. Shepelsky (eds), 117–133, Advances in the Mathematical Sciences 233, Amer. Math. Soc., Providence, 2014.

[23] M. K. Fage and N. I. Nagnibida The problem of equivalence of ordinary linear differential operators. Novosibirsk: Nauka, 1987 (in Russian).

[24] I.S. Gradshtein, I.M. Rizhik, *Table of Integrals, Series, and Products*, FizMatLit, Moscow, 1963 (in Russian).

[25] W. Greiner *Relativistic Quantum Mechanics.* Springer. (2000). 424pgs.

[26] N. J. Gutiérrez, V. V. Kravchenko, S. M. Torba. *Spectral parameter power series for the Dirac system.* In preparation.

[27] N. J. Gutiérrez, V. V. Kravchenko, S. M. Torba. *Analytic approximation of transmutation operators related to one-dimensinal Dirac operators.* In preparation.

[28] J. R. Hiller. *Solution of the one-dimensional Dirac equation with a linear scalar potential*, American Journal of Physics 70 (2002) p. 522.

[29] R. O. Hryniv. *Analyticity and uniform stability in the inverse spectral problem for Dirac operators*, J. Math. Phys. 52 (2011), 063513 (17pp)

[30] R. O. Hryniv and N. Pronska. *Inverse spectral problems for energy-dependent Sturm-Liouville equations*, Inverse Problems. 28 (2012), no. 8, 085008 (21pp)

[31] R. Jackiw and S.-Y Pi, *Persistence of zero modes in a gauged Dirac model for bilayer graphene*, Phys. Rev. B 78 (2008), 132104 (6pp)

[32] K. V. Khmelnytskaya, V. V. Kravchenko and J. A. Baldenebro-Obeso. *Spectral parameter power series for fourth-order Sturm-Liouville problems*, Appl. Math. Comput. 219 (2012) 3610–3624.

[33] K. V. Khmelnytskaya, V. V. Kravchenko, S. M. Torba, S. Tremblay, *Wave polynomials, transmutations and Cauchy's problem for the Klein-Gordon equation.* J. Math. Anal. Appl. 399 (2013). 191-212.

[34] A. N. Kolmogorov, S. V. Fomin, *Elements of Function Theory and Functional Analysis,* Fizmatlit, Moscow, 2004 (in Russian).

[35] V. V. Kravchenko, *A representation for solutions of the Sturm-Liouville equation*, Complex Var. Elliptic Equ., 53 (2008), 775–789.

[36] V. V. Kravchenko, *Applied pseudoanalytic function theory*, Basel: Birkhäuser, Series: Frontiers in Mathematics, 2009.

[37] V. V. Kravchenko, S. Morelos and S. M. Torba, *Liouville transformation, analytic approximation of transmutation operators and solution of spectral problems*, Appl. Math. Comput. 273 (2016), 321–336.

[38] V. V. Kravchenko and R. M. Porter, *Spectral parameter power series for Sturm-Liouville problems*, Math. Methods Appl. Sci., 33 (2010), 459–468.

[39] V. V. Kravchenko and S. M. Torba, *Transmutations for Darboux transformed operators with applications*, J. Phys. A 45 (2012) # 075201. 21 pp.

[40] V. V. Kravchenko and S. M. Torba, *Modified spectral parameter power series representations for solutions of Sturm-Liouville equations and their applications*, Appl. Math. Comput., 238 (2014), 82–105.

[41] V. V. Kravchenko, S. M. Torba, *Transmutations and spectral parameter power series in eigenvalue problems*, Oper. Theory Adv. Appl. 228 (2013). 209-238.

[42] V. V. Kravchenko, S. M. Torba, *Construction of transmutation operators and hyperbolic pseudoanalytic functions*, Complex Anal. Oper. Theory 9, (2015). 379–429.

[43] V. V. Kravchenko, S. M. Torba, *Analytic approximation of transmutation operators and applications to highly accurate solution of spectral problems*, J. Math. Anal. Appl. 275 (2015). 1-26.

[44] V. V. Kravchenko, S. M. Torba, *Analytic approximation of transmutation operators and related systems of functions*, Bol. Soc. Mat. Mex. 22 (2016) No.2, 389-429.

[45] V. V. Kravchenko, L. J. Navarro, S. M. Torba *Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions.* Applied Mathematics and Computation, v. 314 No. 1 (2017), 173-192.

[46] V. V. Kravchenko, S. M. Torba and R. Castillo, *A Neumann series of Bessel functions representation for solutions of perturbed Bessel equations.* Applicable Analysis, 97 (2018), issue 5, 677-704.

[47] V. V. Kravchenko, S. M. Torba and U. Velasco-García, *Spectral parameter power series for polynomial pencils of Sturm-Liouville operators and Zakharov-Shabat systems.* J. Math. Phys., 56 (2015), 073508 (20pp)

[48] B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987.

[49] B. M. Levitan, I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer, Dordrecht, 1991.

[50] F. J. López-Rodríguez and G. G. Naumis, *Graphene under perpendicular incidence of electromagnetic waves: Gaps and band structure*, Phil. Mag. 21 (2010), 2977-2988.

[51] V. A. Marchenko. *Sturm-Liouville Operators and Applications.* Birkhuser Verlag. (1986). 367pgs.

[52] G. Pólya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc., 24 (1924), 312–324.

[53] B. W. Roos , W. C. Sangren *Spectral theory of Dirac's radial relativistic wave equation.* J. Math. Phys. 3 (1962). 882-890.

[54] A. M. Savchuk and A. A. Shkalikov *The Dirac Operator with Complex-Valued Summable Potential*, Math Notes. 96 (2014). 777-810.

[55] L. I., Schiff. *Quantum Mechanics* McGraw-Hill Book Company, Inc., (1950).

[56] S. M. Sitnik. Transmutations and applications: a survey. arXiv:1012.3741v1 (2010). In: Korobeinik, Yu.F., Kusraev, A.G. (eds.) Advances in Modern Analysis and Mathematical Modeling, pp. 226293. Vladikavkaz Scientific Center of the Russian Academy of Sciences and Republic of North Ossetia- Alania, Vladikavkaz (2008).

[57] P., Strange. *Reiativistic Quantum Mechanics*, cambridge university press.(1998). 594pgs

[58] G., Teschl. *Renormalized oscillation theory for Dirac operators* . Proceedings of the American Mathematical Society. Vol. 126, No. 6 (1998), pp. 1685-1695

[59] G. Teschl. *Deforming The Point Spectra Of One-Dimensional Dirac Operators.* Proceedings of the American Mathematical Society, 126 (1997) 2873-2881.

[60] B. Thaller. *"The Dirac Equation."* Springer-Verlag . (1992). 357pgs.

[61] M. Yamamoto. *Inverse eigenvalue problem for a vibration of a string with viscous drag*, J. Math. Anal. Appl. 152 (1990). 20-34.

[62] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP 34 (1972) 62–69.

[63] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs V. 121, American Mathematical Society, 2005.