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Construcción de Campos Cuánticos p-Ádicos en el Marco del Análisis de Ruido Blanco

Tesis presentada

Por

Msc. **Edilberto Arroyo Ortiz**

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Director de la Tesis:

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Construction of p-Adic Quantum Fields in the Framework of White Noise Analysis

A thesis presented
by

Msc. **Edilberto Arroyo Ortiz**

in partial fulfillment of the requirements
for the degree of

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Advisor

Dr. Wilson Alvaro Zúñiga Galindo

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Summary

The main result of this dissertation is a mathematically rigorous construction of a large class of interacting Euclidean quantum field theories, over a p-adic space time, by using white noise calculus. In contrast to the classical Euclidean quantum field theory, our basic objects are probability measures on the space $\mathcal{H}_{\infty}^{*}(\mathbb{R})$, which is a non-Archimedean analog of the distribution space of Schwartz $\mathcal{S}'(\mathbb{R}^n)$.

We introduce p-adic versions of the Kondratiev and Hida spaces and characterize the Kondratiev-type disrributions via its $S-$transformation, in order to use the tools provided by the Wick calculus on the Kondratiev spaces.

We introduce a non-Archimedan free covariance function and the non-Archimedan free Euclidean Bose field. We also define the Schwinger functions corresponding to a distribution $\mathbf{\Phi} \in (\mathcal{H}_{\infty})^{-1}$ and the truncated Schwinger functions. We construct a class of Euclidean invariant distributions Φ_H^G indexed by function H, which is holomorphic at zero, where Φ^G is a well defined Kondratiev-type distributions.

The quantum fields introduced here fulfill all the Osterwalder-Schrader axioms, except the reflection positivity.

Resumen

El resultado principal de esta tesis es una construcción matemática rigurosa de una clase amplia de teorías de campos cuánticos Euclidianos con interacciones, sobre un espacio tiempo p-ádico, usando el cálculo de ruido blanco. En contraste con la teoría clásica de campos cuánticos Euclidiana, nuestros objetos básicos son medidas de probabilidad sobre el espacio $\mathcal{H}_{\infty}^{*}(\mathbb{R})$, que es un análogo no arquímediano del espacio de distribucines de Schwartz $\mathcal{S}'(\mathbb{R}^n)$.

Introducimos las versiones p-ádicas de los espacios de Kondratiev y de Hida, y caracterizamos las distribuciones del tipo Kondratiev vía su $S-$transformación, para utilizar las herramientas proporcionadas por el cálculo de Wick en los espacios de Kondratiev.

Introducimos una función covarianza libre no arquimediana y un campo bosónico libre no arquimediano. También definimos las funciones de Schwinger correspondientes a una distribución $\mathbf{\Phi} \in (\mathcal{H}_{\infty})^{-1}$ y las funcines truncadas de Schwinger. Construimos una clase de distribuciones invariantes Euclidianas Φ_H^G indizadas por la función H holomorfa en cero, donde Φ^G es una distribución bien definida de tipo Kondratiev.

Los campos cuánticos introducidos aquí satisfacen todos los axiomas de Osterwalder-Schrader, excepto el de positividad de reflexión.

Contents

Overview

In this thesis, we construct interacting Euclidean quantum field theories, over a p-adic spacetime, in arbitrary dimension, which satisfy all the Osterwalder-Schrader axioms [46] except for reflection positivity. More precisely, we present a p-adic analogue of the interacting field theories constructed by Grothaus and Streit in [19]. The basic objects of an Euclidean quantum field theory are probability measures on distributions spaces, in the classical case, on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. In conventional quantum field theory (QFT) there have been some studies devoted to the optimal choice of the space of test functions. In [23], Jaffe discussed this topic (see also [38] and [53]); his conclusion was that, rather than an optimal choice, there exists a set of conditions that must be satisfied by the candidate space, and any class of test functions with these properties should be considered as valid. The main condition is that the space of test functions must be a nuclear countable Hilbert one. This fact constitutes the main mathematical motivation the study of QFT on general nuclear spaces.

A physical motivation for studying QFT in the $p-$adic setting comes from the conjecture of Volovich stating that spacetime has a non-Archimedean nature at the Planck scale, [63], see also [56]. The existence of the Planck scale implies that below it the very notion of measurement as well as the idea of 'infinitesimal length' become meaningless, and this fact translates into the mathematical statement that the Archimedean axiom is no longer valid, which in turn drives to consider models based on p-adic numbers. In the p-adic framework, the relevance of constructing quantum field theories was stressed in [62] and [57]. In the last 35 years p-adic QFT has attracted a lot of attention of physicists and mathematicians, see e.g. [1], [14]-[16], [20], [24]-[25], [31]-[37], [40]-[43], [50]-[51], [56]-[67], and the references therein.

A p-adic number is a sequence of the form

$$x = x_{-k}p^{-k} + x_{-k+1}p^{-k+1} + \ldots + x_0 + x_1 p + \ldots, \text{ with } x_{-k} \neq 0, \tag{1}$$

where p denotes a fixed prime number, and the x_js are p-adic digits, i.e. numbers in the set $\{0, 1, \ldots, p-1\}$. There are natural field operations, sum and multiplication, on series of form (1). The set of all possible p-adic sequences constitutes the field of p-adic numbers $\mathbb{Q}_p$. The field $\mathbb{Q}_p$ can not be ordered. There is also a natural norm in $\mathbb{Q}_p$ defined as $|x|_p = p^k$, for a nonzero p-adic number x of the form (1). The field of p-adic numbers with the distance induced by $|\cdot|_p$ is a complete ultrametric space. The ultrametric property refers to the fact that $|x-y|_p \leq \max\left\{|x-z|_p, |z-y|_p\right\}$ for any x, y, z in $\mathbb{Q}_p$. As a topological space, $\left(\mathbb{Q}_p, |\cdot|_p\right)$ is completely disconnected, i.e. the connected components are points. The field of p-adic numbers has a fractal structure, see e.g. [2], [62]. All these results can be extended easily to $\mathbb{Q}_p^N$, see Appendix A.

In [66], see also [35, Chapter 11], Zúñiga-Galindo introduced a class of non-Archimedean massive Euclidean fields, in arbitrary dimension, which are constructed as solutions of certain covariant p-adic stochastic pseudodifferential equations, by using techniques of white noise calculus. In particular a new non-Archimedean Gel'fand triple was introduced. By using this new triple, here we introduce non-Archimedean versions of the Kondratiev and Hida spaces, see Section 2. The non-Archimedean Kondratiev spaces, denoted as $(\mathcal{H}_\infty)^1$, $(\mathcal{H}_\infty)^{-1}$, play a central role in this work.

Formally an interacting field theory with interaction V has associated a measure of the form

$$d\mu_V = \frac{\exp\left(-\int_{\mathbb{Q}_p^N} V\left(\mathbf{\Phi}\left(x\right)\right) d^N x\right) d\mu}{\int \exp\left(-\int_{\mathbb{Q}_p^N} V\left(\mathbf{\Phi}\left(x\right)\right) d^N x\right) d\mu}, \tag{2}$$

where μ is the Gaussian white noise measure, $\mathbf{\Phi}(x)$ is a random process at the point $x \in \mathbb{Q}_p^N$. In general $\mathbf{\Phi}(x)$ is not an integrable function rather a distribution, thus a natural problem is how to define $V(\mathbf{\Phi}(x))$. For a review about the techniques for regularizing $V(\mathbf{\Phi}(x))$ and the construction of the associated measures, the reader may consult [18], [19], [49], [53] and the references therein.

Following [19], we consider the following generalized white functional:

$$\mathbf{\Phi}_H = \exp^{\diamond}\left(-\int_{\mathbb{Q}_p^N} H^{\diamond}\left(\mathbf{\Phi}\left(x\right)\right) d^N x\right), \tag{3}$$

where H is analytic function at the origin satisfying $H(0) = 0$. The Wick analytic function $H^{\diamond}(\mathbf{\Phi}(x))$ of process $\mathbf{\Phi}(x)$ coincides with the usual Wick ordered function $: H(\mathbf{\Phi}(x)) :$ when H is a polynomial function. It turns out that $H^{\diamond}(\mathbf{\Phi}(x))$ is a distribution from the Kondratiev space $(\mathcal{H}_\infty)^{-1}$, and consequently, its integral belong to $(\mathcal{H}_\infty)^{-1}$, if it exists. In general we cannot take the exponential of $-\int H^{\diamond}(\mathbf{\Phi}(x))\, d^N x$, however, by using the Wick calculus in $(\mathcal{H}_\infty)^{-1}$, see Section 2.7.1, we can take the Wick exponential $\exp^{\diamond}(\cdot)$.

In certain cases, for instance when H is linear or is a polynomial of even degree, see [24], and if we integrate only over a compact subset K of $\mathbb{Q}_p^N$ (the space cutoff), the function $\mathbf{\Phi}_H$ is integrable, and we have a direct correspondence between (2) and (3), i.e.

$$\mathbf{\Phi}_H d\mu = \left\{\frac{\exp\left(-\int_K H^{\diamond}\left(\mathbf{\Phi}\left(x\right)\right) d^N x\right)}{\int \exp\left(-\int_K H^{\diamond}\left(\mathbf{\Phi}\left(x\right)\right) d^N x\right) d\mu}\right\} d\mu.$$

In general the distribution $\mathbf{\Phi}_H$ is not necessarily positive, and for a large class of functions H, there are no measures representing $\mathbf{\Phi}_H$. It turns out that $\mathbf{\Phi}_H$ can be represented by a measure if and only if $-H(it)+\frac{1}{2}t^2$, $t\in\mathbb{R}$, is a Lévy characteristic, see Theorem 13. These measures are called generalized white noise measures.

Generalized white measures were considered in [66], in the p-adic framework, and in the Archimedean case in [3]-[5]. Euclidean random fields over $\mathbb{Q}_p^N$ were constructed by convolving generalized white noise with the fundamental solutions of certain p-adic pseudodifferential equations. These fundamental solutions are invariant under the action of a p-adic version of the Euclidean group, see Section 3.7.

For all convoluted generalized white noise measures such that their Lévy characteristics have an analytic extension at the origin, we can give an explicit formula for the generalized density with respect to the white noise measure, see Theorem 16. In addition, there exists a large class of distributions $\boldsymbol{\Phi}_H$ of type (3) that do not have an associated measure, see Remark 25. We also prove that the Schwinger functions corresponding to convoluted generalized functions satisfy Osterwalder-Schrader axioms (axioms OS1, OS2, OS4, OS5 in the notation used in [19]) except for reflect positivity, see Lemma 7, Theorems 16, 24, just like in the Archimedean case presented in [19].

The p-adic spacetime $\left(\mathbb{Q}_p^N,\mathfrak{q}\left(\xi\right)\right)$ is a $\mathbb{Q}_p$-vector space of dimension N with an elliptic quadratic form $\mathfrak{q}\left(\xi\right)$, i.e. $\mathfrak{q}\left(\xi\right)=0\Leftrightarrow\xi=0$. This spacetime differs from the classical spacetime $\left(\mathbb{R}^N,\xi_1^2+\cdots+\xi_N^2\right)$ in several aspects. The p-adic spacetime is not an 'infinitely divisible continuum', because $\mathbb{Q}_p^N$ is a completely disconnected topological space, the connected components (the points) play the role of 'spacetime quanta'. Since $\mathbb{Q}_p$ is not an ordered field, the notions of past and future do not exist, then any p-adic QFT is an acausal theory. The reader may consult the introduction of [40] for an in-depth discussion of this matter. Consequently, the reflection positivity, if it exists in the p-adic framework, requires a particular formulation, that we do not know at the moment. The study of the p-adic Wightman functions via the reconstruction theorem is an open problem.

Another important difference between the classical case and the p-adic one comes from the fact that in the p-adic setting there are no elliptic quadratic forms in dimension $N\geq 5$. We replace $\mathfrak{q}\left(\xi\right)$ by an elliptic polynomial $\mathfrak{l}\left(\xi\right)$, which is a homogeneous polynomial satisfying $\mathfrak{l}\left(\xi\right)=0\Leftrightarrow\xi=0$. For any dimension N there are elliptic polynomials of degree $d\geq 2$. We use $\left|\mathfrak{l}\left(\xi\right)\right|_p^{\frac{2}{d}}$ as a replacement of $\left|\mathfrak{q}\left(\xi\right)\right|_p$. This approach is particularly useful to define the p-adic Laplace equation that the (free) covariance function $C_p\left(x-y\right)$ satisfies, this equation has the following form:

$$\left(\boldsymbol{L}_\alpha+m^2\right)C_p\left(x-y\right)=\delta\left(x-y\right),\ \ x,y\in\mathbb{Q}_p^N,$$

where $\alpha>0$, $m>0$ and $\boldsymbol{L}_\alpha$, is the pseudodifferential operator

$$\boldsymbol{L}_\alpha\varphi\left(x\right)=\mathcal{F}_{\xi\to x}^{-1}(\left|\mathfrak{l}\left(\xi\right)\right|_p^{\alpha}\mathcal{F}_{x\to\xi}\varphi),$$

here $\mathcal{F}$ denotes the Fourier transform. The QFTs presented here are families depending on several parameters, among them, p, α, m, $\mathfrak{l}(\xi)$.

The p-adic free covariance $C_p(x-y)$ may have singularities at the origin depending on the parameters α, d, N, and has a 'polynomial' decay at infinity, see Section 3.5.3. The p-adic cluster property holds under the condition $\alpha d > N$. Under this hypothesis the covariance function does not have singularities at the origin. Since α is a 'free' parameter, this condition can be satisfied in any dimension. We think that the condition $\alpha d > N$ is completely necessary to have the cluster property due to the fact that our test functions do not decay exponentially at infinity, see Remark 21.

Let us briefly describe the contents of this work. The chapters from one to four correspond to the article [6] written in collaboration with professor Zúñiga-Galindo. In Chapter 1, we recall the construction of the spaces $\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{C})$ and $\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{R})$ were introduced in [66], see also [35], we also give a description of the corresponding dual spaces $\mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{C})$ and $\mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{R})$, see section 1.1.1.

Following the standard literature on white noise analysis, see e.g. [21], [22], [36],[44], we introduce a probability measure μ on $\left(\mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{R}), \mathcal{B}\right)$ see section 1.2. Later we introduce the Wick-ordered polynomials (see section 1.3 or Appendix D.1) and establish the Wiener-Itô-Segal isomorphism, see section 1.4 or Theorem 56.

In Chapter 2, we present the construction of non-Archimedean versions of Kondratiev-type spaces of test functions $(\mathcal{H}_\infty(\mathbb{C}))^1$ and distributions $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, see section 2.1 and 2.2. We also introduce the $S-$transform which is our main analytical tool in working with non-Archimedean Kondratiev spaces, see section 2.3.1. The $S-$transform allow us to characterize the distribution space $(\mathcal{H}_\infty)^{-1}$, see Theorem 60 . We finish this Chapter by giving a brief description of the holomorphic functions on $\mathcal{H}_\infty(\mathbb{C})$ (see section 2.4 or appendix D.2) and defining the the Wick product in the spaces of distributions $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, see section 2.7 or appendix D.3.1.

In Chapter 3 is devoted to the Euclidean quantum field theory in the non-Archimedean framework. We show that the Schwinger functions satisfy axioms (OS1) and (OS4), see Lemma 7 . We also present the non-Archimedean free Euclidean Bose field, see section 3.6. We also introduce the symmetries exhibit for the quantum fields introduced hare, see section 3.7.

In Chapter 4 is dedicated to the study of the truncated Schwinger functions and the cluster property (axioms (OS5)), see Lemma 23, and Theorem 24.

In order to make this dissertation self contained, we summarize some basic notions and results on $p-$adic analysis in Appendix A. The appendix B contains the general concepts of countably-Hilbert spaces and nuclear spaces. We recall the theorem Bochner-Minlos. In Appendix C, we give a Wavelet basis for the spaces $\mathcal{H}_l(\mathbb{C})$, see Theorem 47. This is an unpublished material. Finally, the Appendix D is dedicated to recall the Wiener-Itô-segal isomorphism. We also to recall the some definitions and properties of holomorphic functions in locally convex topological vector spaces. We recall the characterization theorem of the Kondratiev Distributions, see Theorem 60. We finish this appendix with the Wick product in the spaces of distributions in $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, see appendix D.3.1.

1

A class of non-Archimedean nuclear spaces

1.1 $\mathcal{H}_\infty$, a non-Archimedean analog of the Schwartz space

We denote the set on non-negative integers by $\mathbb{N}$, and set $[\xi]_p := [\max(1, \|\xi\|_p)]$ for $\xi \in \mathbb{Q}_p^N$. We define for φ, $\theta \in \mathcal{D}(\mathbb{Q}_p^N)$, and $l \in \mathbb{N}$, the following scalar product:

$$\langle \varphi, \theta \rangle_l = \int_{\mathbb{Q}_p^N} [\xi]_p^l \, \overline{\widehat{\varphi}(\xi)} \widehat{\theta}(\xi) \, d^N\xi,$$

where the overbar denotes the complex conjugate. We also set $\|\varphi\|_l := \langle \varphi, \varphi \rangle_l$. Notice that $\|\cdot\|_l \leq \|\cdot\|_m$ for $l \leq m$. We denote by $\mathcal{H}_l(\mathbb{C}) := \mathcal{H}_l(\mathbb{Q}_p^N, \mathbb{C})$ the complex Hilbert space obtained by completing $\mathcal{D}(\mathbb{Q}_p^N)$ with respect to $\langle \cdot, \cdot \rangle_l$. Then $\mathcal{H}_m(\mathbb{C}) \hookrightarrow \mathcal{H}_l(\mathbb{C})$ for $l \leq m$. Now we set

$$\mathcal{H}_\infty(\mathbb{C}) := \mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{C}) = \bigcap\nolimits_{l \in \mathbb{N}} \mathcal{H}_l(\mathbb{C}).$$

Notice that $\mathcal{H}_\infty(\mathbb{C}) \subset L^2$. With the topology induced by the family of seminorms $\{\|\cdot\|_l\}_{l \in \mathbb{N}}$, $\mathcal{H}_\infty(\mathbb{C})$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}, \text{ with } f,\, g \in \mathcal{H}_\infty(\mathbb{C}),$$

is a metric for the topology of $\mathcal{H}_\infty(\mathbb{C})$. The projective topology τ_P of $\mathcal{H}_\infty(\mathbb{C})$ coincides with the topology induced by the family of seminorms $\{\|\cdot\|_l\}_{l \in \mathbb{N}}$. The space $\mathcal{H}_\infty(\mathbb{C})$ endowed with the topology τ_P is a countably Hilbert space in the sense of Gel'fand-Vilenkin (Appendix B.1.1) . Furthermore, $(\mathcal{H}_\infty(\mathbb{C}), \tau_P)$ is metrizable and complete and hence a Fréchet space, cf. [35, Lemma 10.3], see also [66].

The space $(\mathcal{H}_\infty(\mathbb{C}), d)$ is the completion of $(\mathcal{D}(\mathbb{Q}_p^N), d)$ with respect to d, and since $\mathcal{D}(\mathbb{Q}_p^N)$ is nuclear, then $\mathcal{H}_\infty(\mathbb{C})$ is a nuclear space, which is continuously embedded in $C_0(\mathbb{Q}_p^N, \mathbb{C})$, the space of complex-valued bounded functions vanishing at infinity. In addition, $\mathcal{H}_\infty(\mathbb{C}) \subset L^1 \cap L^2$, cf. [35, Theorem 10.15].

Remark 1 *(i) We denote by $\mathcal{H}_l(\mathbb{R}) := \mathcal{H}_l(\mathbb{Q}_p^N, \mathbb{R})$ the real Hilbert space obtained by completing $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ with respect to $\langle \cdot, \cdot \rangle_l$. We also set $\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{R}) := \mathcal{H}_\infty(\mathbb{R}) = \cap_{l\in\mathbb{N}} \mathcal{H}_l(\mathbb{R})$. In the case in which the ground field ($\mathbb{R}$ or $\mathbb{C}$) is clear, we shall use the simplified notation $\mathcal{H}_l$, $\mathcal{H}_\infty$. All the above announced results for the spaces $\mathcal{H}_l(\mathbb{C})$, $\mathcal{H}_\infty(\mathbb{C})$ are valid for the spaces $\mathcal{H}_l(\mathbb{R})$, $\mathcal{H}_\infty(\mathbb{R})$. In particular, $\mathcal{H}_\infty(\mathbb{R})$ is a nuclear countably Hilbert space (Appendix B.1.1).*

(ii) The following characterization of the space $\mathcal{H}_\infty(\mathbb{C})$ is very useful:

$$\begin{aligned}\mathcal{H}_\infty(\mathbb{C}) &= \left\{ f \in L^2\left(\mathbb{Q}_p^N\right); \|f\|_l < \infty \text{ for any } l \in \mathbb{N} \right\} \\ &= \left\{ W \in \mathcal{D}'\left(\mathbb{Q}_p^N\right); \|W\|_l < \infty \text{ for any } l \in \mathbb{N} \right\},\end{aligned}$$

cf. [35, Lemma 10.8]. An analog result is valid for $\mathcal{H}_\infty(\mathbb{R})$.

(iii) The spaces $\mathcal{H}_l(\mathbb{R})$, $\mathcal{H}_l(\mathbb{C})$, for any $l \in \mathbb{N}$, are nuclear and consequently they are separable, cf. [17, Chapter I, Section 3.4].

The spaces $\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{C})$ and $\mathcal{H}_\infty(\mathbb{Q}_p^N, \mathbb{R})$ were introduced in [66], see also [35]. These spaces are invariant under the action of a large class of pseudodifferential operators.

1.1.1 The dual space of $\mathcal{H}_\infty$

For $m \in \mathbb{N}$, and $W \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ such that $\widehat{W}$ is a measurable function, we set

$$\|W\|_{-m}^2 := \int_{\mathbb{Q}_p^N} [\xi]_p^{-m} \left|\widehat{W}(\xi)\right|^2 d^N\xi.$$

Then

$$\mathcal{H}_{-m}(\mathbb{C}) := \mathcal{H}_{-m}(\mathbb{Q}_p^N, \mathbb{C}) = \left\{ W \in \mathcal{D}'\left(\mathbb{Q}_p^N\right); \|W\|_{-m} < \infty \right\} \tag{1.1}$$

is a complex Hilbert space. If $\mathcal{X}$ is a locally convex, we denote by $\mathcal{X}^*$ the dual space endowed with the strong dual topology or the topology of the bounded convergence. We denote by $\mathcal{H}_m^*(\mathbb{C})$ the dual of $\mathcal{H}_m(\mathbb{C})$ for $m \in \mathbb{N}$, we identify $\mathcal{H}_m^*(\mathbb{C})$ with $\mathcal{H}_{-m}(\mathbb{C})$, by using the bilinear form:

$$\langle W, g \rangle = \int_{\mathbb{Q}_p^N} \overline{\widehat{W}(\xi)} \widehat{g}(\xi)\, d^N\xi \text{ for } W \in \mathcal{H}_{-m}(\mathbb{C}) \text{ and } g \in \mathcal{H}_m(\mathbb{C}). \tag{1.2}$$

Then

$$\mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{C}) := \mathcal{H}_\infty^*(\mathbb{C}) = \bigcup_{m\in\mathbb{N}} \mathcal{H}_{-m}(\mathbb{C})$$
$$= \left\{ W \in \mathcal{D}'\left(\mathbb{Q}_p^N\right); \|W\|_{-m} < \infty \text{ for some } m \in \mathbb{N} \right\}.$$

We consider $\mathcal{H}_\infty^*(\mathbb{C})$ endowed with the strong topology. We use (1.2) as pairing between $\mathcal{H}_\infty^*(\mathbb{C})$ and $\mathcal{H}_\infty(\mathbb{C})$. By a similar construction one obtains the space $\mathcal{H}_\infty^*(\mathbb{R}) := \mathcal{H}_\infty^*(\mathbb{Q}_p^N, \mathbb{R})$. The above announced results are also valid for $\mathcal{H}_\infty^*(\mathbb{R})$. If there is no danger of confusion we use $\mathcal{H}_\infty^*$ instead of $\mathcal{H}_\infty^*(\mathbb{C})$ or $\mathcal{H}_\infty^*(\mathbb{R})$.

Remark 2 *(i) For complex and real spaces, $\|\cdot\|_{\pm l}$ denotes the norm on $\mathcal{H}_l$ and $\mathcal{H}_{-l}$. We denote by $\langle\cdot,\cdot\rangle$ the dual pairings between $\mathcal{H}_{-l}$ and $\mathcal{H}_l$ and between $\mathcal{H}_\infty$ and $\mathcal{H}_\infty^*$. We preserve this notation for the norm and pairing on tensor powers of these spaces.*

(ii) If $\{\mathcal{X}_l\}_{l\in A}$ is a family of locally convex spaces, we denote by $\varprojlim_{l\in\mathbb{N}}\mathcal{X}_l$ the projective limit of the family, and by $\varinjlim_{l\in\mathbb{N}}\mathcal{X}_l$ the inductive limit of the family.

(iii) If $\mathcal{N}$ is a nuclear space, which is the projective limit of the Hilbert spaces H_l, $l \in \mathbb{N}$,the n-th symmetric tensor product of $\mathcal{N}$ is defined as $\mathcal{N}^{\widehat{\otimes}n} = \varprojlim_{l\in\mathbb{N}} H_l^{\widehat{\otimes}n}$. This is a nuclear space. The dual space is $\mathcal{N}^{*\widehat{\otimes}n} = \varinjlim_{l\in\mathbb{N}} H_{-l}^{\widehat{\otimes}n}$.

1.2 Non-Archimedean Gaussian measures

The spaces

$$\mathcal{H}_\infty(\mathbb{R}) \hookrightarrow L^2_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \hookrightarrow \mathcal{H}_\infty^*(\mathbb{R})$$

form a Gel'fand triple, that is, $\mathcal{H}_\infty(\mathbb{R})$ is a nuclear countably Hilbert space which is densely and continuously embedded in $L^2_{\mathbb{R}}$ and $\|g\|_0^2 = \langle g, g\rangle_0$ for $g \in \mathcal{H}_\infty(\mathbb{R})$. This triple was introduced in [66], see also [35, Chapter 10]. The inner product and the norm of $\left(L^2_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)\right)^{\otimes m} \simeq L^2_{\mathbb{R}}\left(\mathbb{Q}_p^{Nm}\right)$ are denoted by $\langle\cdot,\cdot\rangle_0$ and $\|\cdot\|_0$. From now on, we consider $\mathcal{H}_\infty^{\widehat{\otimes}n}(\mathbb{R})$ as subspace of $\mathcal{H}_\infty^{\otimes n}(\mathbb{R})$, then $\langle\cdot,\cdot\rangle_{\mathcal{H}_\infty^{\widehat{\otimes}n}(\mathbb{R})} = n!\,\langle\cdot,\cdot\rangle_0$.

We denote by $\mathcal{B} := \mathcal{B}(\mathcal{H}_\infty^*(\mathbb{R}))$ the σ-algebra generated by the cylinder subsets of $\mathcal{H}_\infty^*(\mathbb{R})$ (Appendix B.2). The mapping

$$\begin{array}{rccc} \mathcal{C}: & \mathcal{H}_\infty(\mathbb{R}) & \to & \mathbb{C} \\ & f & \to & e^{-\frac{1}{2}\|f\|_0^2} \end{array} \tag{1.3}$$

defines a characteristic functional, i.e. $\mathcal{C}$ is continuous, positive definite and $\mathcal{C}(0) = 1$. By the Bochner-Minlos theorem, see e.g. [7], [21], (Appendix B.2), there exists a probability measure μ, called *the canonical Gaussian measure* on $(\mathcal{H}_\infty^*(\mathbb{R}), \mathcal{B})$, given by its characteristic functional as

$$\int_{\mathcal{H}_{\infty}^{*}(\mathbb{R})} e^{i\langle W,f\rangle} d\mu(W) = e^{-\frac{1}{2}\|f\|_0^2},\ \ f \in \mathcal{H}_{\infty}(\mathbb{R}).$$

We set $(L_{\mathbb{C}}^2) := L^2(\mathcal{H}_{\infty}^{*}(\mathbb{R}), \mu; \mathbb{C})$ to denote the complex vector space of measurable functions $\Psi : \mathcal{H}_{\infty}^{*}(\mathbb{R}) \to \mathbb{C}$ satisfying

$$\|\Psi\|_{(L_{\mathbb{C}}^2)}^2 = \int_{\mathcal{H}_{\infty}^{*}(\mathbb{R})} |\Psi(W)|^2 \, d\mu(W) < \infty.$$

The space $(L_{\mathbb{R}}^2) := L^2(\mathcal{H}_{\infty}^{*}(\mathbb{R}), \mu; \mathbb{R})$ is defined in a similar way. The pairing $\mathcal{H}_{\infty}^{*}(\mathbb{R}) \times \mathcal{H}_{\infty}(\mathbb{R})$ can be extended to $\mathcal{H}_{\infty}^{*}(\mathbb{R}) \times L^2(\mathbb{Q}_p^N)$ as an $(L_{\mathbb{C}}^2)$-function on $\mathcal{H}_{\infty}^{*}(\mathbb{R})$, this fact follows from

$$\int_{\mathcal{H}_{\infty}^{*}(\mathbb{R})} |\langle W, g\rangle|^2 \, d\mu(W) = \|g\|_0^2, \tag{1.4}$$

see e.g. [44, Lemma 2.1.5]. If $g \in L_{\mathbb{R}}^2$, then $W \to \langle W, g\rangle$ belongs to $(L_{\mathbb{R}}^2)$.

Let $f \in \mathcal{H}_{\infty}(\mathbb{R})$ and $W_f(J) := \langle J, f\rangle$, $J \in \mathcal{H}_{\infty}^{*}(\mathbb{R})$. Then W_f is a Gaussian random variable on $(\mathcal{H}_{\infty}^{*}(\mathbb{R}), \mu)$ satisfying

$$\mathbb{E}_{\mu}(W_f) = 0,\ \ \mathbb{E}_{\mu}(W_f^2) = \|f\|_0^2.$$

Then the linear map

$$\begin{array}{ccc} \mathcal{H}_{\infty}(\mathbb{R}) & \to & (L_{\mathbb{R}}^2) \\ & & \\ f & \to & W_f \end{array}$$

can be extended to a linear isometry from $L^2(\mathbb{Q}_p^N)$ to $(L_{\mathbb{C}}^2)$.

1.3 Wick-ordered polynomials

Let $\mathcal{P}_n(\mathbb{R})$, respectively $\mathcal{P}_n(\mathbb{C})$, be the vector space of finite linear combinations of functions of the form

$$W \to \langle W, f\rangle^n = \left\langle W^{\otimes n}, f^{\otimes n}\right\rangle, \text{ with } W \in \mathcal{H}_{\infty}^{*}(\mathbb{R}),$$

where f runs over $\mathcal{H}_{\infty}(\mathbb{R})$, respectively $\mathcal{H}_{\infty}(\mathbb{C})$. Notice that $\mathcal{P}_n(\mathbb{C}) = \mathcal{P}_n(\mathbb{R}) + i\mathcal{P}_n(\mathbb{R})$. An element of the direct algebraic sums

$$\mathcal{P}(\mathbb{R}) := \bigoplus_{n=0}^{\infty} \mathcal{P}_n(\mathbb{R}),\ \ \mathcal{P}(\mathbb{C}) := \bigoplus_{n=0}^{\infty} \mathcal{P}_n(\mathbb{C}) \tag{1.5}$$

is called a *polynomial* on the Gaussian space $\mathcal{H}_{\infty}^{*}(\mathbb{R})$. These functions are not very useful because they do not satisfy orthogonality relations. This is the main motivation to introduce and utilize the Wick-ordered polynomials (Appendix D.1).

For $W \in \mathcal{H}_{\infty}^{*}(\mathbb{R})$ and $f \in \mathcal{H}_{\infty}$, we define *the Wick-ordered monomial* as

$$\begin{aligned}\left\langle : W^{\otimes n} :, f^{\otimes n} \right\rangle &= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!\,(n-2k)!} \left(\frac{-1}{2} \left\langle f, f \right\rangle_0 \right)^k \left\langle W, f \right\rangle^{n-2k} \\ &= \|f\|_0^n \, \boldsymbol{H}_n \left(\|f\|_0^{-1} \left\langle W, f \right\rangle \right),\end{aligned}$$

where $\boldsymbol{H}_n$ denotes the n-th Hermite polynomial. Then $: W^{\otimes n} :\in \mathcal{H}_{\infty}^{*\widehat{\otimes} n}$, in addition, any polynomial $\Phi \in \mathcal{P}(\mathbb{R})$, respectively $\mathcal{P}(\mathbb{C})$, is expressed as

$$\Phi(W) = \sum_{n=0}^{\infty} \left\langle : W^{\otimes n} :, \phi_n \right\rangle, \tag{1.6}$$

where ϕ_n belong to the symmetric n-fold algebraic tensor product $\left(\mathcal{H}_{\infty}(\mathbb{R})\right)^{\widehat{\otimes} n}$ of $\mathcal{H}_{\infty}(\mathbb{R})$, respectively of $\mathcal{H}_{\infty}(\mathbb{C})$, and the sum symbol involves only a finite number of non-zero terms. A function of type (1.6) is called a *Wick-ordered polynomial.* For two polynomials Φ, $\Psi \in \mathcal{P}(\mathbb{C})$ given respectively by (1.6) with $\phi_n \in \left(\mathcal{H}_{\infty}(\mathbb{C})\right)^{\widehat{\otimes} n}$, and by

$$\Psi(W) = \sum_{n=0}^{\infty} \left\langle : W^{\otimes n} :, \psi_n \right\rangle, \text{ with } \psi_n \in \left(\mathcal{H}_{\infty}(\mathbb{C})\right)^{\widehat{\otimes} n}, \tag{1.7}$$

it holds that

$$\int_{\mathcal{H}_{\infty}^{*}(\mathbb{R})} \Phi(W)\,\Psi(W)\,d\mu(W) = \sum\nolimits_{n=0}^{\infty} n!\,\langle \varphi_n, \psi_n \rangle_0 ,$$

where $\langle \cdot, \cdot \rangle_0$ denotes the scalar product in $\left(L^2\left(\mathbb{Q}_p^N\right)\right)^{\otimes n}$. In particular,

$$\|\Phi\|_{\left(L^2_{\mathbb{C}}\right)}^2 = \sum\nolimits_{n=0}^{\infty} n!\,\|\phi_n\|_0^2 ,$$

where $\|\cdot\|_0$ denotes the norms of $\left(L^2\left(\mathbb{Q}_p^N\right)\right)^{\otimes n}$, see e.g. [44, Proposition 2.2.10]. Consequently, each $\Psi \in \mathcal{P}(\mathbb{C})$ is uniquely expressed as a Wick-ordered polynomial.

Remark 3 *We denote by $I_n(f_n)$ the linear extension to $\left(L^2\left(\mathbb{Q}_p^N\right)\right)^{\widehat{\otimes} n}$ of the map $f_n \rightarrow \langle : W^{\otimes n} :, f_n \rangle$, $W \in \mathcal{H}_{\infty}^{*}(\mathbb{R})$, then*

$$I_n(f^{\otimes n}) = \|f\|_0^n \, \boldsymbol{H}_n(\|f\|_0^{-1} W_f), \;\; f \in L^2,$$

and

$$\int_{\mathcal{H}_\infty^*(\mathbb{R})} I_n(f_n)I_m(g_m)d\mu = \delta_{nm} n! \langle f_n, g_m\rangle_0 \,, \quad f_n \in L^{2\widehat{\otimes} n},\ g_m \in L^{2\widehat{\otimes} m}.$$

We shall also use $\langle : W^{\otimes n} :, f_n\rangle$ *to denote* $I_n(f_n)$ *formally. In this case the symbol* $\langle\cdot,\cdot\rangle$ *should not be confused with the bilinear form on* $\mathcal{H}_\infty^* \times \mathcal{H}_\infty$*.*

1.4 Wiener-Itô-Segal isomorphism

Let $\Gamma\left(L^2\left(\mathbb{Q}_p^N\right)\right)$ be the space of sequences $\boldsymbol{f} = \{f_n\}_{n\in\mathbb{N}}$, $f_n \in \left(L^2\left(\mathbb{Q}_p^N\right)\right)^{\widehat{\otimes} n}$, such that

$$\|\boldsymbol{f}\|^2_{\Gamma\left(L^2\left(\mathbb{Q}_p^N\right)\right)} := \sum_{n=0}^{\infty} n! \, \|f_n\|_0^2 < \infty.$$

The Hilbert space $\Gamma\left(L^2\left(\mathbb{Q}_p^N\right)\right)$ is called *the Boson Fock Space on* $L^2\left(\mathbb{Q}_p^N\right)$. The Wiener-Itô-Segal theorem asserts that for each $\Phi \in (L^2_{\mathbb{C}})$ there exists a sequence $\boldsymbol{\phi} = \{\phi_n\}_{n\in\mathbb{N}}$ in $\Gamma\left(L^2\left(\mathbb{Q}_p^N\right)\right)$ such that (1.6) holds in the $(L^2_{\mathbb{C}})$-sense, but with $\phi_n \in \left(L^2\left(\mathbb{Q}_p^N\right)\right)^{\widehat{\otimes} n}$, see Remark 3. Conversely, for any $\boldsymbol{\phi} = \{\phi_n\}_{n\in\mathbb{N}} \in \Gamma\left(L^2_{\mathbb{C}}\left(\mathbb{Q}_p^N\right)\right)$, (1.6) defines a function in $(L^2_{\mathbb{C}})$. In this case

$$\|\Phi\|^2_{\left(L^2_{\mathbb{C}}\right)} = \sum_{n=0}^{\infty} n! \, \|\phi_n\|_0^2 = \|\boldsymbol{\phi}\|^2_{\Gamma\left(L^2\left(\mathbb{Q}_p^N\right)\right)} \,,$$

see e.g. [44, Theorem 2.3.5], [48], (Appendix D.1, Theorem 56)

2

Non-Archimedean Kondratiev Spaces of Test Functions and Distributions

In this section we introduce non-Archimedean versions of Kondratiev-type spaces of test functions and distributions.

2.1 Kondratiev-type spaces of test functions

We define for l, $k \in \mathbb{N}$, and $\beta \in [0,1]$ fixed, the following norm on $(L^2_{\mathbb{C}})$:

$$\left\|\Phi\right\|^2_{l,k,\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta}\, 2^{nk} \left\|\phi_n\right\|^2_l ,$$

where Φ is given in (1.6), and $\left\|\cdot\right\|_l$ denotes the norm on $\mathcal{H}_l^{\widehat{\otimes} n}$.

We now define

$$\mathcal{H}_{l,k,\beta} = \left\{ \Phi\left(W\right) = \sum_{n=0}^{\infty} \left\langle : W^{\otimes n} :, \phi_n \right\rangle \in \left(L^2_{\mathbb{C}}\right); \left\|\Phi\right\|^2_{l,k,\beta} < \infty \right\}.$$

The space $\mathcal{H}_{l,k,\beta}$ is a Hilbert space with inner product

$$\left\langle \Phi, \Psi \right\rangle_{l,k,\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta}\, 2^{nk} \left\langle \phi_n, \psi_n \right\rangle_l ,$$

where Φ, $\Psi \in (L^2_{\mathbb{C}})$ are as in (1.6)-(1.7), and $\left\langle \cdot, \cdot \right\rangle_l$ denotes the inner product on $\mathcal{H}_l^{\widehat{\otimes} n}$.

The Kondratiev space of test functions $\left(\mathcal{H}_{\infty}\right)^{\beta}$ is defined to be the projective limit of the spaces $\mathcal{H}_{l,k,\beta}$:

$$(\mathcal{H}_\infty)^\beta = \varprojlim_{l,k\in\mathbb{N}} \mathcal{H}_{l,k,\beta}.$$

As a vector space $(\mathcal{H}_\infty)^\beta = \cap_{l,k\in\mathbb{N}} \mathcal{H}_{l,k,\beta}$. The space of test functions $(\mathcal{H}_\infty)^\beta$ is a nuclear countable Hilbert space, which is continuously and densely embedded in $(L^2_\mathbb{C})$. Moreover, $(\mathcal{H}_\infty)^\beta$ and its topology do not depend on the family of Hilbertian norms $\{\|\cdot\|_l\}_{l\in\mathbb{N}}$, see e.g. [26, Theorem 1], [22, Chapter IV, Theorem 1.4].

The construction used to obtain the spaces $(\mathcal{H}_\infty)^\beta$ can be carried out starting with an arbitrary nuclear space $\mathcal{N}$. For $0 \le \beta \le 1$, the spaces $(\mathcal{N})^\beta$ were studied by Kondratiev, Leukert and Streit in [29], [27], [26], see also [22, Chapter IV]. In the case $\beta = 0$ and $\mathcal{N} = \mathcal{S}$, the Schwartz space in $\mathbb{R}^n$, the space $(\mathcal{N})^0$ is the Hida space of test functions, see e.g. [21].

2.2 Kondratiev-type spaces of distributions

Let $\mathcal{H}_{-l,-k,-\beta}$ be the dual with respect to $(L^2_\mathbb{C})$ of $\mathcal{H}_{l,k,\beta}$ and let $(\mathcal{H}_\infty)^{-\beta}$ be the dual with respect to $(L^2_\mathbb{C})$ of $(\mathcal{H}_\infty)^\beta$. We denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the corresponding dual pairing which is given by the extension of the scalar product on $(L^2_\mathbb{C})$. We define the expectation of a distribution $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-\beta}$ as $\mathbb{E}_\mu(\mathbf{\Phi}) = \langle\langle \mathbf{\Phi}, 1\rangle\rangle$.

The dual space of $(\mathcal{H}_\infty)^{-\beta}$ is given by

$$(\mathcal{H}_\infty)^{-\beta} = \underset{l,k\in\mathbb{N}}{\cup} \mathcal{H}_{-l,-k,-\beta},$$

see [22, Chapter IV, Theorem 1.5]. We will consider $(\mathcal{H}_\infty)^{-\beta}$ with the inductive limit topology. In particular, we know that every distribution is of finite order, i.e. for any $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-\beta}$ there exist $l, k \in \mathbb{N}$ such that $\mathbf{\Phi} \in \mathcal{H}_{-l,-k,-\beta}$. The chaos decomposition introduces a natural decomposition of $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-\beta}$ into generalized kernels $\Phi_n \in (\mathcal{H}^*_\infty(\mathbb{C}))^{\widehat{\otimes} n}$. Let $\Phi_n \in (\mathcal{H}^*_\infty(\mathbb{C}))^{\widehat{\otimes} n}$ be given. Then there is a distribution, denoted as $\langle \Phi_n, : W^{\otimes n} :\rangle$, in $(\mathcal{H}_\infty)^{-\beta}$ acting on $\Psi \in (\mathcal{H}_\infty)^\beta$ ($\Psi = \sum\limits_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, \psi_n\rangle$, with $\psi_n \in (\mathcal{H}_\infty(\mathbb{C}))^{\widehat{\otimes} n}$) as

$$\left\langle\left\langle \left\langle \Phi_n, : W^{\otimes n} : \right\rangle, \Psi \right\rangle\right\rangle = n! \left\langle \Phi_n, \psi_n \right\rangle.$$

Any $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-\beta}$ has a unique decomposition of the form

$$\mathbf{\Phi} = \sum_{n=0}^{\infty} \left\langle \Phi_n, : W^{\otimes n} : \right\rangle, \ \Phi_n \in (\mathcal{H}^*_\infty(\mathbb{C}))^{\widehat{\otimes} n},$$

where the series converges in $(\mathcal{H}_\infty)^{-\beta}$, in addition, we have

$$\langle\langle \mathbf{\Phi}, \Psi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \Phi_n, \psi_n \rangle , \ \Psi \in (\mathcal{H}_\infty)^{\beta}.$$

Now, $\mathcal{H}_{-l,-k,-\beta}$ is a Hilbert space, that can be described as follows:

$$\mathcal{H}_{-l,-k,-\beta} = \left\{ \mathbf{\Phi} \in (\mathcal{H}_\infty)^{-\beta}; \ \ \|\mathbf{\Phi}\|_{-l,-k,-\beta} < \infty \right\},$$

where

$$\|\mathbf{\Phi}\|^2_{-l,-k,-\beta} = \sum_{n=0}^{\infty} (n!)^{1-\beta}\, 2^{-nk} \|\Phi_n\|^2_{-l}, \tag{2.1}$$

see [22, Chapter IV, Theorem 1.5].

Remark 4 *Notice that*

$$\begin{aligned}(\mathcal{H}_\infty)^1 \subset \cdots \subset (\mathcal{H}_\infty)^{\beta} \subset \cdots \subset (\mathcal{H}_\infty)^0 \subset (L^2_{\mathbb{C}}) \\ \subset (\mathcal{H}_\infty)^{-0} \subset \cdots \subset (\mathcal{H}_\infty)^{-\beta} \subset \cdots \subset (\mathcal{H}_\infty)^{-1}.\end{aligned}$$

Following Kondratiev, Leukert and Streit, in this article we work with the Gel'fand triple $(\mathcal{H}_\infty)^1 \subset (L^2_{\mathbb{C}}) \subset (\mathcal{H}_\infty)^{-1}$.

2.3 The S-transform and the characterization of $(\mathcal{H}_\infty)^{-1}$

2.3.1 The S-transform

We first consider the Wick exponential:

$$:\exp\langle W, g\rangle := \exp\left(\langle W, g\rangle - \frac{1}{2}\|g\|_0^2\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle : W^{\otimes n} :, g^{\otimes n} \right\rangle,$$

for $W \in \mathcal{H}^*_\infty(\mathbb{R})$, $g \in \mathcal{H}_\infty(\mathbb{C})$. Then $:\exp\langle W, g\rangle :\in (L^2_{\mathbb{C}})$ and its l, k, 1-norm is given by

$$\|:\exp\langle \cdot, g\rangle :\|^2_{l,k,1} = \sum_{n=0}^{\infty} (n!)^2 2^{nk} \left\| \frac{1}{n!} g^{\otimes n} \right\|^2_l = \sum_{n=0}^{\infty} \left(2^k \|g\|_l^2\right)^n.$$

This norm is finite if and only if $2^k \|g\|_l^2 < 1$, i.e. $:\exp\langle W, g\rangle :\in \mathcal{H}_{l,k,\beta}$ if and only if g belongs to the following neighborhood of zero:

$$\mathcal{U}_{l,k} = \left\{ f \in \mathcal{H}_\infty(\mathbb{C});\, \|f\|_l < \frac{1}{2^{\frac{k}{2}}} \right\}.$$

Therefore the Wick exponential does not belong to $(\mathcal{H}_\infty)^1$, i.e. it is not a test function, in contrast to usual white noise analysis.

Let $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$, then there exist l, k such that $\mathbf{\Phi} \in \mathcal{H}_{-l,-k,-1}$. For all $f \in \mathcal{U}_{l,k}$, we define the (local) S-transform of $\mathbf{\Phi}$ as

$$S\mathbf{\Phi}\,(f) = \langle\langle \mathbf{\Phi}, :\exp\langle \cdot, f\rangle :\rangle\rangle = \sum_{n=0}^{\infty} \left\langle \Phi_n, f^{\otimes n}\right\rangle. \tag{2.2}$$

Hence, for $\mathbf{\Phi} \in \mathcal{H}_{-l,-k,-1}$, (2.2) defines the S-transform for all $f \in \mathcal{U}_{l,k}$.

2.4 Holomorphic functions on $\mathcal{H}_\infty(\mathbb{C})$

Let $\mathcal{V}_{l,\epsilon} = \{f \in \mathcal{H}_\infty(\mathbb{C})\,; \|f\|_l < \epsilon\}$ be a neighborhood of zero in $\mathcal{H}_\infty(\mathbb{C})$. A map $F : \mathcal{V}_{l,\epsilon} \to \mathbb{C}$ is called *holomorphic* in $\mathcal{V}_{l,\epsilon}$ (Appendix D.2), if it satisfies the following two conditions: (i) for each $g_0 \in \mathcal{V}_{l,\epsilon}$, $g \in \mathcal{H}_\infty(\mathbb{C})$ there exists a neighborhood $V_{g_0,g}$ in $\mathbb{C}$ around the origin such that the map $z \to F(g_0 + zg)$ is holomorphic in $V_{g_0,g}$. (ii) For each $g \in \mathcal{V}_{l,\epsilon}$ there exists an open set $\mathcal{U} \subset \mathcal{V}_{l,\epsilon}$ containing g such that $F(\mathcal{U})$ is bounded.

By identifying two maps F_1 and F_2 coinciding in a neighborhood of zero, we define $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ as the space of germs of holomorphic maps around the origin.

2.5 Characterization of $(\mathcal{H}_\infty)^{-1}$

A key result is the following: the mapping

$$\begin{array}{rccc} S: & (\mathcal{H}_\infty)^{-1} & \to & Hol_0(\mathcal{H}_\infty(\mathbb{C})) \\ & \mathbf{\Phi} & \to & S\mathbf{\Phi} \end{array}$$

is a well-defined bijection, see [26, Theorem 3], [22, Chapter IV, Theorem 2.13].

2.6 Integration of distributions

Let $(\mathfrak{L}, \mathcal{A}, \nu)$ be a measure space, and

$$\begin{array}{ccc} \mathfrak{L} & \to & (\mathcal{H}_\infty)^{-1} \\ \mathfrak{l} & \to & \mathbf{\Phi}_\mathfrak{l} \end{array}.$$

Assume that there exists an open neighborhood $\mathcal{V} \subset \mathcal{H}_\infty(\mathbb{C})$ of zero such that (i) $S\mathbf{\Phi}_\mathfrak{l}$, $\mathfrak{l} \in \mathfrak{L}$, is holomorphic in $\mathcal{V}$; (ii) the mapping $\mathfrak{l} \to S\mathbf{\Phi}_\mathfrak{l}\,(g)$ is measurable for every $g \in \mathcal{V}$;

and (iii) there exists a function $C(\mathfrak{l}) \in L^1(\mathfrak{L}, \mathcal{A}, \nu)$ such that $|S\mathbf{\Phi}_{\mathfrak{l}}(g)| \leq C(\mathfrak{l})$ for all $g \in \mathcal{V}$ and for ν-almost $\mathfrak{l} \in \mathfrak{L}$. Then there exist l_0, $k_0 \in \mathbb{N}$ such that $\int_{\mathsf{L}} \mathbf{\Phi}_{\mathfrak{l}} d\nu(\mathfrak{l})$ exists as a Bochner integral in $\mathcal{H}_{-l_0,-k_0,-1}$, in particular,

$$S\left(\int_{\mathsf{L}} \mathbf{\Phi}_{\mathfrak{l}} d\nu(\mathfrak{l})\right)(g) = \int_{\mathsf{L}} S\mathbf{\Phi}_{\mathfrak{l}}(g)\, d\nu(\mathfrak{l}), \text{ for any } g \in \mathcal{V}, \tag{2.3}$$

cf. [26, Theorem 6], [22, Chapter IV, Theorem 2.15], (Appendix D.3, Theorem 63).

2.7 The Wick product

Given $\mathbf{\Phi}$, $\mathbf{\Psi} \in (\mathcal{H}_\infty)^{-1}$, we define the *Wick product* of them as

$$\mathbf{\Phi}\Diamond\mathbf{\Psi} = S^{-1}(S\mathbf{\Phi}S\mathbf{\Psi}).$$

This product is well-defined because $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ is an algebra. The map

$$\begin{array}{ccc} (\mathcal{H}_\infty)^{-1} \times (\mathcal{H}_\infty)^{-1} & \to & (\mathcal{H}_\infty)^{-1} \\ (\mathbf{\Phi}, \mathbf{\Psi}) & \to & \mathbf{\Phi}\Diamond\mathbf{\Psi} \end{array}$$

is well-defined and continuous. Furthermore, if $\mathbf{\Phi} \in \mathcal{H}_{-l_1,-k_1,-1}$, $\mathbf{\Psi} \in \mathcal{H}_{-l_2,-k_2,-1}$, and $l := \max\{l_1, l_2\}$, $k := k_1 + k_2 + 1$, then

$$\|\mathbf{\Phi}\Diamond\mathbf{\Psi}\|_{-l,-k,-1} \leq \|\mathbf{\Phi}\|_{-l_1,-k_1,-1} \|\mathbf{\Psi}\|_{-l_2,-k_2,-1},$$

cf. [26, Proposition 11]. The Wick product leaves $(\mathcal{H}_\infty)$ invariant. By induction on n, we can define the Wick powers:

$$\mathbf{\Phi}^{\Diamond n} = S^{-1}((S\mathbf{\Phi})^n) \in (\mathcal{H}_\infty)^{-1}.$$

Consequently $\sum_{n=0}^{m} a_n \mathbf{\Phi}^{\Diamond n} \in (\mathcal{H}_\infty)^{-1}$, see Appendix D.3.1.

2.7.1 Wick analytic functions in $(\mathcal{H}_\infty)^{-1}$

Assume that F is an analytic function in a neighborhood of the point $z_0 = \mathbb{E}_\mu(\mathbf{\Phi})$ in $\mathbb{C}$, with $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$. Then $F^{\Diamond}(\mathbf{\Phi}) = S^{-1}(F(S\mathbf{\Phi}))$ exists in $(\mathcal{H}_\infty)^{-1}$, cf. [26, Theorem 12]. In addition, if F is analytic in $z_0 = \mathbb{E}_\mu(\mathbf{\Phi})$, with power series $F(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, then the Wick series $\sum_{n=0}^{\infty} c_n (\mathbf{\Phi} - z_0)^{\Diamond n}$ converges in $(\mathcal{H}_\infty)^{-1}$ and $F^{\Diamond}(\mathbf{\Phi}) = \sum_{n=0}^{\infty} c_n (\mathbf{\Phi} - z_0)^{\Diamond n}$.

3

Schwinger Functions and Euclidean Quantum Field Theory

3.1 Schwinger functions

Definition 5 *Let $f_1, \ldots, f_n \in \mathcal{H}_\infty(\mathbb{R})$, $n \in \mathbb{N}$. The n-th Schwinger function corresponding to $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$, with $\mathbb{E}_\mu(\mathbf{\Phi}) = 1$, is defined as*

$$\mathcal{S}_n^{\mathbf{\Phi}}(f_1 \otimes \cdots \otimes f_n)(W) = \begin{cases} 1 & \textit{if} \quad n = 0 \\ \\ \langle\langle \mathbf{\Phi}, \langle W, f_1\rangle \cdots \langle W, f_n\rangle\rangle\rangle & \textit{if} \quad n \geq 1, \end{cases} \tag{3.1}$$

for $W \in \mathcal{H}_\infty^*(\mathbb{R})$.

The pairing in (3.1) is well-defined because the Wick polynomials $\mathcal{P}(\mathcal{H}_\infty^*(\mathbb{R}))$ are dense in $(\mathcal{H}_\infty)^1$.

The T-transform of a distribution is defined as

$$T\mathbf{\Phi}(g) = \exp\left(\frac{-1}{2}\|g\|_0^2\right) S\mathbf{\Phi}(ig) \tag{3.2}$$

for $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$ and $g \in \mathcal{U}$, where $\mathcal{U}$ is neighborhood of zero in $\mathcal{H}_\infty(\mathbb{C})$. The Schwinger functions can be computed by using the T-transform:

Lemma 6 ([19, Proposition III.3]) *Let $f_1, \ldots, f_n \in \mathcal{H}_\infty(\mathbb{R})$, $n \in \mathbb{N}$. The n-th Schwinger function corresponding to $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$ is given by*

$$\mathcal{S}_n^{\mathbf{\Phi}}(f_1 \otimes \cdots \otimes f_n) = (-i)^n \frac{\partial^n}{\partial t_1 \cdots \partial t_n} T\mathbf{\Phi}(t_1 f_1 + \cdots + t_n f_n)\,|_{t_1 = \cdots = t_n = 0}.$$

Lemma 7 *For each distribution $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$, with $\mathbb{E}_\mu(\mathbf{\Phi}) = 1$, the Schwinger functions $\{\mathcal{S}_n^{\mathbf{\Phi}}\}_{n\in\mathbb{N}}$ satisfy the following conditions:*

(OS1) the sequence $\{\mathcal{S}_n^{\mathbf{\Phi}}\}_{n\in\mathbb{N}}$, with $\mathcal{S}_n^{\mathbf{\Phi}} \in (\mathcal{H}_\infty^*(\mathbb{C}))^{\otimes n}$, satisfies

$$\left|\mathcal{S}_n^{\mathbf{\Phi}}(f_1 \otimes \cdots \otimes f_n)\right| \leq KC^n n! \prod_{i=1}^n \|f_i\|_l ,$$

for some l, $k \in \mathbb{N}$, where $K = \sqrt{I_0(2^{-k})}\,\|\Phi\|_{-l.-k-1}$, here I_0 is the modified Bessel function of order zero, which satisfies $I_0(2^{-k}) < 1.3$, $C = e2^{\frac{k}{2}}$, and for any $f_1, \cdots, f_n \in \mathcal{H}_\infty(\mathbb{R})$;

(OS4) for $n \geq 2$ and all $\sigma \in \mathfrak{S}_n$, the permutation group of order n, it holds that

$$\mathcal{S}_n^{\mathbf{\Phi}}(f_1 \otimes \cdots \otimes f_n) = \mathcal{S}_n^{\mathbf{\Phi}}\left(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}\right),$$

for any $f_1, \cdots, f_n \in \mathcal{H}_\infty(\mathbb{R})$.

Proof. Estimation (OS1) is given in the proof of Theorem 2 in [28]. The Schwinger functions $\left(\mathcal{S}_n^{\mathbf{\Phi}}\right)$ are symmetric by definition. ■

3.2 A white-noise process

For $t \in \mathbb{Q}_p$, $\overrightarrow{x} \in \mathbb{Q}_p^{N-1}$, we set $x = (t, \overrightarrow{x})$. We denote by $\delta_x := \delta_{(t,\overrightarrow{x})}$, the Dirac distribution at $(t, \overrightarrow{x})$.

Lemma 8 $\delta_{(t,\overrightarrow{x})} \in (\mathcal{H}_\infty)^{-1}$.

Proof. We first notice that

$$\left\|\delta_{(t,\overrightarrow{x})}\right\|_{-l}^2 = \int_{\mathbb{Q}_p^N} \frac{d^N\xi}{[\xi]_p^l} < \infty \text{ for } l > N,$$

which implies that $\delta_{(t,\overrightarrow{x})} \in \mathcal{H}_{-l}(\mathbb{C})$ for all $l > N$, see (1.1). Now, we define $\{\Phi_n\}_{n\in\mathbb{N}}$, with $\Phi_n \in (\mathcal{H}_\infty^*(\mathbb{C}))^{\widehat{\otimes} n}$, as $\Phi_n = 0$ if $n \neq 1$ and $\Phi_1 = \delta_{(t,\overrightarrow{x})}$. Then

$$\sum_n \left\langle \Phi_n, : W^{\otimes n} : \right\rangle = \left\langle \delta_{(t,\overrightarrow{x})}, : W : \right\rangle \in (\mathcal{H}_\infty)^{-1}.$$

In addition, for $\psi \in \mathcal{H}_\infty(\mathbb{C})$, we have

$$\left\langle\left\langle\left\langle \delta_{(t,\overrightarrow{x})}, : W : \right\rangle, \psi\right\rangle\right\rangle = \left\langle \delta_{(t,\overrightarrow{x})}, \psi \right\rangle = \int_{\mathbb{Q}_p^N} \chi_p(-\xi \cdot x)\, \widehat{\psi}(x)\, d^N\xi$$
$$= \psi(t, \overrightarrow{x}),$$

where we used that ψ is a continuous function in $L^1 \cap L^2$, see Section 1.1 and [35, Theorem 10.15]. ■

We now set

$$\mathbf{\Phi}(t, \overrightarrow{x}) := \left\langle \delta_{(t,\overrightarrow{x})}, : W : \right\rangle \in (\mathcal{H}_\infty)^{-1}.$$

Then $\mathbf{\Phi}(t, \overrightarrow{x})$ is a white-noise process with $\mathbb{E}_\mu(\mathbf{\Phi}(t, \overrightarrow{x})) = 0$.

Assume that

$$H(z) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k z^k,\ z \in U \subset \mathbb{C},$$

is a holomorphic function in U, an open neighborhood of $0 = \mathbb{E}_\mu(\mathbf{\Phi}(t, \overrightarrow{x}))$. By [26, Theorem 12], see also Section 2.7.1, we can define

$$H^\Diamond(\mathbf{\Phi}(t, \overrightarrow{x})) = \sum_{k=0}^{\infty} \frac{1}{k!} H_k \mathbf{\Phi}(t, \overrightarrow{x})^{\Diamond k}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} H_k \left\langle \delta_{(t,\overrightarrow{x})}^{\otimes k}, : W^{\otimes k} : \right\rangle \in (\mathcal{H}_\infty)^{-1}.$$

Our next goal is the construction of the potential

$$\int_{\mathbb{Q}_p^N} H^\Diamond(\mathbf{\Phi}(x)) d^N x \tag{3.3}$$

as a white-noise distribution. This goal is accomplished through the following result:

Theorem 9 *(i) Let H be a holomorphic function at zero such that $H(0) = 0$. Then (3.3) exists as a Bochner integral in a suitable subspace of $(\mathcal{H}_\infty)^{-1}$.*
(ii) The distribution

$$\mathbf{\Phi}_H := \exp^\Diamond\left(-\int_{\mathbb{Q}_p^N} H^\Diamond(\mathbf{\Phi}(x)) d^N x\right)$$

is an element of $(\mathcal{H}_\infty)^{-1}$.
(iii) The T-transform of $\mathbf{\Phi}_H$ is given by

$$T\mathbf{\Phi}_H(g) = \exp\left(-\int_{\mathbb{Q}_p^N} H(ig(x)) + \frac{1}{2}(g(x))^2\ d^N x\right)$$

for all g in a neighborhood $\mathcal{U} \subset \mathcal{H}_\infty(\mathbb{C})$ of the zero. In particular, $\mathbb{E}_\mu(\mathbf{\Phi}_H) = 1$.

Proof. (i) The result follows from the discussion presented in Section 2.5, see also [26, Theorem 6], as follows. Let $r>0$ be the radius of convergence of the Taylor series of H at the origin. We set $C(N):=\sqrt{\int_{\mathbb{Q}_p^N}\frac{d^N\xi}{[\xi]_p^l}}$, for a fixed $l>N$, and

$$\mathcal{U}_0:=\left\{g\in\mathcal{H}_\infty(\mathbb{C})\,;\left\|g\right\|_l<\frac{r}{C(N)}\right\}.$$

Then, for $g\in\mathcal{U}_0$ we have

$$\begin{aligned}SH^\Diamond(\mathbf{\Phi}(x))(g)&=\sum_{k=1}^\infty\frac{1}{k!}H_k\left\langle\delta_x^{\otimes k},g^{\otimes k}\right\rangle=\sum_{k=1}^\infty\frac{1}{k!}H_kg(x)^k\\&=\sum_{k=1}^\infty\frac{1}{k!}H_k\left\{\frac{g(x)}{r}\right\}^kr^k.\end{aligned}\tag{3.4}$$

By Claim A, $\left|\frac{g(x)}{r}\right|<1$, and from (3.4) we obtain that

$$\left|SH^\Diamond(\mathbf{\Phi}(x))(g)\right|\leq|g(x)|\sum_{k=1}^\infty\frac{1}{k!}\left|H_k\right|r^{k-1}\in L^1\left(\mathbb{Q}_p^N\right),\tag{3.5}$$

because $\mathcal{H}_\infty(\mathbb{C})\subset L^1\left(\mathbb{Q}_p^N\right)$, cf. [35, Theorem 10.15]. Estimation (3.5) implies the holomorphy of $SH^\Diamond(\mathbf{\Phi}(x))(g)$ for any $g\in\mathcal{U}_0$. Since $SH^\Diamond(\mathbf{\Phi}(x))(g)$ is measurable by [26, Theorem 6], we conclude that (3.3) is an element of $(\mathcal{H}_\infty)^{-1}$.

Claim A. $\mathcal{U}_0\subset\mathcal{U}:=\left\{g\in\mathcal{H}_\infty(\mathbb{C})\,;\left\|g\right\|_{L^\infty}<r\right\}.$

The Claim follows from the fact that

$$\left\|g\right\|_{L^\infty}\leq C(N)\left\|g\right\|_l,\text{ for }g\in\mathcal{H}_\infty(\mathbb{C}).$$

This last fact is verified as follows: by using that $g\in L^1\left(\mathbb{Q}_p^N\right)\cap L^2\left(\mathbb{Q}_p^N\right)$, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|g(x)|&=\left|\int_{\mathbb{Q}_p^N}\chi_p(-\xi\cdot x)\,\widehat{g}(\xi)\,d^N\xi\right|\leq\int_{\mathbb{Q}_p^N}\left|\widehat{g}(\xi)\right|d^N\xi\\&=\int_{\mathbb{Q}_p^N}\frac{1}{[\xi]_p^{\frac{l}{2}}}\left\{[\xi]_p^{\frac{l}{2}}\left|\widehat{g}(\xi)\right|\right\}d^N\xi\leq C(N)\left\|g\right\|_l.\end{aligned}$$

(ii) Since exp is analytic in a neighborhood of $0=\mathbb{E}_\mu(\mathbf{\Phi}(t,\overrightarrow{x}))$, then

$$\exp^\Diamond\left(-\int_{\mathbb{Q}_p^N}H^\Diamond(\mathbf{\Phi}(x))d^Nx\right)=S^{-1}\left(\exp\left(S\left(-\int_{\mathbb{Q}_p^N}H^\Diamond(\mathbf{\Phi}(x))d^Nx\right)\right)\right),$$

and by (i), $-\int_{\mathbb{Q}_p^N} H^{\diamond}(\mathbf{\Phi}(x))d^N x \in (\mathcal{H}_\infty)^{-1}$, and then its S-transform is analytic at the origin, and its composition with exp gives again an analytic function at the origin, whose inverse S-transform gives an element of $(\mathcal{H}_\infty)^{-1}$, cf. [26, Theorem 12].

(iii) The calculation of the T-transform uses (3.2), $\exp^{\diamond}(\cdot) = S^{-1}(\exp(S(\cdot)))$, and (3.4) as follows:

$$\begin{aligned}
(T\mathbf{\Phi}_H)(g) &= \exp\left(-\frac{1}{2}\|g\|_0^2\right)\exp\left(S\left(-\int_{\mathbb{Q}_p^N} H^{\diamond}(\mathbf{\Phi}(x))\,d^N x\right)(ig)\right) \\
&= \exp\left(-\frac{1}{2}\|g\|_0^2\right)\exp\left(-\int_{\mathbb{Q}_p^N} \left\langle\left\langle H^{\diamond}(\mathbf{\Phi}(x)), :\exp\langle\cdot, ig\rangle:\right\rangle\right\rangle d^N x\right) \\
&= \exp\left(-\frac{1}{2}\|g\|_0^2\right)\exp\left(-\int_{\mathbb{Q}_p^N} SH^{\diamond}(\mathbf{\Phi}(x))(ig)\,d^N x\right) \\
&= \exp\left(-\int_{\mathbb{Q}_p^N} H(ig(x)) + \frac{1}{2}g(x)^2\ \ d^N x\right).
\end{aligned}$$

In particular $\mathbb{E}_\mu(\mathbf{\Phi}_H) = T\mathbf{\Phi}_H(0) = 1$. ■

3.3 Pseudodifferential Operators and Green Functions

A non-constant homogeneous polynomial $\mathfrak{l}(\xi) \in \mathbb{Z}_p[\xi_1, \cdots, \xi_N]$ of degree d is called *elliptic* if it satisfies $\mathfrak{l}(\xi) = 0 \Leftrightarrow \xi = 0$. There are infinitely many elliptic polynomials, cf. [67, Lemma 24]. A such polynomial satisfies

$$C_0(\alpha)\|\xi\|_p^{\alpha d} \le |\mathfrak{l}(\xi)|_p^{\alpha} \le C_1(\alpha)\|\xi\|_p^{\alpha d}, \tag{3.6}$$

for some positive constants $C_0(\alpha)$, $C_1(\alpha)$, cf. [67, Lemma 25]. We define an *elliptic pseudodifferential operator with symbol* $|\mathfrak{l}(\xi)|_p^{\alpha}$, with $\alpha > 0$, as

$$(\mathbf{L}_\alpha h)(x) = \mathcal{F}^{-1}_{\xi\to x}\left(|\mathfrak{l}(\xi)|_p^{\alpha}\mathcal{F}_{x\to\xi}h\right), \tag{3.7}$$

for $h \in \mathcal{D}(\mathbb{Q}_p^N)$. We define $G := G(x; m, \alpha) \in \mathcal{D}'(\mathbb{Q}_p^N)$, with $\alpha > 0$, $m > 0$, to be the solution of

$$\left(\boldsymbol{L}_\alpha + m^2\right)G = \delta \text{ in } \mathcal{D}'(\mathbb{Q}_p^N).$$

We will say that the *Green function* $G(x; m, \alpha)$ is a *fundamental solution* of the equation

$$\left(\boldsymbol{L}_\alpha + m^2\right)u = h, \text{ with } h \in \mathcal{D}(\mathbb{Q}_p^N),\ m > 0. \tag{3.8}$$

As a distribution from $\mathcal{D}'(\mathbb{Q}_p^N)$, the Green function $G(x;m,\alpha)$ is given by

$$G(x;\alpha,m)=\mathcal{F}_{\xi\rightarrow x}^{-1}\left(\frac{1}{|\mathfrak{l}(\xi)|_p^{\alpha}+m^2}\right). \tag{3.9}$$

Notice that by (3.6), we have

$$\frac{1}{|\mathfrak{l}(\xi)|_p^{\alpha}+m^2}\in L^1\left(\mathbb{Q}_p^N,d^N\xi\right) \quad \text{for} \quad \alpha d>N,$$

and in this case, $G(x;\alpha,m)$ is an L^{∞}-function.

There exists a Green function $G(x;\alpha,m)$ for the operator $\boldsymbol{L}_{\alpha}+m^2$, which is continuous and non-negative on $\mathbb{Q}_p^n\smallsetminus\{0\}$, and tends to zero at infinity. The equation

$$\left(\boldsymbol{L}_{\alpha}+m^2\right)u=g, \tag{3.10}$$

with $g\in\mathcal{H}_{\infty}(\mathbb{R})$, has a unique solution $u(x)=G(x;\alpha,m)*g(x)\in\mathcal{H}_{\infty}(\mathbb{R})$, cf. [35, Theorem 11.2].

As a consequence one obtains that the mapping

$$\begin{array}{llll}\mathcal{G}_{\alpha,m}: & \mathcal{H}_{\infty}(\mathbb{R}) & \rightarrow & \mathcal{H}_{\infty}(\mathbb{R})\\ & g(x) & \rightarrow & G(x;\alpha,m)*g(x),\end{array} \tag{3.11}$$

is continuous, cf. [35, Corollary 11.3].

Remark 10 *For $\alpha>0$, $\beta>0$, $m>0$, we set*

$$\left(\mathbf{L}_{\alpha,\beta,m}h\right)(x)=\mathcal{F}_{\xi\rightarrow x}^{-1}\left(\left(|\mathfrak{l}(\xi)|_p^{\alpha}+m^2\right)^{\beta}\mathcal{F}_{x\rightarrow\xi}h\right),$$

for $h\in\mathcal{D}(\mathbb{Q}_p^N)$. We denote by $G(x;\alpha,\beta,m)$ the associated Green function. By using the fact that

$$C_0(\alpha,\beta,m)\,[\xi]_p^{\alpha\beta d}\leq\left(|\mathfrak{l}(\xi)|_p^{\alpha}+m^2\right)^{\beta}\leq C_1(\alpha,\beta,m)\,[\xi]_p^{\alpha\beta d},$$

all the results presented in this section for operators $\mathbf{L}_{\alpha}+m^2$ can be extended to operators $\mathbf{L}_{\alpha,\beta,m}$. In particular,

$$\begin{array}{llll}\mathcal{G}_{\alpha,\beta,m}: & \mathcal{H}_{\infty}(\mathbb{R}) & \rightarrow & \mathcal{H}_{\infty}(\mathbb{R})\\ & g(x) & \rightarrow & G(x;\alpha,\beta,m)*g(x),\end{array} \tag{3.12}$$

gives rise to a continuous mapping. As operators on $\mathcal{H}_{\infty}(\mathbb{R})$, we can identify $\mathcal{G}_{\alpha,\beta,m}$ with the operator $(\boldsymbol{L}_{\alpha}+m^2)^{-\beta}$, which is a pseudodifferential operator with symbol $\left(|\mathfrak{l}(\xi)|_p^{\alpha}+m^2\right)^{-\beta}$.

Remark 11 *The mapping*

$$\begin{array}{rccc} \mathcal{G}_{\alpha,m}^{\otimes 2}-1: & \mathcal{H}_{\infty}^{\otimes 2} & \rightarrow & \mathcal{H}_{\infty}^{\otimes 2} \\ & f\otimes g & \rightarrow & \mathcal{G}_{\alpha,m}(f)\otimes\mathcal{G}_{\alpha,m}(g)-f\otimes g \end{array}$$

is well-defined and continuous. By using [44, Proposition 1.3.6], any element h *of* $\mathcal{H}_{\infty}^{\otimes 2}$ *can be represented as an absolutely convergent series of the form* $h=\sum_i f_i\otimes g_i$*, consequently,* $\sum_i \mathcal{G}_{\alpha,m}(f_i)\otimes\mathcal{G}_{\alpha,m}(g_i)$ *is an element of* $\mathcal{H}_{\infty}^{\otimes 2}$*, which implies that* $\mathcal{G}_{\alpha,m}^{\otimes 2}-1$ *is a well-defined mapping. On the other hand, the space* $\mathcal{H}_{\infty}^{\otimes 2}$ *is locally convex, the topology is defined by the seminorms*

$$\|h\|_{l,k}=\inf\sum_i \|f_i\|_l\otimes\|g_i\|_k\,,\quad h\in\mathcal{H}_{\infty}\otimes_{alg}\mathcal{H}_{\infty},$$

where the infimum is taken over all the pairs (f_i,g_j) *satisfying* $h=\sum_j f_j\otimes g_j$*. The continuity of* $\mathcal{G}_{\alpha,m}^{\otimes 2}-1$ *is equivalent to*

$$\left\|\left(\mathcal{G}_{\alpha,m}^{\otimes 2}-1\right)h\right\|_{l,k}\leq C\,\|h\|_{l',k'}\,,$$

where the indices l'*,* k' *depend on* l*,* k*. This condition can be verified easily using the continuity of* $\mathcal{G}_{\alpha,m}$*.*

Remark 12 *We denote by* Tr *(the trace), which is the unique element of* $\mathcal{H}_{\infty}^{*\widehat{\otimes}2}$ *determined by the formula*

$$\langle Tr, f\otimes g\rangle=\langle f,g\rangle_0\,,\ \text{for } f,g\in\mathcal{H}_{\infty}.$$

We define $\left(\mathcal{G}_{\alpha,m}^{\otimes 2}-1\right)Tr\in\mathcal{H}_{\infty}^{*\widehat{\otimes}2}$ *as*

$$\left\langle\left(\mathcal{G}_{\alpha,m}^{\otimes 2}-1\right)Tr, f\otimes g\right\rangle=\left\langle Tr,\left(\mathcal{G}_{\alpha,m}^{\otimes 2}-1\right)(f\otimes g)\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ *is the pairing between* $\mathcal{H}_{\infty}^{*\widehat{\otimes}2}$ *and* $\mathcal{H}_{\infty}^{\widehat{\otimes}2}$*. For a general construction of this type of operators the reader may consult [36, Theorem 9.11].*

3.4 Lévy characteristics

We recall that an infinitely divisible probability distribution P is a probability distribution having the property that for each $n\in\mathbb{N}\setminus\{0\}$ there exists a probability distribution P_n such that $P=P_n*\cdots*P_n$ (n-times). By the Lévy-Khinchine Theorem, see e.g. [39], the characteristic function C_P of P satisfies

$$C_P(t)=\int_{\mathbb{R}}e^{ist}dP(s)=e^{F(t)},\ t\in\mathbb{R}, \tag{3.13}$$

where $\digamma : \mathbb{R} \to \mathbb{C}$ is a continuous function, called the *Lévy characteristic of* P, which is uniquely represented as follows:

$$\digamma(t) = iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\smallsetminus\{0\}} \left(e^{ist} - 1 - \frac{ist}{1+s^2} \right) dM(s),\ t \in \mathbb{R},$$

where a, $\sigma \in \mathbb{R}$, with $\sigma \geq 0$, and the measure $dM(s)$ satisfies

$$\int_{\mathbb{R}\smallsetminus\{0\}} \min\left(1, s^2\right) dM(s) < \infty. \tag{3.14}$$

On the other hand, given a triple (a, σ, dM) with $a \in \mathbb{R}$, $\sigma \geq 0$, and dM a measure on $\mathbb{R}\smallsetminus\{0\}$ satisfying (3.14), there exists a unique infinitely divisible probability distribution P such that its Lévy characteristic is given by (3.13).

Let $\digamma$ be a Lévy characteristic defined by (3.13). Then there exists a unique probability measure $\mathrm{P}_\digamma$ on $\left(\mathcal{H}^*_\infty(\mathbb{R}), \mathcal{B}\right)$ such that the 'Fourier transform' of $\mathrm{P}_\digamma$ satisfies

$$\int_{\mathcal{H}^*_\infty(\mathbb{R})} e^{i\langle W,f\rangle} \mathrm{dP}_\digamma(W) = \exp\left\{ \int_{\mathbb{Q}_p^N} \digamma(f(x))\, d^N x \right\},\ f \in \mathcal{H}_\infty(\mathbb{R}), \tag{3.15}$$

cf. [66, Theorem 5.2], alternatively [35, Theorem 11.6].

We will say that a distribution $\mathbf{\Theta} \in (\mathcal{H}_\infty)^{-1}$ is *represented by a probability measure* P on $\left(\mathcal{H}^*_{\mathbb{R}}(\infty), \mathcal{B}\right)$ if

$$\langle\langle \mathbf{\Theta}, \Psi \rangle\rangle = \textstyle\int_{\mathcal{H}^*_\infty(\mathbb{R})} \Psi(W)\, d\mathrm{P}(W) \text{ for any } \Psi \in (\mathcal{H}_\infty)^1. \tag{3.16}$$

We will denote this fact as $d\mathrm{P} = \mathbf{\Theta} d\mu$. In this case $\mathbf{\Theta}$ may be regarded as the generalized Radon-Nikodym derivative $\frac{d\mathrm{P}}{d\mu}$ of P with respect to μ.

By using this result, Theorem 9-(iii), and assuming that

$$\digamma(t) = -H(it) - \frac{1}{2}t^2,\ t \in \mathbb{R} \tag{3.17}$$

is a Lévy characteristic, there exists a probability measure P_H on $\left(\mathcal{H}^*_{\mathbb{R}}(\infty), \mathcal{B}\right)$ such that

$$T\mathbf{\Phi}_H(f) = \textstyle\int_{\mathcal{H}^*_\infty(\mathbb{R})} \exp\left(i\langle W, f\rangle\right) d\mathrm{P}_H(W),\ \ f \in \mathcal{H}_\infty(\mathbb{R}). \tag{3.18}$$

Theorem 13 *Assume that H is a holomorphic function at the origin satisfying $H(0) = 0$. Then $d\mathrm{P}_H = \mathbf{\Phi}_H d\mu$ if and only if $\digamma(t)$ is a Lévy characteristic.*

Proof. Assume that $\digamma(t)$ is a Lévy characteristic. By (3.18), we have

$$T\mathbf{\Phi}_H(\lambda f)=\int_{\mathcal{H}^*_\infty(\mathbb{R})}\exp(\lambda i\langle W,f\rangle)\,d\mathrm{P}_H(W)=\langle\langle\mathbf{\Phi}_H,\exp(\lambda i\langle W,f\rangle)\rangle\rangle, \tag{3.19}$$

for any $\lambda\in\mathbb{R}$.

In order to establish (3.16), it is sufficient to show that (3.16) holds for Ψ in a dense subspace of $(L^2_{\mathbb{C}})$, we can choose the linear span of the exponential functions of the form $\exp\alpha\langle W,f\rangle$ for $\alpha\in\mathbb{C}$, $f\in\mathcal{H}_\infty(\mathbb{R})$, cf. [21, Proposition 1.9]. On the other hand, since $\mathbf{\Phi}_H\in\mathcal{H}_{-l,-k.-1}(\mathbb{C})$ for some l, $k\in\mathbb{N}$, and $(L^2_{\mathbb{C}})$ is dense in $\mathcal{H}_{-l,-k.-1}(\mathbb{C})$, it is sufficient to establish (3.16) when $\mathbf{\Phi}_H\in(L^2_{\mathbb{C}})$. Now the result follows from (3.19) by using the fact that

$$\lambda\to T\mathbf{\Phi}_H(\lambda f)=\int_{\mathcal{H}^*_\infty(\mathbb{R})}\exp(\lambda i\langle W,f\rangle)\,d\mu(W),\ \lambda\in\mathbb{R},$$

has an entire analytic extension, cf. [21, Proposition 2.2].

Conversely, assume that $d\mathrm{P}_H=\mathbf{\Phi}_H d\mu$, then by Theorem 9-(iii), we have

$$\begin{aligned}&\int_{\mathcal{H}^*_\infty(\mathbb{R})}e^{i\langle W,f\rangle}d\mathrm{P}_H(W)=\int_{\mathcal{H}^*_\infty(\mathbb{R})}e^{i\langle W,f\rangle}\mathbf{\Phi}_H(W)\,d\mu(W)\\&=\left\langle\left\langle\mathbf{\Phi}_H,e^{i\langle\cdot,f\rangle}\right\rangle\right\rangle=T\mathbf{\Phi}_H(f)=\exp\left\{\int_{\mathbb{Q}_p^N}\digamma(f(x))\,d^Nx\right\},\end{aligned}\tag{3.20}$$

for $f\in\mathcal{H}_\infty(\mathbb{R})$. We now take $f(x)=t1_{\mathbb{Z}_p^N}(x)$, where $t\in\mathbb{R}$ and $1_{\mathbb{Z}_p^N}$ is the characteristic function of $\mathbb{Z}_p^N$. By using that $H(0)=0$, we have

$$\exp\left\{\int_{\mathbb{Q}_p^N}\digamma(f(x))\,d^Nx\right\}=\exp\digamma(t). \tag{3.21}$$

Now, we consider the random variable:

$$\begin{aligned}\left\langle\cdot,1_{\mathbb{Z}_p^N}\right\rangle:\ \ (\mathcal{H}^*_\infty(\mathbb{R}),\mathcal{B},\mathrm{P}_H)\ &\to\ (\mathbb{R},\mathcal{B}(\mathbb{R}))\\ W\ &\to\ \left\langle W,1_{\mathbb{Z}_p^N}\right\rangle,\end{aligned}$$

with probability distribution $\nu_{\left\langle\cdot,1_{\mathbb{Z}_p^N}\right\rangle}(A)=\mathrm{P}_H\left\{W\in\mathcal{H}^*_\infty(\mathbb{R});\left\langle W,1_{\mathbb{Z}_p^N}\right\rangle\in A\right\}$, where A is a Borel subset of $\mathbb{R}$. Then, by (3.20)-(3.21),

$$\int_{\mathcal{H}^*_\infty(\mathbb{R})}e^{it\langle W,f\rangle}d\mathrm{P}_H(W)=\int_{\mathbb{R}}e^{itz}d\nu_{\left\langle\cdot,1_{\mathbb{Z}_p^N}\right\rangle}(z)=\exp\digamma(t). \tag{3.22}$$

■

We call these measures *generalized white noise measures*. The moments of the measure P_H are the Schwinger functions $\left\{\mathcal{S}_n^{\mathbf{\Phi}_H}\right\}_{n\in\mathbb{N}}$.

Since $\left(\mathcal{G}_{\alpha,m}f\right)(x) := G\left(x;\alpha,m\right)*f\left(x\right)$ gives rise to a continuous mapping from $\mathcal{H}_{\mathbb{R}}(\infty)$ into itself, then, the conjugate operator $\widetilde{\mathcal{G}}_{\alpha,m} : \mathcal{H}_{\mathbb{R}}^{*}(\infty) \to \mathcal{H}_{\mathbb{R}}^{*}(\infty)$ is a measurable mapping from $\left(\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right),\mathcal{B}\right)$ into itself. For the sake of simplicity, we use $\mathcal{G}$ instead of $\mathcal{G}_{\alpha,m}$ and G instead of $G\left(x;\alpha,m\right)$. We set P_H^G to be the image probability measure of P_H under $\widetilde{\mathcal{G}}$, i.e. P_H^G is the measure on $\left(\mathcal{H}_{\mathbb{R}}^{*}\left(\infty\right),\mathcal{B}\right)$ defined by

$$\mathrm{P}_H^G\left(A\right) = \mathrm{P}_H\left(\widetilde{\mathcal{G}}^{-1}\left(A\right)\right), \text{ for } A\in\mathcal{B}. \tag{3.23}$$

The Fourier transform of P_H^G is given by

$$\int_{\mathcal{H}_{\infty}^{*}(\mathbb{R})} e^{i\langle W,f\rangle} d\mathrm{P}_H^G\left(W\right) = \exp\left\{\int_{\mathbb{Q}_p^N} F\left\{\int_{\mathbb{Q}_p^N} G\left(x-y;\alpha,m\right)f\left(y\right)d^N y\right\}d^N x\right\}, \tag{3.24}$$

for $f\in\mathcal{H}_{\mathbb{R}}\left(\infty\right)$, where F is given as in (3.17), cf. [66, Proposition 6.2], alternatively [35, Proposition 11.12]. Finally, (3.24) is also valid if we replace $G = G\left(x;\alpha,m\right)$ by $G\left(x;\alpha,\beta,m\right)$.

3.5 The free Euclidean Bose field

An important difference between the real and p-adic Euclidean quantum field theories comes from the 'ellipticity' of the quadratic form $\mathfrak{q}_N\left(\xi\right) = \xi_1^2+\cdots+\xi_N^2$. In the real case $\mathfrak{q}_N\left(\xi\right)$ is elliptic for any $N\geq 1$. In the p-adic case, $\mathfrak{q}_N\left(\xi\right)$ is not elliptic for $N\geq 5$. In the case $N=4$, there is a unique elliptic quadratic form, up to linear equivalence, which is $\xi_1^2 - s\xi_2^2 - p\xi_3^2 + s\xi_4^2$, where $s\in\mathbb{Z}\smallsetminus\{0\}$ is a quadratic non-residue, i.e. $\left(\frac{s}{p}\right)=-1$.

3.5.1 The Archimedean free covariance function

The free covariance function $C(x-y;m) := C(x-y)$ is the solution of the Laplace equation

$$\left(-\Delta+m^2\right)C(x-y) = \delta\left(x-y\right),$$

where $\Delta = \sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}$. As a distribution from $\mathcal{S}'(\mathbb{R}^N)$, the free covariance is given by

$$C(x-y) = \frac{1}{\left(2\pi\right)^N}\int\limits_{\mathbb{R}^N}\frac{\exp\left(-ik\cdot\left(x-y\right)\right)}{k^2+m^2}d^N k,$$

where $k,\, x,\, y \in \mathbb{R}^N$, $d^N k$ is the Lebesgue measure of $\mathbb{R}^N$, $k^2 = k \cdot k$, and $k \cdot x = \sum_{i=1}^{N} k_i x_i$. Notice that the quadratic form used in the definition of the Fourier transform k^2 is the same as the one used in the propagator $\frac{1}{k^2+m^2}$, this situation does not occur in the p-adic case. In particular the group of symmetries of $C(x-y)$ is the $SO(N,\mathbb{R})$. The function $C(x-y)$ has the following properties (see [18, Proposition 7.2.1].):
(i) $C(x-y)$ is positive and analytic for $x-y \neq 0$;
(ii) $C(x-y) \leq \exp\left(-m\left\|x-y\right\|\right)$ as $\left\|x-y\right\| \to \infty$;
(iii) for $N \geq 3$ and $m\left\|x-y\right\|$ in a neighborhood of zero,

$$C(x-y) \sim \left\|x-y\right\|^{-N+2},$$

(iv) for $N = 2$ and $m\left\|x-y\right\|$ in a neighborhood of zero,

$$C(x-y) \sim -\ln\left(m\left\|x-y\right\|\right).$$

3.5.2 The Archimedean free Euclidean Bose field

Take H_m to be the Hilbert space defined as the closure of $\mathcal{S}(\mathbb{R}^N)$ with respect to the norm $\left\|\cdot\right\|_m$ induced by the scalar product

$$(f,g)_m := \int_{\mathbb{R}^N} f\left(x\right)\left(-\Delta+m^2\right)^{-1} g\left(x\right) d^N x = \left(f, \left(-\Delta+m^2\right)^{-1} g\right)_{L^2(\mathbb{R}^N)}.$$

Then $\mathcal{S}(\mathbb{R}^N) \hookrightarrow H_m \hookrightarrow \mathcal{S}'(\mathbb{R}^N)$ form a Gel'fand triple. The probability space $\left(\mathcal{S}'(\mathbb{R}^N), \mathcal{B}, \nu\right)$, where ν is the centered Gaussian measure on $\mathcal{B}$ (the σ-algebra of cylinder sets) with covariance

$$\int_{\mathcal{S}'(\mathbb{R}^N)} \langle W, f\rangle \langle W, g\rangle \, d\nu\left(W\right) = \left(f, \left(-\Delta+m^2\right)^{-1} g\right)_{L^2(\mathbb{R}^N)},$$

for $f,\, g \in \mathcal{S}(\mathbb{R}^N)$, jointly with the coordinate process $W \to \langle W, f\rangle$, with fixed $f \in \mathcal{S}(\mathbb{R}^N)$, is called the free Euclidean Bose field of mass m in N dimensions.

3.5.3 The non-Archimedean free covariance function

The p-adic free covariance $C_p(x-y;m) := C_p(x-y)$ is the solution of the pseudodifferential equation

$$\left(\boldsymbol{L}_\alpha + m^2\right) C(x-y) = \delta\left(x-y\right),$$

where $\boldsymbol{L}_\alpha$ is the pseudodifferential operator defined in (3.7). As a distribution from $\mathcal{D}'(\mathbb{Q}_p^N)$, the free covariance is given by

$$C_p(x-y) = \int\limits_{\mathbb{Q}_p^N} \frac{\chi_p\left(-\xi\cdot(x-y)\right)}{\left|\mathfrak{l}(\xi)\right|_p^\alpha + m^2} d^N\xi,$$

where k, x, $y \in \mathbb{Q}_p^N$, $d^N\xi$ is the Haar measure of $\mathbb{Q}_p^N$, $\mathfrak{l}(k)$ is an elliptic polynomial of degree d, and $k\cdot x = \sum_{i=1}^N k_i x_i$. In this case $\mathfrak{l}(k) \neq k\cdot k$, and then the symmetries of $C_p(x-y)$ form a subgroup of the p-adic orthogonal group attached to the quadratic form $k\cdot k$. There are other possible propagators, for instance

$$\frac{1}{\left(\left|\mathfrak{l}(k)\right|_p + m^2\right)^\alpha},\ \alpha > 0.$$

For a discussion on the possible scalar propagators, in the p-adic setting, the reader may consult [50].

The function $C_p(x-y)$ satisfies (see [66, Proposition 4.1], or [35, Proposition 11.1]):
(i) $C_p(x-y)$ is positive and locally constant for $x-y\neq 0$;
(ii) $C_p(x-y) \leq C\left\|x-y\right\|_p^{-\alpha d - N}$ as $\left\|x-y\right\|_p \to \infty$;
(iii) for $0<\alpha d<N$ and $\left\|x-y\right\|_p \leq 1$,

$$C_p(x-y) \leq C\left\|x-y\right\|_p^{\alpha d - N};$$

(iv) for $N = \alpha d$ and $\left\|x-y\right\|_p \leq 1$,

$$C_p(x-y) \leq C_0 - C_1 \ln\left\|x-y\right\|_p.$$

3.6 The non-Archimedean free Euclidean Bose field

Take H_m to be the Hilbert space defined as the closure of $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ with respect to the norm $\left\|\cdot\right\|_m$ induced by the scalar product

$$(f,g)_m := \int_{\mathbb{Q}_p^N} \overline{\widehat{f}(\xi)}\widehat{g}(\xi)\,\frac{d^N\xi}{\left|\mathfrak{l}(\xi)\right|_p^\alpha + m^2} = \left(f, \left(\boldsymbol{L}_\alpha + m^2\right)^{-1} g\right)_{L^2_{\mathbb{R}}(\mathbb{Q}_p^N)}.$$

By using that

$$C_0\left[\xi\right]_p^{\lfloor d\alpha\rfloor} \leq \left|\mathfrak{l}(\xi)\right|_p^\alpha + m^2 \leq C_1\left[\xi\right]_p^{\lceil d\alpha\rceil},$$

where $\lceil t\rceil = \min\{m \in \mathbb{Z}; m \geq x\}$ and $\lfloor t\rfloor = \max\{m \in \mathbb{Z}; m \leq x\}$, we have

$$\mathcal{H}_{-\lfloor d\alpha\rfloor}(\mathbb{R}) \hookrightarrow H_m \hookrightarrow \mathcal{H}_{-\lceil d\alpha\rceil}(\mathbb{R}).$$

Then $\mathcal{H}_\infty(\mathbb{R}) \hookrightarrow H_m \hookrightarrow \mathcal{H}_\infty^*(\mathbb{R})$ from a Gel'fand triple. The probability space $(\mathcal{H}_\infty^*(\mathbb{R}), \mathcal{B}, \nu_{d,\alpha})$, where $\nu_{d,\alpha}$ is the centered Gaussian measure on $\mathcal{B}$ (the σ-algebra of cylinder sets) with covariance

$$\int_{\mathcal{H}_\infty^*(\mathbb{R})} \langle W, f\rangle \langle W, g\rangle \, d\nu_{d,\alpha}(W) = \left(f, \left(\boldsymbol{L}_\alpha + m^2\right)^{-1} g\right)_{L^2_{\mathbb{R}}(\mathbb{Q}_p^N)},$$

for f, $g \in \mathcal{H}_\infty(\mathbb{R})$, jointly with the coordinate process $W \to \langle W, f\rangle$, with fixed $f \in \mathcal{H}_\infty(\mathbb{R})$), is called the non-Archimedean free Euclidean Bose field of mass m in N dimensions.

If $N = 4$ and $d = 2$, then there is a unique elliptic quadratic form up to linear equivalence. If $N \geq 5$ and $\mathfrak{l}(\xi)$ is an elliptic polynomial of degree d, then $|\mathfrak{l}(\xi)|_p^{\frac{2}{d}}$ is a homogeneous function of degree 2 that vanishes only at the origin. We can use this function as the symbol for a pseudodifferential operator, such operator is a p-adic analogue of $-\Delta$ in dimension N.

If we use the propagator $\frac{1}{\left(|\mathfrak{l}(k)|_p + m^2\right)^\alpha}$ instead of $\frac{1}{|\mathfrak{l}(k)|_p^\alpha + m^2}$, similar results are obtained due to the fact that

$$\text{and } C_0' \left[k\right]_p^{\lfloor d\alpha\rfloor} \leq \left(|\mathfrak{l}(k)|_p + m^2\right)^\alpha \leq C_1' \left[k\right]_p^{\lceil d\alpha\rceil}.$$

We prefer using propagator $\frac{1}{|\mathfrak{l}(k)|_p^\alpha + m^2}$ because the corresponding 'Laplace equation' has been studied extensively in the literature. On the other hand, $\frac{\partial u(x,t)}{\partial t} + \boldsymbol{L}_\alpha u(x,t) = 0$, with $x \in \mathbb{Q}_p^N$, $t > 0$, behaves like a 'heat equation', i.e. the semigroup associated to this equation is a Markov semigroup, see [67, Chapter 2], which means that $-\boldsymbol{L}_\alpha$ can be considered as p-adic version of the Laplacian.

3.7 Symmetries

Given a polynomial $\mathfrak{a}(\xi) \in \mathbb{Q}_p[\xi_1, \cdots, \xi_n]$ and $\Lambda \in GL_N(\mathbb{Q}_p)$, we say that Λ *preserves* $\mathfrak{a}$ if $\mathfrak{a}(\xi) = \mathfrak{a}(\Lambda\xi)$, for all $\xi \in \mathbb{Q}_p^N$. By simplicity, we use Λx to mean $[\Lambda_{ij}]\, x^T$, $x = (x_1, \cdots, x_N) \in \mathbb{Q}_p^N$, where we identify Λ with the matrix $[\Lambda_{ij}]$.

Let $\mathfrak{q}_N(\xi) = \xi_1^2 + \cdots + \xi_N^2$ be the elliptic quadratic form used in the definition of the Fourier transform, and let $\mathfrak{l}(\xi)$ be the elliptic polynomial that appears in the symbol of the operator $\boldsymbol{L}_\alpha$. We define the homogeneous Euclidean group of $\mathbb{Q}_p^N$ relative to $\mathfrak{q}(\xi)$ and

$\mathfrak{l}(\xi)$, denoted as $E_0\left(\mathbb{Q}_p^N\right) := E_0\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right)$, as the subgroup of $GL_N\left(\mathbb{Q}_p\right)$ whose elements preserve $\mathfrak{q}(\xi)$ and $\mathfrak{l}(\xi)$ simultaneously. Notice that if $\mathbb{O}(\mathfrak{q}_N)$ is the orthogonal group of $\mathfrak{q}_N$, then $E_0\left(\mathbb{Q}_p^N\right)$ is a subgroup of $\mathbb{O}(\mathfrak{q}_N)$. We define the inhomogeneous Euclidean group, denoted as $E\left(\mathbb{Q}_p^N\right) := E\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right)$, to be the group of transformations of the form $(a, \Lambda)\, x = a + \Lambda x$, for $a, x \in \mathbb{Q}_p^N$, $\Lambda \in E_0\left(\mathbb{Q}_p^N\right)$.

In the real case $\mathfrak{q}_N = \mathfrak{l}(\xi)$ and thus the homogeneous Euclidean group is $SO(N, \mathbb{R})$. In the p-adic case, $E_0\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right)$ is a subgroup of $\mathbb{O}(\mathfrak{q}_N)$, in addition, it is not a straightforward matter to decide whether or not $E_0\left(\mathbb{Q}_p^N; \mathfrak{q}, \mathfrak{l}\right)$ is non trivial. For this reason, we approach the Green kernels in a different way than do in [19], which is based on [52].

Notice that $(a, \Lambda)^{-1} x = \Lambda^{-1}(x - a)$. Let (a, Λ) be a transformation in $E\left(\mathbb{Q}_p^N\right)$, the action of (a, Λ) on a function $f \in \mathcal{H}_\infty$ is defined by

$$((a, \Lambda) f)(x) = f\left((a, \Lambda)^{-1} x\right), \text{ for } x \in \mathbb{Q}_p^N,$$

and on a functional $W \in \mathcal{H}_\infty^*$, by

$$\langle (a, \Lambda) W, f \rangle := \left\langle W, (a, \Lambda)^{-1} f \right\rangle, \text{ for } f \in \mathcal{H}_\infty(\mathbb{R}).$$

These definitions can be extended to elements of the spaces $\mathcal{H}_\infty^{\otimes n}$ and $\mathcal{H}_\infty^{*\otimes n}$, by taking

$$(a, \Lambda)(f_1 \otimes \cdots \otimes f_n) := (a, \Lambda)^{-1} f_1 \otimes \cdots \otimes (a, \Lambda)^{-1} f_n.$$

In general, if $F : \mathcal{H}_\infty^{\otimes n} \to \mathcal{X}$ is linear $\mathcal{X}$-valued functional, where $\mathcal{X}$ is a vector space, we define

$$((a, \Lambda) F)(f_1 \otimes \cdots \otimes f_n) = F((a, \Lambda)(f_1 \otimes \cdots \otimes f_n)),$$

and we say that F is *Euclidean invariant* if and only if $(a, \Lambda) F = F$ for any $(a, \Lambda) \in E\left(\mathbb{Q}_p^N\right)$.

Definition 14 *We call a distribution* $\mathbf{\Phi} = \sum_{n=0}^{\infty} \langle \Phi_n, : \cdot^{\otimes n} : \rangle \in (\mathcal{H}_\infty)^{-1}$, *with* $\Phi_n \in \mathcal{H}_\infty^{*\widehat{\otimes} n}$, *Euclidean invariant if and only if the functional* $\langle \Phi_n, \cdot \rangle$ *is Euclidean invariant for any* $n \in \mathbb{N}$.

It follows from this definition that $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$ is Euclidean invariant if and only if $S\mathbf{\Phi}$ and $T\mathbf{\Phi}$ are Euclidean invariant.

4

Schwinger functions and convoluted white noise

We set $G := G(x; m, \alpha)$ for the Green function (3.9). For $\mathbf{\Phi} \in (\mathcal{H}_\infty)^{-1}$, we define $\mathbf{\Phi}^G$ as

$$\left(T\mathbf{\Phi}^G\right)(g) = (T\mathbf{\Phi})(G * g), \ \ g \in \mathcal{U}, \tag{4.1}$$

where $\mathcal{U}$ is an open neighborhood of zero. Since $\mathcal{G} : \mathcal{H}_\infty(\mathbb{R}) \to \mathcal{H}_\infty(\mathbb{R})$, see (3.11), is linear and continuous, cf. [35, Corollary 11.3], by the characterization theorem, cf. [26, Theorem 3], or Section 2.5, $\mathbf{\Phi}^G$ is a well-defined and unique element of $(\mathcal{H}_\infty)^{-1}$.

Remark 15 *By using that $\langle \delta_x, G * f\rangle = \langle G * \delta_x, f\rangle$ for any $f \in \mathcal{H}_\infty$, we have that the white-noise process introduced in Section 3.2 satisfies*

$$\langle\langle G * \mathbf{\Phi}(x), \Psi\rangle\rangle = \langle\langle \mathbf{\Phi}(x), G * \Psi\rangle\rangle,$$

because $\langle\langle \mathbf{\Phi}(x), G * \Psi\rangle\rangle = \langle \delta_x, G * \psi_1\rangle$, where $\Psi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, \psi_n\rangle$, $\psi_n \in \mathcal{H}_\infty^{\widehat{\otimes} n}$.

We denote by $\left\{S_n^{H,G}\right\}_{n\in N}$ the Schwinger functions attached to $\mathbf{\Phi}_H^G$.

Theorem 16 *With H and $\mathbf{\Phi}_H$ as in Theorem 9, then distribution $\mathbf{\Phi}_H^G \in (\mathcal{H}_\infty)^{-1}$ is Euclidean invariant and is given by*

$$\mathbf{\Phi}_H^G = \exp^{\Diamond}\left(-\int_{\mathbb{Q}_p^N} H^{\Diamond}(G * \mathbf{\Phi}(x))\, d^N x + \frac{1}{2}\left\langle \left(\mathcal{G}^{\otimes 2} - 1\right) Tr, : \cdot^{\otimes 2} :\right\rangle\right), \tag{4.2}$$

where $Tr \in \left(\mathcal{H}_\infty^*\left(\mathbb{Q}_p^N, \mathbb{C}\right)\right)^{\widehat{\otimes} 2}$ denotes the trace kernel defined by $\langle Tr, f \otimes g\rangle = \langle f, g\rangle_0$, f, $g \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$. The Schwinger functions $\left\{S_n^{H,G}\right\}_{n\in N}$ satisfy the conditions (OS1) and (OS4) given in Lemma 7, and

$$\text{(Euclidean invariance)} \quad S_n^{H,G}((a, \Lambda) f) = S_n^{H,G}(f), \ f \in (\mathcal{H}_\infty(\mathbb{C}))^{\otimes n}, \tag{OS2}$$

for any $(a,\Lambda)\in E\left(\mathbb{Q}_p^N\right)$.

Proof. By definition (4.1) and Theorem 9-(iii), we have

$$\left(T\mathbf{\Phi}_H^G\right)(g)=\exp\left(-\int_{\mathbb{Q}_p^N} H(iG*g\left(x\right))+\frac{1}{2}\left(G*g\left(x\right)\right)^2\ d^Nx\right). \tag{4.3}$$

On the other hand, by taking the T-transform in (4.2) and using (3.4) and Remarks 11-15, we obtain

$$\left(T\mathbf{\Phi}_H^G\right)(g)=\exp\left(\frac{-1}{2}\left\|g\right\|_0^2\right)\times$$
$$\exp\left\{-S\left(\int_{\mathbb{Q}_p^N} H^{\Diamond}\left(G*\mathbf{\Phi}\left(x\right)\right)d^Nx\right)(ig)-\frac{1}{2}S\left(\left\langle\left(\mathcal{G}^{\otimes 2}-1\right)Tr,:\cdot^{\otimes 2}:\right\rangle\right)(ig)\right\}$$

$$=\exp\left(\frac{-1}{2}\left\|g\right\|_0^2\right)\times$$
$$\exp\left\{-\int_{\mathbb{Q}_p^N} H\left(iG*g\left(x\right)\right)d^Nx\right\}\exp\left\{\frac{1}{2}S\left(\left\langle\left(\mathcal{G}^{\otimes 2}-1\right)Tr,:\cdot^{\otimes 2}:\right\rangle\right)(ig)\right\}$$
$$=\exp\left(\frac{-1}{2}\left\|g\right\|_0^2\right)\exp\left\{-\int_{\mathbb{Q}_p^N} H\left(iG*g\left(x\right)\right)d^Nx-\frac{1}{2}\left\langle\left(\mathcal{G}^{\otimes 2}-1\right)Tr,g\otimes g\right\rangle\right\}$$
$$=\exp\left(\frac{-1}{2}\left\|g\right\|_0^2\right)\exp\left\{-\int_{\mathbb{Q}_p^N} H\left(iG*g\left(x\right)\right)d^Nx-\frac{1}{2}\left\langle Tr,G*g\otimes G*g-g\otimes g\right\rangle\right\}$$

$$=\exp\left\{-\int_{\mathbb{Q}_p^N} H\left(iG*g\left(x\right)\right)d^Nx-\frac{1}{2}\left\langle G*g\otimes G*g\right\rangle_0\right\} \tag{4.4}$$

Formula (4.2) follows from (4.3)-(4.4). Since $\mathbb{E}_\mu(\mathbf{\Phi}_H^G)=1$, conditions (OS1) and (OS4) follow from Lemma 7, and condition (OS2) follows from Lemma 6 by using the Euclidean invariance of $\mathbf{\Phi}_H^G$. ■

Remark 17 *(i) Set* $\mathcal{G}_{\frac{1}{2}}:=\mathcal{G}_{\alpha,\frac{1}{2},m}=\left(\boldsymbol{L}_\alpha+m^2\right)^{-\frac{1}{2}}$, *and* $\mathcal{G}_{\frac{1}{2}}\left(f\right):=G_{\frac{1}{2}}*f$ *for* $f\in\mathcal{H}_\infty(\mathbb{R})$. *By taking* $H\equiv 0$, *we obtain the free Euclidean field. Indeed,* $f\rightarrow\exp\left\{-\frac{1}{2}\left\langle G_{\frac{1}{2}}*f,G_{\frac{1}{2}}*f\right\rangle_0\right\}$ *defines a characteristic functional. Let denote by* $\nu_{G_{\frac{1}{2}}}$ *the probability measure on* $(\mathcal{H}_\infty^*(\mathbb{R}),\mathcal{B})$

provided by the Bochner-Minlos theorem. Then

$$\left(T\mathbf{\Phi}_0^{G_{\frac{1}{2}}}\right)(g) = \exp\left\{-\frac{1}{2}\left\langle G_{\frac{1}{2}} * g, G_{\frac{1}{2}} * g\right\rangle_0\right\}$$
$$= \left\langle\left\langle \mathbf{\Phi}_0^{G_{\frac{1}{2}}}, \exp i\langle\cdot, g\rangle\right\rangle\right\rangle = \int_{\mathcal{H}_\infty^*(\mathbb{R})} \exp i\langle W, g\rangle \, d\nu_{G_{\frac{1}{2}}}(W).$$

(ii)Assuming that $\digamma(t)$, see (3.17), is a Lévy characteristic, Theorem 16 implies that the probability measure P_H^G, see (3.23), admits $\mathbf{\Phi}_H^G$ as a generalized density with respect to to white noise measure μ, i.e. $\mathrm{P}_H^G = \mathbf{\Phi}_H^G \mu$. Indeed, by (3.24) and (4.4), we have

$$\int_{\mathcal{H}_\infty^*(\mathbb{R})} e^{i\langle W, f\rangle} d\mathrm{P}_H^G(W) = \exp\left\{\int_{\mathbb{Q}_p^N} \digamma\left(G(x;\alpha,m) * f(x)\right) y d^N x\right\}$$
$$= \exp\left\{-\int_{\mathbb{Q}_p^N} H\left(iG * f(x)\right) d^N x - \frac{1}{2}\langle G * f, G * f\rangle_0\right\} = \left(T\mathbf{\Phi}_H^G\right)(f)$$
$$= \left\langle\left\langle \mathbf{\Phi}_H^G, \exp i\langle\cdot, f\rangle\right\rangle\right\rangle.$$

4.1 Truncated Schwinger functions and the cluster property

We denote by $P^{(n)}$ the collection of all partitions I of $\{1, \ldots, n\}$ into disjoint subsets.

Definition 18 *Let $\left\{S_n^{H,G}\right\}_{n\in\mathbb{N}}$ be a sequence of Schwinger functions, with $S_0^{H,G} = 1$, and $S_n^{H,G} \in \mathcal{H}_\infty^*\left(\mathbb{Q}_p^{Nn}, \mathbb{C}\right)$ for $n \geq 1$. The truncated Schwinger functions $\left\{S_{n,T}^{H,G}\right\}_{n\in\mathbb{N}}$ are defined recursively by the formula*

$$S_n^{H,G}(f_1 \otimes \cdots \otimes f_n) = \sum_{I\in P^{(n)}} \prod_{\{j_1,\ldots,j_l\}} S_{l,T}^{H,G}\left(f_{j_1} \otimes \cdots \otimes f_{j_l}\right),$$

for $n \geq 1$. Here for $\{j_1, \ldots, j_l\} \in I$ we assume that $j_1 < \ldots < j_l$.

Remark 19 *By the kernel theorem, the sequence $\left\{S_n^{H,G}\right\}_{n\in\mathbb{N}}$ uniquely determines the sequence $\left\{S_{n,T}^{H,G}\right\}_{n\in\mathbb{N}}$ and vice versa. All the $S_n^{H,G}$ are Euclidean (translation) invariant if and only if all the $S_{n,T}^{H,G}$ are Euclidean (translation) invariant. The same equivalence holds for 'temperedness' (i.e. membership to $(\mathcal{H}_\infty)^{-1}$).*

Definition 20 *Let $a \in \mathbb{Q}_p^N$, $a \neq 0$, and $\lambda \in \mathbb{Q}_p$. Let $T_{a\lambda}$ denote the representation of the translation by $a\lambda$ on $\mathcal{H}_\infty\left(\mathbb{Q}_p^{Nn}, \mathbb{R}\right)$. Take n, $m \geq 1$, $f_1, \cdots, f_n \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$.*

*(OS5)(**Cluster property**) A sequence of Schwinger functions* $\left\{S_n^{H,G}\right\}_{n\in\mathbb{N}}$ *has the cluster property if for all* n, $m\geq 1$, *it verifies that*

$$\lim_{|\lambda|_p\to\infty}\left\{S_{m+n}^{H,G}\left(f_1\otimes\cdots\otimes f_m\otimes T_{a\lambda}\left(f_{m+1}\otimes\cdots\otimes f_{m+n}\right)\right)\right\} \tag{4.5}$$
$$=S_m^{H,G}\left(f_1\otimes\cdots\otimes f_m\right)S_n^{H,G}\left(f_{m+1}\otimes\cdots\otimes f_{m+n}\right).$$

*(**Cluster property of truncated Schwinger functions**) A sequence of truncated Schwinger functions* $\left\{S_{n,T}^{H,G}\right\}_{n\in\mathbb{N}}$ *has the cluster property, if for all* n, $m\geq 1$, *it verifies that*

$$\lim_{|\lambda|_p\to\infty}S_{m+n,T}^{H,G}\left(f_1\otimes\cdots\otimes f_m\otimes T_{a\lambda}\left(f_{m+1}\otimes\cdots\otimes f_{m+n}\right)\right)=0. \tag{4.6}$$

Remark 21 *In the Archimedean case, it is possible to replace* $\lim_{\lambda\to\infty}(\cdot)$ *in (4.5) and (4.6) by* $\lim_{\lambda\to\infty}|\lambda|^m(\cdot)$ *for arbitrary* m, *cf. [3, Remark 4.4]. This is possible because Schwartz functions decay at infinity faster than any polynomial function. This is not possible in the p-adic case, because the elements of our 'p-adic Schwartz space* $\mathcal{H}_\infty\left(\mathbb{Q}_p^N,\mathbb{R}\right)$*' only have a polynomial decay at infinity. For instance, consider the one-dimensional p-adic heat kernel* $Z(x;t)=\mathcal{F}^{-1}_{\xi\to x}\left(e^{-t|\xi|_p^\alpha}\right)$, *for* $t>0$, *and* $\alpha>0$, *which is an element of* $\mathcal{H}_\infty\left(\mathbb{Q}_p,\mathbb{R}\right)$. *The Fourier transform* $e^{-t|\xi|_p^\alpha}$ *of* $Z(x;t)$ *decays faster that any polynomial function in* $|\xi|_p$. *However,* $Z(x;t)$ *has only a polynomial decay at infinity, more precisely,*

$$Z(x;t)\leq C\frac{t}{\left(t^{\frac{1}{\alpha}}+|x|_p\right)^{\alpha+1}},\ t>0,\ x\in\mathbb{Q}_p,$$

cf. [25, Lemma 4.1].

Lemma 22 *Let* $H(z)=\sum_{n=0}^\infty H_nz^n$, $z\in U\subset\mathbb{C}$, *and* G *as in Theorem 16, and* $f_1,\cdots,f_n\in\mathcal{H}_\infty\left(\mathbb{Q}_p^N,\mathbb{R}\right)$. *Assume that* $F(t)=-H(it)-\frac{1}{2}t^2$, $t\in\mathbb{R}$ *is a Lévy characteristic, then the truncated Schwinger functions are given by*

$$S_{n,T}^{H,G}\left(f_1\otimes\cdots\otimes f_n\right)=\begin{cases}-H_n\int_{\mathbb{Q}_p^N}\prod_{i=1}^n G*f_i(x)\,d^Nx & \text{for}\quad n\geq 2\\ \\ (-H_2+1)\int_{\mathbb{Q}_p^N}G*f_1(x)\,G*f_2(x)\,d^Nx & \text{for}\quad n=2.\end{cases} \tag{4.7}$$

Proof. The result follows from the formula for the Schwinger functions given in Theorem 7.7 in [66], and the uniqueness of the truncated Schwinger functions. The coefficients in front of the integrals in (4.7) are the n-th derivatives of the Lévy characteristic divided by i^n. For the general H as in Theorem 16 these coefficients are the n-th derivatives of $-\left(H(iz)+\frac{1}{2}z^2\right)$, $z\in U$. ■

Lemma 23 *Assume that $\alpha d > N$. Let $\mathbf{\Phi}$, H, G as in Theorem 16. Then the sequence of truncated Schwinger functions $\left\{S_{n,T}^{H,G}\right\}_{n\in\mathbb{N}}$ has the cluster property.*

Proof. Fix $a \in \mathbb{Q}_p^N$ and take $\lambda \in \mathbb{Q}_p$, m, $n \geq 1$, $f_1, \cdots, f_{m+n} \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$. By Lemma 22, we have

$$\begin{aligned}&\left|S_{n,T}^{H,G}\left(f_1 \otimes \cdots \otimes f_n\right) \otimes T_{a\lambda}\left(f_{m+1} \otimes \cdots \otimes f_{m+n}\right)\right| \\ &= \left|-H_{m+n}\right| \left|\int_{\mathbb{Q}_p^N} \prod_{i=1}^{m} \left(G * f_i\right)(x) \prod_{i=m+1}^{m+n} T_{a\lambda}\left(G * f_i\right)(x)\right|\end{aligned}$$

We now use that $G * f_i \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$ and that $\mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right) \subset \mathcal{C}_0\left(\mathbb{Q}_p^N, \mathbb{R}\right)$ to get

$$\begin{aligned}&\left|S_{n,T}^{H,G}\left(f_1 \otimes \cdots \otimes f_n\right) \otimes T_{a\lambda}\left(f_{m+1} \otimes \cdots \otimes f_{m+n}\right)\right| \\ &\leq \left|H_{m+n}\right| \prod_{i=1}^{m} \left\|G * f_i\right\|_{L^\infty} \prod_{i=m+1}^{m+n-1} \left\|T_{a\lambda}\left(G * f_i\right)\right\|_{L^\infty} \int_{\mathbb{Q}_p^N} \left|T_{a\lambda}\left(G * f_{m+n}\right)(x)\right| d^N x.\end{aligned}$$

Now, the announced result follows from the following fact:

Claim. If $\alpha d > N$, for any $f \in \mathcal{H}_\infty\left(\mathbb{Q}_p^N, \mathbb{R}\right)$, it verifies that

$$\lim_{|\lambda|_p \to \infty} \int_{\mathbb{Q}_p^N} G\left(x - \lambda a - y\right)\left|f_{m+n}(y)\right| \; d^N y = 0.$$

Since $\alpha d > N$, by the Riemann-Lebesgue theorem, $G \in \mathcal{C}_0\left(\mathbb{Q}_p^N, \mathbb{R}\right)$, and consequently $G\left(x - \lambda a - y\right)\left|f_{m+n}(y)\right| \leq \left\|G\right\|_{L^\infty}\left|f_{m+n}(y)\right| \in L_{\mathbb{R}}^1\left(\mathbb{Q}_p^N\right)$. Now the Claim follows by applying the dominated convergence theorem. ■

Theorem 24 *With H, G and $\mathbf{\Phi}_H^G \in \left(\mathcal{H}_\infty\right)^{-1}$ as in Theorem 16. If $\alpha d > N$, then the sequence of Schwinger functions $\left\{S_n^{H,G}\right\}_{n\in\mathbb{N}}$ has the cluster property (OS5).*

Proof. In [3, Theorem 4.5] was established that the cluster property and the truncated cluster property are equivalent. By using this result, the announced result follows from Lemma 23. ■

Remark 25 *The class of Schwinger functions $\left\{S_n^{H,G}\right\}_{n\in\mathbb{N}}$ corresponding to a distribution $\mathbf{\Phi}_H^G \in \left(\mathcal{H}_\infty\right)^{-1}$ as in Theorem 16 differs of the class of Schwinger functions corresponding to the convoluted generalized white noise introduced in [66]. In order to explain the differences, let us compare the properties of the Levy characteristic used in [66] with the properties of the function H used in this article, where $\digamma(t) = -H(it) - \frac{1}{2}t^2$, $t \in U \subset \mathbb{R}$. We require only that function H be holomorphic at zero and $H(0) = 0$, as in [19]. This* *only impose a restriction in choosing the coefficients in front of the integrals corresponding to the n-th truncated Schwinger function, see (4.7). On the other hand in [66], the author requires the condition that the measure M has finite moments of all orders. This implies that $\digamma$ belongs to $C^\infty(\mathbb{R})$, but $\digamma$ does not have to have a holomorphic extension. Furthermore, since $\exp s\digamma(t)$ is positive definite for any $s > 0$, cf. [66, Proposition 5.5], and by using $\digamma(0) = 0$ and a result due Schoenberg, cf. [8, Theorem 7.8], we have $-\digamma(t) : \mathbb{R} \to \mathbb{C}$ is a negative definite analytic function. Since $|-\digamma(t)| \leq C|t|^2$ for any $|t| \geq 1$, [8, Corollary 7.16], we conclude that $-\digamma(t)$ is a polynomial of the degree at most 2, and then $H_n = 0$ for $n \geq 3$.*

A

A brief review of the $p-$adic analysis

The p-adic numbers were discovered by the German mathematician Kurt Hensel in 1897. The construction the field of p-adic numbers $\mathbb{Q}_p$ is very similar to the construction of the field of real numbers $\mathbb{R}$ (here p is a fixed prime number), both number fields are completions of the field of rational numbers $\mathbb{Q}$. While the real numbers $\mathbb{R}$, are the completion of $\mathbb{Q}$ with respect to the usual absolute value, denoted by $|\cdot|_\infty$, the p-adic numbers $\mathbb{Q}_p$ are obtained by completing $\mathbb{Q}$ with respect to the p-adic absolute value, denoted by $|\cdot|_p$.
The theory of p-adic numbers has received a lot of attention into several areas of mathematics, including number theory, algebraic geometry, algebraic topology and analysis. Recently in the literature, there are many articles where p-adic analysis is applied to other branches of the science such as physics, mathematical physics, biology and psychology, among others.

The conventional description of the physical space-time uses the field $\mathbb{R}$ of real numbers, and there are many mathematical models based on $\mathbb{R}$ that successfully describe physical reality. Nevertheless, there are general arguments that suggest that one cannot make measurements in regions of extent smaller than the Planck length $\approx 10^{-33}$ cm, see e.g.[56] . On the other hand, by Ostrowski theorem, see e.g. [2], it is natural to use he field $\mathbb{Q}_p$ of p-adic numbers instead of the real field $\mathbb{R}$, as a possible alternative to describe the structure of space-time. In [63]-[64], I. Volovich posed the conjecture of the non-Archimedean nature of the space-time at the level of the Planck scale. This conjecture has originated a lot research, for instance, in quantum mechanics, see e.g. [34], [60], [61], [65], in string theory, see e.g. [10], [17], [58], [59], and in quantum field theory, see e.g. [24],[41]. For a further discussion on non-Archimedean mathematical physics, the reader may consult [15], [62] and the references therein.

In this section we collect some basic results about p-adic analysis that will be used along this thesis. For an in-depth review of the p-adic analysis the reader may consult [2], [54], [62].

A.1 The field of p-adic numbers

Let p a fixed prime number. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases} \tag{A.1}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of x*.

Any $p-$adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j, \tag{A.2}$$

where $x_j \in \{0, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of $x \in \mathbb{Q}_p$*, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0 \\ \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j & \text{if} \quad ord(x) < 0. \end{cases} \tag{A.3}$$

In addition, any non-zero $p-$adic number can be represented uniquely as $x = p^{ord(x)} ac\,(x)$ where $ac\,(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the *angular component* of x. Notice that $|ac\,(x)|_p = 1$.

We extend the $p-$adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \text{ for } x = (x_1, \dots, x_N) \in \mathbb{Q}_p^N. \tag{A.4}$$

We define $ord(x) = \min_{1 \leq i \leq N}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a complete ultrametric space. For $r \in \mathbb{Z}$, denote by $B_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x-a||_p \leq p^r\}$ *the ball of radius p^r with center at $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$*, and take $B_r^N(0) := B_r^N$. Note that $B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^N equals the product of N copies of $B_0 = \mathbb{Z}_p$, *the ring of $p-$adic integers of $\mathbb{Q}_p$*. We also denote by $S_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x-a||_p = p^r\}$ *the sphere of radius p^r with center at $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$*, and take $S_r^N(0) := S_r^N$. We notice that $S_0^1 = \mathbb{Z}_p^{\times}$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^{\times}\right)^N \subsetneq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$, see e.g. [62, Section 1.3], or [2, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a locally compact topological space.

We will use $\Omega\left(p^{-r}||x-a||_p\right)$ to denote the characteristic function of the ball $B_r^N(a)$. We will use the notation 1_A for the characteristic function of a set A. Along the article $d^N x$ will denote a Haar measure on $\left(\mathbb{Q}_p^N, +\right)$ normalized so that $\int_{\mathbb{Z}_p^N} d^N x = 1$.

A.2 The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p^N$ is *called locally constant* if for any $x \in \mathbb{Q}_p^N$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x+x') = \varphi(x) \text{ for } x' \in B_{l(x)}^N. \tag{A.5}$$

Denote by $\mathcal{E}\left(\mathbb{Q}_p^N\right)$ the linear space of locally constant $\mathbb{C}$-value functions on $\mathbb{Q}_p^N$. A function $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D} := \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N) := \mathcal{D}(\mathbb{Q}_p^N)$. Note that one cannot define differential operators acting on complex-functions over $\mathbb{Q}_p^N$.

Remark 26 *Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls.*

Definition 27 *For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (A.5) is called the parameter of constancy of the function φ. Let us denote by $\mathcal{D}_M^l(\mathbb{Q}_p^N)$ the finite-dimensional space of test functions having supports in the ball B_M^N and with parameters of constancy $\geq l$.*

Remark 28 *The following embedding holds: $\mathcal{D}_L^l(\mathbb{Q}) \subset \mathcal{D}_M^{l'}(\mathbb{Q})$, $\quad M \geq L, l \geq l'$. These representations give us the inductive limit topology on the corresponding spaces:*

$$\mathcal{D}_M(\mathbb{Q}^{\mathbb{N}}) = \varinjlim_{-l,\ l\in\mathbb{N}} \mathcal{D}_M^l(\mathbb{Q}^N),\ \mathcal{D}(\mathbb{Q}_p^N) = \varinjlim_{M\in\mathbb{N}} \mathcal{D}_M(\mathbb{Q}^N).$$

The convergence in $\mathcal{D}(\mathbb{Q}_p^N)$ is defined in the following way: $\varphi_j \to 0$ in $\mathcal{D}(\mathbb{Q}_p^N)$ as $j \to \infty$ if and only if there are M, l that such that $\varphi_j \in \mathcal{D}_M^l(\mathbb{Q}_p^N)$, where M, l are two numbers, not depending on j, and $\varphi_j \to 0$ uniformly in $\mathcal{D}_M^l(\mathbb{Q}^N)$.

A.2.1 The Fourier transform of test functions

Set $\chi_p(y) := \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0+x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$, the isomorphism is given by $\xi \to \chi_p(\xi x)$, see e.g. [2, Section 2.3].

Given $\xi = (\xi_1, \ldots, \xi_N)$ and $y = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$, we set $\xi \cdot x := \sum_{j=1}^{N} \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x)\varphi(x)d^N x \quad \text{for } \xi \in \mathbb{Q}_p^N, \tag{A.6}$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [2], [62]. Moreover,

$$\varphi \in \mathcal{D}_M^l(\mathbb{Q}^N) \Leftrightarrow \mathcal{F}[\varphi] \in \mathcal{D}_{-l}^{-M}(\mathbb{Q}^N). \tag{A.7}$$

We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

If $f \in L^1$ its Fourier transform is defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x)f(x)d^N x, \quad \text{for } \xi \in \mathbb{Q}_p^N. \tag{A.8}$$

If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||_p \le p^k} \chi_p(\xi \cdot x)f(x)d^N x, \quad \text{for } \xi \in \mathbb{Q}_p^N, \tag{A.9}$$

where the limit is taken in L^2. We recall that the Fourier transform is unitary on L^2, i.e. $||f||_{L^2} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2$ and $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$ is also valid in L^2, see e.g. [54, Chapter III, Section 2].

For a more details on p-adic analysis we invite the reader to consult [2], [62]

A.3 The Bruhat–Schwartz distributions

Let $\mathcal{D}' := \mathcal{D}'(\mathbb{Q}_p^N)$ denote the set of all continuous functional (distributions) on $\mathcal{D}$. $\mathcal{D}'(\mathbb{Q}_p^N)$ is a complete topological space. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^N) \times \mathcal{D}(\mathbb{Q}_p^N) \to \mathbb{C}$ is denoted as (ψ, φ) for $\psi \in \mathcal{D}'(\mathbb{Q}_p^N)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. The convergence in $\mathcal{D}'(\mathbb{Q}_p^N)$ is defined in the following way: $\psi_j \to 0$ in $\mathcal{D}'(\mathbb{Q}_p^N)$ as $j \to \infty$ if and only if $(\psi_j, \varphi) \to 0$ for any $\varphi \in \mathcal{D}$ see e.g. [2], [62].

We will denote by $\mathcal{D}_{\mathbb{R}} := \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$, the $\mathbb{R}$-vector space of test functions, and by $\mathcal{D}'_{\mathbb{R}} := \mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p^N)$, the $\mathbb{R}$-vector space of distributions.

Every $f \in \mathcal{E}\left(\mathbb{Q}_p^N\right)$, or more generally in L^1_{loc}, defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^N} f(x)\, \varphi(x)\, d^m x. \tag{A.10}$$

Such distributions are called *regular distributions*.

A.3.1 The Fourier transform of a distribution

The Fourier transform $\mathcal{F}[W]$ of a distribution $W \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ is defined by

$$(\mathcal{F}[W], \varphi) = (W, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^N). \tag{A.11}$$

The Fourier transform $W \to \mathcal{F}[W]$ is a linear isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$ onto itself. Furthermore, $W(\xi) = \mathcal{F}[\mathcal{F}[W](-\xi)]$. We also use the notation $\mathcal{F}_{x\to\xi}W$ and $\widehat{W}$ for the Fourier transform of W.

A.4 Some function Spaces

Given $\rho \in [0, \infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^N\right) := L^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, the $\mathbb{C}-$vector space of all the complex valued functions g satisfying $\int_{\mathbb{Q}_p^N} |g(x)|^\rho\, d^N x < \infty$, and $L^\infty := L^\infty\left(\mathbb{Q}_p^N\right) = L^\infty\left(\mathbb{Q}_p^N, d^N x\right)$ denotes the $\mathbb{C}-$vector space of all the complex valued functions g such that the essential supremum of $|g|$ is bounded. The corresponding $\mathbb{R}$-vector spaces are denoted as $L^\rho_{\mathbb{R}} := L^\rho_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) = L^\rho_{\mathbb{R}}\left(\mathbb{Q}_p^N, d^N x\right)$, $1 \leq \rho \leq \infty$.

Set

$$\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{C}) := \left\{ f : \mathbb{Q}_p^N \to \mathbb{C};\ f \text{ is continuous and } \lim_{||x||_p \to \infty} f(x) = 0 \right\}, \tag{A.12}$$

where $\lim_{||x||_p\to\infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in \mathbb{Q}_p^N \backslash B(\epsilon)$. We recall that $(\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{C}), || \cdot ||_{L^\infty})$ is a Banach space. The corresponding $\mathbb{R}$-vector space will be denoted as $\mathcal{C}_0(\mathbb{Q}_p^N, \mathbb{R})$.

B

A brief review of the white noise analysis

The Lebesgue measure plays a fundamental role in the integration theory in $\mathbb{R}^N$. Recall that this measure is uniquely determined (up to some constant) by the following conditions:

1. it assigns finite values to bounded Borel sets and positive numbers to non-empty open sets;

2. it is translation invariant.

It is well-known that these is no a Lebesgue measure on infinite dimensional vector spaces. For instance, if we consider a separable Hilbert space H and μ be a Borel measure in H. Assume that μ satisfies the above conditions 1 , 2. Let $\{\xi_1, \xi_2, \cdots\}$ be orthonormal basis of H. Let B_n be ball of radius $1/2$ centered at ξ_n, and B the ball of radius 2 centered at the origin. Then $0 < \mu(B_1) = \mu(B_2) = \mu(B_3) = \cdots < \infty$. Note that the B_ns are disjoint and contained in B. Therefore, we must have $\mu(B) \geq \sum_n \mu(B_n) = \infty$. This contradicts 1.

The study of Gaussian white noise as a mathematically rigorous object was initiated by T. Hida in 1975. This can be considered an infinite-dimensional analogue of the Schwartz distribution theory, where the role of the Lebesgue measure on $\mathbb{R}^N$ is played by the Gaussian measure μ on the dual of certain nuclear space $\mathcal{N}$.

The starting point of white noise analysis [21], [44] and [26] is a real separable Hilbert space H with inner product $(\cdot, \cdot)$ and the corresponding norm $\|\cdot\|$, and a Gel'fand triple $\mathcal{N} \subset H \subset \mathcal{N}^*$, where $\mathcal{N}$ is a nuclear space densely and continuously embedded in H. By the Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathfrak{B}(\mathcal{N}^*)$ ($\sigma-$algebra on $\mathcal{N}^*$ generated by the cylinder sets) with the following property:

$$\mathbb{E}\left(e^{i\langle W,\varphi\rangle}\right) = \int_{\mathcal{N}^*} e^{i\langle W,\varphi\rangle} d\mu(W) = e^{\left(-\frac{1}{2}\|\varphi\|^2\right)} . \tag{B.1}$$

The probability space $(\mathcal{N}^*, \mathfrak{B}(\mathcal{N}^*), \mu)$ is called Gaussian space associated with $\mathcal{N}$ and H.

In recent years Gaussian white noise analysis has become into a useful tool in applied mathematics and mathematical physics. For a detailed exposition of the theory and for many examples of applications we refer the reader to the monograph [21].

In this section we give a brief introduction into the concepts and results of white noise analysis used throughout this work, for an in-depth exposition the reader may consult [21], [22], [29] and [36].

B.1 Locally convex spaces

In this section, we review some of the basic properties of the locally convex spaces and the countably Hilbert spaces, for an in-depth exposition the reader may consult [45], [47], [55]. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$.

Definition 29 *A topological vector space, $\mathcal{V}$, is a $\mathbb{K}-$vector space equipped with a topology for which addition $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\cdot : \mathbb{K} \times \mathcal{V} \to \mathcal{V}$ are continuous.*

Let $V\ (0)$ be a neighborhood system of the origin. By using the continuity of the addition operation on $\mathcal{V}$, a neighborhood system a point a is obtained by translating V (0) to a, i.e. $V\ (a) = a + V\ (0)$.

Definition 30 *A locally convex space, $\mathcal{V}$, is a topological vector space $\mathcal{V}$ whose topology is generated by translations of balanced, absorbent, convex sets. Equivalently they can be defined as a topological vector space whose topology is defined by a family of seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$.*

We recall a set G is said to be balanced if for all $g \in G$ and all scalars λ, with $|\lambda| \leq 1$ we have $\lambda g \in G$. The set G is said to be absorbent if to every $v \in \mathcal{V}$, there is a number $\rho > 0$ such that $v \in \lambda G$ for all scalars λ such that $|\lambda| < \rho$. Assuming that G is balanced, in order that G be absorbent it suffices that to every $v \in \mathcal{V}$ there is $\rho > 0$ such that $v \in \rho G$.

Without changing the topology we may choose a directed family of seminorms for $\mathcal{V}$, which means that for any $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\|v\|_\alpha \leq \|v\|_\gamma$ and $\|v\|_\beta \leq \|v\|_\gamma$ for all $v \in \mathcal{V}$.

Theorem 31 ([45, Theorem V.2]) *Let $\mathcal{V}$ and $\mathcal{W}$ be locally convex spaces with families of semi-norms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$ and $\{\|\cdot\|_\beta\}_{\beta\in B}$. Then a linear map $T:\mathcal{V}\to\mathcal{W}$, is continuous if and only if for all $\beta\in B$, there are $\alpha_1,\alpha_2,\cdots,\alpha_n\in A$ and $C>0$ with*

$$\|Tv\|_\beta \le C\left\{\|v\|_{\alpha_1}+\|v\|_{\alpha_2}+\cdots+\|v\|_{\alpha_2}\right\}.$$

If the $\{\|\cdot\|_\alpha\}_{\alpha\in A}$ are directed, then T is continuous if and only if for all $\beta\in B$

$$\|Tv\|_\beta \le C\,\|v\|_\alpha\,,$$

for some $\alpha\in A$, and $C>0$.

Remark 32

1. *A locally convex space is metrizable if and only if it admits a countable set of semi-norms, see [45, Theorem V.5].*

2. *A locally convex space is called Fréchet if it is metrizable and complete.*

Remark 33 *For any topological linear space $\mathcal{V}$ on $\mathbb{R}$, we denote by $\mathcal{V}_\mathbb{C}$ its complexification, i.e., $\mathcal{V}_\mathbb{C}=\mathcal{V}+i\mathcal{V}$. By definition every element $\upsilon\in\mathcal{V}_\mathbb{C}$ can be decomposed into $\upsilon=\xi+i\eta$; $\xi,\eta\in\mathcal{V}$. The canonical bilinear form on $\mathcal{V}^*\times\mathcal{V}$, which is an $\mathbb{R}$-bilinear form, can be naturally extended to a $\mathbb{C}$-bilinear form on $\mathcal{V}^*_\mathbb{C}\times\mathcal{V}_\mathbb{C}$, which is denoted by the same symbol. Namely,*

$$\langle x+iy,\xi+i\eta\rangle=\langle x,\xi\rangle-\langle y,\eta\rangle+i(\langle x,\eta\rangle+\langle y,\xi\rangle),\ \ x,y\in\mathcal{V}^*,\xi,\eta\in\mathcal{V}. \tag{B.2}$$

If $\mathcal{H}$ is a real Hilbert space, then the inner product on $H_\mathbb{C}=H+i\mathcal{H}$ of f_1+ig_1 and f_2+ig_2 with $f_1,g_1f_2,g_2\in H$ is

$$(\ f_1+ig_1,f_2+ig_2)=(f_1,f_2)+(g_1,g_2)+i\left[(f_2,g_1)-(f_1,g_2)\right].$$

The canonical bilinear form on $H_\mathbb{C}\times H_\mathbb{C}$ is

$$\langle f_1+ig_1,f_2+ig_2\rangle=(f_1,f_2)-(g_1,g_2)+i\left[(f_2,g_1)+(f_1,g_2)\right],$$

it is related to the inner product on $H_\mathbb{C}$ as follows:

$$\langle F,G\rangle=(F,\overline{G}),\quad F,g\in H_\mathbb{C}.$$

B.1.1 Countably-Hilbert spaces and Nuclear spaces

For an in-depth discussion about countably-Hilbert space may consult [17], [44].

Definition 34 *A seminorm* $\|\cdot\|$ *on a vector space* $\mathcal{V}$ *over* $\mathbb{R}$ *(resp.* $\mathbb{C}$*) is called Hilbertian if it is induced by some non-negative, symmetric bilinear (resp. Hermitian sesquilinear) form* $(\cdot,\cdot)$ *on* $\mathcal{V}\times\mathcal{V}$*, namely if* $\|v\|^2=(v,v)$ *for all* $v\in\mathcal{V}$*.*

Definition 35 *Two norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$*, defined in vector space* $\mathcal{V}$ *are said to be compatible if every sequence* $\{v_n\}_{n\in\mathbb{N}}$ *with* $v_n\in\mathcal{V}$*, which is Cauchy with respect to both norms and converges to the zero element with respect one of them, also converges to zero elements with respect to the second.*

Definition 36 *A locally convex space* $\mathcal{V}$ *is called a countably Hilbert space or a* CH*-space for brevity, if it admits a countable set of compatible Hilbertian norms.*

Let $\mathcal{V}$ be a countably Hilbert space associated with an increasing sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, $\|\cdot\|_n=\sqrt{(\cdot,\cdot)_n}$ of Hilbertian norms. Denote $\mathcal{V}_n$ the completion of $\mathcal{V}$ with respect to the norm $\|\cdot\|_n$. In each of these spaces the set $\mathcal{V}$ is dense. By hypothesis, if $m\le n$ then $(v,v)_m\le(v,v)_n$ $\forall v\in\mathcal{V}$. From this it follows that the function maps en element $v\in\mathcal{V}$ from $\mathcal{V}_n$ to $\mathcal{V}_m$ (i.e. the same element v considered in two different spaces) is a continuous function of an everywhere dense set in $\mathcal{V}_n$ onto an everywhere dense set in $\mathcal{V}_m$, so it can be extended to a continuous linear operator T_m^n which maps the space $\mathcal{V}_n$ onto an everywhere dense set of $\mathcal{V}_m$ ($T_m^n:\mathcal{V}_n\to\mathcal{V}_m$). Note that $T_m^p=T_m^nT_n^p$ if $m\le n\le p$.

Definition 37 *A linear operator* T *which maps a Hilbert space* $\mathcal{V}$ *into a Hilbert space* $\mathcal{W}$ *is called Hilbert-Schmidt, if* T *admits a representation of the form*

$$Tv=\sum_{k=1}^{\infty}\lambda_k\,(v,v_k)\,w_k,\quad v\in\mathcal{V}, \tag{B.3}$$

where $\{v_k\}$ *and* $\{w_k\}$ *are orthonormal systems of vectors in the spaces* $\mathcal{V}$ *and* $\mathcal{W}$ *respectively,* $\lambda_k>0$ *and* $\sum_{k=1}^{\infty}\lambda_k^2<\infty$*, see [17, Chapter I, section 2.2, Theorem 3].*

Remark 38 *Every series of the form (B.3), in which* v_k,w_k,λ_k *have the aforementioned properties, defines a completely continuous linear operator (see [17, Chapter I, section 2.1]). This means that carries any bounded set in a set whose closure is compact.*

We can equivalently say that the operator T be of Hilbert-Schmidt type, if the series $\sum_{k=1}^{\infty}\|Tv_k\|^2$ converge for at least one orthonormal basis $\{v_k\}$ of $\mathcal{V}$, see [17, Chapter I, section 2.2, Theorem 2.].

Definition 39 *A countably Hilbert space $\mathcal{N}$ is called nuclear, if for any m there is an $n \geq m$ such that the operator T_m^n is nuclear, i.e. has the form*

$$T_m^n v = \sum_{k=1}^{\infty} \lambda_k (v, v_k)_n w_k, \quad v \in \mathcal{N},$$

where$\{v_k\}$ and $\{w_k\}$ are orthonormal systems of vectors in the spaces $\mathcal{N}_n$ and $\mathcal{N}_m$ respectively, $\lambda_k > 0$ and $\sum_{k=1}^{\infty} \lambda_k < \infty$. Where $\mathcal{N}_n, \mathcal{N}_m$ denotes the completion of $\mathcal{N}$ with respect to the norm $\|\cdot\|_n$ and $\|\cdot\|_m$, respectively.

Remark 40 *The product ST any two Hilbert-Schmidt operator is a nuclear operator. Conversely, every nuclear operator is a product of two Hilbert-Schmidt operators see [17, Chapter I, section 2.3 Theorem 4]. Besides like the convergence of series $\sum_{k=1}^{\infty} \lambda_k^2$ follows from the convergence of $\sum_{k=1}^{\infty} \lambda_k$, then every nuclear operator is of Hilbert-Schmidt type.*

We note further that instead of the nuclearity of the operator T_m^n one can require that it be of Hilbert-Schmidt type. Indeed, since the product of any two Hilbert-Schmidt operators is a nuclear operator. If the operators T_n^p and T_m^n are of Hilbert-Schmidt type, then $T_m^p = T_m^n T_n^p$ is a nuclear operator.

Remark 41 *Every nuclear space $\mathcal{N}$ is perfect. In others words, every bounded closed set in a nuclear space $\mathcal{N}$ is compact, see [17, Chapter I, section 3.4]. The basic properties of nuclear spaces $\mathcal{N}$ are the following*

1. $\mathcal{N}$ is separable (contains an everywhere dense countable set), see [17, Chapter I, section 3.4].
2. $\mathcal{N}$ is complete relative to weak convergence, see [17, Chapter I, section 3.4].
3. Both in $\mathcal{N}$ and its dual $\mathcal{N}^*$, strong and weak convergence coincide, see [17, Chapter I, section 3.4].
4. $\mathcal{N}$ is perfect relative to the topology of weak and strong convergence, see [17, Chapter I, section 3.4].
5. A linear subspace of a nuclear space is nuclear, see [55, Part III, Proposition 50.1].
6. The quotient of a nuclear space modulo a closed linear subspace is nuclear, see [55, Part III, Proposition 50.1].
7. A product of nuclear spaces is nuclear, see [55, Part III, Proposition 50.1].

8. A countable topological direct sum of nuclear spaces is nuclear, see [55, Part III, Proposition 50.1].

Remark 42 *A Fréchet space $\mathcal{V}$ is nuclear if and only if so is $\mathcal{V}^*$, see [44, Proposition 1.3.3].*

Theorem 43 *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{l\in\mathbb{N}} \mathcal{N}_l,$$

where $\{\mathcal{N}_{l,l\in\mathbb{N}}\}$ is a family of Hilbert spaces such that for all $l_1, l_2 \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that the embeddings $\mathcal{N}_l \to \mathcal{N}_{l_1}$ and $\mathcal{N}_l \to \mathcal{N}_{l_2}$ are of Hilbert-Schmidt type. The topology of $\mathcal{N}$ is given by the projective limit topology. i.e., the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \to \mathcal{N}_l$ are continuous for all $l \in \mathbb{N}$, See [47, Chapter III, section 7.3], or [9, Corollary 2.9.7].

B.2 Bochner-Minlos theorem

Definition 44 *A $\mathbb{C}$-valued function $\mathcal{C}$ on $\mathcal{N}$ a nuclear space, is called a characteristic functional if it satisfies the following conditions :*

1. *$\mathcal{C}$ is continuous;*
2. *$\mathcal{C}$ is positive definite, i.e.,*

$$\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \mathcal{C}(\xi_j - \xi_k) \geq 0$$

for any choice of $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$, $\xi_1, \cdots, \xi_n \in \mathcal{N}$ and $j = 1, 2, \cdots, n$;

3. *$\mathcal{C}(0) = 1$.*

A characteristic functional is a generalization of a characteristic function of a probability distribution on $\mathbb{R}^N$.

Let $\mathcal{N}$ be a real nuclear space, $\mathcal{N}^*$ its dual space and $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $\mathcal{N}^* \times \mathcal{N}$. Let $\mathfrak{B}$ be the so-called cylindrical σ-field on $\mathcal{N}^*$, i.e., the smallest σ-field such that the function

$$x \to (\langle x, \xi_1 \rangle, \cdots, \langle x, \xi_N \rangle) \in \mathbb{R}^N, \quad x \in \mathcal{N}^*,$$

is a measurable for choice of $\xi_1, \cdots, \xi_N \in \mathcal{N}$ and $N = 1, 2, \cdots$, where $\mathbb{R}^N$ is equipped whit the Borel σ- field.

Theorem 45 (Bochner-Minlos) *[44, Theorem 1.5.2] Let $\mathcal{N}$ be a real nuclear space. If μ is a probability measure on $\mathcal{N}^*$, its Fourier transform*

$$\widehat{\mu}(\xi) = \int_{\mathcal{N}^*} e^{i\langle x,\xi\rangle} \mu(dx), \quad \xi \in \mathcal{N}, \tag{B.4}$$

is a characteristic function. Conversely, for a characteristic function $\mathcal{C}$ on a nuclear space $\mathcal{N}$ there exist a unique probability measure μ on $\mathcal{N}^*$ such that $\mathcal{C} = \widehat{\mu}$.

C

A Wavelet basis for the spaces $\mathcal{H}_l(\mathbb{C})$

In this Appendix we extend the Kozyrev basis, originally proved for L^2, to the spaces $\mathcal{H}_l(\mathbb{C})$, for l a non-negative integer.

Let us consider the N-dimensional basis $\{\psi_{\gamma\eta J}\}$ of p-adic wavelets of $\mathbb{Q}^N$, introduced by Albeverio and Kozyrev in [4]:

$$\psi_{\gamma\eta J}(x) = p^{\frac{-N\gamma}{2}}\chi(p^{-1}J\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-\eta\|_p), x\in\mathbb{Q}_p^N, \quad \gamma\in\mathbb{Z}, \quad \eta\in\mathbb{Q}_p^N/\mathbb{Z}_p^N, \tag{C.1}$$

$$\eta=(\eta^{(1)},\eta^{(2)},\ldots,\eta^{(N)}), \quad \eta^{(l)}=\sum_{i=\beta_l}^{-1}\eta_{(i)}^{l}p^{i}, \quad \eta_i^{(l)}=0,1,\ldots,p-1, \quad \beta_l\in\mathbb{Z}^{-},$$

$$J=(j_1,j_2,\ldots,j_N), j_l=0,1,\ldots,p-1, \text{ where at least one of } j_l \text{ is not equal to zero.}$$

Remark 46 *We compute the Fourier transform of $\psi_{\gamma\eta J}$ for further use:*

$$\begin{aligned}\widehat{\psi}_{\gamma\eta J}(\xi) &= \int_{\mathbb{Q}_p^N}\chi(\xi\cdot x)\psi_{\gamma\eta J}(x)d^Nx\\ &= p^{\frac{-N\gamma}{2}}\int_{\mathbb{Q}_p^N}\chi(\xi\cdot x)\chi(p^{-1}J\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-\eta\|_p)d^Nx\\ &= p^{\frac{-N\gamma}{2}}\int_{\mathbb{Q}_p^N}\chi[\xi\cdot x+p^{\gamma-1}J\cdot x-p^{-1}J\cdot\eta]\Omega(\|p^{\gamma}x-\eta\|_p)d^Nx.\end{aligned}$$

Making the change variable $z=p^{\gamma}x-\eta, \quad d^Nx=p^{N\gamma}d^Nz$, in the previous integral we *obtain*

$$
\begin{aligned}
&\widehat{\psi}_{\gamma\eta J}(\xi)\\
&= p^{\frac{-N\gamma}{2}} \int_{\mathbb{Q}_p^N} \chi[p^{-\gamma}\xi\cdot(z+\eta) + p^{\gamma-1}J\cdot(z+\eta)p^{-\gamma} - p^{-1}J\cdot\eta)]\Omega(\|z\|_p)p^{N\gamma}d^Nz\\
&= p^{\frac{N\gamma}{2}} \int_{\mathbb{Q}_p^N} \chi[p^{-\gamma}\xi\cdot z + p^{-\gamma}\xi\cdot\eta + p^{-1}J\cdot z + p^{-1}J\cdot\eta - p^{-1}J\cdot\eta]\Omega(\|z\|_p)d^Nz\\
&= p^{\frac{N\gamma}{2}} \int_{\mathbb{Q}_p^N} \chi(p^{-\gamma}\xi\cdot\eta)\chi(p^{-\gamma}\xi\cdot z + p^{-1}J\cdot z)\Omega(\|z\|_p)d^Nz\\
&= p^{\frac{N\gamma}{2}} \chi(p^{-\gamma}\xi\cdot\eta) \int_{\mathbb{Q}_p^N} \chi[(p^{-\gamma}\xi + p^{-1}J)\cdot z]\Omega(\|z\|_p)d^Nz\\
&= p^{\frac{N\gamma}{2}} \chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p).
\end{aligned}
$$

Theorem 47 *The set of functions*

$$\psi_{\gamma\eta J}(x) = p^{\frac{-N\gamma}{2}}\chi(p^{-1}J\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-\eta\|_p) \quad \textit{with} \quad \gamma, J, \ \eta \ \textit{as before,}$$

is an orthonormal basis in $\mathcal{H}_l(\mathbb{C})$ (see section 1.1) consisting of eigenvectors of the operator

$$(A(\theta))(x) = \mathcal{F}^{-1}_{\xi\to x}\{\alpha(\|\xi\|_p)\mathcal{F}_{x\to\xi}\theta\} \tag{C.2}$$

where α is a radial function.

Proof. *let us prove that the functions in (C.1) are eigenvectors of the operator (C.2) i.e. $(A(\psi_{\gamma\eta J}))(x) = \alpha(p^{1-\gamma})\psi_{\gamma\eta J}$.*

$$
\begin{aligned}
(A(\psi_{\gamma\eta J}))(x) &= \mathcal{F}^{-1}_{\xi\to x}\{\alpha(\|\xi\|_p)\mathcal{F}_{x\to\xi}\psi_{\gamma\eta J}\} = \{\alpha(\|\xi\|_p)\widehat{\psi}_{\gamma\eta J}(\xi)\}^{\vee}(x)\\
&= \int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N} \chi((y-x)\cdot\xi)\alpha(\|\xi\|_p)\psi_{\gamma\eta J}(y)d^N\xi d^Ny\\
&= \int_{\mathbb{Q}_p^N} \chi(-\xi\cdot x)\alpha(\|\xi\|_p)\widehat{\psi}_{\gamma\eta J}(\xi)d^Nx\\
&= \int_{\mathbb{Q}_p^N} \chi(-\xi\cdot x)\alpha(\|\xi\|_p)p^{\frac{N\gamma}{2}}\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)d^Nx\\
&= p^{\frac{N\gamma}{2}} \int_{\mathbb{Q}_p^N} \chi((p^{-\gamma}\eta - x)\cdot\xi)\alpha(\|\xi\|_p)\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)d^Nx
\end{aligned}
$$

Suppose that $\|p^{-\gamma}\xi + p^{-1}J\|_p \leq 1$. Then $\xi \in -p^{\gamma-1}J + p^{\gamma}\mathbb{Z}_p^N$, and $\|\xi\|_p = p^{1-\gamma}$.

$$(A(\psi_{\gamma\eta J}))(x) = p^{\frac{N\gamma}{2}}\alpha(p^{1-\gamma}) \int_{\mathbb{Q}_p^N} \chi((p^{-\gamma}\eta - x)\cdot\xi)\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)d^Nx$$

Making the change variable $z = p^{-\gamma}\xi + p^{-1}J, \quad d^N\xi = p^{-N\gamma}d^Nz$, in the previous integral we obtain

$$
\begin{aligned}
(A(\psi_{\gamma\eta J}))(x) &= p^{\frac{N\gamma}{2}}\alpha(p^{1-\gamma})\int_{\mathbb{Q}_p^N}\chi[p^{\gamma}(p^{-\gamma}\eta - x)\cdot(z - p^{-1}J)]\Omega(\|z\|_p)p^{-N\gamma}d^Nz\\
&= p^{\frac{-N\gamma}{2}}\alpha(p^{1-\gamma})\int_{\mathbb{Q}_p^N}\chi[p^{\gamma}(p^{-\gamma}\eta - x)\cdot z - p^{\gamma-1}(p^{-\gamma}\eta - x)\cdot J)]\Omega(\|z\|_p)d^Nz\\
&= p^{\frac{-N\gamma}{2}}\alpha(p^{1-\gamma})\chi[-p^{\gamma-1}(p^{-\gamma}\eta - x)\cdot J]\int_{\mathbb{Q}_p^N}\chi[p^{\gamma}(p^{-\gamma}\eta - x)\cdot z]\Omega(\|z\|_p)d^Nz\\
&= p^{\frac{-N\gamma}{2}}\alpha(p^{1-\gamma})\chi[-p^{\gamma-1}(p^{-\gamma}\eta - x)\cdot J]\Omega(\|p^{\gamma}(p^{-\gamma}\eta - x)\|_p)\\
&= \alpha(p^{1-\gamma})p^{\frac{-N\gamma}{2}}\chi[p^{-1}J\cdot(p^{\gamma}x - \eta)]\Omega(\|p^{\gamma}x - \eta\|_p)\\
&= \alpha(p^{1-\gamma})\psi_{\gamma\eta J}(x).
\end{aligned}
$$

Let us see now that functions (C.1) are orthogonal:

$$
\begin{aligned}
\langle\psi_{\gamma,\eta,J},\psi_{\gamma',\eta',J'}\rangle_l &= \int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\widehat{\psi}_{\gamma,\eta,J}(\xi)\overline{\widehat{\psi}}_{\gamma',\eta',J'},(\xi)d^N\xi\\
&= p^{\frac{N(\gamma+\gamma')}{2}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)\\
&\qquad\times\chi(-p^{-\gamma'}\xi\cdot\eta')\Omega(\|p^{-\gamma'}\xi + p^{-1}J'\|_p)d^N\xi.\\
&= p^{\frac{N(\gamma+\gamma')}{2}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\chi(-p^{-\gamma'}\xi\cdot\eta')\\
&\qquad\times\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)\Omega(\|p^{-\gamma'}\xi + p^{-1}J'\|_p)d^N\xi.
\end{aligned}
$$

Suppose that $l, k \in \mathbb{Z}$ with $l \leq k$. Then following product of indicators is either an indicator or zero see e.g. [2] and [62]:

$$
\Omega(\|p^lx - a\|_p)\Omega(\|p^kx - b\|_p) = \Omega(\|p^lx - a\|_p)\Omega(\|p^{k-l}a - b\|_p), \text{ with } x, a, b \in \mathbb{Q}_p^N. \quad \text{(C.3)}
$$

If $\gamma \leq \gamma'$, by the above

$$
\begin{aligned}
\langle\psi_{\gamma,\eta,J},\psi_{\gamma',\eta',J'}\rangle_l \;=\; & p^{\frac{N(\gamma+\gamma')}{2}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\chi(-p^{-\gamma'}\xi\cdot\eta')\\
&\times\Omega(\|p^{-\gamma'}\xi + p^{-1}J'\|_p)\Omega(\| - p^{\gamma'-\gamma-1}J' + p^{-1}J\|_p)d^N\xi.
\end{aligned}
$$

Suppose that $\gamma < \gamma'$. Then

$$
\|p^{\gamma'-\gamma-1}J' + p^{-1}J\|_p = \max(\|p^{\gamma'-\gamma-1}J'\|_p, \|p^{-1}J\|_p) = p > 1.
$$

Thus $\Omega(\| - p^{\gamma^{'}-\gamma-1}J^{'} + p^{-1}J\|_p) = 0$ *and* $\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l,\alpha} = 0$.
Consequently the scalar product $\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l,\alpha}$, *can be nonzero only is* $\gamma = \gamma^{'}$. *In this case we obtain*

$$\begin{aligned}\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l} &= \delta_{\gamma,\gamma^{'}}p^{N\gamma}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\chi(-p^{-\gamma}\xi\cdot\eta^{'})\\ &\quad\times\Omega(\|p^{-\gamma}\xi + p^{-1}J^{'}\|_p)\Omega(\| - p^{-1}J^{'} + p^{-1}J\|_p)d^N\xi.\end{aligned}$$

Since $\Omega(\| - p^{-1}J^{'} + p^{-1}J\|_p) = 1$ *only if* $J = J^{'}$, *we have* $\Omega(\| - p^{-1}J^{'} + p^{-1}J\|_p) = \delta_{J,J^{'}}$. *Hence the scalar product can be non-zero only when* $\gamma = \gamma^{'}$, *and* $J = J^{'}$, *then the previous integral is equal to*

$$\begin{aligned}&\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l} = \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}p^{N\gamma}\\ &\times\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot(\eta-\eta^{'}))\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)d^N\xi.\end{aligned}$$

Suppose that $\|p^{-\gamma}\xi + p^{-1}J\|_p \le 1$. *Then* $\xi \in -p^{\gamma-1}J + p^{\gamma}\mathbb{Z}_p^N$, *and* $\|\xi\|_p = p^{1-\gamma}$.

$$\begin{aligned}\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l} &= \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}p^{N\gamma}[\max(1,p^{1-\gamma})]^{2l}\\ &\quad\times\int_{\mathbb{Q}_p^N}\chi(p^{-\gamma}\xi\cdot(\eta-\eta^{'}))\Omega(\|p^{-\gamma}\xi + p^{-1}J\|_p)d^N\xi.\end{aligned}$$

Making the change variable $z = p^{-\gamma}\xi + p^{-1}J$, $d^N\xi = p^{-N\gamma}d^Nz$, *in the previous integral we obtain*

$$\begin{aligned}&\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l}\\ &\quad= \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}p^{N\gamma}[\max(1,p^{1-\gamma})]^{2l}\int_{\mathbb{Q}_p^N}\chi(p^{-\gamma}(\eta-\eta^{'})\cdot(z-p^{-1}J)p^{\gamma})\Omega(\|z\|_p)p^{-N\gamma}d^Nz\\ &\quad= \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}[\max(1,p^{1-\gamma})]^{2l}\chi(-p^{-1}(\eta-\eta^{'})\cdot J)\int_{\mathbb{Q}_p^N}\chi[(\eta-\eta^{'})\cdot z]\Omega(\|z\|_p)d^Nz\\ &\quad= \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}[\max(1,p^{1-\gamma})]^{2l}\chi(-p^{-1}(\eta-\eta^{'})\cdot J)\Omega(\|\eta-\eta^{'}\|_p).\end{aligned}\tag{C.4}$$

If $\eta \ne \eta^{'}$. *Then* $\|\eta - \eta^{'}\|_p \ge p > 1$ *and the previous integral is zero.*
Consequently we have to the product is

$$\langle\psi_{\gamma,\eta,J},\psi_{\gamma^{'},\eta^{'},J^{'}}\rangle_{l} = \delta_{\gamma,\gamma^{'}}\delta_{J,J^{'}}\delta_{\eta,\eta^{'}}\ [\max(1,p^{1-\gamma})]^{2l}.$$

Thus the system of functions (C.1) is orthogonal. Without loss of generality we can *consider the orthonormal system*

$$\psi^{(l)}_{\gamma,\eta,J} = \frac{p^{\frac{-N\gamma}{2}}\chi(p^{-1}J\cdot(p^{\gamma}x-\eta))\Omega(\|p^{\gamma}x-n\|_p)}{[\max(1,p^{1-\gamma})]^{2l}}. \tag{C.5}$$

To prove the completeness of the system of functions (C.1), remember that the space $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^N)$ *is dense in* $\mathcal{H}_l(\mathbb{C})$ *and besides the set of functions* $\psi^{(l)}_{\gamma,j,n}$ *is invariant under dilations and translations, therefore is sufficient to verify the Parseval identity for the characteristic function* $\Omega(\|x\|_p)$*:*

$$\begin{aligned}\langle\Omega(\|x\|_p),\psi^{(l)}_{\gamma,\eta,J}\rangle_l &= \frac{1}{[\max(1,p^{1-\gamma})]^{2\alpha l}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\widehat{\Omega}(\|\xi\|_p),\overline{\widehat{\psi}}_{\gamma,\eta,J}(\xi)d^N\xi\\ &= \frac{1}{[\max(1,p^{1-\gamma})]^{2l}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\Omega(\|\xi\|_p)p^{\frac{N\gamma}{2}}\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)d\xi.\end{aligned} \tag{C.6}$$

Suppose that $0\le-\gamma$ $(\gamma\le 0)$*, according to (C.3) we obtain that the product of indicators is zero:*

$$\Omega(\|\xi\|_p)\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)=\Omega(\|\xi\|_p)\Omega(\|-p^{-1}J\|_p).$$

Suppose that $-\gamma\le 0$ $(0\le\gamma)$*, according to (C.3), we obtain that the product of indicators is non-zero*

$$\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)\Omega(\|\xi\|_p)=\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)\Omega(\|-p^{\gamma-1}J\|_p),\ \ \textit{if}\ \gamma\ge 1$$

$$\begin{aligned}\langle\Omega(\|x\|_p),\psi^{(l)}_{\gamma,\eta,J}\rangle_l &= \frac{p^{\frac{N\gamma}{2}}}{[\max(1,p^{1-\gamma})]^{2l}}\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\Omega(\|\xi\|_p)\\ &\qquad\times\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)d^N\xi\\ &= \frac{p^{\frac{N\gamma}{2}}[\max(1,p^{1-\gamma})]^{2l}}{[\max(1,p^{1-\gamma})]^{2l}}\Omega(\|-p^{\gamma-1}J\|_p)\int_{\mathbb{Q}_p^N}[\max(1,\|\xi\|_p)]^{2l}\chi(p^{-\gamma}\xi\cdot\eta)\\ &\qquad\times\Omega(\|p^{-\gamma}\xi+p^{-1}J\|_p)d^N\xi\\ &= p^{\frac{N\gamma}{2}}\Omega(\|-p^{\gamma-1}J\|_p)\int_{\mathbb{Q}_p^N}\chi[p^{-\gamma}(p^{\gamma}(z-p^{-1}J))\cdot\eta]\Omega(\|z\|_p)p^{-N\gamma}d^N\xi\\ &= p^{-\frac{N\gamma}{2}}\Omega(\|-p^{\gamma-1}J\|_p)\chi(-p^{-1}J\cdot\eta)\int_{\mathbb{Q}_p^N}\chi(z\cdot\eta)\Omega(\|z\|_p)d^N\xi\\ &= p^{-\frac{N\gamma}{2}}\chi(-p^{-1}J\cdot\eta)\Omega(\|\eta\|_p),\end{aligned} \tag{C.7}$$

for $\gamma\ge 1$.

If $\eta \neq 0$, then the previous product is zero. Therefore if $\eta = 0$, and $\gamma \geq 1$ then

$$\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,J} \rangle_l = p^{-\frac{N\gamma}{2}}.$$

Finally,

$$\begin{aligned}
\sum_{\gamma\in\mathbb{Z},\ \eta\in\mathbb{Q}_p^N/\mathbb{Z}_p^N,\ J=1}^{p^N-1} |\langle \Omega(\|x\|_p), \psi^{(l)}_{\gamma,\eta,J} \rangle_{l,\alpha}|^2 &= \sum_{\gamma=1}^{\infty} \sum_{J=1}^{p^N-1} (p^{-\frac{N\gamma}{2}})^2 \\
&= \sum_{\gamma=1}^{\infty} (p^N - 1) p^{-N\gamma} \\
&= 1 = \|\Omega(\|x\|_p)\|_l^2.
\end{aligned}$$

■

D

Properties of the generalized stochastic process W_g

By a generalized stochastic process, we mean a mapping $W(g)$ depending linearly on test functions g such that for g, $W(g)$ is random variable. A white noise is a generalized stochastic process W_g such that for each test function $g \in \mathcal{H}_\infty(\mathbb{R})$ (see Section 1.1) the random variable $W_g = \langle \cdot, g \rangle$ is Gaussian with mean 0 and variance $\int_{\mathbb{Q}_p^N} |g(x)|^2 \, d^N x = \|g\|_0^2$.

Remark 48 *From the definition of the Gaussian measure μ given by the formula (1.3), it follows straightforwardly that for every $g \in \mathcal{H}_\infty(\mathbb{R})$, $\langle \cdot, g \rangle$ is a normally distributed random variable with variance $\|g\|_0^2$. Thus, for all $g \in \mathcal{H}_\infty(\mathbb{R})$, $g \neq 0$,*

$$\|\langle \cdot, g \rangle\|_{(L^2)}^2 = \int_{\mathcal{H}_\infty^*(\mathbb{R})} \langle W, g \rangle^2 \, d\mu(W) = \|g\|_0^2 . \tag{D.1}$$

Moreover, again by the formula (1.3), the real process W defined on $\mathcal{H}_\infty^*(\mathbb{R}) \times \mathcal{H}_\infty(\mathbb{R})$ by $W_g = \langle W, g \rangle$ is centered Gaussian with covariance

$$\begin{aligned} & \int_{\mathcal{H}_\infty^*(\mathbb{R})} \langle W, g_1 \rangle \langle W, g_2 \rangle \, d\mu(W) \\ = & \frac{1}{2} \left\{ \|\langle \cdot, g_1 + g_2 \rangle\|_{(L^2)}^2 - \|\langle \cdot, g_1 \rangle\|_{(L^2)}^2 - \|\langle \cdot, g_2 \rangle\|_{(L^2)}^2 \right\} \\ = & \langle g_1, g_2 \rangle_0 . \end{aligned} \tag{D.2}$$

We recall that if $g_1, \cdots, g_N \in \mathcal{H}_\infty(\mathbb{R})$ is an orthonormal set in $L^2_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$, then the image of the Gaussian measure under the map

$$\begin{aligned} \mathcal{H}_\infty^*(\mathbb{C}) &: \to \mathbb{R}^n \\ W &\to (\langle W, g_1 \rangle, \cdots \langle W, g_n \rangle), \end{aligned} \tag{D.3}$$

is a product of the standard Gaussian measure on $\mathbb{R}$, namely,

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{(t_1^2+\cdots+t_n^2)/2} dt_1 \cdots dt_n, \tag{D.4}$$

see e.g. [44, , Chapter 2]. In addition, if $f_1, \cdots f_n$ are integrable functions on $\mathbb{R}$ with respect to the 1−dimensional standard Gaussian measure, we have

$$\int_{\mathcal{H}_\infty^*(\mathbb{R})} f_1(\langle W, g_1\rangle) \cdots f_n(\langle W, g_n\rangle) d\mu(W) = \prod_{k=1}^{n} \int_{\mathcal{H}_\infty^*(\mathbb{R})} f_k(\langle W, g_k\rangle) d\mu(W), \tag{D.5}$$

see e.g. [44, , Chapter 2].

Proposition 49 *The Gaussian measure μ is quasi-invariant under the translation by any $g \in L^2_{\mathbb{R}}(\mathbb{Q}_p^N)$ and the Radon-Nikodym derivative is given by*

$$\frac{d\mu(W-g)}{d\mu(W)} = e^{\langle W,g\rangle-(g,g)/2}, \quad W \in \mathcal{H}_\infty^*(\mathbb{R}). \tag{D.6}$$

For more further details the reader may consult [44, Proposition 2.1.6].

D.1 Wick-ordered polynomial and Wiener-Itô-segal Isomorphism

Definition 50 *Let $W \in \mathcal{H}_\infty^*(\mathbb{R})$ and $n \in \mathbb{N}_0$ we define the so-called Wick power of order n denoted by $: W^{\otimes n} :\in (\mathcal{H}_\infty^*(\mathbb{R}))^{\widehat{\otimes} n}$ inductively as follows:*

$$\begin{cases} : W^{\otimes 0} : & = 1 \\ : W^{\otimes 1} : & = W \\ : W^{\otimes n} : & = W \widehat{\otimes} : W^{\otimes(n-1)} : - (n-1)\, Tr \widehat{\otimes} : W^{\otimes(n-2)} :, \ n \geq 2, \end{cases}$$

where $Tr \in (\mathcal{H}_\infty^*(\mathbb{R}))^{\widehat{\otimes} 2}$ is the trace, see Remark 12.

Remark 51 *Due the canonical correspondence between the bilinear forms $B(\mathcal{H}_\infty(\mathbb{R}) \times \mathcal{H}_\infty(\mathbb{R}))$ and $(\mathcal{H}_\infty(\mathbb{R}) \times \mathcal{H}_\infty(\mathbb{R}))^*$ (see [44, proposition 1.3.9]) the trace operator Tr is uniquely defined via the formula*

$$\langle Tr, f \otimes g\rangle = \langle f, g\rangle_0, \ \text{for } f, g \in \mathcal{H}_\infty(\mathbb{R}).$$

The trace operator can be represented by

$$Tr=\sum_{n=0}^{\infty}\xi_n\otimes\xi_n,$$

with respect to the strong dual topology of $(\mathcal{H}_\infty(\mathbb{R})\times\mathcal{H}_\infty(\mathbb{R}))^*$, where (ξ_n) is an arbitrary orthonormal basis of $L^2_{\mathbb{R}}(\mathbb{Q}_p^N)$. We recall that the space $L^2_{\mathbb{R}}(\mathbb{Q}_p^N)$ is is separable since it is nuclear space. For further details, the reader may consult [44, Proposition 2.2.1].

Proposition 52 ([44, proposition 2.2.3]) *Let* $W\in\mathcal{H}^*_\infty(\mathbb{R})$ *and* $g\in\mathcal{H}_\infty(\mathbb{R})$. *Then*

$$\langle :W^{\otimes n}:,g^{\otimes n}\rangle=\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!}\left(-\frac{1}{2}\langle g,g\rangle\right)^k\langle W,g\rangle^{n-2k}, \tag{D.7}$$

$$\langle W,g\rangle^n=\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!}\left(-\frac{1}{2}\langle g,g\rangle\right)^k\left\langle :W^{\otimes(n-2k)}:,g^{\otimes(n-2k)}\right\rangle. \tag{D.8}$$

Corollary 53 ([44, Corollary 2.2.4]) *For* $W\in\mathcal{H}^*_\infty(\mathbb{R})$ *we have*

$$:W^{\otimes n}:=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{(n-2k)!k!2^k}Tr^{\widehat{\otimes}k}\widehat{\otimes}W^{\otimes(n-2k)}, \tag{D.9}$$

$$W^{\otimes n}=\sum_{k=0}^{[n/2]}\frac{n!}{(n-2k)!k!2^k}Tr^{\widehat{\otimes}k}\widehat{\otimes}:W^{\otimes(n-2k)}:. \tag{D.10}$$

Wiener-Itô-Segal isomorphism and Fock space

Remark 54 *The linear space of the so-called polynomial on the Gaussian space* $\mathcal{H}^*_\infty(\mathbb{R})$ *(see the formula 1.5), is denoted by*

$$\mathcal{P}(\mathbb{C}):\left\{\phi(W)=\sum_{n=0}^{M}\langle W^{\otimes n},\phi_n\rangle,\phi_n\in\mathcal{H}_\infty^{\widehat{\otimes}n}(\mathbb{C}),W\in\mathcal{H}^*_\infty(\mathbb{R}),M\in\mathbb{N}\right\}. \tag{D.11}$$

The linear space of the so-called Wick-ordered polynomials (see formula 1.6) is denoted by

$$\mathcal{P}(\mathcal{H}^*_\infty(\mathbb{R})):=\left\{\phi(W)=\sum_{n=0}^{M}\langle :W^{\otimes n}:,\phi_n\rangle,\phi_n\in\mathcal{H}_\infty^{\widehat{\otimes}n}(\mathbb{C}),W\in\mathcal{H}^*_\infty(\mathbb{R}),M\in\mathbb{N}\right\}. \tag{D.12}$$

Proposition 55 ([44, Proposition 2.3.2]) *The polynomials* $\mathcal{P}(\mathbb{R})$ *and* $\mathcal{P}(\mathbb{C})$ *are dense of subspaces of* $L^2(\mathcal{H}^*_\infty(\mathbb{R}),\mu;\mathbb{R})$ *and* $L^2(\mathcal{H}^*_\infty(\mathbb{R}),\mu;\mathbb{C})$, *respectively.*

Theorem 56 ([44, Theorem 2.3.5], Wiener-Itô-Segal) *For each* $\Psi \in L^2(\mathcal{H}^*_\infty(\mathbb{R}), \mu; \mathbb{C})$ *there exist a unique* $\psi = (\psi_n)_{n=0}^\infty \in \Gamma(L^2_\mathbb{C}(\mathbb{Q}_p^N))$ *such that*

$$\Psi(W) = \sum_{n=0}^{\infty} \langle : W^{\otimes n} :, \psi_n \rangle, \tag{D.13}$$

in the (L^2)*-sense. Conversely, for any* $\psi = (\psi_n)_{n=0}^\infty \in \Gamma(L^2_\mathbb{C}(\mathbb{Q}_p^N))$*, (D.13) defines a function in* $L^2(\mathcal{H}^*_\infty(\mathbb{R}), \mu; \mathbb{C})$*. In that case,*

$$\|\psi\|^2 = \sum_{n=0}^{\infty} n! \|\psi_n\|_0^2 = \|\psi\|^2_{\Gamma(L^2_\mathbb{C}(\mathbb{Q}_p^N))}. \tag{D.14}$$

A similar assertion is also true for the real case. In short, we have canonical isomorphisms:

$$L^2(\mathcal{H}^*_\infty(\mathbb{R}), \mu; \mathbb{R}) \cong \Gamma(L^2_\mathbb{R}(\mathbb{Q}_p^N)) \quad L^2(\mathcal{H}^*_\infty(\mathbb{R}), \mu; \mathbb{C}) \cong \Gamma(L^2_\mathbb{C}(\mathbb{Q}_p^N)). \tag{D.15}$$

Definition 57 *The canonical isomorphism established in above theorem is called the Wiener-itô-Segal isomorphism. The expression as in (D.13) is called the Wiener-Itô expansion of* $\Psi \in L^2(\mathcal{H}^*_\infty(\mathbb{R}), \mu; \mathbb{C})$*.*

D.1.1 Bosonic Fock space

Tensor products of Hilbert spaces

Let H_1 and H_1 be Hilbert spaces with inner products $(.,.)_1$ and $(.,.)_2$ respectively. For $h_1 \in H_1$ and $h_2 \in H_2$, we define their tensor product as a conjugate bilinear form on $H_1 \times H_1$:

$$h_1 \otimes h_2 \left(\xi_1, \xi_2\right) = \left(h_1, \xi_1\right)_1 \left(h_2, \xi_2\right)_2 . \tag{D.16}$$

Denote by $\mathcal{E}$ the linear span of $\{h_1 \otimes h_2 : h_1 \in H_1 \, , h_2 \in H_2\}$. For $h_1 \otimes h_2$, $j_1 \otimes j_2 \in \mathcal{E}$, we define

$$b\left(h_1 \otimes h_2, j_1 \otimes j_2\right) = \left(h_1, j_1\right)_1 \left(h_2, j_2\right)_2 , \tag{D.17}$$

and linearly extend it to $\mathcal{E}$. The equation D.17 defines a strictly positive Hermitian form on $\mathcal{E} \times \mathcal{E}$, hence $(\mathcal{E}, b)$ is an inner product space.

Definition 58 *The Hilbert space obtained by completion of* $(\mathcal{E}, b)$ *is called the Hilbertian tensor product (tensor product for short) of* H_1 *and* H_2 *and denoted by* $H_1 \otimes H_2$*.*

By induction we can define tensor products of any finite number of Hilbert spaces i.e. the $n-$fold tensor product of n Hilbert spaces $\{H_i\}_{i=1}^{n}$ is defined recursively as

$$H_1 \otimes H_2 \otimes \cdots \otimes H_n = \{H_1 \otimes H_2 \otimes \cdots \otimes H_{n-1}\} \otimes H_n.$$

Let H be a real or complex Hilbert space with norm $\|\cdot\|_0$. Let $\Gamma(H)$ be the space of all sequences $\boldsymbol{f} = (f_n)_{n\in\mathbb{N}}$, $f_n \in H^{\widehat{\otimes} n}$ with $H^{\widehat{\otimes} 0} = \mathbb{C}$, such that $\sum_{n=0}^{\infty} n!\|f_n\|_0^2 < \infty$. With the norm

$$\|\boldsymbol{f}\|_{\Gamma(H)}^2 = \sum_{n=0}^{\infty} n!\|f_n\|_0^2,$$

$\Gamma(H)$ becomes Hilbert, which is called the Bosonic Fock space or the symmetric Fock space over H.

D.2 Holomorphic functions in locally convex topological vector spaces.

We give brief review of some results of the theory of holomorphic functions in locally convex topological vector spaces (see [11], [12]). Denote by $\mathcal{L}(\mathcal{H}_\infty^n(\mathbb{C}))$ be the space of n-linear mappings from $\mathcal{H}_\infty^n(\mathbb{C})$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{H}_\infty^n(\mathbb{C}))$ the subspace of symmetric $n-$linear forms from $\mathcal{H}_\infty^n(\mathbb{C})$ to $\mathbb{C}$.

For $L \in \mathcal{L}_s(\mathcal{H}_\infty^n(\mathbb{C}))$, put

$$\widehat{L}(\varphi) = L(\varphi, \cdots, \varphi), \quad \varphi \in \mathcal{H}_\infty(\mathbb{C}).$$

$\widehat{L}$ is called the n-homogeneous polynomial corresponding to L. Denote by $\mathcal{P}_n(\mathcal{H}_\infty(\mathbb{C}))$ the set of all n-homogeneous polynomials on $\mathcal{H}_\infty(\mathbb{C})$. As a consequence of the polarization formula (D.20), the map$\widehat{}$: $\mathcal{L}_s(\mathcal{H}_\mathbb{C}^n(\infty)) \to \mathcal{P}_n(\mathcal{H}_\mathbb{C}(\infty))$ is a linear bijection (see e.g. [12, Corollary 1.6]).

We consider a function $F : \mathcal{U} \to \mathbb{C}$ defined on open set $\mathcal{U} \subset \mathcal{H}_\infty(\mathbb{C})$ is said to be $G-$holomorphic or Gâteaux-holomorphic if for each $\xi_0 \in \mathcal{U}$ and each $\xi \in \mathcal{H}_\infty(\mathbb{C})$ the complex-valued function

$$\mathbb{C} \ni \lambda \rightarrowtail F(\xi_0 + \lambda\xi) \in \mathbb{C},$$

is holomorphic on some neighborhood of $0 \in \mathbb{C}$. If $F : \mathcal{U} \to \mathbb{C}$ is G-holomorphic and given any $\eta \in \mathcal{U}$, then there exists a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{\mathrm{d}^n F}(\eta)$ such that

$$F(\xi + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!}\widehat{\mathrm{d}^n F}(\eta)(\xi),$$

for all ξ belongs to some open neighborhood of the origin of $\mathcal{H}_\infty(\mathbb{C})$. Let $H_G(\mathcal{U})$ denote the vector space of all $G-$holomorphic function on $\mathcal{U}$.

A G-holomorphic function $F:\mathcal{U}\rightarrow\mathbb{C}$ is said to be holomorphic, if for all η in $\mathcal{U}$ there is an open neighborhood V of zero such that $\sum_{n=0}^{\infty}\frac{1}{n!}\widehat{\mathrm{d}^n}F(\eta)(\xi)$ converges uniformly on V to a continuous function. We will denote by $H(\mathcal{U})$ the vector space of all holomorphic function on $\mathcal{U}$. A G-holomorphic function $F:\mathcal{U}\rightarrow\mathbb{C}$ is holomorphic if and only if it is locally bounded. For more details and proofs (see [12, Chapther 2]).

We say that F is holomorphic at ξ_0 if it is holomorphic on some neighborhood of $\mathcal{U}\subset\mathcal{H}_\infty(\mathbb{C})$ of ξ_0, such that $F:\mathcal{U}\rightarrow\mathbb{C}$ is holomorphic.

Let us explicitly consider a function G-holomorphic at zero, then

1. there exist l and $\varepsilon>0$ such that for all $\xi_0\in\mathcal{H}_\infty(\mathbb{C})$ with $\|\xi_0\|_l<\varepsilon$ and for all $\xi\in\mathcal{H}_\infty(\mathbb{C})$ the function of one complex variable $\lambda\rightarrowtail F(\xi_0+\lambda\xi)$ is analytic at $0\in\mathbb{C}$, and

2. there exists $c>0$ such that for all $\xi\in\mathcal{H}_\infty(\mathbb{C})$ with $\|\xi\|_l<\varepsilon$, $|F(\xi)|\leq\varepsilon$.

Remark 59 *The following assumption will be needed to identify the between different restrictions of one holomorphic function, we consider germs of holomorphic functions, i.e. we identify F and G if there exists an open neighborhood of zero $\mathcal{U}$ such that $F(\xi)=G(\xi)$ for all $\xi\in\mathcal{U}$. Thus, we define $Hol_0(\mathcal{H}_{\mathbb{C}}(\infty))$ as the algebra of germs of functions holomorphic at zero. Algebraically, it is clear that $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ endowed with the pointwise multiplication of functions is an algebra.*

D.3 The characterization Theorem of the Kondratiev Distributions.$(\mathcal{H}_\infty(\mathbb{C}))^{-1}$

Theorem 60 *[26, Theorem 3]*

1. If $\phi\in(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, then the function $S\phi$ is holomorphic on some open neighborhood of zero in $\mathcal{H}_\infty(\mathbb{C})$ (i.e. $S\phi\in Hol_0(\mathcal{H}_\infty(\mathbb{C})))$.

2. Given any $F\in Hol_0(\mathcal{H}_\infty(\mathbb{C}))$, there exists a unique $\phi\in(\mathcal{H}^n_\infty(\mathbb{C}))^{-1}$ such that $S\phi=F$.

Remark 61 *The following result about the degree of singularity of a distribution, it is a consequence of the previous theorem (see e.g. [26, Corollary 4.]). If $F\in Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ be holomorphic on $\mathcal{U}_{l,k}$ and $|F(\xi)|\leq C$ for some $C>0$ and all $\xi\in\mathcal{U}_{l,k}$. Let $l'>l$*

be such that $I_{l'l} : \mathcal{H}_{l'}(\mathbb{C}) \to \mathcal{H}_l(\mathbb{C})$ *is a Hilbert-Schmidt operator and* k' *be such that* $\rho = 2^{-\left(k'-2k+2\right)}\rho^2 \left\|I_{l'l}\right\|_{HS}^2 < 1$. *Then* Φ *corresponding to* F *belongs to* $\mathcal{H}_{-l,-k,-\beta}$, *and we have the estimate*

$$\|\Phi\|_{-l',-k',-1} \le C\,(1-\rho)^{-1/2}\,. \tag{D.18}$$

The following two important corollaries are a consequence of Theorem 60. One concerns the convergence of sequences of distributions the second one the Bochner integration of families of the same type of distributions (or generalized functions).

Corollary 62 *[26, Theorem 5] Let* $(F_n)_{n\in\mathbb{N}}$ *be a sequence in* $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$. *Assume that the following two conditions are satisfied:*

1. there exists $\mathcal{U}_{l,k}$ and $C > 0$ such that all F_n are holomorphic on $\mathcal{U}_{l,k}$, and $|F_n(\xi)| \le C$, $\forall \xi\ \mathcal{U}_{l,k}$;
2. $(F_n(\eta))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\eta \in \mathcal{U}_{l,k}$.

Then $S^{-1}F_n$ converges strongly in $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$.

Theorem 63 *[26, Theorem 5] Let* $(\Omega, \mathcal{F}, \upsilon)$ *be a measure space and* $\omega \to \Phi_\omega$ *be a mapping from* Ω *to* $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$. *We assume that there is* $\mathcal{U}_{l,k}$, *such that*

1. $S\Phi_\omega(\eta) : \Omega \to \mathbb{C}$ is measurable for all $\eta \in \mathcal{U}_{l,k}$, and
2. there exists $C \in L^1(\Omega, \upsilon)$ such that $|S\Phi_\omega(\eta)| \le C$ for all $\eta \in \mathcal{U}_{l,k}$ and for υ-almost all ω.

Then there are $l', k' \in \mathbb{N}$, (l', k' as in remark 61), such that

$$\int_\Omega \Phi_\omega d\upsilon(\omega)\,,$$

exists as a Bochner integral in $\mathcal{H}_{-l,-k,-\beta}$. In particular

$$\int_\Omega \Phi_\omega d\upsilon(\omega) \in (\mathcal{H}_\infty(\mathbb{C}))^{-1},$$

and

$$\left\langle\left\langle \int_\Omega \Phi_\omega d\upsilon(\omega)\,, \varphi \right\rangle\right\rangle = \int_\Omega \langle\langle \Phi_\omega, \varphi\rangle\rangle\, d\upsilon(\omega)\,, \quad \varphi \in (\mathcal{H}_\infty(\mathbb{C}))^1. \tag{D.19}$$

D.3.1 Wick products of Distributions

In general, the product of distributions is not well-defined, let us recall that $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ is the algebra of germs of functions holomorphic at zero (Remark 59), then the S-transform is a bijection of $Hol_0(\mathcal{H}_\infty(\mathbb{C}))$ onto $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$. By Theorem 60 for each pair $\Phi_1\Phi_2 \in (\mathcal{H}_\infty(\mathbb{C}))^{-1}$ we define the so-called Wick product by

$$\Phi_1 \diamond \Phi_2 = S^{-1}\left((S\Phi_1)\,(S\Phi_2)\right).$$

It turns out that this multiplication is clearly associative. Furthermore, the above characterization of test functions leaves $(\mathcal{H}_\infty(\mathbb{R}))^1$ invariant, i.e.

$$\Phi_1\Phi_2 \in (\mathcal{H}_\infty(\mathbb{C}))^1 \Longrightarrow \Phi_1 \diamond \Phi_2 \in (\mathcal{H}_\infty(\mathbb{C}))^1.$$

The Wick product can be described in terms of chaos decomposition as well. If $\Phi_i \in (\mathcal{H}_\infty(\mathbb{C}))^{-1}, i = 1, 2,$ such that $\Phi_i = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \phi_i^{(n)} \right\rangle$, then

$$\Phi_1 \diamond \Phi_2 = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \Xi^{(n)} \right\rangle, \ \Xi^{(n)} = \sum_{k=0}^{n} \phi_1^{(k)} \widehat{\otimes} \phi_2^{(n-k)}.$$

Proposition 64 ([26, Proposition 11]) *The Wick product is continuous on $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, in particular, for*

$$\Phi_1 \in \mathcal{H}_{-l_1,-k_1,-1}, \ \Phi_2 \in \mathcal{H}_{-l_2,-k_2,-1},$$

and $l = \max(l_1, l_2), k = k_1 + k_2 + 1,$ the following estimate holds

$$\|\Phi_1 \diamond \Phi_2\|_{-l,-k,-1} \le \|\Phi_1\|_{-l_1,-k_1,-1} \|\Phi_2\|_{-l_2,-k_2,-1}$$

For further details see [26].

Clearly, we can also define Wick powers

$$\Phi^{\diamond n} := S^{-1}\left((S\Phi)^n\right) = \underbrace{\Phi \diamond \diamond \diamond \Phi}_{n-\text{times}}, \quad \Psi \in (\mathcal{H}_\infty(\mathbb{C}))^{-1}.$$

in $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$, and thus finite linear combinations of the form

$$\sum_{n=0}^{M} a_n \Phi^{\diamond n} \left(\Phi^{\diamond 0} := 1\right).$$

Remark 65 *Given a function $F:\mathbb{C}\to\mathbb{C}$ analytic on some neighborhood of the point $\mathbb{E}(\Phi)=(S\Phi)(0)=z_0\in\mathbb{C}$, we can define $F^{\diamond}(\Phi)\in(\mathcal{H}_\infty(\mathbb{R}))^{-1}$ by $F^{\diamond}(\Phi):=S^{-1}(F(S\Phi))$. If F has a power series representation $F(z)=\sum_n a_n(z-z_0)^n$, $z\in\mathbb{C}$, then it is easy to see that the Wick series $\sum_n a_n(\Phi-\mathbb{E}(\Phi))^{\diamond n}$ converges in $(\mathcal{H}_\infty(\mathbb{C}))^{-1}$. In particular, we can define $\exp^{\diamond}\Phi=S^{-1}(\exp(S\Phi))=\sum_{n=0}^{\infty}\frac{1}{n!}\Phi^{\diamond n}$. Furthermore we have the functional equation*

$$\exp^{\diamond}\Phi_1\diamond\exp^{\diamond}\Phi_2=\exp^{\diamond}(\Phi_1+\Phi_2).$$

For further details, see [26, Theorem 12, Example 3].

In an analogous way the $T-$transform of $\Phi\in(\mathcal{H}_\infty(\mathbb{C}))^{-1}$ is the mapping defined as

$$T\Phi(\xi):=\exp\left(-\frac{1}{2}\|\xi\|_0^2\right)\cdot S\Phi(i\xi),\quad \xi\in\mathcal{H}_\infty(\mathbb{C})\ \text{ with } 2^k\|\xi\|_l^2<1.$$

The $T-$transform has properties similar to the $S-$transform and Kondratiev distributions can be characterized through their $S-$and $T-$transform.

D.4 Polarization formula

Let Ψ and $\mathfrak{D}$ be vector spaces. Let F be a symmetric n-linear map from $\Psi\times\cdots\times\Psi$ (n-times) to $\mathfrak{D}$ and put

$$A(\varphi)=F(\underbrace{\varphi,\cdots,\varphi}_{n-times}),\quad \varphi\in\Psi,$$

$$F(\varphi_1,\cdots,\varphi_n)=\frac{1}{2^n n!}\sum_{\epsilon}\epsilon_1\cdots\epsilon_n A(\epsilon_1\varphi_1+\cdots\epsilon_n\varphi_n), \tag{D.20}$$

where $\sum_{\epsilon}$ means the summation over $\epsilon_1=\pm1,\cdots\epsilon_n=\pm1$. As an immediate consequence, we obtain

$$\varphi_1\hat{\otimes}\cdots\hat{\otimes}\varphi_n=\frac{1}{2^n n!}\sum_{\epsilon}\epsilon_1\cdots\epsilon_n(\epsilon_1\varphi_1+\cdots\epsilon_n\varphi_n)^{\otimes n} \tag{D.21}$$

for any $\varphi_1,\cdots,\varphi_n\in\Psi$.

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