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Resumen

Es una conjetura de Moerdijk que cualquier orbidad compacta no efectiva es presentable, esto es, puede ser obtenida como el cociente global de una variedad suave dividida por la acción de un grupo de Lie compacto. El objetivo central de esta tesis es demostrar la conjetura de Moerdijk mediante las técnicas del análisis armónico sobre groupoides de Lie.

Abstract

It is a conjecture of Moerdijk that every non-effective compact orbifold is presentable, that is, can be obtain as the global quotient of a smooth manifold divided by the action of a compact Lie group. The main goal of this thesis is to prove Moerdijk's conjecture through the techniques of Harmonic analysis over Lie groupoids.

Contents

Chapter 1

Preliminaries

1.1 A Brief Outline

An orbifold $\mathscr{X}$ consists of a topological space X together with a local presentation as a quotient of an open set of euclidean space divided by a local finite group, $U_x/G_x \cong V_x$, for every point $x \in X$ (where V_x is some open neighborhood of x), so that all such local presentations glue nicely (see, for instance [1]). Given a compact Lie group Γ, and due to *Koszul's slice theorem* [10], every almost free Γ-manifold M (so that for every point m in M the stabilizer Γ_m is finite) produces an orbifold $\mathscr{X} = M /\!\!/ \Gamma$[1] with underlying orbit space $X = M/\Gamma$. This motivates the following basic definition: We say that an orbifold $\mathscr{X}$ is *presentable* if $\mathscr{X} = M /\!\!/ \Gamma$ as above.

Proposition. *An orbifold $\mathscr{X}$ is called effective iff every one of its charts $U_x /\!\!/ G_x$ comes from an effective action of G_x on U_x. Every effective orbifold is presentable.*

[1] We will denote by $M /\!\!/ \Gamma$ to denote the orbit space M/Γ together with its orbifold or stack structure where we remember ths stabilizers of Γ at the various points of M.

The proof of this proposition is due to Satake, and it's very simple. Since, for every chart effectiveness ensures that G_x acts freely on the frame bundle of the tangent bundle $V_n(T(U_x))$, then the global frame bundle of the tangent bundle $M := V_n(T\mathscr{X})$ is a manifold, and therefore the global presentation

$$\mathscr{X} = M \mathbin{/\!\!/} O(n)$$

is what we were looking for. It is a conjecture of Moerdijk that every non-effective compact orbifold is presentable. In [8] Henriques and Metzler have partially proved this conjecture for the class of orbifolds whose ineffective groups have trivial center, and reduced the general problem to the case of a purely ineffective orbifold groupoid with equivariantly trivial abelian stabilizers. The main goal of this thesis is to prove Moerdijk's conjecture in full generality[2]. Let us briefly sketch the contents of the thesis. In Section 1.2 we follow Moerdijk [15] in formally defining an orbifold (or C^∞-Deligne-Mumford stack) as a Morita equivalence class of a proper étale groupoid. Then in Section 1.3 we recall the basic aspects of the representation theory of smooth groupoids following Kalivsnik [9]. In [17] Totaro has proved a special case of an algebraic version of Moerdijk's conjecture by proving the equivalence of two fundamental properties of algebraic stacks: being a quotient stack in a strong sense, and the *algebraic resolution property*, which says that every coherent sheaf is a quotient of some vector bundle. In Section **??** we rephrase the resolution property for the case of C^∞-orbifolds by means of the theory of universal Hilbert bundles over groupoids developed in [5]. In Section 2.2 we prove that every compact orbifold has the resolution property using the classical spectral theorem for self-adjoint compact operators along with some general facts about Hilbert bundles and C^*-algebras developed by Dixmier and Douady [4]. Finally, in Section 2.3 we show that to prove the resolution property in this case is equivalent to prove Moerdijk's conjecture.

[2] It should be noted that our method to approach this conjecture is totally independent of [8].

1.2 Differentiable Stacks

In this section we briefly recall how the notion of orbifold fits within the general formalism of differentiable stacks. In order to keep things simple, we will take a shortcut and describe differentiable stacks as Morita equivalence classes of Lie groupoids. This will be enough for the purposes we have in mind. For a detailed review about the theory of smooth stacks we refer the reader to Metzler [14] and for a discussion about orbifolds within this formalism we recommend Moerdijk [15] (see also Lerman [13]).

Roughly speaking, a Lie groupoid is a groupoid object inside the category of smooth manifolds[3]. More precisely, this amounts to specify the following data. A **smooth groupoid** G is of a pair (G_0, G_1) of smooth manifolds whose points are called the **objects** and the **arrows** of G, respectively, together with the following structural smooth maps:

- A pair of submersions $s, t : G_1 \to G_0$ called the **source** and **target** maps, respectively. We will usually represent an arrow $f \in G_1$ by the notation $f : s(f) \to t(f)$. Note that the submersivity condition guarantees that the set of **composable arrows** defined by

  $$G_1 \times_{s,G_0,t} G_1 := \{(f, g) \in G_1 \times G_1 | s(f) = t(g)\}.$$

  is endowed with a smooth manifold structure induced by the product manifold $G_1 \times G_1$.

- A **composition** map $m : G_1 \times_{s,G_0,t} G_1 \to G_1$ which takes a pair of composable arrows

  $$z \xleftarrow{f} y \xleftarrow{g} x$$

  into the composite arrow $f \cdot g := m(f, g)$. Moreover, this composition of arrows is supposed to be associative, i.e. whenever one has composable

[3]For us smooth manifolds will always be Hausdorff and second countable differentiable manifolds of C^∞-class.

arrows $z \xleftarrow{f} y \xleftarrow{g} x \xleftarrow{h} w$ the equality $f \cdot (g \cdot h) = (f \cdot g) \cdot h$ is satisfied.

- A **unit** map $u : G_0 \to G_1$ which takes each object $x \in G_0$ into the unit arrow $1_x := u(x) : x \to x$ satisfying $f \cdot 1_{s(f)} = 1_{t(f)} \cdot f = f$ for every other arrow $f \in G_1$.
- An **inverse** map $\iota : G_1 \to G_1$ which takes each arrow $f \in G_1$ into the inverse arrow $f^{-1} := \iota(f) : t(f) \to s(f)$ satisfying the following pair of equalities $f \cdot f^{-1} = 1_{t(f)}$ and $f^{-1} \cdot f = 1_{s(f)}$.

We sometimes use the notation $G = [G_1 \rightrightarrows G_0]$ to denote a Lie groupoid. Note that the image of the **anchor map**

$$a_G := (s, t) : G_1 \to G_0 \times G_0$$

$$f \mapsto (s(f), t(f))$$

determines an equivalence relation on G_0, i.e. a pair of objects $x, y \in G_0$ will be related if there is an arrow $f : x \to y$ in G_1. The corresponding set of equivalence classes, $|G| := G_0/G_1$, can be endowed with the quotient topology induced by the canonical projection map

$$\pi_G : G_0 \to |G|,$$

and is called the **coarse** quotient space of G. The core intuition in the theory of differentiable stacks is that the groupoid G acts as an enhacement for the above equivalence relation in the sense that it allows to relate a pair of objects $x, y \in G_0$ in more than one way; namely one for each arrow $f : x \to y$. In particular, any single object $x \in G_0$ comes equipped with a **stabilizer** group

$$G_x := \{f \in G_1 | s(f) = x = t(f)\}$$

controlling the variety of ways in which it can be related to itself. The following are the simplest examples of Lie groupoids.

Example 1.2.1. *For any smooth manifold M we have its corresponding* **unit** *groupoid $[M \rightrightarrows M]$ which has for objects the points of M and has only identity arrows. In particular, its coarse quotient is M itself.*

Example 1.2.2. *For any Lie group G one has a groupoid $[G \rightrightarrows \bullet]$ with a single object whose composition of arrows is given by the product of G. Note that the corresponding coarse quotient space is reduced to a singleton.*

Example 1.2.3. *More generally, let G be a Lie group and suppose that M is a smooth G-manifold[4]. Then we define the* **translation** *groupoid*

$$M \ltimes G := [M \times G \rightrightarrows M],$$

whose arrows have the form $(m,g) : m \to m \cdot g$ and where the composition is defined by means of $(m \cdot g, m \cdot gh) \cdot (m, m \cdot g) := (m, m \cdot gh)$. Moreover for each object $m \in M$ we define the identity $1_m := (m,e)$ and the inverse of an arrow $(m,g) \in M \times G$ is defined by $(m,g)^{-1} := (m \cdot g, g^{-1})$, where e denotes the identity of G. Thus the groupoid $M \ltimes G$ provides a finer model for the orbit space $M/G = |M \ltimes G|$.

The natural notion for morphisms between Lie groupoids is that of smooth functors[5], namely a **smooth functor** $F : G \to H$ consists of a pair of smooth maps $F_0 : G_0 \to H_0$ and $F_1 : G_1 \to H_1$ such that

$$F_1(\, x \xrightarrow{f} y \,) = \; F_0(x) \xrightarrow{F_1(f)} F_0(y)$$

for each arrow $f \in G_1$ and that $F_1(1_x) = 1_{F_0(x)}$ for each object $x \in G_0$ as well that $F_1(f \circ g) = F_1(f) \circ F_1(g)$ for each pair $z \xleftarrow{f} y \xleftarrow{g} x$ of composable arrows. One can organize Lie groupoids together with smooth functors into a category, denoted by Groupoids, which can in turn be enhanced into a strict bicategory[6] by means of smooth natural transformations, namely a **smooth**

[4]Than is, a smooth manifold equipped with a smooth (right) action $M \circlearrowright G$.

[5]These are sometimes called **strict morphisms**.

[6]For a survey about bicategories see Leinster [12].

natural transformation $\theta : F \to F'$ between a pair of smooth functors $F, F' : G \to H$ is a smooth map $\theta : G_0 \to H_1$ such that for each arrow $f : x \to y$ in G_1 the square below commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F_1(f)} & F(y) \\ {\scriptstyle \theta(x)}\downarrow & & \downarrow{\scriptstyle \theta(y)} \\ F'(x) & \xrightarrow[F_1'(f)]{} & F'(y) \end{array}$$

It turs out that strict isomorphisms as well as strict equivalences are too restrictive notions in the sense that two Lie groupoids could constitute equally good fine models for the same coarse quotient space but without being neither strictly isomorphic nor strictly equivalent. The appropriate notion of equivalence for such matters is that of weak equivalence, namely a smooth functor $F : G \to H$ is said to be a **weak equivalence**[7] if:

- The square below is cartesian[8]:

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ {\scriptstyle a_G}\downarrow & & \downarrow{\scriptstyle a_H} \\ G_0 \times G_0 & \xrightarrow[F_0 \times F_0]{} & H_0 \times H_0 \end{array}$$

- The map $t \circ \pi_2 : G_0 \times_{F_0, H_0, s} H_1 \to H_0$ is a surjective submersion.

In other words a weak equivalence $F : G \xrightarrow{\sim} H$ is a categorical equivalence which takes the relevant smooth structures into account. In particular it induces a homeomorphism $|G| \cong |H|$ beween the coarse quotients. We say that a pair of Lie groupoids G and H are **Moerdijk-Morita equivalent** if there is a third Lie groupoids L together with a pair of weak equivalences

$$G \xleftarrow{\sim} L \xrightarrow{\sim} H.$$

[7]We denote this by $F : G \xrightarrow{\sim} H$

[8]That is, a fiber product of smooth manifolds.

In fact this is an equivalence relation between Lie groupoids. We could define a **differentiable stack** $\mathscr{X}$ simply as a Morita equivalence class of Lie groupoids and any particular Lie groupoid G belonging to that class is said to be a **presentation**[9], denoted by $\mathscr{X} = [G]^{\mathrm{st}}$. In order to define morphism between differentiable stacks we can proceed as follows. The idea is that to get the bicategory of differentiable stacks, denoted by $\underline{\mathrm{Stacks}}$ we only need to formally invert the class $W \subset \mathrm{Mor}(\underline{\mathrm{Groupoids}})$ of weak equivalences in the bicategory of Lie groupoids, of course, in the bicategorical sense of the term[10]. Thus we would get a lax localization functor

$$\underline{\mathrm{Groupoids}} \to \underline{\mathrm{Stacks}} := \underline{\mathrm{Groupoids}}[W^{-1}]$$

with the property that sends a smooth functor $F : G \to H$ into a equivalence iff $F : G \xrightarrow{\sim} H$ is a weak equivalence of Lie groupoids. This provides a complete characterization for the bicategory of differentiable stacks[11]. There are various ways to realize such a localization procedure. One of them is thorugh the notion of Hilsum-Skandalis map (see, e.g. [13]).

Let us recall that a smooth (left) action of a Lie groupoid G on a smooth manifold M along a smooth map $p : M \to G_0$ is a smooth map

$$\theta : G_1 \times_{s,G_0,p} M \to M$$

$$(f, m) \mapsto f \cdot m := \theta(f, m)$$

such that $1_x \cdot m = m$ and $(f \cdot h) \cdot m = f \cdot (h \cdot m)$, whenever makes sense.

Remark 1.2.1. *Note that for any such smooth action $G \circlearrowleft M$ one can also define a corresponding translation groupoid $G \rtimes M$.*

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[9]We also say that $\mathscr{X}$ is the **stacky quotient** of G.

[10]See Pronk-Moerdijk [16].

[11]Also this provides a more rigorous formulation of the naive idea that a differentiable stack is a Morita equivalence class of Lie groupoids.

We conclude this section by saying a few words about orbifolds. Let us recall that a Lie groupoid G is called an **orbifold groupoid** if its source and target maps are *étale*[12] and if the anchor map $a_G : G_1 \to G_0 \times G_0$ is proper. It turns out that in such a case, for every point $x \in G_0$ we can find an open neighborhood $x \in U_x \subseteq G_0$ such that the **restricted groupoid** G_{U_x} is is strictly isomorphic to the translation groupoid $U_x \ltimes G_x$ where for a suitably defined action of the stabilizer group at x on U_x. In particular, this implies that the coarse quotient $|G|$ has an orbifold structure. An **orbifold** is a differentiable stack which admits a presentation $\mathscr{X} = [G]^{\mathrm{st}}$ by an orbifold groupoid G.

1.3 Representations of Groupoids

Here we briefly recall the basic definitions from the representation theory of Lie groupoids (cf. [9]). A **unitary representation** of a Lie groupoid G is a (locally trivial) Hilbert bundle $\pi_{\mathcal{H}} : \mathcal{H} \to G_0$ over its manifold of objects together with a continuous action $\theta : G_1 \times_{s,G_0,\pi_{\mathcal{H}}} \mathcal{H} \to \mathcal{H}$ on the total space along the bundle projection which is fiberwise unitary, i.e. for each arrow $f \in G_1$ the corresponding map

$$\mathcal{H}(s(f)) \xrightarrow{\theta_f} \mathcal{H}(t(f))$$

on the fibers[13] given by $v \mapsto \theta(f, v)$, where $(f, v) \in G_1 \times_{s,G_0,\pi_{\mathcal{H}}} \mathcal{H}$, is a unitary isomorphism. Thus, for each point $x \in G_0$ we have $\theta_{1_x} = 1_{\mathcal{H}(x)}$, and for each pair $(f, g) \in G_1 \times_{s,G_0,t} G_1$ of composable arrows the equality $\theta_{f\cdot g} = \theta_f \circ \theta_g$ holds. In particular, one has defined a unitary representation $G_x \circlearrowleft \mathcal{H}(x)$ of the stabilizer group at every point $x \in G_0$.

Example 1.3.1. *For every smooth manifold M, a unitary representation of the unit groupoid $[M \rightrightarrows M]$ is nothing but a Hilbert bundle $\mathcal{H} \to M$.*

[12] i.e. local diffeomorphisms.

[13] We write $\mathcal{H}(x)$ to denote the fiber above $x \in G_0$

Example 1.3.2. *For every Lie group G, a unitary representation of the corresponding groupoid $[G \rightrightarrows \bullet]$ with one object is the same thing as a unitary group representation $G \circlearrowleft \mathcal{H}(\bullet)$ on a single Hilbert space.*

Example 1.3.3. *Given a Lie group G and a smooth G-manifold, a unitary representation of the translation groupoid $M \ltimes G$ is just a G-equivariant Hilbert bundle $\mathcal{H} \to M$.*

Let Let us be a little bit more precise about the notion of Hilbert bundle used in the above definition. A **continuous field of Hilbert spaces** over a topological space X (cf. [4]) consists of a family $\{\mathcal{H}(x)\}_{x\in X}$ of complex Hilbert spaces together with a linear subspace $\Gamma \subseteq \prod_{x\in X} \mathcal{H}(x)$[14], satisfying:

- For each point $x \in X$ and each vector $v_0 \in H_x$ there is a $v \in \Gamma$ such that $v_x = v_0$.
- For each pair $v, w \in \Gamma$[15] the **braket** function $\langle v|w\rangle : X \to \mathbb{C}$ defined by $x \mapsto \langle v(x)|w(x)\rangle_x$[16] is continuous.
- If $v \in \prod_{x\in X} H_x$ verifies that the function $||v - w|| : X \to \mathbb{R}_{\geq 0}$[17] is continuous for every $w \in \Gamma$, then $v \in \Gamma$.

Whenever the pair $(\{\mathcal{H}_x\}_{x\in X}, \Gamma)$ is a continuous field of Hilbert spaces over the space X, for each point $x \in X$ the Hilbert space $\mathcal{H}(x)$ is called the **fiber** above x, and the elements of the linear space Γ[18] are called its **continuous fields of vectors**. This terminology is justified since we can define a suitable

[14]The linear operations on $\prod_{x\in X} \mathcal{H}(x)$ are defined pointwise.

[15]We usually denote an element $v \in \prod_{x\in X} \mathcal{H}(x)$ by means of $v = \{v(x)\}_{x\in X}$.

[16]Here we are using Dirac's bracket notation $\langle\cdot|\cdot\rangle_x : \mathcal{H}(x) \times \mathcal{H}(x) \to \mathbb{C}$ to denote the inner product.

[17]For each $v \in \prod_{x\in X} \mathcal{H}(x)$ the **norm** function $||v|| : X \to \mathbb{R}_{\geq 0}$ is defined by means of $x \mapsto ||v(x)||_x := \sqrt{\langle v(x)|v(x)\rangle_x}$.

[18]It turns out that Γ is, in fact, a module over the algebra of complex valued continuous functions $X \to \mathbb{C}$ by means of pointwise scalar product.

topology on the **total space** of the field, defined as the disjoint union

$$\mathcal{H} := \coprod_{x \in G_0} \mathcal{H}(x)$$

of the fibers, such that the obvious projection $\pi_{\mathcal{H}} : \mathcal{H} \to X$ becomes an open continuous map establishing an identification $\Gamma \cong \Gamma(X, \mathcal{H})$ between continuous fields of vectors and continuous sections of $\pi_{\mathcal{H}}$. Moreover, for each point $x \in X$ the topology induced on the fiber $\mathcal{H}(x)$ coincides with the norm topology. On the other hand, the sum $+ : \mathcal{H} \times_X \mathcal{H} \to \mathcal{H}$ and the scalar product $\cdot : \mathbb{C} \times \mathcal{H} \to \mathcal{H}$ operations also becomes continuous. In short, the map $\pi_{\mathcal{H}} : \mathcal{H} \to G_0$ is a (not necessarily locally trivial) Hilbert bundle[19] over the space X.

Example 1.3.4. *We define the* **constant** *field over X with fiber given by the standard separable Hilbert space $l^2(\mathbb{Z})$ in the following way*[20]*. For each point $x \in X$ take $\mathcal{H}(x) := l^2(\mathbb{Z})$ and let $\Gamma := C(X, l^2(\mathbb{Z}))$ be the collection of continuous functions $X \to l^2(\mathbb{Z})$ in the norm topology of $l^2(\mathbb{Z})$. It turns out that $(\{\mathcal{H}(x)\}_{x \in X}, \Gamma)$ defines a continuous field of Hilbert spaces over X and that there is a canonical bundle homeomorphism $\mathcal{H} \cong (l^2(\mathbb{Z}) \times X)$ with the trivial bundle.*

Example 1.3.5. *Suppose that $\pi_{\mathcal{E}} : \mathcal{E} \to X$ is a locally trivial complex vector bundle of finite rank over X endowed with a Hermitian metric. Hence, it follows that $(\{\mathcal{E}(x)\}_{x \in X}, \Gamma(X, \mathcal{E}))$, where $\mathcal{E}(x) := \pi_{\mathcal{E}}^{-1}(x)$ and $\Gamma(X, \mathcal{E})$ is the space of continuous sections of $\pi_{\mathcal{E}}$, is a continuous field of finite dimensional Hilbert spaces over X.*

There is the following useful way to construct continuous fields of Hilbert spaces. Let $\{\mathcal{H}(x)\}_{x \in X}$ be a family of complex Hilbert spaces parametrized

[19] We will usually denote a continuous field of Hilbert spaces $(\{\mathcal{H}_x\}_{x \in X}, \Gamma)$ simply by means of its total space $\mathcal{H}$.

[20] The same construction works for any finite dimensional Hilbert space.

by the points of a topological space X. Suppose that $\Lambda \subseteq \prod_{x\in X}\mathcal{H}(x)$ is a linear subspace such that for each point $x \in X$ the subset

$$\Lambda(x) := \{v(x)|v \in \Lambda\} \subseteq \mathcal{H}_x$$

is dense in $\mathcal{H}(x)$ and for each $v \in \Lambda$ the norm function $||v|| : X \to \mathbb{C}$ is continuous. Then there exist a unique linear subspace $\Gamma \subseteq \prod_{x\in X}\mathcal{H}(x)$ such that $\Lambda \subseteq \Gamma$ for which the pair $(\{\mathcal{H}(x)\}_{x\in X}, \Gamma)$ becomes a continuous field of Hilbert spaces over X. Namely, the linear subspace $\Gamma \subseteq \prod_{x\in X}\mathcal{H}(x)$ consists of those elements $v \in \prod_{x\in X}\mathcal{H}(x)$ such that $||v-w|| : X \to \mathbb{R}_{\geq 0}$ is continuous whenever $w \in \Lambda$. When this happens, we will say that the continuous field of Hilbert spaces $(\{\mathcal{H}(x)\}_{x\in X}, \Gamma)$ is generated by Λ.

Example 1.3.6. *Let $p : Y \to X$ be a continuous map between topological spaces. If $(\{\mathcal{H}(x)\}_{x\in X}, \Gamma)$ if a continuous field of Hilbert spaces over X, then the linear subspace*

$$\Lambda := \{v = \{v(p(y))\}_{y\in Y}|v \in \Gamma\} \subseteq \prod_{y\in Y}\mathcal{H}(f(y))$$

generates a continuous field of Hilbert spaces over Y, called the **pull-back** *of $\mathcal{H}$ along p and denoted by $p^*(\mathcal{H})$. It turns out that there is a canonically homeomorphism $p^*(\mathcal{H}) \cong \mathcal{H} \times_{\pi_{\mathcal{H}},X,p} Y$. Thus, in particular, we can define the* **restriction** *of $\mathcal{H}$ over an open subset $U \subseteq X$ as its pull-back along the inclusion map $U \hookrightarrow X$, usually denoted by $\mathcal{H}|_U$.*

Example 1.3.7. *Let $\{\mathcal{H}_i\}_{i\in I}$ be a collection of continuous fields of Hilbert spaces over X indexed by a set I. Consider the family of Hilbert spaces*

$$\{\mathcal{H}(x) := \widehat{\bigoplus}_{i\in I}\mathcal{H}_i(x)\}_{x\in X}$$

obtained by taking orthogonal sums. Then, it turns out that the linear subspace $\Lambda \subset \prod_{x\in X}\mathcal{H}(x)$ consisting of those

$$v = \sum_{i\in I_0\subset I} v_i$$

such that I_0 finite and $v_i \in \Gamma(X, \mathcal{H}_i)$ for very $i \in I_0$ generates a continuous field of Hilbert spaces over X. This is called the **orthogonal sum** *of the family $\{\mathcal{H}_i\}_{i\in I}$ and is usually denoted by $\widehat{\bigoplus}_{i\in I}\mathcal{H}_i$.*

Now let's discuss morphisms between continuous field of Hilbert spaces. A **bounded operator** $\mathcal{H} \to \mathcal{H}'$ between a pair of continuous fields of Hilbert spaces over X is a family

$$\{\Psi(x) : \mathcal{H}(x) \to \mathcal{H}'(x)\}_{x \in X}$$

of bounded operators between the fibers (in the usual sense) such that the induced map $\Psi : \mathcal{H} \to \mathcal{H}'$ on the total spaces is continuous[21]. For an arbitrary bounded operator $\Psi : \mathcal{H} \to \mathcal{H}'$ the corresponding family of adjoints

$$\{\Psi^*(x) : \mathcal{H}'(x) \to \mathcal{H}(x)\}_{x \in X}$$

does not necessarily defines a bounded operator $\Psi^* : \mathcal{H}' \to \mathcal{H}$. Whenever it actually does, we will say that Ψ is **adjointable** and Ψ^* will be called its **adjoint**. It is easy to see that the composition of two adjointable operators is again adjointable.

Remark 1.3.1. *It can be prove that for any bounded operator $\Psi : \mathcal{H} \to \mathcal{H}'$ the* **norm function** $||\Psi|| : X \to \mathbb{R}_{\geq 0}$ *defined by means of $x \mapsto ||\Psi(x)||_x$*[22] *is locally bunded.*

We will say that an adjointable operator $\Psi : \mathcal{H} \to \mathcal{H}'$ is **unitary** if the identities $\Psi^*\Psi = 1_{\mathcal{H}}$ and $\Psi\Psi^* = 1_{\mathcal{H}'}$ are satisfied. In such a case we will say that $\mathcal{H}$ and $\mathcal{H}'$ are unitarily isomoprhic, denoted by $\mathcal{H} \cong \mathcal{H}'$.

Example 1.3.8. *Let $\mathcal{H}$ be a continuous field of Hilbert spaces over X. A* **subfield** $\mathcal{H}' \subseteq \mathcal{H}$ *is another continuous field of Hilbert spaces over X such that for each point $x \in X$ one has that $\mathcal{H}'(x) \subseteq \mathcal{H}(x)$ is a closed subspace and $\Gamma(X, \mathcal{H}') \subseteq \Gamma(X, \mathcal{H})$ is a linear subspace. It turns out that the canonical inclusion $\iota_{\mathcal{H}'} : \mathcal{H}' \hookrightarrow \mathcal{H}$ defines a bounded operator.*

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[21] Note that such $\Psi : \mathcal{H} \to \mathcal{H}'$ induces a linear map $\Gamma \to \Gamma'$ between continuous fields of vectors by $v \mapsto \Psi v := \{\Psi(x)v(x)\}_{x \in X}$. It turns out that this linear map determines Ψ completely.

[22] Here $||\Psi(x)||_x$ denotes the operator norm.

Example 1.3.9. *Let $\mathcal{H}'$ and $\mathcal{H}''$ be a pair of continuous fields of Hilbert spaces over X. Then we can canonically identify $\mathcal{H}'$ and $\mathcal{H}''$ with subfields of the orthogonal sum $\mathcal{H} := \mathcal{H}' \bigoplus \mathcal{H}''$. On the other hand, the family of orthogonal projectors $P_{\mathcal{H}'} := \{P_{\mathcal{H}'(x)} : \mathcal{H}(x) \to \mathcal{H}'(x)\}_{x \in X}$ is an adjointable bounded operator $P_{\mathcal{H}'} : \mathcal{H} \to \mathcal{H}'$ whose adjoint is given by the corresponding inclusion $P^*_{\mathcal{H}'} = \iota_{\mathcal{H}'}$.*

Example 1.3.10. *It turns out that the inclusion $\iota_{\mathcal{H}'} : \mathcal{H}' \hookrightarrow \mathcal{H}$ of a subfield is adjointable iff $\mathcal{H}'$ is* **complementable**, *i.e. if the* **orthogonal complement** *of $\mathcal{H}$ defines a subfield $\mathcal{H}'^{\perp} \subseteq \mathcal{H}$ such that $\mathcal{H} \cong \mathcal{H}' \oplus \mathcal{H}'^{\perp}$.*

Example 1.3.11. *Let $\mathcal{H}$ be a continuous field of Hilbert spaces over X. For any pair of continuous fields of vectors $v, w \in \Gamma(X, \mathcal{H})$ we define the* **ketbra** *operator $|v\rangle\langle w| : \mathcal{H} \to \mathcal{H}$ by means of $u \mapsto \langle u|v\rangle \cdot w$ for $u \in \Gamma(X, \mathcal{H})$. It turs out that given any family of continuous fields of vectors*

$$\{v_k, w_k \in \Gamma(X, \mathcal{H})\}_{k=1}^n$$

the corresponding **finite rank** *operator defined by $\sum_{k=1}^n |v_k\rangle\langle w_k|$ is an adjointable bounded operator on $\mathcal{H}$. More generally, we say that an adjointable bunded operator $\Psi : \mathcal{H} \to \mathcal{H}$ is* **compact** *if it can be locally approximated by finite rank operators, i.e. if for each point $x \in X$ and every $\epsilon > 0$ there is an open neighborhood $x \in U \subseteq X$ together with an operator $\Phi : \mathcal{H} \to \mathcal{H}$ of finite rank such that $||\Psi - \Phi|| < \epsilon$ on U. In such a case, it is not hard to see that, for each point $x \in X$ the operator $\Psi(x) : \mathcal{H}(x) \to \mathcal{H}(x)$ is a compact operator in the usual sense, and that the norm function $||\Psi|| : X \to \mathbb{R}_{\geq 0}$ is continuous. Moreover, whenever $\mathcal{H} = (l^2(\mathbb{Z}) \times X)$ is the constant field of fiber $l^2(\mathbb{Z})$, to have a compact operator $\Psi : \mathcal{H} \to \mathcal{H}$ is just the same thing as to have a continuous function $\Psi : X \to \mathcal{K}$ where $\mathcal{K}$ denotes the spaces of compact operators on $l^2(\mathbb{Z})$ endowed with the operator norm topology.*

We say that a continuous field of (separable) Hilbert spaces $\mathcal{H}$ over X is a **Hilbert bundle** if is locally trivial, i.e. if for every point $x \in X$ there is an open neighborhood $x \in U \subseteq X$ such that the restriction $\mathcal{H}|_U$ is isomorphic to a constant field.

Remark 1.3.2. *It turns out that every infinite dimensional Hilbert bundle over a smooth manifold is globally trivial (cf. [4] Théorème 5.). This is a consequence of Kuiper's theorem*[23] *On the other hand, if $\mathcal{E}$ is a continuous family of finite dimensional Hilbert spaces over X, then an easy argument using the continuity of the determinant shows that the function $x \mapsto dim_{\mathbb{C}}(\mathcal{E}(x))$ is lower semi-continuous. Moreover, it is not hard to see that $\mathcal{E}$ is locally trivial iff this function is constant.*

Let's come back to groupoid representations. Suppose that G is a Lie groupoid and let $\mathcal{H}$ and $\mathcal{H}'$ be a pair of unitary G-representations. We will say that an adjointable bounded operator $\Psi : \mathcal{H} \to \mathcal{H}'$ between the undelying Hilbert bundles is G-equivariant if for each arrow $f : x \to y$ in G_1 the square below commutes:

$$\begin{array}{ccc} \mathcal{H}(x) & \xrightarrow{\theta_f} & \mathcal{H}(y) \\ {\scriptstyle \Psi(x)}\downarrow & & \downarrow{\scriptstyle \Psi(y)} \\ \mathcal{H}'(x) & \xrightarrow[\theta'_f]{} & \mathcal{H}'(y) \end{array}$$

One can organize unitary representations of G together with G-equivariant adjointable bounded operators between them into a $*$-category[24], which we will denoted by $\mathrm{Rep}(G)$. It turns out that any equivalence $G \simeq H$ between Lie groupoids induces an equivalence $\mathrm{Rep}(G) \simeq \mathrm{Rep}(H)$ between its categories of representations.

A representations $\mathcal{H}$ of a Lie groupoid G is said to be **universal** if every other representation $\mathcal{H}'$ can be G-equivariantly embedded as $\mathcal{H}' \hookrightarrow \mathcal{H}$. By means of the *Eilenberg swindle trick* (cf. [5] Lemma A.24) one can show that a representation $\mathcal{H}$ is universal iff for every other $\mathcal{H}'$ we have a G-equivariant

[23]cf. Kuiper [11].
[24]That is, a complex linear category with an antilinear involution (see Baez [2]).

isomorphism $\mathcal{H} \oplus \mathcal{H}' \cong \mathcal{H}$, when this happens we also say that $\mathcal{H}$ has the **absortion property**. It follows that universal representations, whenever exist, are unique up to isomorphism. It turs out that universality is a local property (cf. [5] Lemma A.28). More precisely, we have that $\mathcal{H}$ is universal iff there is a cover $G = \cup_{i \in I} G_i$ by open subgroupoids $G_i \subseteq G$ such that for each $i \in I$ the restricted respresentation $\mathcal{H}|_{U_i}$ is universal.

Remark 1.3.3. *Note that whenever $\mathcal{H}$ is universal the underlying Hilbert bundle $\mathcal{H} \to G_0$ must be infinite dimensional and, therefore, it is globally trivial (in a non equivariant way).*

Example 1.3.12. *It is easy to see that for any smooth manifold M the trivial bundle $l^2(\mathbb{Z}) \times M \to M$ has the absortion property. For instance, if $\mathcal{H}$ is a Hilbert bundle over M one has $\mathcal{H} \oplus (l^2(\mathbb{Z}) \otimes M) \cong l^2(\mathbb{Z}) \otimes M$, since every infinite dimensional Hilbert bundle over M is globally trivial.*

Example 1.3.13. *By the Peter-Weyl theorem it follows that for any compact Lie group G the left regular representation $L^2(G)$ is a universal.*

Example 1.3.14. *By combining the above two examples it can be prove that for any compact Lie group G and any smooth G manifold M the trivial Hilbert bundle $L^2(G) \times M \to M$ endowed with the obvious G action is a universal representation of the translation groupoid $M \ltimes G$ (cf. [5] Lemma A.32)*

Let us briefly sketch how one can construct a universal representation of an orbifold groupoid.

Proposition 1.3.1. *Every orbifold groupoid has a universal representation.*

Proof: Firstly, let us show that every translation groupoid $U \ltimes \Gamma$ where U is contractible and Γ is a finite group has a universal representation. Let us denote by $\mathbb{C}[\Gamma]$ the left regular representation of Γ. Note that the trivial bundle $\mathbb{C}[\Gamma] \times U \to U$ endowed with the obvious Γ-action defines a finite rank representation of $U \ltimes \Gamma$. We claim that the corresponding stabilized representation $(l^2(\mathbb{Z}) \otimes \mathbb{C}[\Gamma]) \times U \to U$ is universal. Let us prove that for any representation $\mathcal{H} \in \mathrm{Rep}(U \ltimes \Gamma)$ of infinite rank there is a Γ-equivariant emmbeding $\mathcal{H} \hookrightarrow l^2(\mathbb{Z}) \otimes U$ (the finite rank case is similar). Since the underlying Hilbert bundle $\mathcal{H} \to U$ is trivial we can choose an homogeneous orthogonal frame $\{e_n \in \Gamma(U, \mathcal{H})\}_{n\in\mathbb{N}}$, that is $\langle e_i | e_j \rangle \equiv \delta_{i,j}$. Note that there is a linear action $\Gamma \circlearrowleft \Gamma(G_0, \mathcal{H})$ defined by $g \cdot v := (v(x \cdot g) \cdot g^{-1})_{x\in U}$ for any pair $(g, v) \in \Gamma \times \Gamma(G_0, \mathcal{H})$. Hence, it is not hard to see that the map

$$\Gamma(U, \mathcal{H}) \to \Gamma(U, l^2(\mathbb{Z}) \otimes \mathbb{C}[\Gamma])$$

$$v \mapsto \sum_{n\in\mathbb{N}} \langle v | e_n \rangle \otimes \sum_{g\in\Gamma} \langle v \cdot g | v \rangle$$

induces a unitary Γ-equivariant embedding $\mathcal{H} \hookrightarrow l^2(\mathbb{Z}) \otimes \mathbb{C}[\Gamma]$. Now suppose that G is an arbitrary orbifold groupoid. Hence, there is a cover

$$G = \cup_{n\in\mathbb{N}} (U_n \ltimes \Gamma_n)$$

by open subgroupoids $(U_n \ltimes \Gamma_n) \subseteq G$, where U_n is a contractible smooth manifold and Γ_n is a finte group. We knows that for each $n \in \mathbb{N}$ there exist a universal representation $\mathcal{H}_n \in \mathrm{Rep}(U_n \ltimes \Gamma_n)$, and we will construct from those a universal representation of G as follows. For each $k \in \mathbb{N}$ let us define the open subgroupoid $G_k \subseteq G$ as

$$G_k := \cup_{n=1}^{k} (U_n \ltimes \Gamma_n).$$

Now suppose that for some particular $k \in \mathbb{N}$ we know that there exist a universal representation $\mathcal{H} \in \mathrm{Rep}(U_k \ltimes \Gamma_k)$. Then the restriction of $\mathcal{H}_k$ as well as the restriction of $\mathcal{H}$ defines a universal representation of the intersection groupoid $G_k \cap (U_{k+1} \ltimes \Gamma_{k+1})$. In particular, there exist an isomorphism

$$\mathcal{H}|_{G_k \cap (U_{k+1} \ltimes \Gamma_{k+1})} \cong \mathcal{H}_k|_{G_k \cap (U_{k+1} \ltimes \Gamma_{k+1})}$$

of representations, and we can glue $\mathcal{H}$ and $\mathcal{H}_k$ along $G_k \cap (U_{k+1} \ltimes \Gamma_{k+1})$ by means of this isomorphism to get a universal representation of G_{k+1}. We can obtain inductively a universal representation of G through the precedent construction.$\square$

Remark 1.3.4. *More generally, the above proof can be generalized to the case of locally presentable Lie groupoids (cf. [5] Corollary A.33). In particular it follows that every proper Lie groupoid admits a universal representation.*

Chapter 2

The Generalized Moerdijk's Conjecture

In this second chapter we will prove a generalization of the Moerdijk's conjecture about the presentation problem of compact orbifolds. In order to approach this conjecture we will introduce the resolution property for differentiable stacks and discuss some basic aspects of the harmonic analysis on Lie groupoids .

2.1 The Resolution Property

(1) Classically, in the realm of algebraic geometry, the resolution property for algebraic stacks reads as follows: We say that an algebraic stack $\mathscr{X}$ has the resolution property if any coherent sheaf $\mathcal{E} \in \mathrm{Coh}(\mathscr{X})$ is the quotient of a locally free sheaf $\mathcal{E}' \in \mathrm{Vect}(\mathscr{X})$, i.e. there is an epimorphism $\mathcal{E}' \twoheadrightarrow \mathcal{E}$. In other words, the algebraic stack $\mathscr{X}$ has the resolution property if its category of coherent sheaves is generated by the full subcategory

$$\mathrm{Vect}(\mathscr{X}) \subset \mathrm{Coh}(\mathscr{X})$$

of vector bundles. Roughly, what this means is that there are enough vector bundles on $\mathscr{X}$. In [17], Totaro shows how the resolution property is related with the problem of presenting an algebraic stack as a global quotient. On the other hand, in the topological or the smooth frameworks, one can find suitable analogues of the above resolution property. For instance in [7] it is proposed the following definition: We say that an orbispace $\mathscr{X}$ has enough vector bundles if every vector bundle $\mathcal{E}' \in \mathrm{Vect}(\mathscr{X}')$ of finite rank defined over a sub-orbispace $\mathscr{X}' \subseteq \mathscr{X}$ can be extended to the whole $\mathscr{X}$, i.e. one can find a vector bundle $\mathcal{E} \in \mathrm{Vect}(\mathscr{X})$ such that $\mathcal{E}|_{\mathscr{X}'} \cong \mathcal{E}'$. As in the algebraic situation, the existence of enough vector bundles over orbispaces turns out to be closely related to the presentation problem. In this section, we will introduce a variation of the resolution property for differentiable stacks.

(2) Suppose that $\mathscr{X} = [G]^{\mathrm{st}}$ is a differentiable stack presented by some Lie groupoid G. Hence, one can thought in the corresponding category $\mathrm{Rep}(G)$ of unitary representations as a replacement of the category of (quasi-)coherent sheaves on $\mathscr{X}$ within the smooth realm. So, we would like to say that $\mathscr{X}$ has the resolution property if the full subcategory $\mathrm{Vect}(G) \subset \mathrm{Rep}(G)$ of finite dimensional representations generates, in some sense, the whole category of unitary representations. However, the categorical notion of generation is no so well suited in this context, and we must rephrase it in another way.

Definition 2.1.1. *We say that a Lie groupoid G has the* **resolution property** *if it admits a universal representation $\mathcal{H}_G$ and the finite part*

$$\Gamma(G_0, \mathcal{H}_G)^{fin} \subset \Gamma(G_0, \mathcal{H}_G)$$

consisting of those sections $v \in \Gamma(M, \mathcal{H}_G)$ for which there is a sub-representation $\mathcal{E} \subseteq \mathcal{H}$ of finite rank with $v \in \Gamma(G_0, \mathcal{E})$ is total[1]

This resolution property can be thought as a sort of Peter-Weyl theorem for Lie groupoids. It turns that the resolution property is invariant under Morita equivalence. Thus, in fact, it's a property for differentiable stacks.

[1]Recall that a subset of section $\Lambda\Gamma(M, \mathcal{H}_G)$ is called total if for each section $w \in \Gamma(G_0, \mathcal{H}_G)$ and every positive number $\epsilon > 0$ there is a section $v \in \Lambda$ such that $||w - v|| < \epsilon$.

Roughly speaking, a differentiable stack has the resolution property if there are enough finite dimensional vector bundles over it (cf. Totaro [17]). It follows from the proof of Proposition 1.3.1 that locally every orbifold has the resolution property. However, the resolution property itself is not local in character. In fact, as we shall see, it is closely related to the presentation problem.

2.2 Harmonic Analysis on Groupoids

This section is devoted to the proof of the following theorem, which is the firs main original result of this thesis.

Theorem 2.2.1. *Every proper Lie groupoid with compact and connected coarse quotient space has the resolution property.*

We will prove Theorem 2.2.1 by generalizing a classical argument used for proving the Peter-Weyl theorem (see, e.g. [3]). Let us fix throughout this section a proper Lie groupoid G with compact and connected coarse quotient space $|G|$. We denote by $\mathcal{K}_G$ the $*$-algebra of G-equivariant compact operators on its universal representation $\mathcal{H}_G$. For any such $\Psi \in \mathcal{K}_G$ the norm function

$$||\Psi|| : G_0 \to \mathbb{R}_{\geq 0}$$

is continuous and G-invariant and, it therefore, descents to a continuous function $|G| \to \mathbb{R}_{\geq 0}$ on the coarse quotient space. Hence, it follows that the **supremum norm** defined below is finite:

$$||\Psi||_G := \sup_{x \in G_0}\{||\Psi(x)||_x\}$$

Proposition 2.2.1. *The pair $(\mathcal{K}_G, || \cdot ||_G)$ defines a C^*-algebra.*

Proof: It is clear that $|| \cdot ||_G$ is a submultiplicative norm for the composition product of $\mathcal{K}_G$. Moreover, the norm $|| \cdot ||_G$ satisfy the C^*-identity since for each point $x \in G_0$ we have

$$||\Psi(x)^*\Psi(x_0)||_x = ||\Psi(x)||_x^2$$

Now we will prove that the norm $||\cdot||_G$ is complete. Let $\{\Psi_n \in \mathcal{K}_G\}_{n\in\mathbb{N}}$ be a Cauchy sequence relative to $||\cdot||_G$. Then for each point $x \in G_0$ one has that

$$\{\Psi_n(x) : \mathcal{H}_G(x) \to \mathcal{H}_G(x)\}_{n\in\mathbb{N}}$$

is a Cauchy sequence of compact operators on the Hilbert space $\mathcal{H}_G(x)$ in the operator norm. Hence, then the limit

$$\Psi(x) := \lim_n(\Psi(x))$$

is a compact operator on $\mathcal{H}_G(x)$ and $x \mapsto \Psi(x)$ defines a G-equivariant family of compact operators over G_0. Since the sequence of norm functions

$$\{||\Psi_n|| : G_0 \to \mathbb{R}_{\geq 0}\}_{n\in\mathbb{N}}$$

converges uniformly, it follows that $||\Psi|| : G_0 \to \mathbb{R}_{\geq 0}$ is continuous. This implies that $\Psi \in \mathcal{K}_G$ and $\lim_n ||\Psi - \Psi_n|| = 0$, which proves that $||\cdot||_G$ is complete.□

The following proposition can be thought as a sort of Serre-Swan theorem for differentiable stacks.

Proposition 2.2.2. *If $P \in \mathcal{K}_G$ is a projection*[2] *then $Im(P) \subset \mathcal{H}_G$ is a finite dimensional G-subbundle. Conversely, if $\mathcal{E} \subseteq \mathcal{H}_G$ is a finite dimensional G-subbundle there is a unique projection $P_\mathcal{E} \in \mathcal{K}_G$ such that $\mathcal{E} \cong Im(P_\mathcal{E})$.*

Proof: Let $P \in \mathcal{K}_G$ be a projection. For each point $x \in X$ choose an open neighborhood $x \in U \subseteq X$ with compact closure. Since the underlying Hilbert bundle $\mathcal{H}_G \to G_0$ is globally trivial, it follows that the restriction $P|_{\overline{U}}$ defines a projection in the stabilized algebra $\mathcal{K} \otimes C^0(\overline{U})$. Therefore, by the classical

[2]Recall that $P \in \mathcal{K}_G$ is called a projection if $P^2 = P = P^*$.

Serre-Swan theorem (see, e.g. [6]) the image $\mathrm{Im}(P|_U) \subset \mathcal{H}_G|_U$ is a finite dimensional subbundle. Moreover, since P is G-equivariant and the coarse quotient space $|G|$ is assumed to be connected, the image $\mathrm{Im}(P) \subset \mathcal{H}_G$ is a subrepresentation of finite rank. On the other hand, let $\mathcal{E} \subset \mathcal{H}_G$ be a subrepresentation of finite rank. Then, by the absorption property, one has that $\mathcal{E} \oplus \mathcal{H}_G \cong \mathcal{H}_G$. Let $P_{\mathcal{E}} : \mathcal{H}_G \to \mathcal{H}_G$ be the corresponding orthogonal projection onto $\mathcal{E}$. Then $P_{\mathcal{E}}$ is compact and G-equivariant.□

As an immediate consequence of Proposition 2.2.2 we can reformulate the resolution property in the following way.

Corollary 2.2.1. *One has that G has the resolution property iff $\mathcal{K}_G$ has an approximation of the identity by projections.*

One can produce a good supply of compact operators on $\mathcal{H}_G$ by means of the following standard smearing construction. Fix a normalized right Haar system $\mu = \{\mu^x : C^0_c(G_1) \to \mathbb{C}\}_{x \in G_0}$ on G. Hence, for any $a \in C^0_c(G_1)$ and every section $v \in \Gamma(G_0, \mathcal{H}_G)$ the formula

$$\Psi_a v := \iint_G a(g)(h^{-1}gh) \cdot v_{s(h)} dg dh$$

provides a well defined $C^0_c(G_0)$-linear map $\Psi_a : \mathcal{H}_G \to \mathcal{H}_G$, which is called the **smearing operator** associated to a. Note that for each $x \in G_0$ the induced linear operator $\Psi_a(x) : \mathcal{H}_G(x) \to \mathcal{H}_G(x)$ on the fiber above x is compact since it takes the form

$$v_x \mapsto \Psi_a(x) v_x = \iint_{g \in t^{-1}(x), h \in s^{-1}(x)} a(g)(h^{-1}gh) v_x dg dh.$$

Moreover, the inequality $||\Psi_a v|| \leq ||v|| \int_G |a(g)| dg$ implies that Ψ_a actually defines a compact operator on $\mathcal{H}_G$. On the other hand, note that for any arrow $f : x \to y$ in G_1 the equality $f \cdot \Psi_a(x) = \Psi_a(y) \cdot f$ holds since

$$\iint_{g \in t^{-1}(x), h \in s^{-1}(x)} a(g)(fh^{-1}gh) v_x dg dh = \iint_{g \in t^{-1}(y), h \in s^{-1}(y)} a(g)(h^{-1}ghf) v_y dg dh$$

which follows from the right invariance of the Haar system μ, i.e. we have that $\Psi_a \in \mathcal{K}_G$. Similarly, it is not hard to see that $\Psi_{a^*} = \Psi_a^*$.

Poof of Theorem 2.2.1: Let $\{a_n \in C^0_c(G_1)\}_n$ be a sequence of non negative functions with $\int_G a_n(g)dg = 1$ such that for any $b \in C^0_c(G_1)$ one has that

$$\lim_n \int_G b(g)a_n(g)dg = b|_{G_0}$$

Note that for any section $v \in \Gamma(G_0, \mathcal{H}_G)$ we have

$$||\Psi_{a_n}v - v|| = || \iint_G a_n(g)(h^{-1}gh) \cdot v_{s(h)}dgdh - \iint_G a_n(g)v_{s(h)}dgdh|| \leq b_n||v||$$

where the function $b_n = \iint_G a_n(g)||h^-1gh - 1_{s(h)}||_{s(h)}dgdh$ is independent of the section v. By the compactness of $|G|$ it follows that there is a positive number $\epsilon > 0$ such that $\mathrm{sp}(\Psi_{a_n}) \cap (0, \epsilon) = \emptyset$ for $n >> 0$, where $\mathrm{sp}(\Psi_{a_n})$ is the spectrum of Ψ_{a_n} as an element of the C^*-algebra $\mathcal{K}_G$. Thus, for $n >> 0$ the indicator function $1_{\mathrm{sp}(\Psi_{a_n})-\{0\}}$ is continuous on the spectrum

$$\mathrm{sp}(\Psi_{a_n}) \subset [0, ||\Psi_{a_n}||_G] \subset \mathbb{R}$$

By using continuous functional calculus, we obtain a sequence of projections $P_n := 1_{\mathrm{sp}(\Psi_{a_n})-\{0\}}(\Psi_{a_n}) \in \mathcal{K}_G$ such that $\mathrm{Im}(P_a) = \mathrm{Im}(\Psi_a)$. Finally, note that $\lim_n b_n = 0$ since the norm function $||h^{-1}gh - 1_{s(g)}||$ approaches to zero as the arrows $g, h \in G_1$ come closed to the subspace $G_0 \hookrightarrow G_1$. Thus, the sequence $\{P_n\}_n$ is an approximation of the identity in $\mathcal{K}_G$, and the theorem follows from corollary 2.2.1.□

2.3 The Presentation Problem

We conclude this chapter by discussing the presentation problem for proper differentiable stacks in the light of Theorem 2.2.1. Indeed, our second main original result reads as follows.

Theorem 2.3.1. *Every compact and connected proper differentiable stack can be presented as the global quotient of a compact smooth manifold divided by the action of a compact Lie group.*

We will prove Theorem 2.3.1 by means of the following standard *frame bundle trick* (cf. [9] Proposition 5.1 and [5] A.37).

Proposition 2.3.1. *A proper Lie groupoid G is Morita equivalent to the translation groupoid of a differentiable manifold divided by a smooth action of a compact Lie group iff there exist a representation $\mathcal{E} \in Rep(G)$ of finite rank such that for every point $x \in G_0$ the corresponding linear representation of the stabilizer group G_x on the fiber $\mathcal{E}(x)$ above x is faithful (when this happens we say that $\mathcal{E}$ is* **effective***).*

Proof: Suppose that G is Morita equivalent to a translation groupoid $M \ltimes \Gamma$ where Γ is a compact Lie group and M is a smooth Γ-manifold. By the Peter-Weyl theorem we know that there exist a finite dimensional faithful representation V of Γ. Hence, the trivial bundle $V \times M \to M$ endowed with the obvious Γ-action defines an effective representation of $M \ltimes \Gamma$. On the other hand, suppose that $\mathcal{E} \in \mathrm{Rep}(G)$ is an effective representation of finite rank $n < \infty$. We can assume that $\mathcal{E}$ is smooth. Hence, the induced action $G \circlearrowleft \mathrm{Fr}(\mathcal{E})$ on the frame bundle is free, proper, and commutes with the action $\mathrm{Fr}(\mathcal{E}) \circlearrowright U(n)$ of the corresponding unitary group. It turns out that the coarse quotient space $M := |G \ltimes \mathrm{Fr}(\mathcal{E})|$ is a smooth $U(n)$-manifold such that the translation groupoid $M \ltimes U(n)$ is Morita equivalent to G.□

Proposition 2.3.2. *Every proper Lie groupoid with compact and connected coarse quotient space admits an effective representation of finite rank.*

Proof: Let G be a proper Lie groupoid with compact and connected coarse quotient space. Then one can find a finite covering $G = \cup_{k=1}^{n} G_k$ by open subgroupoid $G_k \subseteq G$ such that G_k is equivalent to a global quotient. Then for each k there is an effective representation $\mathcal{E}_k \in \mathrm{Rep}(G_k)$ of finite rank. Hence, since the Lie groupoid G has the resolution property, there is a representation $\mathcal{E}'_k \in \mathrm{Rep}(G)$ of finite rank such that the restriction $\mathcal{E}'_k|_{G_k}$ approximates $\mathcal{E}_k$ close enough in order to guarantee that $\mathcal{E}'_k|_{G_k}$ is effective. It follows that $\mathcal{E} := \bigoplus_{k=1}^{n} \mathcal{E}'_k \in \mathrm{Rep}(G)$ is an effective representation of finite rank.□

Finally, the proof of theorem 2.3.1 is a direct consequence of propositions 2.3.1 and 2.3.2.

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