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Abstract

This thesis consists of three parts. In the first part, motivated by the boundary crossing problem for Brownian motion, we study the heat equation with a moving boundary. Making use of the heat polynomials and Fourier analysis, we develop a procedure to solve the heat equation with a moving boundary for a family of boundaries, including quadratic, square root, and cubic boundaries. Furthermore, we use the solution to the heat equation with quadratic moving boundary to finding the density of the hitting time of the Brownian motion up to a quadratic boundary.

In the second part, we use heat polynomials and tools from stochastic analysis, to introduce a family of Ito processes with Bessel-like properties. We show that these processes have positive sample paths, we compute their transition probabilities, and their hitting time densities. Besides, we show that these processes are Schroedinger bridges that, a fixed time $T > 0$, behave as a Bessel process of odd dimension. Finally, we propose an application of these processes to the modelling of stochastic volatility for timer options.

Finally, in the third part, we propose a Stackelberg model corresponding to a resource allocation problem on an urban region. The problem is solved as a bilevel optimization problem. The existence of solutions to the problem is obtained by means of optimal transport techniques.

Resumen

Esta tesis consiste de tres partes. En la primera parte, motivados por el problema de cruce de fontera para el movimiento browniano, estudiamos el problema de la ecuación del calor con frontera móvil. Usando los polinomios del calor y elementos básicos de anális de Fourier, desarrollamos un método para encontrar soluciones a la ecuación del calor con frontera móvil para una familia de fronteras, incluyendo las fronteras cuadrática, raíz cuadrada y cúbica. Además, usando la solución a la ecuación del calor con frontera cuadrática calculamos la densidad del cruce de frontera del movimiento browniano para una frontera cuadrática.

En la segunda parte, usando los polinomios del calor y herramientas de análisis estocástico, construimos una familia de procesos de Ito con propiedades tipo Bessel. Mostramos que estos procesos tienen trayectorias positivas, calculamos sus probabilidades de transición y densidades de cruces de frontera. Además, mostramos que estos procesos resultan ser puentes de Schrödinger que en un tiempo fijo $T > 0$ se comportan como un procesos de Bessel de dimensión impar. Finalmente, proponemos una aplicación de estos procesos modelando volatilidad estocástica en un problema de opciones "timer".

Finalmente, en la tercera parte proponemos un modelo de Stackelberg para la asignación óptima de recursos en una región urbana. El modelo se resuelve como un problema de optimización de dos niveles. La existencia de soluciones del problema se obtiene por medio de técnicas de transporte óptimo.

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1 Introduction

The first part of this thesis deals with a classic and challenging problem in stochastic analysis: the hitting time problem for Brownian motion. Consider a standard one-dimensional Brownian motion starting at zero $\{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$ and a continuous function $f : [0, \infty) \to \mathbb{R}$ such that $f(0) \neq 0$. The hitting time up to the curve f, called a moving boundary, is defined as

$$T_f(t) := \inf\{t > 0 : B_t = f(t)\}.$$

Finding the distribution of the stopping time T_f is the hitting time problem. Hitting time problems also are known as boundary crossing problems.

The hitting time problem can be traced back (at least) to Bachelier's doctoral thesis [5]. Hitting time problems appear in applications to finance, economics, optimal control, among others as can be seen in [26] and references therein. They are also related to important problems in statistics, such as sequential analysis ,and the law of the iterated logarithm, [30]. Different approaches have been used to study hitting time problems. A modern survey on some of these techniques is [3].

In the case of the Brownian motion it is known that if f is a C^1 function, then the stopping time T_f admits a density that is continuous [42]. Moreover, the relation between the boundary crossing problem for Brownian motion and the heat equation has been studied ([26] p. 262). In particular, in [20] the hitting time problem is studied in the case in which f is a convex function that satisfies some technical conditions (see Section 2.2 below). In [20] it is shown that the problem of finding a density for the hitting time is equivalent to finding a solution to the heat equation with some non-standard initial and boundary conditions. The relation between differential equation and the boundary crossing problem holds in more general settings [2].

For completeness, we begin in Chapter 2 with a brief review of [20]. The main aim of this chapter is to establish the relationship between hitting time problems for Brownian motion and the heat equation with a moving boundary as well as a distributional initial condition. The main result of this chapter is obtained with the initial and boundary conditions (2.20) and (2.21).

Motivated by hitting time problems, Chapter 3 is devoted to study the heat equation with a moving boundary. The heat equation with a moving boundary appears in several theoretical and applied problems in mathematics and physics mentioned in the introduction of Chapter 3. In the particular case of the boundary crossing problem the relationship between hitting times and the heat equation with a moving boundary appears in Theorem 1.1. of Lerche [30].

In Chapter 3 we develop a procedure, with minimal tools, for solving the heat equation with a moving boundary for a family of boundaries. Using heat polynomials and tools from Fourier analysis, a solution to the heat equation with a moving boundary is obtained as the convolution between the heat kernel and a function ϕ that solves an ordinary differential equation. Using this approach, we find solutions to the heat equation with a moving boundary for quadratic and cubic cases, among others. Besides, with the solution to the heat equation with a quadratic moving boundary and the theory developed in Chapter 2, we compute the density of the hitting time for Brownian motion up to a quadratic boundary.

In Chapter 4 we introduce a family of Ito processes with Bessel-like properties. Using heat polynomials we make a change of measure on a Brownian motion absorbed at zero. For these processes we show that they have positive sample paths, we compute their transition probabilities as well as their hitting time densities. We also show that these processes solve the stochastic control version of the classical Schroedinger problem formulated by Dai Pra [12]. In particular, these processes can be considered as Schroedinger bridges given that a fixed (but arbitrary) time $T > 0$ they behave as Bessel processes of odd dimension. This bridge property could be useful to model, for instance, a portfolio which may vary the amount of assets in a fixed time $T > 0$. Finally, we propose an application of these processes too model stochastic volatility in timer options.

The study of Schroedinger bridges leads us naturally to the optimal transportation problem [29]. The origins of the optimal transport problem can be traced back up to the 18th century and it was not until the relaxed formulation by Kantorovich in 1942 that the problem was completely solved [45]. The optimal transport problem is related to several branch in mathematics such as partial differential equation, Riemannian geometry and probability [44]. In addition, in the last few years the optimal transport problem has been widely used in applications to economics, statistics and operations research [39].

Strongly inspired by [32] and [10], in Chapter 5 we propose an Stackelberg model corresponding to a resource allocation problem on an urban region. The model takes into account transportation costs, distribution costs ,and utility functions. In this model, a social planner wishes to minimizing total costs while maximizing the welfare obtained by the population that receives the resource. The problem is solved as a bilevel optimization problem (also known as a Stackelberg game in game theory). The existence of the best-response function is guaranteed thanks to the optimal transportation problem.

2 On hitting times for Brownian motion

2.1 Introduction

Consider a standard one dimensional Brownian motion $\{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ starting at zero, and a real-valued function $f \in C([0,\infty))$. The hitting time problem is to find the distribution for

$$T_f := \inf\{t > 0 : B_t = f(t)\}. \tag{2.1}$$

A hitting time problem is also known as a boundary crossing problem and it is a fundamental and challenging problem in stochastic analysis. The study of hitting time problems may be traced back to Bachelier's doctoral thesis [5] and nowadays it is a problem with deep applications in pure and applied mathematics as well as in physics.

This chapter is based on [20] where it is shown a relationship between the hitting time problem and the heat equation with particular initial and (moving) boundary conditions.

2.2 Hitting time problems

Let $\{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a standard Brownian motion. Consider the hitting time to a fixed level a

$$T_a := \inf\{t > 0 : B_t = a\}.$$

Using the reflection principle (see [26] p. 81) it can be shown that, for $a > 0$ fixed, T_a has a density with respect to the Lebesgue measure given by

$$h(t,a) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} \quad \text{for} \quad t > 0. \tag{2.2}$$

Note that if $\omega(t,a)$ is the heat kernel (also known as the fundamental solution to the heat equation), that is,

$$\omega(t,a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \tag{2.3}$$

then $\omega_a(t,a) = -h(t,a)$ where h is as in (2.2), and ω_a denotes partial derivative.

Let $C_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions with compact support, also known as test functions.

It is known that ([17] p. 208)

$$\lim_{t\downarrow 0} \omega(t,a) = \delta_0(a)$$

where $\delta_0(a)$ is the Dirac mass at 0 with respect to the variable a and the limit is in the distributional sense, that is,

$$\lim_{t\downarrow 0} \int_{\mathbb{R}} \omega(t,a)\phi(a)da = \phi(0) \tag{2.4}$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Then, given that differentiation is continuous respect to the distributional convergence ([17] p. 315) we have

$$\lim_{t\downarrow 0} h(t,a) = -\frac{1}{2}\delta_0'(a) \tag{2.5}$$

where the limit is in distributional sense, that is,

$$\lim_{t\downarrow 0} \int_0^\infty h(t,a)\varphi(a)da = \frac{1}{2}\varphi'(0) \tag{2.6}$$

for all φ in the set of test functions $\mathcal{E}$ defined as

$$\mathcal{E} := \{\varphi \in C_c^\infty([0,\infty)) : \varphi(0) = 0\}. \tag{2.7}$$

We will study the hitting time problem (2.1) for a function f that satisfies the following conditions.

Assumption 2.1. *Let $f \in C^2([0,\infty))$ be a real-valued function such that, for all $t > 0$,*

$$f(0) = 0, \quad f''(t) \geq 0, \quad \text{and} \quad \int_0^t (f'(s))^2 ds < \infty. \tag{2.8}$$

We denote as $f(t,a)$ the family of translations of f given by

$$f(t,a) = a + f(t) = a + \int_0^t f'(s)ds \quad \text{for} \quad a > 0. \tag{2.9}$$

Making use of Girsanov's theorem ([26] p. 190) in [20] it is set a relationship between the problem of finding the density of hitting times (2.1) and 3-dimensional Bessel bridges. For completeness thesis, we next recall some of these facts.

A 3-dimensional Bessel process represents the Euclidean norm of a Brownian motion on $\mathbb{R}^3$. On the other hand, a 3-dimensional Bessel bridge X_u starting at $a > 0$ is a 3-dimensional Bessel process that is conditioned to hit level zero (for the first time) at a fixed time $t > 0$. In [36] p. 463 it is shown that a 3-dimensional Bessel bridge X_u satisfies the stochastic differential equation

$$dX_u = \left(\frac{1}{X_u} - \frac{X_u}{t-u} \right) du + dW_u \quad \text{with } X_0 = a > 0, \quad 0 \le u \le t, \tag{2.10}$$

where W_u is a standard Brownian motion.

We denote by $T_f(t, a)$ the density of the hitting time (2.1) when it hits a boundary that satisfies Assumption 2.1. Theorem 3.1 in [20] states that $T_f(t, a)$ is given as

$$T_f(t,a) = \mathbb{E}\Big[\exp\Big\{ - \int_0^t f''(u) X_u du \Big\} \Big] e^{-\frac{1}{2}\int_0^t (f'(u))^2 du - f'(0)a} h(t,a), \tag{2.11}$$

where h is as in (2.2) and X_u is a 3-dimensional Bessel bridge that satisfies (2.10).

2.3 Hitting times problems and the heat equation

Note that the most complicated term in (2.11) is

$$\mathbb{E}\Big[\exp\Big\{ - \int_0^t f''(u) X_u du \Big\} \Big],$$

which is related to the main contribution of Theorem 4.1 in [20].

Theorem 2.2 (Hernández-del-Valle [20]). *Suppose that $v : [0, s] \times [0, \infty) \to [0, \infty)$ is continuous of class $C^{1,2}([0, s) \times [0, \infty))$, and satisfies the Cauchy problem*

$$-\frac{\partial v}{\partial t} + f''(t) a v = \frac{1}{2}\frac{\partial^2 v}{\partial a^2} + \left(\frac{1}{a} - \frac{a}{s-t} \right) \frac{\partial v}{\partial a} \quad \text{with} \quad v(s,a) = 1, \tag{2.12}$$

as well as $0 \le v(t, a) \le 1$ for $0 \le t \le s$. Then $v(t, a)$ admits the representation

$$v(t,a) = \mathbb{E}^{t,a}\Big[\exp\Big\{ - \int_t^s f''(u) X_u du \Big\} \Big] \tag{2.13}$$

and this representation is unique. Here, $\mathbb{E}^{t,x}$ denotes expectation with respect to the 3-dimensional Bessel bridge with initial conditions $X_t = x$.

The proof to this result lies on the Feynman-Kac formula ([26]p. 366) and it yields a relationship between the hitting time problem for Brownian motion and some partial differential equations. The main objective in the rest of this chapter is to find a solution to (2.12) in terms of the heat equation.

To obtain a solution v to (2.12) consider a function w defined as

$$v(t,a) = \frac{w(t,a)}{h(s-t,a)}, \tag{2.14}$$

where h is given in (2.2). If $w(t,x)$ satisfies that

$$-w_t(t,a) + f''(t)aw(t,a) = \frac{1}{2}\frac{\partial^2 w}{\partial a^2} \quad \text{on} \quad [0,s)\times(0,\infty), \tag{2.15}$$

then $v(t,a)$ in (2.14) satisfies (2.12). Furthermore, from the boundary condition (2.5) and the fact that $v(s,a)=1$ we obtain

$$\lim_{t\uparrow s} w(t,a) = -\frac{1}{2}\delta_0'(a) \tag{2.16}$$

and, from the function h it follows that

$$\lim_{a\downarrow 0} w(t,a) = 0. \tag{2.17}$$

So far, if we have a function w that satisfies the partial differential equation (2.15) on $[0,s)\times(0,\infty)$ as well as the conditions (2.16) and (2.17), then the density $T_f(s,a)$ is $w(0,a)$. Note the important role played by s.

Using Fourier transforms, Theorem 6.1 in [20] establishes that a solution to (2.15) on $[0,s)\times(0,\infty)$ is

$$w(t,a) = e^{\frac{1}{2}\int_t^s (f'(u)^2 du + af'(t)}\kappa(s-t, a+\int_t^s f'(u)du) \tag{2.18}$$

where $\kappa(t,x)$ is given by

$$\kappa(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(y)e^{\frac{1}{2}y^2t+iyx}dy \tag{2.19}$$

for every function $\hat{g}(x)$ for which the integral in (2.19) is defined and where $i=\sqrt{-1}$.

Note that κ is the convolution between a function g with Fourier transform $\hat{g}$ and the heat kernel (2.3). Then, if $g(x)$ satisfies some growth condition (see for instance [26] p. 254), then the function κ is a solution to the heat equation.

For a solution κ to the heat equation and considering the initial condition (2.16) we obtain

$$\lim_{t\uparrow s}\kappa\Big(s-t,a+\int_t^s f'(u)du\Big)=-\frac{1}{2}\delta_0'(a) \tag{2.20}$$

where the limit is in the distributional sense on the set of test functions $\mathcal{E}$ defined at (2.7).

The condition (2.17) yields

$$\lim_{a\downarrow 0}\kappa\Big(s-t,a+\int_t^s f'(u)du\Big)=\kappa(s-t,f(s)-f(t))=0. \tag{2.21}$$

Summarizing, if κ is a solution to the heat equation on $[0,s)\times(0,\infty)$ that satisfies (2.20) and (2.21), then the density of the hitting time at time s, $T_f(s,a)$, is

$$T_f(s,a)=\kappa\Big(s,a+\int_0^s f'(u)du\Big)=\kappa(s,a+f(s)). \tag{2.22}$$

The following chapter es devoted to deal with a solution to the heat equation that satisfies (2.21). Moreover, in Section 3.6 of Chapter 3 we study the hitting time for the Brownian motion up to a family of quadratic boundaries. To this end, we will use Sturm-Liouville theory to satisfies condition (2.20).

3 Solution to the heat equation with a moving boundary

3.1 Introduction

Solutions to the heat equation absorbed at a moving boundary $f(t)$, that is, functions $\nu(t,x)$ such that

$$\nu_t(t,x) = \frac{1}{2}\nu_{xx}(t,x), \tag{3.1}$$

$$\nu(t, f(t)) = 0, \tag{3.2}$$

$$(t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad f \in C^2$$

appear prominently in applications. For instance, see [8, 20, 38] in the construction of first hitting time densities of Brownian motion; [9, 22, 28] in the valuation of barrier options; [13, 11] in the quantification of counterparty risk; [19, 33] for applications of the quadratic boundary in biology and other fields. In fact, explicit solutions to the problem (3.1)-(3.2) are well known in some particular cases. For instance:

(a) *Linear boundary.* For $b \in \mathbb{R}$, the function

$$\nu(t,x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\} + b\frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}. \tag{3.3}$$

solves (3.1)-(3.2) in the case in which $f(t) = -bt$. [See, for instance; Karatzas & Shreve (1991) for an example of this function in the first hitting time of Brownian motion to a linear boundary.]

(b) *Quadratic boundary.* Given that A_i denotes an Airy function and $\xi \in \mathbb{R}^- := \{x \in \mathbb{R}; x < 0\}$ is any of its roots, then

$$\nu(t,x) = \exp\left\{\frac{t^3}{12} + \frac{tx}{2}\right\} A_i\left(x + \frac{t^2}{4}\right) \tag{3.4}$$

is a solution of problem (3.1)-(3.2) when $f(t) = \xi - t^2/4$. (See, for instance, [43] for the general theory and applications of Airy functions)

(c) *Rayleigh type equation.* Let

$$\nu(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{ i\lambda x - \frac{1}{2}\lambda^2 t - \frac{\lambda^4}{4} \right\} d\lambda$$

be the so-called Pearcey function. This function solves problem (3.1)-(3.2) when f satisfies

$$f''(t) = 2[f'(t)]^3 - \frac{1}{2}tf'(t) - \frac{1}{4}f(t).$$

See [21].

(d) In general, one could try to find a function f that solves (3.2) whenever ν is a linear combination of n solutions $\{\nu_j\}_{j=1,\ldots,n}$ to the heat equation (3.1), that is,

$$\nu(t,x) = a_1\nu_1(t,x) + \cdots + a_n\nu_n(t,x).$$

Such is the case of the density of the first time a Wiener process hits a quadratic boundary as in [19, 33]. Or the first time it reaches a square root boundary as in [8, 38]

Let us distinguish the following problems:

(a) Given that ν solves the heat equation (3.1) then, find f that solves (3.2).

(b) For a given moving boundary f, there exists a solution ν to the heat equation (3.1), such that (3.2) holds.

In this chapter we are mainly concerned with problem (b). That is, to find a solution to (3.1)-(3.2) in the case in which the moving boundary $f(t)$ is at most cubic, that is, $f(t) = bt^3$ for $b \in \mathbb{R}$ and $t \geq 0$.

The technique used to achieve our goal is remarkably straightforward, and is based on analyzing the convolution between the *fundamental* solution of the heat equation and some real-valued and sufficiently smooth function ϕ. In Hernández-del-Valle [21], the author uses similar arguments, but in that work the problem is of the type (a). In contrast, in the present work the problem is as the one posed in (b). We will show that making use of the technique briefly described above, we may find a function ϕ which convolved with the fundamental solution to the heat equation leads to solutions of the type (b) for a family of boundaries that include quadratic and cubic boundaries. We suspect that this technique can be generalized to the case of a polynomial boundary. Finally, we apply the solution to the heat equation with a quadratic moving boundary, together with the theory developed in Chapter 2, for computing explicitly the density of the hitting time of a Brownian motion up to a family of quadratic boundaries.

The remainder of this chapter is organized as follows. In Section 3.2 we state some notation, define heat polynomials, and recall some of their properties. In Section 3.3 the technique used to link solutions ν of the heat equation with moving boundaries f is introduced in the case in which the linking function $\phi(x)$ is $C^2(\mathbb{R})$. Furthermore, in Section 3.4 we study in detail the case of absorption at the linear, quadratic, and square root boundaries with our approach. Subsequently, in Section 3.5, we derive the solution of the heat equation with a cubic absorbing boundary. Section 3.6 is devoted to compute the density of the hitting time up to a family of quadratic boundaries. Finally, we close the chapter in Section 3.7 with some concluding remarks.

3.2 Preliminary results

In this section we introduce the notation that will be used in the chapter. Furthermore, we define the so-called heat polynomials and state some of their properties.

Remark 3.1. *(a) For the remainder of this chapter, given a function $\nu(t,x)$, its n-th partial derivative with respect to the state variable x will be denoted as $\nu^{(n)}(t,x)$.*

(b) Recall that the fundamental solution to the heat equation (3.1) (also known as the heat kernel) is given by

$$\begin{aligned}\omega(t,x) &:= \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2t}\right\}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\frac{1}{2}\lambda^2 t}d\lambda, \quad i := \sqrt{-1}\end{aligned}$$

where the last identity is expressed in terms of the inverse Fourier transform.

(c) *For a function f its Fourier transform is denoted as $\mathcal{F}[f]$. We recall the following properties of the Fourier transform*

$$\mathcal{F}\Big[\frac{d^n f(x)}{dx^n}\Big](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda)$$
$$\mathcal{F}[x^n f(x)](\lambda) = i^n \frac{d^n \mathcal{F}[f](\lambda)}{d\lambda^n}.$$

For short, the Fourier transform $\mathcal{F}[f](\lambda)$ will be also denoted as $\overline{f}(\lambda)$.

Now we will define the so-called heat polynomials and write some of their properties. We will make use of heat polynomials in lemma 3.3 and in the proof of proposition 3.4 bellow.

Definition 3.2 (Rosembloom and Widder [37]). *We define the heat polynomials as functions $v_n(t,x)$ such that*

$$e^{i\lambda x - \lambda^2 t/2} = \sum_{n=0}^{\infty} v_n(t/2, x) \frac{(i\lambda)^n}{n!}.$$

The heat polynomials satisfy the following properties whose proofs can be seen in [37].

- The recurrence relation

$$v_{n+1}(t,x) = x v_n(t,x) + 2nt v_{n-1}(t,x) \tag{3.5}$$

  holds for $n = 1, 2, \dots$, with $v_0(t,x) = 1, v_1(t,x) = x$.

- For $n = 1, 2, \dots$,

$$v_n^{(1)}(t,x) = n v_{n-1}(t,x). \tag{3.6}$$

Finally, we prove a technical lemma that will be used in the proof of proposition 3.4.

Lemma 3.3. *Let $v_n(t,x)$ be the heat polynomials. Then*

$$\frac{d^n}{d\lambda^n}\left[e^{i\lambda x - \lambda^2 t/2}\right] = v_n(-t/2, ix - \lambda t) e^{i\lambda x - \lambda^2 t/2} \quad \textit{for} \quad n = 0, 1 \cdots . \tag{3.7}$$

Proof. The proof is by induction. For $n = 1$, from (3.5) direct calculations give the result. Now, assume that (3.7) holds for n; we will prove that it holds for $n+1$. The induction hypothesis yields

$$
\begin{aligned}
\frac{d^{n+1}}{d\lambda^{n+1}}\left[e^{i\lambda x-\lambda^2 t/2}\right] &= \frac{d}{d\lambda}\left[\frac{d^n}{d\lambda^n}\left[e^{i\lambda x-\lambda^2 t/2}\right]\right] \\
&= \frac{d}{d\lambda}\Big[v_n(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}\Big],
\end{aligned}
$$

which together with (3.6) gives

$$
\begin{aligned}
\frac{d^{n+1}}{d\lambda^{n+1}}\left[e^{i\lambda x-\lambda^2 t/2}\right] &= \Big[-tnv_{n-1}(-t/2, x-\lambda t) + v_n(-t/2, x-\lambda t)(ix-\lambda t)\Big]e^{i\lambda x-\lambda^2 t/2} \\
&= v_{n+1}(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}.
\end{aligned}
$$

The last equality follows from (3.5). The proof is complete. □

3.3 Main results

In this section we derive some algebraic properties of the convolution between the fundamental solution of the heat equation and a $C^2(\mathbb{R})$ function ϕ .

Proposition 3.4. *For positive integers p, q, r and constant coefficients $a, b \in \mathbb{R}$, consider the differential equation*

$$
x^p\phi^{(2)}(x) = ax^q\phi^{(1)}(x) + bx^r\phi^{(0)}(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{3.8}
$$

In addition, let v_n be the heat polynomials. If $\overline{\phi}$ denotes the Fourier transform of a solution ϕ to (3.8), then the following holds

$$
\begin{aligned}
(-i)^p \int (i\lambda)^2\overline{\phi}(\lambda)v_p(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda \\
= (-i)^q a \int i\lambda\overline{\phi}(\lambda)v_q(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda \\
+(-i)^r b \int \overline{\phi}(\lambda)v_r(-t/2, ix-\lambda t)e^{i\lambda x-\lambda^2 t/2}d\lambda.
\end{aligned} \tag{3.9}
$$

Proof. Applying the Fourier transform to both sides of (3.8) yields

$$
i^p\frac{d^p}{d\lambda^p}\left[(i\lambda)^2\overline{\phi}(\lambda)\right] = ai^q\frac{d^q}{d\lambda^q}\left[i\lambda\overline{\phi}(\lambda)\right] + bi^r\frac{d^r}{d\lambda^r}\left[\overline{\phi}(\lambda)\right].
$$

Next, we multiply both sides of the previous expression by $e^{i\lambda x-\lambda^2 t/2}$, and then integrate to obtain

$$\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^p \frac{d^p}{d\lambda^p}\left[(i\lambda)^2\overline{\phi}\right] d\lambda \tag{3.10}$$
$$= a\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^q \frac{d^q}{d\lambda^q}\left[i\lambda\overline{\phi}\right] d\lambda + b\int_{-\infty}^{+\infty} e^{i\lambda x-\lambda^2 t/2} i^r \frac{d^r}{d\lambda^r}\left[\overline{\phi}\right] d\lambda.$$

Note that the function $e^{i\lambda x-\lambda^2 t/2}$ vanishes as $|\lambda|\to\infty$. Hence, integration by parts gives

$$\begin{aligned}(-i)^p \int (i\lambda)^2\overline{\phi}\frac{d^p}{d\lambda^p}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda \\ = (-i)^q a\int (i\lambda)^1\overline{\phi}\frac{d^q}{d\lambda^q}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda \\ +(-i)^r b\int (i\lambda)^0\overline{\phi}\frac{d^r}{d\lambda^r}\left[e^{i\lambda x-\lambda^2 t/2}\right] d\lambda.\end{aligned}$$

Thus, equation (3.9) follows from the latter equality and (3.7). □

We will use Proposition 3.4 in combination with the following facts.

Remark 3.5. *(a) Suppose there exists a pair of functions ν and f that solve the moving boundary problem (3.1)-(3.2). Then, direct calculations give the following*

$$f'(t)\nu^{(1)}(t,f(t)) + \frac{1}{2}\nu^{(2)}(t,f(t)) = 0 \tag{3.11}$$

and

$$f''(t)\nu^{(1)} + f'(t)\left(f'(t)\nu^{(2)} + \nu^{(3)}\right) + \frac{1}{4}\nu^{(4)} = 0. \tag{3.12}$$

(b) Consider the function $\nu(t,x)$ defined as the convolution between a solution ϕ to (3.8) and the fundamental solution to the heat equation, i.e.

$$\nu(t,x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\phi}(\lambda)e^{i\lambda x-\lambda^2 t/2}d\lambda \quad \textit{for} \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}.$$

If ϕ satisfies some growth condition (see [26] p. 254), Then $\nu(t,x)$ is a solution to the heat equation; Furthermore, by properties of Fourier transforms,

$$\nu^{(n)}(t,x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\lambda)^n\overline{\phi}(\lambda)e^{i\lambda x-\lambda^2 t/2}d\lambda \tag{3.13}$$

for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

We are now ready to present the main result of this section which will make use of Proposition 3.4 and Remark 3.5. Theorem 3.6 links functions ν and f that solve (3.1)-(3.2), through a specific C^2 function ϕ.

Theorem 3.6. *For given fixed coefficients $d_0, d_1, c_0, c_1, c_2 \in \mathbb{R}$, let ϕ be a real-valued solution of the following second order ODE*

$$\phi^{(2)}(x) = \sum_{j=0}^{1} d_j x^j \phi^{(1)}(x) + \sum_{j=0}^{2} c_j x^j \phi(x) \tag{3.14}$$

for $x \in \mathbb{R}$, with Fourier transform $\overline{\phi}$. Let

$$\nu(t,x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\phi}(\lambda) e^{i\lambda x - \lambda^2 t/2} d\lambda \tag{3.15}$$

be the convolution between ϕ and the fundamental solution of the heat equation. If there exists f such that $\nu(t, f(t)) = 0$ and $\nu^{(1)}(t, f(t) \neq 0$, it follows that

1. *If $d_1 = c_2 = 0$ in (3.14), then we have that*

$$f(t) = -\frac{d_0}{2}t - \frac{c_1}{4}t^2. \tag{3.16}$$

2. *If at least one of the coefficients d_1, c_2 are different from zero, then, for an arbitrary constant $\mathcal{C}$, the function f is of the form*

$$f(t) = \frac{-d_0 d_1 - 2c_1 - 2c_2 d_0 t + c_1 d_1 t}{d_1^2 + 4c_2} + \sqrt{-1 + d_1 t + c_2 t^2} \cdot \mathcal{C}. \tag{3.17}$$

Proof. If the function ϕ is a solution of (3.14), it follows from Proposition 3.4 and (3.13) that its convolution with the fundamental solution of the heat equation satisfies that

$$\begin{aligned}(1 - d_1 t - c_2 t^2)\nu^{(2)}(t,x) = \\ (d_0 + d_1 x + c_1 t + c_2 2tx)\nu^{(1)}(t,x) \\ +(c_0 + c_1 x + c_2 x^2 + c_2 t)\nu^{(0)}(t,x).\end{aligned} \tag{3.18}$$

Now, suppose that there exists a function f such that $\nu(t, f(t)) = 0$ for all $t \geq 0$. It follows from (3.18) that

$$\begin{aligned}(1 - d_1 t - c_2 t^2)\nu^{(2)}(t, f(t)) \\ = (d_0 + c_1 t + [d_1 + 2c_2 t]f(t))\nu^{(1)}(t, f(t))\end{aligned} \tag{3.19}$$

and, from (3.11),

$$\nu^{(2)}(t, f(t)) = -2f'(t)\nu^{(1)}(t, f(t)). \tag{3.20}$$

Thus, from (3.19) and (3.20), we have that

$$-2f'(t)(1 - d_1 t - c_2 t^2) = (d_0 + c_1 t + [d_1 + 2c_2 t]f(t)) . \tag{3.21}$$

This implies that f, which solves the latter ODE, has the following general solution (by standard techniques) as long as at least one of the coefficients d_1, c_2 are different from zero

$$f(t) = \frac{-d_0 d_1 - 2c_1 - 2c_2 d_0 t + c_1 d_1 t}{d_1^2 + 4c_2} + \sqrt{-1 + d_1 t + c_2 t^2} \cdot \mathcal{C}.$$

In turn, if $d_1 = c_2 = 0$, it follows from (3.21) that

$$f(t) = -\frac{d_0}{2} t - \frac{c_1}{4} t^2,$$

as claimed. □

3.4 Applications of Theorem 3.6

In this section we will show how Theorem 3.6 works to solve (3.1)-(3.2).

3.4.1 The linear boundary

From the proof of Theorem 3.6 we know that if all the coefficients are zero except d_0, then we will recover the linear boundary.

Next we will proceed to construct a solution. To this end recall that, for arbitrary constants C_1 and C_2, the equation

$$\phi^{(2)}(x) = d_0 \phi^{(1)}(x) + c_0 \phi^{(0)}(x),$$

has the general solution

$$\phi(x) = e^{1/2\left(d_0 - \sqrt{4c_0 + d_0^2}\right)x} C_1 + e^{1/2\left(d_0 + \sqrt{4c_0 + d_0^2}\right)x} C_2.$$

If we take convolution between this solution with the fundamental solution of the heat equation, we obtain

$$\begin{aligned}\nu(t,x) &= e^{1/2\left(d_0-\sqrt{4c_0+d_0^2}\right)x+\frac{1}{2}t\left[1/2\left(d_0-\sqrt{4c_0+d_0^2}\right)\right]^2}C_1\\ &\quad +e^{1/2\left(d_0+\sqrt{4c_0+d_0^2}\right)x+\frac{1}{2}t\left[1/2\left(d_0+\sqrt{4c_0+d_0^2}\right)\right]^2}C_2. \end{aligned}\tag{3.22}$$

Thus if $C_1 = -C_2$ we verify that $\nu\left(t, f(t)\right) = 0$ for all $t \geq 0$ in the case in which the boundary is

$$f(t) = -\frac{d_0}{2}t.$$

3.4.2 The quadratic boundary

Next we study a quadratic boundary. Taking $c_0, c_1 \neq 0$ in (3.14) we have that

$$\phi^{(2)}(x) = c_1 x\phi^{(0)}(x) + c_0\phi^{(0)}(x), \tag{3.23}$$

which is the Airy differential equation. To solve (3.23) first consider the homogeneous Airy equation

$$\phi''(x) - x\phi(x) = 0. \tag{3.24}$$

Using Fourier transform we can find a solution $\phi(x)$ to (3.24) such that $\lim_{x\to+\infty}\phi(x) = 0$. It is given by

$$A_i(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xz+z^3/3)}dz. \tag{3.25}$$

Direct calculations show that a solution $\phi(x)$ to (3.23) is given by $A_i^{c_1}(x+\frac{c_0}{c_1})$, where

$$A_i^{c_1}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(xz+\frac{z^3}{3c_1})}dz = c_1^{1/3}A_i(c_1^{1/3}x). \tag{3.26}$$

In particular, the convolution of $A_i^{c_1}(x+\frac{c_0}{c_1})$ with the fundamental solution of the heat equation is

$$\nu(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big(i\lambda x - \frac{\lambda^2 t}{2} + ic_0\frac{\lambda}{c_1} + i\frac{\lambda^3}{3c_1}\Big)d\lambda \tag{3.27}$$

$$= \exp\Big(\frac{c_1^2t^3}{12} + \frac{xc_1t}{2} + \frac{tc_0}{2}\Big)A_i^{c_1}\Big(x+\frac{c_0}{c_1}+\frac{c_1t^2}{4}\Big) \tag{3.28}$$

Hence, if we take $c_0 = c_1^{2/3}c_n$, where c_n is a root of $A_i(x)$ we obtain

$$\nu(t,x) = \exp\Big(\frac{c_1^2t^3}{12} + \frac{xc_1t}{2} + \frac{tc_1^{2/3}c_n}{2}\Big)A_i^{c_1}\Big(x+\frac{c_n}{c_1^{1/3}}+\frac{c_1t^2}{4}\Big) \tag{3.29}$$

From (3.29) we note that $\nu(t, f(t)) = 0$ for all $t \geq 0$ in the case in which

$$f(t) = -\frac{c_1}{4}t^2.$$

We know that the zeros of $A_i(x)$ are $c_n \in \mathbb{R}^- := \{x \in \mathbb{R}; x < 0\}$, and countable; see [43] p. 15.

3.4.3 The square root boundary

Next we analyze a square root boundary. To this end, from Theorem 3.6 we study the equation

$$\phi^{(2)}(x) = xd_1\phi^{(1)}(x) + c_0\phi^{(0)}(x),$$

whose solution has Fourier transform

$$\overline{\phi}(\lambda) = C_1\lambda^{\frac{c_0-d_1}{d_1}}e^{\frac{\lambda^2}{2d_1}}.$$

In particular, if we let $s = -1/d_1 > 0$ and $z_0 = c_0/d_1$, then

$$\nu(t, x) = \frac{1}{\pi}2^{\frac{z_0}{2}-1}(s+t)^{-\frac{z_0+1}{2}}I,$$

where

$$I = \Bigg[\sqrt{2}x\cos\left[\tfrac{z_0\pi}{2}\right]\Gamma\left[\tfrac{1+z_0}{2}\right]\ {}_1F_1\left[\tfrac{1+z_0}{2}, \tfrac{3}{2}, -\tfrac{x^2}{2(s+t)}\right] + \sqrt{s+t}\,\Gamma\left[\tfrac{z_0}{2}\right]\ {}_1F_1\left[\tfrac{z_0}{2}, \tfrac{1}{2}, -\tfrac{x^2}{2(s+t)}\right]\sin\left[\tfrac{z_0\pi}{2}\right]\Bigg],$$

where ${}_1F_1$ is the confluent hypergeometric function (see [1] p. 503) and $\Gamma(\cdot)$ is the gamma function ([1] p. 253). To verify that with $f(t) = \sqrt{s+t}$ we obtain $\nu(t, f(t)) = 0$ for all $t \geq 0$, we use the previous expression to obtain

$$\nu(t, \sqrt{s+t}) = \frac{1}{\pi}2^{\frac{z_0}{2}-1}(s+t)^{-\frac{z_0}{2}}I$$

where

$$I = \Bigg[\sqrt{2}\cos\left[\tfrac{z_0\pi}{2}\right]\Gamma\left[\tfrac{1+z_0}{2}\right]\ {}_1F_1\left[\tfrac{1+z_0}{2}, \tfrac{3}{2}, -\tfrac{1}{2}\right] + \Gamma\left[\tfrac{z_0}{2}\right]\ {}_1F_1\left[\tfrac{z_0}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right]\sin\left[\tfrac{z_0\pi}{2}\right]\Bigg].$$

We conclude by noticing that I is independent of s and t. Furthermore, by properties of the hypergeometric functions one can check that it has countably many roots as a function of c_0.

3.4.4 Remarks

1. In Theorem 3.6 the coefficient c_0 is independent of the boundaries. In section 3.6 below we will use c_0 as an eigenvalue in the standard Sturm-Liouville theory to find densities of hitting times for Brownian motion.

2. We note that the coefficient of $\nu^{(2)}$ in equation (3.18) is given by

$$(1 - d_1 t - c_2 t^2).$$

   In turn, d_0 and c_1 corresponded to the linear and quadratic boundaries respectively. By analogy, we are tempted to study an ODE that leads to a solution of the heat equation involving a coefficient where t is of cubic order. We will do so in the next section.

3.5 Derivation of the cubic boundary

In this section we derive the function f which corresponds to a solution of the heat equation (3.1)-(3.2) with cubic absorbing boundary. We will make use of part 2 in Remark 3.4.4.

Theorem 3.7. *Suppose that the moving boundary f in (3.2) is $f(t) = -\frac{b}{8}t^3$. Furthermore, take $b \in \mathbb{R}\backslash\{0\}$ and let ϕ be a real-valued function that satisfies*

$$\phi'''(x) = bx^2\phi(x). \tag{3.30}$$

Then there exists a real-valued solution of (3.30), that convolved with the heat kernel yields a function ν that solves problem (3.1)-(3.2).

Before proving the theorem, we provide an example.

Example 3.8. *For $b = -1$, the function*

$$\phi(x) := \frac{x}{5^{3/5}} \cdot {}_0F_2\left[\{\}, \left\{\frac{4}{5}, \frac{6}{5}\right\}, -\frac{x^5}{125}\right],$$

defined in terms of the generalized hypergeometric function ${}_0F_2$ (see [41]), solves (3.30). Then the convolution of the heat kernel ω and ϕ,

$$\nu(t,y) = \int_{-\infty}^{\infty} \omega(t,x)\phi(y-x)dx,$$

is a solution to the problem (3.1)-(3.2) when $f(t) = t^3/8$. That is,

$$\nu(t, t^3/8) = 0 \qquad \forall t \geq 0.$$

The proof of Theorem 3.7 is in the spirit of the proof of Theorem 3.6. The only difference is that the function ϕ that links ν and the boundary f in problem (3.1)-(3.2) is now C^3 instead of C^2.

Proof of Theorem 3.7. If ϕ is a solution to (3.30) and $\nu(t,x)$ denotes the convolution of ϕ and the fundamental solution to the heat equation, then a direct application of Proposition 3.4 to the ODE (3.30) yields

$$\nu^{(3)}(t,x) = bt^2\nu^{(2)}(t,x) + 2btx\nu^{(1)}(t,x) + (bx^2 + bt)\nu(t,x). \tag{3.31}$$

Now, if f is such that $\nu(t, f(t)) = 0$, then from (3.11) we obtain

$$\nu^{(3)}(t, f(t)) = (-2bt^2 f'(t) + 2btf(t))\nu^{(1)}(t, f(t)). \tag{3.32}$$

Moreover, differentiating (3.31) with respect to x we have

$$\begin{aligned}\nu^{(4)}(t,x) &= bt^2\nu^{(3)}(t,x) + 2btx\nu^{(2)}(t,x)\\ &\qquad + (2bt + bx^2 + bt)\nu^{(1)}(t,x) + 2bx\nu(t,x)\end{aligned}$$

and again, if $\nu(t, f(t)) = 0$, from this last expression and (3.32) it follows that

$$\nu^{(4)} = (-2b^2t^4 f'(t) + 2b^2t^3 f(t) - 4btf(t)f'(t) + 3bt + bf^2(t))\nu^{(1)}. \tag{3.33}$$

On the other hand, (3.12) reads

$$\nu^{(4)} = -4f''(t)\nu^{(1)} - 4f'(t)(f'(t)\nu^{(2)} + \nu^{(3)}).$$

From this latter equation, if $\nu(t, f(t)) = 0$, then (3.32) yields

$$\nu^{(4)} = (-4f'' + 8(f')^3 + 8bt^2(f')^2 - 8btff')\nu^{(1)}. \tag{3.34}$$

Thus equating (3.33) and (3.34) we obtain

$$- 4f''(t) + 2f'(t)[4(f'(t))^2 + 4b_2t^2f'(t) - 2b_2tf(t) + b_2^2t^4] \\ - b_2f(t)[2b_2t^3 + f(t)] - 3b_2t = 0. \tag{3.35}$$

Finally, to verify the statement of the theorem, let $f(t) = \delta t^3$, so $f'(t) = 3\delta t^2$ and $f''(t) = 6\delta t$. Then substitute the values of f, f' and f'' in (3.35) to obtain

$$-24\delta t - 3b_2t + 2(3\delta t^2)(4 \cdot 9\delta^2t^4 + 4b_2t^2 \cdot 3\delta t^2 - 2b_2t\delta t^3 + b_2^2t^4) \\ -b_2\delta t^3(2b_2t^3 + \delta t^3) = 0,$$

or equivalently

$$-3t(8\delta + b_2) + 6\delta t^2(36\delta^2t^4 + 12b_2\delta t^4 - 2b_2\delta t^4 + b_2^2t^4) \\ -b_2\delta t^3(2b_2t^3 + \delta t^3) = 0.$$

Factorizing in terms of t and t^6 we have

$$-3t(8\delta + b_2) \\ +\delta t^6(216\delta^2 + 72b_2\delta - 12b_2\delta + 6b_2^2 - 2b_2^2 - b_2\delta) = 0.$$

Thus, for the latter equality to hold for all $t \geq 0$ we should have

$$\delta = -b_2/8.$$

But this also yields

$$216b_2^2/64 - 59b_2^2/8 + 4b_2^2 = 0,$$

and thus the proof is complete. □

3.6 Hitting time of a Brownian motion to a quadratic boundary

The use of the heat equation with a moving boundary for finding the density of a hitting time appeared for the first time in the method of images developed by Lerche, [30] Theorem 1.1. A concise statement of this theorem appears in Proposition 3.3 in [2]. Among the type of boundaries ψ for which Lerche's approach holds are:

1. ψ is a concave function,
2. $\psi(t)/t$ is monotone decreasing.

In this section we will compute explicitly the density of the hitting time of a standard Brownian motion up to a quadratic boundary $f(t) = a + \frac{k}{4}t^2$ for $a > 0, k > 0$. We will use the solution to the heat equation with quadratic moving boundary computed above, in (3.29). An interesting fact is that a quadratic boundary does not fit to the setting of Lerche's image method. However, the heat equation with moving boundary is still useful to compute this density making use of our tools developed in Chapter 2.

Recall that we are looking for a solution $\kappa(t, x)$ to the heat equation on $[0, s) \times [0, \infty)$ that satisfies the initial condition (2.20), that is,

$$\lim_{t \uparrow s} \kappa\Big(s - t, a + \int_t^s f'(u)du\Big) = -\frac{1}{2}\delta_0'(a) \tag{3.36}$$

where the limit is in the distributional sense on the set of test functions $\mathcal{E}$ defined at (2.7). Besides, κ also has to holds boundary condition (2.21)

$$\lim_{a \downarrow 0} \kappa\Big(s - t, a + \int_t^s f'(u)du\Big) = \kappa(s - t, f(s) - f(t)) = 0, \tag{3.37}$$

where $f(t) = \frac{k}{4}t^2$. We will use the solution $\nu(t, x)$ to the heat equation with quadratic moving boundary computed above (see (3.29)),

$$\nu(t, x) = \exp\left\{\frac{k^2t^3}{12} + \frac{ktx}{2} + \frac{k^{2/3}c_nt}{2}\right\} A_i^k\left(x + \frac{c_n}{k^{1/3}} + \frac{kt^2}{4}\right) \tag{3.38}$$

where c_n is a zero of the Airy function $A_i(x)$. To fulfill the condition (3.36) we will use c_n as an eigenvalue in the Sturm-Liouville theory as was pointed out in Remark 3.4.4. To this end recall that the Airy function

$$A_i(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xz+z^3/3)}dz \tag{3.39}$$

has countably many zeros (see [43] p. 20) on the negative real axis.

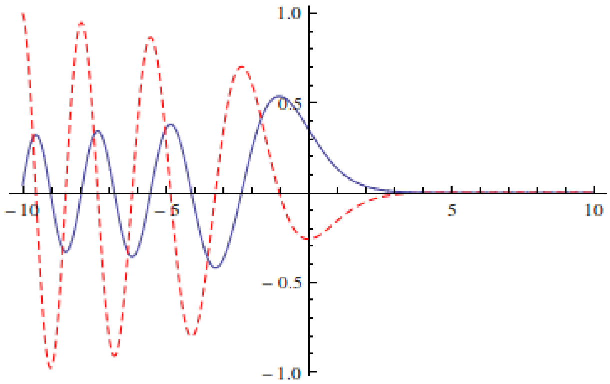

Figure 3.1: Functions $A_i(x)$ (solid line), and $A_i'(x)$ (dashed line).

Let

$$\mathcal{A} := A_i^{-1}(\{0\}) = \{c_n \in \mathbb{R}^- : n = 0, 1, \cdots, A_i(c_n) = 0, c_{n+1} < c_n\} \tag{3.40}$$

be the set of zeros of $A_i(x)$; see Figure 3.1. It is known ([43] p. 88) that

$$\int_0^\infty A_i^2(x + c_n)dx = A_i'^2(c_n). \tag{3.41}$$

Consider the regular Sturm-Liouville problem

$$\phi''(x) - kx\phi(x) = \lambda\phi(x), \quad \text{with} \quad \phi(0) = 0, \quad \lim_{x\to\infty}\phi(x) = 0, \tag{3.42}$$

for $x \in [0, \infty)$, defined on the ideal domain of C^2 functions that satisfy the boundaries conditions. From (3.42), letting $h(x) = \phi(x - \frac{\lambda}{k})$ we obtain

$$h''(x) - kxh(x) = 0, \quad \text{with} \quad h\Big(\frac{\lambda}{k}\Big) = 0, \quad \text{for} \quad x \in [0, \infty). \tag{3.43}$$

Hence, given that we know the solution to the ODE in (3.43) and its zeros, it follows that eigenvalues and eigenfunctions to the Sturm-Liouville problem are

$$\lambda = k^{2/3}c_n, \quad A_i^k\Big(x + \frac{c_n}{k^{1/3}}\Big) \tag{3.44}$$

respectively, where $c_n \in \mathcal{A}$ and $A_i^k(x) := k^{1/3}A_i(k^{1/3}x)$ was defined at (3.26).

From classical Sturm-Liouville theory (see [17], chapter 3) it follows that the eigenvalues (3.44) form a discrete unbounded set and that the eigenfunctions in (3.44) form a complete orthogonal set in $L^2[0,\infty)$.

To compute the norm in $L^2[0,\infty)$ of $A_i^k(x+\frac{c_n}{k^{1/3}})$, note that

$$\int_0^\infty (A_i^k(x+\frac{c_n}{k^{1/3}}))^2 dx = |k|^{1/3}\int_0^\infty A_i^2(y+c_n)dy = |k|^{1/3}A_i'^2(c_n)$$

where the last integral follows from (3.41). From these facts we have that

$$\left\{\frac{A_i^k(x+\frac{c_n}{k^{1/3}})}{|k|^{1/6}|A_i'(c_n)|}\right\}_{n=0}^\infty \tag{3.45}$$

is a complete orthonormal set on $L^2[0,\infty)$.

The so-called closure representation to Dirac's delta in term of a complete orthonormal family $\{\varphi_n\}$ in $L^2(\mathbb{R})$ is give by

$$\delta_0(x-t) = \sum_{i=0}^\infty \varphi_n(t)\varphi_n(x), \tag{3.46}$$

where this equality is in the distributional sense; see [4], p. 89.

The Fourier coefficients associated to (3.45) for the derivative of Dirac's delta are

$$\langle -\delta_0'(x), \frac{A_i^k(x-c_n/k^{1/3})}{k^{1/6}A_i'(c_n)}\rangle = (-1)^n|k|^{1/2}$$

Therefore, in the distributional sense we have that

$$-\frac{\delta_0'(x)}{2} = \sum_{n=0}^\infty |k|^{1/3}\frac{A_i^k(x+c_n/k^{1/3})}{2A_i'(c_n)}. \tag{3.47}$$

Taking convolution with respect to the heat kernel we obtain

$$\eta(t,x) = \sum_{n=0}^\infty \frac{|k|^{1/3}}{2A_i'(c_n)} e^{\frac{k^2t^3}{12}+\frac{ktx}{2}+\frac{k^{2/3}c_nt}{2}} A_i^k\left(x+\frac{c_n}{k^{1/3}}+\frac{kt^2}{4}\right). \tag{3.48}$$

Note that $\eta(t,x)$ is a solution to the heat equation that satisfies

$$\lim_{t\downarrow 0}\eta(t,x) = -\frac{1}{2}\delta_0'(x),$$

hence, using η we can find a solution to (3.36). On the other hand, to satisfies the boundary condition (3.37) we use the following well known transformation

$$\kappa(t,x) = \exp\left(\mu x+\frac{1}{2}\mu^2 t\right)\eta(t,x+\mu t) \tag{3.49}$$

where μ is a constant given by $\mu = -f'(s) = -\frac{ks}{2}$. Note that κ is a solution to the heat equation and furthermore

$$\kappa(s-t, a+\int_t^s f'(u)du) = \sum_{n=0}^{\infty} \frac{|k|^{1/3}}{2A_i'(c_n)} I_n \tag{3.50}$$

where I_n is given by

$$I_n = \exp\Big(\frac{k^2}{12}(s-t)^3 - \frac{kta}{2} - \frac{k^2s^3}{8} + \frac{k^2t^3}{8} + \frac{k^2s^2t}{4} - \frac{k^2st^2}{4}\Big) A_i^k\Big(a + \frac{c_n}{k^{1/3}}\Big). \tag{3.51}$$

Direct calculations show that κ satisfy initial and boundary conditions (3.37)-(3.36). Hence, the density of the hitting time up to the quadratic boundary $a + \frac{k}{4}t^2$ is

$$T(t,a) = \kappa(t, a + \frac{k}{4}t^2) = \sum_{n=0}^{\infty} \frac{k^{1/3}}{2A_i'(c_n)} e^{-\frac{k^2t^3}{24} + \frac{k^{2/3}tc_n}{2}} A_i^k\Big(a + \frac{c_n}{k^{1/3}}\Big). \tag{3.52}$$

This formula coincides with that obtained in Pierre Patie's thesis [34], Lemma 2.3.3.

3.7 Concluding Remarks

In this chapter we present a general framework to study the problem of the heat equation with absorbing moving boundaries. We first analyze the case of some known boundaries and then we show that the procedure extends at least to the cubic case. As an application, we used the problem of the heat equation with a quadratic moving boundary to compute the density of the hitting time up to a quadratic boundary. A more general description of our procedure, as well as applications, is work in progress.

4 Heat polynomials and Bessel-like processes

4.1 Introduction

In this chapter we consider again the heat polynomials in Definition 3.2, above. These polynomials were introduced by Rosembloom and Widder [37] as a way to express solutions to the heat equation in terms of series. They are related to Hermite polynomials and satisfy certain orthogonal properties. In this chapter it will be shown that they can be used, together with some space-time transformation, to construct a family of Ito processes with Bessel-like properties. This family of processes can be seen as Bessel processes with varying dimension, as in [40].

Let us recall that if d is a positive integer, a Bessel process of dimension d is the dynamics of the Euclidean norm of a Brownian motion in $\mathbb{R}^d$.

One reason why Bessel processes are widely studied is their relationship with well known models in mathematical finance such as the geometric Brownian motion and the Cox-Ingersoll-Ross (CIR) processes. A detailed study of Bessel processes can be found for instance in [18] and [36].

In this chapter we construct a family of Ito processes with a given initial condition and which at a fixed time T behave as a Bessel process of odd dimension. It is shown that this family have Bessel-like properties; in particular it has positive sample paths. We compute explicit formulas for their transition probabilities and the density of their hitting times. We also show that the method to build this family of processes is optimum in the sense that it minimizes a Fisher energy functional.

As an example where this family of Ito processes can be used is a closed monopolistic economy that at some future time will be open to more competitors. Or, alternatively, a portfolio which may vary the amount of assets in a fixed time $T > 0$.

In terms of applications, doing a space-time transformation and following [31] we propose an application of this family in the modelling of stochastic volatility. In particular, these processes are employed to model stochastic variance in timer options. As described by Li [31]: " a timer option can be viewed as a call option with random maturity, where the maturity occurs at the first time a prescribed variance budget is exhausted". We can characterize the variance process stopped at a time S_a (where a is the accumulated variance) by an Ito process, with the advantage that the last one can be simulated numerically

This chapter is organized as follows: In Section 4.2 we review the concepts and main properties of heat polynomials. In Section 4.3 we construct the family of Ito processes we are concerned with. Using Girsanov's theorem and heat polynomials we make a change of measure on the space of trajectories of the Brownian motion absorbed at zero. Also, making use of a comparison theorem ([36] chapter IX) we can see that the paths of these processes are positive. Section 4.4 is devoted to show that this change of measure is reached with minimum cost, that is, the Kullback distance (or entropy) between these probability measures is minimized. In Section 4.5 using Radon-Nikodym derivatives (of the change of measures) and the optimal sampling theorem we can compute hitting-times densities up to a fixed level. Finally, in section 4.6, after a space-time transformation, we build a family of CIR-like processes and then, making use of a change of time (with the cross variation of a martingale), we can characterize the CIR-like processes stopped at a certain stopping time by Ito processes.

4.2 Heat polynomials

For future reference, in this section we review some properties of the so-called heat polynomials.

Recall that heat polynomial were introduced in Definition 3.2. However, for the propose of this chapter we define the heat polynomials in the following manner (see [37]). For each

$n = 0, 1, \ldots$ we define the heat polynomial

$$v_n(t,x) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \left(\frac{t}{2}\right)^k \frac{1}{k!}, \tag{4.1}$$

where $[\cdot]$ denotes the floor function. Note that, as a function of x, $v_n(t,x)$ is a monic polynomial of degree n and, moreover, it is even or odd depending on n. An easy calculation shows that v_n solves the heat equation, i.e.

$$\frac{\partial}{\partial t} v_n(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} v_n(t,x) \quad \text{for all} \quad n = 0, 1, \ldots.$$

Furthermore, if $\omega(t,x)$ denotes the fundamental solution of the heat equation (2.3) $u_t = \frac{1}{2} u_{xx}$, i.e.,

$$\omega(t,x) = \frac{\exp(-\frac{x^2}{2t})}{\sqrt{2\pi t}},$$

one can see that v_n can be expressed as the following convolution [37]

$$v_n(t,x) = \omega(t,x) * x^n := \int_{-\infty}^{\infty} \omega(t, x-y) y^n dy, \quad \text{for} \quad n = 0, 1 \ldots. \tag{4.2}$$

From (4.1) one can deduce that, for all $n = 0, 1, \cdots$,

$$v_{2n+1}(t,0) = 0, \quad \text{and} \quad v_{2n}(t,0) = \frac{(2n)!}{n!} \left(\frac{t}{2}\right)^n. \tag{4.3}$$

Also, for any complex number z we have

$$\exp\left(xz + \frac{t}{2} z^2\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} v_n(t,x) \quad \text{for} \quad -\infty < x < \infty, \quad 0 < t < \infty. \tag{4.4}$$

Therefore, from (4.4) one can see that, as in (3.6),

$$\frac{\partial}{\partial x} v_n(t,x) = n v_{n-1}(t,x) \quad \text{for} \quad n = 1, 2, \ldots. \tag{4.5}$$

4.3 Ito processes with heat polynomials

In this section we construct the processes we are concerned with. First, since the heat polynomials are solutions to the heat equation, we can use them together with Girsanov's theorem to make a change of measure on the space of trajectories of a Brownian motion absorbed at zero. We show that these processes have positive trajectories and we compute explicit formulas for their transition probabilities.

For notational ease, from now on we will write

$$v'_{2n+1}(T-t,x) := \frac{\partial}{\partial x} v_{2n+1}(T-t,x).$$

We defined in Chapter 1 the hitting time of a Brownian motion up to a moving boundary f. In this chapter we will work just with the hitting time up to a fixed level $a > 0$. For completeness we restate here the definition of hitting time and of Brownian family given in [26] p. 73.

Definition 4.1. *A Brownian family is an adapted, one-dimensional process $W = \{W_t, \mathcal{F}_t; t \geq 0\}$ on a measurable space $(\Omega, \mathcal{F})$, and a family of probability measures $\{P\}_{x\in\mathbb{R}}$, such that*

i for each $F \in \mathcal{F}$, the mapping $x \to P^x(F)$ is measurable;

ii for each $x \in \mathbb{R}^d$, $P^x[W_0 = x] = 1$;

iii under each P^x, the process W is a one-dimensional Brownian motion starting at x.

The process W under P^0 is known as the standard Brownian motion.

Definition 4.2. *For a stochastic process X and a real number a we define the hitting time of a by*

$$T_a = \inf\{t > 0 : X_t = a\}.$$

Remark 4.3. *Let $(W_t, \mathcal{F}_t)$,$\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ be a Brownian family.*

1. *Let T_0 be the hitting time of 0 for a standard Brownian motion. If $x, y > 0$, it follows from the reflection principle ([26] p. 79) that*

$$\mathbb{P}^x(W_t \in dy, T_0 > t) = p_t(x,y) - p_t(x,-y), \tag{4.6}$$

where

$$p_t(x,y) := \frac{\exp\left(-\frac{(x-y)^2}{2t}\right)}{\sqrt{2\pi t}}.$$

See [26] p. 96.

2. *Girsanov's theorem: Consider the stochastic differential equation*

$$Z_t(X) = 1 + \int_0^t Z_s(X) X_s dW_s \tag{4.7}$$

where W_s is a standard Brownian motion. Ito's formula shows that a solution to (4.7) is given by the Dóleans-Dade exponential (and then Z_t is a local martingale)

$$Z_t(X) = \exp\Big(\int_0^t X_s dW_s - \frac{1}{2}\int_0^s X_s^2 ds\Big) \tag{4.8}$$

where $Z_0(X) = 1$.

Theorem 4.4. *Assume that Z_t in (4.7) is a martingale. Consider the measure $\mathbb{Q}$ given by*

$$\mathbb{Q}(A) := \mathbb{E}[\mathbb{1}_A Z_T(X)]; \quad A \in \mathcal{F}_T.$$

Define a process $\tilde{W} = (\tilde{W}_t, \mathcal{F}_t; 0 \le t \le T)$ by

$$\tilde{W}_t = W_t - \int_0^t X_s ds; \quad \text{for} \quad 0 \le t \le T, \tag{4.9}$$

then, the process $\{\tilde{W}_t, \mathcal{F}_t; 0 \le t \le T\}$ is a one-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

Now we state the main theorem in this section.

Theorem 4.5. *Let T_0 be as in Definition 4.2, and $T > 0$ fixed. For each $n = 0, 1, \dots,$ the stochastic differential equation*

$$dX_t = \frac{v'_{2n+1}(T-t, X_t)}{v_{2n+1}(T-t, X_t)}dt + dW_t, \quad X_0 = x > 0, \quad 0 \le t < T_0$$

admits a weak solution. Furthermore, if $(X, W), (\Omega, \mathcal{F}, \mathbb{Q}^x)$ is such a solution (for fixed n) , then we have

$$\mathbb{Q}^x(X_t \in dy) = \frac{v_{2n+1}(T-t, y)}{v_{2n+1}(T, x)}\Big(\frac{\exp(\frac{(x-y)^2}{2t})}{\sqrt{2\pi t}} - \frac{\exp(\frac{(x+y)^2}{2t})}{\sqrt{2\pi t}}\Big).$$

Proof. Consider the process $Y_t := v_{2n+1}(T-t, W_t)$ where W_t is a Brownian motion started at x. By Ito's rule we have

$$dY_t = \Big(\frac{\partial}{\partial t}v_{2n+1}(T-t, W_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}v_{2n+1}(T-t, W_t)\Big)dt + \frac{\partial}{\partial x}v_{2n+1}(T-t, W_t)dW_t,$$

for $0 < t < T$. Since the polynomial $v_n(T-t, x)$ is a solution to the backward heat equation, it follows that

$$dY_t = \frac{v'_{2n+1}(T-t, W_t)}{v_{2n+1}(T-t, W_t)}v_{2n+1}(T-t, W_t)dW_t, \quad 0 < t < T_0.$$

This shows that the process

$$Y_t = \frac{v_{2n+1}(T-t, W_t)}{v_{2n+1}(T, x)}$$

is a Dóleans-Dade exponential and hence a local martingale. The process $Y_{t \wedge T_0}$ is a positive martigale (by proposition 4.6). Taking $\mathcal{F}$ the filtration generated by W, if we define

$$\mathbb{Q}^x(A) := \mathbb{E}\Big[\frac{v_{2n+1}(T-t, W_t)}{v_{2n+1}(T, x)} \mathbb{1}_A \mathbb{1}_{\{T_0 > t\}}\Big] \quad \text{for} \quad A \in \mathcal{F}_{t \wedge T_0},$$

then $\mathbb{Q}^x$ is a probability measure. Thus using Girsanov's theorem one can see that the process $W_{t \wedge T_0}$ satisfies the stochastic differential equation (under $\mathbb{Q}^x$)

$$dX_t = \frac{v'_{2n+1}(T-t, X_t)}{v_{2n+1}(T-t, X_t)} dt + dW_t, \quad X_0 = x > 0, \quad 0 \leq t < T_0.$$

The result follows. □

Now we show an interesting property of processes generated by heat polynomials. They have positive sample paths, which is an important property for applications in finance and economics.

Proposition 4.6. *Fix $T > 0$, and let v_n be as in (4.1). The Ito processes given by*

$$dX_t = \frac{v'_{2n+1}(T-t, X_t)}{v_{2n+1}(T-t, X_t)} dt + dW_t, \quad \text{with} \quad X_0 = x, \quad 0 \leq t < T_0, \tag{4.10}$$

for $n = 0, 1, \cdots$, have positive trajectories.

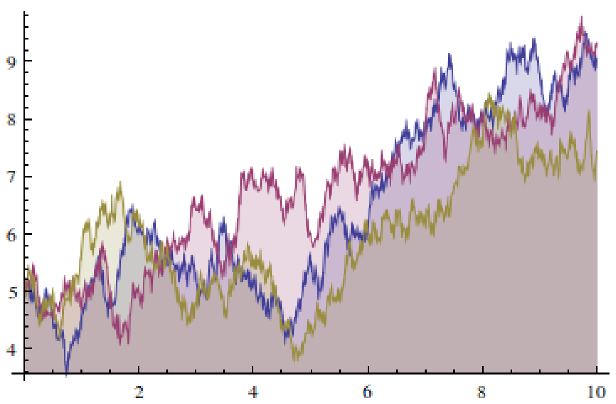

Figure 4.1: Simulation of 3 sample paths corresponding to $v_3(t, x)$

Proof. Using (4.5) and (4.1) we have

$$\frac{v'_{2n+1}(T-t,x)}{v_{2n+1}(T-t,x)} = \frac{2n+1}{x}\frac{(2n)!}{(2n+1)!}\frac{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{(2(n-k))!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}}{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{(2(n-k)+1)!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}}. \tag{4.11}$$

Now note that, for all n,

$$\frac{x^{2(n-k)}}{(2(n-k)+1)!} < \frac{x^{2(n-k)}}{(2(n-k))!}, \qquad k = 0, 1, \ldots, n.$$

Therefore

$$\frac{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{2(n-k)!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}}{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{(2(n-k)+1)!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}} > 1. \tag{4.12}$$

Hence, from (4.11) and (4.12), we have

$$\frac{2n+1}{x}\frac{(2n)!}{(2n+1)!}\frac{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{2(n-k)!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}}{\sum_{k=0}^{n}\frac{x^{2(n-k)}}{(2(n-k)+1)!}\left(\frac{T-t}{2}\right)^k\frac{1}{k!}} > \frac{1}{x}.$$

The desired result follows from the comparison theorem in [26], p. 293, and the fact that the Bessel process of dimension 3 has positive trajectories. □

Remark 4.7. *1. From Proposition 4.6 it follows that for every $T > 0$*

$$\mathbb{Q}^x(T_0 < T) = 0;$$

hence, we can consider, for every $T > 0$ and $n = 0, 1, \cdots$, the processes

$$dX_t = \frac{v'_{2n+1}(T-t,X_t)}{v_{2n+1}(T-t,X_t)}dt + dW_t, \quad X_0 = x > 0, \quad 0 \le t < T.$$

2. From now on we refer to the processes that introduced in Theorem 4.5 as Ito processes generated by heat polynomials.

Using a technique similar to that in Theorem 4.5, we can build a family of processes akin to the processes generated by the heat polynomials $v_{2n+1}(t,x)$, but with lineal drift. Now we illustrate how to do that.

Definition 4.8. *We define the hitting time of a line $l := \{(t,x) : x = at + b, t \in \mathbb{R}\}$ by*

$$T_l = \inf\{t > 0 : W_t = at + b\}. \tag{4.13}$$

Let $(W_t, \mathcal{F}_t)$, $\mathbb{P}^x, x \in \mathbb{R}$ be a Brownian family. One can see in [26] p. 197 that

$$\mathbb{P}^x(T_l \in dt) = \frac{(x-b)}{\sqrt{2\pi t^3}} \exp\Big(-\Big(\frac{(x-b)^2}{2t} - ab + \frac{1}{2}a^2 t\Big)\Big) \tag{4.14}$$

Using (4.14) we can compute the density

$$\mathbb{P}^x(W_t \in dy, T_l > t) = \frac{\exp\big(-\frac{(x-y)^2}{2t}\big)}{\sqrt{2\pi t}} - \int_0^t \frac{\exp\big(-\frac{(y-b-at)^2}{2(t-u)}\big)}{\sqrt{2\pi(t-u)}} \mathbb{P}^x(T_l \in du)du. \tag{4.15}$$

Proposition 4.9. *Let $T > 0$ be fixed. The processes given by*

$$dY_t = \left[\alpha + \frac{v'_{2n+1}(T-t, Y_t + \alpha(T-t))}{v_{2n+1}(T-t, Y_t + \alpha(T-t))}\right] dt + dW_t,$$

for $X_0 = x > 0, 0 \le t < T, n = 0, 1, \ldots,$ have transition probabilities given by

$$\mathbb{Q}^x(Y_t \in dy) = \exp(\alpha(y-x) - \frac{1}{2}\alpha^2 t)\frac{v_{2n+1}(T-t, y+\alpha(T-t))}{v_{2n+1}(T, x+\alpha T)}\mathbb{P}^x(W_t \in dy, T_l > t),$$

where $\mathbb{P}^x(W_t \in dy, T_l > t)$ is given in (4.15) and, in this case,

$$T_l = \inf\{t > 0 : W_t = \alpha(t-T)\}.$$

Proof. For $\alpha \in \mathbb{R}$, we define

$$u_n(t,x) := \exp\big(\alpha x + \frac{1}{2}\alpha^2 t\big) v_n(t, x+\alpha t), \quad n = 0, 1, \ldots. \tag{4.16}$$

we can see that $u_n(T-t, x)$ is a solution to the backward heat equation. Now, doing a change of measure as in Theorem 4.5, but in this case the initial measure is that which supports the Brownian motion absorbed the first time that it hits the line $y = \alpha(T-t)$. In this case the Radon-Nikodym derivative is given by the martingale

$$\begin{aligned} Y_t &= \frac{u_{2n+1}(T-t, W_t)}{u_{2n+1}(T, x)} \\ &= \exp(\alpha(W_t - x) + \frac{1}{2}\alpha^2 t)\frac{v_{2n+1}(T-t, W_t + \alpha(T-t))}{v_{2n+1}(T, x+\alpha T)}. \end{aligned}$$

Note that

$$\frac{u'_{2n+1}(T-t, x)}{u_{2n+1}(T-t, x)} = \alpha + \frac{v'_{2n+1}(T-t, x+\alpha(T-t))}{v_{2n+1}(T-t, x+\alpha(T-t))}.$$

The result follows.

□

4.4 Steering towards a Bessel process of dimension $2(2n+1)+1$

In this section we will show that Ito process generated by heat polynomial v_{2n+1} behaves as a Bessel process of dimension $2(2n+1)+1$ at time $T>0$ fixed. Furthermore, we show that the procedure developed to construct the Ito processes in Theorem 4.5 minimizes the energy functional below, see (4.18). Actually, minimizing the Fisher functional (4.18) is a stochastic control version of a classical problem proposed by Schroedinger, as can be seen in [12].

Consider the following stochastic control problem:

$$dX_t = u(t, X_t)dt + dW_t \qquad 0 \le t \le T. \tag{4.17}$$

Let

$$F(u) = \frac{1}{2}u^2$$

be a cost function. If we set

$$J(t,x,u) := \mathbb{E}_{t,x}\left[\int_t^T F(u(r,X_r))dr + \psi(X(T))\right], \tag{4.18}$$

where $\psi(x) = (2n+1)\log(x)$, our goal is to find

$$V(t,x) = \inf_u J(t,x;u)$$

We can see from [16] that the Hamilton-Jacobi-Bellman equation associated to the control problem is given by

$$\frac{\partial V}{\partial t}(t,x) + \inf_u \left[F(u) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x) + u\frac{\partial V}{\partial x}(t,x)\right] = 0$$

with boundary condition $V(T,x) = (2n+1)\log(x)$. Therefore,

$$u = V_x(t,x) \tag{4.19}$$

or

$$\frac{\partial V}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x) + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2(t,x) = 0.$$

Differentiating with respect to x the last line and setting $h(t,x) = V_x(t,x)$, we have

$$\frac{\partial h}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + h(t,x)\frac{\partial h}{\partial x}(t,x) = 0,$$

which is to be solved with the final boundary condition $h(T,x) = V_x(T,x) = \frac{2n+1}{x}$. From (4.17), (4.19), and recalling that $v_{2n+1}(T-t,x)$ is a solution to the backward heat equation we have that

$$\begin{aligned} dX_t &= V_x(t,X_t)dt + dW_t \\ &= h(t,X_t)dt + dW_t \\ &= \frac{v'_{2n+1}(T-t,X_t)}{v_{2n+1}(T-t,X_t)}dt + dW_t. \end{aligned}$$

The last equality follows from the Hopf-Cole transform [23] that relates solutions to the Burgers equation with solutions to the heat equation.

Remark 4.10. 1. *Recall that a Bessel process of dimension d follows the dynamics*

$$dX_t = \frac{(d-1)/2}{X_t}dt + dW_t$$

where W_t is a standard Brownian motion.

2. *Note that*

$$\lim_{t\uparrow T}\frac{v'_{2n+1}(T-t,x)}{v_{2n+1}(T-t,x)} = \frac{2n+1}{x} = \frac{(2(2n+1)+1-1)/2}{x}$$

therefore, an Ito process generated by the heat polynomial v_{2n+1} behaves as a Bessel process of dimension $2(2n+1)+1$ at time T.

To give a suitable interpretation of the previous procedure we firs recall the Kullback distance.

Definition 4.11. *Let μ and ν be two σ-finite measures defined on the same measurable space. If $\mu \ll \nu$ we define the Kullback (or Kullback-Leibler) distance [27] between μ and ν by*

$$K(\mu,\nu) = \int \log\left(\frac{d\mu}{d\nu}\right)d\mu. \tag{4.20}$$

Although the Kullback distance is not really a metric, it is frequently used as a measure of loss of information when the measure μ is approximated by the measure ν. The Kullback distance between μ and ν is also known as the entropy of μ relative to ν.

Now we discuss the connection between the solution to the optimal control problem described above and entropy. To this end, let us start by describing the dynamics of the following pair:

$$\begin{cases} (\mathbb{Q}) \quad dY(t) = \frac{h_x}{h} dt + dW_t, \quad 0 \le t \le T \\ (\mathbb{P}) \quad dX(t) = dW_t, \quad 0 \le t \le T \end{cases} \tag{4.21}$$

where h is a solution to the backward heat equation and $Y_T \sim \mu_T$ for some μ_T fixed. Furthermore, let

$$Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{h(t, X_t)}{h(0, x_0)}$$

be the Radon-Nikodym derivative associated to the change of measure from X to Y. From Ito's rule we have

$$d\log(Z_t) = -\frac{1}{2}\Big(\frac{h_x(t, X_t)}{h(t, X_t)}\Big)^2 dt + \frac{h_x(t, X_t)}{h(t, X_t)} dW_t \tag{4.22}$$

Taking expectation in (4.22) with respect to $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}\Big[\log\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \mathbb{E}^{\mathbb{Q}}\Big[-\frac{1}{2}\int\Big(\frac{h_x}{h}\Big)^2 dt\Big]$$

. Note that the left-hand side in the last line is precisely (4.20). Thus, the procedure developed to building Ito processes with heat polynomials minimizes the energy functional (4.18) and it reaches the entropy of $\mathbb{Q}$ relative to $\mathbb{P}$.

4.5 Hitting times for processes generated by heat polynomials

By the way we built this family of processes we can compute the density for the hitting times using known results in the case of Brownian motion.

Remark 4.12. *Let (W_t, F_t), $\mathbb{P}^x, x \in \mathbb{R}$ be a Brownian family and let T_0 and T_a be the hitting times of 0 and a, respectively. Then*

Case I.

$$\mathbb{P}^x(T_a \in dt) = \frac{(x-a)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(x-a)^2}{2t}\Big). \tag{4.23}$$

See [26] p. 80.

Case II.

$$\begin{aligned}\mathbb{P}^x[T_a \wedge T_0 \in dt] &= \tfrac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{+\infty} \Big[(2na+x)\exp\Big(\frac{-(2na+x)^2}{2t}\Big) \\ &+ (2na+a-x)\exp\Big(\tfrac{-(2na+a-x)^2}{2t}\Big)\Big] dt.\end{aligned} \tag{4.24}$$

See [26] p. 99 exercise 8.11.

Theorem 4.13. *Consider the processes given by*

$$dX_t = \frac{v'_{2n+1}(T-t, X_t)}{v_{2n+1}(T-t, X_t)} dt + dW_t, \quad X_0 = x, \quad 0 \le t < T$$

for $n = 0, 1, \cdots$. *If* $\mathbb{Q}^x$ *is the measure, which supports this process (for fixed* n*), we have*

Case I. If $0 < a < x$, *then*

$$\mathbb{Q}^x[T_a \in dt] = \frac{v_{2n+1}(T-t, a)}{v_{2n+1}(T, x)} \mathbb{P}^x[T_a \in dt]. \tag{4.25}$$

Case II. If $0 < x < a$, *then*

$$\mathbb{Q}^x[T_a \in dt] = \frac{v_{2n+1}(T-t, a)}{v_{2n+1}(T, x)} \Big[\mathbb{P}^x[T_a \wedge T_0 \in dt] - \mathbb{P}^x[T_0 \in dt]\Big]. \tag{4.26}$$

Proof. Case I. As $0 < a < x$, by the continuity of the sample paths of Brownian motion we have $T_a < T_0$. Hence the event $\{T_a \le t\}$ is in F_{T_0}. If $\mathbb{Q}^x$ denotes the probability of the process X and $\mathbb{P}^x$ denotes the probability of Brownian motion started at x, then we have

$$\begin{aligned}\mathbb{Q}^x[T_a < t] &= \int_{\{T_a < t\}} \frac{v_{2n+1}(T-t, X_t)}{v_{2n+1}(T, x)} d\mathbb{P}^x \\ &= \int_0^{+\infty} \mathbb{E}_{\mathbb{P}^x}\Big[\frac{v_{2n+1}(T-t, X_t)}{v_{2n+1}(T, x)} \mathbb{1}_{\{T_a < t\}} | T_a = s\Big] \mathbb{P}^x(T_a \in ds) \\ &= \int_0^t \frac{v_{2n+1}(T-s, a)}{v_{2n+1}(T, x)} \mathbb{P}^x(T_a \in ds).\end{aligned}$$

In the last line we applied the optimal sampling theorem. Differentiating with respect to t and using (4.23) we have the desired result.

Case II. We follow the proof of the first case. However, we must take into account that the process could hit 0 before hitting a. Let $\mathbb{Q}^x$ and $\mathbb{P}^x$ be as in the first case. Then

$$\begin{aligned}\mathbb{Q}^x[T_a < t] &= \mathbb{E}_{\mathbb{Q}^x}\left[\mathbb{1}_{\{T_a<t\}}\mathbb{1}_{\{T_0>t\}}\right] \\ &= \mathbb{E}_{\mathbb{Q}^x}\left[\mathbb{1}_{\{T_a<t,T_0>t\}}\right] \\ &= \mathbb{E}_{\mathbb{P}^x}\left[\frac{v_{2n+1}(T-t,X_t)}{v_{2n+1}(T,x)}\mathbb{1}_{\{T_a<t,T_0>t\}}\right] \\ &= \mathbb{E}_{\mathbb{P}^x}\left[\frac{v_{2n+1}(T-t,a)}{v_{2n+1}(T,x)}\mathbb{1}_{\{T_a<t,T_0>t\}}\right],\end{aligned}$$

where in the last line we used again the optimal sampling theorem. Finally note that $\mathbb{1}_{\{T_a<t,T_0>t\}} = \mathbb{1}_{\{T_a\wedge T_0<t\}} - \mathbb{1}_{\{T_0<t\}}$.

□

4.6 Example. The process with $v_3(t,x)$. Stochastic volatility

The aim now is to put into action the results obtained in the previous sections. Let us first recall that, in general, Bessel processes are used in a wide range of applications. In particular, there exists a space-time transformation that transforms Bessel processes into CIR (or square-root) diffusions. We will follow this analogy and will point out some possible applications. Take

$$dX_t = \frac{3X_t^2 + 3(T-t)}{X_t^3 + 3(T-t)X_t}dt + dB_t. \tag{4.27}$$

Let

$$Y_t := e^{-t/2}X_{e^t}, \qquad W_t := \int_0^{e^t} e^{-s/2}dB_s.$$

Then we have

$$\begin{aligned}dX_{e^t} &= \frac{3X_{e^t}^2 + 3(e^T - e^t)}{X_{e^t}^3 + 3(e^T-e^t)X_{e^t}}e^t dt + e^{t/2}dB^{e^t} \\ &= \frac{3e^tY_t^2 + 3(e^T-e^t)}{e^{3/2\cdot t}Y_t^3 + 3(e^T-e^t)e^{1/2\cdot t}Y_t}e^t dt + e^{t/2}dB^{e^t} \\ &= e^{t/2}\left(\frac{3e^tY_t^2 + 3(e^T-e^t)}{e^tY_t^3 + 3(e^T-e^t)Y_t}dt + dB^{e^t}\right) \\ &= e^{t/2}\left(\frac{3Y_t^2 + 3(e^{T-t}-1)}{Y_t^3 + 3(e^{T-t}-1)Y_t}dt + dB^{e^t}\right).\end{aligned}$$

It follows that

$$dY_t = \left(\frac{3Y_t^2 + 3(e^{T-t} - 1)}{Y_t^3 + 3(e^{T-t} - 1)Y_t} - \frac{Y_t}{2}\right) dt + dW_t.$$

Thus, if we set $U = Y^2$, then

$$dU_t = \left(6\frac{U_t + (e^{T-t} - 1)}{U_t + 3(e^{T-t} - 1)} - \frac{U_t}{2} + 1\right) dt + 2\sqrt{U_t} dW_t.$$

Note that this last process is a CIR-like process. If process U_t is used to model stochastic volatility in timer options, we can get an analogous result to Proposition 4.1 in [31]. We will next explain what we mean. If we rewrite the process U_t in integral form, then we have

$$U_t = U_0 + \int_0^t \Big(6\frac{U_v + (e^{T-v} - 1)}{U_v + 3(e^{T-v} - 1)} - \frac{U_v}{2} + 1\Big) dv + 2\int_0^t \sqrt{U_v} dW_v. \tag{4.28}$$

Letting $M_t := \int_0^t \sqrt{U_s} dW_s$, then $\langle M\rangle_t = \int_0^t U_s ds$. Now, we define

$$S_a := \inf\{t > 0 : \int_0^t U_v dv = a\}$$

S_a is the first time that the total realized variance reaches the level a. We will find the joint distribution of the random variable (U_{S_a}, S_a), which is used in the valuation of timer options. From (4.28) it follows that

$$U_{S_a} = U_0 + \int_0^{S_a} \Big(6\frac{U_v + (e^{T-v} - 1)}{U_v + 3(e^{T-v} - 1)} - \frac{U_v}{2} + 1\Big) dv + 2\int_0^{S_a} \sqrt{U_v} dW_v.$$

By Theorem 4.6 in [26] p. 174 we have that $B_{\langle M\rangle_{S_a}} := \int_0^{S_a} \sqrt{U_v} dW_v$ is an standard Brownian motion. Then

$$U_{S_a} = U_0 + \int_0^{S_a} \Big(6\frac{U_v + (e^{T-v} - 1)}{U_v + 3(e^{T-v} - 1)} - \frac{U_v}{2} + 1\Big) dv + 2B_{\langle M\rangle_{S_a}},$$

and application of the inverse function theorem gives us

$$S_t = \int_0^t \frac{1}{U_{S_v}} dv.$$

Using Exercise 4.5 in [26] p. 174 we have

$$U_{S_a} = U_0 + \int_0^a \frac{1}{U_{S_u}} \Big(6\Big(\frac{U_{S_u} + (e^{T-S_u} - 1)}{U_{S_u} + 3(e^{T-S_u} - 1)}\Big) - \frac{U_{S_u}}{2} + 1\Big) du + 2B_a.$$

If $X_a := \frac{U_{S_a}}{2}$, then

$$X_a = \frac{U_0}{2} + \int_0^a \frac{1}{4X_u}\Big(6\Big(\frac{2X_u + (e^{T-S_u} - 1}{2X_u + 3(e^{T-S_u} - 1}\Big) - X_u + 1\Big)du + B_a. \tag{4.29}$$

Note that

$$e^{-S_u} = \exp\big(-\int_0^u \frac{1}{U_{S_r}}dr\big) = \exp\big(-\int_0^u \frac{1}{2X_r}dr\big).$$

From (4.29) we have

$$dX_a = \frac{1}{4X_a}\Big(6\frac{2X_a + e^T e^{-\int_0^a \frac{1}{2X_r}dr} - 1}{2X_a + 3(e^T e^{-\int_0^a \frac{1}{2X_r}dr} - 1)} - X_a + 1\Big)da + dB_a;$$

hence, it follows that

$$(U_{S_a}, S_a) \stackrel{\mathcal{D}}{=} \Big(X_a, \int_0^a \frac{ds}{2X_s}\Big).$$

Hence, we have found the distribution of the joint random variable (U_{S_a}, S_a) in terms of the process (4.27) generated by $v_3(t,x)$. In this case S_a represents the random maturity and U_t is the stochastic volatility. As was pointed out above, the joint distributions of (U_{S_a}, S_a), is used in the valuation of timer options.

4.7 Concluding remarks

In this chapter we derive and study some properties of a Bessel-like family of stochastic processes. Processes generated by the heat polynomial v_{2n+1} can be interpreted as steering a 3-D Bessel process into a $2(2n+1)+1$ Bessel process by time T. For example, think of an industry within a given economy which is a monopoly and by some future time T will have n competitors. This could have effects on the policy makers.

5 Optimal allocation of resources via optimal transport

5.1 Introduction

In this chapter we study an optimal resource allocation problem on a urban region $\Omega \subset \mathbb{R}^n$ which has a population distribution $\mu \in P(\Omega)$. The set $\Omega \subset \mathbb{R}^n$ is assumed to be compact. The optimization problem comes from the necessity of a social planner to distribute an infinitely divisible resource on the region Ω in the following setting: we have n fixed facilities on Ω denoted by $y_1, \cdots, y_n$ from where the resource will be delivered directly to the people. The social planner has to find the resource distribution, which will be modeled as a probability measure $\nu \in P(\Omega)$, and the part of the population that will be served by every facility y_i. Therefore, the social planner's problem is to minimize transportation costs and distribution costs while maximizing the welfare obtained by the population that receives the resource.

Following ideas of Carlier and Mallozzi [10] we solve the problem as a bi-level optimization problem.

5.2 Preliminaries

For completeness, in this section we recall some basic facts from convex analysis and optimal transport. A function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called a proper convex function if it is not identically infinity and

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) \quad \forall x, y \in \mathbb{R}^n, \quad \forall t \in [0,1]. \tag{5.1}$$

For dealing with (possibly) non-differentiable convex functions we define the subdifferential of a convex function.

Definition 5.1. *The subdifferential of a convex function ϕ, denoted as $\partial\phi$, is a set-valued application given by*

$$y \in \partial\phi(x) \iff \Big[\forall z \in \mathbb{R}^n, \phi(z) \geq \phi(x) + \langle y, z - x\rangle\Big] \tag{5.2}$$

where $\langle\cdot,\cdot\rangle$ is the canonical inner product in $\mathbb{R}^n$.

For a convex function ϕ we define its Legendre transform by

$$\phi^*(y) = \sup_{x\in\mathbb{R}^n}\{\langle x, y\rangle - \phi(x)\}. \tag{5.3}$$

In [45] p. 55 it is shown the following characterization of the subdifferential.

Proposition 5.2. *Let ϕ be a proper lower semi-continuous convex function on $\mathbb{R}^n$. Then, for all $x, y \in \mathbb{R}^n$,*

$$\langle x, y\rangle = \phi(x) + \phi(y) \iff y \in \partial\phi(x) \iff x \in \partial\phi^*(y). \tag{5.4}$$

Let X be a complete and separable metric space (also known as Polish space). We denote by $P(X)$ the set of probability measures on X. Given two probability measures $\mu, \nu \in P(X)$ and a function $c : X \times X \to \mathbb{R}$ the Monge-Kantorovich problem is

$$MK_c(\mu, \nu) := \inf_{\gamma\in\Pi(\mu,\nu)}\Big\{\int_{X\times X} c(x, y)d\gamma(x, y)\Big\} \tag{5.5}$$

where $\Pi(\mu, \nu)$ is the set of probability measures on $X \times X$ with marginals given by μ and ν respectively. In [39] p. 5 it is shown that the set $\Pi(\mu, \nu)$ is a compact subset of $P(X \times X)$ with the weak-* topology. Then, under mild conditions on the cost function $c(x, y)$, we can guarantee the existence of minimizers for (5.5). A minimizer for (5.5) is called an optimal plan. If there exists a minimizer of (5.5) that is supported on the graph of a function $T : X \to X$, then we say that T solves the Monge problem

$$M_c(\mu, \nu) = \inf_{T_{\#}\mu=\nu}\Big\{\int_X c(x, T(x))d\mu(x)\Big\} \tag{5.6}$$

where $T_{\#}\mu := \mu(T^{-1})$. In this case T is called an optimal map and it will be denoted $(Id, T)_{\#\mu}$.

Consider now the dual problem

$$DP_c(\mu,\nu) := \sup\Big\{ \int_X \varphi d\mu + \int_X \psi d\nu : \varphi, \psi \in C(\Omega), \phi \oplus \psi(x,y) \leq c(x,y) \Big\} \tag{5.7}$$

where $\phi \oplus \psi(x,y) := \phi(x) + \psi(y)$. The duality theorem for the Monge-Kantorovich problem establishes that, under suitable assumptions,

$$MK_c(\mu,\nu) = DP_c(\mu,\nu). \tag{5.8}$$

Given a function $c(x,y)$ we define the c-transform of a function $\phi(x)$ by

$$\phi^c(x) = \min_y \{c(x,y) - \phi(y)\}. \tag{5.9}$$

The dual problem (5.7) is equivalent to

$$DP_c(\mu,\nu) = \sup\Big\{ \int_X \psi^c d\mu + \int_X \psi d\nu : \psi \in C(\Omega) \Big\}. \tag{5.10}$$

Technical details of optimal transportation problems can be read at [45] and [39].

In our case, the set $X = \Omega$, where $\Omega \subset \mathbb{R}^n$ is assumed to be compact. In this chapter it will be very important the semi-discrete case in which $\nu = \sum_{i=1}^n \omega_i \delta_{y_i}$ where $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$. In this case, the duality formula (5.10) takes the form

$$DP_c(\mu, \sum_{i=1}^n \omega_i \delta_{y_i}) = \sup_{b \in \mathbb{R}^n} \Big\{ \int_\Omega \min_{i=1,\cdots,n} (c(x,y_i) - b_i) d\mu + \sum_{i=1}^n \omega_i b_i \Big\}. \tag{5.11}$$

5.3 The model

In our model, there is a social planner that wishes to distribute a resource (health services, help for victims of a natural disaster) on a city $\Omega \subset \mathbb{R}^n$ that has a population density $\mu \in P(\Omega)$. The resource is transported first to n fixed facilities $y_1, \cdots, y_n$ and then every location y_i distributes the resource to a part of the population ω_i. The social planner wants to find a manner to distribute the resource minimizing the distribution costs but also maximizing the utility (or welfare) obtained by the population that receives the resource.

With the city $\Omega \subset \mathbb{R}^n$, population density μ, and fixed locations $y_1, \cdots, y_n \in \mathbb{R}^n$ the problem data includes a cost of transportation $c(x,y)$, costs of distribution $F_i(x)$ from facility y_i, and profit functions $u_i(x)$ by distribution of the resource from facility y_i.

Definition 5.3. *The unit simplex* S *in* $\mathbb{R}^n$ *is*

$$S := \{\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n : \omega_i \geq 0, \quad \sum_{i=1}^{n} \omega_i = 1\}$$

The cost incurred by the social planner for delivering the resource at resident $x \in \Omega$ from facility y_i is $c(x, y_i) + 2F_i(x)$, where 2 is just a normalizing constant. On the other side a resident $x \in \Omega$ that receives the resource from facility y_i obtain a utility $u_i(x)$. The main unknown of this model are the distribution of the resource in the city, which will be modelled as a probability measure $\nu \in P(\Omega)$, and a partition of population $\omega = (\omega_1, \cdots, \omega_n) \in S$. In this case ω_i is the proportion of population that will be served from facility y_i. Given a resource distribution $\nu \in P(\Omega)$ and $\omega \in S$ we define

$$A_i(\nu, \omega) := \bigcap_{j \neq i} \{x \in \Omega : c(x, y_i) + 2\omega_i \int_\Omega F_i d\nu < c(x, y_j) + 2\omega_j \int_\Omega F_j d\nu\}. \tag{5.12}$$

Note that given $\nu \in P(\Omega)$ and $\omega \in S$, the set $A_i(\nu, \omega)$ is the part of the population for which it is cheaper to deliver the resource from facility y_i. In order to state rigorously the problem we will assume the following.

- The city $\Omega \subset \mathbb{R}^n$ is compact, connected and $c(x, y) : \Omega \times \Omega \to \mathbb{R}^+$ is continuous,
- the population density $\mu \in P(\Omega)$ is absolutely continuously with respect to the Lebesgue measure L on Ω,
- for $i = 1, \cdots, n$, the distribution costs $F_i : \Omega \to \mathbb{R}^+$ are continuous,
- for $i = 1, \cdots, n$, the profit functions $u_i : \Omega \to \mathbb{R}^+$ are upper semi-continuous on Ω,
- for technical reasons, we impose the following condition on the population density and the transportation costs:

$$\mu\{x \in \Omega : c(x, y_i) = k\} = \mu\{x \in \Omega : c(x, y_i) - c(x, y_j) = k\} = 0 \tag{5.13}$$

  holds for all $i, j = 1 \cdots, n$ and $k \in \mathbb{R}$.

Condition (5.13) ensures that the set of the population for which is indifferent deliver the resource from two (or more) different facilities is negligible.

Now, given a fixed resource distribution $\nu \in P(\Omega)$, the social planner has to find $\omega \in S$ such that

$$\omega_i = \mu(A_i(\nu, \omega)) \quad \text{for} \quad i = 1 \cdots, n \tag{5.14}$$

where $A_i(\nu, \omega)$ is defined in (5.12). Condition (5.14) ensures that we split the population in such a way that every facility will deliver exactly to the fraction of people for which the costs are minimum. On the other hand the profit (or welfare) function by the distribution of the resources is given by

$$U(\nu, \omega) = \sum_{i=1}^{n} \omega_i \int_\Omega u_i(x) d\nu(x). \tag{5.15}$$

Summarizing, we have to find $\nu \in P(\Omega)$ and $\omega \in S$ such that (5.14) holds while (5.15) is maximized. Strongly inspired by ideas in [32] and [10] we will solve the coupled problem (5.14)-(5.15) as a bi-level optimization problem. We will describe next briefly the strategy that will be used to solve this resource-allocation problem.

Let $\nu \in P(\Omega)$ be fixed. the so-called lower level problem (LLP) is to find $\omega \in S$ such that (5.12)-(5.14) are satisfied. It will be shown that for $\nu \in P(\Omega)$ fixed, there exists a unique $\omega \in S$ that satisfies (5.12)-(5.14).

Definition 5.4. *Given $\nu \in P(\Omega)$, we denote by $\Psi(\nu)$ the unique $\omega \in S$ that satisfies (5.12)-(5.14) and it is called the best response function.*

We will show in Lemma 5.9 below that the best response function is continuous with respect to the weak-* topology on $P(\Omega)$. This allows us to define the upper level problem (ULP) by

$$\sup_{\nu \in P(\Omega)} \Big\{ \sum_{i=1}^{n} \Psi(\nu)_i \int_\Omega u_i d\nu \Big\}, \tag{5.16}$$

where $\Psi(\nu)_i$ is ith coordinate of the vector $\Psi(\nu)$. We will show in Theorem 5.10 below that ULP admits at least one maximizer.

5.4 The lower level problem

In this section we show that the LLP is solvable, that is, for fixed $\nu \in P(\Omega)$ there exists a unique $\omega \in S$ such that

$$\omega_i = \mu(A_i(\nu,\omega)) \quad \text{for} \quad i = 1\cdots,n, \tag{5.17}$$

where

$$A_i(\nu,\omega) := \bigcap_{i\neq j}\big\{x\in\Omega : c(x,y_i) + 2\omega_i\int_\Omega F_i d\nu < c(x,y_j) + 2\omega_j\int_\Omega F_j d\nu\big\}. \tag{5.18}$$

As in [10] this result is obtained by establishing a relationship between (5.17) and optimal transportation problems.

Theorem 5.5. *Let $\nu \in P(\Omega)$ be fixed. Then $\omega \in S$ solves (5.17)-(5.18) if and only if ω minimizes*

$$G^\nu(\omega) := MK_c(\mu, \sum_{i=1}^n \omega_i\delta_{y_i}) + \sum_{i=1}^n \omega_i^2\int_\Omega F_i(x)d\nu(x). \tag{5.19}$$

Remark 5.6. *We note that, by the duality formula (5.8), in the semi-discrete case (5.11) we have*

$$MK_c(\mu, \sum_{i=1}^n \omega_i\delta_{y_i}) = \sup_{b\in\mathbb{R}^n}\Big\{\int_\Omega \min_{i=1,\cdots,n}(c(x,y_i)-b_i)d\mu + \sum_{i=1}^n \omega_i b_i\Big\}.$$

Given that MK_c is the supremum of affine functionals (linear functionals plus a constant), then it is convex and lower semi-continuous on S. Besides, for $\nu \in P(\Omega)$ fixed, $\sum_{i=1}^n \omega_i^2\int_\Omega F_i(x)d\nu(x)$ is continuous and strictly convex on S which is compact. It follows that $G(\omega)$ is a strictly convex and lower semi-continuous function on S and so G reaches the minimun which is unique by strict convexity.

We need some lemmas to prove Theorem 5.5. First, we give the explicit solution to the Monge-Kantorovich problem in terms of a vector $\mathbf{b}$ that solves (5.11) and sets (5.18).

Lemma 5.7. *Let $\mathbf{b}$ be a vector that solves the dual problem in the left hand side of (5.11). Then the solution to the optimal transport problem $MK_c(\mu, \sum_{i=1}^n \omega_i\delta_{y_i})$ is given by*

$$\gamma(dx,dy) = (Id, \mathbb{1}_{A_i(\mathbf{b})}(x)y_i)_{\#\mu}, \tag{5.20}$$

where $A_i(\mathbf{b}) = \cap_{j\neq i}\{x\in\Omega; c(x,y_i)+b_i < c(x,y_j)+b_j\}$ and $\mathbb{1}$ denotes the indicator function.

Proof. Let $\gamma(dx,dy) \in \Pi(\mu,\nu)$ be such that $MK_c(\mu,\nu) = \int_\Omega c(x,y)d\gamma(x,y)$ where $\nu = \sum_{i=1}^n \omega_i \delta_{y_i}$. By disintegration of measures it follows that

$$\gamma(dx,dy) = \mu(dx) \otimes (\sum_{i=1}^n a_i(x)\delta_{y_i}(dy)) \tag{5.21}$$

where $\sum_{i=1}^n a_i(x) = 1$ μ-a.e. Furthermore,

$$\omega_i = \nu(\{y_i\}) = \gamma(\{y_i\} \times \Omega) = \int_\Omega a_i(x)d\mu(x), \quad i = 1, \cdots, n. \tag{5.22}$$

From (5.21) and the duality formula (5.11) it follows that

$$\begin{aligned}\int_{\Omega\times\Omega} c(x,y)d\gamma(x,y) &= \sum_{i=1}^n \int_\Omega a_i(x)c(x,y_i)d\mu(x) \\ &= \int_\Omega \min_{i=1,\cdots,n}(c(x,y_i) - b_i)d\mu(x) + \sum_{i=1}^n b_i\omega_i.\end{aligned}$$

Equivalently, using (5.22),

$$\sum_{i=1}^n \int_\Omega (c(x,y_i) - b_i)a_i(x)d\mu(x) = \int_\Omega \min_{i=1,\cdots,n}(c(x,y_i) - b_i)d\mu(x).$$

By (5.13) it follows that

$$a_i(x) = \mathbb{1}_{A_i(\mathbf{b})}(x) \quad \mu\text{-a.e.}$$

□

Lemma 5.8. *Let $J : S \to \mathbb{R}$ be given by $J(\omega) = MK_c(\mu, \sum_{i=1}^n \omega_i\delta_{y_i})$. Then*

$$\mathbf{p} \in \partial J(\omega) \iff \mathbf{p} \quad \textit{solves} \quad \sup_{b\in\mathbb{R}^n}\Big\{\int_\Omega \min_{i=1,\cdots,n}(c(x,y_i) - b_i) + \langle \mathbf{b}, \omega\rangle\Big\}. \tag{5.23}$$

Proof. By Remark 5.6 we know that J is a convex function and then we can work with the subdifferential of J. If we define

$$H(\mathbf{b}) := -\int_\Omega \min_{i=1,\cdots,n}(c(x,y_i) - b_i)d\mu(x) \quad \forall \mathbf{b} \in \mathbb{R}^n,$$

then the Legendre transform (5.3), $H^*(\omega)$, becomes

$$H^*(\omega) = \begin{cases} \sup_{b\in\mathbb{R}^n}\Big\{\int_\Omega \min_{i=1,\cdots,n}(c(x,y_i) - b_i) + \langle \mathbf{b}, \omega\rangle\Big\} & \omega \in S, \\ +\infty & \omega \notin S. \end{cases}$$

By the duality formula (5.8) we have

$$H^*(\omega) = \begin{cases} MK_c(\mu, \sum_{i=1}^n \omega_i \delta_{y_i}) & \omega \in S, \\ +\infty & \omega \notin S. \end{cases}$$

This fact and (5.4) yield

$$\omega \in \partial H(\mathbf{b}) \iff \mathbf{b} \in \partial H^*(\omega) \iff \langle \omega, \mathbf{b} \rangle = H^*(\omega) + H(\mathbf{b}).$$

Then we have

$$\mathbf{b} \in \partial J^*(\omega) \iff \omega \in S \quad \text{and} \quad J(\omega) = \langle \omega, \mathbf{b} \rangle - H(\mathbf{b}).$$

Hence, our result follows. □

Now we have enough tools for proving Theorem 5.5.

Proof Theorem 5.5. Note that

$$\mathbf{0} \in \partial G^\nu(\omega) \iff \Big(2\omega_i \int_\Omega F_i d\nu\Big)_{i=1}^n \in \partial MK_c(\mu, \sum_{i=1}^n \omega_i \delta_{y_i}).$$

By Lemma 5.8, this is equivalent to

$$\Big(-2\omega_i \int_\Omega F_i d\nu\Big)_{i=1}^n \quad \text{solves} \quad \sup_{b \in \mathbb{R}^n} \Big\{ \int_\Omega \min_{i=1,\cdots,n} (c(x, y_i) - b_i) + \langle \mathbf{b}, \omega \rangle \Big\},$$

which by (5.20) also is equivalent to

$$\omega_i = \mu(A_i(\nu, \omega)), \quad i = 1, \cdots, n.$$

□

5.5 Best response function and the upper level problem

Recall that S is the unit simplex in $\mathbb{R}^n$ defined in (5.3). We define the best response function as (compare with (5.4))

$$\Psi : P(\Omega) \longrightarrow S \quad \text{with} \quad \Psi(\nu) = \arg\min G^\nu(\omega) \tag{5.24}$$

where $G^\nu(\omega)$ is given in (5.19). The goal of this section is to show that the best response function is well defined and continuous. Furthermore, we show that upper level problem is solvable.

Lemma 5.9. *The best response function (5.24) is well defined and continuous with respect to weak-* topology on $P(\Omega)$.*

Proof. The best response function is well defined thanks to Theorem 5.5 and Remark 5.6. To show continuity let $\nu_n \in P(\Omega)$ be such that $\nu_n \to \nu$ in the weak-* topology. Let $\omega_n = \Psi(\nu_n)$ and $\omega = \Psi(\nu)$ be best response values. The unit simplex S is a compact set and then $\{\omega_n\}_{n=1}^{\infty}$ has a accumulation point ω^*. Without loss of generality we can assume that $\omega_n \to \omega^*$. By definition we have

$$G^{\nu}(\omega) \leq G^{\nu}(\omega^*). \tag{5.25}$$

On the other hand

$$G^{\nu_n}(\omega_n) \leq G^{\nu_n}(\omega).$$

Now, given that the F_i are continuous functions and $\nu_n \to \nu$, we have $G^{\nu_n}(\omega_n) \to G^{\nu}(\omega^*)$ as $n \to \infty$. Hence

$$G^{\nu}(\omega^*) \leq G^{\nu}(\omega). \tag{5.26}$$

From (5.25) and (5.26) we have

$$G^{\nu}(\omega^*) = G^{\nu}(\omega),$$

but then, by strict convexity, $\omega = \omega^*$. The result follows. □

Making use of Theorem 5.5 we can state the upper level problem as

$$\sup_{\nu \in P(\Omega)} \Big\{ \sum_{i=1}^{n} \Psi(\nu)_i \int_{\Omega} u_i d\nu \Big\}, \tag{5.27}$$

where $\Psi(\nu)_i$ is ith coordinate of the vector $\Psi(\nu)$. Now we show that the upper level problem (5.27) is solvable.

Theorem 5.10. *The upper level problem (5.27) admits at least one solution.*

Proof. We know that $\nu \to \int_{\Omega} u_i d\nu$ is upper semi-continuous (usc) in the weak-* topology if u_i is usc; see [39] p. 250. Besides, by Lemma 5.9 we know that the best response function is continuous on $P(\Omega)$. Therefore the function

$$\nu \to \sup_{\nu \in P(\Omega)} \Big\{ \sum_{i=1}^{n} \Psi(\nu)_i \int_{\Omega} u_i d\nu \Big\},$$

is the composition of a continuous and a usc function on $P(\Omega)$. Then it is usc functions on $P(\Omega)$ and given that $P(\Omega)$ is compact in the weak-* topology (see [7]), we prove the desired result. □

5.6 Concluding remarks

In this chapter we propose a Stackelberg model corresponding to an optimal allocation of resources on an urban region $\Omega \in \mathbb{R}^n$. In the problem we minimize transportation costs and distribution costs while maximizing the welfare obtained by the population that receives the resource. The problem is solved as a bilevel optimization problem where the existence of the best response function is shown by means of the Monge-Kantorovich problem. Providing examples, as well as the relationship between the model and equilibrium theory is work in progress.

6 Conclusions and future work

We developed a procedure for solving the heat equation with a moving boundary for a family of boundaries. We showed that the procedure extends up to a cubic boundary. As immediate future work we have to try to extend this procedure to arbitrary polynomial boundaries.

In the second part of this thesis we study a class of Ito Processes that are constructed in terms of the so-called heat polynomials. In particular the elements of this family can be viewed as processes which are being steered towards Bessel processes of odd dimension by a fixed, future time T starting from a 3D-Bessel process. An example could be a closed monopolistic economy that at some future time will be open to more competitors.

Finally, we propose a Stackelberg model in the optimal allocation of resources. The population as well as the distribution of the resource are modelled as probability measures. The model takes into account cost of transportation and distribution as well as profit function getting by population that receives the resource. A solution to the model is gotten by means of optimal transport techniques. In future work we want to study the relationship of this model with equilibrium theory.

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