CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

Campus Zacatenco

Department of Mathematics

Characterization of some model commutative C*-algebras generated by Toeplitz operators

A DISSERTATION PRESENTED BY:

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TO OBTAIN THE DEGREE OF

DOCTOR IN SCIENCE

IN THE SPECIALITY OF MATHEMATICS

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MÉXICO, D.F. JULY 28, 2015.

Abstract

The aim of this work is to give explicit descriptions of the C*-algebras generated by Toeplitz operator whose defining symbols are invariant under dilations, rotations and translations. Therefore, we have divided this thesis in three chapters:

In Chapter 2 we consider Toeplitz operators acting on the weighted Bergman space $\mathscr{A}^2_\lambda(\Pi)$ (Π denotes the upper half-plane) with defining symbol invariant under dilations, i.e., functions $\phi \in L_\infty(\Pi)$, such that for every $h > 0$ the equality $\phi(z) = \phi(hz)$ holds a.e. $z \in \Pi$. This class of defining symbols is called *angular*, and we give a criterion for a function to be angular:

$$\phi \textit{ is angular, if and only if there exists } a \in L_\infty(0,\pi) \textit{ such that } \phi(z) = a(\arg z),\ a.e.\ z \in \Pi.$$

The main result states that the uniform closure of the set of all Toeplitz operators acting on the weighted Bergman space over the upper half-plane whose L_∞-symbols are angular coincides with the C*-algebra generated by the above Toeplitz operators and is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}$) of bounded functions that are very slowly oscillating on the real line in the sense that they are uniformly continuous with respect to the arcsinh-metric $\rho(x,y) = |\mathrm{arcsinh} x - \mathrm{arcsinh} y|$ on the real line.

Chapter 3 is devoted to the study of Toeplitz operators acting on the Fock space $\mathscr{F}^2(\mathbb{C})$ with defining symbol invariant under rotations, i.e., functions $\varphi \in L_\infty(\mathbb{C})$, such that for every $t \in [0,2\pi)$ the equality $\varphi(z) = \varphi(e^{-it}z)$ holds a.e. $z \in \mathbb{C}$. This class of defining symbols is called *radial*, and we give a criterion for a function to be radial:

$$\varphi \textit{ is radial, if and only if there exists } b \in L_\infty(\mathbb{R}_+) \textit{ such that } \varphi(z) = b(|z|),\ a.e.\ z \in \mathbb{C}.$$

The principal theorem shows that the C*-algebra generated by radial Toeplitz operators with L_∞-symbols acting on the Fock space is isometrically isomorphic to the C*-algebra RO($\mathbb{Z}_+$) of bounded sequences uniformly continuous with respect to the square-root-metric $\rho(j,k) = |\sqrt{j} - \sqrt{k}\,|$. More precisely, we prove that the sequences of eigenvalues of radial Toeplitz operators form a dense subset of the latter C*-algebra of sequences.

Finally, in Chapter 4 the Toeplitz operators on the Fock space $\mathscr{F}^2(\mathbb{C}^n)$ are taken with defining symbol invariant under imaginary translations, i.e., functions $\varphi \in L_\infty(\mathbb{C}^n)$, such that for every $h \in \mathbb{R}^n$ the equality $\varphi(z) = \varphi(z - ih)$ holds a.e. $z \in \mathbb{C}^n$. This class of defining symbols is called *horizontal*, and we give a criterion for a function to be horizontal:

$$\varphi \textit{ is horizontal, if and only if there exists } c \in L_\infty(\mathbb{R}^n) \textit{ such that } \varphi(z) = c(\mathrm{Re}\, z),\ a.e.\ z \in \mathbb{C}^n.$$

Let $\mathscr{L}$ be any Lagrangian plane of the symplectic real space $(\mathbb{R}^{2n}, \omega_0)$. We show that the C*-algebra $\mathcal{T}_{\mathscr{L}}(L_\infty)$ generated by Toeplitz operators acting on the Fock space whose defining L_∞– symbols are $\mathscr{L}$–invariant is isometrically isomorphic to the C*-algebra $\mathcal{T}_{hor}(L_\infty)$ generated by Toeplitz operators acting on the Fock space with horizontal L_∞–symbols. Here, a function $\psi \in L_\infty(\mathbb{R}^{2n})$ is said to be *$\mathscr{L}$–invariant* if for every $h \in \mathscr{L}$ one gets that $\psi(z-h) = \psi(z)$ a.e. $z \in \mathbb{R}^{2n}$, in particular, the horizontal case corresponds to $\mathscr{L} = \{0\} \times \mathbb{R}^n$. The main result of this part states that $\mathcal{T}_{hor}(L_\infty)$ is isometrically isomorphic to the C*-algebra $C_{b,u}(\mathbb{R}^n)$ of bounded functions that are uniformly continuous with respect to the usual metric $d(x,y) = |x-y|$ on the n-dimensional real plane. More precisely, we prove that the corresponding spectral functions form a dense subset in $C_{b,u}(\mathbb{R}^n)$.

The results of the Chapter 2 were published in the Journal of Communications in Mathematical Analysis, Volume 17, Number 2 (2014), 151-162, http://projecteuclid.org/euclid.cma/1418919761 and online first in the Journal of Integral Equation and Operator Theory. http://dx.doi.org/10.1007/s00020-015-2243-4 The results of Chapter 3 have been published online arXiv:1505.07906 and submitted to the Journal of Complex Analysis and Operator Theory.

Dedication and acknowledgements

Every work needs self efforts as well as a guidance of God and the aid of each person in your life, especially those who were very close to my heart. For that, I would like to acknowledge with gratitude, the support and love of my family, my parents *Nicanor* and *Elvis* (R.I.P), my brothers *Kelmar* and *Kenny*, my wife and my friends, with a special acknowledgement to the *Profesor Osmin Ferrer* who have helped me in all moment.

I would like to thank *Dr. Nikolai Vasilevski* and *Dr. Egor Maximenko* for their enthusiasm, their encouragement and by showing me the way forward to achieve my objective. They were always available for my questions and they were positive and gave me generously of their time and vast knowledge. They always knew where to look for the answers to obstacles while leading me to the right source, theory and perspective.

Finally, I am also grateful to CINVESTAV, and CONACyT by their support all these years, and every person mentioned or not previously by make possible this Ph.D. thesis.

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Introduction

In linear algebra an infinite Toeplitz matrix T is defined by the rule:

$$T = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$. In 1911 *Otto Toeplitz* proved that the matrix T defines a bounded operator on $\ell_2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, if and only if the numbers a_n are the Fourier coefficients of a function $a \in L_\infty(S^1)$, where S^1 is the unit circle.

The classical Hardy space $\mathscr{H}^2$ can be viewed as the closed linear span in $L_2(S^1)$ of $\{z^n : n \geq 0\}$. For $g \in L_\infty(S^1)$, the Toeplitz operator T_g defined by $T_g h = B(gh)$, where B denotes the orthogonal projection from $L_2(S^1)$ onto $\mathscr{H}^2$, is bounded and satisfies $\|T_g\| \leq \|g\|_\infty$. The matrix of T_g with respect to the orthonormal basis $\{z^n : n \geq 0\}$ is the Toeplitz matrix T with a_n being the Fourier coefficients of g. Thus, the Toeplitz operators are a generalization of the Toeplitz matrices T.

Let $\mathscr{X}$ be a function space and let B be a projection of $\mathscr{X}$ onto some closed subspace $\mathscr{Y}$ of $\mathscr{X}$. Then the Toeplitz operator $T_g : \mathscr{Y} \longrightarrow \mathscr{Y}$ with defining symbol g is given by $T_g f = B(gf)$. The most studied cases are when $\mathscr{Y}$ is either the Bergman space, the Hardy space, or the Fock space. More recently Toeplitz operators have been also studied on many other spaces, for example on the harmonic Bergman space [38].

The Toeplitz operators have been extensively studied in several branches of mathematics: complex analysis, theory of normed algebras, operator theory [5, 28, 36, 53], harmonic analysis [1, 17], and mathematical physics, particularly in connection with quantum mechanics [9, 11], etc.

Recently, Jingbo Xia [51] showed that the norm closure of the set of Toeplitz operators acting on Bergman spaces (and Fock spaces) with general L_∞-symbols coincides with the C*-algebra generated by them. It is a depper result but unfortunately it is know very little about the properties of Toeplitz operators with general L_∞-symbols; though some general results are collected, for example, in [54] the common strategy here is to study Toeplitz operators with symbols from certain special subclasses of L_∞.

The most complete results were obtained for the families of symbols that generate commutative C*-algebras of Toeplitz operators acting on the weighted Bergman space. They were described in a series of papers summarized in the book [50], see also [26]. These families of defining symbols lead to the following three model cases: *radial* symbols, functions on the unit disk depending only on $|z|$, *vertical* symbols, functions on the upper half-plane depending on $\operatorname{Im} z$, and *angular* symbols defined on the upper half-plane and depending only on $\arg z$. Unlike the Bergman case, on the Fock space there are only two model cases that generate commutative C*-algebras of Toeplitz operators: *radial symbols*, functions on the complex plane $\mathbb{C}$ depending only on $|w|$, and *horizontal symbols*, functions on $\mathbb{C}$ depending only on $\operatorname{Re} w$.

In each one of the above models (for the horizontal model is proved in this Ph.D. thesis), the corresponding Toeplitz operator admit an explicit diagonalization, i.e. there exists an isometric isomorphism that transforms all Toeplitz operators of the selected type to the multiplication operators by some specific functions (we call them *spectral functions*, in the radial case they are just the sequences of eigenvalues). Of course, such a diagonalization immediately reveals all the main properties of the corresponding Toeplitz operators [23–25, 28].

Then the next natural problem emerges: give an explicit and independent description of the class of spectral functions (and of the algebra generated by them) for each one of the above cases. First step in this direction was made by Suárez [46, 47]. He proved that the sequences of eigenvalues of Toeplitz operators acting on the Bergman space with bounded *radial* symbols form a dense subset in the ℓ_∞-closure of the class d_1 of bounded sequences $(\sigma_k)_{k=1}^\infty$ satisfying

$$\sup_k (k+1)|\sigma_{k+1} - \sigma_k| < +\infty.$$

As a consequence, the C*-algebra generated by Toeplitz operators with bounded radial symbols is isometrically isomorphic to this ℓ_∞-closure of d_1. The results of Suárez have been complemented and generalized to the weighted Bergman space on the unit ball in [3, 4, 27, 40]. The above ℓ_∞-closure of d_1 was characterized in [27]. As it turned out, this closure coincides with the C*-algebra $\mathrm{VSO}(\mathbb{N})$ of bounded functions (sequences) $\mathbb{N} \to \mathbb{C}$ that are uniformly continuous with respect to the logarithmic metric $|\ln(j) - \ln(k)|$. Surprisingly this class of sequences was already introduced by Schmidt [43, § 9] in the beginning of the 20th century in connection with Tauberian theory. It is worth mentioning that the above description shows the room that radial Toeplitz operators occupy amongst all bounded radial operators (the set of which is isomorphic to $\ell_\infty(\mathbb{N})$).

Herrera Yañez, Hutník, Maximenko and Vasilevski [30–32] continue this program and give a description of the commutative algebra generated by Toeplitz operators acting on the Bergman space with bounded *vertical* symbols. The result states that their spectral functions form a dense subset in the C*-algebra $\mathrm{VSO}(\mathbb{R}_+)$ of *very slowly oscillating functions* on $\mathbb{R}_+$, i.e. the bounded functions $\mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous with respect to the metric $|\ln(x) - \ln(y)|$. This, in particular, means that the C*-algebra generated by Toeplitz operators with bounded vertical symbols is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

This thesis is devoted to the study of remaining model cases. That is, we give an explicit description of the commutative C*-algebras generated by Toeplitz operator acting on the weighted Bergman space whose defining symbols are *angular*, and of the commutative C*-algebras generated by Toeplitz operator acting on the Fock space whose defining symbols are *radial* and *horizontal*. With this we complete thus the intrinsic description of the commutative C*-algebras generated by Toeplitz operators with bounded symbols for each one of the model classes that appear for the weighted Bergman space and the Fock space.

The work is divided into three chapters. In Chapter 2 we are interested in the Toeplitz operator with angular symbols acting on the weighted Bergman space $\mathscr{A}^2_\lambda(\Pi)$ over the upper half-plane Π, which consists of all analytics functions in $L_2(\Pi, d\nu_\lambda)$, where

$$d\nu_\lambda(z) = (\lambda+1)(2y)^\lambda dy\,dx, \quad z = x + iy, \quad \lambda \in (-1, +\infty).$$

A function $g \in L_\infty(\Pi)$ is said to be *homogeneous of order zero* or *angular* if for every $h > 0$ the equality $g(hz) = g(z)$ holds for a.e. $z \in \Pi$, or, equivalently, if there exists a function a in $L_\infty(0,\pi)$ such that $g(z) = a(\arg z)$ for a.e. z in Π. We denote by $\mathscr{A}_\infty$ this class of functions, and introduce the set $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ of all Toeplitz operators acting on $\mathscr{A}^2_\lambda(\Pi)$ with defining symbols in $\mathscr{A}_\infty$.

As was shown in [25], the uniraty operator $R_\lambda : \mathscr{A}^2_\lambda(\Pi) \to L_2(\mathbb{R})$, where

$$(R_\lambda\varphi)(x) = \frac{1}{\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix-\left(\frac{\lambda+2}{2}\right)} \varphi(z)\, d\mu_\lambda(z), \quad x \in \mathbb{R}, \tag{1}$$

with

$$c_\lambda(x) = \int_0^\pi e^{-2x\theta} \sin^\lambda\theta\, d\theta, \quad x \in \mathbb{R}. \tag{2}$$

diagonalizes each Toeplitz operator T_g with angular symbol $g(z) = a(\arg z)$; that is $R_\lambda T_g R_\lambda^* = \gamma_{a,\lambda} I$, where the spectral function $\gamma_{a,\lambda} : \mathbb{R} \to \mathbb{C}$ is given by

$$\gamma_{a,\lambda}(x) = \frac{1}{c_\lambda(x)} \int_0^\pi a(\theta) e^{-2x\theta} \sin^\lambda \theta \, d\theta, \quad x \in \mathbb{R}. \tag{3}$$

In particular, this implies that the algebra generated by $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ is isometrically isomorphic to the C*-algebra generated by

$$\Gamma_\lambda = \left\{\gamma_{a,\lambda} : \ a \in L_\infty(0,\pi)\right\}. \tag{4}$$

Chapter 2 describes explicitly this C*-algebra. We denote by VSO($\mathbb{R}$) the C*-algebra of *very slowly oscillating* functions on the real line [18], which consists of all bounded functions that are uniformly continuous with respect to the *arcsinh-metric*

$$\rho(x,y) = |\operatorname{arcsinh} x - \operatorname{arcsinh} y|, \quad x, y \in \mathbb{R}. \tag{5}$$

The main result here (Theorem 2.4) states that the uniform closure of Γ_λ coincides with the C*-algebra VSO($\mathbb{R}$). As a consequence, the C*-algebra $\mathscr{T}_\lambda(\mathscr{A}_\infty)$ generated by the set $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ coincides just with the closure of this set of its initial generators, and is isometrically isomorphic to VSO($\mathbb{R}$). Note that the result does not depend on a value of the weight parameter $\lambda > -1$.

As a by-product of the main result, we show that the closure of the set $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ in the strong operator topology coincides with the C*-algebra of all angular operators.

With this work we finish the explicit descriptions of the above mentioned commutative C*-algebras of Toeplitz operators on the unit disk and upper half-plane. In all three cases the spectral functions oscillate at infinity with the logarithmic speed. The C*-algebras VSO($\mathbb{R}_+$) and VSO($\mathbb{R}$) corresponding to the vertical and angular cases, respectively, are isometrically isomorphic (via the change of variables $v \mapsto \sinh(\ln(v))$), and both of them are isometrically isomorphic to the C*-algebra $C_{b,u}(\mathbb{R})$ consisting of all bounded functions on $\mathbb{R}$ that are uniformly continuous with respect to the usual metric (via the changes of variables $v \mapsto \ln(v)$ and $v \mapsto \operatorname{arcsinh}(v)$, respectively). The sequences from VSO($\mathbb{N}$) are nothing but the restrictions to $\mathbb{N}$ of the functions from VSO($\mathbb{R}_+$).

Note that the proof in the vertical case was the simplest one because the corresponding spectral functions $\gamma^v_{a,\lambda}$ admit representations in terms of the Mellin convolutions, and the result about density was obtained just by using a convenient Dirac sequence. Unfortunately, in the angular case this simple approach does not work.

The key idea of the proof presented in this work is to approximate functions from VSO($\mathbb{R}$) by $\gamma^v_{a,\lambda}$ near $+\infty$ and $-\infty$. After that, the problem is reduced to the approximation of $C_0(\mathbb{R})$ functions by appropriate $\gamma_{a,\lambda}$; the latter problem was solved using the duality and the analyticity arguments (Theorem 2.3).

This chapter is organized as follows: Section 2.1 contains criteria of angular operator and of angular Toeplitz operator. In Section 2.2 we introduce formally the class of functions VSO($\mathbb{R}$) and is showed that the functions of the class Γ_λ are Lipschitz continuous with respect the arcsinh metric ρ. That is $\Gamma_\lambda \subsetneq \mathrm{VSO}(\mathbb{R})$. In Section 2.3 we prove the density of Γ_λ in VSO($\mathbb{R}$), and we finish this chapter showing in Section 2.4 that the closure of $\{T_g : g \in \mathscr{A}_\infty\}$ in the strong operator topology coincides with $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$.

Chapter 3 focuses on the study of the C*-algebra generated by radial Toeplitz operators acting on Fock spaces. It is well known [53] that the normalized monomials $e_n(z) = z^n/\sqrt{n!}$, $n \in \mathbb{Z}_+$, form an orthonormal basis of $\mathscr{F}^2(\mathbb{C})$, and that the Toeplitz operators with bounded radial symbols are diagonal with respect to this basis [28]. Namely, if $a \in L_\infty(\mathbb{R}_+)$ and $\varphi(z) = a(|z|)$ a.e. $z \in \mathbb{C}$, then $T_\varphi e_n = \gamma_a(n) e_n$, where

$$\gamma_a(n) = \frac{1}{\sqrt{n!}} \int_{\mathbb{R}_+} a(\sqrt{r}) e^{-r} r^n \, dr, \quad n \in \mathbb{Z}_+. \tag{6}$$

From this diagonalization we have that the C*-algebra $\mathscr{T}_{rad}$ generated by Toeplitz operators with radial L_∞-symbols is isometrically isomorphic to the C*-algebra $\mathscr{G}$ generated by the set $\mathfrak{G}$ of all sequences of the eigenvalues:

$$\mathfrak{G} = \left\{ \gamma_a : a \in L_\infty(\mathbb{R}_+) \right\}. \tag{7}$$

The main result of this part states that the uniform closure of $\mathfrak{G}$ coincides with the C*-algebra RO($\mathbb{Z}_+$) consisting of all bounded sequences $\sigma : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ that are uniformly continuous with respect to the *square-root-metric*

$$\varrho(m,n) = \left| \sqrt{m} - \sqrt{n} \right|, \quad m,n \in \mathbb{Z}_+.$$

As a consequence, we obtain an explicit description of the C*-algebra $\mathscr{G}$ generated by $\mathfrak{G}$:

$$\mathscr{G} = \mathrm{RO}(\mathbb{Z}_+).$$

Surprisingly for us, the C*-algebra $\mathscr{G}$ turns out to be wider than the class of sequences VSO($\mathbb{N}$) obtained for the radial case on weighted Bergman spaces. The results obtained here can be generalized to radial Toeplitz operators on the multi-dimensional Fock space $\mathscr{F}^2(\mathbb{C}^n, (\alpha/\pi)^n e^{-\alpha|z|^2} dv_n(z))$; in this case the eigenvalue associated to the element e_β of the canonical basis depends only on the length of the multi-index β, as in [27].

This chapter is organized as follows: In Section 3.1 we have compiled some basic facts about radial Toeplitz operators in Fock space. In Section 3.2 we introduce the class $\mathrm{RO}(\mathbb{Z}_+)$ and prove that $\mathfrak{G}$ is contained in $\mathrm{RO}(\mathbb{Z}_+)$.

The major part of Chapter 3 is occupied by a proof that $\mathfrak{G}$ is *dense* in $\mathrm{RO}(\mathbb{Z}_+)$, see a scheme in Figure 0.1. Given a sequence $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$, we extend it to a sqrt-oscillating function f on $\mathbb{R}_+$ (Proposition 3.5). After the change of variables $h(x) = f(x^2)$ we obtain a bounded and uniformly continuous function h on $\mathbb{R}$. In Section 1.5, using Dirac sequences and Wiener's division lemma, we show that functions from $C_{b,u}(\mathbb{R})$ can be uniformly approximated by convolutions $k * b$, where k is a fixed $L_1(\mathbb{R})$-function whose Fourier transform does not vanish on $\mathbb{R}$. This construction will be applied when k is the heat kernel H (Section 3.3) and we examine the asymptotic behavior of the sequences of eigenvalues γ_a. It is shown there that after change of variables $\sqrt{n} = x$, the function $x \mapsto \gamma_a(x^2)$, for x sufficiently large, is close to the convolution of the symbol a with the heat kernel H. In Section 3.3 also we show that $c_0(\mathbb{Z}_+)$ coincides with the uniform closure of the set $\{\gamma_a\colon\ a \in L_\infty(\mathbb{R}_+),\ \lim_{r\to\infty} a(r) = 0\}$. After that, gathering together all the pieces, we obtain the main result of this chapter (Theorem 3.4). Finally, in Section 3.4 we describe a class of generating symbols bigger than $L_\infty(\mathbb{R})$, with eigenvalues' sequences still belonging to $\mathrm{RO}(\mathbb{Z}_+)$, and construct an unbounded generating symbol a such that $\gamma_a \in \ell_\infty(\mathbb{Z}_+) \setminus \mathrm{RO}(\mathbb{Z}_+)$.

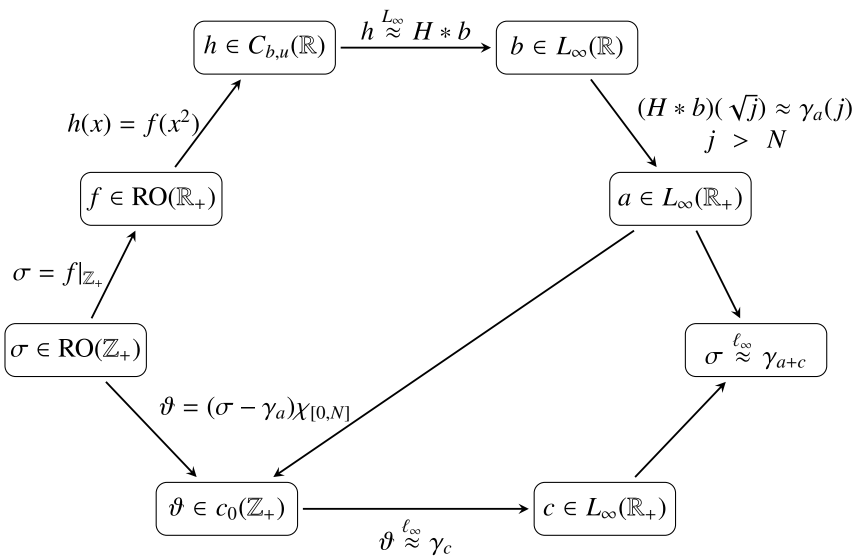


Figure 0.1: Scheme of the proof of density: the upper chain represents the approximation of $\sigma(j)$ for large values of j ($j > N$), and the lower one corresponds to the uniform approximation of the sequence $\sigma - \gamma_a$ multiplied by the characteristic function $\chi_{[0,N]}$.

Finally Chapter 4 provides a detailed description of the Toeplitz operators with horizontal symbols acting on the Fock space $\mathscr{F}^2(\mathbb{C}^n)$, where a function $f \in L_\infty(\mathbb{C}^n)$, is called *horizontal* if for every $h \in \mathbb{R}^n$ the equality $f(z) = f(z - ih)$ holds a.e. $z \in \mathbb{C}^n$. Equivalently, f is a horizontal function, if there exists $a \in L_\infty(\mathbb{R}^n)$ such that $f(z) = a(\mathrm{Re}\, z)$, $a.e.\ z \in \mathbb{C}^n$.

It is well-known that the Bargmann transform is an isometric isomorphism from $L_2(\mathbb{R}^{2n})$ onto the $\mathscr{F}^2(\mathbb{C}^n)$ [53], and hence plays an important role in the description of Toeplitz operators. Furthermore, this transformation relates the Toeplitz operators acting on $\mathscr{F}^2(\mathbb{C}^n)$ with pseudo-differential operators acting on $L_2(\mathbb{R}^{2n})$. The Bargmann transform is important in our approach, because the main idea as in the above model cases is to get spectral functions γ_a^H such that the Toeplitz operators are unitary equivalent to the multiplication operators $M_{\gamma_a^H}$ acting on $L_2(\mathbb{R}^n)$.

First of all, we find a decomposition of the Bargmann transform B^* as composition of two unitary operators which allows us to diagonalize the Toeplitz operators with horizontal symbols. More precisely, if $\varphi(z) = a(\mathrm{Re}\, z)$ is a horizontal function, then the Toeplitz operator T_φ is unitary equivalent to the multiplication operator $\mathrm{B}\, T_\varphi \mathrm{B}^* = \gamma_a^H \mathrm{I}$, where the spectral function $\gamma_a^H : \mathbb{R}^n \to \mathbb{C}$ is given by the formula

$$\gamma_a^H(x) = \pi^{-n/2} \int_{\mathbb{R}^n} a\left(\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2} dy, \quad x \in \mathbb{R}^n. \tag{8}$$

A consequence of the latter diagonalization is that the C*-algebra $\mathscr{T}_{hor}(L_\infty)$ of Toeplitz operators whose defining symbols are horizontal is isometrically isomorphic to the C*-algebra $\mathscr{G}^H$ generated by

$$\mathfrak{G}^H = \left\{\gamma_a^H : a \in L_\infty(\mathbb{R}^n)\right\}. \tag{9}$$

This result is generalized for functions invariant under translations over certain kind of subspaces of $\mathbb{C}^n$. Recall that $\mathbb{R}^{2n}$ has the standard symplectic form

$$\omega_0(z, w) = y \cdot x' - y' \cdot x,$$

for $z = (x, y)$ and $w = (x', y')$. It is well-known that a linear subspace $\mathscr{L}$ of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ is a *Lagrangian plane* if for every pair $(z, w) \in \mathscr{L} \times \mathscr{L}$, one has that $\omega_0(z, w) = 0$. The simplest examples of Lagrangian planes of $(\mathbb{R}^{2n}, \omega_0)$ are both coordinates planes: $\mathscr{L}_x = \mathbb{R}^n \times \{0\}$ and $\mathscr{L}_y = \{0\} \times \mathbb{R}^n$.

If we identify $i\mathbb{R}^n$ with $\{0\} \times \mathbb{R}^n$, then the horizontal functions can be viewed as functions invariant under translations on the Lagrangian plane $\mathscr{L}_y = \{0\} \times \mathbb{R}^n$. Thus, it is natural to study Toeplitz operators with defining symbols invariant under translation on

Lagrangian planes $\mathscr{L}$ of $(\mathbb{R}^{2n},\omega_0)$, for brevity we call them *$\mathscr{L}$–invariant symbols*. That is, φ is a *$\mathscr{L}$–invariant function* if for every $h \in \mathscr{L}$ one gets that $\varphi(z-h) = \varphi(z)$ a.e. $z \in \mathbb{C}^n$.

Using the symplectic rotations $\mathrm{U}(2n,\mathbb{R})$ we give a criterion for a function to be $\mathscr{L}$–invariant, and thus relate Toeplitz operators with this class of defining symbols with Toeplitz operators having horizontal symbols. In fact, given any Lagrangian plane $\mathscr{L}$ we can find a symplectic rotation $B \in \mathrm{U}(2n,\mathbb{R})$ with $B\mathscr{L} = \{0\} \times \mathbb{R}^n$ and a unitary operator V_B such that for every $\mathscr{L}$–invariant function $\varphi \in L_\infty(\mathbb{R}^{2n})$ there exists a horizontal function $\psi_B \in L_\infty(\mathbb{R}^{2n})$ (depending on φ and $\mathscr{L}$) such that

$$V_{B^{-1}} T_\varphi V_B = T_{\psi_B}.$$

Therefore, the C*-algebra $\mathscr{T}_{\mathscr{L}}(L_\infty)$ generated by Toeplitz operators with $\mathscr{L}$–invariant symbols is isometrically isomorphic to $\mathscr{T}_{hor}(L_\infty)$. Thus, both C*-algebras $\mathscr{T}_{\mathscr{L}}(L_\infty)$ and $\mathscr{T}_{hor}(L_\infty)$ are isometrically isomorphic to the C*-algebra $\mathscr{G}^H$ generated by $\mathfrak{G}^H$, see (9).

The main result of this chapter states that the uniform closure of $\mathfrak{G}^H$ coincides with the C*-algebra $C_{b,u}(\mathbb{R}^n)$ consisting of all bounded functions $\sigma\colon \mathbb{R}^n \longrightarrow \mathbb{C}$ that are uniformly continuous with respect to the *usual metric* on $\mathbb{R}^n$. As a consequence, we obtain an explicit description of $\mathscr{G}^H$:

$$\mathscr{G}^H = C_{b,u}(\mathbb{R}^n).$$

Notice that, unlike the angular and radial case, here we do not need to approximate functions $\sigma \in C_{b,u}(\mathbb{R}^n)$ at the infinity by spectral function γ_a^H. It follows from the structure of the functions γ_a^H, that they can be written directly as convolutions of functions $a * H$, where H is the n-dimensional heat kernel $H(x) = \pi^{-n/2} e^{-x^2}$.

This chapter is organized as follows: in Section 4.1 we write the Bargmann transform as a composition of two unitary operators. In Section 4.2 we introduce the horizontal operators and horizontal Toeplitz operators, give some basic properties of them, including a criterion of a operator to be horizontal. Also, we prove that the Toeplitz operators with horizontal symbols are unitary equivalent to the multiplication operator $M_{\gamma_a^H}$ acting on $L_2(\mathbb{R}^n)$. In Section 4.3 we introduce the $\mathscr{L}$–invariant functions, and establish a criterion of a function to be $\mathscr{L}$–invariant. We show that the C*-algebra $\mathscr{T}_{\mathscr{L}}(L_\infty)$ generated by the Toeplitz operators whose defining symbols are $\mathscr{L}$-invariant is isometrically isomorphic to $\mathscr{T}_{hor}(L_\infty)$. Finally, in Section 4.4 we prove the main result of this chapter. The proof is based on approximations of bounded uniformly continuous functions by convolutions used in Chapter 3. That is, using Dirac sequences and Wiener's division lemma, we show that the functions from $C_{b,u}(\mathbb{R}^n)$ can be uniformly approximated by convolutions $k * b$, where k is a fixed $L_1(\mathbb{R}^n)$-function whose Fourier transform does not vanish on $\mathbb{R}^n$. This construction is applied when k is the n-dimensional heat kernel H.

CHAPTER 1

PRELIMINARIES

In this chapter we collect several preliminary results, the main purpose here is to fix notation and to facilitate references later on. All the results are well known [14, 20, 29, 42, 50, 53, 54].

1.1 Weighted Bergman space

Let Π be the upper half-plane of the complex plane $\mathbb{C}$:

$$\Pi := \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}. \tag{1.1}$$

Given the *weight parameter* $\lambda \in (-1, +\infty)$, we introduce the following standard measure on the upper half-plane Π:

$$dv_\lambda(z) = (\lambda+1)(2y)^\lambda \, dy\, dx, \quad z = x + iy.$$

The weighted Bergman space $\mathscr{A}_\lambda^2(\Pi)$ consists of all analytic functions belonging to $L_2(\Pi, dv_\lambda)$. An important property of the Bergman space is contained in the following lemma.

Lemma 1.1. *Let $n \in \{0, 1, 2, \dots\}$. Given a compact set $K \subset \Pi$, there is a constant $C = C_{n,K,\lambda}$, depending on n, K and $\lambda \in (-1, +\infty)$, such that*

$$\sup_{z \in K} |f^{(n)}(z)| \le C \|f\|_{\mathscr{A}_\lambda^2(\Pi)}, \tag{1.2}$$

for all $f \in \mathscr{A}_\lambda^2(\Pi)$.

Proposition 1.1. *The Bergman space $\mathscr{A}^2_\lambda(\Pi)$ is a closed subspace of $L_2(\Pi, d\nu_\lambda)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a fundamental sequence of analytic functions from $\mathscr{A}^2_\lambda(\Pi)$ converging on $L_2(\Pi, d\nu_\lambda)$ to certain function $f \in L_2(\Pi, d\nu_\lambda)$. By Lemma 1.1 we see that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on every compact subset K of Π to certain analytic function g. However, by (1.2) we have for each compac subset K of Π that

$$|f(z) - g(z)| \leq \lim_{m\to\infty} |f(z) - f_m(z)| \leq C_{K,\lambda} \lim_{m\to\infty} \|f - f_m\|_{\mathscr{A}^2_\lambda(\Pi)} = 0, \quad z \in K.$$

Therefore, f is analytic and belongs to $\mathscr{A}^2_\lambda(\Pi)$. ∎

From Lemma 1.1 is follows as well that for any fixed point $z \in \Pi$ the evaluation functional $\psi_z(f) = f(z)$ is linear and bounded. Thus by the Riesz-Fréchet representation theorem there exists a unique element $K_{z,\lambda} \in \mathscr{A}^2_\lambda(\Pi)$ such that $\psi_z = \langle \cdot, K_{z,\lambda} \rangle$. The function $K_{z,\lambda}$ is the so-called *Bergman kernel* at a point z, and it is well known that is given by the formula

$$K_{z,\lambda}(w) = \left(\frac{i}{w - \overline{z}}\right)^{\lambda+2} \qquad w \in \Pi. \tag{1.3}$$

Observe that from the above definition it follows that the Bergman kernel function $K_{z,\lambda}(w)$ is analytic in w and anti-analytic in z (analytic in $\overline{z}$). On the other hand, since the Bergman space $\mathscr{A}^2_\lambda(\Pi)$ is a closed subspace of $L_2(\Pi, d\nu_\lambda)$, there exists the unique orthogonal projection B_λ from $L_2(\Pi, d\nu_\lambda)$ onto $\mathscr{A}^2_\lambda(\Pi)$. This projection is called the *Bergman projection* and has the integral representation

$$(B_\lambda f)(z) = i^{\lambda+2} \int_\Pi \frac{f(w)}{(z - \overline{w})^{\lambda+2}}\, d\nu_\lambda(w), \quad f \in L_2(\Pi, d\nu_\lambda).$$

Representations of the weighted Bergman space

The Bergman space can be characterized as the set of $L_2(\Pi, d\nu_\lambda)$ functions which satisfy the Cauchy-Riemann equation

$$\frac{\partial}{\partial \overline{z}} f(z) = 0. \tag{1.4}$$

Passing to polar coordinates we have the tensor decomposition

$$L_2(\Pi, d\nu_\lambda) := L_2\left(\mathbb{R}^+, r^{\lambda+1} dr\right) \otimes L_2\left([0,\pi], \frac{2^\lambda}{\pi}(\lambda+1)\sin^\lambda\theta\, d\theta\right), \tag{1.5}$$

and rewriting (1.4) we have that the Bergman space $\mathscr{A}^2_\lambda(\Pi)$ is the set of all functions satisfying the equation

$$\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)\varphi(r,\theta) = 0. \tag{1.6}$$

Let $U_{1,\lambda} = \frac{1}{\sqrt{\pi}}(M \otimes \mathrm{Id})$ be the unitary operator from

$$L_2\left(\mathbb{R}^+, r^{\lambda+1}\,dr\right) \otimes L_2\left([0,\pi], \frac{2^\lambda}{\pi}(\lambda+1)\sin^\lambda\theta\,d\theta\right)$$

onto $L_2(\mathbb{R}) \otimes L_2\left([0,\pi], \frac{2^\lambda}{\pi}(\lambda+1)\sin^\lambda\theta\,d\theta\right)$, where $M : L_2\left(\mathbb{R}^+, r^{\lambda+1}\,dr\right) \longrightarrow L_2(\mathbb{R})$ is the Mellin transform defined by the rule

$$\left(M\psi\right)(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} r^{-ix+\frac{\lambda}{2}}\,\psi(r)\,dr. \tag{1.7}$$

Its inverse Mellin transform $M^{-1} : L_2(\mathbb{R}) \longrightarrow L_2\left(\mathbb{R}^+, r^{\lambda+1}\,dr\right)$ is given by

$$\left(M^{-1}\psi\right)(r) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} r^{i\xi-\frac{\lambda}{2}-1}\,\psi(\xi)\,d\xi. \tag{1.8}$$

Observe that

$$(M \otimes \mathrm{Id})\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right)\left(M^{-1} \otimes \mathrm{Id}\right) = \left(i\xi - \left(\frac{\lambda}{2}+1\right)\right)\mathrm{Id} + i\frac{\partial}{\partial\theta} = i\left(\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right)\mathrm{Id} + \frac{\partial}{\partial\theta}\right).$$

Hence, the image of the weighted Bergman space $\mathscr{A}^2_\lambda(\pi)$ under the unitary operator $U_{1,\lambda}$

$$\mathscr{A}_{1,\lambda} = U_{1,\lambda}\left(\mathscr{A}^2_\lambda(\Pi)\right)$$

it is the closed subspace of $L_2(\mathbb{R}) \otimes L_2\left([0,\pi], \frac{2^\lambda}{\pi}(\lambda+1)\sin^\lambda\theta\,d\theta\right)$, consisting of all functions $\psi(\xi,\theta)$ satisfying the equation

$$\left(\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right)\mathrm{Id} + \frac{\partial}{\partial\theta}\right)\psi(\xi,\theta) = 0.$$

The general $L_2(\mathbb{R}) \otimes L_2\left([0,\pi], \frac{2^\lambda}{\pi}(\lambda+1)\sin^\lambda\theta\,d\theta\right)$-solution of this equation has the form

$$\psi(\xi,\theta) = \frac{f(\xi)\,e^{-\left(\xi+\left(\frac{\lambda+2}{2}\right)i\right)\theta}}{\sqrt{2^\lambda(\lambda+1)\,c_\lambda(\xi)}}, \quad f \in L_2(\mathbb{R}),$$

where the function $c_\lambda : \mathbb{R} \longrightarrow \mathbb{R}_+$ is of the form

$$c_\lambda(x) = \int_0^\pi e^{-2x\theta}\sin^\lambda\theta\,d\theta, \quad x \in \mathbb{R}. \tag{1.9}$$

and

$$\left\|\psi(\lambda,\theta)\right\|_{L_2(\mathbb{R})\otimes L_2\left([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta\,d\theta\right)} = \|f\|_{L_2(\mathbb{R})}$$

Lemma 1.2. *Let $c_\lambda(x)$ be the function given in (1.9). The following properties hold.*

(i) *The function c_λ can be written as*

$$c_\lambda(x) = \frac{\pi\Gamma(\lambda+1)e^{-\pi x}}{2^\lambda\left|\Gamma\left(\frac{\lambda+2}{2}+ix\right)\right|^2}, \qquad x\in\mathbb{R}. \tag{1.10}$$

(ii) *The function c_λ is infinitely smooth, for every $p=0,1,2,\ldots$ its p-th derivative is given by the integral*

$$\frac{d^p c_\lambda(x)}{dx^p} = (-2)^p\int_0^\pi \theta^p e^{-2x\theta}\sin^\lambda\theta\, d\theta, \tag{1.11}$$

moreover, it has the following asymptotic behavior at $+\infty$:

$$\frac{d^p c_\lambda(x)}{dx^p} \sim \frac{(-2)^p\Gamma(\lambda+p+1)}{(2x)^{\lambda+p+1}}, \quad as \quad x\to+\infty. \tag{1.12}$$

Proof. (i) By definition of Beta function and by [22, Eq: 3.892-1] one gets that:

$$\begin{aligned}
c_\lambda(x) &= \int_0^\pi e^{-2x\theta}\sin^\lambda\theta\, d\theta = \int_0^\pi e^{i\beta\theta}\sin^{v-1}\theta\, d\theta, \quad \text{where} \quad \beta=2ix,\, v=\lambda+1,\\
&= \frac{\pi e^{\frac{i\beta\pi}{2}}}{2^{v-1}v\,\mathbb{B}\left(\frac{v+\beta+1}{2},\frac{v-\beta+1}{2}\right)} = \frac{\pi e^{x\pi}}{2^\lambda(\lambda+1)\mathbb{B}\left(\frac{\lambda+2}{2}+ix,\frac{\lambda+2}{2}-ix\right)} = \frac{\pi\Gamma(\lambda+2)e^{-\pi x}}{2^\lambda(\lambda+1)\left|\Gamma\left(\frac{\lambda+2}{2}+ix\right)\right|^2}\\
&= \frac{\pi\Gamma(\lambda+1)e^{-\pi x}}{2^\lambda\left|\Gamma\left(\frac{\lambda+2}{2}+ix\right)\right|^2}, \quad x\in\mathbb{R}.
\end{aligned}$$

(ii) Since $c_\lambda(x)=e^{-2x\pi}c_\lambda(-x)$ for each $x\in\mathbb{R}$, we give the proof of the equality for the case $x\ge 0$. Given $p\in\{0,1,2,\ldots\}$ note that $\theta^p e^{-2x\theta}\sin^\lambda\theta \le \theta^p\sin^\lambda\theta$ for each $\theta\in(0,\pi)$ and each $x\ge 0$. Thus, by the Leibniz's rule c_λ is infinitely smooth and

$$\frac{d^p c_\lambda(x)}{dx^p} = (-2)^p\int_0^\pi \theta^p e^{-2x\theta}\sin^\lambda\theta\, d\theta, \quad x\in\mathbb{R}.$$

On the other hand, the asymptotic behavior is easily analized using the Watson's Lemma (Proposition A.1). Writing $\theta^p\sin^\lambda\theta$ as $\theta^{\lambda+p}\left(\frac{\sin\theta}{\theta}\right)^\lambda$, where $\left(\frac{\sin\theta}{\theta}\right)^\lambda$ is infinitely smooth near 0, we obtain:

$$\int_0^\pi \theta^p e^{-2x\theta}\sin^\lambda\theta\, d\theta \sim \frac{\Gamma(\lambda+p+1)}{(2x)^{\lambda+p+1}}, \quad \text{as } x\to+\infty,$$

thus the proof is completed. ∎

Example 1.1. Let $z\in\Pi$ and $K_{z,\lambda}$ the Bergman kernel. Consider

$$K_{z,\lambda}(w) = \left(\frac{i}{re^{i\alpha}-\overline{z}}\right)^{\lambda+2}, \quad w=re^{i\alpha},$$

by [22, Eq: 3.194.3] we have that

$$\begin{aligned}(U_{1,\lambda}K_{z,\lambda})(x,\alpha) &= \frac{i^{\lambda+2}}{\pi\sqrt{2}}\int_{\mathbb{R}_+}\frac{r^{-ix+\frac{\lambda}{2}}}{(re^{i\alpha}-\overline{z})^{\lambda+2}}dr = \frac{i^{\lambda+2}}{\pi(-\overline{z})^{\lambda+2}\sqrt{2}}\int_{\mathbb{R}_+}\frac{r^{u-1}}{(r\beta+1)^{v}}dr\\ &= \frac{i^{\lambda+2}}{\pi(-\overline{z})^{\lambda+2}\sqrt{2}}\beta^{-u}\mathbb{B}(u,v-u),\end{aligned}$$

where $u=-ix+\frac{\lambda+2}{2}, \beta=-\frac{e^{i\alpha}}{\overline{z}}=\frac{e^{i(\alpha-\pi)}}{\overline{z}}$, $v=\lambda+2$ and $\mathbb{B}$ is the Beta function. Hence

$$\begin{aligned}(U_{1,\lambda}K_{z,\lambda})(x,\alpha) &= \frac{i^{\lambda+2}}{\pi(-\overline{z})^{\lambda+2}\sqrt{2}}\left[\frac{e^{i(\alpha-\pi)}}{\overline{z}}\right]^{ix-\left(\frac{\lambda+2}{2}\right)}\mathbb{B}\left(\frac{\lambda+2}{2}-ix,\frac{\lambda+2}{2}+ix\right)\\ &= \frac{i^{\lambda+2}}{\pi(-\overline{z})^{\lambda+2}\sqrt{2}}\left[\frac{e^{i(\alpha-\pi)}}{\overline{z}}\right]^{ix-\left(\frac{\lambda+2}{2}\right)}\frac{\left|\Gamma\left(\frac{\lambda+2}{2}+ix\right)\right|^2}{\Gamma(\lambda+2)}\\ &= \left(\frac{(-i)^{\lambda+2}(e^{i\pi})^{\frac{(\lambda+2)}{2}}e^{-x\alpha-i-\left(\frac{\lambda+2}{2}\right)}}{(\overline{z})^{ix+\frac{\lambda+2}{2}}\sqrt{2}}\right)\left(\frac{e^{x\pi}\left|\Gamma\left(\frac{\lambda+2}{2}+ix\right)\right|^2}{\pi\Gamma(\lambda+2)}\right).\end{aligned}$$

Now, since $(e^{i\pi})^{\frac{(\lambda+2)}{2}}=i^{\lambda+2}$, by (1.10) item (i) we have for each $\alpha\in(0,\pi)$ that

$$(U_{1,\lambda}K_{z,\lambda})(x,\alpha)=\frac{e^{-x\alpha-i\left(\frac{\lambda+2}{2}\right)\alpha}(\overline{z})^{-ix-\left(\frac{\lambda+2}{2}\right)}}{2^{\lambda+\frac{1}{2}}(\lambda+1)c_\lambda(x)},\quad x\in\mathbb{R}_+. \tag{1.13}$$

Lemma 1.3. *The unitary operator $U_{1,\lambda}$ is an isometric isomorphism of the space $L_2\left(\Pi,d\mu_\lambda\right)$ onto $L_2(\mathbb{R})\otimes L_2\left([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta\,d\theta\right)$ under which the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is mapped onto*

$$\mathcal{A}_{1,\lambda}=\left\{\frac{f(\xi)e^{-\left(\left(\xi+\left(\frac{\lambda}{2}+1\right)i\right)\theta\right)}}{\sqrt{2^\lambda(\lambda+1)c_\lambda(\xi)}},\quad f\in L_2(\mathbb{R})\right\}. \tag{1.14}$$

Let $R_{0,\lambda}:L_2(\mathbb{R})\longrightarrow\mathcal{A}_{1,\lambda}\subset L_2(\mathbb{R})\otimes L_2\left([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta\,d\theta\right)$ be the isometric embedding given by

$$\left(R_{0,\lambda}f\right)(\xi,\theta)=\frac{f(\xi)e^{-\left(\xi+\left(\frac{\lambda+2}{2}\right)i\right)\theta}}{\sqrt{2^\lambda(\lambda+1)c_\lambda(\xi)}}. \tag{1.15}$$

The adjoint operator $R^*_{0,\lambda}:\mathcal{A}_{1,\lambda}\subset L_2(\mathbb{R})\otimes L_2\left([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta\,d\theta\right)\longrightarrow L_2(\mathbb{R})$ has the form

$$\left(R^*_{0,\lambda}\psi\right)(\xi)=\sqrt{\frac{2^\lambda(\lambda+1)}{c_\lambda(\xi)}}\int_0^\pi\psi(\xi,\theta)e^{-\left(\xi-\left(\frac{\lambda+2}{2}\right)i\right)\theta}\sin^\lambda\theta\,d\theta. \tag{1.16}$$

Note that

$$R_0^* R_0 = \mathrm{I}_{L_2(\mathbb{R})} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R});$$
$$R_0 R_0^* = B_{1,\lambda} : L_2(\mathbb{R}) \otimes L_2\left([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta\, d\theta\right) \longrightarrow \mathscr{A}_{1,\lambda}$$

where $B_{1,\lambda} = U_{1,\lambda} B_{\Pi,\lambda} U_{1,\lambda}^{-1}$ is the orthogonal projection from

$$L_2(\mathbb{R}) \otimes L_2\left([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta\, d\theta\right)$$

onto $\mathscr{A}_{1,\lambda}$.

Now the operator $R_\lambda = R_{0,\lambda}^* U_{1,\lambda}$ maps the space $L_2(\Pi, d\mu_\lambda)$ onto $L_2(\mathbb{R})$, and its restriction

$$R_\lambda|_{\mathscr{A}_\lambda^2(\Pi)} = R_{0,\lambda}^* U_{1,\lambda} : \mathscr{A}_\lambda^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R_\lambda^* = U_{1,\lambda}^* R_{0,\lambda} : L_2(\mathbb{R}) \longrightarrow \mathscr{A}_\lambda^2(\Pi) \subset L_2(\Pi, d\mu_\lambda)$$

is an isometric isomorphism from $L_2(\mathbb{R})$ onto the Bergman space $\mathscr{A}_\lambda^2(\Pi)$.

Remark 1.1.

$$R_\lambda R_\lambda^* = \mathrm{Id}_{L_2(\mathbb{R})} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R});$$
$$R_\lambda^* R_\lambda = B_\lambda : L_2\left(\Pi, d\mu_\lambda\right) \longrightarrow \mathscr{A}_\lambda^2(\Pi);$$

Theorem 1.2. *The isometric isomorphism $R_\lambda^* = R_{0,\lambda}^* U_{1,\lambda} : \mathscr{A}_\lambda^2(\Pi) \longrightarrow L_2(\mathbb{R})$ is given by*

$$\left(R_\lambda^* f\right)(z) = \int_{\mathbb{R}} z^{i\xi - \left(\frac{\lambda+2}{2}\right)} \frac{f(\xi)}{\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(\xi)}}\, d\xi. \tag{1.17}$$

Corollary 1.1. *The inverse isomorphism $R_\lambda = R_{0,\lambda}^* U_{1,\lambda} : \mathscr{A}_\lambda^2(\Pi) \longrightarrow L_2(\mathbb{R})$ is given by*

$$(R_\lambda \varphi)(x) = \frac{1}{\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix-\left(\frac{\lambda+2}{2}\right)} \varphi(z)\, d\mu_\lambda(z), \quad x \in \mathbb{R}, \tag{1.18}$$

where the Lebesgue measure μ_λ is given by

$$d\mu_\lambda(z) = \frac{2^\lambda}{\pi} r^{\lambda+1} \sin^\lambda\theta\, dr d\theta, \quad z = re^{i\theta}. \tag{1.19}$$

Example 1.2. Let $z \in \Pi$ and $K_{z,\lambda}$ be the Bergman kernel of $\mathcal{A}^2_\lambda(\Pi)$. Since $R_\lambda = R^*_{0,\lambda} U_{1,\lambda}$, by (1.13) and (1.16) we get

$$
\begin{aligned}
(R_\lambda K_{z,\lambda})(x) = (R^*_{0,\lambda} U_{1,\lambda} K_{z,\lambda})(x) &= \sqrt{\frac{2^\lambda(\lambda+1)}{c_\lambda(x)}} \int_0^\pi (U_{1,\lambda} K_{z,\lambda})(x,\theta)\, e^{-\left(x - \left(\frac{\lambda+2}{2}\right) i\right)\theta} \sin^\lambda \theta \, d\theta \\
&= \sqrt{\frac{2^\lambda(\lambda+1)}{c_\lambda(x)}} \int_0^\pi \frac{e^{-x\theta - i\left(\frac{\lambda+2}{2}\right)\theta} (\overline{z})^{-ix - \left(\frac{\lambda+2}{2}\right)}}{2^{\lambda+\frac{1}{2}} (\lambda+1)\, c_\lambda(x)}\, e^{-\left(x - \left(\frac{\lambda+2}{2}\right) i\right)\theta} \sin^\lambda \theta \, d\theta \\
&= \frac{(\overline{z})^{-ix - \left(\frac{\lambda+2}{2}\right)}}{c_\lambda^{3/2}(x) \sqrt{2^{\lambda+1}(\lambda+1)}} \int_0^\pi e^{-2x\theta} \sin^\lambda \theta \, d\theta, \quad x \in \mathbb{R}.
\end{aligned}
$$

Therefore

$$
(R_\lambda K_{z,\lambda})(x) = \frac{(\overline{z})^{-ix - \left(\frac{\lambda+2}{2}\right)}}{\sqrt{2^{\lambda+1}(\lambda+1)\, c_\lambda(x)}}, \quad x \in \mathbb{R}. \tag{1.20}
$$

Example 1.3. By (1.18) we have for each $h > 0$ that

$$
\begin{aligned}
\left(R_\lambda D_{h,\lambda} \varphi\right)(x) &= \frac{1}{\sqrt{2^{\lambda+1}(\lambda+1)\pi\, c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix - \left(\frac{\lambda+2}{2}\right)} h^{\frac{\lambda+2}{2}} \varphi(hz)\, d\mu_\lambda(z) \\
&= \frac{h^{ix}}{\sqrt{2^{\lambda+1}(\lambda+1)\pi\, c_\lambda(x)}} \int_\Pi (\overline{z})^{-ix - \left(\frac{\lambda+2}{2}\right)} \varphi(z)\, d\mu_\lambda(z) \\
&= h^{ix} (R_\lambda \varphi)(x) = (M_{E_{ih}} R_\lambda \varphi)(x).
\end{aligned}
$$

This clearly forces

$$
R_\lambda D_{h,\lambda} R^*_\lambda = M_{E_h}, \quad \forall h \in \mathbb{R}_+. \tag{1.21}
$$

1.2 Fock spaces

In this section we recall some elementary results about Fock spaces. First, we consider the entire functions on the n-dimensional complex plane and some basic properties of them. For more detail see for example [41, 53]

Let $\mathbb{C}^n$ be the n-dimensional complex plane. The addition, scalar multiplication and conjugation are defined on $\mathbb{C}^n$ componentwise. We will use the following standard notation: $z = x + iy = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$;

$$
z \cdot w = \sum_{k=1}^n z_k w_k; \qquad z^2 = z \cdot z = \sum_{k=1}^n z_k^2; \qquad |z|^2 = z \cdot \overline{z} = \sum_{k=1}^n |z_k|^2.
$$

Consider the following partial differential operators

$$
\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right); \qquad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right) \tag{1.22}
$$

Definition 1.1. Let $\mathcal{O} \subset \mathbb{C}^n$ be a open set. A function $f : \mathcal{O} \to \mathbb{C}$ is called *holomorphic* (on $\mathcal{O}$) if $f \in C^1(\mathcal{O})$ and satisfies the system of partial differential equations

$$\frac{\partial f}{\partial \overline{z}_j}(z) = 0, \quad \text{for } 1 \leq j \leq n \text{ and } z \in \mathbb{C}^n. \tag{1.23}$$

We denote by $\mathcal{H}(\mathcal{O})$ *the space of holomorphic functions* on $\mathcal{O}$.

The following proposition is a generalization of Theorem 2.2 in [45] and was proved by Miroslav Engliš, [16, Proposition 1] for $\mathcal{O} \subsetneq \mathbb{C}^n$, and by Folland [21, Proposition 1.69] for $\mathcal{O} = \mathbb{C}^n$.

Proposition 1.2. *Let $\mathcal{O}$ be a domain in $\mathbb{C}^n$, $\overline{\mathcal{O}} = \{\overline{z} \in \mathbb{C}^n : z \in \mathcal{O}\}$, and F be an analytic function on $\mathcal{O} \times \overline{\mathcal{O}}$ such that*

$$F(z, \overline{z}) = 0, \quad z \in \mathcal{O}.$$

Then F vanishes identically on $\mathcal{O} \times \overline{\mathcal{O}}$.

A function f is said to be *entire* when it is analytic on the whole complex plane $\mathbb{C}^n$. i.e., $f \in \mathcal{H}(\mathbb{C}^n)$. For $n = 1$, it is well known that a function f is entire if and only if f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with infinite radius of convergence. Also, it is well known that every bounded entire function is a constant function (Liouville's Theorem) which is a consequence of the most fundamental result in complex analysis, namely, *the identity theorem*.

Proposition 1.3 (identity theorem). *Suppose that f is an entire function. If there is a point $z \in \mathbb{C}$ such that $f^{(n)}(z) = 0$ for all $n = 0, 1, 2, \ldots$, then $f \equiv 0$ on $\mathbb{C}$.*

A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an element of $\mathbb{Z}^n_+$. Next, we fix the standard multi-index notation:

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}; \qquad \overline{z}^\alpha = \overline{z_1}^{\alpha_1} \overline{z_2}^{\alpha_2} \cdots \overline{z_n}^{\alpha_n};$$

$$\left(\frac{\partial}{\partial z}\right)^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}; \qquad \left(\frac{\partial}{\partial \overline{z}}\right)^\alpha = \left(\frac{\partial}{\partial \overline{z_1}}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \overline{z_n}}\right)^{\alpha_n};$$

$$\alpha! = \alpha_1! \cdots \alpha_n!; \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j;$$

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_n^{\alpha_n}}; \qquad \overline{\partial}^\alpha f = \frac{\partial^{|\alpha|}}{\partial \overline{z_1}^{\alpha_1} \partial \overline{z_2}^{\alpha_2} \cdots \partial \overline{z_n}^{\alpha_n}}.$$

The lack of bounded entire functions is one of the key differences between the theory of Fock spaces and the more classical theories of Hardy and Bergman spaces. Next, we introduce the Fock spaces and give some their basic properties. For any $\varsigma > 0$ let us denote by $g_{n,\varsigma}$ the normed Gaussian measure on $\mathbb{C}^n$ with respect to the density

$$dg_{n,\varsigma}(z) = \left(\frac{\varsigma}{\pi}\right)^n e^{-\varsigma|z|^2} d\mu_n(z), \tag{1.24}$$

where μ_n is the usual Lebesgue measure on $L_2(\mathbb{R}^{2n})$. Observe that $dg_{n,\varsigma}$ is a probability measure, in effect

$$g_{n,\varsigma}(\mathbb{C}^n) = \int_{\mathbb{C}^n} dg_{n,\varsigma}(z) = \left(\frac{\varsigma}{\pi}\right)^n \prod_{j=1}^{n} \int_0^{2\pi} \int_0^{+\infty} e^{-\varsigma r_j^2} r_j dr_j d\theta_j = (2\varsigma)^n \prod_{j=1}^{n} \int_0^{+\infty} e^{-\varsigma r_j^2} r_j dr_j = 1.$$

Definition 1.2. The *Fock space* (also known as the Segal–Bargmann space, see [2, 8, 19, 44]) $\mathscr{F}_\varsigma^2(\mathbb{C}^n)$ consists of all entire functions that are square integrable on $\mathbb{C}^n$ with respect to the Gaussian measure (1.24).

Let $\{e_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ be the system of functions in $\mathscr{F}_\varsigma^2(\mathbb{C}^n)$ given by the rule

$$e_\alpha(z) = \sqrt{\frac{\varsigma^{|\alpha|}}{\alpha!}}\, z^\alpha, \quad \alpha \in \mathbb{Z}_+^n. \tag{1.25}$$

By the integral formula of the Gamma function one can easily check that $\langle z^\alpha, z^\beta\rangle = \delta_{\alpha,\beta}$. Since every function $f \in \mathscr{F}_\varsigma^2(\mathbb{C}^n)$ has an expansion in Taylor series

$$f(z) = \sum_{\alpha\in\mathbb{Z}_+^n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha, \quad z \in \mathbb{C}^n,$$

it is easy to see that $span\left\{e_\alpha : \alpha \in \mathbb{Z}_+^n\right\}$ is a dense subset of $\mathscr{F}_\varsigma^2(\mathbb{C}^n)$. Thus the system $\{e_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ form an orthonormal basis for $\mathscr{F}_\varsigma^2(\mathbb{C}^n)$.

Proposition 1.4. *The Fock space $\mathscr{F}_\varsigma^2(\mathbb{C}^n)$ is a closed subspace of $L_2(\mathbb{C}^n, dg_{n,\varsigma})$.*

For any fixed $z \in \mathbb{C}^n$, the evaluation functional $\psi_z(f) = f(z)$ is linear and bounded. Thus, by the Riesz representation theorem there exists a unique element $k_{z,\varsigma} \in \mathscr{F}_\varsigma^2(\mathbb{C}^n)$ such that $\psi_z = \langle\cdot, k_{z,\alpha}\rangle$; that is

$$f(z) = \int_{\mathbb{C}^n} f(w)\overline{k_{z,\varsigma}(w)}\, dg_{n,\varsigma}(w).$$

The function $k_{z,\varsigma}$ is given by the formula

$$k_{z,\varsigma}(w) = e^{\varsigma\bar{z}w} \quad w \in \mathbb{C}^n. \tag{1.26}$$

Since the Fock space $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$ is a closed subspace of $L_2(\mathbb{C}^n, dg_{n,\varsigma})$, there exists the unique orthogonal projection $\mathrm{P}_{n,\varsigma}$ from $L_2(\mathbb{C}^n, dg_{n,\varsigma})$ onto $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$. This projection is called the *Bargmann projection* and has the integral representation

$$(\mathrm{P}_{n,\varsigma} f)(z) = \int_{\mathbb{C}^n} f(w) e^{\varsigma \overline{w} z}\, dg_{n,\varsigma}(w), \quad f \in L_2(\mathbb{C}^n, dg_{n,\varsigma}). \tag{1.27}$$

1.3 Toeplitz operators

Next, we introduce the Toeplitz operators acting on either the Bergman spaces or the Fock spaces and give some their basic properties.

Let $\mathscr{H}L_2(\mathscr{O})$ be the closed subspace of $L_2(\mathscr{O})$ consisting of holomorphic functions on the non-empty open set $\mathscr{O} \subseteq \mathbb{C}$, with reproducing kernel K and let $P : L_2(\mathscr{O}) \to \mathscr{H}L_2(\mathscr{O})$ be the orthogonal projection from $L_2(\mathscr{O})$ onto $\mathscr{H}L_2(\mathscr{O})$ given in terms of the reproducing kernel K as

$$(Pf)(z) = \int_{\mathscr{O}} f(w) K(z,w)\, dmu(w), \quad f \in L_2(\mathscr{O}).$$

Now, suppose that $\phi \in L_\infty(\mathscr{O})$, define a linear operator $T_\phi : \mathscr{H}L_2(\mathscr{O}) \longrightarrow \mathscr{H}L_2(\mathscr{O})$ by

$$T_\phi f = P(\phi f), \quad f \in \mathscr{H}L_2(\mathscr{O}).$$

This is called the *Toeplitz operator*, with defining symbol ϕ, acting on $\mathscr{H}L_2(\mathscr{O})$. So, the Toeplitz operator is one of the form "multiply then project", that is, multiply by ϕ and then project back into the holomorphic subspace. The following proposition summarizes some the most important properties of Toeplitz operators.

Proposition 1.5. *[50, Theorem 2.81] Let $\alpha, \beta \in \mathbb{C}$, and $f, g \in L_\infty(\mathscr{O})$, then*
(a) the operator T_f is bounded on $\mathscr{H}L_2(\mathscr{O})$ and $\|T_f\| \leq \|f\|_\infty$,
(b) $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$,
(c) $T^*_f = T_{\overline{f}}$.

If ϕ is an unbounded function, then we can still define the Toeplitz operator T_ϕ in the same way, except that T_ϕ may be undounded.

Example 1.4. (i). If $\mathscr{H}L_2(\mathscr{O}) = \mathscr{A}^2_\lambda(\Pi)$, then the reproducing kernel K is given in (1.3) and P is the Bergman projection from $L_2(\Pi, dv_\lambda)$ onto $\mathscr{A}^2_\lambda(\Pi)$. Thus the Toeplitz operator T_φ with defining symbol $\varphi \in L_\infty(\Pi)$ acting on $\mathscr{A}^2_\lambda(\Pi)$ has the integral form

$$(T_\varphi f)(z) = i^{\lambda+2} \int_\Pi \frac{\varphi(w) f(w)}{(w - \overline{z})^{\lambda+2}} dv_\lambda(w), \quad z \in \Pi.$$

(ii). If $\mathscr{H}L_2(\mathscr{O}) = \mathscr{F}^2_\varsigma(\mathbb{C}^n)$, then the reproducing kernel K is given in (1.26) and P is the Bargmann projection from $L_2(\mathbb{C}^n, dg_{n,\varsigma})$ onto $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$. Thus, the Toeplitz operator T_ϕ with defining symbol $\phi \in L_\infty(\mathbb{C}^n)$ acting on $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$ has the following integral form

$$(T_\phi f)(z) = \int_{\mathbb{C}^n} \phi(w) f(w) e^{\varsigma \overline{w} z} dg_{n,\varsigma}(w), \quad z \in \mathbb{C}^n.$$

The next theorem shows that there is a one-to-one-correspondence between the Toeplitz operators and their bounded defining symbols.

Proposition 1.6. *Let $\phi \in L_\infty(\mathscr{O})$. Then T_ϕ is zero if and only if $\phi \equiv 0$ almost everywhere in $\mathscr{O}$.*

The corresponding result for Toeplitz operators acting on Bergman spaces over the unit disk is well known (see for example [50, Theorem 2.8.2]). To extend it to the upper half-plane case, we introduce the Caley transform

$$\Psi : \Pi \longrightarrow \mathbb{D}, \qquad w \mapsto \frac{w-i}{w+i},$$

the corresponding unitary operator $U : \mathscr{A}^2_\lambda(\mathbb{D}) \longrightarrow \mathscr{A}^2_\lambda(\Pi)$ given by the rule

$$\varphi \mapsto (\varphi \circ \Psi)\Psi'$$

and observe that $U^* T_\varphi U = T_{\varphi \circ \Psi^{-1}}$.

On the other hand, the corresponding result (Proposition 1.6) for Toeplitz operators acting on Fock spaces was proved by Berger and Coburn in [9, Theorem 4]. There are other classes of functions such that Proposition 1.6 is true. For example, Folland [20, P. 140] extended Proposition 1.6 for Toeplitz operators T_a acting on Fock space $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$ whose defining symbols a belong to the class of unbounded functions which satisfy the inequality

$$|a(z)| \leq const\, e^{\delta |z|^2}, \quad \text{for some } \delta < 1. \tag{1.28}$$

1.4 Berezin transform

The *Berezin* transform [7, 8, 52] associates smooth functions with operators on Hilbert spaces of analytic functions. The Berezin transform plays an important role in the description of properties of bounded operators, particularly for Toeplitz operators.

Definition 1.3. $\mathcal{H}L_2(\mathcal{O})$ be the closed subspace of $L_2(\mathcal{O}, d\mu)$ consisting of holomorphic functions on the non-empty open set $\mathcal{O} \subseteq \mathbb{C}$ (or $\mathbb{C}^n$), with reproducing kernel K. Let S be a bounded linear operator on $\mathcal{H}L_2(\mathcal{O})$, the *Berezin transform* of S is defined by

$$\widetilde{S}(z) = \frac{\langle SK_z, K_z\rangle}{\langle K_z, K_z\rangle}, \quad z \in \mathcal{O}. \tag{1.29}$$

In particular, the Berezin transform of a function $\phi \in L_\infty(\mathcal{O})$ (denoted by $\widetilde{\phi}$) is definded as the Berezin transform of the Toeplitz operator T_ϕ. Likewise, $\widetilde{\phi} = \widetilde{T_\phi}$.

For each bounded operator, the Berezin transform is a bounded real-analytic function on a domain of $\mathbb{C}^n$. Indeed, observe that if $S \in \mathcal{B}(\mathcal{H}L_2(\mathcal{O}))$, then $\widetilde{S} \in L_\infty(\mathcal{O})$, and by the Cauchy-Schwarz inequality it satisfies the relation

$$\|\widetilde{S}\|_\infty \le \|S\|. \tag{1.30}$$

Example 1.5. The *Berezin transform* of an operator $V \in \mathcal{B}(\mathcal{A}^2_\lambda(\Pi))$ is the function $\widetilde{V} : \Pi \to \mathbb{C}$ defined by

$$\widetilde{V}(z) := (2\operatorname{Im} z)^{\lambda+2} i^{\lambda+2} \int_\Pi \frac{(VK_{z,\lambda})(w)}{(\overline{w} - z)^{\lambda+2}} dv_\lambda(w), \quad z \in \Pi, \tag{1.31}$$

where $K_{z,\lambda}$ is the Bergman kernel (1.3). Thus for $V = T_\varphi$ with $\varphi \in L_\infty(\Pi)$ one gets for every $z \in \Pi$ that

$$\widetilde{\varphi}(z) = (2\operatorname{Im} z)^{\lambda+2} \int_\Pi \frac{\varphi(w)}{|\overline{z} - w|^{2(\lambda+2)}} dv_\lambda(w).$$

Example 1.6. The Berezin transform of a bounded operator S on the Fock space $\mathcal{F}^2_\varsigma(\mathbb{C}^n)$ is the function $\widetilde{S} : \mathbb{C}^n \to \mathbb{C}$ defined by

$$\widetilde{S}(z) = e^{-\varsigma|z|^2} \int_{\mathbb{C}^n} (Sk_z)(w) e^{\varsigma \overline{w} z} dg_{n,\varsigma}(w), \quad z \in \mathbb{C}^n. \tag{1.32}$$

Thus, for $S = T_\phi$ with $\phi \in L_\infty(\mathbb{C}^n)$ one has for every $z \in \mathbb{C}^n$ that

$$\begin{aligned}\widetilde{\phi}(z) &= e^{-\varsigma|z|^2} \int_{\mathbb{C}^n} \phi(w) e^{\varsigma(z\overline{w} + \overline{z}w)} dg_{n,\varsigma}(w) = \int_{\mathbb{C}^n} \phi(w) e^{-\varsigma(z-w)\cdot\overline{(z-w)}} d\mu_n(w) \\ &= \int_{\mathbb{C}^n} \phi(w) e^{-\varsigma|z-w|^2} d\mu_n(w).\end{aligned}$$

Proposition 1.7 (injectivity of the Berezin transform). *Let $\mathcal{H}L_2(\mathcal{O})$ be the closed subspace of $L_2(\mathcal{O}, d\mu)$ consisting of holomorphic functions on the non-empty open set $\mathcal{O} \subseteq \mathbb{C}^n$, and let S be a bounded operator on $\mathcal{H}L_2(\mathcal{O})$. $S = 0$ if and only if $\widetilde{S}(w) = 0$ for all $w \in \mathcal{O}$.*

Proof. If $S=0$, then $\widetilde{S}(w)=0$ for all $w\in\mathscr{O}$ trivially. Conversely, suppose that $\widetilde{S}(w)=0$ for all $w\in\mathscr{O}$. Define $F:\mathscr{O}\times\mathscr{O}\to\mathbb{C}$ by $F(z,w)=\langle SK_{\overline{w}},K_z\rangle$, $z,w\in\mathscr{O}$. The function F has an integral representation as

$$F(z,w)=\int_{\mathscr{O}}(SK_{\overline{w}})(\zeta)\overline{K_z(\zeta)}d\mu(\zeta)=\int_{\mathscr{O}}(SK_{\overline{w}})(\zeta)K_\zeta(z)d\mu(\zeta),\quad z,w\in\mathscr{O}.$$

From this observe that F is analytic in the first variable, that is, for fixed $\overline{w}\in\overline{\mathscr{O}}$ the function $z\mapsto F(z,w)$ is analytic on $\mathscr{O}$. Furthermore, using that

$$F(z,w)=\left\langle K_{\overline{z}},S^*K_z\right\rangle=\overline{\langle S^*K_z,K_{\overline{w}}\rangle},$$

we see that F is analytic in the second variable. It follows from the fact for an analytic function g on $\mathscr{O}$ the function $w\mapsto\overline{g(\overline{w})}$ is analityc on $\overline{\mathscr{O}}$. On the other hand, $F(z,\overline{z})=0$ for all $z\in\mathscr{O}$, thus by Lemma 1.2 one gets that $F(z,w)=0$ on $\mathscr{O}\times\overline{\mathscr{O}}$. Equivalently, $(SK_w)(z)=\langle SK_w,K_z\rangle=0$ for all $z,w\in\mathscr{O}$, and hence $SK_w=0$ for $w\in\mathscr{O}$. Now, given an arbitrary $f\in\mathscr{H}(\mathscr{O})$ and $w\in\mathscr{O}$ we have

$$(S^*f)(w)=\left\langle S^*f,K_w\right\rangle=\langle f,SK_w\rangle=0.$$

Thus $S^*=0$, and therefore $S=0$. ■

1.5 Approximation of uniformly continuous functions by convolutions

In this section we recall a technique that permits us to approximate bounded uniformly continuous functions by convolutions with a fixed kernel satisfying Wiener's condition. We could not find Proposition 1.9 in the literature, but it is based on well-known ideas and can be considered as a variation of the Wiener's Tauberian theorem. The constructions of this section can be generalized to abelian locally compact groups.

Let $f:\mathbb{R}^n\to\mathbb{C}$. Recall that the function $\Omega_f:[0,+\infty]\longrightarrow[0,+\infty]$ defined by

$$\Omega_f(\delta):=\sup\left\{|f(x)-f(y)|:\ x,y\in\mathbb{R},\ d(x,y)\le\delta\right\}. \tag{1.33}$$

is the *modulus of continuity* of f with respect to the usual metric d on $\mathbb{R}$ (or simply *modulus of continuity*). Hence, if $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ is a bounded function, then f is said to be *uniformly continuous* whenever $\lim\limits_{\delta\to0}\Omega_f(\delta)=0$. The set of all such functions is denoted by $C_{b,u}(\mathbb{R})$. In fact, it is well-known that $C_{b,u}(\mathbb{R}^n)$ is a C*-algebra with pointwise operations.

The *convolution* of two complex-valued functions on $\mathbb{R}^n$ is itself a complex-valued function on $\mathbb{R}^n$ defined by:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \quad x \in \mathbb{R}^n. \tag{1.34}$$

It is well known that the convolution of f and g exists if $f \in L_1(\mathbb{R}^n)$ and $g \in L_p(\mathbb{R}^n)$ where $1 \le p \le +\infty$. In this case $f * g \in L_p(\mathbb{R}^n)$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Proposition 1.8. $L_1(\mathbb{R}^n) * L_\infty(\mathbb{R}^n) \subset C_{b,u}(\mathbb{R}^n)$.

Proof. Let $f \in L_1(\mathbb{R}^n)$ and $g \in L_\infty(\mathbb{R}^n)$. Then for $h \in \mathbb{R}^n$ one obtains

$$\begin{aligned} |(f * g)(x+h) - (f * g)(x)| &\le \int_{\mathbb{R}^n} |f(x-h-t) - f(x-t)|\, |g(t)| dt \\ &\le \|g\|_\infty \int_{\mathbb{R}^n} |f(x-h-t) - f(x-t)|\, dt \\ &\overset{y=x-t}{=} \|g\|_\infty \int_{\mathbb{R}^n} |f(y-h) - f(y)|\, dy \\ &= \|g\|_\infty \|L_h f - f\|_1, \quad x \in \mathbb{R}^n. \end{aligned}$$

Here τ_h denotes the translation operator by h,(see (A.18)) . Thus, since for each $f \in L_1(\mathbb{R}^n)$ the mapping $h \mapsto \tau_h f$ from $\mathbb{R}^n$ into $L_1(\mathbb{R}^n)$ is uniformly continuous (Corollary A.1), we have that $f * g$ belongs to $C_{b,u}(\mathbb{R}^n)$. ■

In Proposition 1.8 we can change by an equality. i.e., $L_1(\mathbb{R}^n) * L_\infty(\mathbb{R}^n) = C_{b,u}(\mathbb{R}^n)$ (see [33, p. 283, 32-45]). Therefore, any function in $\{k * f : f \in L_\infty(\mathbb{R}^n)\}$ belongs to $C_{b,u}(\mathbb{R}^n)$.

Proposition 1.9. *If $k \in L_1(\mathbb{R}^n)$ and $\hat{k}(t) \neq 0$ for each $t \in \mathbb{R}^n$, then $\{k * f : f \in L_\infty(\mathbb{R}^n)\}$ is a dense subset of $C_{b,u}(\mathbb{R}^n)$.*

Proof. By Proposition 1.8 every function in $\{k * f : f \in L_\infty(\mathbb{R})\}$ belongs to $C_{b,u}(\mathbb{R})$. Next, the density is proved by means of Wiener's Division Lemma and Lemma A.1 as follows: Let $(h_n)_{n\in\mathbb{N}}$ be a Dirac sequence such that the functions $\hat{h}_n$ have compact supports. For example, $(h_n)_{n\in\mathbb{N}}$ can be defined by (A.22). Since $\widehat{k}(t) \neq 0$ for each $t \in \mathbb{R}$, by Wiener's Division Lemma (Theorem A.4) for every $n \in \mathbb{N}$ there exists $q_n \in L_1(\mathbb{R})$ such that $h_n = k * q_n$. Now, given $\psi \in C_{b,u}(\mathbb{R})$, we construct a sequence $(w_n)_{n\in\mathbb{N}}$ by the rule $w_n = q_n * \psi$. Then $w_n \in L_\infty$ and the sequence $(k * w_n)_{n\in\mathbb{N}}$ takes values in the set $\{k * f : f \in L_\infty(\mathbb{R})\}$. Finally, applying the identities

$$k * w_n = k * q_n * \psi = h_n * \psi$$

and Lemma A.1, we conclude that this sequence converges uniformly to ψ. ■

1.6 Symplectic spaces and Lagrangian planes

This section contains a brief summary of basic concepts of the theory of the symplectic group and related topics.

Definition 1.4 (symplectic group). A real $2n \times 2n$ matrix B is said to be symplectic if it satisfies the conditions:

$$B^T JB = BJB^T = J, \tag{1.35}$$

where J is the " standard symplectic matrix" given by

$$J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}. \tag{1.36}$$

Here 0 and I_n are the $n \times n$ zero and identity matrices. The set of all symplectic matrices is denoted by $\mathrm{Sp}(2n, \mathbb{R})$.

A symplectic matrix is invertible and has determinant 1. In fact, if $B \in \mathrm{Sp}(2n, \mathbb{R})$, then $B^{-1} \in \mathrm{Sp}(2n, \mathbb{R})$. It is well-known that $\mathrm{Sp}(2n, \mathbb{R})$ form a group, and is called *the (real) symplectic group.*

Recall that a *skew-symmetric bilinear form ω* is a bilinear form such that

$$\omega(z, w) = -\omega(w, z)$$

for all $z, w \in \mathbb{R}^{2n}$. Notice that if ω is a skew-symmetric bilinear form, then all vectors z are isotropic. i.e., for every $z \in \mathbb{R}^{2n}$ one gets that

$$\omega(z, z) = 0.$$

Definition 1.5. A bilinear form on $\mathbb{R}^{2n}$ is called a *symplectic form* if it is a non-degenerate skew-symmetric bilinear form.

The special skew-symmetric bilinear form ω_0 on $\mathbb{R}^{2n}$ defined by

$$\omega_0(z, w) = y \cdot x' - y' \cdot x, \tag{1.37}$$

for $z = (x, y)$ and $w = (x', y')$ is symplectic; it is called *the standard symplectic form* of $\mathbb{R}^{2n}$.

The standard symplectic form ω_0 can be re-written in a convenient way using the symplectic standard matrix J given in (1.36):

$$\omega_0(z, w) = Jz \cdot w, \quad z, w \in \mathbb{R}^{2n}. \tag{1.38}$$

Let $\psi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping. The condition

$$\omega_0(\psi(z),\psi(w)) = \omega_0(z,w),$$

is equivalent to $B^T JB = J$, where B is the matrix of ψ in the canonical basis of $\mathbb{R}^{2n}$, that is, $B \in \mathrm{Sp}(2n,\mathbb{R})$. We can thus redefine the symplectic group by saying that it is the group of all linear automorphism of $\mathbb{R}^{2n}$ which preserve the standard symplectic form ω_0.

Definition 1.6 (Lagrangian plane). A n-dimensional linear subspace $\mathscr{L}$ of $\mathbb{R}^{2n}$ is said to be a *Lagrangian plane* of the symplectic space $(\mathbb{R}^{2n},\omega_0)$ if $\omega_0(z,w) = 0$, for every $z,w \in \mathscr{L}$. The set of all Lagrangian planes in $(\mathbb{R}^{2n},\omega_0)$ is denoted by $\mathrm{Lag}(2n,\mathbb{R})$.

Example 1.7. The simplest examples of Lagrangian planes of $(\mathbb{R}^{2n},\omega_0)$ are both coordinates planes: $\mathscr{L}_x = \mathbb{R}^n \times \{0\}$ and $\mathscr{L}_y = \{0\} \times \mathbb{R}^n$, and so is the diagonal $\Delta = \{(x,x)\colon\ x \in \mathbb{R}^n\}$.

Let $\mathscr{M}(n,\mathbb{C})$ be the algebra of complex matrices of dimension n. Denote by $\mathrm{U}(n,\mathbb{C})$ the unitary subgroup of $\mathscr{M}(n,\mathbb{C})$ consisting of all complex matrices U of $n \times n$ such that

$$U^*U = UU^* = \mathrm{I}_n,$$

where $U^* = \overline{U}^T$. Define the mapping $\iota\colon \mathscr{M}(n,\mathbb{C}) \to \mathscr{M}(n,\mathbb{C})$ by the rule

$$\iota(C) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{1.39}$$

where the matrix $C \in \mathscr{M}(n,\mathbb{C})$ is the form $C = A + iB$ with A and B real matrices. It is easy to see from the definition of ι that it is an injective mapping.

Lemma 1.4. *The mapping ι given in (1.39) satisfies the following properties:*

- *For every $U,V,\in \mathscr{M}(n,\mathbb{C})$ and $\alpha,\beta \in \mathbb{C}$ the mapping ι is lineal. That is, $\iota(\alpha U + \beta V) = \alpha\,\iota(U) + \beta\,\iota(V)$.*
- *For every $U,V,\in \mathscr{M}(n,\mathbb{C})$ the mapping ι is multiplicative. That is, $\iota(UV) = \iota(U)\iota(V)$.*
- *For every $U \in \mathscr{M}(n,\mathbb{C})$ one gets that $\iota(U^*) = \iota(U)^T$.*

Define the set $\mathrm{U}(2n,\mathbb{R})$ of real matrices $n \times n$ by the rule

$$\mathrm{U}(2n,\mathbb{R}) = \iota(\mathrm{U}(n,\mathbb{C}))\,. \tag{1.40}$$

$\mathrm{U}(2n,\mathbb{R})$ is called symplectic rotations of $(\mathbb{R}^{2n},\omega_0)$.

Recall that if $C = A + iB \in \mathrm{U}(n,\mathbb{C})$, then $CC^* = \mathrm{I}_n$. Equivalently

$$\mathrm{I}_n = (A+iB)(A^T - iB^T) = AA^T + BB^T + i(BA^T - AB^T).$$

That is, $AA^T + BB^T = \mathrm{I}_n$ and $BA^T = AB^T$. Thus by (1.39) one can see the symplectic rotations $\mathrm{U}(2n,\mathbb{R})$ as:

$$\mathrm{U}(2n,\mathbb{R}) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : AA^T + BB^T = \mathrm{I}_n \quad \text{and} \quad AB^T = BA^T, \quad A,B \in \mathscr{M}(n,\mathbb{R}) \right\} \tag{1.41}$$

Proposition 1.10.

$$\mathrm{U}(2n,\mathbb{R}) = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R}).$$

Proof. Let $U \in \mathrm{U}(2n,\mathbb{R})$. Then by (1.41) there are $A,B \in \mathscr{M}(n,\mathbb{R})$ with $AA^T + BB^T = \mathrm{I}_n$ and $AB^T = BA^T$ such that

$$U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Therefore

$$UU^T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} = \begin{pmatrix} AA^T + BB^T & AB^T - BA^T \\ BA^T - AB^T & AA^T + BB^T \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{2n}.$$

Analogously, $U^T U = \mathrm{I}_{2n}$. This implies that $U \in \mathrm{O}(2n,\mathbb{R})$. On the other hand,

$$UJU^T = \begin{pmatrix} B & A \\ -A & B \end{pmatrix} \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} = \begin{pmatrix} BA^T - AB^T & AA^T + BB^T \\ -(AA^T + BB^T) & BA^T - AB^T \end{pmatrix} = J.$$

In the same way is showed that $U^T JU = J$. Thus $U \in \mathrm{Sp}(2n,\mathbb{R})$ and hence

$$\mathrm{U}(2n,\mathbb{R}) \subset \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R}).$$

Now, if $V \in \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R})$, then $VJV^T = J$ and $VV^T = V^T V = \mathrm{I}_{2n}$. We thus get that $VJ = JV$, from this equality it is easy to see that the matrix V belongs to $\mathrm{U}(2n,\mathbb{R})$. ∎

By (1.35) and Proposition 1.10 a $2n \times 2n$ matrix B belongs to $\mathrm{U}(2n,\mathbb{R})$ if it commutes with the standard symplectic matrix J, that is:

$$BJ = JB. \tag{1.42}$$

Proposition 1.11 ([14, Section 4.3]). *The group of symplectic rotations* $\mathrm{U}(2n,\mathbb{R})$ *acts transitively on* $\mathrm{Lag}(2n,\mathbb{R})$. *That is, for every pair* $(\mathscr{L},\mathscr{L}') \in \mathrm{Lag}(2n,\mathbb{R}) \times \mathrm{Lag}(2n,\mathbb{R})$, *there exists* $B \in \mathrm{U}(2n,\mathbb{R})$ *such that* $\mathscr{L} = B\mathscr{L}'$.

CHAPTER 2

C*-ALGEBRA OF ANGULAR TOEPLITZ OPERATORS ON WEIGHTED BERMAN SPACES OVER THE UPPER HALF–PLANE

In this chapter we show that the uniform closure of the set of all Toeplitz operators acting on the weighted Bergman space over the upper half-plane whose L_∞-symbols are angular coincides with the C*-algebra generated by the above Toeplitz operators and is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}$) of bounded functions that are very slowly oscillating on the real line in the sense that they are uniformly continuous with respect to the arcsinh-metric $\rho(x,y) = |\mathrm{arcsinh} x - \mathrm{arcsinh} y|$ on the real line.

2.1 Angular Toeplitz operators

In this section we characterize the angular Toeplitz operators acting on weighted Bergman spaces. The characterization is based on the notion of angular operator. So, we will first introduce the angular operators and study their basic properties, including a simple criterion for an operator to be angular.

Let $\mathscr{B}(\mathscr{A}_\lambda^2(\Pi))$ be the algebra of all linear bounded operators acting on the Bergman space $\mathscr{A}_\lambda^2(\Pi)$. Given $h \in \mathbb{R}_+$, let $D_{\lambda,h} \in \mathscr{B}(\mathscr{A}_\lambda^2(\Pi))$ be the *dilation* operator defined by

$$D_{\lambda,h} f(z) = h^{\frac{\lambda+2}{2}} f(hz). \tag{2.1}$$

Next, we introduce the angular operators acting on the Bargman space $\mathscr{A}_\lambda^2(\Pi)$.

Definition 2.1 (angular operators). An operator $V \in \mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$ is said to be *angular* or *invariant under dilations* if it commutes with all dilation operators. That is, for every $h \in \mathbb{R}_+$

$$VD_{\lambda,h} = D_{\lambda,h}V. \tag{2.2}$$

We denote by $\mathfrak{A}_\lambda$ the set of all angular operators:

$$\mathfrak{A}_\lambda := \left\{V \in \mathscr{B}(\mathscr{A}^2_\lambda(\Pi)) : \quad \forall h \in \mathbb{R}_+ \quad D_{\lambda,h}V = VD_{\lambda,h}\right\}. \tag{2.3}$$

Proposition 2.1. *$\mathfrak{A}_\lambda$ is a* C*-*subalgebra of* $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$.

Proof. Let $S, V \in \mathfrak{A}_\lambda$ and $h > 0$. Then $(S+V)D_{\lambda,h} = SD_{\lambda,h} + VD_{\lambda,h} = D_{\lambda,h}S + D_{\lambda,h}V = D_{\lambda,h}(S+V)$, we thus have that $S+V \in \mathfrak{A}_\lambda$. On the other hand, $TSD_{\lambda,h} = TD_{\lambda,h}S = D_{\lambda,h}TS$, hence $TS \in \mathfrak{A}_\lambda$, this implies that $\mathfrak{A}_\lambda$ is a subalgebra of $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$. The mapping $V \mapsto V^*$, where V^* is the adjoint operator of V, defines an involution on $\mathfrak{A}_\lambda$, furthermore, for each $V \in \mathfrak{A}_\lambda$ one has that

$$V^* D_{\lambda,h} = \left(D^{-1}_{\lambda,h}V\right)^* = \left(D_{\lambda,h^{-1}}V\right)^* = \left(VD_{\lambda,h^{-1}}\right)^* = D_{\lambda,h}V^*.$$

Thus $\mathfrak{A}^*_\lambda = \mathfrak{A}_\lambda$. Now, given $V \in \overline{\mathfrak{A}_\lambda}$, there exists $(V_n)_{n\in\mathbb{N}} \subset \mathfrak{A}_\lambda$ such that $V_n \xrightarrow{n\to\infty} V$, but since $D_{\lambda,h}V_n = V_nD_{\lambda,h}$ and $D_{\lambda,h}$ is a unitary operator, we get that $D_{\lambda,h}V_n = V_nD_{\lambda,h}$ converges to $D_{\lambda,h}V$. Therefore, by uniqueness of the limit we conclude $D_{\lambda,h}S = VD_{\lambda,h}$. That is $V \in \mathfrak{A}_\lambda$. ■

The following theorem gives a criterion for an operator to be angular and is analogous to the Zorboska's result [52] for radial operators and Herrera Yañez, Maximenko, Vasilevski [31] for vertical operators.

Theorem 2.1 (criterion of angular operators).
Let $V \in \mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$, $h \in \mathbb{R}_+$ and M_{E_h} be the multiplication operator by the function $E_h(x) = h^{ix}$. The following conditions are equivalent:

(i) $V \in \mathfrak{A}_\lambda$,

(ii) $R_\lambda V R^*_\lambda M_{E_h} = M_{E_h} R_\lambda V R^*_\lambda$ *for all* $h \in \mathbb{R}_+$,

(iii) *there exists* $\phi \in L_\infty(\mathbb{R})$ *such that* $V = R^*_\lambda M_\phi R^*_\lambda$,

(iv) *the Berezin transform* $\widetilde{V}$ *depends on* $\arg z$ *only.*

Proof. (i)⟶(ii) Let $V \in \mathfrak{A}_\lambda$, by (1.21) one gets that:

$$R_\lambda V R^*_\lambda M_{E_h} = R_\lambda V D_{\lambda,h} R^*_\lambda = R_\lambda D_{\lambda,h} V R^*_\lambda = M_{E_h} R_\lambda V R^*_\lambda, \quad h > 0.$$

(ii)$\longrightarrow$(iii) Observe that for every $\eta \in \mathbb{R}$ we have $E_{e^\eta}(x) = \Theta_\eta(x) = e^{ix\eta}$. Therefore, by (ii) one gets that

$$R_\lambda V R_\lambda^* M_{\Theta_\eta} = R_\lambda V R_\lambda^* M_{E_{e^\eta}} = M_{E_{e^\eta}} R_\lambda V R_\lambda^* = M_{\Theta_\eta} R_\lambda V R_\lambda^*.$$

Thus, by Proposition A.4, there exists $\phi \in L_\infty(\mathbb{R})$ such that $R_\lambda V R_\lambda^* = M_\phi$.

(iii)$\longrightarrow$(iv). By (1.20) we have for every point $w = \rho e^{i\beta}$,

$$(R_\lambda K_{w,\lambda})(x) = \frac{e^{\left(i\left(\frac{\lambda+2}{2}\right)-x\right)\beta}}{\rho^{\left(\frac{\lambda+2}{2}\right)+ix}\sqrt{2^{\lambda+1}(\lambda+1)c_\lambda(x)}}, \quad x \in \mathbb{R}.$$

Therefore, if the operator V is diagonalized by R_λ, like in (iii), then its Berezin transform may be written in terms of ϕ and depends only on the angle β of w:

$$\widetilde{V}(w) = \frac{1}{K_{w,\lambda}(w)} \int_{\mathbb{R}} \phi(x) |R_\lambda K_{w,\lambda}(x)|^2 \, dx = \frac{2\sin^{\lambda+2}\beta}{\lambda+1} \int_{\mathbb{R}} \frac{\phi(x) e^{-2x\beta}}{c_\lambda(x)} \, dx.$$

(iv)$\longrightarrow$(i) Given $z, w \in \Pi$, and $h \in \mathbb{R}_+$, by (1.3) and (2.1)

$$(D_{\lambda,h} K_{w,\lambda})(z) = h^{\frac{\lambda+2}{2}} K_{w,\lambda}(hz) = i^{\lambda+2} h^{\frac{\lambda+2}{2}} (\overline{w} - hz)^{-(\lambda+2)} = \frac{i^{\lambda+2}}{h^{\frac{\lambda+2}{2}}} \left(\frac{\overline{w}}{h} - z\right)^{-(\lambda+2)} = h^{-\left(\frac{\lambda+2}{2}\right)} K_{\frac{w}{h},\lambda}(z).$$

Using this formula we calculate the Berezin transform of the operator $D_{h^{-1},\lambda} V D_{h,\lambda}$:

$$\widetilde{D_{\lambda,h^{-1}} V D_{\lambda,h}}(w) = \frac{\langle V D_{\lambda,h} K_{w,\lambda}, D_{\lambda,h} K_{w,\lambda}\rangle}{\langle D_{\lambda,h} K_{w,\lambda}, D_{\lambda,h} K_{w,\lambda}\rangle} = \frac{\langle V K_{\frac{w}{h},\lambda}, K_{\frac{w}{h},\lambda}\rangle}{\langle K_{\frac{w}{h},\lambda}, K_{\frac{w}{h},\lambda}\rangle} = \widetilde{V}\left(\frac{w}{h}\right) = \widetilde{V}(w).$$

Since the Berezin transform is injective (Proposition 1.7), we have $D_{\lambda,h^{-1}} V D_{\lambda,h} = V$. ■

Definition 2.2 (angular functions). A function $g \in L_\infty(\Pi)$ is said to be a *homogeneous of order zero* or *angular* if for every $h > 0$ the equality $g(hz) = g(z)$ holds for a.e. $z \in \Pi$.

To describe Toeplitz operators with angular symbols we need a simple criterion of angular functions.

The following lemma gives a criterion for a function on $\mathbb{R}$ to be almost everywhere constant. We use there the Lebesgue measure, which is denoted by $\mu_{\mathbb{R}}$ to indicate on $\mathbb{R}$ and $\mu_{\mathbb{R}^2}$ to indicate on $\mathbb{R}^2$. The proof can be found in [31, Section 3, Lemma 3.2].

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. Then the following conditions are equivalent:*

(i) *There exists a constant $c \in \mathbb{C}$ such that $f(x) = c$ for almost all $x \in \mathbb{R}$.*

(ii) $\mu_{\mathbb{R}^2}(D) = 0$*; where $D := \{(x,y) \in \mathbb{R}^2 : f(x) \neq f(y)\}$.*

(iii) $\mu_{\mathbb{R}}(D_x) = 0$ *for almost all $x \in \mathbb{R}$; where $D_x := \{y \in \mathbb{R} : f(x) \neq f(y)\}$.*

Lemma 2.2 (criterion for a function to be angular). *Let $g \in L_\infty(\Pi)$. Then the following two conditions are equivalent:*
(i) *for every $h \in \mathbb{R}_+$, the equality $g(hz) = g(z)$ holds for a.e. z in Π,*
(ii) *there exists a in $L_\infty(0,\pi)$ such that $g(z) = a(\arg z)$ for a.e. z in Π.*

Proof. (i) → (ii) Suppose that for all $h \in \mathbb{R}_+$ the equality $g(z) = g(hz)$ holds for a. e. $z \in \Pi$, that is,

$$\mu_\Pi(\Delta_h) = 0, \quad \text{where} \quad \Delta_h = \left\{(x,\theta) \in \mathbb{R}_+ \times (0,\pi) : g(xe^{i\theta}) \neq g(hxe^{i\theta})\right\}. \tag{2.4}$$

Define $\Phi : \mathbb{R}_+^2 \times (0,\pi) \to \mathbb{C}$ by

$$\Phi(x,y,\theta) = \begin{cases} 0, & \text{if } g(xe^{i\theta}) = g(ye^{i\theta}); \\ 1, & \text{if } g(xe^{i\theta}) \neq g(ye^{i\theta}), \end{cases}$$

and note that for all $h \in \mathbb{R}_+$

$$\{(x,\theta) \in \Pi : \Phi(x,hx,\theta) \neq 0\} = \left\{(x,\theta) \in \mathbb{R}_+ \times (0,\pi) : g(xe^{i\theta}) \neq g(hxe^{i\theta})\right\} = \Delta_h. \tag{2.5}$$

Accordingly, by (2.4) for all $h \in \mathbb{R}_+$ we get $\Phi(x,hx,\theta) = 0$ a. e. $(x,\theta) \in \Pi$, and by Tonelli's theorem

$$\begin{aligned}\int_{\mathbb{R}_+^2 \times (0,\pi)} \Phi(x,y,\theta) xy \, d\theta \, dx \, dy &\overset{y=hx}{=} \int_{\mathbb{R}_+^2 \times (0,\pi)} \Phi(x,hx,\theta) x^3 h d\theta \, dx \, dh \\ &= \int_{\mathbb{R}_+} h \left(\int_\Pi \Phi(x,hx,\theta) x^2 d\mu_\Pi(x,\theta) \right) dh = 0.\end{aligned}$$

Therefore, for almost $\theta \in (0,\pi)$

$$\begin{aligned}0 &= \mu_{\mathbb{R}_+^2}\left(\left\{ (x,y) \in \mathbb{R}_+^2 : g(xe^{i\theta}) \neq g(ye^{i\theta}) \right\} \right) = \int_{\mathbb{R}_+^2} \Phi(x,y,\theta) xy \, dx \, dy \\ &= \int_{\mathbb{R}^2} \Phi(e^t, e^u, \theta) e^{2t} e^{2u} \, dt \, du.\end{aligned}$$

It follows that

$$0 = \mu_{\mathbb{R}^2}\left(\left\{ (t,u) \in \mathbb{R}^2 : \Phi(e^t,e^u,\theta) \neq 0 \right\} \right) = \mu_{\mathbb{R}^2}\left(\left\{ (t,u) \in \mathbb{R}^2 : g \circ \exp(t+i\theta) \neq g \circ \exp(u+i\theta) \right\} \right)$$

a. e. $\theta \in (0,\pi)$. Now, by Lemma 2.1 there exists a constant $c(\theta)$ such that $g \circ \exp(t+i\theta) = c(\theta)$ for almost $t \in \mathbb{R}$, for this reason the bounded function $a : (0,\pi) \to \mathbb{C}$ given by

$$a(\theta) = \begin{cases} c(\theta), & \text{if } \mu_{\mathbb{R}^2}\left(\{(t,u) \in \mathbb{R}^2 : g \circ \exp(t+i\theta) \neq g \circ \exp(u+i\theta)\} \right) = 0; \\ 0, & \text{otherwise,} \end{cases}$$

satisfies the equality $g(z) = a(\arg z)$ for almost all $z \in \Pi$.

Conversely, let $g \in L_\infty(\Pi)$, if there exists $a \in L_\infty(0,\pi)$ such that $g(z) = a(\arg z)$ for almost every $z \in \Pi$, then for all $h \in \mathbb{R}_+$ we get $g(hz) = a(\arg(hz)) = a(\arg z) = g(z)$ for almost all $z \in \Pi$. ■

Denote by $\mathscr{A}_\infty$ the C*-algebra generated by all L_∞-functions which are angular on Π, and introduce the set $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ of all Toeplitz operators acting on $\mathscr{A}^2_\lambda(\Pi)$ with defining symbol in $\mathscr{A}_\infty$.

Proposition 2.2. *[50, Theorem 10.4.16] Given $g \in \mathscr{A}_\infty$, with $g(z) = a(\arg z)$, the Toeplitz operator T_a acting on $\mathscr{A}^2_\lambda(\Pi)$ is unitarily equivalent to the multiplication operator $\gamma_{a,\lambda}\mathrm{Id} = R_\lambda T_a R^*_\lambda$ acting on $L_2(\mathbb{R})$. The function $\gamma_{a,\lambda}(x)$ is*

$$\gamma_{a,\lambda}(x) = \frac{1}{c_\lambda(x)} \int_0^\pi a(\theta) e^{-2x\theta} \sin^\lambda \theta \, d\theta, \quad x \in \mathbb{R}, \tag{2.6}$$

and the operators R^*_λ and R_λ are given by (1.17) and (1.18), respectively.

Example 2.1. For $\lambda = 0$ and $a(\theta) = e^{2i\theta}$ we get that

$$\gamma_a(x) = \frac{2x}{1-e^{-2x\pi}} \int_0^\pi e^{i2\theta} e^{-2x\theta} \, d\theta = \frac{2x}{1-e^{-2x\pi}} \left. \frac{e^{2\theta(i-x)}}{2(i-x)} \right|_0^\pi = \frac{x}{i-x} = \frac{x}{1+x^2} + i\frac{x^2}{1+x^2}, \quad x \in \mathbb{R}.$$

Therefore, $\gamma_{\cos(2\cdot),0}(x) = \mathrm{Re}(\gamma_{a,0}(x)) = \frac{x}{1+x^2}$, and $\gamma_{\sin(2\cdot),0}(x) = \mathrm{Im}(\gamma_{a,0}(x)) = \frac{x^2}{1+x^2}$. In particular, $\|\gamma_a\|_\infty \leq 1$.

The following result provides a criterion for a Toeplitz operator to be angular.

Proposition 2.3. *Let $g \in L_\infty(\Pi)$. The Toeplitz operator T_g is angular if and only if g is angular.*

Proof. If T_g is angular, then for every $h > 0$ one gets that

$$T_g = D_{\lambda,h} T_g D_{\lambda,h^{-1}} = T_{g_h}, \quad \text{where } g_h(z) = g(hz)\,.$$

Thus, by Proposition 1.6 we obtain that $g(z) = g_h(z) = g(hz)$ almost every $z \in \Pi$. Therefore g is an angular function by Lemma 2.2.

Conversely, if g is an angular function on Π, then by Lemma 2.2 there is $a \in L_\infty(0,\pi)$ such that $g(z) = a(\arg z)$ a. e. $z \in \Pi$. Hence by Theorem 2.2 and the criterion of angular operators (Theorem 2.1) we conclude that the Toeplitz operator T_g is angular. ■

Denote by Γ_λ the set of all “ spectral functions"

$$\Gamma_\lambda = \left\{\gamma_{a,\lambda} : a \in L_\infty(0,\pi)\right\}. \tag{2.7}$$

Corollary 2.1. *The* C*-*algebra* $\mathscr{T}_\lambda(\mathscr{A}_\infty)$ *generated by all Toeplitz operators* T_a *with symbols* $a \in \mathscr{A}_\infty$ *is commutative and is isometrically isomorphic to the* C*-*algebra generated by* Γ_λ.

2.2 Very slowly oscillating property of the spectral functions

In this section we start with the definition of the class VSO($\mathbb{R}$), we also introduce a metric ζ_λ on $\mathbb{R}$ which is the most "natural" metric for the functions $\gamma_{a,\lambda}$(Proposition 2.5) and which is given by formulas (2.10) and (2.11). Unfortunately we could not find any simple and manageable expression for this metric. Thus, we need to find another metric which, first, is equivalent to the above one, and, second, admits as simple as possible expression, preferably in elementary functions. This is the way how the arcsinh metric appeared. It is uniformly equivalent to the first metric, i.e. they induce the same uniform structure. In Proposition 2.6 we prove the upper estimate only, as the proof of the lower estimate is more complicated (and the lower estimate holds only for small values of the metrics). In consequence, for every $a \in L_\infty(0,\pi)$ the corresponding spectral functions $\gamma_{a,\lambda}$ is Lipschitz continuous with respect to the arcsinh metric ρ, where $\gamma_{a,\lambda}$ is given in (2.6). We finish this chapter showing that the set of all spectral functions Γ_λ is dense in VSO($\mathbb{R}$). As was mentioned in the introduction the key idea of the proof is to approximate functions from VSO($\mathbb{R}$) by $\gamma^{v}_{a,\lambda}$ near $+\infty$ and $-\infty$. After that, the problem is reduced to the approximation of $C_0(\mathbb{R})$ functions by appropriate $\gamma_{a,\lambda}$; the latter problem is solved using the duality and the analyticity arguments (Theorem 2.3).

Very slowly oscillating functions on the real line

Definition 2.3. Let $f : \mathbb{R} \to \mathbb{C}$. The function $\Omega_{\rho,f} : [0,+\infty] \longrightarrow [0,+\infty]$ defined by

$$\Omega_{\rho,f}(\delta) := \sup\big\{|f(x) - f(y)| : \ x, y \in \mathbb{R}, \ \rho(x,y) \le \delta\big\}. \tag{2.8}$$

is called the *modulus of continuity* of f with respect to the arcsinh-metric ρ, see (5).

Definition 2.4 (very slowly oscillating functions). Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a bounded function. We say that f is *very slowly oscillating* if it is uniformly continuous with respect to ρ, i.e. if $\lim\limits_{\delta\to 0} \Omega_{\rho,f}(\delta) = 0$.

In other words, f is very slowly oscillating if and only if the composition $f \circ \sinh$ is uniformly continuous with respect the standard Euclidean metric on $\mathbb{R}$. The set of all such functions is denoted by $\mathrm{VSO}(\mathbb{R})$.

Example 2.2.
The function sin(arcsinh) belongs to $\mathrm{VSO}(\mathbb{R})$. In fact, since the *Sine* function is Lipschitz continuous with respect to the usual Euclidean metric on $\mathbb{R}$, we have for every $x, y \in \mathbb{R}$ that

$$|\sin(\operatorname{arcsinh} x) - \sin(\operatorname{arcsinh} y)| \leq \rho(x, y).$$

Also, by the same argument applied above, it is easy to see that the function cos(arcsinh) belongs to $\mathrm{VSO}(\mathbb{R})$.

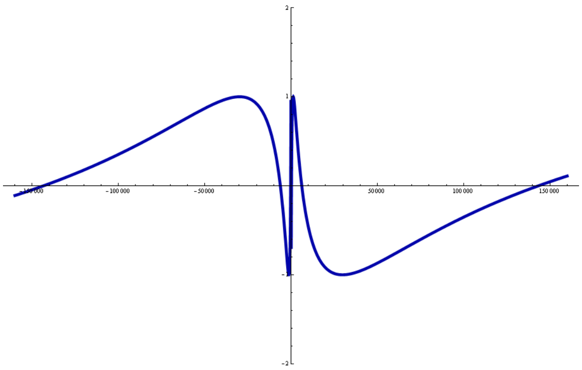


Figure 2.1: The graph shows the slow oscillation of the function sin(arcsinh)

Proposition 2.4. $\mathrm{VSO}(\mathbb{R})$ *is a closed* C**-subalgebra of* $C_{b,u}(\mathbb{R})$ *with pointwise operations.*

Proof. Using the following elementary properties of the modulus of continuity one can see that $\mathrm{VSO}(\mathbb{R})$ is closed with respect to the pointwise operations:

$$\Omega_{\rho,f+g} \leq \Omega_{\rho,f} + \Omega_{\rho,g}, \qquad \Omega_{\rho,\lambda f} = |\lambda|\Omega_{\rho,f},$$
$$\Omega_{\rho,fg} \leq \|g\|_\infty \Omega_{\rho,f} + \|f\|_\infty \Omega_{\rho,g}, \qquad \Omega_{\rho,\overline{f}} = \Omega_{\rho,f},$$

The inequality $\Omega_{\rho,f}(\delta) \leq 2\|f - g\|_\infty + \Omega_{\rho,g}(\delta)$ and the usual "$\frac{\varepsilon}{3}$-argument" show that the space $\mathrm{VSO}(\mathbb{R})$ is topologically closed in $C_{b,u}(\mathbb{R})$. ∎

VSO–property of the spectral functions

It is useful to write the spectral functions $\gamma_{a,\lambda}$ given in (2.6) as the values of the integral operator

$$\gamma_{a,\lambda}(x) = \int_0^\pi a(\theta) K_\lambda(x,\theta)\, d\theta,$$

where

$$K_\lambda(x,\theta) = \frac{e^{-2x\theta} \sin^\lambda \theta}{c_\lambda(x)}, \quad (x,\theta) \in \mathbb{R} \times (0,\pi). \tag{2.9}$$

Proposition 2.5. *Let* $\zeta_\lambda : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$ *be given by*

$$\zeta_\lambda(x,y) = \sup_{\substack{a\in L_\infty(0,\pi)\\ \|a\|_\infty=1}} |\gamma_{a,\lambda}(x)-\gamma_{a,\lambda}(y)|. \tag{2.10}$$

Then for every $x,y\in\mathbb{R}$

$$\zeta_\lambda(x,y) = \int_0^\pi |K_\lambda(x,\theta)-K_\lambda(y,\theta)|\,d\theta. \tag{2.11}$$

Proof. Let $x,y\in\mathbb{R}$. Then for every $a\in L_\infty(0,\pi)$ such that $\|a\|_\infty = 1$ we have

$$\left|\gamma_{a,\lambda}(x)-\gamma_{a,\lambda}(y)\right| = \left|\int_0^\pi a(\theta)[K_\lambda(x,\theta)-K_\lambda(y,\theta)]\,d\theta\right| \le \int_0^\pi |K_\lambda(x,\theta)-K_\lambda(y,\theta)|\,d\theta.$$

On the other hand, taking $b_0(\theta) = \operatorname{sign}(K_\lambda(x,\theta)-K_\lambda(y,\theta))$ we get that $b_0\in L_\infty(0,\pi)$ with $\|b_0\|_\infty = 1$ and

$$\zeta_\lambda(x,y) \ge \left|\gamma_{b_0}(x)-\gamma_{b_0}(y)\right| = \int_0^\pi |K_\lambda(x,\theta)-K_\lambda(y,\theta)|\,d\theta. \qquad \blacksquare$$

Let us mention some symmetry properties of K_λ and ζ_λ.

Lemma 2.3. *For every* $x\in\mathbb{R}$ *and every* $\theta\in(0,\pi)$*,*

$$K_\lambda(-x,\theta) = K_\lambda(x,\pi-\theta). \tag{2.12}$$

Proof. First we make the change of variables $\eta = \pi-\theta$ in the integral (1.9) defining c_λ:

$$c_\lambda(-x) = \int_0^\pi e^{2x\theta}\sin^\lambda\theta\,d\theta = e^{2\pi x}\int_0^\pi e^{-2x\eta}\sin^\lambda(\pi-\eta)\,d\eta = e^{2\pi x}c_\lambda(x). \tag{2.13}$$

Hence, given $x\in\mathbb{R}$ by (2.13) one gets for every $\theta\in(0,\pi)$ that

$$K_\lambda(-x,\theta) = \frac{e^{2x\theta}\sin^\lambda\theta}{e^{2x\pi}\,c_\lambda(x)} = \frac{e^{-2x(\pi-\theta)}\sin^\lambda\theta}{c_\lambda(x)} = \frac{e^{-2x(\pi-\theta)}\sin^\lambda(\pi-\theta)}{c_\lambda(x)} = K_\lambda(x,\pi-\theta). \qquad \blacksquare$$

Lemma 2.4. (i) $\zeta_\lambda(x,y) = \zeta_\lambda(-x,-y)$ *for every* $x,y\in\mathbb{R}$.

(ii) *If* $x\le 0\le y$*, then* $\zeta_\lambda(x,y)\le\zeta_\lambda(x,0)+\zeta_\lambda(0,y)$.

Proof. (i) Given $x,y\in\mathbb{R}$ by Lemma 2.3, we have

$$\zeta_\lambda(-x,-y) = \int_0^\pi |K_\lambda(-x,\theta)-K_\lambda(-y,\theta)|\,d\theta = \int_0^\pi |K_\lambda(x,\pi-\theta)-K_\lambda(y,\pi-\theta)|\,d\theta \overset{\beta=\pi-\theta}{=} \zeta_\lambda(x,y).$$

(ii) follows from the Triangle Inequality. $\blacksquare$

Lemma 2.5. *Let $a_0(\theta)=\theta$. The function $\kappa:[0,+\infty)\to(0,\infty)$, given by the formula*

$$\kappa(x)=2\gamma_{a_0,\lambda}(x)\sqrt{x^2+1}, \tag{2.14}$$

is continuous and bounded.

Proof. The function κ, being the product of two continuous functions, is obviously continuous. In [50] it is proved that $\gamma_{a,\lambda}(-x)+\gamma_{a,\lambda}(x)=\pi$ for every $a\in L_\infty(0,\pi)$ and every $x\in\mathbb{R}$. In particular, $\gamma_{a_0,\lambda}(0)=\pi/2$ and thus $\kappa(0)=\pi$. Since

$$\kappa(x)=-\frac{c'_\lambda(x)\sqrt{x^2+1}}{c_\lambda(x)},\qquad x\in\mathbb{R},$$

the asymptotical behavior of $\kappa(x)$ as $x\to+\infty$ follows from Lemma 1.2. By (1.12) with $p=0$ and $p=1$ we obtain:

$$c_\lambda(x)\sim\frac{\Gamma(\lambda+1)}{(2x)^{\lambda+1}},\qquad c'_\lambda(x)\sim-\frac{2\Gamma(\lambda+2)}{(2x)^{\lambda+2}},\qquad \text{as } x\to+\infty.$$

Thus,

$$\lim_{x\to+\infty}\kappa(x)=\lim_{x\to+\infty}\frac{2\Gamma(\lambda+2)(2x)^{\lambda+1}\sqrt{x^2+1}}{\Gamma(\lambda+1)(2x)^{\lambda+2}}=\lambda+1,$$

and κ is bounded. ■

Lemma 2.6. *If $a_0(\theta)=\theta$, then*

$$\int_0^\pi\left|\frac{\partial K_\lambda(x,\theta)}{\partial x}\right|d\theta\le 4\gamma_{a_0,\lambda}(x),\qquad x\in\mathbb{R}. \tag{2.15}$$

Proof. By (1.11) and (2.9) we have that

$$\begin{aligned}\frac{\partial K_\lambda(x,\theta)}{\partial x}&=-\frac{e^{-2x\theta}\sin^\lambda\theta}{c_\lambda^2(x)}\left[2c_\lambda(x)\theta+c'_\lambda(x)\right]=-K_\lambda(x,\theta)\left[2\theta-\frac{2\int_0^\pi\beta e^{-2x\beta}\sin^\lambda\beta\,d\beta}{c_\lambda(x)}\right]\\&=2K_\lambda(x,\theta)\left[\gamma_{a_0,\lambda}(x)-\theta\right].\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\pi\left|\frac{\partial K_\lambda(x,\theta)}{\partial x}\right|d\theta&\le 2\left(\gamma_{a_0,\lambda}(x)\int_0^\pi K_\lambda(x,\theta)d\theta+\int_0^\pi\theta K_\lambda(x,\theta)d\theta\right)\\&=2\left(\gamma_{a_0,\lambda}(x)+\gamma_{a_0,\lambda}(x)\right)=4\gamma_{a_0,\lambda}(x).\end{aligned}$$

■

Proposition 2.6. *There exists $C>0$ such that $\zeta_\lambda(x,y)\le C\rho(x,y)$ for every $x,y\in\mathbb{R}$.*

Proof. Due to Lemma 2.4, we only have to consider the case $y > x \geq 0$. By Cauchy's Mean-Value Theorem there exists $c \in (x, y)$ such that

$$\frac{\zeta_\lambda(x,y)}{\rho(x,y)} = \int_0^\pi \left| \frac{K_\lambda(x,\theta) - K_\lambda(y,\theta)}{\operatorname{arcsinh}(y) - \operatorname{arcsinh}(x)} \right| d\theta \leq 4\gamma_{a_0,\lambda}(c)\sqrt{c^2+1},$$

and the result yields by Lemma 2.5. ∎

Theorem 2.2. $\Gamma_\lambda \subsetneq \mathrm{VSO}(\mathbb{R})$.

Proof. Let $a \in L_\infty(0,\pi)$. The inequality $\|\gamma_{a,\lambda}\|_\infty \leq \|a\|_\infty$ shows that the function $\gamma_{a,\lambda}$ is bounded, and Proposition 2.6 implies that $\gamma_{a,\lambda}$ is Lipschitz continuous with respect to ρ:

$$|\gamma_{a,\lambda}(x) - \gamma_{a,\lambda}(y)| \leq \|a\|_\infty \zeta_\lambda(x,y) \leq C\,\|a\|_\infty \rho(x,y), \quad x, y \in \mathbb{R}.$$

Observe that the function $\eta(x) = \frac{x^{1/3}}{x^2+1}$ is uniformly continuous, but not Lipschitz on $\mathbb{R}$. Consequently, the composition $\eta \circ \operatorname{arcsinh}$ belongs to $\mathrm{VSO}(\mathbb{R})$, but it is not Lipschitz continuous with respect to ρ and therefore does not belong to Γ_λ. ∎

2.3 Density of Γ_λ in VSO($\mathbb{R}$)

In this section we show the uniform density of the set of all spectral functions Γ_λ in $\mathrm{VSO}(\mathbb{R})$. That is, we will prove that the closure of Γ_λ in the topology genereted by the uniform metric $d(f,g) = \|f - g\|_\infty$ coincides with $\mathrm{VSO}(\mathbb{R})$. To do that, we need the main result of the paper [30] and some technical lemmas.

Note that given $x, y > 0$, by Cauchy's mean value theorem the arcsinh-metric ρ given by (5) satisfies the inequality

$$\rho(x,y) \leq |\ln(x) - \ln(y)|\,. \tag{2.16}$$

Therefore, if $f \in \mathrm{VSO}(\mathbb{R})$, then $f|_{\mathbb{R}_+} \in \mathrm{VSO}(\mathbb{R}_+)$, where the class of functions $\mathrm{VSO}(\mathbb{R}_+)$ was defined in [31] and mentioned in Introduction. Furthermore, Herrera Yañez, Hutník, and Maximenko [30] have shown that for every $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and $\varepsilon > 0$ there exists $b \in L_\infty(\mathbb{R}_+)$ such that

$$\sup_{x \in \mathbb{R}_+} \left| \sigma(x) - \gamma^v_{b,\lambda}(x) \right| < \varepsilon,$$

where

$$\gamma^v_{b,\lambda}(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty b(t) e^{-2xt} t^\lambda \, dt, \quad x \in \mathbb{R}_+. \tag{2.17}$$

The above considerations lead to the following lemma.

Lemma 2.7. *Let $f \in \mathrm{VSO}(\mathbb{R})$. Given $\varepsilon > 0$ there exists $b \in L_\infty(\mathbb{R}_+)$ such that*

$$\sup_{x\in\mathbb{R}_+} \left| f(x) - \gamma^v_{b,\lambda}(x) \right| < \varepsilon. \tag{2.18}$$

Lemma 2.8 (approximation of γ by γ^v at $+\infty$). *If $b \in L_\infty(\mathbb{R}_+)$ and $a = \chi_{(0,\pi/2)} b$, then*

$$\lim_{x\to+\infty} \left| \gamma^v_{b,\lambda}(x) - \gamma_{a,\lambda}(x) \right| = 0. \tag{2.19}$$

Proof. Given $x \geq 0$, we get

$$\begin{aligned} \left| \gamma^v_{b,\lambda}(x) - \gamma_{a,\lambda}(x) \right| &\leq \|b\|_\infty \int_0^\infty \left| \chi_{(0,\pi/2)} K_\lambda(x,\theta) - \frac{(2x)^{\lambda+1} \theta^\lambda e^{-2x\theta}}{\Gamma(\lambda+1)} \right| d\theta \\ &\leq \|b\|_\infty (\mathrm{I}_1(x) + \mathrm{I}_2(x)), \end{aligned}$$

where

$$\mathrm{I}_1(x) = \int_0^{\pi/2} \left| K_\lambda(x,\theta) - \frac{(2x)^{\lambda+1} e^{-2x\theta} \sin^\lambda\theta}{\Gamma(\lambda+1)} \right| d\theta,$$

$$\mathrm{I}_2(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty e^{-2x\theta} \left| \theta^\lambda - \chi_{(0,\pi/2)} \sin^\lambda\theta \right| d\theta.$$

By (2.9) we have

$$\mathrm{I}_1(x) \leq \int_0^\pi \left| \frac{e^{-2x\theta} \sin^\lambda\theta}{c_\lambda(x)} - \frac{(2x)^{\lambda+1} e^{-2x\theta} \sin^\lambda\theta}{\Gamma(\lambda+1)} \right| d\theta = \left| 1 - \frac{(2x)^{\lambda+1} c_\lambda(x)}{\Gamma(\lambda+1)} \right|,$$

where c_λ is given in (1.9). By (1.12) with $p = 0$, we obtain $\lim_{x\to+\infty} \mathrm{I}_1(x) = 0$.

On the other hand, the integral I_2 can be written as

$$\mathrm{I}_2(x) = \frac{(2x)^{\lambda+1}}{\Gamma(\lambda+1)} \left(\int_0^{\pi/2} e^{-2x\theta} \theta^\lambda \left| \left(\frac{\sin\theta}{\theta} \right)^\lambda - 1 \right| d\theta \right) + \frac{\Gamma(\lambda+1, x\pi)}{\Gamma(\lambda+1)},$$

where $\Gamma(\alpha, x)$ is the incomplete Gamma function. We see for every $\theta \in (0, \pi/2)$ that the function

$$\left| \left(\frac{\sin\theta}{\theta} \right)^\lambda - 1 \right| = \begin{cases} \left(\frac{\sin\theta}{\theta}\right)^\lambda - 1 & \text{if } \lambda \geq 0; \\ 1 - \left(\frac{\sin\theta}{\theta}\right)^\lambda & \text{if } -1 < \lambda \leq 0 \end{cases}$$

is infinitely smooth near 0 and vanishes in 0. Then by Watson's Lemma (Proposition A.1) and by definition of $\Gamma(\alpha, x)$ we get $\lim\limits_{x\to+\infty} \mathrm{I}_2(x) = 0$, which yields (2.19). ∎

The above lemmas permit us to show that each $\sigma \in \mathrm{VSO}(\mathbb{R})$ can be approximated by functions from the class Γ_λ for large values of $|x|$.

Proposition 2.7. *Let $\sigma \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$. Then there exist a generating symbol $a \in L_\infty(0,\pi)$ and a number $L > 0$ such that*

$$\sup_{|x|\geq L} \left|\sigma(x) - \gamma_{a,\lambda}(x)\right| \leq \varepsilon. \tag{2.20}$$

Proof. Given $\sigma \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$ there exist $b \in L_\infty(\mathbb{R}_+)$ such that (2.18) holds. By Lemma 2.8 there exist $c \in L_\infty(0,\pi)$ with $c(\theta) = 0$ for each $\theta \in [\pi/2,\pi)$, and $L_1 > 0$ such that

$$\sup_{x\geq L_1} \left|\sigma(x) - \gamma_{c,\lambda}(x)\right| \leq \sup_{x\geq L_1} \left(\left|\sigma(x) - \gamma^v_{b,\lambda}(x)\right| + \left|\gamma^v_{b,\lambda}(x) - \gamma_{c,\lambda}(x)\right|\right) \leq \varepsilon. \tag{2.21}$$

For large negative values of x, we consider the function $x \mapsto \sigma(-x)$ that also belongs to $\mathrm{VSO}(\mathbb{R})$. Applying the previous arguments to this function we find a function $g \in L_\infty(0,\pi)$ and a number $L_2 > 0$ such that g vanishes in $[\pi/2,\pi)$ and

$$\sup_{x\geq L_2} |\sigma(-x) - \gamma_{g,\lambda}(x)| \leq \varepsilon. \tag{2.22}$$

Now define $d \in L_\infty(0,\pi)$ by $d(\theta) = g(\pi-\theta)$. Then d vanishes on $(0,\pi/2]$, and the identity $\gamma_{d,\lambda}(x) = \gamma_{g,\lambda}(-x)$ holds. Hence (2.22) can be rewritten as

$$\sup_{x\leq -L_2} |\sigma(x) - \gamma_{d,\lambda}(x)| \leq \varepsilon. \tag{2.23}$$

Since c vanishes near π and d vanishes near 0, the corresponding spectral functions fulfill the limit relations $\gamma_{c,\lambda}(-\infty) = 0$ and $\gamma_{d,\lambda}(+\infty) = 0$ (see [50, Chapter 14]), and there are constants $L_3, L_4 > 0$ such that

$$\sup_{x\leq -L_3} |\gamma_{c,\lambda}(x)| \leq \frac{\varepsilon}{2}, \qquad \sup_{x\geq L_4} |\gamma_{d,\lambda}(x)| \leq \frac{\varepsilon}{2}.$$

Taking $a = d + c \in L_\infty(0,\pi)$ and $L = \max\{L_1, L_2, L_3, L_4\}$, we get (2.19). ■

Now we are going to show that the continuous functions on $\mathbb{R}$ vanishing at the infinity can be approximated by spectral functions. To do that, we need some technical lemmas.

Lemma 2.9. *Let $(X,\mathscr{A})$ be a measurable space, $\mu\colon \mathscr{A} \to \mathbb{C}$ be a complex measure, D be a domain in $\mathbb{C}$ and $K\colon D \times X \to \mathbb{C}$ be a function such that for every $\omega \in X$ the function $z \mapsto K(z,\omega)$ is analytic, and for every compact C in D*

$$\sup_{z\in C} \int_X |K(z,\omega)|\, d|\mu|(\omega) < +\infty. \tag{2.24}$$

Then the function $g: D \to \mathbb{C}$,

$$g(z) = \int_X K(z,\omega)\, d\mu(\omega),$$

is analytic on D.

Proof. Since the function $z \mapsto K(z,w)$ is analytic, we get for any triangle Δ in D that

$$\int_{\partial\Delta} K(z,w)dz = 0, \quad w \in X. \tag{2.25}$$

Now, by (2.24) we can interchange the integrals by Fubini's theorem, thus

$$\int_{\partial\Delta} g(z)dz = \int_{\partial\Delta}\int_X K(z,w)d\mu(w)dz = \int_X\int_{\partial\Delta} K(z,w)dz d\mu(w) = 0.$$

Therefore by Morera's theorem g is an analytic function on D. ∎

From now on, we write K_λ as

$$K_\lambda(x,\theta) = F_\lambda(x,\theta)\sin^\lambda\theta,$$

with

$$F_\lambda(x,\theta) = \frac{e^{-2x\theta}}{c_\lambda(x)}, \quad (x,\theta) \in \mathbb{R}\times(0,\pi). \tag{2.26}$$

Lemma 2.10. *Let* $p \in \mathbb{Z}_+$, *and let* ν *be a finite regular Borel complex measure on* $\mathbb{R}$. *If* $\delta \in (0,\pi/2)$, *then*

$$\sup_{\delta\le\alpha\le\pi-\delta} \int_{\mathbb{R}} \left|x^p F_\lambda(x,\alpha)\right| d|\nu|(x) < +\infty. \tag{2.27}$$

Proof. By (1.12) the function $F_\lambda : \mathbb{R}\times(0,\pi) \to (0,+\infty)$ given in (2.26) has an asymptotic behavior

$$F_\lambda(x,\theta) \sim \frac{(2x)^{\lambda+1}e^{-2x\theta}}{\Gamma(\lambda+1)}, \quad \text{as}\, x \to +\infty. \tag{2.28}$$

Hence, given $p \in \{0,1,2,\dots,\}$, $\alpha \in [\delta,\pi-\delta]$ with $\delta \in (0,\pi/2)$ and $x \in [0,+\infty)$, we get

$$\left|x^p F_\lambda(x,\alpha)\right| \sim \frac{(2x)^{\lambda+1+p}e^{-2x\alpha}}{2^p\Gamma(\lambda+1)} \le \frac{(2x)^{\lambda+1+p}e^{-2x\delta}}{2^p\Gamma(\lambda+1)} \le M_{\lambda,p,\delta} \quad \text{as}\, x \to +\infty.$$

However, from Lemma 2.3 we have $F_\lambda(x,\alpha) = F_\lambda(-x,\beta)$, where $\beta = \pi-\alpha \in [\delta,\pi-\delta]$. Therefore $F_\lambda(x,\alpha)$ is bounded for all $\alpha \in [\delta,\pi-\delta]$. ∎

Proposition 2.8 (Leibniz integral rule for differentiation under the integral sign: complex case). *Let X be an open subset of $\mathbb{R}$, $(\Omega, \mathscr{A})$ be a measurable space and $\mu\colon \mathscr{A} \to \mathbb{C}$ be a complex measure. Suppose $f : X \times \Omega \to \mathbb{R}$ satisfies the following conditions:*

(i) *For every $x \in X$, the function $\omega \mapsto f(x,\omega)$ is $|\mu|$-integrable.*

(ii) *For almost all ω in Ω, the derivative f_x exists for all x in X.*

(iii) *There is a $|\mu|$-integrable function $\theta\colon \Omega \to \mathbb{R}$ such that $|f_x(x,\omega)| \leq \theta(\omega)$ for all $x \in X$.*

Then for all x in X

$$\frac{d}{dx}\int_\Omega f(x,\omega)\,d\mu(\omega) = \int_\Omega f_x(x,\omega)\,d\mu(\omega).$$

Proof. The Leibniz's rule is well known in the case of a non-negative measure, but every complex measure μ can be written as a linear combination of four non-negative measures $\mu_1, \mu_2, \mu_3, \mu_4$, with $\mu_j \leq |\mu|$. The conditions (i) and (iii) justify the application of the Leibniz's rule for each one of the measures μ_j. ∎

Lemma 2.11. *Let ν be a regular complex Borel measure of finite total variation on $\mathbb{R}$. Define a function $\psi_\lambda\colon \mathbb{R} \to \mathbb{C}$ by*

$$\psi_\lambda(x) = F_\lambda(x, \pi/2), \qquad x \in \mathbb{R}. \tag{2.29}$$

Denote by Δ the domain $\Delta = \{w \in \mathbb{C} : \ |\mathrm{Im}\, w| < \pi\}$ and define $\Phi_\lambda\colon \Delta \to \mathbb{C}$ by

$$\Phi_\lambda(w) = \int_\mathbb{R} e^{-ixw}\psi_\lambda(x)\,d\nu(x). \tag{2.30}$$

Then Φ_λ is analytic on Δ and for every $p \in \{0,1,2,\ldots\}$

$$\Phi_\lambda^{(p)}(0) = (-i)^p \int_\mathbb{R} x^p \psi_\lambda(x)\,d\nu(x). \tag{2.31}$$

Proof. By Lemma 2.3, ψ_λ is an even function: $\psi_\lambda(-x) = \psi_\lambda(x)$.

Every compact subset of Δ is contained in a strip of the form $\mathbb{R} + i[-L, L]$, where $0 < L < \pi$. For every $w \in \mathbb{C}$ with $|\mathrm{Im}\, w| \leq L$ and every $x \in \mathbb{R}$,

$$|e^{-ixw}\psi_\lambda(x)| = e^{x\,\mathrm{Im}(w)}\psi_\lambda(|x|) \leq e^{|x|L}\psi_\lambda(|x|) = \frac{1}{e^{|x|(\pi-L)}c_\lambda(|x|)}.$$

The condition $\pi - L > 0$ and Lemma 1.2 guarantee that the latter expression defines a bounded function on $\mathbb{R}$.

Since the complex measure ν has a finite total variation, Lemma 2.9 assures that Φ_λ is analytic in Δ. Thus, by (2.27) and the Leibniz's rule (Proposition 2.8) we get

$$\Phi_\lambda^p(z) = \frac{d^p}{dz^p}\left(\int_\mathbb{R} \psi_\lambda(x)e^{-ixz}\,d\nu(x)\right) = (-i)^p \int_\mathbb{R} x^p \psi_\lambda(x) e^{-ixz}\,d\nu(x).$$ ∎

Denote by $\mathscr{Y}$ the Banach subspace of $L_\infty(0,\pi)$ consisting of all bounded functions a having limits 0 at 0 and π.

Lemma 2.12. *If $a \in \mathscr{Y}$, then $\gamma_{a,\lambda} \in C_0(\mathbb{R})$ and*

$$\lim_{x\to+\infty} \gamma_{a,\lambda}(x) = \lim_{\theta\to 0} a(\theta) = 0, \quad \textit{and} \quad \lim_{x\to-\infty} \gamma_{a,\lambda}(x) = \lim_{\theta\to\pi} a(\theta) = 0.$$

Proof. The proof of this fact follows easily from Lemma 7.2.3. of [50] ∎

Theorem 2.3. *The set of functions*

$$\Gamma^\lambda_{(0,\pi)} = \{\gamma_{a,\lambda} : a \in \mathscr{Y}\} \tag{2.32}$$

is dense in $C_0(\mathbb{R})$.

Proof. First we note that if $a \in \mathscr{Y}$, then $\gamma_{a,\lambda} \in C_0(\mathbb{R})$ by Lemma 2.12. Thus $\Gamma^\lambda_{(0,\pi)} \subseteq C_0(\mathbb{R})$. By Hahn–Banach theorem, the density of $\Gamma^\lambda_{(0,\pi)}$ in $C_0(0,\pi)$ will be shown if we prove that any continuous linear functional φ on $C_0(0,\pi)$ that vanishes on $\Gamma^\lambda_{(0,\pi)}$ is the zero functional. Thus, let $\varphi \in C_0(\mathbb{R})^*$ be a linear functional such that $\varphi(\gamma_{a,\lambda}) = 0$ for each $a \in L_\infty(0,\pi)$. By Riesz-Markov representation theorem, there is a regular complex Borel measure ν of finite total variation on $\mathbb{R}$ such that

$$0 = \varphi(\gamma_{a,\lambda}) = \int_{\mathbb{R}} \gamma_{a,\lambda}(x)\, d\nu(x), \quad a \in L_\infty(0,\pi).$$

In particular, if $a_0 = \chi_{[\beta,\theta]} \in \mathscr{Y}$ with $0 < \beta < \theta < \pi$, then by

$$\int_{\mathbb{R}} \int_0^\pi |a_0(\theta) K_\lambda(x,\theta)|\, d\theta\, d|\nu|(x) \le \int_{\mathbb{R}} \int_0^\pi K_\lambda(x,\theta)\, d\theta\, d|\nu|(x) = |\nu|(\mathbb{R}),$$

we can apply Fubini's theorem and get

$$\int_{\mathbb{R}} \gamma_{a_0,\lambda}(x)\, d\nu(x) = \int_{\mathbb{R}} \int_\beta^\theta K_\lambda(x,\alpha)\, d\alpha\, d\nu(x) = \int_\beta^\theta \int_{\mathbb{R}} K_\lambda(x,\alpha)\, d\nu(x)\, d\alpha = 0.$$

The function $\alpha \to \int_{\mathbb{R}} K_\lambda(x,\alpha)\, d\nu(x)$ is continuous (in fact, it is differentiable, see below), therefore by the first fundamental theorem of calculus we obtain that for every θ in $(0,\pi)$

$$\int_{\mathbb{R}} K_\lambda(x,\theta)\, d\nu(x) = 0.$$

Since $K_\lambda(x,\theta) = \digamma_\lambda(x,\theta) \sin^\lambda \theta$ and $\sin\theta > 0$, this is equivalent to

$$\int_{\mathbb{R}} \digamma_\lambda(x,\theta)\, d\nu(x) = 0. \tag{2.33}$$

By Lemma 2.10 and Leibniz's rule, the function in the left-hand side of (2.33) is differentiable, and the derivation with respect to θ commutes with the integral sign. Derivating (2.33) with respect to θ we obtain for every θ in $(0,\pi)$ and every p in $\{0,1,2,\ldots\}$

$$\int_{\mathbb{R}} x^p F_\lambda(x,\theta)\,d\nu(x) = 0. \tag{2.34}$$

Putting $\theta = \pi/2$ in (2.34) we obtain for every p in $\{0,1,2,\ldots\}$:

$$\int_{\mathbb{R}} x^p\,\psi_\lambda(x)\,d\nu(x) = 0, \tag{2.35}$$

where $\psi_\lambda\colon \mathbb{R} \to \mathbb{C}$ is given by $\psi_\lambda(x) = F_\lambda(x,\pi/2)$. Denote by Φ_λ the Fourier transform of the measure $\psi_\lambda\,d\nu$:

$$\Phi_\lambda(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\psi_\lambda(x)\,d\nu(x).$$

Lemma 2.11 shows that the function Φ_λ is analytic on a domain containing $\mathbb{R}$, and (2.35) means that $\Phi_\lambda^{(p)}(0) = 0$ for every $p \in \{0,1,2,\ldots\}$. Therefore $\Phi_\lambda = 0$. By the injective property of the Fourier transform of Borel measures (Proposition A.2), we conclude that $\nu = 0$ and hence $\varphi = 0$. That implies the density of $\Gamma^\lambda_{(0,\pi)}$ in $C_0(\mathbb{R})$. ∎

Proposition 2.7 and Theorem 2.3 imply together the main result on density.

Theorem 2.4. *The set Γ_λ is dense in* $\mathrm{VSO}(\mathbb{R})$.

Proof. Let $f \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$. Our aim is to find a function c in $L_\infty(0,\pi)$ such that $\|f - \gamma_{c,\lambda}\|_\infty \le \varepsilon$. First, using Lemma 2.7 we find a function $a \in L_\infty(0,\pi)$ and a number $L > 0$ such that $\sup_{|x|\ge L} \left|f(x) - \gamma_{a,\lambda}(x)\right| \le \frac{\varepsilon}{2}$. In general, the function $f - \gamma_{a,\lambda}$ may not belong to the class $C_0(\mathbb{R})$, and we will slightly modify it. Let $g\colon \mathbb{R} \to [0,1]$ be a continuous function such that $g(x) = 1$ for each $x \in [-2L,2L]$ and $g(x) = 0$ for each $x \in \mathbb{R}\setminus[-2L,2L]$. Define $h \in C_0(\mathbb{R})$ by

$$h(x) = (f-\gamma_{a,\lambda})(x)g(x) = \begin{cases} f(x) - \gamma_{a,\lambda}(x), & \text{if } |x| \le L; \\ (f(x)-\gamma_{a,\lambda}(x))g(x), & \text{if } L < |x| \le 2L; \\ 0, & \text{if } |x| > 2L. \end{cases}$$

Second, applying Theorem 2.3 we choose $b \in L_\infty(0,\pi)$ such that $\|h - \gamma_{b,\lambda}\|_\infty \le \varepsilon/2$. Now define $c \in L_\infty(0,\pi)$ by $c = a + b$. Then for every x in $[-L,L]$ we obtain

$$|f(x) - \gamma_{c,\lambda}(x)| = |f(x) - \gamma_{a,\lambda}(x) - \gamma_{b,\lambda}(x)| = |h(x) - \gamma_{b,\lambda}(x)| \le \varepsilon/2,$$

and for every x in $\mathbb{R}\setminus[-L,L]$

$$\begin{aligned}|f(x)-\gamma_{c,\lambda}(x)| &= |(f(x)-\gamma_{a,\lambda}(x))(1-g(x))+(f(x)-\gamma_{a,\lambda}(x))g(x)-\gamma_{b,\lambda}(x)| \\ &\le |f(x)-\gamma_{a,\lambda}(x)|(1-g(x))+|h(x)-\gamma_{b,\lambda}(x)|\le\varepsilon.\end{aligned}$$

Therefore $\|f-\gamma_{c,\lambda}\|_\infty\le\varepsilon$. ■

Corollary 2.2. *The C^*-algebra generated by Γ_λ coincides with* VSO($\mathbb{R}$)*, and the C^*-algebra $\mathcal{T}_\lambda(\mathcal{A}_\infty)$ generated by angular Toeplitz operators is isometrically isomorphic to* VSO($\mathbb{R}$).

Recall that the hyperbolic metric in the upper half-plane Π is given by

$$\eta(z_1,z_2)=\ln\frac{|z_1-\overline{z_2}|+|z_1-z_2|}{|z_1-\overline{z_2}|-|z_1-z_2|},\quad z_1,z_2\in\Pi.$$

Now, if we restrict it to the upper half-circle $\{z=e^{i\theta},\theta\in(0,\pi)\}$, then this is given by the formula

$$\eta(e^{i\theta_1},e^{i\theta_2})=|\ln\tan(\theta_1/2)-\ln\tan(\theta_2/2)|,\quad \theta_1,\theta_2\in(0,\pi).$$

The change of variable $q=\ln\tan(\theta/2)=-\operatorname{arcsinh}(\cot\theta)$ inspires the following example.

Example 2.3. We present an example of a function $\gamma_{a,\lambda}$ that has a typical "very slow oscillation" at $\pm\infty$. Consider the generating symbol

$$a(\theta)=\cos(\ln(\tan(\theta/2))).$$

Then $a(\pi-\theta)=a(\theta)$ and $\gamma_{a,\lambda}(-x)=\gamma_{a,\lambda}(x)$. Watson's lemma implies that the asymptotical behavior of $\gamma_{a,\lambda}(x)$ as $x\to+\infty$ is determined by the behaviour of a near the point 0, and $\tan(\theta/2)\sim\theta/2$ as $\theta\to0$. Using arguments similar to those in the proof of Lemma 2.8 we see that as $x\to+\infty$,

$$\gamma_{a,\lambda}(x)=\frac{1}{c_\lambda(x)}\int_0^{+\infty}\theta^\lambda e^{-2x\theta}\cos\left(\ln\frac{\theta}{2}\right)d\theta+o(1)=\frac{\operatorname{Re}((2x)^i\,\Gamma(1-i+\lambda))}{\Gamma(\lambda+1)}+o(1).$$

With the change of variables $x=\sinh(u)$ and applying the limit relation $|\sinh(u)|\sim\exp(|u|)/2$ we obtain that:

$$\gamma_{a,\lambda}(\sinh(u))=\frac{|\Gamma(1-i+\lambda)|}{\Gamma(\lambda+1)}\cos(|u|+\ln2+\arg\Gamma(1-i+\lambda))+o(1),$$

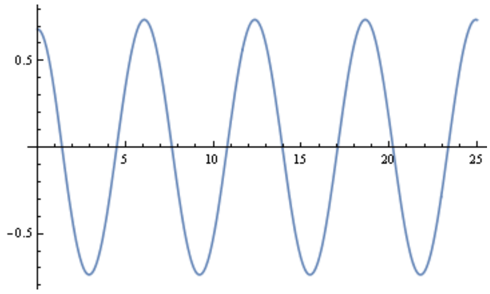
as $u\to\pm\infty$.

Numerical Proof using Wolfram Mathematica: the following numerical experiment shows the approximation illustrated in the above example. The first graph represents the function $\gamma_{a,\lambda}(\sinh x)$ with $a(t) = \cos(\ln(\tan(t/2)))$. The second one represents the function given by the rule

$$\text{gamSinhApprox}(x,\lambda) = \frac{1}{c_\lambda(x)} \int_0^\infty \theta^\lambda e^{-2x\theta} \cos(\ln(\theta/2))d\theta,$$

for $\lambda = 1$ and the last one represents the difference between these functions.

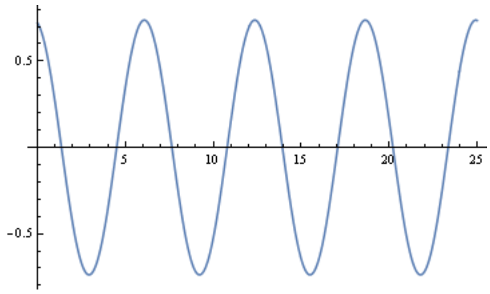
```
c[x_,la_]:=Pi * Gamma[la + 1] * Exp[−Pi * x]/(2^la * Abs[Gamma[1 + I * x + la/2]]^2)
gamExact[x_,la_]:=
NIntegrate[Exp[−2 * x * t] * (Sin[t]^la) * Cos[Log[Tan[t/2]]], {t, 0, Pi}, WorkingPrecision → 60,
MaxRecursion → 40, PrecisionGoal → 6]/c[x, la]
gamSinhTable[xmin_, xmax_, npoints_, la_]:=
Module[{xs, ys, j}, xs = Range[0, npoints − 1] * (xmax − xmin)/(npoints − 1);
Monitor[ys = Table[gamExact[Sinh[xs[[j]]], la], {j, 1, npoints}], j];
Transpose[{xs, ys}]]
tab = gamSinhTable[0, 25, 101, 1];
gamSinhInterpol = Interpolation[tab];
Plot[gamSinhInterpol[x], {x, 0, 25}]
```



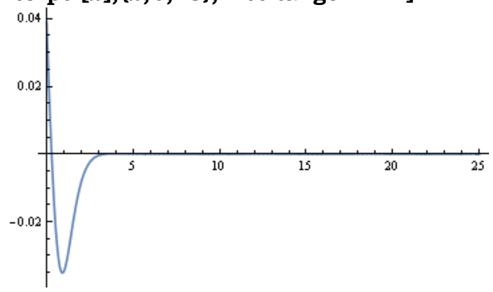

```
Expand[Integrate[Exp[−2 * x * v] * v^la * Cos[Log[v/2]], {v, 0, Infinity}, Assumptions → {x > 0, la > −1}]]
```

$2^{(-2+2i)-\text{la}}x^{(-1+i)-\text{la}}\text{Gamma}[(1-i)+\text{la}] + 2^{(-2-2i)-\text{la}}x^{(-1-i)-\text{la}}\text{Gamma}[(1+i)+\text{la}]$

```
(*multiplyby(2x)^(la + 1)andcomposewithExp(u)/2*)
gamSinhApprox[u_,la_]:=Abs[Gamma[1 + la − I]] * Cos[u + Log[2] + Arg[Gamma[1 + la − I]]]/Gamma[la + 1]
Plot[gamSinhApprox[u, 1], {u, 0, 25}]
```




```
Plot[gamSinhApprox[x, 1] − gamSinhInterpol[x], {x, 0, 25}, PlotRange → All]
```



2.4 Strong density of Toeplitz operators in the C*-algebra of angular operators

Let $\mathscr{H}$ be a Hilbert space. It is well known that the space of all bounded operators $\mathscr{B}(\mathscr{H})$ has various topologies. For example, the uniform operator topology , the strong operator topology and the weak operator topology (see Section A.4). In the particular case of $\mathscr{H} = \mathscr{A}^2_\lambda(\Pi)$ (or $\mathscr{H} = \mathscr{F}^2(\mathbb{C}^n)$) we can characterize $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$ ($\mathscr{B}(\mathscr{F}^2(\mathbb{C}^n))$) by means Toeplitz operators in the strong operator topology. The next proposition states that the set of all Toeplitz operators with L_∞-symbols is dense in $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$ with respect to the strong operator topology, this result was proved for Toeplitz operators acting on the Bergman spaces over the unit disk by Miroslav Engliš, see [16].

Proposition 2.9. *Let $\mathscr{H}L_2(\mathscr{O}) = \mathscr{F}^2(\mathbb{C}^n)$ or $\mathscr{A}^2_\lambda(\Pi)$. Then the closure of $\{T_f : f \in L_\infty(\mathscr{O})\}$ in the strong operator topology coincides with $\mathscr{B}(\mathscr{H}L_2(\mathscr{O}))$, where $\mathscr{O} = \mathbb{C}^n$ or Π.*

Huang [35] proved that if $T \in \mathscr{B}(L_2(\mathbb{R}))$ commutes with the multiplication operator M_φ, where φ is a bounded strictly increasing (or decreasing) function on $\mathbb{R}$, then $T = M_\psi$, for some $\psi \in L_\infty(\mathbb{R})$. Now, since each angular Toeplitz operator T_a is unitarily equivalent to the multiplication operator M_{γ_a}, the above result of Huang implies that the von Neumann algebra $W^*(\mathrm{T}_\lambda(\mathscr{A}_\infty))$ generated by $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ is maximal. In fact, $W^*(\mathrm{T}_\lambda(\mathscr{A}_\infty))$ is the closure of $\mathscr{T}_\lambda(\mathscr{A}_\infty)$ with respect to the strong operator topology (SOT) in $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$.

The space $L_\infty(\mathbb{R})$ may be identified with the dual space of $L_1(\mathbb{R})$. We denote by $\mathscr{W}$ the corresponding weak-* topology on $L_\infty(\mathbb{R})$. Since by Proposition A.7 the space $C_0(\mathbb{R})$ is dense in $(L_\infty(\mathbb{R}), \mathscr{W})$, the main result about density complement the Huang's result providing an explicit description of the SOT-closure of $\mathrm{T}_\lambda(\mathscr{A}_\infty)$. To be more precise, in this section we will show that the closure of $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ coincides with $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$ in the strong operator topology.

Lemma 2.13. *The closure of $\{M_{\gamma_{a,\lambda}} : a \in L_\infty(0,\pi)\}$ in the weak operator topology coincides with $\{M_f : f \in L_\infty(\mathbb{R})\}$*

Proof. Let $M_\varphi \in \{M_f : f \in L_\infty(\mathbb{R})\}$. By Lemma A.7 for each $h \in L_1(\mathbb{R})$ and each $\varepsilon > 0$ there exists $\psi \in C_0(\mathbb{R})$ such that

$$\left|\phi_\psi(h) - \phi_\varphi(h)\right| \le \frac{\varepsilon}{2}. \tag{2.36}$$

On the other hand, by Theorem 2.3, given $\varepsilon > 0$ there exists $a \in \mathscr{Y}$ such that

$$\|\gamma_{a,\lambda} - \psi\|_\infty \le \frac{\varepsilon}{2\|h\|_1}. \tag{2.37}$$

Therefore, by (2.36) and (2.37) one gets for all $h \in L_1(\mathbb{R})$ that

$$\begin{aligned}\left|\phi_{\gamma_{a,\lambda}}(h)-\phi_{\varphi}(h)\right| &\leq \left|\phi_{\gamma_{a,\lambda}}(h)-\phi_{\psi}(h)\right|+\left|\phi_{\psi}(h)-\phi_{\varphi}(h)\right| \leq \frac{\varepsilon}{2}+\|\gamma_{a,\lambda}-\psi\|_{\infty}\|h\|_1\\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Now, for any $f,g \in L_2(\mathbb{R})$ we have

$$\left|\langle (M_{\gamma_{a,\lambda}}-M_{\varphi})f,g\rangle\right| = \left|\int_{\mathbb{R}}(\gamma_{a,\lambda}-\varphi)(x)f(x)\overline{g(x)}dx\right| = |\phi_{\gamma_{a,\lambda}}(h)-\phi_{\varphi}(h)| \leq \varepsilon,$$

where $h=f\overline{g}\in L_1(\mathbb{R})$. This implies that

$$\left\{M_f : f\in L_\infty(\mathbb{R})\right\} = \text{WOT-closure}\left(\{M_{\gamma_{a,\lambda}} : a \in \mathscr{Y}\}\right).$$

Now, the proof holds by the following relation

$$\left\{M_{\gamma_{a,\lambda}} : a\in\mathscr{Y}\right\} \subset \left\{M_{\gamma_{a,\lambda}} : a\in L_\infty(0,\pi)\right\} \subset \left\{M_f : f\in L_\infty(\mathbb{R})\right\}. \quad \blacksquare$$

The next proposition states the density of $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ in the C^*-algebra $\mathfrak{A}_\lambda$ with respect to the strong operator topology in $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$.

Proposition 2.10. SOT-closure$(\mathrm{T}_\lambda(\mathscr{A}_\infty)) = \mathfrak{A}_\lambda$.

Proof. By criterion of angular operators (Theorem 2.1) given $V \in \mathfrak{A}_\lambda$ there exists $\varphi \in L_\infty(\mathbb{R})$ such that $V = R^*_\lambda M_\varphi R_\lambda$. If T_a is an angular Toeplitz operator acting on $\mathscr{A}^2_\lambda(\Pi)$ one gets for each $F,G \in \mathscr{A}^2_\lambda(\Pi)$ that

$$\langle (T_a - V)F, G\rangle = \langle (M_{\gamma_{a,\lambda}} - M_\varphi)R_\lambda F, R_\lambda G\rangle, \tag{2.38}$$

Now, due to a net in $(L_\infty(\mathbb{R}),\mathscr{W})$ converges if and only if its respective multiplication operator in $\mathscr{B}(L_2(\mathbb{R}))$ converges in the weak operator topology (Proposition A.5) we conclude by (2.38) and Lemma 2.13 that WOT-closure$(\mathscr{T}_\lambda(\mathscr{A}_\infty)) = \mathfrak{A}_\lambda$. Furthermore, note that $\mathrm{T}_\lambda(\mathscr{A}_\infty)$ is a convex subset of $\mathscr{B}(\mathscr{A}^2_\lambda(\Pi))$, thus the SOT-closure and the WOT-closure coincide by Theorem A.2. $\blacksquare$

CHAPTER 3

C*-ALGEBRA OF RADIAL TOEPLITZ OPERATORS ACTING ON FOCK SPACES

In this chapter we study the radial Toeplitz operators acting on the Fock space $\mathscr{F}^2(\mathbb{C})$. The principal theorem shows that the C*-algebra generated by radial Toeplitz operators with L_∞-symbols acting on the Fock space is isometrically isomorphic to the C*-algebra $\mathrm{RO}(\mathbb{Z}_+)$ of bounded sequences uniformly continuous with respect to the square-root-metric $\varrho(j,k) = |\sqrt{j} - \sqrt{k}\,|$. More precisely, we prove that the sequences of eigenvalues of radial Toeplitz operators form a dense subset of the latter C*-algebra of sequences.

3.1 Radial Toeplitz operators acting on Fock spaces

In this section we compile some basic facts on Toeplitz operators with radial symbols from $L_\infty(\mathbb{C})$ acting on the Fock space $\mathscr{F}^2(\mathbb{C})$. Essentially we repeat for the Fock space the facts stated by Zorboska [52] for the Bergman space over the unit disk, adding some ideas from [28]. The results of this chaptert can be generalized to radial Toeplitz operators on the multi-dimensional Fock space $\mathscr{F}^2(\mathbb{C}^n, (\varsigma/\pi)^n e^{-\varsigma|z|^2} dv_n(z))$; in this case the eigenvalue associated to the element e_β of the canonical basis depends only on the length of the multi-index β as in [27].

Let $t \in \mathbb{R}$, and $U_t \colon \mathscr{F}^2(\mathbb{C}) \to \mathscr{F}^2(\mathbb{C})$ be the unitary operator given by the composition of functions with the rotation by the angle t around the origin in the negative direction:

$$(U_t f)(z) = f(e^{-it} z), \quad z \in \mathbb{C}. \tag{3.1}$$

For $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$ we denote by $\mathrm{Rad}(S)$ the *radialization* of S defined by

$$\mathrm{Rad}(S) = \frac{1}{2\pi}\int_0^{2\pi} U_{-t} S U_t \, dt, \tag{3.2}$$

where the integral is understood in the weak sense.

Definition 3.1 (radial operator). Let $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$. The operator S is said to be *radial* if it is invariant under rotations, that is, if for every $t \in [0,2\pi)$

$$SU_t = U_t S. \tag{3.3}$$

Observe that $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$ is radial if and only if $\mathrm{Rad}(S) = S$. The set of all radial operators is denoted by $\mathfrak{R}$:

$$\mathfrak{R} := \left\{ S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C})) : \quad \forall t \in [0,2\pi) \quad U_t S = S U_t \right\}. \tag{3.4}$$

Proposition 3.1. *$\mathfrak{R}$ is a* C*-*subalgebra of $\mathscr{B}(\mathscr{F}^2(\mathbb{C}))$.*

Proof. Let $S, V \in \mathfrak{R}$ and $t \in [0,2\pi)$. Then $(S+V)U_t = SU_t + VU_t = U_t S + U_t V = U_t(S+V)$, we thus have that $S+V \in \mathfrak{R}$. On the other hand, $TSU_t = TU_t S = U_t TS$, hence $TS \in \mathfrak{R}$, this implies that $\mathfrak{R}$ is a subalgebra of $\mathscr{B}(\mathscr{F}^2(\mathbb{C}))$. The mapping $S \mapsto S^*$, where S^* is the adjoint operator of S, defines an involution on $\mathfrak{R}$, furthermore, for each $S \in \mathfrak{R}$ one has that

$$S^* U_t = \left(U_{-t} S\right)^* = \left(S U_{-t}\right)^* = U_t S^*.$$

Thus $\mathfrak{R}^* = \mathfrak{R}$. Now, given $S \in \overline{\mathfrak{R}}$, there exists $(S_n)_{n\in\mathbb{N}} \subset \mathfrak{R}$ such that $S_n \xrightarrow{n\to\infty} S$, but since $U_t S_n = S_n U_t$ and U_t is a unitary operator, we get that $U_t S_n = S_n U_t$ converges to $U_t S$ and SU_t. Therefore by uniqueness of the limit we conclude $U_t S = SU_t$. That is $S \in \mathfrak{R}$. ■

Example 3.1. Given $\theta \in \mathbb{R}$, one gets that

$$U_t U_\theta = U_{t+\theta} = U_\theta U_t, \quad t \in [0,2\pi].$$

That is, U_θ is a radial operator.

Remark 3.1. Let $\{e_n\}_{n\in\mathbb{N}}$ be the monomial basis of $\mathscr{F}^2(\mathbb{C})$. Therefore, given $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$U_t e_n(z) = e_n(e^{it}z) = \frac{e^{int} z^n}{\sqrt{n!}} = e^{int} e_n(z), \quad t \in \mathbb{R}. \tag{3.5}$$

This implies that U_t is a diagonal operator for each $t \in \mathbb{R}$.

Definition 3.2 (radial function). A function $\varphi \in L_\infty(\mathbb{C})$ is called *radial* if there exists $a \in L_\infty(\mathbb{R}_+)$ such that $\varphi(z) = a(|z|)$ a.e. $z \in \mathbb{C}$.

Definition 3.3 (the radialization of a function). Let $\varphi \in L_\infty(\mathbb{C})$. The function $\mathrm{rad}(\varphi)$ given by

$$\mathrm{rad}(\varphi)(z) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it}z)\,dt \tag{3.6}$$

is called the *radialization* of φ.

By the periodicity of the mapping $t \mapsto e^{it}$, the formula (3.6) can rewritten as

$$\mathrm{rad}(\varphi)(z) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it}|z|)\,dt. \tag{3.7}$$

Lemma 3.1 (criterion for a function to be radial). *A function $\varphi \in L_\infty(\mathbb{C})$ is radial if and only if $\varphi(z) = \mathrm{rad}(\varphi)(z)$ a.e. $z \in \mathbb{C}$.*

Proof. Suppose that $\varphi \in L_\infty(\mathbb{C})$ is radial, i.e. there exists $a \in L_\infty(\mathbb{R}_+)$ such that $\varphi(z) = a(|z|)$ a.e. $z \in \mathbb{C}$. Therefore, by (3.7) and by Fubini's theorem one gets that

$$\begin{aligned}
\int_{\mathbb{C}} \mathrm{rad}(\varphi)(w) w^n \overline{w}^m \,d\mathrm{g}(w) &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}_+}\int_0^{2\pi} r^{n+m+1}e^{-r^2}\varphi(e^{i\alpha}r)e^{i\beta(n-m)}\,d\alpha\,dr\,d\beta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}_+}\int_0^{2\pi} r^{n+m+1}e^{-r^2} a\left(|e^{i\alpha}r|\right)e^{i\beta(n-m)}\,d\alpha\,dr\,d\beta\\
&= \left(\int_{\mathbb{R}_+} a(r) r^{n+m+1}e^{-r^2}\,dr\right)\left(\int_0^{2\pi} e^{i\beta(n-m)}\,d\beta\right)\\
&= \int_{\mathbb{C}} \varphi(w) w^n \overline{w}^m \,d\mathrm{g}(w), \qquad n,m \in \mathbb{Z}_+.
\end{aligned}$$

Now, since $\mathrm{rad}(\varphi) - \varphi$ belongs to $L_2(\mathbb{C}, dg)$ and the span of $\{w^m \overline{w}^n : m,n \in \mathbb{Z}_+\}$ is dense in $L_2(\mathbb{C}, dg)$, we obtain that $\mathrm{rad}(\varphi)(z) = \varphi(z)$ a.e. $z \in \mathbb{C}$.

Conversely, if $\varphi(z) = \mathrm{rad}(\varphi)(z)$ a.e $z \in \mathbb{C}$, then by (3.7) one gets that $\varphi(z) = \mathrm{rad}(\varphi)(|z|)$ a.e. $z \in \mathbb{C}$, which means that the condition of Definition 3.2 holds with $a(r) = \mathrm{rad}(\varphi)(r)$. ∎

From Lemma 3.1 and (3.7) it is easy to see that a function $\varphi \in L_\infty(\mathbb{C})$ is radial if and only if for every $t \in [0, 2\pi)$ one get that

$$\varphi(e^{-it}z) = \mathrm{rad}(\varphi)(e^{-it}z) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}|e^{-it}z|)d\theta = \varphi(z), \quad a.e. z \in \mathbb{C}.$$

Lemma 3.2. *If $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$ is a diagonal operator with respect to the monomial basis $\{e_n\}_{n\in\mathbb{N}}$, then $S = \mathrm{Rad}(S)$.*

Proof. Let $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$. Then given $m,n \in \mathbb{Z}_+$ by (3.5) one gets that

$$\begin{aligned}
\langle \mathrm{Rad}(S)e_n, e_m\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle U_{-t}SU_t e_n, e_m\rangle\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle SU_t e_n, U_t e_m\rangle\, dt = \frac{1}{2\pi}\int_0^{2\pi} \langle S(e^{int} e_n), e^{imt} e_m\rangle\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t} \langle S e_n, e_m\rangle\, dt \\
&= \begin{cases} 0, & \text{whenever } m \neq n. \\ \langle S e_n, e_n\rangle & \text{whenever } m = n. \end{cases}
\end{aligned}$$

Thus, if S is a diagonal operator with respect to the monomial basis $\{e_n\}_{n\in\mathbb{Z}_+}$, then for every $n,m \in \mathbb{Z}_+$

$$\langle (S - \mathrm{Rad}(S))e_n, e_m\rangle = 0.$$

Therefore, $S = \mathrm{Rad}(S)$. ∎

The next result provides a criterion for an operator to be radial. It mimics a result given by Zorboska [52] for operators on the Bergman space over the unit disk.

Theorem 3.2. *Let $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}))$. The following conditions are equivalent.*

(i) $S \in \mathfrak{R}$.

(ii) *S is a diagonal operator with respect to the monomial basis.*

(iii) *The Berezin transform $\widetilde{S}$ is a radial function.*

Proof. (i)$\longrightarrow$(ii) Let $\{e_n\}_{n\in\mathbb{N}}$ be the monomial basis of $\mathscr{F}^2(\mathbb{C})$, and let S be a radial operator. Thus $S = \mathrm{Rad}(S)$ and for every $m,n \in \mathbb{Z}_+$

$$\langle S e_n, e_m\rangle = \langle \mathrm{Rad}(S)e_n, e_m\rangle = a_n \delta_{n,m} = a_n \langle e_n, e_m\rangle,$$

where $a_n = \langle S e_n, e_n\rangle$.

(ii)$\longrightarrow$(iii) Let S be a diagonal operator. Then

$$\begin{aligned}
\widetilde{\mathrm{Rad}(S)}(z) &= \frac{\langle \mathrm{Rad}(S)K_z, K_z\rangle}{\langle K_z, K_z\rangle} = \frac{1}{2\pi\langle K_z, K_z\rangle}\int_0^{2\pi} \langle U_{-t}SU_t K_z, K_z\rangle\, dt \\
&= \frac{1}{2\pi\langle K_z, K_z\rangle}\int_0^{2\pi} \langle SU_t K_z, U_t K_z\rangle\, dt = \frac{1}{2\pi\langle K_z, K_z\rangle}\int_0^{2\pi} \langle SK_{ze^{-it}}, K_{ze^{-it}}\rangle\, dt \\
&= \frac{1}{2\pi\langle U_{-t}K_z, U_{-t}K_z\rangle}\int_0^{2\pi} \langle SK_{ze^{-it}}, K_{ze^{-it}}\rangle\, dt = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{S}(e^{-it}z)dt \\
&= \mathrm{rad}(\widetilde{S})(z), \quad z \in \mathbb{C}.
\end{aligned}$$

Thus, $\widetilde{\mathrm{Rad}(S)} = \mathrm{rad}(\widetilde{S})$, but by Lemma 3.2 we have $S = \mathrm{Rad}(S)$. Hence the Berezin transform of S is a radial function.

(iii)⟶(i) If $\widetilde{S}$ is a radial function, then $\widetilde{S} = \mathrm{rad}(\widetilde{S}) = \widetilde{\mathrm{Rad}(S)}$. Since the Berezin transform is injective, we get that $S = \mathrm{Rad}(S)$, hence by (3.7) one has for $n,m \in \mathbb{Z}_+$ that

$$\begin{aligned}\langle U_{-\alpha} S U_\alpha e_n, e_m\rangle = \langle U_{-\alpha}\mathrm{Rad}(S) U_\alpha e_n, e_m\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle S U_t U_\alpha e_n, U_t U_\alpha e_m\rangle\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle S U_{t+\alpha} e_n, U_{t+\alpha} e_m\rangle\, dt = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} \langle S U_\beta e_n, U_\beta e_m\rangle\, d\beta \\ &= \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{i(n-m)\beta}\langle S e_n, e_m\rangle\, d\beta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\beta}\langle S e_n, e_m\rangle\, d\beta \\ &= \langle S e_n, e_m\rangle, \quad \alpha \in [0,2\pi].\end{aligned}$$

Therefore, since $\mathrm{span}\{e_n : n \in \mathbb{Z}_+\}$ is a dense subset of the Fock space $\mathscr{F}^2(\mathbb{C})$, we have $U_{-t} S U_t = S$ for all $t \in [0,2\pi]$. ■

Proposition 3.2. *Let $\varphi \in L_\infty(\mathbb{C})$. The Toeplitz operator T_φ is radial if and only if φ is a radial function.*

Proof. If φ is radial, then by Lemma 3.1 we have $\mathrm{rad}(\varphi) = \varphi$. Hence

$$\langle T_\varphi e_n, e_m\rangle = \langle \mathrm{rad}(\varphi) e_n, e_m\rangle = \int_{\mathbb{C}} \mathrm{rad}(\varphi)(w) w^n \overline{w}^m\, dg(w) = \gamma_a(n)\delta_{n,m}, \quad m,n \in \mathbb{Z}_+.$$

Thus, by Theorem 3.2 the Toeplitz operator T_φ is a radial operator.

Conversely, if T_φ is a radial operator, then by Fubini's theorem for every $f,g \in \mathscr{F}^2(\mathbb{C})$ we obtain

$$\begin{aligned}\langle T_\varphi f, g\rangle = \langle \mathrm{Rad}(T_\varphi) f, g\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle U_{-t} T_\varphi U_t f, g\rangle\, dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \varphi U_t f, U_{-t} g\rangle\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} \varphi(w) f(e^{-it}w)\overline{g(e^{-it}w)}\, dg(w) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} \varphi(e^{it}w) f(w)\overline{g(w)}\, dg(w) \\ &= \langle \mathrm{rad}(\varphi) f, g\rangle = \langle T_{\mathrm{rad}(\varphi)} f, g\rangle.\end{aligned}$$

Therefore, $T_\varphi = T_{\mathrm{rad}(\varphi)}$ and by Proposition 1.6 and Lemma 3.1 we have $\varphi = \mathrm{rad}(\varphi)$. ■

3.2 Square-root-oscillating property of the eigenvalues' sequences

In this section we introduce the set of sequences $\mathrm{RO}(\mathbb{Z}_+)$ and functions $\mathrm{RO}([0,+\infty))$. We also show that the sequences of the class $\mathrm{RO}(\mathbb{Z}_+)$ can be extended to functions of the class $\mathrm{RO}([0,+\infty))$. We finish this section showing that the sequences of eigenvalues belongs to $\mathrm{RO}(\mathbb{Z}_+)$.

square-root-oscillating sequences

Definition 3.4 (square root metric on $\mathbb{Z}_+$). Define $\varrho\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to [0,+\infty)$ by

$$\varrho(m,n) = \left|\sqrt{m} - \sqrt{n}\right|. \tag{3.8}$$

The function ϱ is a metric on $\mathbb{Z}_+$ because it is obtained from the usual metric

$$d\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty), \quad d(t,u) := |t-u|,$$

via the injective function $\mathbb{Z}_+ \to \mathbb{R}_+$, $m \mapsto \sqrt{m}$.

Definition 3.5 (modulus of continuity of a sequence with respect to the square–root-metric). Let $x = (x_n)_{n\in\mathbb{Z}_+}$ be a complex sequence. The *modulus of continuity* of x with respect to the square-root-metric ϱ is the function $\Omega_{\varrho,x}\colon [0,+\infty) \to [0,+\infty]$ given by the rule

$$\Omega_{\varrho,x}(\delta) = \sup\left\{|x_n - x_m|\colon\ m,n \in \mathbb{Z}_+,\ \varrho(m,n) \le \delta\right\}. \tag{3.9}$$

We denote by $\mathrm{RO}(\mathbb{Z}_+)$ the set of the bounded sequences that are uniformly continuous with respect to the square-root-metric:

$$\mathrm{RO}(\mathbb{Z}_+) = \left\{x \in \ell_\infty(\mathbb{Z}_+)\colon\ \lim_{\delta\to 0} \Omega_{\varrho,x}(\delta) = 0\right\}. \tag{3.10}$$

Proposition 3.3. $\mathrm{RO}(\mathbb{Z}_+)$ *is a closed* C*-*subalgebra of* $\ell_\infty(\mathbb{Z}_+)$.

Proof. Using the following elementary properties of the modulus of continuity one can see that $\mathrm{VSO}(\mathbb{R})$ is closed with respect to the pointwise operations:

$$\Omega_{\varrho,\sigma+\varpi} \le \Omega_{\varrho,\sigma} + \Omega_{\varrho,\varpi}, \qquad \Omega_{\varrho,\lambda\sigma} = |\lambda|\Omega_{\varrho,\sigma},$$
$$\Omega_{\varrho,\sigma\varpi} \le \|\varpi\|_\infty \Omega_{\varrho,\sigma} + \|\sigma\|_\infty \Omega_{\varrho,\varpi}, \qquad \Omega_{\varrho,\overline{\sigma}} = \Omega_{\varrho,\sigma},$$

The inequality $\Omega_{\varrho,\sigma}(\delta) \le 2\|\sigma - \varpi\|_\infty + \Omega_{\varrho,\varpi}(\delta)$ and the usual "$\frac{\varepsilon}{3}$-argument" show that the space $\mathrm{RO}(\mathbb{Z}_+)$ is topologically closed in $L_\infty(\mathbb{Z}_+)$. ■

The following simple criterion shows that the Lipschitz-continuity of sequences (with respect to the metric ϱ) can be described in terms of the differences between the adjacent elements.

Proposition 3.4. *A sequence* $\sigma\colon \mathbb{Z}_+ \to \mathbb{C}$ *is Lipschitz continuous with respect to* ϱ *if and only if*

$$\sup_{n\in\mathbb{Z}_+} \left(\sqrt{n+1}\,|\sigma(n+1) - \sigma(n)|\right) < +\infty. \tag{3.11}$$

Proof. Suppose that σ is Lipschitz continuous with respect to ϱ, that is, there exists $M > 0$ such that $|\sigma(m) - \sigma(n)| \le M\varrho(m,n)$ for each $m,n \in \mathbb{Z}_+$. Applying this inequality in the particular case $m = n+1$, one gets

$$\sqrt{n+1}\,|\sigma(n+1) - \sigma(n)| \le M\left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n+1} = \frac{M\sqrt{n+1}}{\sqrt{n+1}+\sqrt{n}} \le M.$$

Conversely, suppose that $\sup_n \left(\sqrt{n+1}\,|\sigma(n+1) - \sigma(n)|\right) = M < +\infty$. Hence, if $n > m$, then we "join" m with n by the chain of the intermediate elements and estimate the differences of the adjacent elements using the hypothesis:

$$\begin{aligned}
|\sigma(m) - \sigma(n)| &\le \sum_{k=m}^{n-1} |\sigma(k+1) - \sigma(k)| = \sum_{k=m}^{n-1} \frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}\,|\sigma(k+1) - \sigma(k)| \\
&\le 2\sum_{k=m}^{n-1} \frac{\sqrt{k+1}}{\sqrt{k+1}+\sqrt{k}}\,|\sigma(k+1) - \sigma(k)| \le 2M \sum_{k=m}^{n-1} \frac{1}{\sqrt{k+1}+\sqrt{k}} \\
&= 2M \sum_{k=m}^{n-1} \left(\sqrt{k+1} - \sqrt{k}\right) = 2M\left(\sqrt{n} - \sqrt{m}\right) = 2M\varrho(n,m).
\end{aligned}$$

The same upper estimate can be drawn for $m \ge n$. Thus, σ is Lipschitz continuous with respect to ϱ. ■

Example 3.2.
The sequence $\left(\sqrt{\frac{n}{n+1}}\right)_{n\in\mathbb{Z}_+}$ belongs to $\mathrm{RO}(\mathbb{Z}_+)$. In fact, note for each $n \in \mathbb{Z}_+$ that

$$\begin{aligned}
\left|\sqrt{n+1}\left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n+1}{n+2}}\right)\right| &= \frac{n+1}{\sqrt{n+2}} - \sqrt{n} \\
&\le \sqrt{n+2} - \sqrt{n} \\
&= \frac{2}{\sqrt{n+2}+\sqrt{n}}.
\end{aligned}$$

Thus, by Proposition 3.4 the sequence $\left(\sqrt{\frac{n}{n+1}}\right)_{n\in\mathbb{Z}_+}$ is Lipschitz with respect to the metric ϱ,

$$\left|\sqrt{\frac{n}{n+1}} - \sqrt{\frac{m}{m+1}}\right| \le 2\varrho(m,n), \quad m,n \in \mathbb{Z}_+.$$

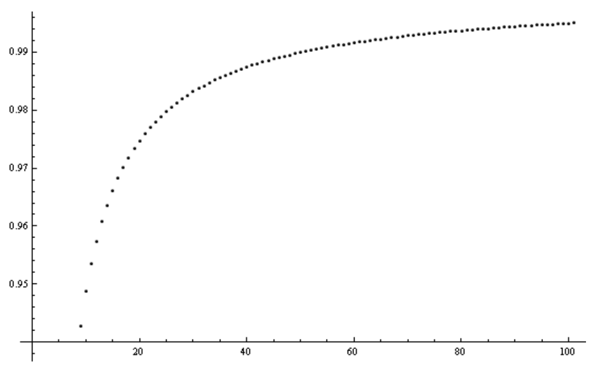


Figure 3.1: The first 100 values of the sequence $\sigma(n) = \sqrt{\frac{n}{n+1}}$.

Sqrt-oscillating functions on $\mathbb{R}_+$

The square root metric ϱ can be extended to the set $[0,+\infty)$:

$$\varrho(x,y) = \left|\sqrt{x} - \sqrt{y}\right|.$$

We denote by $\mathrm{RO}([0,+\infty))$ the C*-algebra of all bounded and uniformly continuous functions on $[0,+\infty)$ with respect to the extended square root metric ϱ:

$$\mathrm{RO}([0,+\infty)) = \left\{ f \in C_{b,u}([0,+\infty)) \colon \lim_{\delta\to 0} \Omega_{\varrho,f}(\delta) = 0 \right\}. \tag{3.12}$$

Here $\Omega_{\varrho,f}$ is the modulus of continuity of the function f with respect to the metric ϱ (Definition 2.3). In other words, $f \in \mathrm{RO}([0,+\infty))$ if and only if the function $h(x) = f(x^2)$ is uniformly continuous with respect the standard Euclidean metric on $\mathbb{R}$. An simply example is the function $f(x) = \cos(\sqrt{x})$.

If f is a function of the class $\mathrm{RO}([0,+\infty))$, then, obviously, its restriction to $\mathbb{Z}_+$ is a sequence of the class $\mathrm{RO}(\mathbb{Z}_+)$. We are going to show that *every* sequence of the class $\mathrm{RO}(\mathbb{Z}_+)$ can be obtained in this manner. Our extension of sequences to functions is just the piecewise-linear interpolation with respect to the parameter $\tau(x) = \sqrt{x}$.

Lemma 3.3. *Let $\sigma\colon \mathbb{Z}_+ \to \mathbb{C}$. Define $f\colon [0,+\infty) \to \mathbb{C}$ by*

$$f(x) = \sigma(n) + \frac{\tau(x)-\tau(n)}{\tau(n+1)-\tau(n)}(\sigma(n+1)-\sigma(n)), \tag{3.13}$$

where $n = \lfloor x \rfloor$ and $\tau(x) = \sqrt{x}$. Then $f|_{\mathbb{Z}_+} = \sigma$, $\|f\|_\infty = \|\sigma\|_\infty$ and for every $\delta \in (0,1]$

$$\Omega_{\varrho,f}(\delta) \le 3\max(\Omega_{\varrho,\sigma}(\delta), \sqrt{\delta}\,\Omega_{\varrho,\sigma}(1)). \tag{3.14}$$

Proof. Note that $f(x)$ is a convex combination of $\sigma(n)$ and $\sigma(n+1)$, where $n = \lfloor x \rfloor$:

$$f(x) = \frac{\tau(n+1)-\tau(x)}{\tau(n+1)-\tau(n)}\sigma(n) + \frac{\tau(x)-\tau(n)}{\tau(n+1)-\tau(n)}\sigma(n+1). \tag{3.15}$$

The first two assertions of the proposition are obvious. Let us prove (3.14). Fix $\delta \in (0,1]$ and suppose that $x, y \ge 0$, $\varrho(x,y) \le \delta$.

Case I: $n \le x \le y \le n+1$ for some $n \in \mathbb{Z}_+$. In this case

$$|f(x)-f(y)| = \frac{\tau(y)-\tau(x)}{\tau(n+1)-\tau(n)}|\sigma(n+1)-\sigma(n)| \le \frac{\varrho(x,y)\Omega_{\varrho,\sigma}(\varrho(n,n+1))}{\varrho(n,n+1)}.$$

If $\varrho(n,n+1) \le \sqrt{\delta}$, then

$$|f(x)-f(y)| \le \frac{\varrho(x,y)}{\varrho(n,n+1)}\Omega_{\varrho,\sigma}(\sqrt{\delta}) \le \Omega_{\varrho,\sigma}(\sqrt{\delta}).$$

If $\varrho(n,n+1) \geq \sqrt{\delta}$, then

$$|f(x)-f(y)| \leq \frac{\delta}{\sqrt{\delta}}\Omega_{\varrho,\sigma}(1) = \sqrt{\delta}\,\Omega_{\varrho,\sigma}(1).$$

In both subcases,

$$|f(x)-f(y)| \leq \max(\Omega_{\varrho,\sigma}(\sqrt{\delta}),\ \sqrt{\delta}\,\Omega_{\varrho,\sigma}(1)). \tag{3.16}$$

Case II: $\lfloor x \rfloor = n < m = \lfloor y \rfloor$. Then $\varrho(n+1,m) \leq \varrho(x,y) \leq \delta$, and

$$|f(x)-f(y)| \leq |f(x)-f(n+1)| + |f(n+1)-f(m)| + |f(m)-f(y)|.$$

Applying the inequality $\varrho(n+1,m) \leq \varrho(x,y) \leq \delta$ and the result of Case I, we obtain

$$\begin{aligned} |f(x)-f(y)| &\leq \Omega_{\varrho,\sigma}(\delta) + 2\max(\Omega_{\varrho,\sigma}(\sqrt{\delta}),\ \sqrt{\delta}\,\Omega_{\varrho,\sigma}(1)) \\ &\leq \Omega_{\varrho,\sigma}(\sqrt{\delta}) + 2\max(\Omega_{\varrho,\sigma}(\sqrt{\delta}),\ \sqrt{\delta}\,\Omega_{\varrho,\sigma}(1)). \end{aligned}$$

In both cases, (3.14) is true. ■

Proposition 3.5. *Let $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ and $f\colon [0,+\infty) \to \mathbb{C}$ be the extension of σ defined by (3.13). Then $f \in \mathrm{RO}([0,+\infty))$.*

Proof. The assumption $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ guarantees that the right-hand side of (3.14) tends to 0 as $\delta \to 0$. ■

Note that Lemma 3.3 and Proposition 3.5 remain true for every metric ρ of the form $\rho(x,y) = |\tau(x)-\tau(y)|$, where $\tau\colon [0,+\infty) \to [0,+\infty)$ is a strictly increasing function satisfying $\tau(n+1)-\tau(n) \leq 1$ for every $n \in \mathbb{Z}_+$. In particular, applying this construction with $\tau(n) = \ln(n+1)$ we obtain another proof of [31, Theorem 2.3] about the class $\mathrm{SO}(\mathbb{Z}_+)$; the proof in [31] was based on the usual piecewise-linear interpolation.

Square-root-oscillating property of the eigenvalues' sequences

In what follows we show that $\gamma_a \in \mathrm{RO}(\mathbb{Z}_+)$ for all $a \in L_\infty(\mathbb{R}_+)$. From now on, we write the eigenvalues' sequence γ_a as follows:

$$\gamma_a(n) = \int_{\mathbb{R}_+} a(\sqrt{r})K(n,r)\,dr, \quad \text{where} \quad K(n,r) = \frac{r^n e^{-r}}{n!}, \quad n \in \mathbb{Z}_+. \tag{3.17}$$

The following proposition introduces a metric on $\mathbb{Z}_+$ which is, in a certain sense, the most "natural" for the functions γ_a.

Proposition 3.6. *Let $\kappa\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to [0,+\infty)$ be the function given by*

$$\kappa(m,n) = \sup_{\substack{a\in L_\infty(\mathbb{R}_+)\\ \|a\|_\infty=1}} \left|\gamma_a(m)-\gamma_a(n)\right|. \tag{3.18}$$

Then

$$\kappa(m,n) = \int_{\mathbb{R}_+} |K(m,r)-K(n,r)|\,dr. \tag{3.19}$$

Proof. For every $a \in L_\infty(\mathbb{R}_+)$ and $m,n \in \mathbb{Z}_+$ we have

$$\left|\gamma_a(m)-\gamma_a(n)\right| \le \|a\|_\infty \int_{\mathbb{R}_+} |K(m,r)-K(n,r)|\,dr.$$

On the other hand, if m and n are fixed and $m \neq n$, we define $a_0\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$a_0(r) = \operatorname{sign}(K(m,r)-K(n,r))$$

thus $a_0 \in L_\infty(\mathbb{R}_+)$, with $\|a_0\|_\infty = 1$, and

$$\kappa(x,y) \ge \left|\gamma_{a_0}(m)-\gamma_{a_0}(n)\right| = \int_{\mathbb{R}_+} |K(m,r)-K(n,r)|\,dr. \qquad \blacksquare$$

Lemma 3.4. *For every $n \in \mathbb{N}$ we get*

$$\kappa(n-1,n) = \frac{2n^n e^{-n}}{n!} \tag{3.20}$$

Proof. Given $n \in \mathbb{N}$, we write $\kappa(n-1,n)$ using (3.19):

$$\kappa(n-1,n) = \int_0^{+\infty} \left|\frac{r^{n-1}e^{-r}}{(n-1)!} - \frac{r^n e^{-r}}{n!}\right| dr = \int_0^{+\infty} \frac{r^{n-1}e^{-r}}{(n-1)!}\left|1-\frac{r}{n}\right| dr.$$

Now the integral falls naturally into two parts:

$$\begin{aligned}
\kappa(n-1,n) &= \frac{1}{(n-1)!}\left[\int_0^n r^{n-1}e^{-r}\left(1-\frac{r}{n}\right)dr + \int_n^{+\infty} r^{n-1}e^{-r}\left(\frac{r}{n}-1\right)dr\right]\\
&= \frac{1}{(n-1)!}\left[\int_0^{+\infty} e^{-r}\left(\frac{r^n}{n}-r^{n-1}\right)dr + 2\int_0^n e^{-r}\left(r^{n-1}-\frac{r^n}{n}\right)dr\right]\\
&= \frac{2}{(n-1)!}\int_0^n e^{-r}\left(r^{n-1}-\frac{r^n}{n}\right)dr\\
&= \frac{2}{(n-1)!}\left[\int_0^n e^{-r}r^{n-1}dr - \int_0^n e^{-r}\frac{r^n}{n}dr\right].
\end{aligned}$$

Integrating by parts in the latter integral one gets (3.20). $\blacksquare$

Lemma 3.5. *For each $n \in \mathbb{N}$ we have*

$$\kappa(n-1,n) \le \sqrt{\frac{2}{\pi n}}. \tag{3.21}$$

Moreover,

$$\lim_{n\to\infty} \left(\kappa(n-1,n)\sqrt{n}\right) = \sqrt{\frac{2}{\pi}}. \tag{3.22}$$

Proof. The upper bound (3.21) follows from the left part of the well-known estimates

$$n^n e^{-n}\sqrt{2n\pi} \le n! \le n^n e^{-n}\sqrt{2n\pi}\, e^{\frac{1}{12n}}. \tag{3.23}$$

The limit relation (3.22) is a consequence of Stirling formula. ■

Proposition 3.7. $\mathfrak{G} \subseteq \mathrm{RO}(\mathbb{Z}_+)$.

Proof. Let $a \in L_\infty(\mathbb{R}_+)$. Then for every $n \in \mathbb{Z}_+$

$$|\gamma_a(n)| \le \|a\|_\infty \int_{\mathbb{R}_+} K(n,r)\,dr = \|a\|_\infty.$$

Furthermore, by definition (3.18) of κ and Lemma 3.5, for every $n \in \mathbb{N}$

$$\left|\sqrt{n}\left(\gamma_a(n)-\gamma_a(n-1)\right)\right| \le \|a\|_\infty \kappa(n,n-1)\sqrt{n} \le \sqrt{\frac{2}{\pi}}\,\|a\|_\infty.$$

Thus γ_a is Lipschitz continuous with respect to ϱ by Proposition 3.4. ■

Example 3.3 (square-root-oscillating eigenvalues' sequence). Consider the Toeplitz operator generated by the radial symbol $a(r) = \cos r$. The corresponding eigenvalues are

$$\gamma_a(n) = {}_1F_1(1+n, 1/2, -1/4),$$

where ${}_1F_1$ is the Kummer's confluent hypergeometric function. Using Proposition 3.8 one can deduce an asymptotic formula for $\gamma_a(n)$, as $n \to \infty$:

$$\gamma_a(n) = e^{-1/8}\cos\sqrt{n} + o(1).$$

Figure 3.2 shows a plot of $\gamma_a(n)$ for $n = 0, 1, \ldots, 300$.

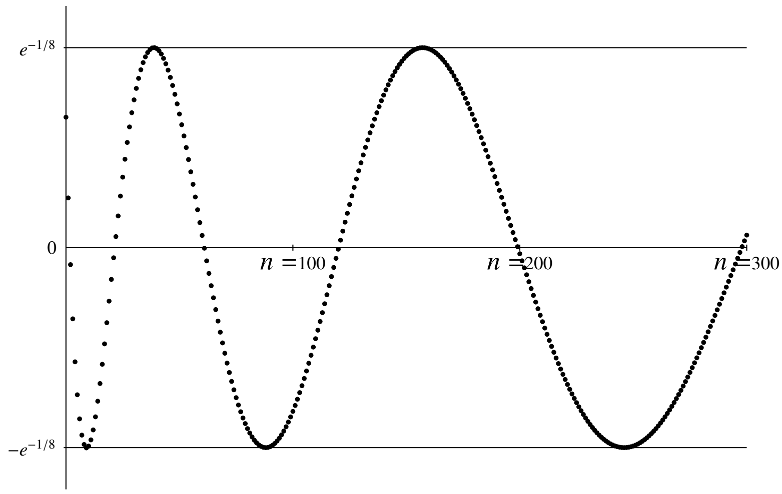


Figure 3.2: The first 301 values of the sequence γ_a from Example 3.3.

3.3 Density of $\mathfrak{G}$ in $\mathrm{RO}(\mathbb{Z}_+)$

In this section we prove the main result. First, we prove that every sequence $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ can be approximated by some eigenvalues' sequence γ_a for large values of n (Proposition 3.9). After that, we prove that the sequences vanishing at the infinity can be approximated by eigenvalues' sequences (Theorem 3.3). Finally, combining these result we show that the uniform closure of $\mathfrak{G}$ coincides with $\mathrm{RO}(\mathbb{Z}_+)$ (Theorem 3.4).

Approximation of the eigenvalues' sequences by convolutions

The idea of this subsection is to approximate $\gamma_a(n)$ by a certain convolution for n large enough. Using the change of variables $r = y^2$ in (3.17) we rewrite $\gamma_a(n)$ in the form

$$\gamma_a(n) = \int_{\mathbb{R}_+} K(n, y^2) 2y\, a(y)\, dy. \tag{3.24}$$

By Stirling formula, $K(n,r)$ has the following asymptotic behavior as $n \to +\infty$:

$$K(n,r) = \frac{r^n e^{-r}}{n!} \sim \frac{r^n e^n}{\sqrt{2\pi}\, n^{n+1/2} e^r}.$$

Using this limit relation and Lebesgue's dominated convergence theorem it is easy to see that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \left| \frac{r^n e^{-r}}{n!} - \frac{r^n e^n}{\sqrt{2\pi}\, n^{n+1/2} e^r} \right| dr = 0. \tag{3.25}$$

We pass from integer n to real $x = \sqrt{n}$ and from $r \geq 0$ to $r = y^2$. Consider the function F defined on $[0,+\infty) \times [0,+\infty)$ by

$$F(x,y) = \frac{y^{2x^2+1} e^{x^2}}{x^{2x^2+1} e^{y^2}} = \exp\left((2x^2+1)(\ln y - \ln x) + x^2 - y^2\right).$$

Then (3.25) can be rewritten in the form

$$\lim_{n\to\infty} \int_{\mathbb{R}+} \left| K(n,y^2)2y - \left(\frac{2}{\pi}\right)^{1/2} F(\sqrt{n},y) \right| dy = 0. \tag{3.26}$$

With the change of variables $u = y - x$ we have

$$\int_{\mathbb{R}^+} F(x,y)a(y)\,dy = \int_{[-x,+\infty)} F(x,x+u)a(x+u)\,dy, \tag{3.27}$$

where

$$\ln F(x,x+u) = (2x^2+1)\ln\left(1+\frac{u}{x}\right) - 2xu - u^2. \tag{3.28}$$

Next, we proceed with some technical lemmas which permit us to analyze the asymptotic behavior of the eigenvalues' sequences at the infinity.

Lemma 3.6 (the integral of the kernel far from the diagonal). *For every $\varepsilon > 0$ there exists $h > 1$ such that the following estimation holds for every $x \geq 1$:*

$$\int_{[-x,+\infty)\setminus[-h,h]} F(x,x+u)\,du \leq \varepsilon.$$

Proof. Apply the elementary inequality $\ln(1+t) \leq t$ which holds for every $t \geq 0$:

$$\ln F(x,x+u) \leq (2x^2+1)\frac{u}{x} - 2xu - u^2 = \frac{u}{x} - u^2 \leq \frac{|u|}{x} - u^2. \tag{3.29}$$

Suppose that $x \geq 1$, $h \geq 2$ and $|u| \geq h$. Since $|u| \geq h \geq 2$, we have that $\frac{|u|}{2} \geq 1$. Thus by (3.29) we get

$$\ln F(x,x+u) \leq \frac{|u|}{x} - u^2 \leq |u|\left(\frac{|u|}{2}\right) - u^2 = \frac{u^2}{2} - u^2 = -\frac{u^2}{2}.$$

It follows that

$$\int_{[-x,+\infty)\setminus[-h,h]} F(x,x+u)\,du \leq \int_{[-x,+\infty)\setminus[-h,h]} e^{-\frac{u^2}{2}}\,du \leq \int_{\mathbb{R}\setminus[-h,h]} e^{-\frac{u^2}{2}}\,du \leq 2\int_h^{+\infty} e^{-\frac{u^2}{2}}\,du.$$

The latter integral tends to zero as h tends to $+\infty$. ∎

Lemma 3.7. *Let $L, x \geq 0$, $h \geq 1$ and $|u| \leq h$. If $h < L < x$, then*

$$\left|F(x,x+u)e^{2u^2} - 1\right| \leq e^{5\frac{h^3}{L}} - 1. \tag{3.30}$$

Proof. Since $\ln(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}$, $\quad t \in (-1,1)$, we obtain for $t = \left|\frac{u}{x}\right| \leq 1$ that

$$\begin{aligned}
\ln F(x,x+u) + 2u^2 &= (2x^2+1)\ln\left(1+\frac{u}{x}\right) - 2xu + u^2 \\
&\leq (2x^2+1)\left(\frac{u}{x} - \frac{u^2}{2x^2} + \frac{u^3}{3x^3}\right) - 2xu + u^2 \\
&= \frac{u}{x} - \frac{u^2}{2x^2} + \frac{u^3}{3x^3} + \frac{2u^3}{3x} \leq \frac{u}{x} + \frac{u^3}{x} \leq \frac{5h^3}{L}.
\end{aligned}$$

On the other hand, by $t - \frac{t^2}{2} \leq \ln(1+t)$ for each $t \in [0,1)$, taking $t = \frac{u}{x}$ with $u \in [0,h]$

$$\begin{aligned}
\ln F(x,x+u) + 2u^2 &= (2x^2+1)\ln\left(1+\frac{u}{x}\right) - 2xu + u^2 \\
&\geq (2x^2+1)\left(\frac{u}{x} - \frac{u^2}{2x^2}\right) - 2xu + u^2 = \frac{u}{x} - \frac{u^2}{2x^2} \geq -\frac{5h^3}{L}.
\end{aligned}$$

Since $\ln(1-t) \geq -t - \frac{t^2}{2} - t^3$ for each $t \in [0,2/3]$, we take $x > 0$ sufficiently large such that $t = -\frac{u}{x} \in [0,2/3]$, with $u \in [-h,0]$. Therefore

$$\begin{aligned}
\ln F(x,x+u) + 2u^2 &= (2x^2+1)\ln\left(1-\left(-\frac{u}{x}\right)\right) - 2xu + u^2 \\
&\geq (2x^2+1)\left(\frac{u}{x} - \frac{u^2}{2x^2} + \frac{u^3}{x^3}\right) - 2xu + u^2 \\
&= \frac{u}{x} - \frac{u^2}{2x^2} + \frac{u^3}{x^3} + \frac{2u^3}{x} \geq -\frac{5h^3}{L}.
\end{aligned}$$

Combining this calculations we get for all $|u| \leq h$ that

$$e^{5\frac{h^3}{L}} - 1 \geq F(x,x+u)e^{u^2} - 1 \geq e^{-5\frac{h^3}{L}} - 1 \geq -(e^{5\frac{h^3}{L}} - 1). \qquad \blacksquare$$

Lemma 3.8 ("convoluzation" of the integral operator near the diagonal). *Given $\varepsilon > 0$ and $h \geq 1$, there exists $L \geq h$ such that for every $x \geq L$*

$$\int_{[-h,h]} |F(x,x+u) - e^{-2u^2}|\,du \leq \varepsilon.$$

Proof. Suppose that $x \geq L$ and $|u| \leq h$. By Lemma 3.7 for $h \geq 1$ we get

$$\int_{[-h,h]} |F(x,x+u) - e^{-2u^2}|\,du \leq \int_{[-h,h]} e^{-2u^2}\,|F(x,x+u)e^{2u^2} - 1|\,du \leq 2h\,(e^{5h^3/L} - 1).$$

The last expression tends to 0 as L tends to $+\infty$. $\blacksquare$

Lemma 3.9.

$$\lim_{x\to+\infty}\int_0^{\infty}\left|F(x,y)-e^{-2(x-y)^2}\right|dy=0, \tag{3.31}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}_+}\left|K(n,y^2)2y-\left(\frac{2}{\pi}\right)^{1/2}e^{-2(\sqrt{n}-y)^2}\right|dr=0. \tag{3.32}$$

Proof. We are going to prove (3.31), then (3.32) will follow by (3.25). Let $\varepsilon>0$. Using Lemma 3.6 choose $h>0$ such that

$$\int_{[-x,+\infty)\setminus[-h,h]}F(x,x+u)\,du\le\frac{\varepsilon}{3\|a\|_\infty},\qquad\int_{[-x,+\infty)\setminus[-h,h]}e^{-2u^2}\,du\le\frac{\varepsilon}{3\,\|a\|_\infty}.$$

After that using Lemma 3.8 choose $L\ge h$ such that for every $x\ge L$

$$\int_{[-h,h]}|F(x,x+u)-e^{-2u^2}|\,du\le\frac{\varepsilon}{3\,\|a\|_\infty}.$$

Then for every $x\ge L$ the left-hand side of (3.31) is less or equal to ε. ∎

The proofs of this subsection have many technical details. To be more confident in formula (3.32), we tested it numerically in Wolfram Mathematica. The numerical experiments showed that for every $n\in\{1,\dots,1000\}$ the integral in the left-hand side of (3.32) is less than $0.54/\sqrt{n}$.

Proposition 3.8. *Let $a\in L_\infty(\mathbb{R}^+)$. Then*

$$\lim_{n\to+\infty}\left|\gamma_a(n)-\left(\frac{2}{\pi}\right)^{1/2}\int_{\mathbb{R}^+}a(y)e^{-2(y-\sqrt{n})^2}\,dy\right|=0. \tag{3.33}$$

Proof. Write $\gamma_a(n)$ as in (3.24), factorize $a(y)$ below the sign of the the integral, estimate $|a(y)|$ by $\|a\|_\infty$ and apply (3.32). ∎

There is no surprise that the heat kernel appears in the properties of the eigenvalues' sequences γ_a, because it plays an important role in the theory of Toeplitz operators acting on Fock spaces. In [10] Berger and Coburn characterized some properties of Toeplitz operators T_φ (boundedness, compactness etc.) by means of its Berezin transform

$$\widetilde{\varphi}(z)=\frac{1}{\pi}\int_{\mathbb{R}}\varphi(w)e^{-\frac{|z-w|^2}{2}}\,d\nu(w),\quad z\in\mathbb{C},$$

which is the convolution of the symbol φ with the heat kernel $H(w,t)=(4t\pi)^{-1}e^{-\frac{|w|^2}{4t}}$ at time $t=\frac{1}{2}$. This result holds also for Toeplitz operators with more general symbols (positive Borel measures), see [36].

Our formula (3.33) relates γ_a with the heat kernel at time $t = \frac{1}{8}$, we denote it simply by H:

$$H(x) = H(x,1/8) = (2/\pi)^{1/2} e^{-2x^2}.$$

Lemma 3.10. *If $b \in L_\infty(\mathbb{R})$ and $a = \chi_{\mathbb{R}_+} b$, then*

$$\lim_{x\to+\infty} |H * a(x) - H * b(x)| = 0. \tag{3.34}$$

Proof. The difference in the left-hand side of (3.34) can be estimated as follows:

$$\begin{aligned} |H * a(x) - H * b(x)| &\le \|b\|_\infty (2/\pi)^{1/2} \int_{-\infty}^{0} e^{-2(x-y)^2}\, dy \\ &\overset{t=x-y}{=} \|b\|_\infty (2/\pi)^{1/2} \int_{x}^{+\infty} e^{-2t^2}\, dt, \quad x \in \mathbb{R}_+. \end{aligned}$$ ∎

Proposition 3.9. *Let $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ and $\varepsilon > 0$. Then there exist $a \in L_\infty(\mathbb{R}_+)$ and $N \in \mathbb{N}$ such that*

$$\sup_{n>N} \left|\sigma(n) - \gamma_a(n)\right| \le \varepsilon. \tag{3.35}$$

Proof. By Proposition 3.5 there is $f \in \mathrm{RO}([0,+\infty))$ such that $f|_{\mathbb{Z}_+} = \sigma$ and $\|f\|_\infty = \|\sigma\|_\infty$. Define $h\colon \mathbb{R} \to \mathbb{C}$ as $h(x) = f(x^2)$. Then $h \in C_{b,u}(\mathbb{R})$. Moreover, by Proposition 1.9 there exists $\ell \in L_\infty(\mathbb{R})$ such that

$$\|H * \ell - h\|_\infty \le \frac{\varepsilon}{3}. \tag{3.36}$$

Denote the restriction $\ell|_{\mathbb{R}_+}$ by a. By (3.33) and Lemma 3.10, there are $L_1, L_2 > 0$ such that

$$\left|\gamma_a(n) - H * a(\sqrt{n}\,)\right| \le \frac{\varepsilon}{3}, \quad n \ge L_1, \quad |H * \ell(x) - H * a(x)| \le \frac{\varepsilon}{3}, \quad x \ge L_2. \tag{3.37}$$

Thus, taking $L = \max\{L_1, L_2\}$ by (3.36) and (3.37) one gets for every $n \ge \lceil L^2 \rceil$ that

$$\begin{aligned} \left|\gamma_a(n) - \sigma(n)\right| &\overset{x=\sqrt{n}}{\le} \left|\gamma_a(x^2) - H * a(x)\right| + |H * a(x) - H * \ell(x)| + |H * \ell(x) - h(x)| \\ &\le \left|\gamma_a(x^2) - H * a(x)\right| + |H * a(x) - H * \ell(x)| + \|H * \ell - h\|_\infty \le \varepsilon. \end{aligned}$$ ∎

Density in $c_0(\mathbb{Z}_+)$ of the eigenvalues' sequences

Next, we finish the proof of our main result. By Proposition 3.9 we already know that every sequence $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ can be approximated by some eigenvalues' sequence γ_a for large values of n. Thus, it only remains to prove that the sequences vanishing at the infinity can be approximated by eigenvalues' sequences.

Denote by $\mathscr{X}$ the Banach subspace of $L_\infty(\mathbb{R}_+)$ consisting of all bounded functions a having limit 0 at the infinity.

Lemma 3.11. *If $a \in \mathscr{X}$, then $\gamma_a \in c_0(\mathbb{Z}_+)$.*

Proof. Given $\varepsilon > 0$, there are $L > 0$ and $N_0 \in \mathbb{Z}_+$ such that

$$|a(t)| \le \frac{\varepsilon}{2}, \quad t \ge L, \quad n^{-1/2} \le \sqrt{\frac{\pi}{2}} \frac{\varepsilon}{\|a\|_\infty L^2}, \quad n \ge N_0. \tag{3.38}$$

Thus, by (3.23) and (3.38) we have for every $n \ge N_0$ that

$$\begin{aligned}
|\gamma_a(n)| &\le \frac{1}{n!}\left[\int_0^{L^2} |a(\sqrt{r})| e^{-r} r^n \, dr + \int_{L^2}^{+\infty} |a(\sqrt{r})| e^{-r} r^n \, dr\right] \\
&\le \frac{1}{n!}\left[\int_0^{L^2} |a(\sqrt{r})| e^{-r} r^n \, dr + \frac{\varepsilon}{2}\int_{L^2}^{+\infty} e^{-r} r^n \, dr\right] \\
&\le \frac{1}{n!}\int_0^{L^2} |a(\sqrt{r})| e^{-r} r^n \, dr + \frac{\varepsilon}{2} \le \frac{\|a\|_\infty}{n!}\int_0^{L^2} e^{-r} r^n \, dr + \frac{\varepsilon}{2} \\
&\le \frac{\|a\|_\infty e^{-n} n^n L^2}{n!} + \frac{\varepsilon}{2} \le \frac{\|a\|_\infty L^2}{\sqrt{2n\pi}} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

∎

Theorem 3.3. *$\{\gamma_a : a \in \mathscr{X}\}$ is a dense subset of $c_0(\mathbb{Z}_+)$.*

Proof. The inclusion $\{\gamma_a : a \in \mathscr{X}\} \subseteq c_0(\mathbb{Z}_+)$ was shown in Lemma 3.11. Unfortunately we were not able to prove the density by constructive tools; the next proof uses non-constructive duality arguments. By Hahn–Banach theorem, the density of $\{\gamma_a : a \in X\}$ in $c_0(\mathbb{Z}_+)$ will be shown if we prove that any continuous linear functional φ on $c_0(\mathbb{Z}_+)$ that vanishes on $\{\gamma_a : a \in \mathscr{X}\}$ is the zero functional. Thus, let $\phi \in c_0(\mathbb{Z}_+)^*$ be a linear functional such that $\phi(\gamma_a) = 0$ for each $a \in L_\infty(\mathbb{R}_+)$. Using the well-known description of the dual space of $c_0(\mathbb{Z}_+)$ we find a sequence $p = (p_n)_{n\in\mathbb{Z}_+} \in \ell_1(\mathbb{Z}_+)$ such that

$$\phi(y) = \sum_{n=0}^{\infty} p_n y_n \qquad y \in c_0(\mathbb{Z}_+).$$

Then we have that

$$0 = \phi(\gamma_a) = \sum_{n=0}^{\infty} \gamma_a(n) p_n, \quad a \in L_\infty(\mathbb{R}_+).$$

In particular, substituting $a = \chi_{[0,x]} \in \mathscr{X}$ with $0 \le x < +\infty$, we obtain

$$0 = \sum_{n=0}^{\infty} \gamma_a(n) p_n = \int_0^x \sum_{n=0}^{\infty} p_n K(n,r) \, dr.$$

The function $r \mapsto \sum_{n=0}^{\infty} p_n K(n,r)$, being the sum of a uniformly converging series of continuous functions, is continuous, and by the first fundamental theorem of calculus,

$$\sum_{n=0}^{\infty} p_n K(n,r) = 0, \qquad r \ge 0. \tag{3.39}$$

Now, replace $K(n,r)$ by $r^n e^{-r}/n!$ and factorize e^{-r}:

$$\sum_{n=0}^{\infty} \frac{p_n r^n}{n!} = 0 \qquad r \geq 0. \tag{3.40}$$

Denote by f the function

$$f(z) = \sum_{n=0}^{\infty} \frac{p_n z^n}{n!}.$$

Since $p_n \to 0$, we can find $M > 0$ such that $|p_n| \leq M$ for all $n \in \mathbb{Z}_+$. Hence, by (3.23) one gets that

$$0 < \left(\frac{|p_n|}{n!}\right)^{1/n} \leq \frac{e M^{1/n}}{n^{1/n}(2\pi)^{1/n}} \xrightarrow{n\to\infty} 0.$$

Thus, f has infite radius of convergence. i.e., f is an entire function. The equality (3.40) says that $f(r) = 0$ for every $r \geq 0$. Therefore f is the zero constant, and all coefficients p_n are zero. ■

Now we are ready to prove the main result of this chapter.

Theorem 3.4. *$\mathfrak{G}$ is dense in* $\mathrm{RO}(\mathbb{Z}_+)$.

Proof. Let $\sigma \in \mathrm{RO}(\mathbb{Z}_+)$ and $\varepsilon > 0$. By Proposition 3.9 there are $b \in L_\infty(\mathbb{R}_+)$ and $N \in \mathbb{Z}_+$ such that

$$|\sigma(n) - \gamma_b(n)| \leq \frac{\varepsilon}{2}, \quad n > N.$$

Define $\vartheta = (\vartheta(n))_{n\in\mathbb{Z}_+}$ by

$$\vartheta(n) = \begin{cases} \sigma(n) - \gamma_b(n), & \text{if } n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\vartheta \in c_0(\mathbb{Z}_+)$, and by Theorem 3.3 there exists $c \in L_\infty(\mathbb{R}_+)$ such that

$$\|\vartheta - \gamma_c\|_\infty \leq \frac{\varepsilon}{2}.$$

Taking $a = b + c \in L_\infty(\mathbb{R}_+)$ one gets that

$$\|\sigma - \gamma_a\|_\infty \leq \|\sigma - \gamma_b - \vartheta\|_\infty + \|\vartheta - \gamma_c\|_\infty \leq \sup_{n>N} |\sigma(n) - \gamma_b(n)| + \frac{\varepsilon}{2} \leq \varepsilon.$$ ■

3.4 Beyond the class of bounded generating symbols

In this section we describe the class of functions wider than $L_\infty(\mathbb{R})$, with eigenvalues' sequences belonging to $\mathrm{RO}(\mathbb{Z}_+)$. Furthermore, we give an unbounded generating symbol a such that $\gamma_a \in \ell_\infty(\mathbb{Z}_+) \setminus \mathrm{RO}(\mathbb{Z}_+)$.

Following [28] we denote by $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ the subspace of all measurable functions a on $\mathbb{R}_+$ for which the following integrals are finite for all $n \in \mathbb{Z}_+$:

$$\int_{\mathbb{R}_+} |a(r)|\, e^{-r^2} r^n dr < +\infty. \tag{3.41}$$

First of all, a Toeplitz operator T_φ with radial symbol $\varphi(z) = a(|z|)$ a.e. $z \in \mathbb{C}$, with $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is a well defined linear operator (in general unbounded) and has a dense domain. In fact, since the set $\mathscr{F}_0^2(\mathbb{C})$ of all polynomials on z form a dense subset on the Fock space, we have for the monomials $p(z) = z^n \in \mathscr{F}_0^2(\mathbb{C})$ that $(T_\varphi p)(z) = \gamma_a(n) p(z)$. Thus

$$T_a p \in \mathscr{F}_0^2(\mathbb{C}) \subset \mathscr{F}^2(\mathbb{C}),$$

and the set $\mathscr{F}_0^2(\mathbb{C})$ is a domain for each Toeplitz operator T_a with symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. That is, the operator T_a has a bounded extension to the whole space $\mathscr{F}^2(\mathbb{C})$ if and only if the sequence $\gamma_a = (\gamma_a(n))_{n\in\mathbb{Z}_+}$ is bounded.

Grudsky and Vasilevski in [28] proved that if $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, then the Toeplitz operator T_a acting on the Fock space $\mathscr{F}^2(\mathbb{C})$ is unitary equivalent to the multiplication operator $\gamma_a \mathrm{Id}$ acting on $\ell_2(\mathbb{Z}_+)$, where the sequence $\gamma_a = (\gamma_a(n))_{n\in\mathbb{Z}_+}$ is given by (3.17). From this fact every Toeplitz operator T_a with radial symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is bounded if and only if the corresponding eigenvalue sequence γ_a is bounded. Also they proved that for each sequence $\gamma \in \ell_\infty(\mathbb{Z}_+)$ there exists a symbol $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that the Toeplitz operator T_a is unitary equivalent to the multiplication operator by this sequence γ; i.e.,$\gamma_a = \gamma$. However, in this class of symbols there exists a nontrivial subspace $\mathscr{V}$ for which $T_a = 0$ for each $a \in \mathscr{V}$. Therefore, we only consider subspaces of $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ where the linear transformation $a \mapsto T_a$ is injective. More details see [28]

For $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$, we consider the following averages [28]:

$$\mathscr{B}_{(j)} a(r) = \int_r^{+\infty} \mathscr{B}_{(j-1)} a(u) e^{r-u} du, \quad j = 1, 2, \ldots, \tag{3.42}$$

where $\mathscr{B}_{(0)} a(r) = a(\sqrt{r})$. Integrating by parts j times one can express γ_a through $\mathscr{B}_{(j)} a$:

$$\gamma_a(n) = \frac{1}{(n-j)!} \int_{\mathbb{R}_+} \mathscr{B}_{(j)} a(r) r^{n-j} e^{-r} dr = \gamma_{\mathscr{B}_{(j)} a \circ \mathrm{sq}}(n-j), \quad n \geq j, \tag{3.43}$$

where $\mathrm{sq}(x) = x^2$, $x \in \mathbb{R}_+$. It is easily seen that if $\mathscr{B}_{(j)}a \in L_\infty(\mathbb{R}_+)$ for some $j \in \mathbb{Z}_+$, then the eigenvalues' sequence (and the corresponding Toeplitz operator) is bounded. The definition of the averages $\mathscr{B}_{(j)}a$ and the facts mentioned above are taken from [28, Section 4].

Next, we summarize some conditions that guarantee the boundedness of Toeplitz operators with radial symbols from $L_1^\infty(\mathbb{R}_+, e^{-r^2})$.

Proposition 3.10. *Let $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$. The Toeplitz operator T_a is bounded on $\mathscr{F}^2(\mathbb{C})$ (the corresponding eigenvalue sequence γ_a is bounded) if one of the following conditions holds:*

(a) $a \in L_\infty(\mathbb{R}_+)$

(b) $\mathscr{B}_{(j)}a \in L_\infty(\mathbb{R}_+)$ *for some* $j \in \mathbb{Z}_+$.

(c) $\widetilde{\gamma_a} = (\widetilde{\gamma_a}(n))_{n \in \mathbb{Z}_+} \in \ell_\infty(\mathbb{Z}_+)$, *where*

$$\widetilde{\gamma_a}(n) = \frac{1}{n!} \int_{\mathbb{R}_+} |a(\sqrt{r})| r^n e^{-r} dr. \tag{3.44}$$

Let us denote by $\mathscr{M}$ the class of symbols $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that the average (3.42) is bounded for some $j \in \mathbb{Z}_+$:

$$\mathscr{M} := \left\{ a \in L_1^\infty(\mathbb{R}_+, e^{-r^2}) \colon \mathscr{B}_{(j)}a \in L_\infty(\mathbb{R}_+) \quad \text{for some } j \in \mathbb{Z}_+ \right\}. \tag{3.45}$$

Proposition 3.11. *If $a \in \mathscr{M}$, then $\gamma_a \in \mathrm{RO}(\mathbb{Z}_+)$.*

Proof. Let $j \in \mathbb{Z}_+$ and $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ such that $\mathscr{B}_{(j)}a \in L_\infty(\mathbb{R}_+)$. By (3.43) we have $\gamma_a(n) = \gamma_{\mathscr{B}_{(j)}a \circ \mathrm{sq}}(n - j)$, with $\mathrm{sq}(x) = x^2$, $x \in \mathbb{R}_+$, hence by (3.18) and (3.21) one gets for every $n > j$ that

$$\begin{aligned}
\sqrt{n+1}\,\left|\gamma_a(n) - \gamma_a(n+1)\right| &= \sqrt{n+1}\,\left|\gamma_{\mathscr{B}_{(j)}a \circ \mathrm{sq}}(n-j+1) - \gamma_{\mathscr{B}_{(j)}a \circ \mathrm{sq}}(n-j)\right| \\
&\le \|\mathscr{B}_{(j)}a \circ \mathrm{sq}\|_\infty \kappa(n-j, n-j+1)\sqrt{n+1} \\
&\le \|\mathscr{B}_{(j)}a \circ \mathrm{sq}\|_\infty \sqrt{\frac{2(n+1)}{\pi(n-j+1)}}.
\end{aligned}$$

Hence $\sup_{n \in \mathbb{Z}_+} \left(\sqrt{n+1}\,\left|\gamma_a(n) - \gamma_a(n+1)\right|\right) < \infty$ and, by Proposition 3.4, the eigenvalue sequence γ_a is Lipschitz continuous with respect ϱ. ∎

Folland [21, Lemma 2.95] proved that for the class of unbounded measurable symbols $a \in L_1^\infty(\mathbb{R}_+, e^{-r^2})$ which satisfy the inequality

$$|a(r)| \le const\, e^{\delta r^2}, \quad \text{for some} \quad \delta < 1, \tag{3.46}$$

the linear mapping $a \mapsto T_a$ is injective. However, this class contains defining symbols which generate eigenvalues' sequences do not belonging to $\mathrm{RO}(\mathbb{Z}_+)$.

Example 3.4. Let $\delta = 1 - \frac{1}{\sqrt{2}}$. Then the function a defined by the rule

$$a(r) = e^{\left(\delta - \frac{i}{\sqrt{2}}\right) r^2}$$

satisfies (3.46) and belongs to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$. Let us calculate the corresponding eigenvalue's sequence using the change of variables $t = \sqrt{2}\, r$ and the formula [22, Eq. 3.381-5]:

$$\gamma_a(n) = \frac{1}{n!} \int_{\mathbb{R}_+} e^{\left(1 - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) r} e^{-r} r^n \, dr = \frac{2^{\frac{n+1}{2}}}{n!} \int_{\mathbb{R}_+} e^{-(1+i)t} t^n \, dt = e^{-i(n+1)\frac{\pi}{4}}.$$

The sequence of its consecutive differences is given by

$$\gamma_a(n+1) - \gamma_a(n) = e^{-i(n+2)\frac{\pi}{4}} \left(1 - e^{i\frac{\pi}{4}}\right)$$

and does not converge to 0, though $\rho(n+1, n) \to 0$. Thus $\gamma_a \in \ell_\infty(\mathbb{Z}_+) \setminus \mathrm{RO}(\mathbb{Z}_+)$.

CHAPTER 4

C*-ALGEBRA OF HORIZONTAL TOEPLITZ OPERATORS ACTING ON FOCK SPACES

In this chapter, we characterize the horizontal Toeplitz operators acting on the Fock spaces. The characterization is based on the decomposition of the Bargmann transform in two operators and the definition of horizontal operator. First of all, we give such decomposition of the Bargmann transform, after that we introduce the horizontal operators and study their basic properties, including a simple criterion for an operator to be horizontal. Also we introduce the $\mathscr{L}$–invariant operators, give some of their basic properties and using symplectic rotations of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ we finish this chapter with a explicit description of the C*-algebra generated by $\mathscr{L}$–invariant Toeplitz operators.

Let $\varsigma > 0$. Then the Fock $\mathscr{F}^2_\varsigma(\mathbb{C}^n)$ consists of all entire functions that are square integrable on $\mathbb{C}^n$ with respect to the Gaussian measure (1.24):

$$dg_{n,\varsigma}(z) = \left(\frac{\varsigma}{\pi}\right)^n e^{-\varsigma|z|^2} d\mu_n(z),$$

where μ_n is the usual Lebesgue measure on $\mathbb{C}^n$. In the present chapter for simplicity of calculations we consider $\varsigma = 1$ and we will write g_n instead of $g_{n,1}$.

4.1 Bargmann transform

As was mentioned in the introduction, the *Bargmann transform* is an isometric isomorphism from $L_2(\mathbb{R}^n)$ onto the Fock space $\mathscr{F}^2(\mathbb{C}^n)$ (see for example [53]), and hence plays an important role in the description of Toeplitz operators. In this section we construct two operators B_0 and U such that the Bargmann transform is the composition of them. By the multiplicativity of the Lebesgue and the Gaussian measure on $\mathbb{C}^n$ for each $m, l \in \mathbb{N}$ with $n = m + l$ we have the following isometries

$$L_2(\mathbb{C}^n, dg_n) \simeq L_2(\mathbb{C}^m, dg_m) \otimes L_2(\mathbb{C}^l, dg_l),$$
$$\mathscr{F}^2(\mathbb{C}^n) \simeq \mathscr{F}^2(\mathbb{C}^m) \otimes \mathscr{F}^2(\mathbb{C}^l),$$

where $\otimes$ is the usual (completed) tensor product of Hilbert spaces, and $\mathrm{P}_n = \mathrm{P}_m \otimes \mathrm{P}_l$.

Introduce the unitary operator $U_1 : L_2(\mathbb{C}^n, dg_n) \to L_2(\mathbb{R}^{2n}, dxdy)$ defined by

$$(U_1\varphi)(z) = \pi^{-n/2} e^{-\frac{z\cdot\bar{z}}{2}} \varphi(z). \tag{4.1}$$

Let $\mathscr{H}_0 = U_1(\mathscr{F}^2(\mathbb{C}^n))$. Then for each $f \in \mathscr{H}_0$ the function $\varphi(z) = \pi^{n/2} e^{\frac{z\cdot\bar{z}}{2}} f(z)$ belongs to $\mathscr{F}^2(\mathbb{C}^n)$ and thus

$$0 = \frac{\partial \varphi}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \left(\pi^{n/2} e^{\frac{z\cdot\bar{z}}{2}} f \right) = \pi^{n/2} e^{\frac{z\cdot\bar{z}}{2}} \left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2} \right) f.$$

It is easy to see that the subspace $\mathscr{H}_0$ of $L_2(\mathbb{R}^{2n})$ can be described as closure of the set of all smooth functions in $L_2(\mathbb{R}^{2n})$ which satisfy to the equation

$$D_0 f = 2\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2} \right) f = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + x + iy \right) f = 0 \tag{4.2}$$

The unitary operator $U_2 = \mathrm{Id} \otimes \mathrm{F}$, where

$$(\mathrm{F} f)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\eta\cdot y} f(\eta) d\eta$$

is the Fourier transformation, maps isometrically the space

$$L_2(\mathbb{R}^{2n}, dxdy) = L_2(\mathbb{R}^n, dx) \otimes L_2(\mathbb{R}^n, dy)$$

onto it self. The image $\mathscr{H}_0' = (\mathrm{Id} \otimes \mathrm{F})\mathscr{H}_0$ of the space $\mathscr{H}_0$ under the mapping $\mathrm{Id} \otimes \mathrm{F}$ is the closure of the set of all smooth functions in $L_2(\mathbb{R}^{2n})$ which satisfy to the equation

$$D_0' f = (\mathrm{Id} \otimes \mathrm{F}) D_0 (\mathrm{Id} \otimes \mathrm{F}^{-1}) f = \left(\frac{\partial}{\partial x} + x - \frac{\partial}{\partial y} - y \right) f = 0.$$

Now, introduce the isomorphism $U_3 = U_3^* = U_3^{-1} : L_2(\mathbb{R}^{2n}) \to L_2(\mathbb{R}^{2n})$ given by the rule

$$(U_3 f)(x, y) = f\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}} \right). \tag{4.3}$$

The isomorphism U_3 maps the space $\mathcal{H}_0'$ onto the space $\mathcal{H}$, which is the closure of the set of the smooth functions satisfying to the equation

$$Df = U_3 D_0' U_3 f = \sqrt{2}\left(\frac{\partial}{\partial y} + y\right) f = 0.$$

This last equation can be easily solved. Its general solution has the form

$$\pi^{-n/4} e^{-\frac{y^2}{2}} h(y),$$

where h is an arbitrary function from $L_2(\mathbb{R}^n)$. The function $l(y) = \pi^{-n/4} e^{-\frac{y^2}{2}}$ belongs to $L_2(\mathbb{R}^n)$ and has unit norm. Denote by L_0 the one–dimensional subspace of $L_2(\mathbb{R}^n)$ generated by the function l. Then, obviously $\mathcal{H} = L_2(\mathbb{R}^n) \otimes L_0$, and the operator $Q = \mathrm{Id} \otimes P_0$ gives the orthonormal projection of the space $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$ onto $\mathcal{H}$. Here

$$(P_0 f)(y) = \pi^{-n/2} \int_{\mathbb{R}^n} f(t) e^{-\frac{1}{2}(y^2+t^2)} dt \tag{4.4}$$

is the one–dimensional orthogonal projection of $L_2(\mathbb{R}^n)$ onto L_0. The following theorem summarizes the obtained results.

Theorem 4.1. *The unitary operator $U = U_3 U_2 U_1$ provides an isometric isomorphism of the space $L_2(\mathbb{C}^n, dg_n)$ onto the space $L_2(\mathbb{R}^{2n}) = L_2(\mathbb{R}^n) \otimes L_2(\mathbb{R}^n)$ under which*

(i) the space $\mathcal{F}^2(\mathbb{C}^n)$ is mapped onto $\mathcal{H} = L_2(\mathbb{R}^n) \otimes L_0$

(ii) for the Bargmann projection (1.27) we have

$$U \mathrm{P}_n U^{-1} = \mathrm{Id} \otimes P_0,$$

where P_0 is the one–dimensional orthogonal projection (4.4) onto the one–dimensional subspace L_0 in $L_2(\mathbb{R}^n, dy)$ generate by the function $l(y)$.

Introduce the isometric embedding $\mathrm{B}_0 : L_2(\mathbb{R}^n, dx) \to L_2(\mathbb{R}^n, dy) \otimes L_2(\mathbb{R}^n, dy)$ defined by

$$(\mathrm{B}_0 h)(x) = h(x) l(y). \tag{4.5}$$

The image of B_0 is exactly $\mathcal{H}$. Then the adjoint operator $\mathrm{B}_0^* : L_2(\mathbb{R}^{2n}) \to L_2(\mathbb{R}^n)$ is defined by

$$(\mathrm{B}_0^* f)(x) = \pi^{-n/4} \int_{\mathbb{R}^n} f(x,y) e^{-\frac{y^2}{2}} dy. \tag{4.6}$$

It is easy to see that

$$\begin{aligned} \mathrm{B}_0^* \mathrm{B}_0 &= \mathrm{Id} \colon L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \\ \mathrm{B}_0 \mathrm{B}_0^* &= Q = \mathrm{Id} \otimes P_0 \colon L_2(\mathbb{R}^{2n}) \to \mathcal{H}. \end{aligned}$$

The linear operator $\mathrm{B}^* = U^* \mathrm{B}_0 \colon L_2(\mathbb{R}^n) \longrightarrow \mathcal{F}^2(\mathbb{C}^n)$ has the form

$$(\mathrm{B}^* f)(z) = \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{\sqrt{2}x\cdot z - \frac{x^2}{2} - \frac{z^2}{2}}\, dx, \quad z \in \mathbb{C}, \tag{4.7}$$

and is an isometric isomorphism from $L_2(\mathbb{R}^n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$ known as the *Bargmann transform*. Its inverse $\mathrm{B}\colon \mathcal{F}^2(\mathbb{C}^n) \longrightarrow L_2(\mathbb{R}^n)$ is given by

$$(\mathrm{B} f)(x) = \pi^{-n/4} \int_{\mathbb{C}^n} f(z) e^{\sqrt{2}x\cdot \overline{z} - \frac{x^2}{2} - \frac{\overline{z}^2}{2}}\, dg_n(z), \quad x \in \mathbb{R}. \tag{4.8}$$

The operators B and B^* provide the following decompositions of the Bargmann projection P_n and of the identity operator on $L_2(\mathbb{R}^n)$

$$\begin{aligned} \mathrm{B}\mathrm{B}^* &= \mathrm{Id}_{L_2(\mathbb{R}^n)} \colon L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n) \\ \mathrm{B}^* \mathrm{B} &= \mathrm{P}_n \colon L_2(\mathbb{C}^n, dg_n) \longrightarrow \mathcal{F}^2(\mathbb{C}^n). \end{aligned}$$

Example 4.1. Let $z \in \mathbb{C}^n$ and $k_z(w) = e^{\overline{z}\cdot w}, \quad w \in \mathbb{C}^n$. By Proposition 6.10 of [53] applied n times we get

$$\begin{aligned} (\mathrm{B} k_z)(x) &= (\pi)^{-n/4} \int_{\mathbb{C}^n} e^{\overline{z}\cdot w} e^{\sqrt{2}x\cdot \overline{w} - \frac{x^2}{2} - \frac{\overline{w}^2}{2}}\, dg_n(w) \\ &= \prod_{j=1}^{n} \pi^{-1/4} \int_{\mathbb{C}} e^{\overline{z_j} w_j} e^{\sqrt{2}x_j \overline{w_j} - \frac{x_j^2}{2} - \frac{\overline{w_j}^2}{2}}\, dg(w_j) \\ &= \pi^{-n/4} e^{\sqrt{2}x\cdot \overline{z} - \frac{x^2}{2} - \frac{\overline{z}^2}{2}}, \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.9}$$

4.2 Horizontal Toeplitz operators

In this section we characterize the horizontal Toeplitz operators acting on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$. The characterization is based on the notion of horizontal operator and horizontal functions. So, we will first introduce the horizontal operators, the horizontal functions and study their basic properties, including a simple criterion for an operator and for a function to be horizontal. We finish this section showing that every Toeplitz operator with horizontal symbols is a unitary equivalent to certain multiplication operator.

Definition 4.1 (Weyl operator). Let $h \in \mathbb{C}^n$. The *Weyl operator* $\mathscr{W}_h$ on $L_2(\mathbb{C}^n, dg_n)$ is a weighted translation given by the rule

$$\mathscr{W}_h f(z) = e^{z\cdot\overline{h} - \frac{|h|^2}{2}} f(z-h), \quad z \in \mathbb{C}^n. \tag{4.10}$$

Let $h \in \mathbb{C}^n$, the adjoint operatorof $\mathscr{W}_h$ is $\mathscr{W}_h^* = \mathscr{W}_{-h}$. In effect, given $f \in \mathscr{F}^2(\mathbb{C})$ one has that

$$\begin{aligned}
\langle \mathscr{W}_h f, G\rangle &= \int_{\mathbb{C}^n} e^{z\cdot\overline{h} - \frac{|h|^2}{2}} f(z-h)\overline{G(z)} e^{-|z|^2} d\mu_n(z) \\
&\overset{w=z-h}{=} \int_{\mathbb{C}^n} e^{\overline{h}\cdot(w+h) - \frac{|h|^2}{2}} f(w)\overline{G(w+h)} e^{-|w+h|^2} d\mu_n(w) \\
&= \int_{\mathbb{C}^n} e^{w\cdot\overline{h} + |h|^2 - \frac{|h|^2}{2}} f(w)\overline{G(w+h)} e^{-|w|^2 - |h|^2 - h\cdot\overline{w} - w\cdot\overline{h}} d\mu_n(w) \\
&= \int_{\mathbb{C}^n} f(w)\overline{\left(e^{-w\cdot\overline{h} - \frac{|h|^2}{2}} G(w+h)\right)} e^{-|w|^2} d\mu_n(w) \\
&= \langle f, \mathscr{W}_{-h} G\rangle, \quad G \in \mathscr{F}^2(\mathbb{C}^n).
\end{aligned}$$

In fact, the Weyl operator $\mathscr{W}_h$ is unitary, with $\mathscr{W}_{-h} = \mathscr{W}_h^{-1}$. The following result summarizes some important properties of the Weyl operator.

Proposition 4.1. *Let $h \in \mathbb{C}^n$. The following statements hold:*
(a). If M_φ be the multiplication operator by $\varphi \in L_\infty(\mathbb{C}^n)$, then

$$\mathscr{W}_h M_\varphi \mathscr{W}_{-h} f = M_{\varphi\circ\tau_h} f, \quad f \in \mathscr{F}^2(\mathbb{C}^n). \tag{4.11}$$

(b). If $z \in \mathbb{C}^n$, then

$$\mathscr{W}_h k_z(w) = e^{-\overline{z}\cdot h - \frac{h^2}{2}} k_{z+h}(w), \quad w \in \mathbb{C}^n, \tag{4.12}$$

where $k_z(w) = e^{\overline{z}\cdot w}$.
(c). If $\varphi \in L_\infty(\mathbb{C}^n)$, then

$$\mathscr{W}_h T_\varphi \mathscr{W}_{-h} = T_{\varphi\circ\tau_h}. \tag{4.13}$$

Proof. (a). Let $\varphi \in L_\infty(\mathbb{C}^n)$ and $f \in \mathscr{F}^2(\mathbb{C}^n)$,

$$\begin{aligned}
\mathscr{W}_h M_\varphi \mathscr{W}_{-h} f(z) &= e^{z\cdot\overline{h} - \frac{|h|^2}{2}} \left(M_\varphi \mathscr{W}_h f\right)(z-h) = e^{z\cdot\overline{h} - \frac{|h|^2}{2}} \varphi(z-h)(\mathscr{W}_{-h} f)(z-h) \\
&= e^{z\cdot\overline{h} - \frac{|h|^2}{2}} \varphi(z-h) f(z) e^{-\overline{h}\cdot(z-h) - \frac{|h|^2}{2}} = \varphi\circ\tau_h(z) f(z) \\
&= (M_{\varphi\circ\tau_h} f)(z), \quad z \in \mathbb{C}^n.
\end{aligned}$$

(b). Given $z \in \mathbb{C}^n$, we have

$$\begin{aligned}\mathscr{W}_h k_z(w) &= e^{w\cdot\overline{h}-\frac{|h|^2}{2}}\, k_z(w-h) = e^{w\cdot\overline{h}-\frac{|h|^2}{2}+\overline{z}\cdot(w-h)}\\ &= e^{w\cdot\overline{(z+h)}-\frac{|h|^2}{2}-h\cdot\overline{z}} = e^{-h\cdot\overline{z}-\frac{|h|^2}{2}}\, k_{z+h}(w), \quad w\in\mathbb{C}^n.\end{aligned}$$

(c). Let $\varphi \in L_\infty(\mathbb{C}^n)$. Since for every $f \in \mathscr{F}^2(\mathbb{C}^n)$, the Toeplitz operator satisfies $(T_\varphi f)(z) = \langle M_\varphi f, k_z\rangle$, where $k_z(w) = e^{w\cdot\overline{z}}$, one has by (4.11) and (4.12) that

$$\begin{aligned}\left(T_\varphi \mathscr{W}_{-h} f\right)(z) &= \langle M_\varphi \mathscr{W}_{-h} f, k_z\rangle = \langle \mathscr{W}_h M_\varphi \mathscr{W}_{-h} f, \mathscr{W}_h k_z\rangle = \langle M_{\varphi\circ\tau_h} f, k_{z+h}\rangle\, \overline{e^{-h\cdot\overline{z}-\frac{|h|^2}{2}}}\\ &= e^{-z\cdot\overline{h}-\frac{|h|^2}{2}}\,(T_{\varphi\circ\tau_h} f)(z+h) = (\mathscr{W}_{-h} T_{\varphi\circ\tau_h} f)(z), \quad z\in\mathbb{C}^n.\end{aligned}$$

This clearly forces $\mathscr{W}_h T_\varphi \mathscr{W}_{-h} = T_{\varphi\circ\tau_h}$. ∎

Definition 4.2 (horizontal operators). A bounded operator S on the Fock space $\mathscr{F}^2(\mathbb{C}^n)$ is said to be invariant under Weyl translations if for every $h \in \mathbb{R}^n$ it commutes with $\mathscr{W}_{ih}$. That is, for every $h \in \mathbb{R}^n$

$$\mathscr{W}_{ih} S = S \mathscr{W}_{ih}. \tag{4.14}$$

For brevity, we use the term *horizontal* for such operator.

The set of all horizontal operators will be denoted by $\mathfrak{H}$:

$$\mathfrak{H} := \left\{S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}^n)) : \quad \forall h \in \mathbb{R}^n \quad \mathscr{W}_{ih} S = S\mathscr{W}_{ih}\right\}. \tag{4.15}$$

The simplest example of horizontal operator is itself the Weyl operator. In fact, if $t \in \mathbb{R}^n$, then for every $h \in \mathbb{R}^n$ one gets that

$$\mathscr{W}_{it}\mathscr{W}_{ih} = \mathscr{W}_{i(t+h)}(z) = \mathscr{W}_{ih}\mathscr{W}_{it}.$$

Proposition 4.2. *$\mathfrak{H}$ is a* C*-*subalgebra of* $\mathscr{B}(\mathscr{F}^2(\mathbb{C}^n))$.

Proof. Let $S, V \in \mathfrak{H}$ and $h \in \mathbb{R}^n$. Then $(S+V)\mathscr{W}_{ih} = S\mathscr{W}_{ih} + V\mathscr{W}_{ih} = \mathscr{W}_{ih}S + \mathscr{W}_{ih}V = \mathscr{W}_{ih}(S+V)$, we thus have that $S+V \in \mathfrak{H}$. On the other hand, $TS\mathscr{W}_{ih} = T\mathscr{W}_{ih}S = \mathscr{W}_{ih}TS$, hence $TS \in \mathfrak{H}$, this implies that $\mathfrak{H}$ is a subalgebra of $\mathscr{B}(\mathscr{F}^2(\mathbb{C}^n))$. The mapping $S \mapsto S^*$, where S^* is the adjoint operator of S, defines an involution on $\mathfrak{H}$, furthermore, for each $S \in \mathfrak{H}$ one has that

$$S^*\mathscr{W}_{ih} = \left(\mathscr{W}_{-ih}S\right)^* = \left(S\mathscr{W}_{-ih}\right)^* = \mathscr{W}_{ih}S^*.$$

Thus $\mathfrak{H}^* = \mathfrak{H}$. Now, given $S \in \overline{\mathfrak{H}}$, there exists $(S_n)_{n\in\mathbb{N}} \subset \mathfrak{H}$ such that $S_n \xrightarrow{n\to\infty} S$, but since $\mathscr{W}_{ih}S_n = S_n\mathscr{W}_{ih}$ and $\mathscr{W}_{ih}$ is a unitary operator, we get that $\mathscr{W}_{ih}S_n = S_n\mathscr{W}_{ih}$ converges to $\mathscr{W}_{ih}S$ and $S\mathscr{W}_{ih}$. Therefore by uniqueness of the limit we conclude $\mathscr{W}_{ih}S = S\mathscr{W}_{ih}$. That is $S \in \mathfrak{H}$. ∎

Lemma 4.1. *Let $h \in \mathbb{R}^n$. Then the Weyl operator $\mathcal{W}_{ih}$ is unitary equivalent to the multiplication operator $\mathrm{B}\mathcal{W}_{ih}\mathrm{B}^* = M_{E_h}$, acting on $L_2(\mathbb{R}^n)$, where the function $E_h : \mathbb{R}^n \to \mathbb{C}$ is given by $E_h(x) = e^{-i\sqrt{2}h\cdot x}$.*

Proof. Given $f \in L_2(\mathbb{R})$ and $h \in \mathbb{R}^n$, we calculate

$$\begin{aligned}
(\mathcal{W}_{ih}\mathrm{B}^* f)(z) &= e^{\overline{ih}\cdot z - \frac{h^2}{2}}(\mathrm{B}^* f)(z-ih) = \pi^{-n/4} e^{-ih\cdot z-\frac{h^2}{2}} \int_{\mathbb{R}^n} f(x) e^{\sqrt{2}x\cdot(z-ih)-\frac{x^2}{2}-\frac{(z-ih)^2}{2}}\,dx \\
&= \pi^{-n/4} e^{-ih\cdot z-\frac{h^2}{2}} \int_{\mathbb{R}^n} f(x) e^{\sqrt{2}x\cdot z-\sqrt{2}ix\cdot h-\frac{x^2}{2}-\left(\frac{z^2}{2}-\frac{h^2}{2}-z\cdot ih\right)}\,dx \\
&= \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{\sqrt{2}x\cdot z-\sqrt{2}ix\cdot h-\frac{x^2}{2}-\frac{z^2}{2}}\,dx \\
&= \pi^{-n/4} \int_{\mathbb{R}^n} \left(M_{E_h} f\right)(x) e^{\sqrt{2}x\cdot z-\frac{x^2}{2}-\frac{z^2}{2}}\,dx = (\mathrm{B}^* M_{E_h} f)(z), \quad z \in \mathbb{C}^n. \qquad \blacksquare
\end{aligned}$$

An important tool in the description of properties of bounded operators on Hilbert spaces of analytic functions is the *Berezin transform* (see Section 1.4). The Berezin transform of $S \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$ is the function $\widetilde{S}$ defined by

$$\widetilde{S}(z) = \frac{\langle S k_z, k_z\rangle}{\langle k_z, k_z\rangle}, \tag{4.16}$$

where the function $k_z(w) = e^{w\cdot\bar{z}}$ is the reproducing kernel of $\mathcal{F}^2(\mathbb{C}^n)$. By (4.12) and (4.13) it is easy to see for every $z \in \mathbb{C}^n$ that

$$\begin{aligned}
\widetilde{W_{ih} S \mathcal{W}_{-ih}} &= \frac{\langle \mathcal{W}_h S \mathcal{W}_{-h} k_z, k_z\rangle}{\langle k_z, k_z\rangle} = \frac{\langle S \mathcal{W}_{-h} k_z, \mathcal{W}_{-h} k_z\rangle}{\langle \mathcal{W}_{-h} k_z, \mathcal{W}_{-h} k_z\rangle} \\
&= \frac{\langle S k_{z-ih}, k_{z-ih}\rangle}{\langle k_{z-ih}, k_{z-ih}\rangle} = (\widetilde{S} \circ \tau_h)(z),
\end{aligned}$$

where $\tau_h z = z - ih$. Therefore, the horizontal operators are related with the translation operators acting on bounded functions.

Definition 4.3 (horizontal function). A function $\varphi \in L_\infty(\mathbb{C}^n)$ is said to be *horizontal* if for every $h \in \mathbb{R}^n$

$$\varphi(\zeta - ih) = \varphi(\zeta), \quad a.e.\, \zeta \in \mathbb{C}^n. \tag{4.17}$$

Let us consider the unitary operator $\mathrm{C}_i : \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ given by the rule

$$(\mathrm{C}_i f)(\zeta) = f(i\zeta), \quad \zeta \in \mathbb{C}^n. \tag{4.18}$$

Observe that if $f \in L_\infty(\mathbb{C}^n)$ is invariant under imaginary translations, then

$$(\mathrm{C}_i f)(\zeta - h) = f(i\zeta - ih) \overset{w=i\zeta}{=} f(w - ih) = f(i\zeta) = (\mathrm{C}_i f)(\zeta), \quad h \in \mathbb{R}^n. \tag{4.19}$$

Herrera Yañez, Maximenko and Vasilevski in the proof of Proposition 3.3 of [31] gave a criterion for a function to be invariant under translations on the upper half plane Π. This result can be extended on the whole complex plane $\mathbb{C}$ and its proof runs almost literally as in the Bergman case. Since it is very technical, it will be omitted.

Lemma 4.2. *A function $\varphi \in L_\infty(\mathbb{C}^n)$ is horizontal if and only if there exists $a \in L_\infty(\mathbb{R}^n)$ such that*

$$\varphi(z) = a(\operatorname{Re} z), \quad a.\ e.\ z \in \mathbb{C}^n.$$

Proof. Suppose that there exist $a \in L_\infty(\mathbb{R}^n)$ with $\varphi(z) = a(\operatorname{Re} z)$ a. e. $z \in \mathbb{C}^n$. Therefore for almost every $z \in \mathbb{C}^n$ we have $\varphi(z+ih) = a(\operatorname{Re}(z+ih)) = a(\operatorname{Re} z) = \varphi(z), \quad h \in \mathbb{R}^n$.

Conversely, let $\varphi \in L_\infty(\mathbb{C}^n)$ be a horizontal function. Then by (4.19) the function $\mathrm{C}_i\varphi$ is invariant under translations. In particular, for $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ and $g = \mathrm{C}_i\varphi$ one gets that

$$g(z', z_n + h_n) = g(z', z_n + h_n) = g(z', z_n), \quad h_n \in \mathbb{R},$$

hence the function $g(z', \cdot)$ is invariant under translation by $h_n \in \mathbb{R}$. Thus by Proposition 3.3 of [31] there exists $a_n \in L_\infty(\mathbb{R})$ such that $g(z', z_n) = a_n(z', \operatorname{Im} z_n)$ a. e. $z_n \in \mathbb{C}$. Making this procedure n-times we find a function $b \in L_\infty(\mathbb{R}^n)$ such that $g(z) = b(\operatorname{Im} z)$ a.e. $z \in \mathbb{C}^n$. Equivalently, $\varphi(z) = b(-\operatorname{Im}(iz)) = b(-\operatorname{Re} z)$, thus taking $a(x) = b(-x)$ the proof holds. ∎

Theorem 4.2 (criterion of horizontal operators). *Let $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}^n))$. Then, the following statements are equivalent:*

(i) *S is horizontal.*

(ii) $\mathrm{B} S \mathrm{B}^* M_{E_h} = M_{E_h} \mathrm{B} S \mathrm{B}^*, \quad h \in \mathbb{R}^n.$

(iii) *There exists $\varphi \in L_\infty(\mathbb{R}^n)$ such that* $\mathrm{B} S \mathrm{B}^* = M_\varphi$.

(iv) *The Berezin transform $\widetilde{S}$ is a horizontal function. i.e., there exists $b \in L_\infty(\mathbb{R}^n)$ such that*

$$\widetilde{S}(z) = b(\operatorname{Re} z), \quad a.e.\ z \in \mathbb{C}^n.$$

Proof. (i)⟶(ii) Let $S \in \mathscr{B}(\mathscr{F}^2(\mathbb{C}^n))$ be a horizontal operator , by Lemma 4.1 one gets that $\mathscr{W}_{ih} \mathrm{B}^* = \mathrm{B}^* M_{E_h}$ for each $h \in \mathbb{R}$. Thus

$$\mathrm{B} S \mathrm{B}^* M_{E_h} = \mathrm{B} S \mathscr{W}_{ih} \mathrm{B}^* = \mathrm{B} \mathscr{W}_{ih} S \mathrm{B}^* = M_{E_h} \mathrm{B} S \mathrm{B}^*, \quad h \in \mathbb{R}.$$

(ii)⟶(iii) Notice for every $h \in \mathbb{R}^n$ that $E_h(x) = \Theta_{\ln\left(\frac{h}{\sqrt{2}}\right)}(x)$, where $\Theta_\eta(x) = e^{ix\eta}$ and $\ln(h) = (\ln(n_1), \cdots, \ln(h_n)) \in \mathbb{R}^n$. Therefore, by (ii) one gets that

$$\mathrm{B} S \mathrm{B}^* M_{\Theta_\eta} = M_{\Theta_\eta} \mathrm{B} S \mathrm{B}^*, \quad \eta \in \mathbb{R}^n.$$

Thus, by Proposition A.4 n times, there exists $\varphi \in L_\infty(\mathbb{R})$ such that $\mathrm{B} S \mathrm{B}^* = M_\varphi$.
(iii)⟶(iv) Let $\varphi \in L_\infty(\mathbb{C}^n)$ be such that $S = \mathrm{B}^* M_\varphi \mathrm{B}$. Then

$$\begin{aligned}
\widetilde{S}(z) &= \frac{\langle M_\varphi \mathrm{B} k_z, \mathrm{B} k_z \rangle}{\langle k_z, k_z \rangle} = e^{-|z|^2} \int_{\mathbb{R}} \varphi(x) |(\mathrm{B} k_z)(x)|^2 dx \\
&= e^{-|z|^2} \int_{\mathbb{R}^n} \varphi(x) \left| \pi^{-n/4} e^{\sqrt{2} x \cdot \bar{z} - \frac{x^2}{2} - \frac{\bar{z}^2}{2}} \right|^2 dx \\
&= \pi^{-n/2} e^{-|z|^2} \int_{\mathbb{R}^n} \varphi(x) e^{-x^2} e^{\sqrt{2} x \cdot \bar{z} + \sqrt{2} x \cdot \bar{z} - \frac{z^2 + \bar{z}^2}{2}} dx \\
&= \pi^{-n/2} e^{-|z|^2} \int_{\mathbb{R}^n} \varphi(x) e^{-x^2} e^{2\sqrt{2} x \cdot \mathrm{Re}\, z - \frac{(z+\bar{z}) \cdot (z+\bar{z}) - 2|z|^2}{2}} dx \\
&= \pi^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-x^2} e^{2\sqrt{2} x \cdot \mathrm{Re}\, z - 2(\mathrm{Re}\, z)^2} dx = \pi^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-(x - \sqrt{2}\,\mathrm{Re}\, z)^2} dx, \quad z \in \mathbb{C}^n.
\end{aligned}$$

(iv)⟶(i) Let $h \in \mathbb{R}^n$, since $\mathcal{W}_{ih}$ is a unitary operator, we obtain by (4.12) that

$$\begin{aligned}
\widetilde{\mathcal{W}_{ih} S \mathcal{W}_{-ih}}(z) &= \frac{\langle S \mathcal{W}_{-ih} k_z, \mathcal{W}_{-ih} k_z \rangle}{\langle k_z, k_z \rangle} = \frac{\langle S \mathcal{W}_{-ih} k_z, \mathcal{W}_{-ih} k_z \rangle}{\langle \mathcal{W}_{-ih} k_z, \mathcal{W}_{-ih} k_z \rangle} = \frac{\langle S k_{z-ih}, k_{z-ih} \rangle}{\langle k_{z-ih}, k_{z-ih} \rangle} \\
&= \widetilde{S} \circ \tau_{ih}(z) = \widetilde{S}(z), \quad z \in \mathbb{C}^n.
\end{aligned}$$

Thus by injectivity of Berezin transform (Proposition 1.7) we conclude $\mathcal{W}_{ih} S \mathcal{W}_{-ih} = S$ for each $h \in \mathbb{R}^n$. ■

Proposition 4.3 (diagonalization of horizontal Toeplitz operators). *Let $\varphi(z) = a(\mathrm{Re}\, z)$ be a horizontal L_∞–function. Then the Toeplitz operator T_φ is unitary equivalent to multiplication operator $\mathrm{B} T_\varphi \mathrm{B}^* = \gamma_a^H \mathrm{Id}$ acting on $L_2(\mathbb{R}^n)$, where the function $\gamma_a^H : \mathbb{R}^n \longrightarrow \mathbb{C}$ is given by the rule*

$$\gamma_a^H(x) = \pi^{-n/2} \int_{\mathbb{R}^n} a\left(\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2} dy, \quad x \in \mathbb{R}^n. \tag{4.20}$$

Proof. Let $\varphi(z) = a(\mathrm{Re}\, z)$ be a horizontal L_∞–function. Then we have that

$$\begin{aligned}
\mathrm{B} T_a \mathrm{B}^* &= \mathrm{B} \mathrm{P}_n a \mathrm{P}_n \mathrm{B}^* = \mathrm{B}(\mathrm{B}^* \mathrm{B}) a (\mathrm{B}^* \mathrm{B}) \mathrm{B}^* \\
&= (\mathrm{B}\mathrm{B}^*) \mathrm{B} a \mathrm{B}^* (\mathrm{B}\mathrm{B}^*) = \mathrm{B} a \mathrm{B}^* \\
&= \mathrm{B}_0^* U_3 (I \otimes \mathrm{F}) U_1 a(x) U_1^{-1} (I \otimes \mathrm{F}^{-1}) U_3^{-1} \mathrm{B}_0 \\
&= \mathrm{B}_0^* U_3 a(x) U_3^{-1} \mathrm{B}_0 \\
&= \mathrm{B}_0^* a\left(\frac{x+y}{\sqrt{2}}\right) \mathrm{B}_0.
\end{aligned}$$

Now

$$(\mathrm{B}_0^* a\left(\frac{x+y}{\sqrt{2}}\right)\mathrm{B}_0 f)(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) f(x) l^2(y)\,dy = \gamma_a^H(x)\cdot f(x),$$

where

$$\gamma_a^H(x) = \int_{\mathbb{R}^n} a\left(\frac{x+y}{\sqrt{2}}\right) l^2(y)\,dy = \pi^{-n/2}\int_{\mathbb{R}^n} a\left(\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2}\,dy, \quad x\in\mathbb{R}^n. \qquad \blacksquare$$

Corollary 4.1. *Let $\varphi \in L_\infty(\mathbb{R}^{2n})$. The Toeplitz operator T_φ is horizontal if and only if φ is horizontal.*

Proof. Let $\varphi \in L_\infty(\mathbb{C}^n)$. If T_φ is horizontal, then for every $h \in \mathbb{R}^n$ one gets that

$$T_\varphi = \mathscr{W}_{ih} T_\varphi \mathscr{W}_{-ih} = T_{\varphi\circ\tau_h}.$$

Thus, by Proposition 1.6 we obtain that $\varphi(z) = \varphi\circ\tau_h(z) = \varphi(z-ih)$ almost every $z\in\mathbb{R}^{2n}$. Therefore φ is a horizontal function by Lemma 4.2.

Conversely, if φ is a horizontal function on $\mathbb{C}^n$, then by Lemma 4.2 there is $a \in L_\infty(\mathbb{R}^n)$ such that $\varphi(z) = a(\mathrm{Re}\, z)$ a. e. $z\in\mathbb{C}^n$. Hence by Proposition 4.3 and the criterion of horizontal operators (Theorem 4.2) we conclude that the Toeplitz operator T_φ is horizontal. $\blacksquare$

Denote by $\mathfrak{G}^H$ the set of all " horizontal spectral functions"

$$\mathfrak{G}^H = \{\gamma_a^H : a \in L_\infty(\mathbb{R}^n)\}. \tag{4.21}$$

Corollary 4.2. *The* C**-algebra $\mathscr{T}_{hor}(L_\infty)$ is isometrically isomorphic to the* C**-algebra $\mathscr{G}^H$ generated by $\mathfrak{G}^H$.*

4.3 $\mathscr{L}$–invariant Toeplitz operators

From now on we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by means of the mapping $z = x+iy \mapsto (x,y)$, where $x = (\mathrm{Re}\, z_1,\ldots,\mathrm{Re}\, z_n)$ and $y = (\mathrm{Im}\, z_1,\ldots,\mathrm{Im}\, z_n)$. Thus $\{0\}\times\mathbb{R}^n$ is identified with $i\mathbb{R}^n$, hence we may see the horizontal operators like a bounded operators invariant under Weyl translation over the Lagrangian plane $\{0\}\times\mathbb{R}^n$.

Let $\mathscr{L}$ be any Lagrangian plane of $(\mathbb{R}^{2n},\omega_0)$. In this section we extend the results about the horizontal Toeplitz operators to Toeplitz operators with $\mathscr{L}$-invariant symbols acting on the Fock space $\mathscr{F}^2(\mathbb{C}^n)$. The characterization is based on the symplectic rotations of the symplectic space $(\mathbb{R}^{2n},\omega_0)$. So, we will first introduce the $\mathscr{L}$–invariant functions and study their basic properties, including a simple criterion for a function to be $\mathscr{L}$–invariant.

We finish this section showing that the C*-algebra generated by Toeplitz operator with horizontal symbols is isometrically isomorphic to the C*-algebra generated by Toeplitz operators whose defining symbols are $\mathscr{L}$–invariant.

Definition 4.4 ($\mathscr{L}$–invariant functions). Let $\mathscr{L} \in \mathrm{Lag}(2n,\mathbb{R})$. A function $\varphi \in L_\infty(\mathbb{R}^{2n})$ is said to be invariant under *Lagrangian translations* if for every $h \in \mathscr{L}$ it satisfies

$$\varphi(z-h) = \varphi(z), \quad a.e. z = (x,y) \in \mathbb{R}^{2n}.$$

For brevity, we use the term *$\mathscr{L}$–invariant* for such functions. In the particular case $\mathscr{L} = \{0\} \times \mathbb{R}^n$ we only say horizontal.

Recall that the group of symplectic rotations $\mathrm{U}(2n,\mathbb{R})$ of real matrices $n \times n$ of $(\mathbb{R}^{2n}, \omega_0)$ is isomorphic to the unitary group $\mathrm{U}(n,\mathbb{C})$. In fact, by (1.40) one has that $\mathrm{U}(2n,\mathbb{R}) = \iota(\mathrm{U}(n,\mathbb{C}))$, where the isomorphism $\iota\colon \mathrm{U}(n,\mathbb{C}) \to \mathrm{U}(2n,\mathbb{R})$ is given by the rule

$$\iota(U+iV) = \begin{pmatrix} U & -V \\ V & U \end{pmatrix},$$

and $U, V \in \mathscr{M}(n,\mathbb{R})$ (1.39). Therefore, we may identify every Lagrangian plane $\mathscr{L}$ of $\mathbb{R}^{2n}$ with a subspace of $\mathbb{C}^n$(if there is not confusion it is denoted by the same $\mathscr{L}$).

Let $\mathscr{L}$ be any Lagrangian plane of $\mathbb{R}^{2n}$. By the transitive property of $\mathrm{U}(2n,\mathbb{R})$ (Proposition 1.11) and by the isomorphism $\mathrm{U}(2n,\mathbb{R}) \simeq \mathrm{U}(n,\mathbb{C})$, there is a unitary matrix $B \in \mathrm{U}(n,\mathbb{C})$ such that

$$B\mathscr{L} = i\mathbb{R}^n.$$

From now on, by simplicity of calculations we use this fact and re-write the definition of $\mathscr{L}$–invariant functions: let $\mathscr{L} \in \mathrm{Lag}(2n,\mathbb{R})$. A function $\varphi \in L_\infty(\mathbb{C}^n)$ is said to be $\mathscr{L}$–*invariant* if for every $h \in \mathscr{L}$ it satisfies

$$\varphi(z-h) = \varphi(z), \quad a.e. z \in \mathbb{C}^n.$$

In particular, the horizontal case corresponds to $\mathscr{L} = i\mathbb{R}^n$. Let $B \in \mathrm{U}(n,\mathbb{C})$. Define the linear operator $V_B : L_2(\mathbb{C}^n, dg_n) \to L_2(\mathbb{C}^n, dg_n)$ by the rule

$$(V_B f)(z) = f(B^* z), \quad z \in \mathbb{C}^n. \tag{4.22}$$

Since $B^* = B^{-1} \in \mathrm{U}(n,\mathbb{C})$, it is easy to see that V_B is a unitary operator, with $V_B^* = V_{B^{-1}}$.

Example 4.2. Let $B \in \mathrm{U}(2n,\mathbb{R})$ and k_z be the reproducing kernel of $\mathscr{F}^2(\mathbb{C}^n)$. Then, for every $z \in \mathbb{C}^n$ we have that

$$V_B k_z = k_{Bz}. \tag{4.23}$$

Let $\varphi \in L_\infty(\mathbb{C}^n)$. By (4.22), and (4.23) one gets for every $B \in \mathrm{U}(n,\mathbb{C})$ and $z \in \mathbb{C}^n$ that

$$\begin{aligned}(V_{B^{-1}}T_\varphi V_B f)(z) &= \langle T_\varphi V_B f, V_B k_z\rangle = \langle T_\varphi V_B f, k_{Bz}\rangle = \langle M_\varphi V_B f, k_{Bz}\rangle \\ &= \langle V_{B^{-1}}M_\varphi V_B f, k_z\rangle = \langle M_{\psi_B} f, k_z\rangle = (T_{\psi_B} f)(z), \quad f \in \mathscr{F}^2(\mathbb{C}^n).\end{aligned}$$

Therefore

$$V_{B^{-1}}T_\varphi V_B = T_{\psi_B}, \tag{4.24}$$

where $\psi_B : \mathbb{C}^n \to \mathbb{C}$ is the L_∞-function given by the rule

$$\psi_B(z) = \varphi(Bz), \quad z \in \mathbb{C}^n. \tag{4.25}$$

Lemma 4.3 (criterion for a function to be $\mathscr{L}$–invariant). *Let $\mathscr{L}$ be a Lagrangian plane and $B \in \mathrm{U}(n,\mathbb{C})$ be such that $B^*\mathscr{L} = i\mathbb{R}^n$. The function $\varphi \in L_\infty(\mathbb{C}^n)$ is $\mathscr{L}$–invariant if and only if there exists $a \in L_\infty(\mathbb{R}^n)$ such that*

$$\varphi(Bz) = a(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2, \ldots, \mathrm{Re}\, z_n), \quad a.e.\, z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n.$$

Proof. Suppose that φ is $\mathscr{L}$–invariant, then for every $h \in i\mathbb{R}^n$ one gets that $Bh \in \mathscr{L}$ and $\varphi(Bz - Bh) = \varphi(Bz), \quad z \in \mathbb{C}^n$. That is, the function ψ_B given in (4.25) is horizontal, and by Lemma 4.2 the statement holds.

Conversely, if there exists $a \in L_\infty(\mathbb{R}^n)$ such that $\varphi(Bz) = a(\mathrm{Re}\, z_1, \ldots, \mathrm{Re}\, z_n)$, a.e. $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then for every $h \in \mathscr{L}$ we have $B^*h \in i\mathbb{R}^n$ and

$$\begin{aligned}\varphi(z-h) &= \varphi(B(B^*z - B^*h)) = a(\mathrm{Re}(B^*z - B^*h)) \\ &= a(\mathrm{Re}\, B^*z) = \varphi(z), \quad a.e.\, z \in \mathbb{C}^n.\end{aligned}$$ ■

Proposition 4.4. *Let $\mathscr{L} \in \mathrm{Lag}(2n,\mathbb{R})$. The C*-algebra $\mathscr{T}_{\mathscr{L}}(L_\infty)$ generated by all Toeplitz operators whose L_∞–symbols are $\mathscr{L}$–invariant is isometrically isomophic to $\mathscr{T}_{hor}(L_\infty)$, and hence is isometrically isomorphic to $\mathscr{G}^H$.*

Proof. Let $\varphi \in L_\infty(\mathbb{C}^n)$ and $B \in \mathrm{U}(n,\mathbb{C})$ be such that $B^*\mathscr{L} = i\mathbb{R}^n$. Then by (4.24) and Lemma 4.3 one gets that T_φ belongs to $\mathscr{T}_{\mathscr{L}}(L_\infty)$ if and only if $T_{\psi_B} \in \mathscr{T}_{hor}(L_\infty)$. Thus, the mapping $T_\varphi \mapsto T_{\psi_B}$ generates an isometric isomorphism from $\mathscr{T}_{\mathscr{L}}(L_\infty)$ onto $\mathscr{T}_{hor}(L_\infty)$. ■

Example 4.3. Let $\mathscr{L} = \mathbb{R}^n \times \{0\}$ and φ be a $\mathscr{L}$–invariant function. Then observe that the standard symplectic matrix J rotates from the Lagrangian plane $\{0\} \times \mathbb{R}^n$ to $\mathbb{R}^n \times \{0\}$. i.e.,

$$\begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Therefore, if $B = -i\mathrm{I}_n \in \mathrm{U}(n,\mathbb{C})$, then by Lemma 4.3 we have that the corresponding spectral function is

$$\varphi(z) = \varphi(BB^*z) = \varphi(B(iz)) = a(\mathrm{Re}(iz_1),\dots,\mathrm{Re}(iz_n)) = a(-\mathrm{Im}\, z_1,\dots,-\mathrm{Im}\, z_n), \quad a.e. z \in \mathbb{C}^n.$$

Thus, by (4.20) the corresponding spectral function is:

$$\gamma_a^H(x) = \pi^{-n/2}\int_{\mathbb{R}^n} a\left(-\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2} dy, \quad x \in \mathbb{R}^n.$$

On the other hand, Let $\Delta = \left\{(x,x) \in \mathbb{R}^{2n} : x \in \mathbb{R}^n\right\}$ and φ be a Δ-horizontal functions. Then note that

$$\begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n \\ -\mathrm{I}_n & \mathrm{I}_n \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$$

Therefore, if $B = \frac{\mathrm{I}_n - i\mathrm{I}_n}{2} \in \mathrm{U}(n,\mathbb{C})$, then by Lemma 4.3 we have that

$$\varphi(z - iz) = \varphi\left(BB^*(z - iz)\right) = \varphi(Bz) = a(\mathrm{Re}\, z_1,\dots,\mathrm{Re}\, z_n), \quad a.e. z \in \mathbb{C}^n.$$

4.4 Density of spectral functions in $C_{b,u}(\mathbb{R}^n)$.

Let $a \in L_\infty(\mathbb{R}^n)$. We re-write the spectral functions γ_a^H given in (4.20) as follows:

$$\gamma_a^H(x) = \int_{\mathbb{R}^n} a\left(\frac{y}{\sqrt{2}}\right) H(x-y)dy, \tag{4.26}$$

where the function $H : \mathbb{R}^n \to \mathbb{R}_+$, is the n-dimensional heat kernel at time $t = 1/4$

$$H(x) = \pi^{-n/2} e^{-x^2}, \quad x \in \mathbb{R}^n. \tag{4.27}$$

Theorem 4.3. *The set of spectral functions $\mathfrak{G}^H$ is dense in $C_{b,u}(\mathbb{R}^n)$.*

Proof. By (4.26) we can write the horizontal spectral functions as convolution of the symbol a and the heat kernel H as follows:

$$\gamma_a = b * H, \quad b = a \circ m_{\sqrt{2}} \in L_\infty(\mathbb{R}^n), \tag{4.28}$$

where $m_{\sqrt{2}}(x) = \frac{x}{\sqrt{2}}$, $x \in \mathbb{R}^n$. Therefore, $\mathfrak{G}^H = \{a * H, a \in L_\infty(\mathbb{R}^n)\}$, and by Proposition 1.9 applied to $k = H$ we have that $\mathfrak{G}^H$ is dense in $C_{b,u}(\mathbb{R}^n)$. ■

Corollary 4.3. *The* C**-algebra $\mathcal{T}_{\mathcal{L}}(L_\infty)$ is isometrically isomorphic to $C_{b,u}(\mathbb{R}^n)$.*

As was mentioned in Chapter 3 , Folland [21, Lemma 2.95] proved that for the class of unbounded measurable symbols a which satisfy the inequality

$$|a(x)| \leq const\, e^{\delta x^2}, \quad \text{for some} \quad \delta < 1, \tag{4.29}$$

the linear mapping $a \mapsto T_a$ is injective. However, this class contains defining symbols which generate spectral functions do not belonging to $C_{b,u}(\mathbb{R}^n)$.

Example 4.4 (unidimensional case). Let $a(x) = e^{ix^2}e^{x^2}$, $x \in \mathbb{R}$. Then by formula 3.323-2 of [22] one gets that

$$\begin{aligned}
\gamma_a^H(y) &= \pi^{-1/2}\int_{\mathbb{R}} a\left(\frac{x}{\sqrt{2}}\right)e^{-(y-x)^2}dx = \pi^{-1/2}\int_{\mathbb{R}} e^{i\frac{x^2}{2}+\frac{x^2}{2}-y^2+2xy-x^2}dx \\
&= \pi^{-1/2}e^{-y^2}\int_{\mathbb{R}} e^{-\frac{(1-i)}{2}x^2+2xy}dx = \sqrt{\frac{2}{1-i}}\,e^{-y^2}e^{\frac{2y^2}{1-i}} \\
&= \sqrt{1+i}\,e^{iy^2}, \quad y \in \mathbb{R}.
\end{aligned}$$

Thus, the spectral function γ_a^H (and the corresponding Toeplitz operator) is bounded. Furthermore, for $x, y \in \mathbb{R}$ one has

$$\begin{aligned}
|\gamma_a^H(x) - \gamma_a^H(y)| &= 2^{1/4}\sqrt{([\cos x^2 - \cos y^2]^2 + [\sin x^2 - \sin y^2]^2)} = 2^{1/4}\sqrt{2\left(1-\cos(x^2-y^2)\right)} \\
&= 2^{5/4}\sin\left|\frac{x^2-y^2}{2}\right|.
\end{aligned}$$

Therefore, if $x_n = n$ and $y_n = n + (\pi/2n)$, then, $\lim_{n\to\infty}|x_n - y_n| = \lim_{n\to\infty}\frac{\pi}{2n} = 0$, but

$$\begin{aligned}
\lim_{n\to\infty}\left|\gamma_a^H(x_n) - \gamma_a^H(y_n)\right| &= 2^{5/4}\lim_{n\to\infty}\sin\left|\frac{x_n^2-y_n^2}{2}\right| = 2^{5/4}\lim_{n\to\infty}\sin\left|\frac{(x_n-y_n)(x_n+y_n)}{2}\right| \\
&= 2^{5/4}\lim_{n\to\infty}\sin\left(\frac{\pi}{4n}\left[2n+\frac{\pi}{2n}\right]\right) = 2^{5/4}\lim_{n\to\infty}\sin\left(\frac{\pi}{2}+\frac{\pi^2}{8n^2}\right) = 2^{5/4}.
\end{aligned}$$

Thus $\gamma_a^H \in L_\infty(\mathbb{R}) \setminus C_{b,u}(\mathbb{R})$.

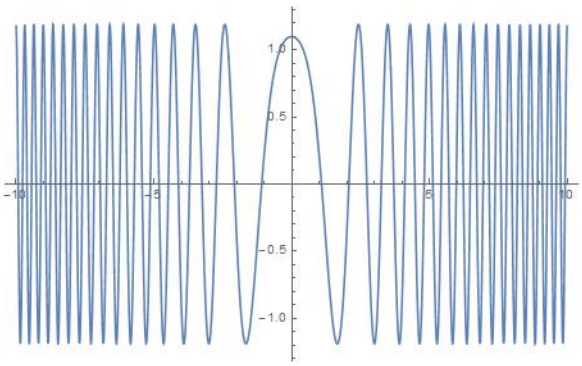


Figure 4.1: The real part of $\gamma_a^H(y) = \sqrt{1+i}\,e^{iy^2}$.

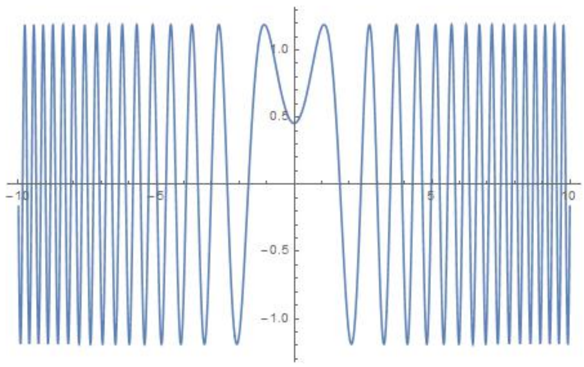


Figure 4.2: The imaginary part of $\gamma_a^H(y) = \sqrt{1+i}\,e^{iy^2}$.

Appendix A

A.1 Watson's Lemma

The technique of substitution of a partial sum of a known series into the integrand and integrating term-by-term while controlling the remainder is the basis of the proof of the following central result in the theory of exponential integrals due to Watson G. More details see [39, Proposition 2.1].

Proposition A.1. *[39, **Watson's Lemma**] Suppose $L > 0$ and φ is complex valued, absolutely integrable function on $[0, L]$:*

$$\int_0^L \left|\varphi(t)\right| dt + \infty.$$

Suppose further that φ is of the form $\varphi(t) = t^{\lambda} g(t)$ where $\lambda > -1$ and g has an infinite number of continuous derivatives in some neighborhood of $t = 0$. Then the exponential integral

$$F(x) := \int_0^L e^{-xt} \varphi(t)\, dt \tag{A.1}$$

is finite for all $x > 0$, and it has the asymptotic expansion

$$F(x) \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda + n + 1)}{n!\, x^{\lambda+n+1}}, \quad \text{as } x \to +\infty \tag{A.2}$$

A.2 Fourier transform of bounded Borel measures

Next, we consider the Fourier transform of bounded Borel measures on $\mathbb{R}$: one of the most important tools in analysis. More details see for example, [12, Section 3.8].

Definition A.1. Let ν be a bounded Borel measure on $\mathbb{R}$. The Fourier transform of ν is the complex function

$$\widetilde{\nu}(y) = \int_{\mathbb{R}} e^{iyx}\, d\nu(x), \quad y \in \mathbb{R}. \tag{A.3}$$

Proposition A.2. *[12, Proposition 3.8.6]*
If two bounded Borel measures have equal Fourier transforms, then they coincide.

A.3 Topology on a Banach space

Next, we are going to consider $(X, \|\cdot\|)$ a Banach space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We denote by X^* the dual space of X, which is the linear space of all continuous linear functional on X. It is a Banach space with norm given by

$$\|\rho\| = \sup_{x \in X\setminus\{0\}} \frac{|\rho(x)|}{\|x\|}. \tag{A.4}$$

Theorem A.1 ($L_\infty(X,\mu)$ **as dual of** $L_1(X,d\mu)$)**.**
Let (X,Ω,μ) be a measure space, where μ is σ-finite measure on X. For each $g \in L_\infty(X,\mu)$, the equation

$$\rho_g(f) = \int_X f(x)g(x)\, d\mu(x), \quad f \in L_1(X,d\mu) \tag{A.5}$$

defines a bounded linear functional ρ_g on the Banach space $L_1(X,d\mu)$, and the mapping $g \mapsto \rho_g$ is an isometric isomorphism from $L_\infty(X,\mu)$ onto the Banach dual space $\big(L_1(X,d\mu)\big)^*$.

Definition A.2 (weak-$*$ Topology).
The *weak-$*$ topology* (denoted by $\mathscr{W}$) of the space $L_\infty(X,\mu)$ considered as the dual of $L_1(X,\mu)$, is the weak topology on $\big(L_1(X,\mu)\big)^*$ induced by the family $\mathscr{F} = \{\varphi_g : g \in L_\infty(X,\mu)\}$, where for each $g \in L_\infty(X,\mu)$ the function $\varphi_g : (L_1(X,\mu))^* \mapsto \mathbb{C}$ is given by (A.5). Note that the finite intersections of the following sets form a local base of a function $a \in L_\infty(X,\mu)$ in the topology $\mathscr{W}$:

$$V(a,\varepsilon,h) = \{b \in L_\infty(X,\mu) \colon\ |\varphi_b(h) - \varphi_a(h)| < \varepsilon\}, \quad \varepsilon > 0, \quad h \in L_1(X,\mu). \tag{A.6}$$

A.4 Topologies on $\mathscr{B}(\mathscr{H})$

We will consider various topologies on $\mathscr{B}(\mathscr{H})$, the space of all bounded linear operators on the Hilbert space $\mathscr{H}$.

Definition A.3 (uniform Topology).
Let $(\mathscr{B}(\mathscr{H}), \|\cdot\|)$ be a normed space, where $\|\cdot\|$ is the norm given by the formula

$$\|T\| = \sup_{x \neq 0 \in \mathscr{H}} \frac{\|Tx\|}{\|x\|}, \quad T \in \mathscr{B}(\mathscr{H}). \tag{A.7}$$

This norm induces a metric, so $\mathscr{B}(\mathscr{H})$ is a metric space. Thus the *uniform topology* is just defined to be the metric topology.

In the uniform topology $T_n \to T$ if and only if $\|T_n - T\| \to 0$.

Definition A.4 (strong-Operator Topology (SOT)).
The *strong-operator topology* has as subbase the collection of all sets of the form

$$\mathscr{O}(T_0, x, \varepsilon) = \{T \in \mathscr{B}(\mathscr{H}) : \|(T - T_0)x\| \leq \varepsilon\} \tag{A.8}$$

We know a base is the collection of all finite intersections of such sets. It follows that a base is the collection of all sets of the form

$$\mathscr{V}(T_0, x_1, x_2, \cdots, x_n, \varepsilon) = \left\{T \in \mathscr{B}(\mathscr{H}) : \|(T - T_0)x_j\| \leq \varepsilon, \quad j = 1, 2, \cdots, n\right\} \tag{A.9}$$

The corresponding concepts of convergence: A net (T_n) converges in the strong-operator topology to T if and only if $\|(T_n - T)x\| \to 0$, for each $x \in \mathscr{H}$.

Definition A.5 (weak-Operator Topology (WOT)).
The *weak-operator topology* has as subbase the collection of all the finite intersections of the following sets:

$$U(T_0, x, y, \varepsilon) = \{T \in \mathscr{B}(\mathscr{H}) : |\langle (T - T_0)x, y\rangle| \leq \varepsilon\} \tag{A.10}$$

The corresponding concepts of convergence: A net (T_n) converges in the weak-operator topology to T if and only if $\langle (T_n - T)x, y\rangle \to 0$, for each $x, y \in \mathscr{H}$.

By the Cauchy-Schwarz inequality, strong-operator convergence implies weak-operator convergence. When the dimension of $\mathscr{H}$ is infinite, the weak-operator topology is strictly weaker than the strong-operator topology.

Example A.1. If T_n is the operator on $L_2(\partial\mathbb{D}, dt)$ defined by

$$T_n f(t) = e^{int} f(t), \quad f \in L_2(\partial\mathbb{D}, dt),$$

then $\langle T_n f, g\rangle = c_n$, where c_n is n-th coefficient of Fourier of $f\overline{g} \in L_1(\partial\mathbb{D}, dt)$, and by the Riemman-Lebesgue's theorem we have that $c_n \to 0$. Thus $T_n \to 0$ in the weak-operator topology, but $T_n \nrightarrow 0$ in the strong-operator topology, since each T_n is unitary.

Theorem A.2. *Suppose that $\mathscr{S}$ is a convex subset of $\mathscr{B}(\mathscr{H})$. Then the closure of $\mathscr{S}$ in the weak-operator topology coincides with the closure of $\mathscr{S}$ in the strong-operator topology in $\mathscr{B}(\mathscr{H})$.*

A.5 About multiplication operator

Let (X, Ω, μ) be a σ-finite measure space. $L_\infty(X, \mu)$ is the set of measurable functions which are bounded almost everywhere in X. $L_2(X, d\mu)$ is a Hilbert space.

Given $\varphi \in L_\infty(X, \mu)$ there is a corresponding linear transformation $\varphi \mapsto M_\varphi$ on $\mathscr{B}\left(L_2(X, d\mu)\right)$, where

$$M_\varphi f = \varphi f, \quad f \in L_2(X, d\mu). \tag{A.11}$$

Define

$$A_\mu = \left\{M_\varphi : \varphi \in L_\infty(X, \mu)\right\}. \tag{A.12}$$

It is an abelian subalgebra of $\mathscr{B}\left(L_2(X, d\mu)\right)$.

Proposition A.3. *Let (X, Ω, μ) be a σ-finite measure space. If $\varphi \in L_\infty(X, \mu)$ the following statements hold:*

i). The operator M_φ is normal, and $M_\varphi^* = M_{\overline{\varphi}}$.

ii). $\varphi \mapsto M_\varphi$ is a $*$-homomorphism from $L_\infty(X, \mu)$ onto A_μ.

iii). $\|M_\varphi\| = \|\varphi\|_\infty$.

Proof. i). Let $\varphi \in L_\infty(X, \mu)$, then

$$\begin{aligned}
\langle M_\varphi f, g\rangle &= \int_X \left(M_\varphi f\right)(x)\overline{g(x)}\, d\mu(x) = \int_X \varphi(x) f(x)\overline{g(x)}\, d\mu(x) \\
&= \int_X f(x)\overline{\left(\overline{\varphi(x)} g(x)\right)}\, d\mu(x) = \int_X f(x)\overline{\left(M_{\overline{\varphi}} g\right)(x)}\, d\mu(x) \\
&= \langle f, M_{\overline{\varphi}} g\rangle, \quad \forall f, g \in L_2(X, d\mu).
\end{aligned}$$

Thus $M_{\varphi}^{*} = M_{\overline{\varphi}}$.

ii). Note that $\varphi+\psi \mapsto M_{\varphi+\psi} = M_{\varphi}+M_{\psi} \in A_{\mu}$, and $\varphi\psi \mapsto M_{\varphi\psi} = M_{\varphi}M_{\psi} \in A_{\mu}$, for every $\varphi,\psi \in L_{\infty}(X,\mu)$. On the other hand, by i we have that $\overline{\varphi} \mapsto M_{\overline{\varphi}} = M_{\varphi}^{*} \in A_{\mu}$, for all $\varphi \in L_{\infty}(X,\mu)$. Hence, $\varphi \mapsto M_{\varphi}$ is a $*$-homomorphism from $L_{\infty}(X,\mu)$ onto A_{μ}.

iii). Let $g \in L_2(X,d\mu)$,

$$\|M_{\varphi}g\|_2^2 = \int_X |\varphi(x)g(x)|^2\, d\mu(x) \le \|\varphi\|_{\infty}^2 \|g\|_2^2.$$

Thus

$$\|M_{\varphi}\| \le \|\varphi\|_{\infty}. \tag{A.13}$$

To show that equality is obtained, let $\varepsilon > 0$ be given, then the subset

$$S_{\varepsilon} = \left\{x \in X : |\varphi(x)| \ge \|\varphi\|_{\infty} - \varepsilon\right\},$$

is measurable, and since μ is σ-finite, this subset has a subset S of finite measure. Let χ_S be the characteristic function of the set S, then $\chi_S \in L_2(X,d\mu)$. Moreover

$$\begin{aligned}\|M_{\varphi}\| \ge \|M_{\varphi}\chi_S\| &= \int_S |\varphi(x)|^2\, d\mu(x) \ge \left(\|\varphi\|_{\infty} - \varepsilon\right)^2 \mu(S)\\ &= \left(\|\varphi\|_{\infty} - \varepsilon\right)^2 \|\chi_S\|_2^2.\end{aligned}$$

Thus, for all $\varepsilon > 0$ we have $\|M_{\varphi}\| \ge \|\varphi\|_{\infty} - \varepsilon$. Hence

$$\|M_{\varphi}\| \ge \|\varphi\|_{\infty}. \tag{A.14}$$

Combining (A.13) and (A.14) the equality is proved. ■

Proposition A.4. *[34, Theorem 2.5.10] Let $S \in \mathscr{B}(L_2(\mathbb{R}))$ and M_{Θ_η} be the multiplication operator by the function $\Theta_\eta(t) = e^{it\eta}, \quad t \in \mathbb{R}$. The following conditions are equivalent:*
(a) S is invariant under M_{Θ_η} for all $\eta \in \mathbb{R}$:

$$SM_{\Theta_\eta} = M_{\Theta_\eta}S.$$

(b) S is the multiplication operator by a bounded measurable function:

$$\exists \varphi \in L_{\infty}(\mathbb{R}) \quad \text{such that} \quad S = M_{\varphi}.$$

Proposition A.5. *[13, Proposition 10.5] Let (X,Ω,μ) be a σ-finite measure space. Consider $L_{\infty}(X,\mu)$ as the dual of $L_1(X,d\mu)$ equipped with the weak-$*$ topology. A net (φ_j) of functions in $L_{\infty}(X,\mu)$ converges in the weak-$*$ topology to a function $\varphi \in L_{\infty}(X,\mu)$, if and only if (M_{φ_j}) is weak-operator convergent to M_{φ} in $\mathscr{B}\left(L_2(X,d\mu)\right)$.*

Proof. Suppose that $\varphi_j \to \varphi$ in the weak-$*$ topology. Therefore for each $h \in L_1(X, d\mu)$ one gets
$$\int_X \varphi_j(x) h(x)\, d\mu(x) \to \int_X \varphi(x) h(x)\, d\mu(x).$$
If $f, g \in L_2(X, d\mu)$, then taking $h = f\overline{g} \in L_1(X, d\mu)$ and by the above remark we have
$$\langle M_{\varphi_j} f, g\rangle = \int_X \varphi_j(x) f(x)\overline{g(x)}\, d\mu \to \int_X \varphi(x) f(x)\overline{g(x)}\, d\mu = \langle M_\varphi f, g\rangle.$$
This implies that $M_{\varphi_j} \to M_\varphi$ in the weak-operator topology.

Conversely, assume that $M_{\varphi_j} \to M_\varphi$ in the weak-operator topology. Given $h \in L_1(X, d\mu)$ there exists $f, g \in L_2(X, d\mu)$ such that $h = f\,\overline{g}$. Therefore
$$\int_X \varphi_j(x) h(x)\, d\mu(x) = \int_X \varphi_j(x) f(x)\overline{g(x)}\, d\mu(x) = \langle M_{\varphi_j} f, g\rangle \to \langle M_\varphi f, g\rangle = \int_X \varphi(x) h(x)\, d\mu.$$
That is, the net (φ_j) converges to φ in the weak-$*$ topology. ■

Proposition A.6. *Let (X, Ω, μ) be a σ-finite measure space. Then the mapping $a \mapsto M_a$ is a homeomorphic embedding of $L_\infty(X, \mu)$ into $(\mathscr{B}\big(L_2(X, d\mu)\big), WOT)$.*

Proof. First we note that if f, g, h are some functions such that $h = f\overline{g}$ and $f, g \in L_2(X, \mu)$, then for every $b \in L_\infty(X, \mu)$ we have
$$|\langle (M_b - M_a) f, g\rangle| = \int_X (b - a) f\,\overline{g}\, d\mu = \int_X (b - a) h\, d\mu = |\phi_b(h) - \phi_a(h)|.$$
Given $a \in L_\infty(X, \mu)$, $\varepsilon > 0$ and $f, g \in L_2(X, \mu)$, we define h as $h = f\overline{g}$ and obtain
$$\{M_b : \ b \in V(a, \varepsilon, h)\} \subseteq U(M_a, \varepsilon, f, g).$$
Conversely, given $a \in L_\infty(X, \mu)$, $\varepsilon > 0$ and $h \in L_1(X, \mu)$, we easily construct two functions $f, g \in L_2(X, \mu)$ such that $h = f\overline{g}$ and obtain
$$\{b \in L_\infty(X, \mu) : \ M_b \in U(M_a, \varepsilon, f, g)\} \subseteq V(a, \varepsilon, h).$$ ■

The space $L_\infty(\mathbb{R})$ may be identified with the dual space of $L_1(\mathbb{R})$. We denote by $\mathscr{W}$ the corresponding weak-$*$ topology on $L_\infty(\mathbb{R})$.

Proposition A.7. *$C_0(\mathbb{R})$ is dense in $(L_\infty(\mathbb{R}), \mathscr{W})$.*

Proof. Let $f \in L_\infty(\mathbb{R})$, $h_1, h_2, \ldots, h_m \in L_1(\mathbb{R})$ and $\varepsilon > 0$. Our goal is to find a function $a \in C_0(\mathbb{R})$ such that for every $j \in \{1, \ldots, m\}$
$$|\phi_a(h_j) - \phi_f(h_j)| \le \varepsilon.$$

If $\|f\|_\infty = 0$, then $a = 0$ do the work, so we suppose that $\|f\|_\infty > 0$. Using the assumption that $h_1, \dots, h_m \in L_1(\mathbb{R})$ and the continuity of the Lebesgue integral, we find a $\delta > 0$ with $\delta < 1/(4\|f\|_\infty)$ such that for every measurable subset Y with $\mu(Y) < \delta$

$$\int_Y |h_j|\, d\mu \le \frac{\varepsilon}{4\|f\|_\infty}, \qquad j \in \{1, \dots, m\}. \tag{A.15}$$

We also choose an open interval A with

$$\int_{\mathbb{R}\setminus A} |h_j|\, d\mu \le \frac{\varepsilon}{4\|f\|_\infty}, \qquad j \in \{1, \dots, m\}. \tag{A.16}$$

Applying Lusin's theorem to the function f on the segment clos(A) we find a continuous function b on clos(A) and compact subset K of A such that $f|_K = b|_K$ and

$$\mu(A \setminus K) < \delta. \tag{A.17}$$

Now, by Urysohn's Lemma, there exists a continuous function u on $\mathbb{R}$ with values in $[0, 1]$ such that $u(x) = 1$ for each $x \in K$ and $u(x) = 0$ for each $x \in \mathbb{R} \setminus A$. Define the function a on $\mathbb{R}$ by

$$a(x) = \begin{cases} b(x)u(x) & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Then a is continuous, $a|_K = f|_K$ and $\|a\|_\infty \le \|f\|_\infty$. Applying (A.15) with $Y = A \setminus K$ and (A.16) we get

$$\int_{\mathbb{R}\setminus K} |h_j|\, d\mu = \int_{\mathbb{R}\setminus A} |h_j|\, d\mu + \int_{A\setminus K} |h_j|\, d\mu \le \frac{\varepsilon}{2\|f\|_\infty}.$$

Combining this with $\|f - a\|_\infty \le \|f\|_\infty + \|a\|_\infty \le 2\|f\|_\infty$ we finally obtain the following upper estimate for each $j \in \{1, \dots, m\}$:

$$\left|\varphi_a(h_j) - \varphi_f(h_j)\right| \le \int_{\mathbb{R}} |a(x) - f(x)|\, |h_j(x)|\, dx \le 2\|f\|_\infty \int_{\mathbb{R}\setminus K} |h_j(x)|\, dx \le \varepsilon. \qquad \blacksquare$$

A.6 Classical harmonic analysis and Wiener's theorem

The algebra $L_1(\mathbb{R}^n)$

The complex-valued functions on $\mathbb{R}^n$ which are Lebesgue integrable over $\mathbb{R}^n$, denoted by $L_1(\mathbb{R}^n)$ form a Banach algebra over the Complex numbers $\mathbb{C}$. The norm is given by

$$\|f\|_1 = \int_{\mathbb{R}^n} |f(x)| dx,$$

multiplication is the convolution $f * g$, defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \quad f,g \in L_1(\mathbb{R}^n),$$

and the inequality

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

holds. It is easily seen that convolution is commutative; thus $L_1(\mathbb{R}^n)$ is a commutative Banach algebra.

In $L_1(\mathbb{R}^n)$ we have two important operators: The *translation* operator L_x given by

$$\tau_x f(y) = f(y-x), \quad x \in \mathbb{R}^n, \tag{A.18}$$

an the *dilation* operator D_h given by

$$D_h f(x) = hf(hx), \quad h \in \mathbb{R}_+.$$

These operators are both isometric:

$$\|\tau_x f\|_1 = \|f\|_1, \quad \|D_h f\|_1 = \|f\|_1. \tag{A.19}$$

The following result due to Lebesgue has many applications.

Proposition A.8. *Let $f \in L_1(\mathbb{R}^n)$. Then for any $\varepsilon > 0$ there is a neighbourhood V_ε of 0 in $\mathbb{R}^n$ such that*

$$\|\tau_x f - f\|_1 \leq \varepsilon, \quad x \in V_\varepsilon.$$

Next, we give an immediate consequence of Proposition A.8 and (A.19).

Corollary A.1. *For each $f \in L_1(\mathbb{R}^n)$ the mapping $x \mapsto \tau_x f$ from $\mathbb{R}^n$ into $L_1(\mathbb{R}^n)$ is uniformly continuous. Likewise, $x \mapsto \|\tau_x f - f\|_1$ is uniformly continuous.*

Wiener's Theorem

Wiener's theorem expresses a fundamental fact concerning Fourier transforms, and has been the starting poitn of many contemporary developments in harmonic analysis. More details see for example [42, Lemma 1.4.2].

Theorem A.3 (Wiener–Lévy)**.** *Let $f \in L_1(\mathbb{R}^n)$ and $K \subset \mathbb{R}^n$ be a compact set. If V is an open neighborhood of $\hat{f}(K)$ and $A : V \to \mathbb{C}$ is an analytic function, then there is a function $u \in L_1(\mathbb{R}^n)$ such that $\hat{g}(t) = A(\hat{f}(t))$ for all $t \in K$.*

Theorem A.4 (Wiener's Division Lemma)**.** *Let $f,g \in L_1(\mathbb{R}^n)$ such that* $\operatorname{supp}(\hat{f})$ *is a compact set, and $\hat{g}(t) \neq 0$ for every $t \in \mathbb{R}^n$. Then there exists $h \in L_1(\mathbb{R}^n)$ such that $f = g * h$.*

Approximate identity

Definition A.6. An *approximate identity* in a normed algebra $\mathscr{A}$ is a net $(e_j)_{j\in J}$ in $\mathscr{A}$ such that for every x in $\mathscr{A}$ it holds

$$\lim_{j\in J} e_j x = \lim_{j\in J} x e_j = x. \tag{A.20}$$

If there is a finite constant $M > 0$ such that $\|e_j\| \leq M$ for every $j \in J$, then the approximate identity is said to be *bounded*.

In fact, in this definition it is sufficient to consider only non-zero elements $x \in \mathscr{A}$. In a similar way left and right approximate identity is defined. Note that unbounded approximate identities are not particularly useful, see [15] for some pathological examples in incomplete normed algebras.

An approximately unital algebra shares some of the properties of a unital algebra. Obviously, if $\mathscr{A}$ is a unital algebra with unit e and J is an arbitrary directed set, then we can define an approximate identity $(e_j)_{j\in J}$ in $\mathscr{A}$ easily by the rule $e_j = e$ for all $j \in J$. Also, from Definition A.6 it follows that if an approximate identity is a divergent net, then the normed algebra is non-unital.

Proposition A.9. *$L_1(\mathbb{R}^n)$ contains an approximate identity $(e_j)_{j\in J}$ such that $\widehat{e_j} \in C_c(\mathbb{R}^n)$ for each $j \in J$.*

Dirac sequences

It is known that any Dirac sequence $(h_m)_{m\in\mathbb{N}}$ behaves like an identity for convolution in the limit as $n \to \infty$. Some people call Dirac sequences "approximate identities" for this reason. Next, we introduce the Dirac sequences and give an example of them.

Definition A.7 (Dirac sequences). A sequence $(h_m)_{m\in\mathbb{N}}$ of functions belonging to $L_1(\mathbb{R}^n)$ is called a *Dirac sequence* if it satisfies the following conditions:

(a) For each $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$, one gets $h_m(x) \geq 0$.

(b) For each $m \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} h_m(t)dt = 1.$$

(c) For every open neighborhood U of 0 in $\mathbb{R}^n$ it holds

$$\lim_{m\to+\infty} \int_{\mathbb{R}^n\setminus U} h_m(t)dt = 0.$$

Example A.2. The sequence $(h_n)_{n\in\mathbb{N}}$ given by

$$h_n(x) = \frac{2\sin^2(nx)}{\pi n x^2}, \quad x \in \mathbb{R}, \tag{A.21}$$

is a Dirac sequence.

Proof. Note that $h_n(-x) = h_n(x) \geq 0$ for each $x \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} h_n(t)dt = 1, \quad n \in \mathbb{N}.$$

On the other hand, let $\delta > 0$, and $n \in \mathbb{N}$, hence

$$\int_{|t|>\delta} h_n(t)dt = 2\int_{t>\delta} h_n(t)dt = \frac{4n}{\pi}\int_{t>\delta} \frac{\sin^2(nt)}{(nt)^2}dt \overset{x=nt}{=} \frac{4}{\pi}\int_{n\delta}^{+\infty} \frac{\sin^2 x}{x^2}dx.$$

Thus $\lim_{n\to+\infty}\int_{|t|>\delta} h_n(t)dt = 0$ for each $\delta > 0$. ■

For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, set $f(x) = \prod_{k=1}^n f_k(x_k)$ where $f_1,\ldots,f_n \in L_1(\mathbb{R})$. Then $f \in L_1(\mathbb{R}^n)$ and its Fourier transform is such that

$$\widehat{f}(t) = \prod_{k=1}^n \widehat{f_k}(t_k), \quad t = (t_1,\ldots,t_n) \in \mathbb{R}^n.$$

In this way we can use functions in $L_1(\mathbb{R})$ with certain properties to obtain function in $L_1(\mathbb{R}^n)$ with analogous properties, especially concerning Fourier transforms.

Example A.3. The sequence $(h_m)_{m\in\mathbb{Z}_+}$ given by

$$h_m(x) = \left(\frac{2}{\pi}\right)^n \prod_{k=1}^n \frac{\sin^2(mx_k)}{m^n x_k^2}, \quad x = (x_1,\ldots,x_n) \in \mathbb{R}^n, \tag{A.22}$$

is a Dirac sequence. In effect, note that $h_n(-x) = h_n(x) \geq 0$ for each $x \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} h_n(t)dt = 1, \quad n \in \mathbb{Z}_+.$$

On the other hand, let $\delta > 0$, and $n \in \mathbb{Z}_+$, hence by Example A.2 one has that

$$\int_{\mathbb{R}^n\setminus\prod_{j=1}^n(-\delta,\delta)} h_m(t)dt = 2^n \prod_{j=1}^n \int_{t_j>\delta} h_m(t_j)dt_j \xrightarrow{m\to+\infty} 0$$

Thus, since for every open neighborhood U of 0 in $\mathbb{R}^n$, we can choose a $\delta > 0$ such that $\prod_{j=1}^n(-\delta,\delta) \subset U$ one has that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^n\setminus U} h_m(t)dt \leq \lim_{n\to+\infty}\int_{\mathbb{R}^n\setminus\prod_{j=1}^n(-\delta,\delta)} h_m(t)dt = 0.$$

On the other hand, the Fourier transform of h_m is

$$\widehat{h_m}(\xi) = \left(\frac{2}{m\pi}\right)^n \prod_{k=1}^n \widehat{\frac{\sin^2(m\xi_k)}{\xi_k^2}} = \left(\frac{2}{\pi}\right)^{n/2} \prod_{k=1}^n \begin{cases} \left(1 - \frac{|\xi_k|}{m}\right) & \text{if } |\xi_k| \leq m \\ 0 & \text{if } |\xi_k| > m. \end{cases}$$

Since the Dirac sequences can be viewed as approximate identities, they provide a powerful tool to approximate functions. The next lemma is a well-known result for uniformly continuous functions, see for example [20, Proposition 2.42].

Lemma A.1. *Let $f \in C_{b,u}(\mathbb{R}^n)$. If $(h_m)_{m\in\mathbb{Z}_+}$ is a Dirac sequence, then*

$$\lim_{m\to\infty} \|f * h_m - f\|_\infty = 0. \tag{A.23}$$

Proof. Due to $f \in C_{b,u}(\mathbb{R}^n)$, given $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists a open neighborhood $\mathscr{V}$ of $0 \in \mathbb{R}^n$ such that $|f(x-t) - f(x)| \le \frac{\varepsilon}{2}$ for all $t \in \mathscr{V}$. On the other hand, for any open neighborhood $\mathscr{O}$ there exists $N_0 \in \mathbb{Z}_+$ such that

$$\int_{\mathbb{R}^n\setminus\mathscr{O}} h_n(t)dt \le \frac{\varepsilon}{2\|f\|_\infty}, \quad \forall n \ge N_0.$$

From this remarks we have for every $n \ge N_0$ that

$$\begin{aligned}|f * h_n(x) - f(x)| &\le \int_{\mathbb{R}^n} |f(x-t) - f(x)|\, h_n(t)dt \\ &= \int_{\mathscr{V}} |f(x-t) - f(x)|\, h_n(t)dt + \int_{\mathbb{R}^n\setminus\mathscr{V}} |f(x-t) - f(x)|\, h_n(t)dt \\ &\le \frac{\varepsilon}{2}\int_{\mathscr{V}} h_n(t)dt + 2\|f\|_\infty \int_{\mathbb{R}^n\setminus\mathscr{V}} h_n(t)dt \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

■

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