On C*-algebras Generated by Toeplitz Operators on the Bergman Space with Certain Classes of Discontinuous Symbols

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August 18, 2015

Contents

Abstract

In this work we give a closer look to the class of functions Q introduced by Kehe Zhu in [27]. The class Q is defined as the largest C^*-algebra of $L^\infty(\mathbb{D})$ for which the map

$$\begin{aligned} \psi \ : \ Q &\rightarrow \mathcal{B}(\mathcal{A}^2(\mathbb{D}))/\mathcal{K} \\ f &\mapsto T_f + \mathcal{K}, \end{aligned}$$

is a C^*-algebra homomorphism.

The characterizations of Q (established by K. Zhu in [27]) using the sets $VMO_\partial(\mathbb{D})$ (bounded functions with vanishing mean oscillation near the boundary of $\mathbb{D}$, Definition 1.3.2), and $ESV(\mathbb{D})$ (functions eventually slowly varying, Definition 1.3.5) lead us to the definitions of the classes of symbols of our interest in this thesis.

The first class of symbols we consider is the class of quasicontinuous symbols QC; the functions in QC characterize the angular behaviour of functions in Q, and their extensions to $\mathbb{D}$ fullfill the definition of $VMO_\partial(\mathbb{D})$. On the other hand, the second class of symbols SO consists of functions with slowly oscillation in $[0, 1)$. The second class of symbols agrees with the radial part of Q, and its extension to $\mathbb{D}$ is a subset of $ESV(\mathbb{D})$. The tensor product $QC \otimes SO$ is a subset of Q but is not the entire set

Q. However, it is an interesting problem to consider Toeplitz operators with these classes of symbols, because (contrary to the class Q), we have much more detailed information on the estructure of its compact set of maximal ideals.

The aim of this thesis is the explicit description of the Calkin algebras generated by Toeplitz operators with symbols in QC, SO, $QC \otimes SO$ and their corresponding piecewise continuous perturbations. We set some standard notation: first of all consider the unit disk $\mathbb{D}$ with its standard measure $dz = dxdy$, let $\mathcal{A}^2(\mathbb{D}) \subset L^2(\mathbb{D})$ denote the Bergman space on the unit disk which consists on all functions analytic on $\mathbb{D}$. The space $\mathcal{A}^2(\mathbb{D})$ has an orthogonal projection $B_{\mathbb{D}} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ called the Bergman projection. For a bounded measurable function a defined on $\mathbb{D}$, the Toeplitz operator T_a acting on $\mathcal{A}^2(\mathbb{D})$ is defined by $T_a(f) = B_{\mathbb{D}}(af)$. Consider a C^*-algebra $\mathcal{A}$ (for example QC or SO), we define the Toeplitz operator algebra $\mathcal{T}_{\mathcal{A}}$ as the C*-algebra generated by Toeplitz operators with symbols in $\mathcal{A}$ acting on $\mathcal{A}^2(\mathbb{D})$.

The main results of this work are established in Theorems 2.1.11 and 3.2.9; they describe the (well defined) Calkin algebras $\hat{\mathcal{T}}_{PQC}$ and $\hat{\mathcal{T}}_{PQCSO}$. Both results are based in the extensive use of the Douglas-Varela local principle machinery (see Section 1.1).

Introduction

A Toeplitz operator is defined as the compression of a multiplication operator acting on the space of square integrable functions on the set in question, onto a suitable subspace (Hardy or Bergman space). This kind of operators first appeared at the beginning of the XX century. Otto Toeplitz defined the Toeplitz matrix as the matrix with constant entries at each diagonal (parallel to the main one)

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_{-1} & a_0 & a_1 & a_2 & \dots \\ a_{-2} & a_{-1} & a_0 & a_1 & \dots \\ \vdots & \ddots & \ddots & \ddots & \dots \end{pmatrix}$$

Consider the sequence space $l^2(\mathbb{Z})$ and the space $L^2(\partial\mathbb{D})$, the discret Fourier transform $\mathcal{F}$ defines a unitary operator between these two Hilbert spaces. The image of the inverse Fourier transform restricted to the subspace $l^2(\mathbb{Z}_+)$ is known as the Hardy space $H^2(\partial\mathbb{D})$. One of the most important results concerning Toeplitz matrices says that a Toeplitz operator matrix define a bounded operator acting on the space $l^2(\mathbb{Z}_+)$ if and only if the sequence $\{a_n\}_{n\in\mathbb{Z}}$ consist of the Fourier coefficients of a bounded function a defined on the unit circle, usually this function a is refered as the defining symbol of the Toeplitz matrix.

The concept of Toeplitz operators was extended to other spaces of functions, in particular, to the Bergman space. In this work we are interested in Toeplitz operators acting on the Bergman space on the unit disk $\mathbb{D}$. Let $\mathcal{A}^2(\mathbb{D}) \subset L^2(\mathbb{D})$ denote the Bergman space on the unit disk which consists of all functions analytic on $\mathbb{D}$. The space $\mathcal{A}^2(\mathbb{D})$ has an orthogonal projection $B_{\mathbb{D}} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ called the Bergman projection. For a bounded measurable function a defined on $\mathbb{D}$, the Toeplitz operator T_a acting on $\mathcal{A}^2(\mathbb{D})$ is defined by $T_a(f) = B_{\mathbb{D}}(af)$. Some properties of the Toeplitz operator T_a are deduced from those of the function a, for instance, a bounded function $(a \in L^\infty(\mathbb{D}))$ generates a bounded operator T_a.

Two natural problems arise concerning Toeplitz operator theory. The first one concerns on what kind of properties can be established for T_a for a given function a; properties like boundness, compactness, Fredholmness, etc . The second problem involves the description of the operator algebra generated by Toeplitz operators with symbols in a prescribed family of functions. The Toeplitz operator algebra generated by a family of symbols $\mathcal{A}$, usually a linear space, is denoted by $\mathcal{T}_\mathcal{A}$. One of the techniques to describe and characterize the C^*-algebra $\mathcal{T}_\mathcal{A}$ uses the Calkin algebra $\hat{\mathcal{T}}_\mathcal{A} := \mathcal{T}_\mathcal{A}/\mathcal{K}_\mathcal{A}$, where $\mathcal{K}_\mathcal{A}$ denotes the ideal of compact operators in $\mathcal{T}_\mathcal{A}$ acting on $\mathcal{A}^2(\mathbb{D})$. If the C^*-algebra $\hat{\mathcal{T}}_\mathcal{A}$ is commutative it is isomorphic to the algebra of continuous functions over a compact space $X_\mathcal{A}$. Thus finding linear spaces (or C^*-algebras) $\mathcal{A}$ such that $\hat{T}_\mathcal{A}$ is commutative, will facilitate the description of $T_\mathcal{A}$.

For Toeplitz operators T_a, T_b we define the semicommutator $[T_a, T_b) = T_a T_b - T_{ab}$ and the commutator $[T_a, T_b] = T_aT_b - T_bT_a$. If the semicommutator (or the commutator) is compact for all $a, b \in \mathcal{A}$ then the Calkin algebra $\hat{\mathcal{T}}_{\mathcal{A}}$ is commutative.

The semicommutator (and the commutator) is compact for the case of continuous symbols. One of the most important result concerning this fact was established by Coburn in [10]; it says that the C^*-algebra $\mathcal{T}_{C(\overline{\mathbb{D}})}$ is irreducible and contains the ideal $\mathcal{K}$ of compact operators acting on $\mathcal{A}^2(\mathbb{D})$. As a further result (due to L. Coburn), the Calkin algebra $\hat{\mathcal{T}}_{C(\overline{\mathbb{D}})}$ is commutative and it is isomorphic and isometric to the continuous functions on the boundary $\partial\mathbb{D}$. By changing the class of symbols from continuous to piecewise continuous ($PC(\overline{\mathbb{D}})$), the commutator $[T_a, T_b]$ is still compact but the semicommutator is not compact in general. This fact implies that the Calkin algebra is commutative but not all the operators in $\mathcal{T}_{PC(\mathbb{D})}$ are compact perturbations of Toeplitz operators (in contrast with the continuous case, see [26]).

As it turns out, $C(\overline{\mathbb{D}})$ is not the largest C^*-algebra for which the semicommutator is compact. The problem of finding this algebra was solve by K. Zhu in [27]. This C^*-algebra is denoted by Q and it is customarily refered as the Kehe Zhu Class. The C^*-algebra Q is characterized by two types of functions, $VMO_\partial(\mathbb{D})$ functions of vanishing mean oscillation near the boundary of $\mathbb{D}$ and $ESV(\mathbb{D})$ functions eventually slowly varying.

In this work we use the definitions of $VMO_\partial(\mathbb{D})$ and $ESV(\mathbb{D})$ to define two C^*-algebras, the first one is the algebra $QC(\partial\mathbb{D})$ of quasicontinuous functions on $\partial\mathbb{D}$; and the second one is the algebra $SO([0,1))$ of slowly oscillating functions on $[0,1)$. The tensor product $QC(\partial\mathbb{D}) \otimes SO([0,1))$ is a subset of Q but is not the entire set Q, even though, we are interested in Toeplitz operators with symbols in QC, SO and the tensor product $QC(\partial\mathbb{D}) \otimes SO([0,1))$. The description of the Toeplitz operator algebras generated by these three classes of symbols (and their piecewise perturbations) are the main objective of this thesis.

The first chapter is devoted to the basic theory needed to understand our work, the general ideas come from [22], [26] and [27]. Section 1 of Chapter 1 contains the details of the principal tool we use to describe a Toeplitz operator algebra, namely, the Douglas-Varela local principle.

Section 1.2 contains the basic theory concerning Toeplitz operators. Basic theorems about Bergman spaces and Toeplitz operators are listed here for reference along this thesis. Most of the results in this section are presented without proofs, for a detailed discution on this topics we refer the reader to [26].

At the end of Section 1.2, we describe two Toeplitz operator algebras. The first one is the algebra generated by Toeplitz operators with picewise continuous symbols acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$ on the unit disk; the second result describes the algebra generated by Toeplitz operators with zero-order homogeneous symbols defined on the upper half plane Π, both results are due to N. Vasilevski [26].

In Section 1.3 we introduce the Kehe Zhu class Q. Characterizations and properties of Q are listed without proofs. The principal aim of this section is to set the theory needed to understand our interest in quasicontinuous symbols and slowly oscillating symbols.

Section 1.4 deals with the theory of QC the quasicontinuous functions, the first class of symbols of our interest. The definitions and results introduced here are due to D. Sarason and follows the approach of [22]. In this section we recall some results from [22] and give proofs of those that will be used in the text. Among the results we are interested in are the characterization of QC as functions of vanishing mean oscillation and the description of the maximal ideal space of QC, denoted by $M(QC)$.

In Section 1.5 we introduce the algebra $SO := SO([0, 1))$ of slowly oscillating functions on $[0, 1)$. The theory presented in this section is focused on the results needed to develop the main properties of Toeplitz operators with quasicontinuous and slowly oscillating symbols. The principal result established here describes the maximal ideal space of SO and gives a first look to the characterzation of the Calkin algebra $\hat{\mathcal{T}}_{SO}$.

The principal results of this work are in Chapter 2 and 3. The results give the description of Toeplitz operator algebras acting on the Bergman space on the unit disk.

In Chapter 2 we use the theory of Section 1.4 as well as the Douglas-Varela local principle to describe the Calkin algebra generated by Toeplitz operators with symbols in an extension of the C^*-algebra PQC of piecewise quasicontinuous functions. PQC is the C^*-algebra generated by piecewise continuous functions and quasicontinuous functions; both spaces of functions defined on $\partial\mathbb{D}$ and extended to $\mathbb{D}$. There are two natural ways to extend these functions; the radial extension and the harmonic extension. After the description of the Calkin algebra $\hat{\mathcal{T}}_{PQC}$ we prove that this result does not depend on the extension choosen for the function space PQC.

In Section 2.3 we give a canonical representation of all operators in $\mathcal{T}_{PQC}$. This canonical representation was inspired by the representation for Toeplitz operators with piecewise continuous functions (Theorem 8.3.5 of [26]).

As a final part of this chapter, in Section 2.5, we give a Fredholm criterion and, in case of Fredholmness, a Fredholm index formula.

In Chapter 3 we define the tensor product $QCSO := QC \otimes SO$ of quasicontinuous functions $QC(\partial\mathbb{D})$ and slowly oscillating function $SO([0,1))$. As an auxiliary result we prove that the Calkin algebra $\hat{\mathcal{T}}_{QCSO}$ is commutative, moreover we describe its maximal ideal space. The main goal of this chapter is the description of the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$, where $PQCSO$ denotes the tensor product of PQC and SO. In order to describe the Calkin algebra of $\mathcal{T}_{PQCSO}$ we use the Douglas-Varela local principle localizing by points in the maximal ideal space of $\hat{\mathcal{T}}_{QCSO}$. The description of the local algebras are obtained following the techniques of the case of piecewise quasicontinuous functions (Section 2.1). As in Chapter 2, in the final section, we give a Fredholm criterion for Toeplitz operators with symbols in $PQCSO$ and, in case of Fredholmness, a Fredholm index formula is provided.

Chapter 1

Preliminaries

In this chapter we present some results regarding Toeplitz operators acting on the Bergman space. We focus on the results that are relevant and necessary to understand our principal problems in Chapter 2 and 3.

In the first section we give a brief exposition of the so-called Douglas-Varela local principle. This principle is a main tool to describe the Toeplitz algebras of our interest in Chapter 2 and 3. A more complete theory can be found in [26].

The second section is devoted to the Bergman space theory. We introduce the Bergman spaces $B(\mathbb{D})$ and $B(\Pi)$; the former on the unit disc and the latter on the upper half plane. We define the Toeplitz operator acting on those spaces and describe two Toeplitz operator algebras that will be used in the solution of our principal problems. Readers that are familiar with concepts like Bergman space, Bergman projection, Toeplitz algebra and Douglas-Varela local principle may omit the first two sections of this chapter.

As a final part of this first chapter we introduce two C^*-algebras of symbols, the C^*-algebra $QC(\partial\mathbb{D})$ of quasicontinuous functions defined on $\partial\mathbb{D}$ and the C^*-algebra $SO([0,1))$ of slowly oscillating functions defined on $[0,1)$. The objective is to establish the preliminary theory needed to describe the Toeplitz operator algebra generated by Toeplitz operators with symbols in $QC(\partial\mathbb{D})$ and $SO([0,1))$ as well as the Toeplitz operator algebra with symbols in their tensor product.

1.1 Local Principle

There are several techniques of localization in operator theory. In the pioneer work [23] I. Simonenko introduced the notion of locally equivalent operators and developed a localization theory. This section is devoted to the Douglas-Varela local principle (DVLP for short), which gives the global description of the algebra under study in terms of continuous sections of a certain canonical C^*-bundle (see, for details, [26]).

The triple $\xi = (p, E, T)$, where E and T are topological spaces, and $p : E \to T$ is a surjective map, is called a *bundle*. The set T is called the *base* of the bundle, and $\xi(t) = p^{-1}(t)$ is called the *fiber over the point* $t \in T$.

Let V be an open set in T. A function $\eta : V \to E$ is called a *local section* of the bundle ξ if $p(\eta(t)) = t$ for all $t \in V$. If $V = T$ the section is called *global* or just *section*. Denote by $\Gamma(\xi)$ the set of all continuous sections of the bundle ξ.

Let $E \vee E = \{(x, y) \in E \times E : p(x) = p(y)\}$. The bundle $\xi = (p, E, T)$ is called a C^*-bundle if each fiber $\xi(t)$ has a structure of a C^*-algebra, and the following conditions hold

i. the functions

$$(x, y) \mapsto x + y : E \vee E \to E,$$
$$(x, y) \mapsto x \cdot y : E \vee E \to E,$$
$$(\alpha, x) \mapsto \alpha x : \mathbb{C} \times E \to E,$$
$$x \mapsto x^* : E \to E$$

are continuous;

ii. the subsets $U_V(\eta, \epsilon) = \{x \in E : p(x) \in V \text{ and } \|x - \eta(p(x))\| < \epsilon\}$, where V is an open subset in T, η is a continuous section of ξ over V and $\epsilon > 0$, form a basis of open sets in the space E.

Given a C^*-bundle $\xi = (p, E, T)$, there is a canonically defined C^*-algebra associated with ξ, namely, the set of all bounded continuous sections η of $\xi = (p, E, T)$ with componentwise operations and the following norm

$$\|\eta\| = \sup_{t \in T} \|\eta(t)\|.$$

We call this algebra the C^*-algebra defined by the C^*-bundle $\xi = (p, E, T)$, and denote it by $\Gamma^b(\xi)$.

Proposition 1.1.1 ([26], Lemma 1.1.1). *1. The function $\|\cdot\| : t \mapsto \|\eta(t)\|$ is upper semi-continuous.*

2. The algebra $\Gamma^b(\xi)$ is a $C^b(T)$-module, where $C^b(T)$ is the C^*-algebra of all bounded continuous functions on T.

3. If the space T is quasi-compact, then $\Gamma^b(\xi) = \Gamma(\xi)$ and $C^b(T) = C(T)$.

Recall that a space T is called *quasi-completely regular* if for each $t_0 \in T$ and each closed set $Y(\subset T)$ which does not contain t_0, there exist a continuous function $f : T \to [0,1]$ such that

$$f(t_0) = 0, \quad f|_Y \equiv 1.$$

Theorem 1.1.2 (Stone-Weierstrass)**.** *Let $\xi = (p, E, T)$ be a C^*-bundle over a quasi-compact, quasi-completely regular space T. Let $\mathcal{A}$ be a closed $C(T)$-submodule of $\Gamma(\xi)$ such that, for each $t \in T$, the set $\mathcal{A}(t) = \{a(t) : a \in \mathcal{A}\}$ is dense in the fiber $\xi(t) = p^{-1}(t)$. Then $\mathcal{A} = \Gamma(\xi)$.*

Let $\mathcal{A}$ be a C^*-algebra and let $J_T = \{J(t) \subset \mathcal{A} : t \in T\}$ be a system of closed two-sided ideals, parameterized by points of a set T. For each $t \in T$ introduce the quotient algebra $\mathcal{A}(t) = \mathcal{A}/J(t)$; we refer to this algebra as the local algebra at the point t. We will denote by $a(t)$ the image of an element $a \in \mathcal{A}$ in the quotient algebra $\mathcal{A}(t)$. Let

$$E = \bigsqcup_{t \in T} \mathcal{A}(t)$$

be the disjoint union of the C^*-algebras $\mathcal{A}(t)$. We define the action of the (additive) group $\mathcal{A}$ on the set E: each element $a \in \mathcal{A}$ generates the mapping $g_a : E \to E$ by the rule $g_a : x(t) \mapsto (x + a)(t)$. The orbit of each point $x = x(t)$ under the action of the group $\mathcal{A}$ coincides with the whole algebra $\mathcal{A}(t)$ and the collection of orbits is parameterized by points of T. The partition of E onto the disjoint orbits generates the projection

$$p : x(t) \in E \mapsto t \in T$$

with $p^{-1}(t) = \mathcal{A}(t)$.

We endow the sets E and T with appropriate topologies in order for the triple $\xi = (p, E, T)$ to be a C^*-bundle. Each element $a \in \mathcal{A}$ generates the section $\tilde{a} : T \to E$

by the rule $\tilde{a} : t \mapsto a(t)$. Denote by $\widetilde{\mathcal{A}}$ the set of all sections $\tilde{a}$. For each $\epsilon > 0$ and each $\tilde{a} \in \widetilde{\mathcal{A}}$ introduce the set

$$U(\tilde{a}, \epsilon) = \{x \in E : \|x - \tilde{a}(p(x))\| < \epsilon\},$$

and endow the set E with the topology whose prebase consists of the all sets $U(\tilde{a}, \epsilon)$. Endow the base T of the bundle $\xi = (p, E, T)$ with the orbit space topology under which the projection $p : E \to T$ is continuous.

Proposition 1.1.3 ([26], Lemma 1.1.6). *1. The mapping $p : E \to T$ is open.*

2. The topology on T coincides with the weakest topology under which all the mappings $\tilde{a} \in \widetilde{A}$ are continuous.

3. A prebase of the topology on T is given by the system of sets

$$V(\tilde{a}, \epsilon) = \{t \in T : \|\tilde{a}(t)\| < \epsilon\}.$$

Given a C^*-algebra $\mathcal{A}$ and a system of closed two-sided ideals $J_T = \{J(t) \subset \mathcal{A} : t \in T\}$, the C^*-bundle $\xi = (p, E, T)$ described above is called the *canonical C^*-bundle defined by the C^*-algebra $\mathcal{A}$ and the system of ideals J_T.*

The next result can be treated as a non commutative generalization of the Gelfand representation of a commutative Banach algebra.

Let $\mathcal{A}$ be a C^*-algebra, $J_T = \{J(t) \subset \mathcal{A} : t \in T\}$ be a system of closed two-sided ideals, $\xi = (p, E, T)$ be the canonical C^*-bundle defined by $\mathcal{A}$ and J_T, $\Gamma^b(\xi)$ be the C^*-algebra defined by the bundle $\xi = (p, E, T)$. Then the mapping

$$\widetilde{\pi} : a \in \mathcal{A} \mapsto \tilde{a} \in \Gamma^b(\xi)$$

is a morphism between the C^*-algebras $\mathcal{A}$ and $\Gamma^b(\xi)$, such that $Ker(\widetilde{\pi}) = \cap_{t\in T} J(t)$. Moreover, the mapping $\widetilde{\pi} : \mathcal{A} \to \widetilde{\mathcal{A}}$ is an isometric $*$-isomorphism if and only if

$$\bigcap_{t\in T} J(t) = \{0\}.$$

Having a C^*-algebra $\mathcal{A}$ and system J_T of its closed two-sided ideals (with or without the property $\bigcap_{t\in T} J(t) = \{0\}$), we *localize by points of the set* T by finding the local algebras $\mathcal{A}(t)$. Two elements a_1 and a_2 of the algebra $\mathcal{A}$ are called *locally equivalent at the point* $t \in T$ ($a_1 \overset{t}{\sim} a_2$) if and only if $a_1 - a_2 \in J(t)$. The natural projections $\pi_t : \mathcal{A} \to \mathcal{A}(t)$ identify the elements locally equivalent at the point t.

Let $\mathcal{A}$ be a C^*-algebra with identity e, and $\mathcal{Z}$ be its central commutative C^*-subalgebra, containing e. Denote by T the compact set of maximal ideals of the algebra $\mathcal{Z}$; then, $\mathcal{Z} \cong C(T)$. For each point $t \in T$ denote by J_t the maximal ideal of the algebra $\mathcal{Z}$ which corresponds to the point t, and denote by $J(t) = \mathcal{A} \cdot J_t$ the closed two-sided ideal generated by J_t in the algebra $\mathcal{A}$. Finally, we introduce the system of ideals $J_T = \{J(t) : t \in T\}$. For these ideals we have

$$\bigcap_{t\in T} J(t) = \{0\}.$$

Theorem 1.1.4 (Douglas-Varela local principle)**.** *The algebra $\mathcal{A}$ is *-isomorphic and isometric to the algebra of all (global) continuous sections of the C^*-bundle, defined by the algebra $\mathcal{A}$ and the system of ideals $J_T = \{J(t) \subset \mathcal{A} : t \in T\}$; moreover, the *-bundle topology on T coincides with hull kernel topology of the compact space T.*

1.2 Toeplitz operators on Bergman spaces

The Bergman space is named in honor of Stefan Bergman, who studied spaces of holomorphic functions. In this section we introduce the notion of the Bergman space, mention some definitions, and well-known results on this subject. The aim of this section is to present two results about Toeplitz operators acting on the Bergman space. The first one on the unit disc $\mathbb{D}$ (with piecewise continuous symbols) and the second one on the upper half plane Π (with angular bounded symbols).

We introduce the Bergman spaces $\mathcal{A}^2(\mathbb{D})$ and $\mathcal{A}^2(\Pi)$ with their corresponding Bergman projections $B_{\mathbb{D}}$ and B_{Π} and define the Toeplitz operator with symbol a (namely T_a). At the end of the section, the description of two Toeplitz algebras is given. The first one is the C^*-algebra generated by Toeplitz operators with piecewise continuous symbols acting on $\mathcal{A}^2(\mathbb{D})$ and the second one is the operator algebra generated by Toeplitz operators with bounded homogeneous symbols of zero-order acting on $\mathcal{A}^2(\Pi)$.

Let D be the unit disc $\mathbb{D}$ or the upper half plane Π, endow D with the area measure $dA(z) = dxdy$, $z = x + iy$. As usual, $L^2(D)$ denotes the space of all measurable and square integrable functions defined in D. Denote by $\mathcal{A}^2(D)$ the closed subspace of $L^2(D)$ consisting of all analytic functions, this space is known as the Bergman space on D. The (bounded) orthogonal projection from $L^2(D)$ onto $\mathcal{A}^2(D)$ is called the Bergman projection and it is denoted by B_D.

For a function $a : D \to \mathbb{C}$, we define the Toeplitz operator T_a with symbol a, acting on $\mathcal{A}^2(D)$, by the formula

$$T_a f = B_D(af).$$

For a general (bounded) symbol a defined on $\mathbb{D}$ the following theorems hold; we refer to [26] for a detailed discussion of these topics.

Theorem 1.2.1 ([26], Theorem 2.8.1)**.** *Let $\alpha, \beta \in \mathbb{C}$ and $a, b \in L^\infty(\mathbb{D})$, then*

- *the operator T_a is bounded on $\mathcal{A}^2(\mathbb{D})$ and $\|T_a\| \le \|a\|_\infty$,*

- $T_{\alpha a+\beta b} = \alpha T_a + \beta T_b$,

- $(T_a)^* = T_{\bar{a}}$, *where* $\bar{a}$ *denotes the complex conjugate function of* a.

The next result shows that there is a one-to-one correspondence between the Toeplitz operators and their defining symbols.

Theorem 1.2.2 ([26], Theorem 2.8.2)**.** *For any* $a \in L^\infty(\mathbb{D})$, $T_a = 0$ *if and only if* $a \equiv 0$ *almost every-where.*

The theorem below establishes that, up to a compact operator, the behaviour of a Toeplitz operator with symbol $a \in L^\infty(\mathbb{D})$ depends only on the behaviour of the function a near the boundary of the unit disk $\partial\mathbb{D}$, thus next theorem contains the basic tools to determine the local behaviour of a Toeplitz operator acting on $\mathcal{A}^2(\mathbb{D})$.

Theorem 1.2.3 ([26], Theorem 2.8.3)**.**

- *If the function* $a(z) \in L^\infty(\mathbb{D})$ *is continuous at all points of* $\partial\mathbb{D}$ *with* $a|_{\partial\mathbb{D}} \equiv 0$, *then the Toeplitz operator* T_a *is compact.*

- *If the functions* $a(z), b(z) \in L^\infty(\mathbb{D})$ *are continuous at all points of* $\partial\mathbb{D}$, *then the so called semicommutator* $[T_a, T_b) = T_aT_b - T_{ab}$ *is compact.*

- *If the functions* $a(z), b(z) \in L^\infty(\mathbb{D})$ *are continuous at all points of* $\partial\mathbb{D}$, *then the commutator* $[T_a, T_b] = T_aT_b - T_bT_a$ *is compact.*

We consider a finite set of n different points $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ on $\partial\mathbb{D}$ and let

$$\delta = \min_{k \neq j}\{|\lambda_k - \lambda_j|, 1\}.$$

We denote by l_k, $k = 1, \ldots, n$, the part of the radius of $\mathbb{D}$ starting at λ_k and having length $\delta/3$, let $\mathbb{L} = \cup l_k$. We denote by $PC(\mathbb{D}, \Lambda)$ the set (algebra) of all piecewise continuous functions on $\mathbb{D}$ which are continuous in $\mathbb{D} \setminus \mathbb{L}$ and have one-sided limit values at each point of $\mathbb{L}$. In particular, every function $a \in PC(\mathbb{D}, \Lambda)$ has at each point $\lambda_k \in \Lambda$ two (different, in general) limit values:

$$a_k^+ := \lim_{\lambda \to \lambda_k^+} a(\lambda) \quad \text{and} \quad a_k^- := \lim_{\lambda \to \lambda_k^-} a(\lambda)$$

following the standard positive orientation of $\partial\mathbb{D}$.

For each $k = 1, \ldots, n$, denote by $\chi_k := \chi_k(z)$ the characteristic function of the half-disk obtained by cutting $\mathbb{D}$ by the diameter passing through $\lambda_k \in \Lambda$, and such that $\chi_k^+ = 1$, and thus $\chi_k^- = 0$.

For each $k = 1, \ldots, n$, we introduce two neighborhoods of the point λ_k:

$$V_k' := \left\{ z \in \mathbb{D} : |z - \lambda_k| < \frac{\delta}{6} \right\}$$

and

$$V_k'' := \left\{ z \in \mathbb{D} : |z - \lambda_k| < \frac{\delta}{3} \right\},$$

and fix a continuous function $\mu_k := \mu_k(z) : \mathbb{D} \to [0, 1]$ such that

$$\mu_k|_{V_k'} \equiv 1, \qquad \mu_k|_{\mathbb{D} \setminus V_k''} \equiv 0.$$

Theorem 1.2.4 ([26], Theorem 2.8.7)**.** *Let* $a(z), b(z) \in PC(\mathbb{D}, \Lambda)$. *Then the commutator* $[T_a, T_b]$ *is compact.*

Let $\mathcal{T}_{PC}$ denote the C^*-algebra generated by all Toeplitz operators T_a with defining symbol $a(z) \in PC(\mathbb{D}, \Lambda)$. We consider $\mathcal{K}$ as the ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$. In [10], L. Coburn proves that the ideal $\mathcal{K}$ is a subset of $\mathcal{T}_{C(\mathbb{D})}$, hence the Calkin algebra $\mathcal{T}_{C(\mathbb{D})}/\mathcal{K}$ is well defined. In order to find the Calkin algebra $\mathcal{T}_{PC}/\mathcal{K}$, we use the Douglas-Varela local principle localizing by points on $\partial\mathbb{D}$. The local algebras at points $\lambda \notin \Lambda$ are isomorphic to $\mathbb{C}$; the local algebras at points $\lambda_k \in \Lambda$ are isomorphic to the algebra of continuous functions on the interval $[0,1]$, as described in Theorem 2.7.3 of [26].

Let $\hat{\partial\mathbb{D}}$ be the set $\partial\mathbb{D}$, cut at the points $\lambda_k \in \Lambda$. The pair of points which correspond to each point λ_k, $k = 1, \ldots, n$, are denoted by λ_k^+ and λ_k^-; following the positive orientation of $\partial\mathbb{D}$. Let $I^n = \bigsqcup_{k=1}^{n} [0,1]_k$ be the disjoint union of n copies of the interval $[0,1]_k$. Denote by Σ the union $\partial\mathbb{D} \bigsqcup I^n$ with the point identification

$$\lambda_k^- \equiv 0_k, \qquad \lambda_k^+ \equiv 1_k,$$

where 0_k and 1_k are the boundary points of $[0,1]_k$, $k = 1, \ldots, n$.

Theorem 1.2.5 ([26], Theorem 2.8.8)**.** *The C^*-algebra $\mathcal{T}_{PC}$ is irreducible and contains the ideal $\mathcal{K}$ of compact operators. The Fredholm symbol algebra* $\mathrm{Sym}(\mathcal{T}_{PC}) = \mathcal{T}_{PC}/\mathcal{K}$ *is isomorphic to the algebra $C(\Sigma)$. Identifying them, the symbol homomorphism*

$$\mathrm{Sym} : \mathcal{T}_{PC} \to C(\Sigma)$$

is generated by the following mapping of generators of $\mathcal{T}_{PC}$,

$$\mathrm{Sym} : T_a \mapsto \begin{cases} a(\lambda), & \lambda \in \partial\mathbb{D} \setminus \Lambda \\ a^-_{\lambda_k}(1-t) + a^+_{\lambda_k} t & t \in [0,1]_k. \end{cases}$$

We analize next a very especial Toeplitz algebra on the upper half plane Π. We consider $\mathcal{A}^2(\Pi)$ as the Bergman space on Π, that is, the (closed) space of square integrable and analytic functions on Π. Let B_Π stand for the Bergman projection $B_\Pi : L^2(\Pi) \to \mathcal{A}^2(\Pi)$.

We denote by $\mathcal{A}_\infty$ the C*-algebra of bounded measurable homogeneous functions on Π of zero-order, or functions depending only in the polar coordinate θ. We introduce the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$ generated by all Toeplitz operators

$$T_a : \phi \in \mathcal{A}^2(\Pi) \mapsto B_\Pi(a\phi) \in \mathcal{A}^2(\Pi)$$

with defining symbols $a(r,\theta) = a(\theta) \in \mathcal{A}_\infty$.

Theorem 1.2.6 ([26], Theorem 7.2.1)**.** *Let $a = a(\theta) \in \mathcal{A}_\infty$. Then the Toeplitz operator T_a acting on $A^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on $L^2(\mathbb{R})$. The function $\gamma_a(s)$ is given by*

$$\gamma_a(s) = \frac{2s}{1 - e^{-2\pi s}} \int_0^\pi a(\theta) e^{-2s\theta} d\theta, \qquad s \in \mathbb{R}.$$

As a corollary of Theorem 1.2.6 we have that the algebra $T(\mathcal{A}_\infty)$ is commutative and is embeded in $C_b(\mathbb{R})$.

The next results describe the Toeplitz algebra generated by the identity and a single Toeplitz operator with piecewise continuous symbol on Π. These results will be used in the proofs of Lemmas 2.1.9 and 2.2.1.

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_{n-1}\}$ be a subset of $[0,\pi]$, denote by $PC([0,\pi],\Theta)$ the C^*-algebra of all piecewise continuous functions on $[0,\pi]$, that is, functions which are continuous on $[0,\pi] \setminus \Theta$ and having one-sided limit values at the points of Θ. Let $PC_0([0,\pi]),\Theta)$ be the subalgebra of $PC([0,\pi],\Theta)$ consisting of all piecewise constant functions. For a function $a_0(\theta)$, denote by $L(1,a_0)$ the linear two-dimensional space generated by 1 and the function a_0.

Denote by $H(PC([0,\pi],\Theta))$ the subset of $\mathcal{A}_\infty$ which consists of all homogeneous functions of zero-order on Π whose restriction onto the upper half of the unit circle (parameterized by $\theta \in [0,\pi]$) belong to $PC([0,\pi],\Theta)$. Denote by $\mathcal{T}(H(PC([0,\pi],\Theta)))$ the C^*-algebra generated by all Toeplitz operators T_a with defining symbols $a \in PC([0,\pi],\Theta)$.

Theorem 1.2.7 ([26], Theorem 7.2.5). *Let $a_0(\theta) \in L^\infty([0,\pi])$ be a real-valued function such that the function $\gamma_{a_0}(s)$ separates the points of $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, the two-point compactification of $\mathbb{R}$. Then the C^*-algebra $T(H(L(1,a_0)))$ is isomorphic and isometric to $C(\bar{\mathbb{R}})$.*

Corollary 1.2.8 ([26], Corollary 7.2.6). *Given a point $\theta_1 \in (0,\pi)$, the C^*-algebra $\mathcal{T}(H(PC_0([0,\pi],\{\theta_1\})))$ is isomorphically isometric to $C(\bar{\mathbb{R}})$.*

We also use the following theorem to state the main result of Section 2.2, namely, Theorem 2.2.4. Let now $a_0(\theta) = \frac{\theta}{\pi}$, this function $a_0(\theta)$ is obviously real-valued and continuous on $[0,\pi]$.

Corollary 1.2.9 ([26], Corollary 7.2.7). *The C^*-algebra $T(H(L(1,a_0)))$ is isomorphic and isometric to $C(\bar{\mathbb{R}})$.*

1.3 The Kehe Zhu class Q

[The Kehe Zhu class]

As we mention in Introduction, K. Zhu ([27]) describes the largest C^*-algebra $Q \subset L^\infty(\mathbb{D})$ such that the map

$$\begin{array}{rcl} \psi \ : \ Q & \to & \mathcal{B}(\mathcal{A}^2(\mathbb{D}))/\mathcal{K} \\ f & \mapsto & T_f + \mathcal{K}, \end{array}$$

is a C^*-algebra homomorphism.

In this section we deal with the basic theory of [27] needed to describe the Kehe Zhu class Q as functions with vanishing mean oscillation near the boundary and functions eventually slowly varying. The details of the theory presented in this section are in [27]. Some tools like Berezin transfom ([4]), Carleson boxes ([9]), and harmonic extension ([2]) are used to characterize Q.

Definition 1.3.1 ([27], Page 633). *Define*

$$\Gamma := \{f \in L^\infty(\mathbb{D}) : T_f T_g - T_{fg} \in \mathcal{K} \ \ \textit{for all } g \in L^\infty(\mathbb{D})\}.$$

Let $Q := \bar{\Gamma} \cap \Gamma$.

For z in $\mathbb{D}$, we define the Carleson Box ([9])

$$S_z := \{w \in \mathbb{D} : \ |w| \geq |z| \ \text{ and } \ \ |\arg(z) - \arg(w)| \leq 1 - |z|\},$$

the area of S_z, denoted by $|S_z|$, is $\pi(1-|z|)^2(1+|z|)$.

Definition 1.3.2 ([27], Page 621). *A function f in $L^1(\mathbb{D})$ belongs to $VMO_\partial(\mathbb{D})$, the space of functions with vanishing mean oscillation near the boundary of $\mathbb{D}$, if*

$$\lim_{|z|\to 1^-} \frac{1}{|S_z|} \int_{S_z} \left| f(w) - \frac{1}{|S_z|} \int_{S_z} f(u) dA(u) \right| dA(w) = 0.$$

Theorem 1.3.3 ([27], Theorem 13). *The algebra Q is the set of bounded functions with vanishing mean oscillation near the boundary, i.e.,*

$$Q = VMO_\partial(\mathbb{D}) \cap L^\infty(\mathbb{D}).$$

Another characterization for Q is given via the Berezin transfom of Toeplitz operators. For this let

$$\tilde{T}_g(z) := \int_{\mathbb{D}} g(w) \frac{1-|w|^2}{(1-z\bar{w})^2} dA(w).$$

denote the *Berezin transform* of the Toeplitz operator T_g (or the *Berezin transform* of g). By simplicity of notation $\tilde{g}(z) := \tilde{T}_g(z)$; note that, for a bounded function g, $\tilde{g}$ belongs to $C_b(\mathbb{D})$ and $\|\tilde{g}\|_\infty \leq \|g\|_\infty$.

A lot of work has been done in operator theory by the extensive use of the Berezin transfom given by Berezin in [4]; papers like [24], [28], [5] are among the most important ones.

We define B as the set of bounded functions on $\mathbb{D}$ such that its Berezin transform goes to zero as z approaches to the boundary of $\mathbb{D}$, that is,

$$B := \{f \in L^\infty(\mathbb{D}) \ : \lim_{|z|\to 1} \tilde{f}(z) = 0\}.$$

In [3], S. Axler and D. Zheng proved that a Toeplitz operator T_g, with bounded symbol g, is compact if and only if g is in B.

The next lemma is due to K. Zhu and it is a combination of some results in [27].

Theorem 1.3.4. *The set Q in Definition 1.3.1 is described as*

$$Q = \{f \in L^\infty(\mathbb{D}) : \ \lim_{|z|\to 1} \widetilde{|f|}^2(z) - |\tilde{f}(z)|^2 = 0\}.$$

The set $B \cap Q$ is an ideal of Q and, for $f \in Q$, the Toeplitz operator T_f is compact if and only if f belongs to $B \cap Q$

The next definitions and theorems are stated here as they are crucial for the main results in Sections 2.2 and 3.1.

For each point z in $\mathbb{D}$ we define

$$S'_z := \left\{ w \in \mathbb{D} \ : \ |w| \geq |z| \quad \text{and} \quad |\arg(z) - \arg(w)| \leq \frac{1 - |z|}{2} \right\}.$$

Definition 1.3.5 ([27], Page 626)**.** *Let f be in $L_\infty(\mathbb{D})$. We say f is in $ESV(\mathbb{D})$ if and only if for any $\epsilon > 0$, and $\eta \in (0,1)$, there exists $\delta_0 > 0$ such that $|f(z)-f(w)| < \epsilon$ whenever $w \in S'_z$ and $|z|, |w| \in [1-\delta, 1-\delta\eta]$, with $\delta < \delta_0$.*

The notation $ESV := ESV(\mathbb{D})$ means eventually slowly varying and was introduced by C. Berger and L. Coburn in [6].

For the next characterization of Q we need to define an average of the function in the box S'_z.

Definition 1.3.6 ([27], Page 627)**.** *For a function f in $L^\infty(\mathbb{D})$ define*

$$\hat{f}(z) := \frac{1}{|S'_z|} \int_{S'_z} f(w) dA(w).$$

Theorem 1.3.7 ([27], Theorem 5)**.** *$Q = ESV + Q \cap B$. A descomposition is given by $f = \hat{f} + (f - \hat{f})$. Moreover*

$$ESV \cap B = \{ f \in L_\infty(\mathbb{D}) \mid \ f(z) \to 0 \text{ as } \ |z| \to 1^- \}.$$

The descriptions of Q given in Theorems 1.3.3 and 1.3.7 are the motivations for the definitions of the spaces of functions in Sections 1.4 and 1.5. If a function f in Q depends only in the angular coordinate we obtain a function defined on the unit circle; this space of functions are treated in Section 1.4. On the other hand, if a function depends only in the radial part of the disk we define a space of functions on $[0,1)$; this C^*-algebra is the main subject of study in Section 1.5.

1.4 The algebra of quasicontinuous functions

In this section we introduce the algebra QC of quasicontinuous functions using the definition of D. Sarason in [22]. The purpose is to state the preliminary theory necessary for the main problem in Chapter 2.

We establish the principal properties of QC via the space of functions with vanishing mean oscillation. Also we describe the topology and properties of its Gelfang space $M(QC)$. For the convenience of the reader we repeat the relevant material from [22], thus making the exposition more self contained.
This section includes some basic facts about the space of Vanishing Mean Oscillation functions on $\partial\mathbb{D}$, denoted here by $VMO(\partial\mathbb{D})$. The importance of this space lies in the fact that $QC = VMO(\partial\mathbb{D}) \cap L^\infty(\partial\mathbb{D})$ (see [21]).

1.4.1 The algebra of vanishing mean oscilating functions.

We define the following spaces of functions on $\partial\mathbb{D}$:

- $L^\infty := L^\infty(\partial\mathbb{D})$, the algebra of bounded measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$,
- $H^\infty := H^\infty(\partial\mathbb{D})$, the algebra of radial limits of bounded analytic functions on $\mathbb{D}$,
- $C := C(\partial\mathbb{D})$ = the algebra of continuous functions on $\partial\mathbb{D}$.

Definition 1.4.1 ([22], Page 818)**.** *The C^*-algebra of quasicontinuous functions QC is defined as the algebra of all bounded functions f on $\partial\mathbb{D}$ such that both, f and its complex conjugate $\bar{f}$, belong to $H^\infty + C$, that is:*

$$QC := (H^\infty + C) \cap \left(\overline{H}^\infty + C\right).$$

Some of the statements below are formulated for segments in the real line but they can also be formulated for arcs in $\partial\mathbb{D}$.

By an interval on $\mathbb{R}$ we always mean a finite interval. The length of the interval I will be denoted by $|I|$.
For $f \in L^1(I)$, the average of f over I is given by:

$$I(f) := |I|^{-1} \int_I f(t)dt. \tag{1.4.1}$$

Remark: We use the fact that $I(f)$ defines a bounded linear nonmultiplicative functional on QC.

For $a > 0$, we set

$$M_a(f, I) := \sup_{J\subset I, |J|<a} \frac{1}{|J|} \int_J |f(t) - J(f)|dt.$$

Note that $0 \leq M_a(f, I) \leq M_b(f, I)$ if $a \leq b$, thus let $M_0(f, I) := \lim_{a\to 0} M_a(f, I)$.

Definition 1.4.2 ([22], Page 818)**.** *A function $f \in L^1(I)$ is of vanishing mean oscillation in the interval I (or the arc I), if $M_0(f, I) = 0$. The set of all vanishing mean oscillation functions on I is denoted by $VMO(I)$.*

In particular, if we replace I by $\partial\mathbb{D}$ in previous definitions we get $VMO := VMO(\partial\mathbb{D})$. Now we establish two equivalents of Definition 1.4.2 for the space $VMO(I)$.

Lemma 1.4.3. *A function f belongs to $VMO(I)$ if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|J|^{-2} \int_J \int_J |f(t) - f(s)|\, dsdt < \epsilon,$$

for all intervals $J \subset I$ with $|J| < \delta$.

Lemma 1.4.4 ([22], Lemma 1)**.** *Let I be an interval on $\mathbb{R}$ and f a function in $L^\infty(I)$. Then the following condition is necessary and sufficient for f to belong to $VMO(I)$:*

for any positive numbers ϵ_1 and ϵ_2, there is a positive number δ such that

$$|\{x \in I : |f(x) - J(f)| > \epsilon_1\} \cap J| \leq \epsilon_2 |J|,$$

whenever J is a subinterval of I satisfying $|J| \leq \delta$. Here $|A|$ denotes de Lebesgue measure of A.

Proof. Let f be in $VMO(I)$ and ϵ_1, ϵ_2 be positive numbers. For $\epsilon = \epsilon_1 \epsilon_2$ there exists a positive number δ such that $M_a(f, I) < \epsilon$ for all $a < \delta$. Then, for every interval $J \subset I$ with $|J| < a < \delta$, we have

$$|J|^{-1} \int_J |f(t) - J(f)| dt < \epsilon. \tag{1.4.2}$$

Let $A = \{t \in I : |f(t) - J(f)| > \epsilon_1\} \cap J$. For $t \in A$,

$$\frac{\epsilon_1}{|J|} < \frac{1}{|J|} |f(t) - J(f)|. \tag{1.4.3}$$

Integrating both sides of the inequality (1.4.3), we get

$$\frac{\epsilon_1 |A|}{|J|} \leq \frac{1}{|J|} \int_A |f(t) - J(f)| dt. \tag{1.4.4}$$

Since the right hand side of inequality (1.4.4) is bounded by (1.4.2), we get we get $|A| \leq \epsilon_2 |J|$.

On the other hand, assume that f is a bounded function (with $\|f\|_\infty \neq 0$) and ϵ is any positive number. Let $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon / \|f\|_\infty$, then there exists $\delta > 0$ such that $|A| \leq \epsilon_2 |J|$ whenever $|J| < \delta$.

Let $B = \{t \in I : |f(t) - J(f)| \leq \epsilon_1\} \cap J$, then,

$$\begin{aligned} |J|^{-1} \int_J |f(t) - J(f)| dt &= |J|^{-1} \int_A |f(t) - J(f)| dt + |J|^{-1} \int_B |f(t) - J(f)| dt \\ &\leq |J|^{-1} \int_A |f(t) - J(f)| dt + \epsilon \\ &\leq |J|^{-2} \int_A \int_J |f(t) - f(s)| ds dt + \epsilon \\ &\leq \frac{2\|f\|_\infty |A|}{|J|} + \epsilon \leq 2\epsilon. \end{aligned}$$

We conclude that $M_a(f, I) \leq \epsilon$ if $a < \delta$, which implies $M_0(f, I) = 0$. □

Definition 1.4.5 ([22], Page 818)**.** *Let f be an integrable function defined in an open interval containing the point λ. The integral gap of f at λ is defined by*

$$\gamma_\lambda(f) := \limsup_{\delta \to 0} \left| \delta^{-1} \int\limits_{\lambda-\delta}^{\lambda} f(t)dt - \delta^{-1} \int\limits_{\lambda}^{\lambda+\delta} f(t)dt \right|.$$

Lemma 1.4.6. *If f belongs to $VMO(I)$, then $\gamma_\lambda(f) = 0$ for each interior point λ of I.*

Proof. Let $\epsilon > 0$ and $f \in VMO(I)$, then there is a $\delta > 0$ such that, for every $J \subset I$ with $|J| \leq \delta$

$$\frac{1}{|J|^2} \int_J \int_J |f(s) - f(t)| ds dt < \epsilon/4.$$

Let J be an interval with center at λ, if $|J| < \delta$ then we can write $J = (\lambda - a, \lambda + a)$ for some a with $2a < \delta$. Then,

$$\left| a^{-1} \int\limits_{\lambda-a}^{\lambda} f(s)ds - a^{-1} \int\limits_{\lambda}^{\lambda+a} f(t)dt \right| \leq \left| a^{-1} \int\limits_{\lambda-a}^{\lambda} (f(t) - a^{-1} \int\limits_{\lambda}^{\lambda+a} f(s)ds)dt \right|$$

$$\leq a^{-2} \int\limits_{\lambda-a}^{\lambda} \int\limits_{\lambda}^{\lambda+a} |f(t) - f(s)| ds dt \leq \frac{4}{|J|^2} \int_J \int_J |f(s) - f(t)| ds dt < \epsilon,$$

which proves the lemma. □

Lemma 1.4.7 ([22], Lemma 2)**.** *Let $I = (a, b)$ be an open interval, λ a point of I, and f a function on I which belongs to both, $VMO((a, \lambda))$ and $VMO((\lambda, b))$. If $\gamma_\lambda(f) = 0$, then f belongs to $VMO(I)$.*

Proof. Fix $\epsilon > 0$, and choose $c > 0$ such that

$$|J|^{-1} \int_J |f(t) - J(f)| dt < \epsilon \tag{1.4.5}$$

whenever J is a subinterval of (a,λ) or (λ,b) satisfying $|J|<c$, and so that

$$\left|\delta^{-1}\int_{\lambda-\delta}^{\lambda} f(t)dt-\delta^{-1}\int_{\lambda}^{\lambda+\delta} f(t)dt\right|<\epsilon, \tag{1.4.6}$$

whenever $\delta<c$. We show that if J is a subinterval of I such that $|J|<c$ then

$$|J|^{-1}\int_J |f(t)-J(f)|dt<\epsilon.$$

Of course we may suppose that J contains λ. We consider first the case where λ is the center of J. Let $J_+=J\cap(\lambda,b)$ and $J_-=J\cap(a,\lambda)$. Then $|J_+(f)-J_-(f)|<\epsilon$, and $J(f)=\frac{1}{2}(J_+(f)+J_-(f))$. From these inequalities we conclude $|J_\pm(f)-J(f)|<\epsilon/2$. Consequently, using the equations (1.4.5) and (1.4.6), we found that

$$|J|^{-1}\int_J |f(t)-J(f)|dt=\frac{1}{2}|J_+|^{-1}\int_{J_+}|f(t)-J(f)|dt+\frac{1}{2}|J_-|^{-1}\int_{J_-}|f(t)-J(f)|dt$$

is bounded by $\frac{3\epsilon}{2}$.

Now suppose that J is any subinterval of I containing λ, and let J_0 be the smallest interval containing J whose center is λ. We can apply the previous argument to J_0. Hence,

$$|J(f)-J_0(f)|\leq|J|^{-1}\int_J|f(t)-J_0(f)|dt\leq 2|J_0|^{-1}\int_{J_0}|f(t)-J_0(f)|dt<3\epsilon. \tag{1.4.7}$$

Equation (1.4.7) implies that $|J|^{-1}\int_J|f(t)-J(f)|dt<\epsilon$ and thus f is in $VMO(I)$. $\square$

For a function defined in a closed interval I, let

$$\omega(f,J):=\sup\{|f(s)-f(t)|:s,t\in J\}$$

denote the oscillation of f over J.

Fix $I = [a, b)$ a half open interval. We say that the function f on I belongs to the class of Slowly Oscillating functions, $SO(I)$, if f is bounded and continuous and if, for each η in $(0, 1)$,

$$\lim_{\delta \to 0} \omega(f, [b - \delta, b - \eta\delta]) = 0.$$

This definition can be extended to the cases $I = (a, b]$, $I = (a, b)$ as well as for half open arcs or open arcs in $\partial\mathbb{D}$. In particular for $I = [0, 1)$ we define $SO([0, 1))$; this algebra and its principal properties are described in Section 1.5.

Lemma 1.4.8 ([22], Lemma 3)**.** *For an open (or half open) interval I we have that $SO(I) \subset VMO(I)$.*

Proof. The proof has been carried out for the case $I = (0, 1]$, the general case follows from this one.

Let f be a bounded continuous function in $SO((0, 1])$. Let $\epsilon > 0$ be given and fix η such that $6\|f\|_\infty \eta < \epsilon$. By definition of $SO(I)$ there is $\delta_0 > 0$ such that $w(f, [\eta\alpha, \alpha]) < \epsilon$ for all $\alpha < 2\delta_0$.

Since f is uniformly continuous on $[\delta_0, 1]$, thus, for $\epsilon > 0$ there is $\delta_1 > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $|s - t| < \delta_1$.

Choose $\delta = \min\{\delta_0, \delta_1\}$. Let $J = (c, d) \subset (0, 1]$ with $|J| < \delta$. We have one of the following two cases for J:

- $J \subset [\delta_0, 1]$;
- $J \subset (0, 2\delta_0)$.

In the first case we use the fact that f is uniformly continuous and then $|J|^{-2} \int_J \int_J |f(t) - f(s)| ds dt < \epsilon$.

In the second case we have that $d < 2\delta < 2\delta_0$ and then $w(f, [\eta d, d]) < \epsilon$. Let J be separated into two pieces $J_1 = (0, \eta d) \cap J$ and $J_2 = [\eta d, d] \cap J$. Now the integral $|J|^{-2} \int_J \int_J |f(t) - f(s)| ds dt$ can be split as follows:

$$\frac{1}{|J|^2}\int\limits_{J_1}\int\limits_{J_1}|f(t)-f(s)|dsdt+\frac{2}{|J|^2}\int\limits_{J_1}\int\limits_{J_2}|f(t)-f(s)|dsdt+\frac{1}{|J|^2}\int\limits_{J_2}\int\limits_{J_2}|f(t)-f(s)|dsdt.$$

These integrals are bounded as

$$\frac{6\|f\|_\infty|J_1|}{|J|^2}(|J_1|+|J_2|)+\epsilon\leq\frac{6\|f\|_\infty|J_1|}{|J|}+\epsilon.$$

The ratio $\frac{|J_1|}{|J|}=\frac{\eta d-c}{d-c}$ is less than η, so the quantity $|J|^{-2}\int_J\int_J|f(t)-f(s)|dsdt$ is bounded by $6\|f\|_\infty\eta+\epsilon\leq 2\epsilon$, thus f belongs to $VMO(I)$. □

Lemma 1.4.9 ([22], Lemma 4)**.** *Let f be a bounded function in $VMO((0,1])$. Let g be defined by*

$$g(t)=(t)^{-1}\int_0^t f(s)ds.$$

Then g belongs to $SO((0,1])$.

Proof. The function g is continuous and bounded. Let $I_d=(0,d)$ for a suitable d, so that $g(t)=I_t(f)$. Let $\epsilon>0$ and $0<\eta<1$. For $\epsilon_1=\epsilon\eta$ there exists $\delta>0$ such that $M_\alpha(f,J)<\epsilon_1$ for all $\alpha\leq\delta$ and any interval J with $|J|<\alpha$. Let $\alpha\leq\delta$ and fix $s<t$ in $[\eta\alpha,\alpha]$

$$\begin{aligned}|g(s)-g(t)| &= |I_s(f)-I_t(f)|\\ &\leq \tfrac{1}{|I_s|}\textstyle\int_{I_s}|f(s)-I_t(f)|ds\\ &\leq (|I_t|/|I_s|)\tfrac{1}{|I_t|}\textstyle\int_{I_t}|f(s)-I_t(f)|ds\\ &\leq |I_t|/|I_s|M_\delta(f,(0,1]).\end{aligned}$$

The value $|I_t|$ is at most δ and the value $|I_s|$ is at least $\eta\delta$, thus $|g(s)-g(t)|\leq\frac{1}{\eta}M_\alpha(f,(0,1])\leq\frac{1}{\eta}\epsilon_1\leq\epsilon$, that is, by definition, g belongs to $SO((0,1])$. □

1.4.2 The maximal ideal space of QC

We denote by $M(QC)$ the space of all non-trivial multiplicative linear functionals on QC, endowed with its Gelfand topology. This space is know as the maximal ideal space of QC. In the same way we define $M(C)$ and identify it with $\partial\mathbb{D}$ via the evaluation functionals. Since C is a subset of QC, every functional in $M(QC)$ induces, by restriction, a functional in C.

Here and subsequently, f_0 denotes the function $f_0(\lambda) = \lambda$. Stone-Weierstrass Theorem implies that f_0 together with $f(\lambda) = 1$ generate the C^*-algebra of all continuous functions on $\partial\mathbb{D}$.

Definition 1.4.10 ([22], Page 822)**.** *For every $\lambda \in \partial\mathbb{D}$, we denote by $M_\lambda(QC)$ the set of all functionals x in $M(QC)$ such that $x(f_0) = \lambda$, that is*

$$M_\lambda(QC) := \{x \in M(QC) : x(f_0) = f_0(\lambda) = \lambda\}.$$

In other words, x belongs to $M_\lambda(QC)$ if the restriction of x to the continuous functions is the evaluation functional at the point λ.

Definition 1.4.11 ([22], Page 822)**.** *We let $M_\lambda^+(QC)$ denote the set of $x \in M_\lambda(QC)$ with the property that $f(x) = 0$ whenever f in QC is a function such that $\lim_{t\to\lambda^+} f(t) = 0$. $M_\lambda^-(QC)$ is defined in an analogous way.*

Let f be a bounded function on $\partial\mathbb{D}$. The Harmonic extension of f to the unit disk is denoted by f_H and is given by the formula (see [2]).

$$f_H(z) := f_H(r,\theta) := \frac{1}{2\pi}\int_{\partial\mathbb{D}} P_r(\theta-\lambda)f(\lambda)d\lambda, \tag{1.4.8}$$

where

$$P_r(\theta) := \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2rcos(\theta)+r^2}$$

is the Poisson Kernel for the unit disk.

For every point z in $\mathbb{D}$ we define a functional in QC by the following rule: $z(f) = f_H(z)$, thus, we consider $\mathbb{D}$ as a subset of the dual space of QC. Under this identification we have that the weak-star closure of $\mathbb{D}$ contains $M(QC)$.

Lemma 1.4.12 ([22], Lemma 7)**.** *The weak-star closure of* $\mathbb{D}$ *contains* $M(QC)$*.*

Proof. Let x be any point of $M(QC)$. Any weak-star neighbourhood of x in QC^* contains a neighbourhood of the form

$$V_x := \{\phi \in QC^* : |\phi(f_r) - f_r(x)| < 1, r = 1, \dots s\},$$

where each f_r is a function in QC. Let $f = |f_1 - f_1(x)| + \cdots + |f_s - f_s(x)|$, f belongs to QC and $f(x) = 0$. f is not invertible in QC, so, it is not invertible in L^∞; this fact implies that the essential infimum of f in $\partial\mathbb{D}$ must be 0. As a consequence, the infimum of the Harmonic extension of f is 0 and then $|f_H(z_0)| \leq 1$ for some z_0 in $\mathbb{D}$. Since $|f_r - f_r(x)| \leq f$ for each r and for every point in $\partial\mathbb{D}$, then, the same inequality holds in the whole disk, $|f_{rH}(z_0) - f_r(x)| \leq |f_H(z_0)| \leq 1$. This means that z_0 is in the neighbourhood described at the beginning of the proof, and thus the weak-star closure of $\mathbb{D}$ contains $M(QC)$. □

Lemma 1.4.13. *Let* f *be a function in* QC *which is continuous at the point* λ_0*. Then* $x(f) = f(\lambda_0)$ *for every functional* x *in* $M_{\lambda_0}(QC)$*.*

Proof. Consider the case where the function f is continuous at λ_0 and such that $f(\lambda_0) = 0$. Let x be a point in $M_{\lambda_0}(QC)$. For $\epsilon > 0$ there is $\delta_0 > 0$ such that

$|f(\lambda)| < \epsilon$ for all λ in the arc $V_{\lambda_0} = (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$. The values taken by the Harmonic extension of f should be small if we evaluate points of $\mathbb{D}$ in an open disk with center at λ_0, *i.e.*, there is a δ_1 such that $|f_H(z)| < \epsilon/2$ if $z \in \mathbb{D}$ and $\text{dist}(z, \lambda_0) < \delta_1$.

We construct a neighbourhood V_x in QC^* with parameters f, f_0, ϵ_1, where $\epsilon_1 = \min\{\delta_0, \delta_1, \epsilon\}$. By Lemma 1.4.12, there is a point z in $\mathbb{D}$ such that $z \in V_x$, that is

$$|f_H(z) - x(f)| < \epsilon_1 < \epsilon \quad \text{and} \quad |f_0(z) - f_0(\lambda_0)| = |z - \lambda_0| < \epsilon_1 < \delta_1.$$

This implies that $\text{dist}(z, \lambda_0) \leq \delta_1$ and then $|f_H(z)| < \epsilon/2$.

Now we estimate $x(f)$,

$$|x(f)| \leq |x(f) - f_H(z)| + |f_H(z)| < \epsilon,$$

consequently $x(f) = 0$.

In the general case, when $f(\lambda_0) \neq 0$, we apply the previous argument to the function $g = f - f(\lambda_0)$. For g we obtain $0 = x(g) = x(f) - f(\lambda_0)$ and thus $x(f) = f(\lambda_0)$ for all $x \in M_{\lambda_0}(QC)$. □

For $z \neq 0$ in $\mathbb{D}$, we let I_z denote the closed arc of $\partial\mathbb{D}$ whose center is $z/|z|$ and whose length is $2\pi(1 - |z|)$. For completeness, $I_0 = \partial\mathbb{D}$.

Lemma 1.4.14 ([22], Lemma 5)**.** *For f in QC and any positive number ϵ, there is a positive number δ, such that $|f_H(z) - I_z(f)| < \epsilon$ whenever $1 - |z| < \delta$.*

Proof. To prove this lemma, we use the fact that the Harmonic extension is asymptotically multiplicative on QC, for which see [19]. Because of that property, there exists, for a given function f in QC and $\epsilon > 0$, a positive number δ such that

$$|f|_H^2(z) - |f_H(z)|^2 < \epsilon^2/9 \text{ if } 1 - |z| < \delta.$$

Using the relation $|\lambda|^2 = \lambda\bar{\lambda}$, we have

$$|f(e^{it}) - f_H(z)|^2 = f(e^{it})\overline{f(e^{it})} - f(e^{it})\overline{f_H(z)} - \overline{f(e^{it})}f_H(z) + |f_H(z)|^2; \qquad (1.4.9)$$

multiplying in both sides of equation (1.4.9) by the Poisson Kernel $P_r(t)$ and then integrating, with respect to the variable t, we obtain

$$|f|_H^2(z) - |f_H(z)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it}) - f_H(z)|^2 P_r(t)dt.$$

Using the Schwarz's inequality we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it}) - f_H(z)|P_r(t)dt \leq |f|_H^2(z) - |f_H(z)|^2. \qquad (1.4.10)$$

Multiplying the inequality $\frac{1}{3|I_z|} \leq \frac{P_r(t)}{2\pi}$, valid for every e^{it} in I_z, by $|f(e^{it}) - f_H(z)|$, and using the definition of $I(f)$, we have

$$|I_z(f) - f_H(z)| = \left|\frac{1}{|I_z|}\int_{I_z}(f(e^{it}) - f_H(z))dt\right| < \epsilon,$$

whenever $1 - |z| < \delta$ as required. □

The average of a function f over an arc I defines a linear functional on QC. Let us identify each arc I with the "averaging" functional in QC, the set of all these functionals is denoted by $\mathcal{G}$. The following lemma is proved using Lemmas 1.4.12 and 1.4.14.

Lemma 1.4.15 ([22], Page 822). *$M(QC)$ is the set of points in the weak-star closure of $\mathcal{G}$ which are not in $\mathcal{G}$ itself.*

Proof. If x is in $M_\lambda(QC)$ then x is not an interval, since the intervals are nonmultiplicative linear functionals in QC^*. Fix f in QC and $\epsilon > 0$; by Lemma 1.4.14 we have $|f_H(z) - I_z(f)| \leq \epsilon/2$ for all z close enough to the boundary of $\mathbb{D}$. Using f, f_0 and ϵ, we construct a neighbourhood of x in QC^*, namely,

$$V_x := \{\phi \in QC^* : |\phi(f) - f(x)| \leq \epsilon/2, \quad \text{and} \quad |\phi(f_0) - f_0(x)| \leq \epsilon/2\}.$$

By Lemma 1.4.12 this neighbourhood contains a point of $\mathbb{D}$, so there is a z such that $|f_0(z) - f_0(x)| = |z - \lambda| \leq \epsilon/2$. This fact implies that such z is close to the boundary, hence, the following inequality holds:

$$|I_z(f) - f(x)| \leq |I_z(f) - f_H(z)| + |f_H(z) - f(x)| \leq \epsilon,$$

which implies $x \in \overline{\mathcal{G}}^*$.

For the other inclusion we prove that if x is in $\overline{\mathcal{G}}^* - \mathcal{G}$ then x is multiplicative. Let $f \in QC$ and $\epsilon > 0$ be fixed. By Lemma 1.4.14 there exists a δ such

$$|f_H(z) - I_z(f)| \leq \epsilon \text{ and } |(f^2)_H(z) - I_z(f^2)| \leq \epsilon, \text{ whenever } 1 - |z| < \delta.$$

We estimate $|x(f^2) - x(f)^2|$

$$\begin{aligned} |x(f^2) - x(f)^2| &\leq |x(f)^2 - f_H(z)x(f)| \\ &+ |f_H(z)x(f) - (f_H(z))^2| \\ &+ |(f_H(z))^2 - (f^2)_H(z)| \\ &+ |(f^2)_H(z) - x(f^2)|. \end{aligned}$$

The summands in the previous inequality are bounded by $K\epsilon$ for an appropriate K in each part of the inequality, so $x(f^2) = x(f)^2$. Using the previous argument to the function $f + g$, we find that $x(fg) = x(f)x(g)$, that is, x is multiplicative and then x belongs to $M(QC)$. □

We denote by $\mathcal{G}_\lambda^0$ the set of all arcs I in $\mathcal{G}$ with center at λ. Let $M_\lambda^0(QC)$ be the set of functionals in $M_\lambda(QC)$ lying in the weak-star closure of $\mathcal{G}_\lambda^0$. By Lemma 1.4.14, the set $M_\lambda^0(QC)$ coincides with the set of functionals in $M_\lambda(QC)$ in the weak-star closure of the radius of $\mathbb{D}$ terminating at λ.

In [22], D. Sarason splits the space $M_\lambda(QC)$ onto three sets: $M_\lambda^+(QC) \setminus M_\lambda^0(QC)$, $M_\lambda^-(QC) \setminus M_\lambda^0(QC)$ and $M_\lambda^0(QC)$. These three sets are mutually disjoint due to next lemma:

Lemma 1.4.16 ([22], Lemma 8)**.** *$M_\lambda^+(QC) \cup M_\lambda^-(QC) = M_\lambda(QC)$ and $M_\lambda^+(QC) \cap M_\lambda^-(QC) = M_\lambda^0(QC)$.*

Proof. First we prove the inclusion $M_\lambda(QC) \subset M_\lambda^+(QC) \cup M_\lambda^-(QC)$. Let x belongs to $M_\lambda(QC)$ such that x does not belong to $M_\lambda^+(QC)$, by definition of $M_\lambda^+(QC)$, there is a function f in QC such that $\lim\limits_{t\to\lambda^+} f(t) = 0$ and $f(x) = 1$.
Let g be a function in QC such that $\lim\limits_{t\to\lambda^-} g(t) = 0$. The product fg is continuous at λ and $fg(\lambda) = 0$. Using Lemma 1.4.13 we have $0 = fg(x) = g(x)$, therefore x is in $M_\lambda^-(QC)$.
By definition, we know that $M_\lambda^\pm(QC)$ is a subset of $M_\lambda(QC)$, so we can conclude the equality $M_\lambda^+(QC) \cup M_\lambda^-(QC) = M_\lambda(QC)$.

For the second relation we only prove the inclusion $M_\lambda^0(QC) \subset M_\lambda^+(QC)$, the inclusion $M_\lambda^0(QC) \subset M_\lambda^-(QC)$ is proved using similar arguments.
Let $x \in M_\lambda^0(QC)$, fix $\epsilon > 0$ and a function f in QC such that $\lim\limits_{t\to\lambda^+} f(t) = 0$. The neighbourhood with parameters f and ϵ has an element of I in $\mathcal{G}_\lambda^0$, such that $|I(f) - x(f)| < \epsilon$, let us use de notation I^+ and I^- for the right part and the left part of I (with respect to λ). By Lemma 1.4.6 we have $\gamma_\lambda(f) = 0$, then, for a small interval

$$|I^-(f) - I^+(f)| < \epsilon.$$

Because $\lim\limits_{t\to\lambda^+} f(t) = 0$ we conclude that the average of f in I^+ is small. Now $|I^-(f)| \leq$

$|I^-(f) - I^+(f)| + |I^+(f)| \leq 2\epsilon$. With this estimate we find

$$|I(f)| = |\frac{I^+(f) + I^-(f)}{2}| < \epsilon.$$

From the previous inequalities follows that $|f(x)| = |x(f)| \leq |I(f) - x(f)| + |I(f)| < 2\epsilon$ and then $f(x) = 0$ which is, by definition, $x \in M_\lambda^+(QC)$.

For the other inclusion we proceed by contradiction. Let x be a point in $M_\lambda^+(QC) \cap M_\lambda^-(QC)$, such that $x \notin M_\lambda^0(QC)$. Then, there is a function f in QC such that $f = 0$ on $M_\lambda^0(QC)$ and $f(x) = 1$.

Choose a function u on $\partial\mathbb{D}$ which is continuous except for a jump discontinuity at λ and such that $u_\lambda^+ = 1$ and $u_\lambda^- = 0$. The function uf belongs to $VMO(\partial\mathbb{D} - \{\lambda\})$, and the gap $\gamma_\lambda(uf)$ is zero, therefore, by Lemma 1.4.7, uf is in QC, and hence is also $f - uf = (1 - u)f$.

The function uf vanishes on $M_\lambda^-(QC)$ because $\lim_{t \to \lambda^-} uf = 0$, by an analogous reason, $(1 - u)f$ vanishes on $M_\lambda^+(QC)$. Therefore, $1 = f(x) = uf(x) + (1 - u)f(x) = 0$ which is a contradiction. We have then the equality $M_\lambda^+(QC) \cap M_\lambda^-(QC) = M_\lambda^0(QC)$. □

The result in Lemma 1.4.16 allows us to draw a sketch of the maximal ideal space $M(QC)$. Consider the unit circle as the interval $[0, 2\pi]$, where the points 0 and 2π represent the same point; at each point λ in $[0, 2\pi]$ we draw a segment representing the fibre $M_\lambda(QC)$. The segment $M_\lambda(QC)$ is split into two parts, the upper part $M_\lambda^+(QC)$ and the lower part $M_\lambda^-(QC)$, their intersection (the bold part) corresponds to $M_\lambda^0(QC)$, the central part of the fiber.

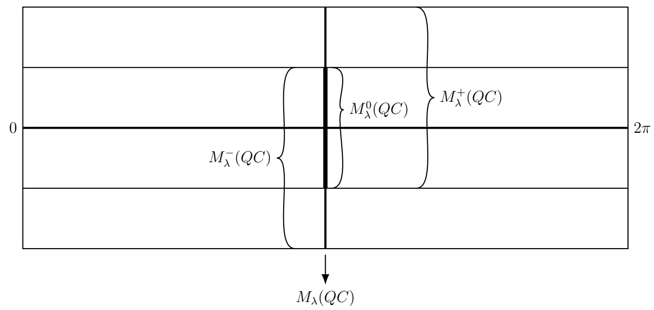

Figure 1.1: The maximal ideal space of QC.

1.5 Slowly oscillating functions

We recall the definition of slowly oscillating functions given in Section 1.4. Our objective is to establish the preliminary theory needed to describe the Toeplitz operator algebra generated by Toeplitz operators with symbols in the tensor product of QC (quasicontinuous symbols) and SO (slowly oscillating symbols).

For a bounded function g defined in a closed interval I, let $\omega(g, I)$ denote the oscillation of g over I

$$\omega(g, I) := \sup_{r,s\in I} |g(s) - g(r)|.$$

Definition 1.5.1. *For the half open interval* $[0, 1)$*, we define the class of* $SO_\infty := SO_\infty([0, 1))$ *which consist of all bounded functions* g *in* $[0, 1)$ *such that for each* $\eta \in (0, 1)$*,*

$$\lim_{\delta\to 0} \omega(g, [1 - \delta, 1 - \eta\delta]) = 0.$$

This is the C^**-algebra of bounded slowly oscillating functions.*

We define a subspace of SO_∞*,* $SO := SO([0, 1))$*, which consist of all continuous functions with slowly oscillation, that is,*

$$SO([0, 1)) := C([0, 1)) \cap SO_\infty([0, 1)).$$

In this section we interested in the theory needed to describe the Calkin algebra $\hat{\mathcal{T}}_{SO_\infty}$; for this end, we prove Lemma 1.5.4 which implies that the Calkin algebras $\hat{\mathcal{T}}_{SO_\infty}$ and $\hat{\mathcal{T}}_{SO}$ are the same. Later, in Section 3.1, we will proved that $\hat{\mathcal{T}}_{SO}$ is isomorphic to the algebra of continuous functions on a suitable compact space.

Let $L_{\infty,0} := L_{\infty,0}([0,1))$ denote the C^*-algebra of bounded functions g on $[0,1)$ such that $\lim_{t\to 1^-} g(t) = 0$. As a consequence of Theorem 1.2.3, the Toeplitz operators with symbols in $L_{\infty,0}$, extended to the unit disk, are compact.

We have shown (in Lemma 1.4.8) that $SO([0,1)) \subset VMO([0,1))$. This fact cannot be proved for functions in SO_∞ because the functions in SO_∞ only have controled behaviour near the point 1 and functions in $VMO([0,1))$ are well behaved on the whole interval.

Lemma 1.5.2. *Let g be fixed a function in SO_∞. For every $\epsilon > 0$ there exists $\delta > 0$, depending on g and ϵ, such that*

$$\frac{1}{|I|^2}\int_I\int_I |g(r) - g(q)|drdq < \epsilon,$$

for all $I \subset [1-2\delta, 1)$.

Previous lemma says that eventually, every function in SO_∞ has vanishing mean oscilation near the point 1; the proof is very similar to the proof of Lemma 1.4.8.

Lemma 1.5.3. *Let g be a function in SO_∞. The function*

$$g_c(t) := \frac{1}{1-t}\int_t^1 g(r)dr$$

belongs to SO.

Proof. Let $\eta \in (0,1)$ and $\epsilon > 0$. By Lemma 1.5.2 there is δ_0 such that $g \in VMO([1-2\delta_0, 1))$, which implies

$$\frac{1}{|I|^2}\int_I\int_I |g(r)-g(q)|drdq < \epsilon,$$

for all intervals $I \subset [1-2\delta_0, 1)$.

Now let $\delta < \delta_0$ and consider $s < t \in [1-\delta, 1-\eta\delta]$. The quantity $|g_c(s)-g_c(t)|$ is bounded by

$$\frac{1}{(1-t)(1-s)}\int_s^1\int_t^1 |g(r)-g(q)|drdq \leq \frac{1-s}{(1-t)(1-s)^2}\int_s^1\int_t^1 |g(r)-g(q)|drdq$$

The factor $\frac{1-s}{1-t}$ is bounded by $\frac{1-\delta}{1-\eta\delta} < 1$, thus,

$$|g_c(s)-g_c(t)| \leq \frac{1}{(1-s)^2}\int_s^1\int_t^1 |g(r)-g(q)|drdq < \epsilon.$$

Sumarizing we found that for $\eta \in (0,1)$,

$$\lim_{\delta\to 0}\omega(g, [1-\delta, 1-\eta\delta]) = 0,$$

which is, by definition, $g_c \in SO$. □

Lemma 1.5.4. *The algebra* $SO_\infty([0,1))$ *is represented as the sum*

$$SO_\infty([0,1)) = SO([0,1)) + L_{\infty,0}([0,1)).$$

Proof. Let g be a function in SO_∞, use the function $g_c(t)$ defined in Lemma 1.5.3 and consider the difference $g_0(t) = g(t) - g_c(t)$. Let $\epsilon > 0$ be fixed, find η and ϵ_1 such that $(1-\eta)\epsilon_1 + 2\|g\|_\infty\eta < 5\epsilon$. For this combination there exists δ_0 such that $|g(s)-g(r)| < \epsilon_1$ for all $s, r \in [1-\delta, 1-\eta\delta]$ whenever $\delta < \delta_0$.

For t in $[1-\delta_0, 1)$

$$|g_0(t)| \leq \frac{1}{1-t} \int_{1-(1-t)}^{1-\eta(1-t)} |g(s) - g(t)| ds + \frac{1}{1-t} \int_{1-\eta(1-t)}^{1} |g(s) - g(t)| ds.$$

The first integral is bounded by $(1-\eta)\epsilon_1$ and the second one is bounded by $2\|g\|_\infty \eta$, thus $\lim_{t \to 1^-} g_0(t) = 0$, that is, g_0 belongs to $L_{\infty,0}$. Obviously $g(t) = g_c(t) + g_0(t)$ which means that $SO_\infty([0,1)) = SO([0,1)) + L_{\infty,0}([0,1))$ as required. □

We extend the functions in SO_∞ (or SO) to the whole disk via the following formula

$$g(z) = g(|z|).$$

Denote this extension by SO_∞ (or SO), and define the Toeplitz operator algebras $\mathcal{T}_{SO_\infty}$ and $\mathcal{T}_{SO}$. By Lemma 1.5.4, up to a compact perturbation, the Toeplitz operator algebras $\mathcal{T}_{SO_\infty}$ and $\mathcal{T}_{SO}$ are the same.

Let $M(SO)$ denote the maximal ideal space of SO. $C([0,1])$ is a C^*-subalgebra of SO, thus every functional in $M(SO)$ induces, by restriction, a functional in $C([0,1])$. The maximal ideal space of $C([0,1])$ is homeomorphic to $[0,1]$, this fact allows us to define a fibration of $M(SO)$ as follows

Definition 1.5.5. *For every point $t \in [0,1]$ we define*

$$M_t(SO) = \{y \in M(SO) : y(g) = g(t) \ \ \textit{for all continuous functions } g \textit{ on } [0,1]\}.$$

The fibers $M_t(SO)$ for points $t \in [0,1)$ are singletons that can be identified with the point t itself, but the fiber $M_1(SO)$ has a very different structure.

Theorem 1.5.6. *Let $M_1'(SO)$ be the subset of $M(SO)$ which consist of all $y \in M(SO)$ with the property that $g(y) = 0$ whenever g in SO is a function such that $\lim_{t \to 1^-} g(t) = 0$. The set $M(SO)$ has the following description*

$$M(SO) = [0,1) \cup M_1'(SO).$$

Proof. We need to prove that $M_1'(SO) = M_1(SO)$. Let f be a function in SO such that $\lim_{t\to 1^-} f(t) = 0$. The function f can be extended continuously to $[0,1]$ defining $f(1) = 0$. In this case, for every y in $M_1(SO)$ we have $y(f) = f(1) = 0$, thus y is in $M_1'(SO)$. This argument proves the inclusion $M_1(SO) \subset M_1'(SO)$. For the other inclusion consider a function f continuous in $[0,1]$, the function $g(t) = f(t) - f(1)$ has limit 0 as t approaches to 1^-. For every y in $M_1'(SO)$, $0 = y(g) = y(f) - f(1)$; this equation implies that $y(f) = f(1)$, that is, y belongs to $M_1(SO)$. $\square$

As a consequence of Theorem 1.5.6, the set $M_1(SO)$ also is characterized as the weak-star closure of the interval $[0,1)$ in the dual space SO^*. With this characterization we have that the value of $y \in M_1(SO)$ at a function $f \in SO$, is obtained via a convergent sequence of points, that is,

$$y(f) = \lim_{n\to\infty} f(t_n),$$

for a sequence t_n (depending on y and f) converging to 1^-.
On the other hand, if there is a sequence $t_n \to 1^-$ such that $\lim_{n\to\infty} f(t_n) = k$, then there is a point $y \in M_1(SO)$ such that $y(f) = k$.

The main application of this theorem relays on the description of the Calkin algebra $\mathcal{T}_{SO_\infty}/\mathcal{K}' \cong \mathcal{T}_{SO}/\mathcal{K}'$, where $\mathcal{K}'$ denotes the ideal $\mathcal{K} \cap \mathcal{T}_{SO}$. The tools needed to obtain this description are not provided yet. In Chapter 3, where we recall the Toeplitz operators with slowly oscillating symbols, we prove this fact among others we need to describe the Calkin algebra $\mathcal{T}_{SO}/\mathcal{K}'$.

Chapter 2

Toeplitz operators with piecewise quasicontinuous symbols

This chapter is devoted to Toeplitz operators with symbols in an extension of the piecewise quasicontinuous C^*-algebra PQC. PQC is defined as the C^*-algebra generated by both, the algebra PC of piecewise continuous functions; and QC the C^*-algebra of quasicontinuous functions, both algebras defined on the unit circle $\partial\mathbb{D}$. The C^*-algebra PQC is defined on $\partial\mathbb{D}$ and can be extended to $\mathbb{D}$ by two standard methods: the first one is the harmonic extension and the second one is the radial extension. Our interest on Toeplitz operators with QC defining symbols lies in Lemma 2.1.3, which implies that the Calkin algebra $\hat{T}_{QC}$ is commutative.

The main goal of this chapter is to describe $\mathcal{T}_{PQC}/\mathcal{K}$, the Calkin algebra of the C^*-algebra generated by Toeplitz operators with symbols in certain extension of PQC to the unit disk. This objective is achieved as follows: first we define the algebras $\mathcal{T}_{PQC}$ and $\mathcal{T}_{QC}$; as a second step, we prove that the Calkin algebra $\mathcal{T}_{QC}/\mathcal{K}$ is a commutative subalgebra of $\mathcal{T}_{PQC}/\mathcal{K}$. Finally, we use the DVLP, explained in Chapter 1, to describe the algebra $\mathcal{T}_{PQC}/\mathcal{K}$ as the C^*-algebra of all continuous sections on the C^*-bundle $\xi_{PQC} = (p, E, M(QC))$. In Section 2.2 we prove that the description of $\mathcal{T}_{PQC}/\mathcal{K}$ does not depend on the extension chosen for the functions in PQC.

In Section 2.3 we give a canonical representation of the operators in $\mathcal{T}_{PQC}$. This canonical representation allows us to determine whether or not an operator $A \in \mathcal{T}_{PQC}$ is a compact perturbation of a Toeplitz operator (Section 2.4).

Final section (Section 2.5) contains the Fredholm theory of the operators in the algebra $\mathcal{T}_{PQC}$. A Fredholm index formula is given for Fredholm operators in $\mathcal{T}_{PQC}$.

2.1 Toeplitz operators with PQC defining symbols

Definition 2.1.1. *Let $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a fixed set of n different points on $\partial\mathbb{D}$. Define $PC := PC_\Lambda$ as the set of continuous functions on $\partial\mathbb{D} \setminus \Lambda$ with one-sided limits at every point λ_k in Λ.*
For a function a in PC we set $a_k^+ := \lim\limits_{\lambda \to \lambda_k^+} a(\lambda)$ and $a_k^- := \lim\limits_{\lambda \to \lambda_k^-} a(\lambda)$, following the standard positive orientation of $\partial\mathbb{D}$.

Definition 2.1.2. *PQC is the C^*-subalgebra of $L^\infty(\partial\mathbb{D})$ generated by PC and QC.*

Our interest is to describe a certain Toeplitz operator algebra acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$. To this end, we extend the functions in PQC to the whole disk. There are two most natural ways of such extensions

- the harmonic extension f_H, given by the Poisson formula in equation (1.4.8),
- the radial extension f_R, defined by $f_R(r, \theta) = f(\theta)$.

In this section we use the radial extension, however, we emphasize that the main result of this chapter does not depend on the extension chosen (Theorem 2.2.4). From now on, and until further notice, we use only the radial extension of a function in PQC. For simplicity of notation, we use PQC to denote functions defined on $\partial\mathbb{D}$ as well as radial extensions of such functions. Moreover f will denote both, the function on $\partial\mathbb{D}$ and its radial extension to $\mathbb{D}$.

Lemma 2.1.3. *Let f be a function in QC. Then, f belongs to Q.*

Proof. According to Definition 1.3.2 and Theorem 1.3.3, we need to estimate

$$\frac{1}{|S_z|}\int_{S_z}\left|f(w)-\frac{1}{|S_z|}\int_{S_z}f(u)dA(u)\right|dA(w). \tag{2.1.1}$$

Using polar coordinates we get that this quantity is equal to

$$\frac{2}{|I_z|^2}\int_{I_z}\int_{I_z}|f(\theta)-f(\phi)|dA(\theta)dA(\phi). \tag{2.1.2}$$

If z is close to the boundary, then the measure of $|I_z|$ is small, hence, the expression in equation (2.1.2) goes to zero because f is in QC. This implies that the expression in (2.1.1) goes to zero if $|z|$ approaches to 1^-, thus f is in Q as required. □

The Harmonic extension f_H also belongs to Q, we prove this fact in the next section.

Recall, from Chapter 1, that the Bergman space $\mathcal{A}^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ which consists of all analytic functions on $\mathbb{D}$.

By $\mathcal{T}_{PQC}$ we denote the C^*-algebra generated by Toeplitz operators with symbols in PQC acting on $\mathcal{A}^2(\mathbb{D})$. We use $\hat{\mathcal{T}}_{PQC}$ to denote the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$. The main goal of this section is to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$.

We use the DVLP to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$. Lemma 2.1.3 and the results in [27] imply that the quotient $\hat{\mathcal{T}}_{QC} = \mathcal{T}_{QC}/\mathcal{K}$ is a commutative C^*-subalgebra of $\hat{\mathcal{T}}_{PQC}$. Thus we use $\hat{\mathcal{T}}_{QC}$ as a central algebra required to apply the DVLP in the description of $\hat{\mathcal{T}}_{PQC}$. The algebra $\hat{\mathcal{T}}_{QC}$ can be identified with QC

$$\hat{\mathcal{T}}_{QC} = \{T_f + \mathcal{K} | f \in QC\},$$

hence we localize by points in $M(QC)$. We construct the system of ideals parametrized by points x in $M(QC)$.

Definition 2.1.4. *For every point $x \in M(QC)$, define the maximal ideal of $\hat{\mathcal{T}}_{QC}$, $J_x := \{f \in QC : f(x) = 0\} = \{T_f + \mathcal{K} : f(x) = 0\}$. The ideal $J(x)$ is defined as the set $J_x \cdot \mathcal{T}_{PQC}/\mathcal{K}$.*

We use the notation $\hat{\mathcal{T}}_{PQC}(x) := \hat{\mathcal{T}}_{PQC}/J(x)$ for the local algebra at the point x. We identify T_f with its class $\hat{T}_f$ via the natural projection $\pi : \mathcal{T}_{PQC} \to \hat{\mathcal{T}}_{PQC}$, thus the local behaviour of T_f actually means the local behaviour of $\pi(T_f) = \hat{T}_f$.

Lemma 2.1.5. *Let f be a function in QC and x a point of $M(QC)$. The Toeplitz operator T_f is locally equivalent, at the point x, to the complex number $f(x)$ (realized as the operator $f(x)I$).*

Proof. Let x be a point in $M(QC)$ and f be a function in QC. The function $f - f(x)$ belongs to $J(x)$, thus the operator $T_f - T_{f(x)} = T_{f-f(x)}$ is zero in $\hat{\mathcal{T}}_{PQC}(x)$. This means that the operator T_f is locally equivalent to the operator $T_{f(x)} = f(x)I$ and then, the operator T_f is locally equivalent to the complex number $f(x)$. □

Lemma 2.1.6. *Let x be a point of $M_\lambda(QC)$ with $\lambda \notin \Lambda$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number $a(\lambda)$ (realized as the operator $a(\lambda)I$).*

The proof is very similar to the proof of Lemma 2.1.5 and it is omitted.

For the case when $x \in M_{\lambda_k}(QC)$, we use Lemma 1.4.16 to split the fiber $M_{\lambda_k}(QC)$ into three disjoints sets: $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and $M^0_{\lambda_k}(QC)$.

Lemma 2.1.7. *Let x be a point of $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a^+_k (realized as the operator $a^+_k I$).*

Proof. Let a be a function for which $a^+_k = 0$. If x belongs to $M^+_{\lambda_k}(QC)\setminus M^0_{\lambda_k}(QC)$, then x belongs to $M^+_{\lambda_k}(QC)$ and does not belong to $M^-_{\lambda_k}(QC)$. This implies the existence of a function g in QC such that $\lim\limits_{\lambda\to\lambda_k^-} g(\lambda) = 0$ and $g(x) = 1$.
The product ag is continuous at λ_k and $ag(\lambda_k) = 0$. The difference $T_a - T_{ag}$ can be rewritten as $T_{(1-g)a} = T_{1-g}T_a + K$ where K is a compact operator. Since the function $1-g$ vanishes at x, T_{1-g} belongs to J_x, and then $T_a - T_{ag}$ belongs to $J(x)$. We conclude that, locally, the Toeplitz operator with symbol a is equivalent to the Toeplitz operator with symbol ag. At the same time, the Toeplitz operator T_{ag} is locally equivalent to the complex number $0 = ag(\lambda_k)$, hence, the operator T_a is locally equivalent to the complex number $a^+_k = 0$.

For the general case, if the function a in PC has limit $a^+_k \neq 0$, we construct the function $b(\lambda) = a(\lambda) - a^+_k$. The function b has lateral limit $b^+_k = 0$, fulfilling the initial assumption of the proof. The Toeplitz operator $T_b = T_a - a^+_k I$ is locally equivalent to the complex number 0, thus the Toeplitz operator T_a is locally equivalent to the complex number a_k^+. $\square$

Similarly the following lemma holds:

Lemma 2.1.8. *Let x be a point of $M_{\lambda_k}^-(QC) \setminus M_{\lambda_k}^0(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a_k^- (realized as the operator $a_k^- I$).*

Now we analize the case when x belongs to central part of the fiber $M_{\lambda_k}(QC)$, *i.e,* $x \in M_{\lambda_k}^0(QC)$. For this case we need the results established in Section 1.2 for Toeplitz operators with zero-order homogeneous symbols defined on Π.

Let $\partial\mathbb{D}_k^+$ denote the upper half of the circunference $\partial\mathbb{D}$ and $\mathbb{D}_k^+$ the upper half of the disk $\mathbb{D}$, both determined by the diameter with end points at λ_k and $-\lambda_k$ and following the standard positive orientation of $\partial\mathbb{D}$. Denote by $\partial\mathbb{D}_k^-$ and $\mathbb{D}_k^-$ the complement of $\partial\mathbb{D}_k^+$ and $\mathbb{D}_k^+$ in $\partial\mathbb{D}$ and $\mathbb{D}$, respectively, and by $\chi_{\partial,k}^+$ and χ_k^+ the characteristic functions of $\partial\mathbb{D}_k^+$ and $\mathbb{D}_k^+$, respectively.

Recall that all the piecewise continuous functions can be written as a linear combination of the characteristic function χ_k^+ and a continuous function, more specifically, for each a in PC there is a continuous function s such that

$$a(\lambda) = a_k^+ \chi_{\partial,k}^+ + a_k^- (1 - \chi_{\partial,k}^+) + s(\lambda),$$

and $s(\lambda_k) = 0$.

Due to the property described in the previous paragraph, we conclude that the local algebra at the point $x \in M_{\lambda_k}^0(QC)$ is isomorphic to the C^*-algebra generated by $T_{\chi_k^+}$

and the identity I.

Let ϕ be a Möbius transformation which sends the upper half plane to the unit disk and such that: $\phi(0) = \lambda_k$, $\phi(i) = 0$ and $\phi(\infty) = -\lambda_k$. Using the function ϕ we construct a unitary transformation W which sends $L^2(\mathbb{D})$ onto $L^2(\Pi)$. Under the unitary transformation W, the Toeplitz operator with symbol h, acting on $\mathcal{A}^2(\mathbb{D})$, is unitary equivalent to the Toeplitz operator $T_{h(\phi(w))}$ acting on $\mathcal{A}^2(\Pi)$.

By Theorem 1.2.6, the Toeplitz operator with symbol $\chi_k^+(\phi(w))$, acting on $\mathcal{A}^2(\Pi)$, is unitary equivalent to the multiplication operator by the function $\gamma_{\chi_k^+(\phi(w))}$, acting on $L^2(\mathbb{R})$. Following the unitary equivalences we conclude the unitary equivalence between $T_{\chi_k^+}$ and $\gamma_{\chi_k^+(\phi(w))}$, thus the algebra generated by $T_{\chi_k^+}$ and I, is isomorphic to the algebra generated by $\gamma_{\chi_k^+(\phi(w))}$ and the function 1.

By Corollary 7.2.2 in [26], the function $\gamma_{\chi_k^+(\phi)}(s) = \frac{1-e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{1+e^{-2s\pi}}$ is continuous in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; thus, as a consequence of Corollary 1.2.8, the function $\gamma_{\chi_k^+(\phi)}(s)$ and the identity generate the algebra of continuous functions on $\bar{\mathbb{R}}$.

Using the change of variable $t = \frac{1}{1+e^{-2s\pi}}$, which is a homeomorphism between $[0,1]$ and $\bar{\mathbb{R}}$, we conclude that the local algebra $\hat{\mathcal{T}}_{PQC}(x)$ is isomorphic and isometric to $C[0,1]$ for every $x \in M^0_{\lambda_k}(QC)$; further, such isomorphism, denoted here by ψ, acts on the generator $T_{\chi_k^+}$ as follows:

$$T_{\chi_k^+} \mapsto t.$$

This implies that the Toeplitz operator with symbol a in PC is sent to $C([0,1])$, via ψ, to the function $a_k^-(1-t) + a_k^+ t$. Thus we come to the following lemma.

Lemma 2.1.9. *If x belongs to $M^0_{\lambda_k}(QC)$, then the local algebra generated by the Toeplitz operators with symbols in PQC is isometric and isomorphic to the algebra of all continuous functions on* $[0,1]$.

With the set $M(QC)$, we construct the C*-bundle $\xi_{\text{PQC}} := (p, E, M(QC))$. We use the description of the local algebras given by Lemmas 2.1.6, 2.1.7, 2.1.8 and 2.1.9 to construct the bundle $E := \bigcup_{x \in M(QC)} E_x$ where

- $E_x = \mathbb{C}$, if $x \in M_\lambda(QC)$ with $\lambda \notin \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$,
- $E_x = C([0,1])$, if $x \in M^0_{\lambda_k}(QC)$, $\lambda_k \in \Lambda$.

The function p is the natural projection from E onto $M(QC)$.

Let $\Gamma(\xi_{PQC})$ denote the algebra of all bounded continuous sections of the bundle ξ_{PQC} (see Section 1.1). By DVLP (Theorem 1.1.4) we come to the following theorem:

Theorem 2.1.10. *The C^*-algebra $\hat{\mathcal{T}}_{PQC}$ is isometric and isomorphic to the C^*-algebra of continuous sections over the C^*-bundle ξ_{PQC}.*

As a corollary of Theorem 2.1.10, the algebra $\hat{\mathcal{T}}_{PQC}$ is commutative, then there is a compact space $X := M(\hat{\mathcal{T}}_{PQC})$, such that $\hat{\mathcal{T}}_{PQC} \cong C(X) = C(M(\hat{\mathcal{T}}_{PQC}))$. The compact space $M(\hat{\mathcal{T}}_{PQC})$ is constructed using the irreducible representations of $\hat{\mathcal{T}}_{PQC}$.

Let $\hat{\partial\mathbb{D}}$ be the set $\partial\mathbb{D}$ cut at the points λ_k of Λ. The pair of points of $\hat{\partial\mathbb{D}}$ which correspond to the point λ_k are denoted by λ_k^+ and λ_k^-, following the positive orientation of $\partial\mathbb{D}$. Let $I^n := \bigsqcup_{k=1}^{n} [0,1]_k$ be the disjoint union of n copies of the interval $[0,1]$.

Denote by Σ the union of $\partial\hat{\mathbb{D}}$ and I_n with the point identification

$$\lambda_k^- \equiv 0_k \qquad \lambda_k^+ \equiv 1_k,$$

where 0_k and 1_k are the boundary points of $[0,1]_k$, $k = 1, \ldots, n$.
Let $M(\hat{\mathcal{T}}_{PQC}) := \bigcup_{\lambda\in\Sigma} M_\lambda(\mathcal{T}_{PQC})$ where each fiber corresponds to

$$\begin{aligned} M_\lambda(\hat{\mathcal{T}}_{PQC}) :=& M_\lambda(QC) \text{ if } \lambda \in \partial\hat{\mathbb{D}}, && \lambda \notin \Lambda, \\ M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}) :=& \big(M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)\big) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda, \\ M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}) :=& \big(M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)\big) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda, \\ M_t(\hat{\mathcal{T}}_{PQC}) :=& M^0_{\lambda_k}(QC) \text{ if } t \in (0,1)_k, && k = 1, \ldots, n. \end{aligned}$$

With the help of Figure 1.1 in Page 42, we draw the maximal ideal space of $\hat{\mathcal{T}}_{PQC}$. The idea is to duplicate the set $M^0_{\lambda_k}(QC)$ and then connect this two copies by the interval $[0,1]$.

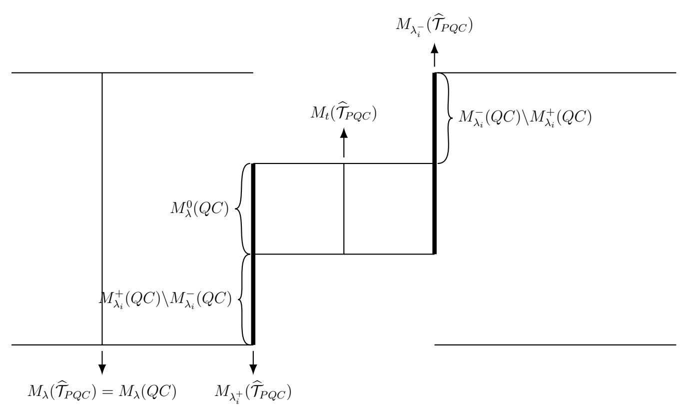

Figure 2.1: The maximal ideal space of $\widehat{\mathcal{T}}_{PQC}$.

We use the topology of $M(QC)$ to describe the topology of $M(\hat{\mathcal{T}}_{PQC})$. We only describe the topology of the fibers $M_{\lambda_k^\pm}(\hat{\mathcal{T}}_{PQC})$ and $M_t(\hat{\mathcal{T}}_{PQC})$, since the topology on the other fibers corresponds to the topology of $M_\lambda(QC)$. For x in $M(QC)$, let $\Omega(x)$ denote the family of open neighbourhoods of x. For $x \in M_\lambda(QC)$ and N in $\Omega(x)$, let $N_\lambda = N \cap M_\lambda(QC)$, and let N_{λ^+} and N_{λ^-} denote the sets of points in N that lie above the semicircles $\partial\mathbb{D}_k^+$ and $\partial\mathbb{D}_k^-$, respectively.

Consider the fiber $M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC})$. The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^+}$ form neighbourhoods of $x \in M_{\lambda_k}^+(QC) \setminus M_{\lambda_k}^-(QC)$.

Let $\Omega_+(x)$ be the set of neighbourhoods N in $\Omega(x)$ satisfying $N = N_\lambda \cup N_{\lambda^+}$. The sets

$$(N_{\lambda_k} \times (1-\epsilon, 1]) \cup N_{\lambda_k^+} \qquad N \in \Omega_+(x), \qquad \text{and} \qquad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M_{\lambda_k}^0(QC)$.

The open neighbourhoods for points in the fiber $M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC})$ are constructed analogously.

The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^-}$ form neighbourhoods of $x \in M_{\lambda_k}^-(QC) \setminus M_{\lambda_k}^+(QC)$.

The sets

$$(N_{\lambda_k} \times [0, \epsilon)) \cup N_{\lambda_k^-} \qquad N \in \Omega_-(x), \qquad \text{and} \qquad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M_{\lambda_k}^0(QC)$.

Finally the set $M_{\lambda_k}^0(QC) \times (0,1)$ carries the product topology.

Theorem 2.1.11. *Let $X := M(\hat{\mathcal{T}}_{PQC})$ with the topology described above. The Calkin algebra $\hat{\mathcal{T}}_{PQC}$ is isomorphic to the C^*-algebra of continuous functions over X. The homomorphism* Sym : $\mathcal{T}_{PQC} \to C(X)$ *acts on the generators of $\mathcal{T}_{PQC}$ in the following way:*

- *For generators which symbols is a function $a \in PC$*

$$\operatorname{Sym}(T_a)(x) = \begin{cases} a(\lambda), & \text{if } x \in M_\lambda(\hat{\mathcal{T}}_{PQC}) \text{ with } \lambda \notin \Lambda; \\ a_k^+, & \text{if } x \in M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-, & \text{if } x \in M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-(1-t) + a_k^+ t, & \text{if } x \in M_t(\hat{\mathcal{T}}_{PQC}), t \in (0,1). \end{cases}$$

- *For generators which symbols are functions f in QC,* $\operatorname{Sym}(T_f)(x) = f(x)$.

2.2 Independence of the result on the extension chosen

In this section we prove that the description of the algebra $\hat{\mathcal{T}}_{PQC}$ (Theorem 2.1.11) does not depend on the extension chosen for functions in PQC.
Recall that PQC is the C^*-subalgebra of $L^\infty(\partial\mathbb{D})$ generated by PC and QC. This algebra is defined on $\partial\mathbb{D}$ and then extended to the whole disk by two different ways:

- the harmonic extension f_H given by the Poisson formula in equation (1.4.8),

- the radial extension f_R, defined by $f_R(r,\theta) = f(\theta)$.

Let a be a function in PC. At the point $x \in M_\lambda(QC)$, for $\lambda \notin \Lambda$; the Toeplitz operator T_{a_R} is locally equivalent to the complex number $a(\lambda)$. The same still true if we use the harmonic extension a_H. For points x in $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ (respectively $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$), the Toeplitz operators T_{a_R} and T_{a_H} are equivalent to the number a_k^+ (respectively a_k^-), and then, the local algebras are the same.

Now, we analize the case when x belongs to $M^0_{\lambda_k}(QC)$. Let $\hat{a}$ be a function in PC, we construct a function a such that $a(\lambda) = \hat{a}^+_k$ on $\partial\mathbb{D}^+_k$ and $a(\lambda) = \hat{a}^-_k$ on $\partial\mathbb{D}^-_k$. The Toeplitz operator with symbol $\hat{a}_H$ is locally equivalent to T_{a_H}. This fact implies that the local behaviour of the class of the operator T_a ($a \in PC$) depends only on the limit values a^+_k and a^-_k, thus, without loss of generality, we can assume that the function a is constant on $\partial\mathbb{D}^+_k$ and $\partial\mathbb{D}^+_k$.

As in Section 2.1, we use a Möbius transformation ϕ to generate a unitary operator between $L^2(\mathbb{D})$ and $L^2(\Pi)$. Consider a function a in PC such that $a|_{\partial\mathbb{D}^+_k} = a^+_k$ and $a|_{\partial\mathbb{D}^-_k} = a^-_k$. The function $a_H(\phi(z))$ is harmonic in Π and corresponds to the harmonic extension of $a(\phi(t))$, moreover, this harmonic extension from $\mathbb{R}$ to Π is $a^\Pi_H := \frac{\theta}{\pi}(a^-_k - a^+_k) - a^+_k$, which is a zero-order homogeneous function on Π. By Theorem 1.2.6, the Toeplitz operator $T_{a^\Pi_H}$ is unitary equivalent to the multiplication operator $\gamma_{a^\Pi_H}$. The function $\gamma_{a^\Pi_H}$ is given by

$$\gamma_{a^\Pi_H} = A\left(\frac{1}{2s\pi} - \frac{1}{e^{-2s\pi} - 1}\right) + B,$$

for suitable complex constants A and B. Corollary 1.2.9 implies that the algebra generated by $\gamma_{a^\Pi_H}$ and the identity is the algebra of continuous functions on $\bar{\mathbb{R}}$.

Following the unitary equivalences from T_{a_H} to $\gamma_{a^\Pi_H}$ and making a change of variables, we have that the algebra generated by T_{a_H} is isomorphic to the algebra of continuous functions over the segment $[0, 1]$.

We already know, by Lemma 2.1.9, that the Toeplitz operator with symbol a_R generates the algebra of continuous functions over $[0, 1]$ as well, so the local algebras, at the point $x \in M^0_{\lambda_k}(QC)$, generated by T_{a_H} and T_{a_R} are the same. We have thus proved

Lemma 2.2.1. *Consider the algebra PC defined on $\partial\mathbb{D}$ and its extensions PC_R and PC_H. The local algebras $\hat{\mathcal{T}}_{PC_R}(x)$ and $\hat{\mathcal{T}}_{PC_H}(x)$ are the same.*

To show the same theorem for functions f in QC we need to use the properties established in Section 1.3. The main tool is the description of Q by means of the Berezin transform (Theorem 1.3.4).

Lemma 2.2.2. *For a function f in QC, the harmonic extension f_H belongs to Q.*

Proof. For this proof we use two facts; the first result is found in [14]; an the second one in [20].

1. The Berezin Transform of a harmonic function is the function itself, in our case, $\tilde{f}_H = f_H$.

2. the harmonic extension is asymptotically multiplicative in QC, which implies

$$\lim_{|z|\to 1^-} |f|^2_H(z) - |f_H(z)|^2 = 0.$$

Now we proceed with the proof:

$$\begin{aligned}\widetilde{|f_H|^2}(z) - |\widetilde{f_H}(z)|^2 &\leq \left|\widetilde{|f_H|^2}(z) - |f|^2_H(z)\right| + \left||f|^2_H(z) - |f_H(z)|^2\right| \\ &\leq \left||f_H|^2\widetilde{(z) - |f|}^2_H(z)\right| + \left||f|^2_H(z) - |f_H(z)|^2\right|.\end{aligned}$$

The two results listed at the beginning of the proof imply that

$$\lim_{|z|\to 1^-} \widetilde{|f_H|^2}(z) - |\widetilde{f_H}(z)|^2 = 0,$$

thus, by Theorem 1.3.4, f_H belongs to Q. □

Lemma 2.2.3. *Consider the function f in QC. The Toeplitz operator with symbol $f_R - f_H$ is compact.*

Proof. For a function f in QC_R, we calculate $\hat{f}_R$ and get $\hat{f}_R(z) = I_z(f)$ (according to Definition 1.3.6) . Then, Theorem 1.3.7 gives us the decomposition $f_R(z) = I_z(f) + (f_R(z) - I_z(f))$.

We write $f_R(z) - f_H(z) = (I_z(f) - f_H(z)) + (f_R(z) - I_z(f))$. The first summand goes to zero as $|z|$ approaches to 1^- by Lemma 1.4.14, then, by Theorem 1.3.7, the function $I_z(f) - f_H(z)$ belongs to $ESV(\mathbb{D}) \cap B$. By the descomposition of Q as $ESV(\mathbb{D}) + Q \cap B$ we have that $(f_R(z) - I_z(f)) = f_R(z) - \hat{f}_R(z)$ belongs to $Q \cap B$.

In summary, the function $f_R(z) - f_H(z)$ belongs to $Q \cap B$ and then the Toeplitz operator with symbol $f_R - f_H$ is compact. □

Now we establish the main result of this section: the algebra described in Theorem 2.1.11 does not depend on the extension chosen for the symbols in PQC.

Theorem 2.2.4. *Let PQC_R and PQC_H denote, respectively, the radial and the harmonic extension to the disk of functions in PQC. Then, the Calkin algebras $\mathcal{T}_{PQC_R}/\mathcal{K}$ and $\mathcal{T}_{PQC_H}/\mathcal{K}$ are the same.*

Proof. The proof follows from Lemma 2.2.1 and 2.2.3. □

2.3 Anatomy of the algebra generated by Toeplitz operators with piecewise quasicontinuous symbols

The objective of this section is to describe the operators in the C^*-algebra $\mathcal{T}_{PQC}$. We follow the approach in Chapter 8 of [26] for Toeplitz operators with piecewise continuous symbols. We introduce the description of the operators in $\mathcal{T}_{PC}$ (Theorem 8.3.5 in [26]) and extend this result to $\mathcal{T}_{PQC}$.

For every point $\lambda_k \in \Lambda$ we choose two open neighbourhoods V_k' and V_k'' such that $V_k' \subset V_k'' \subset \mathbb{D}$ and $V_k'' \cap V_j'' = \emptyset$ for $k \neq j$. For each k we construct a continuous function $\mu_k : \mathbb{D} \to [0,1]$ such that $\mu_k|_{V_k'} = 1$, and $\mu_k|_{\mathbb{D}\setminus V_k''} = 0$. We denote by $\chi_k := \chi_{\mathbb{D}_k^+}$ the characteristic function of the upper half disk determined by the diameter at λ_k; in the case $\lambda_k = 1$ we set $\chi_+ := \chi_{\mathbb{D}_k^+}$.

Theorem 2.3.1 ([26], Theorem 8.3.5)**.** *Every operator A in $\mathcal{T}_{PC}$ admits the canonical representation*

$$A = T_{S_A} + \sum_{k=1}^{n} T_{\mu_k} f_{A,k}(T_{\chi_k}) + K,$$

where S_A is a continuous function on $\overline{\mathbb{D}}$, $f_{A,k}$, $k = 1, 2, \ldots, n$, are continuous functions on $[0,1]$, and K is a compact operator.

The conditions imposed to the functions S_A and $f_{A,k}$ follow from the structure of the maximal ideal space of $\hat{\mathcal{T}_{PC}}$ giving by Theorem 1.2.5 ([26]), thus, in our case, we need to modify these conditions to the space $M(\hat{\mathcal{T}_{PQC}})$ in Theorem 2.1.11. Theorem 2.3.7 establishes the corresponding version of Theorem 2.3.1 for operators in $\mathcal{T}_{PQC}$.

The functions μ_k defined at the beginning of this section permit us to isolate the discontinuities of functions in PC and treat each one separately. Hence the proof of our version of Theorem 2.3.1 for a single discontinuity can be easily extended to the case $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

We split the proof of the principal result of this section (Theorem 2.3.7) into the lemmas presented from now and until the end of this section.

Definition 2.3.2. *Let PQC_0 be the (non closed) algebra generated by PC and QC,* *that is*

$$PQC_0 := \left\{ g \in L^\infty : g = \sum_{j=1}^{m} f_j a_j \text{ where } f_j \in QC \text{ and } a_j \in PC \right\}.$$

Obviously both C^*-algebras, $\mathcal{T}_{PQC}$ and $\mathcal{T}_{PQC_0}$, are the same. The operators of $\mathcal{T}_{PQC}$ are of one of three types:

- Finite sums of Toeplitz operators, $A = \sum_{j=1}^{m} T_{f_j a_j}$ with $f_j \in QC$ and $a_j \in PC$,
- finite sums of finite products of Toeplitz operators, $A = \sum_{j=1}^{m} \prod_{r=1}^{s} T_{f_{rj} a_{rj}}$ with $f_{rj} \in QC$ and $a_{rj} \in PC$,
- and uniform limits of the previous operators.

Recall that a normed space is a Banach space if and only if any absolutely convergent series is convergent (see, for example, [15]). This result allows us to approximate any operator in $\mathcal{T}_{PQC}$ in a very special way.

Lemma 2.3.3. *Any operator A in $\mathcal{T}_{PQC}$ can be represented as an absolutely convergent series*

$$A = \sum_{j=1}^{\infty} A_j,$$

where each operator A_j is a finite product $A_j = \prod_{r=1}^{s} T_{f_{rj} a_{rj}}$, with $f_{rj} \in QC$ and $a_{rj} \in PC$.

Now fix an operator A in $\mathcal{T}_{PQC}$ with an absolutely convergent approximation

$$A = \sum_{j=1}^{\infty} A_j,$$

where $A_j = \prod_{r=1}^{s} T_{f_{rj} a_{rj}}$, with $f_{rj} \in QC$ and $a_{rj} \in PC$. By Lemma 2.1.3,

$$A_j = T_{\prod_{r=1}^{s} f_{rj}} \prod_{r=1}^{s} T_{a_{rj}} + K_j = T_{f_j} B_j + K_j.$$

Applying Theorem 2.3.1 to the operator B_j we obtain

$$B_j = T_{S_{B_j}} + T_{\mu_1} P_{B_j}(T_{\chi+}) + K_j.$$

The symbol of the operator A_j restricted to $M_1^0(QC)\times[0,1]$ is the function $f_j(x)P_{B_j}(t)$.

Lemma 2.3.4. *The sum* $\sum_{j=1}^{\infty} f_j(x)P_{B_j}(t)$ *is an absolutely convergent series in* $C(M_1^0(QC)\times [0,1])$*; moreover it converges to*

$$G_A(x,t) := \sum_{j=1}^{\infty} f_j(x)P_{B_j}(t) = \mathrm{Sym}(A)|_{M_1^0(QC)\times[0,1]}.$$

Proof. The homomorphism Sym is a contraction, thus, in our case,

$$\|f_j P_{B_j}\|_\infty = \|\,\mathrm{Sym}(A_j)|_{M_1^0(QC)\times[0,1]}\|_\infty \le \|A_j\|.$$

The absolute convergence of $\sum_{j=1}^{\infty} A_j$ implies the absolute convergence of $\sum_{j=1}^{\infty} f_j(x)P_{B_j}(t)$. The second assertion of the lemma follows by a simple calculation. □

By Tietze Extension Theorem, the function f_j, which is continuous in $M_1^0(QC)$, can be extended to a continuous function in $M(QC)$ without increasing its norm; we use the same letter f_j to denote this extension. The function f_j belongs to $C(M(QC))$, thus it can be realized as a function in QC. Using this identification we construct the operator $T_{f_j}P_{B_j}(T_{\chi+})$.

Lemma 2.3.5. *The series*

$$A(G) := \sum_{j=1}^{\infty} T_{f_j} P_{B_j}(T_{\chi+})$$

is absolutely convergent.

Proof. For each j, the operator $T_{f_j}P_{B_j}(T_{\chi+})$ is bounded by $\|f_j\|_{M(QC)}\|P_{B_j}\|_{[0,1]} = \|f_j P_{B_j}\|_{M_1^0(QC)\times[0,1]}$, hence the series

$$\sum_{j=1}^{\infty} T_{f_j} P_{B_j}(T_{\chi+})$$

is absolutely convergent. □

Consider the operator $\sum_{j=1}^{s} T_{\mu_1} T_{f_j} P_{B_j}(T_{\chi_+}) \in \mathcal{T}_{PQC}$, its corresponding symbol

$$\mathrm{Sym}\left(\sum_{j=1}^{s} T_{\mu_1} T_{f_j} P_{B_j}(T_{\chi_+})\right) = \sum_{j=1}^{s} S_j$$

generates an absolutely convergent series. Its limit $S := \sum_{j=1}^{\infty} S_j$ is such that

$$S|_{M_1^0(QC)\times[0,1]} = \mathrm{Sym}(A)|_{M_1^0(QC)\times[0,1]}.$$

Now consider the operator $A - T_{\mu_1} A(G)$, passing to the symbol algebra we have

$$S_A := \mathrm{Sym}(A - T_{\mu_1} A(G)) = \mathrm{Sym}(A) - \mathrm{Sym}(\sum_{j=1}^{\infty} T_{\mu_1} T_{f_j} P_{B_j}(T_{\chi_+})) = \mathrm{Sym}(A) - S.$$

The function S_A is continuous in $M(\hat{\mathcal{T}}_{PQC})$ and vanishes at $M_1^0(QC) \times [0,1]$; this implies that S_A is a continuous function in $M(QC)$. This fact allows us to realize S_A as a function in QC. Extend S_A harmonically to obtain a continuous function in $\mathbb{D}$, whose values on $\partial\mathbb{D}$ define a function in QC. The operator T_{S_A} belongs to $\mathcal{T}_{PQC}$, hence, passing from symbols to operators, we have the following lemma:

Lemma 2.3.6. *Let A be an operator in $\mathcal{T}_{PQC}$. The operator A admits a canonical representation given by*

$$A = T_{S_A} + T_{\mu_1} A(G) + K,$$

where S_A is a function in QC, $G_A(x,t)$ is a continuous function in $M_1^0(QC) \times [0,1]$ such that $\mathrm{Sym}(A)|_{M_1^0(QC)\times[0,1]} = G_A(x,t)$, *$A(G)$ is defined in Lemma 2.3.5 and K is a compact operator.*

As we mention before, Lemma 2.3.6 can be easily extended to the general case, thus the principal Theorem of this section has been proved.

Theorem 2.3.7. *Let A be an operator in $\mathcal{T}_{PQC}$. The operator A admits a canonical representation given by*

$$A = T_{S_A} + \sum_{k=1}^{n} T_{\mu_k} A(G_k) + K,$$

where S_A is a function in QC, $G_{A,k}(x,t)$ is a continuous function in $M^0_{\lambda_k}(QC) \times [0,1]_k$ such that $\mathrm{Sym}(A)|_{M^0_{\lambda_k}(QC)\times[0,1]_k} = G_{A,k}(x,t)$, $A(G_k)$ is an operator which depends on the function $G_{A,k}$, and K is a compact operator.
This representation is unique in the sense that any other representation with a different pair S_A and $A(G)$ differs from this one by a compact operator.

2.4 Toeplitz or not Toeplitz

The operator algebra $\mathcal{T}_{PQC}$ contains not only Toeplitz operators with PQC generating symbols but many other operators (Toeplitz and not Toeplitz). This section establishes necessary and sufficient conditions to determine if an operator A in $\mathcal{T}_{PQC}$ is a compact perturbation of a Toeplitz operator. By Toeplitz operator in this section we always mean a Toeplitz operator with bounded measurable defining symbol g (not necessarily in PQC). As in Section 2.3, we follow the approach in [26] (Section 8.4) for Toeplitz operators with piecewise continuous symbols.

By Theorem 2.3.1 we already know that each operator A in $\mathcal{T}_{PC}$ admits a canonical representation

$$A = T_{S_A} + \sum_{k=1}^{n} T_{\mu_k} f_{A,k}(T_{\chi_k}) + K.$$

The main result concerning compact perturbations of Toeplitz operators with piecewise continuous symbols is as follows

Theorem 2.4.1 ([26], Theorem 8.4.3). *An operator* $A \in \mathcal{T}_{PC}$ *is a compact perturbation of a Toeplitz operator if and only if every operator* $f_{A,k}(T_{\chi_k})$ *is a compact perturbation of a Toeplitz operator, where* $f_{A,k} = \mathrm{Sym}(A)|_{[0,1]_k}$ *and* $k = 1, \ldots, n$.

In our case, with piecewise quasicontinuous symbols, one implication is obvious

Lemma 2.4.2. *An operator* A *in* $\mathcal{T}_{PQC}$ *with canonical representation*

$$A = T_{S_A} + \sum_{k=1}^{n} T_{\mu_k} A(G_k) + K,$$

is a compact perturbation of a Toeplitz operator if each operator $A(G_k)$ *is a compact perturbation of a Toeplitz operator as well.*

Lemma 2.4.3. *Let* $A \in \mathcal{T}_{PQC}$ *with canonical representation*

$$A = T_{S_A} + \sum_{k=1}^{n} T_{\mu_k} A(G_k) + K.$$

Each operator $T_{\mu_k} A(G_k)$ *is a compact perturbation of a Toeplitz operator if* A *is a compact perturbation as well.*

Proof. Suppose that Theorem 2.3.7 was proved for $\nu_k = \mu_k^{1/2}$, suppose further that $A = T_a + K$ for some bounded function a. The operator $T_{\nu_k} A$ can be written as

$$T_{\nu_k}(T_a + K) = T_{\nu_k} A = T_{\nu_k} T_{S_A} + T_{\mu_k} A(G_k) + K,$$

then $T_{\mu_k} A(G_k) = T_c + K$ with $c = \nu_k(a - S_A)$. □

Lemma 2.4.4. *For each* $k = 1, \ldots, n$*, the operator* $A(G_k)$ *is a compact perturbation of a Toeplitz operator if the operator* $T_{\mu_k} A(G_k)$ *is a compact perturbation of a Toeplitz operator.*

Proof. Consider $\Lambda = \{\lambda_1 = i\}$ and set the notation $\mu_l := \mu_1$ for functions defined at the beginning of Section 2.3 and $\chi_l := \chi_{\mathbb{D}_1^+}$ as the characteristic function of the left half of $\mathbb{D}$.

We apply Theorem 2.3.7 with $\Lambda = \{i\}$ to the operator $A \in \mathcal{T}_{PQC}$, hence

$$A = T_{S_A} + T_{\mu_l} A(G_l) + K,$$

here $A(G_l) := \sum_{j=1}^{\infty} T_{f_j} P_{B_j}(T_{\chi_l})$.

We introduce the operator

$$(Z\phi)(z) = \overline{\phi(\bar{z})},$$

which is unitary on both $L^2(\mathbb{D})$ and $\mathcal{A}^2(\mathbb{D})$. A simple calculation gives

$$ZT_a Z = T_{\overline{a}(\bar{z})} \qquad \text{and} \qquad ZP_{B_j}(T_{\chi_l})Z = \overline{P}_{B_j}(T_{\chi_l}),$$

for all $a \in L^\infty(\mathbb{D})$ and all polynomials $P_{B_j}(t)$ in $C[0,1]$.

This implies

$$ZT_{f_j} P_{B_j}(T_{\chi_l})Z = T_{\bar{f}(\bar{z})} \overline{P}_{B_j}(T_{\chi_l}).$$

Let $g_j(z) := \bar{f}_j(\bar{z})$; obviously, $g_j \in QC$ and the values of g_j in $M_i^0(QC)$ correspond to those values of f_j in $M_{-i}^0(QC)$ and vice versa.

We construct the operator $A(H_l) := \sum_{j=1}^{\infty} T_{g_j} \overline{P_{B_j}}(T_{\chi_l})$. We have absolute convergence in the series defining $A(H_l)$ and $ZA(G_l)Z = A(H_l)$.

By hypotesis $T_{\mu_l} A(G_l) = T_a + K$ for some bounded function a, thus, using the operator Z, we get

$$ZT_{\mu_l} A(G_l) Z = T_{\mu_l(\bar{z})} A(H_l) = T_{\bar{a}(\bar{z})} + K.$$

The operator

$$B := A(G_l) - T_{\mu_l} A(G_l) - T_{\mu_l(\bar{z})} A(H_l) \tag{2.4.1}$$

belongs to $\mathcal{T}_{PQC}$ with $\Lambda = \{i, -i\}$. The symbol of B in Equation (2.4.1) vanishes on both $M_i^0(QC) \times [0,1]$ and $M_{-i}^0(QC) \times [0,1]$. This fact implies that $\mathrm{Sym}(B)$ can be realized as a function in QC and then the operator B is a compact perturbation of a Toeplitz operator with symbol b in QC, that is

$$B = T_b + K.$$

From Equation (2.4.1)

$$A(G_l) = B + T_{\mu_l} A(G_l) + T_{\mu_l(\bar{z})} A(H_l),$$

which is a compact perturbation of the Toeplitz operator $T_{b+a(z)+\bar{a}(\bar{z})}$. □

Sumarizing Lemmas 2.4.2, 2.4.3, 2.4.4, and returning to $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we conclude the main result of this section.

Theorem 2.4.5. *An operator A in $\mathcal{T}_{PQC}$ is a compact perturbation of a Toeplitz operator if and only if for each $k = 1, 2, \ldots, n$, the operator $A(G_k)$ is a compact perturbation of a Toeplitz operator as well, where $G_{A,k}(x,t) = \mathrm{Sym}(A)|_{M^0_{\lambda_k}(QC)\times[0,1]_k}$.*

2.5 Fredholm Theory

This section is devoted to Fredholm theory of operators in the C^*-algebra T_{PQC}. We present the standard notation and definitions introduced by A. Atkinson in [1], detailed discusion on this subject was the main purpouse of Chapter 5 in [12] and Chapter 11 in [11].

Consider $B(\mathbb{H})$ as the space of bounded linear operators acting on the Hilbert space $\mathbb{H}$. For a given operator $T \in B(\mathbb{H})$ define $\mathrm{Ker}(T) = \{v \in H : T(v) = 0\}$ and $\mathrm{Coker}(T) = \mathbb{H}/Img(T)$.

Definition 2.5.1. *An operator $T \in B(\mathbb{H})$ is a Fredholm operator if T has closed range and the vector spaces* $\mathrm{Ker}(T)$ *and* $\mathrm{Coker}(T)$ *have finite dimension. In case of Fredholmness, the Fredholm index of the operator T is defined by the formula*

$$\mathrm{Index}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Coker}(T)).$$

The next theorem presented here contains the most clasic results concerning Fredholm operators. It is a summary of the theory presented in [11], [12] and [18]. The main goal is to give the basic theory to understand Lemma 2.5.6 which is the principal result in this section.

Theorem 2.5.2.

1. *Consider $\mathcal{K}$ as the ideal of compact operators acting on $\mathbb{H}$. An operator T is Fredholm if and only if the class of T in the Calkin algebra $B(\mathbb{H})/\mathcal{K}$ is invertible.*

2. *Let T be a Fredholm operator and K a compact operator. Then $T + K$ is a Fredholm operator and* $\mathrm{Index}(T + K) = \mathrm{Index}(T)$*; in other words, the function* Index *is invariante under compact perturbations.*

3. *The function index is stable with small perturbations; that is,* $\mathrm{Index}(T + T') = \mathrm{Index}(T)$ *for small operators T'.*

4. *If T is Fredholm then T^* is also Fredholm and* $\mathrm{Index}(T) = -\mathrm{Index}(T^*)$*, moreover,* $\mathrm{Index}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*))$.

In order to calculate $\mathrm{Index}(T)$ for the case of Toeplitz operator , we don't use the difference $\mathrm{Ker}(T) - \mathrm{Ker}(T^*)$ but the the winding number of a function f (a function closely related to T).

Definition 2.5.3. *Let $f : \partial\mathbb{D} \to \mathbb{C}$ be a continuous function. The natural number*

$$\text{wind}(f) := \frac{1}{2\pi}\{arg(f(t))\}_{t\in\partial\mathbb{D}}.$$

is called the winding number of f

The winding number wind(f) counts the number of times that the image of f travels arround the point 0.

For Toeplitz operator with continuous symbols the following result holds ([10]).

Theorem 2.5.4. *The Toeplitz operators T_a with symbol a in $C(\bar{\mathbb{D}})$ is Fredholm if and only if a does not vanish on $\partial\mathbb{D}$. In case of Fredholmness*

$$\text{Index}(T_a) = -\,\text{wind}(a).$$

This result not only holds for the case of continuous symbols but for more general type of symbols including our space of interest PQC (see [22] page 829).

Theorem 2.5.2 and Theorem 2.1.11 give us an obvious characterization of Fredholm operators in $\mathcal{T}_{PQC}$; the objective of this section is to establish this criterion in terms of a bounded function defined in a subset on the complex plane. In case of Fredholmness, an index formula is provided in terms of the winding number of a continuous function.

Theorem 2.5.5. *An operator A in $\mathcal{T}_{PQC}$ is Fredholm if and only if* Sym(A) *never vanishes in $C(M(\hat{\mathcal{T}}_{PQC}))$.*

Recall that the compact space $M(\hat{\mathcal{T}}_{PQC})$ admits a fribration over the set Σ defined in Page 55

$$M(\hat{\mathcal{T}}_{PQC}) := \bigcup_{\lambda\in\Sigma} M_\lambda(\mathcal{T}_{PQC}).$$

The set Σ is obtained after cutting the unit circle $\partial\mathbb{D}$ at the points λ_k and then conecting the points λ_k^+ and λ_k^- with the segment $[0,1]$ in order to avoid the discontinuities of any function in PC. The length of Σ is the length of $\partial\mathbb{D}$ plus the length of the n intervals $[0,1]$, thus we identify the set Σ with the circle of radius $R = 1 + n/2\pi$; denote this circle by $\partial\mathbb{D}_R$, obviously its length is $2\pi + n$.

With the previous identification the set $M(\hat{\mathcal{T}}_{PQC})$ admits a fibration over the circle $\partial\mathbb{D}_R$, thus every function continuous on $M(\hat{\mathcal{T}}_{PQC})$ is realized as a bounded function defined on $\partial\mathbb{D}_R$.

Consider a Fredholm operator $A \in \mathcal{T}_{PQC}$, its symbol $h(x) := \mathrm{Sym}(A)$ is a continuous function on $M(\hat{\mathcal{T}}_{PQC})$, therefore, $h(x)$ is identified with a bounded function on $\partial\mathbb{D}_R$. By Theorem 2.5.5 $h(x) \neq 0$ for all $x \in M(\hat{\mathcal{T}}_{PQC})$, this fact implies that the essential infimum of h in $\partial\mathbb{D}_R$ is not zero. Let h_H denote the Harmonic extension of h to $\mathbb{D}_R$, the disk of radius R. The function h_H is continuous on $\mathbb{D}_R$ and, because $\mathrm{essinf}\{|h(\lambda)|\} > 0$, $h_H(z)$ is bounded away from zero in an annulus of $\mathbb{D}_R$, that is, there is $\epsilon > 0$ and $0 < r_0 < R$ such that $h_H(r,\theta) > \epsilon$ for all $r_0 < r < R$ and $0 \leq \theta \leq 2\pi$.

Let $r_1 \in [r_0, R)$ be fixed, the function $h_H(r_1,\theta)$ is bounded away from zero, hence its winding number is well defined, moreover $\mathrm{wind}(h_H(r_1,\theta)) = \mathrm{wind}(h_H(r_2,\theta))$ provided that $r_0 < r_1, r_2 < R$. Hence, we have established a Fredholm criterion for operators in $\mathcal{T}_{PQC}$ and an index formula in case Fredholmness.

Theorem 2.5.6. *An operator A in $\mathcal{T}_{PQC}$ is Fredholm if and only if* $h(x) := \mathrm{Sym}(A)$ *does not vanishes in $M(\hat{\mathcal{T}_{PQC}})$, if and only if* $\mathrm{essinf}\{|h(\lambda)|\} > 0$ *in $\partial\mathbb{D}_R$. In case of Fredholmness*

$$\mathrm{Index}(A) = -\,\mathrm{wind}(h_H(r_0,\theta)),$$

where h_H *is the harmonic extension of* h *to the disk* $\mathbb{D}_R$ *and* r_0 *is a suitable number such that* $0 < r_0 < R$ *and* $h_H(r, \theta)$ *is bounded away from zero for all* $r_0 < r < R$ *and all* $0 \leq \theta \leq 2\pi$*.*

Chapter 3

Toeplitz operators with piecewise quasicontinuous and slowly oscillating symbols

The main objective of this chapter is the description of a Toeplitz operator algebra. The algebra of our interest acts on the Bergman space on $\mathbb{D}$ and it is generated by Toeplitz operators with symbols in the tensor product of functions defined on $[0, 2\pi]$ and $[0, 1)$. The theory needed to understand the main results in this section was established in Sections 1.3, 1.4, 1.5.

Let $\mathbb{D} = \{z = (r, \theta) | 0 \leq r < 1, 0 \leq \theta \leq 2\pi\}$ be the unit disk in its polar realization with measure $dz = r dr d\theta$. The family of symbols under consideration are: SO_∞ the bounded slowly oscillating functions (in the radial part) and PQC the piecewise quasicontinuous functions (in the angular coordinate).

In the first section we deal with the Toeplitz algebra generated by Toeplitz operators with symbols in a natural extension of the space SO_∞. We characterize its Calkin algebra as the C^*-algebra of continuous functions defined on $M_1(SO)$ (see Theorem

1.5.6). We also describe the algebra of symbols $QCSO := QC \otimes SO$ and describe its Calkin algebra $\hat{\mathcal{T}}_{QCSO} := \mathcal{T}_{QCSO}/\mathcal{K}$ as the algebra of continuous functions defined on a subespace of $M(QC) \times M(SO_\infty)$.

In the second section we deal with the Toeplitz operators with symbols in the tensor product $QC \otimes SO_\infty := QC(\partial\mathbb{D}) \otimes SO_\infty([0,1))$. Even though in the relation $QC \otimes SO_\infty \subset Q$, the inclusion is strict, it is an interesting problem to consider the Toeplitz operator algebra generated by Toeplitz operators with symbols in $QC \otimes SO_\infty$ (and piecewise perturbations).

Section 3.2 also contains the principal result of this chapter, the description of the C^*-algebra $\hat{\mathcal{T}}_{PQCSO}$, where $PQCSO$ denotes the tensor product $PQC \otimes SO_\infty$. This description is established in Theorem 3.2.8 as a consequence of the Douglas-Varela local principle localizing by points in a subset of $M(QC) \times M(SO_\infty)$. As a consequence of this result, the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$ is commutative and thus it is isometric and isomorphic to an algebra of continuous functions on a compact space as is proved in Theorem 3.2.9.

As a final part of this chapter (Section 3.3), we give a Fredholm index formula for all Fredholm operators in $\mathcal{T}_{PQCSO}$.

3.1 Toeplitz operators with slowly oscillating symbols

In this section we are interested in Toeplitz operators with slowly oscillating symbols. We define the function algebra SO_∞ of bounded slowly oscillating functions and its subset SO of continuous oscillating functions. The principal result of this section (Lemma 3.1.3) describes the Calkin algebra $\hat{\mathcal{T}}_{SO}$ as the continuous functions on the compact space $M_1(SO)$ (see Definition 1.5.5).

Definition 3.1.1. *For the half open interval $[0,1)$, we define the class of $SO_\infty := SO_\infty([0,1))$ which consist of all bounded functions g in $[0,1)$ such that for each $\eta \in (0,1)$,*

$$\lim_{\delta \to 0} \omega(g, [1-\delta, 1-\eta\delta]) = 0.$$

This is the C^*-algebra of bounded slowly oscillating functions.
We define a subspace of SO_∞, $SO := SO([0,1)) \subset SO_\infty$, which consist of all continuous functions with slowly oscillation, that is,

$$SO([0,1)) := C([0,1)) \cap SO_\infty([0,1)).$$

We extend this spaces of functions to the whole disk via the formula:

$$g(z) = g(|z|).$$

We have proved, in Section 2.1, that the radial and the harmonic extension of functions in QC belong to Q. We prove that the previous extension of functions in SO generates a subset of Q.

Lemma 3.1.2. *Let g be a function in SO (extended to $\mathbb{D}$ as $g(z) = g(|z|)$), then g belongs to Q.*

Proof. We use Theorem 1.3.7 to prove this fact. According to Definition 1.3.5, let $\eta \in (0,1)$ and $\epsilon > 0$, by hypotesis, g belongs to SO, hence, there exists $\delta_0 > 0$ such that

$$w(g, [1-\delta, 1-\eta\delta]) < \epsilon$$

for all $\delta < \delta_0$.

Now consider $w \in S'_z$ such that $|w|, |z| \in [1-\delta, 1-\eta\delta]$, the diference $|g(z) - g(w)| = |g(|z|) - g(|w|)|$ is bounded by

$$\mathrm{Sup}_{s,t\in[1-\delta,1-\eta\delta]} |g(s) - g(t)| = w(g, [1-\delta, 1-\eta\delta] < \epsilon,$$

hence g belongs to Q. □

Lemma 3.1.2 implies that the semicommutator $[T_g, T_h)$ is compact for all functions g, h in SO. Let $\mathcal{K}'$ denote the ideal of compact operators in $\mathcal{T}_{SO}$. Due to Lemma 1.5.4, the Calkin algebras $\hat{\mathcal{T}}_{SO_\infty}$ and $\hat{\mathcal{T}}_{SO}$ are isomorphic; actually, as a consequence of Lemma 3.1.2, these two algebra are isomorphic to the continuous functions on $M_1(SO)$ (see definition in Theorem 1.5.6).

Lemma 3.1.3. *The Calkin algebra $\hat{\mathcal{T}}_{SO} = \mathcal{T}_{SO}/\mathcal{K}'$ is isomorphic to the space of continuous functions on $M_1(SO)$.*

3.2 Toeplitz operators with piecewise quasicontinuous and slowly oscillating symbols

In this section we consider the tensor product of C^*-algebras. The theory of tensor products of C^*-algebras was one of the subjects contained in books [7], [17] and papers [8], [16], [25].

In order to define the tensor product of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we must consider first the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. In the algebraic tensor product there are defined two standard $*$-norms. The first one is the minimal norm or the spatial norm (introduce by T. Turumaru in [25]), and the second one is the maximal norm or the sup norm defined by $*$-seminorms in $\mathcal{A}\odot\mathcal{B}$ (this one defined by A. Guichardet in 1965).

These two norms are not equals in the general case, thus the completition of $\mathcal{A}\odot\mathcal{B}$ with respect to these norms define different tensor products $\mathcal{A}\otimes_{min}\mathcal{B}$ and $\mathcal{A}\otimes_{max}\mathcal{B}$. For the case of a nuclear C^*-algebras $\mathcal{A}$, the minimal and the maximal $*$-norms are the same for any other C^*-algebra $\mathcal{B}$; actually this is the definition of a nuclear algebra (see [7]).

It is of our interest to apply these definitions to commutative C^*-algebras. This case becomes much more easier and the description of the tensor product of two commutative C^*-algebras $\mathcal{A}$, and $\mathcal{B}$ was established in the next theorem due to T. Takasi [25].

Theorem 3.2.1. *Let $\mathcal{A} = C(X)$ be commutative. Then $\mathcal{A}$ is nuclear, and for any C^*-algebra $\mathcal{B}$, $\mathcal{A}\otimes\mathcal{B}$ can be identified with $C(X, \mathcal{B})$ under the map $(f\otimes b)(x) := f(x)b$. In particular, if $\mathcal{B} = C(Y)$ is also commutative, then $C(X)\otimes C(Y) \cong C(X\times Y)$ under the identification*

$$(f \otimes g)(x, y) = f(x)g(y).$$

Now consider the tensor products $QC \otimes SO$ and $PQC \otimes SO$, these C^*-algebras of functions are denoted by $QCSO$ and $PQCSO$ respectively. For these tensor products we define their corresponding Toeplitz operator algebras $\mathcal{T}_{PQCSO}$ and $\mathcal{T}_{QCSO}$. The algebra $\mathcal{T}_{QCSO}$ can be realized as the algebra generated by $\mathcal{T}_{QC}$ and $\mathcal{T}_{SO}$ (consider the natural extensions of QC and SO to the unit disk).

The description of the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$ is obtained via the DVLP. There are two key points that allow us to use the machinery of the DVLP; the first one (Lemma 3.2.3) is the description of the maximal ideal space of the Calkin algebra $\hat{\mathcal{T}}_{QCSO}$, and the second one is the description of the local algebras showed in Section 2.1.

Lemma 3.2.2. *The ideal of compact operators $\mathcal{K}$ is a subset of $\mathcal{T}_{QCSO}$.*

Proof. It is well known that the ideal of compact operators is a subset of the Toeplitz operator algebra with continuous generating symbols (see for example [10]). Consider the C^*-algebra $C_0([0,1])$ consisting of all continuous functions g on $[0,1]$ such that $\lim_{t\to 0} g(t) = 0$. The Toeplitz operator algebra generated by Toeplitz operators with symbols in the tensor product $C(\partial\mathbb{D}) \otimes (C_0([0,1]) + \mathbb{C})$ contains the ideal of compact operators, thus the following inclusion holds

$$\mathcal{K} \subset \mathcal{T}_{QCSO} \subset \mathcal{T}_{PQCSO}.$$

□

By Lemmas 2.1.3 and 3.1.2, the Calkin algebra $\hat{\mathcal{T}}_{QCSO} := \mathcal{T}_{QCSO}/\mathcal{K}$ is commutative, thus is isomorphic to the continuous functions on a compact space (its maximal ideal space).

Lemma 3.2.3. *The maximal ideal space of* $\hat{\mathcal{T}}_{QCSO}$ *is* $M(QC) \times M_1(SO)$

Proof. To prove this fact we use a known result about homeomorphims between compact spaces Theorem 2.1 of Chapter XI in [13]; the result says that a continuous bijection between compact spaces is an homeomorphism.

Recall that $\hat{\mathcal{T}}_{QCSO}$ is generated by $\hat{\mathcal{T}}_{QC}$ and $\hat{\mathcal{T}}_{SO}$. Let z be a multiplicative functional defined on $\hat{\mathcal{T}}_{QCSO}$; the restriction of z to $\hat{\mathcal{T}}_{QC}$ is a multiplicative funtional, denote this functional by z_x. In an analogous way define z_y as the restriction of z to $\hat{\mathcal{T}}_{SO}$. The pair (z_x, z_y) belongs to $M(QC) \times M_1(SO)$.

Consider (x, y) in $M(QC) \times M_1(SO)$; let $z : \hat{\mathcal{T}}_{QCSO} \to \mathbb{C}$ be defined on the generators as $z(\hat{T}_f) = x(\hat{T}_f)$, for f in QC; and $z(\hat{T}_g) = y(\hat{T}_g)$, for g in SO. By Lemmas 2.1.3 and 3.1.2 we have that $T_{fg} = T_f T_g + K$, for all f in QC and g in SO; this fact allows us to extend z as a multiplicative functional defined on $\hat{\mathcal{T}}_{QCSO}$.

If we consider two differente points z_1 and z_2 in $\hat{\mathcal{T}}_{QCSO}$, then there exist a function f in QC such that $z_1(\hat{T}_f) \neq z_2(\hat{T}_f)$ (or a function g in SO), which implies that $z_{1x}(\hat{T}_f) \neq z_{2x}(\hat{T}_f)$ and thus the points (z_{1x}, z_{1y}) and (z_{2x}, z_{2y}) are different.

The previous paragraphs define a bijective map

$$\psi : M(\hat{\mathcal{T}}_{QCSO}) \to M(QC) \times M(SO).$$

We need to proved that ψ is continuous in order to show that $M(\hat{\mathcal{T}}_{QCSO})$ and $M(QC)\times M_1(SO)$ are homeomorphic. Consider a sequence of points z_n which converges to z, this convergence is in the weak-$*$ topology, thus, for every operator A in $\mathcal{T}_{QCSO}$, the sequence $z_n(\hat{A})$ converges to $z(\hat{A})$; in particular for Toeplitz operators with symbols in QC, $z_n(\hat{T}_f)$ converges to $z(\hat{T}_f) = z_x(\hat{T}_f)$. We can conclude that $\psi(z_n)$ converges to $\psi(z)$ and thus ψ is continuous. $\square$

Recall that $M(QC)$ admits a fibration over $\partial\mathbb{D}$, the same is done for $M(\hat{\mathcal{T}}_{QCSO})$ in the following way

$$M(QCSO) := M(\hat{\mathcal{T}}_{QCSO}) = \bigcup_{\lambda\in\partial\mathbb{D}} M_\lambda(\mathcal{T}_{QCSO}) = \bigcup_{\lambda\in\partial\mathbb{D}} \left(M_\lambda(QC) \times M_1(SO)\right).$$

Our next goal is the description of the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$, for this purpose we use the Douglas-Varela local principle.

The algebra $\hat{\mathcal{T}}_{QCSO}$ is a central commutative subalgebra of $\hat{\mathcal{T}}_{PQCSO}$, hence we localize by points in the maximal ideal space $M(QC) \times M_1(SO)$. The localization techniques used in Section 2.1 to describe the local algebras in the case of $\mathcal{T}_{PQC}$ can be extended to this case.

For each point $(x, y) \in M(QC) \times M_1(SO)$, define $J_{(x,y)}$ as the ideal generated by Toeplitz operators T_{fg} with $f \in QC$ and $g \in SO$ and such that $f(x) = 0$ or $g(y) = 0$. Consider $J(x, y) := J_{(x,y)}\hat{\mathcal{T}}_{PQCSO}$ as the ideal generated by $J_{(x,y)}$ in $\hat{\mathcal{T}}_{PQCSO}$. Further, define the local algebras $\hat{\mathcal{T}}_{PQCSO}(x, y) := \hat{\mathcal{T}}_{PQCSO}/J(x, y)$ parametrized by points $(x, y) \in M(QC) \times M_1(SO)$.

The description of the local algebras $\hat{\mathcal{T}}_{PQCSO}(x, y)$ depends on the fiber where the point (x, y) belongs to, for instance, the local algebras for points $(x, y) \in M_\lambda(QC) \times M_1(SO)$ are easily determined for $\lambda \notin \Lambda$.

Lemma 3.2.4. *For $\lambda \in \partial\mathbb{D}$ such that $\lambda \notin \Lambda$, the local algebras for points $(x, y) \in M_\lambda(QC) \times M_1(SO)$ is isomorphic to $\mathbb{C}$.*

Proof. Let $(x, y) \in M_\lambda(QC) \times M_1(SO)$ and consider an operator T_{fg} where f belongs to PQC and g belongs to SO. The class of the operator $T_{fg} - T_{f(x)g(y)} = T_{fg-f(x)g(y)}$ in the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$ belongs to the ideal $J(x, y)$, thus the operators T_{fg} and $T_{f(x)g(y)}$ are locally equivalent. This fact implies that the local algebra $\hat{\mathcal{T}}_{PQCSO}(x, y)$ is isomorphic to $\mathbb{C}$ for every $(x, y) \in M_\lambda(QC) \times M_1(SO)$ with $\lambda \notin \Lambda$. □

The local algebras corresponding to points $(x, y) \in M_{\lambda_k}(QC) \times M_1(SO)$ are handled separately according to Lemma 1.4.16. Analogous to the case with PQC symbols we have three local algebras. The proof of the description of the local algebras are extended from Lemmas 2.1.7, 2.1.8, and 2.1.9.

Lemma 3.2.5. *Let $(x, y) \in (M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)) \times M_1(SO)$. The local algebra is isomorphic to $\mathbb{C}$.*

Proof. First consider a Toeplitz operator T_{fg} such that $f \in QC$ and $g \in SO$; in this case we follow the argument in the previous proof to show that T_{fg} is locally equivalent to $f(x)g(x)I$.

As a second case consider a Toeplitz operator T_{ag} such that $a \in PC$ and $g \in SO$. Due to Lemma 3.1.2 $T_{ag} = T_a T_g + K$ where K is a compact operator; by Lemma 2.1.7, the class of the operator $T_a - T_{a_k^+}$ belongs to $J(x)$ (see Definition 2.1.4). The difference $T_{ag} - T_{a_k^+ g(y)}$ in the Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$ agrees with the class of

$$T_{ag} - T_{a_k^+ g} + T_{a_k^+ g} - T_{a_k^+ g(y)} = T_{(a-a_k^+)} T_g + T_{a_k^+} T_{g-g(y)},$$

this last expression belongs to $J(x, y)$; hence T_{ag} is locally equivalent, at the point (x, y), to the complex number $a_k^+ g(y)$.

In the general case, for a Toeplitz operator T_{hg} with symbols $h \in PQC$ and $g \in SO$, we approximate the function h by functions in PC and QC and use the previous argument. In conclusion the local algebra corresponding to $(x, y) \in (M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)) \times M_1(SO)$ is isomorphic to $\mathbb{C}$. $\square$

Lemma 3.2.6. *Let* $(x, y) \in (M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)) \times M_1(SO)$. *The local algebra is isomorphic to* $\mathbb{C}$.

Lemma 3.2.7. *Let* $(x, y) \in M^0_{\lambda_k}(QC) \times M_1(SO)$. *The local algebra is the algebra of continuous functions on* $[0, 1]$.

Proof. As we know from Lemma 2.1.9, the Toeplitz operator T_a with symbol $a \in PC$ is locally equivalent to the Toeplitz operator T_{a^0}, where a^0 is a function constant in $\mathbb{D}_k^+$ and $\mathbb{D}_k^-$ (see proof of Lemma 2.1.9). Using the argument of the previous proof, the Toeplitz operator T_{ag} is locally equivalent to the operator $T_{a_0 g(y)}$, hence, the local algebra generated by T_{ag} and $T_{a^0 g(y)}$ are the same. By Lemma 2.1.9, the local algebra $\hat{\mathcal{T}}_{PQCSO}(x, y)$ is isomorphic to the algebra of continuous functions on $[0, 1]$, thus, the local algebra generated by the class of $T_{a^0 g} = T_{a^0} T_{g(y)} + K$ is isomorphic to the C^*-algebra $C([0, 1])$. $\square$

Let $\xi_{\text{PQCSO}} := (p, E, M(QCSO))$ be the C^*-bundle defined by the local algebras described in Lemmas 3.2.4, 3.2.5, 3.2.6, and 3.2.7, parametrized by points in $M(QCSO) = M(QC) \times M_1(SO)$. As a consequence of DVLP we came to the following Theorem.

Theorem 3.2.8. *The Calkin algebra $\hat{\mathcal{T}}_{PQCSO}$ is isometric and isomorphic to the C^*-algebra of continuous sections defined on the C^*-bundle $\xi_{PQCSO} := (p, E, M(QCSO))$.*

As a corollary of this theorem, $\hat{\mathcal{T}}_{PQCSO}$ is commutative and its maximal ideal space is obtained using the irreducible representations implicit in Theorem 3.2.8; moreover each irreducible representation corresponds to a point in $M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO)$.

Theorem 3.2.9. *The maximal ideal space of $\hat{\mathcal{T}}_{PQCSO}$ is the set $M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO)$ with the product topology. Under the identification of $\hat{\mathcal{T}}_{PQCSO}$ with $C(M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO))$, the homomorphism* Sym *acts on the generators of $\mathcal{T}_{PQCSO}$ as follows*

- *For opeartors $A \in \mathcal{T}_{PQC}$,* $\operatorname{Sym} A(x, y) = \operatorname{Sym} A(x)$,
- *For operators $B \in \mathcal{T}_{SO}$,* $\operatorname{Sym} B(x, y) = \operatorname{Sym} B(y)$.

3.3 Fredholm Theory

We know by Theorem 3.2.9 that the maximal ideal space of $\hat{\mathcal{T}}_{PQCSO}$ is the product $M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO)$, hence a Fredholm criterion is easily determine by Theorem 2.5.2. We use the techniques in Section 2.5 to obtain a Fredholm criterion for the algebra $\mathcal{T}_{PQCSO}$.

Theorem 3.3.1. *An operator A in $\mathcal{T}_{PQCSO}$ is Fredholm if and only if $h(x, y) :=$* $\operatorname{Sym}(A)$ *does not vanishes in $M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO)$.*

The fibration of $M(\hat{\mathcal{T}}_{PQC})$ by the set Σ in Page 70, allows us to define a fibration of $M(\hat{\mathcal{T}}_{PQC}) \times M_1(SO)$, thus the identification of Σ with the circle $\partial\mathbb{D}_R$ permit us to identify $C(M(\hat{\mathcal{T}}_{PQCSO}))$ with a subespace of bounded functions defined on $\partial\mathbb{D}_R$ as in Section 2.5.

To determine the Fredholm index of a Fredholm operator $A \in T_{PQCSO}$ we need to prove that the space $M_1(SO)$ is connected.

Theorem 3.3.2. *The set $M_1(SO)$ is connected.*

Proof. Let $A, B \subset M_1(SO)$ be two non-empty disjoint sets such that $A \cup B = M_1(SO)$. The function f such that $f(A) = 0$ and $f(B) = 1$ is continuous on $M_1(SO)$. By the Tietze extension Theorem the function f can be extended continuously to $M(SO)$. This function is identified with a function in SO. Let y_A and y_B be two points in A and B respectively. $y_A(f) = f(y_A) = 0$, hence, there is a sequence of points $t_n^A \to 1^-$ such that $\lim_{n\to\infty} f(t_n^A) = 0$. By an analogous reason there is a sequence $t_n^B \to 1^-$ such that $\lim_{n\to\infty} f(t_n^B) = 1$. Without loss of generality we can assume that $t_1^A < t_1^B < t_2^A < t_2^B < t_3^A < t_3^B < \ldots t_n^A < t_n^B \ldots$ and that $f(t_n^A) < 1/2$ and $f(t_n^B) > 7/8$. By the Intermediate value Theorem there is a sequence of points $t_n^A < t_n < t_n^B$ such that $f(t_n) = 1/2$. The sequence t_n approaches to 1 and $\lim_{n\to\infty} f(t_n) = 1/2$. For the consequence of Theorem 1.5.6 explained at the very end of Section 1.5, there exists $y \in M_1(SO)$ such that $f(y) = y(f) = \lim_{n\to\infty} f(t_n) = 1/2$, which is impossible because $f(M_1(SO)) = \{0, 1\}$. In conclusion the set $M_1(SO)$ is connected. $\square$

Let A be a Fredholm operator in T_{PQCSO}, its symbol $h(x, y) := \text{Sym}(A)$ does not vanishes in $M(\hat{\mathcal{T}}_{PQCSO})$. Fix y_0 in $M_1(SO)$, the function $h_{y_0}(x) := h(x, y_0)$ is identified with a bounded function defined on $\partial\mathbb{D}_R$. As we proved in Section 2.5, the winding number of h_{y_0} is well defined and it coincides with the Fredholm index of the class of operators with symbol h_{y_0}.

Let y_0 and y_1 be two points in $M_1(SO)$, by Theorem 3.3.2, there exists a continuous trajectory $\alpha : [0,1] \to M_1(SO)$ connecting the points y_0 and y_1. This trajectory generates an homotopy between the functions $h_{y_0}(x)$ and $h_{y_1}(x)$, thus the winding numbers of these functions are the same.

The previous arguments led us to the final theorem of this section, a Fredholm criterion and a Fredholm index formula for operators in $\mathcal{T}_{PQCSO}$.

Theorem 3.3.3. *Let A be an operator in $\mathcal{T}_{PQCSO}$. A is a Fredholm operator if and only if $h(x,y) := \mathrm{Sym}(A)$ does not vanishes in $M(\mathcal{T}_{PQCSO})$ if and only if $\mathrm{essinf}_{\lambda\in\ partial\mathbb{D}}\{|h(\lambda)|\} > 0$. In case of Fredholmness the index formula is given by*

$$\mathrm{Index}(A) = \mathrm{wind}(h_H(r_0,\theta)),$$

where h_H is the harmonic extension of h to the disk $\mathbb{D}_R$ and r_0 is a suitable number such that $0 < r_0 < R$ and $h_H(r,\theta)$ is bounded away from zero for all $r_0 < r < R$ and $0 \le \theta \le 2\pi$.

Bibliography

[1] Atkinson, F. V. The normal solubility of linear equations in normed spaces. *Mat. Sbornik N.S. 28(70)* (1951), 3–14.

[2] Axler, S., Bourdon, P., and Ramey, W. *Harmonic function theory*, second ed., vol. 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[3] Axler, S., and Zheng, D. Compact operators via the Berezin transform. *Indiana Univ. Math. J. 47*, 2 (1998), 387–400.

[4] Berezin, F. A. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat. 36* (1972), 1134–1167.

[5] Berezin, F. A. General concept of quantization. *Comm. Math. Phys. 40* (1975), 153–174.

[6] Berger, C. A., and Coburn, L. A. Toeplitz operators and quantum mechanics. *J. Funct. Anal. 68*, 3 (1986), 273–299.

[7] Blackadar, B. *Operator algebras*, vol. 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[8] Blackadar, B. E. Infinite tensor products of C^*-algebras. *Pacific J. Math. 72*, 2 (1977), 313–334.

[9] Carleson, L. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2) 76* (1962), 547–559.

[10] Coburn, L. A. Singular integral operators and Toeplitz operators on odd spheres. *Indiana Univ. Math. J. 23* (1973/74), 433–439.

[11] Conway, J. B. *A course in functional analysis*, second ed., vol. 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1990.

[12] Douglas, R. G. *Banach algebra techniques in operator theory.* Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 49.

[13] Dugundji, J. *Topology.* Allyn and Bacon, Inc., Boston, Mass.-London-Sydney, 1978. Reprinting of the 1966 original, Allyn and Bacon Series in Advanced Mathematics.

[14] Englis, M. Functions invariant under the Berezin transform. *J. Funct. Anal. 121*, 1 (1994), 233–254.

[15] Garnett, J. B. *Bounded analytic functions*, first ed., vol. 236 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[16] Gisarde, A. Tensor products of C^*-algebras. *Dokl. Akad. Nauk SSSR 160*

(1965), 986–989.

[17] Murphy, G. J. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990.

[18] Rudin, W. *Functional analysis.* McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics.

[19] Sarason, D. Approximation of piecewise continuous functions by quotients of bounded analytic functions. *Canad. J. Math. 24* (1972), 642–657.

[20] Sarason, D. Algebras of functions on the unit circle. *Bull. Amer. Math. Soc. 79* (1973), 286–299.

[21] Sarason, D. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc. 207* (1975), 391–405.

[22] Sarason, D. Toeplitz operators with piecewise quasicontinuous symbols. *Indiana Univ. Math. J. 26*, 5 (1977), 817–838.

[23] Simonenko, I. B. A new general method of investigating linear operator equations of singular integral equation type. II. *Izv. Akad. Nauk SSSR Ser. Mat. 29* (1965), 757–782.

[24] Stroethoff, K. The Berezin transform and operators on spaces of analytic functions. In *Linear operators (Warsaw, 1994)*, vol. 38 of *Banach Center Publ.* Polish Acad. Sci., Warsaw, 1997, pp. 361–380.

[25] Turumaru, T. On the direct-product of operator algebras. I. *Tôhoku Math. J. (2) 4* (1952), 242–251.

[26] Vasilevski, N. L. *Commutative algebras of Toeplitz operators on the Bergman space*, vol. 185 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2008.

[27] Zhu, K. H. VMO, ESV, and Toeplitz operators on the Bergman space. *Trans. Amer. Math. Soc. 302*, 2 (1987), 617–646.

[28] Zorboska, N. The Berezin transform and radial operators. *Proc. Amer. Math. Soc. 131*, 3 (2003), 793–800 (electronic).