CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE MATEMÁTICAS

Hipergrafos Simples no Mezclados, Complejos Simpliciales Escalonables y Anillos Cohen-Macaulay

Tesis que presenta

Iván Dario Castrillón Serna

Para Obtener el Grado de

Doctor en Ciencias

En la Especialidad de

Matemáticas

Director de Tesis:
Dr. Enrique Reyes Espinoza

Ciudad de México Febrero 2016

CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTHECHNIC INSTITUTE

CAMPUS ZACATENCO
DEPARTMENT OF MATHEMATICS

Unmixed Clutters, Shellable Simplicial Complexes and Cohen-Macaulay Rings

A dissertation presented by

Iván Dario Castrillón Serna

To Obtain the Degree of

Doctor in Science

In the Speciality of

Mathematics

Thesis Advisor:
Dr. Enrique Reyes Espinoza

Mexico city. February 2016

Resumen

En esta tesis estudiamos complejos simpliciales con las siguientes propiedades: descomponibilidad por vértices, escalonabilidad y Cohen-Macaulay. Se sabe que un complejo simplicial que se descompone por vértices es escalonable, un complejo simplicial escalonable puro es Cohen-Macaulay y un complejo Cohen-Macaulay es puro. La primera y la última inclusión son estrictas, la segunda es un problema abierto para grafos. En este trabajo, damos una caracterización de los grafos Cohen-Macaulay sin 3-ciclos ni 5-ciclos. Además, probamos que esos grafos se descompone por vértices y son escalonables. Otra familia de grafos estudiada en esta tesis son los grafos no mezclados que se descomponen por vértices cuyos 5-ciclos tienen al menos 4 cuerdas. También, caracterizamos la propiedad de ser bien cubierto para los grafos theta-anillados y los grafos sin 3-ciclos, 5-ciclos ni 7-ciclos. En particular, probamos que la segunda familia de grafos tiene un apareamiento perfecto. Por otro lado, mostramos que los complejos simpliciales cuyo ideal de Stanley-Reisner tiene como conjunto minimal de generadores las bases de un matroide es escalonable si y sólo si el matroide es completo. Concluimos la tesis dando algunas condiciones para que un hipergrafo simple con un apareamiento perfecto de tipo König sea escalonable.

Abstract

In this thesis we study simplicial complexes with the following properties: vertex decomposability, shellability and Cohen-Macaulayness. It is known that a vertex decomposable simplicial complex is shellable, a pure shellable simplicial complex is Cohen-Macaulay and a Cohen-Macaulay simplicial complex is pure. The first and third inclusion are strict, while the second inclusion is an open problem for graphs. In this work, we give a characterization of Cohen-Macaulay graphs without 3-cycles and 5-cycles. Furthermore, we prove that these graphs are vertex decomposable and shellable. Another graph family that we study in this thesis are unmixed vertex decomposable graphs whose 5-cycles have at least 4 chords. Also, we give a characterization for well-covered theta-ring graphs and well-covered graphs without 3-cycles, 5-cycles and 7-cycles. In particular, we prove that the second family of graphs has a perfect matching. On the other hand, we show that a simplicial complex whose Stanley-Reisner ideal is a monomial ideal such that its minimal generators are the bases of a matroid is shellable if and only if the matroid is complete. We conclude the thesis giving some conditions for the shellability of a clutter with a perfect matching of König type.

ACKNOWLEDGMENTS

First, I would like to thank Dr. Enrique Reyes for his guidance, for all his comments, remarks and suggestions during these four years. I dedicate this work to my family. Thank to my parents Olga and Dario for their unconditional love and support; particularly, I am deeply grateful with my mom for her help and advices. Also, I wish to thank my daughter Ailyn Nikté for bringing so much joy and happiness to my life. Additionally, I thank my wife Abigail Garcia for her help and patience. I would like to thank Dr. Alejandro Flores and Dr. Leticia Zárate for reviewing this thesis and for their suggestions. I am truly grateful with the Department of Mathematics at CINVESTAV for giving me the opportunity of studying here. I would like to thank my friends Eliecer, Oscar and Lina for their advices. Finally, I thank CONACyT the PhD grant that I received and ABACUS for the financial support to go to some congresses in which I showed the results of this thesis.

PREFACE

Simplicial complexes play an important role in mathematics and they allow a rich interaction between combinatorics, algebra, geometry and topology. Characterizing simplicial complexes with properties like vertex decomposability, shellability and Cohen- Macaulayness is a very active and important research area in combinatorial commutative algebra. Hochster and Stanley start with this study in [29] and [43], respectively. Others works are [3], [6], [7], [8],[15], [42], [45] and [48]. Vertex decomposability was first introduced in [3] by Billeras and Provan in the pure case and extended to nonpure complexes by Björner and Wachs in [6]. In [42] is proven that vertex decomposable simplicial complexes are shellable. Also, Stanley introduced the concept of sequentially Cohen-Macaulay simplicial complexes and he showed that every shellable simplicial complex is sequentially Cohen-Macaulay. Here we take the non-pure definition of shellability introduced by Björner and Wachs in [6]. However, the notion of a pure shellable complex was studied earlier in [33] and [41]. Furthermore, pure shellable implies Cohen-Macaulay (see [24] and [30]). The edge ideal of graphs was introduced by Villarreal in [47]. Furthermore, in [46] Van Tuyl and Villarreal introduced the notion of a shellable graph. A set of vertices without edges is a stable set. A graph is called shellable or Cohen-Macaulay if the simplicial complex of its stables is shellable or Cohen-Macaulay, respectively. Dochtermann, Engstrom (in [15]) and Woodroofe (in [53]) studied vertex decomposable graphs.

In general, we have the following implications (see [42], [48], [53])

$$\begin{array}{c}\text{Pure}\\ \text{vertex decomposable}\end{array} \Rightarrow \begin{array}{c}\text{Pure}\\ \text{shellable}\end{array} \Rightarrow \text{Cohen - Macaulay}$$

The equivalence between the Cohen-Macaulay property and pure vertex decomposability has been studied for some families of graphs like: bipartite graphs (in [18] and [27]); very well-covered graphs (in [13] and [31]); graphs with girth at least 5, block-cactus (in [28]); and graphs without 4-cycles and 5-cycles (in [4]). In this work, we prove the equivalence for graphs without 3-cycles and 5-cycles. Furthermore, we characterize vertex decomposable graphs whose 5-cycles are chorded.

A graph is well-covered if every maximal stable is maximum. Consequently, a graph is well-covered if and only if its simplicial complex is pure. The concept of well-coveredness was introduced by Plummer in [35]. The well-covered property is a necessary condition fof Cohen-Macaulayness (see [8]). The well-covered property has been studied for some families of graphs like: graphs with girth at least 5 (in [21]), graphs without 4 and 5-cycles (in [22]), simplicial, chordal and circular graphs (in [36]), block-cactus graphs (in [37]) and unicyclic graphs (in [44]). We characterize the well-covered property for graphs without 3-cycles, 5-cycles and 7-cycles and theta-ring graphs. The first graph were studied and characterized in [38].

If a bipartite graph is well-covered, pure shellable or Cohen-Macaulay, then it is König and has a perfect matching. This perfect matching is important for the characterization of the Cohen-Macaulay bipartite graphs given by Hibi and Herzog (see [27]) and for the characterization of well-covered bipartite graphs (see [39] and [49]). On the other hand, 3-cycles, 4-cycles, 5-cycles and 7-cycles are the well-covered cycles. Furthermore, only 3-cycles and 5-cycles are vertex decomposable, shellable and Cohen-Macaulay. But 3-cycles, 5-cycles and 7-cycles do not have a perfect matching. Consequently, it is interesting to verify: if well-covered graphs without 3-cycles, 5-cycles and 7-cycles have a perfect matching, and if a Cohen-Macaulay graph without 3-cycles and 5-cycles has a perfect matching. In this thesis, we prove that the answer to this problem is affirmative and we give a combinatorial characterization of these graphs.

Theta-ring graphs were introduced by Gitler, Reyes and Vega in [25]. In this paper they proved that theta-ring graphs are equivalent to CI$\mathcal{O}$ graphs and equivalent to universally signable graphs. Furthermore, chordal graphs, cactus, block-cactus and ring graphs are theta-ring graphs. Some of the families mentioned relate as depicted in the following diagram.

Ring graphs	$\subset$	Theta-rings	$\supset$	Chordal graphs
$\cup$		$\cup$		
Cactus graphs	$\subset$	Block-cactus		
		$\cup$		
		Block		

Well-covered chordal graphs have been characterized in [36] and well-covered cactus and block-cactus graphs in [37]. In this thesis, we characterize well-covered theta-ring graphs.

The study of matroids start in the thirties by Whitney in [51] studying the theory of dependence. This concept was taken from graph and matrix theory. Matroids can be defined in different ways, all of them equivalent. Simplicial complex of a circuit set of a matroid is shellable, since the condition of simplicial vertex in a clutter is exactly the weak circuit exchange property of a matroid (see [52]). Furthermore, the simplicial complex of the independent sets of a matroid is pure shellable (see in [5]). In this thesis we study the simplicial complex of the base set.

Hypergraphs were introduced by Berge to generalize graphs. A clutter $\mathcal{C}$ is a hypergraph such that each edge of $\mathcal{C}$ is not included in another. This concept generalizes the concept of simple graphs. Unmixed clutters play an important role in this thesis. We have that a clutter is unmixed if and only if its Stanley-Reisner simplicial complex is pure. The case of clutters has been studied in [9], [19], [20], [32] and [52]. Simplicial forests were introduced in [19]. These simplicial complexes generalize forest graphs. Facet ideal of Cohen-Macaulay simplicial forests were studied in [20]. These complexes are equivalent to a totally balanced clutters with the König property. Faridy showed in [20] that if a simplicial tree is Cohen-Macaulay, then its Stanley-Reisner complex is shellable. A characterization of unmixed clutters with a perfect matching of König type is given in [32]. Furthermore, Morey, Reyes and Villarreal characterize Cohen-Macaulay König clutters without 3-cycles and 4-cycles (see [32]). In this thesis, we study Cohen-Macaulay and pure shellable clutters with a perfect matching of König type and not having 4-cycles with particular properties.

This thesis is organized as follow: in Chapter 1 we include some definitions, properties and known results that we will use in Chapters 2 and 3, where we show our original results. Particularly, in Chapter 1 we give the preliminaries on commutative algebra and combinatorics. In section 1.1 we review notation and basic definitions on commutative algebra as Cohen-Macaulay ring, Krull dimension and depth. In section 1.2, we review some results and properties of simplicial complexes, in particular, the relation between simplicial complexes, square-free monomial ideals and Stanley-Reisner rings. Section 1.3 offers basic material on clutters including their monomial ideals and the König property. Section 1.4 is dedicated to matroid theory. In Section 1.5 we review some definitions and properties about graphs and their edge ideals. Finally, the definition of theta-ring graphs and some of their properties are given in Section 1.6.

In Chapter 2, we study vertex decomposable, shellable, Cohen-Macaulay and well-covered graphs. In particular, we provide conditions for the equivalences of the first three properties in some graph families. Also, we give necessary and sufficient conditions to characterize some families of well-covered graphs. Chapter 2 is divided as follows: in section 2.2 we give some properties and relations between critical, shedding and extendable vertices that we will use in the following sections. In section 2.3 we prove that a well-covered graph without 3-cycles, 5-cycles and 7-cycles is König and it has a perfect matching. Using this result, in Theorem 2.12 we give a characterization of these graphs. Since bipartite graphs are the graphs without odd cycles, the characterization extends the criterion of well-covered bipartite graph (given in [39] and [49]). Furthermore, we prove that very well-covered graphs and unmixed König graphs are equivalent. In section 2.4 we prove the equivalence between the unmixed vertex decomposable and Cohen-Macaulay properties for König graphs (Theorem 2.25) and graphs without 3-cycles and 5-cycles (Theorem 2.28). Furthermore, we prove that theses properties are equivalent to the following condition: G is an unmixed König graph with a perfect matching $e_1, ..., e_g$ and thee are no 4-cycle with two e_i's. Theorem 2.26 extends the Herzog-Hibi criterion for Cohen-Macaulay bipartite graphs given in [27] and the characterization for Cohen-Macaulay graphs with girth at least 6 (given in [21]). In [46] Van Tuyl and Villarreal proved that the vertex decomposable, shellable (non-pure) and sequentially Cohen-Macaulay properties are equivalent in bipartite graphs. They also gave a criterion that characterizes these graphs. These result and results obtained in sections 2.2 and 2.3 motivate us to study vertex decomposability and shellability (non pure) for graph without 3-cycles and 5-cycles. In particular, in Section 2.5 we prove that the neighborhood of a 2-connected block of G has a free vertex if G is a bipartite shellable graph or if G is a vertex decomposable graph without 3-cycles and 5-cycles. Also, we prove that the criterion of Van Tuyl-Villarreal can be extended for vertex decomposable graphs without 3-cycles and 5-cycles and shellable graphs with girth at least 11. Cohen-Macaulay trees are one of the first Cohen-Macaulay graphs studied (see [18] and [47]). These graphs are bipartite graphs and if we add a new edge, then we obtain the unicyclic graphs. Some properties of unicyclic graphs were studied in [44]. In section 2.6 we characterize unicyclic graphs for each of the following properties: vertex decomposable, shellable, Cohen-Macaulay and well-covered. In section 2.7 we study the well-covered property of a theta-ring graph (see [25]). Chordal, cactus, block-cactus and ring graphs are particular families of theta-ring graphs. We prove that a theta-ring graph is well-covered if and only if it is an induced 7-cycle or its vertex set has a disjoint partition in basic 5-cycles, semi-basic 5-cycles and sun-complete subgraphs. In section 2.8 we characterize well-covered vertex decomposable graphs whose 5-cycles are chorded (have at least 4-chords). Some well-covered graphs have a partition $V_1, ..., V_k$ of their vertex set such that if S and S' are maximal stable sets, then $|S \cap V_i| = |S' \cap V_i|$ for $i = 1, ..., k$. For this reason in section 2.9 we study when a 5-cycle C or a 7-cycle C' of a graph G, satisfies that $|C \cap S| = 2$ and $|C' \cap S| = 3$ for each maximal stable set S of G.

In Chapter 3 we study shellable and the Cohen-Macaulay simplicial complexes associated to clutters. Let $\mathcal{M}$ be a matroid, in Section 3.2 we show that a simplicial complex of the base set of $\mathcal{M}$ is shellable if and only if the base set of $\mathcal{M}$ is a complete clutter (see Theorem 3.5). Finally, in Section 3.3 we give some conditions for shellable clutters with a perfect matching of König type. These result generalize the result obtained in Section 2 about König graphs.

The dissertation is essentially self contained. For details on combinatorial theory or commutative algebra the reader is referred to Chapter 1 and its references.

TABLE OF CONTENTS

CHAPTER 1

PRELIMINARIES

In this chapter we give a review of background material, definitions, known results and properties on commutative algebra and combinatorial theory that we will use in the following chapters.

1.1 COHEN-MACAULAY RINGS

The purpose of this section is to review the definitions and properties of commutative algebra necessary to define Cohen-Macaulay rings. For more details you can see [1], [8], [42] and [48]. For this thesis ”ring” means a commutative ring with an identity element. Throughout this section R denotes a ring.

Definition 1.1 R is a *Noetherian ring* if every ideal I of R is finitely generated, that is, there exists an integer q and $f_1, ..., f_q \in I$ such that

$$I = \left(\sum_{i=1}^{q} a_i f_i \mid a_i \in R, \forall i \right).$$

Definition 1.2 A *chain of prime ideals* of R is a finite strictly increasing sequence of prime ideals $p_0 \subset p_1 \subset \cdots \subset p_n$; n is called the *length* of the chain. Spec(R) denotes the set of prime ideals of R. The *height* of $p \in \mathrm{Spec}(R)$, denoted by ht(p), is the supremum of the lengths of all chains of prime ideals which end at p.

Definition 1.3 If I is an ideal of R, then the height of I, is defined as:

$$\mathrm{ht}(I) = \min\{\mathrm{ht}(p) \mid I \subseteq p \text{ and } p \in \mathrm{Spec}(R)\}.$$

Definition 1.4 The *Krull dimension* of R, denoted by $\dim(R)$, is the supremum of the length of all chains of prime ideals of R.

Proposition 1.5 [48] In general, we have that $\dim(R/I) + \mathrm{ht}(I) \leq \dim(R)$. Equality holds if R is a polynomial ring over a field.

Definition 1.6 Let $M \neq (0)$ be an R-module. The *dimension of the R-module M* is

$$\dim(M) = \dim(R/\mathrm{ann}_R(M)),$$

where $\mathrm{ann}_R(M) = \{x \in R \mid xM = 0\}$ is the *annihilator* of M.

Definition 1.7 Given an R-module M, an element $r \in R$ is a *zero divisor* of M if there is $0 \neq m \in M$ such that $rm = 0$. The set of all zero divisors of M is denoted by $\mathcal{Z}(M)$. If r is not a zero divisor on M, we say that r is a *regular element* of M.

Definition 1.8 A sequence $\underline{\theta} = \theta_1, ..., \theta_n$ in R is called a *regular sequence* of M or an *M-regular sequence* if $(\underline{\theta})M \neq M$ and $\theta_i \notin \mathcal{Z}(M/(\theta_1, ..., \theta_{i-1})M)$ for all i.

Theorem 1.9 (Krull Principal Ideal Theorem) Let I be an ideal of R generated by a sequence $h_1, ..., h_r$. Then

(a) $\mathrm{ht}(p) \leq r$ for any minimal prime p of I.

(b) If $h_1, ..., h_r$ is a regular sequence, then $\mathrm{ht}(p) = r$ for any minimal prime p of I.

Proof. See ([1], Corollary 11.17). □

Definition 1.10 R is called *local ring* if R has only one maximal ideal m. It will be denoted by (R, m).

Definition 1.11 Let $M \neq (0)$ be a module over a local ring (R, m), the *depth* of M, denoted by $\mathrm{depth}(M)$, is the length of any maximal regular sequence on M which is contained in m.

Proposition 1.12 [48] In general, we have $\mathrm{depth}(M) \leq \dim(M)$.

Definition 1.13 An R-module M is called *Cohen-Macaulay* if $M = (0)$ or if

$$\mathrm{depth}(M) = \dim(M).$$

Definition 1.14 Let S be a multiplicative closed subset, i.e., $x, y \in S$ implies $xy \in S$. If $1 \in S$, then the *module of fractions* of M with respect to S, or the *localization* of M with respect to S is $S^{-1}(M) = \{m/s \mid m \in M, s \in S\}$, where $m/s = m_1/s_1$ if and only if there is $t \in S$ such that $t(s_1 m - s m_1) = 0$. If p is a prime ideal of R and $S = R \setminus p$, then $S^{-1}M$ is written as M_p and it is called the *localization of M at p*.

Definition 1.15 A local ring (R, m) is called Cohen-Macaulay if R is Cohen-Macaulay as an R-module. If R is non local and R_p is a Cohen-Macaulay local ring for all $p \in \mathrm{Spec}(R)$, then we say that R is a *Cohen-Macaulay ring*.

Definition 1.16 An ideal I of R is Cohen-Macaulay if R/I is a Cohen-Macaulay R-module.

Definition 1.17 Let $(H, +)$ be an abelian semigroup. An *H-graded ring* is a ring R together with a decomposition

$$R = \bigoplus_{a \in H} R_a$$

(as a $\mathbb{Z}$-module) such that $R_a R_b \subset R_{a+b}$ for all $a, b \in H$.

Remark 1.18 A graded ring is by definition a $\mathbb{Z}$-graded ring.

Definition 1.19 If R is an H-graded ring and M is an R-module such that

$$M = \bigoplus_{a \in H} M_a,$$

where M_a is an additive subgroup and $R_a M_b \subset M_{a+b}$ for all $a, b \in H$, then we say that M is an *H-graded module*. An element $f \in M$ is said to be *homogeneous of degree a* if $f \in M_a$.

Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field k. We set $d = (d_1, ..., d_n) \in \mathbb{N}^n$. For $a = (a_1, ..., a_n) \in \mathbb{N}^n$ we take the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 d_1 + \cdots + a_n d_n$. The induced d-grading of R is: $R = \bigoplus_{i=0}^{\infty} R_i$, where $R_i = \bigoplus_{|a|=i} k x^a$. The *standard grading* or *usual grading* of R is the 1-grading, where $1 = (1, ..., 1)$.

Remark 1.20 [8] Let I be a monomial ideal in $k[x_1, ..., x_n]$. There exists a unique minimal monomial set of generators of I and it is denoted by $G(I)$.

Definition 1.21 Let $R = k[x_1, ..., x_n]$ be a polynomial ring. A graded R-module M is called *sequentially Cohen-Macaulay* (over k) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and their Krull dimensions satisfy

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

1.2 SIMPLICIAL COMPLEXES AND STANLEY-REISNER RINGS

In this section we give some properties of simplicial complexes and their Stanley-Reisner rings.

Definition 1.22 A *simplicial complex* consists of a finite set V of vertices and a collection Δ of subsets of V called *faces* such that: if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Definition 1.23 The maximal faces of Δ are called *facets* and the set of facets of Δ is denoted by $\mathcal{F}(\Delta)$. If $F \in \Delta$, the *dimension of a simplicial complex* of F is $\dim(F) = |F| - 1$. Furthermore, $\dim(\Delta) = \sup\{\dim(F) \mid F \in \Delta\}$. We assume that $\dim(\emptyset) = -1$. A face of dimension q is called a q-face or a q-simplex. $\langle F_1, ..., F_s \rangle$ denotes the simplicial complex whose facets are $F_1, ..., F_s$.

Definition 1.24 Δ is called *pure* if all its facets have the same cardinality.

Definition 1.25 If $F \subseteq V(\Delta)$, then the *deletion* of F in Δ is the subcomplex

$$\mathrm{del}_\Delta(F) = \{G \in \Delta \mid G \cap F = \emptyset\}.$$

Furthermore, if $F \in \Delta$, then the *link* of F in Δ is

$$\mathrm{lk}_\Delta(F) = \{G \in \Delta \mid G \cup F \in \Delta, G \cap F = \emptyset\}.$$

Definition 1.26 A simplicial complex Δ is called *vertex decomposable* if V is the unique facet, or Δ contains a vertex x such that

(a) both $\text{del}_\Delta(x)$ and $\text{lk}_\Delta(x)$ are vertex decomposable, and

(b) no facet of $\text{lk}_\Delta(x)$ is a facet of $\text{del}_\Delta(x)$.

Definition 1.27 A vertex x which satisfies condiction (b) (in the last definition) is called a *shedding vertex* of Δ.

Definition 1.28 If the vertex set of Δ is $V(\Delta) = \{x_1, ..., x_n\}$, the *Stanley-Reisner ring* or *face ring* of Δ over a field k is $k[\Delta] = R/I_\Delta$, where R is the polynomial ring $k[x_1, ..., x_n]$ and I_Δ is the ideal of R generated by

$$\{x_{i_1} \cdots x_{i_k} \mid 1 \leq i_i < \cdots < i_k \leq n, \ \{x_{i_1}, ..., x_{i_k}\} \notin \Delta\}.$$

Proposition 1.29 [48] If Δ is a simplicial complex with vertices $x_1, ..., x_n$, then the primary decomposition of the Stanley-Reisner ideal of Δ is:

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} p_F,$$

where p_F is the ideal generated by all x_i such that $x_i \notin F$.

Proposition 1.30 [48] If $\dim(\Delta) = d$ and $V(\Delta) = \{x_1, ..., x_n\}$, then

$$\dim k[\Delta] = d + 1 = \max\{s \mid x_{i_1} \cdots x_{i_s} \notin I_\Delta \text{ and } i_1 < \cdots < i_s\}.$$

Definition 1.31 [6] Δ is *shellable* if the facets of Δ can be ordered $F_1, ..., F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, ..., j-1\}$ with $F_j \setminus F_l = \{v\}$. In this case, $F_1, ..., F_s$ is called a shelling of Δ.

Remark 1.32 [48] If Δ is pure shellable, then $\text{lk}_\Delta(F)$ is pure shellable.

Theorem 1.33 [42], [48] Let Δ be a simplicial complex.

- If Δ is vertex decomposable, then Δ is shellable.
- If Δ is shellable, then $k[\Delta]$ is sequentially Cohen-Macaulay.
- If $k[\Delta]$ is Cohen-Macaulay, then Δ is pure.

It is known that the above implications are strict. Furthermore, in general Cohen-Macaulay property of $k[\Delta]$ depends of the characteristic of k (see [8],[48]).

Theorem 1.34 [8] Let Δ be a simplicial complex. If k is a field, then the following conditions are equivalent:

(a) $k[\Delta]$ is Cohen-Macaulay over k.

(b) $\tilde{H}_i(\mathrm{lk}_\Delta(F);k) = 0$ for $F \in \Delta$ and $i < \dim(\mathrm{lk}_\Delta(F))$.

Where $\tilde{H}_i(\mathrm{lk}_\Delta(F);k)$ is the ith reduced simplicial homology group of $\mathrm{lk}_\Delta(F)$ with coefficients in k.

Proposition 1.35 [48] If k is a field, then a simplicial complex Δ is Cohen-Macaulay over k if and only if $\mathrm{lk}_\Delta(F)$ is Cohen-Macaulay over k for each $F \in \Delta$.

Definition 1.36 The *Alexander dual* of Δ is the simplicial complex $\Delta^* = \{\overline{\tau} \subseteq V \mid \tau \notin \Delta\}$, where $\overline{\tau} = V \setminus \tau$.

Remark 1.37 [34] The Alexander dual Δ^* is a simplicial complex and $(\Delta^*)^* = \Delta$.

Definition 1.38 If $0 \leq q \leq \dim(\Delta)+1$, the *q-skeleton* of Δ is the simplicial complex Δ^q consisting of all p-simplices of Δ with $p \leq q$, that is, $\Delta^q = \langle \{F \in \Delta \mid |F| \leq q+1\} \rangle$.

Theorem 1.39 ([16], Theorem 3.3) A simplicial complex Δ is sequentially Cohen-Macaulay if and only if Δ^q is Cohen-Macaulay for $0 \leq q \leq \dim(\Delta)$+1.

Definition 1.40 If $\dim(\Delta) = d$, then the *f-vector* of Δ is the $(d+1)$-tuple $f(\Delta) = (f_0, ..., f_d)$, where f_i is the number of i-faces of Δ. Note $f_{-1} = 1$. On the other hand, the *h-vector* of Δ is the $(d+1)$-tuple $h(\Delta) = (h_0, ..., h_{d+1})$ such that

$$\sum_{i=0}^{d+1} f_{i-1}(x-1)^{d+1-i} = \sum_{i=0}^{d+1} h_i x^{d+1-i}.$$

Theorem 1.41 [48] If Δ is a simplicial complex of dimension d and $h(\Delta)$ the h-vector of $k[\Delta]$, then $h_r(\Delta) = 0$ for $r > d+1$ and

$$h_r(\Delta) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d+1-i}{r-i} f_{i-1} \text{ for } 0 \leq r \leq d+1.$$

Theorem 1.42 [48] Let Δ be a simplicial complex of dimension d with n vertices. If $k[\Delta]$ is Cohen-Macaulay and k is an infinite field, then the h-vector of Δ satisfies

$$0 \leq h_i(\Delta) \leq \binom{i+n-d-2}{i} \text{ for } 0 \leq i \leq d+1.$$

1.3 CLUTTERS AND MONOMIAL IDEALS

Definition 1.43 A *clutter* consists of a pair $\mathcal{C} = (V, E)$, where V is a finite set whose elements are called vertices and E is a family of subsets of V called *edges*, none of which is included in another.

Definition 1.44 The *d-complete clutter* with n vertices is the d-uniform clutter with $\binom{n}{d}$ edge set. In some cases, it is called only complete clutter.

Definition 1.45 Let $x \in V(\mathcal{C})$, the *deletion* of x is the clutter $\mathcal{C} \setminus x$ whose vertex set is $V(\mathcal{C}) \setminus \{x\}$ and edge set $\{e \in \mathcal{C} \mid x \notin e\}$ and the *contraction* of x is the clutter $\mathcal{C}/x$ whose vertex set is $V(\mathcal{C}) \setminus \{x\}$ and the edges are the minimal sets of $\{e \setminus \{x\} \mid e \in \mathcal{C}\}$.

Definition 1.46 A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by deletion and/or contraction of a family of vertices is called a *minor* of $\mathcal{C}$. If $\mathcal{D}$ is obtained only by deletions, then $\mathcal{D}$ is called *d-minor* and if $\mathcal{D}$ is obtained only by contractions, then $\mathcal{D}$ is called *c-minor*.

Definition 1.47 We say that v_i is a *free vertex* of $\mathcal{C}$ if v_i appears in exactly one edge of $\mathcal{C}$. $\mathcal{C}$ has the *free vertex property* if all minors of $\mathcal{C}$ have a free vertex. $\mathcal{C}$ has the *c-free vertex property* if all c-minors of $\mathcal{C}$ have a free vertex.

Definition 1.48 A subset $D \subseteq V(\mathcal{C})$ is a *vertex cover* of $\mathcal{C}$ if every edge of $\mathcal{C}$ contains at least one vertex of D. D is *minimal* if each proper subset D is not a vertex cover. The *blocker* $b(\mathcal{C})$ of $\mathcal{C}$ is the set of minimal vertex covers of $\mathcal{C}$. A subset $F \subseteq V(\mathcal{C})$ is called *independent* or *stable* if $e \nsubseteq F$ for each $e \in E(\mathcal{C})$. F is *maximal* if there is no stable F' of $\mathcal{C}$ such that $F \subsetneq F'$.

Remark 1.49 [34] If $\mathcal{C}$ is a clutter, then $b(b(\mathcal{C})) = \mathcal{C}$.

Definition 1.50 The cardinality of a maximum stable set is denoted by $\beta(\mathcal{C})$ and it is called *stability number*. The number of vertices in a minimum vertex cover of $\mathcal{C}$ is called the *covering number* of $\mathcal{C}$ and it is denoted by $\tau(\mathcal{C})$.

Remark 1.51 A subset D of $V(\mathcal{C})$ is a vertex cover of $\mathcal{C}$ if and only if $V(\mathcal{C}) \setminus D$ is a stable set of $\mathcal{C}$. Consequently, $\tau(\mathcal{C}) = n - \beta(\mathcal{C})$.

Definition 1.52 $\mathcal{C}$ is called *well-covered* if all maximal stable sets have the same cardinality. If the minimal vertex covers have the same cardinality, then $\mathcal{C}$ is called *unmixed*.

Remark 1.53 $\mathcal{C}$ is well-covered if and only if $\mathcal{C}$ is unmixed.

Definition 1.54 A collection of edges $e_1, ..., e_g$ is a *matching* of $\mathcal{C}$ if each two edges are disjoint. Furthermore, it is a *perfect matching* if $\bigcup_{i=1}^{g} e_i = V(\mathcal{C})$. The number of elements in a maximum matching is denoted by $\nu(\mathcal{C})$. A clutter $\mathcal{C}$ is *König* if $\nu(\mathcal{C}) = \tau(\mathcal{C})$. A perfect matching $e_1, ..., e_g$ is of *König type* if $g = \tau(\mathcal{C})$.

Lemma 1.55 ([32], Lemma 2.3) If $\mathcal{C}$ is an unmixed clutter with the König property and without isolated vertices, then $\mathcal{C}$ has a perfect matching of König type.

Proposition 1.56 ([32], Proposition 2.9) Let $\mathcal{C}$ be a clutter with a perfect maching $e_1, ..., e_g$ of König type. Then the following are equivalent:

(a) $\mathcal{C}$ is unmixed.

(b) For any two edges $e \neq e'$ and for any distinct vertices $x \in e$, $y \in e'$ contained in some e_i, one has that $(e \setminus x) \cup (e' \setminus y)$ contains an edge.

If $V(\mathcal{C}) = \{x_1, ..., x_n\}$, then we identify each vertex x_i of $\mathcal{C}$ with a variable x_i in a polynomial ring $R = k[x_1, ..., x_n]$ over a field k.

Definition 1.57 The *edge ideal* of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of R generated by all monomials $\prod_{x_i \in e} x_i$ such that $e \in E(\mathcal{C})$.

The assignment $\mathcal{C} \to I(\mathcal{C})$ establishes a natural one to one correspondence between the family of clutters and the family of square-free monomial ideals. (see [8], [48]).

Remark 1.58 We have that $\mathrm{ht}(I(\mathcal{C})) = \tau(\mathcal{C})$. Furthermore, p is a minimal prime of $I(\mathcal{C})$ if and only if $p = (D)$ for some minimal vertex cover D of $\mathcal{C}$. In particular, if $D_1, ..., D_t$ is a complete list of the minimal vertex covers of $\mathcal{C}$, then

$$I(\mathcal{C}) = (D_1) \cap (D_2) \cap \cdots \cap (D_t).$$

Proposition 1.59 ([32], Proposition 2.4) Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, ..., e_g$ of König type and let $C_1, ..., C_r$ be any collection of minimal vertex covers of $\mathcal{C}$. If $\mathcal{C}'$ is the clutter associated to $I = \bigcap_{i=1}^{r}(C_i)$, then $\mathcal{C}'$ has a perfect matching $e'_1, ..., e'_g$ of König type such that:

(a) $e'_i \subseteq e_i$ for all i, and

(b) every vertex of $e_i \setminus e'_i$ is isolated in $\mathcal{C}'$.

Remark 1.60 ([32], Remark 2.5) Let $C_1, .., C_p$ be the minimal vertex covers of $\mathcal{C}$. Since $I(\mathcal{C})$ is equal to $\bigcap_{i=1}^{p}(C_i)$, one has $(I(\mathcal{C}) : x_j) = \bigcap_{x_j \notin C_i}(C_i)$ for any vertex $x_j \notin I(\mathcal{C})$.

Definition 1.61 The Stanley-Reisner complex of $\mathcal{C}$ denoted by $\Delta_{\mathcal{C}}$, is the simplicial complex whose faces are the stable sets of $\mathcal{C}$.

Remark 1.62 F is a facet of $\Delta_{\mathcal{C}}$ if and only if $V(\Delta_{\mathcal{C}}) \setminus F$ is a minimal vertex cover of $\mathcal{C}$. Consequently, $\Delta_{\mathcal{C}}$ is pure if and only if $\mathcal{C}$ is well-covered if and only if $\mathcal{C}$ is unmixed. Furthermore, $I_{\Delta_{\mathcal{C}}} = I(\mathcal{C})$.

Lemma 1.63 ([52], Lemma 2.2) Let $\mathcal{C}$ a clutter and $v \in V(\mathcal{C})$. We have that $\mathrm{lk}_{\Delta_{\mathcal{C}}}(v) = \Delta_{\mathcal{C}/v}$.

Definition 1.64 $\mathcal{C}$ is shellable or vertex decomposable if $\Delta_{\mathcal{C}}$ is shellable or vertex decomposable, respectively. Furthermore, $\mathcal{C}$ is (sequentially) Cohen-Macaulay over k if $k[\Delta_{\mathcal{C}}]$ is (sequentially) Cohen-Macaulay.

Definition 1.65 A family of clutters $\mathcal{F}$ is closed under c-minors if $\mathcal{C} \in \mathcal{F}$ and $\mathcal{C}'$ is a c-minor of $\mathcal{C}$, implies $\mathcal{C}' \in \mathcal{F}$.

Remark 1.66 ([2], [48]) The following properties: shellable, Cohen-Macaulay, sequentially Cohen-Macaulay, vertex decomposable, unmixed and well-covered are closed under c-minors.

Lemma 1.67 ([46], Lemma 5.1) Let $\mathcal{C}$ be a clutter with minimal vertex covers $D_1, ...,$ D_t. If $\Delta_{\mathcal{C}}$ is shellable and $A \subset V(\mathcal{C})$ is a set of vertices, then the Stanley-Reisner complex $\Delta_{I'}$ of the ideal

$$I' = \bigcap_{D_i \cap A = \emptyset} (D_i)$$

is shellable with respect to the linear ordering of the facets of $\Delta_{I'}$ induced by the shelling of the simplicial complex $\Delta_{\mathcal{C}}$.

Definition 1.68 If $V(\mathcal{C}) = \{x_1, ..., x_n\}$ and $E(\mathcal{C}) = \{e_1, ..., e_q\}$, then the *incidence matrix* of $\mathcal{C}$ is the matrix $n \times q$, $A = (a_{ij})$ such that $a_{ij} = 1$ if $x_i \in e_j$ and $a_{ij} = 0$ otherwise. The clutter $\mathcal{C}$ has an s-cycle if A has a submatrix $s \times s$ with exactly two 1's in each row and each column.

Theorem 1.69 ([32], Theorem 2.13) Let $\mathcal{C}$ be a clutter with a perfect matching $e_1, ...,$ e_g of König type. If for any two edges f_1, f_2 of $\mathcal{C}$ and for any e_i, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$. Then $\mathcal{C}$ is unmixed.

Theorem 1.70 ([32], Theorem 2.16) Let $\mathcal{C}$ be a clutter with a perfect matching $e_1, ...,$ e_g of König type. If for any two edges f_1, f_2 of $\mathcal{C}$ and for any e_i of the perfect matching, one has that $f_1 \cap e_i \subset f_2 \cap e_i$ or $f_2 \cap e_i \subset f_1 \cap e_i$, then $\Delta_{\mathcal{C}}$ is pure shellable.

Corollary 1.71 ([32], Corollary 2.19) Let $\mathcal{C}$ be a clutter with the König property without 3-cycles and 4-cycles. Then any of the following conditions are equivalent:

(a) $\mathcal{C}$ is unmixed.

(b) There is a perfect matching $e_1, ..., e_g$ with $g = \text{ht}(I(\mathcal{C}))$, such that for any two edges $f_1, f_2 \in E(\mathcal{C})$ and for any edge e_i of the perfect matching, one has that $f_1 \cap e_i \subseteq f_2 \cap e_i$ or $f_2 \cap e_i \subseteq f_1 \cap e_i$.

(c) $R/I(\mathcal{C})$ is Cohen-Macaulay.

(d) $\Delta_{\mathcal{C}}$ is a pure shellable simplicial complex.

1.4 MATROIDS

Definition 1.72 A *matroid* $\mathcal{M}$ is an ordered pair $(V, \mathcal{I})$ consisting of a finite set V and a collection of subsets $\mathcal{I}$ of V satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then there is $x \in I_2 - I_1$ such that $I_1 \cup x \in \mathcal{I}$.

Definition 1.73 The members of $\mathcal{I}$ are the *independent sets* of $\mathcal{M}$. A subset of V that is not in $\mathcal{I}$ is called *dependent*. Condition (I3) is called the *independence augmentation axiom*. A minimal dependent set in an arbitrary matroid $\mathcal{M}$ will be called a *circuit* of $\mathcal{M}$ and the set of circuits of $\mathcal{M}$ is denoted by $\mathcal{C}(\mathcal{M})$. A maximal element of $\mathcal{I}$ is called *base* of $\mathcal{M}$. The set of bases of $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{M})$.

Proposition 1.74 [34] $\mathcal{C}(\mathcal{M})$ has the following properties:

(C1) $\emptyset \notin \mathcal{C}(\mathcal{M})$.

(C2) If $C_1, C_2 \in \mathcal{C}(\mathcal{M})$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}(\mathcal{M})$ with $C_1 \neq C_2$ and $x \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}(\mathcal{M})$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus x$.

Condition (C_3) is called the *weak circuit exchange property*.

Theorem 1.75 [34] Let V be a set and $\mathcal{C}(\mathcal{M})$ be a collection of subsets of V satisfying (C1)-(C3). If $\mathcal{I}$ is the collection of subsets of V that contain no member of $\mathcal{C}(\mathcal{M})$, then $(V, \mathcal{I})$ is a matroid whose circuit set is $\mathcal{C}(\mathcal{M})$.

Proposition 1.76 $\mathcal{B}(\mathcal{M})$ satisfies the following two conditions:

(B1) $\mathcal{B}(\mathcal{M})$ is not empty.

(B2) If $B_1, B_2 \in \mathcal{B}(\mathcal{M})$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}(\mathcal{M})$.

The condition (B2) is called *basis exchange axiom*.

Remark 1.77 [34] $(B2)$ is equivalent to $(B2)^*$: If $B_1, B_2 \in \mathcal{B}(\mathcal{M})$ and $x \in B_2 \setminus B_1$, then there exists $y \in B_1 \setminus B_2$ such that $(B_1 \setminus y) \cup x \in \mathcal{B}(\mathcal{M})$.

Lemma 1.78 [34] If B_1 and B_2 are bases of a matroid $\mathcal{M}$, then $|B_1| = |B_2|$.

Proposition 1.79 [34] Let $\mathcal{M}$ be a matroid. Then there exists a matroid $\mathcal{M}'$ whose base set is $\mathcal{B}(\mathcal{M}') = \{\overline{\beta} \mid \beta \in \mathcal{B}(\mathcal{M})\}$, where $\overline{\beta} = V \setminus \beta$. $\mathcal{M}'$ is called the *dual matroid* of $\mathcal{M}$.

Remark 1.80 If $\mathcal{M}$ is a matroid, then $(\mathcal{M}')' = \mathcal{M}$.

Remark 1.81 [34] If $\mathcal{M}$ is a matroid, then $b(\mathcal{B}(\mathcal{M})) = \mathcal{C}(\mathcal{M}')$ and $b(\mathcal{C}(\mathcal{M}')) = \mathcal{B}(\mathcal{M})$.

1.5 GRAPHS AND EDGE IDEALS

All graphs considered in this dissertation are finite simple graphs, and for simplicity they will be called graphs. A *graph* is a clutter whose edges have two elements.

Definition 1.82 Let G be a graph. A sequence $\mathcal{L} = (x_1, x_2, ..., x_{l+1})$ is a *walk* in G of length l from x_1 to x_{l+1} if $\{x_1, x_2\}, \{x_2, x_3\}, ..., \{x_l, x_{l+1}\} \in E(G)$. In this case x_1 and x_{l+1} are called the *ends* of $\mathcal{L}$. If the vertices $x_1, x_2, ..., x_{l+1}$ are all distinct, then $\mathcal{L}$ is called *path*. If the length of $\mathcal{L}$ equals zero, then $\mathcal{L}$ is called a *trivial path*.

Definition 1.83 A graph G is *connected* if there is a path joining each pair of distinct vertices of G, otherwise G is called *disconnected*. A maximal connected subgraph of G is called a *connected component* of G. If every connected component of G is a vertex, G is called *totally disconnected*.

Remark 1.84 A connected component of G is a c-minor of G.

Definition 1.85 A walk $\mathcal{L} = (x_1, ..., x_{k+1})$ with $k \geq 3$ such that $x_1, ..., x_k$ are all distinct and $x_1 = x_{k+1}$ is called a *cycle* of length k or *k-cycle*. If G does not contain cycles, then it is called *forest*. A connected forest is a *tree*. The *girth* of G is the length of the smallest cycle or infinite if G is a forest.

Definition 1.86 A *unicyclic* graph is a connected graph with exactly one cycle.

Definition 1.87 If $v \in V(G)$, then the set of *neighbors* of v (in G) is $N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G is $\deg_G(v) = |N_G(v)|$. A vertex of degree one is called *leaf* or *free vertex* and a vertex adjacent to a leaf is called a *stem*. An edge which is incident with a leaf is called *pendant*.

Remark 1.88 [48] If C_n is a n-cycle, then C_n is well-covered if and only if $n = 3, 4, 5$ or 7.

Lemma 1.89 ([21], Lemma 2) Let G be a connected graph of girth ≥ 6, which is neither a 7-cycle or a vertex. Then G is well-covered if and only if its pendant edges form a perfect matching.

Definition 1.90 A *cut vertex* of G is a vertex v such that the number of connected components of $G \setminus v$ is greater than the number of connected components of G. A *bridge* of a connected graph is an edge whose ends are cut vertices. A maximal connected subgraph of G without a cut vertex is called a *block*. A connected graph without cut vertices with at least three vertices is called 2-connected graph.

Lemma 1.91 ([46], Lemma 2.5) If $x \in V(G)$ and $G' = G \setminus N_G[x]$, then $\Delta_{G'} = \mathrm{lk}_{\Delta_G}(x)$.

Remark 1.92 Δ_G is *vertex decomposable* if G is a totally disconnected graph or there is a vertex v such that

(a) $G \setminus v$ and $G \setminus N_G[v]$ are both vertex decomposable, and

(b) each stable set in $G \setminus N_G[v]$ is not a maximal stable set in $G \setminus v$.

Definition 1.93 A *shedding vertex* of G is a vertex with property (b).

Remark 1.94 A vertex v is a shedding vertex of G if and only if v is a shedding vertex of Δ_G. Furthermore, v is a shedding vertex if for every stable set S contained in $G \setminus N_G[v]$, there is some $x \in N_G(v)$ such that $S \cup \{x\}$ is stable.

Definition 1.95 Let H be a subgraph of G, a *chord* of H in G is an edge $e \in E(G) \setminus E(H)$ such that its ends belong to $V(H)$. A graph is *chordal* if each of its cycles of length at least 4 has a chord.

Theorem 1.96 ([53], Theorem 1) If the only chordlees cycles of G are 3-cycles and 5-cycles, then G is vertex decomposable.

Corollary 1.97 Chordal graphs and forests are vertex decomposable. Therefore, they are shellable and sequentially Cohen-Macaulay.

Definition 1.98 A cycle C_k is induced if C_k does not have chords.

Remark 1.99 Let S be a stable set of G. Then $G/S = G \setminus N_G[S]$.

Definition 1.100 Let S be a subset of $V(G)$, the closed neighborhood of S is $N_G[S] = \bigcup\limits_{x\in S} N_G[x]$.

Definition 1.101 Let S be a stable set of G. If x is of degree zero in $G \setminus N_G[S]$, then x is called *isolated vertex* in $G \setminus N_G[S]$, also we say that S *isolates* to x.

Definition 1.102 A graph G with n vertices is called *complete* if for all $u, v \in V(G)$, we have that $\{u, v\} \in E(G)$. This graph is denoted by K_n. A *clique* of a graph G is a maximal complete subgraph of G. A vertex v is called *simplicial* if the induced subgraph $G[N_G[v]]$ is a complete graph. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G.

Lemma 1.103 If $v, w \in V(G)$ such that $N_G[v] \subseteq N_G[w]$, then w is a shedding vertex of G. In particular, if v is a simplicial vertex, then any $w \in N_G(v)$ is a shedding vertex.

Proof. By Lemma 6 and Corollary 7 in [53]. □

Definition 1.104 A vertex x is called *extendable* if G and $G \setminus x$ are well-covered graphs and $\beta(G) = \beta(G \setminus x)$.

Lemma 1.105 ([21], Lemma 2) Let G be a well-covered graph. A vertex $x \in V(G)$ is extendable if and only if $|N_G(x) \setminus N_G(S)| \geq 1$ for every stable set S of $G \setminus N_G[x]$. Furthermore, every $x \in V(G)$ is nonextendable if and only if there is a stable set $S \subseteq V(G)$ such that x is an isolated vertex in $G \setminus N_G[S]$.

Proposition 1.106 ([48], [53]) A graph is unmixed, shellable, vertex decomposable or Cohen-Macaulay if and only if each connected component is unmixed, shellable, vertex decomposable or Cohen-Macaulay, respectively.

Definition 1.107 A graph G is called *very well-covered* if it is well-covered without isolated vertices and $|V(G)| = 2\text{ht}(I(G))$.

Theorem 1.108 ([38], Theorem 5) Let G be well-covered with no isolated vertex and odd girth ≥ 9 (without 3, 5 and 7-cycles), then G is very well-covered.

Remark 1.109 Let G be a graph. If $\tau(G) = \beta(G)$, then G has a perfect matching.

Proposition 1.110 ([23], Proposition 4.1) If C_n is a n-cycle, then C_n is vertex decomposable, shellable or sequentially Cohen-Macaulay if and only if $n = 3$ or 5.

Lemma 1.111 ([50], Lemma 6) If G has a shedding vertex v where $G \setminus v$ and $G \setminus N_G[v]$ are shellables with shelling $F_1, ..., F_k$ and $G_1, ..., G_q$, respectively, then G is shellable with shelling $F_1, ..., F_k, G_1 \cup \{v\}, ..., G_q \cup \{v\}$.

Definition 1.112 A graph G is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge has one end in V_1 and one end in V_2. Furthermore, G is called a *complete bipartite*, denoted by $K_{r,s}$, if $|V_1| = r$, $|V_2| = s$ and every vertex in V_1 is adjacent to every vertex in V_2.

Proposition 1.113 [48] G is bipartite if and only if G does not contain odd cycles.

Lemma 1.114 ([46], Lemma 2.8) Let G be a bipartite with bipartition $\{x_1, ..., x_m\}$, $\{y_1, ..., y_n\}$. If G is shellable and G has non isolated vertices, then there is $v \in V(G)$ with $\deg_G(v) = 1$.

Theorem 1.115 ([46], Theorem 2.9) Let G be a graph and let x_1, y_1 be two adjacent vertices of G with $\deg_G(x_1) = 1$. If $G_1 = G \setminus N_G[x_1]$ and $G_2 = G \setminus N_G[y_1]$, then G is shellable if and only if G_1 and G_2 are shellable.

Definition 1.116 Let X be a subset of $V(G)$, the *induced subgraph* by X in G, denoted by $G[X]$ is the graph with vertex set X and whose edge set is

$$\{\{x, y\} \in E(G) \mid x, y \in X\}.$$

Furthermore, $G \setminus X$ denotes the induced subgraph $G[V(G) \setminus X]$.

Definition 1.117 G is called *whisker* if there exists an induced subgraph H of G such that $V(H) = \{x_1, ..., x_s\}$, $V(G) = V(H) \cup \{y_1, ..., y_s\}$ and $E(G) = E(H) \cup W(H)$ where $W(H) = \{\{x_1, y_1\}, ..., \{x_s, y_s\}\}$. The edges of $W(H)$ are called whiskers and they form a perfect matching.

Theorem 1.118 ([48], Theorem 7.3.17) If G is a tree, then G is a Cohen-Macaulay graph if and only if G is unmixed if and only if G is a whisker graph.

Corollary 1.119 $k[\Delta_G]$ is Cohen-Macaulay if and only if $k[\Delta_G]$ is sequentially Cohen-Macaulay and G is unmixed.

Theorem 1.120 ([46], Theorem 3.3) Let x be a vertex of G and let $G' = G \setminus N_G[x]$. If G is sequentially Cohen-Macaulay, then G' is sequentially Cohen-Macaulay.

Lemma 1.121 ([46], Lemma 3.9) Let G be a bipartite graph. If G is sequentially Cohen-Macaulay, then there is $v \in V(G)$ with $\deg(v) = 1$.

The following criterion classifies the Cohen-Macaulay bipartite graphs.

Theorem 1.122 [27] G is a Cohen-Macaulay bipartite graph if and only if $g = |V_1| = |V_2|$ and we can order the vertices such that:

(h_0) $\{x_i, y_i\} \in E(G)$ for $i = 1, \ldots, g$,

(h_1) if $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and

(h_2) if $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$.

Proposition 1.123 ([37], Proposition 2.3) If G is a well-covered graph, then all its simplexes are pairwise vertex disjoint.

Definition 1.124 A 5-cycle C of G is called *basic 5-cycle* if C does not contain two adjacent vertices of degree three or more in G. A 4-cycle is called *basic* if it contains two adjacent vertices of degree two, and the remaining two vertices belong to a simplex or a basic 5-cycle of G.

Definition 1.125 A graph G is in the family $\mathcal{SQC}$, if $V(G)$ can be partitioned into three disjoint subsets S_G, Q_G and C_G: the subset S_G contains all vertices of the simplexes of G, and the simplexes of G are disjoint vertex; the subset C_G consists of the vertices of the basic 5-cycles and the basic 5-cycles form a partition of C_G; the remaining set Q_G contains all vertices of degree two of the basic 4-cycles.

Theorem 1.126 If $G \in \mathcal{SQC}$, then G is well-covered vertex decomposable.

Proof. By ([28], Theorem 2.3) and ([37], Theorem 3.1). □

Remark 1.127 By Theorem 1.33, we have the following implications for a graph G:

$$\begin{array}{c} \text{unmixed} \\ \text{vertex decomposable} \end{array} \Rightarrow \begin{array}{c} \text{pure} \\ \text{shellable} \end{array} \Rightarrow \text{Cohen - Macaulay} \Rightarrow \text{well - covered}$$

Recently, in [17] authors show an unmixed shellable graph G such that G is not vertex decomposable (see Example 1.128). C_4 is well-covered but it is not Cohen-Macaulay and Villarreal in [48] proved that Cohen-Macaulay property depends of the characteristic of the field k (see Example 1.129).

Example 1.128 [17] The following graph is called circulant graph: $G = C_{16}(1,4,8)$.

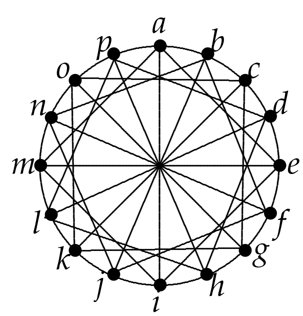


Figure 1.1: Shellable but not vertex decomposable

If $x \in V(G)$, then the h-vector of $G \setminus x$ is $(1, 11, 31, 18, -1)$. Consequently, $G \setminus x$ is not Cohen-Macaulay. Hence, $G \setminus x$ is not vertex decomposable, therefore, G is not vertex decomposable.

A shelling of Δ_G is (from left to right):

$\{i,k,n,p\}$	$\{e,k,n,p\}$	$\{g,i,n,p\}$	$\{c,i,n,p\}$	$\{e,g,n,p\}$	$\{c,e,n,p\}$
$\{f,i,k,p\}$	$\{f,k,m,p\}$	$\{b,k,m,p\}$	$\{b,i,k,p\}$	$\{b,e,k,p\}$	$\{e,g,j,p\}$
$\{c,e,j,p\}$	$\{b,g,i,p\}$	$\{b,g,m,p\}$	$\{g,j,m,p\}$	$\{c,j,m,p\}$	$\{c,f,m,p\}$
$\{c,f,i,p\}$	$\{b,e,g,p\}$	$\{g,i,l,n\}$	$\{c,i,l,n\}$	$\{e,g,l,n\}$	$\{a,g,l,n\}$
$\{c,e,l,n\}$	$\{a,c,l,n\}$	$\{d,i,k,n\}$	$\{e,h,k,n\}$	$\{d,g,i,n\}$	$\{c,e,h,n\}$
$\{a,c,h,n\}$	$\{a,h,k,n\}$	$\{a,d,k,n\}$	$\{a,d,g,n\}$	$\{e,g,j,l\}$	$\{e,j,l,o\}$
$\{a,g,j,l\}$	$\{a,j,l,o\}$	$\{c,e,j,l\}$	$\{a,c,j,l\}$	$\{b,g,i,l\}$	$\{c,f,i,l\}$
$\{b,e,g,l\}$	$\{b,e,l,o\}$	$\{b,i,l,o\}$	$\{f,i,l,o\}$	$\{a,f,l,o\}$	$\{a,c,f,l\}$
$\{d,f,i,k\}$	$\{d,f,i,o\}$	$\{b,d,i,o\}$	$\{d,f,k,m\}$	$\{d,f,m,o\}$	$\{b,d,i,k\}$
$\{b,d,k,m\}$	$\{b,d,m,o\}$	$\{b,e,h,k\}$	$\{b,e,h,o\}$	$\{b,h,m,o\}$	$\{f,h,m,o\}$
$\{e,h,j,o\}$	$\{b,h,k,m\}$	$\{f,h,k,m\}$	$\{c,e,h,j\}$	$\{a,c,h,j\}$	$\{a,h,j,o\}$
$\{a,f,h,o\}$	$\{a,f,h,k\}$	$\{a,d,f,k\}$	$\{a,d,f,o\}$	$\{a,d,j,o\}$	$\{a,d,g,j\}$
$\{b,d,g,i\}$	$\{b,d,g,m\}$	$\{d,g,j,m\}$	$\{d,j,m,o\}$	$\{h,j,m,o\}$	$\{c,h,j,m\}$
$\{c,f,h,m\}$	$\{a,c,f,h\}$				

Example 1.129 [48] Let G be a graph with edges:

$\{x_1,x_3\}$	$\{x_1,x_4\}$	$\{x_1,x_7\}$	$\{x_1,x_{10}\}$	$\{x_1,x_{11}\}$	$\{x_2,x_4\}$	$\{x_2,x_5\}$
$\{x_2,x_8\}$	$\{x_2,x_{10}\}$	$\{x_2,x_{11}\}$	$\{x_3,x_5\}$	$\{x_3,x_6\}$	$\{x_3,x_8\}$	$\{x_3,x_{11}\}$
$\{x_4,x_6\}$	$\{x_4,x_9\}$	$\{x_4,x_{11}\}$	$\{x_5,x_7\}$	$\{x_5,x_9\}$	$\{x_5,x_{11}\}$	$\{x_6,x_8\}$
$\{x_6,x_9\}$	$\{x_7,x_9\}$	$\{x_7,x_{10}\}$	$\{x_8,x_{10}\}$			

Δ_G is the following triangulation of the real projective plane P^2, then the link of any vertex is a cycle. Furthermore, $k[\Delta_G]$ is Cohen-Macaulay if and only if $\mathrm{char}(k) \neq 2$.

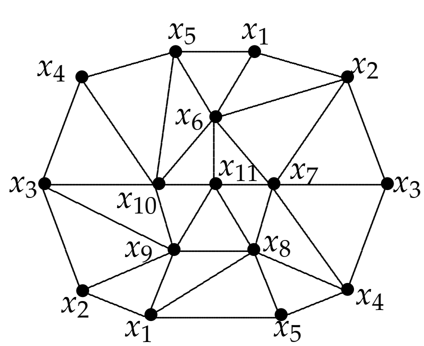


Figure 1.2: Δ_G is a triangulation of P^2.

1.6 THETA-RING GRAPHS

Definition 1.130 Two graphs are *disjoint* if they have no vertex in common. The *union* of the graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Definition 1.131 Let A and B be connected subgraphs of G such that $G = A \cup B$ and $A \cap B$ is a complete graph, then G is called the clique-sum of A and B. In this case, G is denoted by $A \oplus B$. Furthermore, if $|A \cap B| = k$, then G is called *k-clique-sum* of A and B. Let $A_1, ..., A_r$ be subgraphs, the graph $(((A_1 \oplus A_2) \oplus A_3) \cdots) \oplus A_r$ is denoted by $A_1 \oplus A_2 \oplus \cdots \oplus A_r$. The 0-clique-sum is equivalent to the union of two disjoint graphs.

Definition 1.132 [25] A *chorded-theta* T of G is a subgraph induced by three paths $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ each between non adjacent vertices x and y such that $V(\mathcal{L}_i) \cap V(\mathcal{L}_j) = \{x, y\}$ for $1 \leq i < j \leq 3$. The edges of T do not belong to $E(\mathcal{L}_1) \cup E(\mathcal{L}_2) \cup E(\mathcal{L}_3)$ are called the chords of T. A chorded-theta without chords is called a theta graph.

Definition 1.133 Let T be a chorded-theta of G with $\mathcal{L}_1(T) = \{x, x_1, ..., x_{r_1}, y\}$, $\mathcal{L}_2(T) = \{x, y_1, ..., y_{r_2}, y\}$ and $\mathcal{L}_3(T) = \{x, z_1, ..., z_{r_3}, y\}$. A *tranversal triangle* H of T is a triangle in G such that $V(H) = \{x_i, y_j, z_k\}$ for some i, j, k.

Definition 1.134 [25] A graph G is called a *theta-ring graph* if every chorded-theta of G has a transversal triangle. In this case we say that G has the $\forall\theta\exists\Delta$-property.

Remark 1.135 An induced subgraph of a theta-ring graph is a theta-ring graph.

Definition 1.136 A *partial wheel* W is a graph where $V(W) = \{z, z_1, ..., z_k\}$ such that $C = (z_1, ..., z_k)$ is a cycle in W and the edges of W are the edges of C and some edges between z and vertices of C. A partial wheel W is a θ-partial wheel if W is chorded-theta.

Definition 1.137 A *prism* is a graph consisting of two disjoint triangles $C_1 = (x_1, x_2, x_3)$ and $C_2 = (y_1, y_2, y_3)$, and three paths L_1, L_2, L_3 pairwise disjoint, such that each L_i is a path between x_i and y_i for $i = 1, 2, 3$, and the subgraph induced by $V(L_i) \cup V(L_j)$ is a cycle for $1 \leq i < j \leq 3$. A *pyramid* is a graph consisting of a vertex w, a triangle $C = (z_1, z_2, z_3)$, and three paths P_1, P_2, P_3, such that: P_i is a path whose ends are w and z_i for $i = 1, 2, 3$; $V(P_i) \cap V(P_j) = \{w\}$; the subgraph induced by $V(P_i) \cup V(P_j)$ is a cycle for $1 \leq i < j \leq 3$; and at most one of the P_1, P_2, P_3 has only one edge.

Definition 1.138 An orientation $\mathcal{O}$ of the edges of G is an assignment of a direction to each edge of G. Let $G_{\mathcal{O}}$ denote the oriented graph associated to an orientation $\mathcal{O}$ of the edges of G. To each oriented edge $e = (x_i, x_j)$ of $D = G_{\mathcal{O}}$, we associate the vector $v_e \in \{0, 1, -1\}^n$ defined as follows: the ith entry is -1, the jth entry is 1, and the remaining entries are zero. If $v_1, ..., v_q$ are vectors associated to the oriented edges of D, then the *edge subring* of D is $k[D] := k[x^{v_1}, ..., x^{v_q}] \subset k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, where $v_i = (v_i^1, ..., v_i^n)$ and $x^{v_i} = x_1^{v_i^1} \cdots x_n^{v_i^n}$. Let $E(D) = \{t_1, ..., t_q\}$ be the edge set of D. There is an epimorphism of k-algebras given by

$$\varphi : k[t_1, ..., t_q] \to k[D], \text{ where } t_i \to x^{v_i}.$$

The kernel of φ, denoted P_D, is called the *toric ideal* of D. If P_D can be generated exactly by $q - n + r$ binomials it is called a *binomial complete intersection*, where $n = |V(G)|$, $q = |E(G)|$ and r is the number of connected components of D.

Definition 1.139 [25] G is $CI\mathcal{O}$ if the toric ideal $P_{G_{\mathcal{O}}}$ is a binomial complete intersection for each orientation $\mathcal{O}$ of G.

Theorem 1.140 ($\forall\theta\exists\Delta$-Theorem) The following conditions are equivalent:

(i) G is a theta-ring graph.

(ii) G is $CI\mathcal{O}$.

(iii) G can be constructed by 0, 1, 2-clique-sums of chordal graphs and/or cycles.

(iv) G can be constructed by clique-sums of complete graphs and/or cycles.

(v) G does not contain as induced subgraph any graph from the following families: θ-partial wheels, prisms, pyramids and thetas.

Proof. See ([25], Theorem 4) and ([25], Theorem 6). □

Definition 1.141 A graph G is a *block graph* if every block of G is a complete graph. G is a *cactus graph* if G is connected and any two cycles have at most one vertex in common. Finally, G is called *block-cactus* if every block is a complete graph or a cycle.

Definition 1.142 A graph G is a *ring graph* if and only if each block of G, which is not a bridge or a vertex, can be constructed by 2-clique-sums of cycles.

We have the following relations between some families of graphs:

$$\begin{array}{ccccc} \text{Ring} & \subset & \text{Theta-ring} & \supset & \text{Chordal} \\ \cup & & \cup & & \\ \text{Cactus} & \subset & \text{Block-Cactus} & & \\ & & \cup & & \\ & & \text{Block} & & \end{array}$$

CHAPTER 2

WELL-COVERED, VERTEX DECOMPOSABLE AND COHEN-MACAULAY GRAPHS

2.1 INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V(G) = \{x_1, ..., x_n\}$. Let $I = I(G)$ be the edge ideal associated to G in a polynomial ring $R = k[x_1, ..., x_n]$ over a field k, and let Δ_G be the simplicial complex of stable sets of G. In this Chapter we study the vertex decomposability, shellability, Cohen-Macaulayness and well-coveredness in some families of graphs. In general, we have the following implications:

$$\begin{array}{c}\text{Unmixed}\\ \text{vertex decomposable}\end{array} \Rightarrow \begin{array}{c}\text{Pure}\\ \text{shellable}\end{array} \Rightarrow \text{Cohen-Macaulay} \Rightarrow \text{Well-covered}$$

In this Chapter, we give a combinatorial description of the Cohen-Macaulay property for graphs without 3-cycles and 5-cycles, König and unicyclic graphs. In these cases, we prove that the following properties are equivalent: G is unmixed vertex decomposable, Δ_G is pure shellable and R/I is Cohen-Macaulay. Furthermore, we give necessary and sufficient conditions that characterizes when theta-ring graphs is well-covered. We give a characterization for unmixed vertex decomposable graphs whose 5-cycles have at least four chords. Also, we give a new proof of the characterization of well-covered graph without 3-cycles, 5-cycles and 7-cycles. Finally, we study some blocks of well-covered graphs.

The structure of this Chapter is as follows: In Section 2.2, we give some properties and relations between critical, extendable and shedding vertices. In Section 2.3, we prove that all well-covered graphs without 3-cycles, 5-cycles and 7-cycles are König. Furthermore, we give the characterization of these graphs (Theorem 2.12). Through our results we obtain the following contentions:

well-covered König	$\supset$	well-covered $(C_3, C_5, C_7) -$ free	$\supset$	well-covered bipartite
$\shortparallel$				
very well-covered				

In section 2.4, inspired by the classification of Cohen-Macaulay bipartite graphs given by Herzog and Hibi in [27], we study the well-covered König graphs. In Theorem 2.25 and Theorem 2.28 we prove that if G is König or G does not contain 3-cycles and 5-cycles, then the following condition are equivalent: G is unmixed vertex decomposable, Δ_G is pure shellable and $R/I(G)$ is Cohen-Macaulay. Also, we show that these properties are equivalent to the following condition: G is unmixed with a perfect matching $e_1, ..., e_g$ of König type and G does not have 4-cycles with two e_i's. Using this result, we obtain that if the girth of G is at least 6, then the following properties are equivalent: unmixed vertex decomposability, unmixed König property, very well-coveredness, unmixedness with $G \neq C_7$ and G is a whisker graph. In Section 2.5 we characterize the vertex decomposable (non-pure) graphs without 3-cycles and 5-cycles and the shellable graphs with girth at least 11. Furthermore, we show that all 2-connected blocks of a vertex decomposable, shellable or (sequentially) Cohen-Macaulay graph without 3-cycles and 5-cycles have a free vertex in its neighborhood. In Section 2.6 we give a structural description of unicyclic graphs with each one of the following properties: vertex decomposable, shellable, Cohen-Macaulay and well-covered. In Section 2.7, we characterize well-covered theta-ring graphs. In Section 2.8, we give a characterization of unmixed decomposable graphs whose 5-cycles are chorded. Finally, in Section 2.9 we study when a 5-cycle or a 7-cycle are blocks of a well-covered graph.

2.2 CRITICAL, EXTENDABLE AND SHEDDING VERTICES

In this section we study some properties of extendable and shedding vertices. These vertices are defined in Chapter 1.

Lemma 2.1 If x is a vertex of G, then x is a shedding vertex if and only if $|N_G(x) \setminus N_G(S)| \geq 1$ for every stable set S of $G \setminus N_G[x]$.

Proof. $\Rightarrow$) We take a stable set S of $G \setminus N_G[x]$. Since x is a shedding vertex, then there is a vertex $z \in N_G(x)$ such that $S \cup \{z\}$ is stable set of $G \setminus x$. Thus, $z \notin N_G[S]$. Therefore, $|N_G(x) \setminus N_G(S)| \geq 1$.

$\Leftarrow$) We take a stable set S of $G \setminus N_G[x]$. Thus, there exists a vertex $z \in N_G(x) \setminus N_G(S)$. Since $z \in N_G(x)$, we have that $z \notin S$. Furthermore, $z \notin N_G(S)$, then $S \cup \{z\}$ is a stable set of $G \setminus x$. Consequently, S is not a maximal stable set of $G \setminus x$. Therefore, x is a shedding vertex. □

Consequently, x is not a shedding vertex if and only if there exists a stable set S of $G \setminus N_G[x]$ such that $N_G(x) \subseteq N_G(S)$, i.e. x is an isolated vertex in $G \setminus N_G[S]$.

Corollary 2.2 Let S be a stable set of G. If S isolates x in G, then x is not a shedding vertex in $G \setminus N_G[y]$ for all $y \in S$.

Proof. Since S isolates x, then $\deg_{G \setminus N_G[S]}(x) = 0$ and in particular $x \in V(G \setminus N_G[S])$. Thus, $N_G(x) \subseteq N_G[S] \setminus S$. Hence, if $y \in S$ and $G' = G \setminus N_G[y]$, then $x \in V(G')$. Furthermore, since $S \cap N_G[x] = \emptyset$, then $S' = S \setminus y$ is a stable set in $G' \setminus N_{G'}[x]$. Now, since S isolates x, thus if $a \in N_{G'}(x)$, then there exists $s \in S$ such that $\{a, s\} \in E(G)$. But $a \in N_{G'}(x)$, then $a \notin N_G[y]$, consequently $s \in S'$ and $\{a, s\} \in E(G')$. This implies $|N_{G'}(x) \setminus N_{G'}(S')| = 0$. Therefore, by Lemma 2.1, x is not a shedding vertex in G'. □

Lemma 2.3 If x is a shedding vertex of G, then one of the following conditions hold:

(a) There is $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$.

(b) x is in a 5-cycle with at most one chord.

Proof. We take $N_G(x) = \{y_1, y_2, ..., y_k\}$. If G does not satisfy (a), then there is $\{z_1, ..., z_k\} \subseteq V(G) \setminus N_G[x]$ such that $\{y_i, z_i\} \in E(G)$ for $i \in \{1, ..., k\}$. We denote by $L = \{z_1, ..., z_q\} = \{z_1, ..., z_k\}$ and suppose that $z_i \neq z_j$ for $1 \leq i < j \leq q$. By Lemma 2.1, if L is a stable set of G, then $|N_G(x) \setminus N_G(L)| \geq 1$. But $N_G(x) = \{y_1, ..., y_k\} \subseteq N_G(L)$, then L is not a stable set. Hence, $q \geq 2$ and there exist $z_{i_1}, z_{i_2} \in L$ such that $\{z_{i_1}, z_{i_2}\} \in E(G)$. Thus, there exist y_{j_1} and y_{j_2} such that $y_{j_1} \neq y_{j_2}$ and $\{y_{j_1}, z_{i_1}\}, \{y_{j_2}, z_{i_2}\} \in E(G)$. Furthermore, $\{z_{i_1}, y_{j_2}\}, \{z_{i_2}, y_{j_1}\}, \{z_{i_1}, x\}, \{z_{i_2}, x\} \notin E(G)$. Therefore, $(x, y_{j_1}, z_{i_1}, z_{i_2}, y_{j_2})$ is a 5-cycle of G with at most one chord. □

Corollary 2.4 Let G be graph without 4-cycles. If x is a shedding vertex of G, then x is in a 5-cycle or there exists a simplicial vertex z such that $\{x, z\} \in E(G)$ with $|N_G[z]| \leq 3$.

Proof. By Lemma 2.3, if x is not in a 5-cycle, then there is $z \in N_G(x)$ such that $N_G[z] \subseteq N_G[x]$. If $\deg_G(z) = 1$, then z is a simplicial vertex. If $\deg_G(z) = 2$, then $N_G(z) = \{x, w\}$. Consequently, (z, x, w) is a 3-cycle since $N_G[z] \subseteq N_G[x]$. Thus, z is a simplicial vertex. Now, if $\deg_G(z) \geq 3$, then there are $w_1, w_2 \in N_G(z) \setminus x$. Since $N_G[z] \subseteq N_G[x]$, we have that (w_1, z, w_2, x) is a 4-cycle of G. This is a contradiction. Therefore, $|N_G[z]| \leq 3$ and z is a simplicial vertex. □

Remark 2.5 If G is a 5-cycle with $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$, then each x_i is a shedding vertex.

Proof. We can assume that $i = 1$, then $\{x_3\}$ and $\{x_4\}$ are the stable sets in $G \setminus N_G[x_1]$. Furthermore, $\{x_3, x_5\}$ and $\{x_2, x_4\}$ are stable sets in $G \setminus x_1$. Hence, each stable set of $G \setminus N_G[x_1]$ is not a maximal stable set in $G \setminus x_1$. Therefore, x_1 is a shedding vertex. □

Definition 2.6 A vertex v of G is *critical* if $\tau(G \setminus v) < \tau(G)$. Furthermore, G is called a *vertex critical graph* if each vertex of G is critical.

Remark 2.7 If $\tau(G \setminus v) < \tau(G)$, then $\tau(G) = \tau(G \setminus v) + 1$. Moreover, v is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$.

Proof. If t is a minimal vertex cover such that $|t| = \tau(G \setminus v)$, then $t \cup \{v\}$ is a vertex cover of G. Thus, $\tau(G) \leq |t \cup \{v\}| = \tau(G \setminus v) + 1$. Consequently, if $\tau(G) > \tau(G \setminus v)$, then $\tau(G) = \tau(G \setminus v) + 1$.

Now, we have that $\tau(G) + \beta(G) = |V(G)| = |V(G \setminus v)| + 1 = \tau(G \setminus v) + \beta(G \setminus v) + 1$. Hence, $\beta(G) = \beta(G \setminus v)$ if and only if $\tau(G) = \tau(G \setminus v) + 1$. Therefore, v is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$. □

Corollary 2.8 Let G be an unmixed graph and $x \in V(G)$. The following conditions are equivalent:

(a) x is an extendable vertex.

(b) $|N_G(x) \setminus N_G(S)| \geq 1$ for every stable set S of $G \setminus N_G[x]$.

(c) x is a shedding vertex.

(d) x is a critical vertex and $G \setminus x$ is unmixed.

Proof. (a)⇔(b) By Lemma 1.105.

(b)⇔(c) By Lemma 2.1.

(a)⇔(d) Since G is unmixed, then by Remark 2.7, x is extendable if and only if x is a critical vertex and $G \setminus x$ is unmixed. □

2.3 WELL-COVERED GRAPHS WITHOUT 3-CYCLES, 5-CYCLES AND 7-CYCLES

The properties presented in section 2.1 have implications for well-covered graphs without 3-cycles, 5-cycles and 7-cycles. It is known that 3-cycles, 4-cycles, 5-cycles and 7-cycles are the well-covered cycles, but only the 4-cycles have a perfect matching. In this section, we study some properties of well-covered graphs and we give a new proof of Theorem 1.108.

In this thesis we denote by Z_G the set of isolated vertices of G, that is,

$$Z_G = \{x \in V(G) \mid \deg_G(x) = 0\}.$$

Proposition 2.9 Let G be a König graph and $G' = G \setminus Z_G$. Then the following are equivalent:

(a) G is unmixed.

(b) G' is unmixed.

(c) If $V(G') \neq \emptyset$, then G' has a perfect matching $e_1, ..., e_g$ of König type such that for any two edges $f_1 \neq f_2$ and for two distinct vertices $x \in f_1$, $y \in f_2$ contained in some e_i, one has that $(f_1 \setminus x) \cup (f_2 \setminus y)$ is an edge.

Proof. (a)⇔(b) Since $V(G) \setminus V(G') = Z_G$, then C is a vertex cover of G if and only if C is a vertex cover of G'. Therefore, G is unmixed if and only if G' is unmixed.

(b)⇔(c) By Lemma 1.55 and Proposition 1.56. □

Lemma 2.10 G is an unmixed König graph if and only if G is totally disconnected or $G' = G \setminus Z_G$ is very well-covered.

Proof. ⇒) If G is not totally disconnected, then from Proposition 2.9, G' has a perfect matching $e_1, ..., e_g$ of König type. Hence, $|V(G')| = 2g = 2\tau(G') = 2\text{ht}(I(G'))$. Furthermore, G' is unmixed, therefore G' is very well-covered.

⇐) If G is totally disconnected, then $\nu(G) = 0$ and $\tau(G) = 0$. Hence, G is an unmixed König graph. Now, if G is not totally disconnected, then G' is very well-covered. Consequently, by Remark 1.109 G' has a perfect matching. Thus, $\nu(G') = |V(G')|/2 = \text{ht}(I(G')) = \tau(G')$. Hence, G' is König. Furthermore, $\nu(G) = \nu(G')$ and $\tau(G) = \tau(G')$, then G is König. Finally, since G' is unmixed, by Proposition 2.9, G is also unmixed. □

Lemma 2.11 If G is a well-covered graph without 3-cycles, 5-cycles and 7-cycles, then G is a König graph.

Proof. By induction on $|V(G)|$. If $y \in Z_G$, then by induction hypothesis, $G \setminus y$ is König. This implies that G is König. Therefore, we can assume $Z_G = \emptyset$. Now, we take $x \in V(G)$, then by Remark 1.66, $G_1 = G \setminus N_G[x]$ is a well-covered graph. Also, G_1 does not contain 3-cycles, 5-cycles and 7-cycles, so by induction hypothesis, G_1 is König. If $V(G_2) = \emptyset$ with $G_2 = G_1 \setminus Z_{G_1}$, then $V(G) = N_G[x] \cup Z_{G_1}$. Furthermore, $\{x\} \cup Z_{G_1}$ and $N_G(x)$ are stable sets since G does not have 3-cycles. Thus, G is bipartite and consequently G is König. Hence, we can assume that $V(G_2) \neq \emptyset$. By Proposition 2.9, G_2 has a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type. We can assume that $N_G(x) = \{z_1, ..., z_r\}$ and $D = \{x_1, ..., x_g\}$ is a minimal vertex cover of G_2. This implies that $F = \{y_1, ..., y_g\}$ is a maximal stable set of G_2. Now, we take the subsets: $A_1 = N_G(z_1, ..., z_r) \cap D$, $B_1 = \{y_j \in F \mid x_j \in A_1\}$, $B_2 = N_G(z_1, ..., z_r) \cap F$ and $A_2 = \{x_j \in D \mid y_j \in B_2\}$. If there exists $y_i \in B_1 \cap B_2$, then $x_i \in A_1$ and there exist $z_k, z_p \in N_G(x)$ such that $\{x_i, z_k\}, \{y_i, z_p\} \in E(G)$. If $k = p$, then (z_k, x_i, y_i) is a 3-cycle and if $k \neq p$, then (x, z_k, x_i, y_i, z_p) is a 5-cycle, this is a contradiction. Consequently $B_1 \cap B_2 = \emptyset$. Now, we take the subsets $B_3 = (N_G(A_2) \cap F) \setminus B_2$, $A_3 = \{x_j \in D \mid y_j \in B_3\}$, $B_4 = (N_G(A_1) \cap F) \setminus B_1$

and $A_4 = \{x_j \in D \mid y_j \in B_4\}$. If $y_i \in B_1 \cap B_3$, then there exist $x_j \in A_2$ and $z_k, z_p \in N_G(x)$ such that $\{x_i, z_k\}, \{y_i, x_j\}$ and $\{y_j, z_p\} \in E(G)$. Hence, if $k = p$, then $(z_k, x_i, y_i, x_j, y_j)$ is a 5-cycle and if $k \neq p$ implies $(x, z_k, x_i, y_i, x_j, y_j, z_p)$ is a 7-cycle, a contradiction. So $B_1 \cap B_3 = \emptyset$. Now, if $y_i \in B_2 \cap B_4$, then there exist $x_j \in A_1$ and $z_k, z_p \in N_G(x)$ such that $\{x_j, z_k\}, \{x_j, y_i\}$ and $\{y_i, z_p\} \in E(G)$. Consequently, if $k = p$, thus (z_k, x_j, y_i) is a 3-cycle and if $k \neq p$, then (x, z_k, x_j, y_i, z_p) is a 5-cycle, this is a contradiction. Hence $B_2 \cap B_4 = \emptyset$. Now, if $y_i \in B_3 \cap B_4$, then there exist $x_j \in A_1$, $x_q \in A_2$ and $z_k, z_p \in N_G(x)$ such that $\{x_j, y_i\}$, $\{x_q, y_i\}$, $\{x_j, z_k\}$ and $\{y_q, z_p\} \in E(G)$. Thus, if $k = p$ we have that $(z_k, x_j, y_i, x_q, y_q)$ is a 5-cycle and if $k \neq p$, then $(x, z_k, x_j, y_i, x_q, y_q, z_p)$ is a 7-cycle. This implies $B_3 \cap B_4 = \emptyset$. Therefore, B_1, B_2, B_3, B_4, B_5 are pairwise disjoint sets, where $B_5 = F \setminus (B_1 \cup B_2 \cup B_3 \cup B_4)$.

Now, we will prove that if $A' = A_1 \cup A_4 \cup A_5$ and $B' = B_1 \cup B_4 \cup B_5$ where $A_5 = \{x_j \in D \mid y_j \in B_5\}$, then $N_G(B') \subseteq A'$. Since $B_1 \cap B_3 = B_4 \cap B_3 = B_5 \cap B_3 = \emptyset$, we have $N_G(B_1) \cap A_2 = N_G(B_4) \cap A_2 = N_G(B_5) \cap A_2 = \emptyset$. This implies $N_G(B') \cap A_2 = \emptyset$. Furthermore, if $x_i \in N_G(B') \cap A_3$, then there exist $y_j \in B'$ and $x_q \in A_2$ such that $\{x_i, y_j\}, \{x_q, y_i\} \in E(G)$. Since G_1 is well-covered and by Proposition 2.9, $\{x_q, y_j\} = (\{x_q, y_i\} \setminus y_i) \cup (\{x_i, y_j\} \setminus x_i) \in E(G)$. Consequently, $y_j \in B_3 \cap B'$, this is a contradiction. Thus, $N_G(B') \cap A_3 = \emptyset$. Moreover, we have that $N_G(Z_{G_1}) \subseteq N_G(x)$. Therefore, $N_G(B') = A'$.

This implies, $G_3 = G \setminus N_G[B'] = G \setminus (A' \cup B')$. But, B' is a stable set, then G_3 is well-covered without 3-cycles, 5-cycles and 7-cycles. If $G \neq G_3$, then by induction hypothesis, G_3 is König implying $\tau(G_3) = \nu(G_3)$. Furthermore, since $N_G(B') = A'$, if D' is a minimal vertex cover of G_3, then $D' \cup A'$ is a vertex cover of G. Also, $G[A' \cup B']$ has a perfect matching with $|A'|$ elements. Consequently, $\tau(G) \leq \tau(G_3) + |A'| = \nu(G_3) + |A'| \leq \nu(G)$. Hence, $\tau(G) = \nu(G)$ and G is König. Therefore, we can assume $G = G_3$.

Now, if there exist $x_i, x_j \in A_2$ such that $\{x_i, x_j\} \in E(G)$, then there exist $z_k, z_p \in N_G(x)$ such that $\{y_i, z_k\}$ and $\{y_j, z_p\} \in E(G)$. If $k = p$, then $(z_k, y_i, x_i, x_j, y_j)$ is a 5-cycle and if $k \neq p$, then $(x, z_k, y_i, x_i, x_j, y_j, z_p)$ is a 7-cycle, a contradiction. So A_2 is a stable set. Similarly, if $\{x_i, x_j\} \in E(G)$ with $x_i \in A_2$ and $x_j \in A_3$, then there exists $x_q \in A_2$ such that $\{x_q, y_j\} \in E(G)$. If $q = i$, then (x_i, x_j, y_j) is a 3-cycle. Thus, $q \neq i$. Since G_1 is well-covered and by Proposition 2.9, we have that $\{x_i, x_q\} = (\{x_i, x_j\} \setminus x_j) \cup (\{x_q, y_j\} \setminus y_j) \in E(G_1)$. This is a contradiction since A_2 is a stable set of G. Consequently, there are no edges between A_2 and

A_3. Finally, if $\{x_i, x_j\} \in E(G)$ with $x_i, x_j \in A_3$, then there is a vertex $x_q \in A_2$ such that $\{x_q, y_i\} \in E(G)$. Since G_1 is well-covered, then $\{x_j, x_q\} = (\{x_i, x_j\} \setminus x_i) \cup (\{x_q, y_i\} \setminus y_i) \in E(G)$. But, there are no edges between A_2 and A_3. Hence, A_3 is a stable set, implying $A_2 \cup A_3$ is also a stable set in G. Furthermore, since $N_G(Z_{G_1}) \subseteq N_G(x)$, we have that $\{x\} \cup Z_{G_1} \cup B_2 \cup B_3$ and $N_G(x) \cup A_2 \cup A_3$ are stable sets. Therefore G is bipartite, implying that G is a König graph. □

Theorem 2.12 Let G be a graph without 3-cycles, 5-cycles and 7-cycles. If $G' = G \setminus Z_G$, then the following conditions are equivalent:

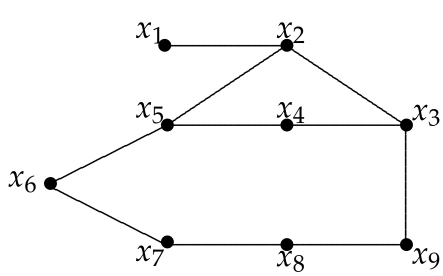
(1) G is well-covered.

(2) If $V(G') \neq \emptyset$, then G' has a perfect matching $e_1, ..., e_g$ of König type such that for any two edges $f_1 \neq f_2$ and for two distinct vertices $a \in f_1$, $b \in f_2$ with $\{a, b\} = e_i$, one has that $(f_1 \setminus a) \cup (f_2 \setminus b) \in E(G)$.

Proof. (1) ⇒ (2) By Lemma 2.11, we have that G is König. Furthermore, G is well-covered. Hence, by Proposition 2.9, G satisfies (2).

(2) ⇒ (1) By Proposition 2.9, G is well-covered. □

Example 2.13 Let G be the following graph:



$$I_G = (x_1x_2, x_2x_3, x_2x_5, x_3x_4, x_3x_9, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_8x_9)$$

Figure 2.1: Well-covered graph without a perfect matching.

G is a well-covered graph without 3-cycles and 5-cycles. If G is König, then by Proposition 2.9, G has a perfect matching. But $|V(G)| = 9$ and G does not have a perfect matching, therefore, G is not König

Lemma 2.14 If G is an unmixed graph and $x \in V(G)$, then $N_G(x)$ does not contain two free vertices.

Proof. We suppose that there exists $x \in V(G)$ such that $y_1, ..., y_s$ are free vertices in $N_G(x)$. Hence, $G_1 = G \setminus N_G[y_1, ..., y_s] = G \setminus \{x, y_1, ..., y_s\}$ is unmixed. Now, we take a maximal stable set S of G_1. Thus, $|S| = \beta(G_1)$ since G_1 is unmixed. Consequently, $S_1 = S \cup \{y_1, ..., y_s\}$ is a stable set in G. We take S_2 a maximal stable in G such that $x \in S_2$. Since G is unmixed, we have that $|S_2| \geq |S_1| = |S| + s$. Furthermore, $S_2 \setminus x$ is a stable set in G_1, then $|S_2| \leq \beta(G_1) + 1$. This implies $\beta(G_1) + \geq |S| + s$. But $|S| = \beta(G_1)$, therefore $s \leq 1$. $\square$

Definition 2.15 If $v, w \in V(G)$, then the *distance* $d(u, v)$ between u and v in G is the length of the shortest path joining them, otherwise $d(u, v) = \infty$. Now, if $H \subseteq G$, then the distance from a vertex v to H is $d(v, H) = \min\{d(v, u) \mid u \in V(H)\}$. Furthermore, if $W \subseteq V(G)$, then we define $d(v, W) = d(v, G[W])$ and $D_i(W) = \{v \in V(G) \mid d(v, W) = i\}$.

Proposition 2.16 Let G be an unmixed connected graph without 3-cycles and 5-cycles. If C is a 7-cycle and H is a c-minor of G with $C \subseteq H$ such that C has three non adjacent vertices of degree 2 in H, then C is a c-minor of G.

Proof. We take a minimal c-minor H of G such that $C \subseteq H$ and C has three non adjacent vertices of degree 2 in H. We can suppose that $C = (x, z_1, w_1, a, b, w_2, z_2)$ with $\deg_H(x) = \deg_H(w_1) = \deg_H(w_2) = 2$. If $\{z_1, b\} \in E(H)$, then (z_1, b, w_2, z_2, x) is a 5-cycle of G. Thus, $\{z_1, b\} \notin E(H)$, similarly $\{z_2, a\} \notin E(H)$. Furthermore, since G does not have 3-cycles, then $\{z_1, z_2\}, \{z_1, a\}, \{z_2, b\} \notin E(H)$. Hence, C is an induced cycle in H. On the other hand, if there exists $v \in V(H)$ such that $d(v, C) \geq 2$, then $H' = H \setminus N_G[v]$ is a c-minor of G and $C \subseteq H' \subset H$. This is a contradiction by the minimality of H. Therefore, $d(v, C) \leq 1$ for each $v \in V(H)$.

Now, if $\deg_H(b) \geq 3$, then there exists $c \in V(H) \setminus V(C)$ such that $\{b, c\} \in E(H)$. If $\{c, z_2\} \notin E(G)$ implies that $N_{H_1}(z_2)$ has two free vertices, w_2 and x, in $H_1 = H \setminus N_H[w_1, c]$, this is a contradiction by Lemma 2.14. Thus $\{c, z_2\} \in E(H)$. Furthermore, $\{a, c\}, \{z_1, c\} \notin E(H)$ since (a, b, c) and (z_1, w_1, a, b, c) are not cycles in G. Hence, if $\deg_H(c) \geq 3$, then there exists $d \in V(H) \setminus V(C)$ such that $\{c, d\} \in E(H)$. Also, $\{d, b\}, \{d, z_2\}, \{d, z_1\} \notin E(H)$ since (c, b, d), (z_2, d, c) and (z_1, x, z_2, c, d) are not cycles of G. But $d(d, C) \leq 1$, so $\{a, d\} \in E(H)$. Consequently, $N_{H_2}(z_1)$ has two free vertices, w_1 and x, in $H_2 = H \setminus N_H[d, w_2]$, a contradiction by Lemma 2.14, then $\deg_H(c) = 2$. This implies, $N_{H_3}(z_2)$ has two free vertices, w_2 and c, in $H_3 = H \setminus N_H[a]$. This is not possible, therefore $\deg_H(b) = 2$. Similarly, $\deg_H(a) = 2$.

Now, if $\deg_H(z_2) \geq 3$ we have that there exists $c' \in V(H) \setminus V(C)$ such that $\{c', z_2\} \in E(H)$. If there exists $d' \in V(H) \setminus V(C)$ such that $\{c', d'\} \in E(H)$, then $\{d', z_1\}$ or $\{d', z_2\} \in E(G)$, since $d(d', C) \leq 1$. But (c', d', z_2) and (x, z_2, c', d', z_1) are not cycles of H, thus, $N_H(c') \subseteq \{z_1, z_2\}$. Consequently, $N_{H_4}(z_2)$ has two free vertices, x and c', in $H_4 = H \setminus N_H[w_1]$, a contradiction. Hence $\deg_H(z_2) = 2$. Similarly, $\deg_H(z_1) = 2$. Furthermore, since H is minimal, then it is connected. Therefore, $H = C$ and C is a c-minor of G. □

2.4 KÖNIG AND COHEN-MACAULAY GRAPHS WITHOUT 3-CYCLES AND 5-CYCLES

It is known that if each chordless cycle of G has length 3 or 5, then G is vertex decomposable (see Theorem 1.96). Furthermore, both a 3-cycle and a 5-cycle are Cohen-Macaulay, but they do not have a perfect matching. In this section we characterize the Cohen-Macaulay property in König graphs and graphs without 3-cycles and 5-cycles. In particular, we prove that these graphs have a perfect matching.

Lemma 2.17 Let G be a graph such that $\{z_1, ..., z_r\}$ is a stable set. If $N = \bigcup_{i=1}^{r} N_G[z_i]$ and $G^{i+1} = G^i \setminus N_{G^i}[z_i]$ with $G^1 = G$, then:

(a) $\{z_{i+1}, ..., z_r\} \subseteq V(G^{i+1})$ and

(b) $G^{r+1} = G \setminus N$.

Proof. (a) By induction on i. If $i = 1$, since $\{z_1, ..., z_r\}$ is a stable set, then $\{z_2, ..., z_r\}$ $\cap N_G[z_1] = \emptyset$ and $\{z_2, ..., z_r\} \subseteq V(G \setminus N_G[z_1]) = V(G^2)$. Now, by induction hypothesis we have that $\{z_i, z_{i+1}, ..., z_r\} \subseteq V(G^i)$. Since $\{z_i, ..., z_r\}$ is a stable set, then $\{z_{i+1}, ..., z_r\} \cap N_{G^i}[z_i] = \emptyset$. Hence, $\{z_{i+1}, ..., z_r\} \subseteq V(G^i \setminus N_{G^i}[z_i]) = V(G^{i+1})$.

(b) By induction on r. If $r = 1$, then $N = N_G[z_1]$ and $G^2 = G \setminus N_G[z_1] = G \setminus N$. Now, if $r \geq 2$, we take $N' = \bigcup_{i=1}^{r-1} N_G[z_i]$ and by induction hypothesis we have that $G^r = G \setminus N'$. Through (a), we have that $\{z_r\} \subseteq V(G^r)$. We will prove that $N' \cup N_{G^r}[z_r] = N$. We have that $N' \subseteq N$. Furthermore, if $y \in N_{G^r}[z_r]$, then $y \in N_G[z_r]$. Consequently, $y \in N$. Now, if $y \in N \setminus N'$, then $y \in N_G[z_r]$ and

$y \notin N_G[z_i]$ for $i \in \{1, ..., r-1\}$. This implies that $y, z_r \in V(G^r)$ and $\{y, z_r\} \in E(G^r)$. Thus, $y \in N_{G^r}[z_r]$. Hence, $N = N^{'} \cup N_{G^r}[z_r]$. Therefore, $G^{r+1} = G^r \setminus N_{G^r}[z_r] = (G \setminus N') \setminus N_{G^r}[z_r] = G \setminus (N' \cup N_{G^r}[z_r]) = G \setminus N$. □

Proposition 2.18 Let G be an unmixed König graph with a perfect matching $e_1 = \{x_1, y_1\}, \ldots, e_s = \{x_s, y_s\}$. Hence, $\beta(G) = s$. Furthermore, if $\{x_i, z\}, \{y_i, z'\} \in E(G)$ for some $1 \leq i \leq s$, then $\{z, z'\} \in E(G)$

Proof. Since $e_1, ..., e_s$ is a perfect matching and G is König, then $s = \nu(G) = \tau(G)$. Consequently, $\beta(G) = |V(G)| - \tau(G) = s$. Now, we suppose that $\{z, z'\} \notin E(G)$. Thus, there exists a maximal stable set S such that $z, z' \in S$. We have that $|S \cap e_j| \leq 1$ for $j = 1, ..., s$, but $|S \cap e_i| = 0$ since $\{x_i, y_i\} \subseteq N_G(z, z') \subseteq N_G(S)$. This implies $|S| \leq s - 1$. This is a contradiction since G is unmixed. Therefore, $\{z, z'\} \in E(G)$. □

Definition 2.19 Let G be a graph. The *neighborhood relation* in $V(G)$ is the equivalence relation given by:

$$z_1 \sim z_2 \Leftrightarrow N_G(z_1) = N_G(z_2).$$

We denote by $[y]$ the equivalence class of y.

Lemma 2.20 Let G be an unmixed graph with a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type. If $\{x_1, ..., x_g\}$ is a minimal vertex cover, then:

(a) There is no triangle with vertices in $\{x_i, y_i, x_j, y_j\}$ for $1 \leq i < j \leq g$.

(b) If $e = \{y_i, x_j\} \in E(G)$, then $N_G(y_j) \subseteq N_G(y_i)$ for $1 \leq i, j \leq g$.

(c) Let $[y_k]$ be an equivalence class such that $N_G(y_k)$ is minimal. If $N_G(y_k) = \{x_{i_1}, ..., x_{i_l}\}$, then $[y_k] \cap \{y_1, ..., y_g\} = \{y_{i_1}, ..., y_{i_l}\}$ and the induced subgraph $G_1 = G[x_{i_1}, ..., x_{i_l}, y_{i_1}, ..., y_{i_l}]$ is a complete bipartite graph.

(d) Let G_1 be the graph in (c). If $x_s, y_s \notin V(G_1)$, then the only possible edges between $V(G_1)$ and $\{x_s, y_s\}$ are $\{x_{i_j}, x_s\}$ or $\{x_{i_j}, y_s\}$ for $j \in \{1, ..., l\}$. Furthermore, if $\{x_{i_k}, x_s\} \in E(G)$, then $\{x_{i_1}, ..., x_{i_l}\} \subseteq N_G(x_s)$.

Proof. Since $\{x_1, ..., x_g\}$ is a vertex cover, then $A = \{y_1, ..., y_g\}$ is a stable set of G.

(a) We suppose that T is a triangle with $V(T) \subseteq \{x_i, y_i, x_j, y_j\}$. Since A is stable, we can assume $e = \{x_i, x_j\}, e_i = \{x_i, y_i\}, e' = \{x_j, y_i\}$ are the edges of T. But, since G is unmixed, then $(e \setminus x_i) \cup (e' \setminus y_i) = \{x_j\}$ contains an edge of G, this is impossible.

(b) If $z \in N_G(y_j)$, then $e' = \{z, y_j\} \in E(G)$. Thus, by Proposition 2.18 $\{y_i, z\} \in E(G)$. Hence $z \in N_G(y_i)$.

(c) If $N_G(y_k) = \{x_k\}$, then we have that G_1 is the complete bipartite graph $K_{1,1}$. Now, if there exists $x_j \in N_G(y_k)$ with $j \neq k$, then $\{y_k, x_j\} \in E(G)$. By (b) we have $N_G(y_j) \subseteq N_G(y_k)$. But, $N_G(y_k)$ is minimal, then $N_G(y_k) = N_G(y_j)$ and $y_j \in [y_k]$. Consequently, $\{y_{i_1}, ..., y_{i_l}\} \subseteq [y_k]$. Furthermore, if $y_i \in [y_k]$, then $x_i \in N_G(y_i) = N_G(y_k)$. Hence, $i \in \{i_1, ..., i_l\}$ implying $[y_k] \cap \{y_1, ..., y_g\} = \{y_{i_1}, ..., y_{i_l}\}$. Now, if $\{x_{i_j}, x_{i_k}\} \in E(G)$, then there exists a triangle whose vertices are in $\{x_{i_j}, y_{i_j}, x_{i_k}, y_{i_k}\}$, a contradiction by (a). Therefore, $\{x_{k_1}, ..., x_{k_l}\}$ is a stable set and $G_1 = G[x_{k_1}, ..., x_{k_l}, y_{k_1}, ..., y_{k_l}]$ is a complete bipartite graph, since $N_G[y_{i_j}] = N_G[y_k] = \{x_{i_1}, ..., x_{i_l}\}$.

(d) Since $\{y_1, ..., y_g\}$ is a stable set, then the only possible edges between $\{x_s, y_s\}$ and $V(G_1)$ are $\{y_{i_j}, x_s\}, \{x_{i_j}, x_s\}$ or $\{x_{i_j}, y_s\}$. But, if $\{y_{i_j}, x_s\} \in E(G)$, then by (b) $N_G(y_s) \subseteq N_G(y_{i_j})$. Thus, $N_G(y_s) = N_G(y_{i_j})$ by the minimality of $N_G(y_{i_j})$. Hence, $y_s \in [y_k]$, a contradiction. Now, assume $\{x_{i_k}, x_s\} \in E(G)$. Since $\{x_{i_k}, y_{i_k}\}, \{x_{i_j}, y_{i_k}\} \in E(G)$, then by Proposition 2.18 $\{x_{i_j}, x_s\} \in E(G)$. Therefore, $\{x_{i_1}, ..., x_{i_l}\} \subseteq N_G(x_s)$. □

The following Proposition generalizes Lemma 1.121.

Proposition 2.21 Let G be a König graph without isolated vertices. If G is Cohen-Macaulay, then G has at least a free vertex.

Proof. If G is a bipartite graph, then by Lemma 1.121, G has a free vertex. Now, we can assume that G is not a bipartite graph. Furthermore, we can assume G is connected, since G has a non bipartite connected component. By Lemma 1.55, there exists a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type. Thus, if D is a minimal vertex cover of G, then $|D \cap e_i| = 1$ for each $i \in \{1, ..., g\}$. Without loss of generality, we can suppose that $D = \{x_1, ..., x_g\}$. Consequently, $\{y_1, ..., y_g\}$ is a stable set of G. We take y_k such that $N_G(y_k)$ is a minimal neighborhood. Hence, by the Lemma 2.20, if $[y_k] \cap \{y_1, ..., y_g\} = \{y_{k_1}, ..., y_{k_l}\}$, then $G_1 = G[x_{k_1}, ..., x_{k_l}, y_{k_1}, ..., y_{k_l}]$ is a complete bipartite graph and $N(y_k) = \{x_{k_1}, ..., x_{k_l}\}$. Since G is not a bipartite graph, then there exists $\{x_s, y_s\} \in E(G)$ such that $x_s, y_s \notin A$, where $A = V(G_1)$. By Lemma 2.20 the only possible edges in G between $\{x_s, y_s\}$ and A are: $\{x_{k_i}, x_s\}$ and $\{x_{k_i}, y_s\}$.

If $\{x_{k_i}, x_s\} \in E(G)$, then by Lemma 2.20 $\{x_{k_1}, ..., x_{k_l}\} \subseteq N_G(x_s)$. Furthermore, since there are no triangles in $G[x_{k_r}, y_{k_r}, x_s, y_s]$, then $\{x_{k_r}, y_s\} \notin E(G)$ for $r \in \{1, ..., l\}$. This implies that $N_G[y_s] \cap A = \emptyset$ since $\{y_1, ..., y_g\}$ is a stable set. We take

$$B_1 = \{y_s \in V(G) \mid x_s, y_s \notin A \text{ and } \{x_{k_1}, ..., x_{k_l}\} \subseteq N_G(x_s)\}.$$

B_1 is a stable since $B_1 \subseteq \{y_1, ..., y_g\}$, then by Remark 1.66, $G_2 = G \setminus N_G[B_1]$ is Cohen-Macaulay and $G_1 \subseteq G_2$. Furthermore, the only possible edges between A and $V(G_2) \setminus A$ are $\{x_{k_i}, y_s\}$. Now, we take

$$B_2 = \{y_s \in V(G_2) \mid x_s, y_s \notin A \text{ and } N_G(y_s) \cap \{x_{k_1}, ..., x_{k_l}\} \neq \emptyset\}.$$

Suppose $\{x_s, x_{s'}\} \in E(G_2)$ with $y_s, y_{s'} \in B_2$. Thus, $\{y_s, x_{k_i}\} \in E(G)$ for some $i \in \{1, ..., l\}$. By Proposition 2.18, $\{x_{s'}, x_{k_i}\} \in E(G)$ implying that $y_{s'} \in B_1$ and $x_{s'} \notin V(G_2)$, a contradiction. Then $\{x_s, x_{s'}\} \notin E(G_2)$. Consequently, if $B_2 = \{y_{u_1}, ..., y_{u'_l}\}$, then $\{x_{u_1}, ..., x_{u'_l}\}$ is a stable set. Hence, by Remark 1.66, $G_3 = G_2 \setminus N_G[x_{u_1}, ..., x_{u'_l}]$ is Cohen-Macaulay. Furthermore, G_1 is a connected component of G_3, then G_1 is a Cohen-Macaulay bipartite subgraph. By Lemma 1.121, G_1 has a free vertex. But G_1 is a complete bipartite graph, therefore $l = 1$ and y_1 is a free vertex. □

Corollary 2.22 Let G be an unmixed shellable graph. If G is König, then G has at least a free vertex.

Proof. If G is unmixed shellable, by Theorem 1.33, G is Cohen-Macaulay. Moreover, by Proposition 2.21, G has at least a free vertex. □

Lemma 2.23 Let G be an unmixed König graph without isolated vertices. If $y \in V(G)$ is a free vertex with $N_G(y) = \{x\}$, then the subgraphs $G \setminus N_G[y]$ and $G \setminus N_G[x]$ have a perfect matching of König type.

Proof. The contractions G/y and G/x are equivalent to the induced subgraphs $G \setminus N[y]$ and $G \setminus N[x]$, respectively. Since G has a perfect matching $e_1, ..., e_g$ of König type and by Proposition 1.59 and Remark 1.60, we have that the subgraphs $G \setminus N[y]$ and $G \setminus N[x]$ have a perfect matching $e'_1, ..., e'_g$ of König type such that $e'_i \subseteq e_i$ for all i. □

Lemma 2.24 Let G be an unmixed König graph with a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$. If G does not have a 4-cycle with two e_i's, then G has at least a free vertex.

Proof. By Proposition 2.18, we have that $\beta(G) = g$. Consequently, if S is a maximal stable set of G, then $|S \cap e_i| = 1$ for each $i \in \{1, ..., g\}$. Hence, we can assume that $\{y_1, ..., y_g\}$ is a stable set. If y_{i_1} is not a free vertex, then there exists a vertex x_{i_2} such that $\{y_{i_1}, x_{i_2}\} \in E(G)$ with $i_1 \neq i_2$. If y_{i_2} is not a free vertex, then there exists a vertex x_{i_3} such that $\{y_{i_2}, x_{i_3}\} \in E(G)$ with $i_2 \neq i_3$. Furthermore, since G does not have a 4-cycle with two $e'_i s$, then $i_3 \neq i_1$. Also, by Proposition 2.18, we have that $\{y_{i_1}, x_{i_3}\} \in E(G)$. Now, we suppose that we have distint vertices $y_{i_1}, y_{i_2}, ..., y_{i_j}$ and $x_{i_1}, x_{i_2}, ..., x_{i_{j+1}}$ such that for each vertex y_{i_l} we have $\{x_{i_k}, y_{i_l}\} \in E(G)$ with $k \in \{l, ..., j+1\}$. If $y_{i_{j+1}}$ is not a free vertex, then there exists $x_{i_{j+2}}$ such that $\{y_{i_{j+1}}, x_{i_{j+2}}\} \in E(G)$. Since there are no 4-cycles with two e_i's, then $i_{j+2} \notin \{i_1, ..., i_{j+1}\}$. Furthermore, by Proposition 2.18, $\{y_{i_k}, x_{i_{j+2}}\}$ for $k = 1, ..., i+1$. This process is finite since $|V(G)| = 2g$, therefore G has a free vertex. □

Theorem 2.25 Let G be a König graph without isolated vertices, then the following conditions are equivalent:

(a) G is unmixed with a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type and G does not have 4-cycles with two e_i's.

(b) Δ_G is pure shellable.

(c) $R/I(G)$ is Cohen-Macaulay.

(d) G is unmixed vertex decomposable.

Proof. (a) $\Rightarrow$ (b) By induction on the number of vertices. Using Lemma 2.24, G has a free vertex. Without loss of generality, we suppose that y_1 is a free vertex, then $N_G(y_1) = \{x_1\}$. Furthermore, by Lemma 2.23 the subgraphs $G \setminus N_G[y_1]$ and $G \setminus N_G[x_1]$ have a perfect matching of König type and they do not contain 4-cycles with two e_i's. Moreover, they are unmixed subgraphs, then by induction hypothesis

$G \setminus N_G[y_1]$ and $G \setminus N_G[x_1]$ are shellables. Therefore, by the Theorem 1.115 G is shellable.

(b) $\Rightarrow$ (a) By induction on the number of vertices. Since Δ_G is pure shellable and G is König, then by Corollary 2.22 we have that G has a free vertex y_1. We can suppose that $N_G(y_1) = \{x_1\}$. Since, $G_1 = G \setminus N_G[y_1]$ is unmixed shellable, by induction hypothesis G_1 has a perfect matching $e_2, ..., e_g$ of König type and G_1 does not contain 4-cycles with two e_i's. Consequently, $e_1 = \{x_1, y_1\}, e_2 ..., e_g$ is a perfect matching of G. Hence, $g = \nu(G) = \tau(G)$ since G is König. Because $\deg_G(y_1) = 1$ there are no 4-cycles with two e_i's.

(b) $\Rightarrow$ (c) By Theorem 1.33.

(c) $\Rightarrow$ (b) Since G is Cohen-Macaulay, then G is unmixed. Now, we will prove that G is shellable by induction. Since G is a König graph, then G has a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type. By Proposition 2.21, there is a free vertex x. We can assume $x = y_1$. By Theorem 1.120 and Lemma 2.23, $G \setminus N_G[x_1]$ and $G \setminus N_G[y_1]$ are König and Cohen-Macaulay. Hence, by induction hypothesis $G \setminus N_G[x_1]$ and $G \setminus N_G[y_1]$ are shellables. Therefore, by Theorem 1.115, G is shellable.

(d)$\Rightarrow$(b) By Theorem 1.33.

(b)$\Rightarrow$(d) By induction on $|V(G)|$. Using Corollary 2.22, G has a free vertex y, i.e., $N_G(y) = \{x\}$. Hence, x is a shedding vertex by Lemma 1.103. On the other hand, $G \setminus N_G[y]$ and $G \setminus N_G[x]$ are pure shellables and by Lemma 2.23 they are König. Thus, by induction hypothesis $G \setminus N_G[x]$ is vertex decomposable. Furthermore, $G \setminus x = (G \setminus N_G[y]) \cup \{y\}$. Consequently, if S is a maximal stable set of $G \setminus x$, then $S = S' \cup \{y\}$ where S' is a maximal stable set of $G \setminus N_G[y]$. Therefore, $G \setminus x$ is pure shellable implying that $G \setminus x$ is vertex decomposable. □

The next result generalizes the classification of Hibi-Herzog about Cohen-Macaulay bipartite graphs (see Theorem 1.122).

Theorem 2.26 G is a Cohen-Macaulay König graph without isolated vertices if and only if there is a partition of $V(G) = V_1 \cup V_2$ with $V_1 = \{x_1, ..., x_g\}$ and $V_2 = \{y_1, ..., y_g\}$ such that

(a) V_2 is a maximal stable set.

(b) $\{x_i, y_i\} \in E(G)$ for $i = 1, ..., g$.

(c) If $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and

(d) If $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$, then $\{x_i, y_k\} \in E(G)$ and if $\{x_i, x_j\}, \{y_j, x_k\} \in E(G)$, then $\{x_i, x_k\} \in E(G)$.

Proof. $\Rightarrow$) By induction on $|V(G)|$. By Proposition 2.21 G has a free vertex y, i.e., $N_G(y) = \{x\}$. If $V(G) = \{x, y\}$, then we have $V_1 = \{x\}$ and $V_2 = \{y\}$ and G satisfies (a), (b), (c) and (d). Now, we assume $V(G) \neq \{x, y\}$. We take $G' = G \setminus N_G[y]$, then G' is Cohen-Macaulay. Furthermore, by Lemma 2.23, G' is König. Moreover, $g - 1 = \tau(G') = \tau(G) - 1$. Hence, by induction hypothesis, $V_1' \cup V_2'$ is a partition of $V(G')$ that satisfies (a), (b), (c) and (d). We can assume $V_1' = \{x_1', ..., x_{g-1}'\}$ and $V_2' = \{y_1', ..., y_{g-1}'\}$. Now, we take $x_1 = x$, $y_1 = y$, $x_i = x_{i-1}'$ and $y_i = y_{i-1}'$ for $i = 2, ..., g$. We set $V_1 = \{x_1, ..., x_g\}$ and $V_2 = \{y_1, ..., y_g\}$. Since V_2' is stable and $N_G(y_1) = \{x_1\}$, then V_2 is stable. Also, $\{x_1, y_1\}, ..., \{x_g, y_g\}$ is a perfect matching. Consequently, by Proposition 2.18, G satisfies (d). Since $\deg_G(y_1) = 1$, thus if $\{x_i, y_1\}$, then $i = 1$. This implies that G satisfies (b) and (c), since G' also satisfies (b) and (c).

$\Leftarrow$) Since $V_1 \cup V_2$ is a partition of $V(G)$ and from (b) we have that $e_1 = \{x_1, y_1\}, ...,$ $e_g = \{x_g, y_g\}$ is a perfect matching. We take a maximal stable S. We suppose that there exists $i \in \{1, ..., g\}$ such that $|S \cap e_i| = \emptyset$. Hence, there exist $z_1, z_2 \in S$ such that $\{z_1, x_i\}, \{z_2, y_i\} \in E(G)$. But, from (d) we have that $\{z_1, z_2\} \in E(G)$, a contradiction since $z_1, z_2 \in S$. Consequently, $|S \cap e_i| = 1$ for each $i \in \{1, ..., g\}$. This implies, G is unmixed and $\tau(G) = g$. Thus, G is König and $e_1, ..., e_g$ is a perfect matching of König type. Now, if G has a 4-cycle Q with two e_i's and since V_2 is a stable set, we can suppose that the edges of Q are $\{x_i, y_i\}, \{x_j, y_j\}, \{x_i, y_j\}$ and $\{x_j, y_i\}$ with $i < j$. By (c), $j \leq i$ since $\{x_j, y_i\} \in E(G)$, this is a contradiction. Then G does not contain 4-cycle with two e_i's. Therefore, by Theorem 2.25, G is Cohen-Macaulay . □

Lemma 2.27 Let G be a connected unmixed graph with a perfect matching e_1, ..., e_g of König type without 4-cycles with two e_i's and $g \geq 2$. For each $z \in V(G)$ we have that:

(a) If $\deg_G(z) \geq 2$, then there exist $\{z, w_1\}, \{w_1, w_2\} \in E(G)$ such that $\deg_G(w_2) = 1$. Furthermore, $e_i = \{w_1, w_2\}$ for some $i \in \{1, ..., g\}$.

(b) If $\deg_G(z) = 1$, then there exist $\{z, w_1\}, \{w_1, w_2\}, \{w_2, w_3\} \in E(G)$ such that $\deg_G(w_3) = 1$. Moreover, $e_i = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$ for some $i, j \in \{1, ..., g\}$.

Proof. Since $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ is a perfect matching of König type we can assume $D = \{x_1, ..., x_g\}$ is a minimal vertex cover. Thus, $F = \{y_1, ..., y_g\}$ is a maximal stable set. By Theorem 2.25 G is Cohen-Macaulay, and by Theorem 2.26 we can assume that if $\{x_i, y_j\} \in E(G)$, then $i \leq j$. Now, we take a vertex $z \in V(G)$.

(a) First, we suppose that $z = x_k$ and there is a vertex x_j in $N_G(x_k)$. If y_j is a free vertex, then we take $w_1 = x_j$ and $w_2 = y_j$, and $e_j = \{w_1, w_2\}$. Now, we can assume $N_G(y_j) \setminus x_j = \{x_{p_1}, ..., x_{p_r}\}$ with $p_1 < \cdots < p_r < j$. If y_{p_1} is not a free vertex, then there is a vertex x_p with $p < p_1$ such that $\{x_p, y_{p_1}\} \in E(G)$. Since G is unmixed, from Proposition 2.9, we obtain that $\{x_p, y_j\} = (\{x_p, y_{p_1}\} \setminus y_{p_1}) \cup (\{y_j, x_{p_1}\} \setminus x_{p_1}) \in E(G)$. But $p < p_1$, a contradiction since p_1 is minimal. Consequently, $\deg_G(y_{p_1}) = 1$. Also, from Proposition 2.9, we have that $\{x_k, x_{p_1}\} = (\{x_k, x_j\} \setminus x_j) \cup (\{x_{p_1}, y_j\} \setminus y_j) \in E(G)$. Hence, we take $w_1 = x_{p_1}$ and $w_2 = y_{p_1}$, and we have that $e_{p_1} = \{w_1, w_2\}$. Now, we assume that $z = x_k$ and $N_G(x_k) \setminus y_k = \{y_{j_1}, ..., y_{j_t}\}$ with $k < j_1 < \cdots < j_t$. We suppose that $\deg_G(x_{j_t}) \geq 2$. If there is a vertex y_r such that $\{x_{j_t}, y_r\} \in E(G)$, then $r > j_t$. Since G is unmixed, $\{x_k, y_r\} = (\{x_k, y_{j_t}\} \setminus y_{j_t}) \cup (\{y_r, x_{j_t}\} \setminus x_{j_t}) \in E(G)$, a contradiction since j_t is maximal. Thus, there exists a vertex x_p such that $\{x_{j_t}, x_p\} \in E(G)$. But, since G is unmixed, then $\{x_k, x_p\} = (\{x_k, y_{j_t}\} \setminus y_{j_t}) \cup (\{x_p, x_{j_t}\} \setminus x_{j_t}) \in E(G)$. This is a contradiction since $N_G(x_k) \setminus y_k = \{y_{j_1}, ..., y_{i_t}\}$. Consequently, $\deg_G(x_{j_t}) = 1$. Therefore, we take $w_1 = y_{j_t}$ and $w_2 = x_{j_t}$, with $e_{j_t} = \{w_1, w_2\}$.

Finally, we assume that $z = y_k$, since y_k is not a free vertex, then $N_G(y_k) \setminus x_k = \{x_{j_1}, ..., x_{j_r}\}$ with $j_1 < \cdots < j_r < k$. If y_{j_1} is not a free vertex, then there is a vertex x_q such that $\{x_q, y_{j_1}\} \in E(G)$ with $q < j_1$. This implies $\{x_q, y_k\} = (\{x_q, y_{j_1}\} \setminus y_{j_1}) \cup (\{x_{j_1}, y_k\} \setminus x_{j_1}) \in E(G)$. But $q < j_1$, a contradiction. Therefore, $\deg_G(y_{j_1}) = 1$ and we take $w_1 = x_{j_1}$ and $w_2 = y_{j_1}$. Hence, $e_{j_1} = \{w_1, w_2\}$.

(b) Since $e_1, ..., e_g$ is a perfect matching, then there exists $i \in \{1, ..., g\}$ such that

$e_i = \{z, z'\}$. Since G is connected, z is a free vertex and $g \geq 2$, then $\deg_G(z') \geq 2$. Thus, by (a) there exist $w'_1, w'_2 \in V(G)$ such that $\{z', w'_1\}$, $\{w'_1, w'_2\} \in E(G)$ where $\deg_G(w'_2) = 1$ and $\{w'_1, w'_2\} = e_j$ for some $j \in \{1, ..., g\}$. Therefore, we take $w_1 = z'$, $w_2 = w'_1$, $w_3 = w'_2$. Consequently, $e_i = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$. □

Theorem 2.28 Let G be a graph without 3-cycles and 5-cycles. If $G_1, ..., G_k$ are the connected components of G, then the following conditions are equivalent:

(a) G is unmixed vertex decomposable.

(b) G is pure shellable.

(c) G is Cohen-Macaulay

(d) G is unmixed and if G_i is not an isolated vertex, then G_i has a perfect matching $e_1, ..., e_g$ of König type without 4-cycles with two $e_i's$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) By Remark 1.127

(d) $\Rightarrow$ (a) Since each component G_i is König, then G is König. Therefore, from Therorem 2.25, G is unmixed vertex decomposable.

(c) $\Rightarrow$ (d) Since G is Cohen-Macaulay, then G is unmixed. By induction on $|V(G)|$. We take $x \in V(G)$ such that $\deg_G(x)$ is minimal and suppose that $N_G(x) = \{z_1, ..., z_r\}$. By Remark 1.66, $G' = G \setminus N_G[x]$ is a Cohen-Macaulay graph. We take $G'_1, ..., G'_s$, the connected components of G'. We can assume that $V(G'_i) = \{y_i\}$ for $i \in \{1, ..., s'\}$. Since $\deg_G(x)$ is minimal, this implies $\{y_i, z_j\} \in E(G)$ for all $i \in \{1, ..., s'\}$ and $j \in \{1, ..., r\}$. Since G does not contain 3-cycles, we have that $N_G(x)$ is a stable set. If $s' = s$, then the only maximal stable sets of G are $\{y_1, ..., y_{s'}, x\}$ and $\{z_1, ..., z_r\}$. Thus, G is a bipartite graph. So, G is König. Hence, by Theorem 2.25, G satisfies (d). Consequently, we can assume $s > s'$, implying that there is a component G'_i with an edge $e = \{w, w'\}$.

Now, we suppose that $r \geq 2$. Since $\deg_G(x)$ is minimal, there exist $a, b \in V(G)$ such that $\{a, w\}, \{b, w'\} \in E(G)$. If $a = b$, then (a, w, w') is a 3-cycle in G. Hence, $a \neq b$. If $a, b \in N_G(x)$, then (x, a, w, w', b) is a 5-cycle in G. Thus, $|\{w, w', a, b\} \cap V(G'_i)| \geq 3$. By induction hypothesis, G' satisfies (d). So, G'_i has a perfect matching and $\tau(G'_i) \geq 2$. Furthermore, by Proposition 2.21, G'_i has a free vertex a'. Then, by Lemma 2.27 (b), there exist edges $\{a', w_1\}, \{w_1, w_2\}, \{w_2, b'\} \in E(G'_i)$ such that

$\deg_{G_i'}(a') = \deg_{G_i'}(b') = 1$. By the minimality of $\deg_G(x)$ we have that a' and b' are adjacent with at least $r-1$ neighbor vertices of x. If $r \geq 3$, then there exists z_j such that $z_j \in N_G(a') \cap N_G(b')$. This implies that (a', w_1, w_2, b', z_j) is a 5-cycle of G. But G does not have 5-cycles, consequently, $r = 2$. We can assume that $\{a', z_1\}, \{b', z_2\} \in E(G)$, implying $C = (x, z_1, a', w_1, w_2, b', z_2)$ is a 7-cycle with $\deg_G(a') = \deg_G(b') = \deg_G(x) = 2$. Hence, by Proposition 2.16, C is a c-minor of G. Thus, by Remark 1.66, C is Cohen-Macaulay. This is a contradiction by Proposition 1.110. Therefore, $\deg_G(x) = r \leq 1$.

If $r = 0$, then the result is clear. Now, if $r = 1$ we can assume that $G_1, ..., G_k$ are the connected components of G and $z_1 \in V(G_1)$. Consequently, the connected components of $G \setminus N_G[x]$ are $F_1, ..., F_l, G_2, ..., G_k$ where $F_1, ..., F_l$ are the connected components of $G_1 \setminus N_{G_1}[x]$. By induction hypothesis $G_2, ..., G_k$ satisfy (d). If $F_j = \{d_j\}$, then $N_G(z_1)$ has two free vertice, d_j and x, a contradiction by Lemma 2.14. Hence, $|V(F_i)| \geq 2$ for $i \in \{1, ..., l\}$. By induction hypothesis, we have that F_i has a perfect matching $M_i = \{e_1^i, ..., e_{g_i}^i\}$ of König type. Thus, $\{e\} \cup (\bigcup_{i=1}^l M_i)$ is a perfect matching of G_1, where $e = \{x, z_1\}$. Also, $\{z_1\} \cup (\bigcup_{i=1}^l X_i)$ is a vertex cover of G_1, where X_i is a minimal vertex cover of F_i. Consequently, $\nu(G_1) \geq 1 + \sum_{i=1}^l |M_i| = 1 + \sum_{i=1}^l g_i = 1 + \sum_{i=1}^l |X_i| \geq \tau(G_1)$. This implies that G_1 is König. Furthermore, by Remark 1.66, we have that G_1 is Cohen-Macaulay. Therefore, by Theorem 2.25, G_1 satisfies (d). □

Corollary 2.29 Let G be a connected graph without 3-cycles and 5-cycles. If G is Cohen-Macaulay, then G has at least an extendable vertex x adjacent to a free vertex.

Proof. From Theorem 2.28, G is König. Thus, by Proposition 2.21 there exists a free vertex x. If $N_G(x) = \{y\}$, then from Lemma 1.103, y is a shedding vertex. Therefore, from Corollary 2.8 y is an extendable vertex, since G is unmixed. □

Corollary 2.30 Let G be a connected graph of girth 6 or more. If G is not an isolated vertex, then the following conditions are equivalent:

(i) G is unmixed vertex decomposable.

(ii) Δ_G is pure shellable.

(iii) $R/I(G)$ is Cohen-Macaulay.

(iv) G is an unmixed König graph.

(v) G is very well-covered.

(vi) G is unmixed with $G \neq C_7$.

(vii) G is a whisker graph.

Proof. (i) ⇒ (ii) ⇒ (iii) By Remark 1.127. (iii) ⇒ (iv) G is unmixed and from Theorem 2.28, G is König. (iv) ⇒ (v) From Lemma 2.10. (v) ⇒ (vi) It is clear, since C_7 is not very well-covered.

(vi) ⇒ (vii) By Lemma 1.89, the pendant edges $\{x_1, y_1\}, ..., \{x_g, y_g\}$ of G form a perfect matching. Since $\{x_i, y_i\}$ is a pendant edge, we can assume that $\deg_G(y_i) = 1$ for each $1 \leq i \leq g$. We take $H = G[x_1, ..., x_n]$. Therefore G is a whisker graph with $W(H) = \{\{x_1, y_1\}, ..., \{x_g, y_g\}\}$.

(vii) ⇒ (i) If G is a whisker graph, then there exists a subgraph H such that $V(H) = \{x_1, ..., x_s\}$, $V(G) = V(H) \cup \{y_1, ..., y_s\}$ and $E(G) = E(H) \cup W(H)$ where $W(H) = \{\{x_1, y_1\}, ..., \{x_s, y_s\}\}$. Consequently, $W(H)$ is a perfect matching and $\nu(G) = s$. Furthermore, $D = \{x_1, ..., x_s\}$ is a vertex cover, then $s = \nu(G) \leq \tau(G) \leq s$. Hence, G is König and $W(H)$ is a perfect matching of König type. Also, there are no 4-cycles with two e_i's since $\deg_G(y_i) = 1$. Therefore, by Proposition 2.9, G is unmixed and by Theorem 2.28, G is unmixed vertex decomposable.

□

2.5 SHELLABLE PROPERTIES IN GRAPHS WITHOUT 3-CYCLES AND 5-CYCLES

In this section we prove that the neighborhood of some 2-connected blocks of a graph G without 3-cycles and 5-cycles have a free vertex if G is unmixed, Cohen-Macaulay, vertex decomposable or shellable. Also, we prove that the criterion of Van Tuyl-Villarreal can be extended for vertex decomposable graphs without 3-cycles and 5-cycles and shellable graphs with girth at least 11.

Lemma 2.31 If G is a graph, then any vertex of degree at least 3 in a basic 5-cycle is a shedding vertex.

Proof. Let $C = (x_1, x_2, x_3, x_4, x_5)$ be a basic 5-cycle. We suppose that $\deg_G(x_1) \geq 3$, since C is a basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_5) = 2$. Also, we can assume that $\deg_G(x_3) = 2$ We take a stable set S of $G \setminus N_G[x_1]$. Since $\{x_3, x_4\} \in E(G)$, then $|S \cap \{x_3, x_4\}| \leq 1$. Hence, $x_3 \notin S$ or $x_4 \notin S$. Consequently, $S \cup \{x_2\}$ or $S \cup \{x_5\}$ is a stable set of $G \setminus x_1$. Therefore, x_1 is a shedding vertex. □

Theorem 2.32 Let G be a connected graph with a basic 5-cycle C. G is a shellable graph if and only if there is a shedding vertex $x \in V(C)$ such that $G \setminus x$ and $G \setminus N_G[x]$ are shellable graphs.

Proof. $\Rightarrow$) We can suppose that $C = (x_1, x_2, x_3, x_4, x_5)$. If $G = C$, then G is shellable. By Remark 2.5, each vertex is a shedding vertex. Furthermore, $G \setminus x_1$ is a path with shelling $\{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}$ and $G \setminus N_G[x_1]$ is an edge. Therefore, $G \setminus x_1$ and $G \setminus N_G[x_1]$ are shellable graphs. Now, we suppose $G \neq C$. We can assume that $\deg_G(x_1) \geq 3$. Since C is a basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_5) = 2$. Also, we can suppose that $\deg_G(x_3) = 2$ and $\deg_G(x_4) \geq 2$. By Lemma 2.31, x_1 is a shedding vertex. Furthermore by Remark 1.66, we have that $G \setminus N_G[x_1]$ is a shellable graph. Now, we will prove that $G_1 = G \setminus x_1$ is shellable. Since G is shellable and since shellability is closed under c-minors, then $G_2 = G \setminus N_G[x_2]$ is shellable. We assume that $F_1, ..., F_r$ is a shelling of Δ_{G_2}. Also, $G_3 = G \setminus N_G[x_3, x_5]$ is shellable. We suppose that $H_1, H_2, ..., H_k$ is a shelling of Δ_{G_3}. We take $F \in \mathcal{F}(\Delta_{G_1})$. If $x_2 \in F$, then $F \setminus x_2 \in \mathcal{F}(\Delta_{G_2})$ and there exists F_i such that $F = F_i \cup \{x_2\}$. If $x_2 \notin F$, then $x_3 \in F$ and $x_4 \notin F$. Thus, $x_5 \in F$. Hence, $F \setminus \{x_3, x_5\} \in \mathcal{F}(\Delta_{G_3})$, then there exists H_j such that $F = H_j \cup \{x_3, x_5\}$. This implies, $\mathcal{F}(\Delta_{G_1}) = \{F_1 \cup \{x_2\}, ..., F_r \cup \{x_2\}, H_1 \cup \{x_3, x_5\}, ..., H_k \cup \{x_3, x_5\}\}$. Furthermore, $F_1 \cup \{x_2\}, ..., F_r \cup \{x_2\}$ and $H_1 \cup \{x_3, x_5\}, ..., H_k \cup \{x_3, x_5\}$ are shellings. Now, $x_3 \in (H_j \cup \{x_3, x_5\}) \setminus (F_i \cup \{x_2\})$ and H_j is a stable set of G without vertices of C. So, $H_j \cup \{x_2, x_5\}$ is a maximal stable set of G_1 since $N_G(x_2, x_5) = V(C)$ and $\{x_2, x_5\} \notin E(G)$. Consequently, $H_j \cup \{x_2, x_5\} = F_l \cup \{x_2\}$ for some $l \in \{1, ..., r\}$ and $(H_j \cup \{x_3, x_5\}) \setminus (F_l \cup \{x_2\}) = \{x_3\}$. Therefore, G_1 is a shellable graph.

$\Leftarrow$) By Lemma 1.111. □

In the following result P is a closed property under c-minors, i.e., if G has property P, then each c-minor of G has property P. Recall that $D_1(B) = \{v \in V(G) \mid v \in N_G(B) \setminus B\}$ for $B \subseteq V(G)$.

Theorem 2.33 Let G be a graph without 3-cycles and 5-cycles with a 2-connected block B. If G satisfies the property P and B does not satisfy P, then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Proof. By contradiction, we assume that if $x \in D_1(B)$, then $|N_G(x)| > 1$. Thus, there exist $a, b \in N_G(x)$ with $a \neq b$. We can suppose that $a \in V(B)$. If $b \in V(B)$, then $G[\{x\} \cup V(B)]$ is 2-connected. But $B \subsetneq G[\{x\} \cup V(B)]$. This is a contradiction since B is a block. Consequently, $V(B) \cap N_G(x) = \{a\}$. Now, we suppose that $b \in D_1(B)$. Since there is no 3-cycle in G, then $a \notin N_G(b)$. Hence, there exists $c \in N_G(b) \cap V(B)$ such that $c \neq a$. This implies $G[\{x, b\} \cup V(B)]$ is 2-connected. But $B \subsetneq G[\{x, b\} \cup V(B)]$, a contradiction. Then $D_1(B) \cap N_G(x) = \emptyset$. Thus, $N_G(x) \cap (V(B) \cup D_1(B)) = \{a\}$ and $b \in D_2(B)$. Now, if $D_1(B) = \{x_1, \ldots, x_r\}$, then there exists a_i such that $V(B) \cap N_G(x_i) = \{a_i\}$. Also, there exists b_i such that $b_i \in N_G(x_i) \cap D_2(B)$. We can suppose that $L = \{b_1, \ldots, b_s\} = \{b_1, \ldots, b_r\}$ with $b_i \neq b_j$ for $1 \leq i < j \leq s$. We will prove that L is a stable set. Suppose that $\{b_i, b_j\} \in E(G)$, if $a_i = a_j$, then $(a_i, x_i, b_i, b_j, x_j, a_i)$ is a 5-cycle in G, this is a contradiction. Consequently $a_i \neq a_j$ and the induced subgraph $G[\{x_i, b_i, b_j, x_j\} \cup V(B)]$ is 2-connected. But B is a block, then $\{b_i, b_j\} \notin E(G)$. Therefore, L is a stable set. Furthermore, $G' = G \setminus N_G[L]$ is a c-minor of G, implying that G' satisfies the property P. Since $D_1(B) \subset N_G(L)$, we have that B is a connected component of G'. But, B does not satisfy P. This is a contradiction since each connected component of G is a c-minor. Therefore, there exists a free vertex in $D_1(B)$. □

Corollary 2.34 Let G be a graph without 3-cycles and 5-cycles and B a 2-connected block. If G is shellable (unmixed, Cohen-Macaulay, sequentially Cohen-Macaulay or vertex decomposable) and B is not shellable (unmixed, Cohen - Macaulay, sequentially Cohen-Macaulay or vertex decomposable), then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Proof. From Remark 1.66 and Theorem 2.33. □

Corollary 2.35 Let G be a bipartite graph and B a 2-connected block. If G is shellable, then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Proof. Since G is bipartite, then B is bipartite. If H is a shellable bipartite graph, then from Lemma 1.114 we have that H has a free vertex. But H is not 2-connected. Hence, B is not shellable. Therefore, by Corollary 2.34, there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$. □

Lemma 2.36 Let G be a graph without 3-cycles and 5-cycles. If G is vertex decomposable, then G has a free vertex.

Proof. Since G is vertex decomposable, then there is a shedding vertex x. Furthermore, there are no 5-cycles in G. Hence, by Lemma 2.3, there exists $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$. If $z \in N_G(y) \setminus x$, then (x, y, z) is a 3-cycle. This is a contradiction. Therefore, $N_G(y) = \{x\}$, implying that y is a free vertex. □

Proposition 2.37 Let G be a graph without 3-cycles and 5-cycles. G is vertex decomposable if and only if there exists a free vertex x with $N_G(x) = \{y\}$ such that $G_1 = G \setminus N_G[x]$ and $G_2 = G \setminus N_G[y]$ are vertex decomposable.

Proof. $\Rightarrow$) By Lemma 2.36, there exists a free vertex x. Furthermore, by Remark 1.66, G_1 and G_2 are vertex decomposable.

$\Leftarrow$) By Lemma 1.103, y is a shedding vertex. Moreover, $G \setminus y = G_1 \cup \{x\}$. Furthermore, since G_1 is vertex decomposable, then $G \setminus y$ is also it. Therefore, G is vertex decomposable, since G_2 is vertex decomposable. □

Corollary 2.38 If G is a 2-connected graph without 3-cycles and 5-cycles, then G is not vertex decomposable.

Proof. Since G is 2-connected, then G does not have a free vertex. Therefore, by Lemma 2.36, G is not vertex decomposable. □

Corollary 2.39 If G is the 2-clique-sum of the cycles C_1 and C_2 with $|V(C_1)| = r_1 \leq r_2 = |V(C_2)|$, then G is vertex decomposable if and only if $r_1 = 3$ or $r_1 = r_2 = 5$.

Proof. $\Leftarrow$) First, we suppose that $r_1 = 3$. Consequently, we can assume $C_1 = (x_1, x_2, x_3)$ and $x_2, x_3 \in V(C_1) \cap V(C_2)$. Thus, x_1 is a simplicial vertex. Hence, by Lemma 1.103, x_2 is a shedding vertex. Furthermore, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are trees. Consequently, by Corollary 1.97, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are vertex decomposable graphs. Therefore, G is vertex decomposable.

Now, we assume that $r_1 = r_2 = 5$ with $C_1 = (x_1, x_2, x_3, x_4, x_5)$ and $C_2 = (y_1, x_2, x_3, y_4, y_5)$. We take a stable set S in $G \setminus N_G[x_5]$. If $x_2 \in S$, then $S \cup \{x_4\}$ is a stable set in

$G_1 = G \setminus x_5$. If $x_2 \notin S$, then $S \cup \{x_1\}$ is a stable set in G_1. Consequently, by Lemma 2.1, x_5 is a shedding vertex. Since x_2 is a neighbor of a free vertex in G_1, then x_2 is a shedding vertex in G_1. Furthermore, since $G_1 \setminus x_2$ and $G_1 \setminus N_{G_1}[x_2]$ are forests, then they are vertex decomposable graphs, by Corollary 1.97. Thus, G_1 is vertex decomposable. Since $G \setminus N_G[x_5] = C_2$, it is vertex decomposable by Proposition 1.110. Therefore, G is vertex decomposable.

$\Rightarrow$) By Corollary 2.38, we have that $\{r_1, r_2\} \cap \{3, 5\} \neq \emptyset$. We suppose $r_1 \neq 3$. So $r_1 = 5$ or $r_2 = 5$. Consequently, we can assume that $\{C_1, C_2\} = \{C, C'\}$ where $C = (x_1, x_2, x_3, x_4, x_5)$ and $x_2, x_3 \in V(C) \cap V(C')$. Thus, $G \setminus N_G[x_5] = C'$ is vertex decomposable. Hence, from Proposition 1.110, $|V(C')| \in \{3, 5\}$. But $r_1 \neq 3$, then $|V(C')| = 5$ and $r_1 = r_2 = 5$. Therefore, $r_1 = 3$ or $r_1 = r_2 = 5$. □

Lemma 2.40 Let G be a 2-connected graph with girth at least 11. Then G is not shellable.

Proof. Since G is 2-connected, then G is not a forest. Consequently, if r is the girth of G, then there exists a cycle $C = (x_1, x_2, ..., x_r)$. If $G = C$, then G is not shellable Proposition 1.110. Hence $G \neq C$, implying $D_1(C) \neq \emptyset$. We take $y \in D_1(C)$, without loss of generality we can assume that $\{x_1, y\} \in E(G)$. If $\{x_i, y\} \in E(G)$ for some $i \in \{2, ..., r\}$, then we take the cycles $C_1 = (y, x_1, x_2, ..., x_i)$ and $C_2 = (y, x_1, x_r, x_{r-1}, ..., x_i)$. Thus, $|V(C_1)| = i + 1$ and $|V(C_2)| = r - i + 3$. Since r is the girth of G, then $i + 1 \geq r$ and $r - i + 3 \geq r$. Consequently, $3 \geq i$ implies $4 \geq r$. But $r \geq 11$, this is a contradiction. This implies that $|N_G(y) \cap V(C)| = 1$. Now, we suppose that there exist $y_1, y_2 \in D_1(C)$ such that $\{y_1, y_2\} \in E(G)$. We can assume that $\{x_1, y_1\}, \{x_i, y_2\} \in E(G)$. Since $r \geq 11$, then there are no 3-cycles in G. In particular, $x_1 \neq x_i$. Now, we take the cycles $C' = (y_1, x_1, ..., x_i, y_2)$ and $C'' = (y_1, x_1, x_r, x_{r-1}, ..., x_i, y_2)$. So, $|V(C')| = i + 2$ and $|V(C'')| = r - i + 4$. Since r is the girth, we have that $i + 2 \geq r$ and $r - i + 4 \geq r$. Hence, $4 \geq i$ and $6 \geq r$, this is a contradiction. Then $D_1(C)$ is a stable set. Now, since G is 2-connected, then for each $y \in D_1(C)$ there exists $z \in N_G(y) \cap D_2(C)$. If there exist $z_1, z_2 \in D_2(C)$ such that $\{z_1, z_2\} \in E(G)$, then there exist $y_1, y_j \in D_1(C)$ such that $\{z_1, y_1\}, \{z_2, y_j\} \in E(G)$. Since there are no 3-cycles in G, we have that $y_1 \neq y_j$. We can assume that $\{x_1, y_1\}, \{x_i, y_j\} \in E(G)$. Since there are no 5-cycles, then $i \neq 1$. Consequently, there exist cycles $C'_1 = (x_1, ..., x_i, y_j, z_2, z_1, y_1)$ and $C'_2 = (x_i, ..., x_r, x_1, y_1, z_1, z_2, y_j)$. This implies $r \leq |V(C'_1)| = i + 4$ and $r \leq |V(C'_2)| = r - i + 6$. Hence, $i \leq 6$ and $r \leq 10$, this is a contradiction. Then $D_2(C)$ is a stable set. Furthermore, C is a connected component of $G \setminus N_G[D_2(C)]$. But C is not shellable, therefore G is not shellable. □

Theorem 2.41 If G has girth at least 11, then G is shellable if and only if there exists $x \in V(G)$ with $N_G(x) = \{y\}$ such that $G \setminus N_G[x]$ and $G \setminus N_G[y]$ are shellables.

Proof. $\Leftarrow$) By Theorem 1.115.

$\Rightarrow$) By Remark 1.66, shellability is closed under c-minors. Consequently, it is only necessary to prove that G has a free vertex. If every block of G is an edge or a vertex, then G is a forest and there exists $x \in V(G)$ with $\deg_G(x) = 1$. Hence, we can assume that there exists a 2-connected block B of G. The girth of B is at least 11, since B is an induced subgraph of G. Thus, by Lemma 2.40, B is not shellable. Therefore, by Theorem 2.33, there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$. □

2.6 UNICYCLIC GRAPHS

In this section we characterize the Cohen-Macaulay, shellable and well-covered unyciclic graphs.

Theorem 2.42 Let G be an unicyclic graph with cycle C. Then the following conditions are equivalent:

(1) G is vertex decomposable.

(2) G is shellable

(3) G is sequentially Cohen-Macaulay.

(4) $|V(C)| \in \{3, 5\}$ or there exists $x \in D_1(C)$ such that $\deg_G(x) = 1$

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) By Remark 1.127.

(3) $\Rightarrow$ (4) If $|V(C)| \notin \{3, 5\}$, then by Proposition 1.110 C is not sequentially Cohen-Macaulay. Furthermore, C is the only 2-connected block of G. Thus, by Theorem 2.33, there exists $x \in D_1(C)$ such that $\deg_G(x) = 1$.

(4) $\Rightarrow$ (1) If $|V(C)| = \{3,5\}$, then G is vertex decomposable by Theorem 1.96. Now, if $|V(C)| \neq 3,5$, then we can assume that $N_G(x) = \{y\}$ for some $x \in D_1(C)$. Hence, by Lemma 1.103, y is a shedding vertex. Since $x \in D_1(C)$ and $\deg_G(x) = 1$, then $y \in V(C)$. Consequently, $G \setminus y$ and $G \setminus N_G[y]$ are forests, so they are vertex decomposable. Therefore, G is vertex decomposable. $\square$

Remark 2.43 Let G be an unicyclic graph with cycle C. If C is a basic 4-cycle, then C contains two adjacent vertices of degree two and the remaining two vertices are joined to a free vertex.

Lemma 2.44 Let x be a free vertex of an unmixed graph G such that $G' = G \setminus N_G[x]$ is a whisker graph. If there is no cycle C of G such that $z \in V(C)$ where $\{x,z\} \in E(G)$, then G is a whisker graph.

Proof. Let $G_1, ..., G_s$ be the connected components of G'. Thus, there exist induced subgraphs $H_1, ..., H_s$ such that $V(H_i) = \{x_1^i, ..., x_{r_i}^i\}$, $V(G_i) = V(H_i) \cup \{y_1^i, ..., y_{r_i}^i\}$ and $E(G_i) = E(H_i) \cup W(H_i)$, where $W(H_i) = \{\{x_1^i, y_1^i\}, ..., \{x_{r_i}^i, y_{r_i}^i\}\}$. If $\{x,z\}$ is a connected component of G, then $G = \{x,z\} \cup G_1 \cup \cdots \cup G_s$. Consequently, G is a whisker graph. Now, suppose there are $w_1^i, w_2^i \in V(G_i)$ such that $\{w_1^i, z\}, \{w_2^i, z\} \in E(G)$. Since G_i is connected, then there exists a path $(w_1^i, v_1, ..., v_m, w_2^i)$ between w_1^i and w_2^i in G_i. Hence, $C = (z, w_1^i, v_1, ..., v_m, w_2^i)$ is a cycle, a contradiction. Then $|N_G(z) \cap V(G_i)| \leq 1$ for $i \in \{1, ..., s\}$. We can suppose that $\{G_1, ..., G_{s_1}\} = \{G_i \mid N_G(z) \cap V(G_i) \neq \emptyset\}$. Thus, $F_1, G_{s_1+1}, ..., G_s$ are the connected components of G, where $F_1 = G[\{x,z\} \cup \bigcup_{i=1}^{s_i} G_i]$. Furthermore, we can assume that $|V(G_i)| = 2$ for $1 \leq i \leq s_2$ and $|V(G_i)| \geq 3$ for $s_2 < i \leq s_1$. Hence, $\deg_{G_i}(x_1^i) = \deg_{G_i}(y_1^i) = 1$ and we can assume that $N_G(z) \cap V(G_i) = \{x_1^i\}$ for $1 \leq i \leq s_2$. We suppose that $\{z, y_j^i\} \in E(G)$ for some $s_2 < i \leq s_1$. Since G_i is connected and $|V(G_i)| \geq 3$, then there is a vertex $w \in \{x_1^i, ..., x_{r_i}^i\} \setminus x_j^i$ such that $\{w, x_j^i\} \in V(G_i)$. Consequently, $G'' = G \setminus N_G[w]$ is unmixed. But $N_{G''}(z)$ has two free vertices, y_j^i and x, a contradiction by Lemma 2.14. Then $N_G(z) \cap V(G_i) \subseteq V(H_i)$ for $1 \leq i \leq s_1$. We take $H = G[\bigcup_{i=1}^{s_1} H_i \cup \{z\}]$, then $E(F_1) = E(H) \cup W(H)$ where $W(H) = \bigcup_{i=1}^{s_1} W(H_i) \cup \{x,z\}$. Hence, F_1 is a whisker graph. Therefore, G is a whisker graph. $\square$

Theorem 2.45 Let G be an unicyclic graph with cycle C. Then G is well-covered if and only if G satisfies one of the following conditions:

(a) $G \in \{C_3, C_4, C_5, C_7\}$.

(b) G is a whisker graph.

(c) C is a simplex 3-cycle or a basic 5-cycle and $G \setminus V(C)$ is a whisker forest graph.

(d) C is a basic 4-cycle with two adjacent vertices a, b of degree 2 in G such that $G \setminus \{a, b\}$ is a whisker graph.

Proof. $\Rightarrow$) By induction on $l = |V(G) \setminus V(C)|$. If $l = 0$, then $G = C$. Since the well-covered cycles are C_3, C_4, C_5 and C_7, then C satisfies (a). Now, we assume $l \geq 1$. Hence, $C \subsetneq G$ and G has a free vertex. We take a free vertex x such that $d(x, C) = \max\{d(a, C) \mid a \in V(G),\ \deg_G(a) = 1\}$. We assume that $N_G(x) = \{z\}$ and $C = (y_1, ..., y_k)$ with $k \geq 3$. By Remark 1.66, we have that $G' = G \setminus N_G[x]$ is well-covered.

If $d(x, C) \geq 2$, then $C \subseteq G'$ and by induction hypothesis G' satisfies (a), (b), (c) or (d). We take $G'_1 = G' \setminus V(C)$. If G' satisfies (a), then $G' = C_r$ with $r \in \{3, 4, 5, 7\}$. Since G is connected, we can assume that $\{z, y_1\} \in E(G)$. Furthermore, G is unicyclic, then $N_G(z) = \{y_1\}$. If $r = 4$ or 7, then $N_{G_2}(z)$ has two free vertices, x and y_1, in $G_2 = G \setminus N_G[y_3, y_{r-1}]$. A contradiction by Lemma 2.14. Consequently, $r = 3$ or 5. We have that $G'_1 = G[\{x, z\}]$, thus G'_1 is a whisker forest. Furthermore, C is a simplex 3-cycle or a basic 5-cycle in G. Hence, G satisfies (c). Now, if G' satisfies (b), since $C \subseteq G'$ and G is unicyclic, then z is not in a cycle of G. Consequently, by Lemma 2.44, G is a whisker graph and G satisfies (b). Now, if G' satisfies (c), then C is a simplex 3-cycle or a basic 5-cycle and G'_1 is a whisker forest. If $C = C_3$ and C_3 is not a simplex in G, then $|N_G(z) \cap V(C)| \geq 1$. Since G is unicyclic, we can assume that $N_G(z) \cap V(C) = \{y_1\}$ and there are $z_2, z_3 \in V(G)$ such that $\{y_2, z_2\}, \{y_3, z_3\} \in E(G)$. Thus, $N_{G_3}(z)$ has two free vertices, x and y_1, in $G_3 = G \setminus N_G[z_2, z_3]$, this is a contradiction. So C_3 is a simplex of G. Now, if $C = C_5$ and C is not basic in G, then we can assume $\deg_G(y_1) \geq 3$ and $\deg_G(y_2) \geq 3$. Since C is a basic 5-cycle in G' and G is unicyclic, we can assume $N_G(z) \cap V(C) = \{y_1\}$. Also, there exists $z_2 \in V(G')$ such that $\{y_2, z_2\} \in E(G)$. Hence, $N_{G_4}(z)$ has two free vertices, y_1 and x, in $G_4 = G \setminus N_G[z_2, y_4]$, a contradiction. Then C is a basic 5-cycle in G. Now, if $H = G \setminus V(C)$, then $H = G[G'_1 \cup \{x, z\}]$ and $H \setminus N_H[x] = G'_1$. Consequently, by Lemma 2.44, H is a whisker forest. This implies that G satisfies (c). Now, if G' satisfies (d), then C is a basic 4-cycle. We can assume that $\deg_{G'}(y_1) = \deg_{G'}(y_2) = 2$ and $G'' = G' \setminus \{y_1, y_2\}$ is a whisker. Since G is unicyclic, we have that $|N_G(z) \cap \{y_1, y_2\}| = 1$. We can suppose $\{y_1, z\} \in E(G)$. But $N_{G_5}(z)$ has two free vertices, x and y_1, in $G_5 = G \setminus N_G[y_3]$, this is a contradiction. So $\deg_G(y_1) = \deg_G(y_2) = 2$. We take $H_1 = G \setminus \{y_1, y_2\}$, therefore $H_1 \setminus N_{H_1}[x] = G''$. By Lemma 2.44, H_1 is a whisker and G satisfies (d).

Now, we assume $d(x, C) = 1$. Thus, $V(G) = V(C) \cup D_1(C)$ and G' is a forest. Consequently by Theorem 1.118, G' is a whisker graph. Also $z \in V(C)$, then we can assume that $z = y_1$. Since G is unicyclic, if $w \in D_1(C)$, then w is a free vertex. If each $y_i \in V(C)$ is adjacent to one free vertex, we have that G is a whisker graph. Hence, G satisfies (b). If there is a vertex $y_j \in V(C)$ such that $\deg_G(y_j) = 2$. Without loss of generality, we can assume $j = 2$. If $r \geq 5$, then $N_{G_6}(y_1)$ has two free vertices, x and y_2, in $G_6 = G \setminus N_G[y_4]$, a contradiction by Lemma 2.14. So $r \leq 4$. If $r = 3$, then $\{y_1, y_3\}$ and $\{y_2, y_3, x\}$ are minimal vertex covers of G. But G is well-covered, implying $r = 4$. If $\deg_G(y_4) = 2$, then $N_{G'}(y_3)$ has two free vertices, y_2 and y_4. Consequently, $\deg_G(y_4) \geq 3$ and there is a free vertex z_4 such that $\{y_4, z_4\} \in E(G)$. Now, if $\deg_G(y_3) \geq 3$, then there is a free vertex z_3 such that $\{y_3, z_3\} \in E(G)$. But, $N_{G_7}(y_3)$ has two free vertices, y_2 and z_3, in $G_7 = G \setminus N_G[x, z_4]$. Hence, $\deg_G(y_3) = 2$ and C is a basic 4-cycle. Furthermore, $G \setminus \{y_2, y_3\}$ is a whisker. Therefore, G satisfies (d).

$\Leftarrow$) Graphs of families (a), (b), (c) and (d) are in $\mathcal{SQC} \cup \{C_4, C_7\}$. Hence, by Theorem 1.126, they are well-covered. □

Corollary 2.46 Let G be an unicyclic graph. Then the following conditions are equivalent:

(1) G is unmixed vertex decomposable.

(2) G is pure shellable

(3) G is Cohen-Macaulay.

(4) G is unmixed and $G \neq C_4, C_7$.

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) By Theorem 2.42.

(3) $\Rightarrow$ (4) Since G is Cohen-Macaulay, then G is unmixed. Furthermore, C_4 and C_7 are not Cohen-Macaulay.

(4) $\Rightarrow$ (1) Let C be the cycle of G. Since $G \neq C_4, C_7$, then by Theorem 2.45, we have that $G \in \mathcal{SQC}$. Therefore, by Theorem 1.126, G is vertex decomposable. □

Corollary 2.47 If G is an unmixed unicyclic graph, then G is vertex decomposable if and only if $G \neq C_4, C_7$.

Proof. By Corollary 2.46. □

2.7 WELL-COVERED THETA-RING GRAPHS

In this Section, we characterize the well-covered property for theta-ring graphs. These graphs can be constructed by clique-sum of cycles and complete graphs (see Theorem 1.140). Also, they contain chordal graphs, cactus graphs and ring graphs (see Section 1.6). We prove that a theta-ring graph is well-covered if and only if it has a reduction in sun-complete subgraphs, basic 5-cycles and semi-basic 5-cycles.

Lemma 2.48 Let G be a theta-ring graph. If $C = (z_1, ..., z_k)$ is an induced cycle and $y_1 \in N_G(C) \setminus V(C)$, then

(1) If there are $z_i, z_j \in V(C) \cap N_G(y_1)$ with $i \neq j$, then $\{z_i, z_j\} \in E(G)$. Furthermore, $|V(C) \cap N_G(y_1)| \leq 2$ or $|V(C) \cap N_G(y_1)| = 3$ with C a 3-cycle.

(2) If $\{y_1, z_i\} \in E(G)$, then there is no path between y_1 and z_l with $l \in \{1, ..., i-2, i+2, ..., k\}$. Furthermore, there are no two paths P_1 between y_1 and z_{i-1} and P_2 between y_1 and z_{i+1}, except if C is a 3-cycle and $\{y_1, z_{i-1}\}, \{y_1, z_{i+1}\} \in E(G)$.

(3) If there is $y_2 \in N_G(C)$ such that $\{y_1, y_2\} \in E(G)$, then for all $z_i \in V(C) \cap N_G(y_1)$ and $z_j \in V(C) \cap N_G(y_2)$, we have that $z_i = z_j$ or $\{z_i, z_j\} \in E(G)$.

Proof. (1) Suppose that $\{z_i, z_j\} \notin E(G)$, then $j \notin \{i-1, i+1\}$. Consequently, $k \geq 4$. We can assume $i < j$. Hence, there exists a chorded-theta H whose principal paths are $L_1 = (z_i, z_{i+1}, ..., z_j)$, $L_2 = (z_i, z_{i-1}, ...z_1, z_k, z_{k-1}, ..., z_j)$ and $L_3 = (z_i, y_1, z_j)$. But C is an induced cycle, then H does not contain transversal triangles. This is a contradiction, since G is a theta-ring graph. Therefore, $\{z_i, z_j\} \in E(G)$. Now, if $|V(C) \cap N_G(y_1)| \geq 3$, then there exist different vertices $z_{l_1}, z_{l_2}, z_{l_3} \in V(C) \cap N_G(y_1)$. Hence, by the last argument $\{z_{l_1}, z_{l_2}\}$, $\{z_{l_1}, z_{l_3}\}$, $\{z_{l_2}, z_{l_3}\} \in E(G)$. Therefore, $C = (z_{l_1}, z_{l_2}, z_{l_3})$ and $|V(C) \cap N_G(y_1)| = 3$, since C is an induced cycle.

(2) We suppose that there is a path $P_1 = \{y_1, c_1, ..., c_m, z_l\}$ with $l \notin \{i-1, i+1\}$, then we have a chorded-theta H' with principal paths $L_1 = \{z_i, z_{i+1}, ..., z_l\}$, $L_2 = \{z_i, z_{i-1}, ..., z_1, z_k, z_{k-1}, ..., z_l\}$ and $L_3 = \{z_i, y_1, c_1, ..., c_m, z_l\}$. Since C is induced, then

H' does not have transversal triangles in G. This is a contradiction. We suppose that there are minimal paths $P_1 = \{y_1, a_1, ..., a_m, z_{i-1}\}$ and $P_2 = \{y_1, b_1, ..., b_n, z_{i+1}\}$. Hence, there is a chorded-theta H'' with principal paths $L_1 = \{z_{i-1}, z_i, z_{i+1}\}$, $L_2 = \{z_{i-1}, z_{i-2}, ..., z_1, z_k, z_{k-1}, ..., z_{i+1}\}$ and $L_3 = \{z_{i-1}, a_m, ..., a_1, y_1, b_1, ..., b_n, z_{i+1}\}$. If $|V(C)| \geq 4$, then H'' does not have transversal triangles since C is induced, this is a contradiction. Thus $|V(C)| = 3$. If $m \geq 1$, we take the cycle $C' = (y_1, a_1, ..., a_m, z_{i-1}, z_i)$, this is a contradiction by the last argument. Thus, $m = n = 0$. Hence, $\{y_1, z_{i-1}\}, \{y_1, z_{i+1}\} \in E(G)$ and C is a 3-cycle.

(3) By (2), there is no path between y_1 and z_l with $l \notin \{i-1, i+1\}$. Furthermore, there is no path between y_1 and z_j, then $j \in \{i+1, i, i-1\}$. Therefore, $z_i = z_j$ or $\{z_i, z_j\} \in E(G)$. □

Definition 2.49 A complete subgraph H of G is called *sun-complete* if each maximal stable set S of G satisfies $|V(H) \cap S| = 1$.

Remark 2.50 Let K be a complete subgraph of G. K is a sun-complete subgraph if and only if $V(K) \nsubseteq N_G(S)$ for each stable set S of G.

Remark 2.51 If K is a simplex of G, then K is a sun-complete subgraph of G.

Proof. By contradiction, suppose that there is a maximal stable set S of G such that $V(K) \cap S = \emptyset$, consequently, $V(K) \subseteq N_G(S)$. Since K is a simplex there exists a simplicial vertex x in $V(K)$. Hence, there is $y \in S$ such that $\{x, y\} \in E(G)$ and $y \notin V(K)$. A contradiction since x is simplicial. □

Lemma 2.52 Let G be a well-covered graph. If K_1 is a simplex and K_2 is a sun-complete subgraph of G, then $V(K_1) \cap V(K_2) = \emptyset$ or $K_1 = K_2$.

Proof. By contradiction, suppose that $K_1 \neq K_2$ and $w \in V(K_1) \cap V(K_2)$, then there is a maximal stable set S such that $w \in S$. We take a simplicial vertex y of K_1. Hence, $S' = S \setminus w \cup \{y\}$ is a stable set of G. Furthermore, S' is maximal since G is well-covered and $|S'| = |S|$. Since $K_1 \neq K_2$, consequently $y \notin V(K_2)$, then $S' \cap V(K_2) = \emptyset$, a contradiction since K_2 is sun-complete. Therefore, $V(K_1) \cap V(K_2) = \emptyset$ or $K_1 = K_2$. □

Corollary 2.53 If G is a well-covered graph, then all its simplexes are pairwise vertex disjoint. In particular, if $x \in V(G)$, then $N_G(x)$ does not contain two free vertices.

Proof. By Remark 2.51 and Lemma 2.52. □

Definition 2.54 A 5-cycle $C' = (a, b, c, d, e)$ is *semi-basic* if $\deg_G(a) = \deg_G(c) = 2$, $\deg_G(d) = \deg_G(e) = 3$ and there exists an induced 4-cycle Q' such that $V(Q') \cap V(C') = \{d, e\}$.

Remark 2.55 If C' is a semi-basic 5-cycle of G, then C' is an induced cycle.

Lemma 2.56 Let G be a well-covered. If C is a basic 5-cycle and C' is a semi-basic 5-cycle, then $V(C) \cap V(C') = \emptyset$.
Proof. We suppose that there is $x_1 \in V(C) \cap V(C')$ and $C = (x_1, x_2, x_3, x_4, x_5)$. By definition $C \neq C'$, then there is $y_2 \in V(C') \setminus V(C)$ such that $\{y_2, x_1\} \in E(G)$. Hence, $\deg_G(x_2) = \deg_G(x_3) = \deg_G(x_5) = 2$. If $|V(C) \cap V(C')| \geq 2$, then $\{y_2, x_4\} \in E(G)$ or there is $y_3 \in V(C)$ such that $\{y_2, y_3\}, \{y_3, x_4\} \in E(G)$. If $\{y_2, x_4\} \in E(G)$, then: if there is $w_1 \neq x_2$ and $w_1 \in N_G(x_1)$, then there is $w_2 \in N_G(x_4)$ with $w_2 \notin \{x_3, x_5\}$. Hence, $\deg_G(x_1) \geq 4$ and $\deg_G(x_4) \geq 4$, a contradiction by definition of semi-basic 5-cycle. Then $w_1 = x_2$. Therefore, $C' = (x_1, x_2, x_3, x_4, y_2)$, then $\deg_G(x_1) = 2$ or $\deg_G(x_4) = 2$ by definition. A contradiction since $\deg_G(x_1) \geq 3$ and $\deg_G(x_4) \geq 3$. Now, if there is $y_3 \in V(C')$ such that $\{y_2, y_3\}, \{y_3, x_4\} \in E(G)$ and $C' = (x_1, y_2, y_3, x_4, x_5)$, then: if there is $w \neq x_5$ and $w \in N_G(x_1)$, then $w \in N_G(x_4)$, hence $C' = (x_1, y_2, y_3, y_4, w)$ and $\deg_G(x_1) \geq 4$ and $\deg_G(x_4) \geq 4$, a contradiction. So, $w = x_5$, then $C' = (x_1, y_2, y_3, y_4, x_5)$. Since $\deg_G(x_1) \geq 3$ and $\deg_G(x_4) \geq 3$, then there exists an induced 4-cycle Q such that $x_1, x_2, y_2 \in V(Q)$ or $x_3, x_4, y_3 \in V(Q)$, but $\deg_G(x_2) = \deg_G(x_3) = 2$, a contradiction. Then C' is not a semi-basic 5-cycle. Therefore, $V(C) \cap V(C') = \{x_1\}$. Hence, $C' = (x_1, y_2, y_3, y_4, y_5)$ with $\deg_G(y_2) = \deg_G(y_5) = 2$ and $\deg_G(y_3) = \deg_G(y_4) = 3$. Furthermore, $\{x_3, y_4\} \in E(G)$. Therefore, $N_G(x_1)$ has two free vertices, x_5 and y_2, in $G \setminus N_G[x_3, y_4]$, a contradiction. Thus, $V(C) \cap V(C') = \emptyset$. □

Lemma 2.57 Let G be a well-covered graph. If $C = (x_1, x_2, x_3, x_4, x_5)$ is a semi-basic 5-cycle with a 4-cycle $C' = (x_1, x_5, y_2, y_1)$, then $\deg_G(y_1) = 2$ if and only if $\deg_G(y_2) = 2$.

Proof. Since C is a semi-basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_4) = 2$ and $\deg_G(x_1) = \deg_G(x_5) = 3$. Now, we suppose that $\deg_G(y_1) = 2$ and $\deg_G(y_2) \geq 3$, then there is $w \in N_G(y_2) \setminus \{y_1, x_5\}$. Consequently, $N_{G'}(x_1)$ has two free vertices, y_1 and x_2, in $G' = G \setminus N_G[w, x_4]$. This is a contradiction by Corollary 2.53. Therefore, $\deg_G(y_2) = 2$. Similarly if $\deg_G(y_2) = 2$, then $\deg_G(y_1) = 2$. □

Lemma 2.58 Let $C = (x_1, x_2, x_3, x_4)$ be an induced 4-cycle and let $C' = (x_4, x_5, x_6, x_7, x_8)$ be an induced 5-cycle in a graph G.

(a) If G is well-covered, $\deg_G(x_5) = \deg_G(x_7) = 2$ and $\{x_1, x_6\} \notin E(G)$, then $\{x_1, x_8\} \notin E(G)$.

(b) If G is theta-ring, then $\{x_1, x_6\}, \{x_1, x_7\} \notin E(G)$.

Proof. (a) We suppose that $\{x_1, x_8\} \in E(G)$, then $N_{G'}(x_6)$ has two free vertices, x_5 and x_7, in $G' = G \setminus N_G[x_1]$. This is a contradiction by Corollary 2.53. Thus, $\{x_1, x_8\} \notin E(G)$.

(b) Since C' is an induced cycle, by Lemma 2.48, $\{x_1, x_6\}, \{x_1, x_7\} \notin E(G)$. □

Lemma 2.59 Let K be a sun-complete subgraph of a graph G. If $x \notin V(K)$ and $K' = K \setminus N_G[x]$, then K' is a sun-complete subgraph in $G' = G \setminus N_G[x]$.

Proof. If $x \in V(G) \setminus N_G[K]$, then $K' = K$. Indeed, if there is a maximal stable set S' in G' such that $V(K) \subseteq N_{G'}(S')$, then S' is a stable set in G and $V(K) \subseteq N_G(S')$. A contradiction since K is sun-complete. Thus, K is a sun-complete subgraph in G'. Now, if $y \in V(K)$ is a simplicial vertex in G, then $\{y, x\} \notin E(G)$. Hence, $y \in V(K')$ is a simplicial vertex in G'. Thus, K' is a sun-complete subgraph in G, by Remark 2.51. Now, we assume that K does not have simplicial vertices. We suppose that K' is not sun-complete, then there is a stable set S' in G' such that $V(K') \subseteq N_{G'}(S')$. Furthermore, $V(K) = V(K') \cup A$ with $A \subseteq N_G(x)$. Consequently, $V(K) = V(K') \cup A \subseteq N_G(S') \cup N_G(x) = N_G(S' \cup \{x\})$. A contradiction since $S' \cup \{x\}$ is stable and K is sun-complete. Therefore, K' is a sun-complete subgraph in G'. □

Lemma 2.60 Let G be a well-covered graph. If K is a sun-complete subgraph and C is a basic or semi-basic 5-cycle, then $V(K) \cap V(C) = \emptyset$.

Proof. We suppose that exists $y \in V(K) \cap V(C)$ with $C = (y, z_1, z_2, z_3, z_4)$. If $V(K) \subseteq V(C)$, then we can suppose $K = \{y, z_1\}$ since C is an induced cycle. But there exists a maximal stable set S such that $z_2, z_4 \in S$. Consequently, $S \cap V(K) =$

$\emptyset$, a contradiction. Then there is $y' \in V(K) \setminus V(C)$ such that $\{y', y\} \in E(G)$. Hence, we can assume $\deg_G(z_1) = 2$. We will prove that $V(K) \cap N_G[z_2] = \emptyset$. Suppose $w \in V(K) \cap N_G[z_2]$, then $\{y, w\} \in E(G)$. Thus, by Lemma 2.48 $w = z_1$ since C is an induced cycle. So, $K = \{y, z_1\}$ since $\deg_G(z_1) = 2$, a contradiction since $V(K) \nsubseteq V(C)$. This implies that $V(K) \cap N_G[z_2] = \emptyset$. Now, we assume that C is a basic 5-cycle, then we can suppose $\deg_G(z_2) = \deg_G(z_4) = 2$. Consequently $\{y, z_4\}$ is a simplex in $G_1 = G \setminus N_G[z_2]$. Furthermore, K is a sun-complete subgraph in G_1 by Lemma 2.59. But, $y \in V(K) \cap \{y, z_4\}$ in G_1, a contradiction from Lemma 2.52. Now, we assume C is a semi-basic 5-cycle. If $\deg_G(y) = \deg_G(z_4) = 3$, then $\deg_G(z_3) = 2$. Furthermore, there exists an induced 4-cycle (y, z_4, a, b). Hence, $K = \{y, b\}$ and there is a maximal stable S' such that $a, z_1 \in S'$. Thus, $V(K) \cap S' = \emptyset$, this is a contradiction. Then, we can suppose $\deg_G(z_4) = 2$ and $\deg_G(z_2) = \deg_G(z_3) = 3$. Consequently, $\{y, z_4\}$ is a simplex in G_1. A contradiction since $y \in V(K) \cap \{y, z_4\}$ in G_1. Therefore, $V(K) \cap V(C) = \emptyset$. □

Lemma 2.61 Let G be a theta-ring graph and let K be a sun-complete subgraph of G with $V(K) = \{y_1, ..., y_m\}$, $z_i \in N_G(y_i) \setminus N_G(y_j)$ and $z_j \in N_G(y_j) \setminus N_G(y_i)$. If $\{z_i, z_j\} \in E(G)$ and $k \in \{1, ..., m\} \setminus \{i, j\}$. Then:

(1) $\{z, z_i\} \notin E(G)$ for all $z \in N_G(y_j) \setminus \{y_i, z_j\}$.

(2) $z_i, z_j \notin N_G(y_k)$.

(3) If $z \in N_G(y_k) \setminus \{y_i, y_j\}$, then $\{z_i, z\}, \{z_j, z\} \notin E(G)$.

(4) $K' = K \setminus y_i$ is a sun-complete subgraph in $G' = G \setminus N_G[z_i]$ and $\deg_G(y_k) = \deg_{G'}(y_k) + 1$ with $k \neq i, j$. Furthermore, $\deg_G(y_j) = \deg_{G'}(y_j) + 2$.

Proof. Since $z_i \notin N_G(y_j)$, $z_j \notin N_G(y_i)$ and $\{z_i, z_j\} \in E(G)$, then $C = (y_i, z_i, z_j, y_j)$ is an induced 4-cycle.

(1) By Lemma 2.48, $\{z, z_i\} \notin E(G)$ for all $z \in N_G(y_j) \setminus \{y_i, z_j\}$.

(2) If $z_i \in N_G(y_k)$, then by Lemma 2.48 $\{y_j, z_i\} \in E(G)$. A contradiction since C is induced. Thus $z_i \notin N_G(y_k)$. Similarly, $z_j \notin N_G(y_k)$.

(3) From (2), $z \neq z_i, z_j$. We suppose that $\{z_i, z\} \in E(G)$, then there are paths $P_1 = \{y_k, z, z_i\}$ and $P_2 = \{y_k, y_j\}$. This is not possible by Lemma 2.48.

(4) From (2), $z_i \notin N_G(y_k)$ and $z_i \notin N_G(y_j)$ by hypothesis, then $K' = K \setminus N_G[z_i]$. Thus, by Lemma 2.59, K' is a sun-complete subgraph in G'. By (3), for all $z \in N_G(y_k) \setminus \{y_i, y_j\}$, then $\{z_i, z\} \notin E(G)$. Hence, $N_G(y_k) \setminus N_{G'}(y_k) = \{y_i\}$. Consequently, $\deg_G(y_k) = \deg_{G'}(y_k) + 1$. Furthermore, by (1) we have that $N_G(y_j) \setminus N_{G'}(y_j) = \{y_i, z_j\}$. Therefore, $\deg_G(y_j) = \deg_{G'}(y_j) + 2$. □

Corollary 2.62 Let G be a theta-ring graph. If K is a sun-complete subgraph of G and there is an induced 4-cycle $C = (a_1, a_2, a_3, a_4)$ such that $a_1, a_4 \in V(K)$ and $a_2, a_3 \notin V(K)$, then $\deg_G(a_2) = 2$ if and only if $\deg_G(a_3) = 2$.

Proof. If $\deg_G(a_2) = 2$ and $\deg_G(a_3) \geq 3$, then there is $c \in N_G(a_3) \setminus \{a_2, a_4\}$. By Lemma 2.61 $\{c, a\} \notin E(G)$ for all $a \in V(K) \setminus a_4$. Consequently, K or $K \setminus a_4$ are sun-complete subgraphs in $G_1 = G \setminus N_G[c]$. Furthermore, $G_1[\{a_1, a_2\}]$ is a a simplex in G_1. But $a_1 \in \{a_1, a_2\} \cap V(K)$ or $a_1 \in \{a_1, a_2\} \cap V(K \setminus a_4)$. A contradiction by Lemma 2.52, therefore, $\deg_G(a_3) = 2$. Similarly, if $\deg_G(a_3) = 2$, then $\deg_G(a_2) = 2$. □

Lemma 2.63 Let G be a theta-ring graph and $K = \{x, y\}$ a subgraph of G. The following condition are equivalent:

(a) K is a sun-complete subgraph.

(b) $N_G(x) \cap N_G(y) = \emptyset$ and if $z \in N_G(x) \setminus y$ and $z' \in N_G(y) \setminus x$, then $\{z, z'\} \in E(G)$.

Proof. (a) $\Rightarrow$ (b) By contradiction, if there exists $w \in N_G(x) \cap N_G(y)$, then there is a stable set S such that $w \in S$, hence $V(K) \cap S = \emptyset$. A contradiction. Now, we suppose that there exist $z \in N_G(x) \setminus y$ and $z' \in N_G(y) \setminus x$ such that $\{z, z'\} \notin E(G)$. Consequently, there exists a maximal stable set S of G such that $\{z, z'\} \subseteq S$. Hence, $x, y \notin S$. Therefore, $V(K) \cap S = \emptyset$ and K is not a sun-complete subgraph.

(b) $\Rightarrow$ (a) We take a maximal stable set S of G. If $V(K) \cap S = \emptyset$, then there exist $z \in N_G(x) \setminus y$ and $z' \in N_G(y) \setminus x$ such that $z, z' \in S$. Furthermore, $N_G(x) \cap N_G(y) = \emptyset$, hence $z \neq z'$ Consequently, $\{z, z'\} \notin E(G)$, a contradiction. Therefore, $V(K) \cap S \neq \emptyset$ and K is a sun-complete subgraph. □

Corollary 2.64 Let G be a theta-ring graph and let $K = \{x, y\}$ be a subgraph of

G. K is a sun-complete subgraph if and only if x or y is a simplicial vertex in G or there are vertices $z, z' \in V(G)$ such that (x, z, z', y) is an induced 4-cycle and $\deg_G(x) = \deg_G(y) = 2$.

Proof. $\Rightarrow$) We suppose that K does not have simplicial vertices. Thus, there exist $z \in N_G(x) \setminus y$, $z' \in N_G(y) \setminus x$ and $z \neq z'$. By Lemma 2.63, $\{z, z'\} \in E(G)$. Hence, (x, z, z', y) is an induced 4-cycle. Now, if there is $w \in N_G(x) \setminus \{y, z\}$, then $w \neq z'$ and $\{w, z'\} \in E(G)$, a contradiction by Lemma 2.61. Thus, $\deg_G(x) = 2$. Similarly, $\deg_G(y) = 2$.

$\Leftarrow$) If K has a simplicial vertex, then K is sun-complete by Remark 2.51. Now, if there is an induced 4-cycle (x, z, z', y) such that $\deg_G(x) = \deg_G(y) = 2$. If $z \in S$, then $y \in S$ since $\deg_G(y) = 2$. Similarly, if $z' \in S$, then $x \in S$. Furthermore, if $z, z' \notin S$, then $|\{x, y\} \cap S| = 1$. Therefore, K is a sun-complete subgraph in G. □

Remark 2.65 In $G = C_4$ each edge is a sun-complete subgraph.

Proof. By Corollary 2.64. □

Proposition 2.66 Let G be a theta-ring graph. If K is a sun-complete subgraph of G, then K is a simplex or there is $y \in V(K)$ such that $\deg_G(y) = |V(K)|$, and (y, z, z', y') is an induced 4-cycle with $y' \in V(K)$.

Proof. By induction on $|V(K)|$. We set $V(K) = \{y_1, ..., y_m\}$ and we assume that K is not a simplex. Hence, there is $z_i \in N_G(y_i) \setminus V(K)$ for each $i \in \{1, ..., m\}$. We take a minimal subset $L \subseteq \{z_1, ..., z_m\}$ such that $V(K) \subseteq N_G(L)$. Since K is a sun-complete subgraph, we have that L is not stable. Consequently, there exist $a_1, a'_1 \in L$ such that $\{a_1, a'_1\} \in E(G)$. Furthermore, by the minimality of L there are $y_i, y_j \in V(K)$ such that $y_i \in N_G(a_1) \setminus N_G(a'_1)$ and $y_j \in N_G(a'_1) \setminus N_G(a_1)$. Without loss of generality, we can take $i = 1$ and $j = 2$. This implies that (y_1, a_1, a'_1, y_2) is an induced 4-cycle. By Lemma 2.61, we have that $K_1 = K \setminus y_1$ is a sun-complete subgraph in $G_1 = G \setminus N_G[a_1]$. Moreover, G_1 is theta-ring since G_1 is an induced subgraph of G. Thus, by induction hypothesis K_1 is a simplex or there exists $w \in V(K_1)$ such that $\deg_{G_1}(w) = |V(K_1)|$ and (w, x, x', w') is an induced 4-cycle with $w' \in V(K_1)$. If $y_k \in V(K) \setminus \{y_1, y_2\}$, then by Lemma 2.61 $\deg_G(y_k) = \deg_{G_1}(y_k) + 1$. Hence, $N_{G_1}(y_k) = N_G(y_k) \setminus y_1$. Similarly, by Lemma 2.61 $\deg_G(y_2) = \deg_{G_1}(y_2) + 2$. Hence, $N_{G_1}(y_2) = N_G(y_2) \setminus \{y_1, a'_1\}$. Consequently, if K_1 is a simplex, then y_2

is the simplicial vertex of K_1 since K is not a simplex. This implies, $\deg_G(y_2) = \deg_{G_1}(y_2)+2 = |V(K_1)\setminus y_2| + |\{y_1, a_1'\}| = |V(K)|$. Hence, we take $y = y_2$, $y' = y_1$, $z' = a_1$ and $z = a_1'$. Now, we assume there is w with $\deg_{G_1}(w) = |V(K_1)|$. If $w = y_k$ with $k \neq 1,2$, then $\deg_G(w) = \deg_{G_1}(w)+1 = |V(K_1)|+1 = |V(K)|$. Hence, $y = w$, $y' = w'$, $z = x$ and $z' = x'$. If $w = y_2$, we can take $a_2 = x$, $a_2' = x'$ and $y_3 = w'$. Furthermore, $\deg_G(y_2) = |V(K)|+1$ and $\{a_1, a_2\}$ is a stable set by Lemma 2.61. Now, we take the sun-complete subgraph $K_2 = K \setminus y_2$ in $G_2 = G \setminus N_G[a_2]$ by Lemma 2.61. By induction hypothesis, K_2 is a simplex or there is $u \in V(K_2)$ such that $\deg_{G_2}(u) = |V(K_2)|$ and (u, q, q', u') is an induced 4-cycle with $u' \in V(K_2)$. If G is a simplex, then y_3 is a simplicial vertex in K_2. Hence, $\deg_G(y_3) = \deg_{G_2}(y_3)+2 = |V(K_2)\setminus y_3| + |\{y_2, a_2'\}| = |V(K)|$ and we can take $y = y_3$, $z = a_2'$, $z' = a_2$ and $y' = y_2$. Now, we assume there is u with $\deg_{G_2}(u) = |V(K_2)|$. If $u = y_r$ with $r \neq 1,2,3$, then $\deg_G(u) = \deg_{G_2}(u)+1 = |V(K_2)|+1 = |V(K)|$. Hence, $y = u$, $y' = u'$, $z = q$ and $z' = q'$. If $u = y_1$, then $\deg_G(y_1) = |V(K)|$. Consequently, $a_1 = q$ and $u' = y_s$ with $s \neq 1,2$, but this is not possible by Lemma 2.61. If $u = y_3$, then we can take $a_3 = q$, $a_3' = q'$ and $y_4 = u'$. Consequently, $\deg_G(y_3) = |V(K)|+1$ and $\{a_1, a_2, a_3\}$ is a stable set in G. Now, we take the sun-complete subgraph $K_3 = K \setminus y_3$ in $G_3 = G \setminus N_G[a_3]$. By induction hypothesis, K_3 is a simplex or there is $v \in V(K_3)$ such that $\deg_G(v) = |V(K_3)|$ and (v, p, p', v') is an induced 4-cycle with $v' \in V(K_3)$. By the last argument, if K_3 is a simplex, then y_4 is the simplicial vertex in K_3. Hence, $\deg_G(y_4) = |V(K)|$. Therefore, $y = y_4$, $y' = y_3$, $z = a_3'$ and $z' = a_3$. Now, we assume there is $v \in V(K_3)$ with $\deg_{G_3}(v) = |V(K_3)|$. Since $\deg_G(y_i) = |V(K)|+1$ with $i = 2,3$, then $v \neq y_i$. Moreover, $v \neq y_1$ since $\deg_G(y_1) \geq |V(K)|+1$If $v = y_s$ with $s \neq 1,2,3,4$, $\deg_G(v) = |V(K)|$, hence $y = v$, $y' = v'$, $z = q$ and $z' = q'$, then $\deg_G(v) = |V(K)|+1$ and $\{a_1, a_2, a_3, a_4\}$ is a stable set in G. If $v = y_4$, then we can take $a_4 = p$, $a_4' = p'$ and $y_5 = v'$. Furthermore, $\deg_G(y_4) = |V(K)|+1$ and $\{a_1, a_2, a_3, a_4\}$ is a stable set. With this process, we have vertices $y_2, y_3, ..., y_r$ such that $\deg_G(y_i) = |V(K)|+1$ for $i \in \{2, ..., r-1\}$. If $r-1 = m$, then $\{y_1, ..., y_m\} = V(K) \subseteq N_G(S_1)$, a contradiction since K is sun-complete. If $r = 1$, that is, $y_2, y_3, ..., y_{r-1}, y_1$ or $r \leq m-1$, we take the stable set $S_1 = \{a_1, a_2, ..., a_{r-1}\}$. Hence, $\{y_1, y_2, ..., y_{r-1}\} \subseteq N_G(S')$ for some maximal stable set S'. Therefore, we have the vertices $y_r, ..., y_m$. We can use the last argument, then we have the stable sets $S_2 = \{a_r, ..., a_k\}, ..., S_l = \{a_t, ..., a_m\}$ such that $S = \bigcup_{j=1}^{l} S_j = \{a_1, ..., a_m\}$ is a stable set in G such that $V(K) \subseteq N_G(S)$, a contradiction. Thus, there is a vertex y_n such that $y = y_n$, $\deg_G(y) = |V(K)|$ and (y, z, z', y') is an induced 4-cycle with $y' \in V(K)$. □

Definition 2.67 Let K be a complete subgraph in G with $|V(K)| = m$. A family of induced 4-cycles $T_1 = (y_1, z_1, z_1', y_2)$, $T_2 = (y_2, z_2, z_2', y_3)$,...,$T_j = (y_j, z_j, z_j', y_{j+1})$ is called a *chain of 4-cycles* in K of size j if $y_1, ..., y_j$ are different vertices in K and

$\deg_G(y_1) = \deg_G(y_{j+1}) = m$. In this case y_1 and y_{j+1} are called *end vertices*. K has a complete chain if $j = m$.

Lemma 2.68 Let K be a complete subgraph of G. If K has a chain of 4-cycles of size j and $\deg_G(y_i) = |V(K)| + 1$ for $i \in \{2, ..., j\}$, then K is a sun-complete subgraph.

Proof. We take the chain of 4-cycles $T_1 = (y_1, z_1, z_1', y_2)$, $T_2 = (y_2, z_2, z_2', y_3)$,..., $T_j = (y_j, z_j, z_j', y_{j+1})$. We suppose that there exists a maximal stable set S in G such that $V(K) \subseteq N_G(S)$. Since $N_G(y_1) \setminus V(K) = \{z_1\}$ and $N_G(y_{j+1}) \setminus V(K) = \{z_j'\}$, then $z_1, z_j' \in S$. Hence, $z_1' \notin S$. Since $\deg_G(y_2) = |V(K)| + 1$, then $N_G(y_2) \setminus V(K) = \{z_1', z_2\}$, hence $z_2 \in S$. Similarly, we have that $N_G(y_i) \setminus V(K) = \{z_{i-1}', z_i\}$ for $i \in \{3, ..., j\}$. Hence, $z_1, ..., z_j \in S$, but $\{z_j, z_j'\} \in E(G)$, a contradiction. Thus, K is a sun-complete subgraph of G. □

Theorem 2.69 Let G be a theta-ring graph. If K is a sun-complete subgraph of G, then K is a simplex or there exists a chain of 4-cycles in K.

Proof. We proceed by induction on $|V(K)|$. If $|V(K)| = 2$, by Corollary 2.64, K is a simplex or there is an induced 4-cycle (y_1, z, z', y_2) with $y_1, y_2 \in V(K)$ and $\deg_G(y_1) = \deg_G(y_2) = 2 = |V(K)|$. Now, if $|V(K)| \geq 3$, by Proposition 2.66 K is a simplex or there exists a vertex y_1 such that $\deg_G(y_1) = |V(K)|$ and (y_1, z, z', y_2) is an induced 4-cycle with $y_2 \in V(K)$. We suppose that K is not a simplex. By Lemma 2.61, $K' = K \setminus y_1$ is a sun-complete subgraph in $G' = G \setminus N_G[z]$. Thus, by induction hypothesis, if K' is a simplex, then y_2 is a simplicial vertex in K', since K is not a simplex. Consequently, $\deg_G(y_2) = |V(K)|$ and (y_1, z, z', y_2) is a chain of 4-cycles in K. Now, if K' has a chain of 4-cycles, then there exist different vertices $y_1', ..., y_s'$ in K' such that $T_1, ..., T_{s-1}$ is a chain of 4-cycles of K' with $T_i = (y_i', z_i, z_i', y_{i+1}')$ such that $\deg_G(y_1') = \deg_G(y_s') = |V(K')|$. If $y_1', y_s' \neq y_2$, then $\deg_G(y_1') = \deg_{G''}(y_1') + 1 = |V(K')| + 1 = |V(K)|$, of the same way $\deg_G(y_s') = |V(K)|$, then $T_1, ..., T_{s-1}$ is a chain of 4-cycles in K. If $y_2 \in \{y_1', y_s'\}$ without loss of generality, we can take $y_s' = y_2$ and $y_{s+1}' = y_1$, then $\deg_G(y_1') = |V(K)|$ and $\deg_G(y_{s+1}') = |V(K)|$ with $T_1, ..., T_{s-1}, T_s$ a chain of 4-cycles in K, where $T_s = (y_s', z', z, y_{s+1}')$. □

Proposition 2.70 Let G be a well-covered theta-ring graph. If K_1 and K_2 are different sun-complete subgraphs, then G has an induced 4-cycle as a connected component or $V(K_1) \cap V(K_2) = \emptyset$.

Proof. By Lemma 2.52, we can suppose that K_1 and K_2 are not simplexes. We suppose that $V(K_1) \cap V(K_2) = \{w_1, ..., w_r\}$ with $r \geq 1$. We take a connected component G' of G such that K_1 and K_2 are subgraphs. From Theorem 2.69, K_1 has a chain of 4-cycles $T_1 = (y_1, b_1, b'_1, y_2), ..., T_j = (y_j, b_j, b'_j, y_{j+1})$. We suppose that $y_1, y_{j+1} \in V(K_1) \cap V(K_2)$. Since $N_G(y_1) \setminus V(K_1) = \{b_1\}$, then $b_1 \in V(K_2)$, hence $\{b_1, y_{j+1}\} \in E(G)$. If $j = 1$, then $j + 1 = 2$, hence (y_1, b_1, b'_1, y_2) is not an induced 4-cycle, a contradiction. Therefore, $j \geq 2$, but $\{b_1, y_{j+1}\} \notin E(G)$ by Lemma 2.61, a contradiction. Now, if $y_1 \in V(K_1) \cap V(K_2)$, then $y_{j+1} \notin V(K_1) \cap V(K_2)$. If $j \geq 2$, then $\{y_1, b_j\}, \{b_1, b_j\} \notin E(G)$ by Lemma 2.61. Hence, $K_1 \setminus y_j$ and K_2 (or $K_2 \setminus y_j$) are sun-complete subgraphs in $G_1 = G \setminus N_G[b_j]$ by Lemma 2.59. But, $y_{j+1} \in V(K_1 \setminus y_j)$ is a simplicial vertex in G_1. Hence, $y_1 \in V(K_1 \setminus y_j) \cap V(K_2)$ (or $y_1 \in V(K_1 \setminus y_j) \cap V(K_2 \setminus y_j)$), a contradiction by Lemma 2.52. Hence, $K_1 \setminus y_j = K_2$ (or $K_1 \setminus y_j = K_2 \setminus y_j$). Therefore, $K_2 \subsetneq K_1$ (or $K_1 = K_2$). This is not possible. Then $j = 1$, if there is $c \in N_G(b'_1) \setminus \{b_1, y\}$, then $\{y_1, c\} \notin E(G)$. Hence, $c \notin V(K_2)$. Therefore, $K'_1 = K_1 \setminus N_G[c]$ and $K'_2 = K_2 \setminus N_G[c]$ are sun-complete subgraphs in $G_2 = G \setminus N_G[c]$, but y_2 is a simplicial vertex in G_2 and $y_1 \in V(K'_1) \cap V(K'_2)$, this is a contradiction by Lemma 2.52. Thus, $\deg_G(b'_1) = 2$. Furthermore, $\deg_G(b_1) = 2$ by Corollary 2.62. Hence, $|V(K_2)| = 2$ since $\deg_G(y_1) = |V(K_1)|$, consequently $V(K_2) = \{y_1, b_1\}$, by Lemma 2.63 $\deg_G(y_1) = \deg_G(b_1) = 2$. Therefore, $N_G(y_1) = \{b_1, y_2\}$. Hence, $V(K_1) = \{y_1, y_2\}$ and $\deg_G(y_1) = 2$ by Lemma 2.63. Thus, $G' = C_4$. Of the same way if $y_{j+1} \in V(K_2)$. Therefore, $G' = C_4$. Now, we take $w \neq y_1, y_{j+1}$. Since $\{b_1, x\} \notin E(G)$ for all $x \in V(K_1) \setminus \{y_1\}$ by Lemma 2.61, then if $b_1 \in V(K_2)$, then $y_1 \in V(K_2)$ since $N_G(b_1) \cap V(K_1) = \{y_1\}$, a contradiction. Then $b_1 \notin V(K_2)$. If there exists $c \in V(G)$ such that $\{b_1, c\} \in E(G)$, then K_1 and K_2 are sun-complete subgraphs in $G_3 = G \setminus N_G[c]$ by Lemma 2.59. But, y_1 is a simplicial vertex in K_1, hence K_1 is a simplex in G_3 and $y_1 \in V(K_1) \cap V(K_2)$, a contradiction by Lemma 2.52. Therefore, $\deg_G(b_1) = 2$. By Corollary 2.62 $\deg_G(b'_1) = 2$. Hence, we take a maximal stable set S of G such that $w \in S$. If $S_1 = S \setminus w \cup \{y_1\}$ is a stable set, then S_1 is maximal since G is well-covered, then $S_1 \cap V(K_2) = \emptyset$, a contradiction. Then $b_1 \in S$, hence $S_2 = S \setminus \{w, b_1\} \cup \{y_1, b'_1\}$ is a maximal stable set and $S_2 \cap V(K_2) = \emptyset$, a contradiction. Thus, $V(K_1) \cap V(K_2) = \emptyset$. □

Lemma 2.71 Let $G = H_1 \oplus H_2$ be a well-covered theta-ring graph. If H_2 is a cycle, then $|V(H_2)| \leq 5$. Furthermore:

(a) If H_2 is a 5-cycle and $\oplus$ is a 2-clique-sum, then H_2 is a semi-basic 5-cycle.

(b) If H_2 is a 4-cycle, then $\oplus$ is a 2-clique-sum. Furthermore, if $H_2 = (x_1, x_2, x_3, x_4)$ with $\deg_G(x_2) = \deg_G(x_3) = 2$, then x_1 is a shedding vertex in G if and only if x_4 is a shedding vertex in G.

Proof. We assume $H_2 = (x_1, x_2, ..., x_k)$ with $\deg_G(x_1) \geq 3$ and $\deg_G(x_i) = 2$ for $i \in \{2, ..., k-1\}$. Hence, there is $y_1 \in V(G) \setminus V(H_2)$ such that $\{x_1, y_1\} \in E(G)$. If $k = 6$, then $N_{G_1}(x_4)$ has two free vertices, x_3 and x_5, in $G_1 = G \setminus N_G[x_1]$. If $k = 7$, then $N_{G_2}(x_3)$ has two free vertices, x_2 and x_4, in $G_2 = G \setminus N_G[y_1, x_6]$. If $k \geq 8$, then $N_{G_3}(x_4)$ has two free vertices, x_3 and x_5, in $G_3 = G \setminus N_G[x_1, x_7]$. This is impossible by Corollary 2.53. Therefore, $k \leq 5$. On the other hand:

(a) If $k = 5$ and $\oplus$ is a 2-clique-sum, then there is $y_2 \notin V(H_2)$ such that $\{x_5, y_2\} \in E(G)$. If $\{y_1, y_2\}$ is a stable set, then $N_{G_4}(x_3)$ has two free vertices, x_2 and x_4, in $G_4 = G \setminus N_G[y_1, y_2]$, a contradiction by Corollary 2.53. Thus, $y_1 \neq y_2$ and $\{y_1, y_2\} \in E(G)$. Hence, $C = (x_1, x_5, y_2, y_1)$ is an induced 4-cycle. Now, if $\deg_G(x_1) \geq 4$, then there exists y' such that $y' \neq y_1$ and $\{x_1, y'\} \in E(G)$. By the last argument $\{y', y_2\} \in E(G)$. Since C is an induced cycle, then by Lemma 2.48, $\{x_1, y_2\} \in E(G)$, a contradiction since C is induced. Thus, $\deg_G(x_1) = \deg_G(x_5) = 3$. Therefore, H_2 is a semi-basic 5-cycle.

(b) We assume $k = 4$. If $\oplus$ is a 1-clique-sum, then $\deg_G(x_4) = 2$ and $N_{G_5}(x_3)$ has two free vertices, x_2 and x_4, in $G_5 = G \setminus N_G[y_1]$. This is impossible by Corollary 2.53. Therefore, $\oplus$ is a 2-clique-sum. Now, we suppose that x_1 is not a shedding vertex, then there is a maximal stable set S in $G \setminus N_G[x_1]$ such that $N_G(x_1) \subseteq N_G(S)$. Since x_3 is an isolated vertex in $G \setminus N_G[x_1]$, then $x_3 \in S$. If there is $w \in N_G(x_4) \setminus \{x_1, x_3\}$ such that $w \in S$, then $N_{G_6}(x_2)$ has two free vertices, x_1 and x_3, in $G_6 = G \setminus N_G[S \setminus x_3]$, a contradiction. Then, for all $w \in N_G(x_4) \setminus \{x_1, x_3\}$ there exists $z \in S$ such that $\{z, w\} \in E(G)$. Hence, $N_G(x_4) \subseteq N_G((S \setminus x_3) \cup x_2)$. Furthermore, $(S \setminus x_3) \cup x_2$ is a stable set in $G \setminus N_G[x_4]$. Therefore, x_4 is not a shedding vertex. Similarly, if x_4 is not a shedding vertex in G, then x_1 is not a shedding vertex in G. $\square$

Definition 2.72 A graph G is in the family $\mathcal{T}$ if $V(G)$ can be partitioned into three disjoint subsets S_G, C_G and C'_G such that S_G contains the vertices of the sun-complete subgraphs and they are a partition of S_G; the subset C_G contains the vertices of the basic 5-cycles and they form a partition of C_G; the subset C'_G contains the vertices of the semi-basic 5-cycles and the semi-basic 5-cycles form a partition of C'_G.

Lemma 2.73 If $G \in \mathcal{T}$, then G is well-covered.

Proof. We take a maximal stable set S of G. Let $C = (a, b, c, d, e)$ be a basic or semi-basic 5-cycle, then $|S \cap V(C)| \leq 2$. We will prove $|S \cap V(C)| = 2$. We can assume that $\deg_G(a) = \deg_G(c) = 2$. If $a, c \in S$, then $|S \cap V(C)| = 2$. Now, we suppose $|\{a, c\} \cap S| \leq 1$. We can assume that $a \notin S$. Consequently, if $b \notin S$, then $e \in S$, $d \notin S$ and $c \in S$. Therefore, $|S \cap V(C)| = 2$. If $b \in S$ and we suppose $S \cap V(C) = \{b\}$, then $\deg_G(d) \geq 3$ and $\deg_G(e) \geq 3$. This implies that C is a semi-basic 5-cycle. Hence, there exists a 4-cycle $Q = (d, e, f, g)$ with $\deg_G(d) = \deg_G(e) = 3$. Consequently, $f, g \in S$, since $a, c, d, e \notin S$. This is a contradiction since $\{f, g\} \in E(G)$, therefore, $|S \cap V(C)| = 2$. Also, $|V(K) \cap S| = 1$ for each sun-complete subgraph K of G. By Lemma 2.60, Proposition 2.70 and Lemma 2.56, $S_G \cup C_G \cup C'_G$ is a partition of $V(G)$. Therefore, G is well-covered. □

Proposition 2.74 Let G be a connected well-covered theta-ring graph with a simplicial vertex x. If $G' = G \setminus y \in \mathcal{T}$ for some $y \in N_G(x)$, then $G \in \mathcal{T}$.

Proof. We will prove that $S_G = S_{G'} \cup \{y\}$ and $C_G \cup C'_G = C_{G'} \cup C'_{G'}$. We take G'_1 a connected component of G'. We take a sun-complete subgraph K of G'_1. If $x \in K$, then $K \cup \{y\}$ is a sun-complete subgraph in G, since x is a simplicial vertex in G. Now, we assume $x \notin K$. If there is a maximal stable set S of G such that $V(K) \cap S = \emptyset$, then $y \in S$ since K is a sun-complete subgraph of G'. This implies, $V(K) \subseteq N_G(S)$. Also, $S \setminus y$ is a stable set of G'. Thus, there exists a maximal stable set S' of G' such that $S \setminus y \subseteq S'$. Consequently, $|S' \cap V(K)| = 1$; we take $w \in S' \cap V(K)$. Hence, $w \notin N_G(S \setminus y)$. Furthermore, if $x \in N_G(S \setminus y)$, then there is $a \in S \setminus y$ such that $\{a, x\} \in E(G)$. Since x is a simplicial vertex, we have that $\{y, a\} \in E(G)$. This is not possible since S is a stable set, hence $x \notin N_G(S \setminus y)$. Consequently, $(S \setminus y) \cup \{x, w\}$ is a stable set of G. This is a contradiction since G is well-covered. Hence, K is a sun-complete graph in G. Therefore, $S_G = S_{G'} \cup \{y\}$.

Now, let $C = (a_1, a_2, a_3, a_4, a_5)$ be an induced basic 5-cycle in G'_1. If C is not a basic 5-cycle in G, then we can assume $\deg_G(a_1) \geq 3$ and $\deg_G(a_2) \geq 3$. Since $\deg_{G'_1}(a_1) = 2$, then $\{y, a_1\} \in E(G)$. Furthermore, we can suppose that $\deg_{G'_1}(x_3) = 2$. Lemma 2.48 implies that $\{y, a_3\}, \{y, a_4\} \notin E(G)$. Consequently, $\{a_4, x\} \notin E(G)$ and $\deg_G (x_3) = 2$. If $\{y, a_2\} \in E(G)$, then $N_{G'_2}[x]$ and $N_{G'_2}[a_1]$ are simplexes in $G'_2 = G \setminus N_G[a_4]$. But $y \in N_{G'_2}[x] \cap N_{G'_2}[a_1]$, a contradiction by Corollary 2.53. Thus, $\{y, a_2\} \notin E(G)$. Similarly, $\{y, a_5\} \notin E(G)$. Now, we suppose there exists $w \in N_{G'_1}(a_2)$ such that $\{w, y\} \notin E(G)$, then $\{x, w\} \notin E(G)$. By Lemma 2.48, $\{w, a_4\} \notin E(G)$. Hence, $N_{G'_3}[x]$ and $N_{G'_3}[a_1]$ are simplexes in $G'_3 = G \setminus N_G[w, a_4]$ with $y \in N_{G'_3}[x] \cap N_{G'_3}[a_1]$. This implies $x = a_1$. This is a contradiction since a_1 is not simplicial in G. Consequently, $\{w, y\} \in E(G)$. Hence, if $\deg_G(a_2) \geq 4$, then there is $w' \in N_G(a_2)$ such that $w \neq w'$ and $\{w', y\} \in E(G)$. Lemma 2.48 implies $\{y, a_2\} \in E(G)$. A contradiction, therefore $\deg_G(a_2) = 3$. Now, if $\deg_G(a_5) \geq 3$, in the same way as for a_2 there exists $z \in N_G(a_5) \setminus \{a_1, a_4\}$ such that (y, a_1, a_5, z) is an induced 4-cycle in G. If $w = z$, by Lemma 2.48 we have that $\{a_2, a_5\} \in E(G)$, a contradiction. Thus, $w \neq z$ and the paths $P_1 = \{y, z, a_5\}$ and $P_2 = \{y, w, a_2\}$ are in G. This is a contradiction by (2) of Lemma 2.48. Hence, $\deg_G(a_5) = 2$. Therefore, C is a semi-basic 5-cycle in G.

Now, let $C = (a_1, a_2, a_3, a_4, a_5)$ be a semi-basic 5-cycle in G'_1 with an induced 4-cycle $C' = (a_4, a_5, a_6, a_7)$. We have that $\deg_{G'_1}(a_1) = \deg_{G'_1}(a_3) = 2$ and $\deg_{G'_1}(a_4) = \deg_{G'_1}(a_5) = 3$. We suppose that C is not a semi-basic 5-cycle in G and that $\{y, a_1\} \in E(G)$. By Lemma 2.48 $\{y, a_3\} \notin E(G)$. If $\{y, a_6\} \in E(G)$, then there are two paths $P_1 = \{a_1, y, a_6\}$ and $P_3 = \{a_1, a_2, a_3, a_4\}$. This is a contradiction by the same Lemma, then $\{y, a_6\} \notin E(G)$. Consequently, $N_{G'_4}[x]$ and $N_{G'_4}[a_1]$ are simplexes in $G'_4 = G \setminus N_G[a_3, a_6]$. But, $y \in N_{G'_4}[x] \cap N_{G'_4}[a_1]$, a contradiction by Corollary 2.53. This implies, $\{y, a_1\} \notin E(G)$, and in the same way, $\{y, a_3\} \notin E(G)$. Now, suppose $\{y, a_5\} \in E(G)$, then by Lemma 2.48 $\{y, a_2\}, \{y, a_7\} \notin E(G)$. Furthermore, since C' is an induced cycle, then by this Lemma $\{a_2, a_7\} \notin E(G)$. Consequently, $N_{G'_5}[x]$ and $N_{G'_5}[a_5]$ are simplexes in $G'_5 = G \setminus N_G[a_2, a_7]$. But $y \in N_{G'_5}[x] \cap N_{G'_5}[a_5]$. So, $\{y, a_5\} \notin E(G)$. Similarly, $\{y, a_4\} \notin E(G)$. Therefore, C is a semi-basic 5-cycle in G. Thus, $S_G = S_{G'} \cup \{y\}$ and $C_G \cup C'_G = C_{G'} \cup C'_{G'}$. □

Theorem 2.75 Let G be a connected theta-ring graph. G is well-covered if and only if G is a 7-cycle or $G \in \mathcal{T}$.

Proof. $\Rightarrow$) Suppose that G is not a 7-cycle. By induction on $|V(G)|$. Since G is a theta-ring graph, then $G = H_1 \oplus_1 H_2 \oplus_2 \cdots \oplus_{s-1} H_s$ where H_i is a cycle or a complete graph.

If H_s is complete, then G has a simplicial vertex $x \in V(H_s)$. By Lemma 1.103, each $y \in N_G(x)$ is a shedding vertex of G. Note that $G' = G \setminus y$ is well-covered since G is well-covered by Corollary 2.8. Furthermore, G' is a theta-ring graph since G' is an induced subgraph of G. We take a connected component G'_1 of G'. By induction hypothesis, $G'_1 \in \{C_7\} \cup \mathcal{T}$. If $G'_1 = C_7$, then we can take $G'_1 = (a_1, ..., a_7)$ with $\{a_1, y\} \in E(G)$. By Lemma 2.48, $a_3, a_6 \notin N_G(y)$. Consequently, $N_G[x]$ and $\{y, a_1\}$ are simplexes in $G \setminus N_G[a_3, a_6]$. But $y \in N_G[x] \cap \{y, a_1\}$, a contradiction. Hence, $G'_1 \in \mathcal{T}$ and Proposition 2.74, $G \in \mathcal{T}$.

Now, if H_s is a cycle, then by Lemma 2.71, we have that $H_s = (x_1, x_2, ..., x_k)$ with $k \in \{4, 5\}$. We can assume $\deg_G(x_i) = 2$ for $i \in \{2, ..., k-1\}$ and $\deg_G(x_1) \geq 3$. Thus, there is $y_1 \in V(G) \setminus V(H_s)$ such that $\{x_1, y_1\} \in E(G)$. If $k = 5$ and $\oplus_{s-1}$ is a 1-clique-sum, then $\deg_G(x_5) = 2$ and H_s is a basic 5-cycle. Furthermore, $G_1 = G \setminus N_G[x_3]$ is a connected well-covered theta-ring graph and $\deg_{G_1}(x_5) = 1$. Hence, by induction hypothesis $G_1 \in \mathcal{T}$ and $x_5 \in S_{G_1}$. Since $\deg_{G_1}(x_5) = 1$, then $\{x_1, x_5\} \subseteq S_{G_1}$. Consequently, $S_G = S_{G_1} \setminus \{x_1, x_5\}$, $C_G = C_{G_1} \cup V(H_s)$ and $C'_G = C'_{G_1}$. This implies, $G \in \mathcal{T}$. Now, we assume $k = 5$ and $\oplus_{s-1}$ is a 2-clique-sum. Hence, by Lemma 2.71, H_s is a semi-basic 5-cycle. Thus, there is an induced 4-cycle (x_1, y_1, y_2, x_5). If there is a stable set S in $G \setminus N_G[x_3]$ such that $N_G(x_3) \subseteq N_G(S)$, then $x_1, x_5 \in S$. But $\{x_1, x_5\} \in E(G)$, this is not possible. Then x_3 is a shedding vertex in G. Thus, $G_2 = G \setminus x_3$ is a connected well-covered theta-ring graph. Furthermore, $\deg_{G_2}(x_2) = \deg_{G_2}(x_4) = 1$, then $G_2 \in \mathcal{T}$ and $x_2, x_4 \in S_{G_2}$. Consequently, $\{x_1, x_2\}$ and $\{x_4, x_5\}$ are sun-complete subgraphs in G_2. This implies $S_G = S_{G_2} \setminus \{x_1, x_2, x_4, x_5\}$, $C_G = C_{G_2}$ and $C'_G = C'_{G_2} \cup V(H_s)$. Therefore, $G \in \mathcal{T}$.

Now, when $k = 4$ we have that $\oplus_{s-1}$ is a 2-clique-sum by Lemma 2.71. We take $G_3 = G \setminus \{x_2, x_3\}$, hence $G_3 = H_1 \oplus_1 H_2 \oplus_2 \cdots \oplus_{s-2} H_{s-1}$, implying that G_3 is a connected theta-ring graph. Now, we take S a maximal stable set of G_3. Since $\{x_1, x_4\} \in E(G_3)$, then $x_1 \notin S$ or $x_4 \notin S$. Consequently, $S \cup \{x_2\}$ or $S \cup \{x_4\}$ are maximal stable sets of G. This implies that $\beta(G_3) = \beta(G) - 1$ and G_3 is a well-covered graph. By induction hypothesis, $G_3 \in \{C_7\} \cup \mathcal{T}$. First, we suppose $G_3 = C_7 = (x_1, y_1, y_2, y_3, y_4, y_5, x_4)$, then $N_{G_4}(x_3)$ has two free vertices, x_2 and x_4, in $G_4 = G \setminus N_G[y_1, y_4]$, a contradiction by Corollary 2.53. Hence, $G_3 \in \mathcal{T}$. Now, $\{x_2, x_3\}$ is a sun-complete subgraph in G by Lemma 2.63. Then, $S_G = S_{G_3} \cup \{x_2, x_3\}$, $C_G = C_{G_3}$ and $C'_G = C'_{G_3}$. Therefore, $G \in \mathcal{T}$.

$\Leftarrow$) If $G \in \mathcal{T}$, then G is well-covered by Lemma 2.73. Furthermore, a 7-cycle is well-covered by Remark 1.88. $\square$

2.8 PURE VERTEX DECOMPOSABLE GRAPHS WHOSE 5-CYCLES ARE CHORDED

In this section, we characterize the pure vertex decomposable graphs whose 5-cycles have at least 4 chords.

Definition 2.76 A 5-cycle C is chorded if C has at least 4 chords.

Lemma 2.77 Let G be a well-covered graph such that each 5-cycle of G is chorded. If $G' = G \setminus v$ is a well-covered subgraph with two different simplexes S_1' and S_2' such that $S_1' \subseteq N_G[v]$ and $|S_2'| \geq 3$, then $[V(S_1') \cup v]$ and S_2' are simplexes in G.

Proof. Let a_j' be a simplicial vertex of S_j' for $j = 1, 2$. By Proposition 1.123, $\emptyset = S_1' \cap S_2' = N_{G'}[a_1'] \cap N_{G'}[a_2']$, then $\{a_1', a_2'\} \notin E(G)$. Since $S_1' \subseteq N_G[v]$, then $S_1 = [V(S_1') \cup \{v\}]$ is a simplex with a_1' a simplicial vertex in G. We take $B = \{x \in V(S_2') \mid \{x, v\} \notin E(G)\}$. If $B = \emptyset$, then $S_2' \subseteq N_G(v)$ and $S_2 = [V(S_2') \cup \{v\}]$ is a simplex with a_2' a simplicial vertex in G. But $v \in V(S_1') \cap V(S_2')$, a contradiction by Proposition 1.123. Hence, we can assume that $B = \{b_1, ..., b_l\}$ with $l \geq 1$. If $a_2' \in B$, then $\{a_2', v\} \notin E(G)$ and S_2' is a simplex in G. Consequently, we can assume $a_2' \notin B$. If there exists $i \in \{1, ..., l\}$ such that $N_G(b_i) \subseteq V(S_2')$, then $N_G[b_i] = S_2'$ and S_2' is a simplex in G. Thus, we can suppose that for each $i \in \{1, ..., l\}$, there exists $c_i \notin V(S_2')$ such that $\{b_i, c_i\} \in E(G)$. We take C a minimal subset of $\{c_1, ..., c_l\}$ such that $B \subseteq N_G(C)$. If there exist $c_{i_1}, c_{i_2} \in C$ such that $\{c_{i_1}, c_{i_2}\} \in E(G)$, then $i_1 \neq i_2$ and $C_1 = (a_2', b_{i_1}, c_{i_1}, c_{i_2}, b_{i_2})$ is a 5-cycle. By the minimality of C, $\{b_{i_1}, c_{i_2}\}, \{b_{i_2}, c_{i_1}\} \notin E(C_1)$. Hence, C_1 is not chorded, a contradiction. So, C is a stable set. Now, we will prove that $C \cap V(S_1') = \emptyset$. By contradiction suppose that $z \in C \cap V(S_1')$, this implies, there exists $b_i \in B$ such that $\{z, b_i\} \in E(G)$. If $z = a_1'$, then $b_i \in N_G(a_1') = V(S_1') \cup \{v\}$. But $b_i \neq v$, then $b_i \in V(S_1') \cap V(S_2')$. This is a contradiction by Proposition 1.123, it implies $z \neq a_1'$. Consequently, $C_2 = (v, a_1', z, b_i, a_2')$ is a 5-cycle. Furthermore, $\{a_1', a_2'\}, \{a_1', b_i\} \notin E(G)$, then C_2 is not chorded, this is a contradiction. Then $C \cap V(S_1') = \emptyset$. Thus, $C \cap N_G[a_1'] = \emptyset$, since $N_G[a_1'] = V(S_1') \cup \{v\}$ and $C \subseteq N_G(B)$. So, $a_1' \notin N_G(C)$. If $v \in N_G(C)$, then there exists $c_i \in C$ such that $\{v, c_i\} \in E(G)$. Furthermore, since $|S_2'| \geq 3$, there exists $w \in V(S_2') \setminus \{b_i, a_2'\}$. Hence, $C_3 = (v, c_i, b_i, w, a_2')$ is a 5-cycle. Since, $c_i \notin V(S_2')$ and $b_i \in B$, then $\{a_2', c_i\}, \{v, b_i\} \notin E(G)$ implying that C_3 is not chorded, a contradiction. Therefore $v \notin N_G(C)$. This implies that $S_1'' = [(S_1' \cup \{v\}) \setminus N_G[C]]$ and $S_2'' = [(S_2' \cup \{v\}) \setminus N_G[C]]$ are simplexes in $G \setminus N_G[C]$ whose simplex vertices are a'_1 and a'_2, respectively. By Remark 1.66, $G \setminus N_G[C]$ is well-covered. But, $v \in V(S''_1) \cap V(S''_2)$. This is a contradiction by Proposition 1.123. Therefore, S'_2 is a simplex in G. □

Lemma 2.78 Let $e_1, ..., e_g$ be a perfect matching of G. If $e_1, ..., e_{g'}$ is a perfect matching without 4-cycles with two e_i's in $H = G[e_1 \cup \cdots \cup e_{g'}]$ and each edge in $\{e_{g'+1}, ..., e_g\}$ has a vertex of degree 1, then $e_1, ..., e_g$ is a perfect matching of König type without 4-cycles with two e_i's in G.

Proof. We take a minimal vertex cover D' of H, then $D = D' \cup \{b_{g'+1}, ..., b_g\}$ is a vertex cover where $e_{g'+j} = \{a_{g'+j}, b_{g'+j}\}$ such that $\deg_G(a_{g'+j}) = 1$. Since H is well-covered König graph, then $|D'| = g'$ and $\tau(G) \leq |D| = g \leq \nu(G)$. Hence $\tau(G) = \nu(G)$ and $e_1, ..., e_g$ is perfect matching of König type. Furthermore, $e_1, ..., e_g$ does not contain 4-cycles with two e_i's since $\deg G(a_{g'+j}) = 1$. □

Theorem 2.79 Let G be a graph such that every 5-cycle is chorded. Δ_G is pure vertex decomposable with $\Delta_G \neq \emptyset$ if and only if G satisfies the following conditions:

(1) If $A = \{a_1, ..., a_l\}$ is the set of simplicial vertices of G, then $l \geq 1$. Furthermore, if $S_i = N_G[a_i]$ for $i \in \{1, ..., l\}$, then $S_{i_1} \cap S_{i_2} = \emptyset$ or $S_{i_1} = S_{i_2}$ for $1 \leq i_1 < i_2 \leq l$.

(2) If $G \setminus N_G[a_1, ..., a_l] \neq \emptyset$, then it has a perfect matching $e_1 = \{x_1, y_1\}, ..., e_g = \{x_g, y_g\}$ of König type without 4-cycles with two e_i's.

(3) If $\{z_1, c_i\}, \{z_2, d_i\} \in E(G)$ with $\{c_i, d_i\} = e_i$ for some $i \in \{1, ..., g\}$ and $\{z_1, z_2\} \cap e_i = \emptyset$, then $z_1 \neq z_2$ and $\{z_1, z_2\} \in E(G)$.

Proof. $\Rightarrow$) By induction on $|V(G)|$. Since Δ_G is pure vertex decomposable, thus by Remark 1.92 and Corollary 2.8, G has a shedding vertex v such that $G' = G \setminus v$ is a well-covered vertex decomposable graph. If $\Delta_{G'} = \emptyset$, then $G = \{v\}$. Consequently, we can assume $\Delta_{G'} \neq \emptyset$. By induction hypothesis, if $\{a'_1, ..., a'_{l'}\}$ is the set of simplicial vertices of G', then $l' \geq 1$. Furthermore, if $S'_i = N_{G'}[a'_i]$, then $S'_i \cap S'_j = \emptyset$ or $S'_i = S'_j$. Also, $G_1 = G' \setminus N_{G'}[a'_1, ..., a'_{l'}]$ has a perfect matching $e_1, ..., e_{g'}$ of König type satisfying (2) and (3). Without loss of generality we can assume that $\{S'_1, ..., S'_{l'}\} = \{S'_1, ..., S'_t\}$ where $t \leq l'$ and $S'_1, ..., S'_t$ are disjoint sets. By Proposition 2.9, G_1 is a well-covered König graph, then $\beta(G_1) = g'$. Furthermore, if A_1 is a maximal stable set of G_1, then $\{a_1, ..., a_t\} \cup A_1$ is a maximal stable set of G'. Since G' is well-covered, then $\beta(G') = t + g'$.

We will prove that $N_G(v)$ contains a simplicial vertex of G. Since G contains only chorded 5-cycles, hence by Lemma 2.3, there is $b \in N_G(v)$ such that $N_G[b] \subseteq N_G[v]$. We can suppose that $\deg_G(b)$ is minimal. If b is not a simplicial vertex, then there are $d_1, d_2 \in N_G(b) \setminus v$ such that $\{d_1, d_2\} \notin E(G)$. This implies, $\deg_{G'}(b) = |N_G(b) \setminus v| \geq 2$. Now, we suppose that $b \in V(G_1)$. Let G'' be the component of G_1 such that $b \in V(G'')$. Thus, G'' has a perfect matching of König type. Without loss of generality, we can assume that this matching is $e_1, ..., e_{g'_1}$. Also, there is no 4-cycle with two e_i's with $i \in \{1, ..., g'_1\}$. If $g'_1 = 1$, then there exists $x \in V(G'')$ such that $e_1 = \{b, x\}$. Furthermore, since $\deg_{G'}(b) \geq 2$, there is a vertex $b_i \in V(S'_i)$ for some $i \in \{1, ..., t\}$ such that $\{b_i, b\} \in E(G)$. If $\deg_{G'}(x) = 1$, then $N_G[x] \subsetneq N_G[b] \subseteq N_G[v]$, a contradiction by the minimality of $\deg_G(b)$. Consequently, there is a vertex $b_j \in V(S'_j)$ for some $j \in \{1, ..., t\}$ such that $\{b_j, x\} \in E(G)$. Since G' satisfies (3), we have that $b_i \neq b_j$ and $\{b_i, b_j\} \in E(G)$. So, $C_1 = (b, v, x, b_j, b_i)$ is a 5-cycle. Furthermore, G' satisfies (3) implying $\{x, b_i\}, \{b_j, b\} \notin E(G)$. Hence, C_1 is not chorded, a contradiction. Then $g'_1 \geq 2$. If $\deg_{G''}(b) = 1$, from Lemma 2.27, there exist $\{b, w_1\}, \{w_1, w_2\}, \{w_2, w_3\} \in E(G'')$ with $e_j = \{b, w_1\}$ for some $j \in \{1, ..., g'_1\}$. Furthermore, since $\deg_{G'}(b) \geq 2$, there is $b_i \in V(S'_i)$ for any $i \in \{1, ..., t\}$ such that $\{b_i, b\} \in E(G)$. Thus, $\{b_i, w_2\} \in E(G)$, since G' satisfies (3) and $\{b, w_1\} = e_j$. Also, $\{w_1, v\} \in E(G)$ since $N_G[b] \subseteq N_G[v]$. This implies that $C_2 = (v, w_1, w_2, b_i, b)$ is a 5-cycle. Since G' satisfies (3), $\{b, w_2\}, \{w_1, b_i\} \notin E(G)$ implying C_2 is not chorded, a contradiction. Then $\deg_{G''}(b) \geq 2$. Consequently, by Lemma 2.27, there exist $\{b, w'_1\}, \{w'_1, w'_2\} \in E(G'')$ with $\deg_{G''}(w'_2) = 1$ and $e_k = \{w'_1, w'_2\}$ for some $k \in \{1, ..., g'_1\}$. Since $\{a'_1, ..., a'_t\}$ is the set of simplicial vertices of G' and $w'_2 \in V(G_1)$ implying w'_2 is not a simplicial vertex of G'. Hence, there is $c \in V(S'_q)$ with $q \in \{1, ..., t\}$ such that $\{c, w'_2\} \in E(G)$. By (3), we have that $\{b, c\} \in E(G)$. Since $N_G[b] \subseteq N_G[v]$, thus $\{v, c\} \in E(G)$ and $C_3 = (v, b, w'_1, w'_2, c)$ is a 5-cycle. But, from (3), $\{c, w'_1\}, \{w'_2, b\} \notin E(G)$ implying that C_3 is not chorded. This is a contradiction, then $b \notin V(G_1)$. Consequently, $b \in N_{G'}[a'_1, ..., a'_t]$ and there is $r \in \{1, ..., t\}$ such that $b \in V(S'_r)$. This implies, $N_{G'}[a'_r] = V(S'_r) \subseteq N_G[b] \subseteq N_G[v]$. Thus, $G[V(S'_r) \cup \{v\}]$ is a simplex and a'_r is a simplicial vertex of G contained in $N_G(v)$. Therefore, there exists a simplicial vertex x of G contained in $N_G(v)$. Hence, x is a simplicial vertex of G'. Without loss of generality we can assume that $x = a'_1$. Consequently, $S'_1 \subseteq N_G[a'_1] \subseteq N_G[v]$. Now, we can assume that $S'_1, ..., S'_t$ satisfy:

(a) $|V(S'_i)| \geq 3$ for $2 \leq i \leq p$;

(b) $|V(S'_i)| = 2$ and S'_i has a simplicial vertex in G for $p + 1 \leq i \leq q$;

(c) $|V(S'_i)| = 2$ and S'_i does not have simplicial vertices in G for $q + 1 \leq i \leq t$.

By Lemma 2.77, $S_1 = [V(S'_1) \cup v], S_2 = S'_2, ..., S_q = S'_q$ are simplexes of G. We assume that a_i is a simplicial vertex of S_i. Let M be the set of simplicial vertices in

G. We take $z \in M$. We will prove that $[N_G[z]] \in \{S_1, ..., S_q\}$. If $z = v$, since $V(S_1) = N_G[a_1] \subseteq N_G[v]$, then $[N_G[z]] = S_1$. Now, assume $z \neq v$, then $N_{G'}[z] = N_G[z] \setminus v$. Furthermore, $[N_G[z]]$ is a simplex in G, consequently $[N_{G'}[z]]$ is a simplex in G'. This implies, $N_G[z] \setminus v = V(S'_i)$. Thus, $N_G[z] = V(S'_i) \cup \{v\}$ or $N_G[z] = V(S'_i)$. If $N_G[z] = V(S'_i) \cup \{v\}$, then $v \in V(S_1) \cap N_G[z]$. By Lemma 1.106, we have that $[N_G[z]] = S_1$. Now, if $N_G[z] = V(S'_i)$, then $v \notin N_G[z]$. If $i = 1$, then $[N_G[z]] = S'_1 \subsetneq S_1 = [N_G[a_1]]$. But S_1 is a simplex, then $\{z, v\} \in E(G)$, a contradiction. Hence $i \neq 1$. Consequently, $[N_G[z]] = S_i$ with $2 \leq i \leq q$. Therefore, the simplexes of G are $S_1, S_2, ..., S_q$.

Now, since $S'_i \cap S'_j = \emptyset$, then $S_1 \cap S_j = (S'_1 \cup \{v\}) \cap S'_j = \emptyset$ and $S_i \cap S_j = \emptyset$ for $i, j \in \{2, ..., q\}$ with $i \neq j$. Furthermore, $G_2 = G \setminus N_G[a_1, ..., a_q] = G \setminus (S_1 \cup \cdots \cup S_q) = G' \setminus (S'_1 \cup \cdots \cup S'_q) = G' \setminus N_{G'}[a'_1 \cup \cdots \cup a'_q]$. Hence, G_2 has a perfect matching $e_1, ..., e_{g'}, e_{g'+1}, ..., e_g$ where $e_{g'+j} = S'_{q+j}$ for $1 \leq j \leq t - q$ where $g = g' + t - q$. Since a'_{q+j} is a simplicial vertex of G', implying $\deg_{G_2}(a'_{q+j}) = 1$. By Lemma 2.78, $e_1, ..., e_g$ is a perfect matching of König type without 4-cycles with two e_i's. Therefore, G satisfies (1) and (2).

Finally, we take $\{z_1, c_i\}, \{z_2, d_i\} \in E(G)$ with $\{c_i, d_i\} = e_i$ for some $i \in \{1, ..., g\}$ and $\{z_1, z_2\} \cap e_i = \emptyset$. If $v \notin \{z_1, z_2\}$, then $\{z_1, c_i\}, \{z_2, d_i\} \in E(G')$. Consequently, e_i is not a simplex in G'. Thus, $i \in \{1, ..., g'\}$ and since G' satisfies (3), we have that $z_1 \neq z_2$ and $\{z_1, z_2\} \in E(G)$. Now, we can assume $z_1 = v$. We suppose $z_2 \in S_j$. If $j = 1$, we will prove that $v \neq z_2$. Suppose $v = z_2$, we take a maximal stable set S of G such that $v \in S$. Consequently, $e_i \cap S = \emptyset$, $|e_j \cap S| \leq 1$ for $j \neq i$ and $|S_i \cap S| \leq 1$ for $i \in \{1, ..., q\}$. So, $|S| \leq t + g' - 1$. A contradiction, since G is well-covered and $\beta(G) = \beta(G') = t + g'$. Hence, $v \neq z_2$ and $\{v, z_2\} \in E(G)$. Now, we can assume $j = 2$. We take $G_4 = G[V(G_2) \cup \{a_1, a_2, v, z_2\}]$. Since $c_i \notin S_1$ and $d_i \notin S_2$, then $a_1 \neq v$ and $a_2 \neq z_2$. This implies, G_4 has a perfect matching $e_1, ..., e_g, f_1 = \{a_1, v\}, f_2 = \{a_2, z_2\}$. Furthermore, $\deg_{G_4}(a_1) = \deg_{G_4}(a_2) = 1$ since a_1 and a_2 are simplicial vertices in G. Hence, by Lemma 2.78 $e_1, ..., e_g, f_1, f_2$ is a perfect matching of König type of G_4. On the other hand, $G_5 = G \setminus N_G[a_3, ..., a_q]$ is well-covered, by Remark 1.66. Furthermore, by Lemma 1.103, if $w \in N_G(a_i)$, then w is a shedding vertex. Thus, $G_6 = G_5 \setminus [(S_1 \setminus \{a_1, v\}) \cup (S_2 \setminus \{a_2, z_2\})]$ is a well-covered subgraph by Corollary 2.8. Since $V(G_4) = V(G_6)$ and G_4, G_6 are induced subgraphs of G, then $G_4 = G_6$. This implies, G_4 is well-covered with a perfect matching of König type. Consequently, by Proposition 2.9, we have that $\{v, z_2\} \in E(G)$. Similarly, if $z_2 \in e_j$ for some $j \in \{1, ..., g\}$. Therefore, G satisfies (3).

$\Leftarrow$) By induction on $|V(G)|$. We suppose that $L = \{S_1, ..., S_l\} = \{S_1, ..., S_t\}$ where $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq t$ with $t \leq l$. By (1), we have that $t \geq 1$. If $|V(S_i)| = 1$ for some $i \in \{1, ..., t\}$, then by induction hypothesis and by Proposition 1.106, Δ_G is pure vertex decomposable. So, we can assume that $|V(S_t)| \geq 2$, then there exists $v \in N_G(a_t)$. By Lemma 1.103, v is a shedding vertex.

We will prove that Δ_{G_2} is pure vertex decomposable where $G_2 = G \setminus v$. Since $S_t \cap S_i = \emptyset$ for $i \neq t$, then $v \in S_t \setminus \bigcup_{i=1}^{t-1} S_i$. Consequently, $S_i = G_2[N_{G_2}[a_i]]$ for $i = 1, ..., t-1$ and $S_t \setminus v = G_2[N_{G_2}[a_t]]$. Hence, $a_1, ..., a_t$ are simplicial vertices in G_2. We take $G_4 = G_2 \setminus N_{G_2}[a_1, ..., a_t] = G \setminus N_G[a_1, ..., a_t]$. Suppose $b \in B_2$, where

$$B_2 = \{b \in V(G_2) \mid b \text{ is a simplicial vertex and } N_{G_2}[b] \notin \{S_1, ..., S_{t-1}, S_t \setminus v\}\}.$$

Thus, $\{v, b\} \in E(G)$. If $b \notin V(G_4)$, then there exists $i \in \{1, ..., t\}$ such that $b \in N_{G_2}[a_i]$. Since a_i is a simplicial vertex of G_2, we have that $S_i \setminus v = N_{G_2}[a_i] \subseteq N_{G_2}[b]$. But, b is a simplicial vertex of G_2, then $S_i \setminus v = N_{G_2}[b]$, a contradiction since $b \in B_2$. Hence $b \in V(G_4)$. Consequently, from (2), there exist $b' \in V(G_4)$ and $j \in \{1, ..., g\}$ such that $e_j = \{b, b'\}$. Since $N_{G_2}[b]$ is a simplex, if $a \in N_G(b) \setminus \{b', v\}$, then $\{b', a\} \in E(G)$. This is a contradiction since G satisfies (3). Thus, $\deg_{G_2}(b) = 1$. Without loss of generality, we can assume that $B_2 = \{b_1, ..., b_p\}$ such that $b_j \in e_j$ for $j = 1, ..., p$. This implies, $S_1, ..., S_t, S_{t+1} = e_1, ..., S_{t+p} = e_p$ are the simplexes in G_2 and G_2 satisfies (1). Furthermore, $G_2 \setminus N_{G_2}[a_1, ..., a_t, b_1, ..., b_p]$ has a perfect matching $e_{p+1}, ..., e_g$ of König type. Hence, G_2 satisfies (2) and (3), since G satisfies them. Therefore, by induction hypothesis, Δ_{G_2} is pure vertex decomposable.

Now, we will prove that Δ_{G_3} is pure vertex decomposable where $G_3 = G \setminus N_G[v]$. We have that $S_i' = N_{G_3}[a_i] = S_i \setminus N_G[v]$. Now, if $\{v, a_i\} \in E(G)$ for some $1 \leq i \leq t-1$, then $v \in V(S_t) \cap V(S_i)$, a contradiction by Proposition 1.123. Then $\{v, a_i\} \notin E(G)$. Hence, $a_1, ..., a_{t-1}$ are simplicial vertices of G_3. We take $b \in B_3$, where:

$$B_3 = \{b \in V(G_3) \mid b \text{ is a simplicial vertex and } N_{G_3}[b] \neq S_i' \text{ for } 1 \leq i \leq t-1\}.$$

Since $b \in V(G_3)$ and $N_G[a_t] \subseteq N_G[v]$, then $\{b, a_t\} \notin E(G)$. Furthermore, if there exists $1 \leq j \leq t-1$ such that $\{a_j, b\} \in E(G)$, then $N_{G_3}[a_j] = N_{G_3}[b]$ since a_j and b are simplicial vertices in G_3, a contradiction. Thus $\{a_j, b\} \notin E(G)$ for $j \in \{1, ..., t\}$. Consequently, $b \in G \setminus N_G[a_1, ..., a_t]$ and there exists $b' \in V(G)$ such that $e_i = \{b, b'\}$ for $i \in \{1, ..., g\}$. If $b' \notin V(G_3)$, then $\{v, b'\} \in E(G)$. Furthermore,

G satisfies (3), then $N_G(b) \subseteq N_G(v)$ and $N_{G_3}[b] = \{b\}$. Now, if $b' \in V(G_3)$ and $b'' \in N_{G_3}(b) \setminus b'$, implying $\{b', b''\} \in E(G)$ since b is a simplicial vertex in G_3. This is a contradiction since G satisfies (3). Then $N_{G_3}(b) = \{b'\}$. Hence, we can assume that $S_1, ..., S_{t-1}, N_{G_3}[b_1], ..., N_{G_3}[b_r]$ are the distinct simplexes of G_3 where $N_{G_3}[b_i] = \{b_i\}$ or $N_{G_3}[b_i] = e_i$ for some $i = 1, ..., r$ with $r \leq g$. Assume $t - 1 + r = 0$, then $t = 1$ and $r = 0$. Furthermore, suppose $N_G(v) \cap e_i = \emptyset$ with $e_i = \{x_i, y_i\}$. We can assume $\{v, x_i\} \in E(G)$. Since G satisfies (2) and (3), then $y_i \notin N_G(v)$ and $N_G(y_i) \subseteq N_G(v)$. Consequently, $y_i \in V(G_3)$ and $\deg_{G_3}(y_i) = 0$, implying $r > 0$, a contradiction. Then $N_G(v) \cap e_i = \emptyset$ for $i = 1, ..., g$. Hence, $G_3 = G[e_1, ..., e_g]$, G_3 is a König Cohen-Macaulay graph without isolated vertex. Thus, G_3 has a free vertex and it is a simplicial vertex of G_3 and $r > 0$, a contradiction. Then $t - 1 + r > 0$ and G_3 satisfies (1). Now, we take $G_3' = G_3 \setminus N_{G_3}[a_1, ..., a_{t-1}, b_1, ..., b_r] = (G \setminus N_G[a_1, ..., a_{t-1}, v]) \setminus \cup_{i=1}^{r} e_i$. Thus, G_3' has a perfect matching $e_{r+1}, ..., e_g$ of König type without 4-cycles with two e_i's. Consequently, G_3 satisfies (2). Furthermore, G_3 satisfies (3), since G satisfies it. Therefore, Δ_{G_3} is pure vertex decomposable.

Finally, we will prove that G is well-covered. We take a maximal stable set S of G. Since S_i is a simplex, then $|S \cap S_i| = 1$ for all $i \in \{1, ..., t\}$. Now, if $G \setminus N_G[a_1, ..., a_t] = \emptyset$, then $|S| = t$. Thus, G is well-covered. If $G \setminus N_G[a_1, ..., a_t] \neq \emptyset$, then by (2) it has a perfect matching $e_1 = \{x_1, y_1\}, ..., \{x_g, y_g\}$ of König type. If $S \cap e_j = \emptyset$ for some i, then there are $w, w' \in S$ such that $\{w, x_j\}, \{w', y_j\} \in E(G)$. By (3), $\{w, w'\} \in E(G)$, but S is stable, a contradiction. Hence $|S \cap e_j| = 1$ for $1 \leq j \leq g$. Consequently, $|S| = t + g$. Therefore, G is well-covered. □

2.9 SOME BLOCKS OF WELL-COVERED GRAPHS

Some well-covered graphs as: graphs with girth at least 5 [21], graphs without 4-cycles and 5-cycles [22], simplicial and chordal graphs [36], block-cactus graphs [37] and unicyclic graphs [44] have a partition $V_1, ..., V_k$ of the vertex set $V(G)$ such that $G[V_i]$ is a special subgraph of G such that if S and S' are maximal stable sets of G, then $|S \cap V_i| = |S' \cap V_i|$ for each $i \in \{1, ..., k\}$. This motivates us to study subgraphs with the same number of elements in any maximal stable set of G. In this section we study 5-cycles C and 7-cycles C' of G such that $|C \cap S| = 2$ and $|C' \cap S| = 3$ for each maximal stable sets of G.

Definition 2.80 A 5-cycle C is *quasi-basic* of G if $|V(C) \cap S| = 2$ for each maximal stable set S of G.

Lemma 2.81 Let $C = (x_1, x_2, x_3, x_4, x_5)$ be an induced quasi-basic 5-cycle of G. Hence, if $z \in N_G(x_i) \setminus (V(C) \cup N_G(x_{i-2}))$ and $z' \in N_G(x_{i+1}) \setminus (V(C) \cup N_G(x_{i-2}))$, then $\{z, z'\} \in E(G)$.

Proof. By contradiction, suppose that there exist $z \in N_G(x_i) \setminus (V(C) \cup N_G(x_{i-2}))$ and $z' \in N_G(x_{i+1}) \setminus (V(C) \cup N_G(x_{i-2}))$ such that $\{z, z'\} \notin E(G)$. Consequently, $S_1 = \{x_{i-2}, z, z'\}$ is a stable set. Hence, there exists a maximal stable set S of G such that $S_1 \subseteq S$. Thus, $S \cap V(C) = \{x_{i-2}\}$. A contradiction, therefore $\{z, z'\} \in E(G)$. □

Proposition 2.82 Let C be a 5-cycle of G such that $|V(C) \cap S| \geq 1$ for every maximal stable set S of G. C is quasi-basic if and only if for every $z \in N_G(x_i) \setminus (V(C) \cup N_G(x_{i-2}))$ and $z' \in N_G(x_{i+1}) \setminus (V(C) \cup N_G(x_{i-2}))$, we have that $\{z, z'\} \in E(G)$, for each $i \in \{1, ..., 5\}$.

Proof. $\Rightarrow$) By Lemma 2.81.

$\Leftarrow$) We take a maximal stable set S of G. Since $|V(C) \cap S| \geq 1$, there exists $x_j \in V(C) \cap S$. We can assume $j = 1$. Suppose that $V(C) \cap S = \{x_1\}$. Thus, there exist $z \in N_G(x_3)$ and $z' \in N_G(x_4)$ such that $\{z, z', x_1\} \subseteq S$. This implies $z, z' \notin N_G(x_1)$. Since C is induced $z \neq x_1$ and $z' \neq x_1$. Therefore, $z \in N_G(x_3) \setminus (V(C) \cup N_G(x_1))$ and $z' \in N_G(x_4) \setminus (V(C) \cup N_G(x_1))$. By hypothesis, $\{z, z'\} \in E(G)$, a contradiction since S is stable. Therefore, $|V(C) \cap S| = 2$. □

Proposition 2.83 Let C be an induced 5-cycle. If there exists $x_i \in V(C)$ such that $\deg_G(x_i) = 2$, then $|V(C) \cap S| \geq 1$ for every maximal stable set S.

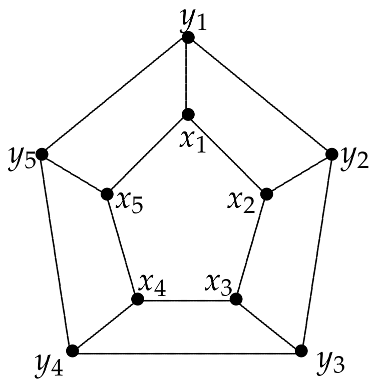
Proof. We take S a maximal stable set of G. If $x_i \in S$, then $|V(C) \cap S| \geq 1$. Now, suppose that $x_i \notin S$, then $N_G(x_i) \cap S \neq \emptyset$. But, $N_G(x_i) = \{x_{i-1}, x_{i+1}\}$. Hence, $x_{i-1} \in S$ or $x_{i+1} \in S$. Therefore, $|V(C) \cap S| \geq 1$. □

Corollary 2.84 Let C be a 5-cycle of G. If C has three non consecutive vertices of degree 2, then C is a quasi-basic 5-cycle.

Proof. We can assume that $C = (x_1, x_2, x_3, x_4, x_5)$ with $\deg_G(x_1) = \deg_G(x_3) = \deg_G(x_4) = 2$. By Proposition 2.83, $|V(C) \cap S| \geq 1$ for each maximal stable S of G.

Furthermore, there are no consecutive vertices of C whose degrees are at least 3. Therefore, by Proposition 2.82, C is a quasi-basic 5-cycle. □

Example 2.85 We take the following graph G:



Stable sets
$\{x_1,x_3,y_2,y_4\},\{x_1,x_3,y_2,y_5\},\{x_1,x_4,y_2,y_5\},$
$\{x_1,x_4,y_3,y_5\},\{x_2,x_4,y_1,y_3\},\{x_2,x_4,y_3,y_5\},$
$\{x_2,x_5,y_1,y_3\},\{x_2,x_5,y_1,y_4\},\{x_3,x_5,y_1,y_4\},$
$\{x_3,x_5,y_2,y_4\}$

(x_1,x_2,x_3,x_4,x_5) and (y_1,y_2,y_3,y_4,y_5) are quasi-basic 5-cycles of G. Its f-vector is $(f_0,f_1,f_2,f_3) = (10,31,31,10)$ and its h-vector is $h(G) = (1,6,7,-5,1)$. Consequently, G is not Cohen-Macaulay.

Definition 2.86 Let $C = (x_1,x_2,...,x_7)$ be an induced 7-cycle in G. C is called a *basic 7-cycle* if for each $y \in D_1(C)$ there exist x_{i_1},x_{i_2},x_{i_3} consecutive vertices in C such that $N_G(y)\cap V(C) = \{x_{i_1},x_{i_3}\}$ and x_{i_2} is an isolated vertex in $G\setminus N_G[y]$.

Lemma 2.87 If $C = (x_1,x_2,...,x_7)$ is a basic 7-cycle, then $N_G(x_i) \subseteq N_G(x_{i-2})\cup N_G(x_{i+2})$.

Proof. We take $z \in N_G(x_i)$. If $z \in V(C)$, then $z \in \{x_{i-1},x_{i+1}\} \subseteq N_G(x_{i-2})\cup N_G(x_{i+2})$. If $z \notin V(C)$, then $z \in D_1(C)$ and $N_G(z)\cap V(C)$ is $\{x_i,x_{i-2}\}$ or $\{x_i,x_{i+2}\}$. Therefore, $z \in N_G(x_{i-2})$ or $z \in N_G(x_{i+2})$. □

Lemma 2.88 Let C be a basic 7-cycle of G. If there exist $a_1,a_2 \in D_1(C)$ such that $L_i = V(C)\cap N_G(a_i) = \{b_1^i,b_3^i\}$ where b_1^i,b_2^i,b_3^i are consecutive vertices in C for $i = 1,2$. Furthermore, we take the 4-cycles $C_i = (a_i,b_1^i,b_2^i,b_3^i)$ for $i = 1,2$. If $L_1 \neq L_2$, then:

(1) $b_2^1 \neq b_2^2$.

(2) If $\{a_1,a_2\} \notin E(G)$, then $|V(C_1)\cap V(C_2)| \leq 1$.

Proof. (1) We suppose $b_2^1 = b_2^2$. Hence, $L_2 = \{b_1^1,b_3^1\} = L_1$. This is a contradiction, therefore $b_2^1 \neq b_2^2$.

(2) We suppose that $b_2^1 \in V(C_1) \cap V(C_2)$. Since $L_1 \neq L_2$, by (1) $b_2^1 \neq b_2^2$. Hence $b_2^1 \in \{b_1^2, b_3^2\}$, implies $\{b_2^1, a_2\} \in E(G)$. Since b_2^1 is an isolated vertex in $G \setminus N_G[a_1]$, thus $\{a_1, a_2\} \in E(G)$. But $\{a_1, a_2\} \notin E(G)$, this is a contradiction. Consequently if $|V(C_1) \cap V(C_2)| > 1$, then $\{b_1^1, b_3^1\} = \{b_1^2, b_3^2\}$. But $L_1 \neq L_2$, therefore $|V(C_1) \cap V(C_2)| \leq 1$. □

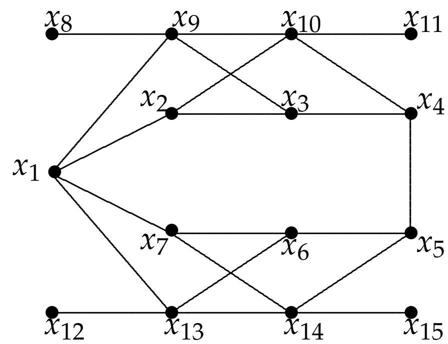
Theorem 2.89 Let C be an induced 7-cycle in a graph G. If C is a basic 7-cycle, then $|V(C) \cap S| = 3$ for all maximal stable set S in G.

Proof. Let S be a maximal stable set of G. If $G = C$, then G is well-covered by Remark 1.88. Thus, $|V(C) \cap S| = |S| = 3$. Now, we assume $G \neq C$, then there exists $y \in D_1(C)$. Since C is a basic 7-cycle, then for each $y_i \in D_1(C)$ there are $b_1^i, b_2^i, b_3^i \in V(C)$ such that $C_i = (y_i, b_1^i, b_2^i, b_3^i)$ is an induced 4-cycle in G and $L_i = V(C) \cap N_G(y_i) = \{b_1^i, b_3^i\}$. We take

$$r = \max\{|A| \mid A \subseteq D_1(C) \cap S \text{ such that if } y_i, y_j \in A \text{ with } y_i \neq y_j, \text{then } L_i \neq L_j\}.$$

We will prove that $r \leq 3$. Suppose that there are $y_1, y_2, y_3, y_4 \in D_1(C) \cap S$ with $L_i \neq L_j$ for $1 \leq i < j \leq 4$. By Lemma 2.88, we have that $|V(C_i) \cap V(C_j)| \leq 1$. Since b_1^i, b_2^i, b_3^i are consecutive in C, then $|V(C_1 \setminus y_1) \cup V(C_2 \setminus y_2) \cup V(C_3 \setminus y_3) \cup V(C_4 \setminus y_4)| \geq 8$. This is a contradiction since $|V(C)| = 7$. Therefore, $r \leq 3$. If $r = 3$, then b_2^1, b_2^2, b_2^3 are isolated vertices in $G \setminus N_G[y_1, y_2, y_3]$. Hence $b_2^1, b_2^2, b_2^3 \in S$. Furthermore, $|V(C) \cap S| \leq 3$, then $|V(C) \cap S| = 3$. Now, if $r = 2$, we have two isolated vertices b_2^1 and b_2^2 in $G \setminus N_G[y_1, y_2]$, then $b_2^1, b_2^2 \in S$. If $|V(C_1) \cap V(C_2)| = 1$, then without loss of generality, we assume $C_1 = (y_1, x_1, x_2, x_3)$ and $C_2 = (y_2, x_3, x_4, x_5)$. Thus, $|\{x_6, x_7\} \cap S| = 1$. Hence $|V(C) \cap S| = 3$. Now, if $|V(C_1) \cap V(C_2)| = 0$, then $|V(C) \setminus N_G[y_1, y_2]| = 1$. Consequently, $|V(C) \cap S| = 3$. Finally, if $r = 1$, then b_2^1 is an isolated vertex in $G \setminus N_G[y_1]$, implies that $b_2^1 \in S$. We can assume $\{b_1', b_2', b_3'\} = \{x_1, x_2, x_3\}$. Consequently, $P = (x_4, x_5, x_6, x_7)$ is a path in $G \setminus N_G[y_1]$, implying $|P \cap S| = 2$. Therefore, $|V(C) \cap S| = 3$. □

Example 2.90 In the following graph $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ is a basic 7-cycle.



Stable sets

$\{x_1, x_4, x_6, x_8, x_{10}, x_{12}, x_{15}\}$, $\{x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{15}\}$, $\{x_1, x_4, x_6, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_2, x_4, x_6, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_3, x_5, x_7, x_9, x_{11}, x_{12}, x_{14}\}$, $\{x_2, x_4, x_7, x_9, x_{11}, x_{12}, x_{15}\}$, $\{x_2, x_5, x_7, x_9, x_{11}, x_{12}, x_{15}\}$, $\{x_3, x_5, x_7, x_9, x_{11}, x_{12}, x_{15}\}$, $\{x_1, x_3, x_6, x_8, x_{10}, x_{12}, x_{15}\}$, $\{x_1, x_3, x_6, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_1, x_3, x_6, x_8, x_{11}, x_{12}, x_{14}\}$, $\{x_1, x_3, x_6, x_8, x_{10}, x_{12}, x_{14}\}$, $\{x_2, x_5, x_7, x_9, x_{11}, x_{13}, x_{15}\}$, $\{x_2, x_4, x_7, x_9, x_{11}, x_{13}, x_{15}\}$, $\{x_2, x_5, x_7, x_8, x_{11}, x_{13}, x_{15}\}$, $\{x_2, x_4, x_7, x_8, x_{11}, x_{13}, x_{15}\}$, $\{x_2, x_4, x_6, x_8, x_{11}, x_{13}, x_{15}\}$, $\{x_2, x_4, x_6, x_8, x_{10}, x_{13}, x_{15}\}$, $\{x_3, x_5, x_7, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_2, x_5, x_7, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_2, x_4, x_7, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_1, x_3, x_5, x_8, x_{11}, x_{12}, x_{15}\}$, $\{x_3, x_5, x_7, x_8, x_{11}, x_{12}, x_{14}\}$, $\{x_1, x_3, x_5, x_8, x_{11}, x_{12}, x_{14}\}$

Chapter 3
Shellable and Cohen-Macaulay Clutters

3.1 Introduction

Let $\mathcal{C} = (V, E)$ be a clutter with vertex set $V(\mathcal{C}) = \{x_1, ..., x_n\}$. $I = I(\mathcal{C})$ is the edge ideal of $\mathcal{C}$ in the polynomial ring $R = k[x_1, ..., x_n]$ over a field k and $\Delta_{\mathcal{C}}$ is the Stanley-Reisner simplicial complex of $\mathcal{C}$; that is, the simplicial complex of the stable sets of $\mathcal{C}$. In this Chapter we study matroids and clutters with a perfect matching of König type with the following properties: vertex decomposable, shellable and Cohen-Macaulay. The Chapter is divided as follows: in Section 3.2, we show that given a matroid $\mathcal{M}$, the simplicial complex $\Delta_{\mathcal{B}(\mathcal{M})}$ is shellable if and only if $\mathcal{B}(\mathcal{M})$ is complete (see Theorem 3.5). In Section 3.3, we study clutters with a perfect matching of König type.

3.2 Matroids

In this section, we denote by $\mathcal{M}$ a matroid with vertex set $V(\mathcal{M}) = \{x_1, ..., x_n\}$. We will prove that $\Delta_{\mathcal{B}(\mathcal{M})}$ is shellable if and only if $\mathcal{B}(\mathcal{M})$ is a complete clutter. We can associate the simplicial complex $\Delta_{\mathcal{B}(\mathcal{M})}$ to $\mathcal{M}$ whose Stanley-Reisner ideal is $I(\mathcal{B}(\mathcal{M}))$. Hence, we have that $I_{\Delta_{\mathcal{B}(\mathcal{M})}} = I(\mathcal{B}(\mathcal{M}))$.

Theorem 3.1 Let $\mathcal{M}$ be a matroid with bases set $\mathcal{B}(\mathcal{M})$. $\mathcal{B}(\mathcal{M})$ satisfies the weak circuit exchange property if and only if $\mathcal{B}(\mathcal{M})$ is a complete clutter.

Proof. $\Rightarrow$) Let $V(\mathcal{M}) = \{x_1, ..., x_n\}$ be the vertex set of $\mathcal{M}$. If $A \subseteq V(\mathcal{M})$ we denote $\mathcal{M} \cap A$ to $\{B \in \mathcal{B}(\mathcal{M}) \mid B \subseteq A\}$. Let B_1 and B_2 be two bases of $\mathcal{M}$. We will prove that the clutter $\mathcal{M} \cap (B_1 \cup B_2)$ is a k-complete clutter. By induction on the number $d = |B_1 \setminus B_2|$. If $d = 0$, then $B_1 = B_2$. Hence, $\mathcal{M} \cap (B_1 \cup B_2) = \{B_1\}$ and it is a k-complete clutter.

Now, if $d \geq 1$, then we can assume $B_1 = \{x_1, ..., x_d, z_{d+1}, ..., z_k\}$ and $B_2 = \{y_1, ..., y_d, z_{d+1}, ..., z_k\}$ with $y_i \neq x_j$ for all $i, j \in \{1, ..., d\}$. Since $x_i \in B_1 \setminus B_2$, then there exists $y_j \in B_2 \setminus B_1$ such that $B_1' = (B_1 \setminus x_i) \cup \{y_j\} \in \mathcal{B}(\mathcal{M})$. Hence, $B_1' = \{x_1, ..., x_{i-1}, y_j, x_{i+1}, ..., x_l, z_{l+1}, ..., z_k\}$. But $|B_1' \setminus B_1| = d - 1$. By induction hypothesis $\mathcal{M} \cap (B_1 \cup B_1') = \mathcal{M} \cap (B_1 \cup B_2 \setminus x_i)$ is a k-complete clutter for $i = 1, ..., d$. Similarly, $\mathcal{M} \cap (B_1 \cup B_2 \setminus y_i)$ is a k-complete clutter for $i = 1, ..., d$. In particular, since $B_3 = \{x_1, y_2, ..., y_l, z_{l+1}, ..., z_k\} \subseteq (B_1 \cup B_2) \setminus y_1$, we have that $B_3 \in \mathcal{M} \cap (B_1 \cup B_2 \setminus y_1) \subseteq \mathcal{B}(\mathcal{M})$. Furthermore, $z_j \in B_2 \cap B_3$ for all $j \in \{l+1, ..., k\}$. Hence, there exist $B_2 \in \mathcal{B}(\mathcal{M})$ and $B_4 \subseteq (B_2 \cup B_3) \setminus z_j$ since $\mathcal{B}(\mathcal{M})$ satisfies the circuit exchange property. But $|B_2 \cup B_3 \setminus z_j| = k$, thus $B_4 = (B_2 \cup B_3) \setminus z_j$. This implies that $B_4 = \{x_1, y_1, ..., y_d, z_{d+1}, ..., z_{j-1}, z_{j+1}, ..., z_k\}$. Now, we have that $|B_1 \setminus B_4| = d$ and $z_j \in B_1 \setminus B_4$, then $\mathcal{M} \cap (B_1 \cup B_4 \setminus z_j)$ is a k-complete clutter. Furthermore, $(B_1 \cup B_4) \setminus z_j = (B_1 \cup B_2) \setminus z_j$. Consequently, $\mathcal{M} \cap (B_1 \cup B_2 \setminus w)$ is k-complete for $w \in \{x_1, ..., x_l, y_1, ..., y_l, z_{l+1}, ..., z_k\} = B_1 \cup B_2$. Therefore, $\mathcal{M} \cup (B_1 \cup B_2)$ is a k-complete clutter.

Now, we will prove that $\mathcal{M} \cap (B_1 \cup \cdots \cup B_s)$ is a k-complete clutter with $B_i \in \mathcal{B}(\mathcal{M})$ for $i = 1, ..., s$. By induction on s. We have proven the result for $s = 2$. We suppose that $s \geq 3$ and we take $B \subseteq B_1 \cup \cdots \cup B_s$ such that $|B| = k$, we denote by $B_i' = B \cap B_i$. We will prove that $B \in \mathcal{B}(\mathcal{M})$. If $B_i' = \emptyset$ for some i, then $B \subseteq B_1 \cup \cdots \cup B_{i-1} \cup B_{i+1} \cup \cdots \cup B_s = V_i$. By induction hypothesis $\mathcal{M} \cap V_i$ is a k-complete clutter, consequently $B \in \mathcal{B}(\mathcal{M})$. Hence, we can assume that $B_i' \neq \emptyset$ for all $i = 1, ..., s$. Since $|B| = |B_i| = |B_j| = k$, then there exists $A \subseteq B_i \cup B_j$ such that $|B_i' \cup B_j' \cup A| = k$. Thus, $H = B_i' \cup B_j' \cup A \in \mathcal{B}(\mathcal{M})$ since $\mathcal{M} \cap (B_i \cup B_j)$ is a k-complete clutter. So, by induction hypothesis if $D = B_1 \cup \cdots \cup B_i' \cup B_{i+1} \cup \cdots \cup B_j' \cup \cdots \cup B_s \cup H$, then $\mathcal{M} \cap D$ is a k-complete clutter. But, $B_i' \cup B_j' \subseteq H$ implies $B \subseteq D$. Hence, $B \in \mathcal{B}(\mathcal{M})$. Finally, we have that $\mathcal{B}(\mathcal{M}) = \mathcal{M} \cap (\bigcup_{B_i \in \mathcal{B}(\mathcal{M})} B_i)$. Therefore, $\mathcal{B}(\mathcal{M})$ is a k-complete clutter.

$\Leftarrow$) We take $B_1, B_2 \in \mathcal{B}(\mathcal{M})$ such that $B_1 \neq B_2$ and $x \in B_1 \cap B_2$, then $|B_1 \cup B_2 \setminus x| \geq |B_1|$. Hence, there exists $B_3 \subseteq (B_1 \cup B_2) \setminus x$ such that $|B_3| = |B_1|$. Since $\mathcal{B}(\mathcal{M})$ is a k-complete clutter, therefore $B_3 \in \mathcal{B}(\mathcal{M})$. $\square$

Lemma 3.2 $\mathcal{H}$ is a k-complete clutter if and only if $\mathcal{F}(\Delta_{\mathcal{H}})$ is a $(k-1)$-complete clutter.

Proof. $\Rightarrow$) We take any facet $F = \{x_{i_1}, ..., x_{i_s}\}$ of $\Delta_{\mathcal{H}}$. If $s \geq k$, then $\{x_{i_1}, ..., x_{i_k}\}$ $\in E(\mathcal{H})$ since $\mathcal{H}$ is a k-complete clutter, a contradiction since F is a stable set of $\mathcal{H}$. Thus $s \leq k-1$. Now, we take $F' \subseteq V(\mathcal{H})$ such that $|F'| = k-1$. If $e \in E(\mathcal{H})$, then $e \nsubseteq F'$ since $|e| = k$. Consequently, F' is a maximal stable set. Therefore, $\mathcal{F}(\Delta_{\mathcal{H}})$ is a $(k-1)$-complete clutter.

$\Leftarrow$) Let e be an edge of $\mathcal{H}$. Since $\mathcal{F}(\Delta_{\mathcal{H}})$ is $(k-1)$-complete clutter, if $F \subseteq V(\mathcal{H})$ with $|F| = k-1$, then $F \in \mathcal{F}(\Delta_{\mathcal{H}})$. Hence, $e \nsubseteq F$. Consequently, $|e| \geq k$. Furthermore, if $e' \subseteq V(\mathcal{H})$ with $|e'| = k$, then $e' \notin \Delta_{\mathcal{H}}$. Also, if $F \subsetneq e'$, then $|F| \leq k-1$; implying $F \in \Delta_{\mathcal{H}}$. Thus, $e' \in E(\mathcal{H})$. Therefore, $\mathcal{H}$ is a k-complete clutter. □

Lemma 3.3 If $\mathcal{H}$ is a complete clutter, then $\Delta_{\mathcal{H}}$ is pure shellable.

Proof. By induction on $|V(\mathcal{H})|$. We take $v \in V(\mathcal{H})$, $\mathcal{H}_2 = \mathcal{H} \setminus v$ and the clutter $\mathcal{H}_1$ where $V(\mathcal{H}_1) = V(\mathcal{H}) \setminus v$ and $E(\mathcal{H}_1) = \{e \setminus v \mid v \in e \text{ and } e \in E(\mathcal{H})\}$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are complete clutters, then by induction hypothesis $\Delta_{\mathcal{H}_1}$ and $\Delta_{\mathcal{H}_2}$ are pure shellables. We assume that $F_1, ..., F_k$ and $L_1, ..., L_r$ are shellings of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We will prove that

$$E_1 = L_1, ..., E_r = L_r, E_{r+1} = F_1 \cup \{v\}, ..., E_{r+k} = F_k \cup \{v\},$$

is a shelling of $\Delta_{\mathcal{H}}$. By Lemma 3.2, $\mathcal{F}(\Delta_{\mathcal{H}})$, $\mathcal{F}(\Delta_{\mathcal{H}_1})$ and $\mathcal{F}(\Delta_{\mathcal{H}_2})$ are complete clutters. This implies $\mathcal{F}(\Delta_H) = \{E_1, ..., E_{r+k}\}$. We take E_i and E_j with $i < j$. Since $E_1, ..., E_r$ and $E_{r+1}, ..., E_{r+k}$ are shellings, we can assume $1 \leq i \leq r < j \leq r+k$, then $E_j = F_p \cup \{v\}$ for some $p \in \{1, ..., k\}$, hence, $v \in E_j \setminus E_i$. Since $\mathcal{F}(\Delta_{\mathcal{H}})$ is a complete clutter, then there exists $1 \leq l \leq r$ such that $F_p \subseteq E_l$. Consequently, $\{v\} = E_j \setminus E_l$ and $l \leq j$. Therefore, $E_1, ..., E_{r+k}$ is a shelling. □

Lemma 3.4 If $\mathcal{M}$ is a matroid and $\Delta = \Delta_{\mathcal{B}(\mathcal{M})}$ with $v \in V(\mathcal{M})$, then $\mathcal{B}(\mathcal{M})/v$ is the bases set of a matroid and $\Delta_{\mathcal{B}(\mathcal{M})/v} = \mathrm{lk}_{\Delta}(v)$.

Proof. If v is an isolated vertex, then $\mathcal{B}(\mathcal{M})/v = \mathcal{B}(\mathcal{M})$. Consequently, we can assume that v is not isolated. We assume that $|B| = k$ for each $B \in \mathcal{B}(\mathcal{M})$. First we will prove that $|B'| = k-1$ for each $B' \in \mathcal{B}(\mathcal{M})/v$. By contradiction, suppose that $B_1 \in \mathcal{B}(\mathcal{M})/v$ with $|B_1| = k$. Thus, $B_1 \in \mathcal{B}(\mathcal{M})$ and $v \notin B_1$. Since v is not an isolated vertex, then there exists $B_2 \in \mathcal{B}(\mathcal{M})$ such that $v \in B_2$. Hence, $v \in B_2 \setminus B_1$

and there is $y \in B_1 \setminus B_2$ such that $B_3 = (B_1 \cup v) \setminus y \in \mathcal{B}(\mathcal{M})$. Consequently, $B_3 \setminus \{v\} \in \mathcal{B}(\mathcal{M})/v$. But, $B_3 \setminus v = B_1 \setminus y \subsetneq B_1$, a contradiction. Implying $|B'| = k-1$ for each $B' \in \mathcal{B}(\mathcal{M}/v)$. Now, we take $B'_1, B'_2 \in \mathcal{B}(\mathcal{M})/v$. Thus, there exists $B_i \in \mathcal{B}(\mathcal{M})$ such that $B'_i = B_i \setminus v$ for $i = 1, 2$. If $x \in B'_1 \setminus B'_2$, it implies $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1 = B'_2 \setminus B'_1$ such that $B_3 = (B_1 \setminus x) \cup \{y\} \in \mathcal{B}(\mathcal{M})$. Since $x \in B'_1$, then $x \neq v$ and $v \in B_3$. Hence, $B_3 \setminus v = ((B_1 \setminus x) \cup \{y\}) \setminus v = (B'_1 \setminus x) \cup \{y\} \in \mathcal{B}(\mathcal{M})/v$. Therefore, there exists a matroid $\mathcal{M}'$ such that $B(\mathcal{M}') = \mathcal{B}(\mathcal{M})/v$. Finally, by Lemma 1.63, we have that $\text{lk}_\Delta(v) = \Delta_{\mathcal{B}(\mathcal{M})/v}$. □

Theorem 3.5 If $\dim(\Delta_{\mathcal{B}(\mathcal{M})}) = k$, then $\Delta_{\mathcal{B}(\mathcal{M})}$ is pure shellable if and only if $\mathcal{B}(\mathcal{M})$ is a (k+2)-complete clutter.

Proof. $\Leftarrow$) By Lemma 3.3, $\Delta_{\mathcal{B}(\mathcal{M})}$ is pure shellable.

$\Rightarrow$) Let V be the vertex set of $\mathcal{M}$ and we set $\Delta = \Delta_{\mathcal{B}(\mathcal{M})}$. By induction on $|V|$. We have that $\dim(\Delta) = k$. Now, we take a subset $F \subseteq V$ such that $|F| = k+1$ and $v \in F$. Since Δ is pure shellable, then by Remark 1.32, $\text{lk}_\Delta(v)$ is pure shellable. Furthermore, by Lemma 3.4, $\text{lk}_\Delta(v) = \Delta_{\mathcal{B}(\mathcal{M}')}$ where $\mathcal{M}'$ is a matroid such that $\mathcal{B}(\mathcal{M}') = \mathcal{B}(\mathcal{M})/v$. By induction hypothesis $\mathcal{B}(\mathcal{M}')$ is a $(k+1)$-complete clutter, since $\dim(\text{lk}_\Delta(v)) = k-1$. By Lemma 3.2, $\mathcal{F}(\Delta_{\mathcal{B}(\mathcal{M}')})$ is a k-complete clutter. This implies, $F \setminus v \in \mathcal{F}(\text{lk}_\Delta(v))$. Consequently, $F \in \mathcal{F}(\Delta_{\mathcal{B}(\mathcal{M})})$. Therefore, $\mathcal{F}(\Delta_{\mathcal{B}(\mathcal{M})})$ is a $(k+1)$-complete clutter and using Lemma 3.2, we obtain that $\mathcal{B}(\mathcal{M})$ is a $(k+2)$-complete clutter. □

Lemma 3.6 $\mathcal{B}(\mathcal{M})$ is a k-complete clutter if and only if $\mathcal{C}(\mathcal{M})$ is a (k+1)-complete clutter.

Proof. $\Rightarrow$) We take $C \subseteq V$ such that $|C| = k+1$. Thus, $C \notin \mathcal{B}(\mathcal{M})$, then C is a dependent set of $\mathcal{M}$. Furthermore, if $C' \subsetneq C$, then $|C'| \leq k$ implying that $C' \in \mathcal{I}(\mathcal{M})$. Therefore, $C \in \mathcal{C}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ is a (k+1)-complete clutter.

$\Leftarrow$) We take $B \subseteq V$ such that $|B| = k$. If $C \in \mathcal{C}(\mathcal{M})$, then $|C| = k+1$. This implies $C \nsubseteq B$. Hence, $B \in \mathcal{I}(\mathcal{M})$. Furthermore, if $B \subsetneq B'$, then $|B'| \geq k+1$. So, there exists B'' such that $B \subsetneq B'' \subseteq B'$ with $|B''| = k+1$. Thus, $B'' \in \mathcal{C}(\mathcal{M})$ and $B' \notin \mathcal{I}(\mathcal{M})$. Consequently, $B \in \mathcal{B}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M})$ is a k-complete clutter. □

Lemma 3.7 $\{x_1, ..., x_k\} \in \Delta^*_{\mathcal{B}(\mathcal{M})}$ if and only if $x_{k+1} \cdots x_n \in I(\mathcal{B}(\mathcal{M}))$.

Proof. We have that $\{x_1, ..., x_k\} \in \Delta^*_{\mathcal{B}(\mathcal{M})}$ if and only if $\{x_{k+1}, ..., x_n\} \notin \Delta_{\mathcal{B}(\mathcal{M})}$. Equivalently, there exists $b \in \mathcal{B}(\mathcal{M})$ such that $b \subseteq \{x_{k+1}, ..., x_n\}$. This is equivalent to have $m \mid x_{k+1} \cdots x_n$, where $m = \prod_{x_i \in b} x_i$. Therefore, $\{x_1, ..., x_k\} \in \Delta^*_{\mathcal{B}(\mathcal{M})}$ if and only if $x_{k+1} \cdots x_n \in I(\mathcal{B}(\mathcal{M}))$. □

Proposition 3.8 Let $\mathcal{M}$ a matroid. Then $\mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M}')}) = \mathcal{B}(\mathcal{M})$.

Proof. We assume $V(\mathcal{M}) = \{x_1, ..., x_n\}$. First, we will prove that: $\{x_1, ..., x_k\} \in \mathcal{F}(\Delta^*)$ if and only if $x_{k+1} \cdots x_n \in G(I)$, where $I = I(\mathcal{B}(\mathcal{M}'))$ and $\Delta^* = \Delta^*_{\mathcal{B}(\mathcal{M}')}$. If $\{x_1, ..., x_k\} \in \mathcal{F}(\Delta^*)$, then by the previous Lemma there exists $m \in G(I)$ such that $m \mid x_{k+1} \cdots x_n$. We can assume that $m = x_{k+1} \cdots x_r$ with $r \leq n$. Hence, $\{x_1, ..., x_k, x_{r+1}, ..., x_n\} \in \Delta^*$. But $\{x_1, ..., x_k\} \in \mathcal{F}(\Delta^*)$, therefore $r = n$ and $x_{k+1} \cdots x_n \in G(I)$. Now, we take $x_{k+1} \cdots x_n \in G(I)$ and suppose that $\{x_1, ..., x_k\} \notin \mathcal{F}(\Delta^*)$. Consequently, there exists a facet F in Δ^* such that $\{x_1, ..., x_k\} \subsetneq F$. We can assume $F = \{x_1, ..., x_k, x_{k+1}, ..., x_s\}$ with $k < s$. Thus, $x_{s+1} \cdots x_n \in I$, a contradiction since $x_{k+1} \cdots x_n \in G(I)$. Therefore, $\{x_1, ..., x_k\} \in \mathcal{F}(\Delta^*)$

On the other hand, $x_{k+1} \cdots x_n \in G(I)$ if and only if $\{x_{k+1}, ..., x_n\} \in \mathcal{B}(\mathcal{M}')$. Equivalently, $\{x_1, ..., x_k\} \in \mathcal{B}(\mathcal{M})$. Therefore, $\{x_1, ..., x_k\} \in \mathcal{F}(\Delta^*)$ if and only if $\{x_1, ..., x_k\} \in \mathcal{B}(\mathcal{M})$. □

Corollary 3.9 $F \in \mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M}')})$ if and only if $V \setminus F \in \mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M})})$.

Proof. By Proposition 3.8, we have that $F \in \mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M}')})$ if and only if $F \in \mathcal{B}(\mathcal{M})$. In the same way, $V \setminus F \in \mathcal{B}(\mathcal{M}')$ if and only if $V \setminus F \in \mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M}'')})$. Therefore, $F \in \mathcal{F}(\Delta^*_{\mathcal{B}(\mathcal{M}')})$ if and only if $V \setminus F \in \mathcal{F}(\Delta_{\mathcal{B}(\mathcal{M})})$ since $\mathcal{M}'' = \mathcal{M}$, and $F \in \mathcal{B}(\mathcal{M})$ if and only if $V \setminus F \in \mathcal{B}(\mathcal{M}')$. □

Lemma 3.10 $x_1 \cdots x_k \in I_{\Delta^*_{\mathcal{B}(\mathcal{M})}}$ if and only if $\{x_1, ..., x_k\}$ is a vertex cover of $\mathcal{B}(\mathcal{M})$.

Proof. We have that $x_1 \cdots x_k \in I_{\Delta^*_{\mathcal{B}(\mathcal{M})}}$ if and only if $F = \{x_1, ..., x_k\} \notin \Delta^*_{\mathcal{B}(\mathcal{M})}$. This is equivalent to have $V \setminus F \in \Delta_{\mathcal{B}(\mathcal{M})}$. That is, $V \setminus F$ is a stable set of $\mathcal{B}(\mathcal{M})$. Therefore, $F = \{x_1, ..., x_k\}$ is a vertex cover of $\mathcal{B}(\mathcal{M})$. □

Proposition 3.11 $I_{\Delta^*_{\mathcal{B}(\mathcal{M})}} = I(b(\mathcal{B}(\mathcal{M})))$ and $I_{\Delta^*_{\mathcal{B}(\mathcal{M})}} = I(\mathcal{C}(\mathcal{M}'))$.

Proof. We take $x_1 \cdots x_k \in G(I_{\Delta^*_{\mathcal{B}(\mathcal{M})}})$. By Lemma 3.10, $\{x_1, ..., x_k\}$ is a vertex cover of $B(\mathcal{M})$. If $\{x_1, ..., x_k\} \notin b(\mathcal{B}(\mathcal{M}))$, then there exists a minimal vertex cover C such that $C \subsetneq \{x_1, ..., x_k\}$. We can assume $C = \{x_1, ..., x_r\}$ with $r < k$. Hence, by the previous result, $x_1 \cdots x_r \in I_{\Delta^*_{\mathcal{B}(\mathcal{M})}}$, this is a contradiction. Then $\{x_1, ..., x_k\} \in b(\mathcal{B}(\mathcal{M}))$.

Now, we take that $\{x_1, ..., x_{k'}\} \in b(\mathcal{B}(\mathcal{M}))$. If $x_1 \cdots x_{k'} \notin G(I_{\Delta^*_{\mathcal{B}(\mathcal{M})}})$, then there exists $m \in G(I_{\Delta^*_{\mathcal{B}(\mathcal{M})}})$ such that $m \mid x_1 \cdots x_{k'}$. We can assume $m = x_1 \cdots x_{r'}$ with $r' < k'$. Thus, by Lemma 3.10, $\{x_1, ..., x_{r'}\}$ is a vertex cover of $\mathcal{B}(\mathcal{M})$. This is a contradiction implying $x_1 \cdots x_{k'} \in G(I^*_{\mathcal{B}(\mathcal{M})})$.

Consequently, $x_1 \cdots x_k \in G(I_{\Delta^*_{\mathcal{B}(\mathcal{M})}})$ if and only if $\{x_1, ..., x_k\} \in b(\mathcal{B}(\mathcal{M}))$. Therefore, $I_{\Delta^*_{\mathcal{B}(\mathcal{M})}} = I(b(\mathcal{B}(\mathcal{M})))$. Furthermore, by Remark 1.81, we have that $I_{\Delta^*_{\mathcal{B}(\mathcal{M})}} = I(\mathcal{C}(\mathcal{M}'))$. □

3.3 CLUTTERS WITH A PERFECT MATCHING OF KÖNIG TYPE

The shellable and unmixed clutters with a perfect matching of König type were studied in [32]. In this section we improve some results given in that paper.

Definition 3.12 Let $\mathcal{C}$ be a clutter. We say that $e \in E(\mathcal{C})$ satisfies the *contention property* if for any two edges $f, f' \in \mathcal{C}$, we have that $f \cap e \subseteq f' \cap e$ or $f' \cap e \subseteq f \cap e$. A subset B of $E(\mathcal{C})$ satisfies the contention property if each member of B satisfies the contention property.

Lemma 3.13 Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, ..., e_g$ of König type. If there exists $i \in \{1, ..., g\}$ such that e_i does not satisfy the contention property, then there is a 4-cycle in $\mathcal{C}$ containing e_i.

Proof. We assume that there are $f_1, f_2 \in E(\mathcal{C})$ such that $f_1 \cap e_i \not\subseteq f_2 \cap e_i$ and $f_2 \cap e_i \not\subseteq f_1 \cap e_i$. Thus, there are $a_1, b_1 \in e_i$ such that $a_1 \in (f_1 \cap e_i) \setminus (f_2 \cap e_i)$ and

$b_1 \in (f_2 \cap e_i) \setminus (f_1 \cap e_i)$. We take

$$A = \{f \in E(\mathcal{C}) \mid f \subseteq (f_1 \setminus a_1) \cup (f_2 \setminus b_1)\}.$$

Since $\mathcal{C}$ is unmixed and by Proposition 1.56 we have that $A \neq \emptyset$. Also, we take $A_1 = \{f \in A \mid (f \setminus f_1) \setminus e_i \neq \emptyset\}$. Suppose that $A_1 = \emptyset$, then $(f \setminus f_1) \subseteq e_i$ for all $f \in A$. Now, we take $f_3 \in A$ such that $|f_3 \setminus f_1|$ is minimal. Since $f_3 \in A$, then $a_1 \notin f_3$ and $f_3 \neq f_1$. Since $\mathcal{C}$ is a clutter, then there is $b_2 \in (f_3 \setminus f_1) \subseteq e_i$. Furthermore, by Proposition 1.56 there is $f_4 \in E(\mathcal{C})$ such that $f_4 \subseteq (f_1 \setminus a_1) \cup (f_3 \setminus b_2) \subseteq (f_1 \setminus a_1) \cup (f_2 \setminus \{b_1, b_2\})$. Consequently, $f_4 \in A$. But $f_4 \setminus f_1 \subseteq [(f_1 \setminus a_1) \cup (f_3 \setminus b_2)] \setminus f_1 = (f_3 \setminus b_2) \setminus f_1 = (f_3 \setminus f_1) \setminus b_2 \subsetneq (f_3 \setminus f_1)$. Hence, $|f_4 \setminus f_1| < |f_3 \setminus f_1|$. This is a contradiction, therefore $A_1 \neq \emptyset$.

Now, we define $A_2 = \{f \in A_1 \mid (f \setminus f_2) \setminus e_i \neq \emptyset\}$. We suppose that $A_2 = \emptyset$, then $(f \setminus f_2) \subseteq e_i$ for all $f \in A_1$. We take $f_5 \in A_1$, then $(f_5 \setminus f_2) \subseteq e_i$. Since $\mathcal{C}$ is a clutter we have that there is $b_3 \in (f_5 \setminus f_2) \subseteq e_i$ and by Proposition 1.56 there is $f_6 \subseteq (f_5 \setminus b_3) \cup (f_2 \setminus b_1)$. We can assume that $|f_6 \setminus f_2|$ is minimal. Since $f_5 \in A$, then $b_1 \notin f_5$. This implies, $f_6 \neq f_2$. Thus, there exists $b_4 \in (f_6 \setminus f_2) \subseteq [(f_5 \setminus b_3) \cup (f_2 \setminus b_1)] \setminus f_2 = (f_5 \setminus b_3) \setminus f_2 = (f_5 \setminus f_2) \setminus b_3 \subseteq e_i$. Since $\mathcal{C}$ is unmixed and $b_4 \in e_i$, then there is $f_7 \subseteq (f_6 \setminus b_4) \cup (f_2 \setminus b_1) \subseteq (f_5 \setminus b_3) \cup (f_2 \setminus b_1)$. Consequently, $(f_7 \setminus f_2) \subseteq [(f_6 \setminus b_4) \cup (f_2 \setminus b_1)] \setminus f_2 = (f_6 \setminus b_4) \setminus f_2 = (f_6 \setminus f_2) \setminus b_4$, implying $|f_7 \setminus f_2| < |f_6 \setminus f_2|$, this is a contradiction. So $A_2 \neq \emptyset$. We can take $f \in A_2$. Hence, there exist $x, y \in V(\mathcal{C})$ such that $x \in f \setminus (f_2 \cup e_i)$ and $y \in f \setminus (f_1 \cup e_i)$. This implies, $x \in f_1$ and $y \in f_2$ since $f \subseteq (f_1 \setminus a_1) \cup (f_2 \setminus b_1)$. Therefore, the incidence matrix of $\mathcal{C}$ has the following submatrix:

	e_i	f_1	f_2	f
a_1	1	1	0	0
b_1	1	0	1	0
x	0	1	0	1
y	0	0	1	1

This is a 4-cycle of $\mathcal{C}$ containing e_i. □

Theorem 3.14 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type. P satisfies the contention property if and only if $\mathcal{C}$ is an unmixed and does not have 4-cycles containing e_i.

Proof. $\Rightarrow$) By Theorem 1.69, $\mathcal{C}$ is unmixed. Now, we suppose that $\mathcal{C}$ has a 4-cycle whose edges are f_1, f_2, f_3 and e_i. Hence, there exist $a, b, c, d \in V(\mathcal{C})$ such that the incidence matrix of $\mathcal{C}$ has the following submatrix:

	e_i	f_1	f_2	f_3
a	1	0	0	1
b	1	1	0	0
c	0	1	1	0
d	0	0	1	1

Hence, $b \in (f_1 \cap e_i) \setminus (f_3 \cap e_i)$ and $a \in (f_3 \cap e_i) \setminus (f_1 \cap e_i)$. This is a contradiction since P satisfies the contention property. Therefore, $\mathcal{C}$ does not have 4-cycles containing e_i.

$\Leftarrow$) Now, we assume that there exist $f_1, f_2 \in E(\mathcal{C})$ such that $f_1 \cap e_i \not\subseteq f_2 \cap e_i$ and $f_2 \cap e_i \not\subseteq f_1 \cap e_i$ for some $i \in \{1, ..., g\}$. Since $\mathcal{C}$ is unmixed, then by Lemma 3.13 we have that $\mathcal{C}$ has at least a 4-cycle containing e_i. This is a contradiction. $\square$

Proposition 3.15 Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, ..., e_g$ of König type. If $\mathcal{C}$ does not have 4-cycles containing e_i, then e_i has a free vertex.

Proof. Since there is no 4-cycle containing e_i, thus by Lemma 3.13, e_i satisfies the contention property. Hence, we can assume that $E(\mathcal{C}) = \{f_1, ..., f_r, e_i\}$ such that $f_1 \cap e_i \subseteq f_2 \cap e_i \subseteq \cdots \subseteq f_r \cap e_i \subsetneq e_i$. Therefore, if $x \in e_i \setminus (f_r \cap e_i)$, then x is a free vertex. $\square$

Example 3.16 Let $\mathcal{C}$ be a clutter with vertex set $V(\mathcal{C}) = \{a, ..., n\}$ and edges:

$$
\begin{array}{llllll}
e_1 = \{a,b\} & e_2 = \{c,d\} & e_3 = \{e,f\} & e_4 = \{g,h\} & e_5 = \{i,j\} & e_6 = \{k,l\} \\
e_7 = \{m,n\} & f_1 = \{a,c\} & f_2 = \{b,f\} & f_3 = \{c,f\} & f_4 = \{c,g,i\} & f_5 = \{f,h,j\} \\
f_6 = \{k,j\} & f_7 = \{j,m\} & f_8 = \{c,m\} & f_9 = \{c,k\} & &
\end{array}
$$

Notice that $e_1, ..., e_7$ is a perfect matching. Now, the clutter has the following of 4-cycles:

	f_6	f_7	f_8	f_9
c	0	0	1	1
j	1	1	0	0
k	1	0	0	1
m	0	1	1	0

	e_1	f_1	f_2	f_3
a	1	1	0	0
b	1	0	1	0
c	0	1	0	1
f	0	0	1	1

	e_4	e_5	f_4	f_5
g	1	0	1	0
h	1	0	0	1
i	0	1	1	0
j	0	1	0	1

First 4-cycle does not contain e_i, the second contains one e_i and the third 4-cycle contains two e_i's.

Corollary 3.17 Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type. If the 4-cycles of $\mathcal{C}$ do not contain e_i's, then P satisfies the contention property.

Proof. By Theorem 3.14, $\mathcal{C}$ satisfies the contention property. □

Theorem 3.18 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type. If $\mathcal{C}$ is an unmixed clutter without 4-cycles with e_i's, then $\Delta_{\mathcal{C}}$ is pure shellable.

Proof. By Corollary 3.17, P satisfies the contention property. Hence, by Theorem 1.70 $\Delta_{\mathcal{C}}$ is pure shellable. □

Example 3.19 The converse of Theorem 3.18 is not true. Let $\mathcal{C}$ be a clutter with vertex set $V(\mathcal{C}) = \{a, b, c, d, e, f, g, h\}$ and edges

$$e_1 = \{a, b, c, d\},\ e_2 = \{e, f\},\ e_3 = \{g, h\},\ f_1 = \{e, g\},\ f_2 = \{d, e\},\ f_3 = \{a, b, g\}.$$

Hence, e_1, e_2, e_3 is a perfect matching of König type. $\Delta_{\mathcal{C}}$ has the following shelling:

$$\begin{array}{llll} F_1 = \{b, c, d, f, h\} & F_2 = \{b, c, d, f, g\} & F_3 = \{a, c, d, f, g\} & F_4 = \{a, c, d, f, h\} \\ F_5 = \{a, b, c, f, h\} & F_6 = \{a, b, c, e, h\} & F_7 = \{a, b, d, f, h\} & \end{array}$$

Therefore, $\Delta_{\mathcal{C}}$ is a pure shellable simplicial complex. But $\mathcal{C}$ has the following 4-cycle containing e_1.

	e_1	f_1	f_2	f_3
a	1	0	0	1
d	1	0	1	0
e	0	1	1	0
g	0	1	0	1

Theorem 3.20 Let $\mathcal{C}$ be a König clutter with a maximum matching $P = \{e_1, ..., e_g\}$ such that it does not have 4-cycles with some e_i. Then the following conditions are equivalent:

(a) $\mathcal{C}$ is unmixed.

(b) P is a perfect matching of $\mathcal{C}$ with $\mathrm{ht}(I(\mathcal{C})) = g$ and every $e_i \in P$ has a free vertex for all i, and P satisfies the contention property.

(c) $\Delta_{\mathcal{C}}$ is a pure shellable simplicial complex.

(d) $R/I(\mathcal{C})$ is Cohen-Macaulay.

Proof. (a) ⇒(b) Using Lemma 1.55 we have that $e_1, ..., e_g$ is a perfect matching and $g = \mathrm{ht}(I(\mathcal{C}))$. By Theorem 3.14, $\mathcal{C}$ satisfies the contention property. Consequently, by Proposition 3.15, every e_i has a free vertex for all $i \in \{1, ..., g\}$.

(b)⇒(a) We have that $e_1, ..., e_g$ is a perfect matching of König type and $\mathcal{C}$ satisfies the contention property. By Theorem 3.14, $\mathcal{C}$ is unmixed.

(b)⇒(c) Since (a) and (b) are equivalent, then $\mathcal{C}$ is unmixed. By Theorem 3.18, $\Delta_{\mathcal{C}}$ is pure shellable.

(c)⇒(d) and (d)⇒(a) By Theorem 1.33. □

Proposition 3.21 Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, ..., e_g$ of König type. If A is a stable set of $\mathcal{C}$, then:

(a) If $I(\mathcal{C}) = \bigcap_{i=1}^{p}(D_i)$, then $I(\mathcal{C}/A) = \bigcap_{A\cap D_i=\emptyset}(D_i)$.

(b) $\mathcal{C}/A$ is unmixed with a perfect matching $e'_1, ..., e'_g$ of König type such that $e'_i \subseteq e_i$ for all i and every vertex of $e_i \setminus e'_i$ is isolated in $\mathcal{C}/A$.

(c) If $\mathcal{C}$ does not have 4-cycles with two e_i's, then $\mathcal{C}/A$ does not have 4-cycles with two e_i's.

(d) If $\mathcal{C}$ is shellable, then $\mathcal{C}/A$ is shellable.

Proof. (a) We take $A = \{x_1, ..., x_k\}$. By induction on k. If $k = 1$, then by Remark 1.60 the edge ideal of $\mathcal{C}/x_1$ is $(I(\mathcal{C}) : x_1) = \bigcap_{x_i \notin D_i}(D_i)$ since $x_1 \notin I(\mathcal{C})$. If $k \geq 2$, we take $A' = \{x_1, ..., x_{k-1}\}$, by induction hypothesis

$$\mathcal{C}/A' = ((\cdots(((I : x_1) : x_2) : x_3)\cdots) : x_{k-1})$$

and $I(\mathcal{C}/A) = \bigcap_{A \cap D_i = \emptyset}(D_i)$ and $x_i \notin I(\mathcal{C})$ for all $x_i \in A'$. By Remark 1.60 the edge ideal of $\mathcal{C}/A = ((\cdots(((\mathcal{C}/x_1)/x_2)/x_3)\cdots)/x_{k-1})/x_k$ is

$$I(\mathcal{C}/A) = \bigcap_{x_i \notin D_i = \emptyset}(D_i) = (((\cdots(((I : x_1) : x_2) : x_3)\cdots) : x_{k-1}) : x_k)$$

and $A' \cap D_i = \emptyset$, then $A' \cup \{x_k\} \cap D_i = \emptyset$. Hence, $I(\mathcal{C}/A) = \bigcap_{A \cap D_i = \emptyset}(D_i)$. Furthermore, for all $f \in E(\mathcal{C})$ we have that $f \notin A$. Consequently, (A) is not generated by $I(\mathcal{C})$.

(b) By (a) the minimal vertex covers of $\mathcal{C}/A$ are $D_1, ..., D_p$ with $A \cap D_k = \emptyset$ for $k \in \{1, ..., p\}$ and $|D_k| = |D_{k'}|$ for $k, k' \in \{1, ..., p\}$. Then, $\mathcal{C}/A$ is unmixed. By Proposition 1.59 $\mathcal{C}/A$ has a perfect matching $e'_1, ..., e'_g$ of König type such that $e'_i \subseteq e_i$ for all i, and every vertex of $e_i \setminus e'_i$ is isolated in $\mathcal{C}/A$.

(c) We assume that $\mathcal{C}/A$ has a 4-cycle with e'_i and e'_j, then there exist $a, b, c, d \in V(\mathcal{C}/A)$ and $f'_1, f'_2 \in E(\mathcal{C}/A)$ with the following incidence submatrix:

	e'_i	e'_j	f'_1	f'_2
a	1	0	1	0
b	1	0	0	1
c	0	1	1	0
d	0	1	0	1

Therefore, $\mathcal{C}$ has a 4-cycle with e_i and e_j with the following incidence matrix:

	$e_i = e_i'$ or $e_i' \cup \sigma_1$	$e_j = e_j'$ or $e_j' \cup \sigma_2$	$f_1 = f_1'$ or $f_1' \cup \sigma_3$	$f_2 = f_2'$ or $f_2' \cup \sigma_4$
a	1	0	1	0
b	1	0	0	1
c	0	1	1	0
d	0	1	0	1

For $\sigma_i \subseteq A$, $i \in \{1, 2, 3, 4\}$. This is a contradiction.

(d) The result follow by (a) and Lemma 1.67. □

Definition 3.22 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$. P has the *quasi contention property* if for every pair of edges f_1, f_2, there exists $i \in \{1, ..., g\}$ such that $f_1 \cap e_j \subseteq f_2 \cap e_j$ or $f_2 \cap e_j \subseteq f_1 \cap e_j$ for each $j \in \{1, ..., g\} \setminus \{i\}$.

Proposition 3.23 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type, then $\mathcal{C}$ does not have 4-cycles with two e_i's if and only if P has the quasi contention property.

Proof. P does not have the quasi contention property if and only if there are $f_1, f_2 \in E(\mathcal{C})$ and e_i, e_j such that $e_i \cap f_1 \nsubseteq e_i \cap f_2$ and $e_i \cap f_2 \nsubseteq e_i \cap f_1$, $e_j \cap f_1 \nsubseteq e_j \cap f_2$ and $e_j \cap f_2 \nsubseteq e_j \cap f_1$. Equivalently, there are $x_1 \in (e_i \cap f_1) \setminus (e_i \cap f_2)$, $x_2 \in (e_i \cap f_2) \setminus (e_i \cap f_1)$, $x_3 \in (e_j \cap f_1) \setminus (e_j \cap f_2)$ and $x_4 \in (e_j \cap f_2) \setminus (e_j \cap f_1)$. This is equivalent to that e_i, e_j, f_1, f_2 form the following 4-cycle.

	e_i	e_j	f_1	f_2
x_1	1	0	1	0
x_2	1	0	0	1
x_3	0	1	1	0
x_4	0	1	0	1

Furthermore, every 4-cycle with two e_i's has the same form, since P is a matching. Therefore, P has the quasi contention property if and only if there are no 4-cycles with two e_i's. □

Definition 3.24 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type and a 4-cycle Q with edges f_1, f_2, e_i and e_j. We say that an edge f_3 is a diagonal of Q if $f_3 \notin P$ and there exist $k \in \{1, ..., g\}$ and $a, b \in e_k$ such that $f_3 \subseteq (f_1 \setminus a) \cup (f_2 \setminus b)$, $a \in (f_1 \setminus f_2)$ and $b \in (f_2 \setminus f_1)$.

Definition 3.25 Let $\mathcal{C}$ be a clutter with a perfect matching $P = \{e_1, ..., e_g\}$ of König type. A *complete 4-cycle of size k* is a set of edges $e_1, ..., e_k, f_1, f_2$ where $e_{i_1}, ..., e_{i_k} \in P$ and $(f_1 \cap e_i) \cup (f_2 \cap e_i) = e_i$ for all $i \in \{1, ..., k\}$. We say $\mathcal{C}$ is a *complete 4-cycle* if $k = g$.

Lemma 3.26 Let $\mathcal{C}$ be an unmixed clutter with a perfect matching $e_1, ..., e_g$ of König type. If $\mathcal{C}$ has a 4-cycle with two $e_i's$ without a diagonal, then $\mathcal{C}$ has a complete 4-cycle.

Proof. Let f_1, f_2, e_i, e_j be edges of a 4-cycle of $\mathcal{C}$, then there are $x_1 \in (e_i \cap f_1) \setminus (e_i \cap f_2)$ and $x_2 \in (e_i \cap f_2) \setminus (e_i \cap f_1)$. Since $\mathcal{C}$ is unmixed there exists an edge f such that $f \subseteq (f_1 \setminus x_1) \cup (f_2 \setminus x_2)$, since the 4-cycle does not have a diagonal, then $f = e_k$ for any $k \in \{1, ..., g\}$. Therefore, $(f_1 \cap e_k) \cup (f_2 \cap e_k) = e_k$. Similarly for e_j, there is e_l with $l \in \{1, ..., g\}$ such that $(f_1 \cap e_l) \cup (f_2 \cap e_l) = e_l$. Therefore, $\mathcal{C}$ has a complete 4-cycle with edges f_1, f_2, e_l, e_k. □

Example 3.27 Let $\mathcal{C}$ be a clutter with vertex set $V(\mathcal{C}) = \{a, ..., l\}$ and edges $e_1 = \{a, b, c\}$, $e_2 = \{d, e, f\}$, $e_3 = \{g, h, i\}$, $e_4 = \{j, k, l\}$, $f_1 = \{a, b, d, e, g\}$, $f_2 = \{c, f, l\}$, $f_3 = \{g, i, l\}$. $\mathcal{C}$ has a complete 4-cycle with edges f_1, f_2, e_1, e_2. $\mathcal{C}$ does not have one diagonal since $i \in f_3$ and $i \notin (f_1 \setminus x) \cup (f_2 \setminus y)$ for all $x \in f_1$ and $y \in f_2$.

The stable sets of $\mathcal{C}$ are: $F_1 = \{a, c, d, e\}$, $F_2 = \{b, c, d, e\}$, $F_3 = \{a, b, d, f\}$, $F_4 = \{a, b, e, f\}$

Hence, $\mathcal{C}$ is not shellable.

With this example we can conjecture that if a clutter $\mathcal{C}$ has a 4-cycle Q and does not have a diagonal, then the clutter is not shellable.

Question 3.28 Are there unmixed shellables clutters that have a 4-cycle without a diagonal?

Remark 3.29 A *pseudoisolated vertex* z, $f = \{z\}$, does not affect the shellability of the clutter $\mathcal{C}$ since $z \in D$ for all minimal vertex covers of $\mathcal{C}$, hence z is not in the stable sets of $\mathcal{C}$, that is, $z \notin V(\mathcal{C}) \setminus D = F$.

Example 3.30 Let $\mathcal{C}$ be a clutter whose edges are:

$$\begin{array}{llllll} e_1 = \{a,b\} & e_2 = \{c,d\} & e_3 = \{e,f\} & & & \\ f_1 = \{a,c\} & f_2 = \{b,d\} & f_3 = \{c,e\} & f_4 = \{d,f\} & f_5 = \{b,e\} & f_6 = \{a,f\} \end{array}$$

$\mathcal{C}$ has maximal stable sets $F_1 = \{a,d,e\}, F_2 = \{b,c,f\}$. Thus, $\mathcal{C}$ is not shellable.

Example 3.31 Let $\mathcal{C}$ be the clutter:

$$\begin{array}{llllll} e_1 = \{a,b\} & e_2 = \{c,d\} & e_3 = \{e,f\} & e_4 = \{g,h\} & & \\ f_1 = \{a,c\} & f_2 = \{b,d\} & f_3 = \{c,e\} & f_4 = \{d,f\} & f_5 = \{b,e\} & f_6 = \{a,f\} \\ f_7 = \{c,g\} & f_8 = \{f,g\} & f_9 = \{b,g\} & f_{10} = \{a,h\} & f_{11} = \{e,h\} & f_{12} = \{d,h\} \end{array}$$

The maximal stable sets of $\mathcal{C}$ are: $F_1 = \{b,c,f,h\}$ and $F_2 = \{a,d,e,g\}$. Thus, $\mathcal{C}$ is not shellable.

BIBLIOGRAPHY

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*. Addison-Wesley, Reading MA, (1969).

[2] J. Biermann, C. A. Francisco, H. Tài Hà, A. Van Tuyl, *Colorings of simplicial complexes and vertex decomposability*. Preprint, 2012, arXiv:1209.3008v1.

[3] L. J. Billera, J. S. Provan, *Decompositions of simplicial complexes related to diameters of convex polyhedra*. Math. Oper. Res. **5**, (1980), 576-594.

[4] T. Bíyíkoglu, Y. Civan, *Vertex-decomposable Graphs, Codismantlability, Cohen-Macaulayness, and Castelnuovo-Mumford Regularity*. Electr. J. Comb. **21** (1), (2014), P1.1.

[5] A. Björner, *The homology and shellability of matroids and geometric lattices*. In Matroid Applications, N. White. Cambridge University, Press, Cambridge, (1992), pp 226-283.

[6] A. Björner, M.L. Wachs, *Shellable nonpure complexes and posets I*. Trans. Amer. Math. Soc. **348**, (1996), 1299-1327.

[7] A. Björner, M.L. Wachs, *Shellable nonpure complexes and posets II*. Trans. Amer. Math. Soc. **349**, (1997), 3945-3975.

[8] A. Bruns, J. Herzog, *Cohen-Macaulay Rings*. Cambridge University Press, Cambridge, (1997).

[9] I. D. Castrillón, R. Cruz, *Escalonabilidad de grafos e hipergrafos simples que contienen vértices simpliciales*. Matemáticas: Enseñanza Universitaria, Vol. XX, (2012), No. 1, 69-80.

[10] I. D. Castrillón, R. Cruz, E. Reyes. *On well-covered, vertex decomposable and Cohen-Macaulay*. ArXiv: 1505.00060v1. May 2015.

[11] I. D. Castrillón, E. Reyes. *Unmixed vertex decomposable graphs whose 5-cycles are chorded*. Preprint.

[12] I. D. Castrillón, E. Reyes. *Well-covered and Cohen-Macaulay theta-ring graphs*. Preprint.

[13] M. Crupi, G. Rinaldo, N. Terai, *Cohen-Macaulay edge ideals whose height is half of the number of vertices*. Nagoya Math. J., **201** (2011), 116-130.

[14] R. Cruz, E. Reyes, *Hipergráficas simples no mezcladas, complejos simpliciales escalonables y anillos de Stanley-Reisner*. Bol. Mat. **18** (2011), 71-84.

[15] A. Dochtermann, A. Engströn *Algebraic properties of edge ideals via combinatorial topology*. Electron. J. Combin. **16:2** (2009), #R2.

[16] A. M. Duval, *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes*. Electron. J. Combin. **3**, (1996), no. 1, Research Paper 21.

[17] J. Earl, K. N. Vander Meulen, A. Van Tuyl, *Independence complexes of well-covered circulant graph*. Preprint, ArXiv: 1505.02837, (2015).

[18] M. Estrada, R. H. Villarreal, *Cohen-Macaulay bipartite graphs*. Arch. Math. **68**, (1997), 124-128.

[19] S. Faridy, *The facet ideal of a simplicial complex*. Manuscripta Math. **109**, (2002), 159-174.

[20] S. Faridy, *Simplicial trees are sequentially Cohen-Macaulay*. J. Pure Appl. Algebra. **190**, (2004), 121-136.

[21] A. Finbow, B. Hartnell, R.J. Nowakowski, *A characterization of well covered graphs of girth 5 or greater*. J. of Combinatorial Theory. **B 57**, (1993), 44-68.

[22] A. Finbow, B. Hartnell, R.J. Nowakowski, *A characterization of well covered graphs that contain neither 4-nor 5-cycles*. J. of Graph Theory. **18**, (1994), 713-721.

[23] C. A. Francisco, A. Van Tuyl, *Sequentially Cohen-Macaulay edges ideals*. Proc. Amer. Math. Soc. (electronic) **135**, (2007), no 8, 2327-2337.

[24] A. Garsia, *Combinatorial methods in the theory of Cohen-Macaulay rings*. Adv. Math. **38**, (1980), 229-266.

[25] I. Gitler, E. Reyes, J. A. Vega, *Complete intersection toric ideals of oriented graphs and chorded-theta subgraphs*. J. Algebr. Comb.

[26] I. Gitler, C. E. Valencia, *On bounds for some graph invariants*. Boletín de la Sociedad Matemática Mexicana 10, (2005).

[27] J. Herzog, T. Hibi, *Distributive lattices, bipartite graphs and Alexander duality*. J. Alg. Combin. **22**, (2005), 289-302.

[28] D. T. Hoang, N. C. Minh, T.N. Trung, *Cohen-Macaulay graphs with large girth*. J. Algebra Appl. **14**, (2015).

[29] M. Hochster, *Cohen-Macaulay rings, combinatorics and simplicial complexes*. In B. R. McDonald and R. A. Morris (eds), Ring theory II. Lect. Notes in Pure and Appl. Math. **26**, M. Dekker, (1977), 171-223.

[30] B. Kind, P. Kleinschmidt, *Schälbare Cohen-Macaulay komplexe und ihre parametrisierung*. Math. Z. **167**, (1979), 173-179.

[31] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai, S. Yassemi, *Vertex decomposability and regularity of very well-covered graphs*. Journal of Pure and Applied Algebra **215**, (2011), 2473-2480.

[32] S. Morey, E. Reyes, R.H. Villarreal, *Cohen-Macaulay, shellable and unmixed clutters with perfect matching of König type*. Journal of Pure and Applied Algebra **212**, (2008), 1770-1786.

[33] P. McMullen, *The maximum numbers of faces of a convex polytope*. Matematika. **17**, (1970), 179-184.

[34] J. G. Oxley, *Matroid Theory*. Oxford university press, (1992).

[35] M. D. Plummer, *Some covering concepts in graphs*. J. Graph Theory. **11**, (1987), 507-515.

[36] E. Prisner, J. Topp, P.D. Vestergaard, *Well covered simplicial, chordal and circular arc graphs*. Journal of Graph Theory. **21**-2, (1996), 113-119.

[37] B. Randerath, L. Volkmann, *A characterization of well covered block-cactus graphs*. Australas. Journal of Combinatorics. **9**, (1994), 307-314.

[38] B. Randerath, P.D. Vestergaard, *On well-covered graphs of odd girth 7 or greater*. Discussiones Mathematicae Graph Theory. **22**, (2002), 159-172.

[39] G. Ravindra, *Well-covered graphs*. J. Combin. Inform. System Sci., **2**, (1977), 20-21.

[40] G. A. Reisner, *Cohen-Macaulay quotients of polynomial rings*. Adv. in Math., **21**, (1976), 30-49.

[41] M. E. Rudin, *An unshellable triangulation of a tetrahedron*. Bull. Amer. Math. Soc. **64**, (1958), 90-91.

[42] R. P. Stanley, *Combinatorics and Commutative Algebra*. Second edition. Progress in Mathematics. **41**. Birkhäuser Boston, Inc., Boston, MA, 1996.

[43] R. P. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*. Stud. Appl. Math. **54**, (1975), 135-142.

[44] J. Topp, L. Volkmann, *Characterization of unicyclic graphs with equal domination and independence number*. Discuss. Math. **11**, (1991), 27-34.

[45] A. Van Tuyl, *Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity*. Preprint, arXiv: 0906.0273v1.

[46] A. Van Tuyl, R. H. Villarreal, *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*. Journal of Combinatorial Theory Series **A 115**, (2008), 799-814.

[47] R. H. Villarreal, *Cohen-Macaulay graphs*. Manuscripta Math. **66** (1990), 277-293.

[48] R. H. Villarreal, *Monomial Algebras*. Second edition, Monographs and Research Notes in Mathematics, Chapman and Hall/CRC, (2015).

[49] R. H. Villarreal, *Unmixed bipartite graphs*. Revista Colombiana de Matematicas., **41** (2), (2007), 393-395.

[50] M. L. Wachs, *Obstructions to shellability*. Discrete Comput. Geom. **22** (1999), no 1, 95-103.

[51] H. Whitney, *On the abstract properties of linear dependence*. American Journal of Mathematics, **57**, No 3, (1935), 509-533.

[52] R. Woodroofe, *Chordal and sequentially Cohen-Macaulay clutters*. Electron. J. Combin., **18**(1), (2011). Paper 208

[53] R. Woodroofe, *Vertex decomposable graphs and obstruction to shellability*. Proc. Amer. Math. Soc. **137**, (2009), 3235-3246.

Index