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Orthogonal systems of contragenic polynomials on spheroidal domains

A dissertation presented by

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Thesis advisors:

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Sistemas ortogonales de polinomios contragénicos en dominios esferoidales

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Abstract

A contragenic function in a domain $\Omega \subseteq \mathbb{R}^3$ is a reduced-quaternion-valued harmonic function, which is orthogonal in $L_2(\Omega)$ to monogenic and antimonogenic functions.

Polynomial bases are constructed for the spaces of contragenic functions defined in spheroids, particularly, oblates and prolates. The notion of contragenic function depends on the domain and, therefore, is not a local property in contrast to the notions of harmonic and monogenic functions.

We present, for spheroidal domains of arbitrary eccentricity, formulas that relate orthogonal bases of harmonic, ambigenic and contragenic functions from one domain to another. This allows showing that there are common contragenic functions to spheroids of any eccentricity.

Resumen

Una función contragénica en un dominio $\Omega \subseteq \mathbb{R}^3$ es una función armónica evaluada en el conjunto de los cuaternios reducidos, que es ortogonal en $L_2(\Omega)$ a las funciones monogénicas y antimonogénicas.

Se construyen bases de polinomios para los espacios de funciones contragénicas definidas en esferoides, particularmente, para esferoides oblatos y prolatos. La noción de función contragénica depende del dominio y, por tanto, no es una propiedad local en contraste con las nociones de funciones armónicas y monogénicas.

Se presentan, para dominios esferoidales de excentricidad arbitraria, fórmulas que relacionan bases ortogonales de funciones armónicas, ambigénicas y contragénicas de un dominio a otro. Esto permite mostrar que existen funciones contragénicas comunes a esferoides de cualquier excentricidad.

Dedication

To my parents.

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Introduction

Background

One of the main objects of study of higher-dimensional analysis, referred to as Clifford analysis, is the partial differential operator

$$\sum_i e_i \partial_i.$$

The origins of Clifford analysis can be found in the work of W. R. Hamilton. Hamilton aimed to represent rotations in three-dimensional space, just as complex numbers do for the two-dimensional case. This was accomplished by the introduction of the quaternionic units i, j, k which satisfy the Hamiltonian relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

The 4-dimensional vector space generated by $1, i, j, k$ with the quaternionic multiplication is denoted $\mathbb{H}$.

In 1878, W. K. Clifford published his first attempt to carry out the transition from Hamilton's quaternion method to what we now know as vector analysis. In his work entitled "Elements of Dynamics", he introduced the vector product, practically as we know it today; a refinement of the theory of determinants eventually shaped it. Clifford also introduced the concept of a geometric product in which the notions of scalar and exterior products converge into a single object through the algebras named in his honor,

$$pq = p \cdot q + p \wedge q.$$

This product is not restricted to four dimensions and reveals the importance of vectorial methods. The considerations that Clifford made were of a geometrical nature. In his work "Applications of Grassmann's Extensive Algebra", he combined Grassmann's extension theory and the Hamiltonian quaternions by constructing a new algebra made up of scalars, vectors and k-vectors $(1 \leq k \leq n)$, elements that are currently known as Clifford numbers.

An important advance was made with the work of R. Fueter [24], who discovered that many properties of holomorphic functions of a complex variable can be generalized in the context of the quaternionic algebra. Fueter defined the concept of left-regular function and proved that these functions lie in the kernel of the operator $\partial = \sum_{\alpha=0}^{3} \partial/\partial x_\alpha$. Moreover, he constructed a collection of polynomials of all degrees that provide a generalization of the concept of expansion in power series of Fueter-regular functions.

In subsequent years, lines of research have been developed in which the analogues of Fueter-regular functions have been studied in other contexts such as Clifford algebras (see, for example, [33]). Taking the polynomials of Fueter as building blocks, H. Leutwiler [52] built a basis for the $\mathbb{R}^3$-valued monogenic polynomials of three variables. However, one of the drawbacks of this system is that it is not orthogonal in the real inner product defined on the unit ball. Due to the instability of the Gram-Schmidt Process, several researchers have chosen to build bases that are numerically and algebraically more accessible. Taking as a starting point the factorization of the Laplacian operator

$$\Delta = \partial\overline{\partial},$$

where ∂ denotes the generalized Cauchy-Riemann operator $\partial = \sum_{\alpha=1}^{3} \partial/\partial x_\alpha$, I. Caçao [13] constructed a basis for homogeneous monogenic polynomials of every degree n defined in terms of the well known spherical harmonics.

In certain physical problems in nonspherical domains, it has been found convenient to replace the classical solid spherical harmonics with harmonic functions better adapted to the domain in question. A technique to obtain bases of monogenic polynomials of degree at most n, is to use the factorization of the Laplacian operator and taking as a starting point a base of harmonic polynomials. P. Garabedian ([25]), obtained a base of harmonic polynomials defined in spheroids of the form

$$\left\{x \in \mathbb{R}^3 | \ \frac{x_0^2}{\cosh^2 \alpha} + \frac{x_1^2 + x_2^2}{\sinh^2 \alpha} < 1\right\} \quad \text{and} \quad \left\{x \in \mathbb{R}^3 | \ \frac{x_0^2}{\sinh^2 \alpha} + \frac{x_1^2 + x_2^2}{\cosh^2 \alpha} < 1\right\}$$

called prolate and oblate, respectively. Note that the spheroids become more and more spherical as $\alpha \to \infty$ (since $\tanh \alpha \to 1$), but the radii also tend to infinity, so that the sphere is not included in the class of functions considered by Garabedian.

Using the polynomials of Garabedian, J. Morais [59, 60] obtained a basis of monogenic polynomials, orthogonal to the spheroidal L_2 inner product.

The notion of contragenic function, an object of study in this thesis, arises from the following considerations. A well known fact from elementary complex analysis says:

> "Every harmonic function in a simply connected plane domain can be expressed as the sum of a holomorphic and an antiholomorphic function."

This result also holds for $\mathbb{H}$; that is, every harmonic function, $\mathbb{H}$-valued, defined on a domain in $\mathbb{H}$, can be expressed as the sum of a monogenic function and an antimonogenic function. Similar statements hold for general Clifford algebras. However, when $\mathbb{R}^3$ is embedded in $\mathbb{H}$ in a natural way, the corresponding generalization for $\mathbb{R}^3$-valued harmonic functions defined on domains in $\mathbb{R}^3$ does not hold. The harmonic functions which are orthogonal to the monogenic and the antimonogenic functions in the sense of L_2 were discovered by Álvarez-Peña and Porter [3] and called *contragenic*. It is necessary to understand the contragenic functions to be able to identify the "monogenic part" of a given harmonic function. Contragenicity, in contrast to harmonicity and monogenicity, is not a local property, since it depends on the domain under consideration.

Results

In this work, by means of a suitable variable change, applying a modification to the works of P. Garabedian and J. Morais, we define a base of monogenic polynomials, such that the spherical harmonics are embedded in a 1-parameter family of spheroidal harmonics. We calculate bases for the collection spheroidal monogenic constants and spheroidal ambigenic polynomials of degree at most n, which are a fundamental piece for the calculation of a basis of contragenic polynomials. In addition, we study the relationship between different systems of harmonic polynomials (Theorem 2.2.4), to obtain, for spheroidal domains of arbitrary eccentricity, relations between the bases of harmonic and contragenic functions by means of explicit expressions (cf. (3.4) and Proposition 4.2.3). This allows us to show that there are common contragenic functions to all spheroids of all eccentricities (Theorem 4.2.1).

In Chapter 1, the basic definitions and well known results for the development of this work are presented, such as monogenic function, antimonogenic function, monogenic constants, ambigenic functions, as well as a summary of the basic properties and necessary recurrence formulas of the special functions known as the associated Legendre functions. Also, we summarize the elementary facts about standard bases for harmonic, monogenic, ambigenic and contragenic polynomials in the unit ball.

In Chapter 2, a 1-parameter basis is constructed for the spheroidal harmonics, and the L_2-norms of the spheroidal harmonic polynomials are explicitly developed. Also, conversion formulas relating the systems of harmonic polynomials for distinct spheroids are calculated.

In Chapter 3, we give a construction for a basis of spheroidal ambigenic polynomials, for which the monogenic constants are calculated. Using expressions of change of basis calculated in Chapter 2, expressions are obtained that relate two spheroidal monogenic systems.

In Chapter 4, we present an explicit construction of a basis for the spheroidal contragenic polynomials, showing that the intersection of contragenic functions defined in spheroids of different eccentricities is a space of infinite dimension, thus presenting the concept of "spheroidal universally contragenic function."

Finally, two appendices are presented. In the first one, the development given in Chapter 1 for prolate spheroidal coordinates is completed, analyzing the case of oblate spheroidal polynomials. In the second appendix, recurrence formulas are presented for the Garabedian spheroidal polynomials that allow us to calculate an expression that relates these polynomials to the solid spherical harmonics.

Chapter 1

Quaternionic Analysis

In this first chapter, we introduce the necessary material of quaternionic analysis that is needed for the development of this thesis. The algebra of real quaternions is an associative but non-commutative field. The generalized Cauchy-Riemann operator ∂ is presented, which generalizes the well-known operator $\partial_{\overline{z}}$ to quaternionic analysis. The null-solutions of this operator are called monogenic. The non-commutative structure of this algebra makes it important to distinguish between an application of the operator ∂ from the left-hand side and from the right-hand side.

It is possible to factor the Laplacian in terms of the quaternionic operator ∂ and its adjoint, in a similar way to the complex case. This factorization gives the possibility to generate monogenic functions from harmonic ones. For detailed information we refer e.g. to [32, 33, 46, 74, 75].

The second part of this chapter begins the discussion of basic properties of the associated Legendre functions, which are used to build the classical system of spherical, and more generally, of spheroidal harmonics. The spheroidal harmonics in question are defined following Garabedian [25], adjusted with a rescaling factor that permits including the sphere as a limit of both the prolate and oblate cases, combined into a single 1-parameter family. We further give computational formulas relating systems of harmonic functions defined in spheroidal domains of differing eccentricity.

There are several works on monogenic polynomial systems defined in spheroidal domains. The last part of this chapter is based on the development by J. Morais [56, 57, 59, 60]. In contrast to the spherical harmonics and monogenics, the basis elements for the spheroid include inhomogeneous polynomials, but is shown that this does not influence the completeness property of that system in the space of solid spheroidal monogenics. In [61] it is shown that the underlying prolate spheroidal monogenics play an important role in studying the monogenic Szegö kernel function for prolate spheroids. In [68], a monogenic polynomial basis for oblate domains is constructed and it is demonstrated that a complete system for these domains can only be either orthogonal or an Appell system.

1.1 Clifford algebras and quaternions

1.1.1 The real Clifford algebra

Inspired by the work of Hamilton, Clifford introduced in 1878 an n-dimensional geometrical algebra in which generalizations of the scalar and vector products to higher dimensions are obtained. This algebra is known as Clifford algebra. For more information on Clifford algebras we refer to [33, 71].

A universal Clifford algebra is an associative but usually non-commutative algebra over the real or the complex field. We consider the Clifford algebra of signature $(0, n)$ denoted by $\mathrm{Cl}_{0,n}$ and $\{e_0, e_1, e_2, \ldots, e_n\}$ stands for the canonical basis of the Euclidean vector space $\mathbb{R}^{n+1}$. The basis elements satisfy the multiplication rules:

$$e_k e_l + e_l e_k = -2\delta_{k,l}, \ (k, l = 1, \ldots, n),$$

where $\delta_{k,l}$ denotes the Kronecker symbol. Moreover, denoting by $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, $e_\emptyset = e_0 = 1$, we obtain a basis for $\mathrm{Cl}_{0,n}$. The addition and multiplication by real numbers are defined coordinatewise. In this way, the multiplication between two elements of $\mathrm{Cl}_{0,n}$ turns out to be associative, anticommutative and has distributive properties, that is, for any $u, v, w \in \mathrm{Cl}_{0,n}$, and for all $\alpha, \beta \in \mathbb{R}$, it follows that

$$u(\alpha v + \beta w) = \alpha u v + \beta u w,$$
$$(\alpha u + \beta v)w = \alpha u w + \beta v w.$$

As $\mathrm{Cl}_{0,n}$ is isomorphic to $\mathbb{R}^{2^n}$ we may provide it with the $\mathbb{R}^{2^n}$-norm $|a|$, and one can verify that for any $a, b \in \mathrm{Cl}_{0,n}$, there holds $|a \cdot b| \leq 2^{n/2} |a|\,|b|$, where $a = \sum_{A \subseteq \{1,\ldots,n\}} a_A e_A$, $b = \sum_{A \subseteq \{1,\ldots,n\}} b_A e_A$.

It is seen that $\mathbb{H}$ can be realized as the Clifford algebra $\mathrm{Cl}_{0,2}$ with the identification $\mathrm{Cl}_{0,2} = \langle \{e_0, e_1, e_2, e_1 e_2\} \rangle$, where $e_1^2 = e_2^2 = -1$, $(e_1 e_2)^2 = -1$. The element e_0 is regarded as the usual unit, that is, $e_0 = 1$.

1.1.2 Quaternionic functions on spatial domains

We summarize here the notation and terminology necessary to developing quaternionic analysis.

We use e_l, $l \in \{1,2,3\}$, instead of i, j, k to denote the generators of $\mathbb{H}$, which are subject to the following rules of multiplication:

$$\begin{aligned} e_i e_j + e_j e_i &= -2\delta_{ij}, \ (i,j = 1,2,3); \\ e_1 e_2 &= e_3. \end{aligned} \tag{1.1}$$

Each quaternion q can be represented in the form

$$q = \sum_{k=0}^{3} q_k e_k, \ q_k \in \mathbb{R} \ (k = 0,1,2,3).$$

Note that two quaternions $p = \sum_{k=0}^{3} p_k e_k$ and $q = \sum_{k=0}^{3} q_k e_k$ are equal if and only if $p_k = q_k$, $k \in \{0,1,2,3\}$.

Now we define the sum and multiplication of two quaternions. The sum is defined by adding the corresponding components, that is

$$p + q = \sum_{k=0}^{3} (p_k + q_k) e_k. \tag{1.2}$$

On the other hand, using relations (1.1) we define the multiplication of two quaternions p and q, as follows:

$$\begin{aligned} pq = {} & (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) \\ & + (p_1q_0 + p_0q_1 + p_2q_3 - p_3q_2)e_1 \\ & + (p_2q_0 + p_0q_2 + p_2q_1 - p_1q_3)e_2 \\ & + (p_3q_0 + p_0q_3 + p_1q_2 - p_2q_1)e_3. \end{aligned} \tag{1.3}$$

The scalar part of q is

$$\operatorname{Sc} q = q_0,$$

and the vector part is

$$\operatorname{Vec} q = \sum_{k=1}^{3} q_k e_k.$$

The conjugate of a quaternion q is given by

$$\overline{q} = q_0 - \sum_{k=1}^{3} q_k e_k.$$

For all $q, r \in \mathbb{H}$, the quaternion conjugation has the following properties:

1. $\overline{q \pm r} = \overline{q} \pm \overline{r}$;

2. $\overline{\overline{q}} = q$;

3. $\overline{qr} = \overline{r}\,\overline{q}$;

4. $q \in \mathbb{R}$ if and only if $q = \overline{q}$; q is a pure quaternion, if and only if, $q = -\overline{q}$.

The norm of a quaternion is defined as

$$|q| := \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{\sum_{k=0}^{3} q_k^2}, \tag{1.4}$$

which coincides with the norm of a vector in $\mathbb{R}^4$. The multiplicative inverse of $q \in \mathbb{H} \setminus \{0\}$ is

$$q^{-1} := \frac{\overline{q}}{|q|^2}.$$

We consider the subset of $\mathbb{H}$ defined as

$$\mathcal{A} := \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H} : x_i \in \mathbb{R}, x_3 = 0\}. \tag{1.5}$$

The elements of the subset $\mathcal{A}$ are known as *reduced quaternions*. Elements of $\mathbb{R}^3$ can be identified with elements of $\mathcal{A}$ by considering $(x_0, x_1, x_2) \in \mathbb{R}^3$. Moreover, since $\mathcal{A}$ is not closed under the quaternionic multiplication, it is clear that $\mathcal{A}$ is only a real vector subspace and not a sub-algebra of $\mathbb{H}$. There are other ways of embedding $\mathbb{R}^3$ in $\mathbb{H}$, for example, by using the subspace of pure quaternions, i.e. by considering $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Throughout the text, let Ω denote an open set of $\mathbb{R}^3$ with a piecewise smooth boundary. A reduced quaternion-valued function $f\colon \Omega \to \mathcal{A}$, is a mapping of the form

$$f(x) = \sum_{k=0}^{2} f_k(x)e_k,$$

where $f_k\colon \Omega \to \mathbb{R}$. Properties such as continuity, differentiability or integrability are ascribed coordinate-wise.

Let $L_2(\Omega, \mathcal{A})$ be the real linear space defined by

$$L_2(\Omega, \mathcal{A}) := \left\{ f : \Omega \to \mathcal{A} | \ \left(\int_\Omega |f(x)|^2 \, dx \right)^{1/2} < \infty \right\}, \tag{1.6}$$

where dx denotes integration respect to volume measure. On this space, a scalar inner product is defined as

$$\langle f, g \rangle_{L_2(\Omega,\mathcal{A})} := \mathrm{Sc} \int_\Omega \overline{f} g \, dx = \int_\Omega \sum_{k=0}^{2} f_k g_k \, dx. \tag{1.7}$$

The L_2-norm $\|f\|$ induced by this inner product coincides with the norm of f considered as a vector-valued function. It is clear that (1.7) does not define an inner product in $L_2(\Omega, \mathcal{A}))$ seen as a quaternionic linear space, because it is not homogeneous with respect to quaternionic scalars.

1.2 Homogeneous harmonic polynomials for the sphere

1.2.1 Decompositions of harmonic polynomials

The *Laplacian* is the differential operator

$$\Delta = \sum \partial / \partial x_i \tag{1.8}$$

in rectangular coordinates. A twice-differentiable function f is *harmonic* when it satisfies the *Laplace equation*

$$\Delta f = 0. \tag{1.9}$$

A special kind of harmonic function is the homogeneous harmonic polynomial. A polynomial $p(x)$ defined in $\mathbb{R}^n$ is called *homogeneous of degree* k, if for a constant s, there holds

$$p(sx) = s^k p(x).$$

(We will be mainly interested in $n = 3$ with $\mathbb{R}^3 = \mathcal{A} \subseteq \mathbb{H} = \mathbb{R}^4$.)

Definition 1.2.1. We denote by $\mathcal{P}_k(\mathbb{R}^n)$ the space of homogeneous polynomials of degree k in $\mathbb{R}^n$ with real coefficients. The subspace $\mathcal{P}_k(\mathbb{R}^n)$ of those polynomials that are harmonic is denoted by $\mathrm{Har}_k(\mathbb{R}^n)$.

Note that any polynomial p of degree k defined in $\mathbb{R}^n$ can be written as

$$p = \sum_{j=0}^{k} p_j, \quad p_j \in \mathcal{P}_j(\mathbb{R}^n).$$

Since $\Delta p = \sum_{j=0}^{k} \Delta p_j$, p is harmonic if and only if each p_j is harmonic. Thus, we can focus our study on the sets $\mathrm{Har}_k(\mathbb{R}^n)$.

To make this work self-contained, we present the following results on the decomposition of homogeneous polynomials; the proofs of these facts can be consulted in [5, Ch. V].

Theorem 1.2.2. *If $k \geq 2$, then $\mathcal{P}_k(\mathbb{R}^n) = \mathrm{Har}_k(\mathbb{R}^n) \oplus |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n)$.*

This theorem states that any homogeneous polynomial $p \in \mathcal{P}_k(\mathbb{R}^n)$ can be decomposed in the form

$$p = p_k + |x|^2 q,$$

where $p_k \in \mathrm{Har}_k(\mathbb{R}^n)$ and $q \in \mathcal{P}_{k-2}(\mathbb{R}^n)$.

Applying successively this result, it follows that for any $p \in \mathcal{P}_k(\mathbb{R}^n)$ there is a unique decomposition of the form

$$p = \sum_{0 \le 2l \le k} |x|^{2l} p_{k-2l}, \tag{1.10}$$

where $p_j \in \mathrm{Har}_j(\mathbb{R}^n)$.

We now consider the dimension of the space $\mathrm{Har}_k(\mathbb{R}^n)$. First we consider $n = 2$, identifying $\mathbb{R}^2$ with the complex numbers. A polynomial $p(z) \in \mathrm{Har}_k(\mathbb{R}^2)$ of the form

$$p(z) = \sum_{l=0}^{k} a_l z^l, \;\; a_l \in \mathbb{C}, \;\; z = x + iy,$$

can be written as $p(x, y) = u(x, y) + iv(x, y)$. Also, if p is an analytic function, then u and v are harmonic functions. In addition, for a homogeneous polynomial of degree k, it is seen that

$$u(z) = \frac{a_k z^k + \overline{a_k z^k}}{2}, \quad v(z) = \frac{a_k z^k - \overline{a_k z^k}}{2i}.$$

In consequence, z^k and $\overline{z}^k$ are homogeneous harmonic polynomials, and any other homogeneous harmonic polynomial p_k can be written as a complex linear combination of the set $\{z^k, \overline{z}^k\}$. Therefore, the dimension of $\mathrm{Har}_k(\mathbb{R}^2)$ over the real numbers is 2, for all $k \geq 1$. If $k = 0$, then p_0 is constant and the dimension of $\mathrm{Har}_0(\mathbb{R}^2)$ is 1. Now, if $n > 2$, we rely on Theorem 1.2.2,

$$\dim(\mathrm{Har}_k(\mathbb{R}^n)) = \dim(\mathcal{P}_k(\mathbb{R}^n)) - \dim(\mathcal{P}_{k-2}(\mathbb{R}^n)).$$

For $k = 0, 1$, every homogeneous polynomial of degree k is harmonic. Therefore, we focus our attention on the case $k \geq 2$. If $x \in \mathbb{R}^n$ and $a = (a_1, a_2, \ldots, a_n)$ is considered as 4-multiindex, we define

$$\begin{aligned} x^a &:= x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \\ |a| &:= a_1 + a_2 + \cdots + a_n. \end{aligned} \tag{1.11}$$

Using this notation, consider the monomials of the form x^a, such that $|a| = k$. Every $p \in \mathcal{P}_k(\mathbb{R}^n)$ is a unique linear combination of these monomials. We have the following result.

Proposition 1.2.3. *If $k \geq 2$, then*

$$\dim \mathrm{Har}_k(\mathbb{R}^n) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}.$$

For $n = 3$, this formula reduces to

$$\dim \mathrm{Har}_k(\mathbb{R}^3) = 2k + 1. \tag{1.12}$$

1.2.2 The associated Legendre functions

Our work focuses on analysis in spheroidal domains. A. M. Legendre in 1783 in his article entitled "Sur l'attraction des sphéroides" stated that if the value of the attraction force of a body of revolution is known at an external point located on its axis, then it is known at all exterior points. With this principle he reduced the problem to the study of the component $P(r, \theta, 0)$, given by the integral over the body

$$P(r, \theta, 0) = \iiint \frac{(r - r')\cos\gamma}{(r^2 - 2rr'\cos\gamma + r'^2)^{3/2}}\, dx,$$

where $\cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos\phi'$, and proved that the integrand can be expressed as a series of powers of r'/r, and finally obtained functions that are currently known as Legendre polynomials and play an important role in the study of spherical harmonics.

The Legendre polynomials satisfy the Rodrigues formula

$$P_n(t) = \frac{1}{2^n\, n!}\frac{d^n}{dt^n}(t^2 - 1)^n, \qquad t \in \mathbb{R}, \quad n = 0, 1, 2, \ldots$$

(see [80, Ch. XV]). Using the binomial expansion and the fact that

$$\frac{d^n}{dt^n}t^r = \begin{cases} 0, & \textit{if } n > r \\ \frac{r!}{(r-n)!}\, t^{r-n} & \textit{if } n \leq r, \end{cases}$$

it follows that

$$P_n(t) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\,(2n - 2k)!}{2^n\, k!\,(n - k)!\,(n - 2k)!}\, t^{n-2k}.$$

Moreover, $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

For each $n \geq 0$, $m = 0, \ldots, n$ and $|t| < 1$, associated Legendre functions are defined as (cf. [69])

$$P_n^m(t) = (-1)^m (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_n(t).$$

Moreover, these functions satisfy the differential equation

$$(1 - t^2)\, y''(t) - 2t\, y'(t) + \left(n(n+1) - \frac{m^2}{1 - t^2} \right) y(t) = 0. \tag{1.13}$$

In addition, when $t \notin [-1, 1]$, the associated Legendre functions are defined by

$$P_n^m(t) = (t^2 - 1)^{m/2} \frac{d^m}{dt^m} P_n(t). \tag{1.14}$$

Next, we present an explicit expression for the associated Legendre functions ([58]). Since the cited result is not universally available, for completeness we reproduce the proof.

Proposition 1.2.4. *Let $n \geq 0$, $0 \leq m \leq n$. Then*

$$P_n^m(t) = \begin{cases} (-1)^m (1 - t^2)^{m/2} \displaystyle\sum_{k=m}^{n} \lambda_k^{n,m} (t-1)^{n-k} (t+1)^{k-m}, & \\ \qquad\qquad\qquad\qquad \textit{if } |t| < 1, & \\ (t^2 - 1)^{m/2} \displaystyle\sum_{k=m}^{n} \lambda_k^{n,m} (t-1)^{n-k} (t+1)^{k-m}, & \\ \qquad\qquad\qquad\qquad \textit{if } t \notin (-1, 1), & \end{cases} \tag{1.15}$$

where

$$\lambda_k^{n,m} = \frac{n!(m+n)!}{2^n(m+n-k)!(n-k)!(k-m)!k!}. \tag{1.16}$$

Proof. We will suppose that $|t| < 1$, the other case is analogous. The associated Legendre functions are defined by

$$\begin{aligned} P_n^m(t) &= \frac{(-1)^m}{2^n n!}(1-t^2)^{m/2}\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n \\ &= \frac{(-1)^m}{2^n n!}(1-t^2)^{m/2}\frac{d^{n+m}}{dt^{n+m}}((t-1)(t+1))^n, \end{aligned}$$

where

$$\frac{d^{n+m}}{dt^{n+m}}((t-1)(t+1))^n = \sum_{k=0}^{n+m}\binom{n+m}{k}\frac{d^{n+m-k}}{dt^{n+m-k}}(t+1)^n\frac{d^k}{dt^k}(t-1)^n. \tag{1.17}$$

Both k and $n+m-k$ must be less or equal to n; otherwise, one of the derivatives is zero. Thus, for $m \geq 0$, the sum runs from $k = m$ to n.

We note that

$$\frac{d^k}{dx^k}(x \pm 1)^n = \frac{n!}{(n-k)!}(x \pm 1)^{n-k}.$$

Substituting in (1.17), we obtain

$$\begin{aligned} &\frac{d^{n+m}}{dt^{n+m}}((t-1)(t+1))^n \\ &= (n!)^2\sum_{k=m}^{n}\binom{n+m}{k}\frac{1}{(k-m)!}(t+1)^{k-m}\frac{1}{(n-k)!}(t-1)^{n-k}. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} P_n^m(t) &= \frac{(-1)^m}{2^n n!}(1-t^2)^{m/2}(n!)^2 \sum_{k=m}^{n} \binom{n+m}{k} \frac{(t+1)^{k-m}(t-1)^{n-k}}{(k-m)!(n-k)!} \\ &= (-1)^m(1-t^2)^{m/2} \sum_{k=m}^{n} \lambda_k^{n,m}(t-1)^{n-k}(t+1)^{k-m}, \end{aligned}$$

with

$$\lambda_k^{n,m} = \frac{n!}{2^n}\binom{n+m}{k}\frac{1}{(k-m)!(n-k)!}$$

□

Note that if $t = \pm 1$ y $m > 0$, $P_n^m(t) = 0$. We now present several recurrence formulas for the associated Legendre functions (cf. [69] or [80]).

Proposition 1.2.5. *For each $n \geq 0$, $m = 0, 1, \ldots, n$ and $t \in \mathbb{R}$, the following identities are satisfied:*

$$(1-t^2)(P_{n+1}^m)'(t) = (n+m+1)P_n^m(t) - (n+1)tP_{n+1}^m(t), \tag{1.18}$$

$$(n-m+1)P_{n+1}^m(t) = (2n+1)tP_n^m(t) - (n+m)P_{n-1}^m(t), \tag{1.19}$$

$$P_{n+1}^n(t) = (2n+1)tP_n^n(t). \tag{1.20}$$

Proposition 1.2.6. *For each $n \geq 0$, $m = 0, 1, \ldots, n$ and $t \in [-1, 1]$, we* *have:*

$$(t^2-1)(P_{n+1}^m)'(t) = (1-t^2)^{1/2}P_{n+1}^{m+1}(t) + mtP_{n+1}^m(t), \tag{1.21}$$

$$(1-t^2)^{1/2}P_{n+1}^m(t) = \frac{1}{2n+3}(P_n^{m+1}(t) - P_{n+2}^{m+1}(t)), \tag{1.22}$$

$$\sqrt{1-t^2}P_n^{m+1}(t) = (n-m)tP_n^m(t) - (n+m)P_{n-1}^m(t), \tag{1.23}$$

$$P_n^n(t) = (-1)^n(2n-1)!!(1-t^2)^{\frac{n}{2}}, \tag{1.24}$$

$$2mtP_{n+1}^m(t) = -(1-t^2)^{1/2}\bigg(P_{n+1}^{m+1}(t) + (n+m+1)(n-m+2)P_{n+1}^{m-1}(t)\bigg). \tag{1.25}$$

Proposition 1.2.7. *For each $n \geq 0$, $m = 0, 1, \ldots, n$ and $|t| > 1$, we have the following identities:*

$$(t^2-1)(P_{n+1}^m)'(t) = (t^2-1)^{1/2}P_{n+1}^{m+1}(t) + mtP_{n+1}^m(t), \tag{1.26}$$

$$(t^2-1)^{1/2}P_{n+1}^m(t) = \frac{1}{2n+3}(P_{n+2}^{m+1}(t) - P_n^{m+1}(t)), \tag{1.27}$$

$$\sqrt{t^2-1}P_n^{m+1}(t) = (n-m)tP_n^m(t) - (n+m)P_{n-1}^m(t), \tag{1.28}$$

$$P_n^n(t) = (2n-1)!!(t^2-1)^{\frac{n}{2}}, \tag{1.29}$$

$$2mtP_{n+1}^m(t) = (t^2-1)^{1/2}\bigg(-P_{n+1}^{m+1}(t) + (n+m+1)(n-m+2)P_{n+1}^{m-1}(t)\bigg). \tag{1.30}$$

Another important property of the associated Legendre functions is their orthogonality in $L_2[-1,1]$.

Proposition 1.2.8. *Let* $n \geq 0,\ m = 0, 1, \ldots, n$ *and* $|t| < 1$*. Then,*

$$\int_{-1}^{1} P_{n_1}^{m_1}(t)\, P_{n_2}^{m_2}(t)\, dt = \frac{2(n_1 + m_1)!}{(2n_1 + 1)(n_1 - m_1)!}\, \delta_{n_1,n_2}\delta_{m_1,m_2}.$$

A more detailed study of the Legendre polynomials and their associated Legendre functions can be found, for example, in [42, 65, 73, 80].

1.2.3 A basis of spherical harmonics

The restriction of harmonic polynomials to the sphere results in important consequences.

Definition 1.2.9. A homogeneous harmonic polynomial of degree k on $\mathbb{R}^n$ restricted to the unit sphere $\mathbb{S}^2$ is called a *spherical harmonic* of degree k. Denote by $\mathrm{Har}_k(\mathbb{S}^2)$ the set of spherical harmonics of degree k.

Since we are interested in studying the Laplace equation (1.9) in terms of spherical coordinates, it is convenient to introduce a spherical coordinate system, given by the following expressions:

$$x_0 = r\cos\theta,\ x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi, \tag{1.31}$$

where $r \in [0, \infty)$ is the radius of the ball, $\theta \in [0, \pi]$ is the polar angle and $\phi \in [0, 2\pi)$ the azimuthal angle.

The Laplace equation $\Delta\Psi(r, \theta, \phi) = 0$ in spherical coordinates is explicitly written in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = 0$$

or, equivalently

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\Delta_{(\theta,\phi)}\Psi=0, \tag{1.32}$$

where

$$\Delta_{(\theta,\phi)}=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

This expression explicitly separates the dependence of the equation on the coordinate r, from the dependence of the equation on the angular coordinates. Applying the method of separation of variables, the function $\Psi(r,\theta,\phi)$ postulated to be the form

$$\Psi(r,\theta,\phi)=\frac{1}{r}R(r)Y(\theta,\phi).$$

Calculating the radial partial derivative, we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=r\frac{\partial^2R(r)}{\partial r^2}Y(\theta,\phi).$$

Substituting this expression in the Laplace equation, we obtain

$$\frac{1}{r^2}\frac{r\partial^2R(r)}{\partial r^2}Y(\Theta,\phi)+\frac{1}{r^2}\frac{1}{r}R(r)\Delta_{(\theta,\phi)}Y(\theta,\phi)=0.$$

Then, multiplying this equation by $r^2/\Psi(r,\theta,\phi)$, we have that

$$\frac{r^2}{R(r)}\frac{\partial^2R(r)}{\partial r^2}+\frac{1}{Y(\theta,\phi)}\Delta_{(\theta,\phi)}Y(\theta,\phi)=0.$$

By introducing the separation variable λ, we have

$$\frac{r^2}{R(r)}\frac{\partial^2R(r)}{\partial r^2}=\lambda=-\frac{1}{Y(\theta,\phi)}\Delta_{(\theta,\phi)}Y(\theta,\phi)$$

In this way, we have obtained the equations

$$\frac{d^2R(r)}{dr^2} = \frac{\lambda}{r^2}R(r),$$
$$\Delta_{\theta,\phi}Y(\theta,\phi) = -\lambda Y(\theta,\phi). \tag{1.33}$$

Using the method of separation of variables again, when proposing the separation $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$, equation (1.33) can be written as

$$\frac{\Phi(\phi)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \frac{\Theta(\theta)}{\sin\theta}\frac{d^2\Phi(\phi)}{d\phi^2} + \lambda\Theta(\theta)\Phi(\phi) = 0.$$

In consequence, multiplying by $\sin^2\theta/(\Theta(\theta)\Phi(\phi))$, we get

$$\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} + \lambda\sin^2\theta = 0.$$

Introducing the separation constant m^2, we obtain the differential equations

$$\frac{d^2\Phi(\phi)}{d\phi^2} + m^2\Phi(\phi) = 0, \tag{1.34}$$
$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + (\lambda\sin^2\theta - m^2)\Theta(\theta) = 0 \tag{1.35}$$

Now, by (1.34),

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -m^2,$$

where the solutions are given by

$$\Phi(\phi) = \{e^{-im\phi}, e^{im\phi}\}.$$

On several occasions, $\Phi(\phi)$ is required to be mono-valued at the azimuthal angle, so m is required to be integer. The functions

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

make up a set of orthonormal functions with respect to the integration over the azimuthal angle. The equation (1.35) is equivalent to

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = 0.$$

Introducing the change of variable $x = \cos\theta$, changing the name of the function $\Theta(\theta)$ for $y(x)$ and multiplying by $(1-x^2)^{-1}$, (1.35) takes the form

$$\frac{d}{dx}\left((1-x^2)\frac{dy(x)}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y(x) = 0. \tag{1.36}$$

So that the solutions do not diverge at $\cos\theta = \pm 1$, we take $\lambda = n(n+1)$ and in this case we get that $\Theta(\theta) = P_n^m(\cos\theta)$.

We now give the definition of the basic spherical harmonics. For each non-negative integer m, we denote by

$$\Phi_m^+(\phi) = \cos(m\phi) \qquad \text{and} \qquad \Phi_m^-(\phi) = \sin(m\phi). \tag{1.37}$$

We will never use Φ_0^- since it is identically zero.

Definition 1.2.10. For each $n \geq 0$ and $m = 0, 1, 2, \ldots, n$, we define the *basic solid spherical harmonics*

$$\widehat{U}_{n,m}[0](r,\theta) := r^n\, P_n^m\,(\cos\,\theta) \;=\; |x|^n\, P_n^m\left(\frac{x_0}{|x|}\right). \tag{1.38}$$

The label [0] indicates that the domain is the ball Ω_0, since later we will work with spheroidal domains of arbitrary eccentricity μ.

Proposition 1.2.11 ([65]). *The functions $U^+_{n,m}[0](r,\theta,\phi)$ and $U^-_{n,l}[0]\,(r,\theta,\phi)$ with $n \geq 0$, $m = 0,1,2,\ldots,n$, $l = 1,2,\ldots,n$, given by*

$$U^{\pm}_{n,m}[0]\,(r,\theta,\phi) := \widehat{U}_{n,m}[0]\,(r,\,\theta)\,\Phi^{\pm}_m(\phi) \tag{1.39}$$

(with the convention that $U^-_{0,0}[0]$ is excluded) conform an orthogonal system in $L_2(\Omega_0)$ of $2n+1$ homogeneous, harmonic polynomials in (x_0,x_1,x_2).

The polynomials defined by (1.39) are usually known as solid spherical harmonics. From this point on, the notation $U^{\pm}_{n,m}[0]$ will be used to indicate that we refer to the solid spherical harmonics, even and odd respectively, meaning that for the case $U^-_{n,m}[0]$, m denotes a non-negative integer which varies between 1 and n. Observe that the identities (1.19), (1.20) and (1.23) allow us to write recursion formulas for the functions $\widehat{U}_{n,m}[0]$.

Proposition 1.2.12 ([13]). *For each $n \geq 1$, $m = 0,1,\ldots,n$,*

$$\widehat{U}_{n+1,m}[0] = \frac{2n+1}{n-m+1}x_0\widehat{U}_{n,m}[0] - \frac{n+m}{n-m+1}|x|^2\widehat{U}_{n-1,m}[0], \tag{1.40}$$

$$\widehat{U}_{n+1,n}[0] = (2n+1)x_0\widehat{U}_{n,n}[0], \tag{1.41}$$

$$\widehat{U}_{n,m+1}[0] = \frac{1}{|x|}\Big((n-m)x_0\widehat{U}_{n,m}[0] - (n+m)|x|^2\widehat{U}_{n-1,m}[0]\Big) \tag{1.42}$$

with $|x| > 0$ in the latter relation.

These relations will be used in Appendix B.

1.3 Homogeneous monogenic polynomials for the sphere

1.3.1 Monogenic functions

In this section, the basic facts concerning about monogenic functions are presented (see [33]). An important issue in quaternionic analysis is to introduce the concept of derivative of a quaternionic-valued function. In the complex plane identified with $\mathbb{R}^2$, this can be defined by the following proposition.

Theorem 1.3.1 ([33]). *Let U be an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{C}$ a function. The following statements are equivalent:*

1. *The limit*
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$
exists for all $z_0 \in U$;

2. *f can be represented, near every point $z_0 \in U$, by a power series*
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n;$$

3. *f is a solution of the Cauchy-Riemann equation on U, namely*
$$\partial_{\overline{z}} f = 0,$$
where $\partial_{\overline{z}} := \dfrac{\partial}{\partial x} + i \dfrac{\partial}{\partial y}$.

For the case of quaternion-valued functions, an analogous condition to 1. of Theorem 1.3.1, would be

Definition 1.3.2. A function $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{H}$ is said to be *quaternion-differentiable on the left* at a point $q \in \Omega$, if the limit

$$\lim_{h \to 0} h^{-1}(f(q+h) - f(q))$$

exists, when h converges to zero along any direction in the quaternionic space.

However, the only quaternionic-valued functions which are quaternionic differentiable in an open set have the form $f(q) = a + qb,\ a, b \in \mathbb{H}$ (see [76]). Therefore, this generates a too restrictive class of functions and, thus, these approaches were abandoned in favor of more useful ones. However, conditions 2. and 3. of Theorem 1.3.1 can be generalized to quaternions.

We denote by $\partial_k := \frac{\partial}{\partial x_k}$ the partial derivative of a function with respect to the variable $x_k,\ k \in \{0, 1, 2\}$. The Cauchy-Riemann (or Fueter) operators are defined as

$$\partial = \partial_0 - \sum_{i=1}^{2} e_i \partial_i, \quad \overline{\partial} = \partial_0 + \sum_{i=1}^{2} e_i \partial_i. \tag{1.43}$$

Now, we are interested in functions $\mathbb{R}^3$-valued, defined in domains on $\mathbb{R}^3$. Regard $\mathbb{R}^3$ as the subset $\mathcal{A}$ defined in (1.5). Although this subspace is not closed under the quaternionic multiplication, it is possible to carry out a great deal of the analysis analogous to that of complex numbers. Thus, a function

f, which is $\mathbb{R}^3$-valued, can be thought as a $\mathbb{H}$-valued function, namely

$$f(x) = \sum_{k=0}^{3} f_k(x) e_k,$$

where $x \in \Omega$ and $f_3(x) = 0$ for all x.

Definition 1.3.3. A function $f \in C^1(\Omega, \mathbb{R}^3)$ is *(left-)monogenic* in Ω if $\partial f = 0$ in Ω, and *(left-)antimonogenic* if $\overline{\partial} f = 0$ identically in Ω.

In consequence, f is (left-)monogenic if it satisfies the following system of differential equations, known as the Riesz system [72]:

$$\begin{cases} \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 &= 0 \\ \partial_1 f_0 - \partial_0 f_1 &= 0 \\ \partial_2 f_0 - \partial_0 f_2 &= 0 \\ \partial_1 f_2 - \partial_2 f_1 &= 0 \end{cases}$$

Similarly f is antimonogenic (from left) if it satisfies

$$\begin{cases} \partial_0 f_0 + \partial_1 f_1 + \partial_2 f_2 &= 0 \\ -\partial_1 f_0 + \partial_0 f_1 &= 0 \\ -\partial_2 f_0 + \partial_0 f_2 &= 0 \\ -\partial_1 f_2 + \partial_2 f_1 &= 0 \end{cases}$$

As

$$-e_3 \,(\overline{\partial}\, f)\, e_3 = \partial \overline{f},$$

it follows that f is monogenic if and only if $\overline{f}$ is antimonogenic. As a consequence of this fact, we have that f is left-monogenic if and only if f it is right-monogenic [57]. This is not true for functions from $\mathbb{R}^4 = \mathbb{H}$ to $\mathbb{R}^4$.

The notion of left (right-) monogenicity provides a powerful generalization of the concept of complex analyticity to quaternionic analysis, since many classical theorems from complex analysis can be generalized to higher dimensions by this approach, for example the Cauchy integral theorem and Cauchy integral formula. We refer for instance to [33, 36, 37, 76].

A first effort to obtain a quaternionic derivative was proposed by R. Fueter in [24]. Later by studying a generalized Bochner-Martinelli integral formula, I. Mitelman and M. Shapiro considered the operator $(1/2)\partial$ as a generalized derivative in quaternions. Following the idea of A. Sudbery's quaternion results, K. Gürlebeck and H. Malonek obtained the so-called hypercomplex derivative in higher dimensions [34].

On the other hand, by a simple calculation, one can verify that

$$\Delta = \partial\overline{\partial} = \overline{\partial}\partial. \tag{1.44}$$

Consequently, if f is an $\mathcal{A}$-valued function defined on Ω, twice differentiable, which is monogenic or antimonogenic, then f is harmonic. The converse is not true.

Definition 1.3.4. We will denote by $\mathcal{M}(\Omega)$ the set of monogenic functions in $\Omega \subseteq \mathbb{R}^3$ and by $\overline{\mathcal{M}}(\Omega)$ the set of antimonogenic functions in Ω. Further,

$$\mathcal{M}^{(n)}(\Omega) = \mathcal{M}(\Omega) \cap \mathcal{P}_n(\mathbb{R}^3),$$

$$\overline{\mathcal{M}}^{(n)}(\Omega) = \overline{\mathcal{M}}(\Omega) \cap \mathcal{P}_n(\mathbb{R}^3).$$

1.3.2 Fueter polynomials

In the decade 1930s, R. Fueter discovered that certain properties of holomorphic functions of a complex variable, can be generalized to $\mathbb{H}$-valued functions defined on domains of $\mathbb{H}$. Fueter began a systematic study of the Dirac operator kernel for quaternions and one of his main contributions was a collection of polynomials (Fueter polynomials), which allow to build the concept of power series expansion of Fueter-regular functions (cf. [24]). The Fueter polynomials may be constructed as follows ([76]). Consider an (unordered) collection of n integers

$$\kappa = \{i_1, i_2, \ldots, i_n\}, \qquad 1 \leq i_r \leq 3.$$

This collection can be written as $\kappa = [n_1, n_2, n_3]$, where n_1 denotes the number of 1's, n_2 the number of 2's and n_3 the number of 3's in κ. Note that $n_1 + n_2 + n_3 = n$. Observe that for each n there are $(n+1)(n+2)/2$ sets κ. Now, consider the "hypercomplex variables"

$$z_i = x_i - x_0 e_i, \quad i = 1, 2, 3.$$

Definition 1.3.5. The *Fueter polynomials* associated to κ are defined as

$$F_\kappa = F_\kappa(x) = \frac{1}{n!} \sum z_{i_1} z_{i_2} \cdots z_{i_n}, \tag{1.45}$$

where the sum is over all the $n!/(n_1!n_2!n_3!)$ different combinations of n_1, n_2 y n_3.

Observe that F_κ is a homogeneous polynomial of degree n. Denote by

$\mathcal{M}_{\mathbb{H}}^{(n)}$ the set of monogenic $\mathbb{H}$-valued functions which are homogeneous of degree n over $\mathbb{R}$. It was demonstrated ([76]) that the dimension of $\mathcal{M}_{\mathbb{H}}^{(n)}$ is $(n+1)(n+2)/2$. It was also shown that the elements of this set are polynomials and the polynomials F_κ form a basis for $\mathcal{M}_{\mathbb{H}}^{(n)}$.

Subsequently, other results for complex-valued functions have been shown to be valid for quaternionic functions (cf. [75]). Note that, as the Fueter-Regular functions are harmonic, there is a very close relationship with potential theory. In the last two decades, these functions have been explored in other areas, particularly in Clifford algebras (cf. [11], [33]). Currently, the regular functions are known as monogenic or hyperholomorphic functions, among other names.

H. Malonek [54] proved that polynomials (1.45) can be written in the form

$$F_\kappa(x) = \frac{1}{\kappa!} z_1^{\gamma_1} \times z_2^{\gamma_2},$$

where $\gamma = (\gamma_1, \gamma_2)$ denotes a multi-index such that $\gamma_1 + \gamma_2 = n$ and $\times$ the permutational product of Clifford numbers. The powers $z_1^{\gamma_1} \times z_2^{\gamma_2}$ are known as generalized powers. Using these functions, expansions analogous to Taylor series are obtained.

Theorem 1.3.6. *([54]) The general form of the Taylor series of a monogenic function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$ in a neighborhood of the origin is given by*

$$f = \sum_{n=0}^{\infty} \sum_{|\gamma|=n} (z_1^{\gamma_1} \times z_2^{\gamma_2}) c_\gamma,$$

where $c_\gamma = (1/\gamma_1!\gamma_2!)\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2} f(x)|_{x=0} \in \mathbb{H}$.

This result, together with the fact that the space of homogeneous monogenic polynomials of fixed degree has finite dimension, allow to study monogenic functions from the perspective of stratification of the space of functions through homogeneous polynomials.

H. Leutwiler [52] built in 2001, using the Fueter polynomials, a basis for monogenic polynomials in $L_2(\mathbb{R}^3, \mathcal{A})$, and in 2007, R. Delanghe generalized these results in the context of Clifford algebras (see [21]). However, one of the drawbacks of the Fueter polynomials is that they are not orthogonal with respect to real inner product on the ball Ω_0 (see [57]). One might think this is not a problem, since the orthonormalization process of Gram-Schmidt can be applied and thus obtain an orthonormal basis. But, it is known that this procedure, in addition to cumbersome, is numerically unstable [13]. For this reason, it is that several researchers have worked in the construction of polynomial basis that are more accessible both in algebraic and numerical management [16, 53, 56, 57, 59].

1.3.3 A basis of spherical monogenics

In [13], an orthogonal basis for $\mathbb{H}$-valued monogenic functions of three real variables was constructed based on spherical harmonics (1.39). In [56, 57], a basis for the case of functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ was calculated. The starting point for these constructions is the factorization of the Laplacian operator given by (1.44). Namely, applying the operator ∂ to the spherical harmonics of degree $n+1$, we obtain $2n+3$ polynomials.

Definition 1.3.7. Let $n \geq 0$ and $0 \leq m \leq n+1$. The *spherical monogenics* are defined by

$$X^{\pm}_{n,m}[0] := \partial U^{\pm}_{n+1,m}[0].$$

An important property of the spherical monogenics, which was proved in [13], is

Proposition 1.3.8 ([13]). *For each $n \geq 1$ and $0 \leq m \leq n+1$*

$$\partial X^{\pm}_{n,m} = 2(n+m+1)X^{\pm}_{n-1,m}.$$

As a consequence of a fact established by Leutwiler it is known that

Proposition 1.3.9 ([52]).

$$\dim_{\mathbb{R}} \mathcal{M}^{(n)}(\Omega_0) = 2n+3.$$

It is also known ([14, 56]) that the collection

$$\{X^{\pm}_{n,m}[0] \,|\, n \geq 0, 0 \leq m \leq n+1\}$$

is a basis for $\mathcal{M}^{(n)}(\Omega_0)$. Moreover, in [56], an explicit representation of the basic elements of $\mathcal{M}^{(n)}(\Omega_0)$ in terms of spherical harmonics, was calculated.

Theorem 1.3.10 ([56]). *For each $n \geq 0$ and $0 \leq m \leq n+1$, the basic* *element for the homogeneous monogenic polynomials of degree n is given by*

$$
\begin{aligned}
&X^{\pm}_{n,m}[0] = (n+m+1)U^{\pm}_{n,m}[0] \\
&+ \frac{1}{2}\bigg(\big((n+m)(n+m+1)U^{\pm}_{n,m-1}[0] - U^{\pm}_{n,m+1}[0]\big)e_1 \\
&\mp \big((n+m)(n+m+1)U^{\mp}_{n,m-1}[0] + U^{\mp}_{n,m+1}[0]\big)e_2 \bigg).
\end{aligned}
$$

In addition, the norms of these polynomials in $L_2(\Omega_0)$ can be calculated explicitly.

Proposition 1.3.11 ([13])**.** *Let $n \geq 0$ and $0 \leq m \leq n+1$.*

$$
||X^{\pm}_{n,m}[0]|| = \sqrt{\frac{(1+\delta_{0,m})\pi(n+1)(n+m+1)!}{2(2n+3)(n-m+1)!}}.
$$

1.4 Spherical ambigenic functions and monogenic constants

In order to discuss contragenic functions it is necessary first to discuss ambigenic functions. While facts about antimonogenic functions are generally trivial modifications of facts about monogenic functions, obtained by taking the conjugate, when one considers both monogenic and antimonogenic functions together, the situation becomes a bit more complicated.

We will freely interchange the notation $\mathcal{A}$ and $\mathbb{R}^3$ when no confusion will arise.

Definition 1.4.1 ([3])**.** Let Ω be a domain in $\mathbb{R}^3$ and let $f \in C^1(\Omega, \mathbb{R}^3)$ be a harmonic function. It is said that f is an *ambigenic* function if

$$f \in \mathcal{M}(\Omega) + \overline{\mathcal{M}}(\Omega),$$

i.e. f can be represented as the sum of a monogenic and an antimonogenic function.

Definition 1.4.2. A *monogenic constant* is a function which is simultaneously monogenic and antimonogenic.

The decomposition of an ambigenic function as a sum of a monogenic and an antimonogenic function is not unique, because the set

$$\mathcal{M}(\Omega) \cap \overline{\mathcal{M}}(\Omega)$$

of monogenic constants in the domain $\Omega \subseteq \mathbb{R}^3$ is nontrivial. Monogenic constants do not depend on x_0 (see [57]) and can be expressed as

$$f = a_0 + f_1 e_1 + f_2 e_2,$$

where $a_0 \in \mathbb{R}$ is a constant, and $f_1 - i f_2$ is an ordinary holomorphic function of the complex variable $x_1 + i x_2$.

There are natural projections of $\mathcal{M}(\Omega)$ onto the subspaces

$$\operatorname{Sc} \mathcal{M}(\Omega) = \{\operatorname{Sc} f |\ f \in \mathcal{M}(\Omega)\} \quad \subseteq \operatorname{Har}_{\mathbb{R}}(\Omega),$$
$$\operatorname{Vec} \mathcal{M}(\Omega) = \{\operatorname{Vec} f |\ f \in \mathcal{M}(\Omega)\} \subseteq \operatorname{Har}_{\{0\}\oplus\mathbb{R}^2}(\Omega),$$

where $\operatorname{Har}_{\mathbb{R}}(\Omega)$ denotes the space of real-valued harmonic functions defined in Ω. Note that $\operatorname{Sc}\mathcal{M}(\Omega) = \operatorname{Sc}\overline{\mathcal{M}}(\Omega)$ and $\operatorname{Vec}\mathcal{M}(\Omega) = \operatorname{Vec}\overline{\mathcal{M}}(\Omega)$.

Proposition 1.4.3 ([57]). *When Ω is simply-connected, $\operatorname{Sc}\mathcal{M}(\Omega) = \operatorname{Har}_{\mathbb{R}}(\Omega)$ (every harmonic function is the scalar part of a monogenic function). The corresponding vector part is unique up to the addition of a monogenic constant.*

A construction of an orthogonal basis for ambigenic functions defined on the unit ball Ω_0 is as follows.

Proposition 1.4.4 ([3]). *Let $n > 0$ and $m = 0, 1, \ldots, n$. Write*

$$\gamma_{n,m}[0] = \frac{n - 2m^2 + 1}{(n+1)(2n+1)}, \quad 0 \le m \le n,$$

and $\gamma_{n,n+1}[0] = 0$. The functions given by

$$\begin{aligned}
Y^{++}_{n,m}[0] =& X^{+}_{n,m}[0], \qquad m = 0, \ldots, n+1, \\
Y^{-+}_{n,m}[0] =& X^{-}_{n,m}[0], \qquad m = 1, \ldots, n, \\
Y^{+-}_{n,m}[0] =& \overline{X}^{+}_{n,m}[0] - \gamma_{n,m}[0] X^{+}_{n,m}[0], \qquad m = 0, \ldots, n, \\
Y^{--}_{n,m}[0] =& \overline{X}^{-}_{n,m}[0] - \gamma_{n,m}[0] X^{-}_{n,m}[0], \qquad m = 1, \ldots, n+1,
\end{aligned}$$

form an orthogonal basis for the space of square integrable ambigenic functions on Ω_0 which are homogeneous of degree n. In particular, the dimension of the homogeneous ambigenic polynomials of degree $n \ge 1$ is

$$\dim(\mathcal{M}^{(n)}(\Omega_0) + \overline{\mathcal{M}^{(n)}(\Omega_0)}) = 4n + 4. \tag{1.46}$$

1.5 Spherical contragenic polynomials

As was explained in Section 1.2.1, a harmonic function in a domain in $\mathbb{C}$ splits in a natural way into a holomorphic and an antiholomorphic part, up to an additive constant. There exist many generalizations for monogenic functions on quaternions [76], Clifford algebras (see [11], for example) and monogenic functions from $\mathbb{R}^3$ to $\mathbb{H}$ ([33], [13]).

However, it was shown [3] that a natural generalization in the context of monogenic functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ does not hold. In consequence, it is possible to find harmonic functions which are orthogonal to the monogenic and antimonogenic functions in the sense of L_2, which are now known as *contragenics*.

One way to quantify the failure of a harmonic function to be ambigenic is via orthogonal complements. We will write

$$\mathcal{M}_2(\Omega) = \mathcal{M}(\Omega) \cap L_2(\Omega). \tag{1.47}$$

Since scalar-valued (i.e. $\mathbb{R}e_0$-valued) functions are by definition orthogonal in $L_2(\Omega)$ to functions which take values in $\mathbb{R}e_1 + \mathbb{R}e_2$, there is a natural orthogonal direct sum decomposition of the space of square-integrable ambigenic functions, namely

Proposition 1.5.1.

$$\mathcal{M}_2(\Omega) + \overline{\mathcal{M}}_2(\Omega) = \operatorname{Sc}\mathcal{M}_2(\Omega) \oplus \operatorname{Vec}\mathcal{M}_2(\Omega),$$
$$\mathcal{M}_2(\Omega) \cap \overline{\mathcal{M}}_2(\Omega) \subseteq \operatorname{Vec}\mathcal{M}(\Omega).$$

It was also observed in [3] that when an $\mathbb{R}$-valued harmonic homogeneous polynomial is completed as the scalar part of a monogenic function (unique up to adding a monogenic constant), the vector part can also be taken to be a homogeneous polynomial of the same degree.

In the following, recall further Definition 1.2.1:

Definition 1.5.2 ([3])**.** In any domain $\Omega \subseteq \mathbb{R}^3$, a square-integrable harmonic function $h \in \mathrm{Har}(\Omega) \cap L_2(\Omega, \mathbb{R}^3)$ is called *contragenic* when it is orthogonal to all square-integrable ambigenic functions, that is, if it lies in

$$\mathcal{N}(\Omega) \,=\, (\mathcal{M}_2\,(\Omega) \,+\, \overline{\mathcal{M}}_2\,(\Omega))^{\perp},$$

where the orthogonal complement is taken in $\mathrm{Har}(\Omega) \cap L_2(\Omega, \mathcal{A})$.

In [3] an orthogonal basis of contragenics for the ball Ω_0 was calculated.

Theorem 1.5.3 ([3])**.** *Let $n \geq 1$. The $2n-1$ functions*

$$\begin{aligned} &Z^{\pm}_{n,m}[0] = \big(a_{n,m}[0]U^{\mp}_{n,m-1}[0] + U^{\mp}_{n,m+1}[0]\big)e_1 \\ &\pm \big(a_{n,m}[0]U^{\pm}_{n,m-1}[0] - U^{\pm}_{n,m+1}[0]\big)e_2, \qquad 1 \leq m \leq n-1 \\ &Z_{n,0}[0] = U^{-}_{n,1}[0]e_1 - U^{+}_{n,1}[0]e_2, \end{aligned}$$

where $a_{n,m}[0] = (n-m)(n-m+1)$ and $U^{\mp}_{n,m}[0]$ as in (1.39), form an orthogonal basis for the space of homogeneous contragenic polynomials of degree n.

One purpose of this thesis is to generalize Theorem 1.5.3 to spheroidal domains.

Chapter 2

Conversions Among Orthogonal Spheroidal Harmonics

In this chapter, we will discuss only families of harmonic functions, with a view to subsequent application to monogenic functions.

We will use the results obtained in [25] to calculate a general expression for the basis changes between different orthogonal sets of spheroidal harmonic polynomials, which will be relevant for the general context of the contragenic polynomials studied in a forthcoming chapter. We relate the systems of harmonic functions associated with the spheroid Ω_μ (defined in (2.1)) to those associated with the ball Ω_0.

Expressions of basis change between different systems of spheroidal harmonic functions are then calculated, obtaining as a main result a basis change expression between a system of harmonic functions associated with a spheroid Ω_μ and a system defined in a spheroid $\Omega_{\widetilde{\mu}}$.

2.1 Spheroidal harmonics

Spheres are commonly used as the reference domain for modeling physical problems. However, in many cases, a spheroidal domain may offer a better approximation to reality. A *spheroid* is a quadric surface generated by rotating an ellipse about one of its principal axes. Assume throughout that the spheroid is oriented so that its axis of rotational symmetric is along the x-axis, and whose focal length equals $2(1-e^{2\nu})^{\frac{1}{2}}$; that is,

$$\Omega_\mu := \left\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + \frac{x_1^2 + x_2^2}{e^{2\nu}} = 1,\ \nu \in \mathbb{R}\right\}. \tag{2.1}$$

where $\nu \in \mathbb{R}$ is arbitrary, and $\mu = (1-e^{2\nu})^{1/2}$ by convention is in the interval $(0,1)$ when $\nu < 0$ (prolate spheroid), and in $i\mathbb{R}^+$ when $\nu > 0$ (oblate spheroid); the intermediate value $\nu = 0$, $\mu = 0$ gives the unit ball $\Omega_0 = \{x\colon\ |x|^2 < 1\}$. The convenience of the parameter μ will become evident later. Here we note that in the prolate case, we obtain Ω_μ by setting $e^\nu = \tanh\alpha$ and rescaling x by a factor of μ^{-1}, while for the oblate case, we set $e^\nu = \coth\alpha$ and rescale by a factor of $(\mu/i)^{-1}$.

A systematic analysis of harmonic functions on spheroidal domains was initiated by Szegö [77], followed by Garabedian [25] who produced orthogonal bases with respect to certain natural inner products associated to prolate and oblate spheroids, among them the L_2-Hilbert space structures on the interior and on the spheroid. However, some properties, such as the relationships between these systems, were not studied. In addition, spherical harmonics were not considered as part of this kind of systems.

2.1.1 Spheroidal coordinates

There are several equivalent ways to introduce spheroidal coordinates. For the sake of convenience, we begin by introducing cylindrical coordinates in order to further analyze prolate and oblate geometries simultaneously. The cylindrical coordinates (x_0, ρ, φ) are related to the Cartesian coordinates by the equations

$$x_0 = x_0, \quad x_1 = \rho \cos \varphi, \quad x_2 = \rho \sin \varphi$$

such that $\rho = \sqrt{x_1^2 + x_2^2}$ and $\varphi = \arctan x_2/x_1$, where $x_0 \in \mathbb{R}$, $\rho \in [0, \infty)$ and $\varphi \in [0, 2\pi)$. Accordingly, the spheroid (2.1) has the equation

$$x_0^2 + \frac{\rho^2}{e^{2\nu}} = 1. \tag{2.2}$$

Now suppose that $\nu < 0$ (the other case $\nu > 0$ is explained in Appendix A). At this point, it is appropriate to introduce prolate spheroidal coordinates (u, v, φ) defined by the relations

$$x_0 = \mu \cos u \cosh v, \quad \rho = \mu \sin u \sinh v, \quad \varphi = \varphi, \tag{2.3}$$

where $u \in (0, \pi]$ is the asymptotic angle with respect to the major axis, $v \in (0, \operatorname{arctanh} e^{\nu}]$ is the radial term with scale factor μ and $\varphi \in [0, 2\pi)$ is the rotation term. To proceed with, we observe that

$$|x|^2 + \mu^2 = \mu^2(\cos^2 u + \cosh^2 v),$$

from which it follows that

$$\cos u = \frac{2x_0}{\omega(\mu)}$$

and

$$\cosh v = \frac{\omega(\mu)}{2\mu},$$

where

$$\omega(\mu) := \sqrt{(x_0+\mu)^2 + x_1^2 + x_2^2} + \sqrt{(x_0-\mu)^2 + x_1^2 + x_2^2}. \tag{2.4}$$

2.1.2 Construction of spheroidal harmonics

When making the change of coordinates (2.3), the *Laplace equation* takes the form

$$\frac{1}{\mu^2\left(\sin^2 u + \sinh^2 v\right)}\left(\frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} + \cot u \frac{\partial U}{\partial u} + \coth v \frac{\partial U}{\partial v}\right) + \frac{1}{\mu^2 \sin^2 u \sinh^2 v}\frac{\partial^2 U}{\partial \varphi^2} \;=\; 0; \quad U \in \mathcal{C}^2(\Omega_\mu).$$

Then applying the method of separation of variables in prolate spheroidal coordinates, we obtain the following three ordinary differential equations

$$\frac{d^2\Theta(u)}{du^2} + \cot u \frac{d\Theta(u)}{du} + \left[n(n+1) - \frac{m^2}{\sin^2 u}\right]\Theta(u)\sin u = 0, \tag{2.5}$$

$$\frac{d^2\Upsilon(v)}{dv^2} + \coth v \frac{d\Upsilon(v)}{dv} - \left[\frac{m^2}{\sinh^2 v} + n(n+1)\right]\Upsilon(v)\sinh v = 0,$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0, \tag{2.6}$$

where n is a constant and m is a parameter introduced during the method of separation of variables. Now, the periodicity of Φ suggest that m is a positive integer or zero. In consequence, solutions to the equation (2.6) are either $\cos(m\varphi)$ or $\sin(m\varphi)$. Also, note that (2.5) and the equation (1.35) are the same. Therefore, $\Theta(u) = P_n^m(\cos u)$ and $\Upsilon(v) = P_n^m(\cosh v)$. So the solutions to Laplacian equation in spheroidal coordinates, which are known as *prolate spheroidal harmonics* [25, 42], are a linear combination of product solutions, namely

$$U(u, v, \varphi) := \Theta(u)\Upsilon(v)\Phi(\varphi). \tag{2.7}$$

Accordingly, we define the required solutions (2.7) to be employed for the space interior to the prescribed spheroid (2.2) as follows.

Definition 2.1.1. The *basic spheroidal harmonics* are

$$U_{n,m}^{\pm}[\mu](x) := \widehat{U}_{n,m}[\mu](x)\Phi_m^{\pm}(\varphi) \tag{2.8}$$

where, for $\mu \neq 0$,

$$\widehat{U}_{n,m}[\mu](x) = \alpha_{n,m}\mu^n P_n^m(\cos u) P_n^m(\cosh v). \tag{2.9}$$

In the above formulas we use the notation

$$\alpha_{n,m} := \frac{(n-m)!}{2^n (1/2)_n}, \tag{2.10}$$

which uses the (rising) Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$.

Except for the constant factor $\alpha_{n,m}$, the basic spheroidal harmonics were introduced in [25]. We proceed by checking how the functions (2.8) behave at infinity. The use of the particular coefficient $\alpha_{n,m}$ is for the following.

Proposition 2.1.2. *For every* $x \in \mathbb{R}^3$, *the limit* $\lim_{\mu\to 0} U^{\pm}_{n,m}[\mu](x)$ *exists and is given by the spherical harmonic* (1.39).

Proof. Since φ in (1.39) and (2.3) does not depend on x_0, we examine the factors $P_n^m(2x_0/\omega)P_n^m(\omega/(2\mu))$ in (2.8), with ω again given by (2.4). Note that if $a \in \mathbb{C}$, $(1+a)^{\frac{1}{2}} = 1 + \frac{1}{2}a + \mathcal{O}(a^2)$ when $a \to 0$. Since

$$\begin{aligned}
\sqrt{(x_0 \pm \mu)^2 + x_1^2 + x_2^2} &= \sqrt{|x|^2 \pm 2x_0\mu + \mu^2} \\
&= |x|\sqrt{1 \pm \frac{2x_0}{|x|^2}\mu + \frac{\mu^2}{|x|^2}} \\
&= |x|\left(1 \pm \frac{x_0}{|x|^2}\mu + \mathcal{O}(\mu^2)\right) \\
&= |x| \pm \frac{x_0}{|x|}\mu + \mathcal{O}(\mu^2),
\end{aligned}$$

we have $\omega = 2|x| + \mathcal{O}(\mu^2)$ as $\mu \to 0$ A direct computation using (2.4) shows that $2x_0/\omega = x_0/|x| + O(\mu)$, so $P_n^m(2x_0/\omega) \to P_n^m(x_0/x)$ as $\mu \to 0$. It can be shown inductively that

$$\frac{1}{\alpha_{n,m}} = 2^{-n} n!(n+m)! \sum_{k=m}^{n} \lambda_k^{n,m},$$

where $\lambda_k^{n,m} = ((n+m-k)!(n-k)!(k-m)!k!)^{-1}$. From the explicit representation

$$P_n^m(t) = \frac{n!(m+n)!}{2^n}(t^2-1)^{m/2} \sum_{k=m}^{n} \lambda_k^{n,m}(t-1)^{n-k}(t+1)^{k-m}$$

valid for real $|t| > 1$, we have the required asymptotic behavior

$$\alpha_{n,m} P_n^m(t) \simeq t^n$$

as $t = \omega/2\mu$ tends to infinity, which corresponds to $\mu \to 0$ for fixed x.

□

Proposition 2.1.2 says that solutions of the form (2.8) have an asymptotic property similar to the spherical harmonics in a neighborhood of infinity. This is very important not only to ensure that those functions are well-defined, but also it gives an evidence of the completeness of the underlying spheroidal harmonics.

Further, the polynomials $U^{\pm}_{n,m}[\mu]$ form an orthogonal complete system for the domain Ω_μ, in the sense of the Dirichlet inner product (see [25]). However, it is not an orthogonal basis under the inner product in $L_2(\Omega_\mu)$, namely, if $n_1 \neq n_2$, we have that

$$\begin{aligned}
\langle U^{+}_{n_1,m}[\mu], U^{+}_{n_2,m}[\mu] \rangle_{L_2(\Omega_\mu)} &= \int\int\int_{\Omega_\mu} U^{+}_{n_1,m}[\mu]\, U^{+}_{n_2,m}[\mu]\, dx \\
= \mu^{n_1+n_2} \alpha_{n_1,m} \alpha_{n_2,m} &\int_0^{\pi} \int_0^{2\pi} \int_0^{\operatorname{arctanh} e^{\nu}} P^m_{n_1}(\cos\, u) P^m_{n_1}(\cosh\, v) \\
\times\, P^m_{n_2}(\cos u) P^m_{n_2}(\cosh v) &\cos^2(m\varphi) \mu^3 \sin u \sinh v (\cos^2 u - \cosh^2 v)\, dv d\varphi du.
\end{aligned}$$

Taking the variable change given by $s = \cosh v$ and $t = \cos u$, it is obtained that

$$\langle U^+_{n_1,m}[\mu], U^+_{n_2,m}[\mu]\rangle_{L_2(\Omega_\mu)} = \mu^{n_1+n_2+3}\alpha_{n_1,m}\alpha_{n_2,m}\pi\left(\int_{-1}^{1} t^2 P^m_{n_1}(t)P^m_{n_2}(t)dt \int_1^{\frac{1}{\mu}} P^m_{n_1}(s)\,P^m_{n_2}(s)\,ds\right).$$

Then, by equation (1.19), we have that

$$\begin{aligned} t^2 P^m_{n_1}(t)P^m_{n_2}(t) = {} & \frac{1}{(2n_1+1)(2n_2+1)} \\ & \times [(n_1-m+1)(n_2-m+1)P^m_{n_1+1}(t)P^m_{n_2+1}(t) \\ & + (n_1-m+1)(n_2+m)P^m_{n_1+1}(t)P^m_{n_2-1}(t) \\ & + (n_2-m+1)(n_1+m)P^m_{n_2+1}(t)P^m_{n_1-1}(t) \\ & + (n_1+m)(n_2+m)P^m_{n_1-1}(t)P^m_{n_2-1}(t)]. \end{aligned}$$

In consequence, if $n_2 = n_1 + 2$, we have

$$\begin{aligned} \langle U^+_{n_1,m}[\mu], U^+_{n_2,m}[\mu]\rangle_{L_2(\Omega_\mu)} = {} & \frac{2\pi\mu^{2n_1+5}(\alpha_{n_1+2,m})^2(n_1+m+2)!}{(n_1-m+2)!(2n_1+5)} \\ & \times\left(\int_1^{\frac{1}{\mu}} P^m_{n_1}(s)P^m_{n_1+2}(s)ds\right). \end{aligned}$$

For example, taking $n_1 = 0$ and $m = 0$, we obtain that

$$\mu^5\int_1^{1/\mu} P^0_0(s)P^0_2(s)\,ds = \mu^5\left(\frac{1-\mu^2}{2\mu^3}\right).$$

Therefore,

$$\langle U^+_{0,0}[\mu], U^+_{2,0}[\mu]\rangle = \frac{4\pi\mu^5(\alpha_{2,0})^2}{(2!)5}\left(\frac{1-\mu^2}{2\mu^3}\right) \neq 0.$$

In conclusion, the collection

$$\{U^{\pm}_{n,m} \,|\, n \geq 0,\ 0 \leq m \leq n\} \tag{2.11}$$

is not an orthogonal system under the inner product in $L_2(\Omega_\nu)$.

Now, when μ tends to 0,

$$\mu^{2n_1+5} P^m_{n_1}(1/\mu) P^m_{n_1+2}(1/\mu) \simeq \frac{n_1!(n_1+2)!(n_1+m)!(n_1+m+2)!}{2^(2n_1+2)}\, \mu^{2n_1+3} \sum_{k=m}^{n_1} \lambda^{n_1,m}_k \sum_{k=m}^{n_1+2} \lambda^{n_1+2,m}_k .$$

In consequence,

$$\mu^{2n_1+5} \int_1^{\frac{1}{\mu}} P^m_{n_1}(s) P^m_{n_1+2}(s) ds \to 0 \tag{2.12}$$

when μ tends to 0 and therefore $\langle U^+_{n_1,m}[\mu], U^+_{n_2,m}[\mu] \rangle_{L_2\,(\Omega_\mu)}$ tends to zero also. Analogously, we obtain that

$$\langle U^-_{n_1,m}[\mu], U^-_{n_2,m}[\mu] \rangle_{L_2(\Omega_\mu)} \to 0$$

when μ tends to zero. Therefore, the functions $U^{\pm}_{n,n}[\mu]$ $(n \geq 0)$ do not form an orthogonal basis of $L_2(\Omega_\mu) \cup \mathrm{Har}(\Omega_\mu)$ except when $\mu = 0$.

2.1.3 Orthogonal families

One wishes to have an orthogonal basis of spheroidal harmonic polynomials in the space $L_2\,(\Omega_\mu)$. It is possible to apply the Gram-Schmidt method to the $\{\widehat{U}_{n,m}\}$-system, however it is a slow and numerically unstable method, so it is necessary to approach the problem using a different technique.

In [59], the problem was solved in the following way: consider the operator

$$\partial_0 = \frac{1}{\mu(\cos^2 u - \cosh^2 v)}(\sin u \cosh v \partial_u - \cos u \sinh v \partial_v),$$

It was shown that

$$\partial_0 \widehat{U}_{n+1,m} = \frac{\mu^n (n+m+1)\alpha_{n+1,m}}{\cosh^2 v - \cos^2 u}[\cosh v P_{n+1}^m(\cosh v) P_n^m(\cos u) - \cos u P_{n+1}^m(\cos u) P_n^m(\cosh v)]. \quad (2.13)$$

Define, for each $n \geq 0$, $0 \leq m \leq n$,

$$\widehat{V}_{n,m}[\mu](x) = \partial_0 \widehat{U}_{n+1,m}[\mu](x). \quad (2.14)$$

When μ is considered fixed, we will denote these functions simply by $\widehat{V}_{n,m}$.

We recall from [56] that for spherical harmonics, there is a formula analogous to Appell differentiation of monomials,

$$\frac{\partial}{\partial x_0} U^{\pm}_{n+1,m}[0](x) = (n+m+1) U^{\pm}_{n,m}[0](x). \quad (2.15)$$

However, $V^{\pm}_{n,m}[\mu]$ is not so simply related to $U^{\pm}_{n,m}[\mu]$ for $\mu \neq 0$, as was explained in [59]. We examine such relations in the next section.

Multiplying (2.14) by the functions $\Phi^{\pm}_m(\varphi)$, we obtain the functions

$$V^{\pm}_{n,m}[\mu](x) = \widehat{V}_{n,m} \Phi^{\pm}_m(\varphi). \quad (2.16)$$

Since the functions (2.16), except for the constant factors $\alpha_{n,m}$ and the rescaling of the x variable, are the functions defined in [25], the main result of that paper can be restated as follows.

Proposition 2.1.3 ([25]). *The functions $V^{\pm}_{n,m}[\mu]$ ($n \geq 0$) are harmonic polynomials in x_0, x_1, x_2 of degree n. They form a complete orthogonal family in the closed subspace $L_2(\Omega_\mu) \cap \mathrm{Har}(\Omega_\mu)$ of $L_2(\Omega_\mu)$. Furthermore,*

$$\|V^{\pm}_{n,m}[\mu]\|^2 = (1+\delta_{0,m})\beta_{n,m}\mu^{2n+3}I_n[\mu], \tag{2.17}$$

$$\beta_{n,m}[\mu] = \frac{\pi}{3(2^{2n-1})}\frac{(n+m+1)(n-m+2)!(n+m+1)!}{(3/2)_n(5/2)_n}, \tag{2.18}$$

and

$$I_{n,m}[\mu] = \int_1^{\frac{1}{\mu}} P_n^m(t)P_{n+2}^m(t)\,dt. \tag{2.19}$$

Now, we present an explicit expression for the integral $I_{n,m}[\mu]$, where $\eta \in [1,\infty)$, which gives an explicit expression for $\|V^{\pm}_{n,m}\|^2$.

Proposition 2.1.4. *For each $n \geq 0$, $0 \leq m \leq n$ and $\eta > 1$,*

$$\int_1^{\eta} P_n^m(t)P_{n+2}^m(t)dt = \sum_{k=0}^{n+m} C_k^{n,m}\big(\eta-1\big)^{2n-2k+3}\,{}_2F_1(a,b;c;d),$$

with ${}_2F_1$ *denoting the classical Gaussian hypergeometric function, where*

$$\begin{aligned}
a &= -(2n-2k+3),\\
b &= -(2n+m-2k+2),\\
c &= m,\\
d &= \frac{\eta+1}{\eta-1},\\
C_k^{n,m} &= \frac{(4n+2m-4k+5)(k-m-1)!(n-k-1)!(n+m)!(n+m+2)!}{\sqrt{\pi}\Gamma(1/2-m)k!(m-1)!(n-k+1)!(n-m)!}\\
&\quad\times\frac{(n+m-k-1)!(2n-k+2)!(1/2)_{k-m}(1/2)_{n-k}(3/2)_n+m-k}{(n-m+2)!(n+m-k+2)!(2n+m-k-1)!(7/2)_{2n+m-k}}.
\end{aligned}$$

Proof. In the first instance, in [2] it was obtained that

$$(x^2-1)^{m/2}P_n^m(t)P_{n+2}^m(t)=\sum_{k=0}^{n+m}\widetilde{C}_k^{n,m}P_{2n+m+2-2k}^m(t),$$

where

$$\begin{aligned}
\widetilde{C}_k^{n,m} &= \frac{(4n+2m-4k+5)(n+m)!(n+m+2)!(2n-k+2)!}{2^m(4n+2m-2k+5)k!(n-m)!(n-m+2)!(n+m-k)!}\\
&\quad\times\frac{(1/2)_{n-k}(1/2)_{n-k+2}(1/2-m)_k}{(n+m-k+2)!(1/2)_{2n+m-k+2}}.
\end{aligned}$$

Then, by expression (1.14), it follows that

$$P_n^m(t)P_{n+2}^m(t)=\sum_{k=0}^{n+m}\widetilde{C}_k^{n,m}\frac{d^m}{dt^m}P_{2n+m+2-2k}(t).$$

In consequence, we have that

$$\int P_n^m(t) P_{n+2}^m(t)\, dt = \sum_{k=0}^{n+m} \widetilde{C}_k^{n,m} \frac{d^{m-1}}{dt^{m-1}} P_{2n+m+2-2k}(t).$$

Also, note that

$$F(t) := \frac{d^{m-1}}{dt^{m-1}} P_{l_k}(t) = \frac{(l_k)!}{2^{l_k}} \sum_{j=m-1}^{l_k} \binom{l_k+m-1}{j} \frac{(t+1)^{j-m+1}(t-1)^{l_k-j}}{(j-m+1)!(l_k-j)!},$$

where $l_k = 2n+m+2-2k$. Therefore, $F(1)=0$ and

$$\begin{aligned}
&\int_1^\eta P_n^m(t) P_{n+2}^m(t) dt = \sum_{k=0}^{n+m} \widetilde{C}_k^{n,m} F(\eta) \\
&= \sum_{k=0}^{n+m} \widetilde{C}_k^{n,m} \frac{(l_k)!}{2^{l_k}} \sum_{j=m-1}^{l_k} \binom{l_k+m-1}{j} \frac{(\eta+1)^{j-m+1}(\eta-1)^{l_k-j}}{(j-m+1)!(l_k-j)!} \\
&= \sum_{k=0}^{n+m} \widetilde{C}_k^{n,m} \frac{(l_k)!}{2^{l_k}\mu^{2n+3-2k}} \sum_{j=m-1}^{l_k} \binom{l_k+m-1}{j} \frac{(\frac{1}{\eta}+1)^{j-m+1}(1-\frac{1}{\eta})^{l_k-j}}{(j-m+1)!(l_k-j)!}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{j=m-1}^{l_k} \binom{l_k+m-1}{j} \frac{(\frac{1}{\eta}+1)^{j-m+1}(1-\frac{1}{\eta})^{l_k-j}}{(j-m+1)!(l_k-j)!} \\
&= \frac{(1-\frac{1}{\eta})^{l_j-m+1}}{(l_k-m+1)!} \binom{l_k+m-1}{m-1} {}_2F_1\Big(-1+m-l_k, -l_k, m, \frac{\eta+1}{\eta-1}\Big).
\end{aligned}$$

By straightforward calculations it is obtained that

$$\widetilde{C}_k^{n,m} \frac{(l_k)!}{2^{l_k}(l_k-m+1)!} \binom{l_k+m-1}{m-1} = C_k^{n,m}$$

and the result follows. □

The following result establishes the asymptotic behavior of the functions (2.16) and it is a direct consequence of Proposition 2.1.2.

Proposition 2.1.5. *For every $x \in \mathbb{R}^3$, the limit $\lim_{\mu\to 0} V^{\pm}_{n,m}[\mu](x)$ exists and is given by $V^{\pm}_{n,m}[0](x) = (\partial/\partial x_0)U^{\pm}_{n+1,m}[0](x)$, where $U^{\pm}_{n+1,m}[0]$ is defined in (1.39).*

The spherical harmonics are embedded in this 1-parameter family of spheroidal harmonics. In contrast, in treatments such as [25, 57, 68], the spheroidal harmonics degenerate to a segment as the eccentricity of the spheroid decreases.

Finally, in [25] the orthogonality of the polynomials $V^{\pm}_{n,m}[\mu]$ over the surface of ellipsoids was studied. To make this work self-contained, we present the proof.

Theorem 2.1.6 ([25])**.** *The polynomials $V^{\pm}_{n,m}[\mu]$ are orthogonal over the surface of the spheroid Ω_μ in the sense of the scalar product*

$$\{f, g\}_\mu = \iint_{\partial\Omega_\mu} fg \left|\mu^2 - (x + i\rho)^2\right|^{1/2} d\sigma.$$

Proof. First, by the expressions (2.3), over the surface of the spheroid Ω_μ, we have that

$$\begin{aligned}\left|\mu^2 - (x+i\rho)^2\right|^{1/2} &= |\mu| \left|1 - \left(\frac{x+i\rho}{\mu}\right)^2\right|^{1/2} \\ &= |\mu| \left|1 - (\cos u \cosh\alpha + i \sin u \sinh\alpha)^2\right|^{1/2} \\ &= |\mu| \,|\sin(u - i\alpha)|,\end{aligned}$$

where $\alpha = \operatorname{arctanh} e^{\nu}$.

On the other hand, if

$$\overline{x} = (\mu \cos u \cosh \alpha, \mu \sin u \sinh \alpha \cos \varphi, \mu \sin u \sinh \alpha \sin \varphi),$$

it follows that

$$\frac{\partial \overline{x}}{\partial u} \times \frac{\partial \overline{x}}{\partial \varphi} =$$
$$\mu^2 (\sin u \cos u \sinh^2 \alpha, \sin^2 u \cos \varphi \sinh \alpha \cosh \alpha, \sin^2 u \sin \varphi \sinh \alpha \cosh \alpha).$$

Thus,

$$\begin{aligned} \left\| \frac{\partial \overline{x}}{\partial u} \times \frac{\partial \overline{x}}{\partial \varphi} \right\| &= \mu^2 \sin u \sinh \alpha \, (\cos^2 u \sinh^2 \alpha + \sin^u \cosh^2 \alpha)^{1/2} \\ &= \mu^2 \sin u \sinh \alpha \, |\sin (u - i\alpha)|. \end{aligned}$$

In consequence, denoting by $\widetilde{V}_{n,m} = \widehat{V}_{n,m}|_{\partial \Omega_\mu}$, we obtain that

$$\begin{aligned} \left\{ V^+_{n_1,m_1}, V^+_{n_2,m_2} \right\}_\mu = &\int_0^{2\pi} \int_0^{\pi} \widetilde{V}_{n_1,m_1} \widetilde{V}_{n_2,m_2} \Phi^+_{m_1} \Phi^+_{m_2} \\ &\times |\mu|^3 \sin u \sinh \alpha \, |\sin (u - i\alpha)|^2 \, du \, d\varphi \\ &= |\mu|^3 (1 + \delta_{0,m_1}) \sinh \alpha \, \pi \delta_{m_1,m_2} \\ &\times \int_0^{\pi} \widetilde{V}_{n_1,m_1} \widetilde{V}_{n_2,m_1} \sin u \, (\cosh^2 \alpha - \cos^2 u) \, du. \end{aligned}$$

Now, by (2.13) and Theorem 2.2.2, it follows that

$$\widetilde{V}_{n,m} = \frac{\mu^n (n+m+1)\alpha_{n+1,m}}{\cosh^2\alpha - \cos^2 u}[\cosh\alpha P_{n+1}^m(\cosh\alpha)P_n^m(\cos u)$$
$$- \cos\, u\, P_{n+1}^m\,(\cos\, u)\, P_n^m\,(\cosh\,\alpha)]$$
$$= \sum_{0\leq 2k\leq n-m} \tau_{n,m,k}\,\mu^{2k}\,\widetilde{U}_{n-2k,m},$$

where

$$\widetilde{U}_{k,m} = \widehat{U}_{k,m}\,|_{\partial\Omega_\mu} = \mu^k \alpha_{k,m} P_k^m(\cosh\alpha)P_k^m(\cos u).$$

So, denoting by $\widetilde{\psi}_{n_1}^{m_1} = \psi_{n_1}^{m_1}\,|_{\partial\Omega_\mu}$, with

$$\psi_n^m = \cosh v P_{n+1}^m(\cosh v)P_n^m(\cos\, u) - \cos u P_{n+1}^m(\cos u)P_n^m(\cosh v),$$

it is obtained that

$$\left\{V_{n_1,m_1}^+, V_{n_2,m_2}^+\right\}_\mu = \pi|\mu|^3\mu^{n_1}(1+\delta_{0,m_1})(n_1+m_1+1)\sinh\alpha$$
$$\times\left(\sum_{0\leq 2k\leq n_2-m_1} \tau_{n_2,m_1,k}\mu^{2k}\int_0^\pi \widetilde{\psi}_{n_1}^{m_1}\,\widetilde{U}_{n_2-2k,m_1}\,\sin\, u\,du\right)\alpha_{n_1+1,m_1}\delta_{m_1,m_2}.$$

Also, for any q

$$\int_0^\pi \widetilde{\psi}_{n_1}^{m_1}\widetilde{U}_{q,m_1}\sin u du = \mu^q\alpha_{q,m_1}P_q^{m_1}(\cosh\alpha)$$
$$\times\Bigg\{\cosh\alpha P_{n_1-1}^{m_1}(\cosh\alpha)\int_0^\pi P_q^{m_1}(\cos u)P_{n_1}^{m_1}(\,\cos u)\sin u du$$
$$-\frac{P_{n_1}^{m_1}(\cosh\alpha)}{2n_1+3}\int_0^\pi \big[(n_1-m_1+2)P_q^{m_1}(\cos u)P_{n_1+2}^{m_1}(\cos u)$$
$$+(n_1+m_1+1)P_q^{m_1}(\cos u)P_{n_1}^{m_1}(\cos u)\big]\sin u du\Bigg\}.$$

Thus

$$\left\{V^{+}_{n_1,m_1}, V^{+}_{n_2,m_2}\right\}_{\mu} = 2\pi|\mu|^{2n_1+3}\frac{(n_1+m_1+1)(n_1+m_1+1)!\,(\alpha_{n_1+1,m})^2}{(2n_1+1)(n_1-m_1)!}$$
$$\times(1+\delta_{0,m_1})\,\sinh\alpha\,P^{m_1}_{n_1}(\cosh\alpha)\,[\cosh\alpha\,P^{m_1}_{n_1-1}(\cosh\alpha)$$
$$-\left(\frac{n_1+m_1+1}{2n_1+3}\right)P^{m_1}_{n_1}(\cosh\alpha)]\delta_{n_1,n_2}\,\delta_{m_1,m_2}.$$

Analogously, it is obtained that

$$\left\{V^{-}_{n_1,m_1}, V^{-}_{n_2,m_2}\right\}_{\mu} = 2\pi\,|\mu|^{2n_1+3}\frac{(n_1+m_1+1)\,(n_1+m_1+1)!\,(\alpha_{n_1+1,m})^2}{(2n_1+1)(n_1-m_1)!}$$
$$\times\sinh\alpha\,P^{m_1}_{n_1}(\cosh\alpha)\,[\cosh\alpha P^{m_1}_{n_1-1}(\cosh\alpha)$$
$$-\left(\frac{n_1+m_1+1}{2n_1+3}\right)\,P^{m_1}_{n_1}(\cosh\alpha)]\delta_{n_1,n_2}\,\delta_{m_1,m_2}.$$

Finally, it is clear that

$$\left\{V^{+}_{n_1,m_1}, V^{-}_{n_2,m_2}\right\}_{\mu} \;=\; 0.$$

Therefore, the system $\{V^{\pm}_{n,m}\}$ is orthogonal over the surface of Ω_μ, as we wanted to show. □

2.2 Relations among spheroidal harmonic systems

It is of interest to express the orthogonal basis of harmonic functions for one spheroid Ω_μ in terms of those for another spheroid. It is natural to use the unit ball Ω_0 as a point of reference, which will be the case in the first results.

We begin the calculation of the coefficients for the relationships among the various classes of harmonic functions by presenting various known formulas in a uniform manner.

2.2.1 Garabedian harmonics expressed by classical harmonics

For $n \geq 0$, consider the rational constants

$$u_{n,m,k} = \frac{(1/2)_{n-k}\,(n+m-2k+1)_{2k}}{(-4)^k(1/2)_n\,k!} \tag{2.20}$$

for $0 \leq m \leq n$, $0 \leq 2k \leq n$, and let $u_{n,m,k} = 0$ otherwise. In the present notation, the main result of [12] may be expressed as follows (i.e. the factor $\alpha_{m,n}$ has been incorporated into (2.20)).

Proposition 2.2.1 ([12])**.** *Let $n \geq 0$ and $0 \leq m \leq n$. Then*

$$\widehat{U}_{n,m}[\mu] = \sum_{0\leq 2k\leq n-m} u_{n,m,k}\mu^{2k}\,\widehat{U}_{n-2k,m}[0].$$

An important characteristic of this relation is that the same coefficients $u_{n,m,k}$ work for the "+" and "−" cases (cosines and sines) and, strikingly, for all values of μ. By (2.8), an equivalent form of expressing Proposition 2.2.1 is

$$U^{\pm}_{n,m}[\mu] = \sum_{0\leq 2k\leq n-m} u_{n,m,k}\mu^{2k}\,U^{\pm}_{n-2k,m}[0]. \tag{2.21}$$

Since $\partial/\partial x_0$ in (2.16) is a linear operator, (2.21) gives automatically the corresponding result for the Garabedian harmonics,

$$V^{\pm}_{n,m}[\mu] = \sum_{0\le 2k\le n-m+1} u_{n,m,k}\mu^{2k}\, V^{\pm}_{n-2k,m}[0] \qquad (2.22)$$

This in turn gives via (2.15) the following expression in terms of the spherical harmonics:

Corollary 2.2.1. *Let $n \ge 0$ and $0 \le m \le n$. Then*

$$\widehat{V}_{n,m}[\mu] = \sum_{0\le 2k\le n-m+1} v_{n,m,k}\mu^{2k}\, \widehat{U}_{n-2k,m}[0], \qquad (2.23)$$

where

$$v_{n,m,k} = (n+m-2k+1)u_{n+1,m,k}.$$

The coefficients

$$\tau_{n,m,k} = \frac{(n+m+1)!\,(1/2)_{n-2k+1}}{4^k(n+m-2k)!(1/2)_{n+1}}$$

give a similar expression for the Garabedian basic harmonics $V^{\pm}_{n,m}[\mu]$ in terms of the standard harmonics $U^{\pm}_{n,m}[\mu]$ for the same spheroid, rather than in terms of $\widehat{U}_{n,m}[0]$:

Theorem 2.2.2 ([59])**.** *Let $n \ge 0$ and $0 \le m \le n$. Then*

$$\widehat{V}_{n,m}[\mu] = \sum_{0\le 2k\le n-m} \tau_{n,m,k}\mu^{2k}\, \widehat{U}_{n-2k,m}[\mu].$$

In [12] the inverse relation of (2.21) was also derived, expressing $U^{\pm}_{n,m}[0]$

in terms of $U^{\pm}_{n,m}[\mu]$, via

$$\widehat{U}_{n,m}[0] = \sum_{0\le k\le n-m} \widetilde{u}_{n,m,k}\mu^{2k}\,\widehat{U}_{n-2k,m}[\mu], \tag{2.24}$$

where the coefficients can be written as

$$\widetilde{u}_{n,m,k} = \frac{4^{n-2k}(2n-4k+1)(n-k)!(m+n)!(1/2)_{n-2k}}{k!(2n-2k+1)!(n+m-2k)!}, \tag{2.25}$$

again independent of μ. In consequence, applying the operator $\partial/\partial x_0$ and using (2.15), we have the following result.

Proposition 2.2.2. *Let $n \ge 0$ and $0 \le m \le n$. Then*

$$\widehat{U}_{n,m}[0] = \sum_{0\le 2k\le n-m} \widetilde{v}_{n,m,k}\mu^{2k}\,\widehat{V}_{n-2k,m}[\mu],$$

where

$$\widetilde{v}_{n,m,k} = \frac{\tau_{n+1,m,k}}{n+m+1}.$$

The inverse relation for Theorem 2.2.2 is a much simpler formula, given as follows:

Corollary 2.2.3 ([59])**.** *For $n \ge 0$ and $0 \le m \le n$,*

$$\widehat{U}_{n,m}[\mu] = \frac{1}{n+m+1}\widehat{V}_{n,m}[\mu] + \frac{n+m}{4n^2-1}\mu^2\,\widehat{V}_{n-2,m}[\mu].$$

This uses the convention $\widehat{V}_{n-2,m}[\mu] = 0$ when $m > n$; i.e.

$$\widehat{U}_{n,n-1}[\mu] = \frac{1}{2n}\widehat{V}_{n,n-1}[\mu],$$
$$\widehat{U}_{n,n}[\mu] = \frac{1}{2n+1}\widehat{V}_{n,n}[\mu].$$

2.2.2 Conversion among Garabedian harmonics

The preceding subsection does not include the inverse relation of (2.22) of the form

$$\widehat{V}_{n,m}[0] = \sum_{0\le 2k\le n-m} w_{n,m,k}[0,\mu]\mu^{2k}\widehat{V}_{n-2k,m}[\mu]. \tag{2.26}$$

Instead of deriving it directly, we verify first the following remarkable conversion formula, which relates the spheroidal harmonics associated with Ω_μ to those associated with any other $\Omega_{\widetilde{\mu}}$. Write

$$\widetilde{w}_{n,m,k} = \frac{(n+m+1)!(1/2)_{n-2k+2}}{4^k k!(n+m-2k+1)!(1/2)_{n-k+2}}$$

when $0 \le 2k \le n-m+2$, otherwise $\widetilde{w}_{n,m,k} = 0$.

Theorem 2.2.4. *Let $n \ge 0$, $0 \le m \le n$, and let $\mu, \widetilde{\mu} \in [0,1) \cup i\mathbb{R}^+$ such that $\mu \ne 0$. The coefficients $w_{n,m,k}[\widetilde{\mu},\mu]$ in the relation*

$$\widehat{V}_{n,m}[\widetilde{\mu}] = \sum_{0\le 2k\le n-m} w_{n,m,k}[\widetilde{\mu},\mu]\,\widehat{V}_{n-2k,m}[\mu]$$

are given by

$$w_{n,m,k}[\widetilde{\mu},\mu] = {}_2F_1(-k,-n+k-3/2;-n-1/2;(\widetilde{\mu}/\mu)^2)\,\widetilde{w}_{n,m,k}\,\mu^{2k}.$$

Proof. We begin by replacing μ with $\widetilde{\mu}$ in Corollary 2.2.1 and substituting the terms on the right-hand side according to Proposition 2.2.2. By linear independence of the harmonic basis elements, it follows that

$$w_{n,m,k}[\widetilde{\mu},\mu] = \mu^{2k}\sum_{l=0}^{k} v_{n,m,l}\widetilde{v}_{n-2l,m,k-l}\left(\frac{\widetilde{\mu}}{\mu}\right)^{2l} \tag{2.27}$$

in which we note that all terms are real valued. Using reductions such as $(2n-4k+3)(1/2)_{n-2k+1} = 2(1/2)_{n-2k+2}$ and recalling $0 \le l \le k$, one easily sees that

$$\begin{aligned} v_{n,m,l} &= \frac{(1/2)_{n-l+1}(n+m-2l+1)_{2l+1}}{(-4^l)l!(1/2)_{n+1}}, \\ \widetilde{v}_{n-2l,m,k-l} &= \frac{2\cdot 4^{n-2k+1}(n+m-2l)!(n-k-l+1)!(1/2)_{n-2k+2}}{(k-l)!(2n-2k-2l+3)!(n+m-2k+1)!}. \end{aligned}$$

Therefore the product can be expressed as

$$v_{n,m,l}\widetilde{v}_{n-2l,m,k-l} = \widetilde{w}_{n,m,k}c_{n,k,l}$$

where

$$\begin{aligned} c_{n,k,l} &= \frac{2\cdot 4^{n-2k+1}(n+m+1)!(n-k-l+1)!(1/2)_{n-2k+2}}{(-4^l)l!(k-l)!(2n-2k-2l+3)!(n+m-2k+1)!} \\ &= \frac{(-k)_l(-n+k-3/2)_l}{l!(-n-1/2)_l} \end{aligned}$$

is the coefficient in the polynomial ${}_2F_1(-k,-n+k-3/2;-n-1/2;(\widetilde{\mu}/\mu)^2) = \sum_{l=0}^{k} c_{n,k,l}(\widetilde{\mu}/\mu)^{2l}$. □

Corollary 2.2.5. *For each $n \geq 0$, $0 \leq m \leq n$, the limits*

$$\lim_{\widetilde{\mu}\to 0} w_{n,m,k}[\widetilde{\mu}, \mu], \quad \lim_{\mu\to 0} w_{n,m,k}[\widetilde{\mu}, \mu]$$

exist and are given, respectively, by

$$v_{n,m,k}[0, \mu] = (n+m+1) v^0_{n,m,k} \mu^{2k}, \quad v_{n,m,k}[\widetilde{\mu}, 0] = \frac{v_{n,m,k}}{n+m-2k+1} \widetilde{\mu}^{2k}.$$

Proof. We may write (2.27) as

$$w_{n,m,k}[\widetilde{\mu}, \mu] = \sum_{l=1}^{k-1} v_{n,m,l} \widetilde{v}_{n-2l,m,k-l}\, \mu^{2(k-l)} \widetilde{\mu}^{2l} + v_{n,m,k} \widetilde{v}_{n-2k,m,0} \widetilde{\mu}^{2k} + v_{n,m,0} \widetilde{v}_{n,m,k} \mu^{2k}$$

and then simply take $\mu = 0$ or $\widetilde{\mu} = 0$ to obtain the desired limit. □

Referring to (2.26), we have

$$w_{n,m,k}[0, \mu] = \frac{v_{n,m,k}}{(n+m-2k+1)}.$$

In the course of this investigation, several recurrence formulas relating to the associated Legendre functions were produced. These formulas are presented in Appendix B, and are used to give an alternative proof of Proposition 2.2.1 (as Proposition B.0.6).

Chapter 3

Monogenic Polynomials Adapted to Spheroidal Domains

In this chapter we deal with the first main objective of this thesis, the monogenic functions on spheroidal domains. In the next chapter we will apply these results to the spheroidal contragenic functions.

Our purpose is to generalize the facts of sections 1.3 and 1.4 concerning polynomial bases for Ω_0 to polynomial bases for spheroids Ω_μ of arbitrary eccentricity. These results were published in [26].

3.1 Spheroidal monogenic polynomials

In analogy to Definition 1.3.7, the following definition was introduced in [59].

Definition 3.1.1. The *basic monogenic spheroidal polynomials* are

$$X^{\pm}_{n,m}[\mu] = \partial U^{\pm}_{n+1,m}[\mu]. \tag{3.1}$$

They are indeed monogenic since $U^{\pm}_{n+1,m}[\mu]$ is harmonic, in view of the factorization $\Delta = \overline{\partial}\partial$ of the Laplacian. As derivatives of polynomials, they are also polynomials in x_1, x_2, x_3.

3.1.1 Interrelationships among spheroidal monogenics

Now we will work out explicit expressions in terms of the orthogonal basis of harmonic functions.

The following expression will be important in the context of the construction of the basic spheroidal monogenics, since it allows to define the zero-order monogenic polynomials.

Lemma 3.1.2. *([59]) For each $n \geq 0$,*

$$\widehat{V}_{n,-1} = -\frac{1}{(n+1)(n+2)}\widehat{V}_{n,1}. \tag{3.2}$$

The formulation which follows is in analogy to Theorem 1.3.10, expressing the basic polynomials in terms of their quaternionic components.

Theorem 3.1.1. *For each $n \geq 0$ and $0 \leq m \leq n+1$, the basic spheroidal* *monogenic polynomial* $X^{\pm}_{n,m}[\mu]$ *is equal to*

$$\begin{aligned} X^{\pm}_{n,m}[\mu] = V^{\pm}_{n,m}[\mu] \\ &+ \frac{e_1}{2}\big((n+m+1)V^{\pm}_{n,m-1}[\mu] - \frac{1}{n+m+2}V^{\pm}_{n,m+1}[\mu]\big) \\ &\mp \frac{e_2}{2}\big((n+m+1)V^{\mp}_{n,m-1}[\mu] + \frac{1}{n+m+2}V^{\mp}_{n,m+1}[\mu]\big) \end{aligned} \tag{3.3}$$

where the harmonic polynomials $V^{\pm}_{n,m}[\mu]$ *were defined in* (2.16)*. The* $X^{\pm}_{n,m}[\mu]$ *are polynomials in* μ^2 *as well as* x_0, x_1, x_2*.*

Proof. The proof of expression (3.3) appears in [59]. Now, we proceed to prove that $X^{\pm}_{n,m}[\mu]$ are polynomials in μ^2. Fix a value of μ. Since $\{X^{\pm}_{n,m}[0]\}$ and $\{X^{\pm}_{n,m}[\mu]\}$ are known to be two bases for the same subspace of monogenic functions, for fixed n, m there must exist real coefficients $a^{\pm}_k$ such that $X^{+}_{n,m}[\mu] = \sum a^{+}_k X^{+}_{n,k}[0] + \sum a^{-}_k X^{-}_{n,k}[0]$. By (3.3), the scalar part of this equation expresses the spheroidal harmonics $V^{\pm}_{n,m}[\mu]$ as a linear combination of the spherical harmonics $V^{\pm}_{n,m}[0]$. By the uniqueness of the representation of expression (2.23) we have that $a^{\pm}_k = v_{n,m,k}\mu^{2k}$. A similar statement holds for $X^{-}_{n,m}[\mu]$. □

In the proof of Theorem 3.1.1 we have obtained the following expression, which permits passing from spherical to spheroidal monogenics.

Proposition 3.1.3.

$$X^{+}_{n,m}[\mu] = \sum_{0\le 2k\le n-m} v_{n,m,k}\mu^{2k} X^{\pm}_{n-2k,m}[0]. \tag{3.4}$$

Some examples in low degree of $X^{\pm}_{n,m}[\mu]$ calculated via Theorem 3.1.1 are exhibited in Tables 3.1 and 3.2.

The inverse relation, namely,

$$X^{\pm}_{n,m}[0] = \sum_{0\leq 2k\leq n-m} \widetilde{v}_{n,m,k}\mu^{2k} X^{\pm}_{n-2k,m}[\mu]. \tag{3.5}$$

follows in a similar way. From Theorem 3.1.1, the antimonogenic polynomials (conjugates of monogenics, i.e. annihilated by ∂) satisfy the same relation,

$$\overline{X}^{\pm}_{n,m}[\mu] = \sum_{0\leq 2k\leq n-m} v_{n,m,k}\mu^{2k} \overline{X}^{\pm}_{n-2k,m}[0]. \tag{3.6}$$

3.1.2 Monogenic constants

We have described the monogenic and antimonogenic spheroidal polynomial systems. Now we present the spheroidal monogenic constants, found among the elements of the canonical basis we have constructed. Later, by dimension considerations, we will see that these generate all monogenic constants.

Proposition 3.1.4. *For each $n \geq 0$, the functions $X^{\pm}_{n,n+1}[\mu]$ are monogenic constants. Further, they do not depend on μ^2.*

Proof. By Theorem 3.1.1,

$$X^{\pm}_{n,n+1}[\mu] = (n+1)(V^{\pm}_{n,n}e_1 \mp V^{\mp}_{n,n}e_2).$$

n	m	$X^{\pm}_{n,m}$
0	0	$X^{+}_{0,0} = 1$
	1	$X^{+}_{0,1} = e_1$ $X^{-}_{0,1} = e_2$
1	0	$X^{+}_{1,0} = 2x_0 + x_1e_1 + x_2e_2$
	1	$X^{+}_{1,1} = -3x_1 + 3x_0e_1$ $X^{-}_{1,1} = -3x_2 + 3x_0e_2$
	2	$X^{+}_{1,2} = -6x_1e_1 + 6x_2e_2$ $X^{-}_{1,2} = -6x_2e_1 - 6x_1e_2$
2	0	$X^{+}_{2,0} = \big(3x_0^2 - \frac{3x_1^2}{2} - \frac{3x_2^2}{2} - \frac{3\mu^2}{5}\big) + 3x_0x_1e_1 + 3x_0x_2e_2$
	1	$X^{+}_{2,1} = -12x_0x_1 + \big(6x_0^2 - \frac{9x_1^2}{2} - \frac{3x_2^2}{2} - \frac{6\mu^2}{5}\big)e_1 - 3x_1x_2e_2$ $X^{-}_{2,1} = -12x_0x_2 - 3x_1x_2e_1 + \big(6x_0^2 - \frac{3x_1^2}{2} - \frac{9x_2^2}{2} - \frac{6\mu^2}{5}\big)e_2$
	2	$X^{+}_{2,2} = 15x_1^2 - 15x_2^2 - 30x_0x_1e_1 + 30x_0x_2e_2$ $X^{-}_{2,2} = 30x_1x_2 - 30x_0x_2e_1 - 30x_0x_1e_2$
	3	$X^{+}_{2,3} = \big(45x_1^2 - 45x_2^2\big)e_1 - 90x_1x_2e_2$ $X^{-}_{2,3} = 90x_1x_2e_1 + \big(45x_1^2 - 45x_2^2\big)e_2$

Table 3.1: Spheroidal monogenic basis polynomials $X^{\pm}_{n,m}[\mu]$ of degree $n = 0, 1, 2$. Observe that the parameter μ appears when $|n - m| \geq 2$. For each n, the last two entries (the first entry for $n = 0$) are monogenic constants.

n	m	$X^{\pm}_{n,m}$
3	0	$X^{+}_{3,0} = \big(4x_0^3 - 6x_0x_1^2 - 6x_0x_2^2 - \frac{12x_0\mu^2}{7}\big) + \big(6x_0^2x_1 - \frac{3x_1^3}{2} - \frac{3x_1x_2^2}{2} - \frac{6x_1\mu^2}{7}\big)e_1 + \big(6x_0^2x_2 - \frac{3x_1^2x_2}{2} - \frac{3x_2^3}{2} - \frac{6x_2\mu^2}{7}\big)e_2$
	1	$X^{+}_{3,1} = -30x_0^2x_1 + \frac{15x_1^3}{2} + \frac{15x_1x_2^2}{2} + \frac{30x_1\mu^2}{7} + \big(10x_0^3 - \frac{45x_0x_1^2}{2} - \frac{15x_0x_2^2}{2} - \frac{30x_0\mu^2}{7}\big)e_1 - 15x_0x_1x_2e_2$ $X^{-}_{3,1} = -30x_0^2x_2 + \frac{15x_1^2x_2}{2} + \frac{15x_2^3}{2} + \frac{30x_2\mu^2}{7} - 15x_0x_1x_2e_1 + \big(5x_0^3 - \frac{15x_0x_1^2}{2} - \frac{45x_0x_2^2}{2} - \frac{30x_0\mu^2}{7}\big)e_2$
	2	$X^{+}_{3,2} = 90x_0x_1^2 - 90x_0x_2^2 + \big(-90x_0^2x_1 + 30x_1^3 + \frac{90x_1\mu^2}{7}\big)e_1 + \big(90x_0^2x_2 - 30x_2^3 - \frac{90x_2\mu^2}{7}\big)e_2$ $X^{-}_{3,2} = 180x_0x_1x_2 + \big(-90x_0^2x_2 + 45x_1^2x_2 + 15x_2^3 + \frac{90x_2\mu^2}{7}\big)e_1 + \big(-90x_0^2x_1 + 15x_1^3 + 45x_1x_2^2 + \frac{90x_1\mu^2}{7}\big)e_2$
	3	$X^{+}_{3,3} = -105x_1^3 + 315x_1x_2^2 + \big(315x_0x_1^2 - 315x_0x_2^2\big)e_1 - 630x_0x_1x_2e_2$ $X^{-}_{3,3} = -315x_1^2x_2 + 105x_2^3 + 630x_0x_1x_2e_1 + \big(315x_0x_1^2 - 315x_0x_2^2\big)e_2$
	4	$X^{+}_{3,4} = (-420x_1^3 + 1260x_1x_2^2)e_1 + (1260x_1^2x_2 - 420x_2^3)e_2$ $X^{-}_{3,4} = (-1260x_1^2x_2 + 420x_2^3)e_1 + (-420x_1^3 + 1260x_1x_2^2)e_2$

Table 3.2: Spheroidal monogenic polynomials of degree $n = 3$.

On the other hand, note that

$$V_{n,n}^{\pm} = \frac{\partial}{\partial x_0} U_{n,n+1}^{\pm} = \frac{(-1)^n (2n+1)!}{2^n n!} \mu^n \sin^n u \sinh^n v \, \Phi_n^{\pm}(\varphi).$$

We apply this as follows. Since

$$\Phi_n^{+}(\varphi) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \cos^{n-2k} \varphi \sin^{2k} \varphi,$$
$$\Phi_n^{-}(\varphi) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor - 1} (-1)^k \binom{n}{2k+1} \cos^{n-(2k+1)} \varphi \sin^{2k+1} \varphi,$$

(where $\lfloor \cdot \rfloor$ denotes the greatest integer or floor function), we have

$$\mu^n \sin^n u \sinh^n v \, \Phi_n^{+}(\varphi) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x_1^{n-2k} x_2^{2k},$$

$$\mu^n \sin^n u \sinh^n v \, \Phi_n^{-}(\varphi) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor - 1} (-1)^k \binom{n}{2k+1} x_1^{n-(2k+1)} x_2^{2k+1}.$$

Since $X_{n,n+1}^{\pm}$ does not depend on x_0 and, since by construction, it is monogenic, we conclude that it is a monogenic constant. Furthermore, observe that it does not depend on μ. □

3.1.3 Normalization of monogenic polynomials for the spheroid

Let $\langle \cdot, \cdot \rangle_{[\mu]}$ denote the inner product (1.7) defining $L_2(\Omega_\mu, \mathbb{R}^3)$, and $\| \cdot \|_{[\mu]}$ denote the corresponding norm. The proof of the following result can be consulted in [59]. Recall the integral $I_{n,m}[\mu]$ given in (2.19).

Theorem 3.1.2 ([59])**.** *For fixed μ, the set $\{X^{\pm}_{n,m}[\mu]\colon\ n \geq 0,\ 0 \leq m \leq n+1\}$ is orthogonal in the sense of the scalar product $\langle \cdot, \cdot \rangle_{[\mu]}$. Their norms are given by*

$$\|X^{\pm}_{n,m}\|^2_{[\mu]} = \frac{\pi\mu^{2n+3}}{3(2^{2n})(n+2)(n+m+2)(3/2)_n(5/2)_n} \Big((n+2)(n+m)(n+m+1)(n-m+3)!(n+m+2)!\, I_{n,m-1}[\mu] + 2\delta_{0,m}(n+m+2)(n+1)!(n+2)!\, I_{n,1}[\mu] + (n+2)(n-m+1)!(n+m+2)!\big(I_{n,m+1}[\mu] + 2(n-m+2)(n+m+1)(1+\delta_{0,m})I_{n,m}[\mu]\big)\Big).$$

By the symmetric form taken by $X^{\pm}_{m,n}[\mu]$ in (3.3), we know that when $m \neq 0$,

$$\|X^{+}_{n,m}[\mu]\|_{[\mu]} = \|X^{-}_{n,m}[\mu]\|_{[\mu]}. \tag{3.7}$$

Based on Theorem 2.1.6, we state an analogous result that shows an orthogonality of the system $\{X^{\pm}_{n,m}[\mu]\}$ on the surface of the spheroid with respect to a suitable weight function.

Theorem 3.1.3. *([59]) The set $\{X^{\pm}_{n,m}[\mu] : n \geq 0,\ 0 \leq m \leq n+1\}$ is orthogonal over the surface of the spheroid Ω_μ in the sense of the scalar product*

$$\{f, g\}_\mu = \int_{\partial\Omega_\mu} \mathrm{Sc}(\overline{f}g) \left|\mu^2 - (x + i(x_1^2 + x_2^2)^{1/2})^2\right|^{1/2} d\sigma. \tag{3.8}$$

3.1.4 Dimensions of polynomial subspaces

It is well known (cf. [52]) that the dimension of the space $\mathcal{M}^{(n)}$ of homogeneous monogenic polynomials of degree n in (x_0, x_1, x_2) is $2n+3$ (this does not depend on the domain Ω).

Since the polynomials we are working with are not homogeneous (when $\mu \neq 0$), we consider the space

$$\mathcal{M}_*^{(n)} = \bigcup_{0 \le k \le n} \mathcal{M}^{(k)}$$

of monogenic polynomials of degree $\le n$. Thus

$$\dim \mathcal{M}_*^{(n)} = \sum_{k=0}^{n} (2k+3) = (n+3)(n+1). \tag{3.9}$$

Consider the collections of $2n+3$ polynomials

$$B_k[\mu] = \{X_{k,m}^{+}[\mu],\ 0 \le m \le k+1\} \cup \{X_{k,m}^{-}[\mu], 1 \le m \le k+1\}.$$

By Theorem 3.1.2 and equation (3.9), the union

$$\bigcup_{0 \le k \le n} B_k[\mu]$$

is an orthogonal basis for $\mathcal{M}_*^{(n)}$.

3.2 Spheroidal ambigenic polynomials

The material in this section is of a somewhat technical nature, which will be needed for the discussion of contragenic functions. The purpose is to generalize section 1.4 for functions in spheroids Ω_μ.

It is known [3], that the real dimension of the space $\mathcal{M}^{(n)} + \overline{\mathcal{M}}^{(n)}$ of homogeneous ambigenic polynomials is $4n+4$ when $n \geq 1$. As discussed in the previous section, the basis polynomials for spheroidal functions are not homogeneous. The dimension of the space $\mathcal{M}_*^{(n)} + \overline{\mathcal{M}}_*^{(n)}$ (not a direct sum) of ambigenic polynomials of degree at most n is

$$\begin{aligned}\dim(\mathcal{M}_*^{(n)} + \overline{\mathcal{M}}_*^{(n)}) &= \sum_{k=0}^{n} \dim(\mathcal{M}^{(k)} + \overline{\mathcal{M}}^{(k)}) \\ &= 3 + \sum_{k=1}^{n} (4k+4) = 2n(n+3) + 3. \end{aligned} \tag{3.10}$$

Observe that for $0 \leq k \leq n$, we have by (3.3) that

$$X^{\pm}_{k,k+1}[\mu] = (k+1)(V^{\pm}_{k,k}[\mu]e_1 \mp V^{\mp}_{k,k}[\mu]e_2),$$

and

$$V^{\mp}_{k,k}[\mu] = \frac{(-1)^k(2k+1)((2k-1)!!)^2}{\alpha_{k,k}}\,(x_1^2+x_2^2)^{k/2}$$

where the denominator is defined in (2.10). Note that the $X^{\pm}_{k,k+1}[\mu]$ are monogenic constants. This observation makes it possible to give a basis for the ambigenic polynomials defined in spheroidal domains. In the following we take into account an extension of (3.7):

Lemma 3.2.1. *For $m \neq 0$,*

$$\langle X^+_{k,m}[\mu],\, \overline{X}^+_{k,m}[\mu]\rangle_{[\mu]} = \langle X^-_{k,m}[\mu],\, \overline{X}^-_{k,m}[\mu]\rangle_{[\mu]}.$$

Proof. Indeed,

$$\begin{aligned}
&\langle X^+_{k,m}[\mu],\, \overline{X}^+_{k,m}[\mu]\rangle_{[\mu]} = \int_{\Omega_\mu} \left[X^+_{k,m}[\mu]\right]_0^2 - \left[X^+_{k,m}[\mu]\right]_1^2 - \left[X^+_{k,m}[\mu]\right]_2^2 \, dx \\
&= \int_0^\pi \int_0^{1/\mu} (\widehat{V}_{k,m})^2 dv du \int_0^{2\pi} \cos^2(m\varphi)\, d\varphi \\
&- \frac{1}{4}\int_0^\pi \int_0^{1/\mu} \big((k+m+1)\widehat{V}_{k,m-1}[\mu] - \frac{1}{k+m+2}\widehat{V}_{k,m+1}[\mu]\big)^2 \, dv\, du \\
&\times \int_0^{2\pi} \cos^2(m\varphi)\, d\varphi \\
&- \frac{1}{4}\int_0^\pi \int_0^{1/\mu} \big((k+m+1)\widehat{V}_{k,m-1}[\mu] - \frac{1}{k+m+2}\widehat{V}_{k,m+1}[\mu]\big)^2 \, dv\, du \\
&\times \int_0^{2\pi} \sin^2(m\varphi) d\varphi.
\end{aligned}$$

Since $m \neq 0$, the two values $\int_0^{2\pi} \Phi^\pm_m(\varphi)^2\, d\varphi$ are equal, and therefore

$$\begin{aligned}
\langle X^+_{k,m}[\mu], \overline{X}^+_{k,m}[\mu]\rangle = &\int_{\Omega_\mu} \left[X^-_{k,m}[\mu]\right]_0^2 - \left[X^-_{k,m}[\mu]\right]_1^2 - \left[X^-_{k,m}[\mu]\right]_2^2 \\
= &\langle X^-_{k,m}[\mu], \overline{X}^-_{k,m}[\mu]\rangle.
\end{aligned}$$

□

It is not possible to extract from the list $\{X^\pm_{n,m},\ \overline{X}^\pm_{n,m}\}$ an orthogonal basis of ambigenic functions, but only a small modification is necessary. This is closely analogous to Proposition 1.4.4, but due to the fact that the polynomials are not homogeneous, we continue to work with all degrees up to n. Define the functions

$$\begin{aligned}
Y^{++}_{n,m}[\mu] =& X^{+}_{n,m}[\mu],\\
Y^{-+}_{n,m}[\mu] =& X^{-}_{n,m}[\mu],\\
Y^{+-}_{n,m}[\mu] =& \overline{X}^{+}_{n,m}[\mu] - \gamma_{n,m}[\mu] X^{+}_{n,m}[\mu],\\
Y^{--}_{n,m}[\mu] =& \overline{X}^{-}_{n,m}[\mu] - \gamma_{n,m}[\mu] X^{-}_{n,m}[\mu],
\end{aligned} \tag{3.11}$$

where

$$\gamma_{n,m}[\mu] = \begin{cases} \frac{\langle X^{+}_{n,m}[\mu], \overline{X}^{+}_{n,m}[\mu]\rangle_{[\mu]}}{\|X^{+}_{n,m}[\mu]\|^2_{[\mu]}}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{if } m = n+1. \end{cases}$$

Proposition 3.2.2. *The collection of* $2n(n+3)+3$ *polynomials*

$$\begin{aligned}
&\{Y^{++}_{k,m} : \ 0 \leq m \leq k+1\} \cup \{Y^{-+}_{k,m} : \ 0 \leq m \leq k\}\\
&\cup \{Y^{+-}_{k,m} : \ 0 \leq m \leq k\} \cup \{Y^{--}_{k,m} : \ 0 \leq m \leq k+1\},
\end{aligned}$$

$0 \leq k \leq n$, *is an orthogonal basis in* $L_2(\Omega_\mu)$ *for the subspace of ambigenic polynomials of degree at most* n.

Proof. Throughout this proof, in view of the fact that μ is fixed, we simply write $X^{\pm}_{k,m}$, $Y^{\pm\pm}_{k,m}$, $\gamma_{k,m}$ for $X^{\pm}_{k,m}[\mu]$, $Y^{\pm,\pm}_{k,m}[\mu]$, $\gamma_{k,m}[\mu]$. Since there are $2n(n+3)+3$ ambigenic functions in the given list, it suffices to prove the orthogonality to conclude that they generate the ambigenic polynomials. Because the set

$$\{X^{+}_{k,0}, X^{+}_{k,m}, X^{-}_{k,m} |\ k = 0, \ldots, n, \ m = 1, \ldots, k+1\}$$

is an orthogonal basis of $\mathcal{M}_*^{(n)}$ in Ω_μ, it follows at once that

$$\langle Y_{k,m}^{++}, \overline{Y_{k,m}^{-+}} \rangle_{[\mu]} = \langle Y_{k,m}^{++}, \overline{Y_{k,m}^{--}} \rangle_{[\mu]} = \langle Y_{k,m}^{+-}, \overline{Y_{k,m}^{-+}} \rangle_{[\mu]} = \langle Y_{k,m}^{+-}, \overline{Y_{k,m}^{--}} \rangle_{[\mu]} = 0.$$

Since

$$\begin{aligned}
&\langle Y_{k_1,m_1}^{+-}, Y_{k_2,m_2}^{+-} \rangle_{[\mu]} = \langle \overline{X}_{k_1,m_1}^{+} - \gamma_{k_1,m_1} X_{k_1,m_1}^{+}, \ \overline{X}_{k_2,m_2}^{+} - \gamma_{k_2,m_2} X_{k_2,m_2}^{+} \rangle_{[\mu]} \\
&= \langle \overline{X}_{k_1,m_1}^{+}, \ \overline{X}_{k_2,m_2}^{+} \rangle_{[\mu]} - \gamma_{k_2,m_2} \langle \overline{X}_{k_1,m_1}^{+}, \ X_{k_2,m_2}^{+} \rangle_{[\mu]} \\
&- \gamma_{k_1,m_1} \langle X_{k_1,m_1}^{+}, \ \overline{X}_{k_2,m_2}^{+} \rangle_{[\mu]} + \gamma_{k_1,m_1} \gamma_{k_2,m_2} \langle X_{k_1,m_1}^{+}, X_{k_2,m_2}^{+} \rangle_{[\mu]},
\end{aligned}$$

it will be enough to study $\langle \overline{X}_{k_1,m_1}^{+}, X_{k_2,m_2}^{+} \rangle_{[\mu]}$ and $\langle X_{k_1,m_1}^{+}, \overline{X}_{k_2,m_2}^{+} \rangle_{[\mu]}$:

$$\begin{aligned}
\langle \overline{X}_{k_1,m_1}^{+}, X_{k_2,m_2}^{+} \rangle_{[\mu]} = \int_{\Omega_\mu} & \Big(\big[X_{k_1,m_1}^{+} \big]_0 \big[X_{k_2,m_2}^{+} \big]_0 \\
& - \Big(\big[X_{k_1,m_1}^{+} \big]_1 \big[X_{k_2,m_2}^{+} \big]_1 + \big[X_{k_1,m_1}^{+} \big]_2 \big[X_{k_2,m_2}^{+} \big]_2 \Big) \Big) \, dx,
\end{aligned}$$

but from the proof of Proposition 3.1.2, we obtain that

$$\langle \overline{X}_{k_1,m_1}^{+}, X_{k_2,m_2}^{+} \rangle_{[\mu]} = (\| \operatorname{Sc} X_{k_1,m_1}^{+} \|_{[\mu]}^2 - \| \operatorname{Vec} X_{k_1,m_1}^{+} \|_{[\mu]}^2) \delta_{k_1,k_2} \delta_{m_1,m_2}.$$

Now we note that

$$\langle Y_{k_1,m_1}^{++}, Y_{k_2,m_2}^{+-} \rangle_{[\mu]} = \langle X_{k_1,m_1}^{+}, \overline{X}_{k_2,m_2}^{+} - \gamma_{k_2,m_2} X_{k_2,m_2}^{+} \rangle_{[\mu]}.$$

By the above observations, these functions are orthogonal when $k_1 \neq k_2$ or

$m_1 \neq m_2$, and when the indices coincide,

$$\langle Y^{++}_{k,m}, Y^{+-}_{k,m}\rangle_{[\mu]} = \langle X^+_{k,m}, \overline{X}^+_{k,m}\rangle_{[\mu]} - \frac{\langle X^+_{k,m},\ \overline{X}^+_{k,m}\rangle_{[\mu]}}{\|X^+_{k,m}\|^2_{[\mu]}}\langle X^+_{k,m}, X^+_{k,m}\rangle_{[\mu]} = 0.$$

Moreover, by the orthogonality of the system $\{\Phi^+_k, \Phi^-_l \mid\ k \geq 0,\ l > 0\}$, it is clear that $\langle Y^{++}_{k_1,m_1}, Y^{--}_{k_2,m_2}\rangle_{[\mu]} = 0$, and further $\langle Y^{++}_{k,m}, Y^{--}_{k,m}\rangle_{[\mu]} = 0$. Finally,

$$\langle Y^{-+}_{k,m}, Y^{--}_{k,m}\rangle_{[\mu]} = \langle X^-_{k,m},\ \overline{X}^-_{k,m}\rangle_{[\mu]} - \frac{\langle X^+_{k,m},\ \overline{X}^+_{k,m}\rangle_{[\mu]}}{\|X^+_{k,m}\|^2_{[\mu]}}\|X^-_{k,m}\|^2_{[\mu]}.$$

Note that

$$\langle X^-_{k,m},\ \overline{X}^-_{k,m}\rangle_{[\mu]} = \langle X^+_{k,m},\ \overline{X}^+_{k,m}\rangle_{[\mu]}$$

and $\|X^-_{k,m}\|^2_{[\mu]} = \|X^+_{k,m}\|^2_{[\mu]}$, when $m \neq 0$. Therefore $\langle Y^{-+}_{k,m}, Y^{--}_{k,m}\rangle_{[\mu]} = 0$. □

Chapter 4

Contragenic Polynomials on Spheroidal Domains

In this chapter we calculate a system of mutually orthogonal contragenic polynomials for each spheroid Ω_μ. These basis elements, which are inhomogeneous polynomials of three spatial variables, depend polynomially on the parameter μ. Then we investigate the relations between the systems for spheroids of different eccentricity. This produces the notion of "universal spheroidal contragenic function." The results that appear in the first section were published in [26]. The results of the second section appear in [27].

4.1 Spheroidal contragenic polynomials

The discovery in Álvarez-Peña and Porter [3] that not all L_2-harmonic functions are ambigenic, which was described in Section 1.5, was obtained by a dimension count concerning harmonic and monogenic polynomials of given degree n.

4.1.1 Dimensions of subspaces

This dimension count for Ω_0 is simply the difference of $\dim \mathrm{Har}_n(\Omega_0) \cap L_2(\Omega_0) = 3(2n+1)$ and $\dim(\mathcal{M}_2^{(n)}(\Omega_0) + \overline{\mathcal{M}}_2^{(n)}(\Omega_0)) = 4n+4$ ((1.12) and (1.46)), and is confirmed by the specific basis in Proposition 1.5.3 as follows.

Proposition 4.1.1 ([3])**.** *For $n \geq 1$,*

$$\dim \mathcal{N}^{(n)}(\Omega_0) = 2n - 1.$$

Since constants are ambigenic, $\dim \mathcal{N}^{(0)}(\Omega) = 0$ trivially for any Ω.

Since the polynomials in $\mathcal{M}^{(n)}(\Omega_\mu)$ for $\mu \neq 0$ are in general not homogeneous, it is convenient to work with the following spaces of polynomials of degree no greater than n (recall the notation of Definitions 1.2.1 and 1.3.4):

Definition 4.1.2. We write

$$\mathrm{Har}_*^{(n)}(\mathbb{R}^n) = \bigcup_{k=0}^{n} \mathrm{Har}_k(\mathbb{R}^n),$$
$$\mathcal{M}_*^{(n)}(\Omega) = \bigcup_{k=0}^{n} \mathcal{M}^{(k)}(\Omega),$$
$$\overline{\mathcal{M}}_*^{(n)}(\Omega) = \bigcup_{k=0}^{n} \overline{\mathcal{M}}^{(k)}(\Omega).$$

Similarly, recalling Definition 1.5.2, we let $\mathcal{N}^{(n)}(\Omega) \subset \mathcal{N}(\Omega)$ denote the subspace of contragenic polynomials of degree n, and we write

$$\mathcal{N}_*^{(n)}(\Omega) = \bigcup_{k=0}^{n} \mathcal{N}^{(k)}(\Omega) \tag{4.1}$$

for the subspace of polynomials of degree $\leq n$.

Recall that unlike the spaces of harmonic, monogenic, antimonogenic and ambigenic polynomials, the definitions of $\mathcal{N}^{(n)}(\Omega)$ and $\mathcal{N}_*^{(n)}(\Omega)$ involve the L_2 inner product and thus depend on the domain Ω, which therefore cannot be omitted from the notation without ambiguity. Since we are interested in spheroids, we will write for simplicity

$$\mathcal{N}_*^{(n)}[\mu] = \mathcal{N}_*^{(n)}(\Omega_\mu). \tag{4.2}$$

Thus we have the successive orthogonal complements

$$\mathcal{N}^{(n)}[\mu] = \mathcal{N}_*^{(n)}[\mu] \ominus \mathcal{N}_*^{(n-1)}[\mu] \tag{4.3}$$

and there is a Hilbert space orthogonal decomposition $\mathcal{N}_*[\mu] = \bigoplus_{k=1}^{\infty} \mathcal{N}^{(k)}[\mu]$ of the full collection of contragenic functions in $L_2(\Omega_\mu)$.

Since the dimension of an orthogonal complement within a fixed finite-dimensional vector space does not depend on the inner product used, and since the harmonic and the ambigenic polynomials of degree $\leq n$ do not depend on the domain, we have by Proposition 4.1.1 that $\dim \mathcal{N}^{(k)}[\mu] = 2n-1$ also for $n \geq 1$. Thus

Proposition 4.1.3.

$$\dim \mathcal{N}_*^{(n)}[\mu] = \sum_{k=0}^{n} \dim \mathcal{N}^{(k)}[\mu] = n^2. \tag{4.4}$$

In Table 4.1 we summarize the dimensions of the various spaces of polynomials of degree less than or equal to n.

Space of polynomials	$\dim_{\mathbb{R}}$
$\mathrm{Har}_*^{(n)}(\mathbb{R})$	$(n+1)^2$
$\mathrm{Har}_*^{(n)}(\mathbb{R}^3)$	$3(n+1)^2$
$\mathcal{M}_*^{(n)}$, $\overline{\mathcal{M}}_*^{(n)}$	$(n+3)(n+1)$
$\mathcal{M}_*^{(n)} \cap \overline{\mathcal{M}}_*^{(n)}$	$2n+3$
$\mathcal{M}_*^{(n)} + \overline{\mathcal{M}}_*^{(n)}$	$2(n^2+3n+1)+1$

Table 4.1: Dimensions of spaces of polynomials ($n \geq 0$)

Table 4.1 may be considered as referring to polynomials defined in all of $\mathbb{R}^3$, or equally well as their restrictions to any domain. For compact domains such as the ellipsoids Ω_μ the functions are automatically square-integrable.

4.1.2 Basic contragenic polynomials

It is of interest to have a basis for the contragenic functions, in order to express an arbitrary harmonic function in a calculable way as a sum of an ambigenic function and a contragenic function. We now give an explicit construction of such a basis of the $\mathcal{N}_*^{(n)}$, using as building blocks the components of the monogenic functions. Write

$$a_{n,m}[\mu] = \left(\frac{\|V^+_{n,m+1}[\mu]\|_{[\mu]}}{(n+m+1)_2 \|V^+_{n,m-1}[\mu]\|_{[\mu]}} \right)^2 . \tag{4.5}$$

where $\|\cdot\|_{[\mu]}$ indicates the norm in $L_2(\Omega_\mu)$. Since $\widehat{V}_{n,-1} = -\frac{1}{(n+1)(n+2)} \widehat{V}_{n,1}$ (Lemma 3.1.2), we obtain that $a_{n,0} = 1$.

Definition 4.1.1. Let $n \geq 1$. The *basic contragenic polynomials* for Ω_μ are

$$\begin{aligned} Z^\pm_{n,m}[\mu] = a_{n,m}[\mu] \Big(&\operatorname{Vec} X^\mp_{n,m}[\mu] \mp \operatorname{Vec} X^\pm_{n,m}[\mu] e_3 \Big) \\ &+ \Big(-\operatorname{Vec} X^\mp_{n,m}[\mu] \mp \operatorname{Vec} X^\pm_{n,m}[\mu] e_3 \Big) \end{aligned} \tag{4.6}$$

for $0 \leq m \leq n-1$.

It remains to show that these are indeed contragenic. In what follows, we will write $X^\pm_{n,m}$, $Z^\pm_{n,m}$, $a_{n,m}$ in place of $X^\pm_{n,m}[\mu]$, $Z^\pm_{n,m}[\mu]$, $a_{n,m}[\mu]$ considering that μ is fixed. We will write $[X]_i$ for the i-th component of the quaternion-valued function $X = \sum [X]_i e_i$.

Observe that

$$\begin{aligned} &2a_{k,m} \Big(\operatorname{Vec} X^\mp_{k,m} \mp \operatorname{Vec} X^\pm_{n,m}[\mu] e_3 \Big) \\ &\quad = 2a_{k,m} \Big(\big([X^\mp_{k,m}]_1 \mp [X^\pm_{k,m}]_2 \big) e_1 + \big([X^\mp_{k,m}]_2 \pm [X^\pm_{k,m}]_1 \big) e_2 \Big). \end{aligned}$$

Also,

$$
\begin{aligned}
&- \operatorname{Vec} X^{\mp}_{k,m} \mp \operatorname{Vec} X^{\pm}_{n,m}[\mu] e_3 \\
&= \big(- \left[X^{\mp}_{k,m}\right]_1 \mp \left[X^{\pm}_{k,m}\right]_2 \big) e_1 + \big(- \left[X^{\mp}_{k,m}\right]_2 \pm \left[X^{\pm}_{k,m}\right]_1 \big) e_2.
\end{aligned}
$$

In consequence, the basic contragenic polynomials may be expressed in terms of their e_1, e_2 components. Defining

$$b^{\pm}_{n,m} = a_{n,m} \pm 1; \tag{4.7}$$

we can write

$$
\begin{aligned}
Z^{\pm}_{n,m} = {}& (b^{-}_{n,m}[X^{\mp}_{n,m}]_1 \mp b^{+}_{n,m}[X^{\pm}_{n,m}]_2) e_1 \\
& + (b^{-}_{n,m} \left[X^{\mp}_{n,m}\right]_2 \pm b^{+}_{n,m} \left[X^{\pm}_{n,m}\right]_1) e_2.
\end{aligned} \tag{4.8}
$$

Therefore we have that

$$
\begin{aligned}
Z^{\pm}_{n,m} = {}& \big((n+m+1) a_{n,m} V^{\mp}_{n,m-1} + \frac{1}{n+m+2} V^{\mp}_{n,m+1}\big) e_1 \\
& \pm \big((n+m+1) a_{n,m} V^{\pm}_{n,m-1} - \frac{1}{n+m+2} V^{\pm}_{n,m+1}\big) e_2.
\end{aligned} \tag{4.9}
$$

Examples of $Z^{\pm}_{n,m}$ provided by (4.9) are given in Table 4.2.

The following fact generalizes Theorem 1.5.3 from the sphere to spheroids, and is one of the most important results of this work. It was published in [26].

n	m	$Z^{\pm}_{n,m}$
1	0	$Z_{1,0} = -x_2e_1 + x_1e_2$
2	0	$Z_{2,0} = -3x_0x_2e_1 + 3x_0x_1e_2$
	1	$Z^+_{2,1} = 6x_1x_2e_1 + \frac{3}{30-20\mu^2+6\mu^4}\big(25x_2^2 - 2\mu^2 - 10x_2^2\mu^2 + 4\mu^4 + x_2^2\mu^4 - 2\mu^6 + 10x_0^2(-1+\mu^2)^2 + x_1^2(-35+30\mu^2-11\mu^4)\big)e_2$ $Z^-_{2,1} = \frac{3}{30-20\mu^2+6\mu^4}\big(-35x_2^2 - 2\mu^2 + 30x_2^2\mu^2 + 4\mu^4 - 11x_2^2\mu^4 - 2\mu^6 + x_1^2(-5+\mu^2)^2 + 10x_0^2(-1+\mu^2)^2\big)e_1 + 6x_1x_2e_2$
3	0	$Z_{3,0} = \frac{3}{14}x_2(-28x_0^2 + 7x_1^2 + 7x_2^2 + 4\mu^2)e_1 - \frac{3}{14}x_1(-28x_0^2 + 7x_1^2 + 7x_2^2 + 4\mu^2)e_2$
	1	$Z^+_{3,1} = 30x_0x_1x_2e_1 + \frac{15x_0}{70-84\mu^2+30\mu^4}\big(49x_2^2 - 6\mu^2 - 42x_2^2\mu^2 + 12\mu^4 + 9x_2^2\mu^4 - 6\mu^6 + 14x_0^2(-1+\mu^2)^2 + x_1^2(-91+126\mu^2-51\mu^4)\big)e_2$ $Z^-_{3,1} = \frac{15x_0}{70-84\mu^2+30\mu^4}\big(-91x_2^2 - 6\mu^2 + 126x_2^2\mu^2 + 12\mu^4 - 51x_2^2\mu^4 - 6\mu^6 + x_1^2(7-3\mu^2)^2 + 14x_0^2(-1+\mu^2)^2\big)e_1 + 30x_0x_1x_2e_2$
	2	$Z^+_{3,2} = -\frac{30x_2}{35-14\mu^2+3\mu^4}\big(-21x_2^2 - 2\mu^2 + 14x_2^2\mu^2 + 4\mu^4 - 5x_2^2\mu^4 - 2\mu^6 + x_1^2(-7+\mu^2)^2 + 14x_0^2(-1+\mu^2)^2\big)e_1 - \frac{30x_1}{35-14\mu^2+3\mu^4}\big(49x_2^2 - 2\mu^2 - 14x_2^2\mu^2 + 4\mu^4 + x_2^2\mu^4 - 2\mu^6 + 14x_0^2(-1+\mu^2)^2 + x_1^2(-21+14\mu^2-5\mu^4)\big)e_2$ $Z^-_{3,2} = \frac{60x_1}{35-14\mu^2+3\mu^4}\big(28x_2^2 + \mu^2 - 14x_2^2\mu^2 - 2\mu^4 + 4x_2^2\mu^4 + \mu^6 - 7x_0^2(-1+\mu^2)^2 + x_1^2(-7+\mu^4)\big)e_1 - \frac{60x_2}{35-14\mu^2+3\mu^4}\big(-7x_2^2 + \mu^2 - 2\mu^4 + x_2^2\mu^4 + \mu^6 - 7x_0^2(-1+\mu^2)^2 + 2x_1^2(14-7\mu^2+2\mu^4)\big)e_2$

Table 4.2: Spheroidal contragenic polynomials of low degree, parametrized by the eccentricity μ.

Theorem 4.1.2. *The n^2 contragenic polynomials $Z_{k,0}[\mu]$, $Z^{\pm}_{k,m}[\mu]$, (with $1 \leq k \leq n$, $1 \leq m \leq k-1$) form an orthogonal basis for $\mathcal{N}^{(n)}_{*}[\mu]$.*

Proof. First we prove that $Z_{k,0}[\mu]$ and $Z^{\pm}_{k,m}[\mu]$ are contragenic. As they have no scalar parts, it suffices to show that they are orthogonal to $\operatorname{Vec}\mathcal{M}^{(n)}_{*}$. To do this, we use the basis obtained by dropping the scalar parts of the basis for $\mathcal{M}^{(n)}_{*}$ given in Theorem 3.1.2. Throughout this proof, we shall denote by $\Phi^{\pm}_{m}$ for the functions $\Phi^{\pm}_{m}(\varphi)$ given in (1.37). Since

$$\{\Phi^{+}_{m_1},\ \Phi^{-}_{m_2} \mid m_1 \geq 0, m_2 \geq 1\}$$

is a system of orthogonal functions in $[0,\pi]$, when $1 \leq m_1 \leq k_1$ and $1 \leq m_2 \leq k_2$, we have

$$\begin{aligned}
2\langle Z^{+}_{k_1,m_1}, \operatorname{Vec} X^{+}_{k_2,m_2}\rangle_{[\mu]} &= \big((k_1+m_1+1)a_{k_1,m_1}V^{-}_{k_1,m_1-1} \\
&\quad + \frac{1}{k_1+m_1+2}V^{-}_{k_1,m_1+1}\big)\big((k_2+m_2+1)V^{+}_{k_2,m_2-1} - \frac{1}{k_2+m_2+2}V^{+}_{k_2,m_2+1}\big) \\
&\quad - \big((k_1+m_1+1)a_{k_1,m_1}V^{+}_{k_1,m_1-1} - \frac{1}{k_1+m_1+2}V^{+}_{k_1,m_1+1}\big) \\
&\qquad \times \big((k_2+m_2+1)V^{-}_{k_2,m_2-1} + \frac{1}{k_2+m_2+2}V^{-}_{k_2,m_2+1}\big).
\end{aligned}$$

Expanding the integrands and applying the trigonometric identities

$$\begin{aligned}
\Phi^{+}_{m_2-1}\Phi^{-}_{m_1-1} - \Phi^{+}_{m_1-1}\Phi^{-}_{m_2-1} &= \Phi^{-}_{m_1-m_2}, \\
\Phi^{+}_{m_2+1}\Phi^{-}_{m_1-1} + \Phi^{-}_{m_2+1}\Phi^{+}_{m_1-1} &= \Phi^{-}_{m_1+m_2}, \\
-\Phi^{-}_{m_1+1}\Phi^{+}_{m_2+1} + \Phi^{+}_{m_1+1}\Phi^{-}_{m_2+1} &= \Phi^{-}_{m_2-m_1}, \\
\Phi^{-}_{m_1+1}\Phi^{+}_{m_2-1} + \Phi^{+}_{m_1+1}\Phi^{-}_{m_2-1} &= \Phi^{-}_{m_1+m_2},
\end{aligned}$$

we obtain that

$$\begin{aligned}
&2\langle Z^{+}_{k_1,m_1}, \operatorname{Vec} X^{+}_{k_2,m_2}\rangle_{[\mu]} = \\
&(k_1+m_1+1)(k_2+m_2+1)a_{k_1,m_1}\widehat{V}_{k_1,m_1-1}\widehat{V}_{k_2,m_1-1}\Phi^{-}_{m_1-m_2} \\
&\quad - \frac{(k_1+m_1+1)a_{k_1,m_1}}{k_2+m_2+2}\widehat{V}_{k_1,m_1-1}\widehat{V}_{k_2,m_2+1}\Phi^{-}_{m_1+m_2} \\
&\quad + \frac{k_2+m_2+1}{k_1+m_1+2}\widehat{V}_{k_1,m_1+1}\widehat{V}_{k_2,m_2-1}\Phi^{-}_{m_1+m_2} \\
&\quad + \frac{1}{(k_1+m_1+1)(k_2+m_2+2)}\widehat{V}_{k_1,m_1+1}\widehat{V}_{k_2,m_2+1}\Phi^{-}_{m_2-m_1}.
\end{aligned}$$

Now, for all $m_1, m_2 \geq 0$ we have that

$$\int_0^{2\pi} \Phi^{-}_{m_1-m_2}\, d\varphi = 0 \quad \text{and} \quad \int_0^{2\pi} \Phi^{-}_{m_1+m_2}\, d\varphi = 0.$$

Therefore $\langle Z^{+}_{k_1,m_1}, \operatorname{Vec} X^{+}_{k_2,m_2}\rangle_{[\mu]} = 0$. In the same way we obtain that

$$\langle Z^{-}_{k_1,m_1}, \operatorname{Vec} X^{-}_{k_2,m_2}\rangle_{[\mu]} = 0.$$

Also, when $m_1 > 0$ and $m_2 \geq 0$, we have that

$$\begin{aligned}
&\langle Z^{\pm}_{k_1,m_1}, \operatorname{Vec} X^{\mp}_{k_2,m_2}\rangle_{[\mu]} = \\
&\quad b^{-}_{k_1,m_1}\int_{\Omega_\mu}\left[X^{\mp}_{k_1,m_1}\right]_1\left[X^{\mp}_{k_2,m_2}\right]_1 dx \mp b^{+}_{k_1,m_1}\int_{\Omega_\mu}\left[X^{\pm}_{k_1,m_1}\right]_2\left[X^{\mp}_{k_2,m_2}\right]_1 dx \\
&+ b^{-}_{k_1,m_1}\int_{\Omega_\mu}\left[X^{\mp}_{k_1,m_1}\right]_2\left[X^{\mp}_{k_2,m_2}\right]_2 dx \pm b^{+}_{k_2,m_2}\int_{\Omega_\mu}\left[X^{\pm}_{k_1,m_1}\right]_1\left[X^{\mp}_{k_2,m_2}\right]_2 dx,
\end{aligned}$$

where the coefficients $b^{\pm}_{k,m}$ come from (4.7). Since the system

$$\left\{\operatorname{Vec} X^{+}_{k,m},\ \operatorname{Vec} X^{-}_{j,l} \mid 0 \leq k \leq n,\ 0 \leq m \leq k,\ 1 \leq j \leq n,\ 1 \leq l \leq j\right\}$$

is orthogonal, it follows that

$$
\begin{aligned}
\langle Z^{\pm}_{k_1,m_1}, \operatorname{Vec} X^{\mp}_{k_2,m_2}\rangle_{[\mu]} = b^{-}_{k_1,m_1} & \int_{\Omega_\mu} [X^{\mp}_{k_1,m_1}]_1 [X^{\mp}_{k_2,m_2}]_1 \, dx \\
& \mp b^{+}_{k_1,m_1} \int_{\Omega_\mu} [X^{\pm}_{k_1,m_1}]_2 [X^{\mp}_{k_2,m_2}]_1 \, dx \\
& + b^{-}_{k_1,m_1} \int_{\Omega_\mu} [X^{\mp}_{k_1,m_1}]_2 [X^{\mp}_{k_2,m_2}]_2 \, dx \\
& \pm b^{+}_{k_2,m_2} \int_{\Omega_\mu} [X^{\pm}_{k_1,m_1}]_1 [X^{\mp}_{k_2,m_2}]_2 \, dx \\
= & \bigg(\frac{\pi}{2} b^{+}_{k_1,m_1} \Big((k_1+m_1+1)^2 \int_0^{1/\mu} \int_0^{\pi} (\widehat{V}_{k_1,m_1-1})^2 \, du \, dv \\
& \quad - \frac{1}{(k_1+m_1+2)^2} \int_0^{1/\mu} \int_0^{\pi} (\widehat{V}_{k_1,m_1+1})^2 \, du \, dv \Big) \\
& \quad + b^{-}_{k_1,m_1} \| \operatorname{Vec} X^{\mp}_{k_1,m_1} \|^2_{\mu} \bigg) \delta_{k_1,k_2} \delta_{m_1,m_2}.
\end{aligned}
$$

Then, using the expression for $\| \operatorname{Vec} X^{\pm}_{k,m} \|^2_{\mu}$ given in Theorem 3.1.2 this reduces to

$$
\begin{aligned}
& \langle Z^{\pm}_{k_1,m_1}, \operatorname{Vec} X^{\mp}_{k_2,m_2}\rangle_{[\mu]} = \\
& \quad \frac{\pi}{2} \bigg(2a_{k_1,m_1} (k_1+m_1+1)^2 \int_0^{1/\mu} \int_0^{\pi} (\widehat{V}_{k_1,m_1-1})^2 \, du \, dv \\
& \quad - \frac{2}{(k_1+m_1+2)^2} \int_0^{1/\mu} \int_0^{\pi} (\widehat{V}_{k_1,m_1+1})^2 \, du \, dv \\
& \quad \mp \frac{2\delta_{0,m_1}}{(k_1+2)^2} \int_0^{1/\mu} \int_0^{\pi} (\widehat{V}_{k_1,1})^2 \, du \, dv \bigg) \delta_{m_1,m_2} \delta_{k_1,k_2}.
\end{aligned} \tag{4.10}
$$

Further, using the expression (4.9) and recalling that

$$
\begin{aligned}
2\,\mathrm{Vec}\, X^-_{k,m} =& \big((k+m+1)V^-_{k,m-1} \frac{1}{k+m+2} V^-_{k,m+1}\big) e_1 \\
&+ \big((k+m+1)V^+_{k,m-1} + \frac{1}{k+m+2} V^+_{k,m+1}\big) e_2
\end{aligned}
$$

when $m > 0$, by (4.5), we obtain that

$$
\begin{aligned}
2\langle Z^+_{k,m},\ \mathrm{Vec}\, X^-_{k,m}\rangle_{[\mu]} =& \\
&\int_{\Omega_\mu} \big((k+m+1)a_{k,m}V^-_{k,m-1} + \frac{1}{k+m+2}V^-_{k,m+1}\big) \\
&\quad \times \big((k+m+1)V^-_{k,m-1} - \frac{1}{k+m+2}V^-_{k,m+1}\big)\,dx \\
&+ \int_{\Omega_\mu} \big((k+m+1)a_{k,m}V^+_{k,m-1} - \frac{1}{k+m+2}V^+_{k,m+1}\big) \\
&\quad \times \big((k+m+1)V^+_{k,m-1} + \frac{1}{k+m+2}V^+_{k,m+1}\big)\,dx \\
&= 2a_{k,m}(k+m+1)^2\|V^+_{k,m-1}\|^2_{[\mu]} - \frac{1}{(k+m+2)^2}\|V^+_{k,m+1}\|^2_{[\mu]} \\
&= 0.
\end{aligned}
$$

Similarly, the orthogonality of $\{\Phi^+_m, \Phi^-_l\}$ gives $\langle Z^-_{k,m},\ \mathrm{Vec}\, X^+_{k,m}\rangle_{[\mu]} = 0$. Next, we expand

$$
\begin{aligned}
\langle Z_{k_1,0}, \mathrm{Vec}\, X^\pm_{k_2,m}\rangle_{[\mu]} =& \frac{1}{(k_1+2)}\Bigg((k_2+m+1)\int_{\Omega_\mu} V^-_{k_1,1}V^\pm_{k_2,m-1}\,dx \\
&- \frac{1}{k_2+m+2}\int_{\Omega_\mu} V^-_{k_1,1}V^\pm_{k_2,m+1}\,dx \pm \big((k_2+m+1)\int_{\Omega_\mu} V^+_{k_1,1}V^\mp_{k_2,m-1}\,dx \\
&+ \frac{1}{k_2+m+2}\int_{\Omega_\mu} V^+_{k_1,1}V^\mp_{k_2,m+1}\,dx\big)\Bigg) \\
=& 0,
\end{aligned}
$$

again by orthogonality of $\{\Phi_m^+, \Phi_l^-\}$. For $k_1 \neq k_2$, by the orthogonality of the system

$$\Big\{V^+_{k_1,m_1}, V^-_{k_2,m_2} |\ 0 \le k_1 \le n_1,\ 0 \le k_2 \le n_2,$$
$$0 \le m_1 \le k_1,\ 1 \le m_2 \le k_2, n_1, n_2 \ge 0\Big\}$$

we have

$$\begin{aligned}\langle Z_{k,0},\, \mathrm{Vec}\, X^-_{k,2}\rangle_{[\mu]} &= \frac{(k+m+1)}{(k+2)}\left(\int_{\Omega_\mu} \left(V^-_{k,1}\right)^2\, dx - \int_{\Omega_\mu} \left(V^+_{k,1}\right)^2\, dx\right)\\ &= 0;\end{aligned}$$

the last equality is a consequence of

$$\begin{aligned}\int_{\Omega_\mu} (V^-_{k,1})^2\, dx &= \int_0^\pi \int_0^{1/\mu} (\widehat{V}_{k,1})^2\, dv\, du \int_0^{2\pi} \sin^2\varphi d\varphi\\ &= \int_0^{1/\mu} (\widehat{V}_{k,1})^2\, dv\, du \int_0^{2\pi} \cos^2\varphi\, d\varphi\\ &= \int_{\Omega_\mu} (V^+_{k,1})^2\, dx.\end{aligned}$$

We have verified that the functions $Z^\pm_{k,m}$ are contragenic.

Now we will prove the orthogonality of the system

$$\{Z^\pm_{k,m} |\ k \ge 1,\ 0 \le m \le k\}.$$

Using the expression (4.9), when $1 \leq m_1, m_2$ we have

$$\begin{aligned}
&\langle Z^{\pm}_{k_1,m_1}, Z^{\pm}_{k_2,m_2}\rangle_{[\mu]} = \\
&\qquad (k_1+m_1+1)(k_2+m_2+1)a_{k_1,m_1}a_{k_2,m_2}\int_{\Omega_\mu} V^{\mp}_{k_1,m_1-1}V^{\mp}_{k_2,m_2-1}\,dx \\
&\qquad + \frac{(k_1+m_1+1)a_{k_1,m_1}}{k_2+m_2+2}\int_{\Omega_\mu} V^{\mp}_{k_1,m_1-1}V^{\mp}_{k_2,m_2+1}\,dx \\
&\qquad + \frac{(k_2+m_2+1)a_{k_2,m_2}}{k_1+m_1+2}\int_{\Omega_\mu} V^{\mp}_{k_1,m_1+1}V^{\mp}_{k_2,m_2-1}\,dx \\
&\qquad + \frac{1}{(k_1+m_1+2)(k_2+m_2+2)}\int_{\Omega_\mu} V^{\mp}_{k_1,m_1+1}V^{\mp}_{k_2,m_2+1}\,dx \\
&\qquad + (k_1+m_1+1)(k_2+m_2+1)a_{k_1,m_1}a_{k_2,m_2}\int_{\Omega_\mu} V^{\pm}_{k_1,m_1-1}V^{\pm}_{k_2,m_2-1}\,dx \\
&\qquad - \frac{(k_1+m_1+1)a_{k_1,m_1}}{k_2+m_2+2}\int_{\Omega_\mu} V^{\pm}_{k_1,m_1-1}V^{\pm}_{k_2,m_2+1}\,dx \\
&\qquad - \frac{(k_2+m_2+1)a_{k_2,m_2}}{k_1+m_1+2}\int_{\Omega_\mu} V^{\pm}_{k_1,m_1+1}V^{\pm}_{k_2,m_2-1}\,dx \\
&\qquad + \frac{1}{(k_1+m_1+2)(k_2+m_2+2)}\int_{\Omega_\mu} V^{\pm}_{k_1,m_1+1}V^{\pm}_{k_2,m_2+1}\,dx.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
&\langle Z^{\pm}_{k_1,m_1}, Z^{\pm}_{k_2,m_2}\rangle_{[\mu]} = 2\pi\Bigg(((k_1+m_1+1)a_{k_1,m_1})^2\int_0^\pi\int_0^\mu (\widehat{V}_{k_1,m_1-1})^2 dv du \\
&+ (\frac{1}{k_1+m_1+2})^2\int_0^\pi\int_0^{1/\mu}(\widehat{V}_{k_1,m_1+1})^2\,dv\,du\Bigg)\delta_{m_1,m_2}\delta_{k_1,k_2}.
\end{aligned}$$

On the other hand, when $1 \leq m \leq k_1$, we have that

$$\langle Z_{k_1,0}, Z^{\pm}_{k_2,m}\rangle_{[\mu]} = \frac{2}{k_1+2}\Bigg((k_2+m+1)a_{k_2,m}\int_{\Omega_\mu} V^{-}_{k_1,1}V^{\mp}_{k_2,m-1}\ dx$$
$$+\frac{1}{k_2+m+2}\int_{\Omega_\mu} V^{-}_{k_1,1}V^{\mp}_{k_2,m+1}\ dx \mp (k_2+m+1)a_{k_2,m}\int_{\Omega_\mu} V^{+}_{k_1,1}V^{\pm}_{k_2,m-1}\ dx$$
$$\pm\frac{1}{(k_2+m+2)}\int_{\Omega_\mu} V^{+}_{k_1,1}V^{\pm}_{k_2,m+1}\ dx\Bigg).$$

From this it is clear that $\langle Z_{k_1,0}, Z^{-}_{k_2,m}\rangle_{[\mu]} = 0$. It remains to check that

$$\langle Z_{k_1,0}, Z^{+}_{k_2,m}\rangle_{[\mu]} = 0.$$

Note that

$$\langle Z_{k_1,0}, Z^{+}_{k_2,m}\rangle_{[\mu]} = \frac{2}{k_1+2}\Bigg((k_2+m+1)a_{k_2,m}\int_{\Omega_\mu} V^{-}_{k_1,1}V^{-}_{k_2,m-1}\ dx$$
$$+\frac{1}{k_2+m+2}\int_{\Omega_\mu} V^{-}_{k_1,1}V^{-}_{k_2,m+1}\, dx - (k_2+m+1)a_{k_2,m}\int_{\Omega_\mu} V^{+}_{k_1,1}V^{+}_{k_2,m-1}\ dx$$
$$+\frac{1}{(k_2+m+2)}\int_{\Omega_\mu} V^{+}_{k_1,1}V^{+}_{k_2,m+1}\, dx\Bigg)$$
$$= \frac{2}{k_1+2}\Bigg(-(k_2+m+1)a_{k_2,m}\int_0^{\pi}\int_0^{1/\mu} \widehat{V}_{k_1,1}\widehat{V}_{k_2,m-1}\, dv\, du$$
$$+\frac{1}{k_2+m+2}\int_0^{\pi}\int_0^{1/\mu} \widehat{V}_{k_1,1}\widehat{V}_{k_2,m+1}\, dv\, du\Bigg)\int_0^{2\pi}\Phi^{+}_{m} d\varphi = 0.$$

Finally, by the orthogonality of the system $\{\Phi^{\pm}_{m}\}$, we arrive at the desired conclusion

$$\langle Z^{\pm}_{k_1,m_1}, Z^{\mp}_{k_2,m_2}\rangle_{[\mu]} = 0.$$

□

4.1.3 Density of the basic contragenics

The following result expresses for contragenics the analogy of the well known denseness of the harmonic polynomials and the monogenic polynomials in the corresponding Hilbert spaces of harmonic and monogenic functions.

It is clear that a limit in L_2 of contragenic functions is again contragenic.

Theorem 4.1.3. *The functions $Z^{\pm}_{k,m}[\mu]$ span a dense set in $\mathcal{N}(\Omega_\mu)$. Therefore the functions $Y^{\pm\pm}_{k,m}$, $Z^{\pm}_{k,m}[\mu]$ form an orthogonal basis for the Hilbert space $\mathrm{Har}(\Omega_\mu) \cap L_2(\Omega_\mu)$.*

Proof. We filter $\mathrm{Har}(\Omega_\mu)$ as follows. Let $H_0 \cong \mathbb{R}$ denote the collection of real constants, and for $n \geq 1$, let H_n be the orthogonal component of H_{n-1} in the space of polynomials $\mathrm{Har}^{(n)}_*(\Omega_\mu)$, so we have an orthogonal Hilbert space decomposition

$$\mathrm{Har}(\Omega_\mu) \cap L_2(\Omega_\mu) = \bigoplus_{n=0}^{\infty} H_n.$$

Let $Z \in \mathcal{N}(\Omega_\mu)$ be an arbitrary monogenic function, and express $Z = \sum_0^\infty U_n$, where $U_n \in H_n$. Let $U_n = Y_n + Z_n$ be the decomposition of U_n into ambigenic and contragenic polynomials. Thus $Z = Y + \sum_1^\infty Z_n$ where $Y = \sum_0^\infty Y_n$. Since Y is both ambigenic and contragenic, $Y = 0$, so $Z = \sum_1^\infty Z_n$.

By Theorem 1.4.4, each Y_n is a linear combination of some of the $Y^{\pm\pm}_{k,m}$, and by Theorem 4.1.2, each Z_n is a linear combination of some of the $Z^{\pm}_{k,m}[\mu]$, as required. □

The orthogonal decomposition of square-integrable harmonic functions as $\mathrm{Har}(\Omega) = (\mathcal{M}_2(\Omega) + \overline{\mathcal{M}}_2(\Omega)) \oplus \mathcal{N}(\Omega)$ justifies the idea of referring to the "ambigenic part" and the "monogenic part" of any harmonic function $\Omega \to \mathbb{R}^3$ (the latter being determined up to an additive monogenic constant). Theorem 4.1.3 provides a method of calculation of this part in the case of spheroids Ω_μ, by obtaining the Fourier coefficients as is done in any Hilbert space, and then discarding the contragenic and antimonogenic terms.

The fact that the notion of contragenicity depends on the domain implies that it is not a local property, in contrast to harmonicity and monogenicity. For example, the restriction of a contragenic function to a subdomain need not be contragenic.

In particular, any attempt to seek a condition on the derivatives of a harmonic function to detect whether it is monogenic or not is doomed to failure. It is not known, however, whether such a condition may exist associated to a fixed domain, such as a sphere or spheroid.

4.2 Relations among contragenic functions

In this section we will investigate functions which are contragenic with respect to two spheroids. Most of our attention will be to relate $\mathcal{N}(\Omega_{\widetilde{\mu}})$ to $\mathcal{N}(\Omega_\mu)$ where $\widetilde{\mu} \neq \mu$.

4.2.1 Representation of harmonics with vanishing scalar part

In this section we will prove Lemma 4.2.2, which will permit us to deduce facts about expressions of elements of $\mathcal{N}(\Omega_{\widetilde{\mu}})$ in terms of elements of $\mathcal{N}(\Omega_\mu)$.

For our purposes, we will need the particular ambigenic functions

$$\mathcal{A}^{\pm}_{n,m}[\mu] = 2\,\mathrm{Vec}\, X^{\pm}_{n,m}[\mu] = X^{\pm}_{n,m}[\mu] - \overline{X}^{\pm}_{n,m}[\mu]. \tag{4.11}$$

Observe that $\mathcal{A}^{\pm}_{n,m}[\mu] = -Y^{\pm,-}_{n,m}[\mu] + (1-\gamma_{n,m}[\mu])Y^{\pm,+}_{n,m}[\mu]$.

Lemma 4.2.1. *The collection* $\{\mathcal{A}^{\pm}_{l,m}[\mu]\}$ *is an orthogonal system in the sense of the scalar product* (1.7)*.*

Proof. Denote by $\sigma_{l,m} = 1-\gamma_{l,m}[\mu]$. Note that

$$\begin{aligned}\langle \mathcal{A}^{\pm}_{l_1,m_1}[\mu],\mathcal{A}^{\pm}_{l_2,m_2}[\mu]\rangle_\mu &= \langle Y^{\pm,-}_{l_1,m_1}, Y^{\pm,-}_{l_2,m_2}\rangle_\mu - \sigma_{l_2,m_2}\langle Y^{\pm,-}_{l_1,m_1}, Y^{\pm,+}_{l_2,m_2}\rangle_\mu \\ &\quad - \sigma_{l_1,m_1}\langle Y^{\pm,+}_{l_1,m_1}, Y^{\pm,-}_{l_2,m_2}\rangle_\mu + \sigma_{l_1,m_1}\sigma_{l_2,m_2}\langle Y^{\pm,+}_{l_1,m_1}, Y^{\pm,+}_{l_2,m_2}\rangle_\mu.\end{aligned}$$

By the orthogonality of the system $\{Y^{\pm,\pm}_{l,m}\}$ (again Proposition 3.2.2), this implies that

$$\langle \mathcal{A}^{\pm}_{l_1,m_1}[\mu], \mathcal{A}^{\pm}_{l_2,m_2}[\mu]\rangle_\mu = 0$$

when $l_1 \neq l_2$ or $m_1 \neq m_2$. Analogously, $\langle \mathcal{A}^{\pm}_{l_1,m_1}[\mu], \mathcal{A}^{\mp}_{l_2,m_2}[\mu]\rangle_\mu = 0$. □

We introduce the further notation

$$\begin{aligned}\Psi^{\pm}_{+,m} &= \Phi^{\pm}_{m}(\varphi)e_1 \pm \Phi^{\mp}_{m}(\varphi)e_2, \\ \Psi^{\pm}_{-,m} &= \Phi^{\pm}_{m}(\varphi)e_1 \mp \Phi^{\mp}_{m}(\varphi)e_2.\end{aligned} \tag{4.12}$$

These functions satisfy by definition the relations

$$\begin{aligned} \Psi^{\pm}_{+,m} e_3 &= \pm \Psi^{\mp}_{+,m}, \\ \Psi^{\pm}_{-,m} e_3 &= \mp \Psi^{\mp}_{-,m}, \\ e_1 V^{\pm}_{n,m}[\mu] + e_2 V^{\mp}_{n,m}[\mu] &= \widehat{V}_{n,m}[\mu] \Psi^{\pm}_{\pm,m}, \\ e_1 V^{\pm}_{n,m}[\mu] - e_2 V^{\mp}_{n,m}[\mu] &= \widehat{V}_{n,m}[\mu] \Psi^{\pm}_{\mp,m} \end{aligned} \tag{4.13}$$

(where the $\widehat{V}_{n,m}[\mu]$ are given by (2.14)). Therefore by (4.11),

$$\begin{aligned} \mathcal{A}^{+}_{n,0}[\mu] &= \frac{-2}{n+2} \widehat{V}_{n,1}[\mu] \Psi^{+}_{+,1}, \\ \mathcal{A}^{\pm}_{n,m}[\mu] &= (n+m+1) \widehat{V}_{n,m-1}[\mu] \Psi^{\pm}_{-,m-1} \\ &\quad - \frac{1}{n+m+2} \widehat{V}_{n,m+1}[\mu] \Psi^{\pm}_{+,m+1}, \end{aligned} \tag{4.14}$$

and by (4.6),

$$\begin{aligned} Z^{+}_{n,0}[\mu] &= \frac{2}{n+2} \widehat{V}_{n,1}[\mu] \Psi^{-}_{+,1}, \\ Z^{\pm}_{n,m}[\mu] &= (n+m+1) a_{n,m}[\mu] \widehat{V}_{n,m-1}[\mu] \Psi^{\mp}_{-,m-1} \\ &\quad + \frac{1}{n+m+2} \widehat{V}_{n,m+1}[\mu] \Psi^{\mp}_{+,m+1}, \end{aligned} \tag{4.15}$$

where $1 \le m \le n-1$.

Adding and subtracting instances of (4.14) and (4.15) gives by cancellation decompositions of the harmonic polynomials $\widehat{V}_{n,m-1}[\mu]\Psi^{\pm}_{+,m}$ and $\widehat{V}_{n,m+1}[\mu]\Psi^{\pm}_{-,m}$ as the sum of a contragenic and an ambigenic as follows:

Lemma 4.2.2. *Let $n \geq 1$ and $1 \leq m \leq n+1$. Then*

$$\widehat{V}_{n,m-1}[\mu]\Psi^{\pm}_{-,m-1} = \frac{1}{(n+m+1)(a_{n,m}[\mu]+1)}\big(Z^{\mp}_{n,m}[\mu] + \mathcal{A}^{\pm}_{n,m}[\mu]\big),$$

and

$$\widehat{V}_{n,m+1}[\mu]\Psi^{\pm}_{+,m+1} = \frac{n+m+2}{a_{n,m}[\mu]+1}\big(Z^{\mp}_{n,m}[\mu] - a_{n,m}[\mu]\mathcal{A}^{\pm}_{n,m}[\mu]\big).$$

4.2.2 Intersections of spaces of monogenic polynomials

The definition of contragenic function does not imply that an L_2-function which belongs to the space $\mathcal{N}^{(n)}_*[\widetilde{\mu}]$ should also be in $\mathcal{N}^{(n)}_*[\mu]$ when $\widetilde{\mu} \neq \mu$, because the notion of orthogonality is different for different spheroids. In other words, we may not expect a general formula like

$$\text{“}Z^{\pm}_{n,m}[\widetilde{\mu}] = \sum z_{n,m,k}[\widetilde{\mu},\mu] Z^{\pm}_{n-2k,m}[\mu]\text{”}$$

analogous to the results presented in Chapter 2 for harmonic and for monogenic functions.

The following result will enable us to give many examples for which $Z^{\pm}_{n,m}[\widetilde{\mu}] \notin \mathcal{N}^{(n)}_*[\mu]$ for $m \geq 1$. However, it also shows that the intersection of all of the $\mathcal{N}^{(n)}_*[\mu]$ is nontrivial, giving what may be called *universal contragenic functions* in the context of spheroids.

We will use the coefficients

$$
z^{\mathrm{C}}_{n,0,k}[\widetilde{\mu},\mu] = \frac{n-2k+2}{n+2} w_{n,1,k}[\widetilde{\mu},\mu],
$$
$$
z^{\mathrm{C}}_{n,m,k}[\widetilde{\mu},\mu] = \begin{cases} \dfrac{a_{n,m}[\widetilde{\mu}]+1}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu], & 0 \le 2k \le n-m-1, \\ \dfrac{a_{n,m}[\widetilde{\mu}]}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu], & n-m \le 2k \le n-m+1; \end{cases}
$$
$$
z^{\mathrm{A}}_{n,m,k}[\widetilde{\mu},\mu] = \begin{cases} \dfrac{a_{n,m}[\widetilde{\mu}]-a_{n,m}[\mu]}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu], & 0 \le 2k \le n-m-1, \\ \dfrac{a_{n,m}[\widetilde{\mu}]}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu], & n-m \le 2k \le n-m+1; \end{cases} \tag{4.16}
$$

($1 \le m \le n-1$) to express the decomposition of contragenics for one spheroid in terms of contragenics and ambigenics of any other.

Proposition 4.2.3. *Let $n \ge 1$. Then*

$$
Z^{+}_{n,0}[\widetilde{\mu}] = \sum_{0 \le 2k \le n-1} z^{\mathrm{C}}_{n,k}[\widetilde{\mu},\mu] Z_{n-2k,0}[\mu];
$$

and for $1 \le m \le n-1$,

$$
Z^{\pm}_{n,m}[\widetilde{\mu}] = \sum_{0 \le 2k \le n-m+1} \left(z^{\mathrm{C}}_{n,m,k}[\widetilde{\mu},\mu] Z^{\pm}_{n-2k,m}[\mu] + z^{\mathrm{A}}_{n,m,k}[\widetilde{\mu},\mu] \mathcal{A}^{\pm}_{n-2k,m}[\mu] \right).
$$

Proof. Apply Theorem 2.2.4 to the first formula of (4.15) with $\widetilde{\mu}$ in place of μ to obtain that

$$
Z^{+}_{n,0}[\widetilde{\mu}] = \frac{2}{n+2} \sum_{0 \le 2k \le n-1} w_{n,1,k}[\widetilde{\mu},\mu] \widehat{V}_{n-2k,1}[\mu] \Psi^{-}_{+,1},
$$

which after another application of (4.15) reduces to the first statement. In the same way, for $m \geq 1$,

$$Z^{\pm}_{n,m}[\widetilde{\mu}] = (n+m+1)a_{n,m}[\widetilde{\mu}] \sum_{0\leq 2k\leq n-m+1} w_{n,m-1,k}[\widetilde{\mu},\mu]\widehat{V}_{n-2k,m-1}[\mu]\Psi^{\pm}_{-,m-1} + \frac{1}{n+m+2}\sum_{0\leq 2k\leq n-m-1} w_{n,m+1,k}[\widetilde{\mu},\mu]\widehat{V}_{n-2k,m+1}[\mu]\Psi^{\pm}_{+,m+1}. \tag{4.17}$$

We observe from the definitions leading to Proposition 2.2.2 that

$$v_{n,m-1,l}\,\widetilde{v}_{n-2l,m-1,k-l} = \frac{n+m-2k+1}{n+m+1} v_{n,m,l}\,\widetilde{v}_{n-2l,m,k-l},$$

so (2.27) tells us that

$$\begin{aligned}\frac{n+m+1}{n+m-2k+1} w_{n,m-1,k}[\widetilde{\mu},\mu] &= w_{n,m,k}[\widetilde{\mu},\mu] \\ &= \frac{n+m-2k+2}{n+m+2} w_{n,m+1,k}[\widetilde{\mu},\mu].\end{aligned}$$

From this and Lemma 4.2.2 we have that

$$\begin{aligned}&(n+m+1)w_{n,m-1,k}[\widetilde{\mu},\mu]\widehat{V}_{n-2k,m-1}[\mu]\Psi^{\pm}_{-,m-1} \\ &= \frac{1}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu](Z^{\mp}_{n-2k,m}[\mu] + \mathcal{A}^{\pm}_{n-2k,m}[\mu]),\end{aligned}$$

and

$$\frac{1}{n+m+2} w_{n,m+1,k}[\widetilde{\mu},\mu]\widehat{V}_{n-2k,m+1}[\mu]\Psi^{\pm}_{+,m+1}$$
$$= \frac{1}{a_{n-2k,m}[\mu]+1} w_{n,m,k}[\widetilde{\mu},\mu](Z^{\mp}_{n-2k,m}[\mu] - a_{n-2k,m}[\mu]\mathcal{A}^{\pm}_{n-2k,m}[\mu]).$$

Inserting these two relations into the respective sums of (4.17) gives the desired result. □

Proposition 4.2.3 provides us with some information about the intersection of the spaces of contragenic functions up to degree n.

Theorem 4.2.1. *Let $n \geq 1$. The following statements hold:*

(i) $Z^{+}_{n,0}[\mu] \in \mathcal{N}^{(n)}_{*}[0]$ *for all* μ*;*

(ii) $Z^{\pm}_{n,m}[\mu] \notin N^{(n)}_{*}[0]$ *when* $\mu \neq 0$ *and* $1 \leq m \leq n-1$*.*

Proof. The first statement is an immediate consequence of the first formula of Proposition 4.2.3.

Now consider a basic element $Z^{\pm}_{n,m}[\mu]$ of $\mathcal{N}^{(n)}_{*}[\mu]$, with $\mu \neq 0$ and $1 \leq m \leq n-1$. A particular instance of the second formula of Proposition 4.2.3 is

$$Z^{\pm}_{n,m}[\mu] = \sum_{0\leq 2k\leq n-m+1} \big(z^{\mathrm{C}}_{n,m,k}[\mu,0] Z^{\pm}_{n-2k,m}[0] + z^{\mathrm{A}}_{n,m,k}[\mu,0]\mathcal{A}^{\pm}_{n-2k,m}[0]\big).$$

Suppose that $Z^{\pm}_{n,m}[\mu] \in \mathcal{N}^{(n)}_{*}[0]$. Then since the right hand side is orthogonal to all Ω_0-ambigenics,

$$\sum_{0\leq 2k\leq n-m+1} z^{\mathrm{A}}_{n,m,k}[\mu,0]\mathcal{A}^{\pm}_{n-2k,m}[0] = 0,$$

and so by the linear independence, $z^{\mathrm{A}}_{n,m,k}[\mu,0] = 0$ for all k. The case in (4.16) where $2k$ is $n-m$ or $n-m+1$ tells us that $a_{n,m}[\mu] = 0$, which is manifestly false by (4.5). Consequently, $Z^{\pm}_{n,m}[\mu] \notin \mathcal{N}^{(n)}_*[0]$ as claimed. □

Note that Theorem 4.2.1 does not assert that $Z^{+}_{n,0}[\mu]$ lies in the top-level slice $\mathcal{N}^{(n)}[0]$ of $\mathcal{N}^{(n)}_*[0]$.

Corollary 4.2.2. *Let $n \geq 1$. Then*

$$\dim \bigcap_{\mu\in[0,1)\cup\mathbb{R}^+} \mathcal{N}^{(n)}_*[\mu] \geq n.$$

Proof. The result is an immediate consequence of the fact that Theorem 4.2.1 is applicable to arbitrary μ, so the intersection contains a fixed n-dimensional subspace of $\mathcal{N}^{(n)}_*[0]$. □

It also follows from Theorem 4.2.1 that the common intersection $\mathcal{N}_0 = \bigcap \mathcal{N}_*[\mu]$ of the full spaces of contragenic functions on spheroids is infinite dimensional, containing all of the contragenic polynomials $Z^{+}_{n,m}[\mu]$ for which $m = 0$.

It seems likely that these contragenic polynomials have special characteristics because of their simpler structure, cf. (4.15). This phenomenon is not yet fully understood.

Further questions relating to the exact relations among the spaces $\mathcal{N}^{(n)}_*[\mu]$ introduced in this thesis still remain open. If the method of the proof of Theorem 4.2.1 is applied to linear combinations of the $Z^{\pm}_{n,m}[\mu]$ instead of just to these generators individually, transcendental equations related to (4.5) appear. These equations may be a subject of future investigation.

Another open question is to determine how the angles between the orthogonal complements of the mode-0 subspace $\mathcal{N}_0^n[0]$ in $\mathcal{N}_*^{(n)}[\mu]$, or of their union $\mathcal{N}_0[0]$ in $\mathcal{N}[\mu]$, vary with μ.

Chapter 5

Conclusions and future work

By means of a suitable variable change, the spherical harmonics and monogenics P. Garabedian and J. Morais are embedded in 1-parameter families of spheroidal functions (see (2.8)) which include the spherical functions. The bases of spheroidal harmonics are a fundamental element for the calculation of a base of spheroidal contragenic polynomials.

Relationships among the systems of harmonic polynomials for spheroids of distinct eccentricity were studied (Theorem 2.2.4). This made it possible to find relations between the bases of contragenic functions by means of explicit expressions.

As a result, it was found that there are some contragenic functions common to all spheroids of all eccentricities (Theorem 4.2.1).

It is not possible to define a condition on the derivatives of a harmonic function to detect whether it is contragenic or not, because contragenicity depends on the entire domain and is not a local property. It is not known, however, whether such a condition may exist associated to a fixed domain, such as a sphere or spheroid. It is hoped that the explicit expressions we have found man shed light on this question.

Further questions relating to the exact relations among the spaces $\mathcal{N}_*^{(n)}[\mu]$ introduced in this thesis still remain open.

Appendix A

Appendix: Oblate Spheroidal Harmonics

To complete the development given in Chapter 1 for prolate spheroidal coordinates, we analyze the expression (2.9) for oblate spheroidal polynomials $\widehat{U}_{n,m}[\mu]$, i.e. when $\nu > 0$.

In this case, $\mu = i\sqrt{e^{2\nu} - 1}$, and

$$\omega\left(\mu\right) \;=\; \sqrt{\zeta} \,+\, \sqrt{\overline{\zeta}},$$

with

$$\zeta \;=\; |x|^2 \,+\, 1 \,-\, e^{2\nu} \,+\, 2x_0\,\sqrt{e^{2\nu} \,-\, 1}\,i.$$

Denoting $a = |x|^2 + 1 - e^{2\nu}$ and $b = 2x_0\sqrt{e^{2\nu} - 1}$, we obtain

$$\sqrt{\zeta} = \begin{cases} \pm\left(\sqrt{\dfrac{a + \sqrt{a^2+b^2}}{2}} + i\sqrt{\dfrac{-a + \sqrt{a^2+b^2}}{2}}\right) & \text{if } b > 0, \\ \\ \pm\left(-\sqrt{\dfrac{a + \sqrt{a^2+b^2}}{2}} + i\sqrt{\dfrac{-a + \sqrt{a^2+b^2}}{2}}\right) & \text{if } b < 0. \end{cases}$$

Taking conjugate roots, it is obtained that

$$\sqrt{\zeta} + \sqrt{\bar{\zeta}} = \pm 2\sqrt{\frac{a + \sqrt{a^2+b^2}}{2}},$$

hence, by taking

$$\omega(\mu) = -2\sqrt{\frac{a + \sqrt{a^2+b^2}}{2}},$$

when $\nu > 0$, we define

$$s(\mu, x) := -\frac{\sqrt{2}\, x_0}{\sqrt{a + \sqrt{a^2+b^2}}}$$

and

$$t(\mu, x) := i\sqrt{\frac{a + \sqrt{a^2+b^2}}{2\,(e^{2\nu} - 1)}}.$$

Note that $|s(\mu, x)| \leq 1$.

Also, we claim that $\operatorname{Im}(t(\mu, x))$ takes values in $[0, \infty)$. Indeed, if $\nu \geq \log\sqrt{2}$, then $\sqrt{e^{2\nu} - 1} > 1$. With this in mind, $\operatorname{Im}(t(\mu, x)) \geq 1$ if and only if

$$x_1^2 + x_2^2 \geq 2\,(e^{2\nu} - (1 + x_0^2)).$$

Now, note that

$$e^{2\nu}(1 - x_0^2) < 2(e^{2\nu} - (1 + x_0^2)).$$

Therefore, $\mathrm{Im}(t(\mu, x)) \geq 1$ iff $(x_0, x_1, x_2) \notin \Omega_\mu$. For this reason,

$$\mathrm{Im}(t(\mu, x)) < 1,$$

for all $(x_0, x_1, x_2) \in \Omega_\mu$ and $\nu \geq \log\sqrt{2}$.

Further, if $\nu < \log\sqrt{2}$, then $\sqrt{e^{2\nu} - 1} < 1$ and we have two cases. When $|x_0| \leq \sqrt{e^{2\nu} - 1}$, $2(e^{2\nu} - 1 - x_0^2) > 0$. Furthermore,

$$e^{2\nu}(1 - x_0^2) \geq 2(e^{2\nu} - (1 + x_0^2)).$$

Thus, if $(x_0, x_1, x_2) \in \Omega_\mu$ such that $|x_0| \leq \sqrt{e^{2\nu} - 1}$, then

$$\mathrm{Im}(t(\mu, x)) \geq 1,$$

if and only if $x_1^2 + x_2^2 \geq 2(e^{2\nu} - (1+x_0^2))$. On the other hand, if $\sqrt{e^{2\nu} - 1} < |x_0| \leq 1$, then $2(e^{2\nu} - 1 - x_0^2) < 0$. Consequently, for all $(x_0, x_1, x_2) \in \Omega_\mu$ such that $|x_0| > \sqrt{e^{2\nu} - 1}$, $\mathrm{Im}(t(\mu, x)) \geq 1$. Hence, $\mathrm{Im}(t(\mu, x))$ takes values in $[0, \infty)$.

Therefore, when $\nu > 0$, we can define $s(\mu, x) = \cos u$, $u \in [0, \pi]$ and $t(\mu, x) = i \sinh v$, $v \in [0, \operatorname{arccoth} e^{\nu}]$.

Appendix B

Appendix: Recurrence Formulas for Spheroidal Harmonics

Taking as inspiration the work of [6, 56, 57], we consider obtain recurrence formulas that allow a more detailed study within a computational framework. These expressions play a very important role to make a detailed and concise proof of the closed form expressions for the expansion coefficients, which exhibit spheroidal harmonics defined on a spheroid Ω_μ as linear combinations of spheroidal harmonics associated with a spheroid $\Omega_{\widetilde{\mu}}$.

In this appendix we set forth some recurrence formulas among classical spherical harmonics which were obtained in the study of the basic spheroidal harmonic polynomials, but which were not used in the proofs of the main results.

We follow the notation of prolate spheroidal coordinates defined in Chapter 2. Set

$$s := \cos u, \qquad t := \cosh v.$$

We shall be concerned here with the new functions

$$\phi_{n+1,m}[\mu](s,t) := \begin{cases} 0 & \text{if } n=-1 \text{ or } n=0,\ 0\le m\le n+1, \\ \mu^{n+1}(tP_n^m(t)P_{n-1}^m(s)+sP_n^m(s)P_{n-1}^m(t)) & \\ \qquad \text{if } n>0,\ 0\le m\le n. \end{cases}$$

For simplicity, we will to denote the functions $\phi_{n+1,m}[\mu](s,t)$ simply by $\phi_{n+1,m}[\mu]$ and, even, when μ is considered fixed, by $\phi_{n,m}$. The following results establish recurrence relations, with respect to degree, between the functions $\phi_{n,m}$ and the spheroidal harmonics $\widehat{U}_{n,m}$.

Proposition B.0.1. *For each $\mu \in \mathbb{R}\cup i\mathbb{R}^+$, the following statements are satisfied.*

a) For each $n>0$ and $m=0,1,...,n-2$, the functions $\phi_{n,m}[\mu](s,t)$ satisfy the recurrence formula given by

$$\begin{aligned}\phi_{n+1,m}[\mu](s,t) = &\left(\frac{t^2+s^2}{\alpha_{n,m}}\right)\widehat{U}_{n-1,m}[\mu] \\ &- \frac{2x_0\mu^2(n+m-1)}{\alpha_{n-1,m}(n-m)}\widehat{U}_{n-2,m}[\mu] \\ &+ \frac{\mu^2(n+m-1)(n+m-2)}{(n-m)(n-m-1)}\phi_{n-1,m}[\mu](s,t) \qquad \text{(B.1)}\end{aligned}$$

b) For each $n > 0$, when $m = n - 1$, it is obtained that

$$\phi_{n+1,n-1}[\mu](s,t) = (-1)^{n-1}\left((2n-3)!!\right)^2(2n-1)(x_1^2+x_2^2)^{(n-1)/2}(|x|^2+\mu^2) \tag{B.2}$$

c) For each $n > 0$, if $m \geq n$, then $\phi_{n+1}^m[\mu](s,t) = 0$.

Proof. Throughout this proof we consider μ fixed. Denote by

$$\widetilde{\phi}_{n,m} = \mu^{n+1}(tP_n^m(t)P_{n-1}^m(s) + sP_n^m(s)P_{n-1}^m(t)).$$

By equation (1.19),

$$r\,P_n^m\,(r) \;=\; \frac{2n-1}{n-m}\,r^2\,P_{n-1}^m\,(r) \;-\; \frac{n+m-1}{n-m}\,r\,P_{n-2}^m\,(r).$$

Taking $r = s, t$, it follows that

$$\widetilde{\phi}_{n+1,m} = P_{n-1}^m\,(s)\left(\frac{2n-1}{n-m}\,t^2\,P_{n-1}^m\,(t) \;-\; \frac{n+m-1}{n-m}\,t\,P_{n-2}^m\,(t)\right)$$

$$+\,P_{n-1}^m\,(t)\left(\frac{2n-1}{n-m}\,s^2\,P_{n-1}^m\,(s) \;-\; \frac{n+m-1}{n-m}\,s\,P_{n-2}^m\,(s)\right)$$

$$= \left(\frac{2n-1}{n-m}\right)\left(\frac{t^2\;+\;s^2}{\alpha_{n-1,m}\mu^{n-1}}\right)\widehat{U}_{n-1,m}$$

$$-\,\frac{n+m-1}{n-m}\,(t\,P_{n-2}^m\,(t)\,P_{n-1}^m\,(s)\;+\;s\,P_{n-2}^m\,(s)\,P_{n-1}^m\,(t)).$$

Again, applying equation (1.19), we obtain

$$P^m_{n-1}(r) = \frac{2n-3}{n-m-1}\, r\, P^m_{n-2}(r) - \frac{n+m-2}{n-m-1}\, P^m_{n-3}(r),$$

hence

$$\widetilde{\phi}_{n+1,m} = \left(\frac{2n-1}{n-m}\right)\left(\frac{t^2 + s^2}{\alpha_{n-1,m}\mu^{n-1}}\right)\widehat{U}_{n-1,m}$$

$$- \frac{n+m-1}{n-m}\left[t\, P^m_{n-2}(t)\left(\frac{2n-3}{n-m-1}\, s\, P^m_{n-2}(s) - \frac{n+m-2}{n-m-1}\, P^m_{n-3}(s)\right)\right.$$

$$\left. + \, s\, P^m_{n-2}(s)\left(\frac{2n-3}{n-m-1}\, t\, P^m_{n-2}(t) - \frac{n+m-2}{n-m-1}\, P^m_{n-3}(t)\right)\right].$$

Also, note that

$$\alpha_{n,m} = \left(\frac{n-m}{2n-1}\right)\alpha_{n-1,m}, \tag{B.3}$$

from which it follows that

$$\widetilde{\phi}_{n+1,m} = \left(\frac{t^2 + s^2}{\alpha_{n,m}\mu^{n-1}}\right)\widehat{U}_{n-1,m} - 2\left(\frac{n+m-1}{n-m}\right)\frac{st}{\alpha_{n-1,m}\mu^{n-2}}\,\widehat{U}_{n-2,m}$$
$$+ \frac{(n+m-1)(n+m-2)}{(n-m)(n-m-1)}\,\widetilde{\phi}_{n-1,m}.$$

Therefore, we obtain that

$$\phi_{n+1,m} = \mu^{n+1}\left[\left(\frac{t^2 + s^2}{\alpha_{n,m}\mu^{n-1}}\right)\widehat{U}_{n-1,m} - \frac{(n+m-1)}{(n-m)}\frac{2st}{\alpha_{n-1,m}\mu^{n-2}}\widehat{U}_{n-2,m}\right.$$

$$\left. + \frac{(n+m-1)(n+m-2)}{(n-m)(n-m-1)}\widetilde{\phi}_{n-1,m}\right]$$

$$= \mu^2\frac{(t^2 + s^2)}{\alpha_{n,m}}\widehat{U}_{n-1,m} - 2\left(\frac{n+m-1}{\alpha_{n-1,m}(n-m)}\right)\mu\, st\,\mu^2\,\widehat{U}_{n-2,m}$$

$$+ \frac{(n+m-1)(n+m-2)}{(n-m)(n-m-1)}\mu^2\,\phi_{n-1,m}.$$

Recalling that $x_0 = \mu\, st$ and $\mu^2(s^2+t^2) = x_0^2+\rho^2+\mu^2 = |x|^2+\mu^2$, we obtain a). Now, if $m = n-1$, by equation (1.20), we have that

$$\widetilde{\phi}_{n+1,n-1} = t\,P_n^{n-1}(t)\,P_{n-1}^{n-1}(s) + s\,P_n^{n-1}(s)\,P_{n-1}^{n-1}(t)$$

$$= (2n-1)(t^2 + s^2)\,P_{n-1}^{n-1}(s)\,P_{n-1}^{n-1}(t).$$

By virtue of (2.3),

$$x_0^2 + \rho^2 + \mu^2 = \mu^2(s^2 + t^2).$$

Applying the formulas (1.24) and (1.29), we obtain the expression

$$\widetilde{\phi}_{n+1,n-1} = (-1)^{n-1}\left((2n-3)!!\right)^2(2n-1)(x_1^2 + x_2^2)^{(n-1)/2}\frac{(|x|^2 + \mu^2)}{\mu^{n+1}}.$$

It follows that

$$\phi_{n+1,n-1} = (-1)^{n-1}(2n - 1)((2n - 3)!!)^2(x_1^2 + x_2^2)^{\frac{n-1}{2}} (|x|^2 + \mu^2)$$

and b) is obtained. Note that, by definition, $\widetilde{\phi}_{n+1,m} = 0$, when $n = 0$. Finally, for each $n > 0$, when $m \geq n$, we have $P_{n-1}^m(r) = 0$ for every $r \in \mathbb{R}$, and therefore $\phi_{n+1,m} = 0$, obtaining c). □

Proposition B.0.2. *For each $n \geq 0$, $m = 0, 1, ..., n$, the functions $\widehat{U}_{n,m}[\mu]$ satisfy the following recurrence formula:*

$$\begin{aligned}\widehat{U}_{n+1,m}[\mu] = &\left(\frac{2n+1}{n-m+1}\right) x_0 \widehat{U}_{n,m}[\mu] \\ &+ \frac{\mu^2(n+m)^2(n-m)}{(2n+1)(2n-1)(n-m+1)} \widehat{U}_{n-1,m}[\mu] \\ &- \frac{\alpha_{n+1,m}(n+m)(2n+1)}{(n-m+1)^2} \phi_{n+1,m}[\mu](s,t)\end{aligned}$$

Proof. By equation (1.19),

$$P_{n+1}^m(r) = \frac{1}{n-m+1}((2n+1)\, r\, P_n^m(r) - (n+m)\, P_{n-1}^m(r)).$$

Taking $r = s, t,$ it follows that

$$P_{n+1}^{m}(s)\,P_{n+1}^{m}(t) = \left(\frac{1}{n-m+1}\right)^2 \left((2n+1)\,s\,P_n^m(s) - (n+m)\,P_{n-1}^m(s)\right)$$
$$\times \left((2n+1)\,t\,P_n^m(t) - (n+m)\,P_{n-1}^m(t)\right)$$
$$= \left(\frac{1}{n-m+1}\right)^2 [(2n+1)^2\, st\, P_n^m(s) P_n^m(t)$$
$$- (n+m)\,(2n+1)\,\widetilde{\phi}_{n+1,m} + (n+m)^2\, P_{n-1}^m(s)\,P_{n-1}^m(t)].$$

Then,

$$\widehat{U}_{n+1,m} = \mu^{n+1}\alpha_{n+1,m} P_{n+1}^m(s)\,P_{n+1}^m(t)$$

$$= \left(\frac{1}{n-m+1}\right)^2 [(2n+1)^2\,\mu\, st\, \mu^n \alpha_{n+1,m} P_n^m(s) P_n^m(t)$$

$$- (n+m)\,(2n+1)\mu^{n+1}\alpha_{n+1,m}\,\widetilde{\phi}_{n+1,m}$$
$$+ \mu^2\,(n+m)^2\,\mu^{n-1}\alpha_{n+1,m} P_{n-1}^m(s)\,P_{n-1}^m(t)].$$

By virtue of (B.3) the result follows. □

The following result establishes recursive formulas on the order m of the functions $\widehat{U}_{n,m}$ and $\phi_{n,m}$.

Proposition B.0.3. *Let $\mu \in \mathbb{R} \cup i\mathbb{R}^+$ be fixed.*

a) For each $n \geq 0, \;\; m = 0, ..., n,$

$$\widehat{U}_{n,m+1}[\mu] = \frac{1}{\rho}\Bigg((n-m)\,x_0\,\widehat{U}_{n,m}[\mu] + \frac{\mu^2\,(n+m)^2}{2n-1}\,\widehat{U}_{n-1,m}[\mu]$$

$$- (n+m)(n-m)\alpha_{n,m+1}\phi_{n+1,m}[\mu](s,t)\Bigg),$$

where $\rho = \sqrt{x_1^2 + x_2^2}$

b) For each $n \geq 0, \;\; m = 0, ..., n,$ *we have*

$$\phi_{n+1,m+1}[\mu](s,t) = \frac{1}{\rho}\Bigg(\frac{(n-m)^2}{\alpha_{n,m}}(|x|^2 \;+\; \mu^2)\,\widehat{U}_{n,m}[\mu]$$

$$+ (n+m)\,(n+m-1)\,\mu^2\,\phi_{n,m}[\mu](s,t)$$

$$- (2n-1)\,(n-m)\,x_0\,\phi_{n+1,m}[\mu](s,t)$$

$$- 2\frac{(n+m)(n-m-1)}{\alpha_{n-1,m}}x_0\mu^2\widehat{U}_{n-1,m}[\mu]\Bigg).$$

Proof. By virtue of (1.23) and (1.28), respectively,

$$P_n^{m+1}(s) \;=\; \frac{1}{\sqrt{1 \;-\; s^2}}\,((n-m)\,s\,P_n^m(s) \;-\; (n+m)\,P_{n-1}^m(s))$$

and

$$P_n^{m+1}(t) = \frac{1}{\sqrt{t^2 - 1}}\left((n-m)\, t\, P_n^m(t) - (n+m)\, P_{n-1}^m(t)\right).$$

Furthermore

$$\frac{1}{\sqrt{1 - s^2}}\,\frac{1}{\sqrt{t^2 - 1}} = \frac{\mu}{\rho},$$

so

$$\begin{aligned}
\widehat{U}_{n,m+1} = \frac{\mu^{n+1}\alpha_{n,m+1}}{\rho}\,&\Big[(n-m)^2\, st\, P_n^m(s)\, P_n^m(t) \\
&- (n-m)(n+m)\{t\, P_n^m(t)\, P_{n-1}^m(s) + s\, P_n^m(s)\, P_{n-1}^m(t)\} \\
&+ (n+m)^2\, P_{n-1}^m(s)\, P_{n-1}^m(t)\Big] \\
= \frac{1}{\rho}\,&\Big[(n-m)^2\, x_0 \mu^n \alpha_{n,m+1} P_n^m(s)\, P_n^m(t) \\
&- (n-m)(n+m)\mu^{n+1}\alpha_{n,m+1}\{t\, P_n^m(t)\, P_{n-1}^m(s) + s\, P_n^m(s)\, P_{n-1}^m(t)\} \\
&+ \mu^2 (n+m)^2\, \mu^{n-1}\alpha_{n,m+1} P_{n-1}^m(s)\, P_{n-1}^m(t)\Big].
\end{aligned}$$

Now, note that

$$\alpha_{n,m+1} = \frac{\alpha_{n,m}}{(n-m)}. \tag{B.4}$$

Because of this

$$\widehat{U}_{n,m+1} = \frac{1}{\rho}\,\Big[(n-m)\,x_0\,\widehat{U}_{n,m} - (n-m)(n+m)\alpha_{n,m+1}\phi_{n+1,m}$$

$$+ \frac{(n+m)^2\,\mu^2}{n-m}\mu^{n-1}\alpha_{n,m}P^m_{n-1}\,(s)\,P^m_{n-1}\,(t)\Big].$$

Then, by (B.3), we obtain a).

In addition, when $n > 0$ and $m \geq n-1$, by Lemma B.0.1, it follows that $\phi_{n+1,m+1} = 0$. If $n > 0$ and $0 \leq m \leq n-2$, by equations (1.23) and (1.28)

$$\phi_{n+1,m+1} = \mu^{n+1}\,\Bigg(\frac{(n-m)\,t^2}{\sqrt{t^2\,-\,1}}\,P^m_n\,(t)\,P^{m+1}_{n-1}\,(s)$$

$$- \frac{(n+m)\,t}{\sqrt{t^2\,-\,1}}\,P^m_{n-1}\,(t)\,P^{m+1}_{n-1}\,(s) + \frac{(n-m)\,s^2}{\sqrt{1\,-\,s^2}}\,P^m_n\,(s)\,P^{m+1}_{n-1}\,(t)$$
$$- \frac{(n+m)\,s}{\sqrt{1\,-\,s^2}}\,P^m_{n-1}\,(s)\,P^{m+1}_{n-1}\,(t)\Bigg).$$

Then, applying again formulas (1.23) and (1.28), we obtain

$$
\begin{aligned}
\frac{(n-m)\,s^2}{\sqrt{1\,-\,s^2}}\,P_n^m\,(s)\,P_{n-1}^{m+1}\,(t) &= \frac{(n-m)\,s^2}{\sqrt{1\,-\,s^2}} P_n^m(s) \\
&\times\left[\frac{1}{\sqrt{t^2\,-\,1}}\,((n-m-1)\,t\,P_{n-1}^m\,(t)\right. \\
&\left. -\,(n+m-1)\,P_{n-2}^m\,(t))\right] \\
&= \frac{\mu}{\rho}[(n-m)(n-m-1)\,s^2 t\,P_n^m\,(s)\,P_{n-1}^m\,(t) \\
&-\,(n-m)(n+m-1)\,s^2\,P_n^m\,(s)\,P_{n-2}^m\,(t)].
\end{aligned}
$$

Also, in a similar manner, we have that

$$
\begin{aligned}
\frac{(n-m)t^2}{\sqrt{t^2-1}} P_n^m(t) P_{n-1}^{m+1}(s) &= \frac{\mu}{\rho}[(n-m)(n-m-1)st^2 P_n^m(t) P_{n-1}^m(s) \\
&-\,(n-m)(n+m-1)t^2 P_n^m(t) P_{n-2}^m(s)].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{(n+m)\,t}{\sqrt{t^2\,-\,1}}\,P_{n-1}^m\,(t)\,P_{n-1}^{m+1}\,(s) &= \frac{(n+m)\,t}{\sqrt{t^2\,-\,1}}\,P_{n-1}^m\,(t) \\
&\times\left[\frac{1}{\sqrt{1\,-\,s^2}}\,((n-m-1)\,s\,P_{n-1}^m\,(s)-(n+m-1)\,P_{n-2}^m\,(s))\right] \\
&= \frac{\mu}{\rho}\,[(n+m)(n-m-1)\,st\,P_{n-1}^m\,(s)\,P_{n-1}^m\,(t) \\
&+\,(n+m)(n+m-1)\,t\,P_{n-1}^m\,(t)\,P_{n-2}^m\,(s)].
\end{aligned}
$$

And, analogously, it is obtained that

$$\frac{(n+m)s}{\sqrt{1-s^2}} P_{n-1}^{m}(s) P_{n-1}^{m+1}(t) = \frac{\mu}{\rho}[(n+m)(n-m-1) st P_{n-1}^{m}(s) P_{n-1}^{m}(t)$$
$$+ \ (n+m)(n+m-1)\, s\, P_{n-1}^{m}\,(s)\, P_{n-2}^{m}\,(t)].$$

Consequently, associating terms and recalling that $x_0 \ = \mu\, st,$ it follows that

$$\phi_{n+1,m+1} = \frac{x_0}{\rho}\,(n-m)(n-m-1)\,\phi_{n+1,m}$$
$$-\,2(n+m)(n-m-1)\frac{x_0}{\rho\,\alpha_{n-1,m}}\,\mu^2\,\widehat{U}_{n-1,m}$$
$$+\,(n+m)(n+m-1)\,\frac{\mu^2}{\rho}\,\phi_{n,m}$$
$$-\,(n-m)(n+m-1)\,\frac{\mu^{n+2}}{\rho}\,[t^2\,P_n^m\,(t)\,P_{n-2}^m\,(s) + s^2\,P_n^m\,(s)\,P_{n-2}^m\,(t)].$$

On the other hand, due to (1.19), we have that

$$P_{n-2}^{m}\,(r) \ = \ \frac{1}{n+m-1}\,((2n-1)\,r\,P_{n-1}^{m}\,(r) \ - \ (n-m)\,P_n^m\,(r)).$$

Then,

$$q^2\,P_n^m\,(q)\,P_{n-2}^m\,(r) = \frac{2n-1}{n+m-1}\,q^2\,r\,P_{n-1}^m\,(r)\,P_n^m\,(q)$$

$$-\,\frac{n-m}{n+m-1}\,q^2\,P_n^m\,(r)\,P_n^m\,(q),$$

so that,

$$
\begin{aligned}
t^2 P_n^m(t) P_{n-2}^m(s) + s^2 P_n^m(s) P_{n-2}^m(t) &= \frac{2n-1}{n+m-1} st\{t P_n^m(t) P_{n-1}^m(s) \\
&\quad + s P_n^m(s) P_{n-1}^m(t)\} - \frac{n-m}{n+m-1}(t^2+s^2) P_n^m(t) P_n^m(s) \\
&= \frac{2n-1}{n+m-1} st\widetilde{\phi}_{n+1,m} - \frac{(n-m)(t^2+s^2)}{\mu^n \, \alpha_{n,m}(n+m-1)} \widehat{U}_{n,m}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\phi_{n+1,m+1} &= \frac{1}{\rho}\, ((n-m)(n-m-1)\, x_0\, \phi_{n+1,m} \\
&\quad - (n-m)(n+m-1) \left(\frac{x_0\,(2n-1)}{(n+m-1)}\, \phi_{n+1,m} \right. \\
&\quad \left. - \left(\frac{n-m}{\alpha_{n,m}\,(n+m-1)} \right) \left(\mu^2\, (s^2 \,+\, t^2) \widehat{U}_{n,m} \right) \right) \\
&\quad + (n+m)(n+m-1)\, \mu^2\, \phi_{n,m} \\
&\quad - 2\frac{((n+m)(n-m-1)}{\alpha_{n-1,m}} \mu^2\, x_0\, \widehat{U}_{n-1,m}) \\
&= \frac{1}{\rho} \left(\frac{(n-m)^2(|x|^2 \,+\, \mu^2)}{\alpha_{n,m}}\, \widehat{U}_{n,m} + (n+m)\,(n+m-1)\, \mu^2\, \phi_{n,m} \right. \\
&\quad - (2n-1)\,(n-m)\, x_0\, \phi_{n+1,m} \\
&\quad \left. - 2\frac{(n+m)(n-m-1) x_0 \mu^2}{\alpha_{n-1,m}} \widehat{U}_{n-1,m} \right);
\end{aligned}
$$

hence we obtain b). □

The previous results allow us to establish relations of recurrence between the spheroidal harmonic polynomials

Proposition B.0.4. *Let $\mu \in \mathbb{R} \cup i\mathbb{R}^+$ be fixed.*

---

a) For each $n \geq 0$, $m = 0, 1, ..., n$ *the following recurrence formula holds:*

$$\widehat{U}_{n+1,m}[\mu] = \left(\frac{2n+1}{n-m+1}\right) x_0\, \widehat{U}_{n,m}[\mu] \;-\; \left(\frac{n+m}{n-m+1}\right) |x|^2\, \widehat{U}_{n-1,m}[\mu]$$

$$+\, \mu^2 \left[\frac{(n+m)(n+m-1)}{(2n-1)(n-m+1)}\, x_0\, \widehat{U}_{n-2,m}[\mu]\right.$$

$$\left. -\; \frac{2(n+m)(n(n-1)+m^2-1)}{(n-m+1)(2n-3)(2n+1)}\, \widehat{U}_{n-1,m}[\mu]\right]$$

$$-\; \frac{(n+m)(n+m-1)(n+m-2)^2(n-m-2)}{(n-m+1)(2n-1)(2n-3)^2(2n-5)}\, \mu^4\, \widehat{U}_{n-3,m}[\mu]$$

b) For each $n > 0$, $m = 0, 1, ..., n-1$,

$$\widehat{U}_{n,m+1}[\mu] = \frac{1}{\rho}\left[-(n+m+1)\, x_0\, \widehat{U}_{n,m}[\mu] \right.$$
$$\left. +\; \frac{(n+m+1)(n+m)^2}{(2n-1)(2n+1)}\, \mu^2\, \widehat{U}_{n-1,m}[\mu] + (n-m+1)\, \widehat{U}_{n+1,m}[\mu]\right]$$

Proof. For clarity throughout this proof we denote the functions $\widehat{U}_{n,m}[\mu]$ simply as $\widehat{U}_{n,m}$ and the functions $\phi_{n,m}(s,t)$ as $\phi_{n,m}$. By virtue of Proposition

B.0.2, it follows that

$$\begin{aligned}\phi_{n+1,m} = \frac{1}{\alpha_{n+1,m}} \Bigg[&\left(\frac{n-m+1}{n+m}\right) x_0\, \widehat{U}_{n,m} \\ &+ \frac{\mu^2\,(n+m)(n-m)(n-m+1)}{(2n+1)^2(2n-1)}\,\widehat{U}_{n-1,m} \\ &- \frac{(n-m+1)^2}{(n+m)(2n+1)}\,\widehat{U}_{n+1,m}\Bigg]. \end{aligned} \tag{B.5}$$

Thus,

$$\begin{aligned}\frac{(n+m-1)(n+m-2)}{(n-m)(n-m-1)}\,\mu^2\,\phi_{n-1,m} = \frac{\mu^2}{\alpha_{n-1,m}} \Bigg[&\left(\frac{n+m-1}{n-m}\right) x_0\, \widehat{U}_{n-2,m} \\ &+ \frac{\mu^2\,(n+m-2)^2(n+m-1)(n-m-2)}{(n-m)(2n-3)^2(2n-5)}\,\widehat{U}_{n-3,m} \\ &- \frac{(n+m-1)(n-m-1)}{(n-m)(2n-3)}\,\widehat{U}_{n-1,m}\Bigg]. \end{aligned}$$

Also, by equation (B.3), we have that

$$\begin{aligned}\phi_{n+1,m} = \frac{1}{\alpha_{n-1,m}} \Bigg[&\frac{(2n+1)(2n-1)}{(n+m)(n-m)} x_0 \widehat{U}_{n,m} + \mu^2\left(\frac{n+m}{2n+1}\right)\widehat{U}_{n-1,m} \\ &- \frac{(2n-1)(n-m+1)}{(n+m)(n-m)}\,\widehat{U}_{n+1,m}\Bigg]. \end{aligned}$$

So, by (B.1), we obtain

$$\begin{aligned}\phi_{n+1,m} &= \frac{(|x|^2+\mu^2)}{\alpha_{n,m}}\widehat{U}_{n-1,m} - 2\left(\frac{n+m-1}{(n-m)\alpha_{n-1,m}}\right) x_0\mu^2\widehat{U}_{n-2,m} \\ &\quad + \frac{\mu^2}{\alpha_{n-1,m}}\Bigg[\left(\frac{n+m-1}{n-m}\right)x_0\widehat{U}_{n-2,m} - \frac{(n+m-1)(n-m-1)}{(n-m)(2n-3)}\widehat{U}_{n-1,m} \\ &\quad + \frac{\mu^2(n+m-2)^2(n+m-1)(n-m-2)}{(n-m)(2n-3)^2(2n-5)}\widehat{U}_{n-3,m}\Bigg].\end{aligned}$$

Again, by equation (B.3), it follows that

$$\begin{aligned}\frac{(2n-1)(n-m+1)}{(n+m)(n-m)}\,\widehat{U}_{n+1,m} &= \frac{(2n+1)(2n-1)}{(n+m)(n-m)}\,x_0\,\widehat{U}_{n,m} \\ &\quad + \mu^2\left(\frac{n+m}{2n+1}\right)\widehat{U}_{n-1,m} - \left(\frac{2n-1}{n-m}\right)|x|^2\widehat{U}_{n-1,m} \\ &\quad - \left(\frac{2n-1}{n-m}\right)\mu^2\widehat{U}_{n-1,m} + \mu^2\left(\frac{n+m-1}{n-m}\right)x_0\widehat{U}_{n-2,m} \\ &\quad - \frac{(n+m-2)^2(n+m-1)(n-m-2)}{(2n-3)^2(2n-5)(n-m)}\,\mu^4\,\widehat{U}_{n-3,m} \\ &\quad + \frac{(n+m-1)(n-m-1)}{(n-m)(2n-3)}\,\mu^2\,\widehat{U}_{n-1,m}.\end{aligned}$$

Thus, by associating terms and reducing expressions, a) follows.

Furthermore, by applying equations (B.5) and (B.3), it is obtained that

$$\alpha_{n,m}(n+m)\phi_{n+1,m} = (2n+1)\, x_0\, \widehat{U}_{n,m} + \mu^2 \frac{(n+m)^2\,(n-m)}{(2n+1)(2n-1)}\, \widehat{U}_{n-1,m} \;-\; (n-m+1)\, \widehat{U}_{n+1,m}.$$

Then, by virtue of Proposition B.0.3 a), we conclude that

$$\widehat{U}_{n,m+1} = \frac{1}{\rho}\Bigg((n-m)\, x_0\, \widehat{U}_{n,m} \;+\; \frac{(n+m)^2}{2n-1}\, \mu^2\, \widehat{U}_{n-1,m}$$

$$+\, (n-m+1)\, \widehat{U}_{n+1,m} \;-\; (2n+1)\, x_0\, \widehat{U}_{n,m} - \frac{(n+m)^2(n-m)}{(2n+1)(2n-1)}\, \mu^2\, \widehat{U}_{n-1,m}\Bigg)$$

$$= \frac{1}{\rho}\Bigg(-(n+m+1)\, x_0\, \widehat{U}_{n,m}$$

$$+\; \frac{(n+m+1)(n+m)^2}{(2n-1)(2n+1)}\, \mu^2\, \widehat{U}_{n-1,m} \;+\; (n-m+1)\, \widehat{U}_{n+1,m}\Bigg)$$

hence b) is demonstrated.

Finally, note that

$$\alpha_{n,n}(2n-1)!! = 1.$$

Therefore, for each $n \geq 0$, by equations (1.24) and (1.29), it follows that

$$\widehat{U}_{n,n} = \alpha_{n,n}\mu^n P_n^n(s)\, P_n^n(t) = (-1)^n \mu^n (2n-1)!!\,(1-s^2)^{\frac{n}{2}}\,(t^2-1)^{\frac{n}{2}}$$

$$= (-1)^n (2n-1)!!\,\rho^n = (-1)^n (2n-1)!!\,\left(x_1^2 + x_2^2\right)^{\frac{n}{2}}$$

$$= (-1)^n (2n-1)!!\,|x|^n \left(1 - \left(\frac{x_0}{|x|}\right)^2\right)^{\frac{n}{2}}.$$

Thus, by virtue of (1.24), for each $n \geq 0$,

$$\widehat{U}_{n,n}[\mu] = |x|^n P_n^n\left(\frac{x_0}{|x|}\right).$$

□

Corollary B.0.5. *For each $n > 0$,*

a) When $m = 0, 1, ..., n$,

$$U^{\pm}_{n+1,m}[\mu] = \frac{1}{n-m+1}\Bigg((2n+1)\,x_0\,U^{\pm}_{n,m}[\mu] - (n+m)\,|x|^2\,U^{\pm}_{n-1,m}[\mu]\Bigg] + \mu^2\left(\frac{(n+m-1)(n+m)}{(n-m+1)(2n-1)}x_0\,U^{\pm}_{n-2,m}[\mu] - \frac{2(n+m)(n(n-1)+m^2-1)}{(n-m+1)(2n-3)(2n+1)}\,U^{\pm}_{n-1,m}[\mu]\right) - \frac{\mu^4\,(n+m-2)^2(n+m-1)(n+m)(n-m-2)}{(n-m+1)(2n-1)(2n-3)^2(2n-5)}\,U^{\pm}_{n-3,m}[\mu]. \tag{B.6}$$

Moreover, for all $\mu \in \mathbb{R}$

$$U^{\pm}_{n,n}[\mu] \,=\, U^{\pm}_{n,n}[0] \tag{B.7}$$

b)

$$\begin{aligned} U^{\pm}_{n,m+1}[\mu] = \frac{1}{x_1^2 \,+\, x_2^2} \bigg(& (n+m+1)\, x_0 \, \left(x_1 \, U^{\pm}_{n,m}[\mu] \,\mp\, x_2 \, U^{\mp}_{n,m}[\mu] \right) \\ & - \frac{(n+m)^2 (n+m+1)}{(2n+1)(2n-1)} \, \mu^2 \, \left(x_1 \, U^{\pm}_{n-1,m}[\mu] \,\mp\, x_2 \, U^{\mp}_{n-1,m}[\mu] \right) \\ & - (n-m+1) \, \left(x_1 \, U^{\pm}_{n+1,m}[\mu] \,\mp\, x_2 \, U^{\mp}_{n+1,m}[\mu] \right) \bigg). \end{aligned} \tag{B.8}$$

Proof. Equations (B.6) and (B.7) are a direct consequence of subparagraphs a) and c), respectively , of Proposition B.0.4. On the other hand, by applying Proposition B.0.4, subparagraph b). The formula for sines and cosines of the sum of angles and recalling that

$$\cos\varphi \,=\, \frac{x_1}{\sqrt{x_1^2 \,+\, x_2^2}} \qquad \text{y} \qquad \sin\varphi \,=\, \frac{x_2}{\sqrt{x_1^2 \,+\, x_2^2}},$$

formula (B.8) is obtained. □

Proposition B.0.6. *For each* $n \geq 0$*,* $m = 0, 1, ..., n$*,*

$$\widehat{U}_{n,m}[\mu] \,=\, \sum_{0 \leq 2k \leq n-m} u_{n,m,k} \, \mu^{2k} \, \widehat{U}_{n-2k,m}[0], \tag{B.9}$$

Proof. First, note that $u_{n,m,0} \,=\, 1$ for every $n \,\geq\, 0$ and $m \,\in\, \{0, 1, ..., n\}$.

Also, observe that

$$u_{n+1,m,k} = \frac{(n+m+1)\,(2n-2k+1)}{(2n+1)(n+m-2k+1)}\,u_{n,m,k}, \tag{B.10}$$

$$u_{n,m+1,k} = \frac{n+m+1}{n+m-2k+1}\,u_{n,m,k}, \tag{B.11}$$

and

$$u_{n,m,k+1} = \frac{-(n+m-2k-1)\,(n+m-2k)}{2(k+1)(2n-2k-1)}\,u_{n,m,k}. \tag{B.12}$$

Recall when μ is considered constant, we denote by $\widehat{U}_{n,m}$ to the functions $\widehat{U}_{n,m}[\mu]$.

Consider $m = 0$. Note that

$$\widehat{U}_{0,0} = 1 = \sum_{0\le 2k\le 0} u_{0,0,k}\mu^{2k}\widehat{U}_{0-2k,0}[0]$$

and

$$\widehat{U}_{1,0} = x_0 = \widehat{U}_{1,0}[0] = \sum_{0\le 2k\le 1} u_{1,0,k}\mu^{2k}\widehat{U}_{1-2k,0}[0],$$

so, we have the inductive basis. Now, suppose that the result is true for $0 \le l \le n$, i.e

$$\widehat{U}_{l,0} = \sum_{0\le 2k\le l} u_{l,0,k}\,\mu^{2k}\,\widehat{U}_{l-2k,0}[0], \qquad 0 \le l \le n.$$

By Proposition B.0.4 subparagraph a), it follows that

$$\widehat{U}_{n+1,0} = \frac{1}{n+1} \left((2n+1)\, x_0\, \widehat{U}_{n,0} \,-\, n\,|x|^2\, \widehat{U}_{n-1,0}\right)$$

$$+ \mu^2 \left(\frac{(n-1)n}{(n+1)(2n-1)} x_0\, \widehat{U}_{n-2,0} \,-\, \frac{2n(n(n-1)-1)}{(n+1)(2n-3)(2n+1)}\, \widehat{U}_{n-1,0}\right)$$

$$- \frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)}\, \mu^4\, \widehat{U}_{n-3,0}.$$

Then, by induction hypothesis, it follows that

$$\begin{aligned}
\widehat{U}_{n+1,0} = \frac{1}{n+1} \Bigg(&(2n+1)x_0 \sum_{0\le 2k\le n} u_{n,0,k}\mu^{2k}\widehat{U}_{n-2k,0}[0] \\
&- n|x|^2 \sum_{0\le 2k\le n-1} u_{n-1,0,k}\mu^{2k}\widehat{U}_{n-2k-1,0}[0]\Bigg) \\
&+ \mu^2 \Bigg(\frac{(n-1)n}{(n+1)(2n-1)} x_0 \sum_{0\le 2k\le n-2} u_{n-2,0,k}\mu^{2k}\widehat{U}_{n-2k-2,0}[0] \\
&- \frac{2n(n(n-1)-1)}{(n+1)(2n-3)(2n+1)} \sum_{0\le 2k\le n-1} u_{n-1,0,k}\mu^{2k}\widehat{U}_{n-2k-1,0}[0]\Bigg) \\
&- \frac{\mu^4 n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)} \sum_{0\le 2k\le n-3} u_{n-3,0,k}\mu^{2k}\widehat{U}_{n-2k-3,0}[0].
\end{aligned}$$

Thus, by performing algebraic operations and rearranging indices, we obtain

$$
\begin{aligned}
\widehat{U}_{n+1,0} = \frac{1}{n+1} & \left((2n+1)\, x_0 \sum_{0\le 2k\le n} u_{n,0,k}\, \mu^{2k}\, \widehat{U}_{n-2k,0}[0] \right. \\
& \left. - \, n\, |x|^2 \sum_{0\le 2k\le n-1} u_{n-1,0,k}\, \mu^{2k}\, \widehat{U}_{n-2k-1,0}[0] \right) \\
& + \frac{(n-1)n}{(n+1)(2n-1)} x_0 \sum_{2\le 2k\le n} u_{n-2,0,k-1}\, \mu^{2k}\, \widehat{U}_{n-2k,0}[0] \\
& - \frac{2n(n(n-1)-1)}{(n+1)(2n-3)(2n+1)} \sum_{2\le 2k\le n+1} u_{n-1,0,k-1}\, \mu^{2k}\, \widehat{U}_{n-2k+1,0}[0] \\
& - \frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)} \sum_{4\le 2k\le n+1} u_{n-3,0,k-2}\, \mu^{2k}\, \widehat{U}_{n-2k+1,0}[0].
\end{aligned}
$$

Now, by the equations (B.10), (B.11) and (B.12),

$$
\begin{aligned}
u_{n-3,0,k-2} &= \frac{2k(2k-2)(2n-1)(2n-3)(2n-5)}{n(n-1)(n-2)(n-2k+1)(2n-2k-1)}\, u_{n,0,k}, \\
u_{n-1,0,k-1} &= -\frac{2k(2n-1)}{n(n-2k+1)}\, u_{n,0,k}, \\
u_{n-1,0,k} &= \frac{(n-2k)(2n-1)}{n(2n-2k-1)}\, u_{n,0,k}
\end{aligned}
$$

and

$$
u_{n-2,0,k-1} = -\frac{2k(2n-1)(2n-3)}{n(n-1)(2n-2k-1)}\, u_{n,0,k}.
$$

Therefore,

$$\begin{aligned}
\widehat{U}_{n+1,0} &= \Psi(n) + \frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n,0}[0] - \frac{n}{n+1}\, |x|^2\, \widehat{U}_{n-1,0}[0] \\
&\quad - \left[\frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n-2,0}[0] - \frac{(2n-1)(n-2)}{(n+1)(2n-3)}\, |x|^2\, \widehat{U}_{n-3,0}[0] \right. \\
&\quad \left. - \frac{2}{(n+1)}\, x_0\, \widehat{U}_{n-2,0}[0] + \frac{4(n(n-1)-1)(2n-1)}{(n+1)(n-1)(2n-3)(2n+1)}\, \widehat{U}_{n-1,0}[0]\right] c_1^{n,0}\, \mu^2,
\end{aligned}$$

where

$$\begin{aligned}
\Psi(n) = \sum_{4\le 2k\le n-1} u_{n,0,k}\, \mu^{2k} & \left[\frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n-2k,0}[0]\right. \\
& - \frac{(2n-1)(n-2k)}{(n+1)(2n-2k-1)}\, |x|^2\, \widehat{U}_{n-2k-1,0}[0] \\
& - \frac{2k(2n-3)}{(n+1)(2n-2k-1)}\, x_0\, \widehat{U}_{n-2k,0}[0] \\
& + \frac{4k(n(n-1)-1)(2n-1)}{(2n+1)(2n-3)(n+1)(n-2k+1)}\, \widehat{U}_{n-2k+1,0}[0] \\
& \left. - \frac{2k(2k-2)(n-2)^2}{(n-2k+1)(2n-2k-1)(n+1)(2n-3)}\, \widehat{U}_{n-2k+1,0}[0]\right] \\
& + \delta_{\frac{n}{2},\lfloor\frac{n}{2}\rfloor}\, \frac{2n-1}{(n-1)(n+1)}\, u_{n,0,\lfloor\frac{n}{2}\rfloor}\, \mu^{2\lfloor\frac{n}{2}\rfloor}\, x_0\, \widehat{U}_{n-2\lfloor\frac{n}{2}\rfloor,0}[0] \\
& - \frac{2n(n(n-1)-1)}{(2n+1)(2n-3)(n+1)}\, u_{n-1,0,\lfloor\frac{n-1}{2}\rfloor}\, \mu^{2\lfloor\frac{n-1}{2}\rfloor+1}\, \widehat{U}_{n-2\lfloor\frac{n-1}{2}\rfloor-1,0}[0] \\
& - \frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)}\, u_{n-3,0,\lfloor\frac{n-3}{2}\rfloor}\, \mu^{2\lfloor\frac{n-3}{2}\rfloor+2}\, \widehat{U}_{n-2\lfloor\frac{n-3}{2}\rfloor-3,0}[0],
\end{aligned}$$

with

$$\delta_{\frac{n}{2},\lfloor\frac{n}{2}\rfloor} = \begin{cases} 1 & \text{if } \frac{n}{2} = \lfloor\frac{n}{2}\rfloor \\ \\ 0 & \text{if } \frac{n}{2} \neq \lfloor\frac{n}{2}\rfloor . \end{cases}$$

Now, by virtue of (1.40), it follows that

$$\widehat{U}_{n+1,0}[0] = \frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n,0}[0] - \frac{n}{n+1}\, |x|^2\, \widehat{U}_{n-1,0}[0].$$

On the other hand, applying again the equation (1.40), we obtain that

$$\frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n-2,0}[0] - \frac{(2n-1)(n-2)}{(n+1)(2n-3)}\, |x|^2\, \widehat{U}_{n-3,0}[0]$$

$$-\frac{2}{(n+1)}\, x_0\, \widehat{U}_{n-2,0}[0] = \frac{(2n-1)(n-1)}{(n+1)(2n-3)}\, \widehat{U}_{n-1,0}[0].$$

So,

$$\left(\frac{4(n(n-1)-1)(2n-1)}{(n+1)(n-1)(2n-3)(2n+1)} + \frac{(2n-1)(n-1)}{(n+1)(2n-3)}\right) \widehat{U}_{n-1,0}[0]$$

$$= \frac{(n+1)(2n-1)}{(n-1)(2n+1)}\, \widehat{U}_{n-1,0}[0].$$

Then, by the equation (B.10), it follows that

$$\frac{(n+1)(2n-1)}{(n-1)(2n+1)}\, u_{n,0,1}\, \mu^2\, \widehat{U}_{n-1,0}[0] = u_{n+1,0,1}\, \mu^2\, \widehat{U}_{n-1,0}[0].$$

Also

$$
\begin{aligned}
&\left(\frac{2n+1}{n+1}\right) x_0\, \widehat{U}_{n-2k,0}[0] - \left(\frac{(2n-1)(n-2k)}{(n+1)(2n-2k-1)}\right) |x|^2\, \widehat{U}_{n-2k-1,0}[0] \\
&- \left(\frac{2k(2n-3)}{(n+1)(2n-2k-1)}\right) x_0\, \widehat{U}_{n-2k,0}[0] \\
&= \frac{2n-1}{(n+1)(2n-2k-1)} \Bigg((2n-4k+1) x_0\, \widehat{U}_{n-2k,0}[0] \\
&- (n-2k)\, |x|^2\, \widehat{U}_{n-2k-1,0}[0] \Bigg(.
\end{aligned}
$$

Hence, by applying the equation (B.10), it results that

$$
\begin{aligned}
&\left(\frac{2n+1}{n+1}\right) x_0\, \widehat{U}_{n-2k,0}[0] - \left(\frac{(2n-1)(n-2k)}{(n+1)(2n-2k-1)}\right) |x|^2\, \widehat{U}_{n-2k-1,0}[0] \\
&- \left(\frac{2k(2n-3)}{(n+1)(2n-2k-1)}\right) x_0\, \widehat{U}_{n-2k,0}[0] \\
&= \frac{(2n-1)(n-2k+1)}{(n+1)(2n-2k-1)}\, \widehat{U}_{n-2k+1,0}[0].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left(\frac{2n+1}{n+1}\, x_0\, \widehat{U}_{n-2k,0}[0] - \frac{(2n-1)(n-2k)}{(n+1)(2n-2k-1)}\, |x|^2\, \widehat{U}_{n-2k-1,0}[0]\right.\\
&\qquad - \frac{2k(2k-2)(n-2)^2}{(n-2k+1)(2n-2k-1)(n+1)(2n-3)}\, \widehat{U}_{n-2k+1,0}[0]\\
&\qquad + \frac{4k(n(n-1)-1)(2n-1)}{(n-2k+1)(2n+1)(2n-3)(n+1)}\, \widehat{U}_{n-2k+1,0}[0]\\
&\qquad \left. - \frac{2k(2n-3)}{(2n-2k-1)(n+1)} x_0\, \widehat{U}_{n-2k,0}[0]\right)\, u_{n,0,k}\, \mu^{2k}\\
&= \frac{(n+1)(2n-2k+1)}{(2n+1)(n-2k+1)}\, u_{n,0,k}\, \mu^{2k}\, \widehat{U}_{n-2k+1,0}[0]\\
&= u_{n+1,0,k}\, \mu^{2k}\, \widehat{U}_{n-2k+1,0}[0],
\end{aligned}
$$

so

$$
\begin{aligned}
\Psi(n) = &\sum_{4\le 2k\le n-1} u_{n+1,0,k}\, \mu^{2k}\, \widehat{U}_{n-2k+1,0}[0]\\
&+ \delta_{\frac{n}{2},\lfloor\frac{n}{2}\rfloor}\, \frac{2n-1}{(n-1)(n+1)}\, u_{n,0,\lfloor\frac{n}{2}\rfloor}\, \mu^{2\lfloor\frac{n}{2}\rfloor}\, x_0\, \widehat{U}_{n-2\lfloor\frac{n}{2}\rfloor,0}[0]\\
&- \frac{2n(n(n-1)-1)}{(2n+1)(2n-3)(n+1)}\, u_{n-1,0,\lfloor\frac{n-1}{2}\rfloor}\, \mu^{2\lfloor\frac{n-1}{2}\rfloor+1}\, \widehat{U}_{n-2\lfloor\frac{n-1}{2}\rfloor-1,0}[0]\\
&- \frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)}\, u_{n-3,0,\lfloor\frac{n-3}{2}\rfloor}\, \mu^{2\lfloor\frac{n-3}{2}\rfloor+2}\, \widehat{U}_{n-2\lfloor\frac{n-3}{2}\rfloor-3,0}[0].
\end{aligned}
$$

Now, let

$$\Lambda\,(n) = \delta_{\frac{n}{2},\lfloor\frac{n}{2}\rfloor}\,\frac{2n-1}{(n-1)(n+1)}\,u_{n,0,\lfloor\frac{n}{2}\rfloor}\,\mu^{2\lfloor\frac{n}{2}\rfloor}\,x_0\,\widehat{U}_{n-2\lfloor\frac{n}{2}\rfloor,0}[0]$$

$$-\,\frac{2n(n(n-1)-1)}{(2n+1)(2n-3)(n+1)}\,u_{n-1,0,\lfloor\frac{n-1}{2}\rfloor}\,\mu^{2\lfloor\frac{n-1}{2}\rfloor+1}\,\widehat{U}_{n-2\lfloor\frac{n-1}{2}\rfloor-1,0}[0]$$

$$-\,\frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)}\,u_{n-3,0,\lfloor\frac{n-3}{2}\rfloor}\,\mu^{2\lfloor\frac{n-3}{2}\rfloor+2}\,\widehat{U}_{n-2\lfloor\frac{n-3}{2}\rfloor-3,0}[0].$$

If n is even, by (1.41), it follows that

$$\begin{aligned}\Lambda\,(n) &= \frac{(n+1)^2}{2n+1}\,u_{n,0,\lfloor\frac{n}{2}\rfloor}\mu^{2\lfloor\frac{n+1}{2}\rfloor}\,\widehat{U}_{n-2\lfloor\frac{n+1}{2}\rfloor+1,0}[0]\\ &= u_{n+1,0,\lfloor\frac{n+1}{2}\rfloor}\mu^{2\lfloor\frac{n+1}{2}\rfloor}\,\widehat{U}_{n+1-2\lfloor\frac{n+1}{2}\rfloor,0}[0].\end{aligned}$$

Finally, if n is odd,

$$\begin{aligned}\Lambda\,(n) &= \Bigg(-\frac{2n(n(n-1)-1)}{(2n+1)(2n-3)(n+1)}\,u_{n-1,0,\lfloor\frac{n}{2}\rfloor}\\ &\quad -\,\frac{n(n-2)^3(n-1)}{(n+1)(2n-1)(2n-3)^2(2n-5)}\,u_{n-3,0,\lfloor\frac{n}{2}\rfloor-1}\Bigg)\\ &\quad\times\mu^{2\lfloor\frac{n-1}{2}\rfloor+1}\,\widehat{U}_{n-2\lfloor\frac{n-1}{2}\rfloor-1,0}[0]\\ &= \frac{n^2}{(2n-1)(2n+1)}\,u_{n-1,0,\lfloor\frac{n}{2}\rfloor}\,\mu^{2\lfloor\frac{n+1}{2}\rfloor}\,\widehat{U}_{n-2\lfloor\frac{n+1}{2}\rfloor+1,0}[0]\\ &= \frac{2}{(n+1)(n+2)}\,u_{n+1,0,\lfloor\frac{n}{2}\rfloor}\,\mu^{2\lfloor\frac{n+1}{2}\rfloor}\,\widehat{U}_{n-2\lfloor\frac{n+1}{2}\rfloor+1,0}[0]\\ &= u_{n+1,0,\lfloor\frac{n+1}{2}\rfloor}\,\mu^{2\lfloor\frac{n+1}{2}\rfloor}\,\widehat{U}_{n-2\lfloor\frac{n+1}{2}\rfloor+1,0}[0]\end{aligned}$$

thus

$$\widehat{U}_{n+1,0} \;=\; \sum_{0\le 2k\le n+1} u_{n+1,0,k}\,\mu^{2k}\,\widehat{U}_{n-2k+1,0}[0]$$

Furthermore, suppose that

$$\widehat{U}_{n,j} \;=\; \sum_{0\le 2k\le n-j} u_{n,j,k}\,\mu^{2k}\,\widehat{U}_{n-2k,j}[0] \qquad 0 \;\le\; j \;\le\; m.$$

So, by Proposition B.0.4, subparagraph b) and the induction hypothesis, we have

$$\begin{aligned}
\widehat{U}_{n,m+1} = -\frac{1}{\rho}\Bigg(&(n+m+1)\,x_0 \sum_{0\le 2k\le n-m} u_{n,m,k}\,\mu^{2k}\,\widehat{U}_{n-2k,m}[0]\\
&- \frac{(n+m+1)(n+m)^2}{(2n-1)(2n+1)}\,\mu^2 \sum_{0\le 2k\le n-m-1} u_{n-1,m,k}\,\mu^{2k}\,\widehat{U}_{n-2k-1,m}[0]\\
&- (n-m+1) \sum_{0\le 2k\le n-m+1} u_{n+1,m,k}\,\mu^{2k}\,\widehat{U}_{n-2k+1,m}[0]\Bigg)\\
= -\frac{1}{\rho}\Bigg(&(n+m+1)\,x_0\,\widehat{U}_{n,m}[0] \,-\, (n-m+1)\,\widehat{U}_{n+1,m}[0]\\
&+ \sum_{2\le 2k\le n-m+1} \Big((n+m+1)\,x_0\,u_{n,m,k}\,\mu^{2k}\,\widehat{U}_{n-2k,m}[0]\\
&\qquad - \frac{(n+m+1)(n+m)^2}{(2n-1)(2n+1)}\,u_{n-1,m,k-1}\,\mu^{2k}\,\widehat{U}_{n-2k+1,m}[0]\\
&\qquad - (n-m+1)\,u_{n+1,m,k}\,\mu^{2k}\,\widehat{U}_{n-2k+1,m}[0]\Big)\Bigg).
\end{aligned}$$

Now, by (1.42), it follows that

$$(n+m+1)\,x_0\,\widehat{U}_{n,m}[0] \,-\, (n-m+1)\,\widehat{U}_{n+1,m}[0] \;=\; -\rho\,\widehat{U}_{n,m+1}[0].$$

In addition, by (B.10),(B.11) and (B.12), we have that

$$(n+m+1)\,u_{n,m,k} = (n+m-2k+1)\,u_{n,m+1,k},$$

$$\frac{(n+m+1)(n+m)^2}{(2n-1)(2n+1)}\,u_{n-1,m,k-1} = -\frac{2k(n+m)}{2n+1}\,u_{n,m+1,k},$$

$$(n-m+1)\,u_{n+1,m,k} = \frac{(n-m+1)(2n-2k+1)}{2n+1}\,u_{n,m+1,k}.$$

Thus,

$$\begin{aligned}
\widehat{U}_{n,m+1} &= -\frac{1}{\rho}\Big(-\rho\widehat{U}_{n,m+1}[0] \\
&+ \sum_{2\leq 2k\leq n-m+1} u_{n,m+1,k}\mu^{2k}\big((n+m-2k+1)x_0\widehat{U}_{n-2k,m}[0] \\
&+ \frac{2k(n+m)}{2n+1}\widehat{U}_{n-2k+1,m}[0] - \frac{(n-m+1)(2n-2k+1)}{2n+1}\widehat{U}_{n-2k+1,m}[0]\big)\Big) \\
&= -\frac{1}{\rho}[-\rho\widehat{U}_{n,m+1}[0] \\
&+ \sum_{2\leq 2k\leq n-m+1} u_{n,m+1,k}\mu^{2k}(n+m-2k+1)\{x_0\widehat{U}_{n-2k,m}[0] - \widehat{U}_{n-2k+1,m}[0]\}].
\end{aligned}$$

Then, by (1.40), we have that

$$\begin{aligned}
x_0\widehat{U}_{n-2k,m}[0] - \widehat{U}_{n-2k+1,m}[0] &= -(n-m-2k)x_0\widehat{U}_{n-2k,m}[0] \\
&+ (n+m-2k)|x|^2\widehat{U}_{n-2k-1,m}[0].
\end{aligned}$$

Finally, applying (1.42), we conclude that

$$\widehat{U}_{n,m+1} \;=\; \sum_{0\le 2\le n-m+1} u_{n,m+1,k}\,\mu^{2k}\,\widehat{U}_{n-2k,m+1}[0].$$

Observe that

$$\left\lfloor \frac{n-m+1}{2} \right\rfloor \;=\; \lfloor \frac{n-m-1}{2} \rfloor \;+\; 1$$

and

$$\widehat{U}_{n-2\left(\lfloor \frac{n-m-1}{2} \rfloor + 1\right),m+1}[0] \;=\; 0,$$

for all $n \,\ge\, 0,\;\; 0 \,\le\, m \,\le\, n.$ □

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