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A quaternionic transmutation operator and complete systems for solutions of perturbed Moisil-Teodorescu operators

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Un operador de transmutación cuaterniónica y sistemas completos para soluciones de operadores de Moisil-Teodorescu perturbados

Tesis que presenta

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Abstract

We consider functions of a vector variable in a domain in 3-dimensional space, taking values in the space of complex quaternions. Let D be the Moisil-Teodorescu differential operator and f a separable nonvanishing function.

The first contribution of this thesis is the construction of an invertible quaternionic operator which transforms solutions of the operator $D + M^{Df/f}$ into solutions of the operator D for bounded domains with certain symmetry. This permits giving a complete solution to equations of the form $Du + \lambda u + u\gamma = 0$ with certain restrictions on the coefficients λ and γ. The solutions are represented locally by a new type of Taylor series adapted to the equation under consideration.

The second contribution is the application of these results to exhibit complete solution sets for various physical systems, including Beltrami fields, some important cases of the Maxwell equations, the Helmholtz equation and the free Dirac equation for particles with mass, among others.

Resumen

Consideramos funciones de una variable vectorial en un dominio en el espacio tridimensional, que toman valores en el espacio de los cuaternios complejos. Sea D el operador diferencial de Moisil-Teodorescu y f es una función separable que no se anula.

La primera contribución de esta tesis es la construcción de un operador cuaternionico invertible, que transforma las soluciones del operador $D + M^{Df/f}$ en soluciones del operador D para dominios acotados con cierta simetría. Esto permite dar una solución completa a ecuaciones de la forma $Du + \lambda u + u\gamma = 0$ con ciertas restricciones sobre los coeficientes λ y γ. Las soluciones se representan localmente por un nuevo tipo de serie de Taylor adaptada a la ecuación bajo consideración.

La segunda contribución es la aplicación de estos resultados para exhibir conjuntos completos de soluciones a diversos sistemas físicos, incluyendo campos de Beltrami, ciertos casos importantes de las ecuaciones de Maxwell, la ecuación de Helmholtz y la ecuación de Dirac libre para partículas con masa.

Dedication

To my parents: Cruz Moreira and Angelina Galván.

To my family: Analy Chairez, Leonardo Moreira and Santiago Moreira.

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Contents

Chapter 1

Introduction

The fundamental equations of mathematical physics, such as the Laplace and Helmholtz equations, Maxwell's equations, and others, are second-order differential equations or systems of first-order equations intimately related to equations of second order, in particular to the Schrödinger equation. In many cases the treatment can be simplified by expressing the equations in terms of quaternionic differential operators such as the Moisil-Teodorescu operator

$$D = \sum_{j=1}^{3} \mathbf{e}_j \partial_j,$$

where $\mathbf{e}_j$ denotes the quaternionic unities for $j \in \{1, 2, 3\}$. For example, the Helmholtz operator admits the factorization [5]

$$\Delta + \lambda^2 = -(D + \lambda)(D - \lambda). \tag{A}$$

Here Δ is the Laplacian operator and λ is a complex number. The situation is analogous to the factorization $x^2 + 1 = (x + \mathbf{i})(x - \mathbf{i})$ which cannot be carried out in the field of real numbers.

In this thesis we find a complete set (in various senses) of complex-quaternion valued solutions u of the equation

$$(D + \lambda + M^{\gamma})u = 0 \tag{B}$$

where u is defined in a suitable domain in Euclidean space $\mathbb{R}^3$, and λ, γ are functions of a single spatial variable. In this formula λ refers to the operator of multiplication from the left, while M^{γ} indicates multiplication from the right. The term complete refers to the property that every solution of the equation may be approximated in an appropriate way by some linear combination of the set of functions under consideration.

When $\lambda = 0$ and $\gamma = 0$, we have the simplest case, functions of three space variables which are annihilated by the operator D. In this thesis such functions will be called monogenic. Monogenic functions, first studied as functions of a quaternionic variable, satisfy many properties analogous to those of a complex variable, being subject to a very rich theory initiated by R. Fueter [23], G. C. Moisil [68], N. Teodorescu (also transliterated as Théodoresco) [78] and developed by R. Delanghe [14].

1. Quaternionic representation of differential equations.

We will apply the general solution of (B) to give explicit solutions to many of the equations of mathematical physics. This is possible because the analysis of complex quaternions provides a method for expressing relationships between important equations of physics such as the following.

(a) *Beltrami fields.* Beltrami fields are complex vector fields **F** which are eigenvector of the curl operator. Such fields appear in many branches of physics including astrophysics [1], electromagnetics [63] and plasma physics [72]. The case $\lambda = 0$ corresponds to potential field theory. For the case when the eigenvalue λ is a nonzero constant (Trkalian fields), see [3, 13]. The case when λ depends only on one variable is studied in [45, 65]. The field is quite vast; the survey book by Marsh [65] collects essentially all which had been discovered concerning the construction of solutions of up to the time of publication.

Our study of Beltrami fields applies a result from [45] due to V. V. Kravchenko, where it was shown that **F** is a Beltrami field if and only if the purely vectorial complex quaternionic function $\mathbf{G} = \sqrt{\lambda}\mathbf{F}$ is a solution of the equation of the type of (B).

(b) *Maxwell's equations.* The system of equations developed by J. Maxwell [66] describe the relationship between the electric and magnetic fields of some electromagnetic phenomena including light. For some particular cases (but of great importance), such as an electromagnetic field in a vacuum, these equations can be expressed via complex quaternionic operators of the form (B). V. V. Kravchenko and M. Shapiro investigated this relationship in [54, 55], giving many equivalent conditions for a pair of fields to be a solution of Maxwell's equations, including the question of boundary value problems. Similar results concerning Maxwell's equations in the context of Clifford algebras were obtained in [67] by A. Mcintosh and M. Mitrea. The bibliographies of these references lead to a great deal of research in this direction. In section 5.2 we apply our description of the kernel of the operator (B) to give a complete solution for the time-harmonic Maxwell's equations.

(c) *Dirac equation.* The classical Dirac operator for a free particle with a specified energy is given in equation (127). A first attempt to rewrite the Dirac equation into a complex quaternionic form was carried out by C. Lanczos in [60], and played an important role in the development of the use of Clifford algebras in physics. Further research in this direction can be found in [11, 43, 49]. In section 5.3 we give complete solutions to the complex-quaternionic forms of the Dirac equation in the context of scalar, electrical and pseudoscalar potentials.

(d) *Helmholtz equation.* K. Gürlebeck [32] studied the operator $D + \lambda$ for $\lambda \in \mathbb{R}$ and gave the factorization (A) of the Helmholtz operator. After that S. Bernstein and K. Gürlebeck [5] worked on the factorization of the Schrödinger operator which is based into appropriate perturbed Moisil-Teodorescu operators (i.e., nonconstant λ). Other research on these operators may be found in [7, 37, 54, 69, 81, 82]. By means of the factorization (A) we give a complete solution for the Helmholtz equation in section 5.4.

(e) *Vekua equation.* Vekua equations play an important role in mathematical physics because many partial differential equations can be transformed into this type of equations. The theory of generalized analytic functions by I. Vekua [79] and L. Bers [6] is used in areas like analysis, geometry and mechanics. This is because the theory of generalized analytic functions is in a position to use the advantages of complex analysis for solving more general systems of partial differential equations than is possible in the framework of classical complex analysis. V.V. Kravchenko found the importance of these type of functions because that are closely related to many important equations such as the Dirac, Maxwell, Klein-Gordon among others (see [46]).

There have been efforts to generalize the notion of pseudoanalytic function (see [46, 64, 76]) for higher dimensions. There are works which solve certain types of Vekua equations (see [9, 17, 52, 73]). V.V. Kravchenko and S. Tremblay in [58] found a way to relate the Vekua equation with vectorial solutions to operator of the type (B).

2. Transmutation operators.

The notion of a transmutation operator relating two linear differential operators was introduced in 1938 by J. Delsarte [18] and the idea was extended together with L. Lions [19]. A. Povzner in [71] proved that for some classes of differential operators a transmutation operator can be realized in the form of a Volterra integral operator (see also [36]). The idea of using transmutation operators for obtaining complete systems of solutions of partial differential equations with variable coefficients was studied and developed in numerous publications. Some examples are in the books [4, 12, 25]. Recently this idea was advanced further in [42, 51, 59], where it was shown that complete systems of solutions for equations with variable coefficients quite often can be obtained with the aid of a transmutation operator, even when a closed form of the transmutation operator is not available. It is sufficient to know how the transmutation operator acts on certain complete systems of elementary functions.

For example, in the present work we find the images of basic monogenic polynomials under the action of the transmutation operator relating monogenic function and the equation (B). Thus, in order to obtain a complete system of solutions of the equation (B) it turns out to be sufficient to choose a complete system of monogenic polynomials.

The complete system of monogenic polynomials which is used in this work was constructed in [30], and we call them Grigor'ev's polynomials.

We use as a starting point the transmutation operators defined on spaces of real- or complex-valued functions, as developed by V. V. Kravchenko and S. Torba, the results can be found in [48]. H. Campos, V. V. Kravchenko and L. Méndez in [9] applied such operators to functions taking values in hyperbolic numbers, applying the real operators to projections of the functions onto subspaces of the hyperbolic numbers. In this thesis we introduce an essentially quaternionic transmutation operator $\mathbf{T}$ defined on spaces of complex-quaternionic valued functions. We develop the basic theory of this operator and use it to solve equation (B) explicitly.

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In Chapter 1 we give some basic definitions regarding real quaternion and complex quaternions and their properties. Using the existence of zero divisors in the complex quaternions $\mathbb{H}(\mathbb{C})$, we define the projectors $\mathcal{P}^{\pm}$ as right multiplication by $(1 \pm \mathbf{i}\mathbf{e}_3)/2$, which allows us to simplify the operator $D + \lambda + M^{\gamma \mathbf{e}_3}$. We summarize many known properties of monogenic functions including the expansion in Taylor series in terms of the Grigor'ev polynomials.

In Chapter 2 the notion of transmutation operator between $\frac{d^2}{dx^2}$ and $\frac{d^2}{dx^2} - r(x)$ is defined, where $r(x)$ is a continuous complex-valued function of a real variable. Some of the properties of the transmutation operator are stated such as boundedness, invertibility and how it acts on the polynomials x^n. Also a fundamental relation between two transmutation operators T_f and $T_{1/f}$ due to [56] is stated, one of the main ideas behind this thesis. We introduce a complex quaternion operator $\mathbf{T}_f$ using $T_f, T_{1/f}$ and we prove some of its properties including invertibility, boundedness, and the transmutation of monogenic functions into solutions of the system $D + M^{Df/f}$.

In Chapter 3 we prove a quaternionic analogue of the complex Runge's theorem for the kernel $\mathrm{Ker}(D + M^{Df/f})$ in a bounded set $\Omega \subseteq \mathbb{R}^3$ with a certain type of symmetry necessary for the application of transmutation operators. We show that complex quaternionic function $u \in \mathrm{Ker}(D+M^{Df/f})$ can be approximated (in various senses) by means of images of polynomials under $\mathbf{T}_f$. In the case that Ω is a ball we prove a direct analogue of the complex case of Taylor's theorem; that is, that every solution u of $\mathrm{Ker}(D + M^{Df/f})$ can be expressed a sum of transmuted polynomials. Finally, we use the projectors $\mathcal{P}^{\pm}$ to reduce the solution of operator $D + \lambda + M^{\gamma \mathbf{e}_3}$ into a direct sum of two solutions of the operator $D + M^{(\lambda \pm \mathbf{i}\gamma)\mathbf{e}_3}$.

In Chapter 4, we use the techniques and results developed in the previous chapters to find complete systems of solutions to various physical systems such as the Dirac equation, the Helmholtz equation, the impedance electrical equation and the Vekua equation.

Chapter 2

Summary of results in complex quaternionic analysis

In this chapter we will summarize all the facts that we need concerning monogenic (holomorphic) functions of a quaternionic or biquaternionic variable.

Quaternionic analysis was initiated by W. Hamilton [38] and developed by various researchers including R. Fueter [23], who proposed the notion of *regular* function, often referred to as Fueter-regular or holomorphic [8, 30, 34] and showed many analogies between quaternionic functions and the theory of functions of a complex variable, for example Cauchy integral formula, Laurent expansion, among others. In particular the representation of quaternionic-analytic (monogenic) functions as power series will be essential in this thesis.

2.1 Complex quaternions

For our purposes we define quaternions as follows.

Definition 2.1. A *real quaternion* q is a formal real linear combination of the quaternionic basis elements 1 (also denoted by $\mathbf{e}_0$), $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$,

$$q = q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3, \quad q_j \in \mathbb{R}. \tag{1}$$

The sum of quaternions is the usual sum in $\mathbb{R}^4$, while for the multiplication we define

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3, \ \ \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1, \ \ \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2, \ \ \text{and } \mathbf{e}_j^2 = -1 \text{ for } j = 1, 2, 3, \tag{2}$$

and extend by linearity; that is, applying the associative and distributive laws. The set of $\mathbb{H} = \mathbb{H}(\mathbb{R})$ of real quaternions is isomorphic to $\mathbb{R}^4$ as a real vector space.

The *vector part* of q is $\vec{q} = q - q_0$ where q_0 in (1) is the *scalar part*. Using the rules of multiplication we have that the multiplication of two quaternions p, q can be expressed in terms of the the scalar product $\vec{p}\cdot\vec{q}$ and vector product $\vec{p} \times \vec{q}$ as

$$pq = (p_0q_0 - \vec{p} \cdot \vec{q}) + (p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q}). \tag{3}$$

A feature of quaternions is that the operation multiplication of two quaternions is noncommutative.

Definition 2.2. A *complex quaternion* [27, 33, 47] is defined by taking $q_j \in \mathbb{C}$ to be complex in (1).

Thus the space $\mathbb{H}(\mathbb{C})$ of complex quaternions can be identified with $\mathbb{C}^4$. In the context of $\mathbb{H}(\mathbb{C})$ elements of $\mathbb{C}$ are regarded as scalars. the sum of two complex quaternions and the multiplication by scalars coincide with the usual operation on vectors in $\mathbb{C}^4$. For clarity we list the algebraic rules:

$$\mathbf{e}_0 = 1,\ \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1,\ \ \mathbf{i}\mathbf{e}_j = \mathbf{e}_j\mathbf{i}\ \ j = 1, 2, 3,$$

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3,\ \ \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1,\ \ \mathbf{e}_3\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2,$$

and extended by linearity as in the case of real quaternions.

Another way to represent the complex quaternion $q = q_0 + \vec{q} \in \mathbb{H}(\mathbb{C})$ is

$$q = \operatorname{Re}_{\mathbb{H}} q + \mathbf{i} \operatorname{Im}_{\mathbb{H}} q \tag{4}$$

where $\operatorname{Re}_{\mathbb{H}} q, \operatorname{Im}_{\mathbb{H}} q \in \mathbb{H} = \mathbb{H}(\mathbb{R})$ are real quaternions called the real quaternionic part and imaginary quaternion part respectively. If $\operatorname{Sc} q = 0$ we say that $q = \vec{q}$ is a purely vectorial quaternion (sometimes also called a pure quaternion). The space of all purely vectorial quaternions is

$$\operatorname{Vec} \mathbb{H}(\mathbb{C}) = \{q \in \mathbb{H}(\mathbb{C}) \colon\ \operatorname{Sc} q = 0\} \tag{5}$$

which is a complex linear space.

Definition 2.3. The *quaternionic conjugate* of the quaternion q is

$$\overline{q} = q_0 - \vec{q}. \tag{6}$$

The *norm* of $q \in \mathbb{H}(\mathbb{C})$ is

$$|q| = \sqrt{|q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2}, \tag{7}$$

where $|q_j|$ is the usual complex norm. Thus $|q|$ is the usual norm in $\mathbb{C}^4$.

Quaternionic conjugation satisfies the properties $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$ and $|q|^2 = q\overline{q}$. In general we have $|pq| \neq |p||q|$ but we can estimate $|pq|$ as follows.

Lemma 2.4. *[47] Let $p, q \in \mathbb{H}(\mathbb{C})$. Then*

$$|pq| \leq \sqrt{2}|p||q| \tag{8}$$

The inequality (8) cannot be improved. As an example take $p = q = (1 + \mathbf{i}\mathbf{e}_1)/2$, then $\sqrt{2}|p||q| = \sqrt{2}/2$ and $|pq| = |p| = \sqrt{2}/2$.

A big difference between $\mathbb{H}$ and $\mathbb{H}(\mathbb{C})$ is the existence of zero divisors in $\mathbb{H}(\mathbb{C})$. As an example take $q = (1+\mathbf{i}\mathbf{e}_1)/2$ and $p = \overline{q}$. Then $pq = 0$ and both are different from the zero element.

Let us denote the set of all zero divisors in $\mathbb{H}(\mathbb{C})$ by $\mathcal{Z}$:

$$\mathcal{Z} = \{q \neq 0 \in \mathbb{H}(\mathbb{C}) |\ \exists p \neq 0 \in \mathbb{H}(\mathbb{C}) \colon pq = 0\}$$

Lemma 2.5. *[47] Let $q \neq 0 \in \mathbb{H}(\mathbb{C})$. The following statements are equivalent:*

1. *$q \in \mathcal{Z}$;*

2. *$q\overline{q} = 0$;*

3. *$q_0^2 = (\vec{q})^2$;*

4. *$q^2 = 2q_0 q = 2\vec{q}q$.*

Note that if $q \in \mathcal{Z}$ and $q_0 = \operatorname{Sc} q \neq 0$ then the number $q/(2q_0)$ satisfies $(q/(2q_0))^2 = q/(2q_0)$, so $q/(2q_0)$ is idempotent. Also if $q \notin \mathcal{Z}$ then the quaternion $q^{-1} = \overline{q}/|q|$ is the multiplicative inverse of q. We are going to use these special zero divisors

$$\frac{1+\mathbf{i}\mathbf{e}_3}{2},\ \frac{1-\mathbf{i}\mathbf{e}_3}{2}, \tag{9}$$

which are idempotent zero divisors in $\mathbb{H}(\mathbb{C})$ whose sum is 1, giving the following complementary projection operators.

Definition 2.6. The operators $\mathcal{P}^{\pm}\colon \mathbb{H}(\mathbb{C}) \to \mathbb{H}(\mathbb{C})$ are defined by

$$\mathcal{P}^{\pm} = M^{(1\pm\mathbf{i}\mathbf{e}_3)/2}, \tag{10}$$

where $M^q(p) = pq$ is the operator of multiplication on the right by q.

Note that $\mathcal{P}^{\pm}$ commutes with the differential operator D which will be defined in (11) as well as with multiplication by complex values.

Lemma 2.7. *For $q \in \mathbb{H}(\mathbb{C})$ the numbers $p_{\pm} = 2\operatorname{Re}\mathcal{P}^{\pm}(q) = \operatorname{Re} q \mp \operatorname{Im} q\mathbf{e}_3$ are the unique values in $\mathbb{H}$ (i.e. with vanishing imaginary part) such that $\mathcal{P}^{\pm}(p_{\pm}) = \mathcal{P}^{\pm}(q)$.*

Proof. Let $q = \operatorname{Re} q + \mathbf{i}\operatorname{Im} q$ be a complex quaternion. Then

$$\begin{aligned} 2\mathcal{P}^{\pm}q &= (\operatorname{Re} q + \mathbf{i}\operatorname{Im} q) \pm (-\mathbf{i}\operatorname{Im} q + \mathbf{i}\operatorname{Im} q)\mathbf{e}_3 \\ &= (\operatorname{Re} q \pm \mathbf{i}\operatorname{Re} q\mathbf{e}_3) + (\operatorname{Im} q \mp \mathbf{i}\operatorname{Im} q)\mathbf{e}_3 \\ &= 2\mathcal{P}^{\pm}p_{\pm} \in \mathbb{H}(\mathbb{R}). \end{aligned}$$

Since the operators $\mathcal{P}^{\pm}$ are idempotent and are projectors, the uniqueness is proved. □

2.2 Moisil-Teodorescu differential operator and monogenic functions

The functions regarded as regular by Fueter are the functions in a domain of $\mathbb{H}$ which are annihilated by the operator

$$\partial_0 + \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3,$$

where $\partial_j = \partial/\partial x_j,\ j = 0, 1, 2, 3$. This operator and its theory can be generalized readily to the setting of Clifford algebras [8, 26, 34]. Further operators may be obtained by restricting to a subspace, for example by embedding $\mathbb{C}^3$ in $\mathbb{H}(\mathbb{C})$ in different ways [28, 74]. Since we are interested in equations of mathematical physics, we will use $\mathbb{H}(\mathbb{C})$-valued functions defined in domains in $\mathbb{R}^3$.

From now on we will use the notation $x \in \mathbb{R}^3$.

Definition 2.8. The *Moisil-Teodorescu differential operator* is defined by

$$D = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3. \tag{11}$$

The operator D may be applied to differentiable functions $u\colon \Omega \to \mathbb{H}(\mathbb{C})$, i,e $u(x) = u_0(x) + \sum_{j=1}^{3} \mathbf{e}_j u_j(x) = \operatorname{Sc} u(x) + \operatorname{Vec} u(x)$, where the coordinate functions u_j are complex valued functions defined in Ω. Thus D really refers to two different operators, acting from either side as follows:

$$Du_0 = u_0 D = \operatorname{grad} u_0,$$

$$D\vec{u} = -\operatorname{div}\vec{w} + \operatorname{curl}\vec{u}, \quad \vec{u}D = -\operatorname{div}\vec{w} - \operatorname{curl}\vec{u}.$$

Thus for $u = u_0 + \vec{u}$, then the left and right operators are

$$Du = -\operatorname{div}\vec{u} + \operatorname{grad} u_0 + \operatorname{curl}\vec{u}, \tag{12}$$

$$uD = -\operatorname{div}\vec{u} + \operatorname{grad} u_0 - \operatorname{curl}\vec{u}, \tag{13}$$

We will be concerned exclusively with the left operator (12). We use the standard notation $C^r(\Omega, \mathbb{H}(\mathbb{C}))$ for the set of r-times continuously differentiable functions defined in Ω, taking values in $\mathbb{H}(\mathbb{C})$. Note that $C^r(\Omega, \mathbb{H}(\mathbb{C}))$ is a right-$\mathbb{H}(\mathbb{C})$ module, that is, $u\lambda \in C^r(\Omega, \mathbb{H}(\mathbb{C}))$ when $u \in C^r(\Omega, \mathbb{H}(\mathbb{C}))$ and $\lambda \in \mathbb{H}(\mathbb{C})$. Then we have the associativity relation

$$D(u\lambda) = (Du)\lambda. \tag{14}$$

Using (14) one can verify the following lemma.

Lemma 2.9. $\operatorname{Ker} D$ *is a quaternionic right linear space. Then means that for* $u, v \in \operatorname{Ker} D$ *and* $\lambda \in \mathbb{H}(\mathbb{C})$ *constant, then* $(u\lambda + v) \in \operatorname{Ker} D$.

Proof. Let $\lambda \in \mathbb{H}(\mathbb{C})$ be a complex quaternionic constant. Then due to (14), $u\lambda$ belongs to $\operatorname{Ker} D$. Then we have $D[u\lambda + v] = D[u\lambda] + D[v]$ and these are zero if $u, v \in \operatorname{Ker} D$. □

Equation (14) is a particular case of the following.

Proposition 2.10 ([47]). *(Leibniz Rule) Let* u, v *be functions in* $C^1(\Omega, \mathbb{H})$. *Then*

$$D[uv] = D[u]v + \overline{u}D[v] + 2(\operatorname{Sc}(uD))[v] \tag{15}$$

where we write

$$(\mathrm{Sc}(vD))[v] = -\sum_{j=1}^{3} v_j \partial_j v.$$

Note that when $\vec{u} = 0$, this simplifies to the classical formula $D[uv] = D[u]v + uD[v]$.

Definition 2.11. Let $\Omega \subseteq \mathbb{R}^3$ be an open set. A function $u \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is called *left-monogenic* or *monogenic* in Ω when $Du = 0$ and we denote by $\operatorname{Ker} D$ the set of all monogenic functions in Ω.

Then by (12), the Moisil-Teodorescu system is satisfied:

$$u \in \operatorname{Ker} D \iff \begin{cases} \operatorname{div} \vec{u} = 0, \\ \operatorname{curl} \vec{u} = -\operatorname{grad} u_0. \end{cases} \tag{16}$$

Because $\mathbf{i}$ commutes with $\mathbf{e}_j$ and div, curl and grad are linear operators. Therefore when $u \in \operatorname{Ker} D$, the real quaternionic functions $\mathrm{Re}_{\mathbb{H}}\, u, \mathrm{Im}_{\mathbb{H}}\, u \in \mathbb{H}(\mathbb{R})$ are also monogenic.

In general $u \in \operatorname{Ker} D$ does not imply, λu lies also in $\operatorname{Ker} D$ when $\lambda \in \mathbb{H}(\mathbb{C})$. As an example, let $u = x_2 - x_3\mathbf{e}_1$ and $\lambda = \mathbf{e}_2$. Then $Du = 0$ but $D(\mathbf{e}_2 u) = -2$.

2.3 Harmonic functions and quaternionic integral operators

Recall that the Laplacian is defined as

$$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2. \tag{17}$$

The Laplacian is a second order differential operator, so $\Delta\colon C^k(\Omega) \to C^{k-2}(\Omega)$ for $k \geq 2$. A function $u \in C^2(\Omega, \mathbb{H}(\mathbb{C}))$ is said to be *harmonic* if $\Delta u = 0$. Properties of harmonic functions are given in [2, 21, 41].

Lemma 2.12. *The Laplacian admits the factorization*

$$\Delta = \overline{D}D = D\overline{D} \tag{18}$$

when applied to C^2 functions. Every monogenic function u is harmonic. Moreover since Δ is a real operator each component u_j of u also satisfies that $\Delta u_j = 0$.

Proof. Observe that

$$\begin{aligned} D\overline{D} &= (\partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \partial_3\mathbf{e}_3)(-\partial_1\mathbf{e}_1 - \partial_2\mathbf{e}_2 - \partial_3\mathbf{e}_3) \\ &= \partial_1^2 + \partial_2^2 + \partial_3^2 \\ &= \Delta. \end{aligned}$$

Similarly $\overline{D}D = \Delta$. Suppose that u is monogenic. It is well known [8] that u is infinitely differentiable. Then we have

$$\Delta u = \overline{D}Du = 0.$$

Since Δ is a real operator, this implies that each component is harmonic. □

Define

$$\mathrm{Har}(\Omega, \mathbb{H}(\mathbb{C})) = \{u\colon \Omega \to \mathbb{H}(\mathbb{C})\colon \Delta u = 0\} \tag{19}$$

for the set of harmonic functions defined in Ω.

Definition 2.13. When $u = u_0 + \vec{u} \in \operatorname{Ker} D$, one says that $(u_0, \vec{u})$ form a *hyperconjugate pair* or that $\vec{u}$ is a *harmonic hyperconjugate* of u_0.

Definition 2.14. We say that a set Ω is *star-shaped* respect to the origin if $tx \in \Omega$ whenever $x \in \Omega$ and $t \in [0,1]$.

The next proposition give us a way to construct the hyperconjugate pair in star-shaped domains.

Proposition 2.15. *[35] Let Ω be an open set in $\mathbb{R}^3$ star-shaped with respect to the origin and let $u_0 \colon \Omega \to \mathbb{R}$ be harmonic in Ω. Then the complex quaternionic function*

$$u = u_0 - \operatorname{Vec} \int_0^1 t(Du_0)(tx)x dt \tag{20}$$

is monogenic in Ω.

Given $u_0 \colon \Omega \to \mathbb{R}$, we can apply Proposition 2.15 to the real and imaginary parts of u_0 to construct a purely vector-valued to construct a purely vector-valued function $\vec{u} : \Omega \to \operatorname{Vec}(\mathbb{H}(\mathbb{C}))$ be a hyperconjugate pair given a complex harmonic function.

Corollary 2.16 ([35])**.** *Let Ω be an open set in $\mathbb{R}^3$ and star-shaped with respect to the origin and let $u_0 \colon \Omega \to \mathbb{C}$ be harmonic in Ω. Then*

$$\vec{u} = -\operatorname{Vec} \int_0^1 t(Du_0)(tx)x dt, \tag{21}$$

is a harmonic hyperconjugate of u_0.

The case when $u \in \operatorname{Ker} D$ and $\operatorname{Sc} u = 0$ has an important role in mathematical-physics as known *potential theory*, since $D\vec{u} = 0$ is equivalent to $\operatorname{div} \vec{u} = 0 =$

$\operatorname{curl}\vec{u}$, also in this work in the sections 5.6, 5.2.2, so with aid of (21) we can construct monogenic purely vectorial functions.

Definition 2.17. Let Ω be an open star-shaped set. The operator

$$\mathcal{F}\colon\ \operatorname{Ker} D \cap C^1(\Omega, \mathbb{H}(\mathbb{C})) \to \operatorname{Ker} D \cap C^1(\Omega, \operatorname{Vec}\mathbb{H}(\mathbb{C}))$$

is defined as follows:

$$\begin{aligned}\mathcal{F}[u] &= u_0 + \operatorname{Vec} u - \left(u_0 - \operatorname{Vec}\int_0^1 t(Du_0)(tx)x dt\right),\\ &= \operatorname{Vec}\left(u + \int_0^1 t(Du_0)(tx)x dt\right). \qquad (22)\end{aligned}$$

In Definition 5.5 we will give a similar operator which does not require the domain to be star-shaped. Since $\mathcal{F}^2 = \mathcal{F}$ we have the following decomposition of $\operatorname{Ker} D$.

Lemma 2.18. *Let Ω be an open star-shaped set. Then the following decomposition holds:*

$$\operatorname{Ker} D = \operatorname{Ker}\mathcal{F} \oplus \operatorname{Ker} D \cap C^1(\Omega, \operatorname{Vec}\mathbb{H}(\mathbb{C})). \qquad (23)$$

The following function is a non-trivial function which is left and right monogenic:

Definition 2.19. The *Cauchy kernel* is defined for $x \in \mathbb{R}^3 \setminus \{0\}$ by

$$G(x) = -\frac{1}{4\pi}\frac{x}{|x|^3}. \qquad (24)$$

This function is a fundamental solution of D (see [35]). Let us assume that Ω is a bounded domain in $\mathbb{R}^3$ with a piecewise smooth boundary $\Gamma = \partial\Omega$.

Proposition 2.20 ([34], Theorem 7.12)**.** (Cauchy Integral Formula) Let $\Omega \in \mathbb{R}^3$ be a bounded domain with boundary at least C^2 smooth. Then for every $u \in C^1(\overline{\Omega})$ we have

$$u(x) = \int_{\Gamma} G(y-x)\mathbf{n}(y)u(y)d\Gamma_y \tag{25}$$

where $\mathbf{n}$ is the quaternionic representation of the outward unit normal to the surface Γ.

Definition 2.21. The *Teodorescu* transform in Ω is defined by

$$\mathbb{T}_{\Omega}[u](x) = -\int_{\Omega} G(y-x)u(y)dy, \quad x \in \mathbb{R}^3 \tag{26}$$

The importance of this operator is that it is an right inverse of D.

Proposition 2.22 ([35])**.** *Let* $u \in L^p(\Omega, \mathbb{H})$ *and* $1 < p < \infty$*. Then* $\mathbb{T}_{\Omega}[u] \in W^{1,p}(\Omega, \mathbb{H})$*. Further,*

$$D\mathbb{T}_{\Omega}[u] = u. \tag{27}$$

2.3.1 Taylor series

It is known in complex analysis that every holomorphic function in a disk can be expressed as a power series whose summands are multiples of holomorphic polynomials which have the form $(x+\mathbf{i}y)^n$. Unfortunately this idea cannot be copied directly into quaternion theory since the polynomials $p(x) = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)^n$ in general are not monogenic. This is the reason for using a different type of polynomials. R. Fueter in [23] introduced polynomials in 1936 using the variables $z_j = x_j - x_0\mathbf{e}_j$ and the permutational product (see [35, section 6]). These variables satisfy $(\partial_0 + D)z_j = z_j(\partial_0 + D) = 0$ where D is given by (11). Also it was proved that every function u which satisfies $(\partial_0 + D)u = 0$ can be expanded into a Taylor series of permutational products of z_j, $j = 1, 2, 3$. Since the monogenic functions which we are studying do not depend on the variable x_0, this expansion is not the best for our purposes. Instead we will use the polynomials defined by Y. Grigor'ev and V. V. Naumov in [29]. For further properties and applications see [30, 31].

Definition 2.23. Let l, m integers and we write $n = l + m$. The *Grigor'ev polynomials* $P^{l,m}$ are the homogeneous $\mathbb{H}$-valued polynomials of degree $n = l + m$ in the vector variable $x = (x_1, x_2, x_3)$ constructed by means of the recurrence formulas

$$P^{l,m}(x) = P^{1,0}(x)P^{l-1,m}(x) + P^{0,1}(x)P^{l,m-1}(x), \tag{28}$$

where $l, m \geq 0$, from the initial cases $P^{0,0}$, $P^{1,0}$, $P^{0,1}$ as given in Table 2.1.

$P^{0,0}$	1
$P^{0,1}$	$x_2 - x_3\mathbf{e}_1$
$P^{1,0}$	$x_1 + x_3\mathbf{e}_2$
$P^{0,2}$	$x_2^2 - x_3^2 - 2x_2x_3\mathbf{e}_1$
$P^{1,1}$	$2x_1x_2 - 2x_1x_3\mathbf{e}_1 + 2x_2x_3\mathbf{e}_2$
$P^{2,0}$	$x_1^2 - x_3^2 + 2x_1x_3\mathbf{e}_3$
$P^{0,3}$	$x_2^3 - 3x_2x_3^2 + (x_3^3 - 3x_2^2x_3)\mathbf{e}_3$
$P^{1,2}$	$(3x_1x_2^2 - 3x_1x_3^2) - 6x_1x_2x_3\mathbf{e}_1 + (3x_2^2x_3 - x_3^3)\mathbf{e}_2$
$P^{2,1}$	$(3x_1^2x_2 - 3x_2x_3^2) - (3x_1^2x_3 - x_3^3)\mathbf{e}_1 + 6x_1x_2x_3\mathbf{e}_2$
$P^{3,0}$	$(x_1^3 - 3x_1x_3^2) + (3x_1^2x_3 - x_3^3)\mathbf{e}_2$
$P^{0,4}$	$(x_2^4 - 6x_2^2x_3^2 + x_3^4) + (4x_2x_3^3 - 4x_2^3x_3)\mathbf{e}_1$
$P^{1,3}$	$(4x_1x_2x_2^2 - 12x_1x_2x_3^2) + (4x_1x_3^3 - 12x_1x_2^2x_3)\mathbf{e}_1 + (4x_2^3x_3 - 4x_2x_3^3)\mathbf{e}_2$
$P^{2,2}$	$(-6x_2^2x_3^2 + 2x_3^4 + 6x_1^2x_2^2 - 6x_1^2x_3^2) + (-12x_1^2x_2x_3 + 4x_2x_3^3)\mathbf{e}_1 + (12x_1x_2^2x_3 - 4x_1x_3^3)\mathbf{e}_2$
$P^{3,1}$	$(4x_1^32x_2 - 12x_1x_2x_3^2) - (4x_1^3x_3 - 4x_1x_3^3)\mathbf{e}_1 + 12(x_1^2x_2x_3 - 4x_2x_3^3)\mathbf{e}_2$
$P^{4,0}$	$x_1^4 - 6x_1^2x_3^2 + x_3^4 + (4x_1^3x_3 - 4x_1x_3^3)\mathbf{e}_2$

Table 2.1: Grigor'ev polynomials of low degree

Proposition 2.24. *[30] The polynomials $P^{l,m}$ are left monogenic (but not necessarily right monogenic), and for each $n \geq 0$ the set*

$$\{P^{l,m} : \ l + m = n\}$$

is a basis of the right-vector space over $\mathbb{H}(\mathbb{C})$ of homogeneous left monogenic polynomials of degree n over $\mathbb{R}$.

Indeed the Grigor'ev polynomials have the following representation [30, *Theorem 3.4]*

$$P^{l,m} = P_0^{l,m} + P_1^{l,m}\mathbf{e}_1 - P_1^{l-1,m+1}\mathbf{e}_2 \tag{29}$$

where the $P_0^{l,m}, P_1^{l,m}$ *are homogeneous scalar valued polynomials in* (x_1, x_2, x_3).

We are interested in purely vectorial monogenic functions, so we need to construct a family of monogenic purely vectorial polynomials. It is known that the dimension over $\mathbb{R}$ of monogenic purely vectorial homogeneous polynomials is $2n+3$ (see [62]).

Lemma 2.25. *The system*

$$\mathcal{M}^n = \{P^{l,m}\mathbf{e}_3, P^{l,m}\mathbf{e}_2 + P^{l-1,m+1}\mathbf{e}_1, P^{n,0}\mathbf{e}_1\} \tag{30}$$

forms a basis for the finite dimensional vector space over $\mathbb{R}$ *of monogenic purely vectorial homogeneous polynomials of degree* n.

Proof. Since $P^{l,m}$ are monogenic, is clear that $P^{l,m}\mathbf{e}_j$ is also monogenic for $j = 1, 2, 3$ and using the fact that the system $P^{l,m}$ is linearly independent over the real this implies that the system $\mathcal{M}^n$ is linearly independent over $\mathbb{R}$. Only rests show that are purely vectorial, using (29) is clear that $P^{l,m}\mathbf{e}_3$ and $P^{n,0} = (x_1 + x_2\mathbf{e}_2)^n$ are vectorial polynomials, notice that due to (29) $\mathrm{Sc}(P^{l,m}\mathbf{e}_2) = P_1^{l-1,m+1}$ and $\mathrm{Sc}(P^{l-1,m+1}\mathbf{e}_1) = -P_1^{l-1,m+1}$ this makes their sum purely vectorial. Only rest to show $\dim_{\mathbb{R}} \mathcal{M}^n = 2n+3$, since $l+m = n$, this implies that $\dim_{\mathbb{R}} P^{l,m}\mathbf{e}_3 = n+1 = \dim_{\mathbb{R}} P^{l,m}\mathbf{e}_2 + P^{l-1,m+1}\mathbf{e}_1$ and finally $\dim_{\mathbb{R}} P^{n,0} = 1$. □

There exist explicit formulas to obtain the Grigor'ev polynomials; for details see [30, Theorem 6.1], the importance of these are the Taylor series and Runge Theorem for monogenic functions.

Proposition 2.26. *(Taylor series)[30] Let u be a monogenic function in any domain Ω and let $x_0 \in \Omega$. Then in any ball $B_R(x_0) \subseteq \Omega$ this function can be expressed as a sum of convergent Taylor series*

$$u(x) = \sum_{j=0}^{\infty} \sum_{l+m=j} P^{l,m}(x - x_0)C_{l,m} \tag{31}$$

where the coefficients $C_{l,m}$ are

$$C_{l,m} = \frac{1}{j!} \frac{\partial u}{\partial_1^l \partial_2^m}\bigg|_{x=x_0}, \tag{32}$$

and the convergence is uniform on every closed subset $K \subseteq B_R(x_0)$.

Definition 2.27. Let $E \subseteq C^0(\Omega, \mathbb{H}(\mathbb{C}))$ be a linear subspace which is a right module over $\mathbb{H}(\mathbb{C})$. A collection of functions $E_0 \subseteq E$ is called a *complete system in E in the sense of compact-uniform convergence* if for every $v \in E$, for every compact $K \subseteq \Omega$ and for every $\epsilon > 0$ there exists a finite collection of elements $\{v_n\}_{n=1}^N \subseteq E_0$ and collection of coefficients $\{a_n\}_{n=1}^N \subseteq \mathbb{H}(\mathbb{C})$ such that

$$\|v - \sum_{n=1}^{N} v_n a_n\|_K < \epsilon.$$

Proposition 2.28. *(Runge theorem) [30] Let $\Omega \subseteq \mathbb{R}^3$ be an open subset which has connected complement. Then every monogenic function $u \colon \Omega \to \mathbb{H}$ can be uniformly approximated on each compact subset $K \subseteq \Omega$ by quaternionic right linear combinations of monogenic polynomials $P^{l,m}$. Thus $\{P^{l,m}\}$*

is a complete system in $\operatorname{Ker} D$ *in the sense of compact-uniform convergence.*

Definition 2.29. Let X be a normed right module over $\mathbb{H}$ or $\mathbb{H}(\mathbb{C})$. A system of vectors $\{u_k\}_{k=0}^{\infty} \subseteq X$ is said to be *complete system* in X (with respect to the given norm) if $(\forall u \in X)(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\exists \{\alpha_k\}_{k=0}^{N} \in \mathbb{H})$ such that

$$\left\| u - \sum_{k=0}^{N} u_k \alpha_k \right\| < \epsilon. \tag{33}$$

We will be interested in completeness of sets in subspaces of the Hilbert space $L^2(\Omega, \mathbb{H})$ or $L^2(\Omega, \mathbb{H}(\mathbb{C}))$. The norm given by

$$\|u\|_2^2 = \sum_{i=0}^{3} \iiint_{\Omega} |u_i(x)|^2 \, dx_1 \, dx_2 \, dx_3. \tag{34}$$

The starting point of the theory of holomorphic Bergman spaces is the well known fact that [8, Theorem 24.8] that $L^2(\Omega, \mathbb{H}) \cap \operatorname{Ker} D$ is a closed subspace of $L^2(\Omega, \mathbb{H})$ with respect to the norm (34) and is therefore itself a Hilbert space.

Proposition 2.30. *[34, Theorem 10.4.] Let* Ω, Ω_2 *be bounded domains in* $\mathbb{R}^3$ *whose boundaries* $\partial\Omega$, $\partial\Omega_2$ *are at least* C^2 *surfaces, and* $\overline{\Omega} \subseteq \Omega_2$. *Let* $\{\xi_k\}$ *be a dense subset of* $\partial\Omega_2$. *Then the collection of vector fields* $\{G(x - \xi_k)\}_{k=1}^{\infty}$ *is a complete system in* $L^2(\Omega, \mathbb{H}) \cap \operatorname{Ker} D$.

Corollary 2.31. *Let* Ω, Ω_2 *be bounded domains in* $\mathbb{R}^3$ *whose boundaries* $\partial\Omega$, $\partial\Omega_2$ *are at least* C^2 *surfaces, and* $\overline{\Omega} \subseteq \Omega_2$ *and* Ω *has connected complement. Then the monogenic polynomials* $P^{l,m}$ *form a complete system in* $L^2(\Omega, \mathbb{H}) \cap \operatorname{Ker} D$.

It is immediate that Proposition 2.30 and Corollary 2.31 hold when $\mathbb{H}$ is replaced by $\mathbb{H}(\mathbb{C})$ because the real and imaginary parts of monogenic functions are monogenic.

2.4 Perturbed Moisil-Teodorescu operator

Let $\lambda \in C(\Omega, \mathbb{C})$ be a scalar function. We are interested in studying the kernel of the operators $D \pm \lambda$ since they are related to some physical problems, as we will see in Chapter 5. For later reference we mention the following. Let $u \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. Then using (16) we have the following equivalence:

$$u \in \operatorname{Ker}(D \pm \lambda) \iff \begin{cases} -\operatorname{div} \vec{u} \pm \lambda u_0 = 0, \\ \operatorname{curl} \vec{u} \pm \lambda \vec{u} = -\operatorname{grad} u_0. \end{cases} \tag{35}$$

When $\operatorname{Sc} u = 0$ this means that λ is an eigenvalue of the curl operator.

We will also consider the differential operator $D + M^\lambda$ acting on functions $u \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ as follows:

$$(D + M^\lambda)u = Du + u\lambda. \tag{36}$$

Chapter 3

Transmutation theory

In this chapter we first summarize the definitions and results of [10, 18, 19, 36, 56] referring to transmutation operators. Transmutation operators relate solutions between two linear differential operators, where it may be easier to find solutions for one of them. Applying the transmutation operator we obtain solutions for the other.

The transmutation operators in the references cited above apply to functions in a real interval. In a trivial way one can apply such operators to functions in $\mathbb{R}^3$ by acting on only one of the variables. Our main objective in this chapter is to define an operator $\mathbf{T}_f$ on functions of a complex quaternionic variable. This operator relates monogenic functions to solutions of the differential operator $(D + M^{\frac{Df}{f}})$. Then using the projectors $\mathcal{P}^{\pm}$ defined by (10) we can also find solutions for the differential operator $D + \lambda(x_3)$, where λ is a complex valued function and only depends on a single variable.

3.1 Transmutation operators of functions of a single variable

3.1.1 Construction of transmutation operator

We begin recalling the basic concept of transmutation operator and then details of the particular operator which we will use.

Definition 3.1. [57] Let X be a linear topological space and let $X_1 \subseteq X$ be a linear subspace (not necessarily closed) and let $\mathbf{A}, \mathbf{B} \colon X_1 \to X$ be linear operators. A linear invertible operator $\mathcal{T} \colon X \to X$, such that X_1 is $\mathcal{T}$ invariant, is said to be a *transmutation operator* for the pair $\mathbf{A}$ and $\mathbf{B}$, if the following conditions are fulfilled:

1. Both the operators $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ are continuous in X.

2. The following equality is valid in X_1:

$$\mathbf{A}\mathcal{T} = \mathcal{T}\mathbf{B}. \tag{37}$$

The importance of this type of operators is that if we know any solution $v \in X_1$ of $\mathbf{B}v = 0$, then we have that $u = \mathcal{T}v$ satisfies $\mathbf{A}u = \mathbf{A}\mathcal{T}v = \mathcal{T}\mathbf{B}v = 0$. Our interest concerns when $\mathbf{A} = -\frac{d^2}{dx^2} + r(x)$ and $\mathbf{B} = -\frac{d^2}{dx^2}$ (we know the solutions), where r is a complex valued function, because the transmutation operator is known and their properties (see [56]). We will explain the construction in this section.

The kernels of transmutation operators can be identified with the solutions of partial differential equations. We explain the relationship briefly here.

Definition 3.2. Let r be a continuous complex valued function in an interval $[-a, a]$. Let $h \in \mathbb{C}$. The *Goursat problem* for r with complex parameter h on the region $0 \leq |t| \leq |x| \leq a$ in the (x, t)-plane is:

$$\left(\frac{\partial^2}{\partial x^2} - r(x) \right) \mathbf{K}(x, t; h) = \frac{\partial^2}{\partial t^2} \mathbf{K}(x, t; h), \tag{38}$$

$$\mathbf{K}(x, x; h) = \frac{h}{2} + \frac{1}{2} \int_0^x r(s) ds, \quad \mathbf{K}(x, -x; h) = \frac{h}{2}. \tag{39}$$

Proposition 3.3 ([36])**.** *Let $r\colon [0, a] \to \mathbb{C}$ be a continuous function. Then*

1. *The Goursat problem has a unique solution for every parameter $h \in \mathbb{C}$.*

2. *If r is n times continuously differentiable, then the kernel $\mathbf{K}(x, t; h)$ is $n+1$ times continuously differentiable respect to x and t.*

3. *Let $r \in C^1[-a, a]$. Then the Volterra operator*

$$v(x) \mapsto v(x) + \int_{-x}^{x} \mathbf{K}(x, y, h) v(y)\, dy, \tag{40}$$

is a transmutation operator on the space $C^2[-a, a]$ sending $\operatorname{Ker}(-\frac{d^2}{dx^2} + r(x))$ *to* $\operatorname{Ker}(-\frac{d^2}{dx^2})$ *if and only if the integral kernel $\mathbf{K}(x, t; h)$ satisfies the Goursat problem* (38)*,* (39)*.*

This result of [36] only guarantees the existence of a Volterra operator in the form (40) with a suitable kernel $\mathbf{K}$, but does not indicate how to obtain the kernel. In [10, 57] a construction of operators which convert solutions of one second-order differential operator $-\frac{d^2}{dx^2}+r(x)$ to another $-\frac{d^2}{dx^2}+\tilde{r}(x)$ was given. We summarize the procedure for $\tilde{r}=0$, and for r of a particular form which is all the generality which we require, as follows.

Definition 3.4. A complex-valued function $f\in C^2[-a,a]$ such that $f(x)\neq 0$ for all $x\in[-a,a]$, $f(0)=1$, will be called simply a *nonvanishing coefficient* on the real interval $[-a,a]$.

Let f be a nonvanishing coefficient. Define the following iterated integrals (see [56]) associated to f as the sequences $X^{(n)}$, $\widetilde{X}^{(n)}$ constructed as follows:

$$\begin{aligned} X^{(0)}(x) &\equiv \widetilde{X}^{(0)}(x)\equiv 1,\\ X^{(n)}(x) &= n\int_0^x X^{(n-1)}(s)\left(f^2(s)\right)^{(-1)^n}ds,\\ \widetilde{X}^{(n)}(x) &= n\int_0^x \widetilde{X}^{(n-1)}(s)\left(f^2(s)\right)^{(-1)^{n-1}}ds. \end{aligned} \tag{41}$$

Definition 3.5. The *formal powers* φ_n, $n=0,1,\ldots$ associated to a nonvanishing coefficient f are defined as

$$\varphi_n(x)=\begin{cases} f(x)X^{(n)}(x), & n \text{ odd},\\ f(x)\widetilde{X}^{(n)}(x), & n \text{ even}. \end{cases} \tag{42}$$

Properties of these formal powers may be found in [10, 50, 56].

Example 3.6. The simplest case is when $f\equiv 1$. Then $\widetilde{X}^{(n)}=X^{(n)}$ and $X^{(n)}=n\int_0^x s^{n-1}ds=x^n$, therefore $\varphi_n(x)=x^n$.

Theorem 3.7. *[10] Let f be a nonvanishing coefficient. There exists a Volterra operator T_f of the form*

$$T_f[v](x)=v(x)+\int_{-x}^{x}\mathbf{K}_f(x,t;h)v(t)\,dt, \tag{43}$$

with continuous kernel $\mathbf{K}_f$ *such that*

$$\mathbf{K}_f(x,t;h) = \frac{h}{2} + K(x,t) + \frac{h}{2}\int_{-t}^{x} (K(x,s) - K(x,-s))ds,$$

the function K *satisfies* (38) *and* (39) *and*

$$T_f(x^n) = \varphi_n(x), \tag{44}$$

for $n = 0,1,\ldots$ *Further, for any* $v \in C^2[-a,a]$, T_f *satisfies the transmutation property*

$$\left(-\partial^2 + \frac{f''}{f}\right) T_f[v] = T_f[-\partial^2 v]. \tag{45}$$

In general it is very difficult to find these kernels in an explicit form due to the difficulty of solving the corresponding Goursat problem. In [53, Theorem 3.2] a general representation for the kernel $\mathbf{K}$ is given in terms of Fourier-Legendre series, where the coefficients can be obtained in a recursive way.

Example 3.8. [10, Example 12] Let $f''/f = c \in \mathbb{C}$. Then the kernel $\mathbf{K}_f$ is known and has the following form:

$$\mathbf{K}_f(x,t) = -\frac{1}{2}\frac{\sqrt{c(x^2-t^2)}J_1\left(\sqrt{c(x^2-t^2)}\right)}{x-t} + \frac{f'(0)}{2}J_0\left(\sqrt{c(x^2-t^2)}\right), \tag{46}$$

where J_0, J_1 are the Bessel functions of first kind.

3.1.2 Properties of the basic transmutation operator

Let f be a nonvanishing coefficient. Note that $1/f$ is also a nonvanishing coefficient. When we interchange f by $1/f$ in Theorem 3.7 and (42), the

Volterra operator $T_{1/f}$ with continuous kernel $\mathbf{K}_{1/f}$ satisfies:

$$T_{1/f}(x^n) = \psi_n(x), \tag{47}$$

for $k = 0, 1, \dots,$ where ψ are the formal powers associated with $1/f$,

$$\psi_n(x) = \begin{cases} \frac{1}{f(x)}\widetilde{X}^{(n)}(x), & n \text{ odd}, \\ \\ \frac{1}{f(x)}X^{(n)}(x), & n \text{ even}. \end{cases} \tag{48}$$

Further, for any $v \in C^2[-a, a]$, $T_{1/f}$ satisfies the transmutation property

$$\left(-\partial^2 + \frac{(1/f)''}{1/f}\right) T_{1/f}[v] = T_{1/f}[-\partial^2 v]. \tag{49}$$

For the purpose of this work the following property plays a fundamental role, because it tells us how the operators $T_f, T_{1/f}$ are related.

Proposition 3.9. *[56] On $C^1[-a, a]$ we have the following relation for any nonvanishing coefficient f:*

$$\partial_x \frac{1}{f} T_f = \frac{1}{f} T_{1/f} \partial_x. \tag{50}$$

Using this proposition, in [50] the concept of generalized derivatives or f-derivatives was introduced as follows.

Definition 3.10. Let f be a nonvanishing coefficient. Then the *generalized f-derivatives* d_n^f of a sufficiently differentiable function g are defined by the following relation.

$$d_0^f[g] = g, \tag{51}$$

$$d_k^f[g] = \begin{cases} f \frac{d}{dx}\left(\frac{1}{f} d_{k-1}^f[g]\right), & k \text{ odd}, \\ \frac{1}{f}\frac{d}{dx}\left(f d_{k-1}^f[g]\right), & k \text{ even}. \end{cases} \tag{52}$$

We say that an operator T *fixes values at the origin* if $T[v](0) = v(0)$ for all $v \in C[a,b]$. Note that the transmutation operator T_f fixes values at the origin because of the limits of integration in (43).

Proposition 3.11 ([57]). *Let $u \in C^n[-a,a]$ and $g = T_f[u]$. Then there exist the first n generalized derivatives of g on $[-a,a]$ and the following equalities hold for $0 \le k \le n$:*

$$d_k^f[g] = \begin{cases} T_{1/f}[\partial^k u], & k \text{ odd}, \\ T_f[\partial^k u], & k \text{ even}. \end{cases} \tag{53}$$

Since the operators $T_f, T_{1/f}$ fix the value at the origin we have

$$d_k^f[g](0) = (\partial^k u)(0). \tag{54}$$

3.2 Quaternionic transmutation operator between $D + M^{\frac{Df}{f}}$ and D

The transmutation operator T_f defined in (43) acts on $\mathbb{C}$-valued functions defined on a real interval. In this section we will extend the notion of T_f to complex-quaternionic operators $\mathbf{T}_{f,j}$ for $j = 1,2,3$. The inspiration comes from ideas of [9] acting on $\mathbb{H}(\mathbb{C})$-valued functions in a spatial domain Ω. For each j, $\mathbf{T}_{f,j}$ will essentially act only on the variable x_j. This will lead to the definition of $\mathbf{T}_{f_1 f_2 f_3}$ for a separable function of three variables.

In this chapter Ω will be a bounded open subset of $\mathbb{R}^3$ with connected complement, and satisfying the following symmetry property with respect to at least one of the variables x_j:

Definition 3.12. Let $j \in \{1,2,3\}$. We will say that $\Omega \subseteq \mathbb{R}^3$ is *convex in the x_j direction* if whenever $x=(x_1,x_2,x_3)\in\Omega$, the straight segment from x to x^* lies in Ω, where x^* is the point obtained by replacing x_j with $-x_j$ in x.

We require this so that the complex operators T_f, $T_{1/f}$ can be applied to functions in Ω with respect to the variable x_j when the other two coordinates are fixed. A nonvanishing coefficient for Ω (with respect to x_j) is understood to be defined in $[-a_j, a_j]$ where $a = \sup\{x_j\colon\ (x_1,x_2,x_3)\in\overline{\Omega}\}$.

Definition 3.13. A function of a vector $x=(x_1,x_2,x_3)$ is said to *depend only upon the variable x_1* if it can be expressed as $(x_1,x_2,x_3)\mapsto f(x_1)$ for some function $f(t)$ of a real variable t, for all (x_1,x_2,x_3) in the domain under consideration. Similarly one defines functions which only depend upon x_2 or x_3.

An operator acting on functions $v\in C(\Omega,\mathbb{H}(\mathbb{C}))$ is said to *act only upon the variable x_1* if it can be expressed as sending v to the function $(x_1,x_2,x_3)\mapsto \Psi[v|_{x_2,x_3}](x_1)$ in Ω for some operator Ψ acting on functions of a single real variable. Here $v|_{x_2,x_3}$ refers to the map $x_1\mapsto v(x_1,x_2,x_3)$. Similarly one defines functions which act upon x_2 or x_3. Thus we can extend the real operator T_f of (43) to act on $\mathbb{H}(\mathbb{C})$-valued functions of a single variable by acting term-by-term. This convention is used in the following definition.

Definition 3.14. Let $j\in\{1,2,3\}$. Let f be a nonvanishing coefficient for Ω in the direction x_j. The *quaternionic transmutation operator* $\mathbf{T}_{f,j}$ acts in the variable x_j on $u(x_1, x_2, x_3) = \sum_{i=0}^{4} u_i(x)\mathbf{e}_i \in C(\Omega, \mathbb{H}(\mathbb{C}))$ $(u_i \colon \Omega \to \mathbb{C})$ as follows:

$$\mathbf{T}_{f,j}[u](x) = T_{1/f}[u_{0,j}](x) + T_f[u_{0,j}^{\perp}](x), \tag{55}$$

where T_f, $T_{1/f}$ act on the variable x_j and the decomposition $u = u_{0,j} + u_{0,j}^{\perp}$ is given by

$$u_{0,j}(x) = u_0(x) + u_j(x)\mathbf{e}_j, \quad u_{0,j}^{\perp}(x) = u(x) - u_{0,j}(x). \tag{56}$$

This operator is not strictly right-linear over $\mathbb{H}(\mathbb{C})$, but we have the following properties:

Lemma 3.15. *Let u, v be complex quaternion valued continuous functions of (x_1, x_2, x_3) and let $a \in \mathbb{H}(\mathbb{C})$. Then for all $j, k \in \{1, 2, 3\}$,*

1. $\operatorname{Sc} \mathbf{T}_{f,j}[u] = \mathbf{T}_{f,j}[\operatorname{Sc} u]$, $\operatorname{Vec} \mathbf{T}_{f,j}[u] = \mathbf{T}_{f,j}[\operatorname{Vec} u]$,
2. $\mathbf{T}_{f,j}[u + v] = \mathbf{T}_{f,j}[u] + \mathbf{T}_{f,j}[v]$,
3. $\mathbf{T}_{f,j}[M^{a_{0,j}}[u]] = M^{a_{0,j}}\mathbf{T}_{f,j}[u]$, *where* $a_{0,j} = a_0 + a_j\mathbf{e}_j$,
4. $\mathbf{T}_{f,j}[M^{a_{0,j}^{\perp}}[u]] = M^{a_{0,j}^{\perp}}\mathbf{T}_{1/f,j}[u]$, *where* $a_{0,j}^{\perp} = a - a_{0,j}$,
5. $\mathbf{T}_{f,j}^{-1}[u](x) = T_{1/f}^{-1}[u_{0,j}](x) + T_f^{-1}[u_{0,j}^{\perp}](x)$,
6. $\mathbf{T}_{f,j}\mathbf{T}_{g,k} = \mathbf{T}_{g,k}\mathbf{T}_{f,j}$ *where g is a nonvanishing coefficient in the direction x_k.*

Proof. Consider the decomposition $a = a_{0,j} + a_{0,j}^{\perp} \in \mathbb{H}(\mathbb{C})$ as in (56). Since according to (55) the operator $\mathbf{T}_{f,j}$ acts component by component on the functions u, part 1 follows. Part 2 follows from the fact that the complex operator T_f satisfies $T_f[u_j+v_j] = T_f[u_j]+T_f[v_j]$ for complex valued functions u_j, v_j. Part 3 follows from the fact $M^{a_{0,j}}u_{0,j} = a_0u_0+a_0u_j\mathbf{e}_j-a_ju_j+a_ju_0\mathbf{e}_j = (a_0u_0-a_ju_j)+(a_0u_j+a_ju_0)\mathbf{e}_j$. Therefore using the linearity of $T_{1/f}$, we have

$$
\begin{aligned}
\mathbf{T}_{f,j}[M^{a_{0,j}}u_{0,j}] &= T_{1/f}[(a_0u_0 - a_ju_j) + (a_0u_j + a_ju_0)\mathbf{e}_j] \\
&= a_0T_{1/f}[u_0] - a_jT_{1/f}[u_j] + a_0T_{1/f}[u_j]\mathbf{e}_j + a_jT_{1/f}[u_0]\mathbf{e}_j \\
&= T_{1/f}[u_{0,j}]a_{0,j} \\
&= M^{a_{0,j}}\mathbf{T}_{f,j}[u_{0,j}].
\end{aligned}
$$

In a similar way $\mathbf{T}_{f,}[M^{a_{0,j}}u_{0,j}^{\perp}] = M^{a_{0,j}}\mathbf{T}_{f,j}[u_{0,j}^{\perp}]$, since the operator $M^{(\cdot)}$ satisfies $M^{a_{0,j}}[u+v] = M^{a_{0,j}}u + M^{a_{0,j}}v$; this ends the proof of part 3. Part 4 is proved in the same way as the part 3. Since T_f is a Volterra operator, this implies the existence of its inverse T_f^{-1} also acting component by component. Thus the inverse has the form $\mathbf{T}_{f,j}^{-1}[u](x) = T_{1/f}^{-1}[u_{0,j}](x) + T_f^{-1}[u_{0,j}^{\perp}](x)$. For part 6 it is clear when $j = k$. Let $j \neq k$. Consider two integral operators G_1, G_2 acting on variables x_1 and x_2. Their composition is

$$
\begin{aligned}
G_2[G_1u](x) &= \int_{-x_2}^{x_2} G_2(x_2, s) \int_{-x_1}^{x_1} G_1(x_1, t)u(t, s, x_3)\,dt\,ds, \\
&= \int_{-x_1}^{x1} G_1(x_1, t) \int_{-x_2}^{x_2} G_2(x_2, s)u(t, s, x_3)\,ds\,dt, \\
&= G_1[G_2u](x),
\end{aligned}
$$

by Fubini's theorem. We apply this fact to the real operators T_f, $T_{1/f}$ acting on a single variable. Therefore we have

$$\begin{aligned}\mathbf{T}_{f,j}[\mathbf{T}_{g,k}\, u] &= T_{1/f}[T_{1/g}\, u] + T_{1/f}[T_g\, u]\mathbf{e}_1 \\ &\quad + T_f[T_{1/g}\, u]\mathbf{e}_2 + T_f[T_g\, u]\mathbf{e}_3, \\ &= \mathbf{T}_{g,k}[\mathbf{T}_{f,j}\, u].\end{aligned}$$

□

We have arranged the coordinates in (55) in such a way that the following transmutation property holds.

Theorem 3.16. *For* $v \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ *and any nonvanishing coefficient* f *for* Ω *in the direction* x_j*,*

$$\left(D + M^{\frac{f'}{f}\mathbf{e}_j}\right)\mathbf{T}_{f,j}[v] = \mathbf{T}_{1/f,j}[Dv], \tag{57}$$

$$\left(D - M^{\frac{f'}{f}\mathbf{e}_j}\right)\mathbf{T}_{1/f,j}[v] = \mathbf{T}_{f,j}[Dv], \tag{58}$$

where D *is the Moisil-Teodorescu operator* (11)*.*

In the expression $M^{\frac{f'}{f}\mathbf{e}_j}$ in (57), the function f'/f is also interpreted as acting on the variable x_j in $\mathbb{R}^3$. In the following this convention will always be clear from the context.

Proof. Let $k \neq j$. We are going to use the fact that each limit of integration of all components of the operator T_f defined in (43) (and applied to the variable x_j) does not depend on x_k, so we can interchange the partial derivatives with the integration. Thus we have $\partial_k \mathbf{T}_{f,j} = \mathbf{T}_{f,j}\partial_k$. According to (12) and

Definition 3.14, the scalar part of the left side of (57) is

$$\begin{aligned}
-\operatorname{div}\operatorname{Vec}(\mathbf{T}_{f,j}[v]) - \frac{f'}{f}T_{1/f}[v_j] &= -\operatorname{div}\mathbf{T}_{f,j}[\operatorname{Vec} v] - \frac{f'}{f}T_{1/f}[v_j],\\
&= -\mathbf{T}_{f,j}[\operatorname{div}[v_{0,j}^{\perp}]] - \partial_j\mathbf{T}_{1/f}[v_j] - \frac{f'}{f}T_{1/f}[v_j],\\
&= -\mathbf{T}_{f,j}[\operatorname{div}[v_{0,j}^{\perp}]] - \mathbf{T}_{f,j}[\partial_j v_j],\\
&= \mathbf{T}_{1/f,j}[\operatorname{Sc} Dv],
\end{aligned}$$

where the second equality is due to (50), as can be seen from $T_f\partial_x = (f'/f)T_{1/f} + \partial_x T_{1/f}$. In other words,

$$\operatorname{Sc}\left(\left(D + M^{\frac{f'}{f}\mathbf{e}_3}\right)\mathbf{T}_{f,j}[v]\right) = -T_{1/f_j}[\operatorname{div} v] = \operatorname{Sc}\left(\mathbf{T}_{1/f,j}[Dv]\right).$$

The equality of the vector parts of (57) can be verified in a similar way. □

Definition 3.17. Suppose that Ω is convex in all three coordinate directions. Let f_j be a nonvanishing coefficient for Ω in the direction x_j for $j = 1, 2, 3$. Write $f(x) = f_1(x_1)f_2(x_2)f_3(x_3)$. We denote the compositions of the operators $\mathbf{T}_{f,j}, \mathbf{T}_{1/f,j}$ as follows:

$$\mathbf{T}_f = \mathbf{T}_{f_1,1}\mathbf{T}_{f_2,2}\mathbf{T}_{f_3,3}, \tag{59}$$

$$\widetilde{\mathbf{T}}_f = \mathbf{T}_{1/f_1,1}\mathbf{T}_{1/f_2,2}\mathbf{T}_{1/f_3,3}. \tag{60}$$

Observe that by Lemma 3.15, part 6, the operators defining $\mathbf{T}_f$, $\widetilde{\mathbf{T}}_f$ commute.

Corollary 3.18. *In the notation of Definition 3.17, for every $v \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$*

the following equality holds:

$$\left(D + M^{\frac{Df}{f}}\right)\mathbf{T}_f[v] = \widetilde{\mathbf{T}}_f[Dv], \tag{61}$$

$$\left(D - M^{\frac{Df}{f}}\right)\widetilde{\mathbf{T}}_f[v] = \mathbf{T}_f[Dv]. \tag{62}$$

Proof. By part 4 of Lemma 3.15, we have $\mathbf{T}_{f,j}[u\mathbf{e}_k] = \mathbf{T}_{1/f,j}[u\mathbf{e}_k]$ for $k \neq j$. Therefore

$$\begin{aligned}
\left(D + M^{\frac{f_1'}{f_1}\mathbf{e}_1}\right)\mathbf{T}_f &= \mathbf{T}_{1/f_1,1}\left[D\,\mathbf{T}_{1/f_2,2}T_{1/f_3,3}\right], \\
&= \mathbf{T}_{1/f_1,1}\left[-M^{\frac{f_2'}{f_2}\mathbf{e}_2}\mathbf{T}_{f_2,2}\mathbf{T}_{f_3,3} + \mathbf{T}_{1/f_2,2}[D\,\mathbf{T}_{f_3,3}]\right], \\
&= -M^{\frac{f_2'}{f_2}\mathbf{e}_2}\mathbf{T}_f + \mathbf{T}_{1/f_1,1}\mathbf{T}_{1/f_2,2}\left[-M^{\frac{f_3'}{f_3}\mathbf{e}_3} + \mathbf{T}_{1/f_3,3}[D]\right], \\
&= -M^{\frac{f_2'}{f_2}\mathbf{e}_2 + \frac{f_3'}{f_3}\mathbf{e}_3}\mathbf{T}_f + \widetilde{\mathbf{T}}_f[D].
\end{aligned}$$

□

Corollary 3.19. *Let $v \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. Then*

$$\left(D + M^{\frac{f_j'}{f_j}\mathbf{e}_j}\right)\mathbf{T}_{f_j,j}[v] = 0 \iff v \in \operatorname{Ker} D \iff \left(D - M^{\frac{f_j'}{f_j}\mathbf{e}_j}\right)\mathbf{T}_{1/f_j,j}[v] = 0,$$

$$\left(D + M^{\frac{Df}{f}}\right)\mathbf{T}_f[v] = 0 \iff v \in \operatorname{Ker} D \iff \left(D - M^{\frac{Df}{f}}\right)\widetilde{\mathbf{T}}_f[v] = 0.$$

Proof. By Lemma 3.15 part 5, the inverse of $\mathbf{T}_{f_j,j}$ exists. Then the inverse of $\mathbf{T}_f$ exists, thus $Dv = 0 \iff (\forall j)\ \mathbf{T}_{f_j,j}[Dv] = 0 \iff \mathbf{T}_f[Dv] = 0$. Therefore the first equivalence is given by (57). The second equivalence follows from the observation that when f is replaced by $1/f$, the logarithmic derivative f'/f is replaced by $-f'/f$. □

Using the relations described in (53) and (54) we introduce the quaternionic generalization of d_n^f in the following way:

Definition 3.20. Let f_j be a nonvanishing coefficient for Ω in the direction x_j, $j = 1, 2, 3$, and consider $f = f_1 f_2 f_3$. The nth *quaternionic* f_j*-derivative* $\mathbf{d}_n^{f_j,j}$ is defined as follows acting on the variable x_j, of a differentiable function u by

$$\mathbf{d}_n^{f_j,j}[u] = d_n^{f_j}[u_{0,j}] + d_n^{1/f_j}[u_{0,j}^{\perp}] \tag{63}$$

and the nth *quaternionic* f*-derivative* acting on (x_1, x_2, x_3) is

$$\mathbf{d}_n^f[u] = \mathbf{d}_n^{f_1,1}\mathbf{d}_n^{f_2,2}\mathbf{d}_n^{f_3,3}[u]. \tag{64}$$

Lemma 3.21. *Let $u \in C^n(\Omega, \mathbb{H}(\mathbb{C}))$. Then we have for $j = 1, 2, 3$,*

$$\mathbf{d}_k^{f_j,j}[\mathbf{T}_{f_j,j}[u]] = \begin{cases} \mathbf{T}_{1/f_j,j}[\partial_j^k u], & k \text{ odd}, \\ \mathbf{T}_{f_j,j}[\partial_j^k u], & k \text{ even}, \end{cases} \tag{65}$$

for $0 \le k \le n$. Since the operators $\mathbf{T}_{f_j,j}, \mathbf{T}_{1/f_j,j}$ fix the value at the origin we have

$$\mathbf{d}_k^{f_j,j}[\mathbf{T}_{f_j,j}[u]](0) = (\partial_j^k u)(0). \tag{66}$$

Proof. Let $u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \in C^n(\Omega, \mathbb{H}(\mathbb{C}))$. Then each component u_i is of class C^n. Using Proposition 3.11 we have for each component u_i for $i = 0, 1, 2, 3$,

$$d_k^{f_j,j}[u_i] = \begin{cases} T_{1/f}[\partial_j^k u_i], & k \text{ odd}, \\ T_f[\partial_j^k u_i], & k \text{ even}. \end{cases} \tag{67}$$

and

$$d_k^{1/f_j,j}[u_i] = \begin{cases} T_f[\partial_j^k u_i], & k \text{ odd}, \\ T_{1/f}[\partial_j^k u_i], & k \text{ even}. \end{cases} \tag{68}$$

Since $\mathbf{d}_k^{f_j,j}$ acts componentwise the first part is proved. Equation (66) follows since the operators $T_f, T_{1/f}$ fix the value at the origin. $\square$

Chapter 4

Approximation theory for $\operatorname{Ker}\left(D + M^{Df/f}\right)$

In this chapter we will prove generalized Runge and Taylor theorems for solutions $v \in \operatorname{Ker}(D + M^{Df/f})$. We also give an approximation theorem in the L^2 norm. These results will be consequences of the analogous theorems for monogenic functions $u \in \operatorname{Ker} D$ and the continuity of the transmutation operator $\mathbf{T}_f$ which we introduced in the previous chapter. The Grigor'ev polynomials will be used for the approximations.

In this chapter we assume that $\Omega \subseteq \mathbb{R}^3$ is an open bounded set which is convex with respect to the variable x_j. Note that when Ω is convex with respect to all three of the variables x_j the complement of Ω is connected.

4.1 Runge theorem

In this section we work with uniform approximation on compact subsets of Ω.

4.1.1 Compact-uniform approximation of monogenic functions

Our first result is a generalization of Runge's approximation theorem which, in the original version (see for example [75, Theorem 13.9]) states that a holomorphic function in a plane domain may be approximated uniformly on compact sets of that domain by rational functions whose poles lie in the complement of the domain. This was generalized for quaternionic functions by K. Nôno (see [70]) with a slightly different differential operator. Nôno's operator is as follows. Let

$$x = x_0 + \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3 = z_1 + z_2 \mathbf{e}_2$$

where $z_1 = x_0 + \mathbf{e}_1 x_1$ and $z_2 = x_2 + \mathbf{e}_1 x_3$. He considers the differential operator

$$\frac{\partial}{\partial z^*} = \frac{\partial}{\partial \overline{z}_1} + \mathbf{e}_2 \frac{\partial}{\partial \overline{z}_2}.$$

Any quaternionic function u which satisfies $\dfrac{\partial u}{\partial z^*} = 0$, will be called Nôno-hyperholomophic.

Proposition 4.1 ([70]). *Let Ω be an open subset in $\mathbb{C}^2$ and let K be a compact subset of Ω. Then every function which is Nôno-hyperholomorphic in Ω can* *be approximated uniformly on K by Nôno-hyperholomorphic polynomials.*

The Runge theorem in Clifford algebras was given by Delanghe and Brackx [15, Theorem 4.1]. They consider the generalized Cauchy-Riemann differential operator $D_{\mathrm{Cl}} = \partial_0 + \sum_{j=1}^{n} \mathbf{e}_j \partial_j$ acting on functions u defined in an open subset Ω of $\mathbb{R}^{n+1}$ and with values in a Clifford Algebra $\mathcal{C}_{1,n}$ defined via a quadratic form on an n-dimensional real vector space. A Clifford-valued function u is called a hyperholomorphic function when $D_{\mathrm{Cl}} u = 0$ in Ω.

Proposition 4.2. *[15, Theorem 3.1] Let Ω be a domain in $\mathbb{R}^{m+1}$ whose complement is connected. Let K be a compact subset of Ω. Then the hyperholomorphic functions on Ω can be uniformly approximated on K by hyperholomorphic polynomials.*

In fact Delange and Brackx prove the analogous result for powers D_{Cl}^k of the differential operator D_{Cl}.

Versions of Runge's Theorem for harmonic functions in several variables, can be found for example in [2] and [21]. The common element in all of these theorems is uniform convergence on compact subsets.

4.1.2 Runge theorem for $D + M^{Df/f}$

Definition 4.3. For a compact subset $K \subseteq \Omega$, we introduce the seminorm

$$\|v\|_K = \sup_K \left(\sum_{j=0}^{3} |v_j|^2 \right)^{1/2},$$

where $v = \sum_{j=0}^{3} v_j \mathbf{e}_j \in C^0(\Omega, \mathbb{H}(\mathbb{C}))$.

Definition 4.4. Let $E \subseteq C^0(\Omega, \mathbb{H}(\mathbb{C}))$. A complex-linear operator $T\colon E \to C^0(\Omega, \mathbb{H}(\mathbb{C}))$ will be called *bounded on compact subsets of* Ω if for every compact $K \subseteq \Omega$ there is a constant $c_K > 0$ such that

$$\|Tv\|_K \leq c_K \|v\|_K,$$

for every $v \in E$. We write $\|T\|_K$ for the minimal such c_K,

Lemma 4.5. *Let $T\colon E \to C^0(\Omega, \mathbb{H}(\mathbb{C}))$ be a bounded $\mathbb{C}$-linear operator on compact subsets of Ω, where $E \subseteq C^0(\Omega, \mathbb{H}(\mathbb{C}))$ is an $\mathbb{H}(\mathbb{C})$ submodule. Let $\{u_k\}_{k=0}^{\infty}$ be a complete system in E in the sense of compact-uniform convergence. Then the system*

$$\left\{ T[u_k], T[u_k \mathbf{e}_1], T[u_k \mathbf{e}_2], T[u_k \mathbf{e}_3] \right\}_{k=0}^{\infty} \tag{69}$$

is complete in $T(E)$ in the sense of compact-uniform convergence.

Proof. Let $v \in T(E)$, i.e. $v = T(u)$ for some $u \in E$. Let K be a compact subset of Ω and $\epsilon > 0$. Assume that $\|T\|_K > 0$ since otherwise the argument will be trivial. Since $\{u_k\}$ is complete, we may take $\{\alpha_k\}_{k=0}^{N} \subseteq \mathbb{H}(\mathbb{C})$, such that

$$\left\| u - \sum_{k=0}^{N} u_k \alpha_k \right\|_K < \frac{\epsilon}{\|T\|_K}.$$

Then we apply T,

$$\begin{aligned} \left\| v - T\left[\sum_{k=0}^{N} u_k \alpha_k \right] \right\|_K &= \left\| T\left[u - \sum_{k=0}^{N} u_k \alpha_k \right] \right\|_K, \\ &< \|T\|_K \left\| u - \sum_{k=0}^{N} u_k \alpha_k \right\|_K, \\ &< \epsilon. \end{aligned}$$

Write $\alpha_k = \sum_{i=0}^3 \alpha_{ki}\mathbf{e}_i$ with $\alpha_{ki} \in \mathbb{C}$. Since T is $\mathbb{C}$-linear, $T[u_k\alpha_k] = \sum_{i=0}^3 T[u_k\mathbf{e}_i]\alpha_{ki}$. From this the result follows. □

Given a nonvanishing coefficient f for Ω with in the direction x_j, for simplicity we will abbreviate by $\mathbf{T}$ the operator $\mathbf{T}_{f,j}$. Since the Grigor'ev polynomials are complete in the compact convergence by Proposition 2.28 we have the following corollary. It requires the convexity property of Ω with respect to x_j which was specified at the beginning of this chapter.

Corollary 4.6. *Given a nonvanishing coefficient f for Ω in the single direction x_j, the following system*

$$\left\{\mathbf{T}[P^{l,m}], \mathbf{T}[P^{l,m}\mathbf{e}_1], \mathbf{T}[P^{l,m}\mathbf{e}_2], \mathbf{T}[P^{l,m}\mathbf{e}_3]\right\}_{l+m=0}^{\infty}$$

is complete in $Ker(D + M^{(f'/f)\mathbf{e}_j}) \subseteq C^1(\Omega, \mathbb{H}(\mathbb{C}))$ in compact uniform convergence.

Proof. Note that $\mathbf{T}_{f_j,j}$ is a bounded operator on compact subsets of Ω. Let $K \subseteq \Omega$ be an arbitrary compact subset. We take a compact subset K_2 convex with respect to x_j, such that $K \subseteq K_2 \subseteq \Omega$. Then

$$\begin{aligned} \|\mathbf{T}_{f_j,j}[v]\|_{K_2} &\leq N_{1,j}\|v_{0,j}\|_{K_2} + N_{2,j}\|v_{0,j}^{\perp}\|_{K_2} \\ &\leq N\|v\|_{K_2}, \end{aligned}$$

where the constants $N_{1,j}, N_{2,j}$ depend only on the corresponding kernels of the bounded Volterra operators T_{f_j}, T_{1/f_j}. This implies the boundedness on compact subsets, and Proposition 2.28 can be applied. □

Remark 4.7. Note that if we write $\mathbf{T}$ for $\mathbf{T}_{f_1 f_2 f_3}$ corresponding to a separable nonvanishing coefficient where Ω satisfies the appropriate triple symmetry, the statement of Corollary 4.6 remains valid (replacing $(f'/f)\mathbf{e}_j$ with Df/f) since $\mathbf{T}$ is the composition of bounded operators $\mathbf{T}_{f_j,j}$, and therefore is also bounded.

4.2 L^2 approximation of solutions of $D + M^{Df/f}$

We begin with the following observation.

Lemma 4.8. *Let $T\colon E \to L^2(\Omega, \mathbb{H}(\mathbb{C}))$ be a bounded $\mathbb{C}$-linear operator where $E \subseteq L^2(\Omega, \mathbb{H}(\mathbb{C}))$ is an $\mathbb{H}(\mathbb{C})$ submodule (not necessarily closed). Let $\{u_k\}_{k=0}^{\infty}$ be a complete system in E with respect to $\|\ \|_2$. Then the system* (69) *is complete in $T(E)$ with respect to $\|\ \|_2$.*

The proof is completely analogous to Lemma 4.6.

Let us recall the hypothesis of Proposition 2.30, where $\overline{\Omega} \subseteq \Omega_2$. In this situation the system $\{G(x - \xi_k)\}_{k=0}^{\infty}$ is a complete system in $L^2(\Omega, \mathbb{H}(\mathbb{C})) \cap \operatorname{Ker} D$, whenever ξ_k is a dense subset in $\partial\Omega_2$. In the following we again write $\mathbf{T} = \mathbf{T}_{f,j}$.

Corollary 4.9. *Let Ω, Ω_2 satisfy the hypothesis of Proposition 2.30 and let f be a nonvanishing coefficient for Ω in the direction x_j. The following system $\{\mathbf{T}[P^{l,m}], \mathbf{T}[P^{l,m}\mathbf{e}_1], \mathbf{T}[P^{l,m}\mathbf{e}_2], \mathbf{T}[P^{l,m}\mathbf{e}_3]\}_{l+m=0}^{\infty}$ is complete in $L^2(\Omega, \mathbb{H}(\mathbb{C})) \cap \operatorname{Ker}(D + M^{(f'/f)\mathbf{e}_j})$ with respect to $\|\ \|_2$.*

Proof. Let $(D + M^{(f'/f)\mathbf{e}_j})u = 0$, $u \in L^2(\Omega)$. Take $\epsilon > 0$. By Corollary 3.18 $u = \mathbf{T}[w]$, where $w \in \operatorname{Ker} D$. By Proposition 2.30 take coefficients $q^k \in \mathbb{H}$ such that

$$\left\| v_\pm - \sum_{j=1}^{N} G(x - \xi_k) q^k \right\|_{L^2(\Omega)} < \frac{\epsilon}{2N},$$

with N as in the proof of Theorem 4.14. Consider an open Ω_1 such that $\Omega \subseteq \overline{\Omega}_1 \subseteq \Omega_2$. Since each $G(x-\xi_k)$ is monogenic in $\overline{\Omega}_1$, the sum is monogenic and by Proposition 2.28 can be approximated uniformly in Ω via the Grigor'ev polynomials:

$$|\sum_{k=1}^{N} G(x - \xi_k) q^k - \sum_{0 \le l+m \le n} P^{l,m} c^{l,m}| < \frac{\epsilon}{2N\sqrt{V(\Omega)}},$$

where $c^{l,m} \in \mathbb{H}$ and $V(\Omega)$ is the volume of Ω. Since Ω is a bounded set, this implies L^2 convergence:

$$\| \sum_{k=1}^{N} G(x - \xi_k) q^j - \sum_{0 \le l+m \le n} P^{l,m} c^{l,m} \|_{L^2(\Omega)} < \frac{\epsilon}{2N}.$$

Using the triangle inequality and then applying $\mathbf{T}$, we show the completness of the system. □

As in Remark 4.7, the same statement holds when $\mathbf{T}$ is $\mathbf{T}_{f_1 f_2 f_3}$.

4.3 Taylor theorem

We now consider the situation in which Ω is a ball, which we suppose to be centered at the origin. Recall Proposition 2.26 which expresses every element of $\operatorname{Ker} D$ as a convergent series of "powers" which are the countably many

Grigor'ev polynomials $P^{l,m}$. In a similar way, in this section we will prove a representation in convergent series of any solution $u \in Ker(D + M^{Df/f})$ in a ball $B_R(0)$.

4.3.1 Taylor series for $D + M^{Df/f}$

Here we give an explicit formula for the Taylor coefficients of elements of $\mathrm{Ker}(D + M^{Df/f})$ in terms of a basic set of solutions given by the transmutation operator $\mathbf{T} = T_{f_1 f_2 f_3}$ defined in (59). We will work in a ball, which is automatically symmetric with respect to all variables. Since the components of a monogenic function v satisfy the relation $\partial_3 v = \mathbf{e}_3(\partial_1 v \mathbf{e}_1 + \partial_2 v \mathbf{e}_2)$, the necessary information is contained in the first two partial derivatives.

Theorem 4.10. *Let $f = f_1 f_2 f_3$ be as in Definition 3.17 where $\Omega = B_R(0)$. Let $u \in \mathrm{Ker}(D + M^{Df/f})$ in $B_R(0)$. Then u can be expanded into a Taylor series in the form*

$$u = \sum_{n=0}^{\infty} \sum_{l+m=n} \sum_{i=0}^{4} \mathbf{T}[P^{l,m}\mathbf{e}_i] c_{l,m}^i, \tag{70}$$

converging uniformly on compact subsets of $B_R(0)$. The coefficients

$$c_{l,m} = \sum_{i=0}^{4} c_{l,m}^i \mathbf{e}_i,$$

with $c_{l,m}^i \in \mathbb{C}$ $(l, m \geq 0)$ are calculated as follows:

$$c_{l,m}^i = \frac{1}{n!} \left(\mathbf{d}_l^{f_1,1} \mathbf{d}_m^{f_2,2} u_i \right)(0). \tag{71}$$

Proof. Let $w = \mathbf{T}^{-1}[u]$. Then w is a monogenic function, so we can expand it into a Taylor series by Proposition 31 as follows

$$w = \sum_{n=0}^{\infty} \sum_{l+m=n} P^{l,m} c_{l,m},$$

where the coefficients $c_{l,m}$ given by (32) for w. Application of $\mathbf{T}$ give us a series for u,

$$\begin{aligned} u &= \mathbf{T}\left[\sum_{n=0}^{\infty} \sum_{l+m=n} P^{l,m} c_{l,m}\right] = \sum_{n=0}^{\infty} \sum_{l+m=n} \mathbf{T}\left[P^{l,m} c_{l,m}\right], \\ &= \sum_{n=0}^{\infty} \sum_{l+m=n} \sum_{i=0}^{4} \mathbf{T}\left[P^{l,m} \mathbf{e}_i\right] c_{l,m}^{i}, \end{aligned}$$

converging uniformly in $B_{R'}(0)$ due to uniform boundedness of $\mathbf{T}$. Then it only remains to evaluate the coefficients $c_{l,m}$, by Lemma 3.21 and by (66) we have the following,

$$\begin{aligned} c_{l,m} &= \frac{1}{n!} \left(\frac{\partial^n w}{\partial_1^l \partial_2^m} \right)(0), \\ &= \frac{1}{n!} \left(\mathbf{d}_l^{f_1,1} \mathbf{d}_m^{f_2,2} u \right)(0), \end{aligned}$$

which is equal to (71). □

4.3.2 Decomposition of $\mathrm{Ker}(D + \lambda(x_3) + M^{\gamma(x_3)\mathbf{e}_3})$

Up to now we have considered quaternionic operators of the form $D + M^{Df/f}$ where f is a separable scalar-valued function of three variables. Theorem 3.16 permits us to determine $\mathrm{Ker}(D + M^{\frac{Df}{f}})$ because the transmutation operators are invertible. We use this to find the kernel of $\mathrm{Ker}\left(D + \lambda(x_3) + M^{\gamma(x_3)\mathbf{e}_3}\right)$ where γ, λ are functions of a single space variable, which without loss of generality we are taking to be x_3, for consistency with the definition of the operators $\mathcal{P}^{\pm}$ in (10). Thus Ω will be convex with respect to x_3.

The following decomposition due to V. Kravchenko plays a main role in this thesis. It reduces the study of $D+\lambda(x_3)+M^{\gamma(x_3)\mathbf{e}_3}$ to two operators of the form already solved.

Proposition 4.11. *[45] Let $\lambda(x_3), \gamma(x_3) \in L^1([-a,a])$. Then*

$$D+\lambda+M^{\gamma\mathbf{e}_3} = \mathcal{P}^+\left(D+M^{(\gamma+\lambda\mathbf{i})\mathbf{e}_3}\right)+\mathcal{P}^-\left(D+M^{(\gamma-\lambda\mathbf{i})\mathbf{e}_3}\right), \tag{72}$$

acting on $C^1(\Omega,\mathbb{H}(\mathbb{C}))$. Further we have the following decomposition:

$$\operatorname{Ker}\left(D+\lambda+M^{\gamma\mathbf{e}_3}\right) = \mathcal{P}^+\operatorname{Ker}\left(D+M^{(\gamma+\lambda\mathbf{i})\mathbf{e}_3}\right)\oplus\mathcal{P}^-\operatorname{Ker}\left(D+M^{(\gamma-\lambda\mathbf{i})\mathbf{e}_3}\right). \tag{73}$$

We proceed to give our second main result. It expresses the solution as a combination of transmutation operators applied to monogenic functions. References to $\mathbf{T}_{f^{\pm}}$ in the rest of this chapter will mean $\mathbf{T}_{f^{\pm},3}$.

Theorem 4.12. *Let Ω be convex with respect to x_3. Let $\lambda(x_3), \gamma(x_3) \in L^1(-a,a)$ and let $\Theta_{\pm}\colon(-a,a)\to\mathbb{C}$ be such that*

$$\Theta'_{\pm} = \gamma \pm \mathbf{i}\lambda \tag{74}$$

and consider the nonvanishing coefficients $f_{\pm}(x_1,x_2,x_3) = e^{\Theta_{\pm}(x_3)}$ for Ω in the direction x_3. Then every $u \in \mathbb{C}^1(\Omega,\mathbb{H}(\mathbb{C}))$ in $\operatorname{Ker}(D+\lambda+M^{\gamma\mathbf{e}_3})$ admits a decomposition

$$u = u_+ + u_-, \tag{75}$$

where

$$u_+ = \mathbf{T}_{f_+}[\mathcal{P}^+ v_+], \quad u_- = \mathbf{T}_{f_-}[\mathcal{P}^- v_-], \tag{76}$$

and where $v_+, v_- \in \operatorname{Ker} D$ *are unique* $\mathbb{H}$*-valued monogenic functions. Conversely, every* u *of the form* (75), (76) *for monogenic* $v_\pm$ *is in* $\operatorname{Ker}(D + \lambda + M^{\gamma \mathbf{e}_3})$.

Proof. Let $w_+ = \mathcal{P}^+ \mathbf{T}_{f_+}^{-1}[u]$, $w_- = \mathcal{P}^- \mathbf{T}_{f_-}^{-1}[u]$. Then by (57),

$$\mathbf{T}_{1/f_\pm}[Dw_\pm] = (D + M^{(\gamma \pm \lambda \mathbf{i})\mathbf{e}_3})\, \mathcal{P}^\pm\, u = \mathcal{P}^\pm (D + M^{(\gamma \pm \lambda \mathbf{i})\mathbf{e}_3}) u,$$

because $\lambda \mathbf{i}\mathbf{e}_3$ and $(1 + \mathbf{i}\mathbf{e}_3)/2$ commute, and $\mathcal{P}^\pm$ commutes with $\mathbf{T}_f$ and D. Supposing $(D + \lambda + M^{\gamma \mathbf{e}_3})u = 0$, by Proposition 4.11 the sum of the two functions $\mathbf{T}_{1/f_\pm}[Dw_\pm]$ is zero, so they both vanish as they are in the image of the complementary projectors $\mathcal{P}^\pm$. By invertibility of $\mathbf{T}_{1/f_\pm}$, we have $Dw_\pm = 0$.

Now define $v_\pm = 2 \operatorname{Re} w_\pm$. These satisfy $Dv_\pm = 0$ and by Lemma 2.7 we have $\mathcal{P}^\pm\, v_\pm = \mathcal{P}^\pm\, w_\pm = w_\pm$. By construction,

$$u = \mathcal{P}^+ u + \mathcal{P}^- u = \mathbf{T}_{f_+} w_+ + \mathbf{T}_{f_-} w_-,$$

which gives the desired decomposition.

The uniqueness is verified by writing the decomposition as $u = \mathcal{P}^+ \mathbf{T}_{f_+}[v_+] + \mathcal{P}^- \mathbf{T}_{f_-}[v_-]$ which by Lemma 2.7 shows that $\mathbf{T}_{f_+}[v_+]$ and $\mathbf{T}_{f_-}[v_-]$ are determined by u, so $v_\pm$ are also determined by u. The converse follows immediately from (57). □

Definition 4.13. Given $\lambda(x_3)$, $\gamma(x_3)$, the *associated* (λ, γ)*-powers* in Ω are defined as the $\mathbb{H}(\mathbb{C})$-valued functions

$$\sigma_{\pm}^{l,m} = \mathbf{T}_{f_\pm}[P^{l,m}],$$
$$\tau_{\pm}^{l,m} = \mathbf{T}_{1/f_\pm}[P^{l,m}]\mathbf{e}_1,$$

where the transmutation operators are constructed with the nonvanishing coefficients $f_\pm$ given by (74).

Any $c \in \mathbb{H}(\mathbb{C})$ is naturally decomposed as $c = c' + c''$ with $c' \in \mathbb{C} + \mathbf{e}_3\mathbb{C}$, $c'' \in \mathbf{e}_1\mathbb{C} + \mathbf{e}_2\mathbb{C}$, the natural projections of c on these linear subspaces. Let

$$a^{\pm}(c) = \mathcal{P}^{\pm}\, c', \quad b^{\pm}(c) = -\mathbf{e}_1\, \mathcal{P}^{\pm}\, c'', \tag{77}$$

so $a^{\pm}, b^{\pm}\colon \mathbb{H}(\mathbb{C}) \to \mathbb{C} + \mathbf{e}_3\mathbb{C}$. Since $\mathcal{P}^{\pm}$ commutes with c' while $\mathbf{e}_3$ anticommutes with c'', we find that

$$a^{\pm}(c) + \mathbf{e}_1 b^{\mp}(c) = \left(\frac{1 \pm \mathbf{i}\mathbf{e}_3}{2}\right) c. \tag{78}$$

Theorem 4.14. *Let λ, γ be as in Theorem 4.12. Then every $\mathbb{H}(\mathbb{C})$-valued solution u of $(D + \lambda + M^{\gamma \mathbf{e}_3})u = 0$ in Ω can be uniformly approximated on each compact $K \subseteq \Omega$ by right-linear combinations of the (λ, γ)-powers, more precisely, by expressions of the form*

$$\sum_{0 \le l+m \le n} \big((\sigma_+^{l,m} a_+^{l,m} + \sigma_-^{l,m} a_-^{l,m}) + (\tau_+^{l,m} b_+^{l,m} + \tau_-^{l,m} b_-^{l,m})\big), \tag{79}$$

with constant coefficients $a_{\pm}^{l,m}, b_{\pm}^{l,m} \in \mathbb{C} + \mathbb{C}\mathbf{e}_3$.

Proof. Suppose $(D + \lambda + M^{\gamma \mathbf{e}_3})u = 0$ in Ω and take $\epsilon > 0$, $K \subseteq \Omega$ compact. Express $u = \mathbf{T}_{f_+}[w_+] + \mathbf{T}_{f_-}[w_-]$ as in Theorem 4.12, where $w_\pm = \mathcal{P}^{\pm}\, v_\pm$, $Dv_\pm = 0$. Take any compact subset K_2 such that $K \subseteq K_2 \subseteq \Omega$ and K_2

satisfies the symmetry property with respect to x_3 as in Lemma 3.15. Let $N = \max(\|\mathbf{T}_{f_+}\|_{K_2}, \|\mathbf{T}_{f_-}\|_{K_2})$, and apply Proposition 2.28 to the monogenic $\mathbb{H}$-valued functions $v_\pm$ to obtain coefficients $c_\pm^{l,m} \in \mathbb{H}$ satisfying

$$\Big|v_\pm - \sum_{0\le l+m\le n} P^{l,m} c_\pm^{l,m}\Big| < \frac{\epsilon}{2N}$$

uniformly in K_2. Now apply $\mathcal{P}^\pm\, \mathbf{T}_{f_\pm}$ to the difference appearing in this inequality; since $u = \mathcal{P}^+\, u + \mathcal{P}^-\, u$, we have by (2.4), (75) and the triangle inequality that

$$\Big|u - \sum_{0\le l+m\le n} \mathcal{P}^+\, \mathbf{T}_{f_+}[P^{l,m} c_+^{l,m}] - \sum_{0\le l+m\le n} \mathcal{P}^-\, \mathbf{T}_{f_-}[P^{l,m} c_-^{l,m}]\Big| < \epsilon$$

in K. Note that Lemma 3.15 says also that

$$\mathcal{P}^\pm\, \mathbf{T}_{f_\pm}[vc] = \mathbf{T}_{f_\pm}[v](\mathcal{P}^\pm\, c') + \mathbf{T}_{1/f_\pm}[v](\mathcal{P}^\pm\, c'')$$

for any complex quaternionic c with $c = c' + c''$ as in (77). For this reason we take

$$a_\pm^{l,m} = a^\pm(c_\pm^{l,m}), \quad b_\pm^{l,m} = b^\pm(c_\pm^{l,m}), \tag{80}$$

and refer to Definition 4.13. □

In the case that $\Omega = B_R(0)$ we can calculate the coefficients of the approximating functions explicitly.

Theorem 4.15. *Let $u \in \mathrm{Ker}(D + \lambda + M^{\gamma \mathbf{e}_3})$ in $B_R(0)$. Then u can be expanded into a Taylor series in (λ,γ)-powers of the form*

$$u = \sum_{n=0}^{\infty} \sum_{l+m=n} \big((\sigma_+^{l,m} a_+^{l,m} + \sigma_-^{l,m} a_-^{l,m}) + (\tau_+^{l,m} b_+^{l,m} + \tau_-^{l,m} b_-^{l,m})\big), \tag{81}$$

converging uniformly on compact subsets of $B_R(0)$. The coefficients $a^{l,m}, b^{l,m} \in$

$\mathbb{C} + \mathbb{C}\mathbf{e}_3$ *are given by*

$$a_\pm^{l,m} = a^\pm(\frac{1}{n!}\partial_1^l \partial_2^m\big|_{x=0} u),$$
$$b_\pm^{l,m} = b^\pm(\frac{1}{n!}\partial_1^l \partial_2^m\big|_{x=0} u), \tag{82}$$

with $a^\pm, b^\pm$ *defined in* (77).

Proof. Take $v_\pm \in \operatorname{Ker} D$, $\mathcal{P}^\pm v_\pm = \mathbf{T}_{f_\pm}^{-1}[\mathcal{P}^\pm u]$ as before. By Proposition 2.26, each of these monogenic functions can be expanded into Taylor series

$$v_\pm = \sum_{n=0}^{\infty} \sum_{l+m=n} P^{l,m} c_{l,m}^\pm,$$

with coefficients $c_{l,m}^\pm \in \mathbb{H}$ given by (32) for $v_\pm$. Application of $\mathcal{P}^\pm \mathbf{T}_{f_\pm}$ gives us a series for $u_\pm$,

$$u_\pm = \mathcal{P}^\pm \mathbf{T}_{f_\pm}\big[\sum_{n=0}^{\infty} \sum_{l+m=n} P^{l,m} c_{l,m}^\pm\big] = \sum_{n=0}^{\infty} \sum_{l+m=n} \mathcal{P}^\pm \mathbf{T}_{f_\pm}\big[P^{l,m} c_{l,m}^\pm, \big],$$

converging uniformly in $B_{R'}(0)$ due to uniform boundedness of $\mathbf{T}_{f_\pm}$. The series can be written as (81) as in the proof of Theorem 4.14; it remains to evaluate the coefficients. Since $\mathbf{T}_f$ operates only on the variable x_3,

$$\frac{\partial^n}{\partial_1^l \partial_2^m} u_\pm = \mathbf{T}_{f_\pm}[\mathcal{P}^\pm \frac{\partial^n}{\partial_1^l \partial_2^m} v_\pm]$$

for all $x \in \Omega$. Since Volterra operators preserve values at the origin,

$$\frac{\partial^n}{\partial_1^l \partial_2^m}\bigg|_{x=0} u_\pm = \mathcal{P}^\pm \frac{\partial^n}{\partial_1^l \partial_2^m}\bigg|_{x=0} v_\pm = n!\, \mathcal{P}^\pm c_\pm^{l,m}.$$

Thus $a_\pm^{l,m}$, $b_\pm^{l,m}$ in (81) are in fact given by (80). □

In the next chapter we will give applications of Theorem 4.12 to several specific differential equations.

Chapter 5

Complete system of solutions to physical systems

Now that we have solved the basic complex quaternionic differential equation $(D+\lambda+M^{\gamma})u=0$, we apply it to obtain complete solutions for several basic equations of mathematical physics. As usual Ω is a bounded domain in $\mathbb{R}^3$ and occasionally we will require the convexity property with with respect to one of the variables x_1, x_2, x_3. We will begin by applying our results to the simplest case, that of Beltrami fields, in particular force-free fields. Then we will look at certain cases of the Maxwell, Dirac, Helmholtz, and Schrödinger equations.

5.1 Beltrami fields

Definition 5.1. A *Beltrami field* in Ω with potential $\lambda\colon \Omega \to \mathbb{C}$ is a complex-valued vector field $\mathbf{F}\colon \Omega \to \mathbb{C}^3$ which satisfies the equation

$$\operatorname{curl}\mathbf{F} + \lambda\mathbf{F} = 0. \tag{83}$$

Such fields appear in many branches of physics such as astrophysics [1], electromagnetics [63] and plasma physics [72].

When the Beltrami field $\mathbf{F}$ also satisfies the condition $\operatorname{div}\mathbf{F} = 0$, then the Beltrami field is known as a *force-free field*. This type of vector field appears in areas of physics such as magnetohydrostatics (see [39]); the structure and dynamics of the solar corona are determined by magnetic fields, which are modeled by force free fields (see [80]).

One may verify that a purely vectorial function $\vec{u} \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is a force-free field if and only if $(D+\lambda)\vec{u} = 0$, but this criterion does not apply in the case of Beltrami fields $\mathbf{F}$ since these fields do not necessarily satisfy $\operatorname{div}\mathbf{F} = 0$. Indeed these fields satisfy the following condition:

$$\lambda \operatorname{div}\mathbf{F} + \operatorname{grad}\lambda \cdot \mathbf{F} = 0.$$

In [45] the following relation was found. For this we need λ to be a nonvanishing function and we fix a branch of $\sqrt{\lambda}$. The shape of the domain in question is not relevant.

Proposition 5.2. *[45, Proposition 1] A $\mathbb{C}^3$-valued function $\boldsymbol{F}$ is a solution of* (83) *if and only if the purely vectorial function $\vec{u} = \sqrt{\lambda}\boldsymbol{F}$ is a solution of* *the equation*

$$(D + M^{\lambda+\vec{\lambda}_1})\vec{u} = 0, \tag{84}$$

where $\vec{\lambda}_1 = \frac{\operatorname{grad}\sqrt{\lambda}}{\sqrt{\lambda}}$.

We are interested in solutions of the equation

$$(D + M^{\lambda+\gamma\mathbf{e}_1})u = 0.$$

If we assume that λ, γ depend only on one variable x_3, then we have the following decomposition.

Proposition 5.3. *[45, Proposition 4] The following equality is true:*

$$D + \lambda + M^{\gamma\mathbf{e}_3} = \mathcal{P}^+(D + M^{(\gamma+\mathbf{i}\lambda)\mathbf{e}_3}) + \mathcal{P}^-(D + M^{(\gamma-\mathbf{i}\lambda)\mathbf{e}_3}), \tag{85}$$

where $P^\pm = \frac{1}{2}M^{1\pm\mathbf{i}\mathbf{e}_3}$. *Moreover, every solution* $\vec{u}$ *of* (85) *can be written as* $u = P^+v_1 + P^-v_2$, *where the functions* v_1, v_2 *satisfy the equations*

$$(D + M^{(\gamma+\mathbf{i}\lambda)\mathbf{e}_3})v_1 = 0, \tag{86}$$

$$(D + M^{(\gamma-\mathbf{i}\lambda)\mathbf{e}_3})v_2 = 0. \tag{87}$$

In this section we will consider λ a nonzero complex constant and also when it depends on a single variable. When it is constant there is a direct relation to the operator $D + \lambda$, so we can apply the results of the preceding sections (with $\gamma = 0$) to give an explicit representation of the solutions of (83). The main complication is to find solutions of $(D + \lambda)u = 0$ with vanishing scalar part. For nonconstant λ we give a complete system of solutions and we exhibit a particular solution by a different method.

5.1.1 λ is a complex constant

When the potential $\lambda \neq 0$ of a Beltrami field is a complex constant, solutions of (83) are known as Trkalian fields [13]. In this case the factorization

$$-(D-\lambda)(D+\lambda) = \Delta + \lambda^2 \tag{88}$$

of the Helmholtz operator $\Delta + \lambda^2$ is valid, where Δ is the three-dimensional Laplacian. Therefore whenever $u = u_0 + \vec{u}$ is in $\operatorname{Ker}(D+\lambda)u$, it also satisfies the Helmholtz equation

$$(\Delta + \lambda^2)u = 0, \tag{89}$$

and the scalar nature of λ implies that each quaternionic component of u individually satisfies the Helmholtz equation. We will use the following observation.

Proposition 5.4. *[16] Let $u_0 \colon \Omega \to \mathbb{C}$ be a scalar function that satisfies the Helmholtz equation* (89) *where $\lambda \in \mathbb{C} \setminus \{0\}$ is constant. Let*

$$\vec{u} = -\frac{1}{\lambda} Du_0. \tag{90}$$

Then $\vec{u}$ is a purely vectorial function such that $u_0 + \vec{u} \in \operatorname{Ker}(D+\lambda)$.

Indeed from (89) it follows immediately that

$$(D+\lambda)(u_0 - \frac{1}{\lambda}Du_0) = -\frac{1}{\lambda}(D+\lambda)(D-\lambda)u_0 = 0.$$

With this idea, we can construct purely vectorial solutions of $(D+\lambda)\vec{u} = 0$ from solutions which are not vectorial. Given $u \in \operatorname{Ker}(D+\lambda)$, the scalar part u_0 satisfies the Helmholtz equation (89) and using Proposition 5.4 we can construct a function $\tilde{u} \in \operatorname{Ker}(D+\lambda)$ with scalar part u_0. This makes the difference $u - \tilde{u} \in \operatorname{Ker}(D + \lambda)$ purely vectorial.

We introduce the vector-field valued operator on $\mathbb{H}(\mathbb{C})$-valued functions.

Definition 5.5. Let $\lambda \in \mathbb{C} \setminus \{0\}$. The vectorializing operator $\mathcal{V} = \mathcal{V}_\lambda$ is defined by

$$\mathcal{V}[u] = \operatorname{Vec} u + \frac{1}{\lambda} D \operatorname{Sc} u. \tag{91}$$

Proposition 5.6. *The operator $\mathcal{V}$ sends* $\operatorname{Ker}(D + \lambda)$ *to vector fields in* $\operatorname{Ker}(D + \lambda)$*, more precisely,*

$$\mathcal{V}\colon \operatorname{Ker}(D + \lambda) \cap C(\Omega, \mathbb{H}(\mathbb{C})) \to \operatorname{Ker}(D + \lambda) \cap C(\Omega, \operatorname{Vec} \mathbb{H}(\mathbb{C})).$$

Further, $\mathcal{V}$ fixes every vector field in $\operatorname{Ker}(D + \lambda)$.

Proof. By construction $\mathcal{V}[u]$ is trivially vectorial. When $(D+\lambda)(u_0+\vec{u}) = 0$ with λ constant, we find that $(D + \lambda)\, \mathcal{V}[u_0 + \vec{u}] = (D + \lambda)(\vec{u} + (1/\lambda)Du_0) = -(D+\lambda)u_0 + (D+\lambda)(1/\lambda)Du_0 = (1/\lambda)(D^2 - \lambda^2)u_0 = 0$ since the scalar part u_0 satisfies the Helmholtz equation. $\square$

Observe that

$$\operatorname{Ker} \mathcal{V} = \{u_0 - \frac{1}{\lambda} Du_0\colon\ u_0 \in C^2(\Omega, \mathbb{C}),\ (\Delta + \lambda^2)u_0 = 0\}.$$

i.e. given $\mathbf{F} = \mathcal{V}[u]$ also we have $\mathbf{F} = \mathcal{V}[u + w_0 - \frac{1}{\lambda} Dw_0]$, for any solution w_0 of Helmholtz equation. Also, one checks using (35) that $\mathbf{F}$ is a Beltrami field with potential λ if and only if $(D + \lambda)\mathbf{F} = 0$. Since trivially $\mathcal{V}^2 = \mathcal{V}$, the following decomposition holds.

Corollary 5.7.

$$\operatorname{Ker}(D + \lambda) = \operatorname{Ker} \mathcal{V}\ \oplus\ \operatorname{Ker}(D + \lambda) \cap C(\Omega, \operatorname{Vec} \mathbb{H}(\mathbb{C})). \tag{92}$$

Example 5.8. Suppose Ω is convex in the direction x_3. Let $f(x_3) = e^{\mathbf{i}\lambda x_3}$ for a constant $\lambda \in \mathbb{C}\setminus\{0\}$. Thus $f'/f = \mathbf{i}\lambda$ and $f(0) = 1$. The formal powers $X^{(n)}$, $\widetilde{X}^{(n)}$ of (41) for f coincide with the formal powers $\widetilde{X}^{(n)}$, $X^{(n)}$ for $1/f$, so one finds $\varphi_0 = 1/\psi_0 = e^{\mathbf{i}\lambda x_3}$, $\varphi_1 = \psi_1 = (\sin\lambda x_3)/\lambda$. For illustration let us consider the monogenic functions $v_+ = P^{1,0}$, $v_- = \mathbf{e}_2$. Referring to Table 2.1 and recalling Lemma 3.15, we see (applying $T_{1/f}$ to the polynomial x_1 but with respect to x_3) that

$$\begin{aligned}\mathcal{P}^+\,\mathbf{T}_f[v_+] &= \mathcal{P}^+(T_{1/f}[x_1] + T_f[x_3]\mathbf{e}_2) = (x_1\psi_0(x_3) + \varphi_1(x_3)\mathbf{e}_2)^+\\ &= \frac{1}{2}x_1\psi_0(x_3) + \big((\varphi_1(x_3)\mathbf{e}_2)^+ + \frac{\mathbf{i}}{2}x_1\psi_0(x_3)\mathbf{e}_3\big),\\ \mathcal{P}^-\,\mathbf{T}_{1/f}[v_-] &= \mathcal{P}^-\left(T_{1/f}[1]\mathbf{e}_2\right) = (\psi_0(x_3)\mathbf{e}_2)^-.\end{aligned}$$

Here we have used the abbreviation $(q)^\pm = \mathcal{P}^\pm q$. This leads to vectorial solutions $\mathbf{F}_1, \mathbf{F}_2 \in \mathrm{Ker}(D+\lambda)$ given by

$$\begin{aligned}\mathbf{F}_1 &= \mathcal{V}[\mathcal{P}^+\,\mathbf{T}_f v_+]\\ &= \big((\varphi_1(x_3)\mathbf{e}_2)^+ + \frac{\mathbf{i}}{2}x_1\psi_0(x_3)\mathbf{e}_3\big) + \frac{1}{2\lambda}D(x_1\psi_0(x_3))\\ &= \left(\frac{\sin\lambda x_3}{\lambda}\mathbf{e}_2\right)^+ + \frac{\mathbf{i}}{2}x_1 e^{-\mathbf{i}\lambda x_3}\mathbf{e}_3 + \frac{1}{2\lambda}D\left(x_1 e^{-\mathbf{i}\lambda x_3}\right),\\ \mathbf{F}_2 &= \mathcal{V}[\mathcal{P}^-\,\mathbf{T}_{1/f}v_-] = (\psi_0(x_3)\mathbf{e}_2)^- = (e^{\mathbf{i}\lambda x_3}\mathbf{e}_2)^-.\end{aligned}$$

From here it is a simple matter to carry out the remaining operations. In the evaluation of D in such examples it can be useful to have the following partial derivative formulas, which are obtained from (42):

$$\partial_3\varphi_k = \frac{f'}{f}\varphi_k + k\psi_{k-1}, \quad \partial_3\psi_k = -\frac{f'}{f}\psi_k + k\varphi_{k-1}. \tag{93}$$

The functions $\mathbf{F}_1$, $\mathbf{F}_2$ satisfy (83) and hence are Beltrami fields. Further, this construction leads to a complete system of Beltrami fields.

Our procedure for constructing Beltrami fields with constant potential is the following. Given $\lambda \in \mathbb{C} \setminus \{0\}$, we take arbitrary $u, v \in \operatorname{Ker} D$. Then by (85) with $\gamma = 0$, we have $\mathcal{P}^+\mathbf{T}_f[u] \ + \ \mathcal{P}^-\mathbf{T}_{1/f}[v] \in \operatorname{Ker}(D+\lambda)$. We then apply Proposition 5.6 to confirm that $\mathcal{V}$ sends this element of $\operatorname{Ker}(D+\lambda)$ to a vector field $\mathbf{F}$ in $\operatorname{Ker}(D+\lambda)$. Every vector field $\mathbf{F}$ in $\operatorname{Ker}(D+\lambda)$ is obtained in this way (although not uniquely) since $\mathcal{V}$ is the identity on vector fields in $\operatorname{Ker}(D+\lambda)$.

This procedure for constructing vector fields in $\operatorname{Ker}(D+\lambda)$ in a bounded subset $\Omega \subseteq \mathbb{R}^3$ with convexity respect to x_3 is illustrated in Figure 1. Recall that these solutions represent free force fields.

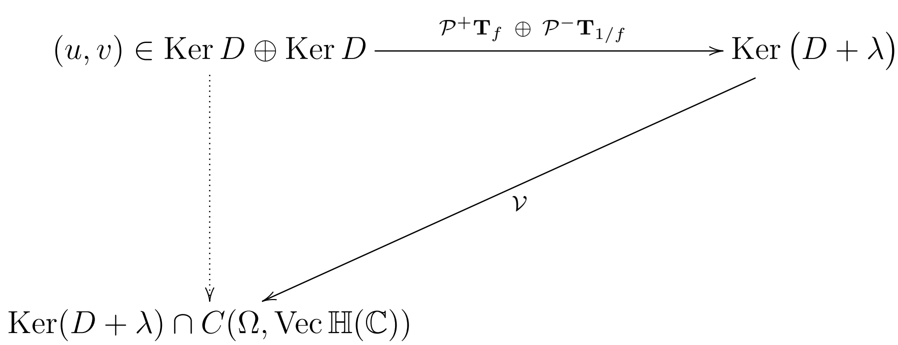

Figure 1: Construction of force-free fields

5.1.2 λ is a function of one variable

Now we deal with the more difficult situation when $\lambda(x_3)$ is a complex valued potential depending only on x_3. We cannot proceed in the same way as in the constant case because when we apply the divergence to the relation (83), the result is

$$\lambda(x_3)\,\mathrm{div}\,\mathbf{F} + (\mathrm{grad}\,\lambda(x_3))\cdot\mathbf{F} = 0,$$

while $(D+\lambda(x_3))\mathbf{F} = 0$ implies $\mathrm{div}\,\mathbf{F} = 0$.

In Proposition 5.2, we cited a general method given in [45] to transform the equation (83) to a form involving D. For the proof of Proposition 5.10 below we rephrase Proposition 5.2 in the particular context of nonvanishing coefficients. This will be the key to solving $(D+\lambda+M^{(\lambda'/\lambda)\mathbf{e}_3})u = 0$ with complex-valued coefficient functions λ. Note the essential use of complex quaternions in the decomposition.

Proposition 5.9. *Let $\lambda(x_3)$ be a $\mathbb{C}$-valued nonvanishing coefficient for Ω. A $\mathbb{C}^3$-valued function $\mathbf{F}$ is a solution of (83) if and only if the purely vectorial complex quaternionic function $\vec{u} = \sqrt{\lambda(x_3)}\mathbf{F}$ is a solution of the equation*

$$\big(D+\lambda(x_3)+M^{\frac{\lambda'(x_3)}{2\lambda(x_3)}\mathbf{e}_3}\big)\vec{u} = 0. \tag{94}$$

Figure 2 shows the method for constructing Beltrami fields via Proposition 5.9 and Proposition 4.11. At the bottom of the diagram are the arbitrary monogenic functions and at the top are the Beltrami fields going through $\mathrm{Ker}\,\big(D+\lambda(x_3)+M^{\gamma\mathbf{e}_3}\big)$.

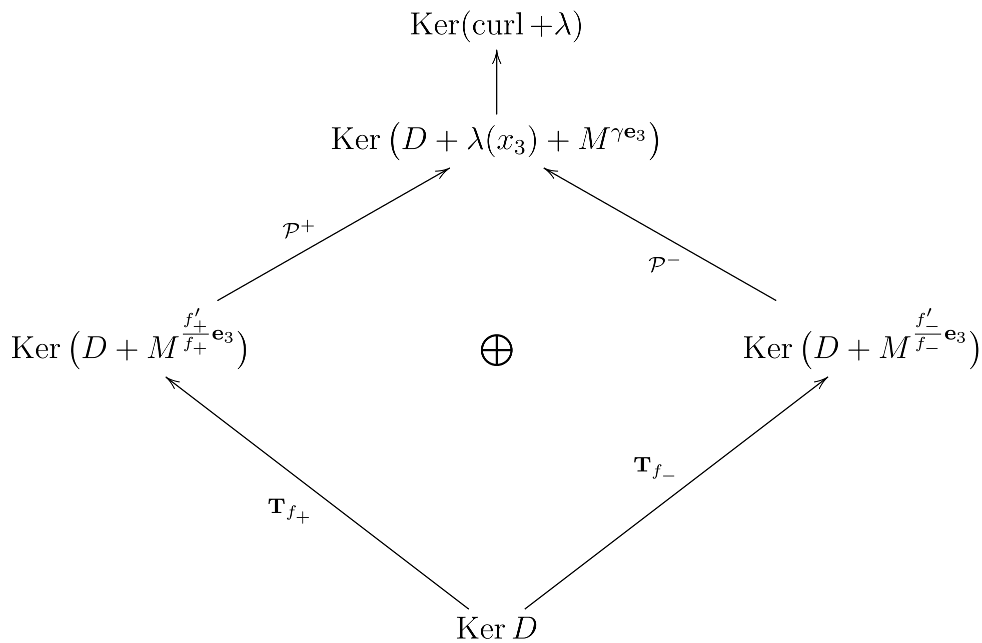

Figure 2: Construction of Beltrami fields.

By Corollary 4.6, every solution of (94) can be uniformly approximated on compact sets by right-linear combinations of (λ, γ)-powers $\sigma_{\pm}^{l,m}, \tau_{\pm}^{l,m}$ which we introduced in Definition 4.13. The approximants obtained for (83) obtained in this way from Proposition 5.9 are not vectorial. The scalar parts of the partial sums will tend to zero, so one may discard them to obtain an approximation of $\vec{u}$ by vectors, although not by solutions of (94). This procedure is illustrated by the upper part of Figure 2.

We construct here a certain class of Beltrami fields by means of functions of a complex variable, without any claim of completeness. The idea is that a transmutation along the variable x_3 does not affect functions of x_1, x_2.

Proposition 5.10. *Let $g = g_1 + \mathbf{i}g_2$, $h = h_1 + \mathbf{i}h_2$ be analytic functions of the complex variable $x_1 + \mathbf{i}x_2$. Then a particular solution of* (83) *is given by*

$$u = \mathcal{P}^+ e^{\mathbf{i}\Theta} (g_1(x_1 + \mathbf{i}x_2)\mathbf{e}_1 - g_2(x_1 + \mathbf{i}x_2)\mathbf{e}_2)$$
$$+ \mathcal{P}^- e^{-\mathbf{i}\Theta} (h_1(x_1 + \mathbf{i}x_2)\mathbf{e}_1 - h_2(x_1 + \mathbf{i}x_2)\mathbf{e}_2),$$

where Θ is any antiderivative of $\lambda(x_3)$.

Proof. Let

$$v_+(x) = g_1(x_1 + \mathbf{i}x_2)\mathbf{e}_1 - g_2(x_1 + \mathbf{i}x_2)\mathbf{e}_2,$$
$$v_-(x) = h_1(x_1 + \mathbf{i}x_2)\mathbf{e}_1 - h_2(x_1 + \mathbf{i}x_2)\mathbf{e}_2.$$

By the Cauchy-Riemann equations, $Dv_\pm = 0$. Let $f_\pm = \sqrt{\lambda(x_3)}e^{\pm\mathbf{i}\Theta(x_3)}$. Then the functions

$$u_+ = \mathcal{P}^+ \mathbf{T}_{f_+}[v_+], \quad u_- = \mathcal{P}^- \mathbf{T}_{f_-}[v_-],$$

are by construction in $\mathrm{Vec}\,\mathbb{H}(\mathbb{C})$ since $\mathbf{T}_{f_\pm}[\mathbf{e}_j] = \sqrt{\lambda(x_3)}e^{\pm\mathbf{i}\Theta}(\mathbf{e}_j)$ for $j = 1, 2$. Further, $u_\pm$ are solutions of (94) by Theorem 4.12. Finally, by Proposition 5.9 we have the result. □

5.1.3 Force-free fields

When the Beltrami field $\mathbf{F}$ also satisfies the condition $\mathrm{div}\,\mathbf{F} = 0$, the Beltrami field is known as a *force-free field*. This type of vector field appears in areas of physics like magnetohydrostatics [39] and the structure and dynamics of the solar corona are determined by magnetic fields, which are modeled by force free fields [80]. It is immediate that a purely vectorial function $\vec{u} \in C^1\big(\Omega, \mathrm{Vec}(\mathbb{H}(\mathbb{C}))\big)$ is a force-free field if and only if $(D+\lambda)\vec{u} = 0$.

Due to Theorem 4.15, we have the following series representation for force-free fields.

Corollary 5.11. *Let $u \in \mathrm{Ker}(D+\lambda(x_3))$ in $B_R(0)$. Then u can be expanded into Taylor series in λ-powers of the form:*

$$u = \sum_{n=0}^{\infty} \sum_{l+m=n} \big(\mathcal{P}^+\,\mathbf{T}_f[P^{l,m}] + \mathcal{P}^-\,\mathbf{T}_{1/f}[P^{l,m}]\big)c_{l,m}, \tag{95}$$

converging uniformly on compact subsets of $B_R(0)$. The coefficients $c_{l,m}$ are the same as in the Taylor series (31) and $f(x_3) = e^{\int_0^{x_3} \mathbf{i}\lambda(s)\,ds}$.

Proof. Note that $f_+ = f = 1/f_-$ because $\gamma = 0$ in (74), so $\tau_-^{l,m} = \sigma_+^{l,m}\mathbf{e}_1$. Applying (77), (78) to the Taylor coefficients $c_{l,m}$ we have

$$\begin{aligned} \sigma_+^{l,m}a_+^{l,m} + \tau_-^{l,m}b_-^{l,m} &= \mathbf{T}_f[P^{l,m}](a_+^{l,m} + \mathbf{e}_1 b_-^{l,m}), \\ &= \mathbf{T}_f[P^{l,m}]\left(\frac{1+\mathbf{i}\mathbf{e}_3}{2}\right)c_{l,m}, \\ &= \big(\mathcal{P}^+\mathbf{T}_f[P^{l,m}]\big)\,c_{l,m}. \end{aligned}$$

The relation $\sigma_-^{l,m}a_-^{l,m} + \tau_+^{l,m}b_+^{l,m} = \big(\mathcal{P}^-\mathbf{T}_{1/f}[P^{l,m}]\big)\,c_{l,m}$ can be obtained in the same way. The result now follows from the series (81). □

Note that when $\lambda = \gamma = 0$, then $f = 1/f = 1$ and the operator $\mathbf{T}_f$ is the identity operator. Then the Taylor series (95) turns into the series (31).

Theorem 5.12. *Let λ be an integrable function in $B_R(0)$ depending only on x_3 and consider $f(x_3) = e^{\int_0^{x_3} \mathbf{i}\lambda(s)\,ds}$. Then every force free field $\mathbf{F}$ with* *potential λ can be expanded in $B_R(0)$ into λ-powers of the form*

$$\mathbf{F} = \sum_{n=0}^{\infty} \sum_{l+m=n} \left(\mathcal{P}^+ \, \mathbf{T}_f[P^{l,m}] + \mathcal{P}^- \, \mathbf{T}_{1/f}[P^{l,m}] \right) \vec{c}_{l,m}, \tag{96}$$

where the purely vectorial coefficients $\vec{c}_{l,m} \in \mathrm{Vec}\,\mathbb{H}(\mathbb{C})$ are given by

$$\vec{c}_{l,m} = \frac{1}{n!} \frac{\partial^n \mathbf{F}(0)}{\partial_1^l \partial_2^m}.$$

The series (96) converges uniformly in every $B_{R'}(0)$ with $R' < R$.

In general, however, the individual summands in (96) are not vector valued.

5.2 Maxwell's equations

Here we apply the solution of the complex quaternionic operator to Maxwell's equations for time-harmonic electromagnetic fields, and fields in nonchiral inhomogeneous media.

5.2.1 Time harmonic equations

Maxwell's equations for time-harmonic electromagnetic fields in a chiral medium have the form

$$\mathrm{div}\, \widetilde{E}(x) = \mathrm{div}\, \widetilde{H}(x) = 0, \tag{97}$$

$$\mathrm{curl}\, \widetilde{E}(x) = \mathbf{i}\omega \widetilde{B}(x), \tag{98}$$

$$\mathrm{curl}\, \widetilde{H}(x) = -\mathbf{i}\omega \widetilde{D}(x), \tag{99}$$

with the constitutive relations (see [61])

$$\widetilde{B} = \mu \left(\widetilde{H}(x) + \beta \operatorname{curl} \widetilde{H}(x) \right), \tag{100}$$

$$\widetilde{D} = \epsilon \left(\widetilde{E}(x) + \beta \operatorname{curl} \widetilde{E}(x) \right), \tag{101}$$

where ω is the frequency, ϵ and μ are complex permittivity and permeability of a medium and β is its chirality measure. As usual x is a point in a spatial domain in $\mathbb{R}^3$ and time does not appear in the equations. The following development is given in [37]. Equations (98)-(99) can be written as follows:

$$\operatorname{curl} \widetilde{E}(x) = \mathbf{i}\omega\mu \left(\widetilde{H}(x) + \beta \operatorname{curl} \widetilde{H}(x) \right), \tag{102}$$

$$\operatorname{curl} \widetilde{H}(x) = -\mathbf{i}\omega\epsilon \left(\widetilde{E}(x) + \beta \operatorname{curl} \widetilde{E}(x) \right). \tag{103}$$

Introducing the notations

$$\widetilde{E}(x) = -\sqrt{\mu}\vec{E}(x), \tag{104}$$

$$\widetilde{H}(x) = \sqrt{\epsilon}\vec{H}(x), \tag{105}$$

we obtain the equations

$$\operatorname{curl} \vec{E}(x) = -\mathbf{i}\alpha \left(\vec{H}(x) + \beta \operatorname{curl} \vec{H}(x) \right), \tag{106}$$

$$\operatorname{curl} \vec{H}(x) = \mathbf{i}\alpha \left(\vec{E}(x) + \beta \operatorname{curl} \vec{E}(x) \right), \tag{107}$$

where $\alpha = \omega\sqrt{\epsilon\mu}$. Let us consider the following purely vectorial complex quaternionic functions:

$$\vec{\zeta}(x) = \vec{E}(x) + \mathbf{i}\vec{H}(x), \tag{108}$$

$$\vec{\eta}(x) = \vec{E}(x) - \mathbf{i}\vec{H}(x). \tag{109}$$

We have that

$$D\vec{\zeta}(x) = \operatorname{curl}\vec{E}(x) + \mathbf{i}\operatorname{curl}\vec{H}(x). \tag{110}$$

Using (106) and (107) we obtain

$$\begin{aligned} D\vec{\zeta}(x) &= -\left(\mathbf{i}\alpha\vec{H}(x) + \alpha\vec{E}(x)\right) - \alpha\beta\left(D\vec{E}(x) + \mathbf{i}D\vec{H}(x)\right), \\ &= -\alpha\vec{\zeta} - \alpha\beta D\vec{\zeta}(x). \end{aligned}$$

Thus the complex quaternionic function $\vec{\zeta}(x)$ satisfies the following equation,

$$\left(D + \frac{\alpha}{1+\alpha\beta}\right)\vec{\zeta}(x) = 0. \tag{111}$$

Analogously we obtain the equation for $\vec{\eta}$,

$$\left(D - \frac{\alpha}{1-\alpha\beta}\right)\vec{\eta}(x) = 0. \tag{112}$$

Obviously the vectors $\vec{E}, \vec{H}$, can be recovered from $\vec{\zeta}, \vec{\eta}$ as

$$\vec{E} = \frac{1}{2}(\vec{\zeta} + \vec{\eta}), \tag{113}$$

$$\vec{H} = \frac{1}{2\mathbf{i}}(\vec{\zeta} - \vec{\eta}). \tag{114}$$

We define the complex numbers

$$\lambda_1 = \frac{\alpha}{1+\alpha\beta}, \ \lambda_2 = \frac{\alpha}{1-\alpha\beta}, \tag{115}$$

so we are looking for vectorial solutions to $D + \lambda_1$ and $D - \lambda_2$. By Theorem 4.11, the following decomposition holds:

$$D - \lambda_2 = \mathcal{P}^+\left(D - M^{\mathbf{i}\lambda_2\mathbf{e}_3}\right) + \mathcal{P}^-\left(D + M^{\mathbf{i}\lambda_2\mathbf{e}_3}\right),$$

and this implies

$$\operatorname{Ker}(D - \lambda_2) = \mathcal{P}^+\operatorname{Ker}\left(D - M^{\mathbf{i}\lambda_2\mathbf{e}_3}\right) \oplus \mathcal{P}^-\operatorname{Ker}\left(D + M^{\mathbf{i}\lambda_2\mathbf{e}_3}\right). \tag{116}$$

We use this result of [37] as follows. In Subsection 5.1.1 we saw how to find vectorial solutions for $D+\lambda_1$ with the aid of $\mathcal{V}$ and the transmutation operators $\mathbf{T}_f$ and $\mathbf{T}_{1/f}$ where the nonvanishing coefficient is $f(x_3)=e^{\mathbf{i}\lambda_1 x_3}$.

In this subsection we denote by $\mathcal{V}^+$, $\mathcal{V}^-$ the operators $\mathcal{V}$ of (91) corresponding to $\lambda=\lambda_1$ and $\lambda=-\lambda_2$ respectively,

$$\mathcal{V}^+\colon\ \mathrm{Ker}(D+\lambda_1)\to\mathrm{Ker}(D+\lambda_1)\cap C(\Omega,\mathrm{Vec}\,\mathbb{H}(\mathbb{C})),$$

$$\mathcal{V}^-\colon\ \mathrm{Ker}(D-\lambda_2)\to\mathrm{Ker}(D-\lambda_2)\cap C(\Omega,\mathrm{Vec}\,\mathbb{H}(\mathbb{C})),$$

Since Maxwell's equations for time-harmonic electromagnetic fields were decomposed into two force-free fields, we have the following statements. We state them without giving the proofs, because they are essentially the same as what we did in Section 5.1 for Beltrami fields.

Every solution $u_1\in\mathrm{Ker}(D+\lambda_1)$ and $u_2\in\mathrm{Ker}(D-\lambda_2)$ in a domain with the convexity property with respect to x_3 can be obtained as

$$\begin{aligned} u_1 &= \mathcal{P}^+\mathbf{T}_f[v_-]+\mathcal{P}^-\mathbf{T}_{1/f}[v_+],\\ u_2 &= \mathcal{P}^+\mathbf{T}_{1/g}[w_-]+\mathcal{P}^-\mathbf{T}_g[w_+], \end{aligned} \tag{117}$$

where $v_\pm, w_\pm\in\mathrm{Ker}\,D$ are arbitrary and $f(x_3)=e^{\mathbf{i}\lambda_1 x_3}$, $g(x_3)=e^{-\mathbf{i}\lambda_2 x_3}$.

Theorem 5.13. *Every $\vec{E},\vec{H}$ which satisfy equations* (106),(107) *in a domain with x_3-convexity can be obtained as*

$$\vec{E}=\frac{1}{2}\left(\mathcal{V}^+[u_1]+\mathcal{V}^-[u_2]\right),$$
$$\vec{H}=\frac{1}{2\mathbf{i}}\left(\mathcal{V}^+[u_1]-\mathcal{V}^-[u_2]\right),$$

where u_1,u_2 are defined by (117).

Example 5.14. Suppose $\beta = 0$ in (100),(101). Then by (115) $\lambda_1 = \lambda_2 = \alpha$. There is a relation between f and g in (117), $g(x_3) = e^{-\mathbf{i}\alpha x_3} = 1/f(x_3)$. Take $v = v_\pm = w_\pm \in \operatorname{Ker} D$. Then the pair of vectors $(\vec{E}, \vec{H})$ has the following form:

$$\vec{E} = \frac{1}{2}\left(\mathbf{T}_{1/f}[\operatorname{Vec} v] + \mathbf{T}_f[\operatorname{Vec} v] + \frac{\mathbf{i}}{\alpha} D\left(T_f[v_1] - T_{1/f}[v_1]\right)\right),$$
$$\vec{H} = \frac{1}{2\mathbf{i}}\left(\mathbf{i}\mathbf{T}_{1/f}[\operatorname{Vec}(v\mathbf{e}_1)] + \mathbf{i}\mathbf{T}_f[\operatorname{Vec}(v\mathbf{e}_1)] + \frac{1}{\alpha} D\left(T_f[v_0] + T_{1/f}[v_0]\right)\right).$$

5.2.2 Nonchiral inhomogeneous medium

In this part we work with separable functions

$$\varepsilon(x) = \varepsilon_1(x_1)\varepsilon_2(x_2)\varepsilon_3(x_3),$$
$$\mu(x) = \mu_1(x_1)\mu_2(x_2)\mu_3(x_3), \tag{118}$$

where ε_j, μ_j are complex valued functions depending on a single variable $x_j,\ j = 1, 2, 3$. We use the notation $\mathbf{T}_\varepsilon$, $\mathbf{T}_\mu$ given in Definition 3.17 to work with the non-vanishing coefficients $\sqrt{\varepsilon_j(x_j)}$, $\sqrt{\mu_j(x_j)}$.

Consider Maxwell's equations for static (time-independent) fields in a nonchiral inhomogeneous medium, which has the form:

$$\operatorname{curl} \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{j}, \tag{119}$$

$$\operatorname{curl} \mathbf{E} = \mu \partial_t \mathbf{H}, \tag{120}$$

$$\operatorname{div}(\varepsilon \mathbf{E}) = \rho, \tag{121}$$

$$\operatorname{div}(\mu \mathbf{H}) = 0. \tag{122}$$

where ε, μ are functions depending on x_1, x_2, x_3 and $\mathbf{j}$ is real vector function which characterizes the distribution of sources of the electromagnetic field. In [47] shows that introducing the notation

$$\vec{\mathcal{E}} := \sqrt{\varepsilon}\mathbf{E}, \quad \vec{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}},$$
$$\vec{\mathcal{H}} := \sqrt{\mu}\mathbf{H}, \quad \vec{\mu} = \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}},$$

the system described by (119)-(122) in sourceless conditions is equivalent to following quaternionic equations:

$$(D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} = 0, \tag{123}$$

$$(D + M^{\vec{\mu}})\vec{\mathcal{H}} = 0. \tag{124}$$

Due to $\mathrm{Vec}(\mathbf{T}[u]) = \mathbf{T}[\vec{u}]$ and Corollary 3.19 we have the following result.

Theorem 5.15. *With ε, μ as in* (118)*, all solutions of* (123)*,* (124) *in a domain Ω with convexity with respect to x_1, x_2, x_3 can be obtained as*

$$\vec{\mathcal{E}} = \mathbf{T}_{\varepsilon}[\vec{u}] \tag{125}$$

$$\vec{\mathcal{H}} = \mathbf{T}_{\mu}[\vec{v}] \tag{126}$$

where $\vec{u}, \vec{v} \in \operatorname{Ker} D$ are arbitrary.

Due to Corollary 4.6 we have the following.

Corollary 5.16. *With ε, μ as before, every solution of* (123)–(124) *in Ω can be approximated arbitrarily closely on any compact subset K of Ω by a finite right linear combination of the images of Grigor'ev polynomials under $\mathbf{T}_{\epsilon}$ and $\mathbf{T}_{\mu}$.*

Example 5.17. Suppose that Ω, in addition to the convexity with respect to x_1, x_2, x_3, is also star-shaped. Then we can obtain every solution of (123)–(124) through Theorem 5.15 from arbitrary $\mathbb{H}(\mathbb{C})$-valued functions in $\operatorname{Ker} D$ by applying $\mathcal{F}$ defined in (22).

5.3 Dirac operators

In this section we present a complete system of solutions for a certain type of Dirac operator. This is possible since the Dirac operator can be transformed into an operator of the form $D+M^{Df/f}$. We summarize the facts we will need concerning the transformation of Dirac operator into a complex quaternionic operator given by V. G. Kravchenko and V. V. Kravchenko in [43]. For more information about Dirac operators see [9, 11, 47, 49, 52].

Paul Dirac [20] published in 1928 the equation

$$\left(\hbar\left(\frac{\gamma_0}{c}\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k\right) + \mathbf{i}mc\right)\Phi = 0, \tag{127}$$

where γ_k, $k = 0, 1, 2, 3$, are the "γ-matrices"

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \mathbf{i} \\ 0 & 0 & -\mathbf{i} & 0 \\ 0 & -\mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and $\gamma_5 = \mathbf{i}\gamma_0\gamma_1\gamma_2\gamma_3$.

In 1929 C. Lanczos in his article [60] rewrote equation (127) in terms of complex quaternions as follows, where $\mathbf{C}$ is the complex conjugation an F is a complex quaternionic function:

$$\left(\mathbf{i}\frac{\hbar}{c}\partial_t + \hbar D - \mathbf{i}mc\mathbf{C}M^{\mathbf{e}_3}\right) F = 0. \tag{128}$$

We will not consider time dependence in the following, but will only consider solutions of the Dirac operator in a spatial domain following [43].

The domain $\widetilde{\Omega} \subseteq \mathbb{R}^3$ is obtained from the domain $\Omega \subseteq \mathbb{R}^3$ by the reflection $x \mapsto \tilde{x}$ replacing x_3 with $-x_3$. A function $\Phi \colon \Omega \to \mathbb{C}^4$ is transformed into a function $F = \mathcal{G}[\Phi] \colon \widetilde{\Omega} \to \mathbb{H}(\mathbb{C})$ by the rule

$$F = \frac{1}{2}\big(-(\widetilde{\Phi}_1 - \widetilde{\Phi}_2) + \mathbf{i}(\widetilde{\Phi}_0 - \widetilde{\Phi}_3)\mathbf{e}_1 - (\widetilde{\Phi}_0 + \widetilde{\Phi}_3)\mathbf{e}_2 + \mathbf{i}(\widetilde{\Phi}_1 + \widetilde{\Phi}_3)\mathbf{e}_3\big) \tag{129}$$

where $\widetilde{\Phi}_i(x) = \Phi_i(\tilde{x})$. The correspondence $\Phi \leftrightarrow F$ is invertible.

5.3.1 Dirac equation with scalar potential

First we will consider the Dirac operator in its covariant form (see [77]). In [43] the classic Dirac operator for a free particle with a specified energy $\omega \in \mathbb{R}$

is expressed as

$$\mathcal{D} = \mathbf{i}\omega\gamma_0 + \sum_{j=1}^{3} \gamma_j \partial_j + \mathbf{i}m \tag{130}$$

and the Dirac operator with scalar potential is written in the form

$$\mathcal{D}^{sc} = \mathcal{D} + \mathbf{i}\varphi_{sc}, \tag{131}$$

where φ_{sc} is a scalar real-valued function of $x = (x_1, x_2, x_3)$ and as usual $\mathbf{i}\varphi_{sc}$ denotes left multiplication by this function.

Proposition 5.18. *[43] Let $\omega, m \in \mathbb{R} \setminus \{0\}$, and φ_{sc} as above. Write*

$$\vec{\alpha}_{sc}(x) = -(\mathbf{i}\omega\mathbf{e}_1 + (m + \widetilde{\varphi}_{sc}(x))\mathbf{e}_2) \in \mathbb{H}(\mathbb{C})$$

for x in a domain Ω in $\mathbb{R}^3$. Then for F, Φ related by (129)*,*

$$\mathcal{D}^{sc}\Phi = 0 \text{ in } \Omega \iff (D + M^{\vec{\alpha}_{sc}})F = 0 \text{ in } \widetilde{\Omega}. \tag{132}$$

Using the relation between the Dirac operator and the solution of $(D + M^{\vec{\alpha}_{sc}})$ by Corollary 3.19, we have the following theorem. As noted in Remark we can apply our results to compositions of quaternionic transformations.

Theorem 5.19. *Let the bounded domain $\Omega \subseteq \mathbb{R}^3$ be convex with respect to the variables x_1, x_2. Let $\varphi_{sc} \colon \Omega \to \mathbb{R}$ be a function depending only on x_2. Consider the nonvanishing coefficients $f_1(x_1) = e^{-\mathbf{i}\omega x_1}$ and $f_2^{sc}(x_2) = e^{-mx_2 - \int_0^{x_2} \widetilde{\varphi}_{sc}(s)ds}$ where $m, \omega \in \mathbb{R} \setminus \{0\}$. Then all elements $\Phi \in \operatorname{Ker} \mathcal{D}^{sc}$ can be obtained as follows: take any $u \in \operatorname{Ker} D$ in $\widetilde{\Omega}$ and let*

$$F = \mathbf{T}_{f_1} \mathbf{T}_{f_2^{sc}}[u] \tag{133}$$

in $\widetilde{\Omega}$. Then let Φ be the corresponding function in Ω given by (129)*.*

Indeed, if $f(x) = f_1(x_1) f_2^{sc}(x_2)$, we have Df/f of the required form to apply Corollary 3.19 to (132).

5.3.2 Dirac equation with electric potential

The Dirac equation with electric potential $\varphi_{el}(x) > 0$ has the form

$$D^{el} = \mathcal{D} + \mathbf{i}\varphi_{el}\gamma_0, \tag{134}$$

acting on $\mathbb{C}^4$-valued functions.

Proposition 5.20. *[43]*

$$\mathcal{D}^{el}\Phi = 0 \ \textit{in}\ \Omega \iff (D + M^{\vec{\alpha}_{el}})F = 0 \ \textit{in}\ \widetilde{\Omega}, \tag{135}$$

where $\vec{\alpha}_{el} = -(\mathbf{i}(\omega + \varphi_{el})\mathbf{e}_1 + m\mathbf{e}_2)$ *and* F, Φ *are related by* (129).

In a similar way we use Corollary 3.19 to deduce the following.

Theorem 5.21. *Let* φ_{el} *be a function depending only on the single variable* x_1*, where the domain* Ω *is convex with respect to* x_1 *and* x_2*. Consider the nonvanishing coefficients* $f_1^{el}(x_1) = e^{-\mathbf{i}\omega x_1 - \int_0^{x_1}\varphi_{el}(s)ds}$ *and* $f_2(x_2) = e^{-mx_2}$*. Then every element* $\Phi \in \operatorname{Ker}\mathcal{D}^{el}$ *can be obtained as the transform via* (129) *of*

$$F = \mathbf{T}_{f_1^{el}}\mathbf{T}_{f_2}[u] \tag{136}$$

where u *is an arbitrary monogenic function in* $\widetilde{\Omega}$*.*

5.3.3 Dirac equation with pseudoscalar potential

The Dirac equation with pseudoscalar potential has the form

$$\mathcal{D}^{ps} = \mathcal{D} + \varphi_{ps}\gamma_0\gamma_5, \tag{137}$$

where φ_{ps} is a real-valued function.

Proposition 5.22. *[43, Proposition 1] Let* $v = -\mathbf{i}\widetilde{\varphi}_{ps}$. *Let* $\vec{\beta} = -(\mathbf{i}\omega\mathbf{e}_1 + m\mathbf{e}_2)$, *with* $m^2 \neq \omega^2$. *We choose a complex number* $\lambda \in \mathbb{C}$ *such that* $\lambda^2 = \vec{\beta}^{\,2}$. *Then we define* $S^{\pm} = \frac{1}{\lambda}M^{(\lambda\pm\beta)}$, $\mathcal{P}_1^{\pm} = \frac{1}{2}M^{1\pm\mathbf{i}\mathbf{e}_1}$. *Then the following equalities are valid in any domain in* $\mathbb{R}^3$*:*

$$D + v + M^{\vec{\beta}} = \mathcal{P}_1^{\pm}S^{+}(D \pm M^{(v+\lambda)\mathbf{i}\mathbf{e}_1}) + \mathcal{P}_1^{-}S^{+}(D \pm M^{(v-\lambda)\mathbf{i}\mathbf{e}_1}), \tag{138}$$

$$\begin{aligned}\operatorname{Ker}(D + v + M^{\vec{\beta}}) &= \mathcal{P}_1^{+}S^{+}\operatorname{Ker}(D + M^{(v+\lambda)\mathbf{i}\mathbf{e}_1}) \oplus \mathcal{P}_1^{-}S^{+}\operatorname{Ker}(D - M^{(v+\lambda)\mathbf{i}\mathbf{e}_1})\\ &\oplus \mathcal{P}_1^{+}S^{-}\operatorname{Ker}(D + M^{(v-\lambda)\mathbf{i}\mathbf{e}_1}) \oplus \mathcal{P}_1^{-}S^{-}\operatorname{Ker}(D - M^{(v-\lambda)\mathbf{i}\mathbf{e}_1}).\end{aligned} \tag{139}$$

By Proposition 5.22, the solution to the Dirac equation (137), can be decomposed into solutions of $D + M^{Df/f}$.

Theorem 5.23. *Let* φ_{sc} *be a function depending on a single variable* x_1, *where* Ω *is bounded and convex with respect to* x_1, *and consider the nonvanishing coefficients*

$$f(x_1) = e^{\int_0^{x_1}(\nu(s)+\lambda)\mathbf{i}ds},$$

$$g(x_1) = e^{\int_0^{x_1}(\nu(s)-\lambda)\mathbf{i}ds}.$$

Under the hypothesis of Proposition 5.22, every element $u \in \operatorname{Ker}(D+v+M^{\vec{\beta}})$

can be obtained as follows:

$$u = \mathcal{P}_1^+ S^+ \mathbf{T}_{f_1}[v^+] + \mathcal{P}_1^- S^+ \mathbf{T}_{1/f_1}[v^-] + \mathcal{P}_1^+ S^- \mathbf{T}_{g_1}[w^+] + \mathcal{P}_1^- S^- \mathbf{T}_{1/g_1}[w^-], \tag{140}$$

where $v^+, v^-, w^+, w^- \in \operatorname{Ker} D$.

The proof is immediate from (139) and Corollary 3.19.

5.4 Helmholtz equations

As mentioned in the Introduction, the Helmholtz equation, or reduced wave equation, has the form

$$(\Delta + \lambda^2)u = 0. \tag{141}$$

The quantity λ is the wave number. It is often real and constant, but it can be complex if the medium of propagation is energy absorbing, or a function of space if the medium is inhomogeneous. For physical considerations one often assumes $\lambda \neq 0$ and $\operatorname{Im} \lambda \geq 0$ whether or not λ is constant. In [54, Section 2] there is a factorization of $\Delta + M^\lambda$ for a general complex quaternionic constant $\lambda \in \mathbb{H}(\mathbb{C})$, which specializes to the following when λ is a complex constant.

Lemma 5.24 ([54])**.** *For* $\lambda \in \mathbb{C} \setminus \{0\}$*, we have a factorization of the Helmholtz operator* $\Delta + \lambda^2 \colon C^2(\Omega, \mathbb{H}(\mathbb{C})) \to C(\Omega, \mathbb{H}(\mathbb{C}))$ *as*

$$\Delta + \lambda^2 = -(D + \lambda)(D - \lambda) = -(D - \lambda)(D + \lambda). \tag{142}$$

Further, there exists the following kernel decomposition:

$$\operatorname{Ker}(\Delta + \lambda^2) = \operatorname{Ker}(D + \lambda) \oplus \operatorname{Ker}(D - \lambda). \tag{143}$$

It is immediate from (142) that for every complex quaternionic valued function $u \in \mathrm{Ker}(D \pm \lambda)$ its components u_j $(j = 0, 1, 2, 3)$ also satisfy $(\Delta + \lambda^2)u_j = 0$ since λ is scalar.

Theorem 5.25. *Let* $\lambda \in \mathbb{C} \setminus \{0\}$. *Then all elements of* $\mathrm{Ker}(\Delta + \lambda^2)$ *in a bounded domain convex with respect to* x_3 *can be obtained in terms of arbitrary monogenic functions* $u^\pm$, $v^\pm$,

$$\mathcal{P}^+\mathbf{T}_f[u^+] \; + \; \mathcal{P}^-\mathbf{T}_{1/f}[u^-] + \mathcal{P}^+\mathbf{T}_{1/f}[v^+] \; + \; \mathcal{P}^-\mathbf{T}_f[v^-],$$

where $f(x_3) = e^{\mathbf{i}\lambda x_3}$.

This follows from the decompositions (117), (143), and is illustrated in Figure 3.

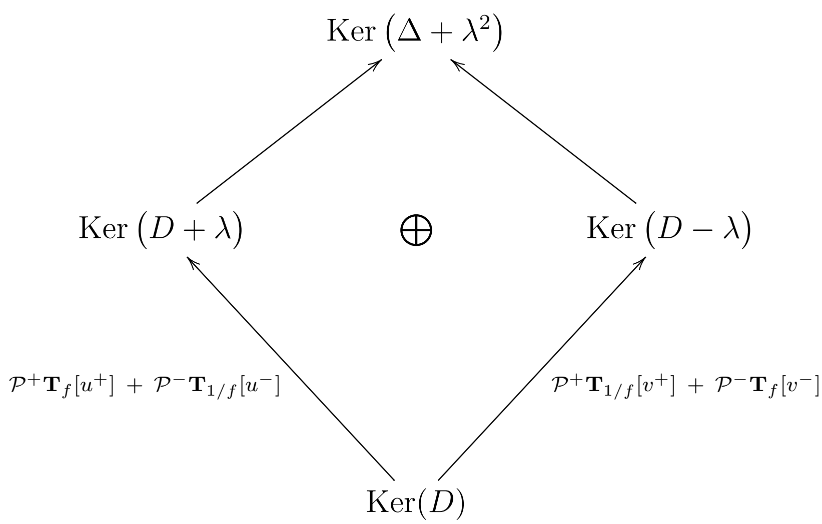

Figure 3: Construction of solutions to the Helmholtz equation

5.5 Scalar Schrödinger operator

Consider the equation

$$(-\Delta + w)g = 0 \tag{144}$$

in $\Omega \subseteq \mathbb{R}^3$ where again Δ is the Laplacian and w and g are complex-valued functions. We assume that $g \in C^2(\Omega, \mathbb{C})$. The operator $-\Delta + w$ is known as the *Schrödinger operator*.

Let $\vec{\lambda}$ be a purely complex quaternion valued function (i.e., a complex quaternion vector field) such that

$$w = -D\vec{\lambda} - (\vec{\lambda})^2. \tag{145}$$

In [5] the decomposition

$$(\Delta - w)u_0 = -(D + M^{\vec{\lambda}})(D - M^{\vec{\lambda}})u_0, \tag{146}$$

was found for C^2 scalar functions u_0. In [44] it was shown that the solution of (146) necessarily has the form Df/f, with f being a solution of (144). Consider $f(x) = f_1(x_1)f_2(x_2)f_3(x_3)$ where each f_j is a nonvanishing coefficient. We are going to use the following elementary equality:

$$\begin{aligned}
D\left(\frac{1}{f}\right) &= -\frac{f_1'(x_1)}{f_1^2(x_1)f_2(x_2)f(x_3)}\mathbf{e}_1 - \frac{f_2'(x_2)}{f_2^2(x_2)f_1(x_1)f(x_3)}\mathbf{e}_2 - \frac{f_3'(x_3)}{f_3^2(x_3)f_2(x_1)f(x_2)}\mathbf{e}_3 \\
&= -\frac{1}{f}\left(\frac{f_1'(x_1)}{f(x_1)}\mathbf{e}_1 + \frac{f_2'(x_2)}{f_2(x_2)}\mathbf{e}_2 + \frac{f_3'(x_3)}{f_3(x_3)}\mathbf{e}_3\right) \\
&= -\frac{1}{f}\frac{Df}{f}.
\end{aligned}$$

Let $\vec{\lambda} = (Df)/f$. Then the vectorial function w defined in (145) turns into $w = \dfrac{\Delta f}{f}$, due to

$$\begin{aligned}
w &= -D\vec{\lambda} - (\vec{\lambda})^2, \\
&= -D\left(\frac{1}{f}Df\right) - \left(\frac{Df}{f}\right)^2, \\
&= -\left(D\frac{1}{f}\right)Df - \frac{1}{f}DDf - \left(\frac{Df}{f}\right)^2, \\
&= \left(\frac{Df}{f}\right)^2 + \frac{\Delta f}{f} - \left(\frac{Df}{f}\right)^2, \\
&= \frac{\Delta f}{f}.
\end{aligned}$$

We thus have another corollary to Theorem 3.16:

Corollary 5.26. *Suppose Ω is bounded and convex with respect to x_1, x_2, x_3. Let $\vec{\lambda} = g_1(x_1)\mathbf{e}_1 + g_2(x_2)\mathbf{e}_2 + g_3(x_3)\mathbf{e}_3$ be a purely vectorial function, $g_i(x_i) \in \mathbb{C}$, and let $w = -D\vec{\lambda} - (\vec{\lambda})^2$. Then we have the following relation for $u_0 \in C^2(\Omega, \mathbb{C})$:*

$$(\Delta - w)\widetilde{\mathbf{T}}_f[u_0] = \widetilde{\mathbf{T}}_f[\Delta u_0], \tag{147}$$

where $\widetilde{\mathbf{T}}_f = \mathbf{T}_{1/f_1,1}\mathbf{T}_{1/f_2,2}\mathbf{T}_{1/f_3,3}$ and the nonvanishing coefficients are given by $f_j(x_j) = e^{\int_0^{x_j} g_j(s)ds}$.

Proof. Let u_0 be a scalar function. Then $\widetilde{\mathbf{T}}_f[u_0]$ by Lemma 3.15 is also scalar. Then due to Corollary 3.18 we have:

$$\begin{aligned}
-(D + M^{\vec{\lambda}})(D - M^{\vec{\lambda}})\widetilde{\mathbf{T}}_f[u_0] &= -(D + M^{\vec{\lambda}})\mathbf{T}_f[Du_0] \\
&= -\widetilde{\mathbf{T}}_f[DDu_0] \\
&= \widetilde{\mathbf{T}}_f[\Delta u_0].
\end{aligned}$$ □

Thus we have found a transmutation operator between harmonic functions and the Schrödinger operator (144).

There is another way to construct solutions of Schrödinger operator (144) using the operator $D + M^{Df/f}$, but first we need the following definition.

Definition 5.27. The *antigradient operator* $\mathcal{A}$ is given by

$$\mathcal{A}[\vec{v}](x_1, x_2, x_3) = \int_{a_1}^{x_1} v_1(t, a_2, a_3)dt + \int_{a_2}^{x_2} v_2(x_1, t, a_3)dt + \int_{a_3}^{x_3} v_3(x_1, x_2, t)dt, \tag{148}$$

where $\vec{v}\colon \Omega \to \mathbb{R}^3$ is any vector field such that $\operatorname{curl}\vec{v} = 0$ and $(a_1, a_2, a_3) \in \Omega$.

The importance of the antigradient operator is the following.

Proposition 5.28 ([46])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected domain. Then the scalar function $\mathcal{A}[\vec{v}]$ satisfies $\operatorname{grad}[\mathcal{A}[\vec{v}]] = \vec{v}$, for any function such that $\operatorname{curl}\vec{v} = 0$.*

Proposition 5.29. *[46, Theorem 158] Let $\boldsymbol{F}$ be a purely vectorial solution to*

$$(D + M^{Df/f})\boldsymbol{F} = 0$$

in a simply connected domain Ω. Then the function $g = f\mathcal{A}[f^{-1}\boldsymbol{F}]$ is a solution of (144).

Theorem 5.30. *Let u be a monogenic function in a star-shaped open set Ω with respect to the origin and let f be a separable scalar function in Ω. Then the function $g = f\mathcal{A}\left[f^{-1}\mathbf{T}_f[\mathcal{F}[u]]\right.$ is a solution of the following Schrödinger equation:*

$$\left(-\Delta + \frac{\Delta f}{f}\right)g = 0.$$

Proof. Let $u \in \operatorname{Ker} D$. Then $\mathcal{F}[u]$ is a purely vectorial monogenic function (recall Definition 2.17). Due to Corollary 3.19, the operator $\mathbf{T}_f[\mathcal{F}[u]]$ is a purely vectorial solution to $D + M^{Df/f}$. Finally, due to Proposition 5.29 we have the result. □

5.6 Impedance equation

Electrical properties such as the electrical conductivity σ and the electric permittivity ϵ, determine the behaviour of materials under the influence of external electric fields. For example, both direct and alternating currents flow easily through materials of high electrical conductivity.

First we summarize a quaternionic reformulation of electrical impedance equation. The following derivation of the equivalence between the impedance equation and the $D+M$ system can be found in [73] and also in [46]. Consider the electrical impedance equation

$$\operatorname{div}(\sigma \operatorname{grad} u) = 0. \tag{149}$$

We define the vector $\vec{E}$ as

$$\vec{E} = -\operatorname{grad} u,$$

so the equation (149) turns into

$$\operatorname{div}\left(\sigma\vec{E}\right) = (\operatorname{grad}\sigma)\cdot\vec{E} + \sigma \operatorname{div}\vec{E} = 0.$$

which is equivalent to

$$\operatorname{div}\vec{E} = -\frac{\operatorname{grad}\sigma}{\sigma}\cdot\vec{E}. \tag{150}$$

Let $\vec{E}$ be a purely vectorial complex quaternionic function. Since σ is a scalar function and $\operatorname{curl}\vec{E} = 0$ we have another equivalent formulation

$$D\vec{E} = -\left(\frac{D\sigma}{\sigma}\right) \cdot \vec{E}. \tag{151}$$

Using the multiplication rule (3), since $D\sigma$ and $\vec{E}$ are purely vectorial functions we have the following relation:

$$\left(\frac{D\sigma}{\sigma}\right) \cdot \vec{E} = \frac{1}{2}\left(\frac{D\sigma}{\sigma}\vec{E} + \vec{E}\frac{D\sigma}{\sigma}\right),$$

Introducing the notation $\vec{\mathcal{E}} = \sqrt{\sigma}\vec{E}$ and $\vec{\sigma} = \frac{D\sqrt{\sigma}}{\sqrt{\sigma}}$, with the aid of the Leibniz rule (Proposition 2.10) and the equality

$$\frac{1}{2}\frac{D\sigma}{\sigma} = \frac{D\sqrt{\sigma}}{\sqrt{\sigma}},$$

we have that (151) becomes

$$\left(D + M^{\vec{\sigma}}\right)\vec{\mathcal{E}} = 0. \tag{152}$$

More generally the following result is known.

Lemma 5.31. *[46, Theorem 159] Let u_0 be a nonvanishing particular solution of the equation*

$$(\operatorname{div}\sigma\operatorname{grad} + w)u = 0 \quad \text{in}\, \Omega \subseteq \mathbb{R}^3, \tag{153}$$

with σ, w and u being complex-valued functions, $\sigma \in C^2(\Omega)$. Then for any scalar function $v_0 \in C^2(\Omega)$ the following equality holds:

$$(\operatorname{div}\sigma\operatorname{grad} + w)v_0 = -\sigma^{1/2}\left(D + M^{Df/f}\right)\left(D - M^{Df/f}\right)\sigma^{1/2}v_0, \tag{154}$$

where $f = \sigma^{1/2}u_0$.

For recovering the function u from $\vec{\mathcal{E}}$ we use the next proposition, which relates purely vectorial solutions to the system $D + M^{Df/f}$ with solutions of (154).

Proposition 5.32. *[46, Theorem 160] Let u_0 be a particular nonvanishing solution of* (153) *where $\sigma \in C^2(\Omega)$ does not vanish. Let $f = \sigma^{1/2} u_0$. Consider a solution $\mathbf{F}$ of the equation*

$$\left(D + M^{Df/f}\right)\mathbf{F} = 0.$$

Then the function

$$u = u_0 \mathcal{A}[f^{-1}\mathbf{F}] \tag{155}$$

is also a solution of (154)*; every solution u is obtained in this way.*

We now apply our results to solve equation (153). This gives a complete solution to the generalized electrical impedance equation (152). Then for (149) we need only take the functions $w \equiv 0$ and $u_0 \equiv 1$.

Theorem 5.33. *Let Ω be bounded and convex with respect to x_1, x_2, x_3. Suppose w is given as well as $\sigma(x) = \sigma_1(x_1)\sigma_2(x_2)\sigma_3(x_3)$, a product of C^2 nonvanishing coefficents. Suppose there exists a separable nonvanishing solution $u_0(x) = \widetilde{u}_1(x_1)\widetilde{u}_2(x_2)\widetilde{u}_3(x_3)$ of* (153) *where each factor is integrable. Let $f = \sigma^{1/2} u_0$. Then every solution u of* (153) *can be obtained as*

$$u = \mathcal{A}\left[\frac{\mathbf{T}_f[\vec{v}]}{\sqrt{\sigma}\, u_0}\right], \tag{156}$$

where $\vec{v} \in \mathrm{Ker}(D) \cap C^1(\Omega, \mathrm{Vec}\,\mathbb{H}(\mathbb{C}))$.

Proof. Given u satisfying (156), set

$$\vec{v} = \mathbf{T}_f^{-1}[\sqrt{\sigma}\, u_0 \,\mathrm{grad}\, u].$$

Since by Lemma 3.15 the operator $\mathbf{T}_f$ sends purely vectorial monogenic functions into purely vectorial solutions of $D + M^{Df/f}$, $\vec{v}$ is a vector field as required. $\square$

In Theorem 5.33 when Ω is star-shaped we can alternatively use the operator $\mathcal{F}$ defined in (22) to obtain a purely vectorial function starting from monogenic functions.

The following is an illustration of the application of Theorem 5.33.

Example 5.34. Consider the equation (149) with electrical conductivity $\sigma(x) = e^{x_1}$. Its quaternionic reformulation is

$$\big(D + M^{\mathbf{e}_1/2}\big)\vec{\mathcal{E}} = 0.$$

Consider the purely vectorial monogenic function $\vec{v} = P^{1,1}\mathbf{e}_3$. In this case $f(x) = e^{x_1/2}$ and $\mathbf{T}_f = \mathbf{T}_{f_1}$. A short calculation using Table 2.1 gives us $\varphi_1(x_1) = -e^{-x_1/2} + e^{x_1/2}$, $\psi_0(x_1) = e^{-x_1/2}$ and

$$\begin{aligned}
\mathbf{T}_f[P^{1,1}\mathbf{e}_3] &= \mathbf{T}_{f_1}[2x_2x_3\mathbf{e}_1 + 2x_1x_3\mathbf{e}_2 + 2x_1x_2\mathbf{e}_3], \\
&= 2\psi_0(x_1)x_2x_3\mathbf{e}_1 + 2\varphi_1(x_1)x_3\mathbf{e}_2 + 2\varphi(x_1)x_2\mathbf{e}_3, \\
\frac{\mathbf{T}_f[P^{1,1}\mathbf{e}_3]}{\sqrt{\sigma}} &= 2x_2x_3e^{-x_1}\mathbf{e}_1 + \big(2x_3 - 2e^{-x_1}x_3\big)\mathbf{e}_2 + \big(2x_2 - e^{-x_1}x_2\big)\mathbf{e}_3,
\end{aligned}$$

and finally

$$u(x) = \mathcal{A}\left[\frac{\mathbf{T}_f[P^{1,1}\mathbf{e}_3]}{\sqrt{\sigma}}\right] = \big(2 - 2e^{-x_1}\big)x_2x_3 + c.$$

It can be verified directly that u satisfies $(\operatorname{div} e^{x_1} \operatorname{grad})u = 0$.

5.7 Quaternionic Vekua equation

In this final section we are going to find solutions of the system $Du = \frac{Df}{f}\overline{u}$ in Ω with the aid of the operator $\mathbf{T}_f$. In addition, using a particular solution u_0 of the three-dimensional Schrödinger equation we are able to construct the vector part $\vec{u}$ such that $u = u_0 + \vec{u}$ is a solution of $Du = (Df/f)\overline{u}$. Viceversa, given a purely vectorial solution vectorial function such that $\operatorname{div}\vec{u} = 0$ and $\operatorname{curl}(f^{-2}\operatorname{curl}\vec{u}) = 0$ we are able to find u_0, such that $u = u_0 + \vec{u}$ is a solution of $Du = (Df/f)\overline{u}$.

The theory of pseudoanalytic functions was developed by Lipman Bers [6] and Ilya Vekua [79] independently. They proved that many properties of analytic functions in the complex plane are still valid for systems more general than the Cauchy-Riemann system. Bers introduced the notion of (F, G)-derivative, which gives a generalization of the notion of holomorphic functions in the sense of complex analysis, since taking $F = 1$ and $G = \mathbf{i}$, we have that the $(1, \mathbf{i})$-derivative coincides with the holomorphic functions.

V.V. Kravchenko discovered that pseudoanalytic functions are closely related to many important equations such as Dirac, Maxwell, Klein-Gordon among others (see [46]).

There have been efforts to generalize the notion of pseudoanalytic function (see [46, 64, 76]). For our purposes we are going to work with the notion introduced by V. V. Kravchenko, which can be found in Section 16 of [46].

5.7.1 The main quaternionic Vekua equation

Definition 5.35. [46] The *Vekua operator* is

$$D + \frac{Df}{f} C_{\mathbb{H}}, \tag{157}$$

acting on $u \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. $C_{\mathbb{H}}$ denotes the quaternionic conjugate (6).

An important fact of the kernel of Vekua operator is the following.

Proposition 5.36. *[46, Theorem 161] Let $W_0 + \mathbf{W}$ be a solution of* (157) *where $\mathbf{W}$ is a vector field. Then W_0 is a solution of the stationary Schrödinger equation*

$$-\Delta W_0 + \frac{\Delta f}{f} W_0 = 0, \tag{158}$$

and the scalar function $u = f^{-1} W_0$ is a solution of the equation

$$\operatorname{div}\left(f^2 \operatorname{grad} u\right) = 0. \tag{159}$$

The vectorial function $\mathbf{V} = f\mathbf{W}$ is a solution of the equation

$$\operatorname{curl}\left(f^{-2} \operatorname{curl} \mathbf{V}\right) = 0. \tag{160}$$

To find solutions of the Vekua equation (157) we need an inverse $\mathcal{B}$ for the double curl operator curl curl. It is known [40, Section 5.7] that

$$\vec{u} \mapsto \frac{1}{4\pi} \int_{R^3} \frac{\vec{u}(y)}{|x-y|} dy \tag{161}$$

is a right-inverse for curl curl for fields $\vec{u}$ on all of $\mathbb{R}^3$. In [17, Corollary A.4], an inverse is given which works for bounded domains Ω with sufficiently smooth boundary. The following proposition refers to the operator (161) but in fact any appropriate $\mathcal{B}$ can be used such that

$$\operatorname{curl}\operatorname{curl}\mathcal{B}[\vec{u}] = \vec{u}$$

for all smooth fields $\vec{u}$ in Ω.

Proposition 5.37. *[58, Theorem 10] Let $\vec{u} \in C^1(\Omega, \operatorname{Vec}\mathbb{H}(\mathbb{C}))$ be a solution of the equation*

$$\left(D + M^{\frac{Df}{f}}\right)\vec{u} = 0.$$

Then an $\mathbb{H}(\mathbb{C})$-valued solution W in Ω of the Vekua equation

$$DW - \frac{Df}{f}\overline{W} = 0,$$

can be constructed as follows:

$$W = \frac{1}{2}\left(f\mathcal{A}\left[\frac{\vec{u}}{f}\right] - \frac{1}{f}\operatorname{curl}\left(\mathcal{B}[f\vec{u}]\right) + \frac{\nabla h}{f}\right), \tag{162}$$

where h is an arbitrary complex valued harmonic function in Ω.

Using the methods developed in this thesis we find purely vectorial solution to $D + M^{Df/f}$ with the aid of the operators $\mathbf{T}_f$ and $\mathcal{F}$ defined in (59) and (22) respectively.

Corollary 5.38. *Let $\Omega \subseteq \mathbb{R}^3$ be a star shaped open subset convex with respect to x_1, x_2, x_3. Let $\mathcal{B}$ be a right inverse for curl curl in Ω. Let f_j be a nonvanishing coefficient in Ω with respect to x_j, $j = 1, 2, 3$ and define $f(x) = f_1(x_1)f_2(x_2)f_3(x_3)$. Then every solution of the Vekua equation*

$$DW - \frac{Df}{f}\overline{W} = 0$$

can be constructed as follows: let $u \in Ker(D)$ be an arbitrary $\mathbb{H}(\mathbb{C})$-valued *monogenic and h an arbitrary complex valued harmonic function in Ω. Write $\vec{v} = \mathbf{T}_f[\mathcal{F}[u]]$. Then*

$$W = \frac{1}{2}\left(f\mathcal{A}\left[\frac{\vec{v}}{f}\right] - \frac{1}{f}\,\mathrm{curl}\,\big(\mathcal{B}[f\vec{v}]\big) + \frac{\nabla h}{f}\right). \tag{163}$$

Proof. Let $u \in \operatorname{Ker} D$. Then the purely vectorial function $\mathcal{F}[u]$ is in $\operatorname{Ker}(D)$. Due to Corollary 3.18 the purely vectorial function $\vec{v}$ is a solution of $D + M^{Df/f}$. Therefore due to Proposition 5.37, W is the general solution of the Vekua equation. □

Our next result will be an application of the following.

Proposition 5.39. *[58, Theorem 11] Let W_0 be a scalar solution of the Schrödinger equation* (158). *Then the complex vector fields $\mathbf{W}$ such that $\mathbf{V} = f\mathbf{W}$ is a solution of* $\operatorname{div}\mathbf{V} = 0 = \operatorname{curl}(f^{-2}\operatorname{curl}\mathbf{V})$ *and the function $W_0 + \mathbf{W}$ is a solution of the Vekua equation* (157), *are constructed according to the formula*

$$\mathbf{W} = -f^{-1}\Big(\mathrm{curl}\,\big(\mathcal{B}[f^2\nabla(f^{-1}W_0)]\big) + \nabla h\Big) \tag{164}$$

where h is an arbitrary complex-valued harmonic function in the domain under consideration and $\mathcal{B}$ is a right inverse for curl curl. Conversely, given a vectorial solution $\mathbf{V}$ of the equations $\operatorname{div}\mathbf{V} = 0 = \operatorname{curl}(f^{-2}\operatorname{curl}\mathbf{V})$, *letting $\mathbf{W} = \frac{1}{f}\mathbf{V}$ and*

$$W_0 = -f\mathcal{A}[f^{-2}\,\mathrm{curl}(f\mathbf{W})], \tag{165}$$

then W_0 is a solution of (158) *such that $W_0 + \mathbf{W}$ is a solution of* (157).

Theorem 5.40. *Let $f(x) = f(x_1)f(x_2)f(x_3)$ for $f_j(x_j)$ be a nonvanishing* *coefficient in the direction x_j for $j = 1, 2, 3$. Assume the same hypotheses of Proposition 5.39 for the Vekua equation $DW = (Df/f)\overline{W}$. Given a harmonic function $u_0 \colon \Omega \to \mathbb{C}$, a solution W of the Vekua equation can be constructed as*

$$W = \widetilde{\mathbf{T}}_f[u_0] + -f^{-1}\bigg(\operatorname{curl}\big(\mathcal{B}[f^2\nabla(f^{-1}\widetilde{\mathbf{T}}_f[u_0])]\big) + \nabla h \bigg). \tag{166}$$

Further, let $\vec{u} \in \operatorname{Ker} D$ be a purely vectorial monogenic function. Then a solution W of the Vekua equation (157) can be constructed as

$$W = -f\mathcal{A}[f^{-2}\operatorname{curl}(\vec{u})] + \frac{\vec{u}}{f}. \tag{167}$$

Proof. Let u_0 be a harmonic function. Then due to Corollary 5.26 we have $\widetilde{\mathbf{T}}_f[u_0]$ is a scalar solution of the Schrödinger equation (158). By Proposition 5.39, equation (166) is proved.

Let $\vec{u}$ be a purely vectorial monogenic function. Then by (12), we have $\operatorname{div}\vec{u} = 0$ and $\operatorname{curl}\vec{u} = 0$. Therefore $\operatorname{curl}(f^{-2}\operatorname{curl}\vec{u}) = 0$. By Proposition 5.39, equation (167) is proved.

□

Chapter 6

Conclusions

We have considered functions of a vector variable in a domain in 3-dimensional space, taking values in the space of complex quaternions.

We have constructed an invertible quaternionic integral operator which transforms solutions of the operator $D + M^{Df/f}$ into solutions of the Moisil-Teodorescu operator D for bounded domains in $\mathbb{R}^3$ with sufficient symmetry for application of classical transmutation operators on real intervals. Previous application of the classical transmutation operators beyond real domains have only been found in few sources, such as [48] for functions taking values in hyperbolic numbers; here we have found a transmutation applicable in the complex quaternions, which opens the possiblity of application to other types of differential equations, including other equations of mathematical physics.

We have given a complete and original solution to equations of the form $Du + \lambda u + u\gamma = 0$ with certain restrictions on the coefficients λ and γ.

In particular, the solutions have been represented locally by a new type of Taylor series adapted to the equation under consideration.

We have applied these results to give complete solution sets for various physical systems, including Beltrami fields, important cases of the Maxwell equations, the Helmholtz equation and the free Dirac equation for particles with mass, among others. For many of these equations, it is common in physics to use solutions of a first-order differential equation (such as involving div or curl) to find some solutions of a related second-order equation (such as the Schrödinger or Helmholtz equations), but not a complete set of solutions.

Future research in this direction would be to find similar factorizations in other number systems such as biquaternions, to apply our results to boundary value problems, and to develop further the applications of our results to other equations of mathematical physics,

Bibliography

[1] M. Aschwanden. Physics of the solar corona an introduction with problems and solutions. *Pour la Science*, 2005.

[2] S. Axler, P. Bourdon and W. Ramey. Harmonic Function Theory. Springer-Verlag, 2001.

[3] C. Athanasiadis, G. Costakis, and I. Stratis. On some properties of Beltrami fields in chiral media. *Reports on Mathematical Physics - REP MATH PHYS*, 45:257–271, 2000.

[4] H. Begehr, R. P. Gilbert. Transformations, transmutations, and kernel functions, Vol. 1. *Harlow: Longman Scientific & Technical,* 1992.

[5] S. Bernstein and K. Gürlebeck. On a higher dimensional Miura transform. *Complex Variables and Elliptic Equations,* 307-319,1999.

[6] L. Bers. Theory of pseudo-analytic functions. *New York University*, 1952.

[7] J. Bory., R. Abreu, L.M Hernández et al. Dirichlet-Type Problems for the Two-Dimensional Helmholtz Operator in Complex Quaternionic Analysis. *Mediterr. J. Math.* 13, 4901–4916 2016. DOI 10.1007/s00009-016-0781-x

[8] F. Brackx, R. Delanghe and F. Sommen. Clifford Analysis. *Pitman Res. Notes. in Math. Ser.,* **76**, 1982.

[9] H. M. Campos, V. V. Kravchenko, and L. M. Mendez. Complete families of solutions for the Dirac equation: an application of bicomplex pseudoanalytic function theory and transmutation operators, *Advances in Applied Clifford Algebras*, 2011.

[10] H. M. Campos, V. V. Kravchenko, and S. M. Torba. Transmutations, L-bases and complete families of solutions of the stationary Schrödinger equation in the plane. *Journal of Mathematical Analysis and Applications*, 389(2):1222–1238, 2012.

[11] A. Castañeda and V. V. Kravchenko. New applications of pseudoanalytic function theory to the Dirac equation. *Journal of Physics A: Mathematical and General*, **38**, 9207-9219, 2005.

[12] D. L. Colton. Solution of boundary value problem by the method of integral operator. *London: Pitman Publishing,* 1976.

[13] S. Chandrasekhar and P. C. Kendall. On force-free magnetic fields. *Astrophysical Journal*, 126:457, 1957.

[14] R. Delanghe. On regular-analytic functions with values in a Clifford algebra. *Math. Ann.*,91-111, 1970.

[15] R. Delanghe and F. Brackx. Runge's theorem in Hypercomplex Function Theory. *Journal of Approximation Theory* **29**, 200-211 (1980)

[16] B. B. Delgado and V. V. Kravchenko. A right inverse operator for curl $+\lambda$ and applications, *Advances in Applied Clifford Algebras, 29:40*, 2019. DOI 10.1007/s00006-019-0958-z

[17] B. B. Delgado and R. M. Porter Hilbert transform for the three-dimensional Vekua equation. *Complex Variables and Elliptic Equations*, 64:11, 1797-1824, 2019. DOI 10.1080/17476933.2018.1555246

[18] J. Delsarte. Sur certaines transformations fonctionnelles rélatives aux équations linéaires aux dérivées partielles du second ordre. *C. R. Acad. Sci., Paris*, 206:1780–1782, 1938.

[19] J. Delsarte and J. L. Lions. Transmutations d'opérateurs différentiels dans le domaine complexe. *Comment. Math. Helv.*, 32:113–128, 1957.

[20] P. A. M. Dirac. The Quantum Theory of the Electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, **117**, 610-624, 1928.

[21] N. Du Plessis. Runge's theorem for harmonic functions, *J. London Math. Soc.* (2)**1** 404-408, (1969).

[22] W. Evans, I. Myron and Stuart A. Rice. Modern Nonlinear Optics Part III. *John Wiley and Sons*, 2001.

[23] R. Fueter. Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen *Varieablen. Comment. Math. Helv. 7:* $307 - 330$, 1935.

[24] S. G. Gal and I. Sabadini. *Quaternionic Approximation.* Birkhäuser Basel, 2019.

[25] R. P. Gilbert, Function Theoretic Methods in Partial Differential Equations, *New York: Academic P,* 1969.

[26] J. Gilbert and M. Murray Clifford Algebras and Dirac Operators in Harmonic Analysis. *Cambridge Studies in advances mathemathics*, 2010.

[27] P. Girard. Quaternions, Clifford Algebras and Relativistic Physics. *Birkhäuser*, 2007.

[28] J. González, J. Luna and M. Shapiro. On the Bergman Theory for Solenoidal and Irrotational Vector Fields *Complex Analysis and Operator Theory*, **5**, 237 -251, 2011.

[29] Yu. M. Grigor'ev and V.V. Naumov, The Laurent series for the Moisil-Teodorescu system. *Dinamika Sploshnoj Sredy*, 1982.

[30] Yu. M. Grigor'ev. Regular quaternionic polynomials and their properties. *Complex Variables and Elliptic Equations*, 62(9):1343–1363, 2017.

[31] Yu. M. Grigor'ev and V. V. Naumov Approximation theorems for the Moisil-Teodorecu system. *Siberian Math. J. (Transl. from Russ.)* 693–701, 1985.

[32] K. Gürlebeck. Hypercomplex factorization of the Hemholtz equation. *Zeitschrift für Analysis und ihre Anwendungen,* **5** 125-131, 1986.

[33] K. Gürlebeck, K. Habetha, and W. Sproessig. *Application of Holomorphic Functions in Two and Higher Dimensions*. Birkhäuser, 2016.

[34] K. Gürlebeck, K. Habetha, and W. Sproessig. *Holomorphic Functions in the Plane and N-Dimensional Space*. Springer, 2008.

[35] K. Gürlebeck and W. Sprössig. *Quaternionic and Clifford Calculus for Physicists and Engineers*. Wiley, 1998.

[36] V. Marchenko. A Sturm-Liouville Operators and Applications. Birkhäuser, Basel, 1986.

[37] K. V. Khmelnytskaya, V. V. Kravchenko and V. Rabinovich. Quaternionic Fundamental solutions for the numerical electromagnetic scattering problems and Application. *Journal of analysis and its applications*, **22**, 147-166, 2003.

[38] W. R. Hamilton. Elements of Quaternions. *London: Longmans Green*, 1866.

[39] A. D. Jette. Force-free magnetic fields in resistive magnetohydrostatics. *Journal of Mathematical Analysis and Applications*, Vol:29, page 109 - 122, 1970.

[40] G. A. Korn and T. M. Korn. Mathematical handbook for scientists and engineers. *McGraw-Hill Book Company*, 1961.

[41] S. G. Krantz. Harmonic and Complex Analysis in Several Variables. *Springer International Publishing.* 2017.

[42] I. V. Kravchenko, V. V. Kravchenko and S. M. Torba. Solution of parabolic free boundary problems using transmuted heat polynomials. *Math. Methods Appl. Sci.,* **42***, 5094-5105*, 2019.

[43] V. G. Kravchenko and V. V. Kravchenko. Quaternionic factorization of the Schrödinger operator and its applications to some first-order systems of mathematical physics. *Journal of Physics A: Mathematical and General,* 2006.

[44] V. G. Kravchenko, V. V. Kravchenko and B. D. Williams. A quaternionic generalization of the Riccati differential equation. *Kluwer Acad. Publ. Cliffords Analysis and its Applications.* Ed. by Brackx et al., 2001.

[45] V. V. Kravchenko. On Beltrami fields with nonconstant proportionality factor. *Journal of Physics A: Mathematical and General*, 36(5):1515–1522, 2003.

[46] V. V. Kravchenko. Applied Pseudoanalytic Function Theory. *Birkhäuser, Basel* , 2009.

[47] V. V. Kravchenko. *Applied Quaternionic Analysis*, volume 28 of *Research and Exposition in Mathematics*. Heldermann: Lemgo, 2003.

[48] V. V. Kravchenko, Direct and Inverse Sturm-Liouville Problems. A Method of Solution, *Frontiers in Mathematics, Birkhäuser,* 2020.

[49] V. V. Kravchenko. On a Biquaternionic Bag Model. *Z. Anal. Anwend.*, 3-14, 1995. DOI 10.4171/ZAA/658

[50] V. V. Kravchenko, S. Morelos and S. Tremblay. Complete systems of recursive integrals and Taylor series for solutions of Sturm-Liouville equations. *Mathematical Methods in the Applied Sciences*, 2012.

[51] V. V. Kravchenko, J. A. Otero and S. M. Torba. Analytic approximation of solutions of parabolic partial differential equations with variable coefficients. *Adv. Math. Phys.,* 2017. DOI 10.1155/2017/2947275

[52] V. V. Kravchenko and M. P. Ramírez. On Bers Generating Functions for First Order Systems of Mathematical Physics. *Adv. Appl. Clifford Algebras* **21**, 547-559, 2011. DOI 10.1007/s00006-010-0261-5.

[53] V. V. Kravchenko, L. J. Navarro, and S. M. Torba. Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions. *Applied Mathematics and Computation*, 314:173–192, 2017.

[54] V. V. Kravchenko and M. Shapiro. Helmholtz Operator with a Quaternionic Wave Number and Associated Function Theory. *Acta Applicandae Mathematicae*, 243-265, 1993.

[55] V. V. Kravchenko and M. Shapiro. Integral representations for spatial modes of mathematical physics. *Pitman Research Notes in Mathematics Series*, 1996.

[56] V. V. Kravchenko and S. M. Torba. Transmutations for Darboux transformed operators with applications. *Journal of Physics A: Mathematical and Theoretical*, 45(7):075201, 2012.

[57] V. V. Kravchenko and S. M. Torba. Transmutations and spectral parameter power series in eigenvalue problems. *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics*, page 209–238, 2012.

[58] V. V. Kravchenko and S. Tremblay. Spatial pseudoanalytic functions arising from the factorization of linear second order elliptic operators. *Math. Meth. Appl. Sci.,* 34: 1999-2010, 2011. DOI 10.1002/mma.1500

[59] V. V. Kravchenko and V. A. Vicente. Transmutation operators and complete systems of solutions for the radial Schrödinger equation. *Math. Methods Appl. Sci.*, 2020. DOI 10.1002/mma.6322

[60] C. Lanczos. Zeits. für Phys. **57**, 447-493, 1929.

[61] A. Lakhtakia, V. K. Varadan and V. V. Varadan *Time-Harmonic electromagnetic fields in chiral media.* Berlin: Springer-Verlag, Lecture Notes in Physics 355, 1989.

[62] H. Leutwiler Quaternionic Analysis on $\mathbb{R}^3$ Versus Its Hyperbolic Modification. NATO Science Series II. Mathematics, Physics and Chemistry. Vol. 25 2001.

[63] T. G. Mackay and A. Lakhtakia. Simultaneous amplification and attenuation in isotropic chiral materials. *Journal of Optics*, 18(5):055104, 2016.

[64] H. R. Malonek. Generalizing the (F, G)-Derivative in the sense of Bers. *In Clifford Algebras and Their Application in Mathematical Physics. Fundamental Theories of Physics (An International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application).* Dietrich V., Habetha K., Jank G. (eds), Springer, Dordrecht , 1998.

[65] Marsh, G. E. Force-free Magnetic Fields: Solutions, Topology And Applications. *World Scientific,* 1996.

[66] J.C. Maxwell. On physical lines of force. *Philosophical Magazine,* 11-23, 2010.

[67] A. Mcintosh and M. Mitrea. Clifford algebras and Maxwell's equations in Lipschitz Domains. Math. Methods Appl., 1599-1620, 1999.

[68] G. C. Moisil. Sur les systèmes d'équations de M. Dirac, du type elliptique. *C.R. Acad. Si. Paris,* 1292-1293, 1930.

[69] J. Morais, H. T. Le and M. A. Pérez. Quaternionic spherical wave functions. *Mathematical Methods in the Applied Sciences.* **39**, 2016.

[70] K. Nôno. Runge's Theorem for Complex valued and Quaternion Valued Hyper-Holomorphic Functions. *Rev. Roumaine Math. Pures Appl.* **32** (1987) 155–158.

[71] A. Y. Povzner. On differential Sturm-Liouville operators on semiaxis. *Math. USSR,* 1948.

[72] V. D. Shafranov. On magnetohydrodynamical equilibrium configurations. *Soviet Phys. JETP*, Vol: 6, 03 , 1958.

[73] M. P. Ramirez, V. D. Sanchez, O. Rodriguez and A. Gutierrez. On the General Solution of the Two-Dimensional Electrical Impedence Equation for a Separable-Variables Conductivity Function. *Proceedings of the Wold Congress on Engineering*, Vol: 2, 2009.

[74] M. Riesz. Clifford numbers and spinors. With the author's private lectures to E. Folke Bolinder, Fundamental Theories of Physics *Kluwer Academic Publishers Group, Dordrecht* **54**, 1993.

[75] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, 1987.

[76] W. Sprössig. On generalizaded Vekua type problems. *Adv. Appl. Clifford Algebras.* **11**, 2001.

[77] B. Thaller. The Dirac Equation.*Springer-Verlag Berlin Heidelberg*, 1992.

[78] N. Théodoresco. La derivée aréolaire. *Ann. Rum, Math.* 1936.

[79] I. N. Vekua. Generalized analytic functions. *Moscow: Nauka (in Russian); English Translation Oxford: Pergamon Press*, 1962.

[80] T. Wiegelmann and T. Sakurai. *Solar Force-free Magnetic Fields.* Springer Science and Business Media LLC, **9**, 2012.

[81] P. Yang, , L. Wang and L. Gao. Some properties and applications of the Teodorescu operator associated to the Helmholtz equation. *J. Inequal. Appl.*, 2017. https://doi.org/10.1186/s13660-017-1537-2

[82] X. Zhenyuan. A function theory for the operator $D - \lambda$. Complex Variables, Theory and Application: An International Journal, 16:1, 27-42, 1991. DOI: 10.1080/17476938208814464