CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO

DEPARTAMENTO DE MATEMÁTICAS

Teoría de excursiones para el proceso de Brox y sus aplicaciones

T E S I S

Que presenta
Mariana Pérez Rojas

Para obtener el grado de
DOCTORA EN CIENCIAS

EN LA ESPECIALIDAD DE MATEMÁTICAS

Director de la Tesis
Dr. Carlos Gabriel Pacheco González

Ciudad de México **Marzo, 2021**

CENTER FOR RESEARCH AND ADVANCED STUDIES OF
THE NATIONAL POLYTECHNIC INSTITUTE

CAMPUS ZACATENCO

DEPARTMENT OF MATHEMATICS

Excursion theory for the Brox diffusion and its applications

T H E S I S

Presented by
Mariana Pérez Rojas

To obtain the degree of
DOCTOR IN SCIENCE

IN THE SPECIALITY OF
MATHEMATICS

Thesis Advisor
Dr. Carlos Gabriel Pacheco González

Mexico City **March, 2021**

EXCURSION THEORY FOR THE BROX DIFFUSION AND ITS APPLICATIONS

Agradecimientos

Agradezco al Dr. Carlos Pacheco por todo su esfuerzo, paciencia y constante apoyo.

A mis padres, por apoyar mis decisiones y siempre brindarme su amor incondicional.

A mi pareja, Rodrigo Romero por estar siempre a mi lado y animarme a seguir adelante.

A mis sinodales, porque sus comentarios y sugerencias sin duda mejoraron el resultado final de esta tesis.

Agradezco al Centro de Investigación y de Estudios Avanzados del Instituto Politecnico Nacional (CINVESTAV) por todo lo que aprendí como estudiante en esta institución y al Consejo Nacional de Ciencia y Tecnología (CONACyT) por el apoyo económico brindado para la realización de mis estudios de doctorado.

Resumen

En esta tesis utilizamos la teoría de excursiones para la difusión de Brox. Lo hacemos relacionando este proceso con el movimiento browniano a través de la representación de Itô-McKean para las difusiones y la representación de su tiempo local. Como primera aplicación de este análisis, obtenemos la distribución de variables aleatorias con respecto al tiempo local en determinados tiempos de paro. Finalmente, utilizamos esta información y un conocido teorema ergódico para proporcionar dos algoritmos diferentes que recuperan el ambiente detrás de una trayectoria de la difusión de Brox.

Abstract

In this thesis, we apply Excursion theory to the so-called Brox diffusion. We do so by relating this process with the Brownian motion through the Itô-McKean's representation for diffusions and the representation of their local time. As a first application of this analysis, we obtain the distribution of random variables regarding the local time at certain hitting times. Finally, we use this information and a well-known ergodic theorem to give two different algorithms that retrieve the environment behind a path of the Brox diffusion.

Introduction

Random processes in a random environment are mathematical models of great interest in the area of probability. Roughly speaking, these objects arise from phenomena that can be described by a random model with a finite number of parameters, but instead of assuming these parameters to be constant, it is assumed that they are subject to random fluctuations, which we call the *environment*. Once the environment is fixed, the evolution of the random process can be performed. Examples of the phenomena that these type of processes can model are of DNA chain replication (see [9]) and turbulent behaviour in fluids through a Lorentz gas description (see [32]).

In discrete time, *Sinai's walk* is an example of such process. In the Sinai's walk, as in the random walk, a particle moves from point $x \in \mathbb{Z}$ to either $x+1$ or $x-1$ according to some parameter α. However, for Sinai's walk α is a function of the position x of the particle, that is,

$$\mathbb{P}[X_n = x+1 | X_{n-1} = x] = \alpha_x,$$
$$\mathbb{P}[X_n = x-1 | X_{n-1} = x] = 1 - \alpha_x.$$

Where $\alpha = \{\alpha_x\}_{x\in\mathbb{Z}}$ is a sequence of i.i.d. random variables taking values in $(0,1)$ called the environment. In Figure 1 a realization of the Sinai's walk is given. In this example, the values of the environment are given in the y-axis and each α_x take the value .25 or .75 according to a Bernoulli law of parameter $1/2$.

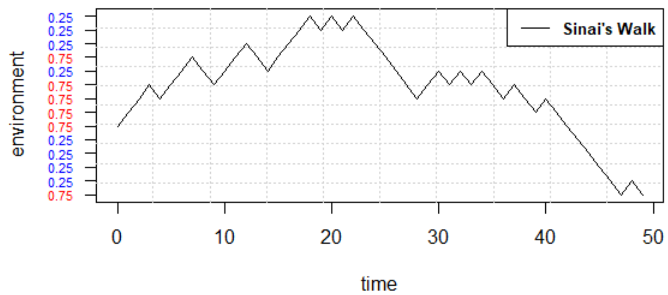

Figure 1: Path of Sinai's walk

In continuous time, an example of a random process in a random environment is the so-called *Brox diffusion*. This process is often considered as the time and space continuous analogue of Sinai's walk. The Brox diffusion is a continuous strong Markov process that has two sources of randomness, the "intrinsic" randomness, which is a Brownian motion, and the "extrinsic" randomness or environment, which is assumed to be a two-sided Brownian motion (a Brownian motion with index $\mathbb{R}$). This process has the peculiarity of spending most of its time near the so-called minima of its environment. Indeed, a known result of the Brox diffusion is that, for t large enough the points where the process spends most of its time are around local minima of the environment [8]. To give an heuristic idea of the reason behind this behavior, consider the following SDE

$$dX_t = dB_t - \frac{1}{2}W'(X_t)dt,$$

where X is the Brox diffusion, B is a Brownian motion and W is a two-sided Brownian motion. Then, X_t can be viewed as a Brownian motion B_t with drift term $-\frac{1}{2}\int_0^t W'(X_s)ds$. Suppose for a moment that W is a nice function, if x_0 is a local minimum of W then for ε small enough, $W'(u)$ is positive for $u \in (x_0, x_0+\varepsilon]$ and $W'(u)$ is negative for $u \in [x_0-\varepsilon, x_0)$. Therefore, when the process X takes values close to this minimum (which will happen with probability 1 as X is recurrent), the drift coefficient will pull the diffusion into the value of $X(x_0)$. Furthermore, this behavior will be more dramatic as the "depth" of the minimum increases.

The main objectives of this thesis are the following:

1. To apply *Excursion theory* to the Brox diffusion in order to find probabilistic properties of random variables of this diffusion. In particular, variables related to the local time which ultimately will help us understand why the particle is attracted to the minima of the environment.

2. To provide methods to infer information of the environment W based on observations from a known trajectory of the Brox diffusion. These methods may be applied for practical purposes in the context of forecasting time series in the future.

Let us further elaborate on our goals. The idea of excursion theory (first developed by K. Itô) is to decompose the path of a stochastic processes into its excursions at a recurrent point $a \in \mathbb{R}$. In the discrete setting, it is easy to verify that the successive excursions of a Markov chain at a recurrent point are independent and identically distributed, and this property plays a major role in the analysis of discrete-time Markov chains. In the continuous setting, it is no longer possible to enumerate the successive excursions at point a in chronological order. The correct point of view, which turns out to be extremely powerful, is to consider *local time* as a way to enumerate these excursions, which gives rise to Itô's point process of excursions.

Itô's excursion theory has proven to be a powerful tool to understand fine aspects of the paths of a stochastic process. In particular, for Brownian motion it has been used to prove results such as the Skorokhod embedding theorem, the Ray-Knight theorem and the arc-sine law. In [28], the proof of the Ray-Knight theorem is described as "a proof so simple as to explain very clearly why the result must take the form it does". For this reason, in Chapter 2 we apply the ideas of excursion theory to the particular case of the Brox diffusion in order to study random variables regarding this process. These random variables will help us understand the rigorous arguments behind the behaviour of the Brox diffusion paths that was heuristically explained above.

With respect to the second point in our list of objectives, when a random process in a random environment is used to model certain phenomenon, recovering information of the environment behind the movement observed can be extremely useful. The idea is that, when conducting an experiment, the realizations observed may depend on a hidden process (or environment) arising from external conditions. Thus, one is interested in inferring this process through the observations. This type of computations have been presented in

[1] for the Sinai's walk, where the logarithm of the local time was used as an estimator of the random environment.

In the last part of this thesis, our aim is to construct an algorithm for recovering the environment from a Brox diffusion path. In order to achieve this goal we use two different approaches. One is through the use of an ergodic theorem from [16] p. 228, which gives us the long term properties of the quotient of functions of the environment. The second approach employs the results of the random variables studied when excursion theory was applied to the Brox diffusion to view the path of the Brox diffusion as samples of these variables, thus obtaining a confidence interval for each point of the environment to be estimated.

In summary, Chapter 1 is concerned with providing the preliminaries that will be used throughout this thesis. We introduce the ideas of Excursion theory and we explain how to calculate certain values of the so-called excursion law of the Brownian motion. In Chapter 2 we apply the ideas seen on Section 1.6 to the Brox diffusion. The main tool used throughout this chapter is Itô and McKean's representation of a diffusion via a time and scale transformation. In Chapter 3 the ideas of recovering the environment from one path of the Brox diffusion are developed, resulting in two different algorithms. The first one gives an a.s. convergence when two parameters (that can be viewed as lengths of partitions of the time and the space) go to zero and the time goes to infinity (Subsection 3.1.2). The second one gives a confidence interval for the value of the environment at a certain point when the length of partitions of time and space go to zero (Subsection 3.2.3).

In addition, for the benefit of the reader we provide the computational coding in R used to carry on the simulations of a Brox diffusion path (Appendix A) and of the estimation of the environment, both for the first and second algorithm (Subsections 3.1.3 and 3.2.4, respectively).

Contents

Chapter 1

Preliminaries

Throughout this thesis, our main object of study will be a mathematical model called a *stochastic process*. Natural phenomena with some kind of randomness and evolving through time can be modeled using stochastic processes, this has be done with great success in areas such as biology [2], physics [12] and finance [22].

Before we introduce the definition of a stochastic process we need to understand where this object "lives". Let Ω be some set and $\mathcal{F}$ be a σ-field (also σ-algebra) on Ω. Then the tuple $(\Omega, \mathcal{F})$ is a measurable space, called the *sample space*. Now we can place on it a *probability measure*, which will be denoted by $\mathbb{P}$, this is a function that takes a measurable set (a set in $\mathcal{F}$) and maps it to a value between 0 and 1, among other properties [6].
Thus, a stochastic process is a collection of random variables indexed by some parameter t. For our purposes, $t \in [0, \infty)$ so that $X = \{X_t, 0 \leq t < \infty\}$ is a stochastic process on $(\Omega, \mathcal{F})$ which takes values in a second measurable space $(S, \mathcal{S})$, called the *state space*. In this thesis, we deal mainly with the state space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It is often convenient to give the index t an interpretation of *time*, thereby obtaining a process evolving through time, in which, at every $t \geq 0$ we can talk about a *past*, *present* and *future*. For a fixed $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ $t \geq 0$ is a *path* (trajectory, realization) of the process X associated with ω.

The notation and conventions used throughout this thesis are the following:
$\sigma(U)$: The σ-field generated by the family of sets U, the smallest σ-field which contains every set in U.
$\mathcal{B}(\mathbb{R})$: Borel sets in $\mathbb{R}$.
B^a: Brownian motion started at $a \in \mathbb{R}$.

$\mathbb{P}^a$: Probability measure under which a stochastic process starts at $a \in \mathbb{R}$.
$\mathbb{E}^a$: Expectation with respect to $\mathbb{P}^a$.
$f \circ g(x) := f(g(x))$ Composition of functions $f : X \to Y$ and $g : Z \to X$.
$\theta_s : \Omega \to \Omega$ Is the operator such that $X_t(\theta_s \omega) = X_{t+s}(\omega)$ $\forall \omega \in \Omega$, $s, t \geq 0$. Is called the shift operator.
$\lambda(A)$: Lebesgue measure of set $A \in \mathcal{B}(\mathbb{R})$.
$\langle X \rangle$: Quadratic variation of process $X = \{X_t\}_{t \geq 0}$.
$a \wedge b := \min(a, b)$, $a, b \in \mathbb{R}$.
$a \vee b := \max(a, b)$, $a, b \in \mathbb{R}$.
$exp(\lambda)$: Exponential distribution with parameter λ.
$Poi(\lambda)$: Poisson distribution with parameter λ.
$X \sim F$: X has distribution F.
$L_X(t, x)$: Local time of the process X until time t at point x.
$\mathcal{L}_X(t, x)$: Modified local time of the process X.
Excursion: A right-continuous with left limits function $f : \mathbb{R}^+ \to \mathbb{R}$ such that for some $a \in \mathbb{R}$ $f(t) = f(H) = a \quad$ for all $t \geq H$, where $H := \inf\{t > 0 : f(t) = a\}$.
U: Excursion space, the set of all excursions.
$N_X^a((0, l] \times A)$: Number of excursions of X at a belonging to the set $A \subseteq U$, and with local time $\leq l$.
$N_{X_\Delta}^u(I)$: Number of crossing of X_Δ defined as $N_{X_\Delta}^u(I) = \#\{t \in I : X_\Delta(t) = u\}$.
$H_X^a := \inf\{t > 0 : X_t = a\}$, $a \in \mathbb{R}$.
$\gamma_t^a := \inf\{u > 0 : L_X(u, a) > t\}$.
$C[0, \infty)$: The space of continuous, real valued functions on $[0, \infty)$.
$\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{Z}^+ := \{0, 1, 2, ...\}$, $\mathbb{R}^+ := [0, \infty)$, $\mathbb{R}^{++} := (0, \infty)$.

Sometimes, if the subscripts or superscripts take the value zero, we may omit them e.g. $B = B^0$ is the Brownian motion started at 0. Let T be a function, then both T_t and $T(t)$ denote the evaluation of the function T at point t.

1.1 Brownian motion, filtrations and stopping times

The first natural example of a stochastic process is the *Brownian motion*. This process satisfies several of the most important properties one can ask from a stochastic process (e.g. independent and stationary increments, continuous paths, self-similarity). Even though the definition of the Brownian motion is very particular, as we will see in the next sections, it is a fundamental element in the theory of much more general processes.

For some given probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, we have the following definition of a Brownian motion.

Definition 1. A real-valued stochastic process $\{B_t, 0 \leq t < \infty\}$ is a Brownian motion if it satisfies the following properties:

(*i*) $B_0 = 0$, $\mathbb{P}$ a.s.

(*ii*) the map $t \mapsto B_t(\omega)$ is a continuous function of $t \in \mathbb{R}^+$, $\mathbb{P}$ a.s.

(*iii*) for every $t, h \geq 0$, $B_{t+h} - B_t$ is independent of $\{B_u : 0 \leq u \leq t\}$, and has a Gaussian distribution with mean 0 and variance h.

In order to keep track of the information known by the observer, we introduce a *filtration* to our measurable space $(\Omega, \mathcal{F})$. A filtration is a nondecreasing family $\{\mathcal{F}_t, t \geq 0\}$ of sub σ-fields of $\mathcal{F}$ i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$. Given a stochastic process X, one can choose the filtration generated by the process itself by taking

$$\mathcal{F}_t^X := \sigma(X_s, 0 \leq s \leq t),$$

this is the smallest σ-field with respect to which X_s is measurable for every $s \in [0, t]$.

Remark 1. Usually, the filtration is part of the definition of a stochastic process. When the filtration is not explicitly stated on the definition, assume we are working with the filtration $\{\mathcal{F}_t^X, t \geq 0\}$.

Definition 2. The stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if, for each $t \geq 0$, X_t is a $\mathcal{F}_t$ measurable random variable.

Definition 3. The stochastic process X is called *progressively measurable* with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if, for each $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$, the set $\{(s, \omega) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

Remark 2. We will only consider stochastic processes $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ where X is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

With the interpretation of a filtration as the information accumulated until time t, we now introduce the idea of a *stopping time*. Intuitively, a stopping time of a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a random event which can be determined if it happened with the information obtained up to time t.

Definition 4. Let $(\Omega, \mathcal{F})$ be a measurable space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A random time T is a *stopping time* of the filtration, if

$$\{T \leq t\} \in \mathcal{F}_t, \quad \text{for every } t \geq 0.$$

A random time T is an *optional time* of the filtration, if

$$\{T < t\} \in \mathcal{F}_t, \quad \text{for every } t \geq 0.$$

Remark 3. Every stopping time is an optional time, and the two elements coincide if the filtration is right-continuous i.e. if $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{u>t} \mathcal{F}_u$ for all $t \geq 0$.

For a process X on $(\Omega, \mathcal{F})$, filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a stopping time T, we define the mapping X_T on the set $\{T < \infty\}$ by

$$X_T(\omega) = X_{T(\omega)}(\omega).$$

Furthermore, we can define the σ-field generated by this process as we see in the next definition.

Definition 5. Let X be a measurable process and T a random time. The collection of all sets of the form $\{X_T \in A\}$, $A \in \mathcal{B}(\mathbb{R})$, together with the set $\{T = \infty\}$ forms a sub-σ-field of $\mathcal{F}$. We call this the σ-field generated by X_T.

The definition of X_T and of its σ-field will be relevant in the next section, when Markov processes are defined.

Definition 6. The space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is said to satisfy the usual conditions if in addition to the filtration property

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad 0 \leq s \leq t,$$

the following properties hold:

(i) The σ-field $\mathcal{F}$ is $\mathbb{P}$-complete (i.e. every subset of every null set is measurable) (*ii*) $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$

(*iii*) $\{\mathcal{F}_t\}$ is right-continuous

Remark 4. From now on, all processes considered are defined on a space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{t\geq 0})$ that satisfy the usual conditions. These conditions are standard in the literature and are required by some fundamental theorems, such as Doob's Supermartingale Convergence Theorem (see [27]).

Let X be a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. with $\mathbb{P}$ almost surely continuous sample paths. Then, X can be regarded as a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(C[0,\infty), \mathcal{B}(C[0,\infty)))$, that is, the space of continous, real valued functions on $[0,\infty)$ together with the Borel σ-field generated by the collection of finite-dimensional cylinders (see [19] ch. 2). Then X induces a probability measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ called the *law* of the stochastic process.

Definition 7. Let X be a continuous stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the law of X is given by

$$\mathcal{P}(B) := \mathbb{P}X^{-1}(B) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\}, \quad B \in \mathcal{B}(C[0,\infty)).$$

1.2 Markov processes

The Markov property is sometimes referred to as the memoryless property. Intuitively speaking, it encapsulates the idea that sometimes in order to make a prediction about the future state of a process we only need information about the present, thus making the past information irrelevant.

An example of a Markov process is the Brownian motion, recall the definition of the Brownian motion from last section as a stochastic process that starts at zero $\mathbb{P}$ a.s. In this section we introduce the concept of a family of measures in order to let our processes start at points other than zero.

Definition 8. Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P}^\mu)$ is said to be a *Markov process with initial distribution* μ if

(*i*) $\mathbb{P}^\mu(X_0 \in \Gamma) = \mu(\Gamma), \forall\, \Gamma \in \mathcal{B}(\mathbb{R})$

(*ii*) for every bounded, measurable map $f : \mathbb{R} \to \mathbb{R}$ and $s, t \geq 0$,

$$\mathbb{E}^\mu[f(X_{s+t})|\mathcal{F}_s] = \mathbb{E}^\mu[f(X_{s+t})|X_s], \quad \mathbb{P}^\mu \text{ a.s.}$$

Remark 5. If μ assigns measure one to some singleton $\{x\}$, we will write $\mathbb{P}^x$ instead of $\mathbb{P}^\mu$.

We will see in the next sections that it is often convenient to have a whole family of probability measures that give rise to a family of Markov processes. Each of these probability measures denotes a different starting point of process X, this will be notationally helpful, especially when studying excursion theory in section 1.6.

Definition 9. A *Markov family* is a process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some $(\Omega, \mathcal{F})$, together with a family of probability measures $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ on $(\Omega, \mathcal{F})$, such that

(*i*) for each $F \in \mathcal{F}$, the mapping $x \mapsto \mathbb{P}^x(F)$ is universally measurable

(*ii*) $\mathbb{P}^x(X_0 = x) = 1\ \forall\, x \in \mathbb{R}$

(*iii*) for every bounded, measurable map $f : \mathbb{R} \to \mathbb{R}$ and $s, t \geq 0$

$$\mathbb{E}^x[f(X_{s+t})|\mathcal{F}_s] = \mathbb{E}^{X_s}[f(X_t)], \quad \mathbb{P}^x \text{ a.s.}$$

Remark 6. Condition (i) is a weaker measurability condition that allows the expansion of $\mathcal{F}$ to larger σ-fields (see [19] p.73). Condition (ii) gives rise to the fact that process X under $\mathbb{P}^x$ is called the Markov process X started at x. Finally, condition (iii) constitute the Markov property.

Our next step is to define the *strong Markov property*. This property is one of the few requirements we will ask from a stochastic process in order to be a *diffusion*. A process satisfies the strong Markov property if the Markov property can be applied to certain random times.

Definition 10. Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some $(\Omega, \mathcal{F}, \mathbb{P}^\mu)$ is said to be a *strong Markov process with initial distribution* μ if

(i) $\mathbb{P}^\mu(X_0 \in \Gamma) = \mu(\Gamma)$, $\forall\, \Gamma \in \mathcal{B}(\mathbb{R})$

(ii) for any stopping time S of $\{\mathcal{F}_t\}$, $t \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ bounded and measurable

$$\mathbb{E}^\mu[f(X_{S+t})|\mathcal{F}_S] = \mathbb{E}^\mu[f(X_{S+t})|X_S], \quad \mathbb{P}^\mu \text{ a.s. on } \{S < \infty\}.$$

Definition 11. A *strong Markov family* is a process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ on some $(\Omega, \mathcal{F})$, together with a family of probability measure $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ on $(\Omega, \mathcal{F})$, such that

(i) for each $F \in \mathcal{F}$, the mapping $x \mapsto \mathbb{P}^x(F)$ is universally measurable

(ii) $\mathbb{P}^x(X_0 = x) = 1\ \forall\, x \in \mathbb{R}$

(iii) for any bounded continuous $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}^x[f(X_{S+t})|\mathcal{F}_S] = \mathbb{E}^{X_s}[f(X_t)] \quad \mathbb{P}^x \text{ a.s. on } \{S < \infty\}.$$

Remark 7. Definitions 10 and 11 can be stated for optional times instead of stopping times (see [19]), in our context these two elements coincide (recall Remark 3 and Definition 6).

1.3 Itô's formula

The main goal of this section is to introduce the single most important formula of Itô calculus, namely *Itô's formula* (also change of variable formula or Itô's rule). This formula provides an integral-differential calculus for the sample paths of certain types of stochastic processes. Before we can state this result, we begin by recalling some basic definitions.

Martingales are stochastic processes associated with the dynamics of "fair" games, where the best guess about the future is the present state of the process. They are characterized by the the fact that $\mathbb{P}$ a.s.

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad \text{for every } 0 \leq s < t < \infty.$$

This definition can be generalized by asking a process to be a martingale only at some sequence of random times as follows:

Definition 12. Let $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ be a (continuous) process with $X_0 = 0$ a.s. If there exists a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathcal{F}_t\}$, such that $\{X^{(n)} := X_{t \wedge T_n}, \mathcal{F}_t : 0 \leq t < \infty\}$ is a martingale for each $n \geq 1$ and $\mathbb{P}(\lim_{n\to\infty} T_n = \infty) = 1$, then we say that X is a (continuous) *local martingale.*

This last definition can be further generalized by introducing the concept of a *semimartingale.* This class of processes is of special importance in stochastic calculus since it constitutes the largest class of processes with respect to which the Itô integral can be defined (see [19] chapter 3 for more information on the Itô integral).

Definition 13. A continuous semimartingale $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is an adapted process which has the decomposition $\mathbb{P}$ a.s.,

$$X_t = X_0 + M_t + A_t; \quad 0 \leq t < \infty, \tag{1.1}$$

where $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous local martingale and $A = \{A_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is the difference of continuous, nondecreasing, adapted processes started at zero.

This definition is $\mathbb{P}$-a.s. unique. Now for the main result of this section, Itô's formula states the rules by which a continuous semimartingale can be manipulated.

Theorem 1 ([18])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and let $X = \{X_t, \mathcal{F}_t; 0 \leq t <$*

$\infty\}$ *be a continuous semimartingale with decomposition (1.1). Then* $\mathbb{P}$ *a.s.,*

$$\begin{aligned} f(X_t) = & f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dA_s \\ & + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s, \quad 0 \le t < \infty, \end{aligned} \tag{1.2}$$

where $\langle M\rangle$ *is the quadratic variation of* M*.*

Remark 8. From the properties of the stochastic integral, we know that the first integral is a continuous, local martingale. The other two integrals of (1.2) are to be understood in the Lebesgue-Stieltjes sense, and so, as function of the upper limit of integration, are of bounded variation. Thus, $\{f(X_t), \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous semimartingale.

Remark 9. Equation (1.2) is often written in differential notation:

$$\begin{aligned} df(X_t) = & f'(X_t)dM_t + f'(X_t)dA_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t \\ = & f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t, \quad 0 \le t < \infty, \end{aligned}$$

This equality is called the "chain rule" for stochastic calculus.

1.4 Local time

Local time is a mathematical object that allows us to measure the amount of time spent by a path of certain stochastic processes around some value $x \in \mathbb{R}$, it was first introduced for the Brownian motion by P. Lévy [21] and proven to be jointly continuous by H.F. Trotter [35]. Local time will be a major tool used throughout this thesis, as we will see later on, it is a fundamental element of excursion theory.

Here, we begin by defining Brownian local time and then we extend the concept of local time for continuous semimartingales.

Theorem 2. *([35]) There exists a process* $\{L_B(t,x) : t \geq 0, x \in \mathbb{R}\}$ *such that*

(i) $(t,x) \to L_B(t,x)$ *is jointly continuous;*

(ii) for any bounded measurable f *and* $t \geq 0$,

$$\int_0^t f(B_s)ds = \int_{-\infty}^{\infty} f(x)L_B(t,x)dx. \tag{1.3}$$

Remark 10. In some literature, equation 1.3 is given with a different normalizing constant. Here we follow [27].

$L_B(t,x)$ is called the local time of B at x up to time t. For a fixed Borel set $A \in \mathcal{B}(\mathbb{R})$, we define the *occupation time* of the Brownian motion B up to time t as

$$\Gamma_B(t,A) = \int_0^t \mathbb{1}_A(B_s)ds = \lambda\{0 \leq s \leq t : B_s \in A\}, \quad 0 \leq t < \infty,$$

where λ is the *Lebesgue measure*. Equation (1.3) with $f(x) = \mathbb{1}_A(x)$, $A \in \mathcal{B}(\mathbb{R})$ indicates that $L_B(t,x)$ can be viewed as the density with respect to the Lebesgue measure of the occupation time. In other words, we have

$$\Gamma_B(t,A) = \int_A L_B(t,x)dx, \quad 0 \leq t < \infty, A \in \mathcal{B}(\mathbb{R}).$$

Later on, we will use this intuition to define a modified local time for diffusions satisfying a certain property.

The next theorem is a well-known result called Tanaka's formula, it has two very important contributions: It introduces the important notion of local time for semimartingales and secondly, this result leads to a generalization of Ito's formula for convex function (recall that Itô's formula was stated for C^2 functions in Theorem 1.2) called the Itô-Tanaka formula (see [34] p.223).

Theorem 3 ([34] p.222). *Let X be a continuous semimartingale. For any real number a, there exists an increasing continuous process $\{L_X(t,a) : t \geq 0\}$ such that:*

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a) dX_s + L_X(t,a),$$

where $sgn(x) = -1$ if $x \leq 0$ and $sgn(x) = 1$ if $x > 0$. The process L_X is called the (semimartingale) local time of X at zero.

Remark 11. The alternative forms of the Tanaka's formula are

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{\{X_s > a\}} dX_s + \frac{1}{2} L_X(t,a),$$
$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbb{1}_{\{X_s \leq a\}} dX_s + \frac{1}{2} L_X(t,a)$$

where $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$.

The next result deals with the question of measurability of the two parameter process $\{L_X(t,a) : a \in \mathbb{R}, t \geq 0\}$.

Theorem 4. *There exists a version of $\{L_X(t,a) : a \in \mathbb{R}, t \geq 0\}$ which is continuous in t and right-continuous with left limits in a.*

Remark 12. From now on, we will assume local time $L_X(t,a)$ to be continuous at t and right-continuous with left limits in a.

1.5 Diffusions and the infinitesimal generator

We will now give a review on diffusion theory, the material in this chapter is mainly based on [28]. We assume the state space to be $I = \mathbb{R}$.

Definition 14. A stochastic process $X = \{X(t), \mathcal{F}_t, t \geq 0\}$ on some $(\Omega, \mathcal{F})$, together with a family of probability measures $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ on $(\Omega, \mathcal{F})$ is said to be a diffusion if it has a.s. continuous paths and it is a strong Markov family.

Remark 13. From Definition 14 we see that a diffusion is in fact a strong Markov family. We use the term diffusion for both the whole family and a single element of it without risk of confusion.

There are several approaches for the construction and study of diffusions, varying from the purely analytical to the probabilistic ones. The methodology of *stochastic differential equations* (SDEs) was first suggested by P.Lévy and carried out by K. Itô. They considered the following equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \tag{1.4}$$

where B is a standard Brownian motion, $\sigma : \mathbb{R} \to (0, \infty)$ and $b : \mathbb{R} \to \mathbb{R}$. Presumably, the most important result of this theory is the theorem by K. Itô which states that if σ and b are Lipschitz functions, then the SDE in 1.4 has a strong solution (a solution on a given probability space, with respect to a given filtration and a given Brownian motion). It can be shown that this solution is in fact a diffusion, moreover it is related to important concepts, such as the Martingale Problem (for further information see [28] chapter 5).

Another probabilistic approach for the study of diffusions is by constructing it from a Brownian motion via a time and space transformation. In this chapter we will be concerned with this approach. As we will see, this method possess a very rich theory where two important elements stand out, namely, the *scale function* and the *speed measure.*

Definition 15. A diffusion X is called regular if for all $x, y \in \mathbb{R}$ we have

$$\mathbb{P}^x(H_X^y < \infty) > 0.$$

The concept of regularity is similar to the concept of irreducibility for Markov chains and gives the diffusion a much more orderly behaviour.

Remark 14. From now on all diffusions considered are regular.

Definition 16. A scale function for a (regular) diffusion X on $\mathbb{R}$ is a continuous strictly increasing function $s : \mathbb{R} \to \mathbb{R}$ such that, for all $x, a, b \in \mathbb{R}$ with x between a and b,

$$\mathbb{P}^x(H_X^b < H_X^a) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

If $s(x) = x$ is the scale function of X, we say that X is in natural scale. The function s is unique up to increasing affine transformations.

The following theorem gives the transformation that has to be applied to a diffusion with scale measure s in order to obtain a diffusion in natural scale.

Theorem 5. *Let X be a diffusion on $\mathbb{R}$ with scale function s, then $Y := s(X)$ is a diffusion in natural scale on $s(\mathbb{R})$.*

The next theorem has two purposes, one is to define the speed measure of a diffusion in natural scale X and the other is to observe that any regular diffusion in natural scale is a Brownian motion with the time changed.

Theorem 6 ([28]). *Let X be a regular diffusion in natural scale on $\mathbb{R}$. Then there exists a measure m on $\mathbb{R}$ such that, for each $y \in \mathbb{R}$, there exists on some enrichment of $(\Omega, \mathcal{F}, \mathbb{P}^y)$ a Brownian motion B started at y such that the diffusion X under $\mathbb{P}^y$, can be expressed as a time change of B:*

$$X_t = B_{T^{-1}(t)}, \tag{1.5}$$

where T^{-1} is the right-continuous inverse of

$$T(t) = \int_{\mathbb{R}} L_B(t, z) m(dz).$$

Definition 17. The measure m appearing in the statement of Theorem 6 is called the speed measure of the diffusion X. The speed measure also has the property:

$$m([a, b]) < \infty \quad \text{for any } a < b \in \mathbb{R}. \tag{1.6}$$

Remark 15. The converse of Theorem 6 also holds. Given a measure m on $\mathbb{R}$ satisfying 1.6 and a Brownian motion B with local time $L_B(t, x)$, we can construct a regular diffusion in natural scale on $\mathbb{R}$ with representation 1.5.

From the results above, we see that any (regular) diffusion is uniquely characterized by its scale function and speed measure, moreover we have that any diffusion is a scale and time-change of Brownian motion. Indeed, if X is a diffusion with scale function s then from Theorem 5, $Y := s(X)$ is in natural scale therefore from Theorem 6 there exists m so that X admits the representation

$$X_t = s^{-1}(B_{T^{-1}(t)}), \tag{1.7}$$

where T^{-1} is the inverse of the time-change function

$$T(t) = \int_{\mathbb{R}} L_B(t,x)m(dx)$$

and B is a standard Brownian motion.

Remark 16. Note that the measure m from last equation is the speed measure of the process in natural scale Y.

From the next lemma we have that any regular diffusion in $\mathbb{R}$ is recurrent. This will greatly simplify the results given on the next subsection about excursion theory for diffusions.

Lemma 1. *Let X be a regular diffusion on $\mathbb{R}$ with scale function s and speed measure m. Then X is recurrent, i.e.,*

$$\mathbb{P}^x(H_X^y < \infty) = 1, \quad \forall x, y \in \mathbb{R}.$$

Definition 18. Consider a Markov family $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$, $(\Omega, \mathcal{F})$, $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$, and assume that X has continuous paths. The *infinitesimal generator* of the Markov family is given by

$$\mathcal{A}f(x) = \lim_{t\downarrow 0} \frac{1}{t}(\mathbb{E}^x[f(X_t)] - f(x)),$$

where f is a function such that the previous limit exists.

From (1.7) and from the fact that Brownian local time exists, it is possible to define the local time of any one-dimensional diffusion. The next theorem provides the explicit form of its local time in terms of the local time for Brownian motion and its interpretation as an occupation density.

Theorem 7. *Let X be a diffusion with representation (1.7), then the local time process of X is*

$$\{L_X(t,a) : a \in \mathbb{R}, t \ge 0\} = \{L_B(T^{-1}(t), s(a)) : a \in \mathbb{R}, t \ge 0\}, \tag{1.8}$$

and the occupation-measure formula says that, for any bounded measurable function f supported in $\mathbb{R}$,

$$\int_0^t f(X_s)ds = \int f(a)L_X(t,a)m(da). \tag{1.9}$$

From equation (1.9) with $f(x) = \mathbb{1}_A(x)$, $A \in \mathcal{B}(\mathbb{R})$ we see that the local time of the diffusion X is the density with respect to the speed measure of X for the occupation time of X, that is

$$\Gamma_X(t, A) = \int_A L_X(t, a) m(da), \quad 0 \leq t < \infty, A \in \mathcal{B}(\mathbb{R}). \tag{1.10}$$

Definition 19. Let X be a diffusion with speed measure m and suppose that m is absolutely continuous with respect to the Lebesgue measure. Then we can define the process

$$\{\mathcal{L}_X(t, a) : a \in \mathbb{R}, t \geq 0\} := \{L_X(t, a)\dot{m}(a) : a \in \mathbb{R}, t \geq 0\}, \tag{1.11}$$

where $m(da) = \dot{m}(a)da$. We call $\mathcal{L}_X(t, a)$ the *modified* local time of X at time t in a.

The process $\mathcal{L}_X(t, a)$ satisfies the following equation

$$\int_0^t f(X_s)ds = \int f(a)\mathcal{L}_X(t, a)da.$$

Note that for $f(x) = \mathbb{1}_A(x)$, the modified local time defined above has the same interpretation of local time for Brownian motion as seen in subsection 1.4. This is the reason why we will give results in terms of this process instead of the local time $L_X(t, a)$.

Remark 17. In some literature, the modified local time as defined above is in fact simply called the local time, we give it a different name in order to be consistent with the definitions found in [28]. See for example [14] and [15].

1.6 Excursion theory

The first explicit appearance of formal excursion theory for Brownian motion was given by K. Itô in [17]. He found that the excursions form a sequence of independent and identically distributed random functions, which together with their local time form in fact a Poisson point process. This discovery has proved to be a powerful computational technique.

The ideas of excursion theory can be applied to a wide variety of processes e.g. for continuous-time Markov process with some recurrent state. Throughout this section

$$X = \{X_t, \mathcal{F}_t, \Omega, \mathcal{F}, \{\mathbb{P}^a\}_{a\in\mathbb{R}}, t \geq 0\}$$

will denote a regular diffusion, the results will be given only for diffusions. For more general results see [28].

Remark 18. We will be using the following notation:

(*i*) γ_u^x is the right-continuous inverse of the local time $L_{X^a}(t,x)$ i.e. $\gamma_u^a := \inf\{t > 0 : L_{X^a}(t,x) > u\}$.

(*ii*) Γ_u^x the right-continuous inverse of the modified local time $\mathcal{L}_{X^a}(t,x)$ i.e. $\Gamma_u^x := \inf\{t > 0 : \mathcal{L}_{X^a}(t,x) > u\}$.

We will now present the following definition in order to give a brief reminder on Poisson random measures.

Definition 20. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(H, \mathcal{H})$ a measurable space, and $\nu(C)$ a $\mathbb{Z}^+ \cup \{\infty\}$-valued random variable, for each fixed $C \in \mathcal{H}$. We say that ν is a *Poisson random measure* if:

(*i*) For every $C \in \mathcal{H}$, either $\mathbb{P}(\nu(C) = \infty) = 1$, or else

$$\psi(C) := \mathbb{E}[\nu(C)] < \infty$$

and $\nu(C)$ is a Poisson random variable:

$$\mathbb{P}(\nu(C) = n) = e^{-\psi(C)} \frac{(\psi(C))^n}{n!}; \quad n \in \mathbb{Z}^+.$$

(*ii*) For any pairwise disjoint sets $C_1, ..., C_m$ in $\mathcal{H}$, the random variables $\nu(C_1), ..., \nu(C_m)$ are independent.

The measure $\psi(C) = \mathbb{E}[\nu(C)]$, $C \in \mathcal{H}$, is called the *intensity measure* of ν.

Definition 21. An excursion is a right-continuous with left limits function $f : \mathbb{R}^+ \to \mathbb{R}$ such that for some $a \in \mathbb{R}$ satisfies the coffin condition:

$$f(t) = f(H) = a \quad \text{for all } t \geq H,$$

where

$$H := \inf\{t > 0 : f(t) = a \text{ or } f(t-) = a\}.$$

The lifetime H of the excursion f must be positive.

We denote the set of all excursions by U and $\mathcal{B}(U)$ the Borel σ-field of the topological space U if we take the Skorokhod metric (for more on this topic see [5] Ch. 3).

Definition 22. The point process of excursions at a of diffusion X is defined as

$$\Pi := \{(l, e_l) : \gamma_l^a \neq \gamma_{l-}^a\},$$

where $e_l \in U$ is defined as follows:

$$e_l(s) = \begin{cases} X(\gamma_{l-}^a + s) & \text{for } 0 \leq s < \gamma_l^a - \gamma_{l-}^a; \\ a & \text{for } s \geq \gamma_l^a - \gamma_{l-}^a. \end{cases}$$

For each l such that $\gamma_l^a \neq \gamma_{l-}^a$, we call e_l the excursion at local time l.

One can think of a point process as a $\mathbb{Z}^+$-valued random measure. We therefore define, for Borel $A \subseteq \mathbb{R}^{++} \times U$, the $\mathbb{Z}^+$-valued random measure

$$N_X^a(A) := |A \cap \Pi| .$$

In Figure 1.1 we see an example of an excursion at a of the Brownian motion. On the left we have the path of a Brownian motion and circled is an excursion at a that starts at time t. On the right we have the excursion e_l where $l = L_B(t, a)$. In Figure 1.2 we illustrate a representation of a point $(l, e_l) \in \Pi^a$ where the x-axis represents the local time and the y-axis the excursion space U.

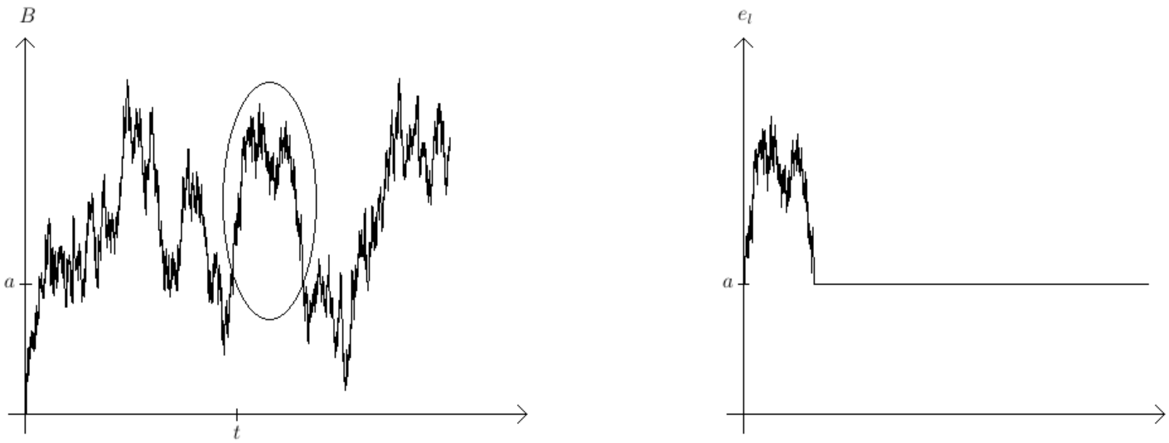

Figure 1.1: Path of Brownian motion (left) and an excursion (right).

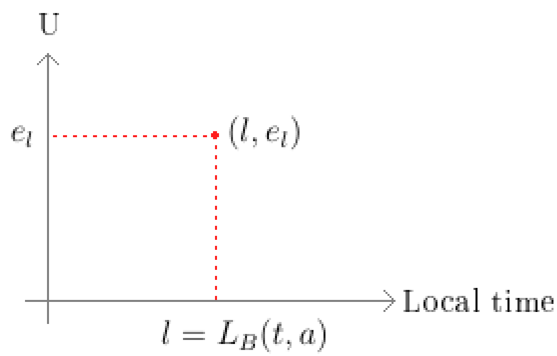

Figure 1.2: Point (l, e_l) that corresponds to the excursion above.

The following result is arguably the most important of Itô's excursion theory.

Theorem 8. *There exists a σ-finite measure η on U with the following property: if N' is a Poisson random measure on $\mathbb{R}^{++} \times U$ with intensity measure $\mu = \lambda \times \eta$, then under $\mathbb{P}^a$,*

$$N_X^a \stackrel{d}{=} N'.$$

The measure η is called the characteristic measure or Itô excursion law of the excursion process.

Remark 19. From now on, η will be the characteristic measure of excursions at zero of the Brownian motion, i.e. $N_B^0((0,t] \times S)$ is a Poisson random variable with intensity measure $\mu = \lambda \times \eta$.

The description of η is somewhat complicated (see [36]), but it is possible to find the value of $\eta(S)$ for sets S of a certain form. The following well known proposition is a very useful result whose proof shows the power of excursion theory, it is also instrumental for our purposes.

Proposition 1. *For $x > 0$,*

(i) $\eta(\{f \in U : \sup_t f(t) > x\}) = (2x)^{-1}$, *and*

(ii) $\eta(\{f \in U : \sup_t |f(t)| > x\}) = x^{-1}$.

Proof. (i) Let $U^{\pm}(x) := \{f \in U : \sup_t \pm f(t) > x\}$ and

$$T := \inf\{t : N_B^0((0,t] \times U^+(x)) > 0\}.$$

Then

$$\mathbb{P}(T > l) = \mathbb{P}(N_B^0((0,l] \times U^+(x)) = 0).$$

By Theorem 8 applied to the Brownian motion, we have that under $\mathbb{P}$, $N_B^0((0,l] \times U^+(x)) \sim Poisson(l\eta(U^+(x)))$, which yields

$$\mathbb{P}(T > l) = \exp(-l\eta(U^+(x))).$$

Thus, T is an exponential random variable with parameter $\eta(U^+(x))$, which gives

$$\mathbb{E}[T] = \eta(U^+(x))^{-1}. \tag{1.12}$$

However, if $\tau := \inf\{t : B_t = x\}$ then $T = L_B(\tau, 0)$. By the Tanaka formula 11 $B_t^+ - \frac{1}{2}L_B(t,0)$ is a martingale, therefore the optional sampling theorem yields $\mathbb{E}[T] = 2x$. This result together with 1.12 establishes (i).

(ii) From the symmetry of the Brownian motion we have that

$$N_B^0((0,l] \times U^+(x)) \stackrel{d}{=} N_B^0((0,l] \times U^-(x)),$$

which together with Theorem 8 implies that $\eta(U^+(x)) = \eta(U^-(x))$. Thus, writing

$$\{f \in U : \sup_t |f(t)| > x\}$$

as the disjoint union of $U^+(x)$ and $U^-(x)$ and applying (i), the result follows (note that $U^+(x)$ and $U^-(x)$ are indeed disjoint as we are looking at excursions at zero). ∎

Remark 20. In some literature the proposition above differs from ours by a constant. This is a consequence of a different normalization of local time.

Chapter 2

Excursion theory for the Brox diffusion

2.1 The Brox diffusion

In [7] , T. Brox considered the following SDE:

$$dX_t = dB_t - \frac{1}{2}W'(X_t)dt, \quad X_0 = 0, \tag{2.1}$$

where $B = \{B_t : t \geq 0\}$ is the standard Brownian motion, and $W = \{W(x) : x \in \mathbb{R}\}$ is a two sided Brownian motion independent of B called the *environment*. W' denotes the derivative of W in the sense of Schwartz distribution and is called the white noise (see [13]).

Assuming that the standard theory of diffusions apply, by taking $b(x) := -\frac{1}{2}W'(x)$ and $\sigma(x) := 1$, one may associate to equation (2.1) the following infinitesimal generator (see [28] p. 163)

$$\mathcal{A}f(x) := \frac{1}{2e^{-W(x)}}\frac{d}{dx}\left(e^{-W(x)}\frac{df(x)}{dx}\right),$$

which is rigorously defined. Moreover, the diffusion associated to $\mathcal{A}$ has scale function

$$s(x) := \int_0^x e^{W(y)}dy,$$

and speed measure

$$m(A) := \int_A e^{-W(y)}dy, \text{ for Borel sets } A \subseteq \mathbb{R}. \tag{2.2}$$

Following (1.7), using the scale function and speed measure above, we can construct a diffusion X as

$$X_t = s^{-1}(B_{T_t^{-1}}), \tag{2.3}$$

where

$$T_t := \int_0^t e^{-2W(s^{-1}(B_u))} du. \tag{2.4}$$

Definition 23. The stochastic process $X = \{X_t, \mathcal{F}_t : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with representation 2.3 is called the *Brox diffusion*.

Remark 21. There are two sources of randomness in (2.1). One coming from B and the other from W.

Remark 22. For a fixed trajectory of W, the process X is a diffusion. From now on we will work with the Brox diffusion with a fixed environment $W(\omega)$, this is known as the *Quenched case*.

Several important relations can be derived from representation 2.3. For instance, the following relationship between the hitting times of the Brownian motion and the Brox diffusion.

Lemma 2. *It holds*

$$T(H_B^a) = H_X^{s^{-1}(a)}. \tag{2.5}$$

Proof. It follows from representation (2.3).

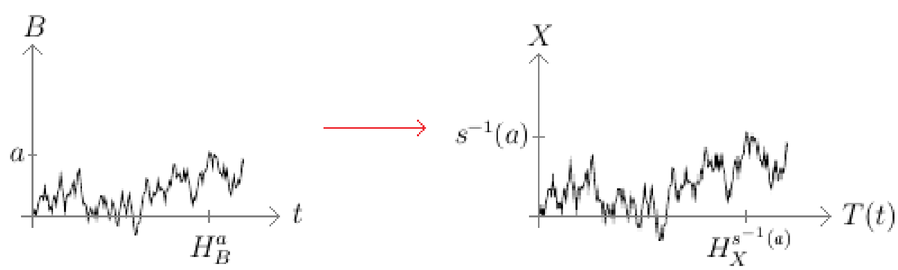

Figure 2.1: Hitting times

■

We now continue by defining the Brox diffusion started at $a \in \mathbb{R}$, we will need this definition later on in order to apply what we saw in section 1.6 to the Brox diffusion.

Fix $a \in \mathbb{R}$, and let us define the process

$$X^a = \{X_{t+H_X^a}\}_{t \geq 0}.$$

We call X^a the Brox diffusion started at a. This process is simply cutting off the first part of the original process and starting when it first visited a. As X satisfies the strong Markov property and H_X^a is an almost surely finite stopping time, this means that X^a is independent of the information on X before H_X^a, X^a also inherits the properties of the original process, therefore it is a diffusion. Our next step is to figure out the analogue of representation 2.3 for this process.

From (2.3) applied to time $t + H_X^a$ we have

$$X_t^a = s^{-1}(B_{T^{-1}(t+H_X^a)}), \quad t \geq 0. \tag{2.6}$$

We define $T_u^a := T_{u+H_B^{s(a)}} - T_{H_B^{s(a)}}$ for $u \geq 0$. The function T^a inherits the strict increasing and continuity properties from T. By relation (2.5),

$$\begin{aligned}
(T^a)^{-1}(t) =& \inf\{u > 0 : T_u^a > t\} \\
=& \inf\{u > 0 : T_{u+H_B^{s(a)}} - T_{H_B^{s(a)}} > t\} \\
=& \inf\{u > 0 : T_{u+H_B^{s(a)}} > t + H_X^a\} \\
=& \inf\{u > 0 : T_u > t + H_X^a\} - H_B^{s(a)} \\
=& T^{-1}(t + H_X^a) - H_B^{s(a)}.
\end{aligned} \tag{2.7}$$

Let $B^{s(a)} := \{B_{t+H_B^{s(a)}}\}_{t\geq 0}$, then $B^{s(a)}$ is a Brownian motion started at $s(a)$. From equations (2.6) and (2.7) we see that the process X^a can be written as

$$X_t^a = s^{-1}(B^{s(a)}_{(T^a)^{-1}(t)}). \tag{2.8}$$

From this representation we can read that the scale function of X^a is the same as the scale function of X, that is

$$s_{X^a}(x) = \int_0^x e^{W(y)} dy.$$

To find the speed measure of X^a, notice that the definition of T^a, equation (2.4) and the change of variable $v = u - H_B^{s(a)}$ yields

$$\begin{aligned}
T_u^a &= \int_0^{t+H_B^{s(a)}} e^{-2W(s^{-1}(B_u))} du - \int_0^{H_B^{s(a)}} e^{-2W(s^{-1}(B_u))} du \\
&= \int_{H_B^{s(a)}}^{t+H_B^{s(a)}} e^{-2W(s^{-1}(B_u))} du \\
&= \int_0^t e^{-2W\left(s^{-1}\left(B_{v+H_B^{s(a)}}\right)\right)} dv \\
&= \int_0^t e^{-2W(s^{-1}(B_v^{s(a)}))} dv.
\end{aligned}$$

As the speed measure is independent of the point in which the Brownian motion starts (see Theorem 6), then the speed measure of X^a is the same as the speed measure of X. That is,

$$m_{X^a}(A) = \int_A e^{-W(y)} dy, \quad A \in \mathcal{B}(\mathbb{R}). \tag{2.9}$$

Definition 24. Let X be a Brox diffusion on $(\Omega, \mathcal{F}, \mathbb{P})$, for each $a \in \mathbb{R}$ we define the following probability measure on $(\Omega, \mathcal{F})$

$$\mathbb{P}^a(X_t \in A) := \mathbb{P}(X_t^a \in A) \quad \forall t \geq 0, A \in \mathcal{B}(\mathbb{R}).$$

Then $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ is a family of probability measures on $(\Omega, \mathcal{F})$.

Remark 23. In [7] the Brox diffusion started at $x \in \mathbb{R}$ together with the family of probabilities $\{\mathbb{P}^x\}_{x \in \mathbb{R}}$ on the canonical space $(C[0,\infty), \mathcal{B}(C[0,\infty)), \mathcal{W})$ is also constructed.

Local time is an important element of excursion theory, using this tool Itô was able to "enumerate" the excursions of Brownian motion. As we have mentioned, we will derive results for some variables of the Brox process using known results for the Brownian motion, this is why it is useful to relate the local time of X^a with Brownian local time. Moreover, as we saw in the previous results, the speed measure of the Brox process is absolutely continuous with respect to the Lebesgue measure, so that the modified local time (as in Definition 19) exists. For this reason we will sometimes give results in terms of this process instead of the local time process, recall discussion at the end of section 1.5.

Proposition 2. *(i) The local time of the process* X^a *is*

$$L_{X^a}(t,x) = L_{B^{s(a)}}((T^a)^{-1}(t), s(x)).$$

(ii) The modified local time of the process X^a is

$$\mathcal{L}_{X^a}(t,x) = e^{-W(x)} L_{B^{s(a)}}((T^a)^{-1}(t), s(x)).$$

Proof. i) Follows from Theorem 7 and ii) follows from Definition 19, Equation 2.9 and i). ∎

Proposition 3. *The right-continuous functions γ_u^x and Γ_u^x of the Brox diffusion satisfy the following relation*

$$\gamma_u^x = \Gamma^x_{e^{-W(x)}u}.$$

Proof. From Remark 18 and Proposition 2:

$$\begin{aligned}
\gamma_u^x &= \inf\{t > 0 : L_{X^a}(t,x) > u\} \\
&= \inf\{t > 0 : e^{-W(x)} L_{X^a}(t,x) > e^{-W(x)}u\} \\
&= \inf\{t > 0 : \mathcal{L}_{X^a}(t,x) > e^{-W(x)}u\} \\
&= \Gamma^x_{e^{-W(x)}u}.
\end{aligned}$$

∎

2.2 The point process of excursions

We continue by linking the excursions of the Brownian motion B and the excursions of the Brox process started at a. Here B is the Brownian motion that helps in the construction (2.8) of the process X^a i.e. we have that

$$X_t^a(\omega) = s^{-1}(B^{s(a)}_{T^{a^{-1}}(t)}(\omega)).$$

Let us begin by relating the times in which excursions of $B^{s(a)}$ and X^a start and end.

Lemma 3. *h_1 and h_2 are, respectively, the times in which an excursion of the Brownian motion $B^{s(a)}$ at b starts and ends if and only if $T^a(h_1)$ and $T^a(h_2)$ are, respectively, the times in which an excursion of the process X^a at $s^{-1}(b)$ starts and ends.*

Proof. By representation (2.8) we have

$$\begin{aligned} X^a_{T^a(t)} &= s^{-1}(B^{s(a)}_{(T^a)^{-1}(T^a(t))}) \\ &= s^{-1}(B^{s(a)}_t). \end{aligned}$$

From this equality we see that $B^{s(a)}_{h_1} = b$ if and only if $X^a_{T^a(h_1)} = s^{-1}(b)$. Analogously, $B^{s(a)}_{h_2} = b$ if and only if $X^a_{T^a(h_2)} = s^{-1}(b)$. Furthermore, $B^{s(a)}_t \neq b$ for $t \in (h_1, h_2)$ if and only if $X^a_u \neq s^{-1}(b)$ for $u \in (T^a(h_1), T^a(h_2))$.

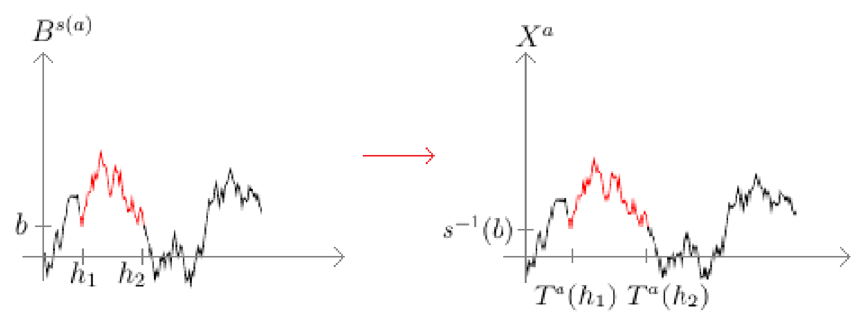

Figure 2.2: Path of BM (left) and path of X^a (right)

∎

With the previous lemma we see that there is a bijection betweeen excursions of $B^{s(a)}$ and X^a. The following lemma tells us how such a bijection works.

Lemma 4. *Let (l, ξ) be a point in the point process of excursions at b of the process X^a, then*

$$(l, s \circ \xi)$$

is the point of the point process of excursions at $s(b)$ of the corresponding Brownian motion $B^{s(a)}$.

Proof. Suppose that $T^a(h_1)$ is the time in which the excursion ξ at b starts. Then, by Lemma 3 the corresponding excursion of the Brownian motion $B^{s(a)}$ at $s(b)$ starts at h_1, therefore the local time when such excursion starts is given by $L_{B^{s(a)}}(h_1, s(b))$. By Proposition 2, using $t = T^a(h_1)$ and $x = b$, we have

$$\begin{aligned} L_{B^{s(a)}}(h_1, s(b)) &= L_{X^a}(T^a(h_1), b) \\ &= l. \end{aligned}$$

Which proves the transformation of the first entry of (l, ξ). For the second one note that if ξ is the excursion of X^a at b that starts on $T^a(h_1)$ then $s \circ \xi$ is the excursion of $B^{s(a)}$ at $s(b)$ that starts on h_1. ■

Corollary 1. *Let $(l_1, l_2] \times S \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(U)$, then*

$$N^b_{X^a}((l_1, l_2] \times S) \quad = N^{s(b)}_B((l_1 + L_B(H^{s(a)}_B, s(b)), l_2 + L_B(H^{s(a)}_B, s(b))] \times (s \circ S)).$$

Proof. It follows from the previous lemma and from the fact that

$$L_{B^{s(a)}}(t, x) = L_B(t + H^{s(a)}_B, x) - L_B(H^{s(a)}_B, x).$$

■

Definition 25. For a fixed $a \in \mathbb{R}$, we define the operator

$$\zeta_a : U \to U,$$

such that for $S \in \mathcal{B}(U)$, $\zeta_a \circ S = \{f - a : f \in S\}$.

Lemma 5. *Let $T \times S \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(U)$, then $N^a_B(T \times S) \stackrel{d}{=} N^0_B(T \times \zeta_a \circ S)$.*

Proof. By properties of the Brownian motion we have that the process

$$B' := \{B_{t+H^a_B}\}_{t \geq 0} \stackrel{d}{=} \{B^a_t\}_{t \geq 0}, \tag{2.10}$$

where $B^a := \{B_t^a\}_{t\geq 0}$ is a Brownian motion started at a. Since the process B does not accumulates local time at a before H_B^a, it follows that

$$\begin{aligned} N_B^a(T \times S) &= N_{B'}^a(T \times S) \\ &\stackrel{d}{=} N_{B^a}^a(T \times S) \\ &\stackrel{d}{=} N_B^0(T \times \zeta_a \circ S). \end{aligned}$$

The second equality is a consequence of equation (2.10) and the medibility of the function N, while the last equality is given by the space homogeneity of the Brownian motion. ■

Now, we can describe the Poisson structure of the point process of excursions of X^a.

Theorem 9. *Let $(0, l] \times S \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(U)$. Then*

$$N_{X^a}^b((0, l] \times S) \sim Poisson(l\eta(\zeta_{s(b)} \circ s \circ S)).$$

Where η is the characteristic measure of the process N_B^0 and s the scale function of X^a.

Proof. By Corollary 1 and Lemma 5,

$$\begin{aligned} N_{X^a}^b((0, l] \times S) &= N_B^{s(b)}((L_B(H_B^{s(a)}, s(b)), l + L_B(H_B^{s(a)}, s(b))] \times (s \circ S)) \\ &\stackrel{d}{=} N_B^0((L_B(H_B^{s(a)}, s(b)), l + L_B(H_B^{s(a)}, s(b))] \times \zeta_{s(b)} \circ s \circ S). \end{aligned}$$

The result follows from Theorem 8 applied to the Brownian motion. ■

From this theorem, we can read the characteristic measure of the process $N_{X^a}^b$ in terms of the characteristic measure of the process N_B^0, both of which are known to exists thanks to Theorem 8. Formally, we have the next corollary:

Corollary 2. *The characteristic measure of the random measure $N_{X^a}^b$ is*

$$\widetilde{\eta} := \eta \circ \zeta_{s(b)} \circ s,$$

where η is the characteristic measure of N_B^0, $\zeta_{s(b)}$ is the operator of Definition 25 and s is the scale function of diffusion X^a.

Proof. If we prove that $\widetilde{\eta}$ is σ-finite, then the result is straightforward from Theorems 8 and 9. As η is σ-finite, we know that there exist sets $\{A_n\}_{n=1}^{\infty} \in \mathcal{B}(U)$ such that $\cup_{n=1}^{\infty} A_n = U$ and $\eta(A_n) < \infty$ for every $n \in N$. Define for $n \in \mathbb{N}$ the sets

$$B_n := s^{-1} \circ \zeta_{s(b)}^{-1} \circ A_n.$$

Then

$$\begin{aligned}\cup_{n=1}^{\infty} B_n &= \cup_{n=1}^{\infty} s^{-1} \circ \zeta_{s(b)}^{-1} \circ A_n \\ &= s^{-1} \circ \zeta_{s(b)}^{-1} \circ \cup_{n=1}^{\infty} A_n \\ &= s^{-1} \circ \zeta_{s(b)}^{-1} \circ U \\ &= U.\end{aligned}$$

Finally,

$$\begin{aligned}\widetilde{\eta}(B_n) &= \eta \circ \zeta_{s(b)} \circ s \circ s^{-1} \circ \zeta_{s(b)}^{-1} \circ A_n \\ &= \eta(A_n) \\ &< \infty.\end{aligned}$$

■

Remark 24. Note that $\widetilde{\eta}$ does not depend on the point a in which the diffusion starts.

2.3 Some random variables

In this section, we will provide the explicit distribution of random variables regarding the Brox diffusion, this results will be helpful in the next chapter, when an algorithm for recovering the environment behind the path of X is given.

Proposition 4. *If $H^c_{X^a} < H^b_{X^a}$ the random variable $\mathcal{L}_{X^a}(H^b_{X^a}, c)$ has exponential distribution with parameter $\frac{e^{W(c)}}{2|s(c)-s(b)|}$. If $H^c_{X^a} \geq H^b_{X^a}$ then $\mathcal{L}_{X^a}(H^b_{X^a}, c) = 0$.*

Proof. When $H^c_{X^a} \geq H^b_{X^a}$ the process X^a reaches b before c so that it does not accumulates local time at c, therefore $\mathcal{L}_{X^a}(H^b_{X^a}, c) = 0$. For the case $H^c_{X^a} < H^b_{X^a}$ we analyze it in separate cases.

Case 1. $b > c$
$L_{X^a}(H^b_{X^a}, c) = \inf\{l > 0 : N^c_{X^a}((0, l] \times U^+(b)) > 0\}$ (recall that $U^{\pm}(x) := \{f \in U : \sup_{t \geq 0} \pm f(t) > x\}$). By Theorem 9 we have

$$\begin{aligned}\mathbb{P}(L_{X^a}(H^b_{X^a}, c) > y) &= \mathbb{P}(N^c_{X^a}((0, y] \times U^+(b)) = 0) \\ &= e^{-y\eta \circ \zeta_{s(c)} \circ s(U^+(b))}.\end{aligned}$$

As $\zeta_{s(c)} \circ s(U^+(b)) = U^+(s(b) - s(c))$, by Proposition 1

$$\begin{aligned}\mathbb{P}(L_{X^a}(H^b_{X^a}, c) > y) &= e^{-y\eta(U^+(s(b)-s(c)))} \\ &= e^{-\frac{y}{2(s(b)-s(c))}}. \end{aligned} \tag{2.11}$$

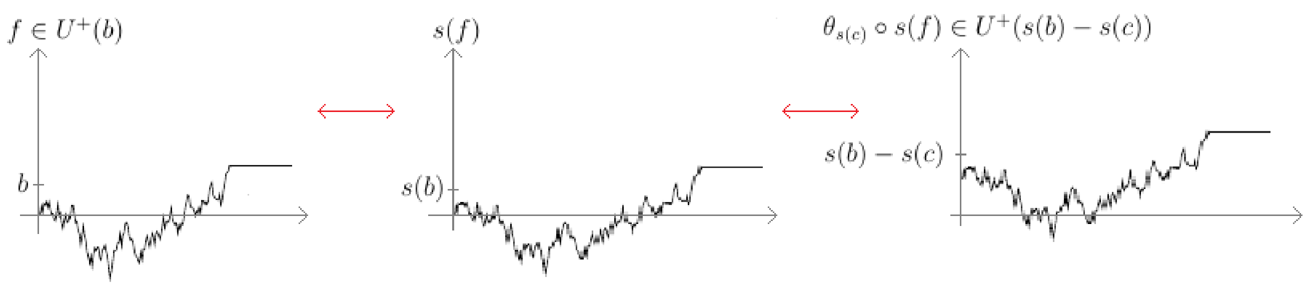

Figure 2.3: Equality $\zeta_{s(c)} \circ s(U^+(b)) = U^+(s(b) - s(c))$

Case 2. $b < c$
$L_{X^a}(H^b_{X^a}, c) = \inf\{l > 0 : N^c_{X^a}((0, l] \times U^-(-b)) > 0\}$. By Theorem 9, Proposition 1 and

by the symmetry of the Brownian motion

$$\begin{aligned}\mathbb{P}(L_{X^a}(H^b_{X^a}, c) > y) &= \mathbb{P}(N^c_{X^a}((0, y] \times U^-(-b)) = 0)\\ &= e^{-y\eta\circ\zeta_{s(c)}\circ s(U^-(-b))}\\ &= e^{-y\eta(U^-(s(c)-s(b)))}\\ &= e^{-y\eta(U^+(s(c)-s(b)))}\\ &= e^{-y\frac{1}{2(s(c)-s(b))}}. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) we have that $L_{X^a}(H^b_{X^a}, c)$ has exponential distribution with parameter $\frac{1}{2|s(c)-s(b)|}$. Applying Proposition 2, the result follows. ■

Corollary 3. *Let E be an exponential random variable with parameter $\frac{e^{W(c)}}{2|s(c)-s(b)|}$ and B a Bernoulli random variable independent of E such that*

$$\mathbb{P}(B = 1) = 1 - \mathbb{P}(B = 0) = \frac{s(a) - s(b)}{s(c) - s(b)}.$$

Then, for $a, b, c \in \mathbb{R}$ and a between b and c we have

$$\mathcal{L}_{X^a}(H^b_{X^a}, c) \stackrel{d}{=} E \cdot B.$$

Proof. From Definition 24 we have that

$$\mathbb{P}(H^c_{X^a} < H^b_{X^a}) = \mathbb{P}^a(H^c_X < H^b_X),$$

and the result follows from Proposition 4 and Definition 16. ■

Before our main result of this chapter, we will need the following definition.

Definition 26. A real-valued process $N = \{N_t : 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *subordinator* if it has stationary, independent increments, and if almost every path of N is nondecreasing, right-continuous, and satisfies $N_0 = 0$.

Proposition 5. *Define the process $Y_t = \mathcal{L}_{X^a}(\Gamma^b_t, c)$, $t \geq 0$, $a, b, c \in \mathbb{R}$. Then*

(i) $\{Y_t\}_{t\geq 0}$ is a subordinator, in fact a compound Poisson process, and

(ii) for $\zeta > \frac{-e^{W(c)}}{2|s(c)-s(b)|}$ we have

$$\mathbb{E}^a[e^{-\zeta Y_t}] = \left(\frac{e^{W(c)}}{e^{W(c)} + 2\zeta\,|s(c) - s(b)|} \cdot \frac{s(a) - s(b)}{s(c) - s(b)} + \frac{s(c) - s(a)}{s(c) - s(b)}\right) \times \exp\left(\frac{-\zeta t e^{W(b)}}{e^{W(c)} + 2\zeta\,|s(c) - s(b)|}\right).$$

Proof. After the process X^a reaches b for the first time, Y_t increases on excursions at b that reach point c during the interval $[0,\Gamma_t^b]$. From Proposition 3 we have that $\Gamma_t^b = \gamma^b_{e^{W(b)}t}$, therefore the number of such excursions is given by the random variable

$$N^b_{X^b}([0,e^{W(b)}t]\times U^+(c)) \quad \text{when } c>b,$$

and by

$$N^b_{X^b}([0,e^{W(b)}t]\times U^-(-c)) \quad \text{when } c<b.$$

From Theorem 9, both have Poisson distribution and by symmetry and Proposition 1, they have parameter $\frac{e^{W(b)}t}{2|s(c)-s(b)|}$.
Furthermore, the modified local time at c of each of these excursions are the i.i.d. random variables $\mathcal{L}_{X^c}(H^b,c)$ which by Proposition 4 have exponential distribution with parameter $\frac{e^{W(c)}}{2|s(c)-s(b)|}$. Thus, considering the time X^a spent on c before reaching b for the first time, we can write

$$Y_t = \mathcal{L}_{X^a}(H^b,c) + \sum_{i=1}^{R_t} e_i,$$

where $R_t \sim Poisson(\frac{e^{W(b)}t}{2|s(c)-s(b)|})$, $e_i \sim exp(\frac{e^{W(c)}}{2|s(c)-s(b)|})$, and

$$\mathcal{L}_{X^a}(H^b,c) \sim \begin{cases} exp(\frac{e^{W(c)}}{2|s(c)-s(b)|}) & \text{if } H^c_{X^a} < H^b_{X^a}, \\ 0 & \text{if } H^c_{X^a} \geq H^b_{X^a}. \end{cases}$$

Moreover, since $\mathcal{L}_{X^a}(H^b,c)$ only depends on the path of X^a before time $H^b_{X^a}$, by the strong Markov property it is independent of e_i, $i=1,2,...$ and R_t. Hence, for $A := \{H^c_{X^a} < H^b_{X^a}\}$, $L := \mathcal{L}_{X^a}(H^b,c)$ and $\zeta > \frac{-e^{W(c)}}{2|s(c)-s(b)|}$, we have

$$\begin{aligned}
\mathbb{E}[e^{-\zeta Y_t}] &= \mathbb{E}[e^{-\zeta(L+\sum_{i=1}^{R_t} e_i)}] \\
&= \mathbb{E}[e^{-\zeta L}(\mathbb{1}_A + \mathbb{1}_{A^c})]\mathbb{E}[e^{-\zeta\sum_{i=1}^{R_t} e_i}] \\
&= \left(\mathbb{E}[e^{-\zeta L}\mathbb{1}_A] + \mathbb{E}[e^{-\zeta L}\mathbb{1}_{A^c}]\right)\mathbb{E}[e^{-\zeta\sum_{i=1}^{R_t} e_i}] \\
&= \left(\mathbb{E}[e^{-\zeta L}|A]\mathbb{P}(A) + \mathbb{P}(A^c)\right)\mathbb{E}[e^{-\zeta\sum_{i=1}^{R_t} e_i}] \\
&= \left(\frac{e^{W(c)}}{e^{W(c)}+2\zeta\,|s(c)-s(b)|}\cdot\frac{s(a)-s(b)}{s(c)-s(b)} + \frac{s(c)-s(a)}{s(c)-s(b)}\right)\exp\left(\frac{-\zeta t e^{W(b)}}{e^{W(c)}+2\zeta\,|s(c)-s(b)|}\right),
\end{aligned}$$

where we used that L is an exponential random variable with parameter $\frac{e^{W(c)}}{2|s(c)-s(b)|}$ and the fact that $\mathbb{P}(A) = \mathbb{P}^a(H^c_X < H^b_X)$ (see Definition 24). One can check that the process

$\{Y_t\}_{t\geq 0}$ is non-decreasing, right-continuous and satisfies $Y_0 = 0$ a.s. In addition, from Lemma 1 we have that the Brox diffusion is recurrent, which implies that

$$\lim_{t\to\infty} Y_t = \infty \quad \text{a.s.}$$

Let us finally see that the increments of Y_t are independent and stationary. For $t_3 > t_2 > t_1 > 0$, we have that $\mathcal{L}_{X^a}(\Gamma^b_{t_3}, c) - \mathcal{L}_{X^a}(\Gamma^b_{t_2}, c)$ and $\mathcal{L}_{X^a}(\Gamma^b_{t_2}, c) - \mathcal{L}_{X^a}(\Gamma^b_{t_1}, c)$ depend on the paths of $\{X^a_s, s \in [\Gamma^b_{t_2}, \Gamma^b_{t_3}]\}$ and $\{X^a_s, s \in [\Gamma^b_{t_1}, \Gamma^b_{t_2}]\}$, respectively, so by the strong Markov property Y has independent increments.
To see that Y_t has stationary increments note that $Y_{t_2} - Y_{t_1}$ increases at excursions at b in $[\Gamma^b_{t_1}, \Gamma^b_{t_2}]$ that reach c, that is

$$Y_{t_2} - Y_{t_1} = \sum_{i=1}^{R} e_i,$$

where $R \sim Poisson(\frac{e^{W(b)}(t_2-t_1)}{2|s(c)-s(b)|})$ and $e_i \sim exp(\frac{e^{W(c)}}{2|s(c)-s(b)|})$, therefore

$$\mathbb{E}[e^{-\zeta(Y_{t_2}-Y_{t_1})}] = \exp\left(\frac{-\zeta(t_2-t_1)e^{W(b)}}{e^{W(c)} + 2\zeta\,|s(c)-s(b)|}\right).$$

That is, the distribution of $Y_{t_2} - Y_{t_1}$ depends only on the difference $t_2 - t_1$. From this we can also conclude that $\{R_t, t \geq 0\}$ is a Poisson process. ■

Corollary 4. *For $b = a$, the Lévy measure of the process $\{Y_t\}_{t\geq 0}$ of Proposition 5 is given by*

$$\sigma(dy) = \frac{e^{W(a)+W(c)}}{4\,|s(c)-s(a)|^2} \exp\left(-\frac{e^{W(c)}y}{2\,|s(c)-s(a)|}\right) dy.$$

Proof. From [29] we have that the moment generating function of a compound Poisson process $\{X_t\}_{t\geq 0}$ is given by

$$\mathbb{E}[e^{-\zeta X_t}] = \exp\left[-tc\int_0^\infty (1-e^{-\zeta x})\sigma(dx)\right], \quad \text{for } \zeta \geq 0,$$

where $c > 0$ and σ is a distribution on $(0,\infty)$ called the Lévy measure of $\{X_t\}_{t\geq 0}$. Then, for $c = 1$, $\zeta \geq 0$ and $\sigma(dx) = \frac{e^{W(a)+W(c)}}{4|s(c)-s(a)|^2}\exp\left(-\frac{e^{W(c)}x}{2|s(c)-s(a)|}\right)dx$, we obtain

$$\begin{aligned}\exp\left[-tc\int_0^\infty (1-e^{-\zeta x})\sigma(dx)\right] &= \exp\left(\frac{-\zeta t e^{W(a)}}{e^{W(c)} + 2\zeta\,|s(c)-s(a)|}\right)\\ &= \mathbb{E}^a[e^{-\zeta Y_t}],\end{aligned}$$

where the last equality is given by Proposition 5 (ii) with $b = a$. Thus, obtaining the result. ■

Corollary 5. *For each* $z \geq 0$, $\{\mathcal{L}_X(\Gamma_t, z) - e^{-W(z)}t : t \geq 0\}$ *is a martingale*

Proof. By Proposition 5 (ii) with $a = b = 0$ and $c = z$ we have that

$$\mathbb{E}[e^{-\zeta \mathcal{L}_X(\Gamma_t, z)}] = e^{\frac{-\zeta t}{e^{W(z)} + 2\zeta |s(z)|}},$$

which implies

$$\mathbb{E}[\mathcal{L}_X(\Gamma_t, z)] = e^{-W(z)}t. \tag{2.13}$$

Then for each z, $\mathcal{L}_X(\Gamma_t, z) - e^{-W(z)}t$ is a zero mean process with stationary independent increments and hence a martingale. ∎

Remark 25. Equation (2.13) gives the expected value of the random variable $\mathcal{L}_X(\Gamma_t, z)$. From this equality we can see in a very intuitive and transparent way that, in average, the Brox diffusion spends more time around a point z where the environment W has a local minimum.

Chapter 3

Estimation of the environment from an excursion

As we know, the Brox diffusion is a random process in a random environment. It becomes a diffusion when such an environment is fixed i.e. for a trajectory $W(\omega)$ where W is a two-sided Brownian motion. The aim of this chapter is to find the path of $W(\omega)$ from a single path of the process X, this type of results are of great interest in many areas such as biology or physics. In [1] an estimation of the environment is given for Sinai's walk, which can be thought of as the discrete analogous of the Brox diffusion (see [25], [30]), in his paper, P. Andreoletti uses properties of the local time to approximate the difference of the random potential in a significant interval.

Suppose that the values of a Brox diffusion path at certain times of length Δ are known. Let X_Δ be the polygonal line going through the points $\{(k\Delta, X(k\Delta)) : k \in \mathbb{Z}^+\}$ then, we call X_Δ the approximation by discretization of process X. In this chapter we will use the information contained in X_Δ in order to find a discrete function W_Δ such that

$$\lim_{\Delta \to 0} W_\Delta(t) = W(t),$$

in some sense, for each $t \in \{k\Delta : k \in \mathbb{Z}^+\}$. Where W is the fixed environment behind the path of X.

In the first section of this chapter we assume that the path of X_Δ is known for every $t \in \{k\Delta : k \in \mathbb{Z}^+\}$ and we use an ergodic result from [16] in order to find such function W_Δ up to a constant factor. We then give a corollary that allows us to approximate the value of the constant that was left unknown.

In the second section we work with the assumption that the values of X_Δ are known for a finite number of points. We then apply the probabilistic results obtained in Chapter 2, more precisely Proposition 5, to learn the distribution of random variables concerning the local time of X, these random variables have parameters that depend on the values of the environment. Then we use a statistical approach, understanding that the values of X_Δ are in fact random samples of these variables, we give a confidence interval for the parameters, i.e. for the value of the environment at a fixed point.

3.1 Algorithm using an Ergodic theorem

3.1.1 Introduction

The first (and most important) component of this first approach for approximating the environment from a Brox diffusion path is a direct consequence of the next theorem. This ergodic theorem yields the long-term behavior of the ratio of two local time integrals, each of them with respect to non-negative measures π_1 and π_2. As we will see, when these measures take a particular form, a result is obtained which will be fundamental for our purposes.

Theorem 10 ([16] p.228)**.** *Let X be a recurrent diffusion with state space I and local time L_X, define the local time integral*

$$\Pi_i(t) = \int_I L_X(t,x)\pi_i(dx), \quad i = 1, 2,$$

for non-negative measures π_1 and π_2. Then

$$\mathbb{P}\left[\lim_{t\uparrow\infty} \frac{\Pi_1(t)}{\Pi_2(t)} = \frac{\pi_1(I)}{\pi_2(I)}\right] = 1 \quad \textit{in case } 0 < \pi_2(I) < \infty. \tag{3.1}$$

Corollary 6. *Let X be as in Theorem 10, then a special case of (3.1) is*

$$\lim_{t\uparrow\infty} \frac{\Gamma_X(t,A)}{\Gamma_X(t,B)} = \frac{m(A)}{m(B)}, \tag{3.2}$$

where Γ_X is the occupation time of diffusion X, $0 < m(B) < \infty$ and $A, B \in \mathcal{B}(I)$.

Proof. Fix $A, B \in \mathcal{B}(I)$ with $0 < m(B) < \infty$. Recalling equation 1.10 of section 1.5, namely that

$$\Gamma_X(t,A) = \int_A L_X(t,x)m(dx), \quad 0 \le t < \infty,$$

the proof follows by taking $\pi_1(C) := \int_C \mathbb{1}_A(x)m(dx)$ and $\pi_2(C) := \int_C \mathbb{1}_B(x)m(dx)$ for $C \in \mathcal{B}(I)$. ■

As we mentioned in the introduction, we will use this result as a tool to approximate the environment of a Brox diffusion path. Corollary 6 shows us how to compute the speed measure from the path of a diffusion up to a constant factor [16], this is very important in our case since for the Brox diffusion, the speed measure is a function of the environment W. Before we can give any such approximation we have to deal with the problem of finding an approximation of the occupation time Γ_X of the diffusion by some function of the discretization X_Δ of X.

To solve this problem we follow the idea in [24]. In his paper H. Ngo presents an approximation of the occupation time for Itô diffusions that satisfy certain conditions on the coefficients σ and b, assuming a discrete sample data $\{X_{k\Delta}, 0 \leq k \leq t/\Delta\}$. As the Brox diffusion does not satisfy the assumptions given in [24], these results can not be applied directly, instead we prove that the same approximation for the occupation time of the Brox diffusion holds for closed intervals of the form $[a, b]$. This assumption, although restrictive, will be sufficient for our purposes.

Proposition 6. *For each closed interval $[a, b]$ $a, b \in \mathbb{R}$ and $a < b$, let us define the following estimator of the occupation time of the Brox diffusion*

$$\Gamma_{X_\Delta}(t, [a, b]) = \Delta \sum_{k=0}^{\lfloor t/\Delta \rfloor} \mathbb{1}_{[a,b]}(X_{k\Delta}),$$

where $\lfloor x \rfloor$ denotes the integer part of x. Then,

$$\Gamma_{X_\Delta}(t, [a, b]) \xrightarrow{a.s.} \Gamma_X(t, [a, b])$$

as $\Delta \to 0$ for any $t > 0$.

Proof. From the definition of the integral for simple functions and applying Fatou's lemma we have that

$$\begin{aligned}
\limsup_{\Delta \to 0} \Delta \sum_{k=0}^{\lfloor t/\Delta \rfloor} \mathbb{1}_{[a,b]}(X_{k\Delta}) &= \limsup_{\Delta \to 0} \int_0^t \mathbb{1}_{[a,b]}(X_{\lfloor s/\Delta \rfloor \Delta}) ds \\
&\leq \int_0^t \limsup_{\Delta \to 0} \mathbb{1}_{[a,b]}(X_{\lfloor s/\Delta \rfloor \Delta}) ds \\
&\leq \int_0^t \mathbb{1}_{[a,b]}(X_s) ds \\
&= \Gamma_X(t, [a, b]) \quad a.s.
\end{aligned} \tag{3.3}$$

Where the last inequality is a consequence of the continuity of X and from the fact that $[a, b]$ is closed (i.e. contains all its limit points). Analogously, we can prove that

$$\liminf_{\Delta \to 0} \Delta \sum_{k=0}^{\lfloor t/\Delta \rfloor} \mathbb{1}_{[a,b]}(X_{k\Delta}) \geq \Gamma_X(t, [a, b]) \quad a.s. \tag{3.4}$$

Finally, equations 3.3 and 3.4 yield the result. ■

Proposition 7. *Let X be the Brox diffusion with speed measure m and occupation measure Γ_X. Fix $a \leq b$ and $c < d$, $a, b, c, d \in \mathbb{R}$ then,*

$$\lim_{t \uparrow \infty} \lim_{\Delta \to 0} \frac{\Gamma_{X_\Delta}(t, [a,b])}{\Gamma_{X_\Delta}(t, [c,d])} = \frac{m([a,b])}{m([c,d])} \quad a.s.$$

Proof. As X is recurrent, for t large enough we will have $\Gamma_{X_\Delta}(t, [c,d]) > 0$, therefore

$$\begin{aligned} \lim_{\Delta \to 0} \frac{\Gamma_{X_\Delta}(t, [a,b])}{\Gamma_{X_\Delta}(t, [c,d])} &= \frac{\lim_{\Delta \to 0} \Gamma_{X_\Delta}(t, [a,b])}{\lim_{\Delta \to 0} \Gamma_{X_\Delta}([c,d])} \\ &= \frac{\Gamma_X(t, [a,b])}{\Gamma_X(t, [c,d])} \quad a.s. \end{aligned} \tag{3.5}$$

Note that from the form of m (equation 2.2), we have that $m([c,d])$ is positive for all $c < d$, applying $\lim_{t \uparrow \infty}$ on both sides of 3.5 and from Corollary 6, the result follows. ■

3.1.2 Algorithm

Now that we have a convergence of functions of the path X_Δ to the ratio of speed measures of the Brox diffusion, we can use this knowledge to give an approximation (up to a constant factor) for the environment.

Theorem 11. *For a given path X_Δ of the Brox diffusion X and fixed $c, d, y \in \mathbb{R}$ with $c < d$, the following approximation of the value of the environment W at y holds.*

For $y \geq 0$,

$$\lim_{\varepsilon \to 0} \lim_{t \uparrow \infty} \lim_{\Delta \to 0} \frac{\Gamma_{X_\Delta}(t, [0, y+\varepsilon]) - \Gamma_{X_\Delta}(t, [0,y])}{\varepsilon \cdot \Gamma_{X_\Delta}(t, [c,d])} = \frac{e^{-W(y)}}{m([c,d])} \quad a.s.$$

And for $y < 0$,

$$\lim_{\varepsilon \to 0} \lim_{t \uparrow \infty} \lim_{\Delta \to 0} \frac{\Gamma_{X_\Delta}(t, [y, 0]) - \Gamma_{X_\Delta}(t, [y+\varepsilon, 0])}{\varepsilon \cdot \Gamma_{X_\Delta}(t, [c,d])} = \frac{e^{-W(y)}}{m([c,d])} \quad a.s.$$

Proof. Fix $y \geq 0$, from the previous proposition we have that

$$\begin{aligned} \lim_{t \uparrow \infty} \lim_{\Delta \to 0} \frac{\Gamma_{X_\Delta}(t, [0, y+\varepsilon]) - \Gamma_{X_\Delta}(t, [0,y])}{\Gamma_{X_\Delta}(t, [c,d])} &= \frac{m([0, y+\varepsilon]) - m([0,y])}{m([c,d])} \\ &= \frac{m((y, y+\varepsilon])}{m([c,d])} \quad a.s. \end{aligned}$$

Taking the limit when $\varepsilon \to 0$ and diving by ε we get that

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{t\uparrow\infty}\lim_{\Delta\to 0}\frac{\Gamma_{X_\Delta}(t,[0,y+\varepsilon])-\Gamma_{X_\Delta}(t,[0,y])}{\varepsilon\cdot\Gamma_{X_\Delta}(t,[c,d])} &= \lim_{\varepsilon\to 0}\frac{m((y,y+\varepsilon])/\varepsilon}{m([c,d])}\\ &= \frac{\dot{m}(y)}{m([c,d])}\\ &= \frac{e^{-W(y)}}{m([c,d])} \quad a.s.\end{aligned}$$

And the result follows for $y \geq 0$. For $y < 0$, the result can be obtained analogously. ■

Towards the idea of approximating the unknown constant $m([c,d])$ of Theorem 11, we have the next corollary.

Corollary 7. *Under the assumptions of Theorem 11, we have that*

$$\lim_{\varepsilon\to 0}\lim_{t\uparrow\infty}\lim_{\Delta\to 0}\frac{\varepsilon\cdot\Gamma_{X_\Delta}(t,[c,d])}{\Gamma_{X_\Delta}(t,[0,\varepsilon])} = m([c,d]) \quad a.s.$$

Proof. The result is a direct consequence of Theorem 11 with $y = 0$ and the fact that $W(0) = 0$ a.s. ■

3.1.3 R code

Code in R for the approximation of the environment with the algorithm of section 3.1.

```
#Set variables
epsilon1 <- round(sqrt(epsilon),3) #epsilon is the step size of X
#[kmin,kmax] are the values to be approximated
kmin <- abs(round_any(min(X)/2,epsilon1, floor)/epsilon1)
kmax <- round_any(max(X)/2,epsilon1, floor)/epsilon1

#Number of points in set B=[0,k*epsilon1]
MedB <- c(1:(kmax+1))
for(k in 1:(kmax+1))
{
  j2 <- 0
  for(i in 1:(Tnew/epsilon+1))
  {
    if (X[i]<=k*epsilon1 & 0<=X[i])
    {
      j2 <- j2+1
    }
  }
```

```
  MedB[k] <- j2
}
#Number of points in set D=[-k*epsilon,0]
MedD <- c(1:kmin)
for(k in 1:(kmin))
{
  h2 <- 0
  for(i in 1:(Tnew/epsilon+1))
  {
    if (X[i]<=0 & -k*epsilon1<=X[i])
    {
      h2 <- h2+1
    }
  }
  MedD[k] <- h2
}
#Approximation of the environment non-negative part
WEst1 <- c(1:(length(MedB)))
WEst1[1] <- 0
for(i in 2:length(MedB))
{
  if(MedB[i]-MedB[i-1]>0)
  {
    WEst1[i] <- -log((MedB[i]-MedB[i-1])/(epsilon1*MedB[length(MedB)]))
  }
  if(MedB[i]-MedB[i-1]==0)
  {
    WEst1[i] <- WEst1[i-1]
  }
}
#Approximation of the environment negative part
WEst2 <- c(1:(length(MedD)))
WEst2[1] <- -log(MedD[1]/(epsilon1*MedB[length(MedB)]))
for(i in 2:length(MedD))
{
  if((MedD[i]-MedD[i-1])>0)
  {
    WEst2[i] <- -log((MedD[i]-MedD[i-1])/(epsilon1*MedB[length(MedB)]))
  }
  if((MedD[i]-MedD[i-1])==0)
  {
    WEst2[i] <- WEst2[i-1]
  }
```

```
}
#Join positive and negative parts
WEst2 <- rev(WEst2)
WEst <- c(1:(length(WEst2)+length(WEst1)))
WEst <- c(WEst2,WEst1)
ejex <- c(1:(length(WEst)))
ejex <- seq((-kmin)*epsilon1,kmax*epsilon1,by=epsilon1)
TrueW <- c(1:length(WEst)) #True values of environment (if known)
for(i in 1:length(WEst))
{
  TrueW[i] <- W[T/delta+1+ejex[i]/delta] #delta is the step size of W
}

#Plot of the path of W and the approximation
plot(ejex,TrueW,type="l",main="Approximation of W(t) (Ergodic theorem)",xlab
    ="time",xlim=c(ejex[1],ejex[length(WEst)]),ylim=c(min(TrueW)-1.5,max(
    TrueW)+ 3.5),col="black",ylab="W(t)")
grid(nx=15,ny=15)
lines(ejex,WEst,type="l",col="red")
legend("topright", legend=c("Environment","Approximation"),col=c("black", "
    red"), lty=1:1, cex=0.65,text.font=2,bg=8)
```

Listing 3.1: Approximation of the environment using an Ergodic theorem

3.1.4 Simulations

The first step to recover an environment is to have a path of an approximation by discretization of the Brox diffusion X_Δ. This path was obtained with the code of Appendix A and the following parameters:

(i) $t = 15$

(ii) $\Delta = .001$

In Figure 3.1 we see the path of X_Δ (black) and the environment (red) behind the path. In this example, the environment has a local minimum at 0.5, so that the path of X_Δ spends most of its time around this minimum.

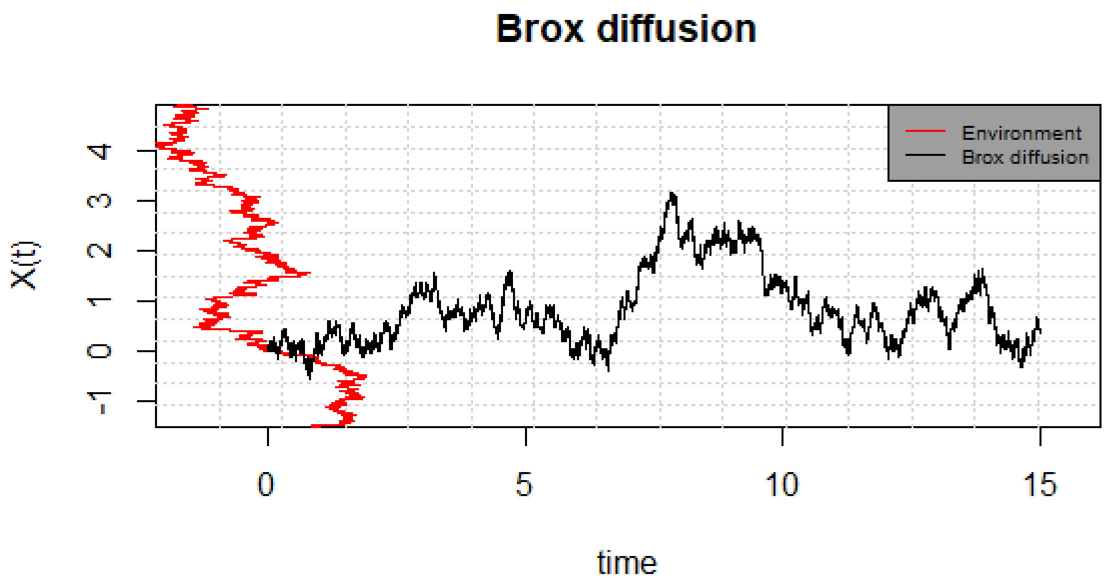

Figure 3.1: Path of the approximation X_Δ and environment W

Assuming that the value of the environment is unknown, from the path of X_Δ we use the code of section 3.1.3 in order to obtain an approximation of the value of $W(y)$ with the following parameters:

(i) $\varepsilon = .001$

(ii) $t = 15$

(iii) $\Delta = \sqrt{\varepsilon}$

(iv) $y \in [-0.28, 1.58]$

(v) $c = 0$

(vi) $d = 1.58$

The interval of the values to be approximated, i.e. $[-0.28, 1.58]$, was chosen as the values where the path of X_Δ spends most of its time. In Figure 3.2 the approximation of the value of the environment up to a constant factor (red) is shown. In black we have the true value of the environment.

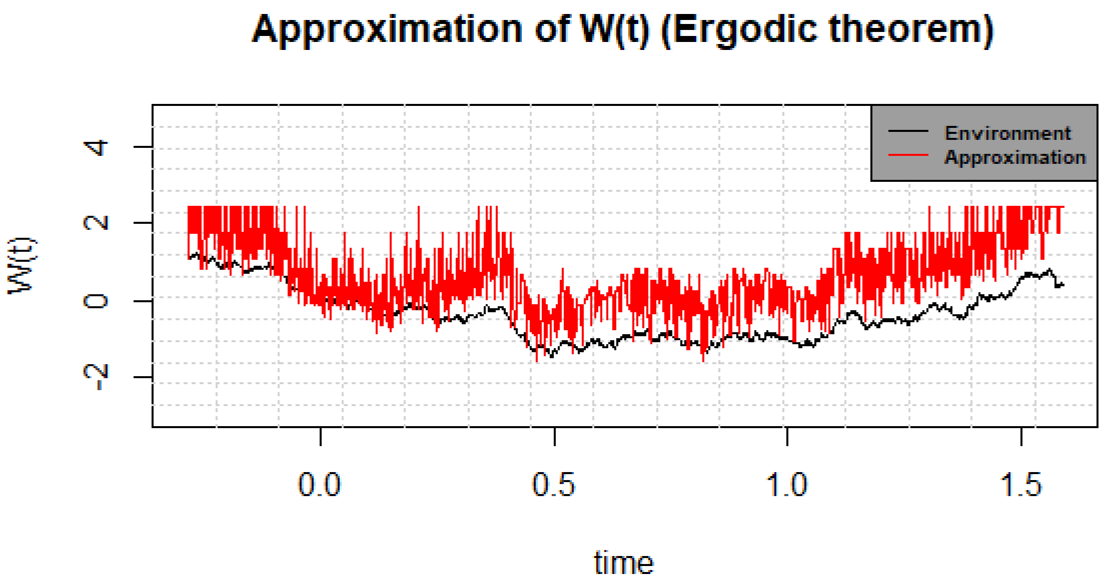

Figure 3.2: Approximation of the environment up to a constant factor

Following Corollary 7, we then computed the value of

$$\frac{\varepsilon \cdot \Gamma_{X_\Delta}(t, [c, d])}{\Gamma_{X_\Delta}(t, [0, \varepsilon])} \approx 1.426$$

with ε, t, c and d as above. Thus, an approximation of the environment with an estimation for the constant factor may now be given. In Figure 3.3 this approximation (red) can be observed.

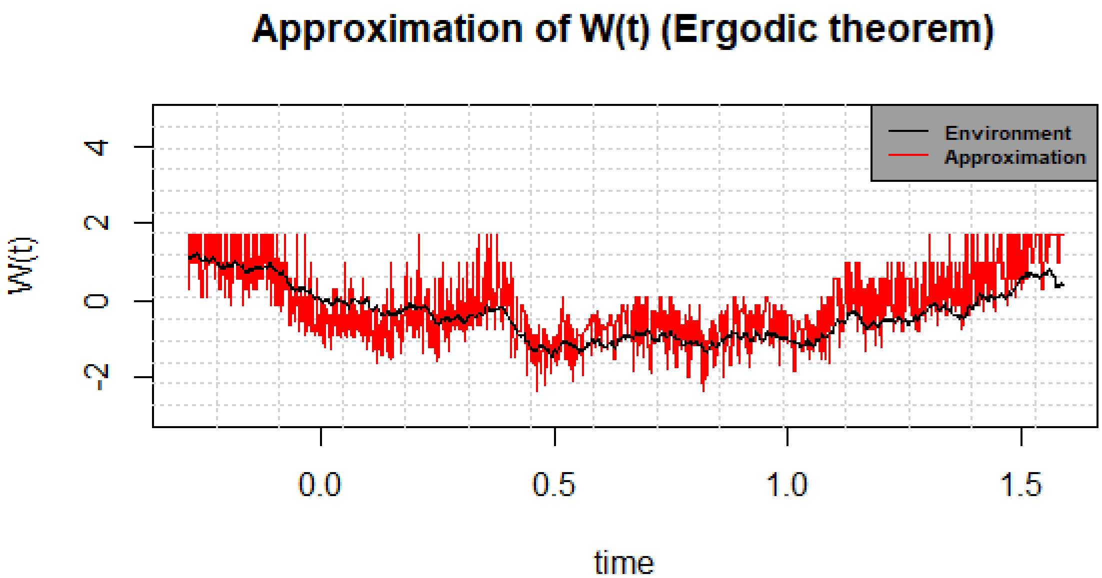

Figure 3.3: Approximation of the environment using an Ergodic theorem

3.2 Algorithm using Excursion theory

3.2.1 Introduction

The results of this section are achieved through the information obtained in section 2.3. We start by stating these results for the local time $L_X(t,x)$ instead of for the modified local time. Recall that from Definition 19, that for the Brox diffusion we have that

$$\mathcal{L}_X(t,a) = e^{-W(a)} L_X(t,a).$$

The results are given without proof since they are a direct consequence of this relationship and the results of section 2.3.

Proposition 8. *If $H^c_{X^a} < H^b_{X^a}$ the random variable $L_{X^a}(H^b,c)$ has exponential distribution with parameter $\frac{1}{2|s(c)-s(b)|}$. If $H^c_{X^a} \geq H^b_{X^a}$ then $L_{X^a}(H^b,c) = 0$.*

Corollary 8. *Let $a,c \in \mathbb{R}$, for each excursion of the Brox diffusion at a that reaches point c, the local time at c are i.i.d. exponential random variables with parameter $\frac{1}{2|s(c)-s(a)|}$.*

This corollary will be very important for developing the algorithm for recovering the environment. As the local time at c of each excursion of X form an i.i.d. collection of random variables, we can view the path of X as samples of these variables. This will allow us to estimate the parameter $\frac{1}{2|s(c)-s(a)|}$, which as we know, is a function of the environment. In order to execute this idea, we will need to approximate the local time through the information provided by X_Δ. This task will be the subject of our next subsection.

3.2.2 Approximation of local time via the number of crossings

In [3], an approximation of the local time of the Brownian motion via the number of crossings through a certain level are found. We will use this result, together with the Itô and McKean representation of the Brox diffusion (recall Equation 1.7), in order to obtain an analogous result for the Brox diffusion.

Definition 27. Let X_Δ be an approximation by discretization of the Brox diffusion X, then the number of crossing of X_Δ at level u in the interval I is defined as

$$N^u_{X_\Delta}(I) = \#\{t \in I : X_\Delta(t) = u\}.$$

Lemma 6. *Let X be the Brox diffusion, X_Δ an approximation by discretization of process X and $u,t \in \mathbb{R}$, then* *(i) X is in u at time t (i.e. $X_t = u$) if and only if the corresponding Brownian motion B is in $s(u)$ at time $T^{-1}(t)$.*

(ii) We have that $N^u_{X_\Delta}(I) = N^{s(u)}_{B_{T^{-1}(\Delta)}}(T^{-1}(I))$, where $B_{T^{-1}(\Delta)}$ is the approximation by discretization of the Brownian motion B that goes through the points

$$\{(T^{-1}(k\Delta), B(T^{-1}(k\Delta))) : k \in \mathbb{Z}^+\}.$$

Proof. a) The proof is straightforward from the fact that

$$X_t = s^{-1}(B(T^{-1}(t))).$$

b) Follows from a). ■

Lemma 7. *([3] p.192) Let B be a standard Brownian motion, then the following convergence holds*

$$\sqrt{\pi/2}\sqrt{\Delta}N^u_{B_\Delta}(I) \xrightarrow[\Delta\to 0]{prob} L_B(I,u),$$

where $L_B(I,u)$ is the local time of B at level u during the interval I.

That is,

$$\forall\varepsilon > 0, \mathbb{P}\left(\left|\sqrt{\pi/2}\sqrt{\Delta}N^u_{B_\Delta}(I) - L_B(I,u)\right| \geq \varepsilon\right) \longrightarrow 0,$$

when $\Delta \to 0$.

Remark 26. Lemma 7 is in fact true for L^k convergence. In this thesis, we will only need to make use of the convergence in probability.

Notice that in Lemma 7, the approximation B_Δ is defined on a partition of the interval I of constant length Δ, while $B_{T^{-1}(\Delta)}$ is defined on a partition of I of variable length

$$T^{-1}(k\Delta) - T^{-1}((k-1)\Delta) \quad \text{for } k = 1, 2, ...$$

However, as $\Delta \to 0$, these two partitions are more likely to count the same number of crossings, this is the reason behind our next conjecture.

Conjecture 1. Let $B_{T^{-1}(\Delta)}$ be as in Lemma 6, then

$$\sqrt{\pi/2}\sqrt{\Delta}N^u_{B_{T^{-1}(\Delta)}}(I) \xrightarrow[\Delta\to 0]{prob} L_B(I,u)$$

The rigorous proof of this convergence is left as an open problem for future work. If we assume Conjecture 1 to be truth, we would have the following result that states the convergence of the count process N_{X_Δ} to the local time of the Brox diffusion.

Conjectural Result 1. Let X_Δ be an approximation by discretization of the Brox diffusion X, then

$$\sqrt{\pi/2}\sqrt{\Delta}N^u_{X_\Delta}(I) \xrightarrow[\Delta\to 0]{prob} L_X(I,u).$$

Proof. From Lemma 6 and Conjecture 1 we have that

$$\begin{aligned}\sqrt{\pi/2}\sqrt{\Delta}N^u_{X_\Delta}(I) &= \sqrt{\pi/2}\sqrt{\Delta}N^{s(u)}_{B_{T^{-1}(\Delta)}}(T^{-1}(I)) \\ &\xrightarrow[\Delta\to 0]{prob} L_B(T^{-1}(I), s(u)) \\ &= L_X(I,u).\end{aligned}$$

■

Lemma 8. *Let $\{N_{i,\Delta}\}_{i=1,..,n}$ be random variables such that $N_{i,\Delta} \xrightarrow[\Delta\to 0]{prob} L_i$ for $i = 1, ..., n$. Then,*

$$\sum_{i=1}^n N_{i,\Delta} \xrightarrow[\Delta\to 0]{prob} \sum_{i=1}^n L_i.$$

Proof. Let $\varepsilon > 0$ be fixed. Then,

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i=1}^n N_{i,\Delta} - \sum_{i=1}^n L_i\right| \geq \varepsilon\right) &\leq \mathbb{P}\left(|N_{1,\Delta} - L_1| \geq \varepsilon/n \cup |N_{2,\Delta} - L_2| \geq \varepsilon/n \cup ... \cup |N_{n,\Delta} - L_n| \geq \varepsilon/n\right) \\ &\leq \sum_{i=1}^n \mathbb{P}\left(|N_{i,\Delta} - L_i| \geq \varepsilon/n\right) \\ &\to 0.\end{aligned}$$

■

3.2.3 Algorithm

Lemma 9. *Fix $c \in \mathbb{R}$ and consider the sequence $\{X_m := \frac{m}{2}L_{X^{c+1/m}}(H^c, c+1/m)\}_{m\geq 1}$ of random variables. Then*

$$X_m \xrightarrow[m\to\infty]{dist} X,$$

where X is an exponential random variable with parameter $e^{-W(c)}$.

Proof. From Proposition 8 we have that $L_{X^{c+1/m}}(H^c, c+1/m) \sim exp\left(\frac{1}{2|s(c+1/m)-s(c)|}\right)$. So that for each $m = 1, 2, ...$ the random variable $X_m \sim exp\left(\frac{1/m}{|s(c+1/m)-s(c)|}\right)$. Then

$$\begin{aligned}\lim_{m\to\infty} \mathbb{P}(X_m \le x) &= \lim_{m\to\infty} (1 - e^{-\frac{1/m}{|s(c+1/m)-s(c)|}x}) \\ &= 1 - e^{-x \lim_{m\to\infty} \frac{1/m}{|s(c+1/m)-s(c)|}} \\ &= 1 - e^{-\frac{x}{s'(c)}} \\ &= 1 - e^{-xe^{-W(c)}} \\ &= \mathbb{P}(X \le x).\end{aligned}$$

From the Portmanteau lemma, the result follows. ■

The next lemma is a well-known result of probability that gives a relation between an i.i.d. collection of exponential random variables and the chi-square distribution. In general, confidence intervals are obtained by finding *pivots*, which are functions of the data and of the parameter of interest and whose distribution do not depend on the parameter (see [10]). This lemma provides a pivot for a random sample of exponential distributions, which is what we will need in order to find the confidence intervals of the environment W.

Lemma 10. *Let $X_1, ..., X_n$ be independent exponential random variables with parameter λ, then $2\lambda \sum_{i=1}^n X_i$ follows a chi-square distribution with $2n$ degrees of freedom.*

Conjectural Result 2. Fix $c \in \mathbb{R}$ and $0 < \alpha < 1$. Suppose we have a path X_Δ, which is an approximation by discretization of process X of size Δ. Let n be the number of excursions at c that reach point $c + \varepsilon$ of X_Δ and let $N_{i,\Delta}(c, c+\varepsilon)$, $i = 1, ..., n$ be the number of crossings of level $c + \varepsilon$ for the process X_Δ at each of these excursions, then

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{\Delta\to 0} \mathbb{P}\left(\ln\left(\frac{\sqrt{\pi\Delta}\sum_{i=1}^n N_{i,\Delta}(c,c+\varepsilon)}{\sqrt{2}\varepsilon\chi^2_{2n}(1-\alpha/2)}\right) \le W(c) \le \ln\left(\frac{\sqrt{\pi\Delta}\sum_{i=1}^n N_{i,\Delta}(c,c+\varepsilon)}{\sqrt{2}\varepsilon\chi^2_{2n}(\alpha/2)}\right)\right) \\ = 1 - \alpha,\end{aligned}$$

where $\chi^2_{2n}(\alpha/2)$ and $\chi^2_{2n}(1-\alpha/2)$ are the $(\alpha/2)\times 100$-th and $(1-\alpha/2)\times 100$-th percentiles of a chi-square distribution with $2n$ degrees of freedom, respectively.

Proof. We will prove the approximation for $c > 0$, the case $c < 0$ can be carried out analogously. Let $T_\Delta := \{k\Delta : k \in \mathbb{Z}^+\}$, for $c > 0$, $\varepsilon > 0$ and $0 < \alpha < 1$ fixed, let the times $0 \le t_1 < t_2 < ... < t_n < T$, $t_i \in T_\Delta$ be the (approximated) starting times of an excursion at c that reaches point $c + \varepsilon$ of the process X_Δ. In other words, $t_1, ..., t_n$ satisfy the following two conditions

(i) $sign(X_\Delta(t_i - \Delta) - c) \neq sign(X_\Delta(t_i) - c)$ and

(ii) $X_\Delta(t_i) \geq c + \varepsilon$, for some $t \in [t_i, t_{i+1}) \cap T_\Delta$.

Now, for each $i = 1, ...n$ and each time $l \in [t_i, t_{i+1}) \cap T_\Delta$, where $t_{n+1} := T$, let us count the number of crossings of level $c+\varepsilon$ of the process X_Δ and call them $N_{i,\Delta}(c, c+\varepsilon)$, $i = 1, .., n$. Furthermore, if $\{L_i(c, c+\varepsilon)\}_{i=1,..,n}$ are n independent copies of the random variable $L_{X^{c+\varepsilon}}(H^c, c+\varepsilon)$. Then, from the Conjectural Result 1 we have that for each $i = 1, ..., n$,

$$\sqrt{\pi/2}\sqrt{\Delta}N_{i,\Delta}(c, c+\varepsilon) \xrightarrow[\Delta\to 0]{prob} L_i(c, c+\varepsilon).$$

Multiplying by $1/2\varepsilon$ and from Lemma 8, we get that

$$\frac{1}{2\varepsilon}\sqrt{\pi/2}\sqrt{\Delta}\sum_{i=1}^{n} N_{i,\Delta}(c, c+1/m) \xrightarrow[\Delta\to 0]{prob} \frac{1}{2\varepsilon}\sum_{i=1}^{n} L_i(c, c+\varepsilon).$$

From Lemma 9 we also have that

$$\frac{1}{2\varepsilon}L_i(c, c+\varepsilon) \xrightarrow[\varepsilon\to 0]{dist} L_{i,c} \quad i = 1, ..., n,$$

where $\{L_{i,c}\}_{i=1,...,n}$ are independent exponential random variables with parameter $e^{-W(c)}$, then, from the continuous mapping theorem and since convergence in probability implies convergence in distribution, we get

$$\frac{1}{2\varepsilon}2e^{-W(c)}\sum_{i=1}^{n} L_i(c, c+\varepsilon) \xrightarrow[\varepsilon\to 0]{dist} 2e^{-W(c)}\sum_{i=1}^{n} L_{i,c}.$$

Finally, from Lemma 10,

$$2e^{-W(c)}\sum_{i=1}^{n} L_{i,c} \sim \chi^2_{2n}.$$

That is, the following identities hold

$$\begin{aligned}
1 - \alpha &= \mathbb{P}\left(\chi^2_{2n}(\alpha/2) \leq 2e^{-W(c)}\sum_{i=1}^{n} L_{i,c} \leq \chi^2_{2n}(1-\alpha/2)\right) \\
&= \lim_{\varepsilon\to 0}\mathbb{P}\left(\chi^2_{2n}(\alpha/2) \leq \frac{1}{2\varepsilon}2e^{-W(c)}\sum_{i=1}^{n} L_i(c, c+\varepsilon) \leq \chi^2_{2n}(1-\alpha/2)\right) \\
&= \lim_{\varepsilon\to 0}\lim_{\Delta\to 0}\mathbb{P}\left(\chi^2_{2n}(\alpha/2) \leq \frac{1}{2\varepsilon}\sqrt{\pi/2}\sqrt{\Delta}2e^{-W(c)}\sum_{i=1}^{n} N_{i,\Delta}(c, c+\varepsilon) \leq \chi^2_{2n}(1-\alpha/2)\right) \\
&= \lim_{\varepsilon\to 0}\lim_{\Delta\to 0}\mathbb{P}\left(\frac{\sqrt{\pi\Delta}\sum_{i=1}^{n} N_{i,\Delta}(c, c+\varepsilon)}{\sqrt{2}\varepsilon\chi^2_{2n}(1-\alpha/2)} \leq e^{W(c)} \leq \frac{\sqrt{\pi\Delta}\sum_{i=1}^{n} N_{i,\Delta}(c, c+\varepsilon)}{\sqrt{2}\varepsilon\chi^2_{2n}(\alpha/2)}\right),
\end{aligned}$$

applying logarithm, the result follows. ■

This result gives us a confidence interval for the value of the environment at some fixed c when $\varepsilon, \Delta \to 0$. As the upper and lower confidence interval from this theorem can be calculated from the information provided by X_Δ, a construction of the approximation of the environment follows. The proof of this result is not yet complete, as it uses the Conjectural Result 1 as part of its proof.

3.2.4 R code

R-code for the approximation of the environment with the algorithm using Excursion theory

```r
#Set variables
epsilon1 <- round_any(sqrt(epsilon),epsilon,floor) #step size of points to be approximated, recall that epsilon is the step size of X
Cmin <- round_any(min(X)/2,epsilon1,floor) #minimum value to approximate
Cmax <- round_any(max(X)/2,epsilon1,floor) #maximum value to approximate
vectordec <- seq(Cmin,Cmax,by=epsilon1) #interval to be approximated
Waproxsup <- rep(0,length(vectordec)) #upper approximation of W
Waproxinf <- rep(0,length(vectordec)) #lower approximation of W
TrueW <- rep(0,length(vectordec)) #true value of W (if known)
alfa <- 0.05 #confidence level

for(j in 1:length(vectordec))
{
  Xinic <- vectordec[j] #Excursions of X at Xinic
  #Distinguish if points of X are above or below Xinic
  signo <- rep(0,length(X))
  if(sign(Xinic) != 0) #case Xinic not zero
  {
    for(i in 1:length(X))
    {
      if(X[i]<Xinic)
      {
        signo[i] <- -1
      }
      if(X[i]>Xinic)
      {
        signo[i] <- 1
      }
    }
    #t0 is the first time X reaches level Xinic
    t0 <- 1
```

```
    if(sign(Xinic)>0)
    {
      while(signo[t0]<0)
      {
        t0 <- t0+1
      }
    }
    if(sign(Xinic)<0)
    {
      while(signo[t0]>0)
      {
        t0 <- t0+1
      }
    }
    #Number of excursions of X from Xinic
    NumExc <- 0
    for(i in t0:(length(X)-1))
    {
      if (signo[i] != signo[i+1])
      {
        NumExc <- NumExc+1
      }
    }
    if(NumExc>1)
    {
      tfinal <- rep(1,NumExc)
      #First excursion
      k <- t0
      while(signo[k] == signo[k+1])
      {
        k <- k+1
      }
      tfinal[1]<- k
      #From second excursion to last excursion
      for(i in 2:NumExc)
      {
        h <- tfinal[i-1]+1
        while(signo[h] == signo[h+1])
        {
          h<-h+1
        }
        tfinal[i]<-h
      }
```

```
    #Count number of excursions that reach level vectordec[j+1]
    if(j<length(vectordec)) #case Xinic < Cmax
    {
      c <- vectordec[j+1]
    }
    if(j==length(vectordec)) #case Xinic = Cmax
    {
      c <- vectordec[j]+epsilon1
    }
    NumCruces <- rep(0,NumExc)
    if(sign(c)>=0)
    {
      #First excursion
      estado <- rep(0,tfinal[1]-t0+1)
      for(i in 1:(tfinal[1]-t0+1))
      {
        if(X[i+t0-1] >= c)
        {
          estado[i] <- 1
        }
        if(X[i+t0-1] < c)
        {
          estado[i] <- 0
        }
      }
      #Count number of crossings (i.e. number of changes in vector estado)
      if(tfinal[1]-t0>1)
      {
        for(k in 1:(tfinal[1]-t0))
        {
          if(estado[k] != estado[k+1])
          {
            NumCruces[1] <- NumCruces[1]+1
          }
        }
      }
      #If the excursion ends in a value over c, there is one more crossing
      if(estado[tfinal[1]-t0+1]>0)
      {
        NumCruces[1] <- NumCruces[1]+1
      }
      #If the excursion started below c and it reached c, an extra
          crossing was counted
```

```
    if(NumCruces[1]>0)
    {
      if(estado[1]==0)
      {
        NumCruces[1] <- NumCruces[1] -1
      }
    }
    #From second excursion to last excursion
    for(h in 2:NumExc)
    {
      if(tfinal[h]-tfinal[h-1]==1)
      {
        if(X[tfinal[h]]>=c)
        {
          NumCruces[h] <- 1
        }
      }
      if(tfinal[h]-tfinal[h-1]>1)
      {
        estado <- rep(0,tfinal[h]-tfinal[h-1])
        for(i in (tfinal[h-1]+1):tfinal[h])
        {
          if(X[i] >= c)
          {
            estado[i-tfinal[h-1]] <- 1
          }
          if(X[i] < c)
          {
            estado[i-tfinal[h-1]] <- 0
          }
        }
        if(estado[tfinal[h]-tfinal[h-1]]>0)
        {
          NumCruces[h] <- NumCruces[h]+1
        }
        for(k in 1:(tfinal[h]-tfinal[h-1]-1))
        {
          if(estado[k] != estado[k+1])
          {
            NumCruces[h] <- NumCruces[h]+1
          }
        }
        if(NumCruces[h]>0)
```

```
            {
              if(estado[1]==0)
              {
                NumCruces[h] <- NumCruces[h]-1
              }
            }
          }
        }
      }
      if(sign(c)<0)
      {
        #First excursion
        estado <- rep(0,tfinal[1]-t0+1)
        for(i in 1:(tfinal[1]-t0+1))
        {
          if(X[i+t0-1] <= c)
          {
            estado[i] <- 1
          }
          if(X[i+t0-1] > c)
          {
            estado[i] <- 0
          }
        }
        #Count the number of crossings
        if(tfinal[1]-t0>1)
        {
          for(k in 1:(tfinal[1]-t0))
          {
            if(estado[k] != estado[k+1])
            {
              NumCruces[1] <- NumCruces[1]+1
            }
          }
        }
        #If the excursion end below c, it crosses one more time
        if(estado[tfinal[1]-t0+1]>0)
        {
          NumCruces[1] <- NumCruces[1]+1
        }
        #If the excursion started above c and it reached c, an extra
            crossing was counted
        if(NumCruces[1]>0)
```

```
        {
          if(estado[1]==0)
          {
            NumCruces[1] <- NumCruces[1] -1
          }
        }
        #Second excursion to last excursion
        for(h in 2:NumExc)
        {
          if(tfinal[h]-tfinal[h-1]==1)
          {
            if(X[tfinal[h]]<=c)
            {
              NumCruces[h] <- 1
            }
          }
          if(tfinal[h]-tfinal[h-1]>1)
          {
            estado <- rep(0,tfinal[h]-tfinal[h-1])
            for(i in (tfinal[h-1]+1):tfinal[h])
            {
              if(X[i] <= c)
              {
                estado[i-tfinal[h-1]] <- 1
              }
              if(X[i] > c)
              {
                estado[i-tfinal[h-1]] <- 0
              }
            }
            if(estado[tfinal[h]-tfinal[h-1]]>0)
            {
              NumCruces[h] <- NumCruces[h]+1
            }
            for(k in 1:(tfinal[h]-tfinal[h-1]-1))
            {
              if(estado[k] != estado[k+1])
              {
                NumCruces[h] <- NumCruces[h]+1
              }
            }
            if(NumCruces[h]>0)
            {
```

```
                if(estado[1]==0)
                {
                  NumCruces[h] <- NumCruces[h]-1
                }
              }
            }
          }
        }
        #Number of excursiones that reached c
        m <- 0
        for(i in 1:NumExc)
        {
          if(NumCruces[i]>0)
          {
            m <- m+1
          }
        }
        #Transform number of crossings into local time
        NumCruces <- NumCruces*sqrt(pi/2)*sqrt(epsilon1)
        #Approximation of the values of W
        if(m>0)
        {
          Waproxinf[j] <- log((1/epsilon1)*sum(NumCruces[1:NumExc])/(qchisq(1-
              alfa/2, df=2*m)))
          Waproxsup[j] <- log((1/epsilon1)*sum(NumCruces[1:NumExc])/(qchisq(
              alfa/2, df=2*m)))
        }
        if(m==0)
        {
          Waproxinf[j] <- 0
          Waproxsup[j] <- 0
        }
      }
      if(sign(Xinic) == 0) #Environment W starts at zero
      {
        Waproxinf[j] <- 0
        Waproxsup[j] <- 0
      }
    }
    if(NumExc <= 1)
    {
      if(j == 1) #Xinic=Cmin
      {
```

```
        Waproxinf[j] <- 0
        Waproxsup[j] <- 0
      }
      if(j != 1)
      {
        Waproxinf[j] <- Waproxinf[j-1]
        Waproxsup[j] <- Waproxsup[j-1]
      }
    }
  TrueW[j] <- W[T/delta+1+Xinic/delta]
}
#Plot of the approximation of the environment and its true value
plot(vectordec,TrueW,type="l",ylim=c(min(TrueW),max(Waproxsup)+.5),xlab="
    time",ylab="W(t)",main="Approximation of W(t) (Excursion theory)")
grid(nx=15,ny=15)
lines(vectordec,TrueW)
lines(vectordec,Waproxinf,type="l",col="blue")
lines(vectordec,Waproxsup,type="l",col="red")
legend("topright", legend=c("Environment","Upper band 95%","Lower band 95%")
    ,col=c("black","red","blue"), lty=1:1, cex=0.65,text.font=2,bg=8)
```

Listing 3.2: Approximation of the environment using Excursion theory

3.2.5 Simulations

First, the path of an approximation by discretization of the Brox diffusion X_Δ was obtained with the code of Appendix A and the following parameters:

(*i*) $t = 15$

(*ii*) $\Delta = .001$

In Figure 3.4 we see the path of X_Δ (black) and the environment (red) behind the path. In this example, the environment has a local minimum at -0.1, so that the path of X_Δ spends most of its time around this minimum.

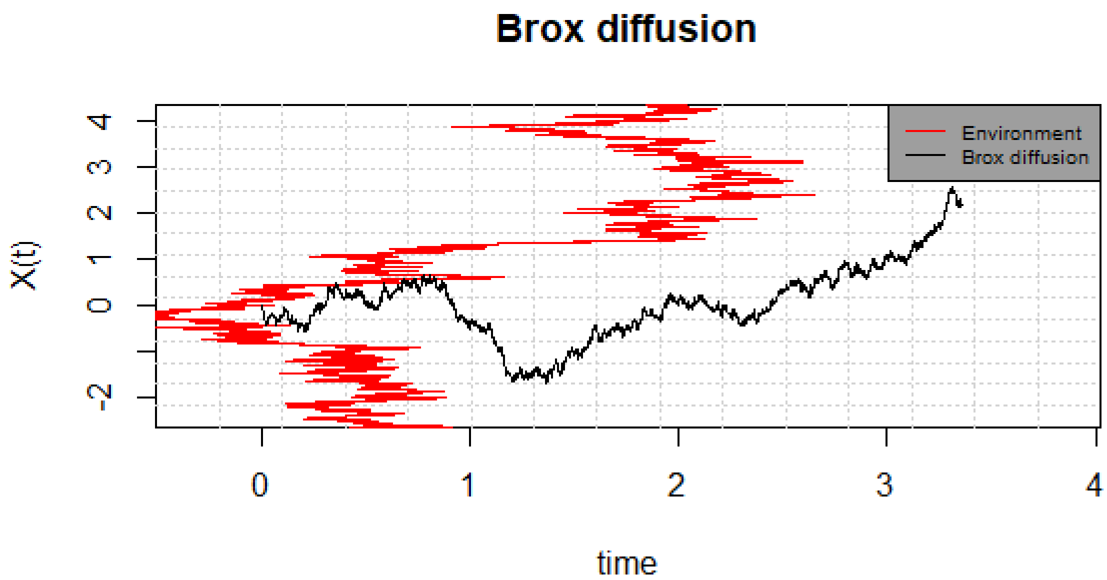

Figure 3.4: Path of the approximation X_Δ and environment W

Assuming that the value of the environment is unknown, from the path of X_Δ we use the code of section 3.2.4 in order to obtain an approximation of the value of $W(y)$ with the following parameters:

(i) $\Delta = .001$

(ii) $\varepsilon = \sqrt{\Delta}$

(iii) $\alpha = .05$

(iv) $c \in [-0.86, 1.28]$

The interval of the values to be approximated, i.e. $[-0.86, 1.28]$, was chosen as the values where the path of X_Δ spends most of its time. In Figure 3.5 the upper (red) and lower (blue) bands with confidence level α is shown. In black we have the true value of the environment.

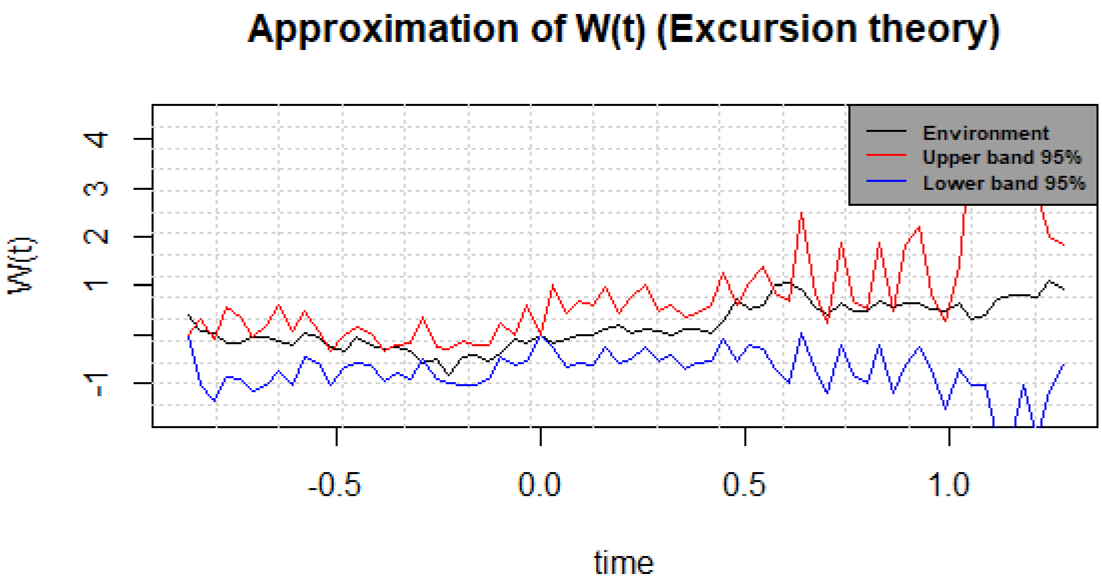

Figure 3.5: Upper and lower bands for the value of W

Appendix A

R-code Brox diffusion

R-code to obtain the path of a Brox diffusion X from a path of the Brownian motion B and a path of the two-sided Brownian motion W (environment) through Itô and McKean's representation of diffusions

$$X_t = s^{-1}(B(T^{-1}(t))).$$

```
#Load libraries
library(sde)
library(plyr)

##Set variables
T <- 15 #time frame
delta <- .001 #step size for W
epsilon <- .001 #step size for X
W1  <- c(1:(T/delta+1)) #positive part of W
W2  <- c(1:(T/delta+1))
W3  <- c(1:(T/delta)) #negative part of W
W   <- c(1:((2*T/delta)+1)) #environment
ExpW  <- c(1:((2*T/delta)+1)) #exponential of environment
B   <- c(1:(T/delta+1)) #Brownian motion B
Spos  <- c(1:(T/delta)) #positive part of scale function
Sneg  <- c(1:(T/delta)) #negative part of scale function
S   <- c(1:(2*T/delta+1)) #scale function
SinvB <- c(1:(T/epsilon+1)) #inverse of scale function
G <- c(1:(T/epsilon+1)) #auxiliar function
A <- c(1:(T/epsilon)) #time-change function
y <- seq(-T,T,by=delta) #x-axis for W and S
h <- seq(0,T,by=epsilon) #x-axis for B and X

#Path of Brownian motion B
```

```
B <- BM(x=0, t0=0, T=T, N=T/delta)

#Path of two-sided Brownian motion W
W1 <- BM(x=0, t0=0, T=T, N=T/delta) #positive part
W2 <- BM(x=0, t0=0, T=T, N=T/delta) #negative part
W3 <- W2[-(1)] #delete W2[1]=0
W3 <- rev(W3) #reverts vector W3
W <- c(W3,W1) #joins W3 with W1 to form W

#Exponential of environment W
ExpW <- exp(W)

#Riemann-integral of vector ExpW
for(i in 1:(T/delta))
{
  Spos[i] <- sum(ExpW[(T/delta+1):(T/delta+i)])*delta
}
for(i in 1:(T/delta))
{
  Sneg[i] <- -sum(ExpW[i:(T/delta)])*delta
}
S <- c(Sneg,0,Spos) #joins Sneg and Spos and adds value s(0)=0

#Compare maximum and minimum of S and B to ensure that the inverse of S
    exists
max(S)
min(S)
max(B) #max(S) must be greater or equal to max(B)
min(B) #min(S) must be less or equal to min(B)

#Point in S where the minimum of B is reached
i <- 1
while(S[i] < min(B)) { i <- i+1 }
MinS <- i

#Compute vector S^{-1}(Bu)
for(i in 1:(T/epsilon+1))
{
  if(B[(i-1)*epsilon/delta+1]<0) #case B(u) negative
  {
    j <- MinS-1 #start from MinS-1
    while(S[j] <= B[(i-1)*epsilon/delta+1])
    {
```

```
      j <- j+1
    }
    SinvB[i] <- j
  }
  if(B[(i-1)*epsilon/delta+1]>0) #case B(u) positive
  {
    j <- T/delta+1 #S is positive from T/delta+1
    while(S[j] <= B[(i-1)*epsilon/delta+1])
    {
      j <- j+1
    }
    SinvB[i] <- j
  }
  if(B[(i-1)*epsilon/delta+1]==0) #case B(u)=0
  {
    SinvB[i] <- T/delta+1 #S^{-1}(0)=0
  }
}

#Riemann-integral of function e^{-2W(S^-1(Bu))}
#Function A(t) is the time-change function T(t) of the thesis
G <- exp(-2*W[SinvB])
for(i in 1:(T/epsilon+1)) { A[i] <- sum(G[1:i])*epsilon }

#Function A must take values greater than the time frame T at some point in
  order for the inverse of A to exist
Tnew <-  min(round_any(max(A),epsilon,floor),T)

#Set variables with new time frame Tnew
x <- seq(0,Tnew,by=epsilon)
Ainv <- c(1:(Tnew/epsilon+1))
BAinv <- c(1:(Tnew/epsilon+1))
X <- c(1:(Tnew/epsilon+1))

#Compute inverse of A
for(i in 1:(Tnew/epsilon+1))
{
  j <- 1
  while(A[j] <= epsilon*(i-1))
  {
    j <- j+1
  }
  Ainv[i] <- j
```

```
}

#Evaluate Ainv in Brownian motion B
BAinv <- B[Ainv]

#Point in S where the minimum of BAinv is reached
i <- 1
while(S[i]<min(BAinv)) { i<-i+1 }
MinBAinv <- i

#Compute function S^{-1}(B(A^{-1}(t)) which gives rise to the path of X
j <- MinBAinv-1
for(i in 1:(Tnew/epsilon+1))
{
  while(S[j] <= BAinv[i])
  {
    j <- j+1
  }
  X[i] <- delta*((j-1)-(T/delta+1))
  j <- MinBAinv-1
}

#Plot of the Brox diffusion X and environment W
plot(W,y,type="l",col="red",xlab="time",ylab="X(t)",xlim=c(min(W)+.5,Tnew
    +.5),ylim=c(min(X)-.7,max(X)+1.5),main="Brox diffusion")
grid(nx=15,ny=15)
lines(x,X,type="l",col="black")
lines(W,y,col="red")
legend("topright", legend=c("Environment", "Brox diffusion"),col=c("red","
    black"), lty=1:1, cex=0.65,text.font=1,bg=8)
```

Listing A.1: Approximation of a Brox diffusion path

Bibliography

[1] Andreoletti P. *On the estimation of the potential of Sinai's RWRE.* Brazilian Journal of Probability and Statistics, Vol. 25, No. 2, pp. 121-144, 2011.

[2] Andrews S.S., Dinh T., Arkin A.P. *Stochastic Models of Biological Processes.* Springer, New York, NY, 2009.

[3] Azaïs J. M. *Approximation des trajectoires et temps local des diffusions.* Annals de l' I.H.P., section B, Vol. 25, No. 2, pp. 175-194, 1989.

[4] Azaïs J. M. *Conditions for convergence of number of crossings to the local time.* Probability and Mathematical Statistics, Vol. 11, pp. 19-36, 1990.

[5] Billingsley P. *Convergence of Probability Measures*, Second Edition, John Wiley & Sons, Inc., 1999.

[6] Billingsley P. *Probability and Measure*, Third Edition, John Wiley & Sons, Inc., 1995.

[7] Brox T. *A one dimensional diffusion process in a Wiener medium*, The Annals of Probability, Vol. 14, No. 4, pp. 1206-1218, 1986.

[8] Cheliotis D. *Localization of favorite points of diffusion in a random environment*, Stochastic Processes and their Applications 118, 1159-1189, 2008.

[9] Chernov A. A. *Replication of a multicomponent chain by the "lightning mechanism"* Biophysics, 12, pp. 336-341, 1967.

[10] DeGroot M., Schervish M. *Probability and Statistics*, Fourth Edition, Pearson, 2012.

[11] Freedman D. *Brownian Motion and Diffusion*, Springer-Verlag New York Heidelberg Berlin, 1983.

[12] Frankze C., O'Kane T., Berner J., Williams P., Lucarini V., *Stochastic climate theory and modeling*, WIREs Clim Change, 6 pp. 63–78, 2015.

[13] Hida T. *Brownian Motion*, pp. 127-131. Springer-Verlag, New York, 1980.

[14] Hu Y., Shi Z. *The problem of the most visited site in random environment*, Probability Theory and Related Fields, vol. 116, pp. 273–302, 2000.

[15] Hu Y., Lê K., and Mytnik L. *Stochastic Differential Equation for Brox Diffusion*, Stochastic Processes and their Applications, vol. 127, 7, pp. 2281-2315, 2017.

[16] Itô K. and McKean H.P. *Diffusion Processes and their Sample Paths*, Springer, Berlin, 1965.

[17] Itô K. *Poisson point process attached to Markov processes*, Proc. Sixth Berkerley Sympos. Math. Statist. Probab. 3, pp. 225-240, 1971.

[18] Itô K. *Stochastic Integral*, Proc. Imp. Acad. 20, no. 8, pp. 519-524, 1944.

[19] Karatzas I., Shreve S. *Brownian Motion and Stochastic Calculus*, Second Edition, Springer-Verlag New York, 1991.

[20] Kawazu K. and Tanaka H. *A diffusion process in a Brownian environment with drift.* J. Math. Soc. 1995.

[21] Lévy P. *Processus stochastiques et mouvement brownien*, Paris, 1948.

[22] Madan B. *Stochastic Processes in Finance*, Annual Review of Financial Economics, vol. 2, number 1, pp. 277-314, 2010.

[23] Marcus M. and Rosen J. *Markov Processes, Gaussian Processes and Local Times*, Cambridge University Press, 2006.

[24] Ngo H. *On the discrete approximation of occupation time of diffusion processes* Electronic Journal of Statistics, Vol. 5, pp. 1374-1393, 2011.

[25] Pacheco C. *From the Sinai's walk to the Brox diffusion using bilinear forms*, arXiv:1605.02826, 2016.

[26] Pacheco C., Pérez-Rojas M., *Excursions of the Brox diffusion*, 2020.

[27] Rogers L.C.G., Williams D., *Diffusions, Markov Processes, and Martingales* Volume 1, Second Edition. Cambridge University Press 2000.

[28] Rogers L.C.G., Williams D., *Diffusions, Markov Processes, and Martingales* Volume 2, Second Edition. Cambridge University Press 2000.

[29] Sato K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, p. 20, 1999.

[30] Seignourel, P. *Discrete schemes for processes in random media*, Springer-Verlag, 2000.

[31] Sinai Y. *The limiting behavior of a one-dimensional random walk in a random medium.* Theory Prob. Appl. 27, pp. 256-268, 1982.

[32] Sinai Y. *Lorentz gas and random walks* Mathematical Problems in Theoretical Physics, Lecture Notes in Physics, 153, pp. 12-14, 1982.

[33] Rogers L.C.G. *Williams' characterization of the Brownian excursion law: proof and applications*, Séminaire de Probabilités XV: Lecture Notes in Mathematics 850, Springer, Berlin, pp. 227-250, 1981.

[34] Revuz D., Yor M., *Continuous Martingales and Brownian Motion*, Springer 3rd edition 1999.

[35] Trotter H.F., *A property of Brownian motion paths*, Illinois J. Math Vol. 2, pp. 425-433, 1958.

[36] Watanabe S. *Itô's theory of excursion point processes and its developments*, Stochastic Processes and their Applications 120, p. 653-677, 2010.