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Neumann series of Bessel functions representations for solutions of linear differential equations

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by

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Dedicatory

To my mother Rosenda

and

To my father's Soul.

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Resumen

En esta tesis se construye una representación en series de Neumann de funciones de Bessel (representación NSBF) para soluciones regulares de ecuaciones diferenciales lineales ordinarias de orden n, $n \geq 2$. Inicialmente, siguiendo a [**35**], por medio de una extensión del kernel de transmutación, se obtiene la representación NSBF para una solución regular $u(\omega, x)$ de la ecuación de Bessel perturbada de la forma

$$-u'' + \left(\frac{l(l+1)}{x^2} + q\right)u = \omega^2 u, \; x \in (0,1], \; \omega \in \mathbb{C},$$

donde $l \geq -\frac{1}{2}$ y q es una función compleja continua de variable real en $[0,1]$. Se demuestra que es una serie uniformemente convergente con respecto al parámetro espectral ω sobre un subconjunto finito del plano complejo de la variable ω. Además, se demuestra que la suma parcial de la representación NSBF aproxima uniformente a la solución u y admite una estimación independiente de ω para todo ω tal que $|\operatorname{Im}\omega| \leq C$, $C \geq 0$. Los resultados están basados en la aplicación de diferentes ideas sobre propiedades de mapeo de operadores de transmutación y de una expansión del kernel de transmutación en series de Fourier-Legendre.

Para el caso que involucra ecuaciones de orden superior, $n > 2$, se considera el siguiente problema de Cauchy

$$y^{(n)} + \sum_{k=2}^{n} p_k y^{(n-k)} = \omega^n y, \tag{0.1}$$

$$y(\omega, 0) = 1, y'(\omega, 0) = \omega, ..., y^{(n-1)}(\omega, 0) = \omega^{n-1}. \tag{0.2}$$

donde p_k, $k = 2, ..., n.$, son funciones de valor complejo y continuas con respecto a la variable x en $[0,b]\,, b > 0$. Se sabe que la solución $y(\omega, x)$ del problema (0.1), (0.2) es una función entera con respecto a ω y de tipo exponencial x, en consecuencia; esta admite la representación integral de Polya, cuyo kernel es la transformada de Borel γ de la solución y. La representación de Polya está definida naturalemente sobre un contorno C_δ contenido en el dominio de regularidad de la transformada de Borel γ. En esta tesis se considera C_δ como un cuadrado centrado en el origen y con uno de sus vértices en $t_1 = (x+\delta) + i(x+\delta)$ con $\delta > 0$, y se define una parametrización en sentido antihorario sobre los lados del cuadrado. Por tanto, se obtienen cuatro integrales definidas en el intervalo $[0,1]$, y se expande el kernel de cada integral en series de Fourier-Legendre.

Por último, se demuestra que la solución $y(\omega, x)$ para un problema de Cauchy de orden arbitrario de la forma (0.1), (0.2) se puede representar como una suma de cuatro series de Neumann de funciones de Bessel que convergen uniformemente con respecto a x en $[0,b]$ con los parámetros ω y δ fijos. Se demuestra que esta representación analítica permite resolver problemas tanto de valor inicial como de valores en la frontera, y también se presenta una explicación sobre la importancia de buscar un valor óptimo para el parámetro δ.

Abstract

In this thesis a Neumann series of Bessel functions representation (NSBF representation) is constructed, for regular solutions of ordinary linear differential equations of order n, $n \geq 2$. Initially, following [**35**], by means of an extension of the transmutation kernel, a NSBF representation for the regular solution $u(\omega, x)$ of the perturbed Bessel equation of the form

$$-u'' + \left(\frac{l(l+1)}{x^2} + q(x)\right) u = \omega^2 u, \ x \in (0,1], \ \omega \in \mathbb{C}$$

is obtained, where $l \geq -\frac{1}{2}$ and q is a continuous complex valued function on the interval $[0,1]$. The series is uniformly convergent with respect to the spectral parameter ω on a finite subset of complex plane of the variable ω. This representation guarantees a uniform approximation of eigendata and admits a $\omega -$ *independent* estimate for ω belonging to the strip $|\mathrm{Im}\, \omega| \leq C$, $C \geq 0$. The results are based on the application of different ideas on mapping properties of transmutation operators and of an expansion of the transmutation kernel into Fourier-Legendre series.

In the case of higher order equations, $n > 2$, the following Cauchy problem is considered

$$y^{(n)} + \sum_{k=2}^{n} p_k y^{(n-k)} = \omega^n y, \tag{0.3}$$

$$y(\omega, 0) = 1, y'(\omega, 0) = \omega, ..., y^{(n-1)}(\omega, 0) = \omega^{n-1}, \tag{0.4}$$

where $p_k, k = 2, ..., n.$, are complex-valued continuous functions on $[0, b]$, $0 < b < \infty$. The solution $y(\omega, x)$ of the problem (0.3), (0.4) is an entire function with respect to ω, and of exponential type equal to x, consequently; this solution admits the Polya integral representation, whose kernel is the Borel transform γ of the solution y. The Polya representation is defined naturally on a contour C_δ lying entirely on the regularity domain of the Borel transform γ. In this thesis C_δ is considered as a square centered at the origin and with one of the vertex being $t_1 = (x+\delta) + i(x+\delta)$ with $\delta > 0$, and a parametrization on counterclockwise oriented boundary of this square is applied. Thus, four integrals, defined on $[0,1]$ are obtained. We prove that the kernel, in each integral, admits an expansion in Fourier-Legendre series.

Finally, we prove that the solution $y(\omega, x)$ of the Cauchy problem of the form (0.3), (0.4) can be represented as a sum of four Neumann series of Bessel functions. For parameters ω and δ fixed the series converge uniformly with respect to $x \in [0, b]$. We show that this analytical representation allows us to solve problems, both initial value, as well as values boundary value, and present an explanation on the importance of finding an optimum value of the parameter δ.

Overview

Introduction

This dissertation is associated with the recent works on the Neumann series of Bessel functions method for regular solutions of second order differential equations, consisting in representing the regular solutions as Neumann series of Bessel functions [**34**], [**35**].

The NSBF representations in [**28**], [**34**], and [**35**] were obtained for the solutions of the Sturm-Liouville equation, and for a regular solution of the perturbed Bessel equation, respectively. The series of Bessel functions obtained in the previous works offer analytical representations of the solutions where the series converge uniformly with respect to the spectral parameter ω on any compact subset of complex plane of the variable ω. Furthermore, these representations guarantee a uniform approximation of eigendata and its partial sum admits an $\omega - \textit{independent}$ error estimate which is of particular importance for solving spectral problems.

In this thesis, a new representation for the regular solution $u(\omega, x)$ of the equation

$$A[u] = -\frac{d^2u}{dx^2} + \left(\frac{l(l+1)}{x^2} + q\right) u = \omega^2 u,\ l \geq -\frac{1}{2}, x \in (0,1] \tag{0.5}$$

is obtained, where $u(\omega, x)$ is normalized by the asymptotic relation $u(\omega, x) \sim x^{l+1}$ when $x \to 0$, and the operator A is a perturbed Bessel operator, also known as a spherical Schrödinger operator, where the potential q is a complex valued continuous function on $[0,1]$ satisfying the following condition

$$\begin{aligned} xq(x) &\in L_1(0,1),\ l > -\frac{1}{2}, \\ x^{1-\varepsilon}q(x) &\in L_1(0,1),\ \text{for some } \varepsilon > 0 \text{ if } l = -\frac{1}{2}. \end{aligned} \tag{0.6}$$

The solution $u(\omega, x)$ of the equation (0.5) is represented as a Neumann series of Bessel functions uniformly convergent with respect to ω on any compact subset of the complex plane of the variable ω and we give two estimates which guarantee that a partial sum of this representation approximates well the solution $u(\omega, x)$ for any $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\operatorname{Im} \omega| \leq C$, $C \geq 0$.

This representation contains new coefficients in the series coinciding with the coefficients obtained in [**35**] only at $x = 1$. The result was obtained using two principal tools. The first, an extension of the transmutation kernel, and the second, an interesting mapping property of the transmutation operators obtained in [**11**], which states that the transmutation operator $\mathcal{T}$ maps the powers x^{2k+l+1} to the functions $(-1)^k\, 2^{2k}k!\left(l+\frac{3}{2}\right)_k \varphi_{2k}$, $k \in \mathbb{N}\cup\{0\}$, where the functions φ_{2k} are called formal powers (see definition in [**8**]) and the operator $\mathcal{T}$ is the transmutation operator for the pair of operators A and B where $B := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}$.

The main result that we consider is developed in this thesis, is a generalization of the NSBF representation for solutions of Cauchy problems associated with ordinary linear differential equations of nth order of the form

$$y^{(n)} + p_2 y^{(n-2)} + \dots + p_n y = \omega^n y,\ x \in (0,b)\,, \tag{0.7}$$

$$y(\omega,0) = 1, y'(\omega,0) = \omega, \dots, y^{n-1}(\omega,0) = \omega^{n-1}, \tag{0.8}$$

where the coefficients p_k, $k = 2,\dots,n$, are complex-valued continuous functions on $[0,b]$, $0 < b < \infty$, and $\omega \in \mathbb{C}$. The solution $y(\omega,x)$ of the problem (0.7), (0.8) is obtained as a sum of four Neumann series of Bessel functions. The formulas for the coefficients of the series are derived by the representation in the form of a uniformly convergent Fourier-Legendre series of the Borel transform γ of the solution y along the boundary of a square centered at the origin that measures $2(x+\delta)$, $\delta > 0$ of side and containing all singularities of γ. In addition, for each fixed x and ω, an estimate for the convergence rate of the approximate solution $y_N(\omega,x)$ to the exact solution $y(\omega,x)$ is obtained. The results were obtained because the solution $y(\omega,x)$ is an entire function with respect to ω and exponential type equal to x, which was proved in [**37**].
Note that the initial conditions (0.8) are sufficient to represent arbitrary Cauchy data for (0.7). One can multiply ω by n roots of 1 resulting in a system of n linearly independent solutions of (0.8).
The NSBF analytical representations obtained in this thesis are of easy numerical implementation that allows us to approximate the solution of both initial value and spectral problems, thus several numerical applications are presented in this work.

This thesis is structured in five chapters. In the preliminary chapter, we present a historical summary of the method implemented in the previous papers and some definitions of the terminology that are used from now on. In the second chapter, a Fourier-Legendre series expansion of an extension of the transmutation kernel of the operator $\mathcal{T}$ is obtained, and we prove that the regular solution $u(\omega,x)$ of the perturbed Bessel equation $A[u] = \omega^2 u$ admits a NSBF representation. In addition, an estimate for the decay rate of the coefficients of the Neumann series of Bessel functions is also obtained in this chapter.

In Chapter 3, a NSBF representation for solutions of linear differential equations of higher order is obtained. We think that this chapter contains the most important result of this dissertation because it presents all the theoretical development of the representation for the solution of the Cauchy problem (0.7), (0.8); we also present as an example all the development for the Cauchy problem for the equation of fourth order allowing us to approach the solution of initial value and spectral problems. An example for equations of fifth order is also presented, for this case, we use a procedure based on variation of parameters as in [**37**, p. 31] in order to obtain the coefficients of the power series representation in ω for the solution $y(\omega, x)$.

Next in Chapter 4, we present a proof of why it is necessary to find an optimal value of the parameter δ in the coefficients of the sums of Neumann series of Bessel functions $\beta_n^k(x;\delta)$ in order to obtain a good numerical approximation of the solution $y(\omega, x)$. Finally, in Chapter 5 the results of the previous chapters are used in order to obtain new representations for solutions of Cauchy problems of arbitrary order. For instance, we prove that the solution $y(\omega, x)$ of the problem (0.7), (0.8) can be represented as a sum of three Neumann series of Bessel functions or as a sum of eight Neumann series of Bessel functions. The results obtained in this dissertation are in collaboration with professors V. Kravchenko and S. Torba. I am deeply grateful for the guidance and support during this process.

State of the Art

The Neumann series of Bessel functions

The Neumann series of Bessel functions were first studied by the German mathematician C. Neumann in 1867 and are named after him. Later, L. Gegenbauer in 1877 developed this theory. Since that date, this subject has been widely studied and used in several areas of mathematics, such as functional analysis and differential equations (see, e.g., [**51**], [**17**], [**16**], [**43**], [**2**]). For a function $f(z)$ analytic inside and on a circle of radius R with centre at the origin, if C denotes de contour formed by this circle and if z is any point inside it, then $f(z)$ admits the form

$$z^{\nu} f(z) = \sum_{n=0}^{\infty} \beta_n J_{\nu+n}(z).$$

This series is named a Neumann series of Bessel functions [**50**, Chap. XVI, Sect. 16.13] where

$$\beta_n = \frac{1}{2\pi i} \int_C f(t) A_{n,\nu}(t)\, dt$$

provided only that ν is not a negative integer; $A_{n,\nu}(t)$ is the Gegenbauer polynomial defined in [**50**, Chap. IX, Sect. 9.2]. The function $J_m(z)$ is a Bessel function of the first kind of order m, defined as

$$J_m(\omega) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+m}}{n!\Gamma(m+n+1)}$$

where m is supposed to be real (see, e.g., [**1**, Chap. 9. Formula 9.1.10], [**42**, Formula 10.2.2, p. 217], and [**50**, Chap. III. Sect. 3.7]).

One of the first works where a representation in series of Bessel functions was obtained for a regular solution of the perturbed Bessel equation was in [**16**]. In [**16**], A. Fitouhi and M. Hamza (1990) proved that the series

$$u_\omega(x)=\sum_{p=0}^{\infty}x^{p+\frac{1}{2}}B_p(x)\,\frac{J_{\alpha+p}(\omega x)}{\omega^{\alpha+p}} \tag{0.9}$$

is a solution of the equation

$$u''-\left(\frac{\alpha^2-\frac{1}{2}}{x^2}-\omega^2-\chi(x)\right)u=0,\ \alpha>-\frac{1}{2},\ x\in(0,b], \tag{0.10}$$

where $\chi(x)$ is an even holomorphic function in a disc centered at the origin, and the coefficients $B_p(x)$ satisfy the following relations

$$\begin{aligned} B_0'(x) &= 0,\\ \left\{x^{p+1}B_{p+1}(x)\right\}' &= -\frac{1}{2}x^p\left\{B_p''(x)+\frac{1-2\alpha}{x}B_p'(x)+\chi(x)B_p(x)\right\}. \end{aligned}$$

The spectral parameter ω is a real or complex parameter. In [**12**], H. Chebli, A. Fitouhi and M. Hamza (1994) used the transmutation operators theory to obtain an expansion of the form (0.9) for regular solutions of the equation (0.10) and proved that if $\chi(x)$ is an even holomorphic function in a disc $D(0,2R)$ then the series (0.9) is the unique solution of this equation satisfying

$$2^\alpha\Gamma(\alpha+1)\,u_\omega(x)\sim x^{\alpha+\frac{1}{2}},\ x\sim 0^+$$

and that this series is uniformly convergent on every subinterval $\left(0,\left(1+|1-2\alpha|^{-\frac{1}{2}}\right)e^{-1}R\right)$ was also proved.

In [**16**] and [**12**], the NSBF representations obtained do not possess the uniformity with respect to the spectral parameter ω, and these also do not guarantee an uniform approximation of eigendata.

Recently, the interest in finding NSBF representations for solutions of differential equations where a uniform approximation of eigendata is guaranteed has grown because these representations offer a simple numerical method to solve spectral problems. In [**28**], V. Kravchenko, L. Navarro and S. Torba (2017) proved that the solutions of the Sturm-Liouville equation,

$$-y''+q(x)y=\omega^2y,\ \omega\in\mathbb{C},$$

assuming q being a complex-valued continuous function of an independent real variable $x\in[0,b]$, admit the following NSBF representations

$$c(\omega,x) = \cos(\omega x)+2\sum_{n=0}^{\infty}(-1)^n\beta_{2n}(x)\,j_{2n}(\omega x), \tag{0.11}$$

$$s(\omega,x) = \sin(\omega x)+2\sum_{n=0}^{\infty}(-1)^n\beta_{2n+1}(x)\,j_{2n+1}(\omega x), \tag{0.12}$$

where β_n admit the form

$$\beta_n(x) = \frac{2n+1}{2}\left(\sum_{k=0}^{n} \frac{l_{k,n}\varphi_k(x)}{x^k} - 1\right),$$

the functions $\varphi_k(x)$ are the formal powers, which were obtained by the SPPS method (see, e.g., [**8**]), $l_{k,n}$ is the corresponding coefficient of x^k of the Legendre polynomial of order n, and j_n stands for the spherical Bessel function of order n. The series (0.11) and (0.12) converge uniformly with respect to x on $[0,b]$ and converge uniformly with respect to ω on any compact subset of the complex plane of the variable ω. Moreover, for the functions

$$\begin{aligned} c_N(\omega,x) &= \cos(\omega x) + 2\sum_{n=0}^{[N/2]} (-1)^n \beta_{2n}(x)\, j_{2n}(\omega x), \\ s_N(\omega,x) &= \sin(\omega x) + 2\sum_{n=0}^{[(N+1)/2]} (-1)^n \beta_{2n+1}(x)\, j_{2n+1}(\omega x), \end{aligned}$$

where $[a]$ denotes the largest integer less or equal to a the following estimates were proved

$$|c(\omega,x) - c_N(\omega,x)| \le 2|x|\varepsilon_N(x) \text{ and } |s(\omega,x) - s_N(\omega,x)| \le 2|x|\varepsilon_N(x),$$

for $\omega \in \mathbb{R}$, $\omega \neq 0$, and

$$|c(\omega,x) - c_N(\omega,x)| \le \frac{2|x|\varepsilon_N(x)\sinh(Cx)}{C}$$

and

$$|s(\omega,x) - s_N(\omega,x)| \le \frac{2|x|\varepsilon_N(x)\sinh(Cx)}{C},$$

for $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\operatorname{Im}\omega| \le C$, $C \ge 0$ [**28**, Theorem 4.1]. $\varepsilon_N(x)$ is a sufficiently small nonnegative function such that $|K(x,t) - K_N(x,t)| \le \varepsilon_N(x)$, $K(x,t)$ is the kernel of the transmutation operator for the operators $-\frac{d^2}{dx^2}$ and $-\frac{d^2}{dx^2} + q(x)$ and $K_N(x,t)$ is its approximation by a polynomial of order N.

In the recent paper [**35**] a NSBF representation for the regular solution u_l of the perturbed Bessel equation,

$$-u'' + \left(\frac{l(l+1)}{x^2} + q(x)\right)u = \omega^2 u,\ x \in (0,1],$$

where l is a real number, $l \ge -\frac{1}{2}$, q is a complex-valued function on $[0,b]$ satisfying the following condition

$$\begin{aligned} xq(x) &\in L_1(0,1),\ l > -\frac{1}{2}, \\ x^{1-\varepsilon}q(x) &\in L_1(0,1),\ \text{for some } \varepsilon > 0 \text{ if } l = -\frac{1}{2}. \end{aligned}$$

was obtained. A NSBF representation for u_l satisfying the asymptotic relation $u_l(\omega, x) \sim x^{l+1}$ when $x \to 0$ was obtained in the following form

$$u_l(\omega, x) = d(\omega) b_l(\omega x) + \sum_{n=0}^{\infty} (-1)^n \beta_n(x) j_{2n}(\omega) \tag{0.13}$$

where β_n are defined by the formula

$$\beta_n(x) = (4n+1) \sum_{k=0}^{n} \frac{l_{2k,2nx}}{x^{2k}} \left(\varphi_k(x) - c_{k,l} x^{2k+l+1} \right),$$

with $d(\omega) := \frac{2^{l+\frac{1}{2}} \Gamma(l+\frac{3}{2})}{\omega^{l+1}}$, and $b_l(\omega x) := \sqrt{\omega x} J_{l+\frac{1}{2}}(\omega x)$. In [**35**], V. Kravchenko, S. Torba and R. Castillo (2018) proved that the series (0.13) converges uniformly with respect to x on $[0, b]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω. And for the approximate solution

$$u_{l;N}(\omega, x) = d(\omega) b_l(\omega x) + \sum_{n=0}^{N} (-1)^n \beta_n(x) j_{2n}(\omega x)$$

the following estimates

$$|u_l(\omega, x) - u_{l;N}(\omega, x)| \leq \sqrt{x} \varepsilon_N(x)$$

for any $\omega \in \mathbb{R}$, $\omega \neq 0$, and

$$|u_l(\omega, x) - u_{l;N}(\omega, x)| \leq \left(\frac{\sin(2Cx)}{2C} \right)^{\frac{1}{2}} \varepsilon_N(x)$$

for any $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\operatorname{Im} \omega| \leq C$, $C \geq 0$, where $\varepsilon_N(x)$ is a sufficiently small nonnegative function such that $\|R(x,t) - R_N(x,t)\|_{L_2[0,x]} \leq \varepsilon_N(x)$ were obtained. $R(x,t)$ is the transmutation kernel.

Transmutation operators

An important tool through which the NSBF representations in this thesis are obtained and that was used to obtain the representations (0.11) and (0.13) is the transmutation operator theory. In the theory of the differential equations the concept of the transmutation operator is also known as transformation operator. The notion of a transmutation operator relating two linear differential operators was introduced in 1938 by J. Delsarte [**13**] and the idea was extended together with L. Lions (see, e.g., [**14**]). For some classes of differential operators a transmutation operator can be realized in the form of a Volterra integral operator. A. Povzner in 1948 proved the existence of a transmutation operator T in the form of a Volterra integral operator of the second kind for the operators $-\frac{d^2}{dx^2}$ and $-\frac{d^2}{dx^2} + q(x)$ [**44**].

Let $\mathbb{T}$ be the transmutation operator for the operators $A = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x)$ and $B = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}$, where the potential q is a continuous complex-valued function on $[0, b]$ and $l \geq -\frac{1}{2}$.

In [**49**] using the aid of a transmutation (transformation) operator in the form of a Volterra integral operator the existence of a unique continuous kernel $V(x,t)$ was proved such that for all $\omega \in \mathbb{C}$, the function

$$u(\omega, x) = \mathbb{T}[b_l(\omega x)] := b_l(\omega x) + \int_0^x V(x,t)\, b_l(\omega x)\, dt$$

is a regular solution of the equation $A[u] = \omega^2 u$, $x \in (0, b]$. The kernel V is ω-independent continuous function with respect to both arguments satisfying the Goursat condition

$$V(x,x) = \frac{1}{2}\int_0^x q(t)\, dt$$

and $b_l(\omega x) := \sqrt{\omega x} J_{l+\frac{1}{2}}(\omega x)$ is a regular solution of the unperturbed Bessel equation $B[b_l] = \omega^2 b_l$, $x \in (0, b]$.

The transmutation operators theory has been associated with forward and inverse problems for linear differential equations (see, e.g., [**9**], [**40**], [**39**]). For differential operators of higher order the transmutation operators have also been studied (see, e.g., [**23**], [**37**]).

In [**8**] and [**11**] mapping properties for transmutation operators T and $\mathbb{T}$ were revealed making possible to apply the transmutation technique even when the integral kernel of the operator is unknown. These mapping properties allow us to know the result of application of the operator T to the non-negative integer powers of the independent variable x and of application of $\mathbb{T}$ to the powers x^{2k+l+1}, $k \in \mathbb{N}_0$, even not knowing the transmutation kernel.

Nowadays it is a widely used tool by the authors V. Kravchenko and S. Torba [**32**], [**21**], [**33**] in order to obtain representations for the solutions of differential equations, these representations allow us to approximate the solution of spectral problems with excellent numerical results.

The two main results of this dissertation are summarized in the following two theorems.

Theorem 2.4. *The regular solution $u_l(\omega, x)$ of the perturbed Bessel equation,*

$$-u'' + \left(\frac{l(l+1)}{x^2} + q(x)\right) u = \omega^2 u, \; x \in (0,1],$$

where l is a real number, $l \geq -\frac{1}{2}$, q is a complex-valued function on $[0,b]$ satisfying the following condition

$$\begin{aligned} xq(x) &\in L_1(0,1), \ l > -\frac{1}{2}, \\ x^{1-\varepsilon}q(x) &\in L_1(0,1), \ \text{for some } \varepsilon > 0 \text{ if } l = -\frac{1}{2}, \end{aligned}$$

can be exppessed in the following form

$$u_l(\omega,x) = d(\omega)b_l(\omega x) + \sum_{n=0}^{\infty}(-1)^n\tilde{\beta}_n(x)j_{2n}(\omega)$$

where $\tilde{\beta}_n$ are defined by the formula

$$\tilde{\beta}_n(x) = (4n+1)\sum_{k=0}^{n} l_{2k,2n}\left(\varphi_k(x) - c_{k,l}x^{2k+l+1}\right),$$

$d(\omega) := \frac{2^{l+\frac{1}{2}}\Gamma(l+\frac{3}{2})}{\omega^{l+1}}$*, and* $b_l(\omega x) := \sqrt{\omega x}J_{l+\frac{1}{2}}(\omega x)$*, the series of Bessel functions converges uniformly with respect to x on $[0,1]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω.*
The approximate solution

$$u_{l,N}(\omega,x) = d(\omega)b_l(\omega x) + \sum_{n=0}^{N}(-1)^n\tilde{\beta}_n(x)j_{2n}(\omega).$$

admits the following estimates for $2N \geq \left[l+\frac{3}{2}-\alpha\right]+1$,
if $\omega \neq 0$ is a real number,

$$\begin{aligned} |u_l(\omega,x) - u_{l,N}(\omega,x)| &= \left|\int_0^1 \left(\tilde{R}(x,t) - \tilde{R}_N(x,t)\right)\cos(\omega t)\,dt\right| \\ &\leq \left\|\tilde{R}(x,t) - \tilde{R}_N(x,t)\right\|_{L_2[0,1]}\int_0^1 \left|\cos^2(\omega t)\right|dt \\ &\leq \frac{c_1(x)}{N^{l+\frac{3}{2}-\alpha}}. \end{aligned}$$

If $\omega \neq 0$ is a number complex,

$$\begin{aligned} |u_l(\omega,x) - u_{l,N}(\omega,x)| &\leq \frac{c_1(x)}{N^{l+\frac{3}{2}-\alpha}}\int_0^1 \left|\cos^2(\omega t)\right|dt \\ &\leq \frac{c_1(x)}{N^{l+\frac{3}{2}-\alpha}}\left(\frac{1}{2} + \frac{2\sinh|2C|}{2|C|}\right), \end{aligned}$$

where $|\mathrm{Im}(\omega)| \leq C$, $C \geq 0$.

Consider a solution of the problem (0.7), with the initial conditions (0.8) in the following form

$$y(\omega, x) = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!}\omega^k \tag{0.14}$$

Theorem 3.7. *The solution $y(\omega, x)$ of the Cauchy problem*

$$y^{(n)} + p_2(x)y^{(n-2)} + ... + p_n(x)y = \omega^n y,\ \ x \in (0, b),$$
$$y(\omega, 0) = 1, y'(\omega, 0) = \omega, ..., y^{n-1}(\omega, 0) = \omega^{n-1},$$

where the coefficients p_k, $k = 2, ..., n$, are complex-valued continuous functions on $[0, b]$, $0 < b < \infty$, and $\omega \in \mathbb{C}$, admits the following representation

$$y(\omega, x) = 2\sum_{n=0}^{\infty}\left(\beta_n^{(1)}(x;\delta)e^{-i\omega(x+\delta)} + (-1)^{n+1}\beta_n^{(3)}(x;\delta)e^{i\omega(x+\delta)}\right)i_n(\omega(x+\delta))$$
$$+2\sum_{n=0}^{\infty}\left(i^{n+1}\beta_n^{(2)}(x;\delta)e^{\omega(x+\delta)} + (-i)^{n+1}\beta_n^{(4)}(x;\delta)e^{-\omega(x+\delta)}\right)j_n(\omega(x+\delta)).$$

where the coefficients $\beta_n^{(k)}$ are defined by the formula

$$\beta_n^{(j)}(x;\delta) = \sum_{k=1}^{\infty}\frac{(-1)^k c_{k,j}\alpha_{k-1}(x)\,(k)_n\,(1-i)^{n+k}}{(x+\delta)^{k-1}2^k}\,{}_2F_1\left(n+k, n+1; 2(n+1); \frac{2}{1+i}\right)$$

*where $b_n = \frac{(2n+1)}{2\pi i (n+1)_{n+1}}$, $\alpha_{k-1}(x)$ is the $k-1$ th coefficient of the series (0.14), δ is a positive real parameter, $(k)_n$ is the Pochhammer symbol, and ${}_2F_1$ is the Gauss hypergeometric function. Here j_n stands for the spherical Bessel function of the first kind of order n, and i_n is the modified spherical Bessel function of the first kind of order n (see the definition, e.g., in [**42**, Chap. 10]). For the approximate solution*

$$y_M(\omega, x) = 2\sum_{n=0}^{M}\left(\beta_n^{(1)}(x;\delta)e^{-i\omega(x+\delta)} + (-1)^{n+1}\beta_n^{(3)}(x;\delta)e^{i\omega(x+\delta)}\right)i_n(\omega(x+\delta))$$
$$+2\sum_{n=0}^{M}\left(i^{n+1}\beta_n^{(2)}(x;\delta)e^{\omega(x+\delta)} + (-i)^{n+1}\beta_n^{(4)}(x;\delta)e^{-\omega(x+\delta)}\right)j_n(\omega(x+\delta))$$

the following estimate holds

$$|y(\omega, x) - y_M(\omega, x)| < \frac{2\sqrt{\pi}\left(M + \frac{1}{2} + 2e\right)}{\Gamma(M + \frac{3}{2})}(x+\delta)\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta, x+\delta]}\sqrt{|\omega|}$$
$$\left(\cosh((x+\delta)\operatorname{Im}(\omega))e^{(x+\delta)|\operatorname{Re}(\omega)|} + \cosh((x+\delta)\operatorname{Re}(\omega))e^{(x+\delta)|\operatorname{Im}(\omega)|}\right).$$

Approbation

The results presented in this dissertation are written in two articles in collaboration with professors V. Kravchenko and S. Torba [**18**], [**19**].

The results contained in this thesis were accepted for presentation in the congress:

XLIX Congreso Nacional de la Sociedad Matemática Mexicana, Villahermosa-Tabasco, México, Octubre 21-26, 2018. Talk: **Nueva representación mediante series de Neumann de la solución regular de la ecuación de Bessel perturbada mediante la extensión del kernel de transmutación.**

International Workshop on TRANSMUTATION OPERATORS AND RELATED TOPICS. I$_W$TORT 2019. Cinvestav, Querétaro, México. September 17-18,2019. Talk: **Neumann series of Bessel functions representations for solutions of linear higher order differential equations.**

Contents

CHAPTER 1

Preliminaries

The aim of this chapter is to introduce the terminology, the notation as well as to explain the motivation for our research. A brief exposition of the methods developed in this thesis focused on the NSBF representation for the solutions of the linear differential equations in question is presented. In Section 1 we present results about the SPPS method focused to solve ordinary linear differential equations. Section 2 contains results on the NSBF representation obtained in previous works for solutions of differential equations of second order. In Section 3 we introduce the notion of transmutation operator. Section 4 contains results about the Borel transform and the Polya representation.

1. Linear differential equations and SPPS representation

Consider the ordinary linear differential equation of order n

$$y^{(n)} + p_1(x)\, y^{(n-1)} + p_2(x)\, y^{(n-2)} + \cdots + p_n(x)\, y = \omega^n y \tag{1.1}$$

where $p_1(x), ..., p_n(x)$ are assumed to be complex-valued functions. The coefficients $p_2(x)$, ..., $p_n(x)$ are continuous with respect to x in the interval $[a,b]$, $p_1 \in C^{n-1}[a,b]$, and $\omega \in \mathbb{C}$, ω is named the spectral parameter. Without loss of generality, we may assume $p_1(x) \equiv 0$. For if $p_1(x) \neq 0$, then by using the substitution

$$y = ue^{\left(-\frac{1}{n}\int p_1(x)dx\right)},$$

we obtain the equation

$$u^{(n)} + q_2(x)\, u^{(n-2)} + ... + q_n(x)\, u = \omega^n u$$

where the coefficients $q_2(x), ..., q_n(x)$ are also continuous with respect to x in the interval $[a,b]$, and ω has not changed (see, e.g.,[**41**, Chap. 2]).

In [**37**, Chap. 1, Sect. 2.5] it was proved that the solution $u(\omega, x)$ of the following Cauchy problem

$$u^{(n)} + q_2 u^{(n-2)} + ... + q_n u = \omega^n u, \tag{1.2}$$

$$u(\omega,0) = 1,\ u'(\omega,0) = \omega, ..., u^{(n-1)}(\omega,0) = \omega^{n-1},$$

as a function of the variable ω is an entire function of the first order of the type equal to $|x|$. Therefore, $u(\omega,x)$ admits the following representation

$$u(\omega, x) = \sum_{n=0}^{\infty} \alpha_n(x)\,\omega^n, \tag{1.3}$$

where the coefficients $\alpha_n(x)$ are obtained using the variation of parameters method. The power series representation (1.3) for the same problem has also been obtained in [**30**], and by a different method the integral recurrent formulas for the coefficients were obtained. The method used to get the representation is named SPPS method. The SPPS method has been used even in cases when the coefficients are not continuous, (see, for example, [**4**], [**11**]).

The SPPS representation of a solution of an ordinary linear differential equation of the form (1.2) consists in representing the solution in form of a power series with respect to the spectral parameter ω. For $n = 2$ the coefficients are obtained by a simple procedure of recurrent integration and requires knowledge of a nonvanishing solution of the equation with one fixed value of the spectral parameter (which can be zero). In contrast to the second order case, for $n > 2$ the coefficients are obtained by a integral procedure in terms of a system of n linearly independent solutions of the equation

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_n y = 0,$$

where is required that n corresponding partial Wronskians do not vanish. In general the system of n linearly independent solutions can be obtained for a fixed value of the spectral parameter. The existence of such a solution system $\{y_1,\ y_{2,\ \ldots,}\ y_n\}$ is established in [**7**], where it can be seen that in fact almost all complex-valued solution sets satisfy this nonvanishing requirement.

This representation was introduced by V. Kravchenko in the work [**26**] for the Sturm-Liouville equation, the result was obtained with the aid of the theory of pseudoanalytic functions [**3**].

In joint work with other researchers, the SPPS representation has been extended to other kinds of equations without depending on pseudoanalytic function theory. For instance, in [**29**] the equation (1.4) was considered

$$(pu')' + qu = \omega r u. \tag{1.4}$$

Assuming that on a finite interval $[a, b]$, the equation $(pu')' + qu = 0$ possesses a particular solution u_0, such that the functions $u_0^2 r$ and $1/(u_0^2 p)$ are continuous on $[a, b]$, then, the general solution of (1.4) on (a, b) has the form

$$u = c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are arbitrary complex constants, and the functions u_1 and u_2 are defined as

$$u_1 = u_0 \sum_{n=0}^{\infty} \frac{\tilde{X}^{(2n)}(x)}{(2n)!}\omega^n \text{ and } u_2 = u_0 \sum_{n=0}^{\infty} \frac{X^{(2n+1)}(x)}{(2n+1)!}\omega^n \tag{1.5}$$

with $\tilde{X}^{(2n)}$ and $X^{(2n+1)}$ being defined by the following procedure

$$\tilde{X}^{(0)} \equiv 1,\ X^{(0)} \equiv 1$$

$$\tilde{X}^{(n)} \equiv \begin{cases} n\displaystyle\int_{x_0}^{x} \tilde{X}^{(n-1)}(s)\, u_0^2(s)\, r(s)\, ds, \ n \ \text{ odd}, \\ n\displaystyle\int_{x_0}^{x} \tilde{X}^{(n-1)}(s)\, \frac{1}{u_0^2(s)p(s)} ds, \ n \ \text{ even}. \end{cases}$$

$$X^{(n)} \equiv \begin{cases} n\displaystyle\int_{x_0}^{x} X^{(n-1)}(s)\, \frac{1}{u_0^2(s)p(s)} ds, \ n \ \text{ odd}, \\ n\displaystyle\int_{x_0}^{x} X^{(n-1)}(s)\, u_0^2(s)\, r(s)\, ds, \ n \ \text{ even}. \end{cases}$$

where x_0 is an arbitrary point in $[a, b]$, such that p is continuous at x_0 and $p(x_0) \neq 0$. Furthermore, both series in (1.5) converge uniformly on $[a, b]$.

Remark 1.1. *By definition the solutions u_1 and u_2 from (1.5) satisfy the following initial conditions*

$$\begin{aligned} u_1(x_0) &= 1,\ u_1'(x_0) = u'(x_0) \\ u_2(x_0) &= 0,\ u_2'(x_0) = \frac{1}{u'(x_0)} \end{aligned}$$

Definition 1.2. [**27**] *The family of functions $\{\varphi_n\}_{n=0}^{\infty}$ constructed according to the rule*

$$\varphi_n(x) \equiv \begin{cases} u_0(s)\, X^{(n)}(s), \ n \ \ odd \\ u_0(s)\, \tilde{X}^{(n)}(s), \ n \ \ even \end{cases}$$

is named the system of formal powers associated with $u_0(s)$.

In [**4**], the SPPS representation was obtained for equations with discontinuous coefficients involving Sturm-Liouville equations. For the perturbed Bessel equation of the form

$$-y'' + \left(\frac{l(l+1)}{x^2} + q\right) y = \omega\left(r_1(x)\, u' + r_0(x)\, u\right),\ x \in (0, b] \tag{1.6}$$

where $l \geq -\frac{1}{2}$ and q is a complex valued continuous function on $(0, b]$ satisfying a bounded growth $|q(x)| \leq Cx^{\alpha}$ at the origin for some $\alpha > -2$, and $r_{0,1} \in C[0, b]$ are complex valued functions, in [**11**], R. Castillo, V. Kravchenko, and S. Torba obtained the SPPS representation for a regular solution. which does not have zeros on $[0, b]$ except at $x = 0$.

Formally this result was proved in the following theorem.

Let u_0 be the regular solution for $\omega = 0$ and suppose it does not have zeros on $[0, b]$. Define

$$\tilde{X}^{(0)} \equiv 1,\ \tilde{X}^{(-1)} \equiv 0$$

$$\tilde{X}^{(n)}(x) \equiv \begin{cases} \int_0^x \Big(u_0(s) R[u_0](s) \tilde{X}^{(n-1)}(s) - r_1(s) \tilde{X}^{(n-2)}(s) \Big) ds, \ n \ \text{odd}, \\ -\int_0^x \tilde{X}^{(n-1)}(s) \frac{1}{u_0^2(s)} ds, \ n \ \text{even}. \end{cases} \tag{1.7}$$

$$Ru = r_0 u + r_1 u'.$$

Theorem 1.3. [**11**] *Let* $-u_0'' + \left(\frac{l(l+1)}{x^2} + q(x) \right) u_0 = 0$ *admit a solution* $u_0 \in [0,b] \cap C^2(0,b]$ *(in general complex valued) which does not have other zeros on* $[0,b]$ *except at* $x=0$ *and satisfies the asymptotic relations*

$$u_0(x) \sim x^{l+1}, \ x \to 0,$$
$$u_0'(x) \sim (l+1) x^l, \ x \to 0.$$

Then, for any $\omega \in \mathbb{C}$ *the function*

$$u = u_0 \sum_{n=0}^{\infty} \omega^n \tilde{X}^{(2n)}$$

is a solution of (1.6) belonging to $C[0,b] \cap C^2(0,b]$ *and the series converges uniformly on* $[0,b]$*. The first derivative of* u *is given by*

$$u' = \frac{u_0'}{u_0} u - \frac{1}{u_0} \sum_{n=1}^{\infty} \omega^n \tilde{X}^{(2n-1)}$$

and the series for the first and second derivatives converge uniformly on an arbitrary compact $K \subset (0,b]$.

In Chapter 3 of this thesis, we give some numerical illustrations of the NSBF representation for the solution of the equation $y^{(4)} + (p(x) y')' = \omega^4 y$, $x \in (0,b)$ where $p \in C^1[0,b]$. We chose this particular equation due to the fact that for the construction of corresponding formal powers and hence of the SPPS representation of the solution an especially simple procedure was developed in [**24**].

Recently, for differential equations of arbitrary order this method has being also applied. The SPPS representation for solutions of linear differential equations of nth order was defined and justified in [**30**] by the researches V. Kravchenko, R. Porter, and S. Torba.

2. Transmutation operator

An important tool in the theory of the differential equations is the concept of the transmutation operator. The main idea of its use consists in relating two linear differential operators and to analyze a more complicated equation in terms of a simpler one. For some classes of differential operators a transmutation operator can be realized in the form of a Volterra integral operator, this result was initially proved by A. Povzner [**44**] for Sturm-Liouville equations. The transmutation operators have been widely studied in a large number of works associated with direct and inverse problems for linear differential equations (see, e.g., [**14**], [**40**], [**9**], [**37**]) and nowadays it is a widely used tool in a practical approach developed by V. Kravchenko and S. Torba (see, e.g., [**32**], [**21**], [**33**], [**34**]) in order to obtain representations of the solutions of differential equations that allow us to approximate the solution of spectral problems.

We keep the definition proposed in [**31**], which is a modification of the definition given by B. Levitan in [**39**], and sufficient for the purposes of this work. Let E be a linear topological space and E_1 its linear subspace (not necessarily closed). Let A, $B : E_1 \to E$ be linear operators.

DEFINITION 1.4. *A linear invertible operator T defined on the whole E such that E_1 is invariant under the action of T is called a transmutation operator for the pair of operators A and B if it fulfills the following two conditions.*

1. Both the operator T and its inverse T^{-1} are continuous in E;

2. The following operator equality is valid

$$AT = TB$$

or which is the same

$$A = TBT^{-1}$$

If for a pair of differential operators the transmutation operator is constructed as a Volterra integral operator of second kind then its integral kernel is obtained as a solution of a certain Goursat problem. For instance, considering the functional space $E = C[-b, b]$ and its subspace $E_1 = C^2[-b, b]$, an operator of transmutation for the operators $A = -\frac{d^2}{dx^2}$ and $B = -\frac{d^2}{dx^2} + q(x)$ where the potential q is a continuously differentiable function is

$$T[u] = u(x) + \int_{-x}^{x} K(x,t)\, u(t)\, dt,$$

where the kernel K is the solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right) K(x,t) = \frac{\partial^2}{\partial t^2} K(x,t),$$

$$K(x,x) = \frac{1}{2}\int_0^x q(t)\, dt, \qquad K(x,-x) = 0,$$

(see [**31**]). Property 2 of transmutation operators from Definition 1.4 is very interesting because if we know the solution u of the equation $B[u] = \omega^2 u$ then $v = T[u]$ is a solution of the equation $A[\nu] = \omega^2 \nu$.

2.1. Transmutation operator for perturbed Bessel operators. Denote the differential operator

$$A=-\frac{d^2}{dx^2}+\frac{l\left(l+1\right)}{x^2}+q\left(x\right),$$

as the perturbed Bessel differential operator, where q is a continuous complex-valued function on $[0,b]$ and $l\geq-\frac{1}{2}$. Let $b_l\left(\omega x\right)$ be a regular solution of the unperturbed Bessel equation

$$B\left[b_l\right]=-\frac{d^2}{dx^2}b_l+\frac{l\left(l+1\right)}{x^2}b_l=\omega^2 b_l,\ x\in\left(0,b\right].$$

The transmutation operator $\mathbb{T}$ for the perturbed Bessel differential operator and the unperturbed Bessel differential operator $(q\equiv 0)$ exists. In [**49**], looking for a transmutation operator in the form of a Volterra integral operator, the existence of a unique continuous kernel $V\left(x,t\right)$ was proved such that for all $\omega\in\mathbb{C}$, the function

$$u\left(\omega,x\right)=\mathbb{T}\left[b_l\left(\omega x\right)\right]:=b_l\left(\omega x\right)+\int_0^x V\left(x,t\right)b_l\left(\omega x\right)dt \tag{1.8}$$

is a regular solution of the equation $A\left[u\right]=\omega^2 u$, $x\in\left(0,b\right]$. The kernel V is ω-independent continuous function with respect to both arguments, satisfying the Goursat condition

$$V\left(x,x\right)=\frac{1}{2}\int_0^x q\left(t\right)dt. \tag{1.9}$$

Remark 1.5. *In* [**11**] *a mapping property for the transmutation operator for the pair of operators A and B was presented.*

$$\mathbb{T}\left[x^{2k+l+1}\right]=\left(-1\right)^k 2^{2k}k!\left(l+\frac{3}{2}\right)_k u_0\left(x\right)\tilde{X}^{(2k)}\left(x\right), \tag{1.10}$$

where $u_0\left(x\right)$ is a non vanishing on $(0,b]$ complex -valued solution of the equation $A\left[u_0\right]=0$ and $\tilde{X}^{(2k)}\left(x\right)$ is defined as in (1.7).

We would like to note here that, in practice this interesting property (1.10) of the operator $\mathbb{T}$ is available even when the kernel V is unknown.

In [**35**], the following representation for $u_l\left(\omega,x\right)$

$$u_l\left(\omega,x\right)=T\left[\cos\omega\right]\left(x\right):=d\left(\omega\right)b_l\left(\omega x\right)+\int_0^x R\left(x,t\right)\cos\left(\omega x\right)dt \tag{1.11}$$

with $d\left(\omega\right)=\frac{2^{l+\frac{1}{2}}\Gamma\left(l+\frac{3}{2}\right)}{\omega^{l+1}}$ and

$$R\left(x,t\right)=\frac{2^{l+1}\Gamma\left(l+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma\left(l+1\right)}\int_s^x V\left(x,t\right)\left(t-\frac{s^2}{t}\right)^l dt$$

was constructed. Also, the following mapping property for the operator defined in (1.11) was given

$$T\left[x^{2k}\right]=\varphi_k\left(x\right)-c_{k,l}x^{2k+l+1},\ k=0,1,2,...$$

where $c_{k,l}:=\frac{\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(k+l+\frac{3}{2}\right)}$. Moreover, that the integral kernel $R\left(x,\cdot\right)$ is square-integrable on $[0,x]$ was also proved in [**35**].

Finally, another important theorem for the subsequent results, is about the following representation of the integral kernel R in terms of the Legendre-Fourier series using the formal powers $\left\{\varphi_k\left(x\right)\right\}_{k\in N_0}$.

Let P_n be the Legendre polynomial of order n, $l_{k,n}$ the corresponding coefficients of x^k, that is $P_n\left(x\right)=\sum_{k=0}^{n}l_{k,n}x^k$. The Legendre polynomials provide a basis for the space $L_2\left[-1,1\right]$, therefore every function R in this space can be represented by the series

$$R=\sum_{n=0}^{\infty}\frac{2n+1}{2}\left\langle R,P_n\right\rangle P_n.$$

From here on P_n stands by Legendre polynomial of of order n unless we give another explanation.

THEOREM 1.6. [**35**, Theorem 4.3.] *If q is a complex valued function on $[0,b]$ satisfying the following condition*

$$xq\left(x\right)\in L_1\left(0,b\right)\ \text{ if }\ l>-\frac{1}{2}, \tag{1.12}$$

$$x^{1-\varepsilon}q\left(x\right)\in L_1\left(0,b\right)\ \text{ for some } \varepsilon>0\ \text{ if }\ l=-\frac{1}{2},$$

then the kernel $R\left(x,t\right)$ has the form

$$R\left(x,t\right)=\sum_{n=0}^{\infty}\frac{\beta_n\left(x\right)}{x}P_{2n}\left(\frac{t}{x}\right) \tag{1.13}$$

with β_n being defined by the equality

$$\beta_n\left(x\right)=\left(4n+1\right)\sum_{k=0}^{n}\frac{l_{2k,2nx}}{x^{2k}}\left(\varphi_k\left(x\right)-c_{k,l}x^{2k+l+1}\right), \tag{1.14}$$

where $c_{k,l}=\frac{\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(k+l+\frac{3}{2}\right)}$. For any $l\geq-\frac{1}{2}$ the series (1.13) converges in the L_2 norm. Let additionally q satisfy

$$x^{\alpha}\tilde{q}\left(x\right)\in L_1\left(0,b\right),\ \text{ for some }\alpha\in\left[0,1\right],\ \alpha<\frac{3}{2}+l. \tag{1.15}$$

where

$$\tilde{q}(x) := \begin{cases} |q(x)|, \ l > -\frac{1}{2}, \\ \left(1 - \log\left(\frac{x}{b}\right)\right)|q(x)|, \ l = -\frac{1}{2}. \end{cases} \tag{1.16}$$

If $l > \alpha - \frac{1}{2}$ *then for any* $x \in (0, b]$ *the series (1.13) converges uniformly with respect to* $t \in [0, x]$*; if* $\alpha - 1 < l \leq \alpha - \frac{1}{2}$*, then for any* $x \in (0, b]$ *the series converges uniformly with respect to* $t \in [0, x'] \subset [0, x)$*. Denote*

$$R_N(x,t) := \sum_{n=0}^{N} \frac{\beta_n(x)}{x} P_{2n}\left(\frac{t}{x}\right).$$

There exist constants C_1 *and* C_2 *dependent on* q *and* l *and independent of* x *and* N*, such that for any* $x > 0$

$$\|R(x,t) - R_N(x,t)\|_{L_2[0,x]} \leq \frac{C_1 x^{l+\frac{3}{2}-\alpha}}{N^{l+\frac{3}{2}-\alpha}}, \ 2N \geq \left[l + \frac{5}{2}\right]$$

and

$$|\beta_N(x)| \leq \frac{C_2 x^{l+2-\alpha}}{(N-1)^{l+1-\alpha}}, \ 2N \geq \left[l + \frac{9}{2}\right].$$

3. NSBF representation for regular solutions of Cauchy problems

The NSBF representation of a function $f(\omega)$ has two main goals. The first is the solution of differential equations even when we do not know the kernel of the transmutation operator. And the second goal is to obtain an expansion in Neumann series of Bessel functions that allows us to obtain uniform convergence of this series with respect to ω in a compact subset of complex plane of the variable ω, and to obtain estimates for the convergence of the partial sums of the exact solutions independent of ω for any strip such that $|\operatorname{Im}\omega| \leq C$, $C > 0$. A relation on convergence of Neumann series of Bessel functions with the Maclaurin expansion of $f(\omega)$ is the following

Remark 1.7. [**50**, Chap. XVI, Sect. 16.2] *If* $f(z)$ *admits the Neumann expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n J_{\nu+n}(z) \tag{1.17}$$

and suppose that the Maclaurin expansion is known

$$f(z) = \sum_{n=0}^{\infty} b_n z^n.$$

It follows that a Neumann series has a circle of convergence, just like a power series, and the circles of convergence of a Neumann series and of the associated power series are identical.

Then, it should be noted that if a solution of a differential equation is obtained by the SPPS representation, then it is possible to know the convergence of this series on any compact subset of the complex plane of the variable z, therefore that the NSBF representation of this solution achieve convergence is also known.

Spherical Bessel functions j_ν, are related with the ordinary Bessel functions J_ν, by the formula

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}} J_{\nu+\frac{1}{2}}(z),$$

(see, e.g. [**1**, p. 437, Formula 10.1.1]). The Bessel functions of the first kind have a number of beautiful properties that we do not describe here (for a good introduction see, e.g., [**50**]). One of the reasons why these functions are convenient and used in this work is due to their behaviour for large values of n. The numbers $j_n(z)$ for fixed z rapidly decrease as $n \to \infty$ (see, e.g., [**1**, Formula 9.1.62]).

The method used to obtain the NSBF representation for a regular solution u of a linear differential equation requires three main mathematical tools. The first is a power series representation of the solution, this can be obtained by the SPPS method. In second place is an integral representation that we can obtain from the transmutation operators theory or by the Borel transform of the solution u. Finally, an expansion in Fourier-Legendre series for the kernel of the integral representation.
An expansion in Fourier-Legendre series for the kernel $R(x,t)$ of the integral representation .was presented in the theorem.
A result that summarizes the main features of the NSBF representation for the regular solution of the perturbed Bessel equation is the following, which was obtained by V. Kravchenko, S. Torba, and R. Castillo (2018).

THEOREM 1.8. [**35**, Theorem 5.1] *Under conditions of Theorem 1.6, the regular solution $u_l(\omega,x)$ of the perturbed Bessel equation satisfying the asymptotic relation $u_l(\omega,x) \sim x^{l+1}$ when $x \to 0$ has the form*

$$u_l(\omega,x) = d(\omega)\, b_l(\omega x) + \sum_{n=0}^{\infty} (-1)^n \beta_n(x)\, j_{2n}(\omega x)$$

where β_n are defined by the equality (1.14) and j_{2n} stands for the spherical Bessel function of order $2n$, the series converges uniformly with respect to x on $[0,b]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω. For the approximate solution,

$$u_{l;N}(\omega,x) = d(\omega)\, b_l(\omega x) + \sum_{n=0}^{N} (-1)^n \beta_n(x)\, j_{2n}(\omega x)$$

the following estimates hold

$$|u_l(\omega,x) - u_{l;N}(\omega,x)| \le \sqrt{x}\,\varepsilon_N(x)$$

for any $\omega \in \mathbb{R}$, $\omega \neq 0$, and

$$\left|u_l(\omega, x) - u_{l;N}(\omega, x)\right| \leq \left(\frac{\sin(2Cx)}{2C}\right)^{\frac{1}{2}} \varepsilon_N(x)$$

for any $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\operatorname{Im}\omega| \leq C$, $C \geq 0$, where $\varepsilon_N(x)$ is a sufficiently small nonnegative function such that $\|R(x,t) - R_N(x,t)\|_{L_2[0,x]} \leq \varepsilon_N(x)$. Moreover, for each fixed x and ω the convergence rate of $u_{l;N}(\omega, x)$ is exponential. To be more precise, let $x > 0$ be fixed and $\omega \in \mathbb{C}$ satisfy $|\omega| \leq \omega_0$. Then for all $N > \omega_0 x/2$ one has

$$\left|u_l(\omega, x) - u_{l;N}(\omega, x)\right| \leq \frac{cxe^{|\operatorname{Im}\omega| x}}{N^{l+1-\alpha}} \cdot \frac{1}{(2N+2)!} \cdot \left|\frac{\omega_0 x}{2}\right|^{2N+2},$$

where c is a constant depending on q and l only and α is the constant from the condition (1.15).

4. The Borel transform

Following the notation from [**5**, Chap. 2] and [**6**] we enunciate the following definition.

Definition 1.9. *An entire function $f(z)$ is of exponential type x if some number δ exist such that for every positive number ε there exists a quantity M depending on ε and δ in general but independent of z, such that for all (finite) values of z we have*

$$|f(z)| < Me^{(\delta+\varepsilon)|z|} \tag{1.18}$$

and that x shall be the least possible value for such numbers δ. Also an alternative definition is

$$x = \limsup_{n\to\infty} \left|f^{(n)}(z)\right|^{\frac{1}{n}}, \tag{1.19}$$

it is immaterial which value of z is used in (1.19).

Let $f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$ be the Maclaurin series of the entire function f, therefore

$$b_n = f^{(n)}(0),$$

thus, we can get $x = \limsup_{n\to\infty} |b_n|^{\frac{1}{n}}$ in the equation (1.19). The growth of the function f in different directions is closely related to the location of the singular points of the Borel associated function. An idea of this is expressed in the next theorem, whose proof is found in [**5**, Chap. 5].

Theorem 1.10. [**5**, Chap. 5. Theorem 5.3.1] *The function*

$$f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n \tag{1.20}$$

is an entire function of exponential type if and only if

$$F(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}} \tag{1.21}$$

is convergent for some (finite) z. If the radius of non convergence of the series in (1.21) is x, $f(z)$ is of type x if $x > 0$, of exponential type 0 if $x = 0$.The function $F(z)$ is often called the Borel transform of $f(z)$.

4.1. The Borel-Laplace transform.

DEFINITION 1.11. *A function $f : \mathbb{R} \to \mathbb{C}$, has exponential order c if there exist a constant $A > 0$ such that for some $t_0 \geq 0$,*

$$|f(t)| \leq Ae^{ct}, \quad t \geq t_0.$$

DEFINITION 1.12. *Let $f(t)$ be a function locally integrable on $(0, \infty)$ and of exponential order c. The function F defined on the set $\{z | \operatorname{Re}(z) > c\}$ and given by the formula*

$$F(z) := \int_0^{\infty} e^{-zt} f(t)\, dt.$$

is the Laplace transform of a function f.

Given $F(z)$, find f such that $\mathcal{L}[f] = F$ we call f the inverse Laplace transform of F.

If the function $F(z)$ is the Borel transform of $f(z)$ as in Theorem 1.10, then $F(z)$ is the Laplace transform of $f(z)$ for z of sufficiently large positive real part, and the analytic continuation of this for the other values of z for which it is regular (see, e.g., [**5**, p. 73]). Therefore, if we consider $z = a + ib$, and $a > x$,

$$\int_0^{\infty} e^{-zt} f(t)\, dt = \sum_{n=0}^{\infty} \frac{b_n}{n!} \int_0^{\infty} e^{-zt} t^n dt = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}} = F(z).$$

The series $\sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ converges for $|z| > x$ where x is defined by formula (1.19).

Following [**5**] and [**37**] we enunciate the following relation for any simple path C lying entirely in the domain of holomorphy of F.

THEOREM 1.13. *If $f(z)$ is an entire function of exponential type x, then the function $f(z)$ can be expressed by the formula*

$$f(z) = \frac{1}{2\pi i} \int_C F(t)\, e^{zt} dt \tag{1.22}$$

where C is a contour containing within itself all singular points of $F(z)$. Note than the contour C can always be taken as a circle $|z| = x + \delta$, $\delta > 0$. The representation (1.22) is called the Polya integral representation.

Proof. The series $\sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$ converges uniformly on C, then we justify the interchange of integration and summation, and the following equality is obtained

$$\int_C F(t)\, e^{zt} dt = \sum_{n=0}^{\infty} b_n \int_C \frac{e^{zt}}{t^{n+1}} dt,$$

by the Cauchy integral formula for derivatives, it is clear that

$$\sum_{n=0}^{\infty} b_n \int_C \frac{e^{zt}}{t^{n+1}} dt = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$$

□

CHAPTER 2

Neumann series representation for the regular solution of perturbed Bessel equation

In this chapter, for the regular solution $u_l(\omega, x)$ of the equation

$$-u'' + \left(\frac{l(l+1)}{x^2} + q(x)\right) u = \omega^2 u, \ x \in (0,1], \tag{2.1}$$

where $l \geq -\frac{1}{2}$, q is a complex-valued function on $[0,1]$ satisfying the following conditions

$$\begin{aligned} xq(x) &\in L_1(0,1) \quad \text{if} \quad l > -\frac{1}{2}, \\ x^{1-\varepsilon} q(x) &\in L_1(0,1) \qquad \text{for some } \ \varepsilon > 0 \qquad if \quad l = -\frac{1}{2}, \end{aligned} \tag{2.2}$$

and $\omega \in \mathbb{C}$ we construct an NSBF representation in the form

$$u_l(\omega, x) = d(\omega) b_l(\omega x) + \sum_{n=0}^{\infty} (-1)^n \tilde{\beta}_n(x) j_{2n}(\omega). \tag{2.3}$$

where $d(\omega) = \frac{2^{l+\frac{1}{2}}\Gamma\left(l+\frac{3}{2}\right)}{\omega^{l+1}}$, $b_l(\omega x) = \sqrt{\omega x} J_{l+\frac{1}{2}}(\omega x)$.

In the recent work [**35**], the use of transmutation operators allowed to show that a regular solution $u_l(\omega, x)$ of the equation (2.1) satisfying the asymptotic relation $u_l(\omega, x) \backsim x^{l+1}$ when $x \to 0$ admits the form

$$u_l(\omega, x) = \frac{2^{l+1}\Gamma(l+\frac{3}{2}) x j_l(\omega x)}{\sqrt{\pi}\omega^l} + \int_0^x R(x,t)\cos(\omega t) dt, \tag{2.4}$$

where the kernel $R(x,t)$ is a continuous function in $0 \leq t \leq x$. In [**35**], the regular solution of the equation (2.1) was obtained as follows

$$u_l(\omega, x) = \frac{2^{l+1}\Gamma(l+\frac{3}{2}) x j_l(\omega x)}{\sqrt{\pi}\omega^l} + \sum_{n=0}^{\infty} (-1)^n \beta_n(x) j_{2n}(\omega x), \tag{2.5}$$

where the Neumann series of Bessel functions converges uniformly with respect to x on $[0,1]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω.

Note that if we need the solution u_l for many different values of x, the representation (2.5) saves the necessity to compute $j_{2n}(\omega x)$, while in the representation (2.3) one need to compute $j_{2n}(\omega)$ only once.

Following [**35**], we obtain the NSBF representation (2.3) for the solution u_l using the extension $\tilde{R}(x,t)$ of kernel R, which is defined as

$$\tilde{R}(x,t) = \begin{cases} R(x,t), 0 \le t \le x, \\ 0, \quad x < t \le 1. \end{cases} \tag{2.6}$$

The results of this chapter are obtained in the following order. In Section 1, a Fourier-Legendre series expansion for the kernel $\tilde{R}(x,t)$ is obtained, for the coefficients $\tilde{\beta}_n$ explicit formulas are obtained, and we present a convergence rate estimate depending on the parameter l and the smoothness of potential q. In Section 2, we construct the NSBF representation for the solution $u_l(\omega,x)$ and we prove that the series converges uniformly with respect to x on $[0,1]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω. Moreover, we obtain estimates of convergence rate of the approximate solution $u_{l,N}(\omega,x)$ to $u_l(\omega,x)$ for any $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\operatorname{Im}\omega| \le C$, $C \ge 0$.

Finally, in Section 3 we find a sequence of differential equations satisfied by the coefficients $\tilde{\beta}_n$ and we use two methods to solve this system of equations. The first method is based on the Polya factorization and the second method is the Green function method. In the last part of the section, we present a numerical example.

1. A Fourier-Legendre representation of the kernel $\tilde{R}$

We consider the function $\tilde{q}(x)$ defined in (1.16) (see, [**25**], [**35**]) and it satisfies additionally that

$$x^{\alpha}\tilde{q}(x) \in L_1(0,1), \text{ for some } \alpha \in [0,1], \ \alpha < \frac{3}{2} + l. \tag{2.7}$$

We recall that the condition (2.7) does not imply additional restrictions on q compared with (2.2), it only specifies the order of the regularity at zero. In [**25**], it was proved that the function

$$g(\omega,x) = u_l(\omega,x) - \frac{2^{l+1}\Gamma(l+\frac{3}{2})xj_l(\omega x)}{\sqrt{\pi}\omega^l}$$

is an entire function and for all $\omega \in \mathbb{C}$ satisfies the following estimate

$$|g(\omega,x)| \le C\left(\frac{x}{1+|\omega|x}\right)^{l+1} e^{|\operatorname{Im}\omega|x}\int_0^x \frac{y\,\tilde{q}(y)}{1+|\omega|y}dy. \tag{2.8}$$

Using the facts that $\frac{y^{1-\alpha}}{(1+|\omega|y)^{1-\alpha}} \le \frac{1}{|\omega|^{1-\alpha}}$ and $\frac{1}{(1+|\omega|y)^{\alpha}} \le 1$ in the estimate (2.8) and under the condition (2.7), the following estimate is obtained

$$|g(\omega, x)| \leq \frac{C}{|\omega|^{l+2-\alpha}},\ \omega \in \mathbb{R}, \tag{2.9}$$

see, [**35**]. This estimate shows that for any fixed x, $g \in L_2(\mathbb{R})$ as a function of ω. Moreover, $g|_{\mathbb{R}}$ on any compact symmetric interval containing zero is bounded because g is analytic in there.

Now, applying the Paley-Wiener theorem [**22**, Theorem VI.7.4], the following representation for the function g is obtained

$$g(\omega, x) = \int_{-x}^{x} \frac{R(x,t)}{2} e^{i\omega t} dt, \tag{2.10}$$

that is, the Fourier transform of the function g, which was denoted by $\frac{R(x,t)}{2}$, is compactly supported on $[-x, x]$. Besides, R is an even function since g is even. Therefore,

$$u_l(\omega, x) = d(\omega)\, b_l(\omega x) + \int_0^x R(x,t) \cos(\omega t)\, dt.$$

Thus, using the extension defined in (2.6), we can represent the solution u_l as

$$u_l(\omega, x) = d(\omega)\, b_l(\omega x) + \int_0^1 \tilde{R}(x,t) \cos(\omega t)\, dt, \tag{2.11}$$

furthermore, we can consider that there exists an even function $K(x,t)$ defined on $(-\infty, \infty)$ such that $2K(x,t) = \tilde{R}(x,t)$, $0 \leq t \leq 1$. In Lemma 2.2 properties of the function $K(x,t)$ are presented.

By $W_2^{\alpha}(\mathbb{R})$ with $\alpha \geq 0$, $Lip_{\alpha}(\mathbb{R})$ and $L_{\alpha}^{*}(\mathbb{R}, 2)$ we denote the fractional order Sobolev space, Lipschitz class of functions and generalized Lipschitz class of functions respectively, see Appendix A.

PROPOSITION 2.1. [**35**, Proposition 4.1] *Let q satisfy the condition (2.7). Let $x > 0$ be fixed. Then there exist an even, compactly supported on $[-x, x]$ function $R'(x,t)$ such that*

1 . $R' \in W_2^{l+\frac{3}{2}-\alpha-\epsilon}(\mathbb{R})$ for any sufficiently small $\epsilon > 0$; if $\alpha < l+1$ then additionally $K \in Lip_{l+1-\alpha-\epsilon}(\mathbb{R})$.
2 . $R' \in L^{*}_{l+\frac{3}{2}-\alpha}(\mathbb{R}, 2)$.
3 . The function R' satisfies

$$R(x,t) = 2R'(x,t) \quad t \leq t \leq x.$$

Considering that $K(x,t)$ is an extension of $\tilde{R}(x,t)$ we obtain the following result.

LEMMA 2.2. *The function $K(x,t)$ as a function of t for any $x \in [0,1]$ satisfies the following properties*

1 . $K \in W_2^{l+\frac{3}{2}-\alpha-\epsilon}(\mathbb{R})$ *for any sufficiently small* $\epsilon > 0$*; if* $\alpha < l+1$ *then additionally* $K \in Lip_{l+1-\alpha-\epsilon}(\mathbb{R})$.

2 . $K \in L^*_{l+\frac{3}{2}-\alpha}(\mathbb{R}, 2)$.

Proof. The proof is similar to the proof of Proposition 2.1 because the kernel $\tilde{R}(x,t)$ inherits these properties of the kernel $R(x,t)$. □

Consider the solution u of the equation (2.1) in the form

$$u_l(\omega, x) = \sum_{k=0}^{\infty} \omega^{2n} \varphi_k(x),$$

where $\varphi_k(x) = u_0 \tilde{X}^{(2n)}(x)$, $\tilde{X}^{(2n)}(x)$ is defined by the recursive procedure (1.7), and u_0 defined in Theorem 1.3. It is a particular case of the equation (1.6), when $r_1(x) \equiv 0$ and $r_0(x) \equiv 1$.

Theorem 2.3. *Let q satisfy (2.2). Then the kernel $\tilde{R}(x,t)$ admits the form*

$$\tilde{R}(x,t) = \sum_{n=0}^{\infty} \tilde{\beta}_n(x) P_{2n}(t), \tag{2.12}$$

with $\tilde{\beta}_n$ being defined by the equality,

$$\tilde{\beta}_n(x) = (4n+1) \sum_{k=0}^{n} l_{2k,2n} \left(\varphi_k(x) - c_{k,l} x^{2k+l+1} \right), \tag{2.13}$$

and $c_{k,l} = \frac{\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(l+k+\frac{3}{2}\right)}$. *For* $l \geq -\frac{1}{2}$*, the series in (2.12) converges in the* L_2 *norm. Let*

$$\tilde{R}_N(x,t) = \sum_{n=0}^{N} \tilde{\beta}_n(x) P_{2n}(t).$$

There exist constants c_1 and c_2, dependent on q and l and independent of x and N, such that for any $x \geq 0$ the inequality holds

$$\left\| \tilde{R}(x,t) - \tilde{R}(x,t)_N \right\|_{L_2[0,1]} \leq \frac{c_1}{N^{l+\frac{3}{2}-\alpha}}, \tag{2.14}$$

and

$$\left| \tilde{\beta}_N(x) \right| \leq \frac{c_2}{(N-1)^{l+1-\alpha}}, \; 2N \geq \left[l + \frac{3}{2} - \alpha \right] + 1. \tag{2.15}$$

Let additionally q satisfy condition (2.7). If $l > \alpha - \frac{1}{2}$ then for any $x \in (0,1]$ the series in (2.12) converges uniformly with respect to $t \in [0,1]$; if $\alpha - 1 < l \leq \alpha - \frac{1}{2}$, $l \geq -\frac{1}{2}$, then for any $x \in (0,1]$ the series converges uniformly with respect to $t \in [0,x'] \subset [0,1)$.

PROOF. For any $x \in (0,1]$ the kernel $\tilde{R}(x,\cdot) \in L_2[0,1]$. Hence it admits a Fourier-Legendre series representation of the form $\sum_{n=0}^{\infty} \beta_n(x) P_{2n}(t)$. Note that

$$\begin{aligned} \int_0^1 \tilde{R}(x,t) P_{2n}(t)\, dt &= \int_0^1 \sum_{m=0}^{\infty} \beta_m(x) P_{2m}(t) P_{2n}(t)\, dt \\ &= \beta_n(x) \frac{1}{4n+1}, \end{aligned}$$

and on the other hand using the equality

$$\varphi_k(x) = c_{k,l} x^{2k+l+1} + \int_0^x R(x,t)\, t^{2k} dt, \tag{2.16}$$

which was obtained in [**35**, Proposition 4.1], we deduce that

$$\begin{aligned} \int_0^1 \tilde{R}(x,t) P_{2n}(t)\, dt &= \sum_{k=0}^{n} l_{2k,2n} \int_0^1 \tilde{R}(x,t)\, t^{2k} dt \\ &= \sum_{k=0}^{n} l_{2k,2n} \int_0^x R(x,t)\, t^{2k} dt \\ &= \sum_{k=0}^{n} l_{2k,2n} \left(\varphi_k(x) - c_{k,l} x^{2k+l+1}\right). \end{aligned}$$

Thus, (2.13) is obtained.

Consider the restriction of the function $K(x,t)$ from lemma 2.2 to the segment $[-1,1]$. We define the functions $h(t) = K(x,t)$ and $h_N(t) = K_N(x,t)$, $t \in [-1,1]$. The function h_N is a partial sum of the Fourier-Legendre of h, that is, h_N coincides with the polynomial of the best approximation in $L_2[-1,1]$ of the function h by polynomials of degree $2N$. Hence, by [**15**, Chap. 7, Theorem 6.3], we have that for any $r \in \mathbb{N}$ there exists a universal constant c_r such that

$$\|h - h_{N}\|_{L_2[-1,1]} \le c_r \omega_r \left(h, \frac{1}{2N}\right)_{L_2[-1,1]}, \quad 2N \ge r \tag{2.17}$$

where ω_r is the rth modulus of smoothness of h defined as in [**15**, Chap. 2, Sect. 7 and Sect. 9]. If a function f belongs to the class of generalized Lipschitz functions $Lip^*_{\alpha}(I,p)$ then $\omega_r(f,t)_{L_p(I)} \le Mt^{\alpha}$ for all $t > 0$ with some constant $M = M(f)$ where $r = [\alpha] + 1$. We take r of the form $r = \left[l + \frac{3}{2} - \alpha\right] + 1$, and by the second condition of Lemma 2.2, we conclude that

$$\omega_r \left(h, \frac{1}{2N}\right)_{L_2[-1,1]} \le \omega_r \left(h, \frac{1}{2N}\right)_{L_2(\mathbb{R})} \le \frac{C(q)}{(2N)^{l+\frac{3}{2}-\alpha}}, \tag{2.18}$$

where the constant $C(q)$ depends neither on x nor on N. Now, note that

$$\left\|\tilde{R} - \tilde{R}_{N}\right\|_{L_2[0,1]} = 2\, \|h - h_{N}\|_{L_2[-1,1]},$$

thus, using the inequalities (2.17) and (2.18) the estimate (2.14) is verified.

To prove the estimate (2.15), we use the fact that $\tilde{R}$ can be represented by a polynomial $\tilde{R}_N$ in even powers of t of degree $2N$ and that the polynomial P_{2N} is orthogonal to the every polynomial $\tilde{R}_{N-1}$ of degree lower than $2N$, we proceed as follows

$$\begin{aligned}
\left|\tilde{\beta}_N(x)\right| &= (4N+1)\left|\int_0^1 \tilde{R}(x,t)\,P_{2N}(t)\,dt\right| \\
&= (4N+1)\left|\int_0^1 \left(\tilde{R}(x,t)-\tilde{R}_{N-1}(x,t)\right)P_{2N}(t)\,dt\right| \\
&\le (4N+1)\left\|\tilde{R}(x,t)-\tilde{R}_{N-1}(x,t)\right\|_{L_2[0,1]}\frac{1}{\sqrt{4N+1}} \\
&\le \frac{c_2}{(N-1)^{l+1-\alpha}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality.

Now, let additionally q satisfy (2.7). Consider the restriction of the function K from Lemma (2.2) to the segment $[-1,1]$. The function K is an even function, then its Fourier-Legendre series contains only even terms and due to the equality

$$\tilde{R}(x,t) = 2K(x,t),\ \ 0 \le t \le 1$$

one has that $K(x,t) = \sum_{n=0}^{\infty}\frac{\tilde{\beta}_n(x)}{2}P_{2n}(t)$, where the series converges in $L_2[-1,1]$. By Lemma 2.2 $K \in Lip_{1+l-\alpha-\varepsilon}(\mathbb{R})$, hence its restriction on $[-1,1]$ belongs to $Lip_{1+l-\alpha-\varepsilon}[-1,1]$, then by [**48**, Theorem 4.10] the uniform convergence of the series (2.12) for any $l > \alpha - \frac{1}{2}$ is established.

For $l > \alpha - 1$, $l \ge -\frac{1}{2}$, we use the Corollary to Theorem $XIII$ from [**20**], which asserts the uniform convergence of the Fourier-Legendre series $K(x,t)$ on any segment $[-1+\varepsilon, 1-\varepsilon] \subset (-1,1)$, i.e., the series (2.12) converges uniformly with respect to $t \in [0, 1-\varepsilon] \subset [0,1)$ for any $\varepsilon > 0$. □

Note that the potential q does not need to be continuous on $[0,1]$. For the equality (2.16), the condition (2.2) is sufficient, see [**4**] and [**11**]. The estimates (2.14) and (2.15) do not depend on the smoothness of the potential q. In [**35**] some improved decay rates for the coefficients β_n requiring q to be sufficiently smooth were given.

2. On the decay rate order of the coefficients $\tilde{\beta}_n(x)$

In this section we give a result that let us conclude that for $x < 1$, the decay rate order of the coefficients of Fourier-Legendre expansion can not be faster than is known in [**35**, Proposition 4.1], that is not be faster than $2l + 3$.

We remember that the solution u_l admits the representation

$$u_l(\omega,x)=\frac{2^{l+1}\Gamma(l+\frac{3}{2})xj_l(\omega x)}{\sqrt{\pi}\omega^l}+\int_0^1\tilde{R}(x,t)\cos(\omega t)dt, \tag{2.19}$$

because the function $f(\omega,x)=u_l(\omega,x)-\frac{2^{l+1}\Gamma(l+\frac{3}{2})xj_l(\omega x)}{\sqrt{\pi}\omega^l}$ as function of ω is an entire function of exponential type and $f(\omega)|_{\mathbb{R}}$ satisfies that $f\in L_2(\mathbb{R})$.

In [**16**], the following asymptotic expansion for $u_l(\omega,x)$ was obtained

$$u_l(\omega,x)=\sum_{k=0}^{m}A_k(x)\frac{\sqrt{x}J_{l+k+\frac{1}{2}}(\omega x)}{\omega^{l+k+\frac{1}{2}}}+R_m(\omega,x), \tag{2.20}$$

when the potential $q(x)$ is analytic on $[0,1]$. The coefficients A_k are defined as follows. Consider the operators H_p, $p=1,2,...$, acting as

$$(H_pf)(x)=\begin{cases}\frac{1}{x^p}\displaystyle\int_0^x t^{p-1}f(t)\,dt, & \text{if } x\neq 0,\\ \frac{1}{p}f(0), & \text{if } x=0.\end{cases}$$

If $f\in C^{(r)}[0,1]$ then $H_pf\in C^{(r)}[0,1]$ and $(H_pf)^{(r)}=H_{p+r}\left(f^{(r)}\right)$, $r=0,1,2,....$
Let $A_k(x)=x^kB_k(x)$. Then the functions B_k satisfy the following recursive relations

$$B_0=2^{l+\frac{1}{2}}\Gamma\left(l+\frac{3}{2}\right)$$

and

$$B_{k+1}=-\frac{1}{2}H_{k+1}\left[B_k''-2lH_1B_k''-qB_k\right],$$

Moreover, their derivatives satisfy the equalities

$$B_{k+1}^{(j)}=-\frac{1}{2}H_{k+j+1}\left[B_k^{(j+2)}-2lH_{j+1}B_k^{(j+2)}-(qB_k)^{(j)}\right],\ j\in\mathbb{N}.$$

For $\omega\in\mathbb{R}$, $|\omega|\geq 1$ the remainder $R_m(\omega,x)$ satisfies the inequality

$$|R_m(\omega,x)|\leq\frac{c(l,m)}{|\omega|^{l+m+2}}\int_0^x\left|\left(t^{m+1}B_{m+1}(t)\right)'\right|dt, \tag{2.21}$$

see [**35**, Proposition 4.5.]. By (2.20) the solution u_l is an even entire function with respect to ω then $R_m(\omega,x)$ also is an even entire function of the complex variable ω, and by inequality (2.21) $R_m(\cdot,x)|_{\mathbb{R}}\in L_2(\mathbb{R})$ and beside it also is an exponential type function. The expression $(t^pB_p(t))'$ is well defined when $q\in C^{2p-1}[0,1]$. Applying the Paley-Wiener theorem we have that

$$R_p(\omega,x)=\int_{-x}^{x}\hat{R}(x,t)e^{i\omega t}dt, \tag{2.22}$$

where the function $\hat{R}(x,\cdot)\in W^{l+p+3/2}(\mathbb{R})\cap Lip_{l+p+1-\varepsilon}(\mathbb{R})\cap Lip^*_{l+p+3/2}(\mathbb{R},2)$ and supp$\hat{R}(x,\cdot)\subset[-x,x]$, for more details see [**35**, Proposition 4.5.].

We define

$$u_l(\omega,x)=A_0(x)\frac{\sqrt{x}J_{l+\frac{1}{2}}(\omega x)}{\omega^{l+\frac{1}{2}}}+A_1(x)\frac{\sqrt{x}J_{l+1+\frac{1}{2}}(\omega x)}{\omega^{l+1+\frac{1}{2}}}+R_1(\omega,x) \tag{2.23}$$

and using (2.22) we obtain

$$R_1(\omega,x)=\int_{-x}^{x}\hat{R}_1(x,t)\,e^{i\omega t}dt=\int_{-x}^{x}\hat{R}_1(x,t)\cos(\omega t)\,dt. \tag{2.24}$$

We use the formula [**38**, (5.10.2)], $J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}\cos(zt)\,dt$ and we have for the second term of the right side of (2.23) the following equality

$$A_1(x)\frac{\sqrt{x}J_{l+1+\frac{1}{2}}(\omega x)}{\omega^{l+1+\frac{1}{2}}}=\frac{A_1(x)\,x^{l+1}}{2^{l+\frac{3}{2}}\sqrt{\pi}\Gamma\left(l+\frac{5}{2}\right)}\int_{-x}^{x}\left(1-\frac{s^2}{x^2}\right)^{l+1}\cos(\omega s)\,ds. \tag{2.25}$$

From the equations (2.19) and (2.23) we obtain

$$\int_0^x R(x,t)\cos(\omega t)dt=A_1(x)\frac{\sqrt{x}J_{l+1+\frac{1}{2}}(\omega x)}{\omega^{l+1+\frac{1}{2}}}+R_1(\omega,x), \tag{2.26}$$

substituting (2.24) and (2.25) in (2.26) we obtain

$$\int_0^x R(x,t)\cos(\omega t)dt=\frac{A_1(x)\,x^{l+1}}{2^{l+\frac{3}{2}}\sqrt{\pi}\Gamma\left(l+\frac{5}{2}\right)}\int_{-x}^{x}\left(1-\frac{s^2}{x^2}\right)^{l+1}\cos(\omega s)\,ds+\int_{-x}^{x}\hat{R}_1(x,t)\cos(\omega t)\,dt. \tag{2.27}$$

We define

$$\hat{\mathcal{R}}_1(x,t)=\begin{cases}\hat{R}_1(x,t), & -x\le t\le x,\\ 0, & x<|t|\le 1,\end{cases}$$

and

$$g(t)=\begin{cases}\left(1-\frac{t^2}{x^2}\right)^{l+1}, & -x\le t\le x,\\ 0, & x<|t|\le 1,\end{cases}$$

then by (2.27) we obtain

$$\frac{\tilde{R}(x,t)}{2}-\hat{\mathcal{R}}_1(x,t)=\frac{\Gamma\left(l+\frac{3}{2}\right)x^{l+2}}{2\sqrt{\pi}\Gamma(l+2)}g(t),\ t\in[-1,1]. \tag{2.28}$$

We need prove that the decay rate order of $\hat{\mathcal{R}}_1(x,t)$ is faster than $l+\frac{3}{2}$, (The order $l+\frac{3}{2}-\alpha$, $0<\alpha<1$ was obtained in the estimate (2.14) for $\tilde{R}(x,t)$) and that there exist a bound for term of the right-hand side of (2.28).

Now, we consider the Fourier-Legendre series for $\hat{\mathcal{R}}_1$,

$$\hat{\mathcal{R}}_1(x,t)=\sum_{n=0}^{\infty}\hat{\beta}_n(x)\,P_{2n}(t),\ \ t\in[-1,1].$$

We know that $\hat{R}(x,\cdot)\in W_2^{l+1+3/2}(\mathbb{R})\cap Lip_{l+1+1-\varepsilon}(\mathbb{R})\cap Lip^*_{l+1+3/2}(\mathbb{R},2)$, then using the same idea that in the proof of theorem (2.3) we obtain the estimate

$$\left\|\hat{\mathcal{R}}_1(x,t)-\hat{\mathcal{R}}_{1,N}(x,t)\right\|_{L_2[-1,1]}\leq\frac{C(q)}{N^{l+\frac{5}{2}}},$$

where $\hat{\mathcal{R}}_{1,N}(x,t)=\sum_{n=0}^{N}\hat{\beta}_n(x)\,P_{2n}(t)$, and

$$\left|\hat{\beta}_N(x)\right|\leq\frac{C_2(q)}{(N-1)^{l+2}},\ \ 2N\geq\left[l+\frac{5}{2}\right]+1. \tag{2.29}$$

On the other hand, using [**36**] (see Appendix B) we will obtain a estimate for the approximate error by polinomials of function g. Consider a weight function

$$W_{(\alpha+m,\beta+m)}(t)=(1-t)^{\frac{\alpha+m}{2}}(1+t)^{\frac{\beta+m}{2}}$$

with $\alpha=\beta=0$, then $W_{(m,m)}(t)=(1-t^2)^{\frac{m}{2}}$. Thus,

$$\int_{-1}^{1}\left|g^{(m)}(t)\,W_{(m,m)}(t)\right|^2dt=\int_{-x}^{x}\left(g^{(m)}(t)\right)^2\left(1-t^2\right)^m dt.$$

We consider the following cases in order to prove that $g(t)\notin S_{l+2}$, ($S_{l+2}:=S_{l+2}^{(0,0)}$ is defined in Appendix B)

(1) When $l\in\mathbb{N}$, the function g has zeros at $t=x$ and $t=-x$, which are of order $l+1$, so the derivatives of order m satisfy

$$g^{(m)}(x)=g^{(m)}(-x)=0,\ \ \forall m=0,...,l.$$

(2) When $l\in\left[-\frac{1}{2},0\right)$, the function g has zeros at $t=x$ and $t=-x$, but does not have continuous derivatives at these points.

(3) When $l\in\mathbb{R}-\mathbb{N}$, the function g has zeros at $t=x$ and $t=-x$, which are of order $\lfloor l+1\rfloor$, so the derivatives of order m satisfy

$$g^{(m)}(x)=g^{(m)}(-x)=0,\ \ \forall m=1,...,\lfloor l+1\rfloor$$

where the function $\lfloor x\rfloor$ is defined as

$$\lfloor x\rfloor=\max\{k\in\mathbb{Z}|\ k\leq x\}.$$

The derivative $l+1$ ($l\in\mathbb{N}$) of g is not continuous, and the derivative $l+2$ does not exist. Using the Inverse Theorem 2 from [**36**], see Appendix B, we conclude that if $g(t)\notin S_{l+2}$ (S_{l+2} is defined in Appendix B), then $\sum_{N=0}^{\infty}(N+1)^{l+2-1}E_N\left(W_{(0,0)};g\right)\to\infty$. Therefore, it is not true that the asymptotic estimate $E_N\left(W_{(0,0)};g\right)=O\left(\frac{1}{N^{l+2+\varepsilon}}\right)$ holds, $0<\varepsilon<1$. Here $E_N\left(W_{(0,0)};g\right)$ is the functional of best approximation of g in the norm L_2 defined as

$$E_N\left(W_{(0,0)};g\right)=\inf_{P\in\Pi}\|g-P\|_2\,,\ N\in\mathbb{N}_0,$$

where Π are the polynomials of order n.

The fourier-Legendre coefficients for the right-hand side of (2.28) can be estimate using that there exist a constant $C>0$ such that $E_N\left(W_{(0,0)};g\right)>\frac{C}{N^{l+2+\varepsilon}}$. Let

$$g\left(t\right)=\sum_{n=0}^{\infty}\check{\beta}_n\left(x\right)P_{2n}(t),\ t\in[-1,1]$$

be the Fourier Legendre expansion of g then

$$\left(\frac{C}{N^{l+2+\varepsilon}}\right)^2<E_N^2\left(W_{(0,0)};g\right)=\sum_{n=N+1}^{\infty}\frac{\left|\check{\beta}_n\left(x\right)\right|^2}{4n+1}\le\frac{C_2}{2rN^{2r}},\tag{2.30}$$

we used the fact that $\frac{\left|\check{\beta}_n(x)\right|^2}{4n+1}\le\frac{C_2}{n^{2r+1}}$. We obtain that $l+2+\frac{\varepsilon}{2}>r$, where r is the decay rate order of the coefficients $\check{\beta}_n\left(x\right)$, therefore by the estimates (2.29) and (2.30) we conclude that the decay rate order of the coefficients $\tilde{\beta}_n\left(x\right)$ does not exceed $2l+3$.

3. Representation in Neumann series of Bessel functions of $u_l\left(\omega,t\right)$

THEOREM 2.4. *Under the conditions of Theorem 2.3, the regular solution $u_l(\omega,x)$ of (2.1) satisfying the asymptotic relation $u(\omega,x)\sim x^{l+1}$ when $x\to0$ has the form*

$$u_l(\omega,x)=d(\omega)b_l(\omega x)+\sum_{n=0}^{\infty}(-1)^n\tilde{\beta}_n(x)j_{2n}(\omega)\tag{2.31}$$

where $\tilde{\beta}_n$ are defined by (2.13), $d(\omega):=\frac{2^{l+\frac12}\Gamma(l+\frac32)}{\omega^{l+1}}$, and $b_l(\omega x):=\sqrt{\omega x}J_{l+\frac12}(\omega x)$, the series converges uniformly with respect to x on $[0,1]$ and converges uniformly with respect to ω on any compact subset of the complex plane of the variable ω. For the approximate solution,

$$u_{l;N}(\omega,x)=d(\omega)b_l(\omega x)+\sum_{n=0}^{N}(-1)^n\tilde{\beta}_n(x)j_{2n}(\omega)$$

the following estimates hold

$$\left|u_l\left(\omega,x\right)-u_{l,N}\left(\omega,x\right)\right|\le\frac{c_1}{N^{l+\frac32-\alpha}}$$

for any $\omega\in\mathbb{R}$, $\omega\neq0$, and

$$\left|u_l\left(\omega,x\right)-u_{l,N}\left(\omega,x\right)\right|\le\frac{c_1}{N^{l+\frac32-\alpha}}\left(\frac12+\frac{2\sinh\left(2C\right)}{2C}\right)$$

for any $\omega \in \mathbb{C}$, $\omega \neq 0$ *belonging to the strip* $|\operatorname{Im}\omega| \leq C$, $C \geq 0$, *where* c_1 *is a constant such that* $\left\|\tilde{R}(x,t) - \tilde{R}(x,t)_N\right\|_{L_2[0,1]} \leq \frac{c_1(x)}{N^{l+\frac{3}{2}-\alpha}}$ *which was obtained in the Theorem 2.3.*

Proof. Using the solution (2.11) and the kernel representation (2.3) we get

$$\begin{aligned} u_l(\omega,x) &= d(\omega)\, b_l(\omega x) + \int_0^1 \tilde{R}(x,t)\cos(\omega t)\,dt \\ &= d(\omega)\, b_l(\omega x) + \sum_{n=0}^{\infty} \tilde{\beta}_n(x) \int_0^1 P_{2n}(t)\cos(\omega t)\,dt \end{aligned}$$

and using the formula 2.17.7 from [**45**, p. 433], the representation (2.31) is obtained. The convergence with respect to ω on any compact subset of the complex plane is established using the fact that for each x the series is considered as a function of the complex variable ω, which is entire and its radius of convergence coincides with the radius of convergence of its associated power series (obtained by the SPPS representation), see [**50**, Chap. XVI, p. 526].

The uniform convergence of the series (2.31) with respect to the variable x is obtained directly from the following estimates. We consider the approximate solution $u_{l,N}(\omega,x)$. If $\omega \neq 0$ is a real number,we obtain

$$\begin{aligned} |u_l(\omega,x) - u_{l,N}(\omega,x)| &= \left|\int_0^1 \left(\tilde{R}(x,t) - \tilde{R}_N(x,t)\right)\cos(\omega t)\,dt\right| \\ &\leq \left\|\tilde{R}(x,t) - \tilde{R}_N(x,t)\right\|_{L_2[0,1]} \int_0^1 \left|\cos^2(\omega t)\right| dt \\ &\leq \frac{c_1}{N^{l+\frac{3}{2}-\alpha}}. \end{aligned}$$

If $\omega \neq 0$ is complex, we get

$$\begin{aligned} |u_l(\omega,x) - u_{l,N}(\omega,x)| &\leq \frac{c_1}{N^{l+\frac{3}{2}-\alpha}} \int_0^1 \left|\cos^2(\omega t)\right| dt \\ &\leq \frac{c_1}{N^{l+\frac{3}{2}-\alpha}} \left(\frac{1}{2} + \frac{\sinh|2\operatorname{Im}(\omega)|}{|\operatorname{Im}(\omega)|}\right). \end{aligned}$$

□

4. Recurrent equations for $\tilde{\beta}_n$

In this section we develop a recurrent procedure for calculating the coefficients $\tilde{\beta}_n$, the procedure lets us find a sequence of recurrent differential equations which must be satisfied by the coefficients $\tilde{\beta}_n$. We use two methods to solve the equation. The first method consists of using the Green function and the second method is based on the Polya factorization.

PROPOSITION 2.5. *Under the conditions of Theorem 2.3, let u_l defined as in (2.31) be the regular solution of equation (2.1) then the coefficients $\tilde{\beta}_n$ satisfy the following recurrent differential equations for $n \geq 1$,*

$$A\left[\frac{\tilde{\beta}_{n+1}(x)}{(4n+3)(4n+5)} - \frac{2\tilde{\beta}_n(x)}{(4n-1)(4n+3)} + \frac{\tilde{\beta}_{n-1}(x)}{(4n-1)(4n-3)}\right] = -\tilde{\beta}_n(x) + A_n x^{l+1} q(x) F, \tag{2.32}$$

where $A_n = \frac{(-1)^n(2n+\frac{1}{2})\Gamma\left(n-\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(n+2)}$ *and* $F = {}_2\mathbf{F}_1\left(n-\frac{1}{2}, -(n+1); l+\frac{3}{2}; x^2\right)$.

PROOF. Substitution of (2.31) into equation (2.1) gives us the equality

$$q(x)2^{l+1}\Gamma(l+\frac{3}{2})\sqrt{x}\frac{J_{l+\frac{1}{2}}(\omega x)}{\omega^{l+\frac{5}{2}}} = \sum_{n=0}^{\infty}(-1)^n\left[\tilde{\beta}(x)j_{2n}(w) - A\left[\tilde{\beta}_n(x)\right]\frac{j_{2n}(\omega)}{\omega^2}\right].$$

Using twice the identity $j_n(z) = \frac{j_{n-1}(z)+j_{n+1}(z)}{2n+1}z$, the linearity of summations, and rewriting the series we obtain

$$\begin{aligned} q(x)2^{l+\frac{1}{2}}\Gamma\left(l+\frac{3}{2}\right)\sqrt{x}\frac{J_{l+\frac{1}{2}}(\omega x)}{\omega^{l+\frac{5}{2}}} &= \left(\tilde{\beta}_0(x) + \frac{A[\tilde{\beta}_1(x)]}{15} + \frac{A[\tilde{\beta_0}(x)]}{3}\right)j_0(x) \\ &\quad + A[\tilde{\beta}_0]j_{-2}(\omega) + \sum_{n=1}^{\infty}(-1)^n\alpha_n(x)j_{2n}(\omega) \end{aligned} \tag{2.33}$$

where

$$\alpha_n(x) = \tilde{\beta}_n(x) + A\left[\frac{\tilde{\beta}_{n+1}(x)}{(4n+3)(4n+5)} - \frac{2\tilde{\beta}_n(x)}{(4n-1)(4n+3)} + \frac{\tilde{\beta}_{n-1}(x)}{(4n-1)(4n-3)}\right]. \tag{2.34}$$

Now, we multiply equality (2.33) by $j_{2m}(\omega)$, $m = 1, 2, \ldots$, integrate with respect to ω over $(0, \infty)$ and utilize the orthogonality of the functions $j_{2m}(\omega)$. Using [**1**, Formula 11.4.34, p. 487] the following equality is obtained

$$\alpha_n(x) = A_n x^{l+1} q(x) {}_2\mathbf{F}_1\left(n-\frac{1}{2}, -(n+1); l+\frac{3}{2}; x^2\right). \tag{2.35}$$

Thus, using (2.34) and (2.35), equation (2.32) is obtained. □

Hence, as can be seen from Proposition 2.5, the construction of the functions $\tilde{\beta}_{n+1}$ for $n = 1, 2, \ldots$ reduces to solution of a recurrent sequence of non-homogeneous equations (2.32) of the following form

$$Au_n = h_n. \tag{2.36}$$

4.1. Formula for coefficients $\tilde{\beta}_n$ using the Green function. A regular solution u_n of (2.36) with the condition $u_n(x) \sim x^{l+1}$, $x \to 0$ can be obtained using the Green function method. We consider a fundamental set of solutions $\{u, v\}$ of the equation $Lu_n = 0$, therefore, a regular solution for (2.36) is given in the following form

$$u_n(x) = \int_0^x G(x,t) h_n(t) dt \tag{2.37}$$

and G is the well known Green function,

$$G(x,t) = \frac{v(x)u(t) - v(t)u(x)}{W(v,u)}, \qquad (x,t) \in (0,1) \times (0,1),$$

.

W is the Wronskian of v and u. The solutions v and u can be constructed by Picard's iteration method, where $\{u_0, v_0\}$ is a fundamental system of solutions for $L_0[u] = 0$ (when $q \equiv 0$),

$$u_0 = x^{l+1}, \qquad v_0(x) = x^{-l}$$

and $l > 0$. Let u and v be defined by

$$u(x) = \sum_{k \geq 0} \phi_k(x,q), \qquad v(x) = \sum_{k \geq 0} \psi_k(x,q)$$

where

$$\begin{cases} \phi_0(x,q) = u_0 \\ \phi_{k+1}(x,q) = \int_0^x G_0(x,t)\, q(t)\, \phi_k(q,t) dt, \ k \in \mathbb{N}; \end{cases}$$

$$\begin{cases} \psi_0(x,q) = v_0 \\ \psi_{k+1}(x,q) = -\int_1^x G_0(x,t)\, q(t)\, \psi_k(q,t)\, dt, \ k \in \mathbb{N}. \end{cases}$$

The function G_0 is the Green function for u_0 and v_0 defined by

$$G_0(x,t) = \frac{x^{-l} t^{l+1} - t^{-l} x^{l+1}}{2l+1}.$$

The series u and v uniformly converge on $[0,1]$ and on bounded sets of $(0,1]$ respectively, (see, e.g., [**46**]).

Remark 2.6. *The functions u and v are constructed in such a way that $u(x) \sim x^{l+1}$ and $v(x) \sim x^{-l}$ when $x \to 0$, therefore, the function (2.36) is solution provided that $|h_n(x)| \leq C x^{l-1+\epsilon}$ in a neighborhood of zero for some positive constants C and ϵ.*

The functions $\tilde{\beta}_n$ from equation (2.32) satisfy $|\tilde{\beta}_n(x)| \leq c_{n,1} x^{l+1}$, $\left|\tilde{\beta}_n'(x)\right| \leq c_{n,2} x^l$ and $\left|\tilde{\beta}_n''(x)\right| \leq c_{n,3} x^{l-1}$, $n \geq 1$ because the functions φ_n in (2.13) satisfy $|\varphi_n(x)| \leq c_{n,4} x^{2n+l+1}$, $|\varphi_n'(x)| \leq c_{n,5} x^{2n+l}$ and $|\varphi_n''(x)| \leq c_{n,6} x^{2n+l-1}$, $n \geq 0$, for some constants $c_{n,i}$.

Thus, one can see that a regular solution for (2.32) is given by

$$\frac{\tilde{\beta}_{n+1}(x)}{(4n+3)(4n+5)} - \frac{2\tilde{\beta}_n(t)}{(4n-1)(4n+3)} + \frac{\tilde{\beta}_{n-1}(t)}{(4n-1)(4n-3)} = \int_0^x G(x,t)\left(\alpha_n(t) - \tilde{\beta}_n(x)\right) dt$$

when $n \geq 1$, where

$$\int_0^x G(x,t)\alpha_n(t)dt = A_n \int_0^x G(x,t)t^{l+1}q(t)_2\mathbf{F}_1\left(n-\frac{1}{2}, -(n+1); l+\frac{3}{2}; t^2\right) dt.$$

Applying the fact that the hypergeometric function ${}_2\mathbf{F}_1\left(n-\frac{1}{2}, -(n+1); l+\frac{3}{2}; x^2\right)$ is reduced to a polynomial

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{\left(n-\frac{1}{2}\right)_k}{\left(l+\frac{3}{2}\right)_k} x^{2k}, \tag{2.38}$$

we obtain

$$\int_0^x G(x,t)\alpha_n(t)dt = A_n \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{\left(n-\frac{1}{2}\right)_k}{\left(l+\frac{3}{2}\right)_k} \int_0^x G(x,t)t^{l+1+2k}q(t)dt.$$

Therefore, the following recurrent formulas for the coefficients $\tilde{\beta}_{n+1}(x)$ are obtained

$$\tilde{\beta}_{n+1}(x) = \frac{2(4n+5)}{4n-1}\tilde{\beta}_n(x) - \frac{(4n+3)(4n+5)}{(4n-1)(4n-3)}\tilde{\beta}_{n-1}(x) + (4n+3)(4n+5)\tilde{\nu}(x), \tag{2.39}$$

where

$$\tilde{\nu}(x) = \int_0^x G(x,t)\left(A_n \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{\left(n-\frac{1}{2}\right)_k}{\left(l+\frac{3}{2}\right)_k} t^{l+1+2k}q(t) - \tilde{\beta}_n(t)\right) dt.$$

The coefficients $\tilde{\beta}_0$ and $\tilde{\beta}_1$ can be obtained from (2.13),

$$\tilde{\beta}_0(x) = \varphi_0(x) - \frac{\Gamma(1/2)}{\sqrt{\pi}}x^{l+1},$$

$$\tilde{\beta}_1(x) = -\frac{5}{2}\tilde{\beta}_0(x) + \frac{15}{2}\left(\varphi_1(x) - \frac{\Gamma(l+3/2)\Gamma(3/2)}{\sqrt{\pi}\Gamma(l+\frac{5}{2})}x^{l+3}\right).$$

On the other hand, the solution $\tilde{\beta}_{n+1}(x)$ of equation (2.32) can be obtained using the Pólya factorization of A, $Au = -\frac{1}{u_0}\partial u_0^2 \partial \frac{u}{u_0}$, where ∂ denotes the derivative with respect to x and u_0 is the solution of $Au = 0$. Following [**11**], in the next section we give the corresponding formulas for the coefficients $\tilde{\beta}_n$.

4.2. Formula for coefficients $\tilde{\beta}_n$ using the Polya factorization. The regular solution u_n of the equation (2.36) admits the form

$$u_n(x) = -u(t)\int_0^x \left(\frac{1}{u^2(x)}\int_0^s u(t)h_n(t)dt\right)ds \tag{2.40}$$

provided that $|h_n(t)| \leq Cx^{l-1+\varepsilon}$ in a neighborhood of zero for some positive C and ε where $u(t)$ is the solution of the equation $Au = 0$. Note that the expression (2.40) gives the unique solution of (2.36) satisfying $u(x) = o\left(x^{l+1}\right)$, $x \to 0$.

We know that the coefficients $\tilde{\beta}_n(x)$ satisfy the equation

$$A\left[\frac{\tilde{\beta}_{n+1}(x)}{(4n+3)(4n+5)} - \frac{2\tilde{\beta}_n(x)}{(4n-1)(4n+3)} + \frac{\tilde{\beta}_{n-1}(x)}{(4n-1)(4n-3)}\right] = -\tilde{\beta}_n(x) + \alpha_n(x)$$

then,

$$\begin{aligned}&\frac{\tilde{\beta}_{n+1}(x)}{(4n+3)(4n+5)} - \frac{2\tilde{\beta}_n(x)}{(4n-1)(4n+3)} + \frac{\tilde{\beta}_{n-1}(x)}{(4n-1)(4n-3)}\\ = \; & u(x)\int_0^x \frac{1}{u^2(s)}\int_0^s u(t)\left(\alpha_n(t) - \tilde{\beta}_n(t)\right)dtds.\end{aligned}$$

Using the formula (2.38) we obtain

$$\int_0^s u(t)\alpha_n(t)dt = A_n\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\frac{\left(n-\frac{1}{2}\right)_k}{\left(l+\frac{3}{2}\right)_k}\int_0^s u(t)\,t^{l+1+2k}q(t)\,dt.$$

Therefore, the following recurrent formulas for the coefficients $\tilde{\beta}_n$, $n \geq 2$ are valid

$$\tilde{\beta}_n(x) = (1-16n^2)\left(\frac{\tilde{\beta}_{n-2}(x)}{(4n-5)(4n-7)} - \frac{2\tilde{\beta}_{n-1}(x)}{(4n-5)(4n-1)} + u(x)\gamma_n(x)\right), \tag{2.41}$$

where

$$\gamma_n = \int_0^x \frac{\kappa_n(s) - \xi_n(s)}{u^2(s)}ds$$

and

$$\kappa_n(x) = -\frac{(-1)^n(2n-\frac{3}{2})\Gamma(n-\frac{3}{2})}{2\sqrt{\pi}\Gamma(n+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\left(n-\frac{3}{2}\right)_k}{\left(l+\frac{3}{2}\right)_k}\int_0^s u(t)\,t^{l+1+2k}q(t)\,dt,$$

$$\xi_n = \int_0^x u(s)\tilde{\beta}_{n-1}(s)ds.$$

Thus, the following statement is proved.

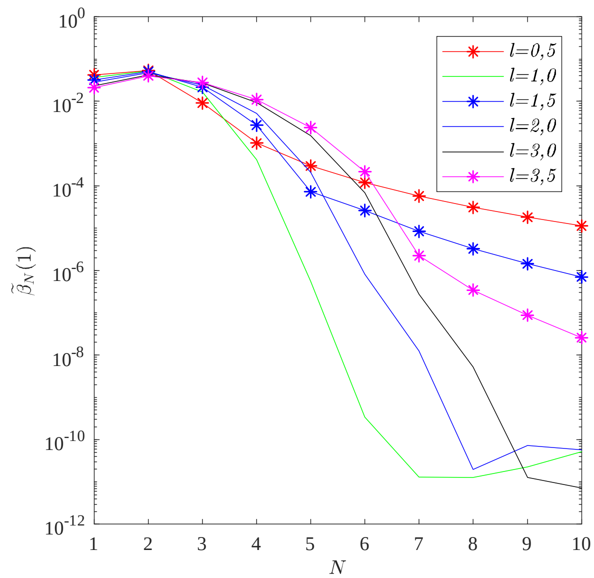

Figure 2.1: The coefficients $\tilde{\beta}_n(1)$ for different values of parameter l are illustrated.

PROPOSITION 2.7. *The coefficients $\tilde{\beta}_n$, $n = 2, 3, ...,$ in (2.12) satisfy the sequence of recurrent differential equations (2.36), satisfy the asymptotic properties in Remark 2.6, and can be obtained from the recursive formulas (2.39) or from formulas (2.41). The coefficients $\tilde{\beta}_n$ obtained by the formulas (2.39) and (2.41) are the same.*

4.3. Numerical results. Absolute values of the coefficients $\tilde{\beta}_n$ decrease as $n \to \infty$, see the estimate (2.15). Unfortunately the formulas (2.39) and (2.41) lead to rapid growth of the error even more than direct formulas (2.13). For the next numerical example we have computed functions $\tilde{\beta}_n$ in Matlab 2018. We compute the functions $\tilde{\beta}_n$ in 2001 points on the interval $[0, 1]$. The integrals of the formulas (2.39) and (2.41) were calculated using the modified six-point Newton-Cotes rule. This rule consists in interpolating the function values at six points by a fifth-order polynomial and using the integral of this polynomial as the approximation for the integral [**35**]. The m-file of this function was provided by Professor S. Torba.

EXAMPLE 2.8. *Consider $q(x) = x^2$ in the equation (2.1). On Figure 2.1 we present the plot of the decay of the values $|\beta_n(1)|$ vs. n for the first ten coefficients for several values of the parameter l. For this illustration we use the direct formula (2.13). As one can see from the graph, the absolute values $|\beta_n(1)|$ obey a power law decay whenever $l \notin \mathbb{N}$, and a faster than polynomial decay for $l \in \mathbb{N}$. On Figure 2.2 we illustrate that the decay rate order of the coefficients at x, such that $x < 1$, does not change when $l \notin \mathbb{N}$ or $l \in \mathbb{N}$.*

EXAMPLE 2.9. *Consider $q(x) = x^2$ and $l = \frac{1}{2}$ in the equation (2.1). On Figures 2.3, 2.4 we illustrate the coefficients β_N obtained by four different methods: The exact formula*

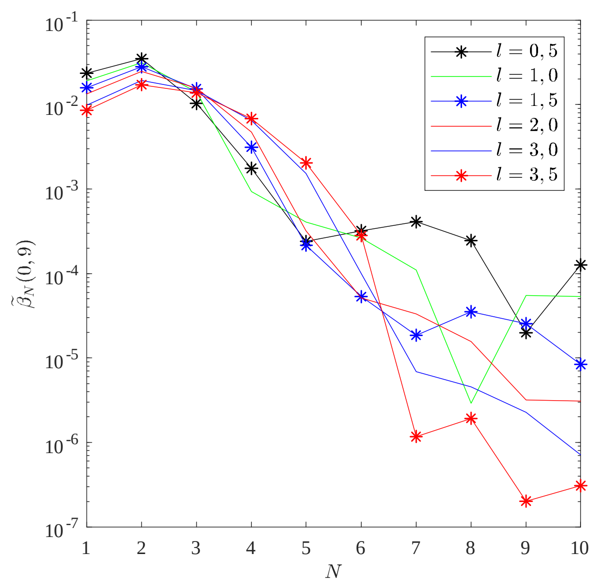

Figure 2.2: The coefficientes $\tilde{\beta}_n\,(0,9)$ are illustrated. In this case the decay rate order is the same both for values $l \in \mathbb{N}$ and $l \notin \mathbb{N}$.

defined in (2.13), Green function method, Polya factorization method and the recurrent formulas obtained in [**35**]. *We conclude that the method that allows to calculate a large number of coefficients is KTC obtained in* [**35**], *unfortunately none of the other methods present a computational advantage compared to this one.*

Example 2.10. *Consider* $q\,(x) = 1$ *and* $l = \frac{5}{2}$ *in the equation (2.1). An exact solution of this equation has the form*

$$u_l\,(\omega, x) = \frac{2^{l+\frac{1}{2}}\Gamma\left(l+\frac{3}{2}\right)x^{\frac{1}{2}}}{\left(\omega^2-1\right)^{\frac{l}{2}+\frac{1}{4}}} J_{l+\frac{1}{2}}\left(\sqrt{\omega^2-1}x\right). \tag{2.42}$$

It satisfies the asymptotic property $u_l\,(\omega, x) \sim x^{l+1}$, $x \to 0$. *On Figure 2.5 we illustrate the plot of the approximate solution* u_N, *for* $N = 1; 2; 5; 6$ *and the exact solution* u_l *defined in (2.42) for* $\omega = 15$. *Note that already when* $N = 6$, *the graph of* u_N *overlaps with the graph of* u_l. *On Figure 2.6 we illustrate the absolute error between the exact solution* u_l *and the approximate solution* u_N *with* $N = 10$. *We compute the coefficients with the direct formula obtained in (2.13). It should be noted that we can not improve the approximation numerically because the computation of more coefficients leads to a growth in the error.*

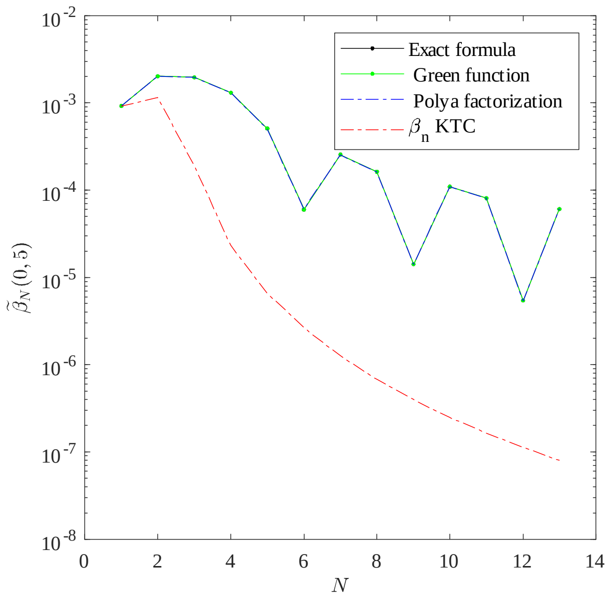

Figure 2.3: *The coefficientes β_n obtained by differents methods at $x = 0,5$ are illustred.*

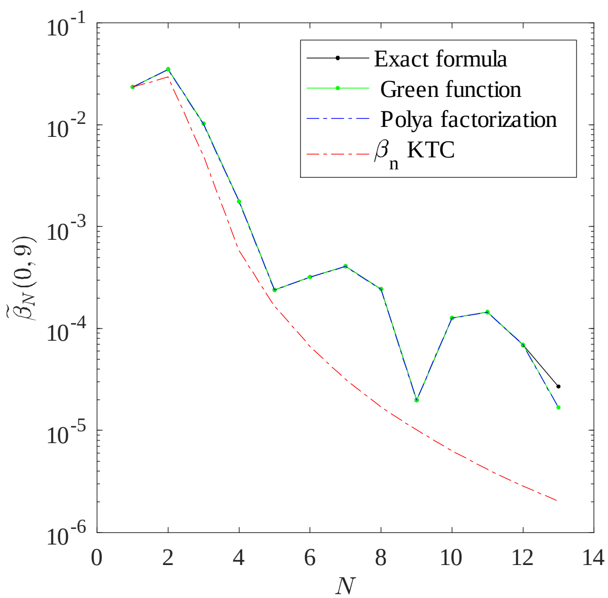

Figure 2.4: *The coefficientes β_n obtained by differents methods at $x = 0,9$ are illustred.*

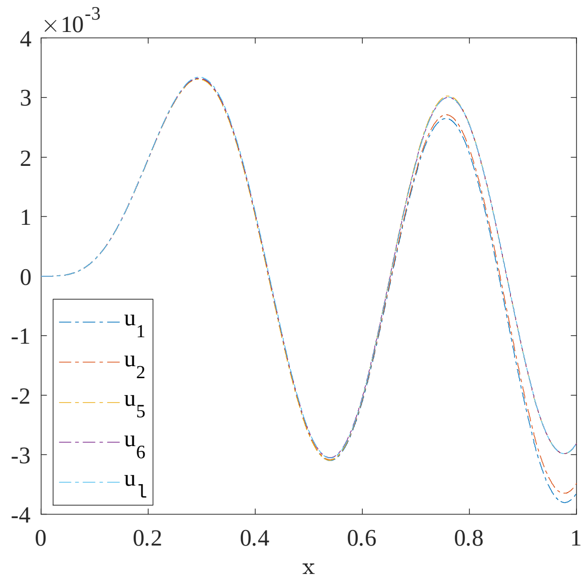

Figure 2.5: The approximate solution u_N converges rapidly to the exact solution u_l.

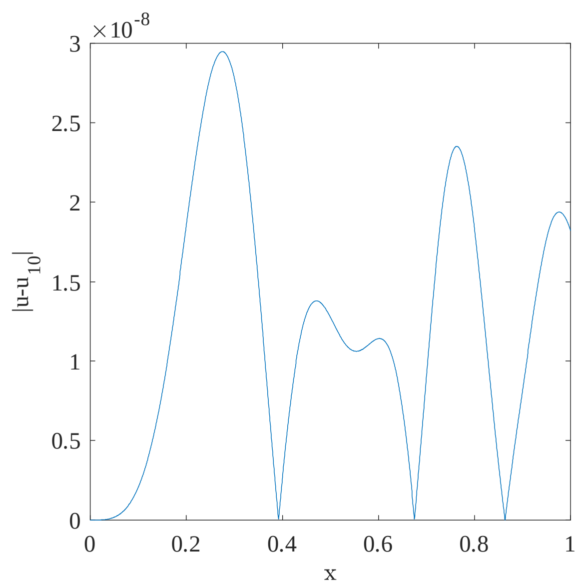

Figure 2.6: Illustration of the absolute error between the exact solution u_l and the approximate solution u_N, when $N = 10$.

CHAPTER 3

On transmutation operators and Neumann series of Bessel functions representation for solutions of linear higher order differential equations

Transmutation operators for linear differential equations of order $n > 2$ have been subject of dozens of publications reflecting the efforts of mathematicians to obtain a satisfactory generalization of this concept, well understood and developed in the case $n = 2$. We refer to a historical review of these efforts in [**47**] and [**21**]. Here we use an idea of such a transmutation operator for $n > 2$ developed in [**37**] and based on the Borel transform of entire functions. The formula for the inverse transform as was pointed out in [**37**] in fact represents a natural transmutation operator transmuting solutions of an elementary equation into solutions of the equation with variable coefficients. Moreover, as was noticed in [**37**, p. 59], this operator reduces to the usual transmutation operator in the case $n = 2$.

Based on the transmutation operator from [**37**] in the present section we obtain a representation of solutions of linear differential equations of order $n > 2$ in terms of Neumann series of Bessel functions. This is an extension of the recent results of [**28**] onto the case $n > 2$. The main result consists in a representation of a solution of a linear differential equation of the form

$$y^{(n)} + p_2(x)y^{(n-2)} + ... + p_n(x)y = \omega^n y,\ x \in (0, b)$$

as a sum of four Neumann series of Bessel functions. The formulas for the coefficients of the series are derived and an estimate for the approximation of the solution by the partial sums of the series is obtained. The result is obtained by representing in the form of a Fourier-Legendre series of the Borel transform γ of the solution y along the boundary of a square centered at the origin and containing all singularities of γ. Additionally we show that the representation obtained is applicable to numerical calculation.

1. A transmutation operator and a representation for the solution

Let $y(\omega, x)$ denote the solution of the following Cauchy problem

$$L[y] = y^{(n)} + p_2(x)y^{(n-2)} + ... + p_n(x)y = \omega^n y,\ x \in (0, b) \tag{3.1}$$

$$y(\omega, 0) = 1,\ y'(\omega, 0) = \omega, ...,\ y^{(n-1)}(\omega, 0) = \omega^{n-1} \tag{3.2}$$

where $p_2, ..., p_n$ are assumed to be complex valued continuous functions on $[0, b]$, $0 < b < \infty$ and $\omega \in \mathbb{C}$ is a spectral parameter.
The following two representations for the solution $y(\omega, x)$ will be used throughout the chapter.

THEOREM 3.1. [**37**, Sect. 3] *The solution $y(\omega, x)$ of the problem 3.1 and 3.2 admits the following spectral parameter power series (SPPS) representation*

$$y(\omega, x) = \sum_{m=0}^{\infty} \frac{\alpha_m(x)\omega^m}{m!} \tag{3.3}$$

where the coefficients α_m can be computed using the simple recursive integration procedure from [**37**, Sect. 3] *or from* [**30**]. *The series converges uniformly with respect to $x \in [0, b]$ and uniformly on any compact subset of the complex plane with respect to ω. The solution $y(\omega, x)$ admits the following representation*

$$y(\omega, x) = \frac{1}{2\pi i} \int_C \gamma(x, t) e^{\omega t} dt \tag{3.4}$$

where for any fixed $x \in [0, b]$ the function $\gamma(x, t)$ as a function of the variable t is analytic outside a regular n -sided polygon Π_x with center at the origin and one of whose vertices being the point $t = x$. C is a circle centered at the origin with a radius greater than x. Moreover, $\gamma(x, t)$ is continuous up to the boundary of the polygon Π_x.

REMARK 3.2. *The function $\gamma(x, t)$ is nothing but the Borel transform of the function $y(\omega, x)$ which is entire with respect to ω (see, e.g.,* [**5**] *and* [**37**]*).*

REMARK 3.3. *Instead of (3.4) the following representation can be used as well*

$$y(\omega, x) = e^{\omega x} + \frac{1}{2\pi i} \int_C \tilde{\gamma}(x, t) e^{\omega t} dt$$

where the function $\tilde{\gamma}(x, t)$ enjoys the same properties as those of the function γ, formulated in the previous theorem.
Both integral representations can be regarded as transmutations of the solution of the elementary Cauchy problem

$$\nu^{(n)} = \omega^n \nu, \qquad x \in (0, b),$$
$$\nu(\omega, 0) = 1,\ \nu'(\omega, 0) = \omega, ..., \nu^{(n-1)}(\omega, 0) = \omega^{n-1}$$

into the solution of (3.1), (3.2).

PROPOSITION 3.4. *Outside Π_x the function $\gamma(x, t)$ admits the following series representation*

$$\gamma(x, t) = \sum_{m=0}^{\infty} \frac{\alpha_m(x)}{t^{m+1}} \tag{3.5}$$

where α_m are the coefficients from (3.3).

PROOF. Let C be a circle centered at the origin with a radius greater than x. Then outside C the function $\gamma(x,t)$ admits the representation

$$\gamma(x,t)=\sum_{m=0}^{\infty}\frac{b_m(x)}{t^{m+1}}.$$

Substitution of this series into (3.4) gives us the equalities

$$2\pi i y(\omega,x)=\sum_{m=0}^{\infty}b_m(x)\int_C\frac{e^{\omega t}}{t^{m+1}}dt=2\pi i\sum_{m=0}^{\infty}b_m(x)\frac{\omega^m}{m!}.$$

From here and from (3.3) we obtain (3.5). ☐

In what follows by C_δ we denote the counterclockwise oriented boundary of the square centered at the origin and with one of the vertex being $t_1=(x+\delta)+i(x+\delta)$ with $\delta>0$.

THEOREM 3.5. *The solution $y(\omega,x)$ of the problem (3.1), (3.2) admits the following representation*

$$\begin{aligned} y(\omega,x) \;=\; & \sum_{k=1}^{\infty}\frac{\alpha_{k-1}(x)}{2\pi i}\Bigg(e^{-i\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\frac{e^{\omega\tau}}{(\tau-i(x+\delta))^k}d\tau \\ & + ie^{\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\frac{(-i)^k e^{i\omega\tau}}{(\tau-i(x+\delta))^k}d\tau-e^{i\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\frac{(-1)^k e^{-\omega\tau}}{(\tau-i(x+\delta))^k}d\tau \\ & - ie^{-\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\frac{i^k e^{-i\omega\tau}}{(\tau-i(x+\delta))^k}d\tau\Bigg) \end{aligned} \tag{3.6}$$

where for any ω and δ fixed each of the four series converges absolutely and uniformly for $x\in[0,b]$.

PROOF. From (3.4) by a natural parametrization of contour C_δ we obtain the equality

$$\begin{aligned} y(\omega,x)=e^{-i\omega(x+\delta)}&\int_{-x-\delta}^{x+\delta}\gamma_1(x,\tau)e^{\omega\tau}d\tau+ie^{\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\gamma_2(x,\tau)e^{i\omega\tau}d\tau \\ -e^{i\omega(x+\delta)}&\int_{-x-\delta}^{x+\delta}\gamma_3(x,\tau)e^{-\omega\tau}d\tau-ie^{-\omega(x+\delta)}\int_{-x-\delta}^{x+\delta}\gamma_4(x,\tau)e^{-i\omega\tau}d\tau \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \gamma_1(x,\tau) &:= \frac{1}{2\pi i}\gamma(x,\tau-i(x+\delta)),\\ \gamma_2(x,\tau) &:= \frac{1}{2\pi i}\gamma(x,(x+\delta)+i\tau),\\ \gamma_3(x,\tau) &:= \frac{1}{2\pi i}\gamma(x,-\tau+i(x+\delta)),\\ \gamma_4(x,\tau) &:= \frac{1}{2\pi i}\gamma(x,-(x+\delta)-i\tau), \end{aligned}$$

for $\tau \in [-x-\delta, x+\delta]$. Due to Proposition 3.4 for the functions γ_j we have the following series representations

$$\begin{aligned}\gamma_1(x,\tau) =& \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}}, \quad \gamma_2(x,\tau) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-i)^{k+1}\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}},\\ \gamma_3(x,\tau) =& \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}}, \quad \gamma_4(x,\tau) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{i^{k+1}\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}}.\end{aligned} \tag{3.8}$$

Let us prove that for any fixed $x \in [0,b]$ the series converge absolutely and uniformly with respect to τ on $[-x-\delta, x+\delta]$.
Consider the series for $\gamma_1(x,\cdot)$. We have

$$\left|\frac{\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}}\right| = \frac{|\alpha_k(x)|}{\left(\tau^2 + (x+\delta)^2\right)^{\frac{k+1}{2}}} \le \frac{|\alpha_k(x)|}{(x+\delta)^{k+1}}.$$

Therefore,

$$\sum_{k=0}^{\infty}\left|\frac{\alpha_k(x)}{\left(\tau - i(x+\delta)\right)^{k+1}}\right| \le \sum_{k=0}^{\infty}\frac{|\alpha_k(x)|}{(x+\delta)^{k+1}}. \tag{3.9}$$

Since for any $x \in [0,b]$, $y(\omega,x)$ is an entire function with respect to ω of order one and exponential type x [**37**, Sect. 3], we have that $x = \limsup_{k\to\infty}|y^{(k)}(\omega,x)|^{1/k}$ where ω is any fixed complex number, see [**5**, Chap. 2, Formula 2.2.12]. From the series representation (3.3) it is obtained that $|\alpha_k(x)| = |y^{(k)}(0,x)|$ and so $\limsup_{k\to\infty}|\alpha_k(x)|^{1/k} = x$. Therefore, by Cauchy's criterion the series on the right hand side of (3.9) converges for any $x \in [0,b]$. By Weierstrass M-test, we conclude that for any $x \in [0,b]$ the series in (3.8) are uniformly absolutely convergent on $[-x-\delta, x+\delta]$. Substitution of the function series for γ_j into equality (3.7) leads then to the representation (3.6).
In order to prove that the function series (3.6) converge absolutely for all x we first obtain the following estimates

$$\begin{aligned}\left|\int_{-x-\delta}^{x+\delta}\frac{e^{\omega\tau - i\omega(x+\delta)}}{(\tau - i(x+\delta))^k}d\tau\right| &\le 2e^{\operatorname{Im}(\omega)(x+\delta)}\frac{\sinh\left(\operatorname{Re}(\omega)(x+\delta)\right)}{\operatorname{Re}(\omega)(x+\delta)^k},\\ \left|\int_{-(x+\delta)}^{(x+\delta)}\frac{e^{i\omega\tau + \omega(x+\delta)}}{(\tau - i(x+\delta))^k}d\tau\right| &\le 2e^{\operatorname{Re}(\omega)(x+\delta)}\frac{\sinh(\operatorname{Im}(\omega)(x+\delta))}{\operatorname{Im}(\omega)(x+\delta)^k},\\ \left|\int_{-x-\delta}^{x+\delta}\frac{e^{-\omega\tau + i\omega(x+\delta)}}{(\tau - i(x+\delta))^k}d\tau\right| &\le 2e^{-\operatorname{Im}(\omega)(x+\delta)}\frac{\sinh(\operatorname{Re}(\omega)(x+\delta))}{\operatorname{Re}(\omega)(x+\delta)^k},\\ \left|\int_{-(x+\delta)}^{(x+\delta)}\frac{e^{-i\omega\tau - \omega(x+\delta)}}{(\tau - i(x+\delta))^k}d\tau\right| &\le 2e^{-\operatorname{Re}(\omega)(x+\delta)}\frac{\sinh(\operatorname{Im}(\omega)(x+\delta))}{\operatorname{Im}(\omega)(x+\delta)^k}.\end{aligned}$$

Considering the first of them we observe that the function $\frac{2e^{\operatorname{Im}(\omega)(x+\delta)}\sinh(\operatorname{Re}(\omega)(x+\delta))}{\operatorname{Re}(\omega)}$ attains a maximum on the interval $[0,b]$. Denote $c_1(\omega,\delta) := \max_{x\in[0,b]}\frac{2e^{\operatorname{Im}(\omega)(x+\delta)}\sinh(\operatorname{Re}(\omega)(x+\delta))}{\operatorname{Re}(\omega)}$.

Hence we have the following estimate for the first series

$$\sum_{k=1}^{\infty}\left|\alpha_{k-1}(x)\int_{-x-\delta}^{x+\delta}\frac{e^{\omega\tau-i\omega(x+\delta)}}{(\tau-i(x+\delta))^k}d\tau\right|\le c_1(\omega,\delta)\sum_{k=1}^{\infty}\frac{|\alpha_{k-1}(x)|}{(x+\delta)^k}.$$

The series on the right converges for all $x\in[0,b]$ because $\limsup|\alpha_{k-1}(x)|^{1/k}=x$. The convergence of the other three series is proved in a similar way. □

2. A Fourier-Legendre series representation for the kernels γ_j

Denote $c_{1,k}=1$, $c_{2,k}=(-i)^k$, $c_{3,k}=(-1)^k$ and $c_{4,k}=i^k$. Then the kernels γ_j, $j=1,2,3,4$ can be written as follows

$$\gamma_j(x,\tau)=\frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{c_{j,k}\alpha_{k-1}(x)}{(\tau-i(x+\delta))^k}. \tag{3.10}$$

Proposition 3.6. *The functions $\gamma_j(x,\tau)$ admit the following representations*

$$\gamma_j(x,\tau)=\sum_{n=0}^{\infty}\frac{\beta_n^{(j)}(x;\delta)}{x+\delta}P_n\left(\frac{\tau}{x+\delta}\right) \tag{3.11}$$

where P_n stands for the Legendre polynomial of order n, and the coefficients $\beta_n^{(j)}$ are defined by the equality

$$\beta_n^{(j)}(x;\delta)=b_n\sum_{k=1}^{\infty}\frac{(-1)^k c_{k,j}\alpha_{k-1}(x)\,(k)_n\,(1-i)^{n+k}}{(x+\delta)^{k-1}2^k}\,{}_2F_1\left(n+k,n+1;2\,(n+1)\,;\frac{2}{1+i}\right)$$

where $b_n=\frac{(2n+1)}{2\pi i(n+1)_{n+1}}$, $(k)_n$ is the Pochhammer symbol, and ${}_2F_1$ is the Gauss hypergeometric function.
The series in (3.11) converges uniformly and for any $x\in[0,b]$ the following estimate is valid

$$|\beta_n^{(j)}(x;\delta)|\le\sqrt{2n+1}(x+\delta)^{1/2}\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta,x+\delta]}. \tag{3.12}$$

Proof. Since $\gamma(x,t)$ as a function of the variable t is analytic outside Π_x the functions $\gamma_j(x,\cdot)$ admit the uniformly convergent Fourier-Legendre series representations of the form (3.11). Multiplying (3.11) by $P_m\left(\frac{\tau}{x+\delta}\right)$ and integrating we obtain

$$\frac{2\beta_n^{(j)}(x;\delta)}{2n+1}=\int_{-x-\delta}^{x+\delta}\gamma_j(x,\tau)P_n\left(\frac{\tau}{x+\delta}\right)d\tau.$$

Now substitution of (3.10) leads to the equalities

$$\begin{aligned}\beta_n^{(j)}(x;\delta) &= \frac{2n+1}{4\pi i}\sum_{k=1}^{\infty} c_{j,k}\alpha_{k-1}(x)\int_{-x-\delta}^{x+\delta}\frac{P_n\left(\frac{\tau}{x+\delta}\right)}{\left(\tau-i(x+\delta)\right)^k}d\tau\\ &= \frac{2n+1}{4\pi i}\sum_{k=1}^{\infty}\frac{c_{j,k}\alpha_{k-1}(x)}{(x+\delta)^{k-1}}\int_{-1}^{1}\frac{P_n(s)}{(s-i)^k}ds\\ &= b_n 2^n\sum_{k=1}^{\infty}\frac{(-1)^k c_{j,k}\alpha_{k-1}(x)\,(k)_n}{(x+\delta)^{k-1}(1+i)^{n+k}}\,{}_2F_1\left(n+k,n+1;2(n+1);\frac{2}{1+i}\right)\end{aligned}$$

where for calculating the integrals

$$p_{n,k} := \int_{-1}^{1}\frac{P_n(s)}{(s-i)^k}ds$$

formula 2.17.1 (12) from [**45**] was used.
The estimate (3.12) follows from the Cauchy-Schwarz inequality,

$$\begin{aligned}|\beta_n^{(j)}(x;\delta)| =& \frac{2n+1}{2}\left|\int_{-x-\delta}^{x+\delta}\gamma_j(x,\tau)P_n\left(\frac{\tau}{x+\delta}\right)d\tau\right|\\ \le& \frac{2n+1}{2}\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta,x+\delta]}\left\|P_n\left(\frac{\tau}{x+\delta}\right)\right\|_{L_2[-x-\delta,x+\delta]}\\ =& \sqrt{2n+1}(x+\delta)^{1/2}\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta,x+\delta]}.\end{aligned}$$

□

3. The Neumann series of Bessel functions representation of the solution

THEOREM 3.7. *The solution $y(\omega,x)$ of (3.1), (3.2) admits the representation*

$$y(\omega,x) = 2\sum_{n=0}^{\infty}\left(\beta_n^{(1)}(x;\delta)e^{-i\omega(x+\delta)} + (-1)^{n+1}\beta_n^{(3)}(x;\delta)e^{i\omega(x+\delta)}\right)i_n(\omega(x+\delta))$$

$$+2\sum_{n=0}^{\infty}\left(i^{n+1}\beta_n^{(2)}(x;\delta)e^{\omega(x+\delta)} + (-i)^{n+1}\beta_n^{(4)}(x;\delta)e^{-\omega(x+\delta)}\right)j_n(\omega(x+\delta)) \tag{3.13}$$

with the coefficients $\beta_n^{(k)}$ from Proposition 3.6. Here j_n stands for the spherical Bessel function of the first kind of order n, and i_n is the modified spherical Bessel function of the first kind of order n (see the definition, e.g., in [**42**, chap. 10]*).*
For the approximate solution
(3.14)

$$y_M(\omega,x) = 2\sum_{n=0}^{M}\left(\beta_n^{(1)}(x;\delta)e^{-i\omega(x+\delta)} + (-1)^{n+1}\beta_n^{(3)}(x;\delta)e^{i\omega(x+\delta)}\right)i_n(\omega(x+\delta))$$

$$+2\sum_{n=0}^{M}\left(i^{n+1}\beta_n^{(2)}(x;\delta)e^{\omega(x+\delta)} + (-i)^{n+1}\beta_n^{(4)}(x;\delta)e^{-\omega(x+\delta)}\right)j_n(\omega(x+\delta))$$

the following estimate holds

(3.15)

$$|y(\omega,x)-y_M(\omega,x)| <\frac{2\sqrt{\pi}\left(M+\frac{1}{2}+2e\right)}{\Gamma(M+\frac{3}{2})}(x+\delta)\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta,x+\delta]}\sqrt{|\omega|}$$
$$\left(\cosh((x+\delta)\operatorname{Im}(\omega))e^{(x+\delta)|\operatorname{Re}(\omega)|}+\cosh((x+\delta)\operatorname{Re}(\omega))e^{(x+\delta)|\operatorname{Im}(\omega)|}\right).$$

PROOF. Substitution of (3.11) into (3.7) with the aid of [**42**, formula 18.17.19] leads to (3.13).
Consider

$$\begin{aligned}|y(\omega,x)-y_M(\omega,x)| \leq & 2\sum_{n=0}^{\infty}\left|\beta^{(1)}_{n+M+1}(x;\delta)e^{-i\omega(x+\delta)}\right|\left|i_{n+M+1}(\omega(x+\delta))\right| \\ &+2\sum_{n=0}^{\infty}\left|\beta^{(3)}_{n+M+1}(x;\delta)e^{i\omega(x+\delta)}\right|\left|i_{n+M+1}(\omega(x+\delta))\right| \\ &+2\sum_{n=0}^{\infty}\left|\beta^{(2)}_{n+M+1}(x;\delta)e^{\omega(x+\delta)}\right|\left|j_{n+M+1}(\omega(x+\delta))\right| \\ &+2\sum_{n=0}^{\infty}\left|\beta^{(4)}_{n+M+1}(x;\delta)e^{-\omega(x+\delta)}\right|\left|j_{n+M+1}(\omega(x+\delta))\right|.\end{aligned}$$

Since

$$|J_\nu(z)|\leq\frac{|z|e^{|Im(z)|}}{2\Gamma(\nu+1)},\quad \left(\nu\geq-\frac{1}{2}\right)$$

(see formula 9.1.62 from [**1**]) we get

$$\begin{aligned}\left|j_{M+1+n}(\omega(x+\delta))\right| &\leq \frac{\sqrt{\pi(x+\delta)}\sqrt{|\omega|}e^{(x+\delta)|\operatorname{Im}(\omega)|}}{2\sqrt{2}\Gamma(M+\frac{3}{2}+n)}, \\ \left|i_{M+1+n}(\omega(x+\delta))\right| &\leq \frac{\sqrt{\pi(x+\delta)}\sqrt{|\omega|}e^{(x+\delta)|\operatorname{Re}(\omega)|}}{2\sqrt{2}\Gamma(M+\frac{3}{2}+n)}.\end{aligned}$$

Using (3.12) we obtain

$$\begin{aligned}|y(\omega,x)-y_M(\omega,x)| \leq & 2\sqrt{\pi}(x+\delta)\left\|\gamma_j(x,\tau)\right\|_{L_2[-x-\delta,x+\delta]}\sqrt{|\omega|} \\ &\times\left(\cosh((x+\delta)\operatorname{Im}(\omega))e^{(x+\delta)|\operatorname{Re}(\omega)|}+\cosh((x+\delta)\operatorname{Re}(\omega))e^{(x+\delta)|\operatorname{Im}(\omega)|}\right) \\ &\times\sum_{n=0}^{\infty}\frac{\sqrt{M+n+3/2}}{\Gamma(M+n+\frac{3}{2})}.\end{aligned}$$

Notice that the series on the right hand side admits the following chain of relations

$$\sum_{n=0}^{\infty}\frac{\sqrt{M+n+\frac{3}{2}}}{\Gamma(M+n+\frac{3}{2})}\leq\sum_{n=0}^{\infty}\frac{M+n+\frac{3}{2}}{\Gamma(M+n+\frac{3}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{M+n+\frac{1}{2}}{\Gamma(M+n+\frac{3}{2})} + \sum_{n=0}^{\infty} \frac{1}{\Gamma(M+n+\frac{3}{2})}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\Gamma(M+n+\frac{1}{2})} + \sum_{n=0}^{\infty} \frac{1}{\Gamma(M+n+\frac{3}{2})}$$

$$= \frac{1}{\Gamma(M+\frac{1}{2})} + 2\sum_{n=0}^{\infty} \frac{1}{\Gamma(M+n+\frac{3}{2})}.$$

Note that

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(M+n+\frac{3}{2})} = \frac{1}{\Gamma(M+\frac{3}{2})} \sum_{n=0}^{\infty} \frac{1}{\left(M+\frac{3}{2}\right)_n},$$

and since $\frac{1}{(1)_n} > \frac{1}{\left(M+\frac{3}{2}\right)_n}$ for any $M \geq 0$, we have that

$$\sum_{n=0}^{\infty} \frac{1}{\left(M+\frac{3}{2}\right)_n} < \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{\sqrt{M+n+\frac{3}{2}}}{\Gamma(M+n+\frac{3}{2})} < \frac{1}{\Gamma(M+\frac{1}{2})} + \frac{2e}{\Gamma(M+\frac{3}{2})} = \frac{M+\frac{1}{2}+2e}{\Gamma(M+\frac{3}{2})}.$$

□

4. An example of application to fourth order ordinary differential equations

In this section we give some numerical illustrations of the representation (3.13) in the case when the equation of the problem (3.1), (3.2) has the following form

$$y^{(4)} + (p(x)y')' = \omega^4 y, \quad x \in (0,b)\,, \tag{3.16}$$

$$y(\omega,0) = 1, \quad y'(\omega,0) = \omega, \quad y''(\omega,0) = \omega^2, \quad y'''(\omega,0) = \omega^3 \tag{3.17}$$

where $p \in C^1[0,b]$. We chose this particular equation due to the fact that for the construction of corresponding formal powers and hence of the SPPS representation of the solution an especially simple procedure was developed in [**24**].

4.1. The SPPS representation. Let f be a particular solution (in general a complex-valued one) of the equation $v'' + pv = 0$ such that f and $1/f \in C[0,b]$. Then following [**24**], we define a system of recursive integrals where $k = 0, 1, 2, ...,$

$$X_1^{(0)} \equiv X_2^{(0)} \equiv X_3^{(0)} \equiv X_4^{(0)} \equiv 1$$

$$X_1^{(m)}(x) = \begin{cases} m\displaystyle\int_0^x X_1^{(m-1)}(s) f(s) ds, & m = 4k+4, \\ m\displaystyle\int_0^x X_1^{(m-1)}(s) \tfrac{1}{f^2(s)} ds, & m = 4k+3, \\ m\displaystyle\int_0^x X_1^{(m-1)}(s) f(s) ds, & m = 4k+2, \\ m\displaystyle\int_0^x X_1^{(m-1)}(s) ds, & m = 4k+1, \end{cases}$$

$$X_2^{(m)}(x) = \begin{cases} m\displaystyle\int_0^x X_2^{(m-1)}(s) \tfrac{1}{f^2(s)} ds, & m = 4k+4, \\ m\displaystyle\int_0^x X_2^{(m-1)}(s) f(s) ds, & m = 4k+3, \\ m\displaystyle\int_0^x X_2^{(m-1)}(s) ds, & m = 4k+2, \\ m\displaystyle\int_0^x X_2^{(m-1)}(s) f(s) ds, & m = 4k+1, \end{cases}$$

$$X_3^{(m)}(x) = \begin{cases} m\displaystyle\int_0^x X_3^{(m-1)}(s) f(s) ds, & m = 4k+4, \\ m\displaystyle\int_0^x X_3^{(m-1)}(s) ds, & m = 4k+3, \\ m\displaystyle\int_0^x X_3^{(m-1)}(s) f(s) ds, & m = 4k+2, \\ m\displaystyle\int_0^x X_3^{(m-1)}(s) \tfrac{1}{f^2(s)} ds, & m = 4k+1, \end{cases}$$

$$X_4^{(m)}(x) = \begin{cases} m\displaystyle\int_0^x X_4^{(m-1)}(s) ds, & m = 4k+4, \\ m\displaystyle\int_0^x X_4^{(m-1)}(s) f(s) ds, & m = 4k+3, \\ m\displaystyle\int_0^x X_4^{(m-1)}(s) \tfrac{1}{f^2(s)} ds, & m = 4k+2, \\ m\displaystyle\int_0^x X_4^{(m-1)}(s) f(s) ds, & m = 4k+1. \end{cases}$$

PROPOSITION 3.8. *The solution of the problem (3.16), (3.17) admits the following representation*

$$y(\omega, x) = \sum_{m=0}^{\infty} \frac{\alpha_m(x)}{m!} \omega^m \tag{3.18}$$

where $\alpha_0 \equiv 1$,

$$\alpha_m(x) = \begin{cases} \frac{1}{f(0)} X_2^{(m)}(x) - f'(0)\frac{X_3^{(m+1)}(x)}{(m+1)} - \frac{f''(0)}{f(0)}\frac{X_4^{(m+2)}(x)}{(m+1)(m+2)}, & m = 4k+1, \\ f(0)X_3^{(m)}(x), & m = 4k+2, \\ X_4^{(m)}(x), & m = 4k+3, \\ X_1^{m}(x), & m = 4k+4, \end{cases} \tag{3.19}$$

$k = 0, 1, \ldots$ *and the series (3.18) converges uniformly with respect to* x *on* $[0, b]$ *and with respect to* ω *on any compact subset of the complex plane.*

PROOF. The proof consists in application of Theorem 3 from [**24**] to the problem (3.16), (3.17). The solution of (3.16), (3.17) is obtained as a linear combination of the four linearly independent solutions from [**24**],

$$u_1(x) = \sum_{m=0}^{\infty} \frac{X_1^{(4m)}(x)}{(4m)!}\omega^{4m} \quad u_2(x) = \sum_{m=0}^{\infty} \frac{X_2^{(4m+1)}(x)}{(4m+1)!}\omega^{4m}$$

$$u_3(x) = \sum_{m=0}^{\infty} \frac{X_3^{(4m+2)}(x)}{(4m+2)!}\omega^{4m} \quad u_4(x) = \sum_{m=0}^{\infty} \frac{X_4^{(4m+3)}(x)}{(4m+3)!}\omega^{4m}.$$

The initial conditions (3.17) are fulfilled by a linear combination of these solutions with the corresponding constants chosen as

$$c_1 = 1, \quad c_2 = \frac{\omega}{f(0)}, \quad c_3 = \omega^2 f(0) - \omega f'(0), \quad c_4 = \omega^3 - \omega\frac{f''(0)}{f(0)}.$$

□

4.2. Numerical illustrations. By $\beta_{n,K}^j$ we denote the approximation of the coefficient β_n^j for $j = 1, 2, 3, 4$, defined by

$$\beta_{n,K}^j(x;\delta) := \frac{2n+1}{4\pi i}\sum_{k=1}^{K} \frac{c_{j,k}\alpha_{k-1}(x)}{(x+\delta)^{k-1}} \int_{-1}^{1} \frac{P_n(s)}{(s-i)^k} ds. \tag{3.20}$$

The coefficients α_k defined by (3.19) are computed using the numerical integration approach explained in [**32**, Section 7], although other approaches (for example, that based on splines [**10**] though considerably slower is also applicable). We emphasize that the computation of a couple of hundreds of the coefficients α_k does not represent any difficulty and can be performed with a remarkable accuracy. The integrals in (3.20) were computed numerically as well, where $n = 0, 1, 2, ..., N$ and N is the number of coefficients used for calculating the approximate solution.

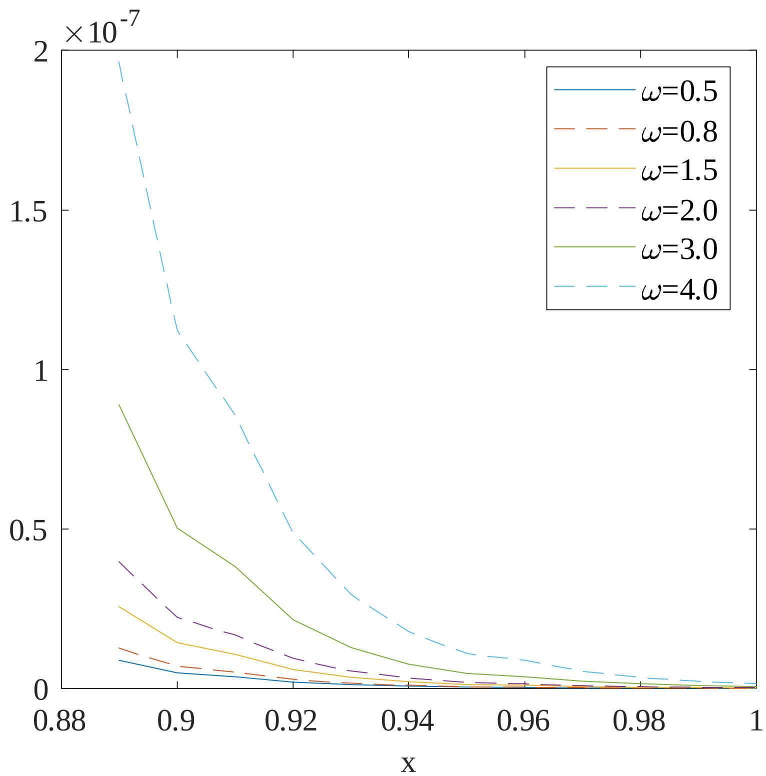

Figure 3.1: Absolute error (3.15) of the approximate solution y_M defined in (3.14) corresponding to the value $\delta = 0$. For the fourth order equation (3.21).

4.2.1. *Solution of initial value problems.*

EXAMPLE 3.9. *Consider the equation*

$$y^{(4)} + 2y'' = \omega^4 y \tag{3.21}$$

subject to conditions (3.17). Figure 3.1 shows the absolute error (3.15) of the approximate solution (3.14) corresponding to the value $\delta = 0$ and computed for several distinct values of ω.

It shows that meanwhile for the end point of the interval the accuracy is reasonably good it deteriorates rapidly for the values of x closer to the origin, and the representation (3.13) is not applicable in the vicinity of $x = 0$. An explanation of this is that the uniform absolute convergence of series (3.8) does not hold with respect to the τ when $x = 0$ and $\delta = 0$.

EXAMPLE 3.10. *The situation is completely different for $\delta > 0$ as shown on Figure 3.2 where the absolute error of the approximate solution is depicted in the case $\delta = 1$. The solution is sufficiently accurate on the whole interval, the accuracy deteriorates to the*

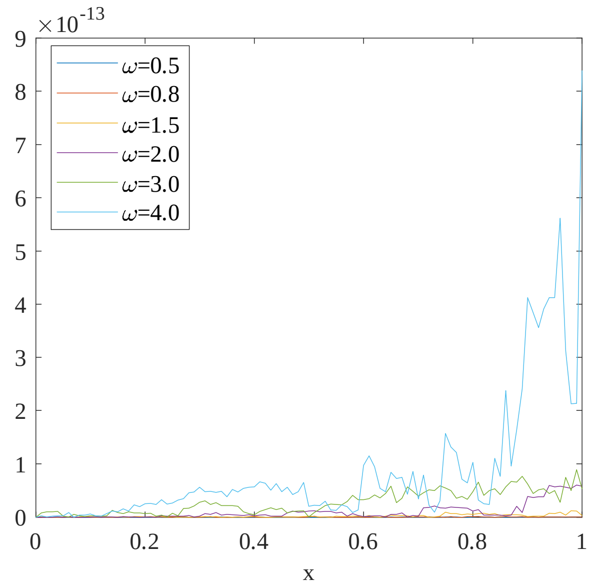

Figure 3.2: *The absolute error of the approximate solution (3.14) computed with $\delta = 1$ for several values of ω.*

right end of the interval, and according to the estimate from Theorem 3.7 the accuracy is better for smaller values of $|\omega|$.

4.2.2. *Solution of eigenvalue problems.*

EXAMPLE 3.11. *Consider the following eigenvalue problem*

$$\begin{aligned} u^{(4)} &= w^4 u(x), \\ u(\omega, 0) &= 0 = u''(\omega, 0) \quad u(\omega, 1) = 0 = u'(\omega, 1). \end{aligned}$$

The exact characteristic equation of the problem has the form

$$(e^{\omega} - e^{-\omega}) \cos(\omega) = (e^{\omega} + e^{-\omega}) \sin(\omega). \tag{3.22}$$

In terms of the solution $y(\omega, x)$ satisfying conditions (3.17) the dispersion equation can be written as follows

$$\begin{aligned} &y(-\omega,1)y'(\omega,1) - y(\omega,1)y'(-\omega,1) + y(\omega,1)y'(-i\omega,1) - y(-i\omega,1)y'(\omega,1) \\ &+ y(-i\omega,1)y'(-\omega,1) - y(-\omega,1)y'(-i\omega,1) + y(i\omega,1)y'(-i\omega,1) - y(-i\omega,1)y'(i\omega,1) = 0. \end{aligned} \tag{3.23}$$

The first six eigenvalues computed by solving this equation with the aid of the representation (3.13) are presented in the table below together with their corresponding absolute errors. The eigenvalues increase rapidly, and similarly to the SPPS [**24**] the representation (3.13) allows one accurate computation of several lower index eigenvalues.

$N=40,\quad K=150\quad$ and $\;\delta=0.25$				$N=180$
n	NSBF	Absolute error	Relative error	E(SPPS) from [**24**]
0	237.721067584372	5.3×10^{-8}	2.2×10^{-10}	1.1×10^{-13}
1	2496.48744387544	6.0×10^{-6}	2.4×10^{-9}	2.1×10^{-13}
2	10867.5824450557	2.2×10^{-4}	2.1×10^{-8}	1.0×10^{-11}
2	31780.0997391812	3.2×10^{-3}	1.0×10^{-7}	2.7×10^{-10}
4	74000.9108835559	6.1×10^{-2}	8.3×10^{-7}	1.9×10^{-9}
5	148630.254375198	4.2	2.8×10^{-5}	9.7×10^{-7}

Table 1: The first six eigenvalues computed for the problem defined in Example 3.11. The columns Absolute error and Relative error present absolute error and relative error between the eigenvalues obtained by the equation (3.22) and by the equation (3.23). The last column presents the absolute error between the eigenvalues obtained by the equation (3.22) and by the SPPS method.

Application to order $n>4$

Example 3.12. *Consider the following equation of order five*

$$y^{(5)}(x)+y(x)=\omega^5 y(x),\quad x\in(0,1)$$

with the initial conditions (3.2). Here we compute the coefficients $A_m(x)=\frac{\alpha_m(x)}{m!}$ *using the method from* [**37**, p. 31]. *Thus, the coefficients are defined by the formula*

$$A_{m+5}(x)=\sum_{k=1}^{5} y_k(x)(-1)^{k+1}\int_0^x A_m(t)\frac{\Delta_k(t)}{\Delta(t)}dt$$

where $y_1,...,y_5$ *is a fundamental system of the equation* $y^{(5)}(x)+y(x)=0$, $\Delta(x)$ *is their Wronskian and* $\Delta_k(x)$ *is the cofactor of the* $(5,k)$ *entry of* $\Delta(x)$.
The absolute error of the approximate solution computed with the aid of (3.13) with $\delta=1$ *is reported on Figure 3.3 for several different values of* ω.

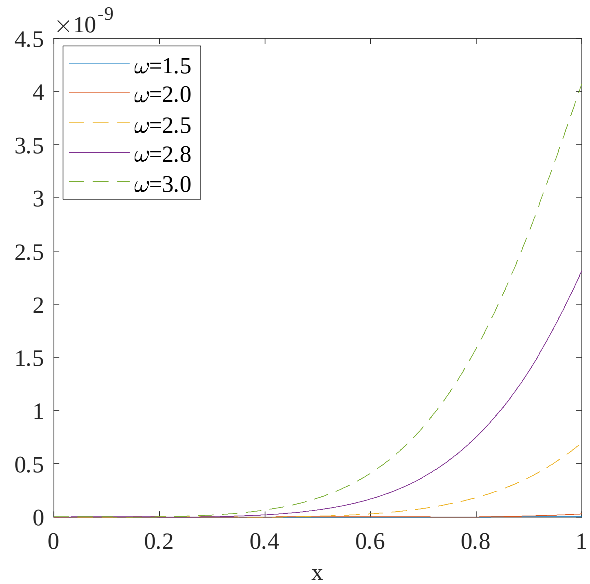

Figure 3.3: *The absolute error between the approximate solution calculated by the representation NSBF and the exact solution.*

CHAPTER 4

Properties of the functions $\gamma_j(x,\tau)$ and of the coefficients $\beta_n^j(x;\delta)$

In Chapter 3, we presented illustrations where the parameter δ has influence on the accuracy of the representation NSBF. In this Chapter, results about the connection between the approximate error by polynomials of order N of function $\gamma_j(x,\tau)$, and the decay rate of the coefficients $\beta_n^j(x,\delta)$ are presented. The results are based on approximation theory.

1. Properties of the functions $\gamma_j(x,\tau)$

From the estimate for the convergence of the approximate solution y_M in the Theorem (3.7). We conclude that the estimate is better when δ takes small values, however in numerical applications we observed that it is important to find an optimal value for δ, especially for problems of eigenvalues. In this section we give an explanation of why this happens.

In the Chapter 3, we defined the function $\gamma(x,t)$ as

$$\gamma(x,t)=\sum_{m=0}^{\infty}\frac{\alpha_m(x)}{t^{m+1}},$$

which is analytic outside of a polygon of n sides with a vertex at x, $t\in\mathbb{C}$. Besides, we considered $\gamma(x,\cdot)$ defined on the sides of a square where one of its vertex is $(x+\delta)+i(x+\delta)$, we illustrate this on Figure 4.1.
The function $\gamma(x,\cdot)$ defined on right side of square is analytic, therefore this admits an extension on a elliptic domain E bounded by a ellipse. A point $t=(u,v)$ on the ellipse that pass at $(x,0)$ and has as foci the points $(x+\delta)+i(x+\delta)$ and $-(x+\delta)-i(x+\delta)$ satisfies the equation

$$\frac{v^2}{\delta^2+(x+\delta)^2}+\frac{(u-x-\delta)^2}{\delta^2}=1. \tag{4.1}$$

On the other hand, following [**15**, Chap. 4, Sect. 2], for fixed ρ and $\rho>1$ an ellipse E_ρ described by the points $\acute{t}$ such that

$$\acute{\tau}=\frac{x+\delta}{2}\left(\frac{1}{\rho}-\rho\right)\sin(t)+i\frac{x+\delta}{2}\left(\rho+\frac{1}{\rho}\right)\cos(t), \tag{4.2}$$

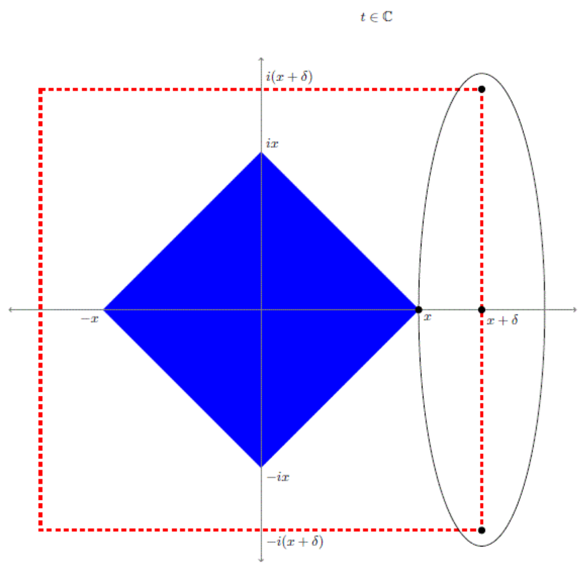

Figure 4.1: Ellipse with semi-axes $a = \frac{x+\delta}{2}\left(\rho + \frac{1}{\rho}\right)$ and $b = \frac{x+\delta}{2}\left|\frac{1}{\rho} - \rho\right|$, and the foci $\pm(x+\delta)\,i$. On the sides of red square is defined the function γ.

i.e. if $\acute{\tau} = (r, s)$ then r and s satisfy

$$\frac{s^2}{\left(\frac{x+\delta}{2}\right)^2\left(\rho+\frac{1}{\rho}\right)^2} + \frac{r^2}{\left(\frac{x+\delta}{2}\right)^2\left|\frac{1}{\rho}-\rho\right|^2} = 1, \tag{4.3}$$

has semi-axes $a = \frac{x+\delta}{2}\left(\rho + \frac{1}{\rho}\right)$ and $b = \frac{x+\delta}{2}\left|\frac{1}{\rho} - \rho\right|$, and the foci $\pm i\,(x+\delta)$. The ellipse (4.1) can be obtained by a translation of ellipse (4.3) when the major and minor axes coincide with those of the ellipse defined by $\acute{\tau}$, thus

$$\frac{x+\delta}{2}\left|\frac{1}{\rho} - p\right| = \delta. \tag{4.4}$$

We need to know about the behaviour of ρ when the length of the minor axis of the ellipse E_p is equal to δ, therefore solving the equation (4.4), we obtain

$$\rho = \frac{\delta}{x+\delta} + \sqrt{1 + \frac{\delta^2}{(x+\delta)^2}} = \frac{\frac{\delta}{x}}{1+\frac{\delta}{x}} + \sqrt{1 + \frac{\frac{\delta^2}{x^2}}{\left(1+\frac{\delta}{x}\right)^2}}, \tag{4.5}$$

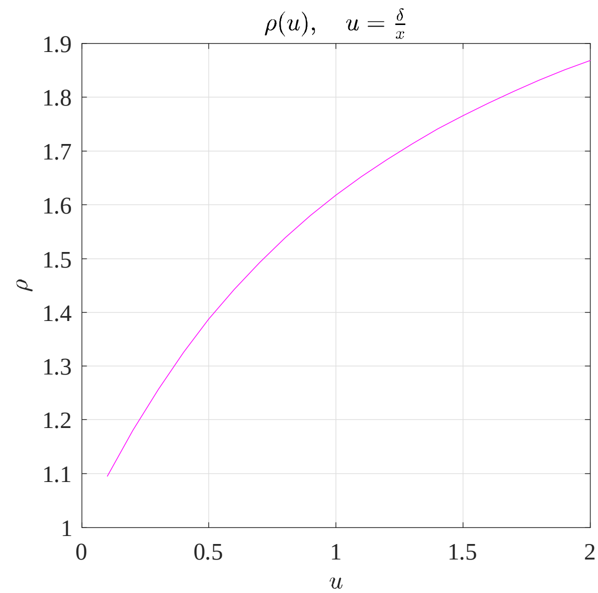

Figure 4.2: The illustration shows the behaviour of $\rho(u)$. When δ is very small, that is, the ellipse widens its minor axis, the parameter ρ is smaller than when δ takes larger values.

when $\rho > 1$, therefore, if $u = \frac{\delta}{x}$ and ρ is understood as a function of variable u, then ρ admits the form

$$\rho(u) = \frac{u}{1+u} + \sqrt{1 + \frac{u^2}{(1+u)^2}}, \tag{4.6}$$

thus, when $u \to \infty$, $\rho(u) \to 1 + \sqrt{2}$. We conclude that for fixed x, if δ takes large values then the value of ρ increases, this fact is illustrated on the Figure 4.2.

Next we will state some results that relate the value of ρ with the approximation error of an analytical function by polynomials.

DEFINITION 4.1. *The error of approximation, $\varepsilon_N(f)$ of an function f by polynomials of order N in the uniform norm on $[-1,1]$ is defined as*

$$\varepsilon_N(f) := \inf_{p\in\Pi} \|f-p\|, \ N = 1,2,...; \ \varepsilon_0(f) = \|f\|$$

where $\|f-p\| := \max_{x\in[-1,1]} |f(x) - p(x)|$ and Π is the set of polynomial of order N.

THEOREM 4.2. [**15**, Chap. 7, Theorem 8.1.] *A function f, defined on $[-1,1]$ is analytic on this interval if and only if* $\limsup \sqrt[N]{\varepsilon_N(f)} < 1$; *and more exactly*

$$\limsup_{N\to\infty} \sqrt[N]{\varepsilon_N(f)} = \frac{1}{\rho_0}, \tag{4.7}$$

where $\rho_0 > 1$ is characterized by the property that f has an analytic extension onto the bounded elliptic domain by the ellipse E_{p_0}, but not onto any of the E_p for $\rho > \rho_0$.

The error $\varepsilon_N(f)$ is a positive sequence such that when $N \to \infty$, $\varepsilon_N(f) \to 0$. By Theorem 4.2 there exist a constant $C(\epsilon)$ such that

$$\sqrt[N]{\varepsilon_N(f)} \leq \frac{\sqrt[N]{C(\epsilon)}}{\rho_0 - \epsilon}, \ \epsilon > 0 \text{ and } \rho_0 - \epsilon > 1,$$

thus if we take ρ, $1 < \rho < \rho_0$ we obtain

$$\varepsilon_N(f) \leq \frac{M}{\rho^N}. \tag{4.8}$$

Consider the function $\gamma_j(x,\tau)$ that in the Chapter 3 was defined as

$$\gamma_j(x,\tau) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{c_{k,j}\alpha_k(x)}{(\tau - i(x+\delta))^{k+1}},$$

the function $\gamma_j(x,\cdot)$ is analytic on $[-x-\delta, x+\delta]$ then it has an analytic extension onto the elliptic domains bounded by ellipses E_ρ. By the Theorem 4.2 we obtain the following result.

PROPOSITION 4.3. *The approximation error $\varepsilon_N(\gamma_j)$ of $\gamma_j(x,\cdot)$ by polynomials of order N in the uniform norm on $[-x-\delta, x+\delta]$ satisfies*

$$E_N(\gamma_j) \leq \frac{C(\delta)}{\rho^N}, \tag{4.9}$$

where C is a constant independent of N.

By proposition 4.3 we conclude even without knowing the value of $C(\delta)$ it can be affirmed that if we take very small values of ρ then the order of decay of the estimation error of γ_j is slower that when ρ takes large values.

THEOREM 4.4. *Let $\hat{\gamma}(x,s)$ be a analytic function on $[-1,1]$ as function of s such that*

$$\hat{\gamma}(x,s) = \gamma_j(x,(x+\delta)s) = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{c_{j,k}\alpha_{k-1}(x)}{((x+\delta)s - i(x+\delta))^k},$$

then coefficients $\beta_N^j(x;\delta)$ (which were defined in Chapter 3, Proposition 3.6) satisfy the following estimate

$$\left|\beta_N^j(x;\delta)\right| \leq \sqrt{2N+1}\frac{M(x+\delta)}{\rho^{N-1}}.$$

where M is constant independent of N such that $\varepsilon_{N-1}(\hat{\gamma}) \leq \frac{M}{\rho^{N-1}}$.

PROOF. Consider the polynomials q_N and q_{N-1}. q_N is the best approximation polynomial of degree N of function $\gamma_j(x,\cdot)$ and q_{N-1} is a polynomial of degree no more than $N-1$. We know that

$$\left|\beta_N^j(x;\delta)\right| = \frac{2N+1}{2}\left|\int_{-x-\delta}^{x+\delta}\gamma_j(x,\tau)\,P_N\left(\frac{\tau}{x+\delta}\right)d\tau\right|,$$

therefore,

$$\begin{aligned}
\left|\beta_N^j(x;\delta)\right| &\le \frac{2N+1}{2}\int_{-x-\delta}^{x+\delta}\left|\gamma_j(x,\tau)-q_{N-1}\left(\frac{\tau}{x+\delta}\right)\right|\left|P_N\left(\frac{\tau}{x+\delta}\right)\right|d\tau\\
&\le \frac{2N+1}{2}(x+\delta)\int_{-1}^{1}\left|\gamma_j(x,(x+\delta)s)-q_{N-1}(s)\right|\left|P_N(s)\right|ds\\
&\le \frac{2N+1}{2}(x+\delta)\,\varepsilon_{N-1}(\hat{\gamma}_j)\left\|P_n\right\|_{L_2[-1,1]}\\
&\le \sqrt{2N+1}\frac{M(x+\delta)}{\rho^{N-1}}.
\end{aligned}$$

where M is a constant independent of N. $\square$

EXAMPLE 4.5. *Consider the differential equation*

$$y^{(4)}+2y''=\omega^4 y$$

with initial conditions

$$y(0,\omega)=1,\ y'(0,\omega)=\omega,\ y^{(2)}(0,\omega)=\omega^2,\ y^{(3)}(0,\omega)=\omega^3.$$

In the Figure 4.3, we illustrate the decay rate of the numbers $\left|\beta_n^j(1,\delta)\right|$, $j=1,2$. *Observe that for large values of* δ, *the number of coefficients that it is possible to calculate is less. And when* $\delta\to 0$ *the decay rate of* $\left|\beta_n^j(1,\delta)\right|$ *is more slow than for large values of* δ. *Thus, it is important to consider an optimum value of* δ *in order to improve the accuracy of the approximate solution.*

Considering once again the previous Cauchy problem of the Example 4.5, we present semi-log graphs of the absolute error between the exact solution $y(\omega,x)$ and the approximate solution $y_N(\omega,x)$, and we illustrate the fact if we use an optimum value of δ then obtain better approximation to the exact solution of problem. We compare with the approximate solution obtained by SPPS method. On Figure 4.4 we illustrate the absolute error between exact solution and the approximate solution y_N obtained by representation NSBF and the absolute error between exact solution and the solution obtained by method SPPS for different values of ω when $\delta=1.0$. The approximate solution was obtained with 40 coefficients β_n^j.

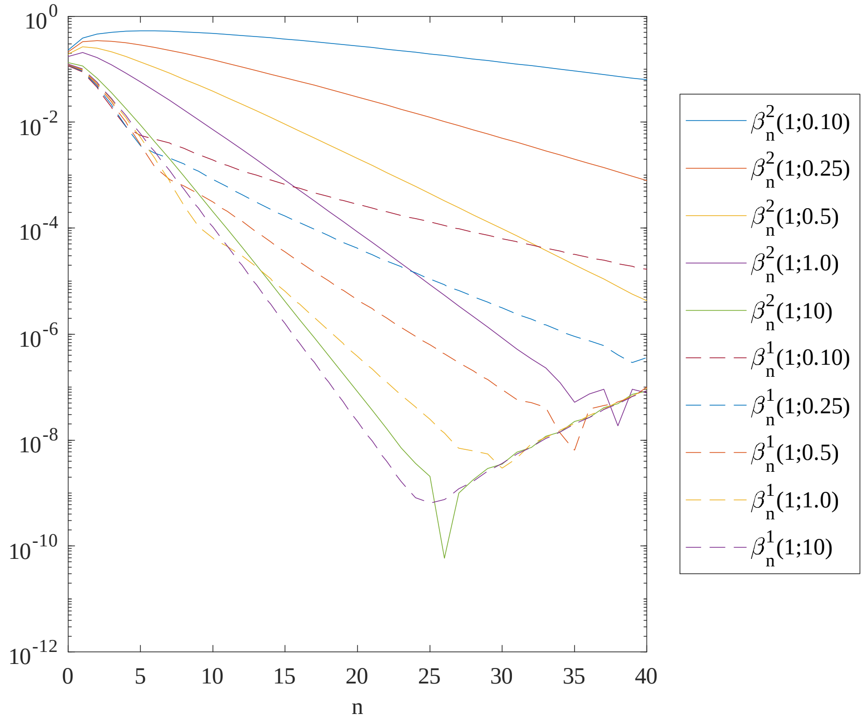

Figure 4.3: *Absolute value of the first* 40 *coefficients* $\beta_n^j(x;\delta)$, $j = 1,2$ *evaluated at* $x = 1$ *for different values of* δ.

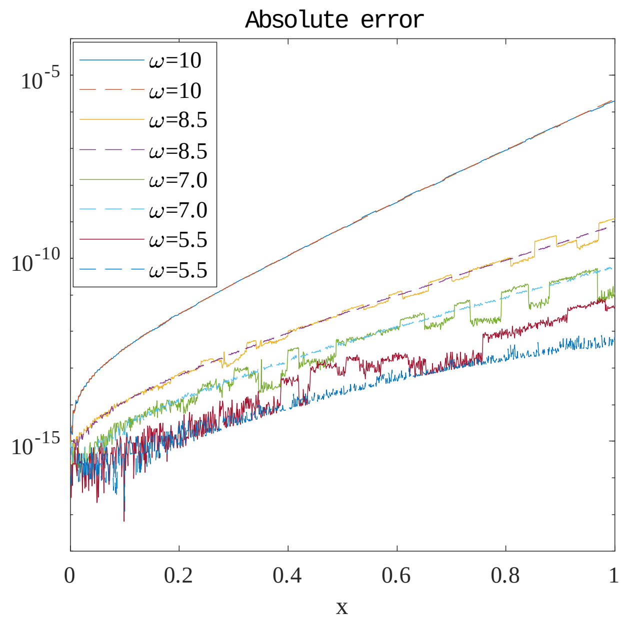

Figure 4.4: Illustration of absolute error between exact solution and approximated solution by the SPPS method (dashed line) and absolute error between exact solution and approximated solution by NSBF for some values of ω when $\delta = 1.0$.

CHAPTER 5

Other representations for the solutions of ordinary linear differential equations of order n

Consider once again the Cauchy problem

$$y^{(n)} + \sum_{i=2}^{n} p_i(x)\, y^{(n-i)} = \omega^n y, \tag{5.1}$$

$$y(\omega, 0) = 1, ..., y^{(n-1)}(\omega, 0) = \omega^{n-1} \tag{5.2}$$

where $p_i(x)$ are continuous complex-valued functions with respect to the variable x, $x \in [0, b]$, $0 < b < \infty$, and $\omega \in \mathbb{C}$. Different NSBF representations can be obtained for the solution $y(\omega, x)$ of the problem (5.1), (5.2) because by the principle of deformation of contours, in the Polya integral representation for $y(\omega, x)$ defined as

$$y(\omega, x) = \frac{1}{2\pi i} \int_C \gamma(x, t)\, e^{\omega t} dt \tag{5.3}$$

we can take any rectifiable curve C_δ on the regularity domain of $\gamma(x, \tau)$.

In this Chapter, we construct two new NSBF representations, taking two different forms of contour C_δ in the Polya representation of the solution $y(\omega, x)$ of the problem (5.1) and (5.2); the first form is obtained by a parametrization on the sides of a triangle and the second form on an 8 -sided polygon Π_x,both centered at the origin. A difference with respect to the results obtained in the Chapter 3 is that in these new representations was used the Legendre shifted polynomials to represent the Borel transform of $y(\omega, 0)$ in a Fourier Legendre expansion.

1. Parametrization on a triangle

We give the following definition that will be used throughout this chapter.

Definition 5.1. *The shifted Legendre polynomials are defined as*

$$\tilde{P}_m(\tau) = P_m(2\tau - 1)$$

where P_m are the Legendre polynomials. The polynomials $\tilde{P}_m(\tau)$ are orthogonal on $[0, 1]$,

$$\int_0^1 \tilde{P}_m(\tau)\tilde{P}_n(\tau)d\tau = \frac{1}{2n+1}\delta_{mn}.$$

Let us consider an equilateral triangle, in which a circle of radius $r = x+\delta$ is inscribed, whose vertices are $z_1 = -\sqrt{3}(x+\delta) - i(x+\delta)$, $z_2 = \sqrt{3}(x+\delta) - i(x+\delta)$ and $z_3 = 2i(x+\delta)$. We suppose that the contour C_δ in the integral representation (5.3) is represented by the counterclockwise oriented perimeter of the triangle $z_1 z_2 z_3$.

Now, the contour C_δ is parametrized by the following functions ,

$$t(\tau) = \sqrt{3}(x+\delta)(2\tau-1) - i(x+\delta), \text{ for the bottom side,}$$

$$t(\tau) = \sqrt{3}(x+\delta)(1-\tau) + i(x+\delta)(3\tau-1), \text{ for the right side,}$$

$$t(\tau) = -\sqrt{3}(x+\delta)\tau + i(x+\delta)(2-3\tau), \text{ for the left side.}$$

where $\tau \in [0,1]$. We conclude the following result

LEMMA 5.2. *The solution $y(\omega, x)$ of (5.1) admits the form*

$$\begin{aligned} y(\omega,x) &= 2\sqrt{3}(x+\delta)e^{-(\sqrt{3}+i)(x+\delta)\omega}\int_0^1 \gamma_1(x,\tau)e^{2\sqrt{3}(x+\delta)\omega\tau}d\tau \\ &+ \left(-\sqrt{3}+3i\right)(x+\delta)e^{(\sqrt{3}-i)(x+\delta)\omega}\int_0^1 \gamma_2(x,\tau)\,e^{\left(\sqrt{3}-3i\right)(x+\delta)\omega\tau}d\tau \\ &- \left(\sqrt{3}+3i\right)(x+\delta)\,e^{2(x+\delta)i\omega}\int_0^1 \gamma_3(x,\tau)\,e^{-\left(\sqrt{3}+3i\right)(x+\delta)\omega\tau}d\tau, \end{aligned} \tag{5.4}$$

where the functions $\gamma_i(x,t)$, $i = 1,2,3,$ are defined as

$$\begin{aligned} \gamma_1(x,\tau) &= \gamma\left(x,(x+\delta)\left(\sqrt{3}\,(2\tau-1)-i\right)\right), \\ \gamma_2(x,\tau) &= \gamma\left(x,(x+\delta)\left(\sqrt{3}\,(1-\tau)+i\,(3\tau-1)\right)\right), \\ \gamma_3(x,\tau) &= \gamma\left(x,(x+\delta)\left(-\sqrt{3}\tau+i\,(2-3\tau)\right)\right). \end{aligned}$$

Here we can note that the functions γ_j, $j = 1,2,3$ inherit the properties of the function $\gamma(x,t)$.

The previous result is obtained by the natural parametrization of contour, thus, we do not present the proof.

LEMMA 5.3. *The functions γ_j, for $j = 1, 2, 3$, take the form*

$$\begin{aligned}
\gamma_1(x,\tau) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{\left((x+\delta)\left(\sqrt{3}\,(2\tau-1)-i\right)\right)^{k+1}},\\
\gamma_2(x,\tau) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{\left((x+\delta)\left(\sqrt{3}\,(1-\tau)+i\,(3\tau-1)\right)\right)^{k+1}},\\
\gamma_3(x,\tau) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{\left((x+\delta)\left(-\sqrt{3}\tau+i\,(2-3\tau)\right)\right)^{k+1}}
\end{aligned}$$

The series γ_j are uniformly absolutely convergent with respect to τ on $[0,1]$.

PROOF. We prove for γ_2, the procedure is similar for all series. Note that

$$\begin{aligned}
\left|\frac{\alpha_k(x)}{\left((x+\delta)\left(\sqrt{3}\,(1-\tau)+i\,(3\tau-1)\right)\right)^{k+1}}\right| &= \frac{|\alpha_k(x)|}{(x+\delta)^{k+1}\left(3\,(1-\tau)^2+(3\tau-1)^2\right)^{\frac{k+1}{2}}}\\
&\leq \frac{|\alpha_k(x)|}{(x+\delta)^{k+1}},
\end{aligned}$$

because $3\,(1-\tau)^2+(3\tau-1)^2 \geq 1$, when $\tau \in [0,1]$. Thus,

$$\sum_{k=0}^{\infty}\left|\frac{\alpha_k(x)}{\left((x+\delta)\left(\sqrt{3}\,(1-\tau)+i\,(3\tau-1)\right)\right)^{k+1}}\right| \leq \sum_{k=0}^{\infty}\frac{|\alpha_k(x)|}{(x+\delta)^{k+1}}.$$

By Weierstrass M-test, we conclude that for any $x \in [0,b]$ the series γ_2 is uniformly absolutely convergent on $[0,1]$. □

PROPOSITION 5.4. *For any fixed x, the functions γ_j, $j = 1, 2, 3$ admit Fourier-Legendre representations*

$$\gamma_j(x,\tau) = \sum_{m=0}^{\infty}\beta_m^j(x;\delta)\tilde{P}_m(\tau) \tag{5.5}$$

respect to τ, where the coefficients β_m^j are defined as

$$\beta_m^j\,(x;\delta) = \frac{(2m+1)}{4\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{(x+\delta)^{k+1}}\int_{-1}^{1}\frac{P_m(u)}{[z_j(\frac{u+1}{2})]^{k+1}}du, \tag{5.6}$$

where

$$\begin{aligned}
z_1(\tau) &= \sqrt{3}\,(2\tau-1)-i,\\
z_2(\tau) &= \sqrt{3}\,(1-\tau)+i\,(3\tau-1)\,,\\
z_3(\tau) &= -\sqrt{3}\tau+i\,(2-3\tau)\,,
\end{aligned}$$

and for β_m^j *the following estimate holds*

$$\left|\beta_m^j(x;\delta)\right| \leq \sqrt{2m+1}\left\|\gamma_j(x,\tau)\right\|_{L[0,1]}. \tag{5.7}$$

The functions $\tilde{P}_m$ *are the shifted Legendre polynomials, which were defined in Definition 5.1 and* P_m *are the ordinary Legendre polynomials.*

Proof. The functions $\gamma_j(x,\cdot)$ are continuous on $[0,1]$, hence admit a Fourier-Legendre series expansion. Multiplying in both sides of (5.5) by $\tilde{P}_m(\tau)$ and using the orthogonality property of shifted Legendre polynomials, the following equality is obtained

$$\beta_m^j(x;\delta) = (2m+1)\int_0^1 \gamma_j(x,\tau)\tilde{P}_m(\tau)d\tau. \tag{5.8}$$

We evaluate $\gamma_j(x,\cdot)$ in (5.8), and using the fact that

$$\tilde{P}_m(x) = P_m(2x-1), \quad x \in [0,1],$$

we obtain

$$\begin{aligned}
\beta_m^j(x;\delta) &= \frac{(2m+1)}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{(x+\delta)^{k+1}}\int_0^1 \frac{\tilde{P}_m(\tau)}{[z_j(\tau)]^{k+1}}d\tau \\
&= \frac{(2m+1)}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{(x+\delta)^{k+1}}\int_0^1 \frac{P_m(2\tau-1)}{[z_j(\tau)]^{k+1}}d\tau \\
&= \frac{(2m+1)}{4\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{(x+\delta)^{k+1}}\int_{-1}^1 \frac{P_m(u)}{[z_j(\frac{u+1}{2})]^{k+1}}du,
\end{aligned}$$

the change of order of summation and integration is justified by the uniform convergence of functions series $\gamma_j(x,\cdot)$.

The estimate (5.7) follows as a consequence of the use of the Cauchy-Schwarz inequality in (5.8) and the fact that $\left\|\tilde{P}_m(\tau)\right\|_{L_2[0,1]} = \frac{1}{\sqrt{2m+1}}$. □

Theorem 5.5. *The solution* $y(\omega,x)$ *of problem (5.1), (5.2) admits the following representation*

$$y(\omega,x) = 2\sqrt{3}(x+\delta)e^{-i(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(1)}(x;\delta)i_m\left(\sqrt{3}(x+\delta)\omega\right)+$$

$$(-\sqrt{3}+3i)(x+\delta)e^{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(2)}(x;\delta))i^m j_m\left(\left(\frac{\sqrt{3}}{2}i+\frac{3}{2}\right)(x+\delta)\,\omega\right)$$

$$-(\sqrt{3}+3i)\,(x+\delta)\,e^{(x+\delta)\left(\frac{i}{2}-\frac{\sqrt{3}}{2}\right)\omega}\sum_{m=0}^{\infty}\beta_m^{(3)}(x;\delta)\,i^m j_m\left(\left(\frac{\sqrt{3}}{2}i-\frac{3}{2}\right)(x+\delta)\,\omega\right)$$

where the coefficients β_n^j *are defined as in (5.6).*

PROOF. Evaluating (5.5) in the solution $y(\omega, x)$ defined in the Lemma 5.2, we obtain

$$\begin{aligned}\frac{y(\omega,x)}{(x+\delta)} &= 2\sqrt{3}e^{-(\sqrt{3}+i)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(1)}(x;\delta)\int_0^1\tilde{P}_m(\tau)\,e^{2\sqrt{3}(x+\delta)\omega\tau}d\tau\\ &\quad+\left(-\sqrt{3}+3i\right)e^{(\sqrt{3}-i)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(2)}(x;\delta)\int_0^1\tilde{P}_m(\tau)e^{(-\sqrt{3}+3i)(x+\delta)\omega\tau}d\tau\\ &\quad-(\sqrt{3}+3i)e^{2(x+\delta)i\omega}\sum_{m=0}^{\infty}\beta_m^{(3)}(x;\delta)\int_0^1\tilde{P}_m(\tau)e^{-(\sqrt{3}+3i)(x+\delta)\omega\tau}d\tau,\end{aligned}$$

the change of variable $u = 2\tau - 1$ gives

$$\begin{aligned}\frac{y(\omega,x)}{(x+\delta)} &= \sqrt{3}e^{-(\sqrt{3}+i)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(1)}(x;\delta)\int_{-1}^1 P_m(u)e^{\sqrt{3}(x+\delta)\omega(u+1)}du+\\ &\quad\left(-\frac{\sqrt{3}}{2}+\frac{3}{2}i\right)e^{(\sqrt{3}-i)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(2)}(x;\delta)\int_{-1}^1 P_m(u)e^{-(\sqrt{3}+3i)(x+\delta)\omega\frac{u+1}{2}}du\\ &\quad-\left(\frac{\sqrt{3}}{2}+\frac{3}{2}i\right)e^{2(x+\delta)i\omega}\sum_{m=0}^{\infty}\beta_m^{(3)}(x;\delta)\int_{-1}^1 P_m(u)e^{-(\sqrt{3}+i)(x+\delta)\omega\frac{u+1}{2}}du,\end{aligned}$$

which is the same as

$$\begin{aligned}\frac{y(\omega,x)}{(x+\delta)} &= \sqrt{3}e^{-i(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(1)}(x;\delta)\int_{-1}^1 P_m(u)e^{\sqrt{3}(x+\delta)\omega u}du+\\ &\quad\frac{1}{2}(-\sqrt{3}+3i)e^{(\frac{\sqrt{3}}{2}+\frac{1}{2}i)(x+\delta)\omega}\sum_{m=0}^{\infty}\beta_m^{(2)}(x;\delta)\int_{-1}^1 P_m(u)e^{(-\sqrt{3}+3i)(x+\delta)\omega\frac{u}{2}}du\\ &\quad-\frac{1}{2}(\sqrt{3}+3i)e^{(x+\delta)(\frac{i}{2}-\frac{\sqrt{3}}{2})\omega}\sum_{m=0}^{\infty}\beta_m^{(3)}(x;\delta)\int_{-1}^1 P_m(u)e^{-(\sqrt{3}+3i)(x+\delta)\omega\frac{u}{2}}du.\end{aligned}\tag{5.9}$$

Using the formula 10.47.1 from [**42**] and the finite Fourier transform of the Legendre polynomial P_m

$$\int_{-1}^1 P_m(x)e^{i\lambda x}dx = i^m\sqrt{\frac{2\pi}{\lambda}}J_{m+\frac{1}{2}}(\lambda)$$

in (5.9) for each integral, thus the result is obtained. □

EXAMPLE 5.6. *Consider the Cauchy problem*

$$y'' = \omega^2 y(x),\ y(\omega,0) = 1,\ y'(\omega,0) = \omega.$$

where $x \in [0,1]$. The Figure 5.1 illustrates the absolute error between the exact solution $y(\omega,x) = e^{\omega x}$ and the approximate solution $y_M(\omega,x)$ of $y(\omega,x)$ obtained in Theorem 5.5 . The figure is illustrated using $\beta_M^{(1)}$, $M = 0,...,19$; $\beta_M^{(2)}$, $M = 0,...,25$, and $\beta_M^{(3)}$, $M = 0,...,16$ for different values of ω.

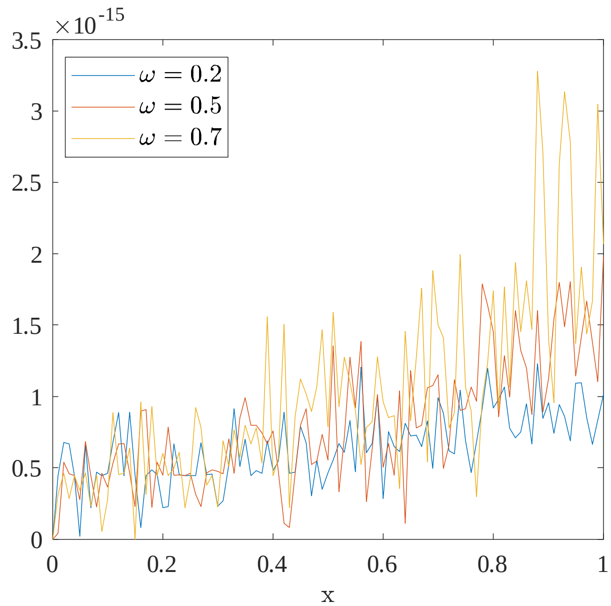

Figure 5.1: Absolute error between exact solution and aproximate solution y_M, when the contour C is a triangle.

2. Parametrization on 8 -sided polygon

In this section, we obtain a NSBF representation for the solution of the problem (5.1), (5.2) when the contour C_δ in the Polya representation is represented by the counterclockwise oriented perimeter of an 8 -sided polygon, which contains a circle of radius length $x + \delta$ inscribed.

Consider the following parametrization $t_k(\tau) : [0, 1] \to \mathbb{C}$. $k = 1, 2, 3, ..., 8$. for the 8 -sided polygon

$$\begin{aligned}
t_1(\tau) &= (x+\delta)\left(1+\tau\frac{\sqrt{2}-2}{2}\right)+i\tau\frac{\sqrt{2}}{2}(x+\delta)\\
t_2(\tau) &= (x+\delta)\frac{\sqrt{2}}{2}(1-\tau)+i(x+\delta)\left(\frac{\sqrt{2}}{2}+\tau\frac{2-\sqrt{2}}{2}\right)\\
t_3(\tau) &= -(x+\delta)\frac{\sqrt{2}}{2}\tau+i(x+\delta)\left(1+\tau\frac{\sqrt{2}-2}{2}\right)\\
t_4(\tau) &= (x+\delta)\left(-\frac{\sqrt{2}}{2}-\tau\left(\frac{2-\sqrt{2}}{2}\right)\right)+i(x+\delta)\frac{\sqrt{2}}{2}(1-\tau)\\
t_5(\tau) &= (x-\delta)\left(-1+\tau\frac{2-\sqrt{2}}{2}\right)-i(x+\delta)\frac{\sqrt{2}}{2}\tau\\
t_6(\tau) &= (x+\delta)\left(-\frac{\sqrt{2}}{2}+\tau\frac{\sqrt{2}}{2}\right)+i(x+\delta)\left(-\frac{\sqrt{2}}{2}+\tau\frac{-2+\sqrt{2}}{2}\right)\\
t_7(\tau) &= (x+\delta)\tau\frac{\sqrt{2}}{2}+i(x+\delta)\left(-1+\tau\frac{2-\sqrt{2}}{2}\right)\\
t_8(\tau) &= (x+\delta)\left(\frac{\sqrt{2}}{2}+\tau\frac{2-\sqrt{2}}{2}\right)+i(x+\delta)\left(-\frac{\sqrt{2}}{2}+\tau\frac{\sqrt{2}}{2}\right)
\end{aligned}$$

Then, the solution $y(\omega,x)$ takes the form

$$\int_C \gamma(x,t)\,e^{\omega t}dt=(x+\delta)\,E$$

where

$$\begin{aligned}
E &= \left(\frac{\sqrt{2}-2}{2}+i\frac{\sqrt{2}}{2}\right)e^{\omega(x+\delta)}\int_0^1 \gamma_1(x,\tau)\,e^{\omega(x+\delta)\left(\frac{\sqrt{2}-2}{2}+i\frac{\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(-\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}\right)e^{\omega(x+\delta)\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}\int_0^1 \gamma_2(x,\tau)\,e^{\omega(x+\delta)\left(-\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}-2}{2}\right)e^{i\omega(x+\delta)}\int_0^1 \gamma_3(x,\tau)\,e^{\omega(x+\delta)\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}-2}{2}\right)\tau}d\tau \\
&+\left(-\frac{2-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)e^{\omega(x+\delta)\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}\int_0^1 \gamma_4(x,\tau)\,e^{\omega(x+\delta)\left(-\frac{2-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(\frac{2-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)e^{-\omega(x+\delta)}\int_0^1 \gamma_5(x,\tau)\,e^{\omega(x+\delta)\left(\frac{2-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(\frac{\sqrt{2}}{2}+i\frac{-2+\sqrt{2}}{2}\right)e^{\omega(x+\delta)\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)}\int_0^1 \gamma_6(x,\tau)\,e^{\omega(x+\delta)\left(\frac{\sqrt{2}}{2}+i\frac{-2+\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}\right)e^{-i\omega(x+\delta)}\int_0^1 \gamma_7(x,\tau)\,e^{\omega(x+\delta)\left(\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}\right)\tau}d\tau \\
&+\left(\frac{2-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)e^{\omega\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)(x+\delta)}\int_0^1 \gamma_8(x,\tau)\,e^{\omega(x+\delta)\left(\frac{2-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\tau}d\tau
\end{aligned}$$

where the functions $\gamma_j(x,\tau)$, $j=1,2,...,8$ admit a Legendre-Fourier series representation defined as

$$\gamma_j(x,\tau)=\sum_{m=0}^{\infty}\beta_m^j(x;\delta)\,\tilde{P}_m(\tau)\,,\quad j=1,2,...,8$$

where we use the orthogonal properties of Legendre polynomials and obtain the relation

$$\beta_n^j(x;\delta)=(2n+1)\int_0^1\gamma_j(x,\tau)\,\tilde{P}_n(\tau)\,d\tau. \tag{5.10}$$

Moreover, the functions $\gamma_j(x,\tau)$ admit an expansion in the form

$$\gamma_j(x,\tau)=\frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{t_j^{k+1}},\ j=1,2,...,8 \tag{5.11}$$

where t_j was defined at the beginning of this section. Thus, evaluating (5.11) in (5.10) leads to the following formula

$$\beta_n^j(x;\delta)=\frac{2n+1}{4\pi i}\sum_{k=0}^{\infty}\alpha_k(x)\int_{-1}^{1}\frac{P_n(s)}{t_j^{k+1}\left(\frac{s+1}{2}\right)}ds. \tag{5.12}$$

THEOREM 5.7. *The solution of the Cauchy problem (5.1), (5.2) admits the following representation*

$$u(\omega, x) = \sum_{j=1}^{8} \sum_{n=0}^{\infty} \beta_n^j(x;\delta)\, e^{\frac{z_j\omega(x+\delta)}{2}} i^m \sqrt{\frac{\pi}{-iz_j\omega(x+\delta)}} J_{n+\frac{1}{2}}\left(-i\frac{z_j}{2}\omega(x+\delta)\right)$$

where $\beta_m^j(x;\delta)$ *are defined as in (5.12) and*

$$z_j = \begin{cases} \frac{\sqrt{2}-2}{2} + i\frac{\sqrt{2}}{2}, & j = 1, \\ \frac{-\sqrt{2}}{2} + i\frac{2-\sqrt{2}}{2}, & j = 2, \\ -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}-2}{2}, & j = 3, \\ \frac{\sqrt{2}-2}{2} - i\frac{\sqrt{2}}{2}, & j = 4, \\ \frac{-\sqrt{2}+2}{2} - i\frac{\sqrt{2}}{2}, & j = 5, \\ \frac{\sqrt{2}}{2} + i\frac{-2+\sqrt{2}}{2}, & j = 6, \\ \frac{\sqrt{2}}{2} + i\frac{2-\sqrt{2}}{2}, & j = 7, \\ \frac{-\sqrt{2}+2}{2} + i\frac{\sqrt{2}}{2}, & j = 8. \end{cases}$$

PROOF. The proof is analogous to the proof of the Theorem (5.5). □

Appendix A

Bessel potential space and generalized Lipschitz class

Definition A1: *The fractional order Sobolev space* $W_2^{\alpha}(\mathbb{R})$, *also called Bessel potential space consists of the functions satisfying* $f \in L_2(\mathbb{R})$ *and* $\left(1+|\omega|^2\right)^{\alpha/2} F[f](\omega) \in L_2(\mathbb{R})$, *where* F *is the Fourier transform operator.*

Definition A2: [**15**, Chap. 2, Sect. 7 and Sect. 9] *The modulus of continuity* $\omega(f,t) =:$ $\omega(t)$ *of a function* f *can be defined when* f *is given on any metric space* A, *we consider* $A = [a,b]$.

$$\omega(t) := \omega(f,t) = \sup_{\substack{|x-y|\leq t \\ x,y\in A}} |f(x) - f(y)|, \; t \geq 0.$$

The function ω *is continuous at* $t = 0$ *if and only if* f *is uniformly continuous on* A.

Definition A3: [**15**, Chap. 2, Sect. 9.] *The Lipschitz space* $Lip_{\beta}(A)$, $0 < \beta \leq 1$, *consists of all continuous functions* f *defined on a set* $A = [a,b]$ *which satisfy*

$$|\Delta_t(f,x)| = |f(x+t) - f(x)| \leq Mt^{\beta}, \; t > 0,$$

or, equivalent,

$$\omega(f,t) \leq Mt^{\beta},$$

where $M > 0$. *One can define* Lip_{β} *on any metric space* X, *for example, for the* L_p *norm, the space* $Lip_{\beta}(L_p)$ *which consists of all* $f \in .L_p$, $0 < p \leq \infty$ *for which*

$$\|\Delta_t(f,x)\|_p = \left[\int_{A_t} |f(x+t) - f(x)|^p \, dx\right]^{\frac{1}{p}} \leq Mt^{\beta}, \, t > 0,$$

whit $A_t := [a, b-t]$, *if* $A = [a,b]$, $t < b - a$.

For $\alpha > 0$ we write $\alpha = r + \beta$,where $r \in \mathbb{N}$ and $0 < \beta < 1$, and say that a function f belongs to $Lip_{\alpha}(I)$ class, with I being either a segment or the whole line, if $f \in C^r(I)$ and $f^{(r)} \in Lip_{\beta}(I)$. Consider the difference operator $\Delta_h : L_p(I) \to L_p(I_h)$ acting on a function f as $\Delta_h f(\cdot) = f(\cdot + h) - f(\cdot)$,here $I_h := [a, b-h]$ if $I = [a, b]$, $h < b-a$ and $I_h := I$ if $I = \mathbb{R}$. Then the r-th modulus of smoothness of f is defined by

$$\omega_r(f,t)_{L_p(I)} := \sup_{0<h\leq t} \|\Delta_h^r(f)\|_{L_p(I_{rh})}.$$

Definition A4: *For $\alpha > 0$ let r be the smallest integer satisfying $r > \alpha$, i.e. $r = [\alpha]+1$. Then the generalized Lipschitz class $Lip_\alpha^*(I,p)$ is defined as the class of functions $f \in L_p(I)$ satisfying $\omega_r(f,t)_{L_p(I)} \leq Mt^\alpha$ for all $t > 0$ with some constant $M(f)$.*

Appendix B

On weighted polynomial approximation

Let

$$W_{(\alpha,\beta)}(x) = (1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}, x \in [-1,1], \alpha, \beta \geq -\frac{1}{2}.$$

For $W_{(\alpha,\beta)}f \in L_2$, we define

$$E_n\left(W_{(\alpha,\beta)}; f\right) = \inf_{P\in\Pi}\left\|(f-P)\,W_{(\alpha,\beta)}\right\|_2, n = 0, 1, ...$$

where Π is the polynomials set of degree n. Consider the known orthonormal Jacobi polynomial $\tilde{P}_n^{(\alpha,\beta)}(x)$. For $W_{(\alpha,\beta)}f \in L_2$, we denote by $S(\alpha,\beta; f, x)$ the orthonormal expansion of f with respect to the system $\left\{\tilde{P}_n^{(\alpha,\beta)}(x)\right\}$ that is

$$f(x) \sim S(\alpha,\beta; f, x) = \sum_{k=0}^{\infty} c_k(\alpha,\beta; f)\,\tilde{P}_n^{(\alpha,\beta)}(x)$$

where

$$c_k(\alpha,\beta; f) = \int_{-1}^{1} f(x)\,\tilde{P}_k^{(\alpha,\beta)}(x)\,W_{(\alpha,\beta)}^2(x), \ k = 0, 1, ...$$

Definition B1: [**36**] *Let* $S_k^{(\alpha,\beta)}$ $(k = 1, 2, ...)$ *be the set of* f *satisfying*

(1) f *is a* k*-times iterated integral function of* $f^{(k)}$ *in* $(-1,1)$.
(2) $f^{(l)}W_{(\alpha+l,\beta+l)} \in L_2$ $(l = 0, 1, 2, ...k)$, $\left(f^{(0)} = f\right)$.

We denote by $S_0^{(\alpha,\beta)}$ *the set of* $f(x)$ *satisfying* $W_{(\alpha,\beta)}f \in L_2$.

Theorem B2: [**36**](Inverse theorem 2) Let k be a positive integer, $f \in S_0^{(\alpha,\beta)}$. If

$$\sum_{v=0}^{\infty}(v+1)^{k-1}E_v\left(W_{(\alpha,\beta)}; f\right) < \infty$$

then $f \in S_k^{(\alpha,\beta)}$; also

$$\begin{aligned} E_n\left(W_{(\alpha+k,\beta+k)}; f^{(k)}\right) &\leq c_2(\alpha,\beta,k)\left[(n+1)^{k-1} E_n\left(W_{(\alpha,\beta)}; f\right)\right. \\ &\left. + \sum_{v=0}^{\infty} (v+1)^{k-1} E_v\left(W_{(\alpha,\beta)}; f\right)\right]; \end{aligned}$$

furthermore

$$\begin{aligned} E_n\left(W_{(\alpha+k,\beta+k)}; f^{(k)}; \frac{1}{n}\right) &\leq c_3(\alpha,\beta,k)\frac{1}{n}\sum_{v=0}^{n}\left[(v+1)^{k-1} E_v\left(W_{(\alpha,\beta)}; f\right)\right. \\ &\left. + \sum_{(s=v+1)}^{\infty} E_s\left(W_{(\alpha,\beta)}; f\right)\right]. \end{aligned}$$

Remark B3: $E_n\left(W_{(\alpha,\beta)}; f\right) = O\left(\frac{1}{n^{k+\alpha}}\right)$, $(k = 0, 1, ..., 0 < 1)$ if and only if $f \in S_k^{(\alpha,\beta)}$ and $\omega\left(W_{(\alpha+k,\beta+k)}; \delta\right) = O\left(\delta^{\alpha}\right)$

Conclusions and future work

Conclusions

In this thesis, we constructed a representation in Neumann series of Bessel functions for solutions of Cauchy problems of order n, $n \geq 2$. One of the main contributions of our work is to represent the solution of this problem in a sum of four Neumann series, which converge uniformly with respect to x on $[0, b]$, $0 < b < \infty$. We also proved that it is possible to represent the solutions of linear differential equations with specified initial conditions in many more ways because this representation depends of the contour C_δ that is considered in the Polya representation . We gave two representations more, when the parametrization is on the sides of a triangle and when it is on an 8 -sided polygon.

A discussion on parameter δ has been provided, using some results of approximation theory we explained the importance of look for an optimal value of δ in order to achieve a better approximation of the problem solution. In particular we presented numerical applications solving initial value problems and spectral problems. Results showed that our representation approaches better for not large values of parameter ω.

Finally another contribution was a Neumann series representation for the solution of the perturbed Bessel equation, the series converge uniformly with respect to both x on $(0, 1]$ and converge uniformly with respect to ω on a finite subset of the complex plane of the ω variable. We also obtained estimates for the Neumann series of Bessel functions that guarantee a uniform approximation of eigendata not depending of ω. We obtained by three ways the formulas to calculate the coefficients of Neumann series of Bessel functions, unfortunately these are not numerically better than coefficients formulas obtained in Theorem 3.1 of Chapter 2.

Future work

Future work concerns deeper analysis of the asymptotic behavior in the plane of ω, this can represent an important advance in the theory of linear differential equations of order n.

Many representations can be constructed depending on the contour that is taken the integral representation of the solution $y(\omega, x)$ and other functions may appear in the series representation then it is interesting to study if this can influence for the convergence with respect to ω.

List of Figures

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Abbreviations & Notation

Abbreviations

SPPS	Spectral parameter power series
NSBF	Neumann Series of Bessel functions

Notation

$\mathcal{T}$	Transmutation operator for unperturbed Bessel operator and perturbed Bessel operator
ω	Spectral parameter, $\omega \in \mathbb{C}$.
$\tilde{R}(x,t)$	Expansion of transmutation kernel $R(x,t)$.
$\beta_n^k(x,\delta)$	n-th coefficient of k-th Fourier-Legendre series.
$\tilde{\beta}_n$	n-th coefficient of Fourier -Legendre series of $\tilde{R}(x,t)$
$J_\nu(z)$	Bessel function of first kind of order ν.
$j_n(z)$	Spherical Bessel function of first kind of order n.
$\langle \cdot, \cdot \rangle$	Scalar product
P_n	Legendre polynomial of order n.
$\tilde{P}_n$	Legendre shifted polynomials of order n.
$u_l(\omega,x)$	Regular solution of the perturbed Bessel equation satisfying the asymptotic relation $u_l(\omega,x) \sim x^{l+1}$ when $x \to 0$.
$W_2^\alpha(\mathbb{R})$	Fractional order Sobolev space.
$Lip_\alpha(\mathbb{R})$	Lipschitz class of functions.
$Lip_\alpha^*(\mathbb{R})$	Generalized Lipschitz class of functions.
$\omega_r(f.t)_{L_p(I)}$	r-th modulus of smoothness of f, respect to L_p norm on $I \subset \mathbb{R}$.
$\mathcal{A}$	Perturbed Bessel operator
$\mathcal{B}$	Unperturbed Bessel operator
L	Linear differential operator of order n.
$y(\omega,x)$	Solution of equation $L[y] = \omega^n y$.
$\gamma(x,t)$	Borel transform of the function $y(\omega,x)$.