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"The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani

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Abstract

In this work we present the construction of a family of quantum scalar fields over a $p-$adic spacetime which satisfy $p-$adic analogues of the Gårding–Wightman axioms, this $p-$adic scalar fields satisfy certain $p-$adic Klein-Gordon pseudo-differential equations. We compute explicitly the fundamental solutions of these equations, we also present the second quantization of the solutions of these Klein-Gordon equations which corresponds exactly to the scalar fields introduced here. Most of the axioms can be formulated the same way in both, the Archimedean and non-Archimedean frameworks; however, the axioms depending on the ordering of the background field must be reformulated, reflecting the acausality of $p-$adic spacetime. The main conclusion is that there seems to be no obstruction to the existence of a mathematically rigorous quantum field theory (QFT) for free fields in the $p-$adic framework, based on an acausal spacetime. This dissertation is based in the article [38], which was written in collaboration with my doctoral supervisors.

Resumen

En este trabajo presentamos la construcción de una familia de campos escalares cuánticos sobre un espacio-tiempo $p-$ádico que satisfacen los análogos $p-$ádicos de los axiomas Gårding–Wightman, estos campos escalares $p-$ádicos satisfacen ciertas ecuaciones pseudodiferenciales $p-$ádicas de tipo Klein-Gordon. Calculamos explícitamente las soluciones fundamentales de estas ecuaciones, también presentamos la segunda cuantización de las soluciones de estas ecuaciones de Klein-Gordon que corresponden exactamente a los campos escalares introducidos aquí. La mayoría de los axiomas se pueden formular de la misma manera, tanto en el marco arquimediano como en el no arquimediano; sin embargo, los axiomas que dependen del orden del campo de fondo deben ser reformulados, reflejando la posibilidad de un espacio-tiempo $p-$ádico. La conclusión principal es que no parece haber ningún obstáculo para la existencia de una teoría de campos cuánticos (TCC) matemáticamente rigurosa para los campos libres en el marco $p-$ádico, basada en un espacio-tiempo acausal. Esta tesis se basa en el artículo [38], que fue escrito en colaboración con mis supervisores de doctorado.

Overview

This work deals with the construction of scalar quantum fields on a $p-$adic spacetime. Although our treatment will be purely mathematical, there are certain physical motivations behind it that we briefly describe next.

There is a increasing amount of research pointing towards the fact that quantum mechanics conflicts with the classical notions relating causality to time ordering[1]. In [44], quantum correlations incompatible with a definite *classical* causal order are constructed (although they prove that a causal order emerges in the classical limit), and the experimental existence of these correlations is reported in [52]. Another experiment reaching similar conclusions is described in [37], using quantum optics, while in [45] quantum gates based on waveplates are used to get acausal superpositions of states. See also [50] for the incompatibility of Quantum Mechanics with some *non-local* causal models. Applications of the absence of a predefined causal structure to quantum computations are given in [8].

Motivated by these considerations, one could wonder whether it is possible to construct a quantum field theory (QFT) on a spacetime devoid of any *a priori* causal structure. The notions of spacelike and timelike intervals which, from an operational point of view, characterize the causal structure, are intimately tied to the existence of a total order on the field number $\mathbb{R}$ compatible with the algebraic field operations, so a possibility is to start from a non-ordered number field. Leaving aside the case of finite fields, the most obvious choice is to consider the non-Archimedean field of $p-$adic numbers $\mathbb{Q}_p$. The corresponding spacetime is $\mathbb{Q}_p^4$. In this way, ($p-$adic) time no longer acts as an ordering parameter. While this is completely consistent with the requirement of covariance, it raises some questions about its meaning in Quantum Mechanics; for some theoretical points of view about the possibility of quantum processes without a time parameter see [70, 51].

The spacetime $\mathbb{Q}_p^4$ is acausal in the broad sense of lacking a causal structure, but also in the particular, technical, sense that for any pair of points on it, there exists no causal curve connecting them (which, in particular, also implies that it is achronal). The question of the intrinsic (a)causality of spacetime has been

[1] Notice that we emphasize the causal character of the time ordering. There are other possible orderings (chronological, horismos) that will be not considered here, although they are related, see [32].

studied sometime ago [33], and is a topic of obligated discussion when dealing with the possibility of 'travels in time' [36, 59]. Acausal (portions of) spacetimes appears often in relation with wormholes in General Relativity [42]. There have been problems in constructing the S matrix for interacting massive scalar fields in this setting [15], but it should be stressed that these are due to the interaction along closed timelike curves, which do not exist at all in the framework of a globally acausal spacetime such as the one presented here, where the very notion of 'timelike' does not make sense.

A problem present in any acausal theory is the characterization of microcausality or local commutativity, that is, the vanishing of the commutator of field operator-valued distributions when the test functions have support in spacelike separated regions. It is not clear *a priori* that a theory without a causal structure will allow for vanishing commutators even restricting the domain of the involved operators, but we will show below that a similar property holds when the test functions are supported in the $p-$adic unit ball. Thus, there is no room for phenomena arising in the non-Archimedean case, such as the connection of spacelike regions by large timelike loops. It is also reasonable to expect that the consideration of $p-$adics numbers could also cure the divergences in $1-$loop effective Lagrangians that appear in the real Euclidean case [6], although no attempt is made here to pursue this direction of research.

Another, different, kind of motivation for studying quantum field theory in the $p-$adic setting comes from the conjecture of Vladimirov and Volovich stating that spacetime has a non-Archimedean nature at the Planck scale, [67], see also [60]. The existence of the Planck scale implies that below it the very notion of measurement as well as the idea of 'infinitesimal length' become meaningless, and this fact translates into the mathematical statement that the Archimedean axiom is no longer valid. Before Volovich, some authors explored the possibility of constructing theories of the spacetime using background fields different from $\mathbb{R}$ and $\mathbb{C}$; for instance, in [13] Everett and Ulam study the Lorentz group over $\mathbb{Q}_p$ in the hope that 'spaces of this sort might be useful in some future models of nuclear or subnuclear theories', see also [60], [61, Chapter 6] and references therein. Volovich's conjecture propelled a wide variety of investigations in cosmology, quantum mechanics, string theory, QTF, etc., and the influence of this conjecture is still relevant nowadays, see e.g. [1], [4]-[12], [11], [10], [20]-[21], [30]-

[41], [62]-[67], [71], [73]. In a completely different framework, that of the physics of complex systems, the paradigm asserting that the space of states of several complex systems has an ultrametric structure has also originated a large amount of research, see [47], [29] and references therein. These two ideas are the main motivations driving the development of $p-$adic mathematical physics. In particular, during the last thirty years $p-$adic QFT has been studied intensively, a topic whose importance has been highlighted by Varadarajan in [61].

The construction of a quantum field theory over a $p-$adic spacetime raises the question about the physical meaning of the prime p. Once a choice for p is made, we can construct $\mathbb{Q}_p^4$ (endowed with the maximum norm) and then give it a geometric structure through a quadratic form $\mathfrak{q}$. The geometry of the resulting spacetime, the quadratic space $(\mathbb{Q}_p^4, \mathfrak{q})$, depends crucially on both, p and $\mathfrak{q}$. We choose the simplest case in which the quadratic form is the unique elliptic form of dimension four and a prime number $p \equiv 1 \bmod 4$. The first choice is motivated by the need for ellipticity when doing the explicit computation of the fundamental solutions (and the corresponding propagators) of the Klein-Gordon equation. Notice that the naive choice $\mathfrak{q}(k) = k_0^2 - (k_1^2 + k_2^2 + k_3^2)$ is excluded because it is not elliptic. It is possible to develop a theory based on this form, but at the cost of facing greater technical difficulties. However, as we will see, our choice for $\mathfrak{q}$ retains all the essential features of a relativistic theory, so it is justifiable from a physical point of view. Regarding the choice of p, the quantum fields introduced here will strongly depend on the geometry of the hypersurface $V = \left\{k \in \mathbb{Q}_p^4; \mathfrak{q}(k) = 1\right\}$, and if we pick $p \equiv 1 \bmod 4$, then we can guarantee that $\sqrt{\omega(\mathbf{k})} \neq 0$ for any $\mathbf{k} \in U_{\mathfrak{q}}$, where $U_{\mathfrak{q}} \subset \mathbb{Q}_p^3$ is a certain open and compact subset (depending on $\mathfrak{q}$) that will be defined later on. Notice that, due to these choices, we are actually defining a family of quantizations, a fact that could be viewed as an advantage over the rigidity of the classical case.

Thus, given a prime number $p \equiv 1 \bmod 4$ and a $p-$adic elliptic quadratic form $\mathfrak{q}$ of dimension 4, we will denote by $\boldsymbol{O}(\mathfrak{q})$ the orthogonal group of $\mathfrak{q}$. As stated, the $p-$adic Minkowski spacetime is, by definition, the quadratic space $(\mathbb{Q}_p^4, \mathfrak{q})$, so the Lorentz group of spacetime is $\boldsymbol{O}(\mathfrak{q})$. In this work, 'time' is a $p-$adic variable, so the notions of past and future are not clearly defined. However, the $p-$adic implicit function theorem allows us to determine k_0, from $\mathfrak{q}\left(k_0, \mathbf{k}\right) = 1$, as $k_0 = \pm\sqrt{\omega\left(\mathbf{k}\right)}$, where $\sqrt{\omega\left(\mathbf{k}\right)}$ is a $p-$adic analytic function defined in $U_{\mathfrak{q}}$, and in this way we can define the mass shells:

$$V^{\pm}=\left\{(k_0,\mathbf{k})\in\mathbb{Q}_p\times\mathbb{Q}_p^3;k_0=\pm\sqrt{\omega\left(\mathbf{k}\right)}\ ,\mathbf{k}\in U_{\mathfrak{q}}\right\}.$$

We will denote by $\mathcal{F}$ the Fourier transform operator associated to the quadratic form $\mathfrak{q}$. The $p-$adic Klein-Gordon operator attached to $\mathfrak{q}$ with unit mass is defined as

$$\square_{\mathfrak{q},\alpha}\varphi=\mathcal{F}^{-1}\left(\left|\mathfrak{q}-1\right|_p^{\alpha}\mathcal{F}\varphi\right)$$

where φ is a test function and α is a fixed positive number.

The $p-$adic Klein-Gordon equations in the form used in this thesis were introduced by Zúñiga-Galindo, see [73, Chapter 6] and references therein, where also the problem of the second quantization of their solutions was posed [73, Chapter 7]. The resulting field theory has a strong number-theoretic flavor. For instance, the calculation of the Green functions is related to the meromorphic continuation of Igusa's local zeta functions, see Theorem 48 and the references [23], [29, Chapter 10], [73, Chapter 5].

The existence of fundamental solutions for $p-$adic pseudodifferential equations with arbitrary polynomial symbol was established by Zúñiga Galindo, see [[73], Theorem 134] by using Igusa's local zeta functions. In this work we show the existence of fundamental solutions for the Klein-Gordon operators $\square_{\mathfrak{q},\alpha}$, which are invariant under the action of $\boldsymbol{O}(\mathfrak{q})$, the orthogonal group of the quadratic form $\mathfrak{q}$, see [Theorem 48, Chapter 3].

The $p-$adic Klein-Gordon equation

$$\square_{\mathfrak{q},\alpha}u\left(t,\mathbf{x}\right)=0 \tag{1}$$

admits solutions of plane wave type, more precisely, the functions

$$\exp 2\pi i\left\{tE^{\pm}-sx_1l_1-px_2l_2+spx_3l_3\right\}_p,$$

where $\{\cdot\}_p$ denotes the $p-$adic fractional part, $\mathbf{l}=(l_1,l_2,l_3)\in\mathbb{Q}_p^3$ is a fixed vector, and $E^{\pm}=\pm\sqrt{\omega\left(\mathbf{l}\right)}$ (here $\sqrt{\omega\left(\mathbf{k}\right)}$ is the $p-$adic dispersion) are weak solutions of (1), see Theorem 67.

The general solution of (1), up to multiplication by a non-zero complex constant, is

$$\int_{U_{\mathfrak{q}}} \left(\chi_p \left(-\sqrt{\omega(\mathbf{k})}t + \mathbf{k} \cdot \mathbf{x} \right) a\left(\mathbf{k}\right) + \chi_p \left(\sqrt{\omega(\mathbf{k})}t - \mathbf{k} \cdot \mathbf{x} \right) a^{\dagger}(-\mathbf{k}) \right) \frac{d^3\mathbf{k}}{\left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p}, \tag{2}$$

where $\chi_p(\cdot) = \exp\left(2\pi i \{\cdot\}_p\right)$ is the standard additive character of $\mathbb{Q}_p$, $U_{\mathfrak{q}} \subset \mathbb{Q}_p^3$ is an open and compact subset, $\mathbf{k} \cdot \mathbf{x}$ denotes a suitable bilinear form, and $a\left(\mathbf{k}\right)$, $a^{\dagger}\left(-\mathbf{k}\right)$ are test functions, see Theorem 67. We consider the inhomogeneous $p-$adic Klein-Gordon equation:

$$\square_{\mathfrak{q},\alpha} u\left(t, \mathbf{x}\right) = h\left(t, \mathbf{x}\right), \tag{3}$$

where $(t, \mathbf{x}) \in \mathbb{Q}_p \times \mathbb{Q}_p^3$ and $h\left(t, \mathbf{x}\right) \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$. We use the techniques and results of [73, Chapter 6]. By a solution (or weak solution) we understand a distribution from $\mathcal{D}'_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$ satisfying (5.14). We denote by $E_{\mathfrak{q}}^0\left(t, \mathbf{x}\right)$, the fundamental solution of (5.14) obtained in Theorem 48.

In the Theorem 67, in Chapter 5, we show that the $p-$adic Klein-Gordon equations admit plane waves as weak solutions, and also we study the Cauchy problem attached to these equations.

Notice that $\left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p A\left(\mathbf{k}\right)$, $\left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p B\left(\mathbf{k}\right)$, are test functions, and also

$$\int_{U_{\mathfrak{q}}} \chi_p \left(\sqrt{\omega\left(\mathbf{k}\right)}t + \mathfrak{B}_0\left(\mathbf{k}, \mathbf{x}\right) \right) B\left(\mathbf{k}\right) \frac{d^3\mathbf{k}}{\left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p} = \int_{U_{\mathfrak{q}}} \chi_p \left(\sqrt{\omega\left(\mathbf{k}\right)}t - \mathfrak{B}_0\left(\mathbf{k}, \mathbf{x}\right) \right) B\left(-\mathbf{k}\right) \frac{d^3\mathbf{k}}{\left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p},$$

so the unique weak solution of $\square_{\mathfrak{q},\alpha} u\left(t, \mathbf{x}\right) = 0$ (with $C = 1/\sqrt{2}$) invariant under $\mathcal{L}_{+}^{\uparrow}$ corresponds to the free scalar field $\mathbf{\Phi}\left(t, \mathbf{x}\right)$, with $a\left(\mathbf{k}\right) = \left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p A\left(\mathbf{k}\right)$, $a^{\dagger}\left(\mathbf{k}\right) = \left| \sqrt{\omega\left(\mathbf{k}\right)} \right|_p B\left(\mathbf{k}\right)$. As we have seen, these solutions can be quantized using the machinery of the second quantization in such a way that Wightman axioms are satisfied.

In conventional QFT there have been some studies devoted to the optimal choice of the space of test functions. In [24], Jaffe discussed this topic (see also [57] and [35]); his conclusion was that, rather than an optimal choice, there exists a set of conditions that must be satisfied by the candidate space, and any class of test functions with these properties should be considered as valid. The main condition is that the space of test functions must be a nuclear countable Hilbert one. In this thesis, we use the following Gel'fand triple: $\mathcal{H}_{\infty}(\mathbb{K}) \subset L^2_{\mathbb{K}} \subset \mathcal{H}^*_{\infty}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$. This triple was introduced in [71]. The space $\mathcal{H}_{\infty}(\mathbb{K})$ is a nuclear countable Hilbert space, which is invariant under the action of a large class of pseudo-differential operators. This space can be considered the 'true' non-Archimedean analogue of the classical Schwartz space, as we will repeatedly justify in what follows. In fact, our results could be summarized by saying that the Gårding–Wightman axioms make sense in the $p-$adic context if we replace the Schwartz space of the classical framework by $\mathcal{H}_{\infty}(\mathbb{C})$.

The solutions (2) can be quantized using the techniques described below, and the corresponding Klein-Gordon fields satisfy the corresponding Wightman axioms, see Theorem 66.

In Chapters 4 and 5 we give the construction of a family of quantum scalar fields over a $p-$adic spacetime which satisfy $p-$adic analogues of the Garding-Wightman axioms. Then at this point it is worth to mention the main continuous operators that we needed for the construction and briebly summarize the construction and state the main theorem obtained in Chapter 5.

For $f \in \mathcal{H}$, the Segal quantum field operator $\mathbf{\Phi}_{\mathrm{S}}$ on F_0 is defined as

$$\mathbf{\Phi}_{\mathrm{S}}(f) = \frac{1}{\sqrt{2}}[a^-(f) + a^-(f)^*]. \tag{4}$$

We define for each $f \in \mathcal{H}_{\infty}(\mathbb{R})$,

$$\mathbf{\Phi}(f) = \mathbf{\Phi}_{\mathrm{S}}(Rf)\,,$$

with R defined as in Lemma 58, and for each $g \in \mathcal{H}_{\infty}(\mathbb{C})$,

$$\mathbf{\Phi}(g) = \mathbf{\Phi}(\mathrm{Re}\, g) + \sqrt{-1}\mathbf{\Phi}(\mathrm{Im}\, g)\,. \tag{5}$$

The main result of this dissertation, (see Theorem 66) says:

(i) the quadruple

$$\left\{\mathfrak{F}_s(L^2_{\mathbb{C}}\left(V^+, d\lambda\right)), \Gamma\left(U\left(\cdot,\cdot\right)\right), \mathbf{\Phi}, F_0\right\}$$

satisfies the p-adic Wightman axioms.

(ii) For each $f \in \mathcal{H}_{\infty}(\mathbb{C})$,

$$\mathbf{\Phi}\left(\square_{\mathfrak{q},\alpha} f\right) = 0.$$

In order to guarantee that the resulting theory has some physical content, in Chapters 4, 5 we show that the corresponding quantum non-Archimedean scalar fields satisfy $p-$adic versions of Gårding–Wightman's axioms. Most of them can be formulated in a way valid in both the Archimedean and non-Archimedean cases, but some of them must be appropriately reformulated in the $p-$adic setting by introducing new mathematical ideas and reinterpreting some classical constructions that are not directly available in the $p-$adic context. For instance, the absence of an ordering in the background number field implies some profound modifications in the usual interpretation of notions such as the timelike or spacelike character of $p-$adic spacetime events, and the introduction of new mathematical objects such as the $p-$adic restricted Lorentz group, that we will discuss below. As another example, our $p-$adic spectral condition does not provide a definition of energy and momentum operators, because this would require a theory of semigroups, with $p-$adic time, for operators acting on complex-valued functions, and such a theory does not exist at the moment. However, the outcomes of our analysis are consistent with the requirement that the mathematical description of physical reality must not depend on the background number field, see [68]. This property is due to the particular nature of the Klein-Gordon field, notice that the same is not true for the Schrödinger equation, as the number i does not have an analog in an arbitrary field.

In the $p-$adic setting the usual geometric notion of cone does not make sense, because it depends on the fact that the real numbers form an ordered field. For this reason, we replace the notion of closed forward light cone by that of 'closed forward semigroup', which is the topological closure of the additive semigroup generated by V^{+}. This notion allow us to construct a spectral measure attached to a strongly continuous unitary representation of the $p-$adic Poincaré group as in the classical case, the detailed proof will appear in the Theorem 66.

In the Archimedean case, the commutator vanishes whenever the test functions f, g are supported on two respective spacelike-separated subsets, that is, $f(x)g(y) = 0$ whenever $x - y$ does not belong to the interior of the light cone.

This subset can be characterized as the 'ball of radius 0' of Minkowski spacetime in the sense of the theory of indefinite quadratic forms (see, e.g., [22] and references therein). Our result can be seen as the equivalent statement in the $p-$adic case, with the unit ball playing this role.

In Chapter 5, we present a second-quantization, based on Segal's formalism, for p-adic free scalar fields whose evolution is described by a certain class of Klein-Gordon type pseudo-differential operators.

We have remarked some features derived from the fact that the spacetime is $p-$adic. Let us now make some comment about those originated in the configuration space of the fields. A key fact is that we work with complex-valued fields. This allow us to use the tools from classical functional analysis, in particular Segal quantization. On the other hand, it is also possible to work with $p-$adic valued fields. In this setting, Khrennikov developed a theory of Gaussian integration of non-Archimedean-valued functions on infinite-dimensional non-Archimedean spaces and a calculus of pseudo-differential operators which is suitable for the second-quantization representation in non-Archimedean quantum field theory, see [25]-[27] and references therein. Mathematically speaking, this is a completely different setting from ours: for instance, $p-$adic Hilbert spaces are radically different to their complex counterparts.

It must be remarked that here we deal with free fields, omitting interactions. The reason for this is that, due to Haag's theorem, interactions require a more technical treatment, but having a consistent theory for the free case is the first step towards a complete $p-$adic QFT.

Finally, let us remark that there are a lot of open questions related to $p-$adic quantum fields and their underlying mathematical techniques that remain to be studied within the present framework. Among them, probably the most important one is the reconstruction theorem, which depends on an appropriate definition of Wightman distributions, and, of course, the inclusion of non-trivial interactions, that will be discussed elsewhere. The corresponding theory for non-elliptic quadratic forms $\mathfrak{q}$, though much more difficult, is also of interest.

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Chapter 1

$p-$adic Analysis: basic aspects

Along this thesis p will denote a prime number different from 2. Due to physical considerations we will formulate all our results in dimension 4, however, many of our results are still valid in arbitrary dimension.

1.1 The field of $p-$adic numbers

In this section we summarize the essential aspects and basic results on $p-$adic analysis that we will use through the thesis. For a detailed exposition of $p-$adic analysis the reader may consult [2, 58, 66].

The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which in turn is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}\,, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order* of x. Any $p-$adic number $x \neq 0$ has a unique expansion of the form

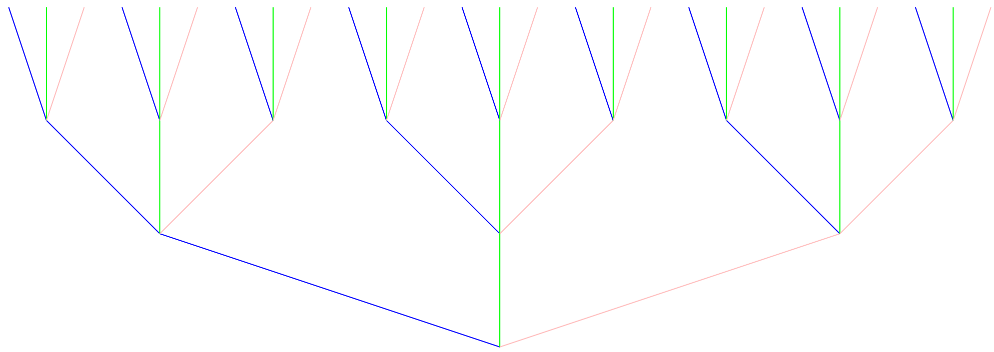
$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j\,, \tag{1.1}$$

where $x_j \in \{0, \dots, p-1\}$ and $x_0 \neq 0$. Any non-zero $p-$adic number x can be written uniquely as $x = p^{ord(x)} ac\,(x)$, with $|ac\,(x)|_p = 1$, $ac\,(x)$ is called *the angular component* of x.

By using expansion (1.1), we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\ p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0 \,. \end{cases}$$

As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [2, 66]. The balls and spheres are compact subsets.



The 3-adic unit ball $\mathbb{Z}_3$, $\quad x_0 + x_1 3 + x_2 3^2 + x_3 3^3 + x_4 3^4 \cdots, \; x_i \in \{0, 1, 2\}$

We extend the $p-$adic norm to $\mathbb{Q}_p^4$ by taking

$$||x||_p := \max_{0 \leq i \leq 3} |x_i|_p, \qquad \text{for } x = (x_0, x_1, x_2, x_3) \in \mathbb{Q}_p^4.$$

We define $ord(x) = \min_{0 \leq i \leq 3} \{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^4, ||\cdot||_p\right)$ is a complete ultrametric space. Thus $(\mathbb{Q}_p^4, \|\cdot\|_p)$ is a locally compact topological space.

For $l \in \mathbb{Z}$, denote by $B_l^4(a) = \{x \in \mathbb{Q}_p^4 : ||x - a||_p \leq p^l\}$ *the ball of radius* p^l *with center at* $a = (a_0, a_1, a_2, a_3) \in \mathbb{Q}_p^4$, and take $B_l^4 := B_l^4(0)$. Note that $B_l^4(a) = B_l(a_0) \times \cdots \times B_l(a_3)$, where $B_l(a_i) := \{x \in \mathbb{Q}_p : |x - a_i|_p \leq p^l\}$ is the one-dimensional ball of radius p^l with center at $a_i \in \mathbb{Q}_p$. The ball B_0^4 equals the product of four copies of $B_0 := \mathbb{Z}_p$, *the ring of* $p-$*adic integers*. For $l \in \mathbb{Z}$, denote by $S_l^4(a) = \{x \in \mathbb{Q}_p^4 : ||x - a||_p = p^l\}$ *the sphere of radius* p^l *with center at* $a \in \mathbb{Q}_p^4$, and take $S_l^4 := S_l^4(0)$.

Remark 1. *The natural map* $\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$, *where* $\mathbb{F}_p$ *is the finite field with* p *elements, is called the reduction modulo* p, *denoted as* $\bar{\cdot}$. *We will identify*

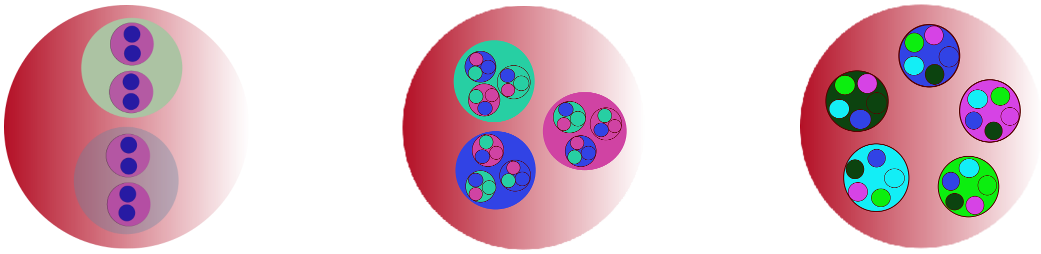


Figure 1.1: The p-adic unit balls $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$.

$\mathbb{F}_p = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$, *where the addition and multiplication are defined modulo* p. *We will distinguish between* $\{0, 1, \ldots, p-1\} \subset \mathbb{Z}_p$ *and* $\mathbb{F}_p$. *Later on, we will also use the symbol '$\bar{\cdot}$' to mean conjugation of complex numbers, but it will be clear from the context which case it is being considered.*

Remark 2. *Let us collect here some conventions.*

(i) We denote by $\Omega(\|x\|_p)$ *the characteristic function of* B_0^4. *For more general sets, say Borel sets, we use* $1_A(x)$ *to denote the characteristic function of* A.

(ii) From now on, we denote by d^4x *the Haar measure of the locally compact group* $(\mathbb{Q}_p^4, +)$ *normalized so that the volume of* $\mathbb{Z}_p^4$ *equals one.*

(iii) We will use the notation $x = (x_0, x_1, x_2, x_3) = (x_0, \boldsymbol{x}) \in \mathbb{Q}_p \times \mathbb{Q}_p^3$ *from now up to Section 5.5.*

1.2 Some function spaces

The theory of generalized functions on any locally compact group was presented by Bruhat in [5], and on a locally compact disconnected field by Gelfand, Graev and Pjatetskii-Shapiro in [16]. Vladimirov exposes the basics of this theory adapted to the field $\mathbb{Q}_p$ and to the space $\mathbb{Q}_p^n$ in [64], [66]. In many aspects, the theory is similar to the corresponding theory on the space $\mathbb{R}^n$, but there are some essential distinctions, that will be stressed in what follows.

1.2.1 The Bruhat-Schwartz space

We take $\mathbb{K}$ to mean $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-valued function φ defined on $\mathbb{Q}_p^4$ is *called locally constant,* if for any $x \in \mathbb{Q}_p^4$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^4. \tag{1.2}$$

A function $\varphi : \mathbb{Q}_p^4 \to \mathbb{K}$ is called a *Bruhat-Schwartz function (or a test function),* if it is locally constant with compact support. The $\mathbb{K}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}_{\mathbb{K}}(\mathbb{Q}_p^4) := \mathcal{D}_{\mathbb{K}}$. Let $\mathcal{D}_{\mathbb{K}}^{'}(\mathbb{Q}_p^4) := \mathcal{D}_{\mathbb{K}}^{'}$ denote the space of all continuous functionals (distributions) on $\mathcal{D}_{\mathbb{K}}$. The space $\mathcal{D}_{\mathbb{K}}^{'}$ coincides with the algebraic dual of $\mathcal{D}_{\mathbb{K}}$, i.e. any linear functional on $\mathcal{D}_{\mathbb{K}}$ is continuous. For an in-depth discussion the reader may consult [2], [58], [66].

Remark 3. *Most of the time we will work in dimension four, with spaces like $\mathcal{D}_{\mathbb{K}}(\mathbb{Q}_p^4)$ and $\mathcal{D}_{\mathbb{K}}^{'}(\mathbb{Q}_p^4)$, in these cases we will use the abbreviated notation $\mathcal{D}_{\mathbb{K}}$, $\mathcal{D}_{\mathbb{K}}^{'}$. In a few occasions we will work in dimensions different from 4, then we will use the notation $\mathcal{D}_{\mathbb{K}}(\mathbb{Q}_p^n)$, $\mathcal{D}_{\mathbb{K}}^{\prime}(\mathbb{Q}_p^n)$. A similar rule will be used for other function spaces.*

1.2.2 The spaces L^r

Given $r \in [1, +\infty)$, we denote by $L_{\mathbb{K}}^r\left(\mathbb{Q}_p^4, d^4x\right) := L_{\mathbb{K}}^r$, the $\mathbb{K}$-vector space of all the $\mathbb{K}$-valued functions g satisfying $\int_{\mathbb{Q}_p^4} |g(x)|^r d^4x < \infty$.

1.3 Fourier transform

Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $\mathbb{Q}_p$ into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$. Using the character $\chi_p(x)$ and the Haar measure d^4x one constructs the Fourier transform for complex valued functions $f(x)$, $\tilde{f}(y) = \int_{\mathbb{Q}_p^4} f(x)\chi_p(x \cdot y) d^4x$. We set

$$\mathfrak{B}(x, y) = x_0y_0 - sx_1y_1 - px_2y_2 + spx_3y_3,$$

where $s \in \mathbb{Z}$ is a quadratic non-residue module p, i.e. the congruence $x^2 \equiv s \mod p$ does not have solution. Then $\mathfrak{B}(x,y)$ is a symmetric non-degenerate $\mathbb{Q}_p-$bilinear form on $\mathbb{Q}_p^4 \times \mathbb{Q}_p^4$, and

$$\mathfrak{q}(x) := \mathfrak{B}(x,x) = x_0^2 - sx_1^2 - px_2^2 + spx_3^2,\ x \in \mathbb{Q}_p^4$$

is a *non-degenerate quadratic form* on $\mathbb{Q}_p^4$. In addition, $\mathfrak{q}(x)$ is the unique (up to linear equivalence) *elliptic quadratic form* in dimension four, here elliptic means that $\mathfrak{q}(x) = 0 \Leftrightarrow x = 0$ (notice that this is not equivalent to the non-degeneracy of $\mathfrak{B}$, as the equation $\mathfrak{q}(x) = 0$ could have its own solutions, not coming from vectors orthogonal to all the vectors in $\mathbb{Q}_p^4$).

In the definition of the Fourier transform, the bilinear form $x \cdot y = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ can be generalized for any symmetric non-degenerate bilinear form $\mathfrak{B}(x,y)$. We will use such Fourier transforms in this work.

We identify the $\mathbb{Q}_p-$vector space $\mathbb{Q}_p^4$ with its algebraic dual $\left(\mathbb{Q}_p^4\right)^*$ by means of $\mathfrak{B}(\cdot,\cdot)$. We now identify the dual group (i.e. the Pontryagin dual) of $\left(\mathbb{Q}_p^4,+\right)$ with $\left(\mathbb{Q}_p^4\right)^*$ by taking $x^*(x) = \chi_p(\mathfrak{B}(x,x^*))$. The Fourier transform is defined by

$$(\mathcal{F}g)(k) = \int_{\mathbb{Q}_p^4} g(x)\,\chi_p(\mathfrak{B}(x,k))\,d\mu(x)\,, \quad \text{for } g \in L^1_{\mathbb{C}},$$

where $d\mu(x)$ is a Haar measure on $\mathbb{Q}_p^4$. Let $\mathcal{L}\left(\mathbb{Q}_p^4\right)$ be the space of complex-valued continuous functions g in $L^1_{\mathbb{C}}$ whose Fourier transform $\mathcal{F}g$ is integrable. The measure $d\mu(x)$ can be uniquely normalized in such a way that

$$(\mathcal{F}(\mathcal{F}g))(x) = g(-x) \text{ for every } g \text{ belonging to } \mathcal{L}\left(\mathbb{Q}_p^4\right).$$

We say that $d\mu(x)$ is *a self-dual measure relative to* $\chi_p(\mathfrak{B}(\cdot,\cdot))$. Notice that $d\mu(x) = C(\mathfrak{q})d^4x$ where $C(\mathfrak{q}) = p^{-2}$ and d^4x is the normalized Haar measure on $\mathbb{Q}_p^4$. For further details about the material presented in this section the reader may consult [69].

We will also use the notation $\mathcal{F}_{x\to\xi}g$ and $\widehat{g}$ for the Fourier transform of g. The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'_{\mathbb{C}}$ is defined by

$$(\mathcal{F}[T],\varphi) = (T,\mathcal{F}\varphi) \text{ for all } \varphi \in \mathcal{D}_{\mathbb{C}}.$$

The Fourier transform $T \rightarrow \mathcal{F}[T]$ is a linear isomorphism from $\mathcal{D}'_{\mathbb{C}}$ onto itself. Furthermore, $T(\xi) = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

Remark 4. *Along this thesis we will use the notation* $\mathfrak{q}(x) = x_0^2 - \mathfrak{q}_0(\boldsymbol{x})$, *where* $\mathfrak{q}_0(\boldsymbol{x}) = sx_1^2 + px_2^2 - spx_3^2$ *is an elliptic quadratic form. The bilinear form corresponding to* $\mathfrak{q}_0$ *will be denoted* $\mathfrak{B}_0(\cdot,\cdot)$. *Then* $\mathfrak{B}(x,y) = x_0y_0 - \mathfrak{B}_0(\boldsymbol{x},\boldsymbol{y})$.

1.4 The $p-$adic Minkowski space

Take $\mathfrak{q}(x)$ as before, and define

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & sp \end{bmatrix}.$$

Then $\mathfrak{q}(x) = x^\top Gx$, where $\top$ denotes the transpose of a matrix, and x is identified with the column vector $[x_0, x_1, x_2, x_3]^\top$. The orthogonal group of $\mathfrak{q}$ is defined as

$$\begin{aligned} \boldsymbol{O}(\mathfrak{q}) &= \{\Lambda \in GL_4(\mathbb{Q}_p); \mathfrak{B}(\Lambda x, \Lambda y) = \mathfrak{B}(x,y)\} \\ &= \{\Lambda \in GL_4(\mathbb{Q}_p); \Lambda^\top G\Lambda = G\}. \end{aligned}$$

Notice that any $\Lambda \in \boldsymbol{O}(\mathfrak{q})$ satisfies $\det\Lambda = \pm 1$. We call the quadratic space $(\mathbb{Q}_p^4, \mathfrak{q})$ *the p-adic Minkowski space*, and we define *the p-adic Lorentz group* to be $\boldsymbol{O}(\mathfrak{q})$. Later on, we will introduce the $p-$adic restricted Lorentz group and the $p-$adic restricted Poincaré group.

Remark 5. *Special relativity in the* $p-$*adic framework was discussed in [13], however, our definitions of Lorentz group and 'light cones' are completely different. In [62]-[63], the authors also investigated the representations of the* $p-$*adic Poincaré group, our notion of Lorentz group agrees with the one used in these works.*

1.5 The Dirac distribution supported on a hypersurface

Take $\mathfrak{f} \in \mathbb{Q}_p[x_0, x_1, x_2, x_3]$ to be a non-constant polynomial. The hypersurface attached to $\mathfrak{f}$ is the set

$$H := H(\mathfrak{f}) = \left\{x \in \mathbb{Q}_p^4; \mathfrak{f}(x) = 0\right\}.$$

We say that H is a *non-singular hypersurface,* if

$$\nabla\mathfrak{f}(x) \neq 0 \text{ for any } x \in H. \tag{1.3}$$

By using the $p-$adic implicit function theorem, see e.g. [23], [53], one shows, as in the case $\mathbb{R}^4$, that H is a $p-$adic manifold embedded in $\mathbb{Q}_p^4$. More exactly, H is a closed submanifold of $\mathbb{Q}_p^4$ (which is a $p-$adic manifold of dimension 4) of codimension 1. For further details about $p-$adic manifolds the reader may consult [23], [53].

The condition (1.3) implies the existence of a 3-form λ (whose restriction to H is unique) satisfying

$$dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = d\mathfrak{f} \wedge \lambda. \tag{1.4}$$

Usually λ is called a *Gel'fand-Leray form* for H. We denote by $d\lambda$ the measure induced by λ on H. For the details about the construction of $d\lambda$, the reader may consult [23, Chapter 7]. This construction is similar to the one in the real case, [17, Chapter III].

The linear functional

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{K}} & \rightarrow & \mathbb{K} \\ \\ \varphi & \rightarrow & (\delta_H, \varphi) = \int\limits_H \varphi(x)\, d\lambda \end{array}$$

gives rise to a distribution $\mathcal{D}'_{\mathbb{K}}$, which is called *the Dirac distribution* δ_H supported on H.

Denote $\mathbb{Q}_p^\times = \mathbb{Q}_p - \{0\}$. For $t \in \mathbb{Q}_p^\times$, we set

$$V_t := V_t(\mathfrak{q}) = \{x \in \mathbb{Q}_p^4; \mathfrak{q}(x) = t\}.$$

Then V_t is a non-singular hypersurface in $\mathbb{Q}_p^4$. The orthogonal group $\boldsymbol{O}(\mathfrak{q})$ acts transitively on V_t. On each non-empty orbit V_t there is a non-zero, positive measure which is invariant under $\boldsymbol{O}(\mathfrak{q})$ and unique up to multiplication by a positive constant, see [46, Proposition 2-2].

For each $t \in \mathbb{Q}_p^\times$, let $d\mu_t$ be a measure on V_t invariant under $\boldsymbol{O}(\mathfrak{q})$. Since V_t is closed in $\mathbb{Q}_p^4$, it is possible to consider $d\mu_t$ as a measure on $\mathbb{Q}_p^4$ supported on V_t, and by the using the Caratheodory theorem, we can identify $d\mu_t$ with a positive distribution, i.e., if ϕ is a non-negative function, then $(d\mu_t, \phi) \geq 0$. The Rallis-Schiffman result above mentioned can be reformulated as follows: on each non-empty orbit V_t there is a non-zero, positive distribution which is invariant under $\boldsymbol{O}(\mathfrak{q})$ and unique up to multiplication by a positive constant.

Now, since δ_{V_t} is invariant under $\boldsymbol{O}(\mathfrak{q})$ (see [73, Lemma 156] for a similar calculation) we conclude that $d\mu_t$ agrees (up to a positive constant) with δ_{V_t}. From now on we identify δ_{V_t} with $d\mu_t$.

Remark 6. *We will denote by $\delta(\mathfrak{f})$ the Dirac distribution supported on the non-singular hypersurface attached to the polynomial $\mathfrak{f}$.*

1.6 The spaces $\mathcal{H}_\infty$

The Bruhat-Schwartz space $\mathcal{D}_{\mathbb{K}}$ is not invariant under the action of pseudodifferential operators; for example, $\mathcal{D}_{\mathbb{K}}$ is not invariant under the action of the Vladimirov and Taibleson Operators. In [71], see also [29, Chapter 10], a class of nuclear countably Hilbert spaces which are invariant under the action of a large class of pseudo-differential operators is introduced. In this section, we review some basic results about these spaces that we will use in the remaining sections.

Remark 7. *We set $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, $[\xi]_p := \max(1, \|\xi\|_p)$ and consider $\mathbb{N}$ to be the set of non-negative integers.*

We define for $f, g \in \mathcal{D}_{\mathbb{K}}$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the following scalar product:

$$\langle f, g\rangle_l := \int_{\mathbb{Q}_p^4} [\xi]_p^l \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d^4\xi,$$

for $l \in \mathbb{N}$, where the bar denotes the complex conjugate. We also set $\|f\|_l^2 = \langle f, f\rangle_l$. Notice that $\|\cdot\|_l \leq \|\cdot\|_m$ for $l \leq m$. Denote by $\mathcal{H}_l(\mathbb{K}) := \mathcal{H}_l(\mathbb{Q}_p^4, \mathbb{K})$ the completion of $\mathcal{D}_{\mathbb{K}}$ with respect to $\langle \cdot, \cdot\rangle_l$. Then $\mathcal{H}_m(\mathbb{K}) \hookrightarrow \mathcal{H}_l(\mathbb{K})$ is a continuous embedding for $l \leq m$. We set

$$\mathcal{H}_\infty(\mathbb{Q}_p^4, \mathbb{K}) := \mathcal{H}_\infty(\mathbb{K}) = \bigcap_{l\in\mathbb{N}} \mathcal{H}_l(\mathbb{K}).$$

Notice that $\mathcal{H}_0(\mathbb{K}) = L^2_{\mathbb{K}}$ and that $\mathcal{H}_\infty(\mathbb{K}) \subset L^2_{\mathbb{K}}$. With the topology induced by the family of seminorms $\|\cdot\|_l$, $\mathcal{H}_\infty(\mathbb{K})$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l\in\mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}, \text{ for } f,\, g \in \mathcal{H}_\infty(\mathbb{K}),$$

is a metric for the topology of the convex topological space $\mathcal{H}_\infty(\mathbb{K})$. A sequence $\{f_l\}_{l\in\mathbb{N}} \in (\mathcal{H}_\infty(\mathbb{K}), d)$ converges to $f \in \mathcal{H}_\infty(\mathbb{K})$, if and only if, $\{f_l\}_{l\in\mathbb{N}}$ converges to f in the norm $\|\cdot\|_l$ for all $l \in \mathbb{N}$. From this observation it follows that the topology of $\mathcal{H}_\infty(\mathbb{K})$ coincides with the projective limit topology τ_P. An open neighborhood base at zero of τ_P is given by the choice of $\epsilon > 0$ and $l \in \mathbb{N}$, and the sets

$$U_{\epsilon,l} := \{f \in \mathcal{H}_\infty(\mathbb{K}) : \|f\|_l < \epsilon\}.$$

The space $\mathcal{H}_\infty(\mathbb{K})$ endowed with the topology τ_P is a countably Hilbert space in the sense of Gel'fand and Vilenkin, see e.g. [18, Chapter I, Section 3.1] or [43, Section 1.2]. Furthermore $(\mathcal{H}_\infty(\mathbb{K}), \tau_P)$ is metrizable and complete and hence a Fréchet space, cf. [71, Lemma 3.3]. In addition, the completion of the metric space $(\mathcal{D}_{\mathbb{K}}(\mathbb{Q}_p^4), d)$ is $(\mathcal{H}_\infty(\mathbb{K}), d)$, and this space is a nuclear countably Hilbert space, see [71, Lemma 3.4, Theorem 3.6] or [29, Chapter 10].

For $m \in \mathbb{N}$ and $T \in \mathcal{D}'_{\mathbb{K}}$, we set

$$\|T\|^2_{-m} := \int_{\mathbb{Q}_p^4} [\xi]_l^{-m} |\widehat{T}(\xi)|^2 d^4\xi.$$

Then $\mathcal{H}_{-m}(\mathbb{K}) := \mathcal{H}_{-m}(\mathbb{Q}_p^4, \mathbb{K}) = \{T \in \mathcal{D}'_{\mathbb{K}}; \|T\|_{-m}^2 < \infty\}$ is a Hilbert space over $\mathbb{K}$. Denote by $\mathcal{H}_m^*(\mathbb{K})$ the strong dual space of $\mathcal{H}_m(\mathbb{K})$. It is useful to suppress the correspondence between $\mathcal{H}_m^*(\mathbb{K})$ and $\mathcal{H}_m(\mathbb{K})$ given by the Riesz theorem. Instead we identify $\mathcal{H}_m^*(\mathbb{K})$ and $\mathcal{H}_{-m}(\mathbb{K})$ by associating $T \in \mathcal{H}_{-m}(\mathbb{K})$ with the functional on $\mathcal{H}_m(\mathbb{K})$ given by

$$[T, g] := \int_{\mathbb{Q}_p^4} \overline{\widehat{T}(\xi)} \widehat{g}(\xi) d^4\xi. \tag{1.5}$$

Notice that $|[T, g]| \leq \|T\|_{-m}\|g\|_m$. Now by a well-known result in the theory of countable Hilbert spaces, see [18], $\mathcal{H}_0^*(\mathbb{K}) \subset \mathcal{H}_1^*(\mathbb{K}) \subset \ldots \subset \mathcal{H}_m^*(\mathbb{K}) \subset \ldots$ and

$$\mathcal{H}_\infty^*(\mathbb{K}) = \bigcup_{m\in\mathbb{N}} \mathcal{H}_{-m}(\mathbb{K}) = \{T \in \mathcal{D}'_{\mathbb{K}}; \|T\|_{-l} < \infty, \text{ for some } l \in \mathbb{N}\} \tag{1.6}$$

as vector spaces. Since $\mathcal{H}_\infty(\mathbb{K})$ is a nuclear space, the weak and strong convergence are equivalent in $\mathcal{H}_\infty^*(\mathbb{K})$, see e.g. [18]. We consider $\mathcal{H}_\infty^*(\mathbb{K})$ endowed with the strong topology. On the other hand, let $B : \mathcal{H}_\infty^*(\mathbb{K}) \times \mathcal{H}_\infty(\mathbb{K}) \to \mathbb{K}$ be a bilinear functional. Then B is continuous in each of its arguments if and only if there exist norms $\|\cdot\|_m^{(a)}$ in $\mathcal{H}_m^*(\mathbb{K})$ and $\|\cdot\|_l^{(b)}$ in $\mathcal{H}_l(\mathbb{K})$ such that $|B(T, g)| \leq M\|T\|_m^{(a)}\|g\|_l^{(b)}$ with M a positive constant independent of T and g, see e.g. [18]. This implies that (1.5) is a continuous bilinear form on $\mathcal{H}_\infty^*(\mathbb{K}) \times \mathcal{H}_\infty(\mathbb{K})$, which we will use as a paring between $\mathcal{H}_\infty^*(\mathbb{K})$ and $\mathcal{H}_\infty(\mathbb{K})$.

Remark 8. *The spaces $\mathcal{H}_\infty(\mathbb{K}) \subset L^2_{\mathbb{K}} \subset \mathcal{H}_\infty^*(\mathbb{K})$ form a Gel'fand triple (also called a rigged Hilbert space), i.e. $\mathcal{H}_\infty(\mathbb{K})$ is a nuclear space which is densely and continuously embedded in $L^2_{\mathbb{K}}$ and $\|g\|^2_{L^2_{\mathbb{K}}} = [g, g]$. This Gel'fald triple was introduced in [71].*

The following result will be used later on:

Lemma 9. *With the above notation, the following assertions hold:*

(i) $\mathcal{H}_l(\mathbb{K}) = \{f \in L^2_{\mathbb{K}}; \|f\|_l < \infty\} = \{T \in \mathcal{D}'_{\mathbb{K}}; \|T\|_l < \infty\}$;

(ii) $\mathcal{H}_\infty(\mathbb{K}) = \{f \in L^2_{\mathbb{K}}; \|f\|_l < \infty,$ *for any* $l \in \mathbb{N}\}$;

(iii) $\mathcal{H}_\infty(\mathbb{K}) = \{T \in \mathcal{D}'_{\mathbb{K}}; \|T\|_l < \infty,$ *for any* $l \in \mathbb{N}\}$.

For the proof the reader may consult ([72, Lemma 3.2]) or [29, Lemma 10.8].

Chapter 2

Second Quantization: basic aspects

2.1 The particle interpretation of fields

The usual quantum mechanical description of a system is based on the Schrödiger equation, which is a one-particle description. When quantum mechanics is coupled to relativity, there exists the possibility of annihilation and creation of particle pairs (due to the mass-energy equivalence), hence a description using a fixed number of particles is not consistent. This fact motivates the introduction of fields into our mathematical framework. Each particle generates a field defined at each point of spacetime, carrying its own energy and momentum. Again, special relativity implies that a field configuration is equivalent to an infinite distribution of particles (the field excitations): photons in the case of the electromagnetic field, gravitons in the case of the gravitational field, etc. Thus, to mathematically describe a quantum field, we need first to give a suitable space of states for a system of infinite particles, a construction known as Fock space. In doing this, we must take into account the celebrated spin-statistics theorem, stating that the state of a collection of bosons (integer spin particles) must be described by a symmetric wavefunction. Our first task, then, is to construct a multiparticle state by tensoring one-particle states, and then to symmetrize the completion of the resulting space.

As mentioned above, the treatment of bosonic and fermionic fields is different, due to the spin-statistics theorem. In the present work we have chosen to work with bosonic fields in the simplest context, the scalar one. In this way, we can explore the mathematical features arising from the p-adic setting without being disturbed by accesory questions, but at the same time we maintain a close connection with the physics behind the problem. Thus, we will work with the $p-$adic analog of the well-known Klein-Gordon equation describing scalar quantum fields with $0-$spin. In classical Minkowski spacetime $\mathbb{R}^{1,3}$, the Klein-Gordon operator (or D'alembertian) is given in cartesian coordinates by

$$\square := \frac{\partial^2}{\partial t^2} + \triangle$$

where $\triangle$ is the $3-$dimensional Laplacian operator. The Klein-Gordon equation for the scalar field $\phi : \mathbb{R}^{1,3} \to \mathbb{R}$ is then $(\square + m^2)\phi(x) = 0$.

In the $p-$adic setting, the corresponding equation must be obtained through the formalism of pseudodifferential operators, and its solutions have a weak character. Details will be provided in Chapters 3 and 4.

For an in-depth discussion of these matters, the reader may consult [9], [14], [48], [49]. Our presentation follows closely the book of Reed and Simon [49].

2.2 Tensor products of Hilbert spaces

We start by reviewing some well-known facts about quantization. Although a functorial construction of tensor products does not exist in the category of Hilbert spaces, it is possible to complete the purely algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, and this will suffice to make sense of the Fock spaces that will be introduced below as the states space for a multiparticle system.

A function from a set X to another set Y, denoted by $f : X \to Y$. $f[X]$ will usually be called the *range* of f and will be denoted $Ran f$. The restriction of f to a subset A of its domain will be denoted by $f \upharpoonright A$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For each $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$, let $\varphi_1 \otimes \varphi_2$ denote the conjugate bilinear form which acts on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$(\varphi_1 \otimes \varphi_2)\langle \psi_1, \psi_2 \rangle = (\psi_1, \varphi_1)(\psi_2, \varphi_2)$$

Let $\mathtt{E}$ be the set of finite linear combinations of such conjugate linear forms; we define an inner product $(\cdot, \cdot)$ on $\mathtt{E}$ by defining

$$(\varphi \otimes \psi, \eta \otimes \mu) = (\varphi, \eta)(\psi, \mu)$$

and extending by linearity to $\mathtt{E}$.

In what follows we summarize some of the main results that we needed for the construction of the main theorem in Chapter 5. For further details, see [48] and [49].

Proposition 10. $(\cdot, \cdot)$ *is well defined and positive definite.*

For the proof we refer the reader to [[48], Proposition 1].

Definition 11. *We define $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be the completion of $\mathtt{E}$ under the inner product $(\cdot, \cdot)$ defined above. $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Proposition 12. *If $\{\varphi_k\}$ and $\{\psi_l\}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then $\{\varphi_k \otimes \psi_l\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

For the proof we refer the reader to [[48], Proposition 2].

Theorem 13. *Let $\langle M_1, \mu_1 \rangle$ and $\langle M_2, \mu_2 \rangle$ be measures spaces so that $L^2(M_1, d\mu_1)$ and $L^2(M_2, d\mu_2)$ are separable. Then*

(a) There is a unique isomorphism from $L^2(M_1, d\mu_1) \otimes L^2(M_2, d\mu_2)$ to $L^2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ so that $f \otimes g \mapsto fg$.

(b) If $\mathcal{H}'$ is a separable Hilbert space, then there is a unique isomorphism from $L^2(M_1, d\mu_1) \otimes \mathcal{H}'$ to $L^2(M_1, d\mu_1; \mathcal{H}')$ so that $f(x) \otimes \varphi \mapsto f(x)\varphi$.

(c) There is a unique isomorphism from $L^2(M_1 \times M_2, d\mu_1 \otimes d\mu_2)$ to $L^2(M_1, d\mu_1; L^2(M_2, d\mu_2))$ such that $f(x, y)$ is taken into the function $x \mapsto f(x, \cdot)$.

See [[48], Theorem II.10] for more details.

Example 14. *The Hilbert space in the quantum-mechanical description of a single Schrödinger particle of spin one-half is $L^2(\mathbb{R}^3, dx; \mathbb{C}^2)$, that is, the set of pairs $\{\psi_1(x), \psi_2(x)\}$ of square-integrable functions (dx is Lebesgue measure). By the above theorem, $L^2(\mathbb{R}^3, dx; \mathbb{C}^2)$ is naturally isomorphic to $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.*

Example 15. *(Fock spaces) Let $\mathcal{H}$ be a Hibert space and denote by $\mathcal{H}^{(n)} = \otimes_{k=1}^{n}\mathcal{H}$ the $n-$fold tensor product. Set $\mathcal{H}^0 = \mathbb{C}$, and define*

$$\mathfrak{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \tag{2.1}$$

$\mathfrak{F}(\mathcal{H})$ is called the Fock space over $\mathcal{H}$; it will be separable if $\mathcal{H}$ is. For example, if $\mathcal{H} = L^2(\mathbb{R}, dx)$, then an element $\psi \in \mathfrak{F}(\mathcal{H})$ is a sequence of functions

$$\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \ldots\}$$

so that

$$|\psi_0|^2 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} |\psi_n(x_1, \ldots, x_n)|^2 dx_1 \cdots dx_n < \infty.$$

There are two subspaces of the Fock space which are used most frequently in quantum field theory. These two subspaces are constructed as follows: Let $\mathcal{P}_n$ be the permutation group on n elements and let $\{\varphi_k\}$ be a basis for $\mathcal{H}$. For each $\sigma \in \mathcal{P}_n$, we define an operator on basis elements of $\mathcal{H}^{(n)}$ by

$$\sigma(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \cdots \otimes \varphi_{k_n}) = \varphi_{k_{\sigma(1)}} \otimes \varphi_{k_{\sigma(2)}} \otimes \cdots \otimes \varphi_{k_{\sigma(n)}} \tag{2.2}$$

σ extends by linearity to a bounded operator on $\mathcal{H}^{(n)}$ so we can define $S_n = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \sigma$, $S_n^2 = S_n$ and $S_n^* = S_n$, so S_n is an orthogonal projection. The range of S_n is called the $n-$fold symmetric tensor product of $\mathcal{H}$. In the case where $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $\mathcal{H}^{(n)} = L^2(\mathbb{R}) \otimes \cdots \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^n, d^n x)$, $S_n\mathcal{H}^{(n)}$ is just the subspace of $L^2(\mathbb{R}^n)$ of all functions left invariant under any permutation of the variables.

We denote by $S_n : \mathcal{H}^{(n)} \to S_n\mathcal{H}^{(n)}$, the *symmetrization operator*, and $S = \bigoplus_{n=0}^{\infty} S_n$ see [48, Section II.4]. The *symmetric Fock space* over $\mathcal{H}$ or *the boson Fock space* over $\mathcal{H}$ is defined as $\mathfrak{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_s^{(n)}$, where $\mathcal{H}_s^{(n)} = S_n\mathcal{H}^{(n)}$. We call $\mathcal{H}_s^{(n)}$ the *n-particle subspace* of $\mathfrak{F}_s(\mathcal{H})$.

Let $\varepsilon(\cdot)$ be the function from $\mathcal{P}_n$ to $\{1, -1\}$ which is one on even permutations and minus one on odd permutations. Define $Alt_n = \frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} \varepsilon(\sigma)\sigma$; then Alt is an orthogonal projection on $\mathcal{H}^{(n)}$, $Alt_n\mathcal{H}^{(n)}$ is called the $n-$fold antisymmetric tensor product of $\mathcal{H}$. In the case where $\mathcal{H} = L^2(\mathbb{R})$, $Alt_n\mathcal{H}^{(n)}$ is just the subspace of $L^2(\mathbb{R}^n)$ consisting of those functions odd under interchange of two coordinates.

The subspace

$$\mathcal{F}_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} Alt_n \mathcal{H}^{(n)} \tag{2.3}$$

is called *the antisymmetric Fock space* over $\mathcal{H}$ or *the Fermion Fock space* over $\mathcal{H}$.

2.3 Symmetric self-adjoint operators

Definition 16. *Let P be an operator on Hilbert space $\mathcal{H}$, if $P^2 = P$, then P is called a projection. If in addition $P = P^*$, then P is called an orthogonal projection.*

Definition 17. *A densely defined operator T on a Hilbert space is called symmetric or Hermitian if $T \subset T^*$, that is, if $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$. Equivalently, T is symmetric if and only if*

$$(T\varphi, \psi) = (\varphi, T\psi) \text{ for all } \varphi, \psi \in D(T). \tag{2.4}$$

Definition 18. *T is called self-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.*

Definition 19. *A symmetric operator T is called essentially self-adjoint if its closure $\overline{T}$ is self-adjoint. If T is closed, a subset $D \subset D(T)$ is called a core for T if $\overline{T \upharpoonright D} = T$.*

Definition 20. *An operator-valued function $U(t)$ satisfying*

(a) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.

(b) If $\varphi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$.

is called a strongly continuous one-parameter unitary group.

Let A and B be densely defined operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We will denote by $D(A) \otimes D(B)$ the set of finite linear combinations of vectors of the form $\phi \otimes \psi$ where $\phi \in D(A)$ and $\psi \in D(B)$. $D(A) \otimes D(B)$ is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. We define $A \otimes B$ on $D(A) \otimes D(B)$ by $(A \otimes B)(\phi \otimes \psi) = A\phi \otimes B\psi$ and extend by linearity.

Proposition 21. *The operator $A \otimes B$ is well defined. Further, if A and B are closable, so is $A \otimes B$.*

Similarly, if A and B are closable then $A \otimes I + I \otimes B$, defined on $D(A) \otimes D(B)$, is closable. For the proof we refer the reader to [[48], Section VIII.10, Proposition 1].

Definition 22. *Let A and B be closable operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The tensor product of A and B is the closure of the operator $A \otimes B$ defined on $D(A) \otimes D(B)$. We will denote the closure by $A \otimes B$ also. Usually $A + B$ will denote the closure of $A \otimes I + I \otimes B$ on $D(A) \otimes D(B)$.*

Proposition 23. *Let $A_1, \ldots, A_N$ be self-adjoint operators on $\mathcal{H}_1, \ldots, \mathcal{H}_N$ and suppose that, for each k, D_k is a domain of essential self-adjointness for A_k. Then the operators $A_\pi = A_1 \otimes \cdots \otimes A_N$ and $A_\Sigma = A_1 + \cdots + A_N$ are essentially self-adjoint on $D = \otimes_{k=1}^N D_k$.*

For the proof we refer the reader to [[48], Section VIII.10, Corollary 1].

Example 24. *(Second quantization) Let $\mathcal{H}$ be a Hilbert space, $\mathcal{F}(\mathcal{H})$ the associated Fock space over H. Suppose that A is a self-adjoint operator on $\mathcal{H}$ with a domain of essential self-adjointness D. Corresponding to each such A we can define an operator $d\Gamma(A)$ on $\mathcal{F}(\mathcal{H})$ as follows. Let*

$$A^{(n)} = A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A,$$

on $\otimes_{k=1}^n D$ where I is the identity operator. Let $D_A \subset \mathcal{F}(\mathcal{H})$ be the set of $\psi = \{\psi_0, \psi_1, \ldots\}$ such that $\psi_n = 0$ for n large enough and $\psi_n \in \otimes_{k=1}^n D$ for each n. D_A is dense in $\mathcal{F}(\mathcal{H})$ since D is dense in $\mathcal{H}$. Define $A^{(0)} = 0$ and $d\Gamma(A) = \sum_{n=0}^\infty A^{(n)}$. $d\Gamma(A)$ makes sense on $D_A \cap \mathcal{H}_s^{(n)}$ and easily seen to be symmetric. $A^{(n)}$ is essentially self-adjoint on $\otimes_{k=1}^n D$. Thus $d\Gamma(A)$ is essentially self-adjoint on D_A. If A is a quantum operator, $d\Gamma(A)$ is called the second quantization of A. $d\Gamma(A)$ commutes with the projections onto the symetric and antisymmetric Fock spaces and it follows that $d\Gamma(A) \restriction \mathcal{F}_s(\mathcal{H})$ and $d\Gamma(A) \restriction \mathcal{F}_a(\mathcal{H})$ are essentially self-adjoint on $D \cap \mathcal{F}_s(\mathcal{H})$ and $D \cap \mathcal{F}_a(\mathcal{H})$ respectively.

Remark 25. *The notation* $d\Gamma$ *arises in the following way.* $\mathcal{F}(\mathcal{H})$ *is an algebra in a natural way with a product defined so that* $(\psi_1 \otimes \cdots \otimes \psi_n) \cdot (\psi_{n+1} \otimes \cdots \otimes \psi_{n+k}) = (\psi_1 \otimes \cdots \otimes \psi_{n+k})$. *This product is denoted by* $\otimes$. *Thus* $\psi \otimes \phi$ *is defined for all* $\psi, \phi \in \mathcal{F}(\mathcal{H})$. *The natural automorphisms of* $\mathcal{F}(H)$ *are invertible linear, norm preserving maps,* V*, obeying* $V(\psi \otimes \phi) = V\psi \otimes V\phi$. *The natural automorphisms of* $\mathcal{H}$ *are just the unitaries. With each unitary,* U*, one can associate uniquely an automorphism,* $\Gamma(U)$ *on* $\mathcal{F}(\mathcal{H})$ *obeying* $\Gamma(U) = U$ *on* $\mathcal{H}$ *by requiring that on* $\mathcal{H}^{(n)} = \otimes_{k=1}^{n} \mathcal{H}$, $\Gamma(U)$ *be just* $U \otimes \cdots \otimes U$ *(*n *times). Thus* Γ *maps the group of unitaries on* $\mathcal{H}$ *into the group of automorphisms on* $\mathcal{F}(\mathcal{H})$ *in a strongly continuous manner.*

Proposition 26. *A closed symmetric operator* A *is self-adjoint if and only if* $D(A)$ *contains a dense set of analytic vectors.*

For more details we refer the reader to [[49], Section X, Corollary 1].

Proposition 27. *Suppose that* A *is a symmetric operator and let* D *be a dense linear set contained in* $D(A)$. *Then, if* D *contains a dense set of analytic vectors if* D *is invariant under* A*, then* A *is essentially self-adjoint on* D.

For the proof we refer the reader to [[49], Section X, Corollary 2].

2.4 Free quantum fields

Our goal is to define the abstract free field on $\mathcal{F}_s(\mathcal{H})$, the boson subspace of $\mathcal{F}(\mathcal{H})$; to do this we need to introduce other families of operators and some terminology. We now fix a vector f in $\mathcal{H}$. For the vectors of the form $\eta = \psi_1 \otimes \ldots \otimes \psi_n$, we define a map $b^-(f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n-1)}$ by $b^-(f)(\eta) = \langle f, \psi_1 \rangle \psi_2 \otimes \ldots \otimes \psi_n$. Then $b^-(f)$ extends to a bounded map (of norm $\|f\|_{\mathcal{H}}$) of $\mathcal{H}^{(n)}$ in to $\mathcal{H}^{(n-1)}$ for each n (except for $n = 0$). In the case $n = 0$, we define $b^-(f) : \mathcal{H}^{(0)} \to 0$. The adjoint $b^+(f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}$ of $b^-(f)$ is defined as $b^+(f)(\psi_1 \otimes \ldots \otimes \psi_n) = f \otimes \psi_1 \otimes \ldots \otimes \psi_n$. The map $f \to b^+(f)$ is linear, but $f \to b^-(f)$ is antilinear.

The boson Fock space is invariant under $b^-(f)$ but not under $b^+(f)$. A vector $\psi = \left\{\psi^{(n)}\right\}_{n\in\mathbb{N}} \in \mathfrak{F}_s(\mathcal{H})$ is called a *finite particle vector* if $\psi_n = 0$ for all but finitely many n. The set of all finite vectors is denote as F_0. We set the vector $\Upsilon_0 = (1, 0, 0, \ldots)$ to be the *vacuum*.

In the case $A = I$, the second quantization $N = d\Gamma(A)$ (*the number operator*) is essentially self-adjoint on F_0 and for $\phi \in \mathcal{H}_s^{(n)}$, $N\phi = n\phi$. If U is a *unitary operator* on $\mathcal{H}$, we define $\Gamma(U)$ to be the unitary operator on $\mathfrak{F}_s(\mathcal{H})$ which equals $\bigotimes_{k=1}^n U$ when restricted to $\mathcal{H}_s^{(n)}$ for $n \geq 0$ and which equals the identity on $\mathcal{H}_s^{(0)}$.

The *annihilation operator* $a^-(f)$ on $\mathfrak{F}_s(\mathcal{H})$ with domain F_0 is given by

$$a^-(f) = \sqrt{N+1}\, b^-(f).$$

$a^-(f)$ is called an annihilation operator because it takes each $(n+1)-$particle subspace into the $n-$particle subspace. For ψ, η in F_0,

$$\left\langle \sqrt{N+1}\, b^-(f)\psi, \eta \right\rangle = \left\langle \psi, Sb^+(f)\sqrt{N+1}\eta \right\rangle,$$

which implies that

$$\left(a^-(f)\right)^* \restriction_{F_0} = Sb^+(f)\sqrt{N+1}.$$

The operator $(a^-(f))^*$ is called a creation operator. Both $a^-(f)$ and $(a^-(f))^* \restriction_{F_0}$ are closable, the corresponding closures are denoted as $a^-(f)$ and as $a^-(f)^*$.

Example 28. *If $\mathcal{H} = L^2(M, d\nu)$, we have*

$$\bigotimes_{j=1}^n L^2(M, d\nu) = L^2(M \times \cdots \times M, d\nu \otimes \cdots \otimes d\nu)$$

and that

$$S \bigotimes_{j=1}^n L^2(M, d\nu) = L_s^2(M \times \cdots \times M, d\nu \otimes \cdots \otimes d\nu)$$

where L_s^2 is the set of functions in L^2 which are invariant under permutations of the coordinates. The operators $a^-(f)$ and $a^-(f)^*$ are given by

$$(a^-(f)\psi)^{(n)}(m_1, \ldots, m_n) = \sqrt{n+1} \int_M \overline{f}(m)\psi^{(n+1)}(m, m_1, \ldots, m_n) d\nu(m)$$

$$(a^-(f)^*\psi)^{(n)}(m_1, \ldots, m_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n f(m_i)\psi^{(n-1)}(m_1, \ldots, \widehat{m}_i, \ldots, m_n) d\nu(m).$$

Where $\widehat{m}_i$ means that m_i is omitted. If A operates on $L^2(M, d\nu)$ by multiplication by the real-valued function $\omega(m)$, then

$$(d\Gamma(A)\psi)^{(n)}(m_1, \dots, m_n) = \left(\sum_{i=1}^{n} \omega(m_i)\right) \psi^{(n)}(m_1, \dots, m_n).$$

The definition of the creation operator implies that the *Segal field operator* $\Phi_S(f)$ on F_0 defined by

$$\Phi_S(f) = \frac{1}{\sqrt{2}}(a^-(f) + a^-(f)^*)$$

is symmetric. In fact, $\Phi_S(f)$ is essentially self-adjoint. The mapping from $\mathcal{H}$ to the self-adjoint operators on $\mathcal{F}_s(\mathcal{H})$ given by $f \mapsto \Phi_S(f)$ is called the *Segal quantization* over $\mathcal{H}$. The following theorem gives the fundamental propierties of the Segal quantization. For further details, see [[49], Theorem X.41].

Theorem 29. *Let $\mathcal{H}$ be a complex Hilbert space; $\Phi_S(\cdot)$ the corresponding Segal quantization. Then:*

(a) (self-adjointness) For each $f \in \mathcal{H}$, $\Phi_S(f)$ is essentially self-adjoint on F_0, the finite particle vectors.

(b) (cyclicity of the vacuum) Υ_0 is in the domain of all finite products $\Phi_S(f_1) \cdots \Phi_S(f_n)$ and the set $\{\Phi_S(f_1) \cdots \Phi_S(f_n)\Upsilon_0 : f_i$ and n arbitrary$\}$ is total in $\mathcal{F}_s(\mathcal{H})$.

(c) (commutation relations) For each $\psi \in F_0$ and $f, g \in \mathcal{H}$,

$$\Phi_S(f)\Phi_S(g)\psi - \Phi_S(g)\Phi_S(f)\psi = iIm(f, g)_{\mathcal{H}}\psi \tag{2.5}$$

Further, if $W(f)$ denotes the unitary operator $e^{i\Phi_S(f)}$, then

$$W(f + g) = e^{\frac{-iIm(f,g)}{2}} W(f)W(g) \tag{2.6}$$

(d) (continuity) If $f_n \to f$ in $\mathcal{H}$, then

$$W(f_n)\psi \to W(f)\psi \quad \text{for all } \psi \in \mathcal{F}_s(\mathcal{H}) \tag{2.7}$$

$$\Phi_S(f_n)\psi \to \Phi_S(f)\psi \quad \text{for all } \psi \in F_0. \tag{2.8}$$

(e) *For every unitary operator* U *on* $\mathcal{H}$, $\Gamma(U) : D(\overline{\Phi_s(f)}) \to D(\overline{\Phi_s(Uf)})$ *and for* $\psi \in D(\overline{\Phi_s(Uf)})$,

$$\Gamma(U)\overline{\Phi_s(f)}\Gamma(U)^{-1}\psi = \overline{\Phi_s(Uf)}$$

for all $f \in \mathcal{H}$. *For the proof we refer the reader to [[49], Theorem X.41].*

Remark 30. *In chapter 5, we work in the* $p-$*adic case. Many of the results presented in this section are still valid in the* $p-$*adic context of* $p-$*adic fields. However, those that involve the notion of order need modifications both in their formulation and its interpretation.*

Chapter 3

Fundamental Solutions for Pseudo-differential Operators of Klein-Gordon Type

3.1 Some preliminary results

For $\alpha > 0$, $m \in \mathbb{Q}_p^{\times}$, and $\mathfrak{q}$ as before, we define the following pseudo-differential operator:

$$\square_{\mathfrak{q},\alpha,m} = \mathcal{F}^{-1} \circ |\mathfrak{q} - m^2|_p^{\alpha} \circ \mathcal{F}, \tag{3.1}$$

where $|\mathfrak{q} - m^2|_p^{\alpha}$ denotes the multiplication operator by the function $|\mathfrak{q} - m^2|_p^{\alpha}$. We call operators of type (3.1), *p-adic Klein-Gordon pseudo-differential operators.* These operators were introduced by Zúñiga-Galindo, see [73, Chapter 6] and the references therein.

In this section, we consider operators $\square_{q,\alpha,m}$ with domain

$$Dom(\square_{\mathfrak{q},\alpha,m}) = \{T \in \mathcal{D}'_{\mathbb{C}} : |\mathfrak{q} - m^2|_p^{\alpha} \mathcal{F}T \in \mathcal{D}'_{\mathbb{C}}\}.$$

Remark 31. *Notice that*

$$\square_{\mathfrak{q},\alpha,m} \left(T\left(mx\right)\right) = |m|_p^{2\alpha} \left(\square_{\mathfrak{q},\alpha,1} T\right)(mx) \text{ for any } T \in Dom(\square_{\mathfrak{q},\alpha,m}).$$

Consequently, we may normalize the mass m to one. From now on we assume that $m = 1$, and we use the notation $\square_{\mathfrak{q},\alpha}$ instead of $\square_{\mathfrak{q},\alpha,1}$.

Definition 32. *We say that* $E_{\mathfrak{q},\alpha} \in \mathcal{D}'_{\mathbb{C}}$ *is a fundamental solution for*

$$\square_{\mathfrak{q},\alpha} u = \varphi, \tag{3.2}$$

if $u = E_{\mathfrak{q},\alpha} * \varphi$ *is a solution of (3.2) in* $\mathcal{D}'_{\mathbb{C}}$*, for any* $\varphi \in \mathcal{D}_{\mathbb{C}}$*.*

From now on, by an abuse of language, we will say that $E_{\mathfrak{q},\alpha}$ is a fundamental solution of $\square_{\mathfrak{q},\alpha}$.

Lemma 33. $E_{\mathfrak{q},\alpha}$ *is a fundamental solution of* $\square_{\mathfrak{q},\alpha}$ *if and only if*

$$|\mathfrak{q} - 1|_p^{\alpha} \mathcal{F}(E_{\mathfrak{q},\alpha}) = 1 \tag{3.3}$$

in $\mathcal{D}'_{\mathbb{C}}$*.*

Proof. If $E_{\mathfrak{q},\alpha}$ is a fundamental solution of $\square_{\mathfrak{q},\alpha}$, then

$$\left(|\mathfrak{q} - 1|_p^{\alpha} \mathcal{F}(E_{\mathfrak{q},\alpha}) - 1\right) \cdot \mathcal{F}\varphi = 0,$$

for any test function in $\mathcal{D}_{\mathbb{C}}$, which implies (3.3). Now, if (3.3) holds, by using the fact that the product of two distributions, if it exists, is commutative and associative (see e.g. [58, p. 127. Theorem 3.19]), we get that

$$\left(|\mathfrak{q} - 1|_p^{\alpha} \mathcal{F}\varphi\right) \cdot \mathcal{F}(E_{\mathfrak{q},\alpha}) = \mathcal{F}\varphi$$

for any test function φ. □

3.2 The $p-$adic submanifold V

Since $\mathfrak{q}(k) = k_0^2 - sk_1^2 - pk_2^2 + spk_3^2$, where $s \in \mathbb{Z}_p^{\times} = \mathbb{Z}_p - \{0\}$ is a quadratic non-residue mod p, is an elliptic quadratic form (i.e. $\mathfrak{q}(k) = 0 \Leftrightarrow k = 0$), we have

$$|\mathfrak{q}(k)|_p \geq \left(\inf_{x \in S_0^4} |\mathfrak{q}(x)|_p\right) \|k\|_p^2, \tag{3.4}$$

see e.g. [73, Lemma 25]. Set

$$V := \{k = (k_0, \boldsymbol{k}) \in \mathbb{Q}_p \times \mathbb{Q}_p^3; \mathfrak{q}(k) = 1\}.$$

By using (3.4), and the fact that $\inf_{x\in S_0^4}|\mathfrak{q}(x)|_p = p^{-1}$, we get that $V \subseteq \mathbb{Z}_p^4$, which implies that V is a compact submanifold of $\mathbb{Z}_p^4$ of codimension 1. Let us emphasize that V is bounded (in contrast to the classical case). Given $(\widetilde{k}_0, \widetilde{\boldsymbol{k}}) \in V$ with $\widetilde{k}_0 \neq 0$, by applying the $p-$adic implicit function theorem, see e.g. [23], there exist open and compact subsets $U_j^0 \subset \mathbb{Z}_p$, $U_j^1 \subset \mathbb{Z}_p^3$ such that $(\widetilde{k}_0, \widetilde{\boldsymbol{k}}) \in U_j = U_j^0 \times U_j^1$, and a $p-$adic analytic function $h_j(\boldsymbol{x}) : U_j^1 \to U_j^0$ such that

$$V \cap U_j = \left\{(k_0, \boldsymbol{k}) \in U_j; k_0 = h_j(\boldsymbol{k})\right\}.$$

Notice that $k_0 = -h_j(\boldsymbol{k})$ is also a 'local parametrization' of V. By using the compactness of V, there exists a finite number of analytic functions $\pm h_j(\boldsymbol{k})$: $U_j^1 \to \pm U_j^0$, $j = 1, \ldots, N$ such that

$$\begin{aligned} V = &\bigsqcup_{j=1}^{N} \left\{(k_0, \boldsymbol{k}) \in U_j^0 \times U_j^1; k_0 = h_j(\boldsymbol{k})\right\} \bigsqcup \\ &\bigsqcup_{j=1}^{N} \left\{(k_0, \boldsymbol{k}) \in -U_j^0 \times U_j^1; k_0 = -h_j(\boldsymbol{k})\right\} \bigsqcup W, \end{aligned}$$

where $W = \{(0, \boldsymbol{k}) : \mathfrak{q}_0(\boldsymbol{k}) = 1\}$. We set $U_{\mathfrak{q}} := \bigsqcup_{j=1}^{N} U_j^1 \subset \mathbb{Z}_p^3$. We now define in $U_{\mathfrak{q}}$, two analytic functions as follows:

$$\begin{array}{ccc} U_{\mathfrak{q}} & \to & \mathbb{Q}_p \\ & & \\ \boldsymbol{k} & \to & \pm\sqrt{1 + sk_1^2 + pk_2^2 - spk_3^2} =: \ \pm\sqrt{\omega\left(\boldsymbol{k}\right)}, \end{array}$$

where $\pm\sqrt{\omega\left(\boldsymbol{k}\right)}\,|_{U_j^1} = \pm h_j(\boldsymbol{k})$.

3.2.1 A notion of positivity

We set $\mathbb{F}_p^\times = \left[\mathbb{F}_p^\times\right]_+ \bigsqcup \left[\mathbb{F}_p^\times\right]_-$, where $\left[\mathbb{F}_p^\times\right]_+ := \left\{\overline{1}, \ldots, \overline{\frac{p-1}{2}}\right\}$ and $\left[\mathbb{F}_p^\times\right]_- = \left\{\overline{\frac{p+1}{2}}, \ldots, \overline{p-1}\right\}$. We define the elements of $\left[\mathbb{F}_p^\times\right]_+$ as *positive* and the elements of $\left[\mathbb{F}_p^\times\right]_-$ as *negative*.

Notice that since $p \neq 2$,

$$\begin{array}{ccc} \left[\mathbb{F}_p^{\times}\right]_+ & \rightarrow & \left[\mathbb{F}_p^{\times}\right]_- \\ \overline{y} & \rightarrow & -\overline{y} \bmod p \end{array}$$

is a bijection. Now, we say that a non-zero $p-$adic number

$$a = p^{-L}\left(a_0 + a_1 p + \ldots\right), \text{ with } L \in \mathbb{Z} \text{ and } a_0 \neq 0,$$

is *positive (denoted as* $a > 0$*)* if $\overline{a}_0 \in \left[\mathbb{F}_p^{\times}\right]_+$, otherwise we say that a is *negative (denoted as* $a < 0$*)*. This is a well-defined and useful notion of 'positivity' in $\mathbb{Q}_p^{\times}$, however, this notion of positivity is not compatible with the field operations, consequently, this notion does not give rise to an order in $\mathbb{Q}_p^{\times}$. We also recall that in the case $p \neq 2$, the equation $x^2 = a$ has two solutions in $\mathbb{Q}_p$ if and only if L is even and the congruence $z^2 \equiv \overline{a_0} \bmod p$ has two solutions, one in $\left[\mathbb{F}_p^{\times}\right]_+$ and the other in $\left[\mathbb{F}_p^{\times}\right]_-$. We denote them as $\pm\sqrt{a_0} \in \mathbb{F}_p^{\times}$; then

$$x = p^{-\frac{L}{2}}\left(\sqrt{a_0} + b_1 p + b_2 p^2 + \ldots\right),$$

where the b's are recursively determined by $\sqrt{a_0}$, i.e. $b_1 = f_1\left(\sqrt{a_0}\right)$, $b_2 = f_2\left(\sqrt{a_0}, b_1\right), \ldots$, and

$$\begin{aligned} -x &= -p^{-\frac{L}{2}}\left(\sqrt{a_0} + b_1 p + b_2 p^2 + \ldots\right) \\ &= p^{-\frac{L}{2}}\left(p - \sqrt{a_0} + (p - 1 - b_1)\, p + (p - 1 - b_2)\, p^2 + \ldots\right). \end{aligned}$$

We now define

$$\begin{aligned} V^+ &= \left\{(k_0, \boldsymbol{k}) \in V; k_0 > 0 \text{ and } k_0 = \sqrt{\omega\left(\boldsymbol{k}\right)}\right\}, \\ V^- &= \left\{(k_0, \boldsymbol{k}) \in V; k_0 < 0 \text{ and } k_0 = -\sqrt{\omega\left(\boldsymbol{k}\right)}\right\}. \end{aligned}$$

We call V^+ *the positive mass shell* and V^- the *negative mass shell.* Therefore

$$V = V^+ \bigsqcup V^- \bigsqcup W.$$

Consequently, W has $d\lambda$-measure zero, so $\int_W \varphi d\lambda \equiv 0$ for any $\varphi \in \mathcal{D}_{\mathbb{C}}$.

3.3 The distributions $\delta_{V^{\pm}}$

Remark 34. *Set* $\mathfrak{q}(k_0, \boldsymbol{k}) := k_0^2 - \mathfrak{q}_0(\boldsymbol{k})$*, then*

$$W = \left\{(k_0, \boldsymbol{k}) \in \mathbb{Z}_p^4; \mathfrak{q}(0, \boldsymbol{k}) = 1\right\} = \left\{\boldsymbol{k} \in \mathbb{Z}_p^3; -\mathfrak{q}_0(\boldsymbol{k}) = 1\right\}.$$

A necessary and sufficient condition to have $W \neq \emptyset$ *is that*

$$-\mathfrak{q}_0(\boldsymbol{k}) \equiv 1 \mod p \quad \textit{i.e.} \quad -sk_1^2 \equiv 1 \bmod p. \tag{3.5}$$

The sufficiency of condition (3.5) follows from the Hensel-Newton lemma, see e.g. [19, Lemma 1]. The existence of solutions for congruence (3.5) requires the computation of the following Legendre symbol:

$$\left(\frac{-s^{-1}}{p}\right) = \begin{cases} 1 \ \textit{if congruence}(3.5) \ \textit{has a solution,} \\ -1 \ \textit{if congruence}(3.5) \ \textit{has no solution.} \end{cases}$$

By using the fact that the Legendre symbol is a multiplicative function and that $\left(\frac{s}{p}\right) = -1$*, we get that*

$$\left(\frac{-s^{-1}}{p}\right) = \begin{cases} -1 \ \textit{if}\ p \equiv 1 \mod 4 \Leftrightarrow W = \emptyset \\ 1 \ \textit{if}\ p \equiv 3 \mod 4 \Leftrightarrow W \neq \emptyset\,. \end{cases}$$

Taking these results into account, we will set $p \equiv 1 \bmod 4$ *from now on, so* $W = \emptyset$*.*

We set $\delta_V = \delta\left(\mathfrak{q} - 1\right)$ as before. The characteristic functions $1_{V^{\pm}}$ are locally constant functions, so the product distributions $1_{V^{\pm}}\delta\left(\mathfrak{q} - 1\right)$ are well-defined. We set $\delta_{V^{\pm}} := 1_{V^{\pm}}\delta\left(\mathfrak{q} - 1\right)$. Then

$$\delta_V = \delta_{V^+} + \delta_{V^-} \ \text{ in } \mathcal{D}'_{\mathbb{C}}.$$

In the open subset of $\mathbb{Q}_p^4$ defined by $k_0 \neq 0$, the 3-form λ satisfying (1.4) (with $\mathfrak{f} = \mathfrak{q}$) is given by

$$\lambda = \frac{dk_1 \wedge dk_2 \wedge dk_3}{2k_0},$$

therefore the corresponding measure is

$$d\lambda=\frac{dk_1dk_2dk_3}{|k_0|_p}=\frac{d^3\boldsymbol{k}}{\left|\sqrt{\omega\left(\boldsymbol{k}\right)}\right|_p}=\frac{d^3\boldsymbol{k}}{\sqrt{|1+\mathfrak{q}_0\left(\boldsymbol{k}\right)|_p}}\text{ for }\boldsymbol{k}\in U_{\mathfrak{q}}.$$

If $p\equiv 1 \bmod 4$, then $\sqrt{\omega\left(\boldsymbol{k}\right)}\neq 0$ or any $\boldsymbol{k}\in U_{\mathfrak{q}}$, and

$$(\delta_{V^{\pm}},\varphi)=\int_{U_{\mathfrak{q}}}\varphi\left(\pm\sqrt{\omega\left(\boldsymbol{k}\right)},\boldsymbol{k}\right)\frac{d^3\boldsymbol{k}}{\left|\sqrt{\omega\left(\boldsymbol{k}\right)}\right|_p}\text{ for any }\varphi\in\mathcal{D}_{\mathbb{C}}.$$

Remark 35. *Take $\overline{a}\in\mathbb{F}_p^4$ satisfying $\mathfrak{q}(\overline{a})\equiv 1 \bmod p$. Since $\nabla\mathfrak{q}(\overline{a})\not\equiv 0 \bmod p$, by the Hensel-Newton lemma, see e.g. [19, Lemma 1], there exists $b\in\mathbb{Z}_p^4$ such that $\mathfrak{q}(b)=1$ and $b\equiv\overline{a} \bmod p$. This b is not unique. We now define the following tubular neighborhood of V:*

$$E_V=\bigsqcup_{\substack{\overline{a}\in\mathbb{F}_p^4\\ \mathfrak{q}(\overline{a})\equiv 1 \ mod\ p}} b+p\mathbb{Z}_p^4,$$

where implicitly we are choosing for each $\overline{a}\in\mathbb{F}_p^4$ a point b in V. Notice that $E_V\neq\emptyset$. Indeed, the solution set of the equation $k_0^2-sk_1^2\equiv 1 \bmod p$ contains the set $A:=\{(1,0,\overline{u},\overline{v})\,;\overline{u},\overline{v}\in\mathbb{F}_p\}$, and the gradient satisfies the condition $\nabla\mathfrak{q}(\overline{y})\not\equiv 0 \bmod p$, for any $\overline{y}$ in A.

Lemma 36. *Let $b=(b_0,b_1,b_2,b_3)\in V$, with $b_0\in\mathbb{Z}_p^{\times}$. Then*

$$(\delta(\mathfrak{q}(k)-1),\phi(k)\Omega(p\|k-b\|_p))=$$
$$p^{-3}\int_{\mathbb{Z}_p^3}\phi(b_0+pf(0,u_1,u_2,u_3),b_1+pu_1,b_2+pu_2,b_3+pu_3)du_1du_2du_3,$$

where $f(0,u_1,u_2,u_3)$ is a $p-$adic analytic function on the ball $\mathbb{Z}_p^3$.

Proof. Recall that

$$(\delta(\mathfrak{q}(k)-1),\phi(k)\Omega(p\|k-b\|_p))=\int_{V\cap\left(b+p\mathbb{Z}_p^4\right)}\phi(k)\frac{dk_1dk_2dk_3}{|k_0|_p}.$$

Now, by changing variables as $k = b + pz$,

$$(\delta(\mathfrak{q}(k)-1), \phi(k)\Omega(p\|k-b\|_p)) = p^{-3}\int_{\{\mathfrak{q}(b+pz)=1\}\cap\mathbb{Z}_p^4} \phi(b+pz)dz_1dz_2dz_3, \quad (3.6)$$

where we are assuming that z_0 is an analytic function of the variables z_1, z_2, z_3. We set

$$u = F(z), \text{ with } u_0 = \frac{\mathfrak{q}(b+pz)-1}{p}, u_i = z_i \text{ for } i = 1, 2, 3. \quad (3.7)$$

Then $Jac_F(z) \equiv 2b_0 + 2pz_0 \equiv \overline{b}_0 \not\equiv 0 \text{ mod } p$, by [23, Lemma 7.4.3], F gives rise to an analytic isomorphism from $\mathbb{Z}_p^4$ into itself which preserves the Haar measure, in this coordinate system $\{\mathfrak{q}(b+pz)=1\}\cap\mathbb{Z}_p^4$ becomes $\{u_0 = 0\}\times\mathbb{Z}_p^3$, and (3.6) takes the form

$$(\delta(\mathfrak{q}(k)-1), \phi(k)\Omega(p\|k-b\|_p)) = \quad (3.8)$$

$$p^{-3}\int_{\mathbb{Z}_p^3} \phi(b_0 + pf(0,u_1,u_2,u_3), b_1+pu_1, b_2+pu_2, b_3+pu_3)du_1du_2du_3,$$

where $f(0,u_1,u_2,u_3) : \mathbb{Z}_p^3 \to \mathbb{Z}_p$ is a $p-$adic analytic function. □

Remark 37. *Let us comment about some related results.*

(i) In the case $b_0 \in p\mathbb{Z}_p$, $b_1 \in \mathbb{Z}_p^\times$, a calculation similar to the one done in the proof of Lemma 36 shows that

$$(\delta(\mathfrak{q}(k)-1), \phi(k)\Omega(p\|k-b\|_p)) =$$

$$p^{-3}\int_{\mathbb{Z}_p^3} \phi(b_0+pu_0, b_1+pg(u_0,0,u_2,u_3), b_2+pu_2, b_3+pu_3)du_0du_2du_3,$$

where $g(u_0,0,u_2,u_3) : \mathbb{Z}_p^3 \to \mathbb{Z}_p$ is a $p-$adic analytic function.

(ii) In the case $b_0 \in p\mathbb{Z}_p$, $b_1 \in p\mathbb{Z}_p$, $b_2 \in \mathbb{Z}_p^\times$, we have

$$\{\mathfrak{q}(k)=1\}\cap[p\mathbb{Z}_p\times p\mathbb{Z}_p\times[b_2+p\mathbb{Z}_p]\times[b_3+p\mathbb{Z}_p]] =$$
$$\left\{p\left(pk_0^2 - spk_1^2 - k_2^2 + sk_3^2\right) = 1\right\}\cap[\mathbb{Z}_p\times\mathbb{Z}_p\times[b_2+p\mathbb{Z}_p]\times[b_3+p\mathbb{Z}_p]] = \emptyset.$$

A similar result is valid in the cases where $b_0 \in p\mathbb{Z}_p$, $b_1 \in p\mathbb{Z}_p$,$b_2 \in p\mathbb{Z}_p$, $b_3 \in \mathbb{Z}_p^\times$, and where $b_0 \in p\mathbb{Z}_p$, $b_1 \in p\mathbb{Z}_p$,$b_2 \in p\mathbb{Z}_p$, $b_3 \in p\mathbb{Z}_p$.

3.4 Fundamental solutions

The existence of fundamental solutions for operators $\square_{\mathfrak{q},\alpha}$ is closely related to the meromorphic continuation of the Igusa local zeta function attached to the polynomial $\mathfrak{q}-1$, which is the distribution defined as

$$\left(|\mathfrak{q}-1|_p^s,\theta\right)=\int_{\mathbb{Q}_p^4\backslash V}|\mathfrak{q}(x)-1|_p^s\theta(x)d^4x \text{ for } \operatorname{Re}(s)>0, \text{ and } \theta\in\mathcal{D}_{\mathbb{C}}. \tag{3.9}$$

Here we use that for $a>0$ and $s\in\mathbb{C}$, $a^s=e^{s\ln a}$. Integrals of type (3.9) admit meromorphic continuations to the whole complex plane as rational functions of p^{-s}, see [23, Theorem 8.2.1].

For further calculations, we rewrite (3.9) as

$$\begin{aligned}(|\mathfrak{q}(x)-1|_p^s,\theta(x))&=\int_{\mathbb{Q}_p^4\backslash E_V}|\mathfrak{q}(x)-1|_p^s\theta(x)d^4x+\int_{E_V\smallsetminus V}|\mathfrak{q}(x)-1|_p^s\theta(x)d^4x\\&=:(I_0\left(s\right),\theta)+(I_1\left(s\right),\theta).\end{aligned}$$

A fundamental solution $E_{\mathfrak{q},\alpha}$ for operator $\square_{\mathfrak{q},\alpha}$ is obtained by computing the Laurent expansion of the local zeta function $|\mathfrak{q}-1|_p^s$ at $s=-\alpha$, see [73, Theorem 134]. Indeed, if

$$|\mathfrak{q}-1|_p^s=\sum_{j=-j_0}^{\infty}c_j(s+\alpha)^j, \text{ where } c_j\in\mathcal{D}'_{\mathbb{C}}, \text{ with } -j_0\in\mathbb{Z}, \tag{3.10}$$

then $\widehat{E}_{\mathfrak{q},\alpha}=c_0$.

Remark 38. *Given two subsets A, B in $\mathbb{Q}_p^4$, we denote the distance between them as*

$$dist(A,B):=\inf_{x\in A,\ y\in B}\left\|x-y\right\|_p.$$

Lemma 39. *For any $\theta\in\mathcal{D}_{\mathbb{C}}$, the function $(I_0\left(s\right),\theta)$ is holomorphic in the whole complex plane.*

Proof. The result follows, by using a well-known result about the analyticity of integrals depending on a complex parameter, see [23, Lemma 5.3.1], from the fact that there exists a positive constant $\varepsilon=\varepsilon\left(\mathfrak{q}\right)$, such that

$$|\mathfrak{q}(x)-1|_p\geq\varepsilon \text{ for any } x\in\mathbb{Q}_p^4\setminus E_V. \tag{3.11}$$

If (3.11) is false, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}_p^4\backslash E_V$ such that $|\mathfrak{q}(y_n)-1|_p \to 0$ as $n\to\infty$, which means that

$$dist\left(V, \mathbb{Q}_p^4 \setminus E_V\right) = 0, \tag{3.12}$$

because, since V is compact, there exists $x_0 \in V$ such that

$$dist\left(V, \mathbb{Q}_p^4 \setminus E_V\right) = \inf_{y\in\mathbb{Q}_p^4\backslash E_V} \|x_0 - y\|_p = \inf_{y\in\mathbb{Q}_p^4\backslash E_V} dist\left(V, y\right).$$

The assertion (3.12) is not true. Indeed, since V is compact and $\mathbb{Q}_p^4 \setminus E_V$ is closed (because E_V is open and closed), we have $dist\left(V, \mathbb{Q}_p^4 \setminus E_V\right) > 0$. □

Remark 40. *Notice the following computation:*

$$\begin{aligned}(I_1(s),\theta) &= \int_{E_V\backslash V} |\mathfrak{q}(x)-1|_p^s \theta(x) d^4x = \sum_{\substack{\overline{b}\in\mathbb{F}_p^4 \\ \mathfrak{q}(\overline{b})\equiv 1 \bmod p}} \int_{b+p\mathbb{Z}_p^4} |\mathfrak{q}(x)-1|_p^s \theta(x) d^4x \\ &= p^{-4} \sum_{\substack{\overline{b}\in\mathbb{F}_p^4 \\ \mathfrak{q}(\overline{b})\equiv 1 \bmod p}} \int_{\mathbb{Z}_p^4} |\mathfrak{q}(b+pz)-1|_p^s \theta(b+pz) d^4z \\ &=: p^{-4} \sum_{\substack{\overline{b}\in\mathbb{F}_p^4 \\ \mathfrak{q}(\overline{b})\equiv 1 \bmod p}} (I_b(s),\theta).\end{aligned} \tag{3.13}$$

Lemma 41. *With the above notations and setting*

$$I_b(s) = \sum_{j=0}^{\infty} c_j\left(I_b,\alpha\right)(s+\alpha)^j, \text{ where } c_j\left(I_b,\alpha\right) \in \mathcal{D}'_{\mathbb{C}},$$

for $\overline{b} \in \mathbb{F}_p^4$, $\mathfrak{q}(\overline{b}) \equiv 1 \bmod p$, *the coefficient* $c_0 \in \mathcal{D}'_{\mathbb{C}}$ *in expansion (3.10) is given by*

$$(c_0,\theta) = \int_{\mathbb{Q}_p^4\backslash E_V} |\mathfrak{q}(x)-1|_p^{-\alpha} \theta(x) d^4x + p^{-4} \sum_{\substack{\overline{b}\in\mathbb{F}_p^4 \\ \mathfrak{q}(\overline{b})\equiv 1 \bmod p}} \left(c_0\left(I_b,\alpha\right),\theta\right).$$

Proof. The formula follows from Lemma 39 and Remark 40. □

We now compute the coefficients $c_0\left(I_b,\alpha\right)$ for some values of b, the calculation of the remaining cases is similar to the one presented here.

Lemma 42. *Assume that* $\overline{b}_0 \not\equiv 0 \bmod p$. *If* $\alpha \neq 1$, *then*

$$\left(c_0\left(I_b,\alpha\right),\theta\right) = p^{\alpha}\int_{\mathbb{Z}_p} |u_0|_p^{-\alpha}(\Theta_b(u_0)-\Theta_b(0))du_0 + \frac{p^{\alpha}(1-p^{-1})}{1-p^{-1+\alpha}}\Theta_b(0),$$

where $\Theta_b = T_{I_b,\alpha}\left(\theta\right) \in \mathcal{D}_{\mathbb{C}}\left(\mathbb{Q}_p\right)$, *and* $T_{I_b,\alpha}$ *is a linear operator from* $\mathcal{D}_{\mathbb{C}}\left(\mathbb{Q}_p^4\right)$ *into* $\mathcal{D}_{\mathbb{C}}\left(\mathbb{Q}_p\right)$, *and*

$$\Theta_b\left(0\right) = p^3(\delta(\mathfrak{q}(k)-1), \theta(k)\Omega(p\|k-b\|_p)).$$

In addition,

$$\left(1_V c_0\left(I_b,\alpha\right),\theta\right) = \frac{p^{\alpha}(1-p^{-1})}{1-p^{-1+\alpha}}\Theta_b(0). \tag{3.14}$$

If $\alpha = 1$, *then*

$$\left(c_0\left(I_b,1\right),\theta\right) = p\int_{\mathbb{Z}_p^4} |u_0|_p^{-1}(\Theta_b(u_0)-\Theta_b(0))du_0 - \frac{p-1}{2}\Theta_b(0).$$

Moreover,

$$\left(1_V c_0\left(I_b,1\right),\theta\right) = -\frac{p-1}{2}\Theta_b(0). \tag{3.15}$$

Proof. By changing variables as $u = F(z)$, see (3.7), we get

$$\begin{aligned}(I_b(s),\theta) &= \int_{\mathbb{Z}_p^4} \left|\mathfrak{q}\left(b+pz\right)-1\right|_p^s \theta\left(b+pz\right) d^4z \\ &= p^{-s}\int_{\mathbb{Z}_p^4} |u_0|_p^s \theta(b_0+pf(u_0,\ldots,u_3), b_1+pu_1, b_2+pu_2, b_3+pu_3)du_0du_1du_2du_3\end{aligned}$$

where $f(u_0,\ldots,u_3)$ is a $p-$adic analytic function on $\mathbb{Z}_p^4$. Set

$$\Theta_b(u_0) := \int_{\mathbb{Z}_p^3\setminus D} \theta(b_0+pf(u_0,\ldots,u_3), b_1+pu_1, b_2+pu_2, b_3+pu_3)du_1du_2du_3,$$

where $D = \{b_0+pf(u_0,\ldots,u_3)=0\}$. Then $\Theta_b(u_0) \in \mathcal{D}_{\mathbb{C}}\left(\mathbb{Q}_p\right)$ and $\Theta_b(0) = p^3(\Omega(p\|k-b\|_p)\delta(\mathfrak{q}(k)-1), \theta(k))$, see (3.8). Notice that for a fixed u_0, the set $\{b_0+pf(u_0,\ldots,u_3)=0\}$ has measure zero, and that $b_0+pf(u_0,\ldots,u_3)$ is locally constant in u_0 on $\mathbb{Z}_p^3 \setminus D$, this last fact is verified by using the $p-$adic Taylor expansion, see e.g. [53]. Therefore

$$(I_b(s),\theta) = p^{-s}\int_{\mathbb{Z}_p^4} |u_0|_p^s(\Theta_b(u_0)-\Theta_b(0))du_0 + \frac{p^{-s}(1-p^{-1})}{1-p^{-1-s}}\Theta_b(0). \tag{3.16}$$

If $\alpha \neq 1$, then $(c_0(I_b,\alpha),\theta)$ is obtained by replacing $s=-\alpha$ in (3.16). In the case $\alpha = 1$, the computation of $(c_0(I_b,1),\theta)$ is achieved by computing the Laurent expansion of $(I_b(s),\theta)$ around $(s+1)$, which follows from the formula:

$$\frac{p^{-s}(1-p^{-1})}{1-p^{-1-s}} = \left(\frac{p-1}{\ln p}\right)\frac{1}{s+1} - \frac{p-1}{2} + O(s+1),$$

where $O(s+1)$ denotes a holomorphic function. Finally formulae (3.14)-(3.15) follow from the fact that in the coordinate system $(u_0,\ldots,u_3)$, $u_0=0$ is a local equation of V. □

Remark 43. *Lemma 42 is valid for general b, but there are small variations in the formulae for the $c_0(I_b,\alpha)$s. In the case $\overline{b}_0 \equiv 0 \mod p$, $\overline{b}_1 \not\equiv 0 \mod p$, the statement of Lemma 42 and the corresponding proof are similar to ones presented here, see Remark 37. We outline the calculations for the case $\overline{b}_0 \equiv 0 \mod p$, $\overline{b}_1 \equiv 0 \mod p$, $\overline{b}_2 \not\equiv 0 \mod p$. In this case, we use the following change of variables:*

$$u = G(z) \text{ with } u_0 = z_0,\quad u_1 = z_1,\quad u_2 = \frac{\mathfrak{q}(b+pz)-1}{p^2},\quad u_3 = z_3.$$

Then

$$Jac_G(z) = \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{p^2}\frac{\partial u_0}{\partial z_0} & \frac{1}{p^2}\frac{\partial u_1}{\partial z_1} & \frac{1}{p^2}\frac{\partial u_2}{\partial z_2} & \frac{1}{p^2}\frac{\partial u_3}{\partial z_3} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{p^2}\frac{\partial u_2}{\partial z_2} = -2\left(b_2 + pz_2\right),$$

and thus $Jac_G(z) \equiv \overline{-2b_2} \equiv \overline{b_2} \not\equiv 0 \mod p$, and by Lemma 7.4.3 in [23], G gives rise to an analytic isomorphism from $\mathbb{Z}_p^4$ to itself which preserves the Haar *measure. By changing variables in integral* $(I_b(s), \theta)$*, we get that*

$$(I_b(s), \theta) =$$
$$p^{-2s} \int_{\mathbb{Z}_p^4} |u_2|_p^s \theta(b_0 + pz_0, b_1 + pu_1, b_2 + ph(u_0, \ldots, u_3), b_3 + pu_3) du_0 du_1 du_2 du_3.$$

Now the calculations proceed as in the proof of Lemma 42 .

Remark 44. *Set* $\delta_k(x) := p^{4k}\Omega(p^k \|x\|_p)$*. We recall the definition of the product of two distributions: given* $F, G \in \mathcal{D}'_{\mathbb{C}}$*, their product is defined as* $(F \cdot G, \varphi) = \lim_{k\to\infty}(G, (F * \delta_k)\varphi)$*, if the limit exist for all* $\varphi \in \mathcal{D}_{\mathbb{C}}$*. If the product* $F \cdot G$ *exists then the product* $G \cdot F$ *exists and they are equal.*

Lemma 45. $(|\mathfrak{q} - 1|_p^\alpha \delta(\mathfrak{q} - 1), \psi) = 0$ *for any* $\psi \in \mathcal{D}_{\mathbb{C}}$ *and for any* $\alpha > 0$*.*

Proof. By Remark 44, $(|\mathfrak{q} - 1|_p^\alpha \delta(\mathfrak{q} - 1), \psi) = \lim_{k\to\infty}(\delta(\mathfrak{q} - 1), (|\mathfrak{q} - 1|_p^\alpha * \delta_k)\psi)$. Now

$$(|\mathfrak{q} - 1|_p^\alpha * \delta_k)(x) = p^{4k} \int_{x+p^k\mathbb{Z}_p^4} |\mathfrak{q}(y) - 1|_p^\alpha d^4y.$$

Since $V \subseteq \mathbb{Z}_p^4$ has measure zero, we may assume without loss of generality that $x \notin V$. Now, if $z \in \mathbb{Z}_p^4$ then $\mathfrak{q}(x + p^k z) - 1 = \mathfrak{q}(x) - 1 + p^k A$, with $A \in \mathbb{Z}_p$ and $\mathfrak{q}(x) - 1 \neq 0$, then by taking k sufficiently large, we have $|\mathfrak{q}(x+p^k z)-1|_p^\alpha = |\mathfrak{q}(x) - 1|_p^\alpha$, consequently $(|\mathfrak{q} - 1|_p^\alpha * \delta_k)(x) = |q(x) - 1|_p^\alpha$ for k sufficiently large. Finally, $(|\mathfrak{q} - 1|_p^\alpha \delta(\mathfrak{q} - 1), \psi) = (\delta(\mathfrak{q} - 1), |\mathfrak{q} - 1|_p^\alpha \psi) = 0$ because supp $\delta(\mathfrak{q} - 1) = V$. □

Remark 46. *For any locally constant function* h*, it holds that* $h|\mathfrak{q}-1|_p^\alpha \delta(\mathfrak{q}-1) \in \mathcal{D}'_{\mathbb{C}}$*, see e.g. [58, p. 126, Proposition 3.16]. Then* $(h|\mathfrak{q} - 1|_p^\alpha \delta(\mathfrak{q} - 1), \psi) = (|\mathfrak{q} - 1|_p^\alpha \delta(\mathfrak{q} - 1), h\psi) = (\delta(\mathfrak{q} - 1), |\mathfrak{q} - 1|_p^\alpha h\psi) = 0$ *for any* $\psi \in \mathcal{D}_{\mathbb{C}}$*.*

Remark 47. *Let us make some comments about orthogonal invariance in this setting.*

(i) Let $\varphi \in \mathcal{D}_{\mathbb{C}}$ *and let* $T \in \mathcal{D}'_{\mathbb{C}}$*. We define the action of* $\Lambda \in \boldsymbol{O}(\mathfrak{q})$*, by putting*

$$(\Lambda\varphi)(x) = \varphi\left(\Lambda^{-1}x\right),$$

and the action of Λ *on* T*, by putting*

$$(\Lambda T, \varphi) = \left(T, \Lambda^{-1}\varphi\right).$$

We say that T is invariant under $\boldsymbol{O}(\mathfrak{q})$, if $\Lambda T = T$ for any $\Lambda \in \boldsymbol{O}(\mathfrak{q})$.

(ii) *T is invariant under $\boldsymbol{O}(\mathfrak{q}) \Leftrightarrow \widehat{T}$ is invariant under $\boldsymbol{O}(\mathfrak{q})$. We first notice that by using $\mathcal{B}(\Lambda^{-1}y, \Lambda^{-1}k) = \mathcal{B}(y,k)$ for any $\Lambda \in \boldsymbol{O}(\mathfrak{q})$, we have*

$$\left(\widehat{\Lambda^{-1}\varphi}\right)(k) = \int_{\mathbb{Q}_p^4} \chi_p\left(\mathcal{B}(x,k)\right)\left(\Lambda^{-1}\varphi\right)(x)\,d\mu(x)$$
$$= \int_{\mathbb{Q}_p^4} \chi_p\left(\mathcal{B}(x,k)\right)\varphi(\Lambda x)\,d\mu(x) = \int_{\mathbb{Q}_p^4} \chi_p\left(\mathcal{B}\left(\Lambda^{-1}y, \Lambda^{-1}(\Lambda k)\right)\right)\varphi(y)\,d\mu(y)$$
$$= \int_{\mathbb{Q}_p^4} \chi_p\left(\mathcal{B}(y,\Lambda k)\right)\varphi(y)\,d\mu(y) = \widehat{\varphi}(\Lambda k),$$

i.e. $\left(\widehat{\Lambda^{-1}\varphi}\right) = \Lambda^{-1}\widehat{\varphi}$. Now, assuming that $\Lambda T = T$ for any $\Lambda \in \boldsymbol{O}(\mathfrak{q})$, we have

$$\left(\Lambda\widehat{T},\varphi\right) = \left(\widehat{T},\Lambda^{-1}\varphi\right) = \left(T,\widehat{\Lambda^{-1}\varphi}\right) = \left(T,\Lambda^{-1}\widehat{\varphi}\right) = (\Lambda T,\widehat{\varphi})$$
$$= (T,\widehat{\varphi}) = (\widehat{T},\varphi).$$

Here, it is worth to mention that our definition of Fourier transform using the bilinear form $\mathcal{B}$ plays a crucial role.

(iii) *By a result of Rallis-Schiffman, the distribution $\delta(\mathfrak{q}-1)$ is the unique (up to multiplication by complex constants) distribution supported on V invariant under $\boldsymbol{O}(\mathfrak{q})$, [46].*

Theorem 48. *There exist fundamental solutions $E_{\mathfrak{q},\alpha}$ for operators $\square_{\mathfrak{q},\alpha}$ which are invariant under the action of $\boldsymbol{O}(\mathfrak{q})$. Furthermore, the distributions $E_{\mathfrak{q},\alpha}$ satisfy the following:*

- $$\mathcal{F}(E_{\mathfrak{q},\alpha}) = \mathcal{F}(E^0_{\mathfrak{q},\alpha}) + C\delta(\mathfrak{q}-1), \tag{3.17}$$

  where C is a non-zero complex constant and $\mathcal{F}(E^0_{\mathfrak{q},\alpha})$, $\delta(\mathfrak{q}-1)$ are distributions invariant under $\boldsymbol{O}(\mathfrak{q})$.

- $$1_V\mathcal{F}(E_{\mathfrak{q},\alpha}) = C\delta(\mathfrak{q}-1). \tag{3.18}$$

In particular, the restriction of $\mathcal{F}(E_{\mathfrak{q},\alpha})$ to V is unique up to multiplication for a non-zero complex constant.

Proof. The existence of fundamental solutions for operators $\square_{\mathfrak{q},\alpha}$ is guaranteed by Theorem 134 in [73]. If $E^0_{\mathfrak{q},\alpha}$ is a fundamental solution for $\square_{\mathfrak{q},\alpha}$, then, by Lemmas 33, 45, $E^0_{\mathfrak{q},\alpha}+C\mathcal{F}^{-1}\left[\delta(\mathfrak{q}-1)\right]$ is also a fundamental solution for any non-zero complex constant C. Therefore, the Fourier transform of any fundamental solution may be written as

$$\mathcal{F}\left[E_{\mathfrak{q},\alpha}\right]=\mathcal{F}\left[E^0_{\mathfrak{q},\alpha}\right]+C\delta(\mathfrak{q}-1), \tag{3.19}$$

for some fundamental solution $E^0_{\mathfrak{q},\alpha}$ and some non-zero complex constant C.

Remark 49. *In fact, if there is another fundamental solution $E'_{\mathfrak{q},\alpha}$ of $\square_{\mathfrak{q},\alpha}$, invariant under $\boldsymbol{O}(\mathfrak{q})$, satisfying*

$$\mathcal{F}\left[E_{\mathfrak{q},\alpha}\right]=\mathcal{F}\left[E'_{\mathfrak{q},\alpha}\right]+C\delta(\mathfrak{q}-1)\,, \tag{3.20}$$

then from (3.19) *and* (3.20) *we get that $\mathcal{F}\left[E'_{\mathfrak{q},\alpha}-E^0_{\mathfrak{q},\alpha}\right]$ is a distribution supported on V and invariant under $\boldsymbol{O}(\mathfrak{q})$, and consequently $\mathcal{F}\left[E'_{\mathfrak{q},\alpha}-E^0_{\mathfrak{q},\alpha}\right]=C_0\delta(\mathfrak{q}-1)\,,$ for some constant C_0.*

By Lemmas 41, 42 and Remark 43, there exists a fundamental solution $E^0_{\mathfrak{q},\alpha}$, such that $\mathcal{F}\left[E^0_{\mathfrak{q},\alpha}\right]$ is a linear combination of distributions of any of the types

$$\int_{\mathbb{Q}_p^4\setminus V}|\mathfrak{q}(x)-1|_p^{-\alpha}\theta(x)\,d^4x \quad \text{or} \quad p^{\alpha}\int_{\mathbb{Z}_p}|u_0|_p^{-\alpha}(\Theta_b(u_0)-\Theta_b(0))\,du_0,$$

with $\Theta_b(u_0)$ defined as in Lemma 42. In addition, we have

$$1_V\mathcal{F}\left[E^0_{\mathfrak{q},\alpha}\right]=0 \text{ in } \mathcal{D}'_{\mathbb{C}}(\mathbb{Q}_p^4).$$

The rest of assertions announced follows from Remark 47 by the following:

Claim. The distribution $E^0_{\mathfrak{q},\alpha}$ is invariant under $\boldsymbol{O}(\mathfrak{q})$.

We first note that

$$\Lambda|\mathfrak{q}-1|_p^s=|\mathfrak{q}-1|_p^s \text{ for any } \Lambda\in\boldsymbol{O}(\mathfrak{q})\,,\text{ and } \ \operatorname{Re}(s)>0, \tag{3.21}$$

because $\mathfrak{q}\left(\Lambda^{-1}y\right) = \mathfrak{q}\left(y\right)$ for any $\Lambda \in \boldsymbol{O}\left(\mathfrak{q}\right)$, and any $y \in \mathbb{Q}_p^4$. Now, we rewrite (3.21) as

$$\left(|\mathfrak{q}-1|_p^s, \Lambda^{-1}\varphi\right) = \left(|\mathfrak{q}-1|_p^s, \varphi\right) \text{ for } \Lambda \in \boldsymbol{O}\left(\mathfrak{q}\right), \varphi \in \mathcal{D}_{\mathbb{C}}, \text{ and } \operatorname{Re}(s) > 0,$$

and use that $\Lambda^{-1}\varphi \in \mathcal{D}_{\mathbb{C}}$ for $\varphi \in \mathcal{D}_{\mathbb{C}}$, and that the distribution $|\mathfrak{q}-1|_p^s$ admits a meromorphic continuation to the whole complex plane to conclude that (3.21) is valid for any s. We now recall that $\mathcal{F}\left[E_{q,\alpha}^0\right] = c_0 \in \mathcal{D}'_{\mathbb{C}}$, where

$$\begin{aligned}\left(|\mathfrak{q}-1|_p^s, \varphi\right) &= \sum_{j=-j_0}^{\infty} \left(c_j, \varphi\right)(s+\alpha)^j = \left(\Lambda|\mathfrak{q}-1|_p^s, \varphi\right) = \left(|\mathfrak{q}-1|_p^s, \Lambda^{-1}\varphi\right) \\ &= \sum_{j=-j_0}^{\infty} \left(c_j, \Lambda^{-1}\varphi\right)(s+\alpha)^j,\end{aligned}$$

then $(c_0, \varphi) = (c_0, \Lambda^{-1}\varphi)$, which implies that c_0 is invariant under $\boldsymbol{O}\left(\mathfrak{q}\right)$, and consequently, $E_{q,\alpha}^0$ is invariant under $\boldsymbol{O}\left(\mathfrak{q}\right)$. □

Chapter 4

Klein-Gordon type operators acting on $\mathcal{H}_\infty$

In Chapter 5 we will construct a family of quantum scalar fields over a $p-$adic spacetime. Here we present a class of continuous operators that will be used in that construction.

4.1 Some technical continuous operator

Lemma 50. *Let $\mathfrak{f}(k) \in \mathbb{Q}_p[k_0, k_1, k_2, k_3]$ be a non-constant homogeneous polynomial of degree e and $\alpha > 0$. Then there exists a positive constant $A = A(\mathfrak{f}, \alpha)$ such that*

$$|\mathfrak{f}(k) - 1|_p^\alpha \leq A[k]_p^{e\alpha} \quad \textit{for } k \in \mathbb{Q}_p^4.$$

Proof. We first note that $|\mathfrak{f}(k) - 1|_p^\alpha \leq \left[\max\{|\mathfrak{f}(k)|_p, 1\}\right]^\alpha$. We now use that $|\mathfrak{f}(k)|_p \leq C(\mathfrak{f})\,[k]_p^e$ for $k \in \mathbb{Q}_p^4$, to obtain

$$\begin{aligned}|\mathfrak{f}(k) - 1|_p^\alpha &\leq \left[\max\{C(\mathfrak{f})\,[k]_p^e, 1\}\right]^\alpha \leq \left[\max\{C(\mathfrak{f}), 1\}\right]^\alpha \left[\max\left[[k]_p^e, 1\right]\right]^\alpha \\ &= A[k]_p^{e\alpha}.\end{aligned}$$

□

Remark 51. *For $\alpha \in \mathbb{R}$, we set $\lceil \alpha \rceil := \min\{\gamma \in \mathbb{Z}; \gamma \geq \alpha\}$, the ceiling function.*

Lemma 52. *The mapping*

$$\begin{array}{cccc}\Box_{\mathfrak{q},\alpha}: & \mathcal{H}_\infty(\mathbb{K}) & \to & \mathcal{H}_\infty(\mathbb{K}) \\ \\ & h & \to & \Box_{\mathfrak{q},\alpha}h\end{array}$$

is a well-defined continuous linear operator between locally convex spaces.

Proof. Take $\mathbb{K}=\mathbb{C}$. Let us first prove that $\Box_{\mathfrak{q},\alpha}$ is a well-defined linear operator. Let $h\in\mathcal{H}_{l+\lceil 4\alpha\rceil}(\mathbb{C})$, then by the Lemma 50, with $e=2$, we have

$$\begin{aligned}\|\Box_{\mathfrak{q},\alpha}h\|_l^2 &= \int_{\mathbb{Q}_p^4}[\xi]_p^l|(\widehat{\Box_{\mathfrak{q},\alpha}h})(k)|^2d^4k=\int_{\mathbb{Q}_p^4}[\xi]_p^l|\mathfrak{q}(k)-1|_p^{2\alpha}|\widehat{h}(k)|^2d^4k \\ &\leq C\int_{\mathbb{Q}_p^4}[\xi]_p^{l+4\alpha}|\widehat{h}(k)|^2d^4k\leq C\int_{\mathbb{Q}_p^4}[\xi]_p^{l+\lceil 4\alpha\rceil}|\widehat{h}(k)|^2d^4k=C\|h\|_{l+\lceil 4\alpha\rceil}^2.\end{aligned}$$

By Lemma 9-(i), $\Box_{\mathfrak{q},\alpha}h\in\mathcal{H}_l(\mathbb{C})$, i.e. $\Box_{\mathfrak{q},\alpha}$ is a well-defined, linear, and continuous operator from $\mathcal{H}_{l+\lceil 4\alpha\rceil}(\mathbb{C})$ into $\mathcal{H}_l(\mathbb{C})$ for any $l\in\mathbb{N}$. In turn, this implies that $\Box_{\mathfrak{q},\alpha}$ is a well-defined linear operator from $\mathcal{H}_\infty(\mathbb{C})$ into $\mathcal{H}_\infty(\mathbb{C})$. To establish the continuity, we use the fact that $(\mathcal{H}_\infty(\mathbb{C}),d)$ is a metric space. Take a sequence $\{\varphi_n\}_{n\in\mathbb{N}}\subset\mathcal{H}_\infty(\mathbb{C})$ such that $\varphi_n\overset{d}{\to}\varphi$, with $\varphi\in\mathcal{H}_\infty(\mathbb{C})$, which is equivalent to say that $\varphi_n\overset{\|\cdot\|_r}{\to}\varphi$, for all $r\in\mathbb{N}$. Take $l\in\mathbb{N}$ and φ, $\varphi_n\in\mathcal{H}_{l+\lceil 4\alpha\rceil}(\mathbb{C})$, then by the continuity of $\Box_{\mathfrak{q},\alpha}:\mathcal{H}_{l+\lceil 4\alpha\rceil}(\mathbb{C})\to\mathcal{H}_l(\mathbb{C})$, we have $\Box_{\mathfrak{q},\alpha}\varphi_n\overset{\|\cdot\|_l}{\to}\Box_{\mathfrak{q},\alpha}\varphi$, and since l is arbitrary in $\mathbb{N}$, we conclude that $\Box_{\mathfrak{q},\alpha}\varphi_n\overset{d}{\to}\Box_{\mathfrak{q},\alpha}\varphi$.

We know turn to the case $\mathbb{K}=\mathbb{R}$. Since $(\Box_{\mathfrak{q},\alpha}\varphi)(x)=\overline{(\Box_{\mathfrak{q},\alpha}\varphi)(x)}$ for $\varphi\in\mathcal{H}_\infty(\mathbb{R})$, the statement is also valid in $\mathcal{H}_\infty(\mathbb{R})$. ☐

Remark 53. *The preceding lemma remains valid if we replace* $|\mathfrak{q}(k)-1|_p^\alpha$ *by* $g([k]_p)|\mathfrak{q}(k)-1|_p^\alpha$, *where* $g:\mathbb{R}_+\to\mathbb{C}$ *is any continuous function.*

Remark 54. *We recall that* V *is a* $p-$*adic compact submanifold of* $\mathbb{Z}_p^4$ *of codimension one. We denote by* $d\lambda$ *the measure corresponding to the distribution* $\delta(\mathfrak{q}-1)$ *as before. Then* $(V,\mathcal{B}(V),d\lambda)$ *is a measure space, where* $\mathcal{B}(V)$ *is the Borel* σ*-algebra generated by the open compact subsets of* V, *and thus the space* $L^2_{\mathbb{K}}(V,d\lambda)$ *is well-defined.*

Proposition 55. *The mapping*

$$\begin{array}{cccc} R: & \mathcal{H}_l(\mathbb{C}) & \rightarrow & L^2_{\mathbb{C}}\left(V^{+}, d\lambda\right) \\ & & & \\ & f & \rightarrow & \widehat{f}\mid_{V^{+}} \end{array}$$

determines a well-defined operator satisfying

$$\|R(f)\|_{L^2_{\mathbb{C}}(V^{+},d\lambda)} \leq C\,\|f\|_l \tag{4.1}$$

for any $l \in \mathbb{N}$*. Consequently,* R *induces a continuous operator from* $\mathcal{H}_\infty(\mathbb{C})$ *into* $L^2_{\mathbb{C}}\left(V^{+}, d\lambda\right)$*.*

Proof. Since $\mathcal{D}_{\mathbb{C}}$ is dense in $\mathcal{H}_l(\mathbb{C})$ for any $l \in \mathbb{N}$, in order to prove (4.1) we may assume without loss of generality that $f \in \mathcal{D}_{\mathbb{C}}$ and that $\widehat{f}\mid_{V^{+}}$ is not the constant function zero. Notice that

$$\|R(f)\|^2_{L^2_{\mathbb{C}}(V^{+},d\lambda)} = \int_{U_{\mathfrak{q}}} \left|\widehat{f}\left(\sqrt{\omega\left(\boldsymbol{k}\right)}, \boldsymbol{k}\right)\right|^2 \frac{d^3\boldsymbol{k}}{\left|\sqrt{\omega\left(\boldsymbol{k}\right)}\right|_p}, \tag{4.2}$$

where $\left|\sqrt{\omega\left(\boldsymbol{k}\right)}\right|_p \neq 0$, cf. Remark 34. For $m \in \mathbb{Q}_p^{\times}$, we set

$$V_m = \left\{(k_0, \boldsymbol{k}) \in \mathbb{Q}_p^4; \mathfrak{q}\left(k_0, \boldsymbol{k}\right) = m\right\}.$$

We recall that $\mathfrak{q}\left(k_0, \boldsymbol{k}\right) = k_0^2 - \mathfrak{q}_0\left(\boldsymbol{k}\right)$. Then V_m is a $p-$adic compact submanifold of $\mathbb{Q}_p^4$ of codimension one. In the case in which $V_m \neq \emptyset$, we denote by $d\lambda\left(m\right)$ the measure on V_m induced by the Gel'fand-Leray form on V_m. Then $dk_0 d^3\boldsymbol{k} = d\lambda\left(m\right)dm$, where dm is the normalized Haar measure of $\mathbb{Q}_p$.

Claim C. For $\widehat{f}\left(k_0, \boldsymbol{k}\right) \in \mathcal{D}_{\mathbb{C}}$, the $\mathbb{R}$-valued function defined by

$$\int_{\mathbb{Q}_p^{\times}} \int_{V_m} \left|\widehat{f}\left(k_0, \boldsymbol{k}\right)\right|^2 d\lambda\left(m\right)dm$$

is in $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ and

$$\|f\|_0^2 = \int_{\mathbb{Q}_p^4} \left|\widehat{f}\left(k_0, \boldsymbol{k}\right)\right|^2 dk_0 d^3\,\boldsymbol{k} = \int_{\mathbb{Q}_p^{\times}} \int_{V_m} \left|\widehat{f}\left(k_0, \boldsymbol{k}\right)\right|^2 d\lambda\left(m\right)dm. \tag{4.3}$$

This claim is a very particular version of a general theorem on integration over the fibers in the framework of $p-$adic manifolds, see [23, Theorem 7.6.1].

Claim D. There exists a positive constant C_0 such that

$$\|f\|_0^2 \geq C_0 \int_{U_{\mathfrak{q}}} \left|\widehat{f}\left(\sqrt{\omega(\boldsymbol{k})}, \boldsymbol{k}\right)\right|^2 \frac{d^3\boldsymbol{k}}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p}. \tag{4.4}$$

Estimation (4.1) follows from (4.2)-(4.4). The fact that operator R extends to $\mathcal{H}_\infty(\mathbb{C})$ follows from (4.1), by using a classical argument based on convergence of sequences due to the fact that the topology of $\mathcal{H}_\infty(\mathbb{C})$ is metrizable.

Proof of Claim D. In order to prove the Claim we proceed as follows. We set $G_M := 1 + p^M\mathbb{Z}_p$, for $M \geq 1$. Then G_M is a multiplicative subgroup of the group of squares of $\mathbb{Q}_p^\times$. This is a compact subgroup so its Haar measure, denoted as $vol(G_M)$, is finite. Now, we notice that

$$\begin{aligned}\int_{\mathbb{Q}_p^\times}\int_{V_m} \left|\widehat{f}(k_0, \boldsymbol{k})\right|^2 d\lambda(m)\, dm &\geq \int_{G_M}\int_{V_m} \left|\widehat{f}(k_0, \boldsymbol{k})\right|^2 d\lambda(m)\, dm \\ &= \int_{G_M}\int_{V_m} \left|\widehat{f}(k_0, \boldsymbol{k})\right|^2 \frac{d^3\boldsymbol{k}dm}{|m + \mathfrak{q}_0(\boldsymbol{k})|_p^{\frac{1}{2}}}. \end{aligned} \tag{4.5}$$

We now use the fact that

Claim E. The mapping

$$\begin{array}{rccc} \sqrt{\cdot}: & G_M & \to & G_M \\ & & & \\ & m & \to & \sqrt{m} \end{array}$$

and its inverse are $p-$adic analytic functions, for M sufficiently large.

We change variables in the last integral in (4.5) as $y_0 = \frac{k_0}{\sqrt{m}}$, $\boldsymbol{y} = \frac{\boldsymbol{k}}{\sqrt{m}}$, then $dk_0 d^3\,\boldsymbol{k} = dy_0 d^3\boldsymbol{y}$ and

$$\begin{aligned}&\int_{G_M}\int_{V_m} \left|\widehat{f}(k_0, \boldsymbol{k})\right|^2 \frac{d^3\boldsymbol{k}dm}{|m + \mathfrak{q}_0(\boldsymbol{k})|_p^{\frac{1}{2}}} \\ &= \int_{G_M}\int_{V} \left|\widehat{f}\left(\sqrt{m}y_0, \sqrt{m}\boldsymbol{y}\right)\right|^2 \frac{d^3\boldsymbol{y}dm}{|1 + \mathfrak{q}_0(\boldsymbol{y})|_p^{\frac{1}{2}}}.\end{aligned}$$

Finally since $\widehat{f}$ is locally constant and $\sqrt{m}$ is a unit for every $m \in G_M$, we have for M sufficiently large that

$$\int_{G_M}\int_V \left|\widehat{f}\left(\sqrt{m}y_0,\sqrt{m}\boldsymbol{y}\right)\right|^2 \frac{d^3\boldsymbol{y}dm}{|1+\mathfrak{q}_0(\boldsymbol{y})|_p^{\frac{1}{2}}} = \int_{G_M}\int_V \left|\widehat{f}(y_0,\boldsymbol{y})\right|^2 \frac{d^3\boldsymbol{y}dm}{|1+\mathfrak{q}_0(\boldsymbol{y})|_p^{\frac{1}{2}}}$$
$$\geq vol\left(G_M\right)\int_{V^+}\left|\widehat{f}(y_0,\boldsymbol{y})\right|^2 \frac{d^3\boldsymbol{y}}{|1+\mathfrak{q}_0(\boldsymbol{y})|_p^{\frac{1}{2}}}.$$

Proof of Claim E.

We first notice that $\left(1+p^M\mathbb{Z}_p\right)^2 = 1+2p^M\mathbb{Z}_p = 1+p^M\mathbb{Z}_p$ for $M\geq 2$, see Lemma 8.4.1 in [23]. This means that the mapping

$$\begin{array}{ccc} G_M & \to & G_M \\ x & \to & x^2 \end{array} \tag{4.6}$$

is well-defined and surjective. Then for any $m\in G_M$, the equation $x^2=m$ has a solution $\sqrt{m}$ in G_M. Notice that there is another solution $-\sqrt{m}=-1+$ (higher order terms) which does not belong to G_M. Consequently the mapping

$$\begin{array}{ccc} G_M & \to & G_M \\ m & \to & \sqrt{m} \end{array} \tag{4.7}$$

is well-defined. The fact that the mappings (4.6)-(4.7) are $p-$adic analytic follows from the implicit function theorem. □

Remark 56. *The preceding Proposition remains valid if we replace $R(f)=\hat{f}|_{V^+}$ by $R(f)(k)=g\left([k]_p\right)\hat{f}(k)|_{V^+}$, where g is any continuous function $g:\mathbb{R}_+\to\mathbb{C}$.*

Lemma 57. *There exist a positive constant C such that*

$$\frac{1}{\left|1+\mathfrak{q}_0(\boldsymbol{k})\right|_p}\leq C \text{ for any } \boldsymbol{k}\in\mathbb{Q}_p^3.$$

Proof. The hypothesis $p\equiv 1 \bmod 4$ implies $W=\left\{\boldsymbol{k}\in\mathbb{Z}_p^3; 1+\mathfrak{q}_0(\boldsymbol{k})=0\right\}=\emptyset$, see Remark 34.

Claim A. $\left|1+\mathfrak{q}_0(\boldsymbol{k})\right|_p > C_1$ for any $C_1\in(0,p)$ and for any $\|\boldsymbol{k}\|_p\geq p$.

We recall that $\mathfrak{q}_0(\boldsymbol{k})$ and $\mathfrak{q}(k_0,\boldsymbol{k})$ are elliptic quadratic forms and that

$$\left|\mathfrak{q}_0(\boldsymbol{k})\right|_p = |\mathfrak{q}(0,\boldsymbol{k})|_p \geq \left(\inf_{\boldsymbol{x}\in S_0^3}|\mathfrak{q}(0,\boldsymbol{x})|_p\right)\|\boldsymbol{k}\|_p^2 = p^{-1}\|\boldsymbol{k}\|_p^2 \text{ for any } \boldsymbol{k}\in\mathbb{Q}_p^3, \tag{4.8}$$

see (3.4). Now, $p^{-1}\|\boldsymbol{k}\|_p^2 > 1$ if and only if $\|\boldsymbol{k}\|_p \geq p$, and by applying the ultrametric property of the norm $\|\cdot\|_p$, we get from (4.8), that for $\|\boldsymbol{k}\|_p \geq p$,

$$|1+\mathfrak{q}_0(\boldsymbol{k})|_p = \max\{1, \mathfrak{q}_0(\boldsymbol{k})\} \geq p^{-1}\|\boldsymbol{k}\|_p^2 \geq p > C_1 \text{ for any } C_1 \in (0,p).$$

Claim B. There exist a constant C_0 such that $\inf_{\boldsymbol{k}\in\mathbb{Z}_p^3} |1+\mathfrak{q}_0(\boldsymbol{k})|_p \geq C_0 > 0$.

This assertion follows from the fact that $|1+\mathfrak{q}_0(\boldsymbol{k})|_p > 0$ for any $\boldsymbol{k} \in \mathbb{Z}_p^3$. The statement of the lemma is a consequence of Claims A and B. □

Lemma 58. *The mapping*

$$\begin{array}{cccc} R: & L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3, d^3\boldsymbol{x}\right) & \to & L^2_{\mathbb{C}}\left(V^+, d\lambda\right) \\ & & & \\ & g & \to & \widehat{g}\mid_{V^+} \end{array}$$

satisfies $\|R(g)\|_{L^2_{\mathbb{C}}(V^+,d\lambda)} \leq C\,\|g\|_{L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)}$. *Here* $\widehat{g}\,(\boldsymbol{k})$ *denotes the* 3*-dimensional Fourier transform is defined with respect to the bilinear form* $-\mathfrak{B}_0\,(\mathbf{x},\mathbf{y}) = -sx_1y_1 - px_2y_2 + spx_3y_3$.

Proof. The results follows from Lemma 57, by using that $|k_0|_p = \left|\sqrt{\omega\,(\boldsymbol{k})}\right|_p = |1+\mathfrak{q}_0(\boldsymbol{k})|_p^{\frac{1}{2}}$ for $\boldsymbol{k} \in U_{\mathfrak{q}}$. □

Remark 59. *Some observations about the functional spaces involved here:*

(i) Let X *be a locally compact totally disconnected space. We denote by* $\mathcal{D}_{\mathbb{C}}(X)$ *the* $\mathbb{C}$*-vector space of locally constant functions with compact support. We recall that* $V^+ \subset \mathbb{Q}_p^4$ *is an open and compact subset, then* $\mathbb{Q}_p^4 \setminus V^+$ *is open and closed subset, and thus* V^+ *and* $\mathbb{Q}_p^4 \setminus V^+$ *are locally compact totally disconnected spaces. The following exact sequence holds:*

$$0 \to \mathcal{D}_{\mathbb{C}}(V^+) \to \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4) \to \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4 \setminus V^+) \to 0, \tag{4.9}$$

see e.g. [23, p. 99].

(ii) It is well-known that the $\mathbb{C}$*-space of finite-valued simple functions is dense in* $L^2_{\mathbb{C}}(V^+, d\lambda)$*. By using the fact that* $d\lambda = \frac{d^3\boldsymbol{k}}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p}$ *is an inner regular*

measure, one can show that any finite-valued simple function can be approximated in the $L^2_{\mathbb{C}}(V^+, d\lambda)$*- norm by an element of* $\mathcal{D}_{\mathbb{C}}(V^+)$*. i.e.* $\mathcal{D}_{\mathbb{C}}(V^+)$ *is dense in* $L^2_{\mathbb{C}}(V^+, d\lambda)$*.*

(iii) The mapping

$$\begin{array}{ccc} L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4k\right) & \underset{\to}{R} & L^2_{\mathbb{C}}(V^+, d\lambda) \\ \\ f & \to & \widehat{f}\mid_{V^+} \end{array}$$

is a well-defined continuous mapping, more precisely,

$$\left\| \widehat{f}\mid_{V^+} \right\|_{L^2_{\mathbb{C}}(V^+, d\lambda)} \leq C \left\| \widehat{f} \right\|_{L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4k\right)} = C \left\| f \right\|_{L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4k\right)}. \tag{4.10}$$

Indeed, (4.10) holds when $\widehat{f} \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$ *and* $\widehat{f}\mid_{V^+} \in \mathcal{D}_{\mathbb{C}}(V^+)$*, see Claim D, then (4.10) follows by the fact that* $\mathcal{D}_{\mathbb{C}}(V^+)$ *is dense in* $L^2_{\mathbb{C}}(V^+, d\lambda)$ *and that* $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$ *is dense in* $L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4k\right)$*.*

Remark 60. *Regarding the spaces of integrable functions introduced in the preceding Remark, we note the following.*

(i) We have the following sequence:

$$L^2_{\mathbb{C}}\left(V^+, d\lambda\right) \overset{J}{\hookrightarrow} L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3\boldsymbol{k}\right) \hookrightarrow L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3, d^3\boldsymbol{k}\right),$$

where '$\hookrightarrow$*' denotes an isometry. The mapping* J *is defined as*

$$f\left(\omega\left(\boldsymbol{k}\right), \boldsymbol{k}\right) \overset{J}{\to} \frac{f\left(\omega\left(\boldsymbol{k}\right), \boldsymbol{k}\right)}{\left|\sqrt{\omega\left(\boldsymbol{k}\right)}\right|_p^{1/2}}, \quad \boldsymbol{k} \in U_{\mathfrak{q}}.$$

Since $U_{\mathfrak{q}} \subset \mathbb{Q}_p^3$ *is open and compact, any function* $f : U_{\mathfrak{q}} \to \mathbb{C}$ *can be extended to* $\mathbb{Q}_p^3$ *by putting* $f \mid_{\mathbb{Q}_p^3 \smallsetminus U_{\mathfrak{q}}} \equiv 0$*. It is known that* $L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3, d^3\boldsymbol{k}\right)$ *admits a countable wavelet basis, see e.g. [2, Theorem 8.12.1], consequently* $L^2_{\mathbb{C}}(V^+, d\lambda)$ *is separable.*

(ii) Since $L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4\boldsymbol{k}\right)$ *and* $L^2_{\mathbb{C}}(V^+, d\lambda)$ *are separable spaces, we have*

$$\bigotimes_{j=1}^{n} L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4x_j\right) = L^2_{\mathbb{C}}\left(\mathbb{Q}_p^{4n}, \prod_{j=1}^{n} d^4x_j\right),$$

and

$$\bigotimes_{j=1}^{n} L^2_{\mathbb{C}}\left(V^+, d\lambda_j\right) = L^2_{\mathbb{C}}\left(\left(V^+\right)^n, \prod_{j=1}^{n} d\lambda_j\right),$$

where each d^4x_j *denotes a copy of normalized Haar measure of* $\mathbb{Q}_p^4$*, and each* $d\lambda_j$ *denotes a copy of the measure* $d\lambda$.

(iii) Take $\theta^{(n+1)}\left(y, x_1, \dots, x_n\right) \in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4, d^4y\right) \bigotimes L^2_{\mathbb{C}}\left(\mathbb{Q}_p^{4n}, \prod_{j=1}^{n} d^4x_j\right)$*, then*

$$\int_{\mathbb{Q}_p^{4n}} \int_{V} \left|\theta^{(n+1)}\left(y, x_1, \dots, x_n\right)\right|^2 d\lambda\left(y\right) \prod_{j=1}^{n} d^4x_j \leq$$

$$C \int_{\mathbb{Q}_p^{4n}} \int_{\mathbb{Q}_p^4} \left|\theta^{(n+1)}\left(y, x_1, \dots, x_n\right)\right|^2 d^4y \prod_{j=1}^{n} d^4x_j = C \left\|\theta^{(n+1)}\right\|^2_{L^2_{\mathbb{C}}\left(\mathbb{Q}_p^{4(n+1)}, \prod_{j=1}^{n+1} d^4x_j\right)}.$$

This result follows from Claim D, by using Fubini's theorem.

Lemma 61. *For* $f \in L^2_{\mathbb{C}}\left(V^+, d\lambda\right)$*, we define* $T_{V^+}\left(f\right) \in \mathcal{D}'_{\mathbb{C}}$ *by*

$$\left(T_{V^+}\left(f\right), \varphi\right) = \int_{V^+} f(x)\varphi\left(x\right) d\lambda\left(x\right) \text{ for } \varphi \in \mathcal{D}_{\mathbb{C}}.$$

Then we have the following sequence of continuous mappings:

$$\mathcal{H}_\infty\left(\mathbb{C}\right) \overset{R}{\to} L^2_{\mathbb{C}}\left(V^+, d\lambda\right) \overset{T_{V^+}}{\to} \mathcal{H}^*_\infty\left(\mathbb{C}\right),$$

where the map R *is defined as in Proposition 55.*

Proof. The support of $T_{V^+}\left(f\right)$ is compact since it is contained in V, which is a compact subset of $\mathbb{Q}_p^4$. The Fourier transform of $T_{V^+}\left(f\right)$ in $\mathcal{D}'_{\mathbb{C}}$ is the locally constant function

$$\widehat{f}\left(k\right) = \int_{V^+} \chi_p\left(\mathcal{B}\left(x, k\right)\right) f(x) d\lambda\left(x\right),$$

for a similar calculation the reader may see, for instance, [2, Theorem 4.9.3]. Now, identifying f with the induced distribution $T_{V^+}f$ on V^+, by using the definition of $\mathcal{H}^*_\infty\left(\mathbb{C}\right)$ (see (1.6)), the Cauchy-Schwartz inequality, and the fact that

$\int_{\mathbb{Q}_p^4} [k]^{-l} d^4k < \infty$ for $l \geq 5$, we have

$$\begin{aligned}\|f\|_{-l}^2 &= \int_{\mathbb{Q}_p^4} [k]^{-l} \left|\widehat{f}(k)\right|^2 d^4k = \int_{\mathbb{Q}_p^4} [k]^{-l} \left|\int_{V^+} \chi_p\left(\mathcal{B}\left(x,k\right)\right) f(x) d\lambda\left(x\right)\right|^2 d^4k \\ &\leq C(l) \int_{V^+} |f(x)|^2 \, d\lambda\left(x\right) = C(l) \, \|f\|_{L^2_{\mathbb{C}}(V^+,d\lambda)}^2 ,\end{aligned}$$

which implies that $T_{V^+}(f) \in \mathcal{H}_\infty^*(\mathbb{C})$. $\square$

Chapter 5

Free non-Archimedean quantum fields

5.1 The Segal quantization

We start by reviewing some well-known fact about quantization. For an in-depth discussion the reader may consult [56, 49], see also [9, 14, 35, 57] for more physically-oriented approaches. Our presentation follows closely the book of Reed and Simon [49]. In particular, our notation mimics the one used in that book. We set $\mathcal{H} = L^2_{\mathbb{C}}(V^+, d\lambda)$ and denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}$. We assume that $\langle f, \alpha g \rangle = \alpha \langle f, g \rangle$, for $\alpha \in \mathbb{C}$, and f, $g \in \mathcal{H}$. We define *the Fock space over* $\mathcal{H}$ as $\mathfrak{F}(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{(n)}$, where $\mathcal{H}^{(n)} = \otimes_{k=1}^{n} \mathcal{H}$, by definition $\mathcal{H}^{(0)} = \mathbb{C}$. We denote by $S_n : \mathcal{H}^{(n)} \to S\mathcal{H}^{(n)}$, the symmetrization operator, and define $S = \oplus_{n=0}^{\infty} S_n$, see [48, Section II.4]. The symmetric Fock space over $\mathcal{H}$ (also called *the boson Fock space* over $\mathcal{H}$) is defined as $\mathfrak{F}_s(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}_s^{(n)}$, where $\mathcal{H}_s^{(n)} = S_n \mathcal{H}^{(n)}$. We call $\mathcal{H}_s^{(n)}$ the *n-particle subspace* of $\mathfrak{F}_s(\mathcal{H})$. We use the same symbol $\langle \cdot, \cdot \rangle$ to denote the inner product of $\mathfrak{F}(\mathcal{H})$.

We now fix a vector f in $\mathcal{H}$. For the vectors of the form $\eta = \psi_1 \otimes \ldots \otimes \psi_n$, we define a map $b^-(f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n-1)}$ by $b^-(f)(\eta) = \langle f, \psi_1 \rangle \psi_2 \otimes \ldots \otimes \psi_n$. Then $b^-(f)$ extends to a bounded map (of norm $\|f\|_{\mathcal{H}}$) of $\mathcal{H}^{(n)}$ in to $\mathcal{H}^{(n-1)}$. In the case $n = 0$, we define $b^-(f) : \mathcal{H}^{(0)} \to 0$. The adjoint $b^+(f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}$ of $b^-(f)$ is defined as $b^+(f)(\psi_1 \otimes \ldots \otimes \psi_n) = f \otimes \psi_1 \otimes \ldots \otimes \psi_n$. The map $f \to b^+(f)$ is linear, but $f \to b^-(f)$ is anti-linear.

The boson Fock space is invariant under $b^-(f)$ but not under $b^+(f)$. A vector

$\psi = \left\{\psi^{(n)}\right\}_{n\in\mathbb{N}} \in \mathfrak{F}_s(\mathcal{H})$ is called a *finite particle vector* if $\psi_n = 0$ for all but finitely many n. The set of all finite vectors is denoted as F_0. We set the vector $\Upsilon_0 = (1, 0, 0, \ldots)$ to be the *vacuum*.

Let A be a self-adjoint operator on $\mathcal{H}$ with domain of essential self-adjointness D. Let $D_A = \left\{\psi \in F_0; \psi^{(n)} \in \otimes_{k=1}^{n} D \text{ for each } n\right\}$. We define the operator $\Gamma(A)$ (*the second quantization of* A) on $D_A \cap \mathcal{H}_s^{(n)}$ as

$$A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A,$$

where I is the identity operator. The operator $\Gamma(A)$ is essentially self-adjoint on D_A. In the case $A = I$, the second quantization $N = \Gamma(A)$ (*the number operator*) is essentially self-adjoint on F_0 and for $\phi \in \mathcal{H}_s^{(n)}$, $N\phi = n\phi$.

The *annihilation operator* $a^-(f)$ on $\mathfrak{F}_s(\mathcal{H})$ with domain F_0 is given by

$$a^-(f) = \sqrt{N+1}\; b^-(f).$$

For ψ, η in F_0,

$$\left\langle \sqrt{N+1}\; b^-(f)\psi, \eta \right\rangle = \left\langle \psi, Sb^+(f)\sqrt{N+1}\eta \right\rangle,$$

which implies that

$$\left(a^-(f)\right)^* \restriction_{F_0} = Sb^+(f)\sqrt{N+1},$$

where '$*$' denotes the adjoint operator. The operator $\left(a^-(f)\right)^*$ is called the *creation operator*. Both $a^-(f)$ and $\left(a^-(f)\right)^* \restriction_{F_0}$ are closable, the corresponding closures are denoted as $a^-(f)$ and as $a^-(f)^*$.

Definition 62. *For $f \in \mathcal{H}$, the Segal quantum field operator $\mathbf{\Phi}_S$ on F_0 is defined as*

$$\mathbf{\Phi}_S(f) = \frac{1}{\sqrt{2}}[a^-(f) + a^-(f)^*]. \tag{5.1}$$

The mapping from $\mathcal{H}$ into the self-adjoint operators on $\mathfrak{F}_s(\mathcal{H})$ given by $f \to \mathbf{\Phi}_{\mathrm{S}}(f)$ is called *the Segal quantization over* $\mathcal{H}$. Notice that the Segal quantization is a real linear map.

Remark 63. *By using the fundamental properties of the Segal quantization, see [49, Theorem X.41], we obtain the following facts (among others):*

(i) *For each* $f \in \mathcal{H}$, $\mathbf{\Phi}_S(f)$ *is essentially self-adjoint on* F_0.

(ii) *The commutation relations: for each* $\psi \in F_0$, *and* f, $g \in \mathcal{H}$,

$$\mathbf{\Phi}_S(f)\mathbf{\Phi}_S(g)\psi - \mathbf{\Phi}_S(g)\mathbf{\Phi}_S(f)\psi = \sqrt{-1}\,\mathrm{Im}\left(\langle f, g\rangle\right)\psi, \tag{5.2}$$

that is, $[\mathbf{\Phi}_S(f), \mathbf{\Phi}_S(g)] = \sqrt{-1}\,\mathrm{Im}\left(\langle f, g\rangle\right) I$, *on* F_0.

5.1.1 The free Hermitian field of unit mass

We define for each $f \in \mathcal{H}_\infty(\mathbb{R})$,

$$\mathbf{\Phi}(f) = \mathbf{\Phi}_{\mathrm{S}}(Rf)\,,$$

with R defined as in Lemma 58, and for each $g \in \mathcal{H}_\infty(\mathbb{C})$,

$$\mathbf{\Phi}(g) = \mathbf{\Phi}(\mathrm{Re}\, g) + \sqrt{-1}\mathbf{\Phi}(\mathrm{Im}\, g)\,. \tag{5.3}$$

We call the mapping $g \to \mathbf{\Phi}(g)$ *the free Hermitian scalar field of unit mass.*

Remark 64. *By extending the mapping* R *as in Remark 56, the field* $f \mapsto \mathbf{\Phi}(f)$ *remains well-defined. We emphasize that the presence of* R *(in any of its forms) means that we are working on-shell.*

5.1.2 The p-adic restricted Poincaré group

As we do not have the structure of light cones available, we must choose a substitute for them. Here we will base our treatment on the mass shells $V^{\pm}$.

We define *the* p*-adic restricted Lorentz* group as

$$\mathcal{L}_+^{\uparrow} = \left\{\Lambda \in \boldsymbol{O}(\mathfrak{q}); \Lambda\left(V^{\pm}\right) = V^{\pm}\right\}.$$

This group is non trivial since transformations of the form

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & F \end{bmatrix} \in \boldsymbol{O}(\mathfrak{q});\ F \in \boldsymbol{O}(\mathfrak{q}_0)\right\},$$

belong to $\mathcal{L}_+^{\uparrow}$. A further justification for choosing $V^{\pm}$ as a replacement for the light cones comes from the fact that the distributions $\delta_{\pm}(\mathfrak{q}-1)$ are invariant under $\mathcal{L}_{+}^{\uparrow}$, see [73, Lemma 163].

We define *the p-adic restricted Poincaré group* $\mathcal{P}_{+}^{\uparrow}$ as the set of pairs (a,Λ), where $a\in\mathbb{Q}_p^4$ and $\Lambda\in\mathcal{L}_{+}^{\uparrow}$, with the group operation

$$(a,\Lambda_1)(b,\Lambda_2)=(a+\Lambda_1 b,\Lambda_1\Lambda_2).$$

The group $\mathcal{P}_{+}^{\uparrow}$ acts naturally on $\mathbb{Q}_p^4$ by setting $(a,\Lambda)x=\Lambda x+a$. With the topology inherited from $\left(\mathbb{Q}_p^4,\|\cdot\|_p\right)$, $\mathcal{L}_{+}^{\uparrow}$ and $\mathcal{P}_{+}^{\uparrow}$ become locally compact topological groups.

On $L^2_{\mathbb{C}}(V^+,d\lambda)$, we define the following projective representation of the restricted Poincaré group:

$$(U(a,\Lambda)\psi)(k)=\chi_p(\mathcal{B}(a,k))\,\psi\left(\Lambda^{-1}k\right). \tag{5.4}$$

5.2 The p-adic Wightman axioms

We present here a p-adic counterpart of the classical Wightman axioms, see e.g. [56, 49], and references therein. We use units where the rationalized Planck's constant and the speed of light are equal to one. We take $H=\mathfrak{F}_s(L^2_{\mathbb{C}}(V^+,d\lambda))$, $\mathfrak{U}=\Gamma(U(\cdot,\cdot))$, with $U(\cdot,\cdot)$ being defined as in (5.4), $\mathbf{\Phi}$ as in (5.3), and $D=F_0$. A *p-adic Hermitian scalar quantum field theory* is a quadruple $\{H,\mathfrak{U},\mathbf{\Phi},D\}$ which satisfies the following properties:

Relativistic invariance of states. H is a separable Hilbert space and $\mathfrak{U}(\cdot,\cdot)$ is a strongly continuous unitary representation on H of the p-adic restricted Poincaré group.

Spectral condition. We define *the closed forward semigroup* $\overline{S(V^+)}$ as the topological closure of the additive semigroup generated by the vectors of V^+. Notice that since $V^+\subset\mathbb{Z}_p^4$, $\overline{S(V^+)}$ is a compact subset of $\mathbb{Z}_p^4$. Furthermore, since $\mathcal{L}_{+}^{\uparrow}(V^+)=V^+$, we have $\mathcal{L}_{+}^{\uparrow}\left(\overline{S(V^+)}\right)=\overline{S(V^+)}$. The p-adic counterpart of the spectral condition is the following: there exists a projection-valued measure E_{V^+} on $\mathbb{Q}_p^4$ corresponding to $\mathfrak{U}(a,I)$ having support in $\overline{S(V^+)}$.

Remark 65. *In the classical case by using a Stone type theorem, see [48, Theorem VIII.12], one shows the existence of four commuting operators P_0, P_1, P_2, P_3, on a* *suitable Hilbert space so that* $\mathfrak{U}(a, I) = e^{i\sum a_j P_j}$. *In the p-adic case, we do not have a complete theory of semigroups, with p-adic time, for operators acting on complex-valued functions. For this reason, at the moment, we do not have a definition for the p-adic counterparts of the operators* P_0, P_1, P_2, P_3, *and consequently, we do not know their spectra.*

Existence and uniqueness of the vacuum. There exists a unique vector $\Upsilon_0 \in H$ such that $U(a, I)\Upsilon_0 = \Upsilon_0$ for all $a \in \mathbb{Q}_p^4$, this vector is called *the vacuum.*

Invariant domains for fields. There exists a dense subspace $D \subset H$ and a map from $\mathcal{H}_\infty(\mathbb{C})$ to the unbounded operators on H such that:

(i) For each $f \in \mathcal{H}_\infty(\mathbb{C})$, we have that $D \subset Dom(\mathbf{\Phi}(f))$, $D \subset Dom(\mathbf{\Phi}(f)^*)$, and $\mathbf{\Phi}(f)^* \upharpoonright D = \mathbf{\Phi}\left(\overline{f}\right) \upharpoonright D$.

(ii) $\Upsilon_0 \in D$, and $\mathbf{\Phi}(f) D \subset D$ for any $f \in \mathcal{H}_\infty(\mathbb{C})$.

(iii) For a fixed $\psi \in D$, the map $f \to \mathbf{\Phi}(f)\psi$ is linear.

Regularity of the field. For any ψ_1 and ψ_2 in D, the map

$$f \to \langle \psi_1, \mathbf{\Phi}(f)\psi_2 \rangle_H$$

is an element of $\mathcal{H}_\infty^*(\mathbb{C})$. In the Archimedean case this is just a tempered distribution, here it turns out to be an element of $\mathcal{H}_\infty^*(\mathbb{C})$, providing yet another argument to consider this space as the correct replacement in the $p-$adic framework of the Schwartz space $\mathcal{S}$.

Poincaré invariance of the field. For each $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, $\mathfrak{U}(a,\Lambda)D \subset D$, and for all $f \in \mathcal{H}_\infty(\mathbb{C})$, $\psi \in D$,

$$\mathfrak{U}(a, \Lambda)\, \mathbf{\Phi}(f)\, \mathfrak{U}(a, \Lambda)^{-1} \psi = \mathbf{\Phi}((a, \Lambda) f)\, \psi,$$

where

$$(a, \Lambda) f(x) = f\left(\Lambda^{-1}(x - a)\right).$$

Local commutativity. The p-adic local commutativity property states that if f, g are in $\mathcal{D}_\mathbb{C}\left(\mathbb{Z}_p^4\right)$, then

$$[\mathbf{\Phi}(f), \mathbf{\Phi}(g)]\, \Psi = (\mathbf{\Phi}(f)\mathbf{\Phi}(g) - \mathbf{\Phi}(g)\, \mathbf{\Phi}(f))\, \Psi = 0,$$

for all $\Psi \in D$. In the Archimedean case, the commutator vanishes whenever the test functions f, g are supported on two respective spacelike-separated subsets, that is, $f(x)g(y) = 0$ whenever $x - y$ does not belong to the interior of the light cone. This subset can be characterized as the 'ball of radius 0' of Minkowski spacetime in the sense of the theory of indefinite quadratic forms (see, e.g., [22] and references therein). Our result can be seen as the equivalent statement in the $p-$adic case, with the unit ball playing this role.

Cyclicity of the vacuum. The set D_0 of finite linear combinations of vectors of the form $\mathbf{\Phi}(f_1)\cdots\mathbf{\Phi}(f_n)\Upsilon_0$ is dense in H.

Theorem 66. *The following holds true:*

(i) The quadruple

$$\left\{\mathfrak{F}_s(L^2_{\mathbb{C}}\left(V^+, d\lambda\right)), \Gamma\left(U\left(\cdot,\cdot\right)\right), \mathbf{\Phi}, F_0\right\}$$

satisfies the p-adic Wightman axioms.

(ii) For each $f \in \mathcal{H}_\infty(\mathbb{C})$,

$$\mathbf{\Phi}\left(\square_{\mathfrak{q},\alpha} f\right) = 0.$$

Proof. In the proof of the first part (66), we use the notation

$$\mathfrak{F}_s = \mathfrak{F}_s(L^2_{\mathbb{C}}\left(V^+, d\lambda\right)) = \oplus_{n=0}^{\infty}\mathcal{H}_s^{(n)}.$$

Relativistic invariance of states. We first note that $\mathfrak{F}_s$ is separable because $L^2_{\mathbb{C}}(V^+, d\lambda)$ is separable, see Remark 60 (i). On the other hand, since V^+ is invariant under $\mathcal{L}_+^{\uparrow}$, $U(\cdot,\cdot)$ is a strongly continuous unitary representation of $\mathcal{P}_+^{\uparrow}$ on $L^2_{\mathbb{C}}(V^+, d\lambda)$, see (5.4). By definition $\Gamma(U)$ is the unitary operator on $\mathfrak{F}_s$ given on $\mathcal{H}_s^{(n)}$ by $\otimes_{k=1}^n U(\cdot,\cdot)$, consequently $\Gamma(U) : \mathcal{H}_s^{(n)} \to \mathcal{H}_s^{(n)}$ determines a strongly continuous unitary representation of $\mathcal{P}_+^{\uparrow}$ on $\mathcal{H}_s^{(n)}$. Notice that $\Gamma(U)$ is strongly continuous in F_0, and since F_0 is dense in $\mathfrak{F}_s$ we conclude that $\Gamma(U)$ is a strongly continuous unitary representation of $\mathcal{P}_+^{\uparrow}$ on $\mathfrak{F}_s$.

Spectral condition. We show that the four parameter group $\Gamma(U(a, I))$ has associated a projection-valued measure supported on $\overline{S(V^+)}$. The argument needed is exactly the classical one, see [49, p. 213]. The notion of closed forward semigroup, which is the $p-$adic counterpart of the closed forward light cone, allows us to carry out the calculations as in the classical case. We first notice that

$L^2_{\mathbb{C}}(V^+, d\lambda)$ is already a spectral representation of $U(a,I)$ since

$$\langle \varphi, U(a,I)\varphi \rangle_{L^2_{\mathbb{C}}(V^+,d\lambda)} = \int\limits_{V^+} \chi_p\left(\mathfrak{B}(a,k)\right) \left|\varphi(k)\right|^2 d\lambda(k). \tag{5.5}$$

Notice that if we define for φ, $\theta \in L^2_{\mathbb{C}}(V^+, d\lambda)$, the set function

$$B \to \int\limits_{V^+} \overline{\varphi(k)} \chi_p\left(\mathfrak{B}(a,k)\right) \theta(k)\, d\lambda(k),$$

B being a Borel set in V^+, and denote the corresponding projection-valued measure as $d(\varphi, E_k\varphi)$, in the case $\varphi = \theta$, then (5.5) can be rewritten as

$$\langle \varphi, U(a,I)\varphi \rangle_{L^2_{\mathbb{C}}(V^+,d\lambda)} = \int\limits_{V^+} \chi_p\left(\mathfrak{B}(a,k)\right) d(\varphi, E_k\varphi).$$

Now, since $\Gamma(U(a,I)) \upharpoonright \mathcal{H}^{(n)}_s = \bigotimes_{k=1}^n U(a,I)$, if $\varphi^{(n)} \in \mathcal{H}^{(n)}_s$ with $n > 0$, then

$$\begin{aligned}
&\left\langle \varphi^{(n)}, U(a,I)\varphi \right\rangle = \\
&\int_{V^+} \cdots \int_{V^+} \chi_p\left(\mathfrak{B}\left(a, \sum_{i=1}^n k_i\right)\right) \left|\varphi^{(n)}(k_1, \ldots, k_n)\right|^2 \prod_{k=1}^n d\lambda(k_i) = \\
&\int_{V^+} \chi_p\left(\mathfrak{B}(a,l)\right) d\mu_{\varphi^{(n)}}(l),
\end{aligned}$$

where

$$\mu_{\varphi^{(n)}}(A) = \int \underset{\sum k_i \in A}{\cdots} \int \left|\varphi^{(n)}(k_1, \ldots, k_n)\right|^2 \prod_{k=1}^n d\lambda(k_i),$$

A being a Borel set in $\overline{S(V^+)}$. Since λ is supported on $V^+ \subset \overline{S(V^+)}$ and $S(V^+)$ is an additive semigroup, then $\mu_{\varphi^{(n)}}$ is supported on $\overline{S(V^+)}$, for any $\varphi^{(n)} \in \mathcal{H}^{(n)}_s$. We now take $\Psi = \left\{\Psi^{(n)}\right\}_{n\in\mathbb{N}}$ in $\mathfrak{F}_s$ and denote by μ_Ψ the spectral measure so that

$$\langle \Psi, \Gamma(U(a,I))\Psi \rangle = \int \chi_p\left(\mathfrak{B}(a,k)\right) d\mu_\Psi(k),$$

then $\mu_\Psi = \sum_{n=0}^\infty \mu_{\Psi^{(n)}}$ since $\Gamma(U(a,I)) : \mathcal{H}^{(n)}_s \to \mathcal{H}^{(n)}_s$.

Existence and uniqueness of the vacuum. The argument in the p-adic case is the same as the Archimedean one, see [49, p. 213].

Invariant domains for fields. By Proposition 55, we have

$$\mathcal{H}_\infty(\mathbb{C}) \overset{R}{\to} L^2_{\mathbb{C}}\left(V^+, d\lambda\right) \to F_0 \to \mathfrak{F}_s(L^2_{\mathbb{C}}\left(V^+, d\lambda\right)), \tag{5.6}$$

where all the arrows denote continuous mappings. By using sequence (4.9), $\mathcal{D}_{\mathbb{C}}(V^+) \subset \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$, and since $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4) \subset \mathcal{H}_\infty(\mathbb{C})$, $\mathcal{F}(\mathcal{D}_{\mathbb{C}}) = \mathcal{D}_{\mathbb{C}}$, and $\mathcal{D}_{\mathbb{C}}(V^+)$ is dense in $L^2_{\mathbb{C}}\left(V^+, d\lambda\right)$, we conclude that $R(\mathcal{H}_\infty(\mathbb{C}))$ is dense in $L^2_{\mathbb{C}}\left(V^+, d\lambda\right)$, and hence $\oplus_{n=0}^{\infty} S_n(R(\mathcal{H}_\infty(\mathbb{C})))$ in $\mathfrak{F}_s(L^2_{\mathbb{C}}(V^+, d\lambda))$.

If f is real-valued, we use that $\mathbf{\Phi}_S(f)$ is essentially self-adjoint on F_0, the fact that $\mathbf{\Phi}_S(f) : F_0 \to F_0$, and sequence (5.6), jointly with the density of $R(\mathcal{H}_\infty(\mathbb{C}))$ to obtain that $\mathbf{\Phi}(f) \upharpoonright_{F_0}$ is essentially self-adjoint, and $\mathbf{\Phi}(f) : F_0 \to F_0$. If f is complex-valued, the results follows from the previous discussion by using the definition of $\mathbf{\Phi}(f)$.

Regularity of the field. Suppose that ψ_1, $\psi_2 \in F_0$ and that $f_n \to f \in \mathcal{H}_\infty(\mathbb{C})$ (i.e. $f_n \overset{\|\cdot\|_l}{\to} f$ for any $l \in \mathbb{N}$), with f_n real-valued. Then (4.1) implies that

$$\widehat{f}_n \mid_{V^+} \overset{L^2_{\mathbb{C}}\left(V^+, d\lambda\right)}{\to} \widehat{f} \mid_{V^+},$$

i.e. $R(f_n) \to R(f)$ in $\mathfrak{F}_s$, see sequence (5.6). Now by using Segal's quantization, cf. Theorem X.41-(d) in [49], we have $\mathbf{\Phi}\left(f_n\right)\psi \to \mathbf{\Phi}\left(f\right)\psi$ for all ψ in F_0, therefore

$$\left\langle \psi_1, \mathbf{\Phi}\left(f_n\right)\psi_2\right\rangle \to \left\langle \psi_1, \mathbf{\Phi}\left(f\right)\psi_2\right\rangle.$$

By treating the real and imaginary parts of f separately, we obtain that $\left\langle \psi_1, \mathbf{\Phi}\left(f\right)\psi_2\right\rangle$ is a complex-valued bilinear form in $F_0 \times F_0$, and that

$$\left|\left\langle \psi_1, \mathbf{\Phi}\left(f\right)\psi_2\right\rangle\right| \leq \left\|\psi_1\right\| \left\|\mathbf{\Phi}\left(f\right)\psi_2\right\|. \tag{5.7}$$

We now estimate $\left\|\mathbf{\Phi}\left(f\right)\psi_2\right\|$. By the definition of $\mathbf{\Phi}\left(f\right)$, it is sufficient to consider that f is real-valued. By taking $\psi_2 = \left\{\psi_2^{(n)}\right\}_{n \in N}$, $x_i \in \mathbb{Q}_p^4$ for $i \in \{1, \ldots, n\}$,

$y \in V^{+}$, and using that

$$\left(\mathbf{\Phi}\left(f\right)\psi_{2}\right)^{(n)}\left(x_{1},\cdots,x_{n}\right)=\frac{\sqrt{n+1}}{\sqrt{2}}\int_{V^{+}}\overline{\widehat{f}(y)}\psi_{2}^{(n+1)}\left(y,x_{1},\cdots,x_{n}\right)d\lambda\left(y\right)$$
$$+\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}\widehat{f}(x_{i})\psi_{2}^{(n-1)}\left(x_{1},\cdots,\widetilde{x}_{i},\cdots,x_{n}\right),$$

where $\widetilde{x}_{i}$ means that x_{i} is omitted, we have

$$\left\|\left(\mathbf{\Phi}\left(f\right)\psi_{2}\right)^{(n)}\right\|_{\mathcal{H}_{s}^{(n)}}^{2}=$$
$$\frac{(n+1)}{2}\int\limits_{\mathbb{Q}_{p}^{4n}}\left|\int_{V^{+}}\overline{\widehat{f}(y)}\psi_{2}^{(n+1)}\left(y,x_{1},\cdots,x_{n}\right)d\lambda\left(y\right)\right|^{2}\prod_{j=1}^{n}d^{4}x_{j}+$$
$$\frac{1}{2n}\int\limits_{\mathbb{Q}_{p}^{4n}}\left|\sum_{i=1}^{n}\widehat{f}(x_{i})\psi_{2}^{(n-1)}\left(x_{1},\cdots,\widetilde{x}_{i},\cdots,x_{n}\right)\right|^{2}\prod_{j=1}^{n}d^{4}x_{j}=:I_{0}+I_{1}.$$

To estimate I_{0}, we use the Cauchy-Schwartz inequality, estimation (4.1), and Remark 60 (iii) to get:

$$\begin{aligned}
I_{0}\leq&\frac{(n+1)}{2}\left\{\int_{V^{+}}\left|\widehat{f\left(y\right)}\right|^{2}d\lambda\left(y\right)\right\}\times\\
&\left\{\int\limits_{\mathbb{Q}_{p}^{4n}}\int_{V^{+}}\left|\psi_{2}^{(n+1)}\left(y,x_{1},\cdots,x_{n}\right)\right|^{2}d\lambda\left(y\right)\prod_{j=1}^{n}d^{4}x_{j}\right\}\\
&\leq C_{1}(n)\left\|f\right\|_{l}^{2}\int\limits_{\mathbb{Q}_{p}^{4n}}\int\limits_{\mathbb{Q}_{p}^{4}}\left|\psi_{2}^{(n+1)}\left(y,x_{1},\cdots,x_{n}\right)\right|^{2}d^{4}y\prod_{j=1}^{n}d^{4}x_{j}\\
&\leq C_{1}(n)\left\|f\right\|_{l}^{2}\left\|\psi_{2}^{(n+1)}\right\|_{\mathcal{H}_{s}^{(n+1)}}^{2},
\end{aligned}$$

for any $l\in\mathbb{N}$. For I_{1}, we have

$$I_{1}\leq\frac{1}{2n}\left(n\left\|f\right\|_{0}\left\|\psi_{2}^{(n-1)}\right\|_{\mathcal{H}_{s}^{(n-1)}}\right)^{2}=n\left\|f\right\|_{0}^{2}\left\|\psi_{2}^{(n-1)}\right\|_{\mathcal{H}_{s}^{(n-1)}}^{2}.$$

Consequently,

$$\left\|\mathbf{\Phi}\left(f\right)\psi_{2}\right\|\leq\sqrt{2}\left\|f\right\|_{l}\left\|\psi_{2}\right\|\text{ for any }l\in\mathbb{N},$$

which implies that

$$f \to \langle \psi_1, \mathbf{\Phi}(f)\, \psi_2 \rangle \text{ is an element of } \mathcal{H}_\infty^*(\mathbb{C}),$$

see (1.6).

Poincaré invariance of the field. The proof is identical to that of Theorem X.42 in [49].

Cyclicity of the vacuum. The cyclicity of the vacuum for $\mathbf{\Phi}(\cdot)$ follows from Theorem X. 41 (parts (b) and (d)) in [49], by using the fact that the mapping

$$\begin{array}{rccc} R: & \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4) & \to & L^2_{\mathbb{C}}(V^+, d\lambda) \\ & f & \to & \widehat{f}\,|_{V^+} \end{array} \tag{5.8}$$

has a dense range. Indeed, by using that $\mathcal{D}_{\mathbb{C}}(V^+)$ is dense in $L^2_{\mathbb{C}}(V^+, d\lambda)$, see Remark 59, and the sequence (4.9), we conclude that $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$ is dense in $L^2_{\mathbb{C}}(V^+, d\lambda)$. Finally, (5.8) follows from the fact that $\mathcal{F}(\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)) = \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$.

Local commutativity. Segal's quantization can be performed on the field $\mathbf{\Phi}(f)$, $f \in \mathcal{H}_\infty(\mathbb{C})$, see [49, Theorem X.41]. Local commutativity in this context means that

$$[\mathbf{\Phi}(f), \mathbf{\Phi}(g)]\, \psi = \mathbf{\Phi}(f)\, \mathbf{\Phi}(g)\, \psi - \mathbf{\Phi}(g)\, \mathbf{\Phi}(f)\, \psi = 0, \tag{5.9}$$

for any f, $g \in \mathcal{H}_\infty(\mathbb{C})$ with support on an appropriate domain, and for all $\psi \in F_0$. Without loss of generality we may suppose that f and g in (5.9) are real-valued since $\mathbf{\Phi}$ is linear. Since the range of $R : \mathcal{D}_{\mathbb{C}} \to L^2_{\mathbb{C}}(V^+, d\lambda)$ is dense in $L^2_{\mathbb{C}}(V^+, d\lambda)$, we may assume that f, g belong to $\mathcal{D}_{\mathbb{C}}$, cf. [49, Theorem X.41-(d)]. By using the Segal quantization, cf. [49, Theorem X.41-(c)], we have

$$\begin{aligned} [\mathbf{\Phi}(f), \mathbf{\Phi}(g)]\, \psi &= \sqrt{-1}\, \mathrm{Im}\, \langle Rf, Rg \rangle_{L^2_{\mathbb{C}}(V^+, d\lambda)}\, \psi \\ &= \frac{1}{2} \left\{ \int_{V^+} \left\{ \overline{\widehat{f}(k)}\, \widehat{g}(k) - \widehat{f}(k)\, \overline{\widehat{g}(k)} \right\} d\lambda(k) \right\} \psi. \end{aligned}$$

Now, we define

$$\Delta(x) = \int_{V^+} \left\{ \chi_p(-\mathcal{B}(x,k)) - \chi_p(\mathcal{B}(x,k)) \right\} d\lambda(k), \tag{5.10}$$

which is a well-defined function in $\mathbb{Q}_p^4$ because V^+ is open and compact. Then

$$[\mathbf{\Phi}(f),\mathbf{\Phi}(g)]\,\psi=\frac{1}{2}\left\{\int\limits_{\mathbb{Q}_p^4}\int\limits_{\mathbb{Q}_p^4}\Delta(x-y)\,f(x)\,g(y)\,d^4xd^4y\right\}\psi. \tag{5.11}$$

Therefore, the study of the local commutativity in the $p-$adic quantum field theory of a scalar field becomes the study of the vanishing of $\Delta(x)$ as a distribution on $\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)\times\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^4)$. It is then enough to observe that $\Delta(x)\equiv 0$ if $x\in\mathbb{Z}_p^4$, because $\chi_p|_{\mathbb{Z}_p}\equiv 1$.

Finally, to prove the second part (66) notice that, since $\square_{\mathfrak{q},\alpha}:\mathcal{H}_\infty(\mathbb{C})\rightarrow\mathcal{H}_\infty(\mathbb{C})$, see Lemma 52 , $\mathbf{\Phi}\ (\square_{\mathfrak{q},\alpha}f)$, $f\in\mathcal{H}_\infty(\mathbb{C})$, is well-defined, and since $\mathcal{H}_\infty(\mathbb{C})\subset L^2_{\mathbb{C}}\left(\mathbb{Q}_p^4,d^4k\right)$, we have $\mathcal{F}(\square_{\mathfrak{q},\alpha}f)=|\mathfrak{q}-1|_p^\alpha\,\mathcal{F}(f)$, so $R(\square_{\mathfrak{q},\alpha}f)=0$, and consequently $\mathbf{\Phi}(\square_{\mathfrak{q},\alpha}f)=0$, for all $f\in\mathcal{H}_\infty(\mathbb{C})$. □

5.3 Conjugated fields

We take $\mathcal{H}=L^2_{\mathbb{C}}(V^+,d\lambda)$ as before. Recall that $(k_0,\boldsymbol{k})\in V^+$ if and only if $(k_0,-\boldsymbol{k})\in V^+$. By using this fact, we define

$$\begin{array}{cccc}\boldsymbol{C}: & \mathcal{H} & \rightarrow & \mathcal{H}\\ & f(k_0,\boldsymbol{k}) & \rightarrow & \overline{f(k_0,-\boldsymbol{k})}.\end{array}$$

Then $\boldsymbol{C}$ induces a *conjugation* on $\mathcal{H}$, i.e. $\boldsymbol{C}$ gives an antilinear isometry satisfying $\boldsymbol{C}^2=I$. We set $\mathcal{H}_{\boldsymbol{C}}:=\{f\in\mathcal{H};\boldsymbol{C}f=f\}$.

We recall that $\omega(\boldsymbol{k}):U_{\mathfrak{q}}\rightarrow\mathbb{Q}_p$ is a non-vanishing analytic function. We define

$$\mu(\boldsymbol{k})=\begin{cases}\sqrt{|\omega(\boldsymbol{k})|_p} & \text{if } \boldsymbol{k}\in U_{\mathfrak{q}}\,,\\ 0 & \text{if } \boldsymbol{k}\in\mathbb{Q}_p^3\backslash U_{\mathfrak{q}}.\end{cases}$$

Then $\mu(\boldsymbol{k})\in\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^3)$.

We now define the canonical fields corresponding to $\boldsymbol{C}$ as follows:

$$\boldsymbol{\varphi}(f)=\frac{1}{\sqrt{2}}\left\{\left(a^{-}(Rf)\right)^{*}+a^{-}(\boldsymbol{C}Rf)\right\},\text{ for }f\in\mathcal{H}_{\infty}(\mathbb{C}),\text{ and}$$

$$\boldsymbol{\pi}(f)=\frac{\sqrt{-1}}{\sqrt{2}}\left\{\left(a^{-}(\mu Rf)\right)^{*}-a^{-}(\boldsymbol{C}\mu Rf)\right\},\text{ for }f\in\mathcal{H}_{\infty}(\mathbb{C}).$$

We call $f\rightarrow\boldsymbol{\varphi}(f)$ the *canonical free field* over $\mathcal{H}_{\boldsymbol{C}}$ of mass 1, and $f\rightarrow\boldsymbol{\pi}(f)$ the *canonical conjugate momentum* over $\mathcal{H}_{\boldsymbol{C}}$ of mass 1. These maps are complex linear and $\boldsymbol{\varphi}(f)$, $\boldsymbol{\pi}(f)$ are self-adjoint if and only if $Rf\in\mathcal{H}_{\boldsymbol{C}}$.

The distribution $\delta(x_0-t_0)\,g(\boldsymbol{x})$ is defined as the direct product of the distributions $\delta(x_0-t_0)$ and $g(\boldsymbol{x})$:

$$\begin{array}{rccc}\delta(x_0-t_0)\times g(\boldsymbol{x}): & \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p)\times\mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^3) & \rightarrow & \mathbb{C}\\ & \sum_i\phi_i(x_0)\,\theta_i(\boldsymbol{x}) & \rightarrow & \sum_i\phi_i(t_0)\int_{\mathbb{Q}_p^3}g(\boldsymbol{x})\,\theta_i(\boldsymbol{x})\,d^3\boldsymbol{x},\end{array}$$

see e.g. [66]. If $g\in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)$, then the Fourier transform of the distribution $\delta(x_0-t_0)\,g(\boldsymbol{x})$ is $\chi_p(k_0t_0)\,\widehat{g}(\boldsymbol{k})$, where $\widehat{g}(\boldsymbol{k})\in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{k}\right)$ is the 3−dimensional Fourier transform with respect to the bilinear form $-\mathfrak{B}_0(\boldsymbol{x},\boldsymbol{k})$. By using Lemma 58, we can extend the projection R to the distributions of the form $\delta(x_0-t_0)\,g(\boldsymbol{x})$, $g\in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)$, and thus we extend the class of functions on which $\boldsymbol{\varphi}(\cdot)$ and $\boldsymbol{\pi}(\cdot)$ are defined to include these distributions.

In the case $t_0=0$, with g real-valued, we have

$$\left(\boldsymbol{C}R\widehat{\delta g}\right)(k_0,\boldsymbol{k})=\overline{R\widehat{\delta g}\,(k_0,-\boldsymbol{k})}=\overline{R\widehat{g}\,(k_0,-\boldsymbol{k})}=\overline{\widehat{g}(-\boldsymbol{k})}=\widehat{g}(\boldsymbol{k})=R\left(\widehat{\delta g}\right).$$

Consequently, $R(\delta g)$ and $\mu R(\delta g)$ are in $\mathcal{H}_{\boldsymbol{C}}$, and $\boldsymbol{\varphi}(\delta g)$, $\boldsymbol{\pi}(\delta g)$ are self-adjoint if $g\in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)$ is real. We call the maps $g\rightarrow\boldsymbol{\varphi}(\delta g)$ and $g\rightarrow\boldsymbol{\pi}(\delta g)$ *the time-zero fields.*

From now on, we will only use 'test functions' of the form δg with $g\in L^2_{\mathbb{C}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)$ in $\boldsymbol{\varphi}(\cdot)$ and $\boldsymbol{\pi}(\cdot)$, and write $\boldsymbol{\varphi}(g)$ and $\boldsymbol{\pi}(g)$ instead of $\boldsymbol{\varphi}(\delta g)$ and $\boldsymbol{\pi}(\delta g)$. If f and g are functions from $L^2_{\mathbb{R}}\left(\mathbb{Q}_p^3,d^3\boldsymbol{x}\right)$, by using Theorem X.43-(c), we have

$$[\boldsymbol{\varphi}(f), \boldsymbol{\pi}(g)]\,\psi = \sqrt{-1}\left\{\int\limits_{V^+} \overline{\widehat{f}(k)}\widehat{g}(k)\mu(k)\,d\lambda(k)\right\}\psi, \text{ for all } \psi \in F_0. \tag{5.12}$$

5.4 Transferring fields from $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(V^+\right)\right)$ to $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}\right)\right)$

We use the notation

$$a^{\dagger}(f) = \left(a^{-}(f)\right)^{*}, \quad a(f) = \left(a^{-}(\boldsymbol{C}f)\right).$$

As we already mentioned, each function $f(\boldsymbol{k}) = f\left(\sqrt{\omega(\boldsymbol{k})}, \boldsymbol{k}\right) \in L^2_{\mathbb{C}}\left(V^+, d\lambda\right)$ is a function on $U_{\mathfrak{q}}$. We take

$$(Jf)(k_0, \boldsymbol{k}) = \frac{f\left(\sqrt{\omega(\boldsymbol{k})}, \boldsymbol{k}\right)}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p^{\frac{1}{2}}}$$

as before. Then J is a unitary isometry of $L^2_{\mathbb{C}}\left(V^+, d\lambda\right)$ onto $L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3\boldsymbol{k}\right)$. The annihilation and creation operators on $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3\boldsymbol{k}\right)\right)$, $\widetilde{a}(\cdot)$, $\widetilde{a}^{\dagger}(\cdot)$ are related to $a(\cdot)$ and $a^{\dagger}(\cdot)$ by the formulas:

$$\begin{aligned} \widetilde{a}(Jf) &= \Gamma(J)\, a(f)\, \Gamma(J)^{-1}, \\ \widetilde{a}^{\dagger}(Jf) &= \Gamma(J)\, a^{\dagger}(f)\, \Gamma(J)^{-1}. \end{aligned}$$

By using the unitary map $\Gamma(J)$, we carry the quantum fields over $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3\boldsymbol{k}\right)\right)$ as follows:

$$\widetilde{\boldsymbol{\Phi}}(f) = \Gamma(J)\,\boldsymbol{\Phi}(f)\,\Gamma(J)^{-1} = \frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\widetilde{\boldsymbol{C}}\frac{Rf}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p^{\frac{1}{2}}}\right) + \widetilde{a}^{\dagger}\left(\frac{Rf}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p^{\frac{1}{2}}}\right)\right\}$$

for $f \in \mathcal{H}_\infty(\mathbb{R})$, and

$$\widetilde{\boldsymbol{\varphi}}(f) = \Gamma(J)\,\boldsymbol{\varphi}(f)\,\Gamma(J)^{-1} = \frac{1}{\sqrt{2}}\left\{\widetilde{a}\left(\frac{R(f\delta)}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p^{\frac{1}{2}}}\right) + \widetilde{a}^{\dagger}\left(\frac{R(f\delta)}{\left|\sqrt{\omega(\boldsymbol{k})}\right|_p^{\frac{1}{2}}}\right)\right\}$$

for $f \in L^2_{\mathbb{C}}(\mathbb{Q}_p^3, d^3\boldsymbol{x})$, where $\widetilde{\boldsymbol{C}} = \Gamma(J)\,\boldsymbol{C}\Gamma(J)^{-1}$ acts by $\left(\widetilde{\boldsymbol{C}}g\right)(\boldsymbol{k}) = \overline{g(-\boldsymbol{k})}$.

We drop the tilde $\widetilde{\cdot}$, and from now on, we work with fields on $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3\boldsymbol{k}\right)\right)$, for f, g real-valued. Then, formula (5.12) becomes

$$[\boldsymbol{\varphi}(f), \boldsymbol{\pi}(f)] = \sqrt{-1}\int_{U_{\mathfrak{q}}} f(\boldsymbol{x})g(\boldsymbol{x})d^3\boldsymbol{x},$$

which is the canonical commutation relation in $L^2_{\mathbb{C}}(U_{\mathfrak{q}}, d^3\boldsymbol{x})$.

5.5 Some classical calculations

In this section, we discuss in a $p-$adic frame the annihilation and creation operators introduced above, to show that they conform to the common usage in the Physics literature. We start by defining

$$D_0 = \left\{\psi; \psi \in F_0, \psi^{(n)} \in \mathcal{D}_{\mathbb{C}}(U_{\mathfrak{q}}^n) \text{ for all } n\right\}$$

and for each $l \in U_{\mathfrak{q}}$ (we do not use bold letters for 3-dimensional vectors) an operator $a(l)$ on $\mathfrak{F}_s\left(L^2_{\mathbb{C}}\left(U_{\mathfrak{q}}, d^3x\right)\right) = \oplus_{n=0}^{\infty}\mathcal{H}_s^{(n)}$ with domain D_0 by

$$(a(l)\,\psi)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\,\psi^{(n+1)}(l,k_1,\ldots,k_n), \quad n \geq 0.$$

The formal adjoint of $a(l)$ is given by

$$\left(a(l)^{\dagger}\psi\right)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(l-k_j)\,\psi^{(n-1)}\left(k_1,\ldots,\widetilde{k}_j,\ldots k_n\right),$$

for $n \geq 1$, and by definition $\left(a(l)^{\dagger}\psi\right)^{(n)}(k_1,\ldots,k_n) = 0$ for $n = 0$. This operator is a well-defined quadratic form on $D_0 \times D_0$: if $\psi_2 = \left\{\psi_2^{(n)}\right\}_{n\in\mathbb{N}}$, $\psi_1 = \left\{\psi_1^{(n)}\right\}_{n\in\mathbb{N}}$

$\in F_0$, then the quadratic form

$$\left\langle \psi_2, a\left(l\right)^\dagger \psi_1 \right\rangle = \sum_{n=1}^{\infty} \left\langle \psi_2^{(n)}, \left(a\left(l\right)^\dagger \psi_1\right)^{(n)} \right\rangle_{\mathcal{H}_s^{(n)}} =$$
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{U_{\mathfrak{q}}^{n-1}} \overline{\psi_2^{(n)}\left(k_1, \ldots, k_{j-1}, l, k_{j+1}, \ldots, k_n\right)} \times$$
$$\psi_1^{(n-1)}\left(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_n\right) \prod_{\substack{i=1 \\ i \neq j}}^{n} d^3 k_i$$

is well-defined. The formulas

$$a\left(g\right) = \int_{U_{\mathfrak{q}}} a\left(k\right) g\left(-k\right) d^3 k \quad \text{and} \quad a^\dagger\left(g\right) = \int_{U_{\mathfrak{q}}} a^\dagger\left(k\right) g\left(k\right) d^3 k, \tag{5.13}$$

hold for all $g\left(k\right) \in \mathcal{D}_{\mathbb{C}}(U_{\mathfrak{q}})$, if the equalities are understood in the sense of quadratic forms, i.e.

$$\left\langle \psi_2, a\left(g\right) \psi_1 \right\rangle := \int_{U_{\mathfrak{q}}} \left\langle \psi_2, a\left(k\right) \psi_1 \right\rangle g\left(-k\right) d^3 k$$

and

$$\left\langle \psi_2, a^\dagger\left(g\right) \psi_1 \right\rangle := \int_{U_{\mathfrak{q}}} \left\langle \psi_2, a^\dagger\left(k\right) \psi_1 \right\rangle g\left(k\right) d^3 k.$$

On the other hand, since $a\left(l\right) : D_0 \to D_0$, the powers of $a\left(l\right)$ are well-defined on D_0. Then

$$\left\langle \psi_1, \left(a\left(l\right)^\dagger\right)^n \psi_2 \right\rangle = \left\langle \left(a\left(l\right)\right)^n \psi_1, \psi_2 \right\rangle \ ,$$

for each n, where the equality is to be understood in the sense of quadratic forms, and

$$\left\langle \psi_1, \left(\prod_{i=N_1+1}^{N_2} a^\dagger\left(l_i\right) \right) \left(\prod_{i=1}^{N_1} a\left(l_i\right) \right) \psi_2 \right\rangle$$

is a well-defined quadratic form on $D_0 \times D_0$. In addition, if $f_i \in \mathcal{D}_{\mathbb{C}}(U_{\mathfrak{q}})$, then the following expressions are well-defined as quadratic forms: The product

$$\left(\prod_{i=N_1+1}^{N_2} a^{\dagger}(f_i)\right)\left(\prod_{i=1}^{N_1} a(f_i)\right) =$$

$$\int\limits_{U_{\mathfrak{q}}^{N_2}} \left(\prod_{i=N_1+1}^{N_2} a^{\dagger}(k_i)\right)\left(\prod_{i=1}^{N_1} a(-k_i)\right)\left(\prod_{i=1}^{N_2} f_i(k_i)\right) d^3k_1 \cdots d^3k_{N_2},$$

the number operator

$$N = \int\limits_{U_{\mathfrak{q}}} a^{\dagger}(k)\, a(k)\, d^3k,$$

and the free Hamiltonian of unit mass,

$$H_0 = \int\limits_{U_{\mathfrak{q}}} \mu(k)\, a^{\dagger}(k)\, a(k)\, d^3k.$$

Finally, by using quadratic forms on D_0 we can express the free scalar field and the time zero fields in terms of $a^{\dagger}(k)$ and $a(k)$ (i.e. by using (5.13) with g real-valued):

$$\boldsymbol{\Phi}(t,x) = \frac{1}{\sqrt{2}}\int\limits_{U_{\mathfrak{q}}} \left\{\chi_p\left(\sqrt{\omega(k)}t - \mathfrak{B}_0(k,x)\right) a^{\dagger}(k) + \chi_p\left(-\sqrt{\omega(k)}t + \mathfrak{B}_0(k,x)\right) a(k)\right\} \times \frac{d^3k}{\left|\sqrt{\omega(k)}\right|_p^{\frac{1}{2}}},$$

$$\boldsymbol{\varphi}(x) = \frac{1}{\sqrt{2}}\int\limits_{U_{\mathfrak{q}}} \left\{\chi_p\left(-\mathfrak{B}_0(k,x)\right) a^{\dagger}(k) + \chi_p\left(\mathfrak{B}_0(k,x)\right) a(k)\right\} \frac{d^3k}{\left|\sqrt{\omega(k)}\right|_p^{\frac{1}{2}}},$$

$$\boldsymbol{\pi}(x) = \frac{\sqrt{-1}}{\sqrt{2}}\int\limits_{U_{\mathfrak{q}}} \left\{\chi_p\left(-\mathfrak{B}_0(k,x)\right) a^{\dagger}(k) - \chi_p\left(\mathfrak{B}_0(k,x)\right) a(k)\right\} \left|\sqrt{\omega(k)}\right|_p^{\frac{1}{2}} d^3k.$$

5.6 A p-adic Klein-Gordon equation

In this section, we consider the inhomogeneous $p-$adic Klein-Gordon equation:

$$\square_{\mathfrak{q},\alpha} u(t,\mathbf{x}) = h(t,\mathbf{x}), \tag{5.14}$$

where $(t,\mathbf{x}) \in \mathbb{Q}_p \times \mathbb{Q}_p^3$ and $h(t,\mathbf{x}) \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$. We use the techniques and results of [73, Chapter 6]. By a solution (or weak solution) we understand a distribution from $\mathcal{D}'_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$ satisfying (5.14). We denote by $E_{\mathfrak{q}}^0(t,\mathbf{x})$, the fundamental solution of (5.14) obtained in Theorem 48.

Theorem 67. *The following hold true:*

(i) The equation

$$\square_{\mathfrak{q},\alpha} u(t,\mathbf{x}) = 0 \tag{5.15}$$

admits plane waves, this means that if $(E^{\pm},\boldsymbol{\kappa}) \in V^{\pm}$, that is, they form a fixed pair of solutions to $E^{\pm} = \pm\sqrt{\omega(\boldsymbol{\kappa})}$, then $\chi_p\{\mp\mathcal{B}((t,\mathbf{x}),(E^{\pm},\boldsymbol{\kappa}))\}$ is a weak solution of (5.15).

(ii) The distributions

$$\int_{U_{\mathfrak{q}}} \chi_p\left\{-\mathcal{B}\left((t,\mathbf{x}),\left(\sqrt{\omega(\mathbf{k})},\mathbf{k}\right)\right)\right\}\frac{d^3\mathbf{k}}{\left|\sqrt{\omega(\mathbf{k})}\right|_p}+$$

$$\int_{U_{\mathfrak{q}}} \chi_p\left\{\mathcal{B}\left((t,\mathbf{x}),\left(-\sqrt{\omega(\mathbf{k})},\mathbf{k}\right)\right)\right\}\frac{d^3\mathbf{k}}{\left|\sqrt{\omega(\mathbf{k})}\right|_p}$$

are the unique weak solutions of (5.15) *(up to the multiplication by a non-zero complex constant) which are invariant under $\mathcal{L}_+^{\uparrow}$.*

(iii) The distributions

$$u(t,\mathbf{x};A,B,C) = E_{\mathfrak{q}}^0(t,\mathbf{x}) * h(t,\mathbf{x}) +$$

$$C\int_{U_{\mathfrak{q}}}\left\{\chi_p\left(-\sqrt{\omega(\mathbf{k})}t + \mathfrak{B}_0(\mathbf{k},\mathbf{x})\right)A(\mathbf{k}) + \chi_p\left(\sqrt{\omega(\mathbf{k})}t + \mathfrak{B}_0(\mathbf{k},\mathbf{x})\right)B(\mathbf{k})\right\}$$

$$\times\frac{d^3\mathbf{k}}{\left|\sqrt{\omega(\mathbf{k})}\right|_p},$$

where C is a non-zero complex number, and $A(\mathbf{k})$, $B(\mathbf{k}) \in \mathcal{D}_{\mathbb{C}}(\mathbb{Q}_p^3)$, are weak solutions of (5.14).

Proof.

(i) Since $\mathcal{F}^{-1}_{\substack{k_0\to t\\ k\to x}}(\delta(k_0 - E^{\pm}, \mathbf{k} - \boldsymbol{\kappa})) = \chi_p\{\mp\mathcal{B}((E^{\pm}, \boldsymbol{\kappa}), (t, \mathbf{x}))\}$, the condition $E^{\pm} = \pm\sqrt{\omega(\boldsymbol{\kappa})}$ implies that $k_0^{\pm} = \pm\sqrt{\omega(\mathbf{k})}$, so $\delta(k_0 - E^{\pm}, \mathbf{k} - \boldsymbol{\kappa})$ is supported on $V^{\pm} \subset V$. The result follows from the fact that the weak solutions of (5.15) are exactly the distributions from $\mathcal{D}'_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$ whose Fourier transform is supported on V, see [73, Lemma 169].

(ii) The distributions of the form $C\delta_V$, for $C \in \mathbb{C}^{\times}$, are the unique solutions of (5.15) which are invariant under $\boldsymbol{O}(\mathfrak{q})$, see [73, Lemma 169] and [46, Proposition 2-2.]. By writing $C\delta_V = C\delta_{V^+} + C\delta_{V^-}$ in $\mathcal{D}'_{\mathbb{C}}(\mathbb{Q}_p \times \mathbb{Q}_p^3)$ and using the fact that $\delta_{V^{\pm}}$ are invariant under $\mathcal{L}_+^{\uparrow} = \{\Lambda \in \boldsymbol{O}(\mathfrak{q}); \Lambda(V^{\pm}) = V^{\pm}\}$, see [73, Lemma 163], we conclude that $C\delta_{V^+} + C\delta_{V^-}$are the unique weak solutions of (5.15) which are invariant under $\mathcal{L}_+^{\uparrow}$. The announced formula follows by computing the inverse Fourier transform of $\delta_{V^{\pm}}$.

(iii) The result follows from the second part by using Theorem 48.

□

Remark 68. *Notice that $\left|\sqrt{\omega(\mathbf{k})}\right|_p A(\mathbf{k})$, $\left|\sqrt{\omega(\mathbf{k})}\right|_p B(\mathbf{k})$, are test functions, and also*

$$\int_{U_{\mathfrak{q}}} \chi_p\left(\sqrt{\omega(\mathbf{k})}t + \mathfrak{B}_0(\mathbf{k}, \mathbf{x})\right) B(\mathbf{k}) \frac{d^3\mathbf{k}}{\left|\sqrt{\omega(\mathbf{k})}\right|_p} = \int_{U_{\mathfrak{q}}} \chi_p\left(\sqrt{\omega(\mathbf{k})}t - \mathfrak{B}_0(\mathbf{k}, \mathbf{x})\right) B(-\mathbf{k}) \frac{d^3\mathbf{k}}{\left|\sqrt{\omega(\mathbf{k})}\right|_p},$$

so the unique weak solution of $\Box_{\mathfrak{q},\alpha} u(t, \mathbf{x}) = 0$ (with $C = 1/\sqrt{2}$) invariant under $\mathcal{L}_+^{\uparrow}$ corresponds to the free scalar field $\boldsymbol{\Phi}(t, \mathbf{x})$, with $a(\mathbf{k}) = \left|\sqrt{\omega(\mathbf{k})}\right|_p A(\mathbf{k})$, $a^{\dagger}(\mathbf{k}) = \left|\sqrt{\omega(\mathbf{k})}\right|_p B(\mathbf{k})$. As we have seen, these solutions can be quantized using the machinery of the second quantization in such a way that Wightman axioms are satisfied.

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