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Minimum Distance Functions and Reed–Muller-Type Codes

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Funciones Distancia Mínima y Códigos Tipo-Reed–Muller

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The beauty of mathematics only shows itself to more patient followers.
Maryam Mirzakhani

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Abstract

Let S be a polynomial ring over the field K and let I be a graded ideal of S. In this thesis we introduce and study two functions associated to I: the minimum distance function δ_I and the footprint function fp_I. To define δ_I and fp_I we use the Hilbert function, the degree (multiplicity), and a Gröbner basis for I. We study these functions from a computational point of view using Gröbner bases methods and implementations in *Macaulay*2. We also study these functions from a theoretical point of view and examine their asymptotic behavior. These functions can be expressed in terms of the algebraic invariants of I. One of our main results shows that fp_I is a lower bound for δ_I. We give formulas to compute fp_I and δ_I in the case of certain complete intersections. In the case of complete intersection monomial ideals δ_I is equal to fp_I and we are able to give an explicit formula in terms of the degrees of a minimal set of generators of I.

Let $K = \mathbb{F}_q$ be a finite field and let $\mathbb{X} \subset \mathbb{P}^{s-1}$ be a finite subset of points in the projective space $\mathbb{P}^{s-1}$ over the field K. We show a formula to compute the number of zeros that a homogeneous polynomial has in $\mathbb{X}$. We use the minimum distance function of the vanishing ideal associated to $\mathbb{X}$ in order to give an algebraic formulation for the minimum distance in coding theory, in particular for projective Reed–Muller-type codes defined on $\mathbb{X}$, we also compute its dimension and length. Following the footprint method, we present bounds for the number of zeros of polynomials in a projective nested Cartesian set $\mathbb{X}$ and for the minimum distance of the corresponding projective nested Cartesian codes.

To show applications of the footprint method we need to study certain monomial ideals that occur as initial ideals of vanishing ideals over finite fields. This leads us to introduce the edge ideal $I = I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$. Using the combinatorial structure of digraphs, we determine the irredundant irreducible decomposition of I. Furthermore, we give a combinatorial characterization for the unmixed property of I, when the digraph is bipartite, a whisker or a cycle. We will also study the Cohen–Macaulay property of I and show that in certain cases I is unmixed if and only if I is Cohen–Macaulay.

Resumen

Sean S un anillo de polinomios sobre el campo K e I un ideal graduado de S. En esta tesis introducimos y estudiamos dos funciones asociadas a I: la función de distancia mínima δ_I y la función huella fp_I. Para definir δ_I y fp_I usamos la función de Hilbert, el grado (multiplicidad) y una base de Gröbner para I. Estudiamos estas funciones desde un punto de vista computacional usando métodos de bases de Gröbner e implementaciones en *Macaulay*2. También estudiamos estas funciones desde un punto de vista teórico y examinamos su comportamiento asintótico. Estas funciones pueden ser expresadas en términos de los invariantes algebraicos de I. Uno de nuestros resultados principales prueba que fp_I es una cota inferior para δ_I. Damos fórmulas para calcular fp_I y δ_I en el caso de ciertas intersecciones completas. En el caso de ideales monomiales que son intersección completa δ_I es igual a fp_I y exhibimos una fórmula explícita en términos de los grados de un conjunto minimal de generadores de I.

Sea $K = \mathbb{F}_q$ un campo finito y $\mathbb{X} \subset \mathbb{P}^{s-1}$ un subconjunto finito de puntos en el espacio proyectivo $\mathbb{P}^{s-1}$ sobre el campo K. Mostramos una fórmula para calcular el número de ceros que un polinomio homogéneo tiene en $\mathbb{X}$. Usamos la función de distancia mínima del ideal anulador asociado a $\mathbb{X}$ para dar una formulación algebraica para la distancia mínima en teoría de códigos, en particular para códigos proyectivos tipo Reed–Muller definidos en $\mathbb{X}$, también calculamos su dimensión y longitud. Siguiendo el método de la huella, presentamos cotas para el número de ceros de un polinomio en un conjunto proyectivo Cartesiano anidado $\mathbb{X}$ y para la distancia mínima de códigos proyectivos Cartesianos anidados.

Para mostrar aplicaciones del método de la huella necesitamos estudiar ciertos ideales monomiales que aparecen como ideales iniciales de ideales anuladores sobre campos finitos. Esto nos lleva a introducir el ideal de aristas $I = I(\mathcal{D})$ de una gráfica orientada pesada $\mathcal{D}$. Usando la estructura combinatoria de digráficas, determinamos la descomposición irreducible irredundante de I. Además, caracterizamos de manera combinatoria cuándo el ideal I es no mezclado para digráficas bipartitas, aristas colgantes o ciclos. También estudiaremos la propiedad Cohen–Macaulay de I y mostraremos que en ciertos casos I es no mezclado si y sólo si I es Cohen–Macaulay.

Introduction

In this thesis we introduce two numerical functions over graded ideals; the minimum distance function and the footprint function, these are defined in terms of algebraic invariants such as the Hilbert function and the degree (multiplicity), these functions have a relevant connection with coding theory that justifies its name. The footprint function of a graded ideal I is obtained by fixing a monomial order and then using the initial ideal of I to find an approximation for the minimum distance function of I.

The interest in these functions is essentially due to the following facts: The minimum distance function is related to the minimum distance in coding theory (Theorem 3.2.1), and the footprint function is much easier to compute. There are significant cases in which either the footprint function is a lower bound for the minimum distance function (Theorem 2.3.2) or the two functions coincide (Theorem 2.5.9, Corollary 3.3.1). Our main interest is to find exact formulas to calculate or to find upper and lower bounds for these functions for some families of graded ideals.

In Chapter 2, we study the minimum distance function and the footprint function of a graded ideal from a theoretical point of view. These functions are defined as follows.

Let $S = K[x_1, \ldots, x_s] = \bigoplus_{d=0}^{\infty} S_d$ be the polynomial ring over a field K with the standard grading and let $I \neq (0)$ be a graded ideal of S of Krull dimension k.

- The *Hilbert function* of S/I is

$$H_I(d) := \dim_K(S_d/I_d),\ d = 0, 1, 2, \ldots,$$

  where $I_d = S_d \cap I$.

- The *degree* or *multiplicity* of S/I is:

$$\deg(S/I) = \begin{cases} (k-1)!\ \lim\limits_{d\to\infty} \dfrac{H_I(d)}{d^{k-1}} & \text{if } k \geq 1, \\ \dim_K(S/I) & \text{if } k = 0. \end{cases}$$

Let $\mathcal{F}_d$ be the set of all polynomials of degree $d \geq 0$ which are zero divisors of S/I:

$$\mathcal{F}_d := \{f \in S_d \mid f \notin I,\ (I\colon f) \neq I\},$$

where $(I\colon f)=\{g\in S\mid gf\in I\}$ is a quotient ideal. The *minimum distance function* of I, denoted δ_I, is the function $\delta_I\colon \mathbb{N}\to\mathbb{Z}$ given by

$$\delta_I(d):=\begin{cases}\deg(S/I)-\max\{\deg(S/(I,f))\,|\,f\in\mathcal{F}_d\} & \text{if } \mathcal{F}_d\neq\emptyset,\\ \deg(S/I) & \text{if } \mathcal{F}_d=\emptyset.\end{cases}$$

For unmixed graded ideals, $\delta_I(d)=\min\{\deg(S/(I\colon f))\,|\,f\in S_d\setminus I\}$ (Theorem 2.1.7).

We are able to show that the minimum distance function satisfies the following properties, which allow us to study the asymptotic behavior of δ_I (Section 2.2).

Theorem 2.1.9. *Let $I\subset S$ be an unmixed graded ideal, let $\prec$ be a monomial order on S, and let $d\geq 1$ be an integer. The following hold.*

(i) $\delta_I(d)\geq 1$.

(ii) *If* $\dim(S/I)\geq 1$ *and* $\mathcal{F}_d\neq\emptyset$ *for* $d\geq 1$, *then* $\delta_I(d)\geq\delta_I(d+1)\geq 1$ *for* $d\geq 1$.

Theorem 2.1.12. *Let $I\subset S$ be an unmixed radical graded ideal. If all the associated primes of I are generated by linear forms, then there is an integer $r_0\geq 1$ such that*

$$\delta_I(1)>\delta_I(2)>\cdots>\delta_I(r_0)=\delta_I(d)=1 \quad \textit{for} \quad d\geq r_0.$$

The integer r_0 where the stabilization occurs is called the *regularity index* of δ_I and is denoted by $\operatorname{reg}(\delta_I)$. If I is the graded vanishing ideal of a set of points in a projective space over a finite field, then $r_0\leq\operatorname{reg}(S/I)$ where $\operatorname{reg}(S/I)$ is the Castelnuovo–Mumford regularity of S/I or simply the *regularity* of S/I (Definition 1.5.9). An excellent reference for this notion is the book of Eisenbud [17]. The *regularity index* of S/I, denoted $\operatorname{ri}(S/I)$, is the least integer $r\geq 0$ such that $H_I(d)$ is equal to $h_I(d)$ for $d\geq r$, where h_I is the Hilbert polynomial of S/I (Theorem 1.5.2, Definition 1.5.5). If I is a graded Cohen–Macaulay ideal of dimension 1, then $\operatorname{reg}(S/I)=\operatorname{ri}(S/I)$ (Remark 1.5.10).

In Section 2.2, we conjecture that $\delta_I(d)=1$ for $d\geq\operatorname{reg}(S/I)$, that is, $r_0\leq\operatorname{reg}(S/I)$ (Conjecture 2.2.2). We show this conjecture when I is the edge ideal (Definition 1.9.14) of a Cohen–Macaulay bipartite graph without isolated vertices.

Proposition 2.2.4. *If $I=I(G)$ is the edge ideal of a Cohen–Macaulay bipartite graph without isolated vertices, then $\delta_I(d)=1$ for $d\geq\operatorname{reg}(S/I)$.*

Conjecture 2.2.2 is still open for square-free monomial ideals. The regularity of S/I can be computed using *Macaulay*2 [25], but r_0 is in general difficult to compute [12].

We use Gröbner bases to study the minimum distance function as we now explain.

Fix a monomial order $\prec$ on S. Let $\Delta_\prec(I)$ be the *footprint* of S/I consisting of all standard monomials of S/I, with respect to $\prec$, and let $G=\{g_1,\ldots,g_r\}$ be a Gröbner basis of I. Then $\Delta_\prec(I)$ is the set of all monomials of S that are not a multiple of any of the leading monomial of $g_1,\ldots,g_r$ (Lemma 1.6.13). We set $\Delta_\prec(I)_d=\Delta_\prec(I)\cap S_d$.

If $\Delta_\prec(I)_d=\{x^{a_1},\ldots,x^{a_n}\}$ and $\mathcal{F}_{\prec,d}=\{f=\sum_i\lambda_ix^{a_i}\mid f\neq 0,\lambda_i\in K,(I\colon f)\neq I\}$, then using the division algorithm (Theorem 1.6.5) we can write:

$$\delta_I(d) \;=\; \deg(S/I)-\max\{\deg(S/(I,f))\,|\,f\in\mathcal{F}_{\prec,d}\}.$$

If $K = \mathbb{F}_q$ is a finite field, using this equality and *Macaulay2* [25], we present an implementation to compute δ_I (Example 2.1.16). Other systems that can be employed are *CoCoA* [1] and *Singular* [21]. To compute δ_I is a difficult problem in commutative algebra, because the number of standard polynomials (Definition 1.6.9) of degree d is $q^n - 1$, where n is the number of standard monomials of degree d. Hence, we can only compute $\delta_I(d)$ for small values of n and q.

Upper bounds for $\delta_I(d)$ can be obtained by fixing a subset $\mathcal{F}'_{\prec,d}$ of $\mathcal{F}_{\prec,d}$ and computing

$$\delta'_I(d) = \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in \mathcal{F}'_{\prec,d}\} \geq \delta_I(d).$$

Typically one use $\mathcal{F}'_{\prec,d} = \{f = \sum_i \lambda_i x^{a_i} \mid f \neq 0, \lambda_i \in \{0,1\}, (I\colon f) \neq I\}$ or a subset of it.

Lower bounds for $\delta_I(d)$ are harder to find. Thus, we seek to estimate $\delta_I(d)$ from below. So, with this in mind, in Section 2.3, we introduce the *footprint function* of I. This is a numerical function defined similarly as δ_I, but here we use a monomial order and the initial ideal of I. The footprint function is defined as follows (Definition 2.3.1).

Let $\mathcal{M}_{\prec,d}$ be the set of all zero divisors of $S/\mathrm{in}_\prec(I)$ of degree $d \geq 1$ that are in $\Delta_\prec(I)$:

$$\mathcal{M}_{\prec,d} := \{x^a \,|\, x^a \in \Delta_\prec(I)_d, (\mathrm{in}_\prec(I)\colon x^a) \neq \mathrm{in}_\prec(I)\},$$

where $\mathrm{in}_\prec(I)$ denotes the initial ideal of I (Definition 1.6.3). The *footprint function* of I, denoted fp_I, is the function $\mathrm{fp}_I\colon \mathbb{N}_+ \to \mathbb{Z}$ given by

$$\mathrm{fp}_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(\mathrm{in}_\prec(I), x^a)) \,|\, x^a \in \mathcal{M}_{\prec,d}\} & \text{if } \mathcal{M}_{\prec,d} \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{M}_{\prec,d} = \emptyset. \end{cases}$$

We come to one of our main results.

Theorem 2.3.2. *Let I be an unmixed graded ideal and let $\prec$ be a monomial order. The following hold.*

(i) $\delta_I(d) \geq \mathrm{fp}_I(d)$ *and* $\delta_I(d) \geq 0$ *for* $d \geq 1$.

(ii) $\mathrm{fp}_I(d) \geq 0$ *if* $\mathrm{in}_\prec(I)$ *is unmixed.*

In particular, the previous theorem tells us that fp_I is a lower bound for δ_I, when both values coincide for $d \geq 1$, we call the ideal I a *Geil–Carvalho ideal*, any unmixed monomial ideal is Geil–Carvalho (Proposition 2.3.3). The first interesting family of ideals where equality holds is due to Geil [18, Theorem 2]. His result essentially shows that $\mathrm{fp}_I(d) = \delta_I(d)$ for $d \geq 1$ when $\prec$ is a graded lexicographical order and I is the homogenization of the vanishing ideal of the affine space $\mathbb{A}^{s-1}$ over a finite field $K = \mathbb{F}_q$. Recently Carvalho [10, Proposition 2.3] extended this result by replacing $\mathbb{A}^{s-1}$ by a Cartesian products of subsets of $\mathbb{F}_q$. In this case the underlying Reed–Muller-type code is called an affine Cartesian code and an explicit formula for the minimum distance was first given in [19, 34]. As an application we show this formula for the minimum distance of an affine Cartesian code by examining the underlying vanishing ideal (Section 3.3).

In Section 2.5, we study the footprint function with respect to a monomial order of a graded ideal I whose initial ideal is a complete intersection. This implies that I is a complete intersection (Proposition 2.5.4(a)). In this case we present an explicit formula for fp_I in terms of the degrees of the generators of the ideal. Now, we present our main results on complete intersections.

Theorem 2.5.6. *Let $I \subset S$ be a graded ideal and let $\prec$ be a monomial order. If $\mathrm{in}_\prec(I)$ is a complete intersection of height $s-1$ generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$ with $d_i = \deg(x^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then $\delta_I(d) \geq \mathrm{fp}_I(d) \geq 1$ and the footprint function in degree $d \geq 1$ is given by*

$$\mathrm{fp}_I(d) = \begin{cases} (d_{k+2} - \ell) d_{k+3} \cdots d_s & \text{if } d \leq \sum\limits_{i=2}^{s} (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum\limits_{i=2}^{s} (d_i - 1), \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are the unique integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

This result is valid if the initial ideal is a complete intersection of dimension greater than or equal to 1. This follows using the next theorem and noticing that Proposition 2.5.4 holds for any height.

Theorem 2.5.9. *Let $I \subset S$ be a complete intersection monomial ideal of dimension ≥ 1 minimally generated by $x^{\alpha_1}, \ldots, x^{\alpha_r}$ and let $d \geq 1$ be an integer. If $d_i = \deg(x^{\alpha_i})$ for $i = 1, \ldots, r$ and $d_1 \leq \cdots \leq d_r$, then*

$$\delta_I(d) = \mathrm{fp}_I(d) = \begin{cases} (d_{k+1} - \ell)\, d_{k+2} \cdots d_r & \text{if } d < \sum\limits_{i=1}^{r} (d_i - 1), \\ 1 & \text{if } d \geq \sum\limits_{i=1}^{r} (d_i - 1), \end{cases}$$

where $0 \leq k \leq r-1$ and ℓ are integers such that $d = \sum_{i=1}^{k} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

In Chapter 3, we show that the minimum distance function of a graded ideal in a polynomial ring with coefficients in a field generalizes the minimum distance of projective Reed–Muller-type codes over finite fields (see the discussion below). This gives an algebraic formulation of the minimum distance of a projective Reed–Muller-type code in terms of the algebraic invariants and structure of the underlying vanishing ideal. Then, we give a method based on Gröbner bases and Hilbert functions, to find lower bounds for the minimum distance of certain Reed–Muller-type codes. This is a very important result because in general computing the minimum distance of linear codes is NP-hard [54].

The study of δ_I was motivated by the notion of minimum distance of linear codes in coding theory. For convenience we mention this notion. Let $K = \mathbb{F}_q$ be a finite field. An $[m, k]-$*linear code* is a linear subspace of K^m of dimension k for some m. The *basic parameters* of a linear code C are *length*: m, *dimension*: $k = \dim_K(C)$, and *minimum* *distance*:

$$\delta(C) := \min\{\|v\| : 0 \neq v \in C\},$$

where $\|v\|$ is the number of nonzero entries of v.

The minimum distance of affine Reed–Muller-type codes has been studied using Gröbner bases techniques; see [10, 18, 19] and the references therein. Of particular interest to us is the footprint technique introduced by Geil [18] to bound from below the minimum distance. In this work we extend this technique to projective Reed–Muller-type codes, a special type of linear codes that generalizes affine Reed–Muller-type codes [35]. These projective codes are constructed as follows.

Let $K = \mathbb{F}_q$ be a finite field with q elements, let $\mathbb{P}^{s-1}$ be a projective space over K, and let $\mathbb{X}$ be a subset of $\mathbb{P}^{s-1}$. The *vanishing ideal* of $\mathbb{X}$, denoted $I(\mathbb{X})$, is the ideal of S generated by the homogeneous polynomials that vanish at all points of $\mathbb{X}$. In this case the Hilbert function of $S/I(\mathbb{X})$ is denoted by $H_{\mathbb{X}}(d)$. We can write $\mathbb{X} = \{[P_1], \ldots, [P_m]\} \subset \mathbb{P}^{s-1}$ with $m = |\mathbb{X}|$.

Fix a degree $d \geq 1$. For each i there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. There is a K-linear map given by

$$\mathrm{ev}_d\colon S_d \to K^m, \qquad f \mapsto \left(\frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_m)}{f_m(P_m)}\right).$$

The image of S_d under ev_d, denoted by $C_{\mathbb{X}}(d)$, is called a *projective Reed–Muller-type code* of degree d on $\mathbb{X}$ [15, 24]. The *basic parameters* of the linear code $C_{\mathbb{X}}(d)$ are:

(a) *length*: $|\mathbb{X}|$,

(b) *dimension*: $\dim_K(C_{\mathbb{X}}(d))$,

(c) *minimum distance*: $\delta_{\mathbb{X}}(d) := \delta(C_{\mathbb{X}}(d))$.

The length and the dimension of $C_{\mathbb{X}}(d)$ are $\deg(S/I(\mathbb{X}))$ and $H_{\mathbb{X}}(d)$, respectively. The Hilbert function and the minimum distance are related by the Singleton bound:

$$1 \leq \delta_{\mathbb{X}}(d) \leq |\mathbb{X}| - H_{\mathbb{X}}(d) + 1.$$

In particular, if $d \geq \mathrm{reg}(S/I(\mathbb{X})) \geq 1$, then $\delta_{\mathbb{X}}(d) = 1$. The converse is not true (Example 3.2.7). Thus, potentially good Reed–Muller-type codes $C_{\mathbb{X}}(d)$ can occur only if $1 \leq d < \mathrm{reg}(S/I(\mathbb{X}))$. There are some families where $d \geq \mathrm{reg}(S/I(\mathbb{X})) \geq 1$ if and only if $\delta_{\mathbb{X}}(d) = 1$ [34, 47, 49], but we do not know of any set $\mathbb{X}$ parameterized by monomials where this fails. If $\mathbb{X}$ is parameterized by monomials we say that $C_{\mathbb{X}}(d)$ is a *projective parameterized code* [46, 52].

A main problem in Reed–Muller-type codes and the theory of algebraic schemes is the following [12, 20, 40]; if $\mathbb{X}$ has nice algebraic or combinatorial structure, find formulas in terms of s, q, d, and the structure of $\mathbb{X}$, for the basic parameters of $C_{\mathbb{X}}(d)$ and $S/I(\mathbb{X})$:

$H_{\mathbb{X}}(d)$, $\deg(S/I(\mathbb{X}))$, $\delta_{\mathbb{X}}(d)$, and $\text{reg}(S/I(\mathbb{X}))$. Our main results can be used to study this problem, especially when $\mathbb{X}$ is parameterized by monomials or when $\mathbb{X}$ is a projective nested Cartesian set (Definition 3.5.1).

The basic parameters of projective Reed–Muller-type codes have been computed in the following cases:

- If $\mathbb{X} = \mathbb{P}^{s-1}$, $C_{\mathbb{X}}(d)$ is the *classical projective Reed–Muller code*. Formulas for its basic parameters were given in [49, Theorem 1].
- If $\mathbb{X}$ is a projective torus (Definition 1.7.7), $C_{\mathbb{X}}(d)$ is the *generalized projective Reed–Solomon code*. Formulas for its basic parameters were given in [47, Theorem 3.5].
- If $\mathbb{X}$ is the image of a Cartesian product of subsets of K, under the map $K^{s-1} \to \mathbb{P}^{s-1}$, $x \to [x, 1]$, then $C_{\mathbb{X}}(d)$ is an *affine Cartesian code* and formulas for its basic parameters were given in [19, 34].

Let f be a homogeneous polynomial of S, the *zero set* of f, denoted by $V_{\mathbb{X}}(f)$, is the set of all $[P] \in \mathbb{X}$ such that $f(P) = 0$, that is, $V_{\mathbb{X}}(f)$ is the set of zeros of f in $\mathbb{X}$. To calculate the minimum distance of a projective Reed–Muller-type code is directly related to computing the number of elements of $V_{\mathbb{X}}(f)$. We give the following nice formula to compute this number (Lemma 3.1.1, Example 3.1.4):

$$|V_{\mathbb{X}}(f)| = \begin{cases} \deg(S/(I(\mathbb{X}), f)) & \text{if } (I(\mathbb{X})\colon f) \neq I(\mathbb{X}), \\ 0 & \text{if } (I(\mathbb{X})\colon f) = I(\mathbb{X}). \end{cases}$$

As a consequence of this formula we derive one of the main results of this thesis.

Theorem 3.2.1. *If* $|\mathbb{X}| \geq 2$, *then* $\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \geq 1$ *for* $d \geq 1$.

If $\prec$ is a monomial order on S, by Proposition 2.1.15, one has:

$$\delta_{\mathbb{X}}(d) = \deg(S/I(\mathbb{X})) - \max\{\deg(S/(I(\mathbb{X}), f))|\, f \in \mathcal{F}_{\prec,d}\}.$$

This description allows us to compute the minimum distance of Reed–Muller-type codes for small values of q and s and it is the first algebraic formulation of the minimum distance in terms of the algebraic properties and invariants of the vanishing ideal. The formula in Theorem 3.2.1 is more interesting from the theoretical point of view than from a computational perspective. Indeed putting together Theorems 2.3.2 and 3.2.1 one has:

$$\delta_{\mathbb{X}}(d) \geq \text{fp}_{I(\mathbb{X})}(d) \geq 0 \quad \text{for} \quad d \geq 1.$$

This inequality gives a lower bound for the minimum distance of any Reed–Muller-type code over a set $\mathbb{X}$ (Example 3.2.6).

As the two most relevant applications of our main results to algebraic coding theory in Section 3.3, we recover the formula for the minimum distance of an affine Cartesian code given in [34, Theorem 3.8] and [19, Proposition 5] and the fact that the homogenization of the corresponding vanishing ideal is a Geil–Carvalho ideal.

Corollary 3.3.1. Let K be a field and let $C_{\mathbb{X}}(d)$ be the projective Reed–Muller-type code of degree d on the finite set $\mathbb{X} = [1 \times A_2 \times \cdots \times A_s] \subset \mathbb{P}^{s-1}$. If $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, with $d_i = |A_i|$, and $d \geq 1$, then the minimum distance of $C_{\mathbb{X}}(d)$ is given by

$$\delta_{\mathbb{X}}(d) = \begin{cases} (d_{k+2} - \ell)\, d_{k+3} \cdots d_s & \text{if } d \leq \sum\limits_{i=2}^{s} (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum\limits_{i=2}^{s} (d_i - 1), \end{cases}$$

where $k \geq 0$, ℓ are the unique integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Then we present an extension of a result of Alon and Füredi [3, Theorem 1] (in terms of the regularity of a vanishing ideal) about coverings of the cube $\{0, 1\}^n$ by affine hyperplanes, that can be applied to any finite subset of a projective space whose vanishing ideal has a complete intersection initial ideal (Example 3.3.4).

Corollary 3.3.3. *Let $\mathbb{X}$ be a finite subset of a projective space $\mathbb{P}^{s-1}$ and let $\prec$ be a monomial order such that $\mathrm{in}_{\prec}(I(\mathbb{X}))$ is a complete intersection generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$ with $d_i = \deg(x^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for all i. If the hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^{s-1}$ avoid a point $[P]$ in $\mathbb{X}$ but otherwise cover all the other $|\mathbb{X}| - 1$ points of $\mathbb{X}$, then $d \geq \mathrm{reg}\,(S/I(\mathbb{X})) = \sum_{i=2}^{s} (d_i - 1)$.*

Finally using *Macaulay2* [25], we exemplify how some of our results can be used in practice, and show that the vanishing ideal of $\mathbb{P}^2$ over $\mathbb{F}_2$ is not Geil–Carvalho by computing all possible initial ideals (Example 3.3.7).

Let $d_1, \ldots, d_s$ be a non-decreasing sequence of positive integers with $d_1 \geq 2$ and $s \geq 2$, and let L be the ideal of S generated by the set of all $x_i x_j^{d_j}$ such that $1 \leq i < j \leq s$. It turns out that the ideal L is the initial ideal of the vanishing ideal of a projective nested Cartesian set (Definition 3.5.1, Proposition 3.5.3). In Section 3.4, we study the ideal L and show some degree equalities as a preparation to show some applications. In particular, we have the following lemma.

Lemma 3.4.1. *The ideal L is Cohen–Macaulay of height $s - 1$, has a unique irredundant primary decomposition given by*

$$L = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s,$$

where $\mathfrak{q}_i = (x_1, \ldots, x_{i-1}, x_{i+1}^{d_{i+1}}, \ldots, x_s^{d_s})$ for $1 \leq i \leq s$, and $\deg(S/L) = 1 + \sum_{i=2}^{s} d_i \cdots d_s$.

In the last chapter of this thesis we recover the previous result from a combinatorial point of view using the notion that L is the edge ideal of a vertex weighted oriented graph (Definition 4.2.2, Corollary 4.4.8).

Projective nested Cartesian codes were introduced and studied in [11]. This type of evaluation codes generalize the classical projective Reed–Muller codes [49]. As an application in Section 3.5, we will give some support for the following interesting conjecture.

Conjecture 3.5.2. (Carvalho, Lopez-Neumann, and López [11]) *Let $A_1, \ldots, A_s$ be subsets of K and let $C_{\mathcal{X}}(d)$ be the d-th projective nested Cartesian code on the set $\mathcal{X} =$*

$[A_1 \times \cdots \times A_s]$ with $d_i = |A_i|$ for $i = 1, \ldots, s$. Then its minimum distance is given by

$$\delta_{\mathcal{X}}(d) = \begin{cases} (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s & \text{if } d \leq \sum_{i=2}^{s} (d_i - 1)\,, \\ 1 & \text{if } d \geq \sum_{i=2}^{s} (d_i - 1) + 1, \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

We find a counterexample where this conjecture fails in general (Example 3.5.8). However the conjecture holds in certain cases. One could ask whether or not the conjecture is valid for $d \leq \operatorname{reg}(\delta_{I(\mathcal{X})})$.

Let $\prec$ be the lexicographical order on S with $x_1 \prec \cdots \prec x_s$. Carvalho et. al. found a Gröbner basis for $I(\mathcal{X})$ whose initial ideal is L, and obtained formulas for the regularity and the degree of the coordinate ring $S/I(\mathcal{X})$ (Proposition 3.5.3).

They showed the conjecture for certain families, and essentially showed that their conjecture can be reduced to:

Conjecture 3.5.4. (Carvalho, Lopez-Neumann, and López [11]) *If $f \in S_d$ is a standard polynomial such that $(I(\mathcal{X}) : f) \neq I(\mathcal{X})$, $1 \leq d \leq \sum_{i=2}^{s} (d_i - 1)$, and $V_{\mathcal{X}}(f)$ is zero set of f in $\mathcal{X}$, then*

$$|V_{\mathcal{X}}(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s,$$

where $0 \leq k \leq s - 2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

We show an explicit upper bound for the number of zeros of f in $\mathcal{X}$:

Theorem 3.5.5. $|V_{\mathcal{X}}(f)| \leq \deg(S/(\operatorname{in}_\prec(I(\mathcal{X})), x^a)) =$

$$\begin{cases} \deg(S/I(\mathcal{X})) - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) & \text{if } a_r \leq d_r, \\ \deg(S/I(\mathcal{X})) - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s) & \text{if } a_r \geq d_r + 1. \end{cases}$$

Then we use Theorem 3.5.5 to show Conjecture 3.5.4 when the variable x_1 divides the leading monomial of f.

Theorem 3.5.6. *If x_1 divides $x^a = \operatorname{in}_\prec(f)$, then*

$$|V_{\mathcal{X}}(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s,$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

As a consequence we show that the minimum distance of $C_{\mathcal{X}}(d)$ of Conjecture 3.5.2 is in fact the minimum distance of certain evaluation linear code (Corollary 3.5.7).

In Chapter 4, we introduce the edge ideal $I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$, for convenience we recall the definition of this notion below. The study of this ideal was motivated because edge ideals of weighted oriented graphs arise in the theory of Reed–Muller codes as initial ideals of vanishing ideals of projective nested Cartesian sets over finite fields [11, 40, 49]. Indeed, the ideal L is the edge ideal of a complete weighted oriented graph and is the initial ideal of $I(\mathcal{X})$. Recall that in Section 3.4, we studied these edge ideals and some of their algebraic invariants from an algebraic point of view. In Chapter 4, we continue this study and develop an algebraic combinatorics theory of these ideals, determine the irredundant irreducible decomposition of $I(\mathcal{D})$ in terms of the notion of strong vertex cover, give a characterization of the associated primes and the unmixed property for certain weighted oriented graphs. Finally, we study the Cohen–Macaulay property of $I(\mathcal{D})$ [45].

A *weighted oriented graph* is a triplet $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}), w)$, where $V(\mathcal{D})$ is a finite set, $E(\mathcal{D}) \subset V(\mathcal{D}) \times V(\mathcal{D})$ and w is a weight function $w\colon V(\mathcal{D}) \to \mathbb{N}$. This set $V(\mathcal{D})$ is the vertex set of $\mathcal{D}$ and $E(\mathcal{D})$ is the edge set of $\mathcal{D}$. Sometimes we will write V and E for the vertex set and edge set of $\mathcal{D}$, respectively. The *weight* of $x \in V(\mathcal{D})$ is $w(x)$. The set $\{x \in V(\mathcal{D}) \mid w(x) \neq 1\}$ is denoted by V^+. If $e = (x, y) \in E(\mathcal{D})$, then x is the tail of e and y is the head of e. The *underlying graph* of $\mathcal{D}$ is the simple graph G whose vertex set is V and whose edge set is $\{\{x, y\} | (x, y) \in E\}$. If $V(\mathcal{D}) = \{x_1, \ldots, x_s\}$, then we consider the polynomial ring $S = K[x_1, \ldots, x_s]$ in s variables over a field K. The *edge ideal* of $\mathcal{D}$ is the ideal of S is given by

$$I(\mathcal{D}) := (x_i x_j^{w(x_j)} : (x_i, x_j) \in E(\mathcal{D})),$$

see Definition 4.2.2. If $w(x) = 1$ for all $x \in V(\mathcal{D})$, we recover the edge ideal of a graph because in this case $I(\mathcal{D})$ is $I(G)$.

In Section 4.1, we study the vertex covers of $\mathcal{D}$ and extend the classical definition in graph theory of minimal vertex cover by introducing the notion of strong vertex cover (Definition 4.1.8) and prove that a minimal vertex cover is strong (Corollary 4.1.10). A set of vertices C of G (resp. $\mathcal{D}$) is called a *vertex cover* of G (resp. of $\mathcal{D}$) if any edge of G (resp. $\mathcal{D}$) contains at least one vertex of C. Note that C is a vertex cover of G if and only if C is a vertex cover of $\mathcal{D}$. A vertex cover C of G or $\mathcal{D}$ is *minimal* if for any other vertex cover C' with $C' \subset C$ one has $C' = C$. Now, we explain some definitions and introduce some more notation.

Let $\mathcal{D}$ be a weighted oriented graph and let x be a vertex of $\mathcal{D}$. The *out-neighbourhood* and the *in-neighbourhood* of x are given by

$$N^+_{\mathcal{D}}(x) = \{y \mid (x, y) \in E(\mathcal{D})\} \text{ and } N^-_{\mathcal{D}}(x) = \{y \mid (y, x) \in E(\mathcal{D})\},$$

respectively. Furthermore, the *neighbourhood* of x is the set $N_{\mathcal{D}}(x) = N^+_{\mathcal{D}}(x) \cup N^-_{\mathcal{D}}(x)$.

Let C be a vertex cover of $\mathcal{D}$, we define the following partition of C:

- $L_1(C) := \{x \in C \mid N^+_{\mathcal{D}}(x) \cap C^c \neq \emptyset\}$,
- $L_2(C) := \{x \in C \mid x \notin L_1(C) \text{ and } N^-_{\mathcal{D}}(x) \cap C^c \neq \emptyset\}$,
- $L_3(C) := C \setminus (L_1(C) \cup L_2(C))$,

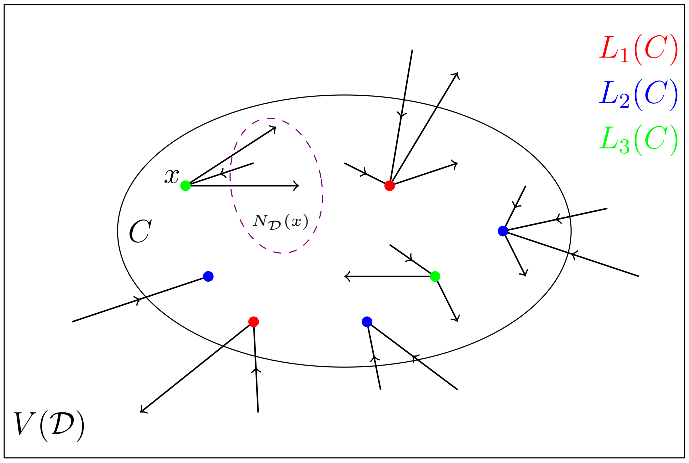


Figure 1: $L_1(C)$, $L_2(C)$ and $L_3(C)$.

where C^c is the complement of C, i.e., $C^c = V(\mathcal{D}) \setminus C$. It is not hard to see that $L_3(C)$ is the set of all $x \in V(\mathcal{D})$ such that $N_{\mathcal{D}}(x) \subset C$ (Proposition 4.1.6). To illustrate $L_1(C)$, $L_2(C)$ and $L_3(C)$ see Figure 1.

A vertex cover C of $\mathcal{D}$ is *strong* if for each $x \in L_3(C)$ there is $(y, x) \in E(\mathcal{D})$ such that $y \in L_2(C) \cup L_3(C)$ and $w(y) > 1$. An important fact is that a strong vertex cover is not always minimal. The vertex set $V(\mathcal{D})$ is clearly a vertex cover that is not minimal. Furthermore since $L_3(V(\mathcal{D})) = V(\mathcal{D})$, $V(\mathcal{D})$ is a strong vertex cover if and only if $N^-_{\mathcal{D}}(x) \cap V^+ \neq \emptyset$ for each $x \in V(\mathcal{D})$ (Remark 4.1.11). In Example 4.2.15, we give a weighted oriented graph $\mathcal{D}$, where $V(\mathcal{D})$ is a strong vertex cover properly containing all other strong vertex covers. We give necessary and sufficient conditions for the vertex set of $\mathcal{D}$ to be a strong vertex cover (Lemmas 4.1.14 and 4.1.15).

We are able to characterize when $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$ in terms of unicycle oriented subgraphs (Definition 4.1.13).

Propositon 4.1.16. *Let $\mathcal{D} = (V, E, w)$ be a weighted oriented graph, hence V is a strong vertex cover of $\mathcal{D}$ if and only if there are $\mathcal{D}_1, \ldots, \mathcal{D}_t$ unicycle oriented subgraphs of $\mathcal{D}$ such that $V(\mathcal{D}_1), \ldots, V(\mathcal{D}_t)$ is a partition of $V = V(\mathcal{D})$.*

The strong vertex covers will determine the irredundant irreducible decomposition of the edge ideal of $\mathcal{D}$ by associating an irreducible ideal to each vertex cover of $\mathcal{D}$. Let C be a vertex cover of $\mathcal{D}$, the *irreducible ideal associated to* C is the ideal of S given by

$$I_C := \big(L_1(C) \cup \{x_j^{w(x_j)} \mid x_j \in L_2(C) \cup L_3(C)\}\big).$$

The following are some of our main results on edge ideals of weighted oriented graphs. The next theorem gives a combinatorial characterization of the minimal irreducible ideals (Definition 4.2.7) of $I(\mathcal{D})$ that would lead us to determine its irredundant irreducible decomposition (Definition 4.2.1).

Theorem 4.2.12. *The following conditions are equivalent:*

(1) $\mathfrak{q}$ *is a minimal irreducible monomial ideal of* $I(\mathcal{D})$.

(2) *There is a strong vertex cover* C *of* $\mathcal{D}$ *such that* $\mathfrak{q} = I_C$.

We are able to show that an irredundant primary decomposition (Definition 1.3.15, Corollary 1.3.17) of $I(\mathcal{D})$ is the irredundant irreducible decomposition of $I(\mathcal{D})$.

Theorem 4.2.13. *If* $S(\mathcal{D})$ *is the set of strong vertex covers of* $\mathcal{D}$, *then the irredundant irreducible decomposition of* $I(\mathcal{D})$ *is given by* $I(\mathcal{D}) = \bigcap_{C \in S(\mathcal{D})} I_C$.

If $C_1, \ldots, C_t$ are the strong vertex covers of $\mathcal{D}$, then by Theorem 4.2.13, $I_{C_1} \cap \cdots \cap I_{C_t}$ is the irredundant irreducible decomposition of $I(\mathcal{D})$. Furthermore, if $\mathfrak{p}_i = \mathrm{rad}(I_{C_i})$, then $\mathfrak{p}_i = (C_i)$. So, $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $1 \leq i < j \leq t$. Thus, $I_{C_1} \cap \cdots \cap I_{C_t}$ is an irredundant primary decomposition of $I(\mathcal{D})$. In particular we have $\mathrm{Ass}(I(\mathcal{D})) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$.

The ideal $I(\mathcal{D})$ is *unmixed* if all its associated primes have height equal to $\mathrm{ht}(I(\mathcal{D}))$ (Definition 1.4.11). The unmixed property is important because Cohen–Macaulay ideals are unmixed [22, Corollary 1.5.14]. In Section 4.3, we use the two previous results to prove the following combinatorial characterization of the unmixed property of $I(\mathcal{D})$.

Theorem 4.3.1. *The following conditions are equivalent:*

(1) $I(\mathcal{D})$ *is unmixed.*

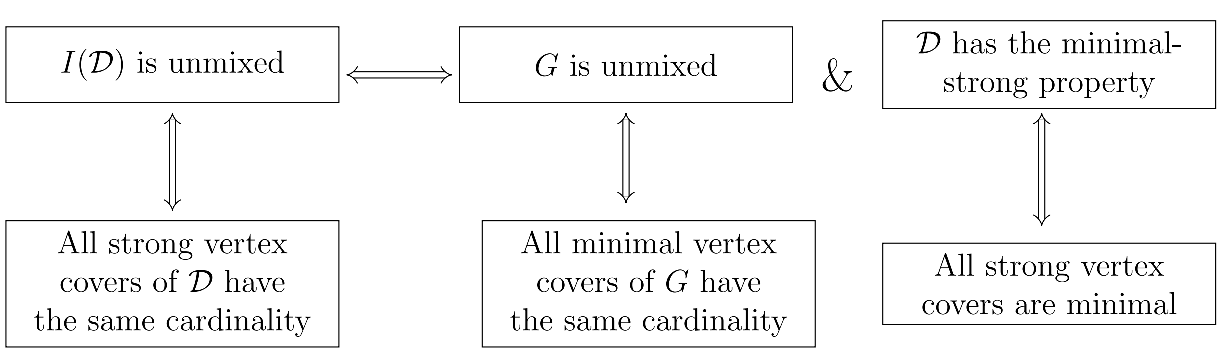
(2) *All strong vertex covers of* $\mathcal{D}$ *have the same cardinality.*

(3) $I(G)$ *is unmixed and* $L_3(C) = \emptyset$ *for each strong vertex cover* C *of* $\mathcal{D}$.

In addition we prove that the unmixed property of a weighted oriented graph is closed under c-minors (Definition 4.3.4).

Theorem 4.3.7. *If* $\mathcal{D}$ *is unmixed, then a c-minor of* $\mathcal{D}$ *is unmixed.*

We say that a weighted oriented graph $\mathcal{D}$ has the *minimal-strong property* if each strong vertex cover is a minimal vertex cover (Definition 4.3.2). The following picture illustrated our results about the unmixed property of $\mathcal{D}$.



Furthermore, if the underlying graph H of a weighted oriented graph $\mathcal{H}$ is a whisker graph, bipartite graph or a cycle, we give an effective combinatorial characterization of the unmixed property of the edge ideal of $\mathcal{H}$. First, we explain the definition of a whisker graph, the origin of the name is clear from a picture where a whisker is added to each vertex of a cycle (see Figure 2), this terminology appears in [48].

To add a whisker to a vertex x of a graph G, one adds a new vertex y and the edge connecting y and x. Then, a *whisker graph* of G is a graph H whose vertex set is $V(H) = V(G) \cup \{y_1, \ldots, y_s\}$ and whose edge set is $E(H) = E(G) \cup \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\}$ (Definition 4.3.13).

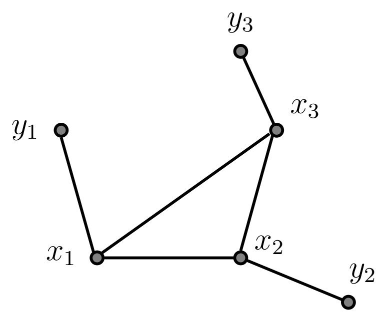


Figure 2: The whisker graph of a 3-cycle.

Let $\mathcal{D}$ and $\mathcal{H}$ be weighted oriented graphs. We say that $\mathcal{H}$ is a *whisker weighted oriented graph* of $\mathcal{D}$ if $\mathcal{D} \subset \mathcal{H}$ and the underlying graph H of $\mathcal{H}$ is a whisker graph of the underlying graph of $\mathcal{D}$ (Definition 4.3.14).

Theorem 4.3.15. *Let $\mathcal{H}$ be a whisker weighted oriented graph of $\mathcal{D}$, where $V(\mathcal{D}) = \{x_1, \ldots, x_s\}$ and $V(\mathcal{H}) = V(\mathcal{D}) \cup \{y_1, \ldots, y_s\}$. The following conditions are equivalents:*

(1) *$I(\mathcal{H})$ is unmixed.*

(2) *If $(x_i, y_i) \in E(\mathcal{H})$ for some $1 \leq i \leq s$, then $w(x_i) = 1$.*

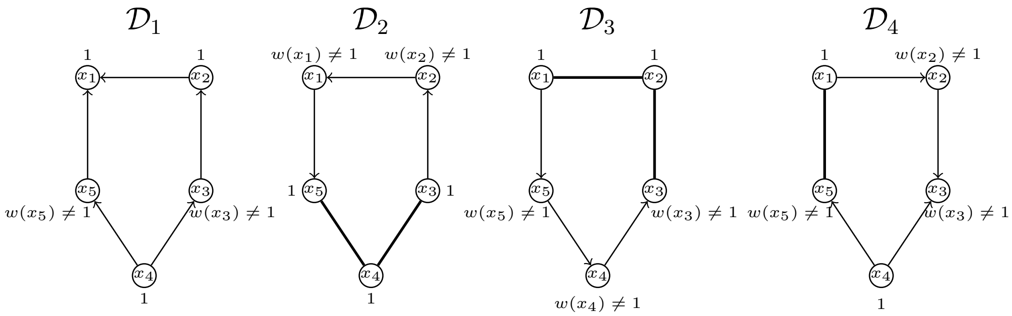
As an important application of Theorem 4.3.1, we give the following characterization for unmixed bipartite weighted oriented graphs. Our result is inspired by a criterion of Villarreal [57, Theorem 1.1] that describe the unmixed property of bipartite graphs in combinatorial terms. A graph G is *bipartite* if its vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2 (Definition 1.9.3). Accordingly, $\mathcal{D}$ is call a *bipartite weighted oriented graph* if its underlying graph G is bipartite.

Theorem 4.3.16. *Let $\mathcal{D}$ be a bipartite weighted oriented graph, then $I(\mathcal{D})$ is unmixed if and only if*

(1) *G has a perfect matching $\{\{x_1^1, x_1^2\}, \ldots, \{x_t^1, x_t^2\}\}$ where $\{x_1^1, \ldots, x_t^1\}$ and $\{x_1^2, \ldots, x_t^2\}$ are stable sets. Furthermore if $\{x_j^1, x_i^2\}$, $\{x_i^1, x_k^2\} \in E(G)$ then $\{x_j^1, x_k^2\} \in E(G)$.*

(2) *If $w(x_j^k) \neq 1$ and $N_{\mathcal{D}}^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$ where $\{k, k'\} = \{1, 2\}$, then $N_{\mathcal{D}}(x_{i_\ell}^k) \subset N_{\mathcal{D}}^+(x_j^k)$ and $N_{\mathcal{D}}^-(x_{i_\ell}^k) \cap V^+ = \emptyset$ for each $1 \leq \ell \leq r$.*

For a cycle C_n (Definition 1.9.3), it is well known that C_n is unmixed if and only if $n = 3, 4, 5, 7$ [22, Exercise 2.4.22]. However, to study unmixed weighted oriented cycles we need to consider the following obstructions:



Then, our characterization for the unmixed property of weighted oriented cycles is the following:

Theorem 4.3.19. *If the underlying graph of a weighted oriented graph $\mathcal{D}$ is a cycle and w is the weight function of $\mathcal{D}$, then $I(\mathcal{D})$ is unmixed if and only if one of the following conditions hold:*

(1) $n = 3$ *and there is* $x \in V(\mathcal{D})$ *such that* $w(x) = 1$.

(2) $n \in \{4, 5, 7\}$ *and the vertices with weight greater than* 1 *are sinks.*

(3) $n = 5$, *there is* $(x, y) \in E(\mathcal{D})$ *with* $w(x) = w(y) = 1$ *and* $\mathcal{D} \not\simeq \mathcal{D}_1, \mathcal{D} \not\simeq \mathcal{D}_2, D \not\simeq \mathcal{D}_3$.

(4) $\mathcal{D} \simeq \mathcal{D}_4$.

Finally in Section 4.4, we study the Cohen–Macaulayness of $I(\mathcal{D})$. We say that a weighted oriented graph $\mathcal{D}$ is *Cohen–Macaulay* over the field K if the ring $S/I(\mathcal{D})$ is Cohen–Macaulay (Definitions 1.4.4 and 4.4.1). And we propose the following interesting conjecture.

Conjecture 4.4.5. *$I(\mathcal{D})$ is Cohen–Macaulay if and only if $I(\mathcal{D})$ is unmixed and $I(G)$ is Cohen–Macaulay.*

In fact, it is clear that $\mathrm{rad}(I(\mathcal{D}))$ is the edge ideal of the underlying graph G (Definition 1.9.14) of $\mathcal{D}$. Then, if $I(\mathcal{D})$ is a Cohen–Macaulay ideal, applying a result of Herzog, Takayama and Terai [30, Theorem 2.6], we have that $I(G)$ is Cohen–Macaulay. Furthermore $I(\mathcal{D})$ is unmixed. This means that to prove Conjecture 4.4.5 we need only show that if $I(\mathcal{D})$ is unmixed and $I(G)$ is Cohen–Macaulay then $I(\mathcal{D})$ is Cohen–Macaulay.

As a support to Conjecture 4.4.5, we characterize the Cohen–Macaulayness when $\mathcal{D}$ is a weighted oriented path or a complete weighted oriented graph.

Proposition 4.4.6. *Let $\mathcal{D}$ be a weighted oriented graph such that $V = \{x_1, \ldots, x_k\}$ and whose underlying graph is a path $G = \{x_1, \ldots, x_k\}$. Then the following conditions are equivalent:*

(1) *$S/I(\mathcal{D})$ is Cohen–Macaulay.*

(2) *$I(\mathcal{D})$ is unmixed.*

(3) *$k = 2$ or $k = 4$. In the second case, if $(x_2, x_1) \in E(\mathcal{D})$ or $(x_3, x_4) \in E(\mathcal{D})$, then $w(x_2) = 1$ or $w(x_3) = 1$ respectively.*

Theorem 4.4.7. *If the underlying graph G of a weighted oriented graph $\mathcal{D}$ is a complete graph, then the following conditions are equivalent:*

(1) *$I(\mathcal{D})$ is unmixed.*

(2) *$I(\mathcal{D})$ is Cohen–Macaulay.*

(3) *There are not $\mathcal{D}_1, \ldots, \mathcal{D}_t$ unicycle oriented subgraphs of $\mathcal{D}$ such that $V(\mathcal{D}_1), \ldots, V(\mathcal{D}_t)$ is a partition of $V(\mathcal{D})$.*

The previous result allows us to recover some the algebraic properties of the initial ideal of the vanishing ideal of a projective nested Cartesian set over a finite field (Lemma 3.4.1, Corollary 4.4.8), which was studied in Section 3.4 from an algebraic point of view.

For all explained terminology and additional information, we refer to [9, 13, 16] (for the theory of Gröbner bases, commutative algebra, and Hilbert functions), and [37, 51] (for the theory of error-correcting codes and linear codes). In the first chapter we present some of the results that will be needed throughout this work and introduce some notation. Some of the results of this chapter are well known. We recall some necessary preliminaries on algebraic geometry, commutative algebra and graph theory. Some of the main topics in this chapter are Noetherian modules, Hilbert functions, Gröbner bases theory, also, we introduce the family of Reed–Muller-type codes and define their basic parameters (length, dimension, minimum distance).

Contents

Chapter 1

Preliminaries

In this chapter we introduced some notion and results from commutative algebra that will be needed throughout this work. For instance we introduce primary decomposition of modules, Cohen–Macaulay modules and rings, Hilbert function. There are very good references to learn commutative algebra, we use mainly [5, 13, 16, 22, 26, 41, 58].

We write a section about graph theory in order to understand relevant results in Section 2.2 and Chapter 4. All results of this section are well-known.

1.1 Noetherian modules

Definition 1.1.1. Let R be a commutative ring and let M be an R-module. M is called *Noetherian* if every submodule N of M is finitely generated, that is $N = Rf_1 + \cdots + Rf_q$, for some $f_1, \ldots, f_q$.

The following theorem gives us a characterization of the definition of Noetherian module.

Theorem 1.1.2. [58, Theorem 2.1.1] *The following conditions are equivalent:*

(a) *M is Noetherian.*

(b) *M satisfies the ascending chain condition for submodules; that is, for every ascending chain of submodules of M*

$$N_0 \subset N_1 \subset \cdots \subset N_n \subset N_{n+1} \subset \cdots \subset M$$

there exists an integer k such that $N_i = N_k$ for every $i \geq k$.

(c) *Any family $\mathcal{F}$ of submodules of M partially ordered by inclusion has a maximal element, i.e, there is $N \in \mathcal{F}$ such that if $N \subset N_i$ and $N_i \in \mathcal{F}$, then $N = N_i$.*

Proposition 1.1.3. [16, Proposition 1.4] *If M is a finitely generated R-module over a Noetherian ring R, then M is a Noetherian module.*

Corollary 1.1.4. *If R is a Noetherian ring and I is an ideal of R, then R/I and R^n are Noetherian R-modules. In particular any submodule of R^n is finitely generated.*

Theorem 1.1.5. (Hilbert's basis theorem [5, Theorem 7.5]) *A polynomial ring $R[x]$ over a Noetherian ring R is Noetherian.*

Definition 1.1.6. Let R be a ring and let $I \subset R$ be an ideal, the set of all prime ideals of R containing I is denoted by $V(I)$ and is called the *variety* of I. And the *minimal primes of I* are the minimal elements of $V(I)$ respect to inclusion.

1.2 Krull dimension and height

In this thesis we will always assume that the rings will be noetherian.

Definition 1.2.1. Let R be a ring.

- The set of all prime ideals of a ring R is called the *spectrum* of R, denoted by $\mathrm{Spec}(R)$.
- A *chain* of prime ideals of R is a finite strictly increasing sequence of primes ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n,$$

  the integer n is called the *length* or the chain.
- The *Krull dimension* of R, denoted by $\dim(R)$, is the supremum of the lengths of all chains of prime ideals in R.
- Let $\mathfrak{p}$ be a prime ideal of R, the *height* of $\mathfrak{p}$, denoted by $\mathrm{ht}(\mathfrak{p})$, is the supremum of the lengths of all chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

  which end at $\mathfrak{p}$.
- If I is an ideal of R, then $\mathrm{ht}(I)$, the *height of I*, is defined as

$$\mathrm{ht}(I) = \min\{\mathrm{ht}(\mathfrak{p}) \mid I \subset \mathfrak{p} \text{ and } \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

In general $\dim(R/I) + \mathrm{ht}(I) \leq \dim(R)$. The difference $\dim(R) - \dim(R/I)$ is called the *codimension of I* and $\dim(R/I)$ is called the *dimension of I*.

Definition 1.2.2. Let M be an R-module.

- The *annihilator of* M is given by

$$\operatorname{ann}_R(M) = \{x \in R \mid xM = 0\},$$

  if $m \in M$, the *annihilator of* m is $\operatorname{ann}(m) = \operatorname{ann}(Rm)$.

- Let N_1 and N_2 be submodules of M, their *ideal quotient* or *colon ideal* is defined as

$$(N_1 :_R N_2) = \{x \in R \mid xN_2 \subset N_1\}.$$

Remark 1.2.3. The *dimension of an* R*-module* M is $\dim(M) = \dim(R/\operatorname{ann}(M))$ and the *codimension of* M is $\operatorname{codim}(M) = \dim(R) - \dim(M)$.

Theorem 1.2.4. [16, Corollary 10.3] *If* $R[x]$ *is a polynomial ring over a Noetherian ring* R*, then* $\dim(R[x]) = \dim(R) + 1$.

1.3 Primary decomposition of modules

Definition 1.3.1. Let R be a ring and let I be an ideal of R.

- The *radical of* I is

$$\operatorname{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n > 0\}.$$

- $\operatorname{rad}(0)$ is called the *nilradical of* R, is the set of *nilpotent elements* of R and is denoted by $\mathfrak{N}_R$ or $\operatorname{nil}(R)$.

- A ring is *reduced* if its nilradical is zero.

- The *Jacobson radical* of R is the intersection of all the maximal ideals of R.

Proposition 1.3.2. [16, Corollary 2.12] *If* I *is a proper ideal of a ring* R*, then* $\operatorname{rad}(I)$ *is the intersection of all prime ideals containing* I.

Definition 1.3.3. Let M be a module over a ring R. The set of *associated primes* of M, denoted by $\operatorname{Ass}_R(M)$, is the set of all prime ideals $\mathfrak{p}$ of R such that there is a monomorphism ϕ of R-modules:

$$R/\mathfrak{p} \hookrightarrow M.$$

Note that $\mathfrak{p} = \operatorname{ann}(\phi(1))$.

Lemma 1.3.4. [58, Lemma 2.1.12] *If* $M \neq 0$ *is an* R*-module, then* $\operatorname{Ass}(M) \neq \emptyset$.

If $M = R/I$, we say that an associated prime ideal of R/I is an *associated prime* ideal of I and we set $\operatorname{Ass}(I) = \operatorname{Ass}(R/I)$.

Definition 1.3.5. Let M be an R-module, the *support* of M, denoted by $\mathrm{Supp}(M)$, is the set of all prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p} \neq 0$, where $M_\mathfrak{p}$ is the localization of M at the prime $\mathfrak{p}$.

Definition 1.3.6. Let M be an R-module. An element $x \in R$ is a *zero divisor* of M if there is $0 \neq m \in M$ such that $xm = 0$. The set of zero divisors of M is denoted by $\mathcal{Z}(M)$. If x is not a zero divisor on M, x is called a *regular element* of M.

Lemma 1.3.7. [58, Lemma 2.1.19] *If M is an R-module, then*

$$\mathcal{Z}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}.$$

Lemma 1.3.8. [9, Lemma 1.5.6] *If M is an $\mathbb{N}$-graded R-module and $\mathfrak{p}$ is in $\mathrm{Ass}(M)$, then $\mathfrak{p}$ is a graded ideal and there is $m \in M$ homogeneous such that $\mathfrak{p} = \mathrm{ann}(m)$.*

Lemma 1.3.9. *Let $V \neq \{0\}$ be a vector space over an infinite field K. Then V is not a finite union of proper subspaces of V.*

Proof. By contradiction. Assume that there are proper subspaces $V_1, \ldots, V_m$ of V such that $V = \bigcup_{i=1}^m V_i$, where m is the least positive integer with this property. Let

$$v_1 \in V_1 \setminus (V_2 \cup \cdots \cup V_m) \text{ and } v_2 \in V_2 \setminus (V1 \cup V_3 \cup \cdots \cup V_m).$$

Pick $m+1$ distinct non-zero scalars $k_0, \ldots, k_m$ in K. Consider the vectors $\beta_i = v_1 - k_i v_2$ for $i = 0, \ldots, m$. By the pigeon–hole principle there are distinct vectors $\beta_r, \beta_s \in V_j$ for some j. Since $\beta_r - \beta_s \in V_j$ we get $v_2 \in V_j$. Thus $j = 2$ by the choice of v_2. To finish the proof observe that $\beta_r \in V_2$ imply $v_1 \in V_2$, which contradicts the choice of v_1. □

Proposition 1.3.10. *Let I be a graded ideal of R. If K is infinite and $\mathfrak{m}$ is not in $\mathrm{Ass}(R/I)$, then there is $h_1 \in R_1$ such that $h_1 \in \mathcal{Z}(R/I)$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the associated primes of R/I. As R/I is graded, by Lemma 1.3.8, $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are graded ideals . We proceed by contradiction. Assume that R_1, the degree 1 part of S, is contained in $\mathcal{Z}(R/I)$. By Lemma 1.3.7, one has that $\mathcal{Z}(R/I) = \bigcup_{i=1}^m \mathfrak{p}_i$. Hence

$$R_1 \subset (\mathfrak{p}_1)_1 \cup (\mathfrak{p}_2)_1 \cup \cdots \cup (\mathfrak{p}_m)_1 \subset R_1,$$

where $(\mathfrak{p}_i)_1$ is the homogeneous part of degree 1 of the graded ideal $\mathfrak{p}_i$. Since K is infinite, from Lemma 1.3.9, we get $R_1 = (\mathfrak{p}_i)_1$ for some i. Hence, $\mathfrak{p}_i = \mathfrak{m}$, a contradiction. □

Definition 1.3.11. Let M be an R-module.

- The *minimal primes* of M are defined to be the minimal elements of $\mathrm{Supp}(M)$ with respect to inclusion.

- A minimal prime of M is called an *isolated associated prime* of M. An associated prime of M which is not isolated is called an *embedded prime*.

Definition 1.3.12. Let M be an R-module. A submodule N of M is said to be a $\mathfrak{p}$-*primary submodule* if $\text{Ass}_R(M/N) = \{\mathfrak{p}\}$. An ideal $\mathfrak{q}$ of a ring R is called a $\mathfrak{p}$-*primary ideal* if $\text{Ass}_R(R/\mathfrak{q}) = \{\mathfrak{p}\}$.

Definition 1.3.13. Let M be an R-module. A submodule N of M is said to be *irreducible* if N cannot be written as an intersection of two submodules of M that properly contain N.

Proposition 1.3.14. [58, Proposition 2.1.24] *Let M be an R-module. If $Q \neq M$ is an irreducible submodule of M, then Q is a primary submodule.*

Definition 1.3.15. Let M be an R-module and let $N \subsetneq M$ be a proper submodule. An *irredundant primary decomposition* of N is an expression of N as an intersection of submodules, say $N = N_1 \cap \cdots \cap N_r$, such that:

(a) (Submodules are primary) $\text{Ass}_R(M/N_i) = \{\mathfrak{p}_i\}$ for all i.

(b) (Irredundancy) $N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r$ for all i.

(c) (Minimality) $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $N_i \neq N_j$.

Theorem 1.3.16. [58, Proposition 2.1.27] *Let M be an R-module. If $N \subsetneq M$ is a proper submodule of M, then N has an irredundant primary decomposition.*

Corollary 1.3.17. *If R is a Noetherian ring and I is a proper ideal of R, then I has an irredundant primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, such that, $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal and* $\text{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Proof. Let $(0) = I/I = (\mathfrak{q}_1/I) \cap \cdots \cap (\mathfrak{q}_r/I)$ be an irredundant decomposition of the zero ideal of R/I. Then $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ and $\mathfrak{q}_i/I$ is $\mathfrak{p}_i$-primary; that is $\text{Ass}((R/I)/(\mathfrak{q}_i/I)) = \text{Ass}(R/\mathfrak{q}_i) = \{\mathfrak{p}_i\}$. Now show that $\mathfrak{q}_i$ is a primary ideal. If $xy \in \mathfrak{q}_i$ and $x \notin \mathfrak{q}_i$, then y is a zero-divisor of $R/\mathfrak{q}_i$, but $\mathcal{Z}(R/\mathfrak{q}_i) = \mathfrak{p}_i$, hence $y \in \mathfrak{p}_i = \text{rad}(\text{ann}(R/\mathfrak{q}_i)) = \text{rad}(\mathfrak{q}_i)$ and y^n is in $\mathfrak{q}_i$ for some $n > 0$. □

Corollary 1.3.18. [58, Corollary 2.1.29] *If M is an R-module, then*

$$\text{rad}(\text{ann}(M)) = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}.$$

Corollary 1.3.19. [58, Corollary 2.1.30] *If $N \subsetneq M$ and $N = N_1 \cap \cdots \cap N_r$ is an irredundant primary decomposition of N with $\text{Ass}_R(M/N_i) = \{\mathfrak{p}_i\}$, then*

$$\text{Ass}_R(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$$

and $\text{ann}(M/N_i)$ is a $\mathfrak{p}_i$-primary ideal for all i.

1.4 Cohen–Macaulay rings and modules

We introduce a some special type of rings and modules called Cohen–Macaulay, this topic is well studied in commutative algebra. The main references for Cohen–Macaulay rings are [9, 16, 58].

Definition 1.4.1. Let M be an R-module.

- M has *finite length* if there is a *composition series*

$$(0) = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_r = M,$$

where M_i/M_{i-1} is a non-zero *simple module* (that is, M_i/M_{i-1} has no proper submodules other than (0)) for all i. Note that M_i/M_{i-1} must be cyclic and thus isomorphic to $R/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$. The number r is independent of the composition series and is called the *length* of M, it is usually denoted by $\ell_R(M)$.

- A sequence $\bar{\theta} := \theta_1, \ldots, \theta_r$ in R is called a *regular sequence* of M or an *M-regular sequence* if $(\bar{\theta})M \neq M$ and $\theta_i \notin \mathcal{Z}(M/(\theta_1, \ldots, \theta_{i-1}))$ for all i.

Theorem 1.4.2. (Dimension theorem [41, Theorem 13.4]) *Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. Set*

$$\delta(M) = \min\{r \mid \text{ there are } x_1, \ldots, x_r \in \mathfrak{m} \text{ with } \ell_R(M/(x_1, \ldots, x_r)M) < \infty\},$$

then $\dim(M) = \delta(M)$.

Lemma 1.4.3. [58, Lemma 2.3.6] *Let M be a module over a local ring $(R, \mathfrak{m})$. If $\theta_1, \ldots, \theta_r$ is an M-regular sequence in $\mathfrak{m}$, then $r \leq \dim(M)$.*

Definition 1.4.4. Let $(R, \mathfrak{m})$ be a local ring and $M \neq 0$ an R-module.

- The *depth* of M, denoted by $\operatorname{depth}(M)$, is the length of any maximal regular sequence on M, which is contained in $\mathfrak{m}$.

- M is called a *Cohen–Macaulay module* (C-M for short) if $\operatorname{depth}(M) = \dim(M)$.

- R is called a *Cohen–Macaulay ring* if R is C-M as an R-module.

- If the dimension of M is d. A *system of parameters* (s.o.p. for short) of M is a set of elements $\theta_1, \ldots, \theta_d$ in $\mathfrak{m}$ such that

$$\ell_R(M/(\theta_1, \ldots, \theta_d)) < \infty.$$

Definition 1.4.5. Let R be a Noetherian ring and M an R-module.

- M is a *Cohen–Macaulay module* if $M_{\mathfrak{m}}$ is a C-M module for all maximal ideals $\mathfrak{m} \in \operatorname{Supp}(M)$. In particular we consider the zero module to be Cohen–Macaulay.

- As in the local case, R is a *Cohen–Macaulay ring* if R is C-M as an R-module.

- An ideal I of R is *Cohen–Macaulay* if R/I is a C-M R-module.

Lemma 1.4.6. (Depth lemma [55, p. 305]) *If* $0 \to N \to M \to L \to 0$ *is a short exact sequence of modules over a local ring* R*, then*

(a) *If* $\operatorname{depth}(M) < \operatorname{depth}(L)$*, then* $\operatorname{depth}(N) = \operatorname{depth}(M)$*.*

(b) *If* $\operatorname{depth}(M) = \operatorname{depth}(L)$*, then* $\operatorname{depth}(N) \geq \operatorname{depth}(M)$*.*

(c) *If* $\operatorname{depth}(M) > \operatorname{depth}(L)$*, then* $\operatorname{depth}(N) = \operatorname{depth}(L) + 1$*.*

Lemma 1.4.7. [58, Lemma 2.3.10] *If* M *is a module over a local ring* $(R, \mathfrak{m})$ *and* $z \in \mathfrak{m}$ *is a regular element of* M*, then*

(a) $\operatorname{depth}(M/zM) = \operatorname{depth}(M) - 1$.

(b) $\dim(M/zM) = \dim(M) - 1$.

Proposition 1.4.8. [58, Proposition 2.3.19] *Let* M *be a module of dimension* d *over a local ring* $(R, \mathfrak{m})$ *and let* $\bar{\theta} = \theta_1, \ldots, \theta_d$ *be a system of parameters of* M*. Then* M *is C-M if and only if* $\bar{\theta}$ *is an* M*-regular sequence.*

Lemma 1.4.9. [58, lemma 2.3.20] *Let* $(R, \mathfrak{m})$ *be a local ring and let* $(f_1, \ldots, f_r)$ *be an ideal of height equal to* r*. Then there are* $f_{r+1}, \ldots, f_d$ *in* $\mathfrak{m}$ *such that* $f_1, \ldots, f_d$ *is a system of parameters of* R*.*

Definition 1.4.10. Let R be a ring and let I be an ideal of R. If I is generated by a regular sequence we say that I is a *complete intersection*.

Definition 1.4.11. An ideal I of a ring R is *height unmixed* or *unmixed* if $\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p})$ for all $\mathfrak{p}$ in $\operatorname{Ass}_R(R/I)$.

Proposition 1.4.12. [58, Proposition 2.3.24] *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring and let* I *be an ideal of* R*. If* I *is a complete intersection, then* R/I *is Cohen–Macaulay and* I *is unmixed.*

Theorem 1.4.13. (Unmixedness theorem [41, Theorem 17.6]) *A ring* R *is Cohen–Macaulay if and only if every proper ideal* I *of* R *of height* r *generated by* r *elements is unmixed.*

1.5 Hilbert function

We introduce the Hilbert function and the notion of degree. In particular, we will recall some results well-known about a standard method to compute the degree using Hilbert series. The main references for Hilbert functions are [4, 13, 16, 21].

Let $S = K[x_1, \ldots, x_s]$ be a polynomial ring over a field K and let $I \subset S$ be an ideal. We will use the following multi-index notation: for $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, set $x^a := x_1^{a_1} \cdots x_s^{a_s}$. The multiplicative group of K is denoted by K^*. As usual, $\mathfrak{m}$ will denote the maximal ideal of S generated by $x_1, \ldots, x_s$. The vector space of polynomials in S (resp. I) of degree at most i is denoted by $S_{\leq i}$ (resp. $I_{\leq i}$).

Definition 1.5.1. Let $S = \bigoplus_{d=0}^{\infty} S_d$ be the polynomial ring with the standard grading and let I be a graded ideal of S.

- The *affine Hilbert function* of S/I, denoted by H_I^a, is given by

$$H_I^a(i) = \dim_K(S_{\leq i}/I_{\leq i}).$$

- The *Hilbert function* of S/I, denoted by H_I, is given by

$$H_I(i) = H_I^a(i) - H_I^a(i-1).$$

Theorem 1.5.2. (Hilbert [9, Theorem 4.1.3]) *Let $S = \bigoplus_{d=0}^{\infty} S_d$ be the polynomial ring with the standard grading and let I be a graded ideal of S with $k = \dim(S/I)$. If S_0 is a field, then there is a unique polynomial $\varphi_I(t) \in \mathbb{Q}[t]$ of degree $k-1$ such that $\varphi_I(i) = H_I(i)$ for $i \gg 0$.*

Let $S[u]$ be a polynomial ring where $u = x_{s+1}$ is a new variable. For $f \in S$ of degree d define

$$f^h = u^d f(x_1/u, \ldots, x_s/u);$$

that is, f^h is the *homogenization* of the polynomial f with respect to u. The *homogenization* of I is the ideal I^h of $S[u]$ given by $I^h = (f^h \mid f \in I)$, and $S[u]$ is given the standard grading.

Lemma 1.5.3. *Let I be a graded ideal of S. Then, $H_I^a(i) = H_{I^h}(i)$ for $i \geq 0$.*

Proof. Fix $i \geq 0$. The mapping $S[u]_i \to S_{\leq i}$ induced by mapping $u \mapsto 1$ is a K-linear surjection. Consider the induced composite K-linear surjection $S[u]_i \to S_{\leq i} \to S_{\leq i}/I_{\leq i}$. An easy check show that this has kernel I_i^h. Hence, we have a K-linear isomorphism of finite dimensional K-vector spaces

$$S[u]_i/I_i^h \simeq S_{\leq i}/I_{\leq i}.$$

Thus, $H_I^a(i) = H_{I^h}(i)$. □

Proposition 1.5.4. *Let $I \subset S$ be an ideal and let k be the Krull dimension of S/I. Then there are unique polynomials*

$$h_I^a(t) = \sum_{j=0}^{k} a_j t^j \in \mathbb{Q}[t] \quad \text{and} \quad h_I(t) = \sum_{j=0}^{k-1} c_j t^j \in \mathbb{Q}[t]$$

of degrees k and $k-1$, respectively, such that $h_I^a(i) = H_I^a(i)$ and $h_I(i) = H_I(i)$ for $i \gg 0$.

Proof. Let I^h be the homogenization of I relative to a new variable u. By Lemma 1.5.3, $H_I^a(i) = H_{I^h}(i)$ for $i \gg 0$, and by Theorem 1.5.2, the Hilbert function of I^h is a polynomial function of degree equal to $\dim(S[u]/I^h) - 1$. Since $\dim(S[u]/I^h) = \dim(S/I) + 1$, we get that H_I^a is a polynomial function of degree k. That H_I is a polynomial function of degree $k-1$ follows recalling that $H_I(i) = H_I^a(i) - H_I^a(i-1)$ for $i \geq 1$. □

Definition 1.5.5. The polynomials h_I^a and h_I are called the *affine Hilbert polynomial* and the *Hilbert polynomial* of S/I. By convention, the zero polynomial has degree -1.

Now, we introduce some algebraic invariants which will be mentioned throughout this thesis.

Definition 1.5.6. The integer $a_k(k!)$, denoted by $\deg(S/I)$, is called the *degree* of S/I.

Remark 1.5.7. Notice that $a_k(k!) = c_{k-1}((k-1)!)$ for $k \geq 1$. If $k = 0$, then $H_I^a(i) = \dim_K(S/I)$ for $i \gg 0$ and the degree of S/I is just $\dim_K(S/I)$.

Definition 1.5.8. The *regularity index* of S/I, denoted by $\operatorname{ri}(S/I)$, is the least integer $r \geq 0$ such that $h_I(d) = H_I(d)$ for $d \geq r$. The *affine regularity index* of S/I, denoted by $\operatorname{ri}^a(S/I)$, is the least integer $r \geq 0$ such that $h_I^a(d) = H_I^a(d)$ for $d \geq r$.

Definition 1.5.9. Let $I \subset S$ be a graded ideal and consider the minimal graded free resolution of $M = S/I$ as an S-module:

$$\mathbb{F}_\star : \; 0 \to \bigoplus_j S(-j)^{b_{gj}} \to \cdots \to \bigoplus_j S(-j)^{b_{1j}} \to S \to S/I \to 0.$$

The *Castelnuovo–Mumford regularity* of M (*regularity* of M for short) is defined as

$$\operatorname{reg}(M) = \max\{j - i | b_{ij} \neq 0\}.$$

Remark 1.5.10. If I is a graded Cohen–Macaulay ideal of S of dimension 1, then $\operatorname{reg}(S/I)$, the Castelnuovo–Mumford regularity of S/I, is equal to the regularity index of S/I (see [17]). In this case we call $\operatorname{ri}(S/I)$ (resp. $\operatorname{ri}^a(S/I)$) the regularity (resp. affine regularity) of S/I and denote this number by $\operatorname{reg}(S/I)$ (resp. $\operatorname{reg}^a(S/I)$).

Definition 1.5.11. Let $I \subset S$ be a graded ideal and let $f_1, \ldots, f_r$ be a minimal generating set of I. The *big degree* of I is defined as $\operatorname{bigdeg}(I) = \max_i\{\deg(f_i)\}$.

If I is graded, its regularity is related to the degrees of a minimal generating set of I. From definition of the regularity of S/I, one has.

Proposition 1.5.12. [17] *Let $I \subset S$ be a graded ideal, then*

$$\operatorname{reg}(S/I) \geq \operatorname{bigdeg}(I) - 1.$$

Remark 1.5.13. The degree or multiplicity of S/I is the positive integer

$$\deg(S/I) = \begin{cases} (k-1)! \lim\limits_{d \to \infty} H_I(d)/d^{k-1} & \text{if } k \geq 1, \\ \dim_K(S/I) & \text{if } k = 0. \end{cases}$$

Remark 1.5.14. If I is graded, $I_d = S_d \cap I$ is a vector subspace of S_d and

$$H_I^a(d) = \sum_{i=0}^{d} \dim_K(S_d/I_d)$$

for $d \geq 0$. Thus, one has $H_I(d) = \dim_K(S/I)_d$ for all d.

Definition 1.5.15. Let $I \subset S$ be a graded ideal. The *Hilbert series* of S/I, denoted by $F_I(t)$, is given by

$$F_I(t) := \sum_{d=0}^{\infty} H_I(d)t^d = \sum_{d=0}^{\infty} \dim_K(S/I)_d t^d.$$

Proposition 1.5.16. [58, Propositions 3.1.33 and 5.1.11] *Let $A = R_1/I_1$, $B = R_2/I_2$ be two standard graded algebras over a field K, where $R_1 = K[\mathbf{x}]$, $R_2 = K[\mathbf{y}]$ are polynomial rings in disjoint sets of variables and I_i is an ideal of R_i. If $R = K[\mathbf{x}, \mathbf{y}]$ and $I = I_1 + I_2$, then*

$$(R_1/I_1) \otimes_K (R_2/I_2) \simeq R/I \quad \textit{and} \quad F(A \otimes_K B, t) = F(A,t)F(B,t),$$

where $F(A,t)$ and $F(B,t)$ are the Hilbert series of A and B, respectively.

Theorem 1.5.17. (Hilbert-Serre [58, Theorem 5.1.4]) *Let $I \subset S$ be a graded ideal. Then there is a unique polynomial $h(t) \in \mathbb{Z}[t]$ such*

$$F_I(t) = \frac{h(t)}{(1-t)^\rho} \quad \textit{and} \quad h(1) > 0,$$

where $\rho = \dim(S/I)$.

Definition 1.5.18. Let $I \subset S$ be a graded ideal. The *a-invariant* of the graded ring S/I, denoted by $a(S/I)$, is the degree of $F_I(t)$ as a rational function, i.e., $a(S/I) = \deg(h(t)) - \rho$.

The $a-$invariant, the regularity, and the depth of M are closely related.

Theorem 1.5.19. [55, Corollary B.4.1] $a(M) \leq \operatorname{reg}(M) - \operatorname{depth}(M)$, *whit equality if M is Cohen–Macaulay.*

We can read of the degree of S/I from its Hilbert series:

Remark 1.5.20. The leading coefficient of the Hilbert polynomial $h_I(t)$ of S/I is equal to $h(1)/(k-1)!$. Thus $h(1)$ is equal to $\deg(S/I)$.

Lemma 1.5.21. [58, p. 177] *If $I \subset S$ is an ideal generated by homogeneous polynomials $f_1, \ldots, f_r$, with $r = \mathrm{ht}(I)$ and $\delta_i = \deg(f_i)$, then Hilbert series, the degree and the regularity of S/I are given by*

$$F_I(t) = \frac{\prod_{i=1}^{r}\left(1 - t^{\delta_i}\right)}{(1-t)^s}\ , \quad \deg(S/I) = \delta_1 \cdots \delta_r \quad \textit{and} \quad \mathrm{reg}(S/I) = \sum_{i=1}^{r}(\delta_i - 1).$$

Lemma 1.5.22. *If $I \subset S$ is a graded ideal and u is a new variable, then $a(S/I) = a(S[u]/I) + 1$.*

Proof. Let $F_1(t)$ and $F_2(t)$ be the Hilbert series of the graded rings S/I and $S[u]/I$ respectively. Using additivity of Hilbert series, from the exact sequence

$$0 \to (S[u]/I)[-1] \stackrel{u}{\to} S[u]/I \to S[u]/(I, u) \to 0,$$

we get $F_2(t) = F_1(t)/(1-t)$, that is, $\deg(F_1) = 1 + \deg(F_2)$. □

Lemma 1.5.23. [58, Corollary 5.1.9] *Let $I \subset S$ be a graded ideal. Then $\mathrm{ri}(S/I) = 0$ if $a(S/I) < 0$, and $\mathrm{ri}(S/I) = a(S/I) + 1$ otherwise.*

Lemma 1.5.24. *Let $I \subset S$ be a graded ideal. If $\dim(S/I) = 1$ and $\deg(S/I) \geq 2$, then $\mathrm{ri}(S/I) = \mathrm{ri}^a(S/I) + 1$.*

Proof. Let u be a new variable. The affine regularity index of S/I is the regularity index of $S[u]/I$ because I is graded. Hence, by Lemmas 1.5.22 and 1.5.23 it suffices to show that $a(S/I) \geq 0$. If $a(S/I) < 0$, the Hilbert series of S/I has the form $F_I(t) = 1/(1-t)$, i.e., $H_I(d) = 1$ for $d \geq 0$ and $\deg(S/I) = 1$, a contradiction. □

Theorem 1.5.25. *Let I be a graded ideal of S. If $\mathrm{depth}(S/I) > 0$, and H_I is the Hilbert function of S/I, then $H_I(i) \leq H_I(i+1)$ for $i \geq 0$.*

Proof. Case 1: If K is infinite, by Proposition 1.3.10, there is $h \in S_1$ a non-zero divisor of S/I. The homomorphism of K-vector spaces

$$(S/I)_i \to (S/I)_{i+1},\ \bar{z} \mapsto \bar{hz}$$

is injective, therefore $H_I(i) = \dim_K(S/I)_i \leq \dim_K(S/I)_{i+1} = H_I(i+1)$.

Case 2: If K is finite, consider the algebraic closure $\bar{K}$ of K. We set $\bar{S} = S \otimes_K \bar{K}$ and $\bar{I} = I\bar{S}$. Hence, from [50, Lemma 1.1], one has that $H_I(I) = H_{\bar{I}}(i)$. This means that the Hilbert function does not change when the base field is extended from K to $\bar{K}$. Applying the previous case to $H_{\bar{I}}$ we obtain the result. □

Lemma 1.5.26. *Let I be a graded ideal of S.The following hold.*

(a) *If $S_i = I_i$ for some i, then $S_\ell = I_\ell$ for all $\ell \geq i$.*

(b) *If $\dim(S/I) \geq 2$, then $\dim_K(S/I)_i > 0$ for $i \geq 0$.*

Proof. a) It suffices to prove the case $\ell = i+1$. As $I_{i+1} \subset S_{i+1}$, we need only show $S_{i+1} \subset I_{i+1}$. Take a non-zero monomial $x^a \in S_{i+1}$. Then, $x^a = x_1^{a_1} \cdots x_s^{a_s}$ with $a_j > 0$ for some j. Thus, $x^a \in S_1 S_i$. As $S_1 I_i \subset I_{i+1}$, we get $x^a \in I_{i+1}$.

b) If $\dim_K(S/I)_i = 0$ for some i, then $S_i = I_i$. Thus, by a), $H_I(j)$ vanishes for $j \geq i$, a contradiction because the Hilbert polynomial of S/I has degree $\dim(S/I) - 1 \geq 1$; see Theorem 1.5.2. □

Theorem 1.5.27. [20] *Let I be a graded ideal with* $\operatorname{depth}(S/I) > 0$. *If* $\dim(S/I) = 1$, *then there is an integer r and a constant c such that:*

$$1 = H_I(0) < H_I(1) < \cdots < H_I(r-1) < H_I(i) = c \text{ for } i \geq r.$$

Proof. Consider the algebraic closure $\bar{K}$ of K. Notice that $|\bar{K}| = \infty$. As in the proof of Theorem 1.5.25, we make a change of coefficients using the functor $(\cdot) \otimes_K \bar{K}$. Hence we may assume that K is infinite. By Proposition 1.3.10, there is $h \in S_1$ a non-zero divisor of S/I. From the exact sequence

$$0 \longrightarrow (S/I)[-1] \stackrel{h}{\longrightarrow} S/I \longrightarrow S/(h,I) \longrightarrow 0,$$

we get $H_I(i+1) - H_I(i) = H_R(i+1)$, where $R = S/(h,I)$.

Let $r \geq 0$ be the first integer such that $H_I(r) = H_I(r+1)$, thus $R_{r+1} = (0)$ and $S_{r+1} = (h,I)_{r+1}$. Then, by Lemma 1.5.26, $R_k = (0)$ for $k \geq r+1$. Hence, the Hilbert function of S/I is constant for $k \geq r$ and strictly increasing on $[0, r-1]$. □

Proposition 1.5.28. ([26, Lemma 5.3.11], [44]) *If I is an ideal of S and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ is a minimal primary decomposition, then*

$$\deg(S/I) = \sum_{\operatorname{ht}(\mathfrak{q}_i) = \operatorname{ht}(I)} \deg(S/\mathfrak{q}_i).$$

Lemma 1.5.29. *Let $I \subset S$ be a radical unmixed graded ideal. If $f \in S$ is homogeneous, $(I\colon f) \neq I$, and $\mathcal{A}$ is the set of all associated primes of S/I that contain f, then* $\operatorname{ht}(I) = \operatorname{ht}(I,f)$ *and*

$$\deg S/(I,f) = \sum_{\mathfrak{p} \in \mathcal{A}} \deg(S/\mathfrak{p}).$$

Proof. As f is a zero divisor of S/I and I is unmixed, there is an associated prime ideal $\mathfrak{p}$ of S/I of height $\operatorname{ht}(I)$ such that $f \in \mathfrak{p}$. Thus $I \subset (I,f) \subset \mathfrak{p}$, and consequently

$\mathrm{ht}(I) = \mathrm{ht}(I, f)$. Therefore the set of associated primes of (I, f) of height equal to $\mathrm{ht}(I)$ is not empty and is equal to $\mathcal{A}$. There is an irredundant primary decomposition

$$(I, f) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}'_{r+1} \cap \cdots \cap \mathfrak{q}'_t, \tag{1.5.1}$$

where $\mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}_{\mathrm{i}}$, $\mathcal{A} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, and $\mathrm{ht}(\mathfrak{q}'_i) > \mathrm{ht}(I)$ for $i > r$. We may assume that the associated primes of S/I are $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Since I is a radical ideal, we get that $I = \bigcap_{i=1}^m \mathfrak{p}_i$. Next we show the following equality:

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}'_{r+1} \cap \cdots \cap \mathfrak{q}'_t \cap \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_m. \tag{1.5.2}$$

The inclusion "$\supset$" is clear because $\mathfrak{q}_i \subset \mathfrak{p}_i$ for $i = 1, \ldots, r$. The equality "$\subset$" follows by noticing that the right hand side of Eq. (1.5.2) is equal to $(I, f) \cap \mathfrak{p}_{r+1} \cap \cdots \cap \mathfrak{p}_m$, and consequently it contains $I = \bigcap_{i=1}^m \mathfrak{p}_i$. Notice that $\mathrm{rad}(\mathfrak{q}'_j) = \mathfrak{p}'_j \not\subset \mathfrak{p}_i$ for all i, j and $\mathfrak{p}_j \not\subset \mathfrak{p}_i$ for $i \neq j$. Hence localizing Eq. (1.5.2) at the prime ideal $\mathfrak{p}_i$ for $i = 1, \ldots, r$, we get that $\mathfrak{p}_i = I_{\mathfrak{p}_i} \cap S = (\mathfrak{q}_i)_{\mathfrak{p}_i} \cap S = \mathfrak{q}_i$ for $i = 1, \ldots, r$. Using Eq. (1.5.1) and the additivity of the degree the required equality follows. □

We can note that the computation of the dimension, degree, a-invariant or index of regularity is reduced to the computation of the Hilbert series of S/I, for this we can help us of different computer algebra systems (*Macaulay2* [25] , *CoCoA* [1], *Singular* [21]) that compute the Hilbert series and the degree of S/I using Gröbner bases. For compute Hilbert series using elimination of variables we can see [6, 7].

1.6 Gröbner theory and footprint of an ideal

In this section we review some basic facts and definitions on Gröbner theory and the footprint of an ideal. The literature on the basics of Gröbner bases theory is numerous we cite for instance [2, 4, 13, 16, 21]. In this thesis we denote by $\mathcal{M}$ the set of monomials in $S = K[x_1, \ldots, x_s]$.

Definition 1.6.1. A total order $\prec$ on $\mathcal{M}$ is called a *monomial order* or *term order* if

(a) $1 \preceq x^a$ for all $x^a \in \mathcal{M}$, and

(b) for all $x^a, x^b, x^c \in \mathcal{M}$, $x^a \preceq x^b$ implies $x^a x^c \preceq x^b x^c$.

Example 1.6.2. Let $S = K[x_1, \ldots, x_s]$ be the polynomial ring over a field K.

(a) The *lexicographic order* (with $x_s \preceq \cdots \preceq x_1$) is defined by setting

- $x^a \preceq x^b$ if $a = b$,
- or the first non-zero entry from the left to the right in $b - a$ is positive.

(b) The *graded lexicographic order* (with $x_s \preceq \cdots \preceq x_1$) is defined by setting

- $x^a \preceq x^b$ if $a = b$ or $\sum_{i=1}^{s} a_i < \sum_{i=1}^{s} b_i$,
- or if $\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i$ then $x^a \preceq_{lex} x^b$.

(c) The *graded reverse lexicographic order* is defined by setting

- $x^a \preceq x^b$ if $a = b$ or $\sum_{i=1}^{s} a_i < \sum_{i=1}^{s} b_i$,
- or if $\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i$ then the first non-zero entry from the right to the left in $b - a$ is negative.

Definition 1.6.3. Let $\prec$ be a monomial order on S and let $(0) \neq I \subset S$ be an ideal. If f is a non-zero polynomial in S. Then one can write

$$f = \lambda_1 x^{\alpha_1} + \cdots + \lambda_r x^{\alpha_r},$$

with $\lambda_i \in K^*$ for all i and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_r}$.

- The *leading monomial*: x^{α_1} of f is denoted by $\mathrm{in}_\prec(f)$.
- The *leading coefficient*: λ_1 of f is denoted by $\mathrm{lc}_\prec(f)$.
- The *leading term*: $\lambda_1 x^{\alpha_1}$ of f is denoted by $\mathrm{lt}_\prec(f)$.
- The *initial ideal* of I, denoted by $\mathrm{in}_\prec(I)$, is the monomial ideal given by

$$\mathrm{in}_\prec(I) = (\{\mathrm{in}_\prec(f) | \, f \in I\}).$$

Definition 1.6.4. To *divide* $f \in S$ by $\{g_1, \ldots, g_r\} \subset S \setminus \{0\}$, with respect to a monomial order $\preceq$, means to find *quotients* $q_1, \ldots, q_r$ and a *remainder* r is S such that $f = q_1 g_1 + \cdots + q_r g_r + r$, and either $r = 0$ or no monomial appearing in r is a multiple of $\mathrm{in}_\prec(g_i)$, for all $i \in \{1, \ldots, r\}$.

Theorem 1.6.5. (Division algorithm [13, Theorem 3, p. 63]) *If $f, g_1, \ldots, g_r$ are polynomials in S, then f can be written as*

$$f = a_1 g_1 + \cdots + a_r g_r + h,$$

where $a_i, h \in S$ and either $h = 0$ or $h \neq 0$ and no term of h is divisible by one of the initial monomials $\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)$. Furthermore if $a_i g_i \neq 0$, then $\mathrm{in}_\prec(f) \geq \mathrm{in}_\prec(a_i g_i)$.

Definition 1.6.6. A subset $\mathcal{G} = \{g_1, \ldots, g_r\}$ of I is called a *Gröbner basis* of I if

$$\mathrm{in}_\prec(I) = (\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)).$$

Proposition 1.6.7. [13, Corollary 6, p. 77] *Fix a monomial order on S. Then every ideal I of S other than $\{0\}$ has a Gröbner basis. Furthermore, any Gröbner basis of an ideal I is a set of generators of I.*

Proposition 1.6.8. [13, Proposition 9,p. 463] *Let I be a homogeneous ideal and let $\prec$ be a monomial order on S. Then the initial ideal $\mathrm{in}_{\prec}(I)$ has the same Hilbert function as I.*

Definition 1.6.9. Let $I \subset S$ be an ideal.

- The *footprint* of I (with respect to a fixed monomial order in $\mathcal{M}$) is the set

$$\Delta(I) = \{M \in \mathcal{M} \mid \text{ is not the leading monomial of any polynomial on } I\}.$$

- The elements of $\Delta(I)$ are called *standard monomials* of I.

- A polynomial f is called *standard* if $f \neq 0$ and f is a K-linear combination of standard monomials.

The footprint of an ideal I has a close relationship with a Gröbner basis for I, both begin defined with respect to the same monomial order on $\mathcal{M}$.

Lemma 1.6.10. *If $I \subset S$ is an ideal and $\mathcal{G} = \{g_1, \dots, g_r\}$ is a Gröbner basis of I. Then a monomial x^a is in $\Delta_{\prec}(I)$ if and only if x^a is not a multiple of $\mathrm{in}_{\prec}(g_i)$ for all $i = 1, \dots, r$.*

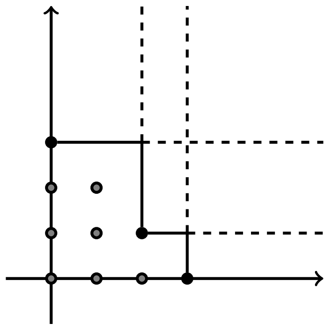
Proof. ($\Leftarrow$) Is obvious from the definition of $\Delta_{\prec}(I)$.

($\Rightarrow$) From the definition of Gröbner basis we know that if x^a is not a multiple of $\mathrm{in}_{\prec}(g_i)$ for all $i = 1, \dots, r$, then x^a is not the leading monomial of any polynomial in I. □

Remark 1.6.11. We can define a Gröbner basis for I as being a set $\{g_1, \dots, g_r\} \subset I$ such that the set of monomial which are multiples of $\mathrm{in}_{\prec}(g_i)$ for some $i \in \{1, \dots, r\}$ is exactly $\mathcal{M} \setminus \Delta_{\prec}(I)$.

In the following example we show how to use the above result to obtain a graphical representation of the footprint.

Example 1.6.12. Let $I = (x^3 - x, y^3 - y, x^2y - y) \subset \mathbb{R}[x, y]$ and endow the monomial set of $\mathbb{R}[x, y]$ with the lexicographic order where $y \leq x$. It is not difficult to check that $\{x^3 - x, y^3 - y, x^2y - y\}$ is a Gröbner basis for I. We have $\mathrm{in}_{\prec}(x^3 - x) = x^3$, $\mathrm{in}_{\prec}(y^3 - y) = y^3$ and $\mathrm{in}_{\prec}(x^2y - y) = x^2y$ and we apply the above lemma to determine $\Delta_{\prec}(I)$.
We can see the footprint of I in the figure below, where we represent a monomial x^ay^b by the pair of non negative integers (a, b).



In fact, the pairs $(3,0),(0,3),(2,1)$ correspond to the leading monomials of the Gröbner basis and from them is easy to determine the monomials which are multiples of at least one of these leading monomials (thus determining the set of monomials of the polynomials in I). Form this set and the above result we get that $\Delta_{\prec}(I)=\{1,x,x^2,y,xy,y^2,xy^2\}$. This graphical representation for the footprint can be generalized for a polynomial ring in n-variables.

This follows from the definition of a Gröbner basis.

Lemma 1.6.13. [10, p. 2] *Let $I\subset S$ be an ideal generated by $\mathcal{G}=\{g_1,\ldots,g_r\}$, then*

$$\Delta_{\prec}(I)\subset\Delta_{\prec}(\mathrm{in}_{\prec}(g_1),\ldots,\mathrm{in}_{\prec}(g_r)),$$

with equality if $\mathcal{G}$ is a Gröbner basis.

Proof. Take x^a in $\Delta_{\prec}(I)$. If $x^a\notin\Delta_{\prec}(\mathrm{in}_{\prec}(g_1),\ldots,\mathrm{in}_{\prec}(g_r))$, then $x^a=x^c\,\mathrm{in}_{\prec}(g_i)$ for some i and some x^c. Thus $x^a=\mathrm{in}_{\prec}(x^cg_i)$, with x^cg_i in I, a contradiction. The second statement holds by the definition of a Gröbner basis. □

Lemma 1.6.14. *Let $\prec$ be a monomial order, let $I\subset S$ be an ideal, and let f be a polynomial of S of positive degree. If $\mathrm{in}_{\prec}(f)$ is regular on $S/\mathrm{in}_{\prec}(I)$, then f is regular on S/I.*

Proof. Let g be a polynomial of S such that $gf\in I$. It suffices to show that $g\in I$. By the Theorem 1.6.5 we may assume that $g=0$ or that g is a standard polynomial of S/I. If $g\neq 0$, then $\mathrm{in}_{\prec}(g)\mathrm{in}_{\prec}(f)$ is in $\mathrm{in}_{\prec}(I)$ and consequently $\mathrm{in}_{\prec}(g)$ is in $\mathrm{in}_{\prec}(I)$, a contradiction. □

Lemma 1.6.15. *Let $\mathcal{G}=\{g_1,\ldots,g_r\}$ be a Gröbner basis of I. If for some i, the variable x_i does not divides $\mathrm{in}_{\prec}(g_j)$ for all j, then x_i is a regular element on S/I.*

Proof. Assume that $x_if\in I$. By the division algorithm we can write $f=g+h$, where $g\in I$ and h is 0 or a standard polynomial. It suffices to show that $h=0$. If $h\neq 0$, then $x_i\mathrm{in}_{\prec}(h)\in\mathrm{in}_{\prec}(I)$. Hence, using our hypothesis on x_i, we get $\mathrm{in}_{\prec}(h)\in\mathrm{in}_{\prec}(I)$, a contradiction. □

This lemma tells us that if x_i is a zero divisor of S/I for all i, then any variable x_i must occur in an initial monomial $\mathrm{in}_{\prec}(g_j)$ for some j.

1.7 Vanishing ideals of finite sets

Definition 1.7.1. Let K be a field. We define the *projective space* of dimension $s-1$ over K, denoted by $\mathbb{P}_K^{s-1}$ or $\mathbb{P}^{s-1}$ if K is understood, to be the quotient space

$$(K^s \setminus \{0\})/\sim$$

where two points α, β in $K^s \setminus \{0\}$ are equivalent under $\sim$ if $\alpha = c\beta$ for some $c \in K$. It is usual to denote the equivalence class of α by $[\alpha]$.

Definition 1.7.2. Let $\mathbb{X}$ be a subset of $\mathbb{P}^{s-1}$.

- The *vanishing ideal* of $\mathbb{X}$ denoted by $I(\mathbb{X})$, is defined as the graded ideal generated by the homogeneous polynomials in S that vanish at all points of $\mathbb{X}$.
- For a graded ideal $I \subset S$ define its *zero set* relative to $\mathbb{X}$ as

$$V_{\mathbb{X}}(I) = \{[\alpha] \in \mathbb{X} |\, f(\alpha) = 0, \forall f \in I \text{ homogeneous}\}.$$

- If $f \in S$ is homogeneous, the *zero set* of f, denoted by $V_{\mathbb{X}}(f)$, is the set of all $[\alpha] \in \mathbb{X}$ such that $f(\alpha) = 0$, that is, $V_{\mathbb{X}}(f)$ is the set of zeros of f in $\mathbb{X}$.

Lemma 1.7.3. [31, Proposition 6.3.3, Corollary 6.3.19] *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $[\alpha]$ be a point in $\mathbb{X}$, with $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $\alpha_k \neq 0$ for some k, and let $I_{[\alpha]}$ be the vanishing ideal of $[\alpha]$. Then $I_{[\alpha]}$ is a prime ideal,*

$$I_{[\alpha]} = (\{\alpha_k x_i - \alpha_i x_k |\, k \neq i \in \{1, \ldots, s\}),\ \deg(S/I_{[\alpha]}) = 1,$$

$\mathrm{ht}(I_{[\alpha]}) = s-1$, *and* $I(\mathbb{X}) = \bigcap_{[\beta]\in\mathbb{X}} I_{[\beta]}$ *is the primary decomposition of* $I(\mathbb{X})$.

Corollary 1.7.4. *If $\mathbb{X} \subset \mathbb{P}^{s-1}$ is a finite set, then* $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$.

Proof. It follows from Lemma 1.7.3 and Proposition 1.5.28. □

If $\mathbb{X}$ is a subset of $\mathbb{P}^{s-1}$ it is usual to denote the Hilbert function of $S/I(\mathbb{X})$ by $H_{\mathbb{X}}$.

Proposition 1.7.5. [20] *If $\mathbb{X} \subset \mathbb{P}^{s-1}$ is a finite set, then*

$$1 = H_{\mathbb{X}}(0) < H_{\mathbb{X}}(1) < \cdots < H_{\mathbb{X}}(r-1) < H_{\mathbb{X}}(d) = |\mathbb{X}|$$

for $d \geq r = \mathrm{reg}(S/I(\mathbb{X}))$.

Proof. It follows from Theorem 1.5.27. □

Lemma 1.7.6. *If $\emptyset \neq \mathbb{X} \subset \mathbb{P}^{s-1}$ and* $\dim(S/I(\mathbb{X})) = 1$, *then* $|\mathbb{X}| < \infty$ *and* $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$.

Proof. Since $\dim(S/I(\mathbb{X}))=1$, the Hilbert polynomial of $S/I(\mathbb{X})$ has degree 0. Then the Hilbert function of $S/I(\mathbb{X})$ is $H_{\mathbb{X}}(d)=a_1$ for $d\gg 0$. If $|\mathbb{X}|>a_1$, pick $[P_1],\ldots,[P_{a_1+1}]$ distinct points in $\mathbb{X}$ and set $I=\bigcap_{i=1}^{a_1+1} I_{[P_i]}$, where $I_{[P_i]}$ is the vanishing ideal of $[P_i]$. Then by Proposition 1.5.28 we have, $\dim(S/I)=1$ and $\deg(S/I)=a_1+1$. Hence, by Corollary 1.7.4, $H_I(d)=a_1+1$ for $d\gg 0$. From the exact sequence

$$0\to I/I(\mathbb{X})\to S/I(\mathbb{X})\to S/I\to 0$$

we get that $a_1=\dim_K(I/I(\mathbb{X}))_d+(a_1+1)$ for $d\gg 0$, a contradiction. Thus $|\mathbb{X}|\le a_1$ and by Corollary 1.7.4 one has equality. □

Definition 1.7.7. The set $\mathbb{T}=\{[(x_1,\ldots,x_s)]\in\mathbb{P}^{s-1}\,|\,x_i\in K^*\,\forall\, i\}$ is called a *projective torus*.

Notice that a torus is a group under componentwise multiplication.

1.8 Reed–Muller-type codes

In this section we introduce the families of projective Reed–Muller-type codes and its connection to vanishing ideals and Hilbert functions. Some references where this codes have been studied are [15, 24, 23].

Let $K=\mathbb{F}_q$ be a finite field and let $\mathbb{X}=\{[P_1],\ldots,[P_m]\}\neq\emptyset$ be a subset of $\mathbb{P}^{s-1}$ with $m=|\mathbb{X}|$. Fix a degree $d\geq 1$. For each i there is $f_i\in S_d$ such that $f_i(P_i)\neq 0$. There is a well-defined K-linear map:

$$\mathrm{ev}_d\colon S_d=K[x_1,\ldots,x_s]_d\to K^{|\mathbb{X}|},\qquad f\mapsto\left(\frac{f(P_1)}{f_1(P_1)},\ldots,\frac{f(P_m)}{f_m(P_m)}\right). \tag{1.8.1}$$

Definition 1.8.1.

- The map ev_d is called an *evaluation map.*
- The image of S_d under ev_d, denoted by $C_{\mathbb{X}}(d)$, is called a *projective Reed–Muller-type code* of degree d over the set $\mathbb{X}$. It is also called an *evaluation code* associated to $\mathbb{X}$.

The kernel of the evaluation map ev_d is $I(\mathbb{X})_d$. Hence there is an isomorphism of K-vector spaces $S_d/I(\mathbb{X})_d\simeq C_{\mathbb{X}}(d)$. If $\mathbb{X}$ is a subset of $\mathbb{P}^{s-1}$ it is usual to denote the Hilbert function $S/I(\mathbb{X})$ by $H_{\mathbb{X}}$. Thus $H_{\mathbb{X}}(d)$ is equal to $\dim_K C_{\mathbb{X}}(d)$.

Definition 1.8.2. By a *linear code* we mean a linear subspace of K^m for some m and for some finite field K.

Definition 1.8.3. Let $0\neq v\in C_{\mathbb{X}}(d)$.

- The *Hamming weight* of v, denoted by $||v||$, is the number of non zero entries of v.
- The *minimum distance* of $C_{\mathbb{X}}(d)$, denoted by $\delta_{\mathbb{X}}(d)$ or $\delta(C_{\mathbb{X}}(d))$, is defined as

$$\delta_{\mathbb{X}}(d):=\min\{||v||\colon 0\neq v\in C)\}.$$

Definition 1.8.4. The *basic parameters* of the linear code $C_{\mathbb{X}}(d)$ are its *length:* $|\mathbb{X}|$, *dimension:* $\dim_K(C_{\mathbb{X}}(d))$ and *minimum distance:* $\delta_{\mathbb{X}}(d)$.

Lemma 1.8.5. *The following hold.*

(a) *The map* ev_d *is well-defined, i.e., it is independent of the set of representatives that we choose for the points of* $\mathbb{X}$.

(b) *The basic parameters of* $C_{\mathbb{X}}(d)$ *are independent of* $f_1,\ldots,f_m$.

Proof. (a): If $P_1',\ldots,P_m'$ is another set of representatives, there are $\lambda_1,\ldots,\lambda_m$ in K^* such that $P_i'=\lambda_iP_i$ for all i. Thus, $f(P_i')/f_i(P_i')=f(P_i)/f_i(P_i)$ for $f\in S_d$ and $1\leq i\leq m$.
(b): Let $f_1',\ldots,f_m'$ be homogeneous polynomials of S of degree d such that $f_i'(P_i)\neq 0$ for $i=1,\ldots,m$, and let

$$\mathrm{ev}_d'\colon S_d\rightarrow K^{|\mathbb{X}|},\qquad f\mapsto\left(\frac{f(P_1)}{f_1'(P_1)},\ldots,\frac{f(P_m)}{f_m'(P_m)}\right)$$

be the evaluation map relative to $f_1',\ldots,f_m'$. Then $\ker(\mathrm{ev}_d)=\ker(\mathrm{ev}_d')$ and $\|\mathrm{ev}_d(f)\|=\|\mathrm{ev}_d'(f)\|$ for $f\in S_d$. It follows that the basic parameters of $\mathrm{ev}_d(S_d)$ and $\mathrm{ev}_d'(S_d)$ are the same. $\square$

Lemma 1.8.6. *Let* $\mathbb{Y}=\{[\alpha],[\beta]\}$ *be a subset of* $\mathbb{P}^{s-1}$ *with two elements. The following hold.*

(i) $\mathrm{reg}(S/I(\mathbb{Y}))=1$.

(ii) *There is* $h\in S_1$, *a form of degree* 1, *such that* $h(\alpha)\neq 0$ *and* $h(\beta)=0$.

(iii) *For each* $d\geq 1$, *there is* $f\in S_d$, *a form of degree* d, *such that* $f(\alpha)\neq 0$ *and* $f(\beta)=0$.

(iv) *If* $\mathbb{X}$ *is a subset of* $\mathbb{P}^{s-1}$ *with at least two elements and* $d\geq 1$, *then there is* $f\in S_d$ *such that* $f\notin I(\mathbb{X})$ *and* $(I(\mathbb{X})\colon f)\neq I(\mathbb{X})$.

Proof. (i): As $H_{\mathbb{Y}}(0)=1$ and $|\mathbb{Y}|=2$, by Proposition 1.7.5, we get that $H_{\mathbb{Y}}(1)=|\mathbb{Y}|=2$. Thus $S/I(\mathbb{Y})$ has regularity equal to 1.

(ii): Consider the evaluation map

$$\mathrm{ev}_1\colon S_1\longrightarrow K^2,\qquad f\mapsto\left(f(\alpha)/f_1(\alpha),f(\beta)/f_2(\beta)\right).$$

By part (i) this map is onto. Thus $(1,0)$ is in the image of ev_1 and the result follows.

(iii): It follows from part (ii) by setting $f=h^d$.

(iv): By part (iii), there are distinct $[\alpha],[\beta]$ in $\mathbb{X}$ and $f\in S_d$ such that $f(\alpha)\neq 0$, $f(\beta)=0$. Then $f\notin I(\mathbb{X})$. Notice that $f(\beta)=0$ if and only if $f\in I_{[\beta]}$. Hence, by Lemma 1.3.7 and Lemma 1.7.3, f is a zero divisor of $S/I(\mathbb{X})$, that is, $(I(\mathbb{X})\colon f)\neq I(\mathbb{X})$. $\square$

Proposition 1.8.7. *There is an integer $r_0 \geq 0$ such that*

$$|\mathbb{X}| = \delta_{\mathbb{X}}(0) > \delta_{\mathbb{X}}(1) > \cdots > \delta_{\mathbb{X}}(d) = \delta_{\mathbb{X}}(r_0) = 1 \quad for \quad d \geq r_0.$$

Proof. Assume that $\delta_{\mathbb{X}}(d) > 1$, it suffices to show that $\delta_{\mathbb{X}}(d) > \delta_{\mathbb{X}}(d+1)$. Pick $g \in S_d$ such that $g \notin I(\mathbb{X})$ and

$$|V_{\mathbb{X}}(g)| = \max\{|V_{\mathbb{X}}(f)| \colon \mathrm{ev}_d(f) \neq 0; f \in S_d\}.$$

Then $\delta_{\mathbb{X}}(d) = |\mathbb{X}| - |V_{\mathbb{X}}(g)| \geq 2$. Thus there are distinct points $[\alpha], [\beta]$ in $\mathbb{X}$ such that $g(\alpha) \neq 0$ and $g(\beta) \neq 0$. By Lemma 1.8.6, there is a linear form $h \in S_1$ such that $h(\alpha) \neq 0$ and $h(\beta) = 0$. Hence the polynomial hg is not in $I(\mathbb{X})$, has degree $d+1$, and has at least $|V_{\mathbb{X}}(g)| + 1$ zeros. Thus $\delta_{\mathbb{X}}(d) > \delta_{\mathbb{X}}(d+1)$, as required. □

The following summarizes the well-known relation between projective Reed–Muller-type codes and the theory of Hilbert functions.

Proposition 1.8.8. ([24], [46]) *The following hold.*

(i) $H_{\mathbb{X}}(d) = \dim_K(C_{\mathbb{X}}(d))$ *for* $d \geq 0$.

(ii) $\deg(S/I(\mathbb{X})) = |\mathbb{X}|$.

(iii) $\delta_{\mathbb{X}}(d) = 1$ *for* $d \geq \mathrm{reg}(S/I(\mathbb{X}))$.

(iv) $S/I(\mathbb{X})$ *is a Cohen–Macaulay graded ring of dimension* 1.

(v) $C_{\mathbb{X}}(d) \neq (0)$ *for* $d \geq 1$.

Proof. (i): The kernel of the evaluation map ev_d, defined in Eq. (1.8.1), is precisely $I(\mathbb{X})_d$. Hence there is an isomorphism of K-vector spaces $S_d/I(\mathbb{X})_d \simeq C_{\mathbb{X}}(d)$. Thus $H_{\mathbb{X}}(d)$ is equal to $\dim_K(C_{\mathbb{X}}(d))$.

(ii): This follows readily from Corollary 1.7.4.

(iii): For $d \geq \mathrm{reg}(S/I(\mathbb{X})))$, one has that $H_{\mathbb{X}}(d) = |\mathbb{X}|$. Thus, by part (i), we get that $C_{\mathbb{X}}(d)$ is equal to $K^{|\mathbb{X}|}$. Consequently $\delta_{\mathbb{X}}(d) = 1$.

(iv): Let $[P]$ be a point in $\mathbb{X}$, with $P = (\alpha_1, \ldots, \alpha_s)$ and $\alpha_k \neq 0$ for some k, and let $I_{[P]}$ be the ideal generated by the homogeneous polynomials of S that vanish at $[P]$. Then $I_{[P]}$ is a prime ideal of height $s-1$,

$$I_{[P]} = (\{\alpha_k x_i - \alpha_i x_k \,|\, k \neq i \in \{1, \ldots, s\}), \; I(\mathbb{X}) = \bigcap_{[Q] \in \mathbb{X}} I_{[Q]}, \tag{1.8.2}$$

and the latter is the primary decomposition of $I(\mathbb{X})$. As $I_{[P]}$ has height $s-1$ for any $[P] \in \mathbb{X}$, we get that the height of $I(\mathbb{X})$ is $s-1$ and the dimension of $S/I(\mathbb{X})$ is 1. Hence $\mathrm{depth}(S/I(\mathbb{X})) \leq 1$. To complete the proof notice that, $\mathfrak{m} = (x_1, \ldots, x_s)$ is not an associated prime of $I(\mathbb{X})$; that is $\mathrm{depth}(S/I(\mathbb{X})) > 0$ and $S/I(\mathbb{X})$ is Cohen–Macaulay.

(v): This follows readily from Proposition 1.7.5. □

1.9 Graph theory and edge ideals of graphs

In this section concepts and facts about graph theory and edge ideals are introduced in order to understand Section 2.2 and Chapter 4. The main references for graph theory are [8, 14, 22] and for edge ideals we cite [22, 28, 58].

A *graph* G is an ordered pair of disjoints finite sets (V, E) where E is a subset of the set of unordered pairs of V. The set V is the set of *vertices* and the set E is called the set of *edges*. Sometimes to refer to the graph G is usually write $V(G)$ and $E(G)$ for the vertex set and the edge set of G.

Let $G := (V, E)$ be a graph and $e = \{v_1, v_2\}$ an edge of G, e is said to join the vertices v_1 and v_2 and we say that the vertices v_1 and v_2 are *adjacent vertices* of G, it is usual to say that e is *incident* with v_1 and v_2. The *degree* of the vertex $v \in V$, denoted by $\deg(v)$ is the number of incident edges with v. A vertex with degree zero is called an *isolated vertex*.

Definition 1.9.1. Let H and G be two graphs.

- H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.
- A subgraph H is called an *induced subgraph* if H contains all the edges $\{v_1, v_j\} \in E(G)$, with $v_i, v_j \in V(H)$.
- A *spanning subgraph* is a subgraph H of G containing all the vertices of G.

Definition 1.9.2. Let G be a graph.

- A *walk of length* n in G is an alternating sequence of vertices and edges

$$w = \{v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n\},$$

  where $e_i = \{v_{i-1}, v_i\}$. A walk may also be written $\{v_0, \ldots, v_n\}$ with the edges understood, or $\{e_1, \ldots, e_n\}$ with the vertices understood.
- If $v_0 = v_n$, the walk w is called a *closed walk*.
- A *path* is a walk all its vertices distinct.

We say that G is *connected* if for every pair of vertices v_1 and v_j there is a path form v_i to v_j. Thus G has a vertex disjoint decomposition

$$G = G_1 \cap \cdots \cap G_r$$

where $G_1, \ldots, G_r$ are the maximal (w.r.t inclusion) connected subgraphs of G, the G_i are called the *connected components* of G. A component is called *even* (resp. *odd*) if its *order* (number of vertices) is even (resp. odd).

Definition 1.9.3. Let G be a graph.

- A *cycle of length* n, denoted by C_n, is a closed path $\{v_0, \ldots, v_n\}$ in which $n \geq 3$.
- If all the vertices of G are isolated, G is called a *discrete graph*.
- A *forest* is an acyclic graph and a *tree* is a connected forest.
- G is a *complete graph* if every pair of its n vertices are adjacent and is denoted by $\mathcal{K}_n$.
- G is *bipartite* if its vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The pair is called a *bipartition* of G.
- G is a *complete bipartite graph* if G is bipartite and we have that V_1 and V_2 are completely joined.

Definition 1.9.4. The *distance* $d(v_1, v_2)$ between two vertices v_1 and v_2 of a graph G is defined to be the minimum of the lengths of all possible paths from v_1 to v_2. If there is no a path joining v_1 and v_2, then $d(v_1, v_2) = \infty$.

Proposition 1.9.5. [22, Proposition 2.1.2] *A graph G is bipartite if and only if all the cycles of G are even.*

If e is an edge, denoted by $G \setminus \{e\}$ the spanning subgraph of G obtained by deleting e and keeping all the vertices of G. The removal of a vertex v from a graph G results in a subgraph $G \setminus \{v\}$ of G consisting of all the vertices of G except v and all the edges not incident with v.

Definition 1.9.6. A set of edges in a graph G is called *independent* or a *matching* if no two of them have a vertex in common.

Definition 1.9.7. Let A be a set of vertices of a graph G. The *neighbor set* of A, denoted by $N_G(A)$ or simply by $N(A)$ if G is understood, is the set of vertices of G that are adjacent with at least one vertex of A.

Definition 1.9.8. Let G be a graph with vertex set V.

- A subset $C \subset V$ is a *minimal vertex cover* of G if:

  (a) Every edge of G is incident with at least one vertex in C.

  (b) There is no proper subset of C with the first property.

  If C only satisfies the condition a), then C is called a *vertex cover* of G and C is said to cover all the edges of G.

- A set of vertices of G is called *independent* or *stable* if no two of them are adjacent.

Remark 1.9.9. A set of vertices in G is a maximal independent set (with respect to inclusion) if and only if its complement is a minimal vertex cover of G.

Theorem 1.9.10. (Marriage theorem [22, Theorem 2.1.9]) *If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), then the following are equivalent*

(a) *G has a perfect matching.*

(b) *$|A| \leq |N(A)|$ for all $A \subset V$ independent set of vertices.*

Definition 1.9.11. An *induced matching* in a graph G is a set of pairwise disjoint edges $f_1, \ldots, f_r$ such that the only edges of G contained in $\bigcup_{i=1}^{r} f_i$ are $f_1, \ldots, f_r$. The *induced matching number*, denoted by $\mathrm{im}(G)$, is the number of edges in the largest induced matching.

Definition 1.9.12. A *directed graph* or *digraph* $\mathcal{D}$ consists of a finite set $V(\mathcal{D})$ of vertices together with a prescribed collection $E(\mathcal{D})$ of ordered pairs of distinct points called edges or arrows. An *oriented graph* is a digraph having no symmetric pair of directed edges. In other words an oriented graph is a graph together with an orientation of its edges. A *tournament* is a complete oriented graph.

Remark 1.9.13. Any tournament has a spanning directed path according to [27].

Edge ideals of graphs. Let G be a graph with vertex set $\{v_1, \ldots, v_s\}$ and let $S = K[x_1, \ldots, x_s]$ be a polynomial ring over a field K, with one variable x_i for each vertex v_i, we will often identify the vertex v_i with the variable x_i.

Definition 1.9.14. The *edge ideal* $I(G)$ associated to the graph G is the ideal of S generated by the set of square-free monomials $x_i x_j$ such that v_i is adjacent to v_j, that is

$$I(G) = (\{x_i x_j \mid \{v_i, v_j\} \in E(G)\}) \subset S.$$

If all the vertices of G are isolated we set $I(G) = (0)$. The ring $S/I(G)$ is called the *edge ring* of G.

The next result establish a one to one correspondence between the minimal vertex covers of a graph and the minimal primes of the corresponding edge ideal.

Proposition 1.9.15. [22, Proposition 2.2.2] *Let $S = K[x_1, \ldots, x_s]$ be a polynomial ring over a field K and let G be a graph with vertices $x_1, \ldots, x_s$. If $\mathfrak{p}$ is an ideal of S generated by $C = \{x_{i_1}, \ldots, x_{i_r}\}$, then $\mathfrak{p}$ is a minimal prime of $I(G)$ if and only if C is a minimal vertex cover of G.*

Corollary 1.9.16. *If G is a graph and $I(G)$ is its edge ideal, then the vertex covering number $\alpha_0(G)$ (that is the number of vertices in a minimum vertex cover in G) is equal to the height of the ideal $I(G)$.*

Proposition 1.9.17. [22, Proposition 2.2.11] *Let G be a graph with n vertices and let $I(G)$ be its edge ideal. Then*

$$s = \alpha_0(G) + \beta_0(G) = \mathrm{ht}(I) + \dim(S/I(G)),$$

where $\beta_0(G)$ is the vertex independent number (that is the number of vertices in a maximum independent set). In particular $\beta_0 = \dim(S/I(G))$.

Definition 1.9.18. A graph G is said to be an *unmixed graph* if any two minimal vertex covers of G have the same cardinality.

Definition 1.9.19. A graph G is said to be *Cohen–Macaulay* over the field K (C-M graph for short) if $S/I(G)$ is a Cohen–Macaulay ring.

Proposition 1.9.20. [22, Proposition 2.2.14] *If G is a Cohen–Macaulay graph, then G is unmixed.*

Proposition 1.9.21. [22, Proposition 2.4.9] *If G is a graph and $G_1, \ldots, G_r$ its connected components, then G is Cohen–Macaulay if and only if G_i is Cohen–Macaulay for all i.*

Proposition 1.9.22. [22, Corollary 2.4.14] *If G is a tree, then G is Cohen–Macaulay if and only if G is unmixed.*

Proposition 1.9.23. [22, Corollary 2.4.15] *The only Cohen–Macaulay cycles are the triangle and the pentagon.*

Lemma 1.9.24. [22, Lemma 2.5.2] *Let G be an unmixed bipartite graph and let $I(G)$ be its edge ideal. If $I(G)$ has height r, then there are disjoints sets of vertices $V_1 = \{v_1, \ldots, v_r\}$ and $V_2 = \{u_1, \ldots, u_r\}$ such that:*

(a) *$\{v_1, u_i\}$ is an edge of G for all i.*

(b) *Every edge of G joins V_1 with V_2.*

Corollary 1.9.25. [22, Corollary 2.5.5] *If G is a Cohen–Macaulay bipartite graph, then $G \setminus \{v\}$ is Cohen–Macaulay for some vertex v in G.*

Theorem 1.9.26. [57, Theorem 1.1] *Let G be a bipartite graph without isolated vertices. Then G is unmixed if and only if G has a bipartition $V_1 = \{v_1, \ldots, v_r\}$, $V_2 = \{u_1, \ldots u_r\}$ such that:*

(a) *$\{v_i, u_i\} \in E(G)$ for all i.*

(b) *If $\{v_i, u_j\}$ and $\{v_j, u_k\}$ are in $E(G)$ and i, j, k are distinct, then $\{v_i, u_k\} \in E(G)$.*

Theorem 1.9.27. [28, Theorem 3.4] *Let G be a bipartite graph without isolated vertices. Then G is a Cohen–Macaulay graph if and only if there is a bipartition $V_1 = \{v_1, \ldots, v_r\}$, $V_2 = \{u_1, \ldots, u_r\}$ of G such that:*

(a) *$\{v_i, u_i\} \in E(G)$ for all i.*

(b) *If $\{v_i, u_j\} \in E(G)$, then $i \leq j$.*

(c) *If $\{v_i, u_j\}$ and $\{v_j, u_k\}$ are in $E(G)$ and $i < j < k$, then $\{v_i, u_k\} \in E(G)$.*

Chapter 2

Minimum Distance and Footprint Functions of Graded Ideals

Let S be a graded polynomial ring over a field K, with a monomial order $\prec$, and let I be a graded ideal of S. In this chapter we study two functions associated to I: the minimum distance function δ_I and the footprint function fp_I. It is shown that δ_I is positive and that fp_I is positive if the initial ideal of I is unmixed. We show that if I is an unmixed radical ideal and its associated primes are generated by linear forms, then δ_I is strictly decreasing until it reaches the asymptotic value 1. If I is the edge ideal of a Cohen–Macaulay bipartite graph, we show that $\delta_I(d) = 1$ for d greater than or equal to the regularity of S/I. For a graded ideal of dimension ≥ 1, whose initial ideal is a complete intersection, we give an exact sharp lower bound for the corresponding minimum distance function.

We study δ_I and fp_I from a theoretical point of view. The functions δ_I and fp_I were introduced in [39, 43]. The interest in these functions is essentially due to the following two facts: the minimum distance function is related to the minimum distance in coding theory (Theorem 3.2.1) and the footprint function is much easier to compute. There are significant cases in which either the footprint function is a lower bound for the minimum distance function (Theorem 2.3.2) or the two functions coincide (Proposition 2.3.3, Theorem 2.5.9).

2.1 Minimum distance function

Let $S = K[x_1, \ldots, x_s] = \bigoplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and let I be a graded ideal of S. We define $\mathcal{F}_d$, the set of all polynomials of degree $d \geq 0$ which are zero divisors of S/I:

$$\mathcal{F}_d := \{f \in S_d \mid f \notin I, (I\colon f) \neq I\},$$

where $(I\colon f) = \{h \in S | hf \in I\}$ is a quotient ideal. Note that $\mathcal{F}_0 = \emptyset$.

Remark 2.1.1. The set $\mathcal{F}_d = \{\, f \in S_d \colon f \notin I,\, (I\colon f) \neq I\}$ could be empty for some values of d. If all the associate primes of S/I are minimally generated by polynomials of degree at least $r \geq 2$, then $\mathcal{F}_d = \emptyset$ for $1 \leq d < r$. On the other hand if I is prime, then $\mathcal{F}_d$ is empty for $d \geq 0$.

Lemma 2.1.2. *Let $I \subset S$ be a radical unmixed graded ideal and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be its associated primes. If $f \in \mathcal{F}_d$ for some $d \geq 1$, then*

$$\deg(S/(I\colon f)) = \sum_{f \notin \mathfrak{p}_i} \deg(S/\mathfrak{p}_i).$$

Proof. Since I is a radical ideal, we get that $I = \bigcap_{i=1}^m \mathfrak{p}_i$. From the equalities

$$(I\colon f) = \bigcap_{i=1}^{m} (\mathfrak{p}_i\colon f) = \cap_{f \notin \mathfrak{p}_i} \mathfrak{p}_i,$$

and using the additivity of the degree (Proposition 1.5.28), the required equality follows. □

Definition 2.1.3. The *minimum distance function* of I, denoted δ_I, is the function $\delta_I \colon \mathbb{N} \to \mathbb{Z}$ given by

$$\delta_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(I,f)) \,|\, f \in \mathcal{F}_d\} & \text{if } \mathcal{F}_d \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{F}_d = \emptyset. \end{cases}$$

The next result will be used to bound the number of zeros of polynomials over finite fields (Corollary 3.1.2) and to study the general properties of δ_I.

Lemma 2.1.4. *Let $I \subset S$ be an unmixed graded ideal and let $\prec$ be a monomial order. If $f \in S$ is homogeneous and $(I\colon f) \neq I$, then*

$$\deg(S/(I,f)) \leq \deg(S/(\mathrm{in}_\prec(I), \mathrm{in}_\prec(f))) \leq \deg(S/I).$$

Proof. To simplify notation we set $J = (I, f)$ and $L = (\mathrm{in}_\prec(I), \mathrm{in}_\prec(f))$. First we show that S/J and S/L have dimension equal to $\dim(S/I)$. As f is a zero divisor of S/I and I is unmixed, there is an associated prime ideal $\mathfrak{p}$ of S/I such that $f \in \mathfrak{p}$ and $\dim(S/I) = \dim(S/\mathfrak{p})$. Since $I \subset J \subset \mathfrak{p}$, we get that $\dim(S/J)$ is $\dim(S/I)$. Since S/I and $S/\mathrm{in}_\prec(I)$ have the same Hilbert function, and so does $S/\mathfrak{p}$ and $S/\mathrm{in}_\prec(\mathfrak{p})$, we obtain

$$\dim(S/\mathrm{in}_\prec(I)) = \dim(S/I) = \dim(S/\mathfrak{p}) = \dim(S/\mathrm{in}_\prec(\mathfrak{p})).$$

Hence, taking heights in the inclusions $\mathrm{in}_\prec(I) \subset L \subset \mathrm{in}_\prec(\mathfrak{p})$, we obtain $\mathrm{ht}(I) = \mathrm{ht}(L)$.

Pick a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_r\}$ of I. Then J is generated by $\mathcal{G} \cup \{f\}$ and by Lemma 1.6.13 one has the inclusions

$$\Delta_\prec(J) = \Delta_\prec(I, f) \subset \Delta_\prec(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r), \mathrm{in}_\prec(f)) = \\ \Delta_\prec(\mathrm{in}_\prec(I), \mathrm{in}_\prec(f)) = \Delta_\prec(L) \subset \Delta_\prec(\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_r)) = \Delta_\prec(I).$$

Thus $\Delta_\prec(J) \subset \Delta_\prec(L) \subset \Delta_\prec(I)$. Recall that $H_I(d)$, the Hilbert function of I at d, is the number of standard monomials of degree d. Hence $H_J(d) \leq H_L(d) \leq H_I(d)$ for $d \geq 0$. If $\dim(S/I)$ is equal to 0, then

$$\deg(S/J) = \sum_{d\geq 0} H_J(d) \leq \deg(S/L) = \sum_{d\geq 0} H_L(d) \leq \deg(S/I) = \sum_{d\geq 0} H_I(d).$$

Assume now that $\dim(S/I) \geq 1$. By the Hilbert-Serre theorem, H_J, H_L, H_I are polynomial functions of degree equal to $k = \dim(S/I) - 1$. Thus

$$k! \lim_{d\to\infty} \frac{H_J(d)}{d^k} \leq k! \lim_{d\to\infty} \frac{H_L(d)}{d^k} \leq k! \lim_{d\to\infty} \frac{H_I(d)}{d^k},$$

that is $\deg(S/J) \leq \deg(S/L) \leq \deg(S/I)$.

□

Lemma 2.1.5. *Let $I \subset S$ be an unmixed graded ideal and let $\prec$ be a monomial order. If $f \in S \setminus I$ is homogeneous and $(I\colon f) \neq I$, then*

$$\deg(S/I) = \deg(S/(I\colon f)) + \deg(S/(I,f)), \text{ in particular } \deg(S/(I,f)) < \deg(S/I).$$

Proof. Using that I is unmixed, it is not hard to see that S/I, $S/(I\colon f)$, and $S/(I,f)$ have the same Krull dimension. There is an exact sequence

$$0 \longrightarrow S/(I\colon f)[-d] \stackrel{f}{\longrightarrow} S/I \longrightarrow S/(I,f) \longrightarrow 0.$$

Hence, by the additivity of Hilbert functions [58, Lemma 5.1.1], we get

$$H_I(i) = H_{(I\colon f)}(i-d) + H_{(I,f)}(i) \quad \text{for } i \geq 0. \tag{2.1.1}$$

If $\dim S/I = 0$, then using Eq. (2.1.1) one has

$$\sum_{i\geq 0} H_I(i) = \sum_{i\geq 0} H_{(I\colon f)}(i) + \sum_{i\geq 0} H_{(I,f)}(i).$$

Therefore, using the definition of degree, the required equality follows. If $k = \dim S/I - 1$ and $k \geq 1$, by Theorem 1.5.2, H_I, $H_{(I,f)}$, and $H_{(I\colon f)}$ are polynomial functions of degree k. Then dividing Eq. (2.1.1) by i^k and taking limits as i goes to infinity, the required equality follows.

□

Remark 2.1.6. Let $I \subset S$ be an unmixed graded ideal of dimension 1. If $f \in S_d$, then $(I\colon f) = I$ if and only if $\dim(S/(I,f)) = 0$. In this case $\deg(S/(I,f))$ could be greater than $\deg(S/I)$.

The next alternative formula for the minimum distance function is valid for unmixed graded ideals. It was pointed out to us by Vasconcelos.

Theorem 2.1.7. *Let $I \subset S$ be an unmixed graded ideal and let $\prec$ be a monomial order on S. If $\Delta_\prec(I)_d^p$ is the set of homogeneous standard polynomials of degree d and $S_d \not\subset I$, then*

$$\begin{aligned}\delta_I(d) &= \min\{\deg(S/(I\colon f))\,|\, f \in S_d \setminus I\}\\ &= \min\{\deg(S/(I\colon f))\,|\, f \in \Delta_\prec(I)_d^p\}.\end{aligned}$$

Proof. The second equality is clear because by the division algorithm any $f \in S_d \setminus I$ can be written as $f = g + h$, where $g \in I$ and $h \in \Delta_\prec(I)_d^p$, and $(I\colon f) = (I\colon h)$. Next we show the first equality. If $\mathcal{F}_d = \emptyset$, $\delta_I(d) = \deg(S/I)$ and for any $f \in S_d \setminus I$, one has that $(I\colon f)$ is equal to I. Thus equality holds. Assume that $\mathcal{F}_d \neq \emptyset$. Take $f \in \mathcal{F}_d$. Using that I is unmixed, it is not hard to see that S/I, $S/(I\colon f)$, and $S/(I, f)$ have the same dimension. There are exact sequences

$$\begin{aligned}&0 \longrightarrow (I\colon f)/I \longrightarrow S/I \longrightarrow S/(I\colon f) \longrightarrow 0, \text{ and}\\ &0 \longrightarrow (I\colon f)/I \longrightarrow (S/I)[-d] \stackrel{f}{\longrightarrow} S/I \longrightarrow S/(I, f) \longrightarrow 0.\end{aligned}$$

Hence, by the additivity of Hilbert functions, we get

$$H_I(i) - H_{(I\colon f)}(i) = H_I(i-d) - H_I(i) + H_{(I,f)}(i) \quad \text{for } i \geq 0. \tag{2.1.2}$$

By definition of $\delta_I(d)$ it suffices to show the following equality

$$\deg(S/(I\colon f)) = \deg(S/I) - \deg(S/(I, f)). \tag{2.1.3}$$

If $\dim(S/I) = 0$, then using Eq. (2.1.2) one has

$$\sum_{i\geq 0} H_I(i) - \sum_{i\geq 0} H_{(I\colon f)}(i) = \sum_{i\geq 0} H_I(i-d) - \sum_{i\geq 0} H_I(i) + \sum_{i\geq 0} H_{(I,f)}(i).$$

Therefore, using the definition of degree, the equality of Eq. (2.1.3) follows. If $k = \dim(S/I) - 1$, by the Hilbert-Serre theorem, H_I, $H_{(I,f)}$, and $H_{(I\colon f)}$ are polynomial functions of degree k. Then dividing Eq. (2.1.2) by i^k and taking limits as i goes to infinity, the equality of Eq. (2.1.3) holds. □

Definition 2.1.8. Let $I \subset S$ be a non-zero proper graded ideal. The *Vasconcelos function* of I is the function $\vartheta_I\colon \mathbb{N}_+ \to \mathbb{N}_+$ given by

$$\vartheta_I(d) = \begin{cases} \min\{\deg(S/(I\colon f))\,|\, f \in S_d \setminus I\} & \text{if } \mathfrak{m}^d \not\subset I,\\ \deg(S/I) & \text{if } \mathfrak{m}^d \subset I.\end{cases}$$

Very little is known about the Vasconcelos function when I is not an unmixed graded ideal. The following results show some properties of δ_I.

Theorem 2.1.9. *Let $I \subset S$ be an unmixed graded ideal, let $\prec$ be a monomial order on S, and let $d \geq 1$ be an integer. The following hold.*

(i) $\delta_I(d) \geq 1$.

(ii) *If* $\dim(S/I) \geq 1$ *and* $\mathcal{F}_d \neq \emptyset$ *for* $d \geq 1$*, then* $\delta_I(d) \geq \delta_I(d+1) \geq 1$ *for* $d \geq 1$.

Proof. (i) If $\mathcal{F}_d = \emptyset$, then $\delta_I(d) = \deg(S/I) \geq 1$, and if $\mathcal{F}_d \neq \emptyset$, then using Lemma 2.1.5 it follows that $\delta_I(d) \geq 1$.

(ii) By part (i), one has $\delta_I(d) \geq 1$. The set $\mathcal{F}_d$ is not empty for $d \geq 1$. Thus, by Theorem 2.1.7, $\delta_I(d) = \deg(S/(I\colon f))$ for some $f \in \mathcal{F}_d$. As I is unmixed and $\dim(S/I) \geq 1$, $\mathfrak{m}$ is not an associated prime of S/I. Thus, since $(I\colon f)$ is a graded ideal, one has $(I\colon f) \subsetneq \mathfrak{m}$. Pick a linear form $h \in S_1$ such that $hf \notin I$. As f is a zero divisor of S/I, so is hf. The ideals $(I\colon f)$ and $(I\colon hf)$ have height equal to $\mathrm{ht}(I)$. Therefore taking the Hilbert functions in the exact sequence

$$0 \longrightarrow (I\colon hf)/(I\colon f) \longrightarrow S/(I\colon f) \longrightarrow S/(I\colon hf) \longrightarrow 0$$

it follows that $\deg(S/(I\colon f)) \geq \deg(S/(I\colon hf))$. Therefore, applying Theorem 2.1.7, we get the inequality $\delta_I(d) \geq \delta_I(d+1)$. □

Theorem 2.1.10. *Let $\prec$ be a monomial order and let $I \subset S$ be an unmixed ideal of dimension ≥ 1 such that x_i is a zero divisor of S/I for $i = 1, \ldots, s$. The following hold.*

(i) *The set* $\mathcal{F}_d = \{f \in S_d\colon f \notin I,\ (I\colon f) \neq I\}$ *is not empty for* $d \geq 1$.

(ii) $\deg(S/(I, x^a)) \leq \deg(S/(\mathrm{in}_\prec(I), x^a)) \leq \deg(S/I)$ *for any* $x^a \in \Delta_\prec(I) \cap S_d$.

(iii) $\delta_I(d) \geq \delta_I(d+1)$ *for* $d \geq 1$.

(iv) *If I is a radical ideal and its associated primes are generated by linear forms, then there is an integer $r_0 \geq 1$ such that*

$$\delta_I(1) > \delta_I(2) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \text{for} \quad d \geq r_0.$$

Proof. (i): Since $\dim(S/I) \geq 1$, there is $1 \leq \ell \leq s$ such that x_ℓ^d is not in I, and $(I\colon x_\ell^d) \neq I$ because x_ℓ^d is a zero divisor of S/I. Thus x_ℓ^d is in $\mathcal{F}_d$.

(ii): Since any standard monomial of degree d is a zero divisor, by Lemma 2.1.4, we get the inequalities in item (ii).

(iii): The set $\mathcal{F}_d$ is not empty for $d \geq 1$ by part (i). From Theorem 2.1.9(ii) we have that $\delta_I(d) \geq \delta_I(d+1)$ for $d \geq 1$.

(iv): By Lemma 2.1.4, $\delta_I(d) \geq 1$ for $d \geq 1$. Assume that $\delta_I(d) > 1$. By part (iii) it suffices to show that $\delta_I(d) > \delta_I(d+1)$. Pick a polynomial F as in part (iii). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the associated primes of I. Then, by Lemma 1.5.29, one has

$$\begin{aligned} \delta_I(d) &= \deg(S/I) - \deg(S/(I, F)) \\ &= \sum_{i=1}^{m} \deg(S/\mathfrak{p}_i) - \sum_{F \in \mathfrak{p}_i} \deg(S/\mathfrak{p}_i) \geq 2. \end{aligned}$$

Hence there are $\mathfrak{p}_k \neq \mathfrak{p}_j$ such that F is not in $\mathfrak{p}_k \cup \mathfrak{p}_j$. Pick a linear form h in $\mathfrak{p}_k \setminus \mathfrak{p}_j$; which exists because I is unmixed and $\mathfrak{p}_k$ is generated by linear forms. Then $hF \notin I$ because $hF \notin \mathfrak{p}_j$, and hF is a zero divisor of S/I because $(I\colon F) \neq I$. Noticing that $F \notin \mathfrak{p}_k$ and $hF \in \mathfrak{p}_k$, by Lemma 1.5.29, we get

$$\deg(S/(I,F)) = \sum_{F\in\mathfrak{p}_i} \deg(S/\mathfrak{p}_i) < \sum_{hF\in\mathfrak{p}_i} \deg(S/\mathfrak{p}_i) = \deg(S/(I,hF)).$$

Therefore $\delta_I(d) > \delta_I(d+1)$. □

Corollary 2.1.11. *If $I \subset S$ is a Cohen–Macaulay square-free monomial ideal, then there is an integer $r_0 \geq 0$ such that*

$$\delta_I(1) > \delta_I(2) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \textit{for} \quad d \geq r_0.$$

Proof. If I is prime, then I is generated by a subset of $\{x_1, \ldots, x_s\}$, $\deg(S/I) = 1$, and $\mathcal{F}_d = \emptyset$ for all d. Hence $\delta_I(d) = 1$ for $d \geq 1$. Thus we may assume that I has at least two associated primes. Any Cohen–Macaulay ideal is unmixed [58]. Thus the degree of S/I is the number of associated primes of I. Hence, we may assume that all variables are zero divisors of S/I and the result follows from Theorem 2.1.10(iv). □

The next result about the asymptotic behavior of the minimum distance function gives a wide generalization of Theorem 2.1.10(iv).

Theorem 2.1.12. *Let $I \subset S$ be an unmixed radical graded ideal. If all the associated primes of I are generated by linear forms, then there is an integer $r_0 \geq 1$ such that*

$$\delta_I(1) > \delta_I(2) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \textit{for} \quad d \geq r_0.$$

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the associated primes of I. As $\mathfrak{p}_i$ is generated by linear forms, then $\deg(S/\mathfrak{p}_i) = 1$ for all i. Indeed if $\mathfrak{p}_i = \mathfrak{m}$, then $\deg(S/\mathfrak{p}_i)$ is $\dim_K(S/\mathfrak{p}_i) = 1$, and if $\mathfrak{p}_i \subsetneq \mathfrak{m}$, then the initial ideal of $\mathfrak{p}_i$, with respect to the GRevLex order $\prec$, is generated by a subset of $x_1, \ldots, x_s$ and $\deg(S/\mathfrak{p}_i)$ is equal to the $\deg(S/\mathrm{in}_\prec(\mathfrak{p}_i)) = 1$. The last equality follows noticing that $S/\mathrm{in}_\prec(\mathfrak{p}_i)$ is a polynomial ring.

If I is prime, then $I = \mathfrak{p}_i$ for some i and $\mathcal{F}_d = \emptyset$ for $d \geq 1$. Thus $\delta_I(d) = \deg(S/\mathfrak{p}_i) = 1$ for $d \geq 1$, and we can take $r_0 = 1$. We may now assume that I has at least two associated primes, that is, $m \geq 2$. As $I \subsetneq \mathfrak{p}_1$, there is a form h of degree 1 in $\mathfrak{p} \setminus I$. Hence, as I is a radical ideal, we get that h^d is in $\mathfrak{p}_1 \setminus I$. Thus $\mathcal{F}_d \neq \emptyset$ for $d \geq 1$. Therefore, by Theorem 2.1.9(ii), one has that $\delta_I(d) \geq \delta_I(d+1) \geq 1$ for $d \geq 1$. Hence, assuming that $\delta_I(d) > 1$, it suffice to show that $\delta_I(d) > \delta_I(d+1)$. By Theorem 2.1.7, there is $f \in \mathcal{F}_d$ such that $\delta_I(d) = \deg(S/(I\colon f))$. Then, by Lemma 2.1.2, one has

$$\delta_I(d) = \deg(S/(I\colon f)) = \sum_{f\notin\mathfrak{p}_i} \deg(S/\mathfrak{p}_i) \geq 2.$$

Hence there are $\mathfrak{p}_k \neq \mathfrak{p}_j$ such that f is not in $\mathfrak{p}_k \cup \mathfrak{p}_j$. Pick a linear form h in $\mathfrak{p}_k \setminus \mathfrak{p}_j$. Then $hf \notin I$ because $hf \notin \mathfrak{p}_j$, and hf is a zero divisor of S/I because $(I\colon f) \neq I$. Noticing that $f \notin \mathfrak{p}_k$ and $hf \in \mathfrak{p}_k$, one obtains the strict inclusion

$$\{\mathfrak{p}_i \mid hf \notin \mathfrak{p}_i\} \subsetneq \{\mathfrak{p}_i \mid f \in \mathfrak{p}_i\}.$$

Therefore, by Lemma 2.1.2, we get

$$\deg(S/(I\colon f)) = \sum_{f\notin\mathfrak{p}_i} \deg(S/\mathfrak{p}_i) > \sum_{hf\notin\mathfrak{p}_i} \deg(S/\mathfrak{p}_i) = \deg(S/(I\colon hf)).$$

Hence, by Theorem 2.1.7, we get $\delta_I(d) > \delta_I(d+1)$. □

The next lemma tells us that one can study the minimum distance function of I for $d \gg 0$ for a wide class of graded ideals.

Lemma 2.1.13. *Let $I \subset S$ be an unmixed graded ideal with at least two associated primes. If K is an infinite field, then the set $\mathcal{F}_d = \{f \in S_d\colon f \notin I,\, (I\colon f) \neq I\}$ is not empty for $d \gg 0$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the associated primes of S/I. As K is infinite, there is $h \in S_1 \setminus \cup_{i=1}^m \mathfrak{p}_i$. As $\mathfrak{p}_1 \not\subset \operatorname{rad}(I)$, there an integer $d \geq 1$ and a homogeneous polynomial $f \in \mathfrak{p}_1 \setminus \operatorname{rad}(I)$ of degree d. Then $h^j f$ is in $\mathfrak{p}_1$ and $h^j f \notin I$ for $j \geq 1$. □

Proposition 2.1.14. *Let $I \subset S$ be a radical unmixed graded binomial ideal of dimension $d \geq 1$ over a field K such that x_ℓ is a zero divisor of S/I for some ℓ. Then $\delta_I(d) \geq 0$ and $\delta_I(d) \geq \delta_I(d+1)$ for $d \geq 1$.*

Proof. The set $\{\, F \in S_d\colon F \notin I,\, (I\colon F) \neq I\}$ is not empty for $d \geq 1$. Indeed x_ℓ^d is not in I because I is a binomial ideal and $(I\colon x_\ell^d) \neq I$ because x_ℓ^d is a zero divisor of S/I. Then, by Lemma 2.1.4, $\delta_I(d) \geq 0$. Pick $G \in S_d$ such that $G \notin I$, $(I\colon G) \neq I$ and

$$\deg(S/(I,G)) = \max\{\deg(S/(I,F))\,|\, F \notin I,\, F \in S_d,\, (I\colon F) \neq I\}.$$

There is $h \in S_1$ such that $hG \notin I$ because otherwise one has that $\mathfrak{m} = (x_1, \ldots, x_s)$ is an associated prime of S/I, a contradiction to the assumption that I is unmixed of dimension ≥ 1. As G is a zero divisor of S/I, so is hG. The ideals (I,G) and (I,hG) have the same height. Therefore taking Hilbert functions in the exact sequence

$$0 \to (I,G)/(I,hG) \to S/(I,hG) \to S/(I,G) \to 0$$

it follows that $\deg(S/(I,hG)) \geq \deg(S/(I,G))$. This proves that $\delta_I(d) \geq \delta_I(d+1)$. □

To compute the minimum distance function, we need the following result.

Proposition 2.1.15. *Fix a monomial order $\prec$. If $\Delta_\prec(I) \cap S_d = \{x^{a_1}, \ldots, x^{a_n}\}$ and $\mathcal{F}_{\prec,d} = \{f = \sum_i \lambda_i x^{a_i} \mid f \neq 0, \lambda_i \in K, (I : f) \neq I\}$, then*

$$\begin{aligned}\delta_I(d) &= \deg(S/I) - \max\{\deg(S/(I,f))\,|\, f \in \mathcal{F}_d\}\\ &= \deg(S/I) - \max\{\deg(S/(I,f))\,|\, f \in \mathcal{F}_{\prec,d}\}.\end{aligned}$$

Proof. It follows using the division algorithm, see Theorem 1.6.5. □

Notice that $\mathcal{F}_d \neq \emptyset$ if and only if $\mathcal{F}_{\prec,d} \neq \emptyset$. If $K = \mathbb{F}_q$ is a finite field, then the number of standard polynomials of degree d is $q^n - 1$, where n is the number of standard monomials of degree d. Hence, we can compute $\delta_I(d)$ for small values of n and q.

Upper bounds for $\delta_I(d)$ can be obtained by fixing a subset $\mathcal{F}'_{\prec,d}$ of $\mathcal{F}_{\prec,d}$ and computing

$$\delta'_I(d) = \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in \mathcal{F}'_{\prec,d}\} \geq \delta_I(d).$$

Typically one use $\mathcal{F}'_{\prec,d} = \{f = \sum_i \lambda_i x^{a_i} \mid f \neq 0, \lambda_i \in \{0,1\}, (I\colon f) \neq I\}$ or a subset of it.

Lower bounds for $\delta_I(d)$ are harder to find. Thus, we seek to estimate $\delta_I(d)$ from below. So, with this in mind, in Section 2.3, we introduce the *footprint function* of I.

In the following example we give an implementation using *Macaulay*2 [25] to explicitly calculate the value of the minimum distance function of the vanishing ideal of a finite projective set of points.

Example 2.1.16. Let K be the field $\mathbb{F}_3$, let $\mathbb{X}$ be the subset of $\mathbb{P}^3$ given by

$$\mathbb{X} = \{[e_1],\ [e_2],\ [e_3],\ [e_4],\ [(1,-1,-1,1)],\ [(1,1,1,1)],\ [(-1,-1,1,1)],\ [(-1,1,-1,1)]\},$$

where e_i is the i-th unit vector, and let $I = I(\mathbb{X})$ be the vanishing ideal of $\mathbb{X}$. Using Lemma 1.7.3 and *Macaulay*2 [25], we get that I is the ideal of $S = K[x_1, x_2, x_3, x_4]$ generated by the binomials $x_1x_2 - x_3x_4$, $x_1x_3 - x_2x_4$, $x_2x_3 - x_1x_4$. Hence, using Theorem 2.1.7 and the procedure below for *Macaulay*2 [25], we get

d	1	2	3	$\cdots$
$\deg(S/I)$	8	8	8	$\cdots$
$H_I(d)$	4	7	8	$\cdots$
$\delta_I(d)$	4	2	1	$\cdots$

```
q=3
S=ZZ/q[x1,x2,x3,x4]
I=ideal(x1*x2-x3*x4,x1*x3-x2*x4,x2*x3-x1*x4)
M=coker gens gb I
h=(d)->min apply(apply(apply(apply(toList
(set(0..q-1))^**(hilbertFunction(d,M))-
(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),x-> degree quotient(I,x))
apply(1..2,h)--this gives the minimum distance in degrees 1,2
```

2.2 Asymptotic behavior of the minimum distance

In this section we study a conjecture about the regularity index of the minimum distance function.

Let $I \subset S$ be an unmixed radical graded ideal whose associated primes are generated by linear forms. According to Theorem 2.1.12, there is an integer $r_0 \geq 1$ such that

$$\delta_I(1) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \text{for} \quad d \geq r_0.$$

Definition 2.2.1. The integer r_0 is called the *regularity index* of δ_I.

If I is the graded vanishing ideal of a set of points in a projective space over a finite field, then $r_0 \leq \operatorname{reg}(S/I)$ [24, 46], but we do not know whether this holds in general. The regularity of S/I can be computed using *Macaulay*2 [25], but r_0 is difficult to compute.

Conjecture 2.2.2. *Let $I \subset S$ be an unmixed radical graded ideal. If all the associated primes of I are generated by linear forms, then $\delta_I(d) = 1$ for $d \geq \operatorname{reg}(S/I)$, that is, $r_0 \leq \operatorname{reg}(S/I)$.*

In this section we give some support for this conjecture. In what follows we focus in the case that I is an unmixed ideal generated by square-free monomial ideals of degree 2.

Conjecture 2.2.2 is open even in the case that I is the edge ideal of an unmixed bipartite graph. Below we prove the conjecture for edge ideals of Cohen–Macaulay graphs.

Proposition 2.2.3. [32, Lemma 2.2] *If G is a graph, then $\operatorname{reg}(S/I(G)) \geq \operatorname{im}(G)$.*

Next we prove Conjecture 2.2.2 for edge ideals of Cohen–Macaulay bipartite graphs.

Proposition 2.2.4. *If $I = I(G)$ is the edge ideal of a Cohen–Macaulay bipartite graph without isolated vertices, then $\delta_I(d) = 1$ for $d \geq \operatorname{reg}(S/I)$.*

Proof. By [33, Theorem 1.1], $\operatorname{reg}(S/I) = \operatorname{im}(G)$. Thus, by Theorem 2.1.12, it suffices to show that $\delta_I(d) = 1$ for some $d \leq \operatorname{im}(G)$. According to [28, Theorem 3.4], there is a bipartition $V_1 = \{x_1, \ldots, x_g\}$, $V_2 = \{y_1, \ldots, y_g\}$ of G such that:

(a) $e_i = \{x_i, y_i\} \in E(G)$ for all i,

(b) if $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and

(c) if $\{x_i, y_j\}$, $\{x_j, y_k\}$ are in $E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$.

Next we construct a sequence $x_{i_1}, \ldots, x_{i_d}$ such that $e_{i_1}, \ldots, e_{i_d}$ form an induced matching and V_2 is a pairwise disjoint union

$$V_2 = N_G(x_{i_1}) \cup \cdots \cup N_G(x_{i_d}), \tag{2.2.1}$$

where $N_G(x_{i_j}) \cap N_G(x_{i_k}) = \emptyset$ for $j \neq k$ and $N_G(x_{i_j})$ is the neighbor set of x_{i_j}, that is, $N_G(x_{i_j})$ is the set of vertices of G adjacent to x_{i_j}. We set $i_1 = 1$. If $N_G(x_{i_1}) \subsetneq V_2$, pick y_{i_2} in $V_2 \setminus N_G(x_{i_1})$. By condition (b), e_{i_1}, e_{i_2} is an induced matching and $N_G(x_{i_1}) \cap N_G(x_{i_2}) = \emptyset$. If $N_G(x_{i_1}) \cup N_G(x_{i_2}) \subsetneq V_2$, pick y_{i_3} in $V_2 \setminus (N_G(x_{i_1}) \cup N_G(x_{i_2}))$. By condition (b), $e_{i_1}, e_{i_2}, e_{i_3}$ form an induced matching and $N_G(x_{i_j}) \cap N_G(x_{i_k}) = \emptyset$ for $j \neq k$. Thus one can continue this process until we get a sequence $x_{i_1},\ldots,x_{i_d}$ such that V_2 is the disjoint union of the $N_G(x_{i_j})$'s and the e_{i_j}'s form an induced matching.

Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ be the associated primes of I. Then, there are minimal vertex covers $C_1,\ldots,C_m$ of G such that $\mathfrak{p}_i$ is generated by C_i for $i=1,\ldots,m$ (see [56, p. 279]). We may assume that $C_m=V_2$. Setting $x^a=x_{i_1}\cdots x_{i_d}$, by Theorem 2.1.7, it suffices to show that x^a is in $\bigcap_{i=1}^{m-1}\mathfrak{p}_i\setminus\mathfrak{p}_m$ and that $\deg(S/(I\colon x^a))=1$, where $S=K[V(G)]$. If $i\neq m$, there is $y_\ell\notin C_i$. From Eq. (2.2.1), there is x_{i_j} such that $y_\ell\in N_G(x_{i_j})$ for some i_j. Hence, as C_i covers the edge $\{x_{i_j},y_\ell\}$, one has that x_{i_j} is in $\mathfrak{p}_i$. Thus x^a is in $\bigcap_{i=1}^{m-1}\mathfrak{p}_i$ and x^a is not in $\mathfrak{p}_m$ because $\mathfrak{p}_m=(y_1,\ldots,y_g)$. Therefore

$$(I\colon x^a)=(\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_m\colon x^a)=(\mathfrak{p}_1\colon x^a)\cap\cdots\cap(\mathfrak{p}_m\colon x^a)=\mathfrak{p}_m.$$

Hence $\deg(S/(I\colon x^a))=1$, as required. □

2.3 Footprint function

In this section we introduce the footprint function of I. This is a numerical function defined similarly as δ_I, but here we use a monomial order and the initial ideal of I.

Let $\mathcal{M}_{\prec,d}$ be the set of all zero divisors of $S/\mathrm{in}_\prec(I)$ of degree $d\geq 1$ that are in $\Delta_\prec(I)$:

$$\mathcal{M}_{\prec,d}:=\{x^a\,|\,x^a\in\Delta_\prec(I)_d,(\mathrm{in}_\prec(I)\colon x^a)\neq\mathrm{in}_\prec(I)\}.$$

Definition 2.3.1. The *footprint function* of I, denoted fp_I, is the function $\mathrm{fp}_I\colon\mathbb{N}_+\to\mathbb{Z}$ given by

$$\mathrm{fp}_I(d):=\begin{cases}\deg(S/I)-\max\{\deg(S/(\mathrm{in}_\prec(I),x^a))\,|\,x^a\in\mathcal{M}_{\prec,d}\} & \text{if } \mathcal{M}_{\prec,d}\neq\emptyset,\\ \deg(S/I) & \text{if } \mathcal{M}_{\prec,d}=\emptyset.\end{cases}$$

We come to one of our main results.

Theorem 2.3.2. *Let I be an unmixed graded ideal and let $\prec$ be a monomial order. The following hold.*

(i) *$\delta_I(d)\geq\mathrm{fp}_I(d)$ and $\delta_I(d)\geq 0$ for $d\geq 1$.*

(ii) *$\mathrm{fp}_I(d)\geq 0$ if $\mathrm{in}_\prec(I)$ is unmixed.*

Proof. If $\mathcal{F}_{\prec,d}=\emptyset$, then clearly $\delta_I(d)=\deg(S/I)\geq 1$, $\delta_I(d)\geq\mathrm{fp}_I(d)$, and if $\mathrm{in}_\prec(I)$ is unmixed, then $\mathrm{fp}_I(d)\geq 0$ (this follows from Lemma 2.1.4). Thus, (i) and (ii) hold. Assume that $\mathcal{F}_{\prec,d}\neq\emptyset$. Pick a standard polynomial $f\in S_d$ such that $(I\colon f)\neq I$ and

$$\delta_I(d)=\deg(S/I)-\deg(S/(I,f)).$$

As I is unmixed, by Lemma 2.1.4, $\deg(S/(I,f))\leq\deg(S/(\mathrm{in}_\prec(I),\mathrm{in}_\prec(f)))$. On the other hand, by Lemma 1.6.14, $\mathrm{in}_\prec(f)$ is a zero divisor of $S/\mathrm{in}_\prec(I)$. Hence $\delta_I(d)\geq\mathrm{fp}_I(d)$. Using the second inequality of Lemma 2.1.4 it follows that $\delta_I(d)\geq 0$, $\mathrm{fp}_I(d)\geq 0$ if $\mathrm{in}_\prec(I)$ is unmixed. □

Proposition 2.3.3. *If I is an unmixed monomial ideal and $\prec$ is any monomial order, then $\delta_I(d) = \mathrm{fp}_I(d)$ for $d \geq 1$, that is, I is a Geil–Carvalho ideal.*

Proof. The inequality $\delta_I(d) \geq \mathrm{fp}_I(d)$ follows from Theorem 2.3.2. To show the reverse inequality notice that $\mathcal{M}_{\prec,d} \subset \mathcal{F}_{\prec,d}$ because one has $I = \mathrm{in}_\prec(I)$. Also notice that $\mathcal{M}_{\prec,d} = \emptyset$ if and only if $\mathcal{F}_{\prec,d} = \emptyset$, this follows from Lemma 1.6.14. Therefore one has $\mathrm{fp}_I(d) \geq \delta_I(d)$. $\square$

Next we show that in certain cases the footprint function can be expressed in terms of the degree of colon ideals.

Corollary 2.3.4. *Let I be a graded ideal and let $\prec$ be a monomial order. If $\mathrm{in}_\prec(I)$ is an unmixed ideal and $\mathcal{M}_{\prec,d} \neq \emptyset$, then*

$$\mathrm{fp}_I(d) = \min\{\deg(S/(\mathrm{in}_\prec(I)\colon x^a)) \,|\, x^a \in S_d \setminus \mathrm{in}_\prec(I)\}.$$

Proof. Take $x^a \in \mathcal{M}_{\prec,d}$. By Lemma 2.1.5(ii) one has the equality:

$$\deg(S/(\mathrm{in}_\prec(I)\colon x^a)) = \deg(S/\mathrm{in}_\prec(I)) - \deg(S/(\mathrm{in}_\prec(I), x^a)).$$

In this case $\deg(S/(\mathrm{in}_\prec(I)\colon x^a)) \leq \deg(S/\mathrm{in}_\prec(I))$. Therefore, noticing that $\deg(S/\mathrm{in}_\prec(I))$ is equal to $\deg(S/I)$, we get

$$\begin{aligned} \mathrm{fp}_I(d) &= \deg(S/I) - \max\{\deg(S/(\mathrm{in}_\prec(I), x^a))|\, x^a \in \mathcal{M}_{\prec,d}\} \\ &= \min\{\deg(S/(\mathrm{in}_\prec(I)\colon x^a)) \,|\, x^a \in \mathcal{M}_{\prec,d}\} \\ &= \min\{\deg(S/(\mathrm{in}_\prec(I)\colon x^a)) \,|\, x^a \in S_d \setminus \mathrm{in}_\prec(I)\}. \quad \square \end{aligned}$$

One can apply the corollary to graded lattice ideals of dimension 1.

Proposition 2.3.5. *Let $I \subset S$ be a graded lattice ideal of dimension 1 and let $\prec$ be a graded monomial order with $x_1 \succ \cdots \succ x_s$. The following hold.*

(a) *If $\mathrm{in}_\prec(I)$ is not prime, then $\mathrm{in}_\prec(I)$ is unmixed and $\mathcal{M}_{\prec,d} \neq \emptyset$ for $d \geq 1$.*

(b) *If $\mathrm{in}_\prec(I)$ is prime, then $I = (x_1 - x_s, \ldots, x_{s-1} - x_s)$ and $\mathcal{M}_{\prec,d} = \emptyset$ for $d \geq 1$.*

Proof. The reduced Gröbner basis of I consists of binomials of the form $x^{a_+} - x^{a_-}$ (see [58, Proposition 8.2.7]). It follows that x_s is a regular element on both S/I and $S/\mathrm{in}_\prec(I)$. Hence I and $\mathrm{in}_\prec(I)$ are Cohen–Macaulay ideals. In particular these ideals are unmixed.

(a): Assume that $\mathrm{in}_\prec(I)$ is not prime. Then there is an associated prime $\mathfrak{p}$ of $S/\mathrm{in}_\prec(I)$ such that $\mathrm{in}_\prec(I) \subsetneq \mathfrak{p}$. Pick a variable x_i in $\mathfrak{p} \setminus \mathrm{in}_\prec(I)$. Then $x_i x_s^{d-1}$ is in $\mathfrak{p}$ and is not in $\mathrm{in}_\prec(I)$ for $d \geq 1$. Thus $x_i x_s^{d-1}$ is in $\mathcal{M}_{\prec,d}$ for $d \geq 1$.

(b): Assume that $\mathrm{in}_\prec(I)$ is prime. This part follows by noticing that $\mathrm{in}_\prec(I)$, being a face ideal generated by variables, is equal to $(x_1, \ldots, x_{s-1})$. $\square$

The following result improves a bit the lower bound for the footprint function given in Theorem 2.3.2.

Proposition 2.3.6. *Let $I \subset S$ be an unmixed graded ideal, let $\prec$ be a monomial order on S, and let $d \geq 1$ be an integer. Then, $\mathrm{fp}_I(d) \geq 1$ if $\mathrm{in}_\prec(I)$ is unmixed.*

Proof. If $\mathcal{M}_{\prec,d} = \emptyset$, then $\mathrm{fp}_I(d) = \deg(S/I) \geq 1$. Next assume that $\mathcal{M}_{\prec,d} \neq \emptyset$. As $\mathrm{in}_\prec(I)$ is unmixed, by Corollary 2.3.4, $\mathrm{fp}_I(d) \geq 1$. □

The following examples shows an implementation in *Macaulay*2 [25] to calculate the values of δ_I and fp_I, also show that fp_I is a lower bound for δ_I.

Example 2.3.7. Let K be the field $\mathbb{F}_3$, let $\mathbb{X}$ be the subset of $\mathbb{P}^3$ given by

$$\begin{aligned}\mathbb{X} \;=\; & \{[(1,0,0)],[(1,1,0)],[(0,1,0)],[(1,0,1)],[(1,1,1)],\\ & [(0,1,1)],[(1,0,2)],[(1,1,2)],[(0,1,2)],[(0,0,1)]\},\end{aligned}$$

and let $I = I(\mathbb{X})$ be the vanishing ideal of $\mathbb{X}$ using Lemma 1.7.3 and *Macaulay*2 [25] , we get that I is the ideal of $K[x_1,x_2,x_3]$ generate by the monomials $x_1^2x_2 - x_1x_2^2, x_2^3x_3 - x_2x_3^3, x_1^3x_3 - x_1x_3^3$. Hence using Theorem 2.1.7 and the procedure below for *Macaulay*2 [25], we get

d	1	2	3	4	$\cdots$
$\lvert\mathbb{X}\rvert$	10	10	10	10	$\cdots$
$H_{\mathbb{X}}(d)$	3	6	9	10	$\cdots$
$\delta_{\mathbb{X}}(d)$	6	3	1	1	$\cdots$
$\mathrm{fp}_{I(\mathbb{X})}(d)$	6	2	1	1	$\cdots$

```
q=3
S=ZZ/q[x_1,x_2,x_3]
I=(a1,a2,a3)->ideal(a1*x_2-a2*x_1,a1*x_3-a3*x_1,a2*x_1-a1*x_2,a2*x_3-
a3*x_2, a3*x_1-a1*x_3,a3*x_2-a2*x_3)
I1=I(1,0,0), I2=I(1,1,0), I3=I(0,1,0), I4=I(1,0,1), I5=I(1,1,1)
I6=I(0,1,1), I7=I(1,0,2), I8=I(1,1,2), I9=I(0,1,2), I10=I(0,0,1)
Ix=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9,I10)
M=coker gens gb Ix
degree M
regularity M
H=(d)->hilbertFunction(d,M)
apply(1..7,H)
h=(d)->min apply(apply(apply(apply(toList(set(0..q-1))^**(hilbertFunction
(d,M))-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
```

```
z->ideal(flatten entries z)), x->degree quotient(Ix,x))
apply(1..2,h)
-----This gives the footprint in degree d------
load "gfaninterface.m2"
universalGroebnerBasis(Ix)
(InL,L)= gfan Ix, #InL
init=ideal(InL_0)
N=coker gens gb init
f=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
fp=(d) ->degree N -max apply(flatten entries basis(d,N),f)
apply(1..regularity(N),fp)
```

Example 2.3.8. Let K be the field $\mathbb{F}_5$, let $\mathbb{X}$ be the subset of $\mathbb{P}^3$ given by

$$\begin{aligned}\mathbb{X} \;=\; &\{[(0,0,1)],[(0,1,0)],[(0,1,1)],[(0,1,2)],[(0,1,3)],[(1,0,0)],[(1,0,1)],\\ &[(1,0,2)],[(1,0,3)],[(1,1,0)],[(1,1,1)],[(1,1,2)],[(1,1,3)]\},\end{aligned}$$

and let $I = I(\mathbb{X})$ be the vanishing ideal of $\mathbb{X}$ using Lemma 1.7.3 and *Macaulay*2 [25], we get that I is the ideal of $K[x_1, x_2, x_3]$ generate by the monomials $x_1^2x_2 - x_1x_2^2, x_2^4x_3 - x_2^3x_3^2 + x_2^2x_3^3 - x_2x_3^4, x_1^4x_3 - x_1^3x_3^2 + x_1^2x_3^3 - x_1x_3^4$. Hence using Theorem 2.1.7 and the procedure below for *Macaulay*2 [25], we get

d	1	2	3	4	5	$\cdots$
$\lvert\mathbb{X}\rvert$	13	13	13	13	13	$\cdots$
$H_{\mathbb{X}}(d)$	3	6	9	12	13	$\cdots$
$\delta_{\mathbb{X}}(d)$	8	4	3	1	1	$\cdots$
$\text{fp}_{I(\mathbb{X})}(d)$	8	3	2	1	1	$\cdots$

```
q=5
S=ZZ/q[x_1,x_2,x_3]
I=(a1,a2,a3)->ideal(a1*x_2-a2*x_1,a1*x_3-a3*x_1,a2*x_1-a1*x_2,a2*x_3-a3*x_2,
a3*x_1-a1*x_3,a3*x_2-a2*x_3)
I1=I(0,0,1), I2=I(0,1,0), I3=I(0,1,1), I4=I(0,1,2), I5=I(0,1,3), I6=I(1,0,0),
I7=I(1,0,1), I8=I(1,0,2), I9=I(1,0,3), I10=I(1,1,0), I11=I(1,1,1),
I12=I(1,1,2), I13=I(1,1,3)
I=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13)
M=coker gens gb I
degree M
regularity M
H=(d)->hilbertFunction(d,M)
apply(1..7,H)
```

```
h=(d)->min apply(apply(apply(apply(toList(set(0..q-1))^**(hilbertFunction(d,M))-
(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),z->ideal
(flatten entries z)), x->degree quotient(I,x))
h(3)
-----This gives the footprint in degree d------
 "gfaninterface.m2"
universalGroebnerBasis(I)
(InL,L)= gfan I, #InL
init=ideal(InL_0)
N=coker gens gb init
f=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
fp=(d) ->degree N -max apply(flatten entries basis(d,N),f)
apply(1..regularity(N),fp)
```

2.4 Two integer inequalities

Lemma 2.4.1. *Let $a_1,\dots,a_r,a,b,e$ be positive integers with $e\geq a$. Then*

(a) $a_1\cdots a_r\geq (a_1+\cdots+a_r)-(r-1)$, *and*

(b) $a(e-b)\geq (a-b)e$.

Proof. Part (a) follows by induction on r, and part (b) is straightforward. □

The next inequality is a generalization of part (a).

Lemma 2.4.2. *Let $1\leq e_1\leq\cdots\leq e_m$ and $0\leq b_i\leq e_i-1$ for $i=1,\dots,m$ be integers. Then*

$$\prod_{i=1}^{m}(e_i-b_i)\geq\left(\sum_{i=1}^{k}(e_i-b_i)-(k-1)-\sum_{i=k+1}^{m}b_i\right)e_{k+1}\cdots e_m \qquad (2.4.1)$$

for $k=1,\dots,m$, where $e_{k+1}\cdots e_m=1$ and $\sum_{i=k+1}^{m}b_i=0$ if $k=m$.

Proof. Fix m and $1\leq k\leq m$. We will proceed by induction on $\sigma=\sum_{i=1}^{k}(e_i-b_i-1)$. If $\sigma=0$, then $e_i-b_i-1=0$ for $i=1,\dots,k$. Thus either $1-\sum_{i=k+1}^{m}b_i<0$ or $1-\sum_{i=k+1}^{m}b_i\geq 1$. In the first case the inequality is clear because the left hand side of Eq. (2.4.1) is positive and in the second case one has $b_i=0$ for $i=k+1,\dots,m$ and equality holds in Eq. (2.4.1). Assume that $\sigma>0$. If $k=m$ or $b_i=0$ for $i=k+1,\dots,m$, the inequality follows at once from Lemma 2.4.1(a). Thus, we may assume $k<m$ and $b_j>0$ for some $k+1\leq j\leq m$. To simplify notation, and without loss of generality, we may assume that $j=m$, that is, $b_m>0$. If the right hand side of Eq. (2.4.1) is negative or zero, the inequality holds. Thus we may also assume that

$$\sum_{i=1}^{k}(e_i-b_i)-\sum_{i=k+1}^{m}b_i\geq k. \qquad (2.4.2)$$

Hence there is $1 \leq \ell \leq k$ such that $e_\ell - b_\ell \geq 2$.

Case (1): Assume $e_\ell - b_\ell - b_m \geq 1$. Setting $a = e_\ell - b_\ell$, $e = e_m$, and $b = b_m$ in Lemma 2.4.1(b), we get

$$(e_\ell - b_\ell)(e_m - b_m) \geq (e_\ell - (b_\ell + b_m))e_m. \tag{2.4.3}$$

Therefore using Eq. (2.4.3), and then applying the induction hypothesis to the two sequences of integers

$$e_1, \ldots, e_{\ell-1}, e_\ell, e_{\ell+1}, \ldots, e_{m-1}, e_m; \quad b_1, \ldots, b_{\ell-1}, b_\ell + b_m, b_{\ell+1}, \ldots, b_{m-1}, 0,$$

we get the inequalities

$$\begin{aligned}
\prod_{i=1}^{m}(e_i - b_i) &= \left(\prod_{i\notin\{\ell,m\}}(e_i - b_i)\right)(e_\ell - b_\ell)(e_m - b_m) \\
&\geq \left(\prod_{i\neq\{\ell,m\}}(e_i - b_i)\right)(e_\ell - (b_\ell + b_m))e_m \\
&\geq \left(\sum_{\ell\neq i=1}^{k}(e_i - b_i) + (e_\ell - (b_\ell + b_m)) - (k-1) - \sum_{i=k+1}^{m-1} b_i\right)e_{k+1}\cdots e_m \\
&= \left(\sum_{i=1}^{k}(e_i - b_i) - (k-1) - \sum_{i=k+1}^{m} b_i\right)e_{k+1}\cdots e_m.
\end{aligned}$$

Case (2): Assume $e_\ell - b_\ell - b_m < 1$. Setting $r_\ell = e_\ell - b_\ell - 1 \geq 1$, one has

$$b_\ell + r_\ell = e_\ell - 1 \geq 1, \quad b_m - r_\ell \geq 1, \quad e_\ell - (b_\ell + r_\ell) = 1.$$

On the other hand by Lemma 2.4.1(a) one has

$$(e_\ell - b_\ell)(e_m - b_m) \geq (e_\ell - b_\ell) + (e_m - b_m) - 1 = (e_\ell - (b_\ell + r_\ell))(e_m - (b_m - r_\ell)). \tag{2.4.4}$$

Therefore using Eq. (2.4.4), and then applying the induction hypothesis to the two sequences of integers

$$e_1, \ldots, e_{\ell-1}, e_\ell, e_{\ell+1}, \ldots, e_{m-1}, e_m; \quad b_1, \ldots, b_{\ell-1}, b_\ell + r_\ell, b_{\ell+1}, \ldots, b_{m-1}, b_m - r_\ell,$$

we get the inequalities

$$\begin{aligned}\prod_{i=1}^{m}(e_i-b_i) &= \left(\prod_{i\notin\{\ell,m\}}(e_i-b_i)\right)(e_\ell-b_\ell)(e_m-b_m)\\ &\geq \left(\prod_{i\neq\{\ell,m\}}(e_i-b_i)\right)(e_\ell-(b_\ell+r_\ell))(e_m-(b_m-r_\ell))\\ &\geq \left(\sum_{\ell\neq i=1}^{k}(e_i-b_i)+(e_\ell-(b_\ell+r_\ell))-(k-1)-\sum_{i=k+1}^{m-1}b_i-(b_m-r_\ell)\right)e_{k+1}\cdots e_m\\ &= \left(\sum_{i=1}^{k}(e_i-b_i)-(k-1)-\sum_{i=k+1}^{m}b_i\right)e_{k+1}\cdots e_m.\ \Box\end{aligned}$$

Proposition 2.4.3. *Let $1\leq e_1\leq\cdots\leq e_m$ and $0\leq b_i\leq e_i-1$ for $i=1,\ldots,m$ be integers. If $b_0\geq 1$, then*

$$\prod_{i=1}^{m}(e_i-b_i)\geq\left(\sum_{i=1}^{k+1}(e_i-b_i)-(k-1)-b_0-\sum_{i=k+2}^{m}b_i\right)e_{k+2}\cdots e_m \tag{2.4.5}$$

for $k=0,\ldots,m-1$, where $e_{k+2}\cdots e_m=1$ and $\sum_{i=k+2}^{m}b_i=0$ if $k=m-1$.

Proof. If $0\leq k\leq m-1$, then $1\leq k+1\leq m$. Applying Lemma 2.4.2, and making the substitution $k\to k+1$ in Eq. (2.4.5), we get

$$\begin{aligned}\prod_{i=1}^{m}(e_i-b_i) &\geq \left(\sum_{i=1}^{k+1}(e_i-b_i)-k-\sum_{i=k+2}^{m}b_i\right)e_{k+2}\cdots e_m\\ &\geq \left(\sum_{i=1}^{k+1}(e_i-b_i)-(k-1)-b_0-\sum_{i=k+2}^{m}b_i\right)e_{k+2}\cdots e_m,\end{aligned}$$

where the second inequality holds because $b_0\geq 1$. $\Box$

2.5 Formulas for complete intersections

In this section we study the footprint function, with respect to a monomial order, of complete intersection graded ideals in a polynomial ring with coefficients in a field. For graded ideals of dimension one, whose initial ideal is a complete intersection, we give a formula for the footprint function and a sharp lower bound for the corresponding minimum distance function.

Let $S=K[x_1,\ldots,x_s]=\bigoplus_{d=0}^{\infty}S_d$ be a polynomial ring over a field K with the standard grading and $s\geq 2$. In what follows, we denote a monomial order by $\prec$, (Definition 1.6.1).

Lemma 2.5.1. *Let $L \subset S$ be an ideal generated by monomials. If $\dim(S/L) = 1$, then L is a complete intersection if and only if, up to permutation of variables, we can write*

(i) $L = (x_2^{d_2}, \ldots, x_s^{d_s})$ *with* $1 \leq d_i \leq d_{i+1}$ *for* $i \geq 2$, *or*

(ii) $L = (x_1^{d_2}, \ldots, x_{p-1}^{d_p}, x_p^{c_p} x_{p+1}^{c_{p+1}}, x_{p+2}^{d_{p+2}}, \ldots, x_s^{d_s})$ *for some* $p \geq 1$ *such that* $1 \leq c_p \leq c_{p+1}$ *and* $1 \leq d_i \leq d_{i+1}$ *for* $2 \leq i \leq s-1$, *where* $d_{p+1} = c_p + c_{p+1}$.

Proof. $\Rightarrow$) Let $x^{\alpha_1}, \ldots, x^{\alpha_{s-1}}$ be the minimal set of generators of L consisting of monomials. By the generalized Krull principal ideal theorem [58, Theorem 2.3.16] it follows that x^{α_i} and x^{α_j} have no common variables for $i \neq j$. Then, either all variables occur in $x^{\alpha_1}, \ldots, x^{\alpha_{s-1}}$ and we are in case (ii), or there is one variable that is not in any of the x^{α_i}'s and we are in case (i).

$\Leftarrow$) In both cases L is an ideal of height $s-1$ generated by $s-1$ elements, that is, L is a complete intersection. □

Proposition 2.5.2. *Let L be the ideal of S generated by $x_2^{d_2}, \ldots, x_s^{d_s}$. If $x^a = x_1^{a_1} x_r^{a_r} \cdots x_s^{a_s}$, $r \geq 2$, $a_r \geq 1$, and $a_i \leq d_i - 1$ for $i \geq r$, then*

$$\deg(S/(L, x^a)) = \deg(S/(L, x_r^{a_r} \cdots x_s^{a_s})) = d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s),$$

where $a_i = 0$ if $2 \leq i < r$.

Proof. In what follows we will use the fact that Hilbert functions and Hilbert series are additive on short exact sequences [58, Lemma 5.1.1]. If $a_1 \geq 1$, then taking Hilbert functions in the exact sequence

$$0 \longrightarrow S/(L, x_r^{a_r} \cdots x_s^{a_s})[-a_1] \stackrel{x_1^{a_1}}{\longrightarrow} S/(L, x^a) \longrightarrow S/(L, x_1^{a_1}) \longrightarrow 0,$$

and noticing that $\dim S/(L, x_1^{a_1}) = 0$, the first equality follows. Thus we may assume that x^a has the form $x^a = x_r^{a_r} \cdots x_s^{a_s}$ and $a_i = 0$ for $i < r$.

We proceed by induction on $s \geq 2$. Assume $s = 2$. Then $r = 2$, $x^a = x_2^{a_2}$, $(L, x^a) = (x_2^{a_2})$, and the degree of $S/(L, x^a)$ is a_2, as required. Assume $s \geq 3$. If $a_i = 0$ for $i > r$, then $(L, x^a) = (L, x_r^{a_r})$ is a complete intersection and the required formula follows from Lemma 1.5.21. Thus we may assume that $a_i \geq 1$ for some $i > r$. There is an exact sequence

$$\begin{aligned} 0 \longrightarrow S/(x_2^{d_2}, \ldots, x_{r-1}^{d_{r-1}}, x_r^{d_r - a_r}, x_{r+1}^{d_{r+1}}, \ldots, x_s^{d_s}, x_{r+1}^{a_{r+1}} \cdots x_s^{a_s})[-a_r] \stackrel{x_r^{a_r}}{\longrightarrow} \\ S/(L, x^a) \longrightarrow S/(x_2^{d_2}, \ldots, x_{r-1}^{d_{r-1}}, x_r^{a_r}, x_{r+1}^{d_{r+1}}, \ldots, x_s^{d_s}) \longrightarrow 0. \end{aligned} \tag{2.5.1}$$

Notice that the ring on the right is a complete intersection and the ring on the left is isomorphic to the tensor product

$$K[x_2, \ldots, x_r]/(x_2^{d_2}, \ldots, x_{r-1}^{d_{r-1}}, x_r^{d_r - a_r}) \otimes_K K[x_1, x_{r+1}, \ldots, x_s]/(x_{r+1}^{d_{r+1}}, \ldots, x_s^{d_s}, x_{r+1}^{a_{r+1}} \cdots x_s^{a_s}). \tag{2.5.2}$$

Hence taking Hilbert series in Eq. (2.5.1), and applying Lemma 1.5.21, Theorem 1.5.17, and Proposition 1.5.16, we get that the Hilbert series of $S/(L,x^a)$ can be written as

$$F(S/(L,x^a),t)=\frac{x^{a_r}(1-x^{d_2})\cdots(1-x^{d_{r-1}})(1-x^{d_r-a_r})}{(1-x)^{r-1}}\frac{g(x)}{(1-x)}+\frac{(1-x^{d_2})\cdots(1-x^{d_{r-1}})(1-x^{a_r})(1-x^{d_{r+1}})\cdots(1-x^{d_s})}{(1-x)^s},$$

where $g(x)/(1-x)$ is the Hilbert series of the second ring in the tensor product of Eq. (2.5.2) and $g(1)$ is its degree (Remark 1.5.20). By induction hypothesis

$$g(1)=d_{r+1}\cdots d_s-(d_{r+1}-a_{r+1})\cdots(d_s-a_s).$$

Therefore writing $F(S/(L,x^a),x)=h(x)/(1-x)$ with $h(x)\in\mathbb{Z}[x]$ and $h(1)>0$, and recalling that $h(1)$ is the degree of $S/(L,x^a)$, we get

$$\begin{aligned}\deg(S/(L,x^a))=h(1)&= d_2\cdots d_{r-1}(d_r-a_r)g(1)+d_2\cdots d_{r-1}a_rd_{r+1}\cdots d_s\\ &= d_2\cdots d_s-(d_2-a_2)\cdots(d_s-a_s). \qquad \square\end{aligned}$$

Lemma 2.5.3. *Let L be the ideal of S generated by $x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p}x_{p+1}^{c_{p+1}},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s}$, where $p\geq 1$, $1\leq c_p\leq c_{p+1}$ and $d_i\geq 1$ for all i. If $x^a=x_1^{a_1}\cdots x_s^{a_s}$ is not in L, $d_{p+1}=c_p+c_{p+1}$, and $a_i\geq 1$ for some i, then the degree of $S/(L,x^a)$ is equal to*

- $d_2\cdots d_s-(c_{p+1}-a_{p+1})\prod_{i=1}^{p-1}(d_{i+1}-a_i)\prod_{i=p+2}^{s}(d_i-a_i)$ *if $a_p\geq c_p$;*
- $d_2\cdots d_s-(c_p-a_p)\prod_{i=1}^{p-1}(d_{i+1}-a_i)\prod_{i=p+2}^{s}(d_i-a_i)$ *if $a_p<c_p$, $a_{p+1}\geq c_{p+1}$;*
- $d_2\cdots d_s-(d_{p+1}-a_p-a_{p+1})\prod_{i=1}^{p-1}(d_{i+1}-a_i)\prod_{i=p+2}^{s}(d_i-a_i)$ *if $a_p<c_p$, $a_{p+1}<c_{p+1}$.*

Proof. Case (i): Assume $a_p\geq c_p$. If $a_i=0$ for $i\neq p$, then $x^a=x_p^{a_p}$, and by the first equality of Proposition 2.5.2 and using Lemma 1.5.21, we get

$$\begin{aligned}\deg(S/(L,x^a))&=\deg(S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{a_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s},x_p^{c_p}x_{p+1}^{c_{p+1}}))\\ &=\deg(S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s}))=d_2\cdots d_pc_pd_{p+2}\cdots d_s\\ &=d_2\cdots d_p(d_{p+1}-c_{p+1})d_{p+2}\cdots d_s=d_2\cdots d_s-c_{p+1}d_2\cdots d_pd_{p+2}\cdots d_s,\end{aligned}$$

as required. We may now assume that $a_i\geq 1$ for some $i\neq p$. As $x^a\notin L$ and $a_p\geq c_p$, one has $a_i<d_{i+1}$ for $i=1,\ldots,p-1$, $a_{p+1}<c_{p+1}$, and $a_i<d_i$ for $i=p+2,\ldots,s$. Therefore

from the exact sequence

$$0 \longrightarrow S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_{p+1}^{c_{p+1}},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s},x_1^{a_1}\cdots x_{p-1}^{a_{p-1}}x_p^{a_p-c_p}x_{p+1}^{a_{p+1}}\cdots x_s^{a_s})[-c_p] \stackrel{x_p^{c_p}}{\longrightarrow}$$
$$S/(L,x^a) \longrightarrow S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s}) \longrightarrow 0,$$

and using Proposition 2.5.2 together with Lemma 1.5.21, the required equality follows.

Case (ii): Assume $a_p < c_p$. If $a_i = 0$ for $i \neq p$, then $x^a = x_p^{a_p}$ and $0 = a_{p+1} < c_{p+1}$. Hence, by Lemma 1.5.21 we get

$$\begin{aligned} \deg(S/(L,x^a)) &= \deg(S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},gp^{a_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s})) \\ &= d_2\cdots d_p a_p d_{p+2}\cdots d_s \\ &= d_2\cdots d_s - (d_{p+1}-a_p)d_2\cdots d_p d_{p+2}\cdots d_s, \end{aligned}$$

as required. We may now assume that $a_i \geq 1$ for some $i \neq p$. Consider the exact sequence

$$0 \longrightarrow S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_{p+1}^{c_{p+1}},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s},x_1^{a_1}\cdots x_{p-1}^{a_{p-1}}x_{p+1}^{a_{p+1}}\cdots x_s^{a_s})[-c_p] \stackrel{x_p^{c_p}}{\longrightarrow}$$
$$S/(L,x^a) \longrightarrow S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s},x_1^{a_1}\cdots x_s^{a_s}) \longrightarrow 0. \quad (2.5.3)$$

Subcase (ii.1): Assume $a_{p+1} \geq c_{p+1}$. As $x^a \notin L$, in our situation, one has $a_i < d_{i+1}$ for $i = 1,\ldots,p-1$, $a_p < c_p$, and $a_i < d_i$ for $i = p+2,\ldots,s$. If $a_i = 0$ for $i \neq p+1$, then taking Hilbert series in Eq. (2.5.3) and noticing that the ring on the right has dimension 0, we get

$$\begin{aligned} \deg(S/(L,x^a)) &= d_2\cdots d_p c_{p+1} d_{p+2}\cdots d_s \\ &= d_2\cdots d_s - c_p d_2\cdots d_p d_{p+2}\cdots d_s, \end{aligned}$$

as required. Thus we may now assume that $a_i \geq 1$ for some $i \neq p+1$. Taking Hilbert series in Eq. (2.5.3), and using Lemma 1.5.21, we obtain

$$\begin{aligned} \deg(S/(L,x^a)) &= d_2\cdots d_p c_{p+1} d_{p+2}\cdots d_s + \\ &\quad \deg(S/(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s},x_1^{a_1}\cdots x_s^{a_s})). \end{aligned}$$

Therefore, using Proposition 2.5.2, the required equality follows.

Subcase (ii.2): Assume $a_{p+1} < c_{p+1}$. If $a_i = 0$ for $i \neq p+1$, taking Hilbert series in Eq. (2.5.3) and noticing that the ring on the right has dimension 0, by Proposition 2.5.2, we get

$$\begin{aligned} \deg(S/(L,x^a)) &= d_2\cdots d_p c_{p+1} d_{p+2}\cdots d_s - d_2\cdots d_p(c_{p+1}-a_{p+1})d_{p+2}\cdots d_s \\ &= d_2\cdots d_p a_{p+1} d_{p+2}\cdots d_s \\ &= d_2\cdots d_s - (d_{p+1}-a_{p+1})d_2\cdots d_p d_{p+2}\cdots d_s, \end{aligned}$$

as required. Thus we may now assume that $a_i \geq 1$ for some $i \neq p+1$. Taking Hilbert series in Eq. (2.5.3) and applying Proposition 2.5.2 to the ends of Eq. (2.5.3) the required equality follows. $\square$

Proposition 2.5.4. *Let $I \subset S$ be a graded ideal and let $\prec$ be a monomial order. Suppose that $\mathrm{in}_\prec(I)$ is a complete intersection of height $s-1$ generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$ with $d_i = \deg(x^{\alpha_i})$ and $d_i \geq 1$ for all i. The following hold.*

(a) *I is a complete intersection and* $\dim(S/I) = 1$.

(b) $\deg(S/I) = d_2 \cdots d_s$ *and* $\mathrm{reg}\,(S/I) = \sum_{i=2}^{s}(d_i - 1)$.

(c) $1 \leq \mathrm{fp}_I(d) \leq \delta_I(d)$ *for* $d \geq 1$.

Proof. (a): The rings S/I and $S/\mathrm{in}_\prec(I)$ have the same dimension. Thus $\dim(S/I) = 1$. As $\prec$ is a graded order, there are $f_2, \ldots, f_s$ homogeneous polynomials in I with $\mathrm{in}_\prec(f_i) = x^{\alpha_i}$ for $i \geq 2$. Since $\mathrm{in}_\prec(I) = (\mathrm{in}_\prec(f_2), \ldots, \mathrm{in}_\prec(f_s))$, the polynomials $f_2, \ldots, f_s$ form a Gröbner basis of I, and in particular they generate I. Hence I is a graded ideal of height $s-1$ generated by $s-1$ polynomials, that is, I is a complete intersection.

(b): Since I is a complete intersection generated by the f_i's, then the degree and regularity of S/I are $\deg(f_2) \cdots \deg(f_s)$ and $\sum_{i=2}^{s}(\deg(f_i) - 1)$, respectively. This follows from the formula for the Hilbert series of a complete intersection given in Lemma 1.5.21.

(c) The ideal I is unmixed because, by part (a), I is a complete intersection. Hence the inequality $\delta_I(d) \geq \mathrm{fp}_I(d)$ follows from Theorem 2.3.2. Let x^a be a standard monomial of S/I of degree d such that $(\mathrm{in}_\prec(I)\colon x^a) \neq \mathrm{in}_\prec(I)$, that is, x^a is in $\mathcal{M}_{\prec,d}$. Using Lemma 2.5.1, and the formulas for $\deg(S/(\mathrm{in}_\prec(I), x^a))$ given in Proposition 2.5.2 and Lemma 2.5.3, we obtain that $\deg(S/(\mathrm{in}_\prec(I), x^a)) < \deg(S/I)$. Thus $\mathrm{fp}_I(d) \geq 1$. □

Remark 2.5.5. Parts (a)-(c) of Proposition 2.5.4 do not need any assumption on the height.

We come to one of the main result of this section.

Theorem 2.5.6. *Let $I \subset S$ be a graded ideal and let $\prec$ be a monomial order. If $\mathrm{in}_\prec(I)$ is a complete intersection of height $s-1$ generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$ with $d_i = \deg(x^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then $\delta_I(d) \geq \mathrm{fp}_I(d) \geq 1$ and the footprint function in degree $d \geq 1$ is given by*

$$\mathrm{fp}_I(d) = \begin{cases} (d_{k+2} - \ell) d_{k+3} \cdots d_s & \text{if } d \leq \sum\limits_{i=2}^{s}(d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum\limits_{i=2}^{s}(d_i - 1), \end{cases}$$

where $0 \leq k \leq s-2$ and ℓ are the unique integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. Let x^a be any standard monomial of S/I of degree d which is a zero divisor of $S/\mathrm{in}_\prec(I)$, that is, x^a is in $\mathcal{M}_{\prec,d}$. Thus $d = \sum_{i=1}^{s} a_i$, where $a = (a_1, \ldots, a_s)$. We set

$r=\sum_{i=2}^{s}(d_i-1)$. If we substitute $-\ell=\sum_{i=2}^{k+1}(d_i-1)-\sum_{i=1}^{s}a_i$ in the expression $(d_{k+2}-\ell)d_{k+3}\cdots d_s$, it follows that for $d<r$ the inequality

$$\mathrm{fp}_I(d)\geq(d_{k+2}-\ell)d_{k+3}\cdots d_s$$

is equivalent to show that

$$\deg(S/I)-\deg(S/(\mathrm{in}_\prec(I),x^a))\geq\left(\sum_{i=2}^{k+2}(d_i-a_i)-k-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s \quad (2.5.4)$$

for any x^a in $\mathcal{M}_{\prec,d}$, where by convention $\sum_{i=k+3}^{s}a_i=0$ and $d_{k+3}\cdots d_s=1$ if $k=s-2$. Recall that by Proposition 2.5.4 one has that $\mathrm{fp}_I(d)\geq 1$ for $d\geq 1$. By Lemma 2.5.1, and by permuting variables and changing I, $\prec$, and x^a accordingly, one has the following two cases to consider.

Case (i): Assume that $\mathrm{in}_\prec(I)=(x_2^{d_2},\ldots,x_s^{d_s})$ with $1\leq d_i\leq d_{i+1}$ for $i\geq 2$. Then, as x^a is in $\mathcal{M}_{\prec,d}$, we can write $x^a=x_1^{a_1}\cdots x_r^{a_r}\cdots x_s^{a_s}$, $r\geq 2$, $a_r\geq 1$, $a_i=0$ if $2\leq i<r$, and $a_i\leq d_i-1$ for $i\geq r$. By Proposition 2.5.2 we get

$$\deg(S/(\mathrm{in}_\prec(I),x^a))=d_2\cdots d_s-(d_2-a_2)\cdots(d_s-a_s) \quad (2.5.5)$$

for any x^a in $\mathcal{M}_{\prec,d}$. If $d\geq r$, setting $x^c=x_1^{d-r}x_2^{d_2-1}\cdots x_s^{d_s-1}$, one has $x^c\in\mathcal{M}_{\prec,d}$. Then, using Eq. (2.5.5), it follows that $\deg(S/(\mathrm{in}_\prec(I),x^c))=d_2\cdots d_s-1$. Thus $\mathrm{fp}_I(d)\leq 1$ and equality $\mathrm{fp}_I(d)=1$ holds. We may now assume $d\leq r-1$. Setting $x^b=x_2^{d_2-1}\cdots x_{k+1}^{d_{k+1}-1}x_{k+2}^{\ell}$, one has $x^b\in\mathcal{M}_{\prec,d}$. Then, using Eq. (2.5.5), we get

$$\deg(S/(\mathrm{in}_\prec(I),x^b))=d_2\cdots d_s-(d_{k+2}-\ell)d_{k+3}\cdots d_s.$$

Hence $\mathrm{fp}_I(d)\leq(d_{k+2}-\ell)d_{k+3}\cdots d_s$. Next we show the reverse inequality by showing that the inequality of Eq. (2.5.4) holds for any $x^a\in\mathcal{M}_{\prec,d}$. By Eq. (2.5.5) it suffices to show that the following equivalent inequality holds

$$(d_2-a_2)\cdots(d_s-a_s)\geq\left(\sum_{i=2}^{k+2}(d_i-a_i)-k-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s$$

for any $a=(a_1,\ldots,a_s)$ such that $x^a\in\mathcal{M}_{\prec,d}$. This inequality follows from Proposition 2.4.3 by making $m=s-1$, $e_i=d_{i+1}$, $b_i=a_{i+1}$ for $i=1,\ldots,s-1$ and $b_0=1+a_1$.

Case (ii): Assume that $\mathrm{in}_\prec(I)=(x_1^{d_2},\ldots,x_{p-1}^{d_p},x_p^{c_p}x_{p+1}^{c_{p+1}},x_{p+2}^{d_{p+2}},\ldots,x_s^{d_s})$ for some $p\geq 1$ such that $1\leq c_p\leq c_{p+1}$ and $1\leq d_i\leq d_{i+1}$ for all i, where $d_{p+1}=c_p+c_{p+1}$.

If $d\geq r$, setting $x^c=x_1^{d_2-1}\cdots x_{p-1}^{d_p-1}x_p^{d-r+c_p}x_{p+1}^{c_{p+1}-1}x_{p+2}^{d_{p+2}-1}\cdots x_s^{d_s-1}$, we get that $x^c\in\mathcal{M}_{\prec,d}$. Then, using the first formula of Lemma 2.5.3, it follows that $\deg(S/(\mathrm{in}_\prec(I),x^c))=d_2\cdots d_s-1$. Thus $\mathrm{fp}_I(d)\leq 1$ and the equality $\mathrm{fp}_I(d)=1$ holds.

We may now assume $d\leq r-1$. First we show the inequality $\mathrm{fp}_I(d)\geq(d_{k+2}-\ell)d_{k+3}\cdots d_s$ by showing that the inequality of Eq. (2.5.4) holds for any x^a in $\mathcal{M}_{\prec,d}$. Take

x^a in $\mathcal{M}_{\prec,d}$. Then we can write $x^a = x_1^{a_1}\cdots x_s^{a_s}$ with $a_i < d_{i+1}$ for $i < p$ and $a_i < d_i$ for $i > p+1$. There are three subcases to consider.

Subcase (ii.1): Assume $a_p \geq c_p$. Then $c_{p+1} > a_{p+1}$ because x^a is a standard monomial of S/I, and by Lemma 2.5.3 we get

$$\deg(S/(\mathrm{in}_\prec(I), x^a)) = d_2\cdots d_s - (c_{p+1} - a_{p+1})\prod_{i=1}^{p-1}(d_{i+1} - a_i)\prod_{i=p+2}^{s}(d_i - a_i).$$

Therefore the inequality of Eq. (2.5.4) is equivalent to

$$\begin{aligned}(c_{p+1} - a_{p+1})&\prod_{i=1}^{p-1}(d_{i+1} - a_i)\prod_{i=p+2}^{s}(d_i - a_i)\\ &\geq \left(\sum_{i=2}^{k+2}(d_i - a_i) - k - a_1 - \sum_{i=k+3}^{s} a_i\right) d_{k+3}\cdots d_s,\end{aligned}$$

and this inequality follows at once from Proposition 2.4.3 by making $m = s-1$, $e_i = d_{i+1}$ for $i = 1,\ldots,m$, $b_i = a_i$ for $1 \leq i \leq p-1$, $b_p = a_{p+1} + c_p$, $b_i = a_{i+1}$ for $p < i \leq m$, and $b_0 = a_p - c_p + 1$. Notice that $\sum_{i=0}^m b_i = 1 + \sum_{i=1}^s a_i$.

Subcase (ii.2): Assume $a_p < c_p$, $a_{p+1} \geq c_{p+1}$. By Lemma 2.5.3 we get

$$\deg(S/(\mathrm{in}_\prec(I), x^a)) = d_2\cdots d_s - (c_p - a_p)\prod_{i=1}^{p-1}(d_{i+1} - a_i)\prod_{i=p+2}^{s}(d_i - a_i).$$

Therefore the inequality of Eq. (2.5.4) is equivalent to

$$\begin{aligned}(c_p - a_p)&\prod_{i=1}^{p-1}(d_{i+1} - a_i)\prod_{i=p+2}^{s}(d_i - a_i)\\ &\geq \left(\sum_{i=2}^{k+2}(d_i - a_i) - k - a_1 - \sum_{i=k+3}^{s} a_i\right) d_{k+3}\cdots d_s,\end{aligned}$$

and this inequality follows from Proposition 2.4.3 by making $m = s-1$, $e_i = d_{i+1}$ for $i = 1,\ldots,m$, $b_i = a_i$ for $1 \leq i \leq p-1$, $b_p = c_{p+1} + a_p$, $b_i = a_{i+1}$ for $p < i \leq m$, and $b_0 = a_{p+1} - c_{p+1} + 1$. Notice that $\sum_{i=0}^m b_i = 1 + \sum_{i=1}^s a_i$.

Subcase (ii.3): Assume $a_p < c_p$, $a_{p+1} \leq c_{p+1} - 1$. By Lemma 2.5.3 we get

$$\deg(S/(\mathrm{in}_\prec(I), x^a)) = d_2\cdots d_s - (d_{p+1} - a_p - a_{p+1})\prod_{i=1}^{p-1}(d_{i+1} - a_i)\prod_{i=p+2}^{s}(d_i - a_i).$$

Therefore the inequality of Eq. (2.5.4) is equivalent to

$$(d_{p+1}-a_p-a_{p+1})\prod_{i=1}^{p-1}(d_{i+1}-a_i)\prod_{i=p+2}^{s}(d_i-a_i)$$
$$\geq\left(\sum_{i=2}^{k+2}(d_i-a_i)-k-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s,$$

and this inequality follows from Proposition 2.4.3 by making $m=s-1$, $e_i=d_{i+1}$ for $i=1,\dots,m$, $b_i=a_i$ for $1\leq i\leq p-1$, $b_p=a_p+a_{p+1}$, $b_i=a_{i+1}$ for $p<i\leq m$, and $b_0=1$. Notice that $\sum_{i=0}^{m}b_i=1+\sum_{i=1}^{s}a_i$.

To complete the proof, we now show the inequality $\mathrm{fp}_I(d)\leq(d_{k+2}-\ell)d_{k+3}\cdots d_s$. It suffices to find a monomial x^b in $\mathcal{M}_{\prec,d}$ such that

$$\deg(S/(\mathrm{in}_\prec(I),x^b))=(d_{k+2}-\ell)d_{k+3}\cdots d_s. \qquad (2.5.6)$$

Subcase (ii.a): $k\geq p+1$. Setting $x^b=x_1^{d_2-1}\cdots x_{p-1}^{d_p-1}x_p^{c_p}x_{p+1}^{c_{p+1}-1}x_{p+2}^{d_{p+2}-1}\cdots x_{k+1}^{d_{k+1}-1}x_{k+2}^{\ell}$, one has that x^b is in $\mathcal{M}_{\prec,d}$. Then, by the first formula of Lemma 2.5.3, we get the equality of Eq. (2.5.6).

Subcase (ii.b): $k=p$. Setting $x^b=x_1^{d_2-1}\cdots x_{p-1}^{d_p-1}x_p^{c_p}x_{p+1}^{c_{p+1}-1}x_{p+2}^{\ell}$, one has that x^b is in $\mathcal{M}_{\prec,d}$. Then, by the first formula of Lemma 2.5.3, we get the equality of Eq. (2.5.6).

Subcase (ii.c): $k\leq p-2$. Setting $x^b=x_1^{d_2-1}\cdots x_k^{d_{k+1}-1}x_{k+1}^{\ell}$, one has that x^b is in $\mathcal{M}_{\prec,d}$. Then, by the third formula of Lemma 2.5.3, we get the equality of Eq. (2.5.6).

Subcase (ii.d): Assume $k=p-1$ and $\ell\geq c_p$. Setting $x^b=x_1^{d_2-1}\cdots x_{p-1}^{d_p-1}x_p^{c_p}x_{p+1}^{\ell-c_p}$, one has that x^b is in $\mathcal{M}_{\prec,d}$. Then, by the first formula of Lemma 2.5.3, we get the equality of Eq. (2.5.6).

Subcase (ii.e): Assume $k=p-1$ and $\ell<c_p$. Setting $x^b=x_1^{d_2-1}\cdots x_{p-1}^{d_p-1}x_p^{\ell}$, one has that x^b is in $\mathcal{M}_{\prec,d}$. Then, by the third formula of Lemma 2.5.3, we get the equality of Eq. (2.5.6). $\square$

It is an open question whether in Theorem 2.5.6 one has the equality $\delta_I(d)=\mathrm{fp}_I(d)$ for $d\geq 1$. The reader is referred to [40] for some interesting applications of this result to algebraic coding theory. In Section 3.3, we give some applications and examples of our main result. A formula for the minimum distance of an affine Cartesian code is given in [34, Theorem 3.8] and in [19, Proposition 5]. A short and elegant proof of this formula was given by Carvalho in [10, Proposition 2.3], where he shows that the best way to study the minimum distance of an affine Cartesian code is by using the footprint. In Section 3.3, we prove this formula.

The lower bound of Theorem 2.5.6 holds for any complete intersection monomial ideal of dimension 1. To show this we need to introduce some results.

Lemma 2.5.7. *Let $I\subset S$ be a complete intersection ideal with minimal set of generators $\{x^{\alpha_1},\dots,x^{\alpha_r}\}$ and let $x^a=x_1^{a_1}\cdots x_s^{a_s}$ be a zero divisor of S/I not in I. The following hold.*

(a) x^{α_i} *and* x^{α_j} *have no common variable for* $i \neq j$.

(b) *If* $x_j^{a_j}$ *is regular on* S/I *and* $x^c = x^a/x_j^{a_j}$, *then* $(I\colon x^a) = (I\colon x^c)$.

(c) *If* x_j *is a zero divisor of* S/I, *then there is a unique* $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,s})$ *such that* $\alpha_{i,j} > 0$, *that is,* x_j *occurs in exactly one* x^{α_i}. *If* $a_j > \alpha_{i,j}$ *and* $x^c = x^a/x_j$, *then* $(I\colon x^a) = (I\colon x^c)$.

(d) *For each* i *there is* x^{β_i} *dividing* x^{α_i} *such that* $\deg(x^{\beta_i}) < \deg(x^{\alpha_i})$ *and* $(I\colon x^a) = (I\colon x^\beta)$, *where* $x^\beta = x^{\beta_1} \cdots x^{\beta_r}$.

Proof. (a): This follows readily from the Krull principal ideal theorem [58, Theorem 2.3.16].

(b): The inclusion "$\supset$" is clear. To show the reverse inclusion take x^δ in $(I\colon x^a)$, that is, $x^\delta x^a = x^\delta x_j^{a_j} x^c$ is in I. Hence $x^\delta x^c$ is in I because $x_j^{a_j}$ is regular on S/I. Thus x^δ is in $(I\colon x^c)$.

(c): If x_j is a zero divisor of S/I, then x_j is in some associated prime of S/I. Hence, by part (a), x_j must occur in a unique x^{α_i} for some i. Thus one has $\alpha_{i,j} > 0$. We claim that $((x^{\alpha_k})\colon x^a) = ((x^{\alpha_k})\colon x^c)$ for all k. If $k \neq i$, by part (a), x_j is regular on $S/(x^{\alpha_k})$. Thus, as in the proof of part (b), we get the asserted equality. Next we assume that $k = i$. The inclusion "$\supset$" is clear. To show the reverse inclusion take x^δ in $((x^{\alpha_i})\colon x^a)$, that is, $x^\delta x^a = x^\gamma x^{\alpha_i}$ for some x^γ. Since $a_j > \alpha_{i,j} > 0$, x_j must divide x^γ. Then we can write $x^\delta x^c = x^\omega x^{\alpha_i}$, where $x^\omega = x^\gamma/x_j$. Thus x^δ is in $((x^{\alpha_i})\colon x^c)$. This completes the proof of the claim. Therefore one has

$$\begin{aligned}(I\colon x^a) &= ((x^{\alpha_1})\colon x^a) + \cdots + ((x^{\alpha_r})\colon x^a)\\ &= ((x^{\alpha_1})\colon x^c) + \cdots + ((x^{\alpha_r})\colon x^c) = (I\colon x^c).\end{aligned}$$

(d): Using part (a) and successively applying parts (b) and (c) to x^a, we get a monomial x^β that divides x^a such that the following conditions are satisfied: (i) all variables that occur in x^β are zero divisors of S/I, (ii) if $x^\beta = x_1^{\gamma_1} \cdots x_s^{\gamma_s}$ and $\gamma_j > 0$, then $\alpha_{i,j} \geq \gamma_j$, where x^{α_i} is the unique monomial, among $x^{\alpha_1}, \ldots, x^{\alpha_r}$, containing x_j, and (iii) $(I\colon x^a) = (I\colon x^\beta)$. We let x^{β_i} be the product of all $x_j^{\gamma_j}$ such that x_j occurs in x^{α_i}. Clearly x^{β_i} divides x^{α_i}, and $\deg(x^{\alpha_i}) > \deg(x^{\beta_i})$ because x^a is not in I by hypothesis. $\square$

The next result gives some additional support to Conjecture 2.2.2.

Proposition 2.5.8. *Let* $I \subset S$ *be a complete intersection monomial ideal of dimension* ≥ 1 *minimally generated by* $x^{\alpha_1}, \ldots, x^{\alpha_r}$. *If* $d_i = \deg(x^{\alpha_i})$ *for* $i = 1, \ldots, r$. *The following hold.*

(a) $\operatorname{reg}(S/I) = \sum_{i=1}^r (d_i - 1)$,

(b) $\delta_I(d) = 1$ *if* $d \geq \operatorname{reg}(S/I)$,

(c) $\delta_I(d) \leq (d_{k+1} - \ell)\, d_{k+2} \cdots d_r$ *if* $d < \operatorname{reg}(S/I)$, *where* $0 \leq k \leq r-1$ *and* ℓ *are integers such that* $d = \sum_{i=1}^k (d_i - 1) + \ell$ *and* $1 \leq \ell \leq d_{k+1} - 1$.

Proof. (a): This follows at once from Proposition 2.5.4.

(b): By Lemma 2.5.7(a) the monomials x^{α_i} and x^{α_j} have no common variables for $i \neq j$. For each i pick x_{j_i} in x^{α_i}. If I is prime, then $I = (x_{j_1}, \ldots, x_{j_r})$, $\operatorname{reg}(S/I) = 0$, $\mathcal{F}_d = \emptyset$ and $\delta_I(d) = 1$ for $d \geq 1$. Thus we may assume that I is not prime. We claim that $\mathcal{F}_d \neq \emptyset$ for $d \geq 1$. As I is not prime, there is m such that x_{j_m} a zero divisor of S/I not in I. If a variable x_n is not in x^{α_i} for any i, then x_n is a regular element on S/I, and $\mathcal{F}_d \neq \emptyset$ because $x_{j_m} x_n^{d-1}$ is in $\mathcal{F}_d$. If any variable x_n is in x^{α_i} for some i, then any monomial of degree d is a zero divisor of S/I because any variable x_n belongs to at least one associated prime of S/I. As $\dim(S/I) \geq 1$, one has $\mathfrak{m}^d \not\subset I$. Pick a monomial x^a of degree d not in I. Then $\mathcal{F}_d \neq \emptyset$ because x^a is in $\mathcal{F}_d$. This completes the proof of the claim. We set $x^{c_i} = x^{\alpha_i}/x_{j_i}$ for $i = 1, \ldots, r$ and $x^c = x^{c_1} \cdots x^{c_r}$. Then it is seen that $(I\colon x^c) = (x_{j_1}, \ldots, x_{j_r})$ and $\deg S/(I\colon x^c) = 1$. Notice that x^c is a zero divisor of S/I, $x^c \notin I$ and $\deg(x^c) = \operatorname{reg}(S/I)$. Hence, by Theorem 2.1.7, we get that $\delta_I(d) = 1$ for $d = \operatorname{reg}(S/I)$. Thus, by Theorem 2.1.9(ii), we get $\delta_I(d) = 1$ for $d \geq \operatorname{reg}(S/I)$.

(c): There is a monomial x^a of degree ℓ that divides $x^{\alpha_{k+1}}$ because ℓ is a positive integer less than or equal to $d_{k+1} - 1$. Setting $x^c = x^{c_1} \cdots x^{c_k} x^a$ and $x^\gamma = x^{\alpha_{k+1}}/x^a$, one has

$$(I\colon x^c) = (x_{j_1}, \ldots, x_{j_k}, x^\gamma, x^{\alpha_{k+2}}, \ldots, x^{\alpha_r}).$$

Hence, by Proposition 2.5.4, we get $\deg(S/(I\colon x^c)) = (d_{k+1} - \ell) d_{k+2} \cdots d_r$ because $(I\colon x^c)$ is a complete intersection. Since $\deg(x^c) = d = \sum_{i=1}^{k} (d_i - 1) + \ell$, x^c is not in I, and x^c is a zero divisor of S/I, by Theorem 2.1.7 we get that $\deg(S/(I\colon x^c)) \geq \delta_I(d)$, as required. $\square$

We are ready to present the other main result of this section, showing that the lower bound of Theorem 2.5.6 holds when I is a complete intersection monomial ideal of dimension 1.

Theorem 2.5.9. *Let $I \subset S$ be a complete intersection monomial ideal of dimension ≥ 1 minimally generated by $x^{\alpha_1}, \ldots, x^{\alpha_r}$ and let $d \geq 1$ be an integer. If $d_i = \deg(x^{\alpha_i})$ for $i = 1, \ldots, r$ and $d_1 \leq \cdots \leq d_r$, then*

$$\delta_I(d) = \operatorname{fp}_I(d) = \begin{cases} (d_{k+1} - \ell)\, d_{k+2} \cdots d_r & \text{if } d < \sum\limits_{i=1}^{r} (d_i - 1)\,, \\ 1 & \text{if } d \geq \sum\limits_{i=1}^{r} (d_i - 1)\,, \end{cases}$$

where $0 \leq k \leq r - 1$ and ℓ are the unique integers such that $d = \sum_{i=1}^{k} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

Proof. The ideal I is unmixed because I is Cohen–Macaulay. Hence, by Proposition 2.3.3, I is Geil–Carvalho, that is, $\delta_I(d) = \operatorname{fp}_I(d)$ for $d \geq 1$. Therefore, by Proposition 2.5.8, it suffices to show that

$$\operatorname{fp}_I(d) \geq (d_{k+1} - \ell) d_{k+2} \cdots d_r \quad \text{for } d < \operatorname{reg}(S/I).$$

Let x^a be a monomial of degree d such that $x^a \notin I$ and $(I\colon x^a) \neq I$. By Lemma 2.5.7(d), for each i there is a monomial x^{β_i} dividing x^{α_i} such that $\deg(x^{\beta_i}) < \deg(x^{\alpha_i})$ and $(I\colon x^a) = (I\colon x^\beta)$, where $x^\beta = x^{\beta_1} \cdots x^{\beta_r}$. One can write

$$x^{\alpha_i} = x_1^{\alpha_{i,1}} \cdots x_s^{\alpha_{i,s}} \quad \text{and} \quad x^{\beta_i} = x_1^{\beta_{i,1}} \cdots x_s^{\beta_{i,s}}$$

for $i = 1, \ldots, r$. According to Lemma 2.5.7(a) the monomials x^{α_i} and x^{α_j} have no common variables for $i \neq j$. As $(I\colon x^\beta)$ is a monomial ideal, it follows that

$$(I\colon x^a) = (I\colon x^\beta) = (\{x_1^{\alpha_{i,1}-\beta_{i,1}} \cdots x_s^{\alpha_{i,s}-\beta_{i,s}}\}_{i=1}^r).$$

Hence, setting $g_i = x_1^{\alpha_{i,1}-\beta_{i,1}} \cdots x_s^{\alpha_{i,s}-\beta_{i,s}}$ for $i = 1, \ldots, r$ and observing that g_i and g_j have no common variables for $i \neq j$, we get that $g_1, \ldots, g_r$ form a regular sequence, that is, $(I\colon x^a)$ is again a complete intersection. Thus, by Proposition 2.5.4, we obtain

$$\deg(S/(I\colon x^a)) = \prod_{i=1}^r \left[\sum_{j=1}^s (\alpha_{i,j} - \beta_{i,j})\right] = \prod_{i=1}^r \left[\deg(x^{\alpha_i}) - \deg(x^{\beta_i})\right].$$

Therefore, setting $b_i = \deg(x^{\beta_i})$ for $i = 1, \ldots, r$, we get

$$\deg(S/(I\colon x^a)) = \prod_{i=1}^r (d_i - b_i).$$

Thus, by Theorem 2.1.7, it suffices to show the inequality

$$\deg(S/(I\colon x^a)) = \prod_{i=1}^r (d_i - b_i) \geq (d_{k+1} - \ell) d_{k+2} \cdots d_r.$$

Noticing that $d = \deg(x^a) = \sum_{i=1}^k (d_i - 1) + \ell \geq \deg(x^\beta) = \sum_{i=1}^r b_i$, one has

$$\left(d_{k+1} + \sum_{i=1}^k (d_i - 1) - \sum_{i=1}^r b_i\right) d_{k+2} \cdots d_r \geq (d_{k+1} - \ell) d_{k+2} \cdots d_r.$$

Hence, we need only show the inequality

$$\prod_{i=1}^r (d_i - b_i) \geq \left(\sum_{i=1}^{k+1} (d_i - b_i) - k - \sum_{i=k+2}^r b_i\right) d_{k+2} \cdots d_r,$$

which follows making $b_0 = 1$ and $m = r$ in Proposition 2.4.3. □

The formula of Theorem 2.5.9, is also valid in dimension zero for $d < \sum_{i=1}^r (d_i - 1)$. Now, for $d \geq \sum_{i=1}^r (d_i - 1)$, the set $\mathcal{F}_d$ is empty simply because $(S/I)_d = (0)$, and so by definition, $\delta_I(d) = \deg(S/I)$.

The most basic application is for complete intersections in $\mathbb{P}^1$.

Corollary 2.5.10. *If* $\mathbb{X}$ *is a finite subset of* $\mathbb{P}^1$ *and* $I(\mathbb{X})$ *is a complete intersection, then*

$$\delta_{I(\mathbb{X})}(d) = \mathrm{fp}_{I(\mathbb{X})}(d) = \begin{cases} |\mathbb{X}| - d & \text{if } 1 \leq d \leq |\mathbb{X}| - 2, \\ 1 & \text{if } d \geq |\mathbb{X}| - 1. \end{cases}$$

Proof. Let f be the generator of $I(\mathbb{X})$. In this case $d_2 = \deg(f) = |\mathbb{X}|$ and $\mathrm{reg}(S/I(\mathbb{X})) = |\mathbb{X}| - 1$. By Proposition 1.8.8 and Theorem 2.5.6 one has

$$\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \geq \mathrm{fp}_{I(\mathbb{X})}(d) = |\mathbb{X}| - d \quad \text{for} \quad 1 \leq d \leq |\mathbb{X}| - 2,$$

and $\delta_{\mathbb{X}}(d) = 1$ for $d \geq |\mathbb{X}| - 1$. Assume that $1 \leq d \leq |\mathbb{X}| - 2$. Pick $[P_1], \ldots, [P_d]$ points in $\mathbb{P}^1$. By Lemma 1.7.3, the vanishing ideal $I_{[P_i]}$ of $[P_i]$ is a principal ideal generated by a linear form h_i. Notice that $V_{\mathbb{X}}(h_i)$, the zero set of h_i in $\mathbb{X}$, is equal to $\{[P_i]\}$. Setting $h = h_1 \cdots h_d$, we get a homogeneous polynomial of degree d with exactly d zeros. Thus $\delta_{\mathbb{X}}(d) \leq |\mathbb{X}| - d$. □

Chapter 3

Minimum Distance of Reed–Muller-type Codes

In this chapter we show that the minimum distance function of a graded ideal in a polynomial ring with coefficients in a field generalizes the minimum distance of projective Reed–Muller-type codes over finite fields. This gives an algebraic formulation of the minimum distance of a projective Reed–Muller-type code in terms of the algebraic invariants and structure of the underlying vanishing ideal. Then we give a method, based on Gröbner bases and Hilbert functions, to find lower bounds for the minimum distance of certain Reed–Muller-type codes. As an application we recover a formula for the minimum distance of an affine Cartesian code and the fact that in this case the minimum distance and the footprint functions coincide. Then we present an extension of a result of Alon and Füredi, about coverings of the cube by affine hyperplanes, in terms of the regularity of a vanishing ideal. Finally we show explicit upper bounds for the number of zeros of polynomials in a projective nested Cartesian set and give some support to a conjecture of Carvalho, Lopez-Neumann and López.

Some of our results rely on degree formulas to compute the number of zeros that a homogeneous polynomial has in any given finite set of points in a projective space.

3.1 Computing the number of zeros using the degree

In this section we give a degree formula to compute the number of zeros that a homogeneous polynomial has in any given finite set of points in a projective space over any field.

Let $\mathbb{P}^{s-1}$ be a projective space over a field K, and let $\mathbb{X}$ be a subset of $\mathbb{P}^{s-1}$. The vanishing ideal of $\mathbb{X}$, denoted by $I(\mathbb{X})$, is the ideal in a polynomial ring $S = K[x_1, \ldots, x_s]$ generated by homogeneous polynomials that vanish at all points of $\mathbb{X}$.

Lemma 3.1.1. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$ over a field K and let $I(\mathbb{X}) \subset S$ be its graded vanishing ideal. If $0 \neq f \in S$ is homogeneous, then the number of zeros of f in $\mathbb{X}$*

is given by

$$|V_{\mathbb{X}}(f)| = \begin{cases} \deg(S/(I(\mathbb{X}), f)) & \text{if } (I(\mathbb{X})\colon f) \neq I(\mathbb{X}), \\ 0 & \text{if } (I(\mathbb{X})\colon f) = I(\mathbb{X}). \end{cases}$$

Proof. Let $[P_1], \ldots, [P_m]$ be the points of $\mathbb{X}$ with $m = |\mathbb{X}|$ and let $[P]$ be a point in $\mathbb{X}$, with $P = (\alpha_1, \ldots, \alpha_s)$ and $\alpha_k \neq 0$ for some k. Then the vanishing ideal $I_{[P]}$ of $[P]$ is a prime ideal of height $s-1$,

$$I_{[P]} = (\{\alpha_k x_i - \alpha_i x_k \,|\, k \neq i \in \{1, \ldots, s\}),\ \deg(S/I_{[P]}) = 1,$$

and $I(\mathbb{X}) = \bigcap_{i=1}^{m} I_{[P_i]}$ is a primary decomposition (Lemma 1.7.3). In particular $I(\mathbb{X})$ is an unmixed radical ideal of dimension 1.

Assume that $(I(\mathbb{X})\colon f) \neq I(\mathbb{X})$. Let $\mathcal{A}$ be the set of all $I_{[P_i]}$ that contain the polynomial f. Then $f(P_i) = 0$ if and only if $I_{[P_i]}$ is in $\mathcal{A}$. Hence, by Lemma 1.5.29, we get

$$|V_{\mathbb{X}}(f)| = \sum_{[P_i] \in V_{\mathbb{X}}(f)} \deg(S/I_{[P_i]}) = \sum_{f \in I_{[P_i]}} \deg(S/I_{[P_i]}) = \deg(S/(I(\mathbb{X}), f)).$$

If $(I(\mathbb{X})\colon f) = I(\mathbb{X})$, then f is a regular element of $S/I(\mathbb{X})$. This means that f is not in any of the associated primes of $I(\mathbb{X})$, that is, $f \notin I_{[P_i]}$ for all i. Thus $V_{\mathbb{X}}(f) = \emptyset$ and $|V_{\mathbb{X}}(f)| = 0$. □

Corollary 3.1.2. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $I(\mathbb{X}) \subset S$ be its vanishing ideal, and let $\prec$ be a monomial order. If $0 \neq f \in S$ is homogeneous and $(I(\mathbb{X})\colon f) \neq I(\mathbb{X})$, then*

$$|V_{\mathbb{X}}(f)| = \deg(S/(I(\mathbb{X}), f)) \leq \deg(S/(\mathrm{in}_\prec(I(\mathbb{X}))), \mathrm{in}_\prec(f)) \leq \deg(S/I(\mathbb{X})),$$

and $\deg(S/(I(\mathbb{X}), f)) < \deg(S/I(\mathbb{X}))$ *if* $f \notin I(\mathbb{X})$.

Proof. It follows from Lemma 3.1.1 and Lemma 2.1.4. □

Corollary 3.1.3. *Let $I = I(\mathbb{X})$ be the vanishing ideal of a finite set $\mathbb{X}$ of a projective points, let $f \in \mathcal{F}_{\prec,d}$, and $\mathrm{in}_\prec(f) = x_i^{a_1} \cdots x_s^{a_s}$. If $\mathrm{in}_\prec(I)$ is generated by $x_2^{d_2} \cdots x_2^{d_s}$, then there is $r \geq 2$ such that $a_r \geq 1$, $a_i \leq d_i - 1$ for $i \geq r$, $a_i = 0$ if $2 \leq i \leq r$, and*

$$|V_{\mathbb{X}}(f)| = d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s).$$

Proof. As f is a zero divisor of S/I, by Lemma 1.6.14, $x^a = \mathrm{in}_\prec(f)$ is a zero divisor of $S/\mathrm{in}_\prec(I)$. Hence, there is $r \geq 2$ such that $a_r \geq 1$ and $a_i = 0$ if $2 \leq i < r$. Using that x^a is a standard monomial of S/I, we get that $a_i \leq d_i - 1$ for $i \geq r$. Therefore, using Lemma 3.1.1 together with Lemma 2.5.2 and Corollary 3.1.2, we get

$$\begin{aligned} |V_{\mathbb{X}}(f)| &= \deg(S/(I(\mathbb{X}), f)) \leq \deg(S/(\mathrm{in}_\prec(I(\mathbb{X}))), \mathrm{in}_\prec(f)) \\ &= d_2 \cdots d_s - (d_2 - a_2) \cdots (d_s - a_s). \end{aligned}$$

□

The following examples shows how to calculate the number of zeros of a polynomial in a finite set of points in a projective space.

Example 3.1.4. Let $V_{\mathbb{X}}(F)$ be the variety in $\mathbb{X} = \mathbb{P}^2$ defined by the polynomial

$$F = x_1^3 + x_2^3 + x_3^3 + x_1x_2x_3$$

over the field $K = \mathbb{F}_{13}$. Using Lemma 3.1.1 and the following procedure for *Macaulay*2 [25] we obtain that F has 18 zeros in $\mathbb{P}^2$. Notice that $I(\mathbb{X}) = (x_1^{13}x_2 - x_1x_2^{13}, x_1^{13}x_3 - x_1x_3^{13}, x_2^{13}x_3 - x_2x_3^{13})$.

```
S=GF(13)[x1,x2,x3];
Ixx=ideal(x1^13*x2-x1*x2^13,x1^13*x3-x1*x3^13,x2^13*x3-x2*x3^13)
F=x1^3+x2^3+x3^3+x1*x2*x3
quotient(Ixx,F)==Ixx
degree ideal(Ixx,F)
```

Example 3.1.5. Let $V_{\mathbb{X}}(F)$ be the variety in $\mathbb{X} = \mathbb{P}^2$ defined by the polynomial

$$F = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 - 3x_1^2x_2 - 3x_2^2x_3 - 3x_1x_3^2$$

over the finite field $K = \mathbb{F}_{13}$. Using *Macaulay*2 [25] with the procedure below we obtain that F has no zeros in $\mathbb{P}^2$. Notice that $I(\mathbb{X}) = (x_1^{13}x_2 - x_1x_2^{13}, x_1^{13}x_3 - x_1x_3^{13}, x_2^{13}x_3 - x_2x_3^{13})$.

```
R=GF(13)[x1,x2,x3];
F=ideal(x1^3+x2^3+x3^3-3*x1*x2*x3-3*x1^2*x2-3*x2^2*x3-3*x1*x3^2)
Ix=ideal(x1^13*x2-x1*x2^13,x1^13*x3-x1*x3^13,x2^13*x3-x2*x3^13)
J=ideal(Ix,F)
quotient(Ix,F)==Ix
degree J
```

3.2 Minimum distance of Reed–Muller-type codes

In this section we give an algebraic formulation of the minimum distance of a projective Reed–Muller-type code in terms of the degree and the structure of the underlying vanishing ideal.

Theorem 3.2.1. *If $K = \mathbb{F}_q$ is a finite field and $|\mathbb{X}| \geq 2$, then $\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \geq 1$ for $d \geq 1$.*

Proof. Setting $I = I(\mathbb{X})$, by Lemma 1.8.6, the set $\mathcal{F}_d := \{ f \in S_d \colon f \notin I, (I\colon f) \neq I\}$ is not empty for $d \geq 1$. Hence, using the formula for $V_{\mathbb{X}}(f)$ of Lemma 3.1.1, we obtain

$$\max\{|V_{\mathbb{X}}(f)| \colon \mathrm{ev}_d(f) \neq 0; f \in S_d\} = \max\{\deg(S/(I,f))| \, f \in \mathcal{F}_d\}.$$

Therefore, using that $\deg(S/I) = |\mathbb{X}|$, we get

$$\begin{aligned}\delta_{\mathbb{X}}(d) &= \min\{\|\mathrm{ev}_d(f)\| \colon \mathrm{ev}_d(f) \neq 0; f \in S_d\}\\ &= |\mathbb{X}| - \max\{|V_{\mathbb{X}}(f)| \colon \mathrm{ev}_d(f) \neq 0; f \in S_d\}\\ &= \deg(S/I) - \max\{\deg(S/(I,f)) | \, f \in \mathcal{F}_d\} = \delta_I(d),\end{aligned}$$

where $\|\mathrm{ev}_d(f)\|$ is the number of non-zero entries of $\mathrm{ev}_d(f)$. □

If I is a graded ideal and $\Delta_\prec(I) \cap S_d = \{x^{a_1}, \ldots, x^{a_n}\}$, recall that the set $\mathcal{F}_{\prec,d}$ is equal to the set of standard polynomials of S/I of degree d which are zero divisors of S/I:

$$\mathcal{F}_{\prec,d} := \left\{f = \textstyle\sum_i \lambda_i x^{a_i} \mid f \neq 0,\ \lambda_i \in K,\ (I \colon f) \neq I\right\}.$$

The next result gives a description of the minimum distance which is suitable for computing this number using a computer algebra system such as *Macaulay*2 [25].

Corollary 3.2.2. *If $K = \mathbb{F}_q$, $|\mathbb{X}| \geq 2$, $I = I(\mathbb{X})$, and $\prec$ is a monomial order, then*

$$\delta_{\mathbb{X}}(d) = \deg S/I - \max\{\deg(S/(I,f)) | \, f \in \mathcal{F}_{\prec,d}\} \geq 1 \quad \text{for} \quad d \geq 1.$$

Proof. It follows from Proposition 2.1.15 and Theorem 3.2.1. □

The expression for $\delta_{\mathbb{X}}(d)$ of Corollary 3.2.2 gives and algorithm that can be implemented in *Macaulay*2 [25] to compute $\delta_{\mathbb{X}}(d)$ (Example 3.2.6). However, in practice, we can only find the minimum distance for small values of q and d. Indeed, if $n = |\Delta_\prec(I) \cap S_d|$, to compute $\delta_{I(\mathbb{X})}$ requires to test the inequality $(I(\mathbb{X}) \colon f) \neq I(\mathbb{X})$ and compute the corresponding degree of $S/(I(\mathbb{X}), f)$ for the $q^n - 1$ standard polynomials of S/I.

Corollary 3.2.3. *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $I(\mathbb{X})$ be its vanishing ideal, and let $\prec$ be a monomial order. If the initial ideal $\mathrm{in}_\prec(I(\mathbb{X}))$ is a complete intersection generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$, with $d_i = \deg(x^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, then*

$$|V_{\mathbb{X}}(f)| \leq \deg(S/(I(\mathbb{X}))) - (d_{k+2} - \ell)d_{k+3} \cdots d_s,$$

for any $f \in S_d$ that does not vanish at all points of $\mathbb{X}$, where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. It follows from Theorems 2.5.6 and 3.2.1. □

Corollary 3.2.4. *Let $K = \mathbb{F}_q$ be a finite field, let $\prec$ be a monomial order, and let $\mathbb{X}$ be a subset of $\mathbb{P}^{s-1}$. Then, $\delta_{\mathbb{X}}(d) \geq \mathrm{fp}_{I(\mathbb{X})}(d) \geq 0$ for $d \geq 1$.*

Proof. The inequalities $\delta_{\mathbb{X}}(d) \geq \mathrm{fp}_{I(\mathbb{X})}(d) \geq 0$ follow from Theorems 2.3.2 and 3.2.1. □

One can use Corollary 3.2.4 to estimate the minimum distance of any Reed–Muller-type code over a set $\mathbb{X}$ parameterized by a set of relatively prime monomials and one has the following result that can be used to compute the vanishing ideal of $\mathbb{X}$ using Gröbner bases and elimination theory.

Theorem 3.2.5. [52] *Let $K = \mathbb{F}_q$ be a finite field. If $\mathbb{X}$ is a subset of $\mathbb{P}^{s-1}$ parameterized by monomials $y^{v_1}, \ldots, y^{v_s}$ in the variables $y_1, \ldots, y_n$, then*

$$I(\mathbb{X}) = (\{x_i - y^{v_i} z\}_{i=1}^{s} \cup \{y_i^q - y_i\}_{i=1}^{n}) \cap S,$$

and $I(\mathbb{X})$ is a binomial ideal.

As an application, Corollary 3.2.4 will be used to study the minimum distance of projective nested Cartesian codes [11] over a set $\mathcal{X}$. In this case one has a Gröbner basis for $I(\mathcal{X})$ [11] (Section 3.5).

In the following example, we present an implementation on *Macaulay2* to calculate the minimum distance function and footprint function. In particular, shows that the footprint is a lower bound for the minimum distance.

Example 3.2.6. Let $\mathbb{X}$ be the set in $\mathbb{P}^3$ parameterized by $y_1y_2, y_2y_3, y_3y_4, y_1y_4$ over the field $\mathbb{F}_3$. Using Theorem 3.2.5, Corollary 3.2.2, and the following procedure for *Macaulay*2 [25] we get

d	1	2	3	$\cdots$
$\lvert\mathbb{X}\rvert$	16	16	16	$\cdots$
$H_{\mathbb{X}}(d)$	4	9	16	$\cdots$
$\delta_{\mathbb{X}}(d)$	9	4	1	$\cdots$
$\text{fp}_{I(\mathbb{X})}(d)$	6	3	1	$\cdots$

```
q=3
R=ZZ/q[y1,y2,y3,y4,z,x1,x2,x3,x4,MonomialOrder=>Eliminate 5];
f1=y1*y2, f2=y2*y3, f3=y3*y4, f4=y4*y1
J=ideal(y1^q-y1,y2^q-y2,y3^q-y3,y4^q-y4,x1-f1*z,x2-f2*z,x3-f3*z,x4-f4*z)
C4=ideal selectInSubring(1,gens gb J)
S=ZZ/q[x1,x2,x3,x4];
I=sub(C4,S)
M=coker gens gb I
h=(d)->degree M - max apply(apply(apply(apply(
toList (set(0..q-1))^**(hilbertFunction(d,M))-
(set{0})^**(hilbertFunction(d,M)), toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),x-> if not
quotient(I,x)==I then degree ideal(I,x) else 0)--The function h(d)
--gives the minimum distance in degree d
h(1), h(2)
f=(a1) -> degree ideal(a1,leadTerm gens gb I)
fp=(d)->degree M - max apply(flatten entries basis(d,M),f)--The
--function fp(d) gives the footprint in degree d
L=toList(1..regularity M)
apply(L,fp)
```

Let $C_{\mathbb{X}}(d)$ be a projective Reed–Muller-type code. If $d \geq \operatorname{reg}(S/I(\mathbb{X}))$, then $\delta_{\mathbb{X}}(d) = 1$. The converse is not true as the next example shows.

Example 3.2.7. Let $\mathbb{X} = \{[(1,1,1)], [(1,-1,0)], [(1,0,-1)], [(0,1,-1)], [(1,0,0)]\}$ and let I be its vanishing ideal over the finite field $\mathbb{F}_3$. Using *Macaulay*2 [25] we obtain that $\operatorname{reg}(S/I) = 3$. Notice that $\delta_{\mathbb{X}}(1) = 1$ because the polynomial $x_1 + x_2 + x_3$ vanishes at all points of $\mathbb{X} \setminus \{[(1,0,0)]\}$.

The next example shows that δ_I is not in general non-increasing. This is why we often require that the dimension of I be at least 1 or that I is unmixed with at least 2 minimal primes.

Example 3.2.8. Let I be the ideal of $\mathbb{F}_5[x_1, x_2]$ generated by x_1^7, x_2^5, $x_1^2x_2$, $x_1x_2^3$. Using Corollary 3.2.2 and *Macaulay*2 [25] we get that the regularity of S/I is 7, that is, $H_I(d) = 0$ for $d \geq 7$, and

d	1	2	3	4	5	6	$\cdots$
$\deg(S/I)$	13	13	13	13	13	13	$\cdots$
$H_I(d)$	2	3	3	2	1	1	$\cdots$
$\delta_I(d)$	6	2	1	1	2	1	$\cdots$

3.3 Minimum distance of affine Cartesian codes

In this section, as an application of Theorem 2.5.6 and Theorem 3.2.1 we recover the formula for the minimum distance of an affine Cartesian code by examining the underlying vanishing ideal.

Let $K = \mathbb{F}_q$ be a finite field, let $A_2, \ldots, A_s$ be a collection of subsets of K, and let

$$\mathcal{X} = [1 \times A_2 \times \cdots \times A_s]$$

be the image of $1 \times A_2 \times \cdots \times A_s \setminus \{0\}$ under the map $K^s \setminus \{0\} \to \mathbb{P}^{s-1}$, $x \to [x]$. $C_{\mathcal{X}}(d)$ denoted the corresponding d-th affine Reed–Muller-type code, and is called the d-th *affine nested Cartesian code.*

Corollary 3.3.1. [19, 34] *Let K be a field and let $C_{\mathbb{X}}(d)$ be the projective Reed–Muller-type code of degree d on the finite set $\mathbb{X} = [1 \times A_2 \times \cdots \times A_s] \subset \mathbb{P}^{s-1}$. If $1 \leq d_i \leq d_{i+1}$ for $i \geq 2$, with $d_i = |A_i|$, and $d \geq 1$, then the minimum distance of $C_{\mathbb{X}}(d)$ is given by*

$$\delta_{\mathbb{X}}(d) = \begin{cases} (d_{k+2} - \ell)\, d_{k+3} \cdots d_s & \text{if } d \leq \sum\limits_{i=2}^{s} (d_i - 1) - 1, \\ 1 & \text{if } d \geq \sum\limits_{i=2}^{s} (d_i - 1), \end{cases}$$

where $k \geq 0$, ℓ are the unique integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. Let $\succ$ be the reverse lexicographical order on S with $x_s \succ \cdots \succ x_2 \succ x_1$. Setting $f_i = \prod_{\gamma \in A_i}(x_i - \gamma x_1)$ for $i = 2, \ldots, s$, one has that $f_2, \ldots, f_s$ is a Gröbner basis of $I(\mathbb{X})$ whose initial ideal is generated by $x_2^{d_2}, \ldots, x_s^{d_s}$ (see [34, Proposition 2.5]). By Theorem 3.2.1 one has the equality $\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d)$ for $d \geq 1$. Thus the inequality "$\geq$" follows at once from Theorem 2.5.6. Assume that $d < \sum_{i=2}^{s}(d_i - 1)$. To show the reverse inequality notice that there is a polynomial $f \in S_d$ which is a product of linear forms such that $|V_{\mathbb{X}}(f)|$, the number of zeros of f in $\mathbb{X}$, is equal to $d_2 \cdots d_s - (d_{k+2} - \ell)d_{k+3} \cdots d_s$ (see [34, p. 15]). As $|\mathbb{X}|$ is equal to $d_2 \cdots d_s$, we get that $\delta_{\mathbb{X}}(d)$ is less than or equal to $(d_{k+2} - \ell)d_{k+3} \cdots d_s$. □

Corollary 3.3.2. [10, Lemma 2.1] *Let $2 \leq d_2 \leq \cdots \leq d_s$ be a sequence of integers with $s \geq 2$. Fix an integer $1 \leq d \leq \sum_{i=2}^{s}(d_i - 1)$. Then*

$$\min\left\{\prod_{i=2}^{s}(d_i - a_i)\,\middle|\, 0 \leq a_i \leq d_i - 1,\ a_i \in \mathbb{N} \text{ for } i \geq 2,\ \sum_{i=2}^{s} a_i \leq d\right\} = (d_{k+2} - \ell)d_{k+3} \cdots d_s,$$

where $k \geq 0$, ℓ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. From Theorem 3.2.1 one has $\delta_{I(\mathbb{X})}(d) = \delta_{\mathbb{X}}(d)$. Hence, the equality follows at once from Proposition 2.5.2 and Corollary 3.3.1. □

The next result is an extension of a result of Alon and Füredi [3, Theorem 1] that can be applied to any finite subset of a projective space whose vanishing ideal has a complete intersection initial ideal relative to a graded monomial order.

Corollary 3.3.3. *Let $\mathbb{X}$ be a finite subset of a projective space $\mathbb{P}^{s-1}$ and let $\prec$ be a monomial order such that $\mathrm{in}_\prec(I(\mathbb{X}))$ is a complete intersection generated by $x^{\alpha_2}, \ldots, x^{\alpha_s}$ with $d_i = \deg(x^{\alpha_i})$ and $1 \leq d_i \leq d_{i+1}$ for all i. If the hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^{s-1}$ avoid a point $[P]$ in $\mathbb{X}$ but otherwise cover all the other $|\mathbb{X}| - 1$ points of $\mathbb{X}$, then $d \geq \mathrm{reg}(S/I(\mathbb{X})) = \sum_{i=2}^{s}(d_i - 1)$.*

Proof. Let $h_1, \ldots, h_d$ be the linear forms in S_1 that define $H_1, \ldots, H_d$, respectively. Assume that $d < \sum_{i=2}^{s}(d_i - 1)$. Consider the polynomial $h = h_1 \cdots h_d$. Notice that $h \notin I(\mathbb{X})$ because $h(P) \neq 0$, and $h(Q) = 0$ for all $[Q] \in \mathbb{X}$ with $[Q] \neq [P]$. By Theorem 2.5.6 $\delta_{\mathbb{X}}(d) \geq \mathrm{fp}_{I(\mathbb{X})} \geq 2$. Hence, h does not vanish in at least two points of $\mathbb{X}$, a contradiction. □

The following examples shows how some of our results can be used in practice, we present implementations in *Macaulay*2 [25] to calculate $H_{\mathbb{X}}(d)$, $\delta_{\mathbb{X}}(d)$ and $\mathrm{fp}_{I(\mathbb{X})}(d)$.

Example 3.3.4. Let S be the polynomial ring $\mathbb{F}_3[x_1, x_4, x_3, x_2]$ with the lexicographical order $x_1 \prec x_4 \prec x_3 \prec x_2$, and let $I = I(\mathbb{X})$ be the vanishing ideal of

$$\begin{aligned}\mathbb{X} = \{&[(1,0,0,0)],\ [(1,1,1,0)],\ [(1,-1,-1,0)],\ [(1,1,0,1)],\\ &[(1,-1,1,1)],\ [(1,0,-1,1)],\ [(1,-1,0,-1)],\ [(1,0,1,-1)],\ [(1,1,-1,-1)]\}.\end{aligned}$$

Using the procedure below in *Macaulay*2 [25] and Theorem 2.5.6, we get that $I(\mathbb{X})$ is generated by $x_2 - x_3 - x_4$, $x_3^3 - x_3x_1^2$, and $x_4^3 - x_4x_1^2$. The regularity and the degree of $S/I(\mathbb{X})$ are 4 and 9, respectively, and $I(\mathbb{X})$ is a Geil–Carvalho ideal whose initial ideal is a complete intersection generated by x_2, x_3^3, x_4^3. The basic parameters of the Reed–Muller-type code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3	4
$\lvert\mathbb{X}\rvert$	9	9	9	9
$H_{\mathbb{X}}(d)$	3	6	8	9
$\delta_{\mathbb{X}}(d)$	6	3	2	1
$\mathrm{fp}_{I(\mathbb{X})}(d)$	6	3	2	1

By Corollary 3.3.3, if the hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^3$ avoid a point $[P]$ in $\mathbb{X}$ but otherwise cover all the other $|\mathbb{X}| - 1$ points of $\mathbb{X}$, then $d \geq \mathrm{reg}(S/I(\mathbb{X})) = 4$.

```
S=ZZ/3[x2,x3,x4,x1,MonomialOrder=>Lex];
I1=ideal(x2,x3,x4),I2=ideal(x4,x3-x1,x2-x1),I3=ideal(x4,x1+x3,x2+x1)
I4=ideal(x4-x1,x4-x2,x3),I5=ideal(x4-x1,x3-x1,x2+x1),
I6=ideal(x2,x1-x4,x3+x1), I7=ideal(x3,x1+x4,x1+x2),
I8=ideal(x2,x4+x1,x3-x1),I9=ideal(x1+x4,x3+x1,x2-x1)
I=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9)
M=coker gens gb I, regularity M, degree M
h=(d)->degree M - max apply(apply(apply(apply (toList
(set(0..q-1))^**(hilbertFunction(d,M))-(set{0})^**(hilbertFunction(d,M)),
toList),x->basis(d,M)*vector x),z->ideal(flatten entries z)),
x-> if not quotient(I,x)==I then degree ideal(I,x) else 0)--this
--gives the minimum distance in degree d
apply(1..3,h)
```

Example 3.3.5. Let S be the polynomial ring $S = \mathbb{F}_3[x_1, x_2, x_3]$ with the lexicographical order $x_1 \succ x_2 \succ x_3$, and let $I = I(\mathbb{X})$ be the vanishing ideal of

$$\mathbb{X} = \{[(1,1,0)],\ [(1,-1,0)],\ [(1,0,1)],\ [(1,0,-1)], [(1,-1,-1)], [(1,1,1)]\}.$$

As in Example 3.3.4, using *Macaulay*2 [25], we get that $I(\mathbb{X})$ is generated by

$$x_2^2x_3 - x_2x_3^2,\ \ x_1^2 - x_2^2 + x_2x_3 - x_3^2.$$

The regularity and the degree of $S/I(\mathbb{X})$ are 3 and 6, respectively, I is a Geil–Carvalho ideal, and $\mathrm{in}_{\prec}(I)$ is a complete intersection generated by $x_2^2x_3$ and x_1^2. The basic parameters of the Reed–Muller-type code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3
$\lvert\mathbb{X}\rvert$	6	6	6
$H_{\mathbb{X}}(d)$	3	5	6
$\delta_{\mathbb{X}}(d)$	3	2	1
$\mathrm{fp}_{I(\mathbb{X})}(d)$	3	2	1

By Corollary 3.3.3, if the hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^2$ avoid a point $[P]$ in $\mathbb{X}$ but otherwise cover all the other $|\mathbb{X}| - 1$ points of $\mathbb{X}$, then $d \geq \mathrm{reg}(S/I(\mathbb{X})) = 3$.

Example 3.3.6. Let S be the polynomial ring $S = \mathbb{F}_3[x_1, x_2, x_3]$ with the lexicographical order $x_1 \succ x_2 \succ x_3$, and let $I = I(\mathbb{X})$ be the vanishing ideal of

$$\mathbb{X} = \{[(1,1,0)], [(1,-1,0)], [(1,0,1)], [(1,0,-1)], [(1,-1,-1)], [(1,1,1)], [(0,1,0)], [(0,0,1)], [(0,1,1)]\}.$$

As in Example 3.3.4, using *Macaulay*2 [25], we get that $I(\mathbb{X})$ is generated by

$$x_2^2x_3 - x_2x_3^2,\ \ x_1^3 - x_1x_2^2 + x_1x_2x_3 - x_1x_3^2.$$

The regularity and the degree of $S/I(\mathbb{X})$ are 4 and 9, respectively, I is a Geil–Carvalho ideal, and $\mathrm{in}_\prec(I)$ is a complete intersection generated by $x_2^2x_3$ and x_1^3. The basic parameters of the Reed–Muller-type code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3	4
$\lvert\mathbb{X}\rvert$	9	9	9	9
$H_{\mathbb{X}}(d)$	3	6	8	9
$\delta_{\mathbb{X}}(d)$	6	3	2	1
$\mathrm{fp}_{I(\mathbb{X})}(d)$	6	3	2	1

By Corollary 3.3.3, if the hyperplanes $H_1, \ldots, H_d$ in $\mathbb{P}^2$ avoid a point $[P]$ in $\mathbb{X}$ but otherwise cover all the other $|\mathbb{X}| - 1$ points of $\mathbb{X}$, then $d \geq \mathrm{reg}(S/I(\mathbb{X})) = 4$.

```
S=ZZ/3[x_1,x_2,x_3,MonomialOrder=>Lex]
I1=ideal(x_2-x_1,x_3)
I2=ideal(x_3,x_2+x_1)
I3=ideal(x_2,x_3-x_1)
I4=ideal(x_2,x_3+x_1)
I5=ideal(x_2+x_1,x_3+x_1)
I6=ideal(x_2-x_1,x_3-x_1)
I7=ideal(x_1,x_3)
I8=ideal(x_1,x_2)
I9=ideal(x_1,x_3-x_2,x_2-x_3)
I=intersect(I1,I2,I3,I4,I5,I6,I7,I8,I9)
```

```
M=coker gens gb I
regularity M
degree M
h=d->hilbertFunction(d,M)
apply(1..4,h)
h1=(d)->degree M-max apply(apply(apply(apply(toList(set(0..2))^**
(hilbertFunction(d,M))-(set{0})^**(hilbertFunction(d,M)), toList),
x->basis(d,M)*vector x),z->ideal(flatten entries z)),
x->if not quotient(I,x)==I then degree ideal(I,x) else 0)
apply(1..4,h1)
```

Next we give an example of a graded vanishing ideal over a finite field, which is not Geil–Carvalho, by computing all possible initial ideals.

Example 3.3.7. Let $\mathbb{X} = \mathbb{P}^2$ be the projective space over the field $\mathbb{F}_2$ and let $I = I(\mathbb{X})$ be the vanishing ideal of $\mathbb{X}$. Using the procedure below in *Macaulay*2 [25] we get that the binomials $x_1x_2^2 - x_1^2x_2$, $x_1x_3^2 - x_1^2x_3$, $x_2x_3^2 - x_2^2x_3$ form a universal Gröbner basis of I, that is, they form a Gröbner basis for any monomial order. The ideal I has exactly six different initial ideals and $\delta_{\mathbb{X}} \neq \mathrm{fp}_I$ for each of them, that is, I is not a Geil–Carvalho ideal. The basic parameters of the projective Reed–Muller code $C_{\mathbb{X}}(d)$ are shown in the following table.

d	1	2	3
$\lvert\mathbb{X}\rvert$	7	7	7
$H_{\mathbb{X}}(d)$	3	6	7
$\delta_{\mathbb{X}}(d)$	4	2	1
$\mathrm{fp}_{I(\mathbb{X})}(d)$	4	1	1

```
load "gfaninterface.m2"
S=ZZ/2[symbol x1, symbol x2, symbol x3]
I=ideal(x1*x2^2-x1^2*x2,x1*x3^2-x1^2*x3,x2*x3^2-x2^2*x3)
universalGroebnerBasis(I)
(InL,L)= gfan I, #InL
init=ideal(InL_0)
M=coker gens gb init
f=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
fp=(d) ->degree M -max apply(flatten entries basis(d,M),f)
apply(1..regularity(M),fp)
```

3.4 Degree formulas of some monomial ideals

Let $S = K[x_1, \ldots, x_s]$ be a polynomial ring over a field K, let $d_1, \ldots, d_s$ be a non-decreasing sequence of positive integers with $d_1 \geq 2$ and $s \geq 2$, and let L be the ideal of S generated by the set of all $x_ix_j^{d_j}$ such that $1 \leq i < j \leq s$. It turns out that the ideal L is the initial ideal of the vanishing ideal of a projective nested Cartesian set. In this section we study the ideal L and show a formula for the degree of $S/(L, x^a)$ for any standard monomial x^a of S/L as a preparation to show some applications.

Lemma 3.4.1. *The ideal L is Cohen–Macaulay of height $s-1$, has a unique irredundant primary decomposition given by*

$$L = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s,$$

where $\mathfrak{q}_i = (x_1, \ldots, x_{i-1}, x_{i+1}^{d_{i+1}}, \ldots, x_s^{d_s})$ *for* $1 \leq i \leq s$, *and* $\deg(S/L) = 1 + \sum_{i=2}^{s} d_i \cdots d_s$.

Proof. Using induction on s and the depth lemma it is seen that L is Cohen–Macaulay (see Lemma 1.4.6). In particular L is unmixed. Since the radical of L is generated by all x_ix_j with $i < j$, the minimal primes of L are $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, where $\mathfrak{p}_i$ is generated by $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s$. The $\mathfrak{p}_i$-primary component of L is uniquely determined and is given by $LS_{\mathfrak{p}_i} \cap S$. Inverting the variable x_i in $LS_{\mathfrak{p}_i}$ it follows that $LS_{\mathfrak{p}_i} = \mathfrak{q}_iS_{\mathfrak{p}_i}$. As $\mathfrak{q}_i$ is an irreducible ideal, it is $\mathfrak{p}_i$-primary and one has the equality $LS_{\mathfrak{p}_i} \cap S = \mathfrak{q}_i$. By the additivity of the degree we obtain the required formula for the degree of S/L. □

Proposition 3.4.2. *Let $x^a = x_r^{a_r} \cdots x_s^{a_s}$ be a standard monomial of S/L with respect to a monomial order $\prec$. If $a_r \geq 1$ and $1 \leq r \leq s$, then $0 \leq a_i \leq d_i - 1$ for $i > r$ and*

$$\deg(S/(L, x^a)) = \begin{cases} \deg(S/L) - \displaystyle\sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) & \text{if} \quad a_r \leq d_r, \\ \deg(S/L) - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s) & \text{if} \quad a_r \geq d_r + 1, \end{cases}$$

where $(d_i - a_i) \cdots (d_s - a_s) = 1$ if $i > s$ and $a_i = 0$ for $i < r$.

Proof. As $f = x^a$ is not a multiple of $x_ix_j^{d_j}$ for $i < j$, we get that $0 \leq a_i \leq d_i - 1$ for $i > r$. To show the formula for the degree we proceed by induction on $s \geq 2$. In what follows we will freely use the additivity of Hilbert series [58, Lemma 5.1.1], a well-known formula for the Hilbert series of a complete intersection [58, p. 177], the formula of Lemma 3.4.1 for the degree of S/L, and the fact any monomial is a zero divisor of S/L (this follows from Lemma 3.4.1). We split the proof of the case $s = 2$ in three easy cases.

Case (1): Assume $s = 2$, $r = 1$. This case is independent of whether $a_1 \leq d_1$ of $a_1 \geq d_1 + 1$ because the two possible values of $\deg(S/(L, f))$ coincide. There are exact sequences

$$0 \longrightarrow S/(x_1)[-d_2] \xrightarrow{x_2^{d_2}} S/(L, f) \longrightarrow S/(x_2^{d_2}, f) \longrightarrow 0,$$

$$0 \longrightarrow S/(x_2^{a_2})[-a_1] \xrightarrow{x_1^{a_1}} S/(x_2^{d_2}, f) \longrightarrow S/(x_2^{d_2}, x_1^{a_1}) \longrightarrow 0.$$

Taking Hilbert series we get

$$F(S/(L, f), x) = \frac{x^{d_2}}{1 - x} + \frac{x^{a_1}(1 + x + \cdots + x^{a_2 - 1})}{1 - x} + \left(\sum_{i=0}^{d_2 - 1} x^i\right)\left(\sum_{i=0}^{a_1 - 1} x^i\right).$$

Writing $F(S/(L,f),x)=h(x)/(1-x)$ with $h(x)\in\mathbb{Z}[x]$ and $h(1)>0$, and noticing that $h(1)$ is the degree of $\deg(S/(L,f))$, we get

$$\deg(S/(L,f))=1+a_2=(d_2+1)-(d_2-a_2)=\deg(S/L)-(d_2-a_2).$$

Case (2): Assume $s=2$, $r=2$, $a_2\leq d_2$. In this case (L,f) is equal to $(x_2^{a_2})$. Thus

$$\deg(S/(L,f))=a_2=(1+d_2)-(d_2-a_2)-1=\deg(S/L)-(d_2-a_2)-1.$$

Case (3): Assume $s=2$, $r=2$, $a_2\geq d_2+1$. Taking Hilbert series in the exact sequence

$$0\longrightarrow S/(x_1,x_2^{a_2-d_2})[-d_2]\stackrel{x_2^{d_2}}{\longrightarrow}S/(L,f)\longrightarrow S/(x_2^{d_2})\longrightarrow 0,$$

we obtain

$$F(S/(L,f),x)=x^{d_2}(1+x+\cdots+x^{a_2-d_2-1})+\frac{(1+x+\cdots+x^{d_2-1})}{1-x}.$$

Thus we may proceed as in Case (1) to get $\deg(S/(L,f))=d_2=\deg(S/L)-1$.

This completes the initial induction step. We may now assume that $s\geq 3$ and split the proof in three cases.

Case (I): Assume $r=s\geq 3$ and $a_s\leq d_s$. Thus $f=x_s^{a_s}$ and $a_i=0$ for $i<s$. Setting L' equal to the ideal generated by the set of all $x_ix_j^{d_j}$ such that $2\leq i<j\leq s$, there is an exact sequence

$$0\longrightarrow S/(x_2^{d_2},\ldots,x_{s-1}^{d_{s-1}},x_s^{a_s})[-1]\stackrel{x_1}{\longrightarrow}S/(L,x_s^{a_s})\longrightarrow S/(L',x_s^{a_s},x_1)\longrightarrow 0.$$

Taking Hilbert series one has

$$F(S/(L,x_s^{a_s}),x)=tF(S/(x_2^{d_2},\ldots,x_{s-1}^{d_{s-1}},x_s^{a_s}),x)+F(S/(L',x_s^{a_s},x_1),x).$$

Hence, setting $S'=K[x_2,\ldots,x_s]$, from the induction hypothesis applied to $S'/(L',x_s^{a_s})$, and using that $\deg(S'/L')=\deg(S/L)-d_2\cdots d_{s-1}d_s$ (see Lemma 3.4.1), we obtain

$$\begin{aligned}\deg(S/(L,f)) &= d_2\cdots d_{s-1}a_s+\deg(S'/L')-\sum_{i=3}^{s+1}d_i\cdots d_{s-1}(d_s-a_s)\\ &= \deg(S/L)-\sum_{i=2}^{s+1}d_i\cdots d_{s-1}(d_s-a_s).\end{aligned}$$

Case (II): Assume $r=s\geq 3$ and $a_s\geq d_s+1$. Using the exact sequence

$$0\longrightarrow S/(x_2^{d_2},\ldots,x_{s-1}^{d_{s-1}},x_s^{d_s})[-1]\stackrel{x_1}{\longrightarrow}S/(L,x_s^{a_s})\longrightarrow S/(L',x_s^{a_s},x_1)\longrightarrow 0,$$

we can proceed as in Case (I) to get $\deg(S/(L,f))=\deg(S/L)-1$.

Case (III): Assume $r < s$. Then, by assumption, $a_s < d_s$. Let L' be the ideal generated by the set of all $x_i x_j^{d_j}$ such that $1 \leq i < j \leq s-1$. Setting $f' = x_r^{a_r} \cdots x_{s-1}^{a_{s-1}}$ and $S' = K[x_1, \ldots, x_{s-1}]$, there are exact sequences

$$0 \longrightarrow S/(x_1, \ldots, x_{s-1})[-d_s] \stackrel{x_s^{d_s}}{\longrightarrow} S/(L, f) \longrightarrow S/(L', f, x_s^{d_s}) \longrightarrow 0,$$
$$0 \longrightarrow S/(L', f', x_s^{d_s - a_s})[-a_s] \stackrel{x_s^{a_s}}{\longrightarrow} S/(L', f, x_s^{d_s}) \longrightarrow S/(L', x_s^{a_s}) \longrightarrow 0.$$

Hence taking Hilbert series, and applying Proposition 1.5.16, we get

$$\text{F(S/(L,f),x)=} \tfrac{x^{d_s}}{1-x} + F(S'/(L', f'), x) F(K[x_s]/(x_s^{d_s - a_s}), x) + F(S'/L', x) F(K[x_s]/(x_s^{a_s}), x).$$

Writing $F(S/(L, f), x) = h(x)/(1-x)$ with $h(x) \in \mathbb{Z}[x]$ and $h(1) > 0$, and noticing that $h(1)$ is the degree of $S/(L, f)$, the induction hypothesis applied to $S'/(L', f')$ yields the equality

$$\begin{aligned} \deg(S/(L,f)) &= 1 + \left(\deg(S'/L') - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_{s-1} - a_{s-1}) \right) (d_s - a_s) + \deg(S'/L') a_s \\ &= 1 + \deg(S'/L') d_s - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) \ \text{ if } \ a_r \leq d_r. \end{aligned}$$

or the equality

$$\begin{aligned} \deg(S/(L,f)) &= 1 + \big(\deg(S'/L') - (d_{r+1} - a_{r+1}) \cdots (d_{s-1} - a_{s-1}) \big) (d_s - a_s) + \deg(S'/L') a_s \\ &= 1 + \deg(S'/L') d_s - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s) \ \text{ if } \ a_r \geq d_r + 1. \end{aligned}$$

To complete the proof it suffices to notice that $\deg(S/L) = 1 + \deg(S'/L') d_s$. This equality follows readily from Lemma 3.4.1. □

Remark 3.4.3. Cases (1), (2), and (3) can also be shown using Hilbert functions instead of Hilbert series, but case (III) is easier to handle using Hilbert series.

Corollary 3.4.4. *Let $x^a = x_r^{a_r} \cdots x_s^{a_s}$ be a standard monomial of S/L with respect to a monomial order $\prec$. If $a_r \geq 1$ and $1 \leq r \leq s$, then $0 \leq a_i \leq d_i - 1$ for $i > r$ and*

$$\deg(S/(L,f)) = \begin{cases} \deg(S/L) - (d_2 - a_2) \cdots (d_s - a_s) & \text{if} \quad r = 1, \\ \deg(S/L) - \displaystyle\sum_{i=2}^{3} (d_i - a_i) \cdots (d_s - a_s) & \text{if} \quad r = 2 \text{ and } a_2 \leq d_2, \\ \deg(S/L) - (d_3 - a_3) \cdots (d_s - a_s) & \text{if} \quad r = 2 \text{ and } a_2 \geq d_2 + 1 \end{cases}$$

where $(d_i - a_i) \cdots (d_s - a_s) = 1$ if $i > s$ and $a_i = 0$ for $i < r$.

Proof. It follows at once from Proposition 3.4.2. □

3.5 Projective nested Cartesian codes

In this section we introduce projective nested Cartesian codes, a type of evaluation codes that generalize the classical projective Reed–Muller codes [49]. As an application we will give some support to a conjecture of Carvalho, Lopez-Neumann and López (Conjecture 3.5.2).

Let $K = \mathbb{F}_q$ be a finite field, let $A_1, \ldots, A_s$ be a collection of subsets of K, and let

$$\mathcal{X} = [A_1 \times \cdots \times A_s]$$

be the image of $A_1 \times \cdots \times A_s \setminus \{0\}$ under the map $K^s \setminus \{0\} \to \mathbb{P}^{s-1}$, $x \to [x]$. Unless otherwise stated $\mathcal{X}^*$ denote the Cartesian product $A_1 \times \cdots \times A_s$ in the affine space.

Definition 3.5.1. [11] The set $\mathcal{X}$ is called a *projective nested Cartesian set* if

(i) $\{0, 1\} \subset A_i$ for $i = 1, \ldots, s$,

(ii) $a/b \in A_j$ for $1 \leq i < j \leq s$, $a \in A_j$, $0 \neq b \in A_i$, and

(iii) $d_1 \leq \cdots \leq d_s$, where $d_i = |A_i|$ for $i = 1, \ldots, s$.

If $\mathcal{X}$ is a projective nested Cartesian set, we call $C_{\mathcal{X}}(d)$ a *projective nested Cartesian code.*

Throughout this section $\prec$ is the lexicographical order on S with $x_1 \prec \cdots \prec x_s$ and $\mathrm{in}_\prec(I(\mathcal{X}))$ is the initial ideal of $I(\mathcal{X})$.

Conjecture 3.5.2. (Carvalho, Lopez-Neumann, and López [11]) *Let $C_{\mathcal{X}}(d)$ be the d-th projective nested Cartesian code on the set $\mathcal{X} = [A_1 \times \cdots \times A_s]$ with $d_i = |A_i|$ for $i = 1, \ldots, s$. Then its minimum distance is given by*

$$\delta_{\mathcal{X}}(d) = \begin{cases} (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s & \text{if } d \leq \sum\limits_{i=2}^{s} (d_i - 1), \\ 1 & \text{if } d \geq \sum\limits_{i=2}^{s} (d_i - 1) + 1, \end{cases}$$

where $0 \leq k \leq s - 2$ and ℓ are the unique integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

This conjecture fails in general (Example 3.5.8). However the conjecture holds in certain cases.

Proposition 3.5.3. [11] *The initial ideal* $\mathrm{in}_\prec(I(\mathcal{X}))$ *is generated by the set of all monomials $x_i x_j^{d_j}$ such that $1 \leq i < j \leq s$,*

$$\deg(S/I(\mathcal{X})) = 1 + \sum_{i=2}^{s} d_i \cdots d_s, \quad \textit{and} \quad \mathrm{reg}(S/I(\mathcal{X})) = 1 + \sum_{i=2}^{s} (d_i - 1).$$

Carvalho, Lopez-Neumann and López, showed that Conjecture 3.5.2 can be reduced to:

Conjecture 3.5.4. (Carvalho, Lopez-Neumann, and López [11]) *If $0 \neq f \in S_d$ is a standard polynomial, with respect to $\prec$, such that $(I(\mathcal{X}) : f) \neq I(\mathcal{X})$ and $1 \leq d \leq \sum_{i=2}^{s}(d_i - 1)$, then*

$$|V_{\mathcal{X}}(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s,$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

We show a formula for $\deg(S/(\mathrm{in}_\prec(I(\mathcal{X})), x^a))$ and use this to show an upper bound for $|V_{\mathcal{X}}(f)|$.

Theorem 3.5.5. *Let $\prec$ be the lexicographical order on S with $x_1 \prec \cdots \prec x_s$ and let $f \neq 0$ be a standard polynomial with $\mathrm{in}_\prec(f) = x_r^{a_r} \cdots x_s^{a_s}$ and $a_r \geq 1$. Then $0 \leq a_i \leq d_i - 1$ for $i > r$ and*

$$\begin{aligned} |V_{\mathcal{X}}(f)| &\leq \deg(S/(\mathrm{in}_\prec(I(\mathcal{X}))), \mathrm{in}_\prec(f)) \\ &= \begin{cases} \deg(S/I(\mathcal{X})) - \displaystyle\sum_{i=2}^{r+1}(d_i - a_i)\cdots(d_s - a_s) & \text{if } a_r \leq d_r, \\ \deg(S/I(\mathcal{X})) - (d_{r+1} - a_{r+1})\cdots(d_s - a_s) & \text{if } a_r \geq d_r + 1, \end{cases} \end{aligned}$$

where $(d_i - a_i)\cdots(d_s - a_s) = 1$ if $i > s$ and $a_i = 0$ for $i < r$.

Proof. By Proposition 3.5.3 the initial ideal of $I(\mathcal{X})$ is generated by the set of all $x_i x_j^{d_j}$ such that $1 \leq i < j \leq s$ and the degree of $S/(\mathrm{in}_\prec(I(\mathcal{X}))$ is equal to the degree of $S/I(\mathcal{X})$. As $\mathrm{in}_\prec(f)$ is a standard monomial, it follows that $0 \leq a_i \leq d_i - 1$ for $i > r$. Notice that if f is a zero divisor of $S/I(\mathcal{X})$, then $V_{\mathcal{X}}(f) \neq \emptyset$. Thus the inequality follows at once from Corollary 3.1.2 and the equality follows from Proposition 3.4.2. □

Theorem 3.5.6. *Let $\prec$ be the lexicographical order on S with $x_1 \prec \cdots \prec x_s$. If $0 \neq f \in S_d$ is a standard polynomial such that $1 \leq d \leq \sum_{i=2}^{s}(d_i - 1)$ and x_1 divides $\mathrm{in}_\prec(f)$, then*

$$|V_{\mathcal{X}}(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s,$$

where $0 \leq k \leq s-2$ and ℓ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. By Lemma 3.1.1 we may assume that $(I(\mathcal{X}) : f) \neq I(\mathcal{X})$. Let $x^a = \mathrm{in}_\prec(f)$ be the initial monomial of f. By Proposition 3.5.3, we can write

$$x^a = x_1^{a_1} \cdots x_s^{a_s},$$

with $a_1 \geq 1$, $0 \leq a_i \leq d_i - 1$ for $i > 1$. By Lemmas 3.1.1 and 2.1.4 it suffices to show that for $r = 1$, the following inequality holds

$$\deg(S/(\mathrm{in}_\prec(I(\mathcal{X})), x^a)) \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1)\, d_{k+3} \cdots d_s. \tag{3.5.1}$$

If we substitute $-\ell = \sum_{i=2}^{k+1}(d_i-1) - \sum_{i=1}^{s} a_i$ in Eq. (3.5.1), and use the formula for the degree of $S/(\mathrm{in}_\prec(I(\mathcal{X})), x^a)$ given in Theorem 3.5.5, we need only show that the following inequalities hold for $r=1$:

$$\sum_{i=2}^{r+1}(d_i-a_i)\cdots(d_s-a_s) \geq \left(\sum_{i=2}^{k+2}(d_i-a_i)-(k-1)-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s \text{ if } a_r \leq d_r, \quad (3.5.2)$$

$$\prod_{i=r+1}^{s}(d_i-a_i) \geq \left(\sum_{i=2}^{k+2}(d_i-a_i)-(k-1)-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s \text{ if } a_r \geq d_r+1, \quad (3.5.3)$$

for $0 \leq k \leq s-2$, where $(d_i-a_i)\cdots(d_s-a_s)=1$ if $i>s$ and $a_i=0$ for $i<r$.

Assume $r=1$. Then Eqs. (3.5.2) and (3.5.3) are the same. Thus we need only show the inequality

$$\prod_{i=2}^{s}(d_i-a_i) \geq \left(\sum_{i=2}^{k+2}(d_i-a_i)-(k-1)-a_1-\sum_{i=k+3}^{s}a_i\right)d_{k+3}\cdots d_s,$$

for $0 \leq k \leq s-2$. This inequality follows making $m=s-1$, $e_i=d_{i+1}$, $b_i=a_{i+1}$ for $i=1,\ldots,m$, and $b_0=a_1$ in Proposition 2.4.3. □

Let $\mathcal{L}_d$ be the K-vector space generated by all $x^a \in S_d$ such that x^a contains x_1 and let C_d be the image of $\mathcal{L}_d$ under the evaluation map ev_d. From the next result it follows that the minimum distance of $C_\mathcal{X}(d)$ proposed in Conjecture 3.5.2, is in fact the minimum distance of the evaluation linear code C_d.

Corollary 3.5.7. *Let $\mathcal{L}_d$ be the K-vector space generated by all $x^a \in S_d$ such that x^a contains x_1. If $1 \leq d \leq \sum_{i=2}^{s}(d_i-1)$, then*

$$\max\{|V_\mathcal{X}(f)| : f \notin I(\mathcal{X}),\, f \in \mathcal{L}_d\} = \deg(S/I(\mathcal{X})) - (d_{k+2}-\ell+1)\,d_{k+3}\cdots d_s,$$

where $0 \leq k \leq s-2$ and ℓ are integers, $d=\sum_{i=2}^{k+1}(d_i-1)+\ell$, and $1 \leq \ell \leq d_{k+2}-1$.

Proof. Take $f \in \mathcal{L}_d \setminus I(\mathcal{X})$. Let $\prec$ be the lexicographical order with $x_1 \prec \cdots \prec x_s$ and let $\mathcal{G}$ be the Gröbner basis of $I(\mathcal{X})$ given in [11, Proposition 2.14]. By the division algorithm, we can write $f=\sum_{i=1}^{r}a_ig_i+g$, where $g_i \in \mathcal{G}$ for all i and g is a standard polynomial of degree d. The polynomial g is again in $\mathcal{L}_d \setminus I(\mathcal{X})$. Indeed if $g \notin \mathcal{L}_d$, there is at least one monomial of g that do not contain x_1, then making $x_1=0$ in the last equality, we get an equality of the form $0=\sum_{i=1}^{r}b_ig_i+h$, where h is a non-zero standard polynomial of $I(\mathcal{X})$, a contradiction. Hence, by Theorem 3.5.6, the inequality $\leq$ follows because $|V_\mathcal{X}(f)|=|V_\mathcal{X}(g)|$. To show equality notice that according to the proof of [11, Lemma 3.1], there is a polynomial f of degree d in $\mathcal{L}_d \setminus I(\mathcal{X})$ whose number of zeros in $\mathcal{X}$ is equal to the right hand side of the required equality. □

The following example shows that Conjecture 3.5.2 is not valid in general.

Example 3.5.8. Let $K = \mathbb{F}_q$ be a finite field with 4 elements and let $K_0 = K_1 = \mathbb{F}_2$, $K_2 = \mathbb{F}_4$ be subfields of K. Then $\mathcal{X} = [K_0 \times K_1 \times K_2]$ is a projective nested Cartesian product, and the minimum distance of the code $C_{\mathcal{X}}(d)$ is:

d	1	2	3	4
$\delta_{\mathcal{X}}(d)$	8	4	3	1

Note that for $d = 4$ we have $d - 1 = (2 - 1) + 2$ and from the formula in Conjecture 3.5.2 $\delta_{\mathcal{X}}(4) = 4 - 3 + 1 = 2$, thus the formula fails. Conjecture 3.5.2 holds for $d = 1, 2, 3$.

The following result shows an upper bound for the minimum distance of projective nested Cartesian codes.

Proposition 3.5.9. [11, Lemma 3.1] *If $\mathcal{X}$ is the projective nested Cartesian set over $A_0, \ldots, A_n$, then the minimum distance of $C_{\mathcal{X}}(d)$ satisfies $\delta_{\mathcal{X}}(d) \leq (d_{k+1} - \ell)d_{k+2} \cdots d_n$ if $1 \leq d \leq \sum_{i=1}^{n}(d_i - 1)$, and $\delta_{\mathcal{X}}(d) = 1$ in otherwise, where $0 \leq k \leq n - 1$ and $0 \leq \ell < d_{k+1} - 1$ are the unique integers such that $d - 1 = \sum_{i=1}^{k}(d_i - 1) + \ell$.*

Chapter 4

Monomial ideals of weighted oriented graphs

In this chapter we introduce the edge ideal $I = I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$. The study of this ideal was motivated because edge ideals of weighted oriented graphs arise in the theory of Reed–Muller codes as initial ideals of vanishing ideals of projective nested Cartesian sets over finite fields [11, 40, 49]. We study and develop an algebraic combinatorics theory of these ideals; determine the irredundant irreducible decomposition of I, characterize the associated primes and the unmixed property of I. Furthermore, we give a combinatorial characterization for the unmixed property of I, when $\mathcal{D}$ is bipartite, a whisker, or a cycle. Finally, we study the Cohen–Macaulay property of I and show that in certain cases I is unmixed if and only if I is Cohen–Macaulay [45].

4.1 Weighted oriented graphs and their vertex covers

In this section we define the weighted oriented graphs, denoted by $\mathcal{D}$ and study their vertex covers. We define the strong vertex covers, this notion extend the classical definition in graph theory of minimal vertex covers and prove that a minimal vertex cover is strong. Furthermore, we characterize when $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$.

Definition 4.1.1. A *weighted oriented graph* is a triplet $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}), w)$, where $V(\mathcal{D})$ is a finite set, $E(\mathcal{D}) \subset V(\mathcal{D}) \times V(\mathcal{D})$ and w is a function $w : V(\mathcal{D}) \to \mathbb{N}$. Sometimes we will write V and E for the vertex set and edge set of $\mathcal{D}$ respectively. The set $\{x \in V(\mathcal{D}) \mid w(x) \neq 1\}$ is denoted by V^+.

Definition 4.1.2. The *underlying graph* of $\mathcal{D}$ is the simple graph G whose vertex set is $V(\mathcal{D})$ and whose edge set is $\{\{x, y\} | (x, y) \in E(\mathcal{D})\}$.

Definition 4.1.3. Let x be a vertex of a weighted oriented graph $\mathcal{D}$, the sets $N^+_{\mathcal{D}}(x) = \{y \mid (x, y) \in E(\mathcal{D})\}$ and $N^-_{\mathcal{D}}(x) = \{y \mid (y, x) \in E(\mathcal{D})\}$ are called the *out-neighbourhood* and the *in-neighbourhood* of v, respectively. Furthermore, the *neighbourhood* of x is the set $N_{\mathcal{D}}(x) = N^+_{\mathcal{D}}(x) \cup N^-_{\mathcal{D}}(x)$.

As usual, a set of vertices C of G (resp. $\mathcal{D}$) is called a *vertex cover* of G (resp. of $\mathcal{D}$) if any edge of G (resp. $\mathcal{D}$) contains at least one vertex of C. A vertex cover C of G or $\mathcal{D}$ is *minimal* if for any other vertex cover C' with $C' \subset C$ one has $C' = C$.

Remark 4.1.4. Is easy to check that; C is a vertex cover of G if and only if C is a vertex cover of $\mathcal{D}$

Definition 4.1.5. Let C be a vertex cover of a weighted oriented graph $\mathcal{D}$, we define:

- $L_1(C) := \{x \in C \mid N_{\mathcal{D}}^{+}(x) \cap C^c \neq \emptyset\}$,
- $L_2(C) := \{x \in C \mid x \notin L_1(C) \text{ and } N_{\mathcal{D}}^{-}(x) \cap C^c \neq \emptyset\}$,
- $L_3(C) := C \setminus (L_1(C) \cup L_2(C))$,

where C^c is the complement of C, i.e. $C^c = V \setminus C$.

The previous subsets form a partition of C. In the following proposition, we give a characterization for the set $L_3(C)$.

Proposition 4.1.6. *If C is a vertex cover of $\mathcal{D}$, then*

$$L_3(C) = \{x \in C \mid N_{\mathcal{D}}(x) \subset C\}.$$

Proof. If $x \in L_3(C)$, then $N_{\mathcal{D}}^{+}(x) \subset C$, since $x \notin L_1(C)$. Furthermore $N_{\mathcal{D}}^{-}(x) \subset C$, since $x \notin L_2(C)$. Hence $N_{\mathcal{D}}(x) \subset C$, since $x \notin N_{\mathcal{D}}(x)$. Now, if $x \in C$ and $N_{\mathcal{D}}(x) \subset C$, then $x \notin L_1(C) \cup L_2(C)$. Therefore $x \in L_3(C)$. □

Proposition 4.1.7. *If C is a vertex cover of $\mathcal{D}$, then $L_3(C) = \emptyset$ if and only if C is a minimal vertex cover of $\mathcal{D}$.*

Proof. $\Rightarrow$) If $x \in C$, then by Proposition 4.1.6 we have $N_{\mathcal{D}}(x) \not\subset C$, since $L_3(C) = \emptyset$. Thus, there is $y \in N_{\mathcal{D}}(x) \setminus C$ implying $C \setminus \{x\}$ is not a vertex cover. Therefore, C is a minimal vertex cover.

$\Leftarrow$) If $x \in L_3(C)$, then by Proposition 4.1.6, $N_{\mathcal{D}}(x) \subset C \setminus \{x\}$. Hence, $C \setminus \{x\}$ is a vertex cover. A contradiction, since C is minimal. Therefore $L_3(C) = \emptyset$. □

Definition 4.1.8. A vertex cover C of $\mathcal{D}$ is *strong* if for each $x \in L_3(C)$ there is $(y, x) \in E(\mathcal{D})$ such that $y \in L_2(C) \cup L_3(C)$ with $y \in V^{+}$ (i.e. $w(y) \neq 1$).

Remark 4.1.9. Let C be a vertex cover of $\mathcal{D}$. Hence, by Proposition 4.1.6 and since $C = L_1(C) \cup L_2(C) \cup L_3(C)$, we have that C is strong if and only if for each $x \in C$ such that $N(x) \subset C$, there exist $y \in N^{-}(v) \cap (C \setminus L_1(C))$ with $y \in V^{+}$.

Corollary 4.1.10. *If C is a minimal vertex cover of $\mathcal{D}$, then C is strong.*

Proof. By Proposition 4.1.7, we have $L_3(C) = \emptyset$, since C. Hence, C is strong. □

The converse of the Corollary 4.1.10 is true if $w(x) = 1$ for all $x \in V(\mathcal{D})$, that is, in the sense of graph theory all strong vertex cover is minimal. An important fact is that a strong vertex cover is not always minimal. In what follows we characterize when $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$.

Remark 4.1.11. The vertex set V of $\mathcal{D}$ is a vertex cover. Also, if $z \in V$, then $N_{\mathcal{D}}(z) \subset V \setminus z$. Hence, by Proposition 4.1.6, $L_3(V) = V$. Consequently, $L_1(V) = L_2(V) = \emptyset$. By Proposition 4.1.7, V is not a minimal vertex cover of $\mathcal{D}$. Furthermore since $L_3(V) = V$, V is a strong vertex cover if and only if $N^-_{\mathcal{D}}(x) \cap V^+ \neq \emptyset$ for each $x \in V$.

Definition 4.1.12. If $\mathcal{D}$ is a cycle with $E(\mathcal{D}) = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ and $V(\mathcal{D}) = \{x_1, \ldots, x_n\}$, then $\mathcal{D}$ is called *oriented cycle*.

Definition 4.1.13. $\mathcal{D}$ is called *unicycle oriented graph* if it satisfies the following conditions:

(1) The underlying graph of $\mathcal{D}$ is connected and it has exactly one cycle C.

(2) C is an oriented cycle in $\mathcal{D}$. Furthermore for each $y \in V(\mathcal{D}) \setminus V(C)$, there is an oriented path from C to y in $\mathcal{D}$.

(3) $w(x) \neq 1$ if $\deg_G(x) > 1$.

Lemma 4.1.14. *If $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$ and $\mathcal{D}_1$ is a maximal unicycle oriented subgraph of $\mathcal{D}$, then $V(\mathcal{D}')$ is a strong vertex cover of $\mathcal{D}' = \mathcal{D} \setminus V(\mathcal{D}_1)$.*

Proof. We take $x \in V(\mathcal{D}')$. Thus, by Remark 4.1.11, there is $y \in N^-_{\mathcal{D}}(x) \cap V^+(\mathcal{D})$. If $y \in \mathcal{D}_1$, then we take $\mathcal{D}_2 = \mathcal{D}_1 \cup \{(y, x)\}$. Hence, if C is the oriented cycle of $\mathcal{D}_1$, then C is the unique cycle of $\mathcal{D}_2$, since $\deg_{\mathcal{D}_2}(v) = 1$. If $u \in C$, then (y, x) is an oriented path from C to x. Now, if $y \notin C$, then there is an oriented path $\mathcal{L}$ form C to y in $\mathcal{D}_1$. Consequently, $\mathcal{L} \cup \{(y, x)\}$ is an oriented path form C to x. Furthermore, $\deg_{\mathcal{D}_2}(x) = 1$ and $w(y) \neq 1$, then $\mathcal{D}_2$ is a unicycle oriented graph. A contradiction, since $\mathcal{D}_1$ is maximal. This implies $y \in V(\mathcal{D}')$, so $y \in N^-_{\mathcal{D}'}(v) \cap V^+(\mathcal{D}')$. Therefore, by Remark 4.1.11, $V(\mathcal{D}')$ is a strong vertex cover of $\mathcal{D}'$. □

Lemma 4.1.15. *If $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$, then there is a unicycle oriented subgraph of $\mathcal{D}$.*

Proof. Let y_1 be a vertex of $\mathcal{D}$. Since $V = V(\mathcal{D})$ is a strong vertex cover, there is $y_2 \in V$ such that $y_2 \in N^-(u_1) \cap V^+$. Similarly, there is $y_3 \in N^-(y_2) \cap V^+$. Consequently, (y_3, y_2, y_1) is an oriented path. Continuing this process, we can assume there exist $y_2, y_3, \ldots, y_k \in V^+$ where $(y_k, y_{k-1}, \ldots, y_2, y_1)$ is an oriented path and there is $1 \leq j \leq k-2$ such that $(y_j, y_k) \in E(\mathcal{D})$, since V is finite. Hence, $C = (y_k, y_{k-1}, \ldots, y_j, y_k)$ is an oriented cycle and $\mathcal{L} = (y_j, \ldots, y_1)$ is an oriented path form C to y_1. Furthermore, if $j = 1$, then $w(y_1) \neq 1$. Therefore, $\mathcal{D}_1 = C \cup \mathcal{L}$ is a unicycle oriented subgraph of $\mathcal{D}$. □

Proposition 4.1.16. *Let $\mathcal{D} = (V, E, w)$ be a weighted oriented graph, hence V is a strong vertex cover of $\mathcal{D}$ if and only if there are $\mathcal{D}_1, \ldots, \mathcal{D}_t$ unicycle oriented subgraphs of $\mathcal{D}$ such that $V(\mathcal{D}_1), \ldots, V(\mathcal{D}_t)$ is a partition of $V = V(\mathcal{D})$.*

Proof. $\Rightarrow$) By Lemma 4.1.15, there is a maximal unicycle oriented subgraph $\mathcal{D}_1$ of $\mathcal{D}$. Hence, by Lemma 4.1.14, $V(\mathcal{D}')$ is a strong vertex cover of $\mathcal{D}' = \mathcal{D} \setminus V(\mathcal{D}_1)$. So, by Lemma 4.1.15, there is $\mathcal{D}_2$ a maximal unicycle oriented subgraph of $\mathcal{D}'$. Continuing this process we obtain unicycle oriented subgraphs $\mathcal{D}_1, \ldots, \mathcal{D}_t$ such that $V(\mathcal{D}_1), \ldots, V(\mathcal{D}_t)$ is a partition of $V(\mathcal{D})$.

$\Leftarrow$) We take $x \in V(\mathcal{D})$. By hypothesis there is $1 \leq j \leq t$ such that $x \in V(\mathcal{D}_j)$. We assume C is the oriented cycle of $\mathcal{D}_j$. If $x \in V(C)$, then there is $y \in V(C)$ such that $(y, x) \in E(\mathcal{D}_j)$ and $w(y) \neq 1$, since $\deg_{\mathcal{D}_j}(x) \geq 2$ and $\mathcal{D}_j$ is a unicycle oriented subgraph. Now, we assume $x \notin V(C)$, then there is an oriented path $\mathcal{L} = (z_1, \ldots, z_r)$ such that $z_1 \in V(C)$ and $z_r = x$. Thus, $(z_{r-1}, x) \in E(\mathcal{D})$. Furthermore, $w(z_{r-1}) \neq 1$, since $\deg_{\mathcal{D}_j}(z_{r-1}) \geq 2$. Therefore V is a strong vertex cover. □

4.2 Edge ideals and their primary decomposition

As usual if I is a monomial ideal of a polynomial ring S, we denote by $\mathcal{G}(I)$ the minimal monomial set of generators of I.

Definition 4.2.1. ([58, Theorem 6.1.17]) There exists a unique decomposition

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r,$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are irreducible monomial ideals such that $I \neq \bigcap_{i \neq j} \mathfrak{q}_i$ for each $j = 1, \ldots, r$. This is called the *irredundant irreducible decomposition of* I.

It is well known that, $\mathfrak{q}_i$ is an *irreducible monomial ideal* of I if and only if $\mathfrak{q}_i = (x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell})$ for some variables x_{i_j} [58, Theorem 6.1.16]. Irreducible ideals are primary, then an irreducible decomposition is a primary decomposition. For more details of primary decomposition of monomial ideals see [29, 58]. In this section, we define the edge ideal $I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$ and characterize its irredundant irreducible decomposition. In particular we prove that this decomposition is an irreducible primary decomposition, i.e, the radicals of the elements of the irredundant irreducible decomposition of $I(\mathcal{D})$ are different.

Definition 4.2.2. Let $\mathcal{D} = (V, E, w)$ be a weighted oriented graph with vertex set $V = \{x_1, \ldots, x_s\}$ and edge set E. The *edge ideal of* $\mathcal{D}$, denote by $I(\mathcal{D})$, is the ideal of $S = K[x_1, \ldots, x_s]$ generated by $\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E\}$.

Definition 4.2.3. A *source* of $\mathcal{D}$ is a vertex x, such that $N_{\mathcal{D}}(x) = N_D^+(x)$. A *sink* of $\mathcal{D}$ is a vertex y such that $N_{\mathcal{D}}(y) = N_{\mathcal{D}}^-(y)$.

Remark 4.2.4. Let $\mathcal{D} = (V, E, w)$ be a weighted oriented graph. We take $\mathcal{D}' = (V, E, w')$ a weighted oriented graph such that $w'(x) = w(x)$ if x is not a source and $w'(x) = 1$ if x is a source. Hence, $I(\mathcal{D}) = I(\mathcal{D}')$. For this reason in this chapter, we will always assume that if x is a source, then $w(x) = 1$.

Definition 4.2.5. Let C be a vertex cover of $\mathcal{D}$, the *irreducible ideal associated to C* is the ideal

$$I_C := \big(L_1(C) \cup \{x_j^{w(x_j)} \mid x_j \in L_2(C) \cup L_3(C)\}\big).$$

Lemma 4.2.6. *$I(\mathcal{D}) \subset I_C$ for each vertex cover C of $\mathcal{D}$.*

Proof. We take $I = I(\mathcal{D})$ and $m \in \mathcal{G}(I)$, then $m = xy^{w(y)}$, where $(x, y) \in \mathcal{D}$. Since C is a vertex cover, $x \in C$ or $y \in C$. If $y \in C$, then $y \in I_C$ or $y^{w(y)} \in I_C$. Thus, $m = xy^{w(y)} \in I_C$. Now, we assume $y \notin C$, then $x \in C$. Hence, $y \in N^+_{\mathcal{D}}(x) \cap C^c$, so $x \in L_1(C)$. Consequently, $x \in I_C$ implying $m = xy^{w(y)} \in I_C$. Therefore $I \subset I_C$. □

Definition 4.2.7. Let I be a monomial ideal. An irreducible monomial ideal $\mathfrak{q}$ that contains I is called a *minimal irreducible monomial ideal of I* if for any irreducible monomial ideal $\mathfrak{p}$ such that $I \subset \mathfrak{p} \subset \mathfrak{q}$ one has that $\mathfrak{p} = \mathfrak{q}$.

Proposition 4.2.8. *If $I = I_1 \cap \cdots \cap I_m$ is the irreducible decomposition of a monomial ideal I, then $I_1, \ldots, I_m$ are the minimal irreducible monomial ideals of I.*

Proof. Let L be an irreducible ideal that contains I. Then $I_i \subset L$ for some i. Indeed if $I_i \not\subset L$ for all i, for each i pick $x_{j_i}^{a_{j_i}} \in I_i \setminus L$. Since $I \subset L$, setting $x_a = \operatorname{lcm}\{x_{a_{j_i}}\}_{i=1}^m$ and writing $L = (x_{k_1}^{c_{k_1}}, \ldots, x_{k_\ell}^{c_{k_\ell}})$, it follows that x^a is in I and $x_{j_i}^{a_{j_i}}$ is a multiple of $x_{k_t}^{c_{k_t}}$ for some $1 \leq i \leq m$ and $1 \leq t \leq \ell$. Thus $x_{j_i}^{a_{j_i}}$ is in L, a contradiction. Therefore if L is minimal one has $L = I_i$ for some i. To complete the proof notice that I_i is a minimal irreducible monomial ideal of I for all i. This follows from the first part of the proof using that $I = I_1 \cap \cdots \cap I_m$ is an irredundant decomposition. □

Lemma 4.2.9. *Let $\mathcal{D}$ be a weighted oriented graph. If $I(\mathcal{D}) \subset (x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell})$, then $\{x_{i_1}, \ldots, x_{i_\ell}\}$ is a vertex cover of $\mathcal{D}$.*

Proof. We take $\mathfrak{q} = (x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell})$. If $(a, b) \in E(\mathcal{D})$, then $ab^{w(b)} \in I(\mathcal{D}) \subset \mathfrak{q}$. Thus, $x_{i_j}^{a_j} | ab^{w(b)}$ for some $1 \leq j \leq \ell$. Hence, $x_{i_j} \in \{a, b\}$ and $\{a, b\} \cap \{x_{i_1}, \ldots, x_{i_\ell}\} \neq \emptyset$. Therefore $\{x_{i_1}, \ldots, x_{i_\ell}\}$ is a vertex cover of $\mathcal{D}$. □

Lemma 4.2.10. *Let $\mathfrak{q}$ be a minimal irreducible monomial ideal of $I(\mathcal{D})$ where $\mathcal{G}(\mathfrak{q}) = \{x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell}\}$. If $a_j \neq 1$ for some $1 \leq j \leq \ell$, then there is $(x, x_{i_j}) \in E(\mathcal{D})$ where $x \notin \mathcal{G}(\mathfrak{q})$.*

Proof. By contradiction suppose there is $a_j \neq 1$ such that if $(x, x_{i_j}) \in E(\mathcal{D})$, then $x \in \mathcal{G}(\mathfrak{q})$. We take the ideal $\mathfrak{q}' = (\mathcal{G}(\mathfrak{q}) \setminus \{x_{i_j}^{a_j}\})$. If $(a, b) \in E(\mathcal{D})$, then $ab^{w(b)} \in I(\mathcal{D}) \subset \mathfrak{q}$. Consequently, $x_{i_k}^{a_k} | ab^{w(b)}$ for some $1 \leq k \leq \ell$. If $k \neq j$, then $ab^{w(b)} \in \mathfrak{q}'$. Now, if $k = j$, then by hypothesis $a_j \neq 1$. Hence, $x_{i_j}^{a_j} | b^{w(b)}$ implying $x_{i_j} = b$. Thus, $(a, x_{i_j}) \in E(\mathcal{D})$, so by hypothesis $a \in \mathcal{G}(\mathfrak{q}) \setminus \{x_{i_j}^{a_j}\}$. This implies $ab^{w(b)} \in \mathfrak{q}'$. Therefore $I(\mathcal{D}) \subset \mathfrak{q}' \subsetneq \mathfrak{q}$. A contradiction, since $\mathfrak{q}$ is minimal. □

Lemma 4.2.11. *Let $\mathfrak{q}$ be a minimal irreducible monomial ideal of $I(\mathcal{D})$ where $\mathcal{G}(\mathfrak{q}) = \{x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell}\}$. If $a_j \neq 1$ for some $1 \leq j \leq \ell$, then $a_j = w(x_{i_j})$.*

Proof. By Lemma 4.2.10, there is $(x, x_{i_j}) \in E(\mathcal{D})$ with $x \notin \mathcal{G}(\mathfrak{q})$. Also, $xx_{i_j}^{w(x_{i_j})} \in I(\mathcal{D}) \subset \mathfrak{q}$, so $x_{i_k}^{a_k} | xx_{i_j}^{w(x_{i_j})}$ for some $1 \leq k \leq \ell$. Hence, $x_{i_k}^{a_k} | x_{i_j}^{w(x_{i_j})}$, since $x \notin \mathcal{G}(\mathfrak{q})$. This implies, $k = j$ and $a_j \leq w(x_{i_j})$. If $a_j < w(x_{i_j})$, then we take $\mathfrak{q}' = (M')$ where $M' = \{\mathcal{G}(\mathfrak{q}) \setminus \{x_{i_j}^{a_j}\}\} \cup \{x_{i_j}^{w(x_{i_j})}\}$. So, $\mathfrak{q}' \subsetneq \mathfrak{q}$. Furthermore, if $(a, b) \in E(\mathcal{D})$, then $m = ab^{w(b)} \in I(\mathcal{D}) \subset \mathfrak{q}$. Thus, $x_{i_k}^{a_k} | ab^{w(b)}$ for some $1 \leq k \leq \ell$. If $k \neq j$, then $x_{i_k}^{a_k} \in M'$ implying $ab^{w(b)} \in \mathfrak{q}'$. Now, if $k = j$ then $x_{i_j}^{a_j} | b^{w(b)}$, since $a_j > 1$. Consequently, $x_{i_j} = b$ and $x_{i_j}^{w(x_{i_j})} | m$. Then $m \in \mathfrak{q}'$. Hence $I(\mathcal{D}) \subset \mathfrak{q}' \subsetneq \mathfrak{q}$, a contradiction since $\mathfrak{q}$ is minimal. Therefore $a_j = w(x_{i_j})$. □

We come to two of our main results.

Theorem 4.2.12. *The following conditions are equivalent:*

(1) *$\mathfrak{q}$ is a minimal irreducible monomial ideal of $I(\mathcal{D})$.*

(2) *There is a strong vertex cover C of $\mathcal{D}$ such that $\mathfrak{q} = I_C$.*

Proof. (2) $\Rightarrow$ (1) By definition $\mathfrak{q} = I_C$ is a monomial irreducible ideal. By Lemma 4.2.6, $I(\mathcal{D}) \subset \mathfrak{q}$. Now, suppose $I(\mathcal{D}) \subset \mathfrak{q}' \subset \mathfrak{q}$, where $\mathfrak{q}'$ is a monomial irreducible ideal. We can assume $\mathcal{G}(\mathfrak{q}') = \{x_{j_1}^{b_1}, \ldots, x_{j_\ell}^{b_\ell}\}$. If $x \in L_1(C)$, then there is $(x, y) \in E(\mathcal{D})$ with $y \notin C$. Hence, $xy^{w(y)} \in I(\mathcal{D})$ and $y^r \notin \mathfrak{q}$ for each $r \in \mathbb{N}$. Consequently $y^r \notin \mathfrak{q}'$ for each r, implying $y \notin \{x_{j_1}, \ldots, x_{j_\ell}\}$. Furthermore $x_{j_i}^{b_i} | xy^{w(y)}$ for some $1 \leq i \leq \ell$, since $xy^{w(y)} \in I(\mathcal{D}) \subset \mathfrak{q}'$. This implies, $x = x_{j_i}^{b_i} \in \mathfrak{q}'$. Now, if $x \in L_2(C)$, then there is $(y, x) \in E(\mathcal{D})$ with $y \notin C$. Thus $y \notin \mathfrak{q}$, so $y \notin \{x_{j_1}^{b_1}, \ldots, x_{j_\ell}^{b_\ell}\}$. Also, $x^{w(x)}y \in I(\mathcal{D}) \subset \mathfrak{q}'$, then $x_{j_i}^{b_i} | x^{w(x)}y$ for some $1 \leq i \leq \ell$. Consequently, $x_{j_i}^{b_i} | x^{w(x)}$ implies $x^{w(x)} \in \mathfrak{q}'$. Finally if $x \in L_3(C)$, then there is $(y, x) \in E(\mathcal{D})$ where $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$, since C is a strong vertex cover. So, $x^{w(x)}y \in I(\mathcal{D}) \subset \mathfrak{q}'$ implies $x_{j_i}^{b_i} | x^{w(x)}y$ for some i. Furthermore $y \notin \mathfrak{q} = I_C$, since $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$. This implies $y \notin \mathfrak{q}'$ so, $x_{j_i}^{b_i} | x^{w(x)}$ then $x^{w(x)} \in \mathfrak{q}'$. Hence, $\mathfrak{q} = I_C \subset \mathfrak{q}'$. Therefore, $\mathfrak{q}$ is a minimal monomial irreducible of $I(\mathcal{D})$.

(1) $\Rightarrow$ (2) Since $\mathfrak{q}$ is irreducible, we can suppose $\mathcal{G}(\mathfrak{q}) = \{x_{i_1}^{a_1}, \ldots, x_{i_\ell}^{a_\ell}\}$. By Lemma 4.2.11, we have $a_j = 1$ or $a_j = w(x_{i_j})$ for each $1 \leq j \leq \ell$. Also, by Lemma 4.2.9, $C = \{x_{i_1}, \ldots, x_{i_\ell}\}$ is a vertex cover of $\mathcal{D}$. We can assume $\mathcal{G}(I_C) = \{x_{i_1}^{b_1}, \ldots, x_{i_\ell}^{b_\ell}\}$, then $b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq \ell$. Now, suppose $b_k = 1$ and $w(x_{i_k}) \neq 1$ for some $1 \leq k \leq \ell$. Consequently $x_{i_k} \in L_1(C)$. Thus, there is $(x_{i_k}, y) \in E(\mathcal{D})$ where $y \notin C$. So, $x_{i_k}y^{w(y)} \in I(\mathcal{D}) \subset \mathfrak{q}$ and $x_{i_r}^{a_r} | x_{i_k}y^{w(y)}$ for some $1 \leq r \leq \ell$. Furthermore $y \notin C$, then $r = k$ and $a_k = a_r = 1$. Hence, $I_C \cap V(\mathcal{D}) \subset \mathfrak{q} \cap V(\mathcal{D})$. This implies, $I_C \subset \mathfrak{q}$, since $a_j, b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq \ell$. Therefore $\mathfrak{q} = I_C$, since $\mathfrak{q}$ is minimal. In particular $a_i = b_i$ for each $1 \leq i \leq \ell$.

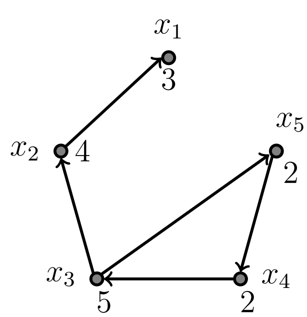
Now, assume C is not strong, then there is $x \in L_3(C)$ such that if $(y,x) \in E(\mathcal{D})$, then $w(y) = 1$ or $y \in L_1(C)$. We can assume $x = x_{i_1}$, and we take $\mathfrak{q}'$ the monomial ideal with $\mathcal{G}(\mathfrak{q}') = \{x_{i_2}^{a_2}, \ldots, x_{i_\ell}^{a_\ell}\}$. We take $(z_1, z_2) \in E(D)$. If $x_{i_j}^{a_j} | z_1 z_2^{w(z_2)}$ for some $2 \le j \le \ell$, then $z_1 z_2^{w(z_2)} \in \mathfrak{q}'$. Now, assume $x_{i_j}^{a_j} \nmid z_1 z_2^{w(z_2)}$ for each $2 \le j \le \ell$. Consequently $z_2 \notin \{x_{i_2}, \ldots, x_{i_\ell}\}$, since $a_j \in \{1, w(x_{i_j})\}$. Also $z_1 z_2^{w(z_2)} \in I(\mathcal{D}) \subset \mathfrak{q}$, then $x_{i_1}^{a_1} | z_1 z_2^{w(z_2)}$. But $x_{i_1} \in L_3(C)$, so $z_1, z_2 \in N_G[x_{i_1}] \subset C$. If $x_{i_1} = z_1$, then there is $2 \le r \le \ell$ such that $z_2 = x_{i_r}$. Thus $x_{i_r}^{a_r} \mid z_1 z_2^{w(z_2)}$. A contradiction, then $x_{i_1} = z_2$, $z_1 \in C$ and $(z_1, x_{i_1}) \in E(\mathcal{D})$. Then, $w(z_1) = 1$ or $z_1 \in L_1(C)$. In both cases $z_1 \in \mathcal{G}(I_C)$. Furthermore $z_1 \neq z_2$ since $(z_1, z_2) \in E(\mathcal{D})$. This implies $z_1 \in \mathcal{G}(\mathfrak{q}')$. So, $z_1 z_2^{w(z_2)} \in \mathfrak{q}'$. Hence, $I(\mathcal{D}) \subset \mathfrak{q}'$. This is a contradiction, since $\mathfrak{q}$ is minimal. Therefore C is strong. □

Theorem 4.2.13. *If $S(\mathcal{D})$ is the set of strong vertex covers of $\mathcal{D}$, then the irredundant irreducible decomposition of $I(\mathcal{D})$ is given by $I(\mathcal{D}) = \bigcap_{C \in S(\mathcal{D})} I_C$.*

Proof. By [29, Theorem 1.3.1], there is a unique irredundant irreducible decomposition $I(\mathcal{D}) = \bigcap_{i=1}^m I_i$. If there is an irreducible ideal I_j' such that $I(\mathcal{D}) \subset I_j' \subset I_j$ for some $j \in \{1, \ldots, m\}$, then $I(\mathcal{D}) = (\bigcap_{i \neq j} I_i) \cap I_j'$ is an irreducible decomposition. Furthermore this decomposition is irredundant. Thus, $I_j' = I_j$. Hence, $I_1, \ldots, I_m$ are minimal irreducible ideals of $I(\mathcal{D})$. Now, if there is $C \in S(\mathcal{D})$ such that $I_C \notin \{I_1, \ldots, I_m\}$, then there is $x_{j_i}^{\alpha_i} \in I_i \setminus I_C$ for each $i \in \{1, \ldots, m\}$. Consequently, $z = \operatorname{lcm}(x_{j_1}^{\alpha_1}, \ldots, x_{j_m}^{\alpha_m}) \in \bigcap_{i=1}^m I_i = I(\mathcal{D}) \subset I_C$. Furthermore, if $C = \{x_{i_1}, \ldots, x_{i_k}\}$, then $I_C = (x_{i_1}^{\beta_1}, \ldots, x_{i_k}^{\beta_k})$ where $\beta_j \in \{1, w(x_{i_j})\}$. Hence, there is $j \in \{1, \ldots, k\}$ such that $x_{i_j}^{\beta_j} | z$. So, there is $1 \le u \le m$ such that $x_{i_j}^{\beta_j} \mid x_{j_u}^{\alpha_u}$. A contradiction, since $x_{j_u}^{\alpha_u} \notin I_C$. Therefore $I(\mathcal{D}) = \bigcap_{C \in S(\mathcal{D})} I_C$ is the irredundant irreducible decomposition of $I(\mathcal{D})$. □

Remark 4.2.14. If $C_1, \ldots, C_t$ are the strong vertex covers of $\mathcal{D}$, then by Theorem 4.2.13, $I_{C_1} \cap \cdots \cap I_{C_t}$ is the irredundant irreducible decomposition of $I(\mathcal{D})$. Furthermore, if $\mathfrak{p}_i = \operatorname{rad}(I_{C_i})$, then $\mathfrak{p}_i = (C_i)$. So, $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $1 \le i < j \le t$. Thus, $I_{C_1} \cap \cdots \cap I_{C_t}$ is an irredundant primary decomposition of $I(\mathcal{D})$. In particular we have $\operatorname{Ass}(I(\mathcal{D})) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$.

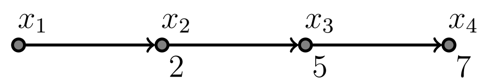
Example 4.2.15. Let $\mathcal{D}$ be the following weighted oriented graph



whose edge ideal is $I(\mathcal{D}) = (x_1^3 x_2, x_2^4 x_3, x_3^5 x_4, x_3 x_5^2, x_4^2 x_5)$. From Theorems 4.2.12 and 4.2.13, the irreducible decomposition of $I(\mathcal{D})$ is:

$$I(\mathcal{D}) = (x_1^3, x_3, x_4^2) \cap (x_1^3, x_3, x_5) \cap (x_2, x_3, x_4^2) \cap (x_2, x_3^5, x_5) \cap (x_2, x_4, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_5) \cap (x_1^3, x_2^4, x_4, x_5^2) \cap (x_2, x_3^5, x_4^2, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_4^2, x_5^2).$$

Example 4.2.16. Let $\mathcal{D}$ be the following weighted oriented graph



Hence, $I(\mathcal{D}) = (x_1x_2^2, x_2x_3^5, x_3x_4^7)$. By Theorems 4.2.12 and 4.2.13, the irreducible decomposition of $I(\mathcal{D})$ is:

$$I(\mathcal{D}) = (x_1, x_3) \cap (x_2^2, x_3) \cap (x_2, x_4^7) \cap (x_1, x_3^5, x_4^7) \cap (x_2^2, x_3^5, x_4^7).$$

In Examples 4.2.15 and 4.2.16, $I(\mathcal{D})$ has embedding primes. Furthermore the monomial ideal $(V(\mathcal{D}))$ is an associated prime of $I(\mathcal{D})$ in Example 4.2.15. Proposition 4.1.16 and Remark 4.2.14 give a combinatorial criterion for to decide when $(V(\mathcal{D})) \in \text{Ass}(I(\mathcal{D}))$.

4.3 Unmixed weighted oriented graphs

Let $\mathcal{D} = (V, E, w)$ be a weighted oriented graph whose underlying graph is G.In this section we characterize the unmixed property of $I(\mathcal{D})$ and we prove that this property is closed under c-minors. In particular if G is a bipartite graph, a whisker, or a cycle, we give an effective (combinatorial) characterization of this property.

The next theorem gives a combinatorial characterization for the unmixed property of weighted oriented graphs.

Theorem 4.3.1. *The following conditions are equivalent:*

(1) *$I(\mathcal{D})$ is unmixed.*

(2) *All strong vertex covers of $\mathcal{D}$ have the same cardinality.*

(3) *$I(G)$ is unmixed and $L_3(C) = \emptyset$ for each strong vertex cover C of $\mathcal{D}$.*

Proof. Let $C_1, \ldots, C_\ell$ be the strong vertex covers of $\mathcal{D}$. By Remark 4.2.14, the associated primes of $I(\mathcal{D})$ are $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$, where $\mathfrak{p}_i = \text{rad}(I_{C_i}) = (C_i)$ for $1 \leq i \leq \ell$.

(1) $\Rightarrow$ (2) Since $I(\mathcal{D})$ is unmixed, $|C_i| = \text{ht}(\mathfrak{p}_i) = \text{ht}(\mathfrak{p}_j) = |C_j|$ for $1 \leq i < j \leq \ell$.

(2) $\Rightarrow$ (3) If C is a minimal vertex cover, then by Corollary 4.1.10, $C \in \{C_1, \ldots, C_\ell\}$. By hypothesis, $|C_i| = |C_j|$ for each $1 \leq i \leq j \leq \ell$, then C_i is a minimal vertex cover of $\mathcal{D}$. Thus, by Lemma 4.1.7, $L_3(C_i) = \emptyset$. Furthermore $I(G)$ is unmixed, since $C_1, \ldots, C_\ell$ are the minimal vertex covers of G.

(3) $\Rightarrow$ (1) By Proposition 4.1.7, C_i is a minimal vertex cover, since $L_3(C_i) = \emptyset$ for each $1 \leq i \leq \ell$. This implies, $C_1, \ldots, C_\ell$ are the minimal vertex covers of G. Since G is unmixed, we have $|C_i| = |C_j|$ for $1 \leq i < j \leq \ell$. Therefore $I(\mathcal{D})$ is unmixed. □

Definition 4.3.2. A weighted oriented graph $\mathcal{D}$ has the *minimal-strong property* if each strong vertex cover is a minimal vertex cover.

Remark 4.3.3. Using Proposition 4.1.7, we have that $\mathcal{D}$ has the minimal-strong property if and only if $L_3(C) = \emptyset$ for each strong vertex cover C of $\mathcal{D}$.

Definition 4.3.4. $\mathcal{D}'$ is a *c-minor* of $\mathcal{D}$ if there is a stable set S of $\mathcal{D}$, such that $\mathcal{D}' = \mathcal{D} \setminus N_G[S]$.

Lemma 4.3.5. *If $\mathcal{D}$ has the minimal-strong property, then $\mathcal{D}' = \mathcal{D} \setminus N_G[x]$ has the minimal-strong property, for each $x \in V$.*

Proof. We take a strong vertex cover C' of $\mathcal{D}' = \mathcal{D} \setminus N_G[x]$ where $x \in V$. Thus, $C = C' \cup N_{\mathcal{D}}(x)$ is a vertex cover of $\mathcal{D}$. If $y' \in L_3(C')$, then by Proposition 4.1.6, $N_{\mathcal{D}'}(y') \subset C'$. Consequently, $N_{\mathcal{D}}(y') \subset C' \cup N_{\mathcal{D}}(x) = C$ implying $y' \in L_3(C)$. Hence, $L_3(C') \subset L_3(C)$. Now, we take $y \in L_3(C)$, then $N_{\mathcal{D}}(y) \subset C$. This implies $y \notin N_{\mathcal{D}}(x)$, since $x \notin C$. Then, $y \in C'$ and $N_{\mathcal{D}'}(y) \cup (N_{\mathcal{D}}(y) \cap N_{\mathcal{D}}(x)) = N_{\mathcal{D}}(y) \subset C = C' \cup N_{\mathcal{D}}(x)$. So, $N_{\mathcal{D}'}(y) \subset C'$ implies $y \in L_3(C')$. Therefore $L_3(C) = L_3(C')$.

Now, if $y \in L_3(C) = L_3(C')$, then there is $z \in C' \setminus L_1(C')$ with $w(z) \neq 1$, such that $(z, y) \in E(\mathcal{D}')$. If $z \in L_1(C)$, then there exist $z' \notin C$ such that $(z, z') \in E(\mathcal{D})$. Since $z' \notin C$, we have $z' \notin C'$, then $z \in L_1(C')$. A contradiction, consequently $z \notin L_1(C)$. Hence, C is strong. This implies $L_3(C) = \emptyset$, since $\mathcal{D}$ has the minimal-strong property. Thus, $L_3(C') = L_3(C) = \emptyset$. Therefore $\mathcal{D}'$ has the minimal-strong property. □

Proposition 4.3.6. *If $\mathcal{D}$ is unmixed and $x \in V$, then $\mathcal{D}' = \mathcal{D} \setminus N_G[x]$ is unmixed.*

Proof. By Theorem 4.3.1, G is unmixed and $\mathcal{D}$ has the minimal-strong property. Hence, by [58], $G' = G \setminus N_G[x]$ is unmixed. Also, by Lemma 4.3.5 we have that $\mathcal{D}'$ has the minimal-strong property. Therefore, by Theorem 4.3.1, $\mathcal{D}'$ is unmixed. □

Theorem 4.3.7. *If $\mathcal{D}$ is unmixed, then a c-minor of $\mathcal{D}$ is unmixed.*

Proof. If $\mathcal{D}'$ is a c-minor of $\mathcal{D}$, then there is a stable $S = \{a_1, \ldots, a_t\}$ such that $\mathcal{D}' = \mathcal{D} \setminus N_G[S]$. Since S is stable, $\mathcal{D}' = (\cdots((\mathcal{D} \setminus N_G[a_1]) \setminus N_G[a_2]) \setminus \cdots) \setminus N_G[a_t]$. Hence, by induction and Proposition 4.3.6, $\mathcal{D}'$ is unmixed. □

Proposition 4.3.8. *If $V(\mathcal{D})$ is a strong vertex cover of $\mathcal{D}$, then $I(\mathcal{D})$ is mixed.*

Proof. By Proposition 4.1.6 $V(\mathcal{D})$ is not minimal, since $L_3(V(\mathcal{D})) = V(\mathcal{D})$. Therefore, by Theorem 4.3.1, $I(\mathcal{D})$ is mixed. □

Remark 4.3.9. If $V = V^+$, then $I(\mathcal{D})$ is mixed.

Proof. If $x_i \in V$, then by Remark 4.2.4 $N^-_{\mathcal{D}}(x_i) \neq \emptyset$, since $V = V^+$. Thus, there is $x_j \in V$ such that $(x_j, x_i) \in E(\mathcal{D})$. Also, $w(x_j) \neq 1$ and $x_j \in V = L_3(V)$. So, V is a strong vertex cover. Hence, by Proposition 4.3.8, $I(\mathcal{D})$ is mixed. □

In the following three results we assume that $\mathcal{D}_1, \ldots, \mathcal{D}_r$ are the connected components of $\mathcal{D}$. Furthermore G_i is the underlying graph of $\mathcal{D}_i$.

Lemma 4.3.10. *Let C be a vertex cover of $\mathcal{D}$, then $L_1(C) = \bigcup_{i=1}^r L_1(C_i)$ and $L_3(C) = \bigcup_{i=1}^r L_3(C_i)$, where $C_i = C \cap V(\mathcal{D}_i)$.*

Proof. We take $x \in C$, then $x \in C_j$ for some $1 \leq j \leq r$. Thus, $N_{\mathcal{D}}(x) = N_{\mathcal{D}_j}(x)$. In particular $N^+_{\mathcal{D}}(x) = N^+_{\mathcal{D}_j}(x)$, so $C \cap N^+_{\mathcal{D}}(x) = C_j \cap N^+_{\mathcal{D}_j}(x)$. Hence, $L_1(C) = \bigcup_{i=1}^r L_1(C_i)$. On the other hand,

$$x \in L_3(C) \Leftrightarrow N_{\mathcal{D}}(x) \subset C \Leftrightarrow N_{\mathcal{D}_j}(x) \subset C_j \Leftrightarrow x \in L_3(C_j).$$

Therefore, $L_3(C) = \bigcup_{i=1}^r L_3(C_i)$. □

Lemma 4.3.11. *Let C be a vertex cover of $\mathcal{D}$, then C is strong if and only if each $C_i = C \cap V(\mathcal{D}_i)$ is strong with $i \in \{1, \ldots, r\}$.*

Proof. $\Rightarrow$) We take $x \in L_3(C_j)$. By Lemma 4.3.10, $x \in L_3(C)$ and there is $z \in N^-_{\mathcal{D}}(x) \cap V^+$ with $z \in C \setminus L_1(C)$, since C is strong. So, $z \in N^-_{\mathcal{D}_j}(x)$ and $z \in V(\mathcal{D}_j)$, since $x \in \mathcal{D}_j$. Consequently, by Lemma 4.3.10, $z \in C_j \setminus L_1(C_j)$. Therefore C_j is strong.

$\Leftarrow$) We take $x \in L_3(C)$, then $x \in C_i$ for some $1 \leq i \leq r$. Then, by Lemma 4.3.10, $x \in L_3(C_i)$. Thus, there is $a \in N^-_{\mathcal{D}_i}(x)$ such that $w(a) \neq 1$ and $a \in C_i \setminus L_1(C_i)$, since C_i is strong. Hence, by Lemma 4.3.10, $a \in C \setminus L_1(C)$. Therefore C is strong. □

Corollary 4.3.12. *$I(\mathcal{D})$ is unmixed if and only if $I(\mathcal{D}_i)$ is unmixed for each $1 \leq i \leq r$.*

Proof. $\Rightarrow$) By Theorem 4.3.7, since $\mathcal{D}_i$ is a c-minor of $\mathcal{D}$.

$\Leftarrow$) By Theorem 4.3.1, G_i is unmixed thus G is unmixed. Now, if C is a strong vertex cover, then by Lemma 4.3.10, $C_i = C \cap V(\mathcal{D}_i)$ is a strong vertex cover. Consequently, $L_3(C_i) = \emptyset$, since $I(\mathcal{D}_i)$ is unmixed. Hence, by Lemma 4.3.10, $L_3(C) = \bigcup_{i=1}^r L_3(C_i) = \emptyset$. Therefore, by Theorem 4.3.1, $I(\mathcal{D})$ is unmixed. □

Definition 4.3.13. Let G be a simple graph whose vertex set is $V(G) = \{x_1, \ldots, x_s\}$ and edge set $E(G)$. A *whisker* of G is a graph H whose vertex set is $V(H) = V(G) \cup \{y_1, \ldots, y_s\}$ and whose edge set is $E(H) = E(G) \cup \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\}$.

Definition 4.3.14. Let $\mathcal{D}$ and $\mathcal{H}$ be weighted oriented graphs. $\mathcal{H}$ is a *whisker weighted oriented graph* of $\mathcal{D}$ if $\mathcal{D} \subset \mathcal{H}$ and the underlying graph H of $\mathcal{H}$ is a whisker of the underlying graph G of $\mathcal{D}$.

In the following results we examine the unmixed property of the edge ideal of $\mathcal{D}$ if its underlying graph G is a whisker graph, bipartite graph or a cycle.

Theorem 4.3.15. *Let $\mathcal{H}$ be a whisker weighted oriented graph of $\mathcal{D}$, where $V(\mathcal{D}) = \{x_1, \ldots, x_s\}$ and $V(\mathcal{H}) = V(\mathcal{D}) \cup \{y_1, \ldots, y_s\}$, then the following conditions are equivalents:*

(1) $I(\mathcal{H})$ *is unmixed.*

(2) *If* $(x_i, y_i) \in E(\mathcal{H})$ *for some* $1 \leq i \leq s$*, then* $w(x_i) = 1$.

Proof. (2) $\Rightarrow$ (1) We take a strong vertex cover C of $\mathcal{H}$. Suppose $x_j, y_j \in C$, then $y_j \in L_3(C)$, since $N_{\mathcal{D}}(y_j) = \{x_j\} \subset C$. Consequently, $(x_j, y_j) \in E(\mathcal{D})$ and $w(x_j) \neq 1$, since C is strong. This is a contradiction by condition (2). This implies, $|C \cap \{x_i, y_i\}| = 1$ for each $1 \leq i \leq s$. So, $|C| = s$. Therefore, by Theorem 4.3.1, $I(\mathcal{H})$ is unmixed.

(1) $\Rightarrow$ (2) By contradiction suppose $(x_i, y_i) \in E(\mathcal{H})$ and $w(x_i) \neq 1$ for some i. Since $w(x_i) \neq 1$ and by Remark 4.2.4, we have that x_i is not a source. Thus, there is $x_j \in V(\mathcal{D})$, such that $(x_j, x_i) \in E(\mathcal{H})$. We take the vertex cover $C = \{V(\mathcal{D}) \setminus x_j\} \cup \{y_j, y_i\}$, then by Proposition 4.1.6, $L_3(C) = \{y_i\}$. Furthermore $N_{\mathcal{D}}(x_i) \setminus C = \{x_j\}$ and $(x_j, x_i) \in E(\mathcal{H})$, then $x_i \in L_2(C)$. Hence C is strong, since $L_3(C) = \{y_i\}$, $(x_i, y_i) \in E(\mathcal{D})$ and $w(x_i) \neq 1$. A contradiction by Theorem 4.3.1, since $I(\mathcal{H})$ is unmixed. □

Theorem 4.3.16. *Let* $\mathcal{D}$ *be a bipartite weighted oriented graph, then* $I(\mathcal{D})$ *is unmixed if and only if*

(1) G *has a perfect matching* $\{\{x_1^1, x_1^2\}, \ldots, \{x_t^1, x_t^2\}\}$ *where* $\{x_1^1, \ldots, x_t^1\}$ *and* $\{x_1^2, \ldots, x_t^2\}$ *are stable sets. Furthermore if* $\{x_j^1, x_i^2\}$, $\{x_i^1, x_k^2\} \in E(G)$ *then* $\{x_j^1, x_k^2\} \in E(G)$.

(2) *If* $w(x_j^k) \neq 1$ *and* $N_{\mathcal{D}}^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$ *where* $\{k, k'\} = \{1, 2\}$*, then* $N_{\mathcal{D}}(x_{i_\ell}^k) \subset N_{\mathcal{D}}^+(x_j^k)$ *and* $N_{\mathcal{D}}^-(x_{i_\ell}^k) \cap V^+ = \emptyset$ *for each* $1 \leq \ell \leq r$.

Proof. $\Leftarrow$) By (1) and [22, Theorem 2.5.7], G is unmixed. We take a strong vertex cover C of $\mathcal{D}$. Suppose $L_3(C) \neq \emptyset$, thus there exist $x_i^k \in L_3(C)$. Since C is strong, there is $x_j^{k'} \in V^+$ such that $(x_j^{k'}, x_i^k) \in E(\mathcal{D})$, $x_j^{k'} \in C \setminus L_1(C)$ and $\{k, k'\} = \{1, 2\}$. Furthermore $N_{\mathcal{D}}^+(x_j^{k'}) \subset C$, since $x_j^{k'} \notin L_1(C)$. Consequently, by (2), $N_{\mathcal{D}}(x_i^{k'}) \subset N_{\mathcal{D}}^+(x_j^{k'}) \subset C$. So, $x_i^{k'} \in L_3(C)$. A contradiction, since by (2) $N_{\mathcal{D}}^-(x_i^{k'}) \cap V^+ = \emptyset$ and C is strong. Hence $L_3(C) = \emptyset$ and $\mathcal{D}$ has the strong–minimal property. Therefore $I(\mathcal{D})$ is unmixed, by Theorem 4.3.1.

$\Rightarrow$) By Theorem 4.3.1, G is unmixed. Hence, by [22, Theorem 2.5.7], G satisfies (1).

If $w(x_j^k) \neq 1$, then we take $C = N_{\mathcal{D}}^+(x_j^k) \cup \{x_i^k \mid N_{\mathcal{D}}(x_i^k) \not\subset N_{\mathcal{D}}^+(x_j^k)\}$ and k' such that $\{k, k'\} = \{1, 2\}$. If $\{x_i^k, x_{i'}^{k'}\} \in E(G)$ and $x_i^k \notin C$, then $x_{i'}^{k'} \in N_{\mathcal{D}}(x_i^k) \subset N_{\mathcal{D}}^+(x_j^k) \subset C$. This implies, C is a vertex cover of $\mathcal{D}$. Now, if $x_{i_1}^k \in L_3(C)$, then $N_{\mathcal{D}}(x_{i_1}^k) \subset C$. Consequently $N_{\mathcal{D}}(x_{i_1}^k) \subset N_{\mathcal{D}}^+(x_j^k)$ implies $x_{i_1}^k \notin C$. A contradiction, then $L_3(C) \subset N_{\mathcal{D}}^+(x_j^k)$. Also, $N_G^-(x_j^k) \neq \emptyset$, since $w(x_j^k) \neq 1$. Thus $x_j^k \in L_2(C)$, since $N_G^-(x_j^k) \cap C = \emptyset$. Hence C is strong, since $L_3(C) \subset N_{\mathcal{D}}^+(x_j^k)$ and $x_j^k \in V^+$. Furthermore $\{x_1^{k'}, \ldots, x_t^{k'}\}$ is a minimal vertex cover, then by Theorem 4.3.1 $|C| = t$, since $\mathcal{D}$ is unmixed. We assume $N_{\mathcal{D}}^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$. Since C is minimal, $x_{i_\ell}^k \notin C$ for each $1 \leq \ell \leq r$. So, $N_{\mathcal{D}}(x_{i_\ell}^k) \subset N_{\mathcal{D}}^+(x_j^k)$. Now, suppose $z \in N_{\mathcal{D}}^-(x_{i_\ell}^k) \cap V^+$, then $z = x_{i_{\ell'}}^{k'}$ for some $1 \leq \ell' \leq r$, since $N_{\mathcal{D}}(x_{i_\ell}^k) \subset N_{\mathcal{D}}^+(x_j^k)$. We take $C' = N_{\mathcal{D}}^+(x_j^k) \cup \{x_i^k \mid i \notin \{i_1, \ldots, i_r\}\} \cup N_{\mathcal{D}}^+(x_{i_{\ell'}}^{k'})$. Since $N_{\mathcal{D}}(x_{i_u}^k) \subset N_{\mathcal{D}}^+(x_j^k)$ for each $1 \leq u \leq r$, we have that C' is a vertex cover. If $\{x_q^k, x_q^{k'}\} \cap L_3(C) \neq \emptyset$, then $\{x_q^k, x_q^{k'}\} \subset C'$, so $x_q^{k'} \in N^+_{\mathcal{D}}(x_j^k)$ implies $q \in \{i_1, \ldots, i_r\}$. Consequently, $x_q^k \in N^+_{\mathcal{D}}(x_{i_{\ell'}}^{k'})$, since $x_q^k \in C'$. This implies, $(x_j^k, x_q^{k'}), (x_{i_{\ell'}}^{k'}, x_q^k) \in E(\mathcal{D})$. Moreover, $N^+_{\mathcal{D}}(x_{i_{\ell'}}^{k'}) \cup N^+_{\mathcal{D}}(x_j^k) \subset C'$, then $x_{i_{\ell'}}^{k'} \notin L_1(C')$ and $x_j^k \notin L_1(C')$. Thus, C' is strong, since $x_j^k, x_{i_{\ell'}}^{k'} \in V^+$. Furthermore, by Theorem 4.3.1, $|C'| = t$. But $x_{i_\ell}^{k'} \in N^+_{\mathcal{D}}(x_j^k)$ and $x_{i_\ell}^k \in N^+_{\mathcal{D}}(x_{i_{\ell'}}^{k'})$, hence $x_{i_\ell}^{k'}, x_{i_\ell}^k \in C'$. A contradiction, so $N^-_{\mathcal{D}}(x_{i_\ell}^k) \cap V^+ = \emptyset$. Therefore $\mathcal{D}$ satisfies (2). □

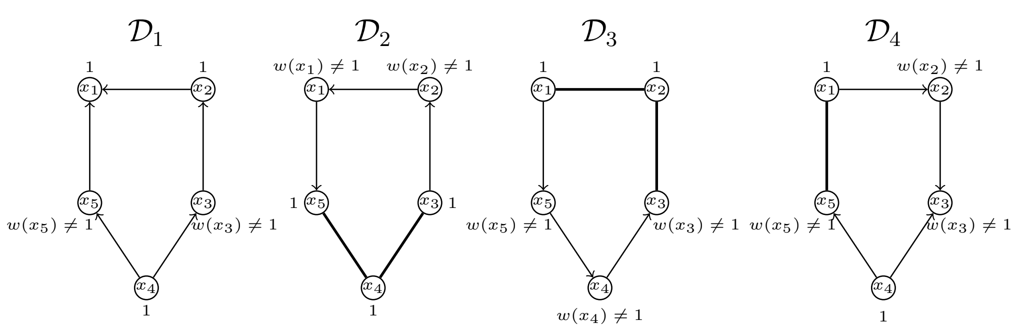
Lemma 4.3.17. *If the vertices of V^+ are sinks, then $\mathcal{D}$ has the minimal-strong property.*

Proof. We take a strong vertex cover C of $\mathcal{D}$. Hence, if $y \in L_3(C)$, then there is $(z, y) \in E(\mathcal{D})$ with $z \in V^+$. Consequently, by hypothesis, z is a sink. A contradiction, since $(z, y) \in E(\mathcal{D})$. Therefore, $L_3(C) = \emptyset$ and C is a minimal vertex cover. □

Lemma 4.3.18. *Let $\mathcal{D}$ be a weighted oriented graph, where $G \simeq C_n$ with $n \geq 6$. Hence, $\mathcal{D}$ has the minimal-strong property if and only if the vertices of V^+ are sinks.*

Proof. $\Leftarrow$) By Lemma 4.3.17.

$\Rightarrow$) By contradiction, suppose there is $(z, y) \in E(\mathcal{D})$, with $z \in V^+$. We can assume $G = (x_1, x_2, \ldots, x_n, x_1) \simeq C_n$, with $x_2 = y$ and $x_3 = z$. We take a strong vertex cover C in the following form: $C = \{x_1, x_3, \ldots, x_{n-1}\} \cup \{x_2\}$ if n is even or $C = \{x_1, x_3, \ldots, x_{n-2}\} \cup \{x_2, x_{n-1}\}$ if n is odd. Consequently, if $x \in C$ and $N_D(x) \subset C$, then $x = x_2$. Hence, $L_3(C) = \{x_2\}$. Furthermore $(x_3, x_2) \in E(\mathcal{D})$ with $x_3 \in V^+$. Thus, x_3 is not a source, so, $(x_4, x_3) \in E(\mathcal{D})$. Then, $x_3 \in L_2(C)$. This implies C is a strong vertex cover. But $L_3(C) \neq \emptyset$. A contradiction, since $\mathcal{D}$ has the minimal-strong property. □



Theorem 4.3.19. *If the underlying graph of a weighted oriented graph $\mathcal{D}$ is a cycle and w is the weight function of $\mathcal{D}$, then $I(\mathcal{D})$ is unmixed if and only if one of the following conditions hold:*

(1) *$n = 3$ and there is $x \in V(\mathcal{D})$ such that $w(x) = 1$.*

(2) *$n \in \{4, 5, 7\}$ and the vertices with weight greater than 1 are sinks.*

(3) *$n = 5$, there is $(x, y) \in E(\mathcal{D})$ with $w(x) = w(y) = 1$ and $\mathcal{D} \not\simeq \mathcal{D}_1, \mathcal{D} \not\simeq \mathcal{D}_2, D \not\simeq \mathcal{D}_3$.*

(4) $\mathcal{D} \simeq \mathcal{D}_4$.

Proof. $\Rightarrow$) By Theorem 4.3.1, $\mathcal{D}$ has the minimal-strong property and G is unmixed. Then, by [22, Exercise 2.4.22], $n \in \{3,4,5,7\}$. If $n = 3$, then by Remark 4.3.9, $\mathcal{D}$ satisfies (1). If $n = 7$, then by Lemma 4.3.18, $\mathcal{D}$ satisfies (2). Now suppose $n = 4$ and $\mathcal{D}$ does not satisfies (2), then we can assume $x_1 \in V^+$ and $(x_1, x_2) \in E(\mathcal{D})$. Consequently, $(x_4, x_1) \in E(G)$, since $w(x_1) \neq 1$. Furthermore, $\mathcal{C} = \{x_1, x_2, x_3\}$ is a vertex cover with $L_3(\mathcal{C}) = \{x_2\}$. Thus, $x_1 \in L_2(\mathcal{C})$ and $(x_1, x_2) \in E(\mathcal{D})$ so $\mathcal{C}$ is strong. A contradiction, since $\mathcal{C}$ is not minimal. This implies $\mathcal{D}$ satisfies (2). Finally suppose $n = 5$. If $\mathcal{D} \simeq \mathcal{D}_1$, then $\mathcal{C}_1 = \{x_1, x_2, x_3, x_5\}$ is a vertex cover with $L_3(\mathcal{C}_1) = \{x_1, x_2\}$. Also $(x_5, x_1), (x_3, x_2) \in E(\mathcal{D})$ with $x_5, x_3 \in V^+$. Consequently, $\mathcal{C}_1$ is strong, since $x_5, x_3 \in L_2(\mathcal{C}_1)$. A contradiction, since $\mathcal{C}_1$ is not minimal. If $\mathcal{D} \simeq \mathcal{D}_2$, then $\mathcal{C}_2 = \{x_1, x_2, x_4, x_5\}$ is a vertex cover where $L_3(\mathcal{C}_2) = \{x_1, x_5\}$ and $(x_2, x_1), (x_1, x_5) \in E(\mathcal{D})$ with $x_2, x_1 \in V^+$. Hence, $\mathcal{C}_2$ is strong, since $x_2, x_1 \notin L_1(\mathcal{C}_2)$. A contradiction, since $\mathcal{C}_2$ is not minimal. If $\mathcal{D} \simeq \mathcal{D}_3$, $\mathcal{C}_3 = \{x_2, x_3, x_4, x_5\}$ is a vertex cover where $L_3(\mathcal{C}_3) = \{x_3, x_4\}$ and $(x_4, x_3), (x_5, x_4) \in E(\mathcal{D})$ with $x_4, x_5 \in V^+$. Thus, $\mathcal{C}_3$ is strong, since $x_4, x_5 \notin L_1(\mathcal{C}_3)$. A contradiction, since $\mathcal{C}_3$ is not minimal. Now, since $n = 5$ and by (3) we can assume $(x_2, x_3) \in E(\mathcal{D})$, $x_2, x_3 \in V^+$ and there are not two adjacent vertices with weight 1. Since $x_2 \in V^+$, $(x_1, x_2) \in E(\mathcal{D})$. Suppose there are not 3 vertices z_1, z_2, z_3 in V^+ such that (z_1, z_2, z_3) is a path in G, then $w(x_4) = w(x_1) = 1$. Furthermore, $w(x_5) \neq 1$, since there are not adjacent vertices with weight 1. So, $\mathcal{C}_4 = \{x_2, x_3, x_4, x_5\}$ is a vertex cover of $\mathcal{D}$, where $L_3(\mathcal{C}_4) = \{x_3, x_4\}$. Also $(x_2, x_3) \in E(G)$ with $w(x_2) \neq 1$. Hence, if $(x_3, x_4) \in E(\mathcal{D})$ or $(x_5, x_4) \in E(\mathcal{D})$, then $\mathcal{C}_4$ is strong, since $x_3, x_5 \in V^+$. But $\mathcal{C}_4$ is not minimal. Consequently, $(x_4, x_3), (x_4, x_5) \in E(\mathcal{D})$ and $\mathcal{D} \simeq \mathcal{D}_4$. Now, we can assume there is a path (z_1, z_2, z_3) in $\mathcal{D}$ such that $z_1, z_2, z_3 \in V^+$. Since there are not adjacent vertices with weight 1, we can suppose there is $z_4 \in V^+$ such that $\mathcal{L} = (z_1, z_2, z_3, z_4)$ is a path. We take $\{z_5\} = V(\mathcal{D}) \setminus V((L))$ and we can assume $(z_2, z_3) \in E(\mathcal{D})$. This implies, (z_1, z_2), $(z_5, z_1) \in E(\mathcal{D})$, since $z_1, z_2 \in V^+$. Thus, $\mathcal{C}_5 = \{z_1, z_2, z_3, z_4\}$ is a vertex cover with $L_3(\mathcal{C}_5) = \{z_2, z_3\}$. Then $\mathcal{C}_5$ is strong, since $(z_1, z_2), (z_2, z_3) \in E(\mathcal{D})$ with $z_2 \in L_3(\mathcal{C}_5)$ and $z_1 \in L_2(\mathcal{C}_5)$. A contradiction, since $\mathcal{C}_5$ is not minimal.

$\Leftarrow$) If $n \in \{3,4,5,7\}$, then by [22, Exercise 2.4.22] G is unmixed. By Theorem 4.3.1, we will only prove that $\mathcal{D}$ has the minimal-strong property. If $\mathcal{D}$ satisfies (2), then by Lemma 4.3.17, $\mathcal{D}$ has the minimal-strong property. If $\mathcal{D}$ satisfies (1) and $\mathcal{C}$ is a strong vertex cover, then by Proposition 4.1.16, $|\mathcal{C}| \leq 2$. This implies $\mathcal{C}$ is minimal. Now, suppose $n = 5$ and $\mathcal{C}'$ is a strong vertex cover of $\mathcal{D}$ with $|\mathcal{C}'| \geq 4$. If $\mathcal{D} \simeq \mathcal{D}_4$, then $x_2, x_5 \notin L_3(\mathcal{C}')$, since $(N^-_{\mathcal{D}}(x_2) \cup N^-_{\mathcal{D}}(x_5)) \cap V^+ = \emptyset$. So $N_{\mathcal{D}}(x_2) \not\subset \mathcal{C}'$ and $N_{\mathcal{D}}(x_5) \not\subset \mathcal{C}'$. Consequently, $x_1 \notin \mathcal{C}'$ implies $\mathcal{C}' = \{x_2, x_3, x_4, x_5\}$. But $x_4 \in L_3(\mathcal{C}')$ and $N^-_{\mathcal{D}}(x_4) = \emptyset$. A contradiction, since $\mathcal{C}'$ is strong. Now assume $\mathcal{D}$ satisfies (3). Suppose there is a path $\mathcal{L} = (x_1, x_2, x_3)$ in G such that $w(x_1) = w(x_2) = w(x_3) = 1$. We can suppose $(x_4, x_5) \in E(\mathcal{D})$ where $V(\mathcal{D}) \setminus V(\mathcal{L}) = \{x_4, x_5\}$. Since $w(x_1) = w(x_3) = 1$, $x_2 \notin L_3(\mathcal{C}')$. If $x_2 \notin \mathcal{C}'$, then $\mathcal{C}' = \{x_1, x_3, x_4, x_5\}$ and $x_4 \in L_3(\mathcal{C}')$. But $N^-_{\mathcal{D}}(x_4) = \{x_3\}$ and $w(x_3) = 1$. A contradiction, hence $x_2 \in \mathcal{C}'$. We can assume $x_3 \notin \mathcal{C}'$, since $x_2 \notin L_3(\mathcal{C}')$. This implies $\mathcal{C}' = \{x_1, x_2, x_4, x_5\}$ and $L_3(\mathcal{C}') = \{x_1, x_5\}$. Thus, $(x_5, x_1) \in E(\mathcal{D})$, $x_5, x_4 \in V^+$. Consequently $(x_3, x_4) \in E(\mathcal{D})$, since $x_4 \in V^+$. A contradiction, since $\mathcal{D} \not\simeq \mathcal{D}_2$. Hence, there are not three consecutive vertices whose weights are 1. Consequently, since $\mathcal{D}$ satisfies (3), we can assume $w(x_1) = w(x_2) = 1$, $w(x_3) \neq 1$ and $w(x_5) \neq 1$. If $w(x_4) = 1$, then $x_3, x_5 \notin L_3(\mathcal{C}')$ since $N_{\mathcal{D}}(x_3, x_5) \cap V^+ = \emptyset$. This implies $N_{\mathcal{D}}(x_3) \not\subset \mathcal{C}'$ and $N_{\mathcal{D}}(x_5) \not\subset \mathcal{C}'$. Then, $x_4 \notin \mathcal{C}'$ and $\mathcal{C}' = \{x_1, x_2, x_3, x_5\}$. Thus, $(x_5, x_1), (x_3, x_2) \in E(\mathcal{D})$, since $L_3(\mathcal{C}') = \{x_1, x_2\}$. Consequently, $(x_4, x_5), (x_4, x_3) \in E(\mathcal{D})$, since $x_5, x_3 \in V^+$. A contradiction, since $\mathcal{D} \not\simeq \mathcal{D}_1$. So, $w(x_4) \neq 1$ and we can assume $(x_5, x_4) \in E(\mathcal{D})$, since $x_4 \in V^+$. Furthermore $(x_1, x_5) \in E(\mathcal{D})$, since $x_5 \in V^+$. Hence, $(x_3, x_4) \in E(\mathcal{D})$, since $\mathcal{D} \not\simeq \mathcal{D}_3$. Then $(x_2, x_3) \in E(\mathcal{D})$, since $x_3 \in V^+$. This implies $x_1, x_2, x_3, x_5 \notin L_3(\mathcal{C}')$, since $N^-_{\mathcal{D}}(x_i) \cap V^+ = \emptyset$ for $i \in \{1, 2, 3, 5\}$. A contradiction, since $|\mathcal{C}'| \geq 4$. Therefore $\mathcal{D}$ has the minimal-strong property. □

4.4 Cohen–Macaulay weighted oriented graphs

In this section we study the Cohen–Macaulayness of $I(\mathcal{D})$. In particular we give a combinatorial characterization of this property when the underlying graph G of $\mathcal{D}$ is a path or a complete graph. Furthermore, we show the Cohen–Macaulay property depends of the characteristic of K.

Definition 4.4.1. The weighted oriented graph $\mathcal{D}$ is *Cohen–Macaulay* over the field K if the ring $S/I(\mathcal{D})$ is Cohen–Macaulay.

Remark 4.4.2. If G is the underlying graph of $\mathcal{D}$, then $\mathrm{rad}(I(\mathcal{D})) = I(G)$.

Proposition 4.4.3. *If $I(\mathcal{D})$ is Cohen–Macaulay, then $I(G)$ is Cohen–Macaulay and $\mathcal{D}$ has the minimal-strong property.*

Proof. By Remark 4.4.2, $I(G) = \mathrm{rad}(I(\mathcal{D}))$, then by [30, Theorem 2.6], $I(G)$ is Cohen–Macaulay. Furthermore $I(\mathcal{D})$ is unmixed, since $I(\mathcal{D})$ is Cohen–Macaulay. Hence, by Theorem 4.3.1, $\mathcal{D}$ has the minimal-strong property. □

Example 4.4.4. In Examples 4.2.15 and 4.2.16 $I(\mathcal{D})$ is mixed. Hence, $I(\mathcal{D})$ is not Cohen–Macaulay, but $I(G)$ is Cohen–Macaulay.

Conjecture 4.4.5. $I(\mathcal{D})$ is Cohen–Macaulay if and only if $I(G)$ is Cohen–Macaulay and $\mathcal{D}$ has the minimal-strong property. Equivalently $I(\mathcal{D})$ is Cohen–Macaulay if and only if $I(\mathcal{D})$ is unmixed and $I(G)$ is Cohen–Macaulay.

Proposition 4.4.6. *Let $\mathcal{D}$ be a weighted oriented graph such that $V = \{x_1, \ldots, x_k\}$ and whose underlying graph is a path $G = (x_1, \ldots, x_k)$. Then the following conditions are equivalent:*

(1) *$S/I(\mathcal{D})$ is Cohen–Macaulay.*

(2) *$I(\mathcal{D})$ is unmixed.*

(3) *$k = 2$ or $k = 4$. In the second case, if $(x_2, x_1) \in E(\mathcal{D})$ or $(x_3, x_4) \in E(\mathcal{D})$, then $w(x_2) = 1$ or $w(x_3) = 1$ respectively.*

Proof. (1) ⇒ (2) By [22, Corollary 1.5.14].

(2) ⇒ (3) By Theorem 4.3.16, G has a perfect matching, since $\mathcal{D}$ is bipartite. Consequently k is even and $\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{k-1}, x_k\}$ is a perfect matching. If $k \geq 6$, then by Theorem 4.3.16, we have $\{x_2, x_5\} \in E(G)$, since $\{x_2, x_3\}$ and $\{x_4, x_5\} \in E(G)$. A contradiction since $\{x_2, x_5\} \notin E(G)$. Therefore $k \in \{2, 4\}$. Furthermore by Theorem 4.3.16, $w(x_2) = 1$ or $w(x_3) = 1$ when $(x_2, x_1) \in E(\mathcal{D})$ or $(x_3, x_4) \in E(\mathcal{D})$, respectively.

(3) ⇒ (1) We take $I = I(\mathcal{D})$. If $k = 2$, then we can assume $(x_1, x_2) \in E(\mathcal{D})$. So, $I = (x_1 x_2^{w(x_2)}) = (x_1) \cap (x_2^{w(x_2)})$. Thus, by Remark 4.2.14, $\mathrm{Ass}(I) = \{(x_1), (x_2)\}$. This implies, $\mathrm{ht}(I) = 1$ and $\dim(S/I) = k - 1 = 1$. Also, $\mathrm{depth}(S/I) \geq 1$, since $(x_1, x_2) \notin \mathrm{Ass}(I)$. Hence, S/I is Cohen–Macaulay. Now, if $k = 4$, then $\mathrm{ht}(I) = \mathrm{ht}(\mathrm{rad}(I)) = \mathrm{ht}(I(G)) = 2$. Consequently, $\dim(S/I) = k - 2 = 2$. Furthermore one of the following sets $\{x_2 - x_1^{w(x_1)}, x_3 - x_4^{w(x_4)}\}$, $\{x_2 - x_1^{w(x_1)}, x_4 - x_3^{w(x_3)}\}$, $\{x_1 - x_2^{w(x_2)}, x_4 - x_3^{w(x_3)}\}$ is a regular sequence of S/I, then $\mathrm{depth}(S/I) \geq 2$. Therefore, I is Cohen–Macaulay. □

Theorem 4.4.7. *If the underlying graph G of a weighted oriented graph $\mathcal{D}$ is a complete graph, then the following conditions are equivalent:*

(1) *$I(\mathcal{D})$ is unmixed.*

(2) *$I(\mathcal{D})$ is Cohen–Macaulay.*

(3) *There are not $\mathcal{D}_1, \ldots, \mathcal{D}_t$ unicycle oriented subgraphs of $\mathcal{D}$ such that $V(\mathcal{D}_1), \ldots, V(\mathcal{D}_t)$ is a partition of $V(\mathcal{D})$.*

Proof. We take $I = I(\mathcal{D})$. Since $I(G) = \mathrm{rad}(I)$ and G is complete, $\mathrm{ht}(I) = \mathrm{ht}(I(G)) = s - 1$.

(1) ⇒ (3) Since $\mathrm{ht}(I) = s - 1$ and I is unmixed, $(x_1, \ldots, x_s) \notin \mathrm{Ass}(I)$. Thus, by Remark 4.2.14, $V(\mathcal{D})$ is not a strong vertex cover of $\mathcal{D}$. Therefore, by Proposition 4.1.16, $\mathcal{D}$ satisfies (3).

(3) ⇒ (2) By Proposition 4.1.16, $V(\mathcal{D})$ is not a strong vertex cover of $\mathcal{D}$. Consequently, by Remark 4.2.14, $(x_1, \ldots, x_s) \notin \mathrm{Ass}(I)$. This implies, $\mathrm{depth}(S/I) \geq 1$. Furthermore, $\dim(S/I) = 1$, since $\mathrm{ht}(I) = s - 1$. Therefore I is Cohen–Macaulay.

(2) ⇒ (1) By [22, Corollary 1.5.14]. □

As an application of some results of this chapter, we have the following corollary.

Let $d_1, \ldots, d_s$ be a non-decreasing sequence of positive integers with $d_1 \geq 2$ and $s \geq 2$, and let L be the ideal of S generated by the set of all $x_i x_j^{d_j}$ such that $1 \leq i < j \leq s$.

Corollary 4.4.8. *The ideal L is Cohen–Macaulay of height $s-1$, has a unique irredundant primary decomposition given by*

$$L = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s,$$

where $\mathfrak{q}_i = (x_1, \ldots, x_{i-1}, x_{i+1}^{d_{i+1}}, \ldots, x_s^{d_s})$ *for* $1 \leq i \leq s$, *and* $\deg(S/L) = 1 + \sum_{i=2}^{s} d_i \cdots d_s$.

Proof. The ideal L corresponds to the edge ideal $I(\mathcal{D})$, where $\mathcal{D}$ is a complete weighted oriented graph whose vertex set is $V(\mathcal{D}) = \{x_1, \ldots, x_s\}$ and edge set is $E(\mathcal{D}) = \{(x_i, x_j) \mid 1 \leq i < j \leq s\}$ and $w(x_i) = d_i$ for $1 \leq i \leq s$.

Since $I(G) = \text{rad}(L)$ and G is complete, $\text{ht}(L) = \text{ht}(I(G)) = s - 1$. As $\mathcal{D}$ is complete, $C_i = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s\}$ is a minimal vertex cover of $\mathcal{D}$ for $1 \leq i \leq s$, by Proposition 4.1.7, $L_3(C_i) = \emptyset$ and by Corollary 4.1.10, C_i is strong for all $i = 1, \ldots, s$. Furthermore, $\mathcal{D}$ has not oriented cycles, since $i < j$ for each $(x_i, x_j) \in E(\mathcal{D})$. Consequently by Proposition 4.1.16, $V(\mathcal{D})$ is not strong, that is, $C_1, \ldots, C_s$ are all the strong vertex cover of $\mathcal{D}$. Therefore, by Theorem 4.3.1, $I(\mathcal{D})$ is unmixed and by Theorem 4.4.7, $I(\mathcal{D})$ is a Cohen–Macaulay ideal.

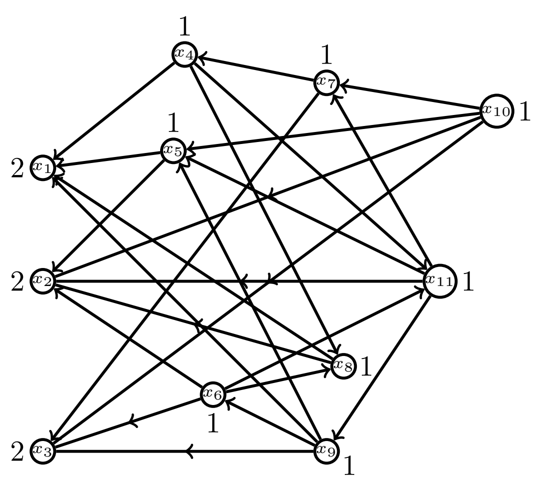
Given that $E(\mathcal{D}) = \{(x_i, x_j) \mid 1 \leq i < j \leq s\}$, then $L_1(C_i) = \{x_1, \ldots, x_{i-1}\}$, since $N^+_{\mathcal{D}}(x_i) = \{x_1, \ldots, x_{i-1}\}$ for all $i = 1, \ldots, s$. Thus, by Theorem 4.2.13 and Remark 4.2.14, we have that the irredundant primary decomposition of L is given by

$$L = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s,$$

where $\mathfrak{q}_i = I_{C_i} = (x_1, \ldots, x_{i-1}, x_{i+1}^{d_{i+1}}, \ldots, x_s^{d_s})$. Finally by additivity of the degree (Proposition 1.5.28), we get that, $\deg(S/L) = 1 + \sum_{i=2}^{s} d_i \cdots d_s$. □

If $\mathcal{D}$ is a complete weighted oriented graph or $\mathcal{D}$ is a weighted oriented path, then the Cohen–Macaulay property does not depends of the field K. This is not true in general, see the following example.

Example 4.4.9. Let $\mathcal{D}$ be the following weighted oriented graph:



Hence,

$$I(\mathcal{D}) = (x_1^2x_4, x_1^2x_8, x_1^2x_5, x_1^2x_9, x_2^2x_{10}, x_2^2x_5, x_2^2x_{11}, x_2^2x_8, x_2^2x_6, x_3^2x_7, x_3^2x_{10}, x_3^2x_6,$$
$$x_3^2x_9, x_4x_8, x_4x_7, x_4x_{11}, x_5x_{10}, x_5x_9, x_5x_{11}, x_6x_8, x_6x_9, x_6x_{11}, x_7x_{10}, x_7x_{11}, x_9x_{11}).$$

By [38, Example 2.3], $I(G)$ is Cohen–Macaulay when the characteristic of the field K is zero but it is not Cohen–Macaulay in characteristic 2. Consequently, $I(\mathcal{D})$ is not Cohen–Macaulay when the characteristic of K is 2. Also, $I(G)$ is unmixed. Furthermore, by Lemma 4.3.17, $I(\mathcal{D})$ has the minimal-strong property, then $I(\mathcal{D})$ is unmixed. Using *Macaulay2* [25] we show that $I(\mathcal{D})$ is Cohen–Macaulay when the characteristic of K is zero.

Conclusions

In this work, we studied two numerical functions, the minimum distance function δ_I and footprint function fp_I, associated to a graded ideal I of a polynomial ring over a field. The first interesting result about the minimum distance is Theorem 2.1.12, where one can find a description of the behavior of the minimum distance function for unmixed radical graded ideals whose associated primes are generated by linear forms. In particular, we show that this function is strictly decreasing until it stabilizes at the value 1. Concerning the footprint function, it is a lower bound for the minimum distance whenever I is unmixed (Theorem 2.3.2). We present some formulas for fp_I of certain ideals that correspond to ideals having a complete intersection initial ideal for some monomial order (Theorem 2.5.6, Theorem 2.5.9).

As an application of the previous results, we studied projective Reed–Muller-type codes, which are evaluation codes associated to a set of points $\mathbb{X}$ of the $(s-1)$-dimensional projective space over a finite field. Theorem 3.2.1, yields that the minimum distance and footprint functions of these codes coincide with the one of the vanishing ideal of $\mathbb{X}$.

Finally, we introduced the edge ideal $I(\mathcal{D})$ of a weighted oriented graph $\mathcal{D}$, and we described the algebraic properties of this monomial ideal in terms of the underlying weighted oriented graph. The main results were:

- All minimal vertex covers are strong.
- The irreducible monomial components of $I(D)$ are in one to one correspondence with the strong vertex covers: as a direct consequence of this, $I(D)$ is unmixed if and only if every strong vertex cover of the graph has the same cardinality.
- $I(D)$ is unmixed when the underlying graph of D is a whisker, bipartite and a cycle.
- $I(D)$ is Cohen–Macaulay when the underlying graph is a path or complete.

Future work: - Give the geometric interpretation of the minimum distance and footprint functions. - Use our theory to study other families of codes. - Study big families of weighted oriented graphs where the Conjecture 4.4.5 holds.

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