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Miguel Ángel Valencia Bucio

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Director de Tesis: Dr. José Guadalupe Martínez Bernal

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Resumen*

Esta tesis trata sobre algunas conexiones entre códigos lineales y demimatroides. Se ha escrito siguiendo un formato de tres artículos separados. Cada capítulo corresponde a un artículo, con su propio resumen, introducción y referencias. Los conceptos preliminares sobre códigos lineales y matroides necesarios para iniciar la lectura del trabajo, tales como pesos de Hamming generalizados y extendidos, identidad de MacWilliams, dualidad de Wei, demimatroides, gráficas signadas, matriz de incidencia de una gráfica signada, biparticidad por aristas, connectividad por aristas, resoluciones libres y números de Betti, se van dando conforme se van necesitando dentro de cada capítulo.

Sea G una gráfica conexa y sea $\mathbb{X}$ el conjunto de puntos proyectivos determinados por los vectores columna de la matriz de incidencia de G sobre un campo finito K de cualquier característica. En el capítulo 1 determinamos los pesos de Hamming generalizados de un código de tipo Reed-Muller, definido sobre el conjunto $\mathbb{X}$, en términos de invariantes de la gráfica. Como aplicación a la teoría de códigos, mostramos que si G es no-bipartita y K es un campo finito de característica distinta de 2, entonces el r-ésimo peso de Hamming generalizado del código lineal generado por los renglones de la matriz de incidencia de G es igual a la r-ésima biparticidad por aristas débil de G, i.e. el menor número de aristas cuya remoción resulta en una gráfica con r componentes bipartitas, y posiblemente algunas componentes no-bipartitas.

Por otro lado, si G es bipartita o la característica de K es 2, entonces probamos que el r-ésimo peso de Hamming generalizado del código es igual a la r-ésima conectividad por aristas de G, i.e. el menor número de aristas cuya remoción resulta en una gráfica con $r+1$ componentes conexas. Una vez obtenidos los pesos de Hamming generalizados

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del código, los de su código dual pueden obtenerse mediante la dualidad de Wei [30]. Ver Teoremas 1.2.8, 1.2.9, 1.2.10.

En el capítulo 2 damos fórmulas, en términos de invariantes de la gráfica, para los pesos de Hamming generalizados del código lineal generado por los renglones de la matriz de incidencia de una gráfica signada sobre un campo finito, y también para aquéllos de su código dual. Después, determinamos la regularidad (de Castelnuovo-Mumford) de los ideales de circuitos y cocircuitos de una gráfica signada, y obtenemos una fórmula algebraica, en términos de la multiplicidad, para el índice de frustación de una gráfica signada no-balanceada. Para esto usamos un resultado de Johnsen y Verdure [19] que da una interpretación de los pesos generalizados de Hamming como los menores corrimientos en la resolución libre minimal del ideal de circuitos del matroide asociado al código, y también utilizamos un resultado de Zaslavsky [33] que determina los circuitos y cocircuitos del matroide asociado a la gráfica signada. Ver Teoremas 2.3.16, 2.3.19, 2.4.7.

Basados en el trabajo de Britz, Johnsen, Mayhew and Shiromoto [1], en el capítulo 3 introducimos demimatroides como una generalización natural de matroides. Como dichos autores lo han mostrado, los demimatroides son los objetos combinatorios adecuados para estudiar la dualidad de Wei. Nuestros resultados aportan más evidencia que confirma la veracidad de esa observación.

Definimos el polinomio de Hamming de un demimatroide M, denotado por $W(x, y, t)$, como una generalización del enumerador de pesos de Hamming extendidos de un matroide. El polinomio $W(x, y, t)$ es una especialización del polinomio de Tutte de M, y de hecho es equivalente a él. Guiados por el trabajo de Johnsen, Rocksvold y Verdure para matroides [10], probamos que los números de Betti de un demimatroide y sus elongaciones determinan al polinomio de Hamming. Los resultados obtenidos en la tesis pueden aplicase a complejos simpliciales, ya que éstos, de una forma canónica, pueden verse como demimatroides.

Adicionalmente, siguiendo el trabajo de Brylawski y Gordon [4], mostramos cómo los demimatroides pueden generalizarse un paso más, a combinatroides. Un combinatroide, o estructura de Brylawski, es una función entero-valuada ρ, definida sobre el conjunto potencia de un conjunto finito, que satisface la única condición $\rho(\emptyset) = 0$. Aún en esta generalidad extrema, vemos cómo muchos conceptos e invariantes de teoría de códigos, pueden llevarse de manera directa a combinatroides, digamos, el polinomio de Tutte, el polinomio característico, la identidad de MacWilliams, el polinomio de Hamming extendido, y el r-ésimo polinomio de pesos de Hamming generalizados; este último, al menos conjeturalmente, guiados por el trabajo de Jurrius y Pellikaan para códigos lineales [11]. Todo ésto resulta en una vasta generalización de las nociones de borrado, contracción, dualidad y códigos a estructuras no-matroidales. Ver Teoremas 3.4.13, 3.5.9, 3.6.2, 3.9.4.

Los resultados de esta tesis han dado lugar a tres pre-artículos (con J. Martínez-Bernal y R.H. Villarreal): "Generalized Hamming weights of projective Reed-Muller-type codes over graphs", "Linear codes over signed graphs" y "Hamming polynomial of a demimatroid".

Summary

This thesis treats on some connections between linear codes and demimatroids. It is written following a format of three separated papers. Each chapter corresponds to a paper with its own abstract, introduction and references. The preliminary concepts on linear codes and matroids needed to start reading the work, such as generalized and extended Hamming weights, MacWilliams identity, Wei's duality, demimatroids, signed graph, incidence matrix of a signed graph, edge biparticity, edge connectivity, free resolutions and Betti numbers, are given as they are needed within each chapter.

Let G be a connected graph and let $\mathbb{X}$ be the set of projective points defined by the column vectors of the incidence matrix of G over a finite field K of any characteristic. In chapter 1 we determine the generalized Hamming weights of the Reed-Muller-type code over the set $\mathbb{X}$ in terms of graph theoretic invariants. As an application to coding theory, we show that if G is non-bipartite and K is a finite field of characteristic different from 2, then the r-th generalized Hamming weight of the linear code generated by the rows of the incidence matrix of G is the r-th weak edge biparticity of G, i.e. the minimum number of edges whose removal results in a graph with r bipartite components, and maybe some non-bipartite components. On the other hand, if G is bipartite or the characteristic of K is 2, then we prove that the r-th generalized Hamming weight of that code is the r-th edge connectivity of G, i.e. the minimum number of edges whose removal results in a graph with $r+1$ connected components. Once obtained the generalized Hamming weights, those of the dual code may be computed by using Wei's duality [30]. See Theorems 1.2.8, 1.2.9, 1.2.10.

In chapter 2 we give formulas, in terms of graph theoretical invariants, for the generalized Hamming weights of the linear code generated by the rows of the incidence matrix of a signed graph over a finite field, and for those of its dual code. Then we

determine the (Castelnuovo-Mumford) regularity of the ideals of circuits and cocircuits of a signed graph, and prove an algebraic formula in terms of the multiplicity, for the frustration index of an unbalanced signed graph. Here we have used a result of Johnsen and Verdure [19], which gives an interpretation of the generalized Hamming weights as the minor shifts in the minimal free resolution of the ideal of circuits of the matroid associated to the code, as well as a result of Zaslavsky [33], which determine the circuits and cocircuits of a vector matroid associated to a signed graph. See Theorems 2.3.16, 2.3.19, 2.4.7.

Following Britz, Johnsen, Mayhew and Shiromoto [1], in chapter 3 we introduce demimatroids as a natural generalization of matroids. As these authors have shown, demimatroids are the appropriate combinatorial objects for studying Wei's duality. Our results here apport further evidence about the trueness of that observation. We define the Hamming polynomial of a demimatroid M, denoted by $W(x, y, t)$, as a generalization of the extended Hamming weight enumerator of a matroid. The polynomial $W(x, y, t)$ is a specialization of the Tutte polynomial of M, and actually is equivalent to it. Guided by work of Johnsen, Roksvold and Verdure for matroids [10], we prove that Betti numbers of a demimatroid and its elongations determine the Hamming polynomial. Our results may be applied to simplicial complexes since in a canonical way they can be viewed as demimatroids. Furthermore, following work of Brylawski and Gordon [4], we show how demimatroids may be generalized one step further, to combinatroids. A combinatroid, or Brylawski structure, is an integer-valued function ρ, defined over the power set of a finite ground set, satisfying the only condition $\rho(\emptyset) = 0$. Even in this extreme generality, we see how many concepts and invariants in coding theory can be carried on directly to combinatroids, say, Tutte polynomial, characteristic polynomial, MacWilliams identity, extended Hamming polynomial, and the r-th generalized Hamming polynomial; this last one, at least conjecturelly, guided by the work of Jurrius and Pellikaan for linear codes [11]. All this largely extends the notions of deletion, contraction, duality and codes to non-matroidal structures. See Theorems 3.4.13, 3.5.9, 3.6.2, 3.9.4.

The results of this thesis have led to three preprints (with J. Martínez-Bernal and R.H. Villarreal): "Generalized Hamming weights of projective Reed-Muller-type codes over graphs”, “Linear codes over signed graphs” and “Hamming polynomial of a demi-matroid”.

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Chapter 1

Reed-Muller-type codes over graphs

Abstract. Let G be a connected graph and let $\mathbb{X}$ be the set of projective points defined by the column vectors of the incidence matrix of G over a field K of any characteristic. We determine the generalized Hamming weights of the Reed-Muller-type code over the set $\mathbb{X}$ in terms of graph theoretic invariants. As an application to coding theory we show that if G is non-bipartite and K is a finite field of characteristic different from 2, then the r-th generalized Hamming weight of the linear code generated by the rows of the incidence matrix of G is the r-th weak edge biparticity of G. If the characteristic of K is 2 or G is bipartite, then we prove that the r-th generalized Hamming weight of that code is the r-th edge connectivity of G.

1.1 Introduction

In this work we study basic parameters of Reed-Muller-type codes on graphs using a geometric approach via graph theory and commutative algebra, and show some applications to linear codes whose generator matrices are incidence matrices of graphs.

Let K be a field of characteristic $p \geq 0$ and let G be a connected graph with vertex set $V(G) = \{t_1, \ldots, t_s\}$ and edge set $E(G) = \{f_1, \ldots, f_m\}$. The *incidence matrix* of G, over the field K, is the $s \times m$ matrix $A = (a_{ij})$ given by $a_{ij} = 1$ if $t_i \in f_j$ and $a_{ij} = 0$ otherwise. The *edge biparticity* of G, denoted $\varphi(G)$, is the minimum number of edges whose removal makes the graph bipartite. The r-th *weak edge biparticity* of G, denoted $\upsilon_r(G)$, is the minimum number of edges whose removal results in a graph with r bipartite components, and maybe some non-bipartite components. If $r = 1$, $\upsilon_1(G)$ is the *weak edge biparticity* of G, and is denoted by $\upsilon(G)$.

The r-th *edge connectivity* of G, denoted $\lambda_r(G)$, is the minimum number of edges whose removal results in a graph with $r+1$ connected components. If $r=1$, $\lambda_1(G)$ is the edge connectivity of G and is denoted by $\lambda(G)$. We will use these invariants to study the minimum distance and the Hamming weights of Reed-Muller-type codes on graphs. The edge biparticity and the edge connectivity are well studied invariants of a graph [16, 26, 33].

The set of columns $\{P_1,\ldots,P_m\}$ of the matrix A can be regarded as a set of points $\mathbb{X}=\{[P_1],\ldots,[P_m]\}$ in a projective space $\mathbb{P}^{s-1}$ over the field K. Consider a polynomial ring $S=K[t_1,\ldots,t_s]=\bigoplus_{d=0}^{\infty}S_d$ over the field K provided with its standard grading. The *vanishing ideal* $I(\mathbb{X})$ of $\mathbb{X}$ is the graded ideal of S generated by the homogeneous polynomials of S that vanish at all points of $\mathbb{X}$. Fix integers $d\geq 1$ and $r\geq 1$. The aim of this chapter is to determine the following number, in terms of the combinatorics of the graph G:

$$\delta_{\mathbb{X}}(d,r):=\min\{|\mathbb{X}\setminus V_{\mathbb{X}}(F)|:\ F=\{f_i\}_{i=1}^{r}\subset S_d,\ \dim_K(\{\overline{f}_i\}_{i=1}^{r})=r\},$$

where $V_{\mathbb{X}}(F)$ is the set of zeros or projective variety of F in $\mathbb{X}$, and $\overline{f}_i=f_i+I(\mathbb{X})$ is the class of f_i modulo $I(\mathbb{X})$. This is equivalent to determine:

$$\text{hyp}_{\mathbb{X}}(d,r):=\max\{|V_{\mathbb{X}}(F)|:\ F=\{f_i\}_{i=1}^{r}\subset S_d,\ \dim_K(\{\overline{f}_i\}_{i=1}^{r})=r\},$$

because $\delta_{\mathbb{X}}(d,r)=|\mathbb{X}|-\text{hyp}_{\mathbb{X}}(d,r)$.

A *projective Reed-Muller-type code* of degree d on $\mathbb{X}$ [8, 13], denoted $C_{\mathbb{X}}(d)$, is the image of the following evaluation linear map

$$\text{ev}_d\colon S_d\to K^m,\quad f\mapsto (f(P_1),\ldots,f(P_m)).$$

The motivation to study $\delta_{\mathbb{X}}(d,r)$ comes from algebraic coding theory because, over a finite field, the r-th generalized Hamming weight of the Reed-Muller-type code $C_{\mathbb{X}}(d)$ of degree d is equal to $\delta_{\mathbb{X}}(d,r)$ [11, Lemma 4.3(iii)].

Generalized Hamming weights were introduced by Wei [18, 22, 30]. For convenience we recall this notion. Let $K=\mathbb{F}_q$ be a finite field and let C be a *linear* $[m,k]$ *code* of *length* m and *dimension* k, that is, C is a linear subspace of K^m with $k=\dim_K(C)$.

Let $1\leq r\leq k$ be an integer. Given a linear subspace D of C, the *support* of D is the set

$$\mathrm{supp}(D):=\{i\,|\,\exists\,(a_1,\ldots,a_m)\in D,\, a_i\neq 0\}.$$

The r-th *generalized Hamming weight* of C, denoted $\delta_r(C)$, is given by

$$\delta_r(C):=\min\{|\mathrm{supp}(D)|\colon D \text{ is a subspace of } C \text{ with } \dim_K(D)=r\}.$$

The set $\{\delta_1(C),\ldots,\delta_k(C)\}$ is called the *weight hierarchy* of the code C. Define the *dual code* of C, denoted by $C^{\perp}$, as the vector space consisting of all those words in $\mathbb{F}_q^n$ orthogonal to C, with respect to the usual inner product. The following *duality* of Wei [30, Theorem 3] is a classical result in this area that shows a strong relationship between the weight hierarchies of C and its dual $C^{\perp}$:

$$\{\delta_i(C)\,|\,i=1,\ldots,k\}=\{1,\ldots,m\}\setminus\{m+1-\delta_i(C^{\perp})\,|\,i=1,\ldots,m-k\}.$$

These numbers are a natural generalization of the notion of minimum distance and they have several applications from cryptography (codes for wire-tap channels of type II), t-resilient functions, trellis or branch complexity of linear codes, and shortening or puncturing structure of codes; see [1, 3, 4, 6, 9, 11, 12, 17, 20, 25, 28, 29, 30, 31, 32] and the references therein. If $r=1$, we obtain the *minimum distance* $\delta(C)$ of C, which is among the most important parameters of a linear code [24]. In this chapter we give combinatorial formulas for the weight hierarchy of $C_{\mathbb{X}}(d)$ for $d\geq 1$.

Our main results are:

Theorems 1.2.8, 1.2.9, 1.2.10 *Let G be a connected graph with s vertices, m edges, r-th weak edge biparticity $\upsilon_r(G)$, r-th edge connectivity $\lambda_r(G)$; and let A be the incidence matrix of G over a field K of $\mathrm{char}(K)=p$. If $\mathbb{X}$ is the set of column vectors of A, then*

$$\delta_{\mathbb{X}}(d,r)=\delta_r(C_{\mathbb{X}}(d))=\begin{cases}\upsilon_r(G), & \text{if } d=1,\ p\neq 2,\ G \text{ is non-bipartite}, 1\leq r\leq s,\\ \lambda_r(G), & \text{if } d=1,\ p=2,\ 1\leq r\leq s-1,\\ \lambda_r(G), & \text{if } d=1,\ G \text{ is bipartite}, 1\leq r\leq s-1,\\ r, & \text{if } d\geq 2 \text{ and } 1\leq r\leq m.\end{cases}$$

Thus computing $\upsilon_r(G)$ and $\lambda_r(G)$ is equivalent to computing the r-th generalized Hamming weight of $C_{\mathbb{X}}(1)$ for $K=\mathbb{F}_2$ or $K=\mathbb{F}_3$. These are the only cases that matter.

The *incidence matrix code* of a graph G over a finite field K of characteristic p, denoted $C_p(G)$, is the linear code generated by the rows of the incidence matrix of G. As an application to coding theory we obtain the following combinatorial formulas for the generalized Hamming weights of $C_p(G)$ when G is connected (Corollary 1.2.11).

$$\delta_r(C_p(G)) = \begin{cases} \upsilon_r(G), & \text{if } p \neq 2,\ G \textit{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } p = 2,\ 1 \leq r \leq s-1, \\ \lambda_r(G), & \text{if } G \textit{ is bipartite}, 1 \leq r \leq s-1. \end{cases}$$

The *minimum distance* of the incidence matrix code of the graph G is defined as

$$\delta(C_p(G)) := \min\{\omega(\alpha)\colon \alpha \in C_p(G) \setminus \{0\}\},$$

where $\omega(\alpha)$ is the Hamming weight of the vector α, that is, the number of nonzero entries of α. The minimum distance of $C_p(G)$ is $\delta_1(C_p(G))$, i.e. the 1st Hamming weight of this code. Then we can recover the combinatorial formulas of Dankelmann, Key and Rodrigues [5, Theorems 1-3] for the minimum distance of $C_p(G)$ in terms of the weak edge biparticity $\upsilon(G)$ and the edge connectivity $\lambda(G)$ of G (Corollary 1.2.12).

In Section 1.2 we address the problem of computing the edge biparticity $\varphi(G)$ of a graph G. One has the following relationships [7, 16]:

$$\kappa(G) \leq \lambda(G) \leq \Delta(G) \text{ and } \max\{\upsilon(G),\, \lambda(G)\} \leq \varphi(G),$$

where $\kappa(G)$ is the vertex connectivity of G and $\Delta(G)$ is the minimum degree of the vertices of G.

Using Macaulay 2 [14], SageMath [27], and Wei's duality [30, Theorem 3], we can compute the weight hierarchy of $C_p(G)$. In Sections 1.3 and 1.4, we illustrate this with some examples and procedures.

1.2 Reed-Muller-type codes over graphs

In this section we present our main results. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.1.

Lemma 1.2.1. *Let G be a graph and let $e_1, \ldots, e_r$ be a minimum set of edges whose removal makes the graph bipartite. Then there is $\omega\colon V(G) \to \{+,-\}$ such that the edges of G whose vertices have the same sign, positive or negative, are precisely $e_1, \ldots, e_r$.*

Proof. If G is bipartite, there is nothing to prove. If G is not bipartite, pick a bipartition V_1, V_2 of the graph $G\setminus\{e_1,\ldots,e_r\}$. Setting $\omega(v)=+$ if $v\in V_1$ and $\omega(v)=-$ if $v\in V_2$, note that the vertices of each e_i have the same sign. Indeed if the vertices of e_i have different sign, then $G\setminus\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_r\}$ is bipartite, a contradiction. $\square$

The edge biparticity of a graph $\varphi(G)$ of a graph G can be easily expressed by considering all possible ways of making G a vertex-signed graph.

Proposition 1.2.2. *Let G be a graph, let $\mathcal{F}$ be the set of surjective functions $\omega\colon V(G)\to\{+,-\}$, and let E_ω be the set of edges of G whose vertices have the same sign. Then*

$$\varphi(G)=\min\{|E_\omega|:\omega\in\mathcal{F}\}.$$

Proof. If G is bipartite, $\varphi(G)=0$ and there is nothing to prove. Assume that G is not bipartite. Then $E_\omega\neq\emptyset$ for $\omega\in\mathcal{F}$. By Lemma 1.2.1, one has the inequality "$\geq$". To show the reverse inequality take ω in $\mathcal{F}$. It suffices to show that $\varphi(G)\leq|E_\omega|$. The vertex set of G can be partitioned as $V(G)=V^+\cup V^-$, where V^+ (resp. V^-) is the set of vertices of G with positive (resp. negative) sign. Then $G\setminus E_\omega$ is bipartite with bipartition V^+, V^-. Thus $\varphi(G)\leq|E_\omega|$. $\square$

The following result is well known.

Proposition 1.2.3. [2, 15, 21] *Let G be a connected graph with s vertices and let A be its incidence matrix over a field K. Then*

$$\operatorname{rank}(A)=\begin{cases} s, & \text{if } \operatorname{char}(K)\neq 2 \text{ and } G \text{ is non bipartite},\\ s-1, & \text{if } \operatorname{char}(K)=2 \text{ or } G \text{ is bipartite}.\end{cases}$$

Corollary 1.2.4. *Let G be a connected graph with s vertices and m edges and let $C=C_p(G)$ (resp. $C^\perp$) be the code (resp. dual code) of G. Then*

(a) C (resp. $C^\perp$) is an $[m,s]$ (resp. $[m,m-s]$) code if $p\neq 2$ and G is non bipartite.

(b) C (resp. $C^\perp$) is an $[m,s-1]$ (resp. $[m,m-s+1]$) code if $p=2$ or G is bipartite.

Lemma 1.2.5. *Let G be a connected graph and let K be a field. The following hold.*

(a) If $\operatorname{char}(K)\neq 2$, G is non-bipartite and h is a linear form in $I(\mathbb{X})$, then $h=0$.

(b) If $\operatorname{char}(K) = 2$ *and* $h \neq 0$ *is a linear form in* $I(\mathbb{X})$*, then* $h = c\sum_{i=1}^{s} t_i$*, for some* $c \in K$*.*

(c) If $\operatorname{char}(K) = 2$ *and* h *is a linear form in* $I(\mathbb{X})$*, in* $s-1$ *variables, then* $h = 0$*.*

Proof. Let ψ be the linear map $\psi : K^s \to K^m$, $x \mapsto xA$. Fix a linear form $h = \sum_{i=1}^{s} a_i t_i$ of S_1 and set $v_h = (a_1, \ldots, a_s)$. Then v_h is in $\ker(\psi)$ if and only if $h \in I(\mathbb{X})$. For use below notice that $s = \dim(\ker(\psi)) + \operatorname{rank}(A)$.

(a): By Proposition 1.2.3, $\ker(\psi) = 0$. Then $v_h = 0$, that is, $h = 0$.

(b): From Proposition 1.2.3 we get that $\ker(\psi)$ has dimension 1, and since $\mathbf{1} = (1, \ldots, 1) \in \ker(\psi)$ the result follows.

(c): It is a consequence of (b). $\square$

Lemma 1.2.6. *Let* G *be a connected bipartite graph with bipartition* V_1*,* V_2*. The following hold.*

(a) If K *is a field and* $h \neq 0$ *is a linear form of* S *that vanishes at all points of* $\mathbb{X}$*, then* $h = c(\sum_{t_i \in V_1} t_i - \sum_{t_i \in V_2} t_i)$ *for some* $c \in K$*.*

(b) If t_i *and* t_j *are in* V_1*, then* $G \cup \{t_i, t_j\}$ *contains an odd cycle.*

Lemma 1.2.7. *Let* G *be a connected non-bipartite graph. If* $\ell = \upsilon_r(G)$ *and* $f_1, \ldots, f_\ell$ *are edges of* G*, then the graph* $H = G \setminus \{f_1, \ldots, f_\ell\}$ *has at most* r *bipartite connected components.*

Proof. Let $H_1, \ldots, H_n$ be the connected components of H. We proceed by contradiction assuming that $H_1, \ldots, H_{r+1}$ are bipartite. Consider the graph $G' = G \setminus \{f_2, \ldots, f_\ell\}$. If $f_1 \subset H_i$ for some i, then G' has r bipartite components, a contradiction. Thus $f_1 \not\subset H_i$ for $i = 1, \ldots, n$. Hence, f_1 joins H_i and H_j for some i, j with $i < j$. If $j \leq r+1$, the graph $H_i \cup H_j \cup \{f_1\}$ is bipartite and connected, and G' has r bipartite components, a contradiction. Thus $j > r+1$ and in this case G' has r bipartite components, a contradiction. $\square$

We come to one of our main results.

Theorem 1.2.8. *Let G be a connected non-bipartite graph with s vertices and m edges, let K be a field of* $\mathrm{char}(K) \neq 2$*, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column vectors of A and $\upsilon_r(G)$ is the r-th weak edge biparticity of G, then*

$$\delta_{\mathbb{X}}(d,r) = \begin{cases} \upsilon_r(G), & \text{if } d = 1 \text{ and } 1 \leq r \leq s = \dim_K(C_{\mathbb{X}}(d)), \\ r, & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_{\mathbb{X}}(d)). \end{cases}$$

Proof. Assume $d = 1$. First we show the inequality $\delta_{\mathbb{X}}(1,r) \geq \upsilon_r(G)$. We proceed by contradiction assuming that $\upsilon_r(G) > \delta_{\mathbb{X}}(1,r)$. Then $\upsilon_r(G) > |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$ for some set F consisting of r linear forms $h_1, \ldots, h_r$ which are linearly independent modulo $I(\mathbb{X})$. Let $[P_1], \ldots, [P_\ell]$ be the points in $\mathbb{X} \setminus V_{\mathbb{X}}(F)$ and let $f_1, \ldots, f_\ell$ be the edges of G corresponding to these points. Consider the graph $H = G \setminus \{f_1, \ldots, f_\ell\}$. Let $H_1, \ldots, H_n$ be the bipartite connected components of H. Since $\upsilon_r(G) > \ell$, n is at most $r-1$. Let $\mathbb{X}_H$ be the set of points corresponding to the columns of the incidence matrix of H. Note that h_i vanishes at all points of $\mathbb{X}_H$ for $i = 1, \ldots, r$. Then, by Lemma 1.2.5, $h_1, \ldots, h_r$ are linear forms in the variables $V(H_1) \cup \cdots \cup V(H_n)$. For each $1 \leq j \leq n$, let A_1^j, A_2^j be the bipartition of H_j and set $g_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i$. Then, by Lemma 1.2.6, $F = \{h_1, \ldots, h_r\}$ is in the K-linear space generated by $g_1, \ldots, g_n$, a contradiction because F is linearly independent over K and $n < r$.

Now we show the inequality $\delta_{\mathbb{X}}(1,r) \leq \upsilon_r(G)$. Note that by Lemma 1.2.5, it suffices to find a set $F = \{h_1, \ldots, h_r\}$ of linearly independent forms of degree 1 such that $\upsilon_r(G) = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. We set $\ell = \upsilon_r(G)$. There are edges $f_1, \ldots, f_\ell$ of G such that the graph

$$H = G \setminus \{f_1, \ldots, f_\ell\}$$

has exactly r connected bipartite components (see Lemma 1.2.7). We denote the connected components of H by $H_1, \ldots, H_n$, where $H_1, \ldots, H_r$ are bipartite. Consider a bipartition A_1^j, A_2^j of H_j for $j = 1, \ldots, r$ and set

$$h_j = \sum_{t_i \in A_1^j} t_i - \sum_{t_i \in A_2^j} t_i.$$

Let P_i be the point in $\mathbb{P}^{s-1}$ that corresponds to f_i for $i = 1, \ldots, \ell$. To complete the proof of the case $d = 1$ we need only show the equality $\{[P_1], \ldots, [P_\ell]\} = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. To show the inclusion "$\subset$" fix an edge f_k with $1 \leq k \leq \ell$ and set

$$H' = \textstyle\bigcup_{i=1}^r H_i, \quad H'' = \bigcup_{i=r+1}^n H_i \ \text{ and } \ G' = G \setminus \{f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_\ell\}.$$

Note that $f_k \not\subset V(H_j)$ for $r < j$, otherwise G' has r bipartite components. As a consequence f_k intersects $V(H')$, otherwise $f_k \subset V(H'')$, f_k joins H_i and H_j for some $r < i < j$, and the graph G' has a r bipartite components, a contradiction.

Case (1): $f_k \subset V(H_j)$ for some $1 \leq j \leq r$. As $V(H_j) = A_1^j \cup A_2^j$, either $f_k \subset A_1^j$ or $f_k \subset A_2^j$, otherwise the graph G' has r bipartite components, a contradiction. Hence, as $\mathrm{char}(K) \neq 2$, we get that $h_j(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

Case (2): $f_k \cap V(H_i) \neq \emptyset$ and $f_k \cap V(H_j) \neq \emptyset$ for some $i < j \leq r$. Then using the bipartitions of H_i and H_j we get $h_i(P_k) \neq 0$ and $h_j(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

Case (3): $f_k \cap V(H_i) \neq \emptyset$ for some $1 \leq i \leq r$ and $f_k \cap V(H'') \neq \emptyset$. Then using the bipartition of H_i we get $h_i(P_k) \neq 0$. Thus $[P_k] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

To show the inclusion "$\supset$" take $[P] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$ and denote by f its corresponding edge in G. Then there is $1 \leq j \leq n$ such that $h_j(P) \neq 0$. We proceed by contradiction assuming $[P] \notin \{[P_1], \ldots, [P_\ell]\}$, that is, $f \neq f_i$ for $i = 1, \ldots \ell$. Then f is an edge of H. Thus f is an edge of H_k for some $1 \leq k \leq n$. If $r < k$, then $h_i(P) = 0$ for $i = 1, \ldots, r$ by construction of the h_i's, a contradiction. Thus $1 \leq k \leq r$. If $f \subset A_1^k$ or $f \subset A_2^k$, then H_k would not be bipartite, a contradiction. Hence f joins A_1^k with A_2^k, and consequently $h_i(P) = 0$ for $i = 1, \ldots, r$ by construction of the h_i's, a contradiction. Thus $P = P_i$ for some $1 \leq i \leq \ell$, as required.

Assume $d \geq 2$. We claim that $\dim_K(S_d/I(\mathbb{X})_d)$ is equal to $m = |E(G)| = |\mathbb{X}|$, the number of edges of G. The set of all squarefree monomials $t_i t_j$ such that $\{t_i, t_j\}$ is an edge of G is K-linearly independent modulo $I(\mathbb{X})$. This follows using that the vanishing ideal of $\mathbb{X}$ is the intersection of the vanishing ideals of the points of $\mathbb{X}$ and using a well known formula for the vanishing ideal of a projective point [23, p. 398, Corollary 6.3.19]. Therefore $\dim_K(S_2/I(\mathbb{X})_2) \geq |\mathbb{X}|$. As $\dim_K(S_d/I(\mathbb{X})_d)$ is a non-decreasing function of d and it is bounded from above by the number of points of $|\mathbb{X}|$ (see [10]), the claim follows. Therefore, since $S_d/I(\mathbb{X})_d \simeq C_{\mathbb{X}}(d)$, one has $C_{\mathbb{X}}(d) = K^m$. Thus $\delta_{\mathbb{X}}(d, r) = r$ for $1 \leq r \leq m$. $\square$

We come to another of our main results.

Theorem 1.2.9. *Let G be a connected graph with s vertices and m edges, let K be a field of $\mathrm{char}(K) = 2$, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column* *vectors of A and $\lambda_r(G)$ is the r-th edge connectivity of G, then*

$$\delta_{\mathbb{X}}(d,r)=\begin{cases}\lambda_r(G), & \text{if } d=1 \text{ and } 1\leq r\leq s-1=\dim_K(C_{\mathbb{X}}(d)),\\ r, & \text{if } d\geq 2 \text{ and } 1\leq r\leq m=\dim_K(C_{\mathbb{X}}(d)).\end{cases}$$

Proof. Assume $d=1$. First we show the inequality $\delta_{\mathbb{X}}(1,r)\geq\lambda_r(G)$. We proceed by contradiction assuming that $\lambda_r(G)>\delta_{\mathbb{X}}(1,r)$. Then $\lambda_r(G)>|\mathbb{X}\setminus V_{\mathbb{X}}(F)|$ for some set F consisting of r linear forms $h_1,\ldots,h_r$ which are linearly independent modulo $I(\mathbb{X})$. We set $\ell=|\mathbb{X}\setminus V_{\mathbb{X}}(F)|$. Let $[P_1],\ldots,[P_\ell]$ be the points in $\mathbb{X}\setminus V_{\mathbb{X}}(F)$ and let $f_1,\ldots,f_\ell$ be the edges of G corresponding to these points. Consider the graph $H=G\setminus\{f_1,\ldots,f_\ell\}$ and denote by $H_1,\ldots,H_n$ its connected components. Since $\lambda_r(G)>\ell$, H cannot have $r+1$ components, that is, $n\leq r$. Let $\mathbb{X}_H$ be the set of points corresponding to the columns of the incidence matrix of H. Note that h_i vanishes at all points of $\mathbb{X}_H$ for $i=1,\ldots,r$. Indeed, take a point $[P]$ in $\mathbb{X}_H$, then its corresponding edge f is in H_k for some k, then $f\neq f_j$ for $j=1,\ldots,\ell$ and $[P]\notin\mathbb{X}\setminus V_{\mathbb{X}}(F)$, that is, $h_i(P)=0$. We set $g_j=\sum_{t_i\in V(H_j)}t_i$ for $j=1,\ldots,n$. As $h_i\in I(\mathbb{X}_H)$, by Lemma 1.2.5, h_i is a linear combination of $g_1,\ldots,g_n$ for $i=1,\ldots,r$. Therefore

$$Kh_1\oplus\cdots\oplus Kh_r\subset Kg_1\oplus\cdots\oplus Kg_n,$$

and consequently $r\leq n$. Thus $r=n$ and the inclusion above is an equality. Therefore taking classes modulo $I(\mathbb{X})$, we get

$$K\overline{h}_1\oplus\cdots\oplus K\overline{h}_r=K\overline{g}_1\oplus\cdots\oplus K\overline{g}_n.$$

As $\overline{h}_1,\ldots,\overline{h}_r$ are linearly independent, so are $\overline{g}_1,\ldots,\overline{g}_n$ because $r=n$, a contradiction because by construction of the g_i's and since $\operatorname{char}(K)=2$, one has $\sum_{i=1}^n\overline{g}_i=\sum_{i=1}^s\overline{t}_i=\overline{0}$.

Next we show the inequality $\delta_{\mathbb{X}}(1,r)\leq\lambda_r(G)$. Note that by Lemma 1.2.5. It suffices to find a set $F=\{h_1,\ldots,h_r\}$ of forms of degree 1 whose image $\overline{F}=\{\overline{h}_1,\ldots,\overline{h}_r\}$ in $S/I(\mathbb{X})$ is linearly independent over K and $\lambda_r(G)=|\mathbb{X}\setminus V_{\mathbb{X}}(F)|$. We set $\ell=\lambda_r(G)$. There are edges $f_1,\ldots,f_\ell$ of G such that the graph

$$H=G\setminus\{f_1,\ldots,f_\ell\}$$

has exactly $r+1$ connected components $H_1, \ldots, H_{r+1}$. For $j = 1, \ldots, r$, we set

$$h_j = \sum_{t_i \in V(H_j)} t_i.$$

Note that h_i and h_j have no common variables for $i \neq j$ and any sum of the polynomials $h_1, \ldots, h_r$ is a linear form in $s-1$ variables. Hence, by Lemma 1.2.6, $\overline{F}$ is linearly independent.

Let P_i be the point in $\mathbb{P}^{s-1}$ that corresponds to f_i for $i = 1, \ldots, \ell$. To complete the proof of the case $d = 1$ we need only show the equality $\{[P_1], \ldots, [P_\ell]\} = |\mathbb{X} \setminus V_{\mathbb{X}}(F)|$. To show the inclusion "$\subset$" fix an edge f_k with $1 \leq k \leq \ell$ and set

$$G' = G \setminus \{f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_\ell\}.$$

Note that $f_k \not\subset V(H_j)$ for $j = 1, \ldots, r+1$, otherwise G' has $r+1$ components, a contradiction. As a consequence f_k joins H_i and H_j for some $i < j$. Thus $h_i(P_k) \neq 0$ and $P_k \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$.

To show the inclusion "$\supset$" take $[P] \in \mathbb{X} \setminus V_{\mathbb{X}}(F)$ and denote by f its corresponding edge in G. Then there is $1 \leq j \leq r$ such that $h_j(P) \neq 0$. We proceed by contradiction assuming $[P] \notin \{[P_1], \ldots, [P_\ell]\}$, that is, $f \neq f_i$ for $i = 1, \ldots \ell$. Then f is an edge of H. As $\operatorname{char}(K) = 2$, we get $h_i(P) = 0$ for $i = 1, \ldots, r$ by construction of h_i, a contradiction.

If $d \geq 2$, the equality $\delta_{\mathbb{X}}(d, r) = r$ for $1 \leq r \leq m = \dim_K(C_{\mathbb{X}}(d))$ follows from the proof of Theorem 1.2.8. □

The next result is a hybrid of Theorems 1.2.8 and 1.2.9 and is characteristic free.

Theorem 1.2.10. *Let G be a connected bipartite graph with s vertices and m edges, let K be a field of any characteristic, and let A be the incidence matrix of G. If $\mathbb{X}$ is the set of column vectors of A and $\lambda_r(G)$ is the r-th edge connectivity of G, then*

$$\delta_{\mathbb{X}}(d, r) = \begin{cases} \lambda_r(G), & \text{if } d = 1 \text{ and } 1 \leq r \leq s-1 = \dim_K(C_{\mathbb{X}}(d)), \\ r, & \text{if } d \geq 2 \text{ and } 1 \leq r \leq m = \dim_K(C_{\mathbb{X}}(d)). \end{cases}$$

Proof. Let V_1, V_2 be the bipartition of G. Consider the set $\mathbb{Y}$ of all points $[e_i - e_j]$ in $\mathbb{P}^{s-1}$ such that $\{t_i, t_j\}$ is an edge of G with $t_i \in V_1$ and $t_j \in V_2$, where e_i is the i-th unit vector in K^s. Noticing that the polynomial $h = t_1 + \cdots + t_s$ vanishes at all points of $\mathbb{Y}$ and the equality $C_{\mathbb{X}}(1) = C_{\mathbb{Y}}(1)$, the result follows adapting Lemma 1.2.5 and the proof of Theorem 1.2.9 with $\mathbb{Y}$ playing the role of $\mathbb{X}$. $\square$

Corollary 1.2.11. *Let $C_p(G)$ be the code of a connected graph G with s vertices, m edges, r-th weak edge biparticity $\upsilon_r(G)$, r-th edge connectivity $\lambda_r(G)$, over a finite field K of $\operatorname{char}(K) = p$. Then the r-th generalized Hamming weight of $C_p(G)$ is given by*

$$\delta_r(C_p(G)) = \begin{cases} \upsilon_r(G), & \text{if } p \neq 2,\ G \text{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } p = 2,\ 1 \leq r \leq s-1, \\ \lambda_r(G), & \text{if } G \text{ is bipartite}, 1 \leq r \leq s-1. \end{cases}$$

Proof. Note that the linear code $C_p(G)$ is the image of S_1—the vector space of linear forms of S—under the evaluation map $\operatorname{ev}_1\colon S_1 \to K^m$, $f \mapsto (f(P_1), \ldots, f(P_m))$. Note that the image of the linear function t_i, under the map ev_1, gives the i-th row of the incidence matrix of G. This means that $C_p(G)$ is the Reed-Muller-type code $C_{\mathbb{X}}(1)$. Hence, the result follows using the equality $\delta_{\mathbb{X}}(1, r) = \delta_r(C_{\mathbb{X}}(1))$ [11, Lemma 4.3(iii)] and Theorems 1.2.8, 1.2.9, and 1.2.10. $\square$

Corollary 1.2.12. [5, Theorems 1–3] *Let $C_p(G)$ be the code of a connected graph G with s vertices, m edges, weak edge biparticity $\upsilon(G)$, edge connectivity $\lambda(G)$, over a finite field K of $\operatorname{char}(K) = p$. Then the minimum distance of $C_p(G)$ is given by*

$$\delta(C_p(G)) = \begin{cases} \upsilon(G), & \text{if } p \neq 2,\ G \text{ is non-bipartite}, 1 \leq r \leq s, \\ \lambda(G), & \text{if } p = 2,\ 1 \leq r \leq s-1, \\ \lambda(G), & \text{if } G \text{ is bipartite}, 1 \leq r \leq s-1. \end{cases}$$

Proof. It follows from Corollary 1.2.11 making $r = 1$. $\square$

1.3 Examples

Let G be a connected graph and let $C_p(G)$ be the incidence matrix code of G over a finite field $\mathbb{F}_q$ of characteristic p. Using Macaulay 2 [14], SageMath [27], and Wei's duality [30, Theorem 3], we can compute the weight hierarchy of $C_p(G)$. We illustrate this with some examples.

The weight hierarchy of $C_p(G)$ can also be computed using a nice formula of Johnsen and Verdure [19] for the generalized weights in terms of the Betti numbers of the

Stanley-Reisner ring of the representable matroid determined by the incidence matrix of G.

Note that, by Corollary 1.2.11, we can compute the corresponding higher weak biparticity and edge connectivity numbers of the graph. Conversely any algorithm that computes these graph invariants can be used to compute the weight hierarchy of $C_p(G)$.

Example 1.3.1. Let G be the graph of Figure 1.1. Recall that the dimension of $C_p(G)$ is 6 if $p = 3$ and is 5 if $p = 2$ (Corollary 1.2.4). For use below we denote the dual code by $C_p(G)^\perp$.

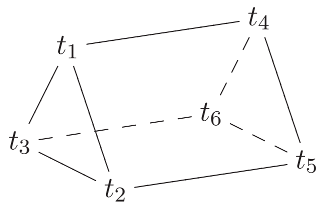

Figure 1.1: Non-bipartite graph G.

Using Procedure 1.4.1, together with Wei's duality [30, Theorem 3], we obtain the following table with the weight hierarchy of $C_p(G)$. The edge biparticity of this graph is 2, the weak edge biparticity is 2, and the edge connectivity is 3.

r	1	2	3	4	5	6
$\delta_r(C_2(G))$	3	5	6	8	9	
$\delta_r(C_2(G)^\perp)$	3	6	8	9		
$\delta_r(C_3(G))$	2	4	5	7	8	9
$\delta_r(C_3(G)^\perp)$	4	7	9			

Table 1.1: Weight hierarchy of $C_p(G)$ for the graph of Figure 1.1.

Example 1.3.2. Let G be the Petersen graph of Figure 1.2. Recall that the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 9 (resp. 6) if $p = 2$, and the dimension of $C_p(G)$ (resp. $C_p(G)^\perp$) is 10 (resp. 5) if $p \neq 2$; (Corollary 1.2.4).

Using Procedure 1.4.2, together with, Wei's duality [30, Theorem 3] we obtain the following table with the weight hierarchy of $C_p(G)$ for $p = 2$. The edge biparticity, the weak edge biparticity , and the edge connectivity of the Petersen graph are equal to 3.

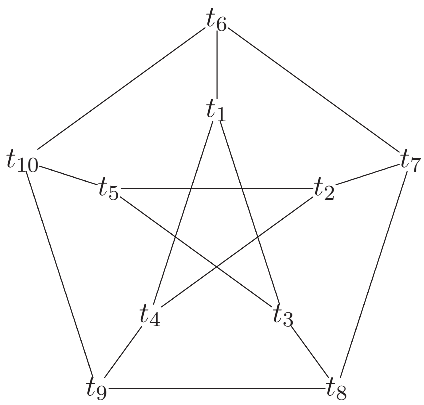

Figure 1.2: Petersen graph G.

r	1	2	3	4	5	6	7	8	9	10
$\delta_r(C_2(G))$	3	5	7	9	10	12	13	14	15	
$\delta_r(C_2(G)^\perp)$	5	8	10	12	14	15				
$\delta_r(C_3(G))$	3	5	7	8	9	11	12	13	14	15
$\delta_r(C_3(G)^\perp)$	6	10	12	14	15					

Table 1.2: Weight hierarchy of $C_p(G)$ for the graph of Figure 1.2.

1.4 Procedure for Macaulay2 and SageMath

Procedure 1.4.1. Computing the weight hierarchies using *Macaulay*2 [14], SageMath [27], and Wei duality [30]. This procedure corresponds to Example 1.3.1. It could be applied to any connected graph G to obtain the generalized Hamming weights of $C_p(G)$. The next procedure for *Macaulay*2 uses the algorithms of [11] to compute generalized minimum distance functions.

```
--Procedure for Macaulay2
input "points.m2"
q=3, R=ZZ/q[t1,t2,t3,t4,t5,t6]--p=char(K)=3
A=transpose(matrix{{1,1,0,0,0,0},{0,1,1,0,0,0},{1,0,1,0,0,0},
{0,0,0,1,1,0},{0,0,0,0,1,1},{0,0,0,1,0,1},{1,0,0,1,0,0},
{0,1,0,0,0,1},{0,0,1,0,1,0}})
I=ideal(projectivePointsByIntersection(A,R)), M=coker gens gb I
```

```
genmd=(d,r)->degree M-max apply(apply(subsets(apply(apply(apply
(toList(set(0..q-1))^**(hilbertFunction(d,M))
-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),r),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==r and not quotient(I,x)==I
then degree(I+x) else 0)
--The following are the first two generalized Hamming weights
genmd(1,1),genmd(1,2)
```

```
#Procedure for SageMath
A=transpose(matrix(GF(3),[[1,1,0,0,0,0],[0,1,1,0,0,0],[1,0,1,0,0,0],
[0,0,0,1,1,0],[0,0,0,0,1,1],[0,0,0,1,0,1],[1,0,0,1,0,0],[0,1,0,0,0,1],
[0,0,1,0,1,0]]))
C = codes.LinearCode(A)
C.parity_check_matrix()
C.generator_matrix()
#the next line Gives the minimum distance of the dual code
C.dual_code().minimum_distance()
```

Procedure 1.4.2. Computing the weight hierarchies using *Macaulay*2 [14], SageMath [27], and Wei duality [30]. This procedure corresponds to Example 1.3.2. The next procedure for *Macaulay*2 uses the algorithms of [11] to compute generalized footprint functions. The footprint gives lower bounds for the generalized weights.

```
--Procedure for Macaulay2 for Petersen graph
input "points.m2"
R=QQ[t1,t2,t3,t4,t5,t6,t7,t8,t9,t10]
--Incidence matrix to compute biparticity
A=transpose matrix{{1,1,0,0,0,0,0,0,0,0},{0,1,1,0,0,0,0,0,0,0},
{0,0,1,1,0,0,0,0,0,0},{0,0,0,1,1,0,0,0,0,0},{1,0,0,0,1,0,0,0,0,0},
{1,0,0,0,0,1,0,0,0,0},{0,1,0,0,0,0,1,0,0,0},{0,0,1,0,0,0,0,1,0,0},
{0,0,0,1,0,0,0,0,1,0},{0,0,0,0,1,0,0,0,0,1},{0,0,0,0,0,1,0,1,0,0},
{0,0,0,0,0,0,0,1,0,1},{0,0,0,0,0,0,1,0,0,1},{0,0,0,0,0,0,1,0,1,0},
```

```
{0,0,0,0,0,1,0,0,1,0}}
q=2, R = ZZ/q[t1,t2,t3,t4,t5,t6,t7,t8,t9]
--Generator matrix computed with Sage to find Hamming weights.
A1=matrix({{1,0,0,0,1,0,0,0,0,0,1,0,0,0,1},
{0,1,0,0,1,0,0,0,0,0,1,0,1,1,1},{0,0,1,0,1,0,0,0,0,0,0,1,1,1,1},
{0,0,0,1,1,0,0,0,0,0,0,1,1,0,0},{0,0,0,0,0,1,0,0,0,0,1,0,0,0,1},
{0,0,0,0,0,0,1,0,0,0,0,0,1,1,0},{0,0,0,0,0,0,0,1,0,0,1,1,0,0,0},
{0,0,0,0,0,0,0,0,1,0,0,0,0,1,1},{0,0,0,0,0,0,0,0,0,1,0,1,1,0,0}})
q=2, R = ZZ/q[t1,t2,t3,t4,t5,t6]
--parity check matrix computed with Sage to find
--the Hamming weights of dual code
A2=matrix({{1,0,0,0,0,1,1,0,0,0,0,0,0,1,1},
{0,1,0,0,0,0,1,1,0,0,0,1,1,0,0},{0,0,1,0,0,0,0,1,1,0,0,1,1,1,0},
{0,0,0,1,0,0,0,0,1,1,0,0,1,1,0},{0,0,0,0,1,1,0,0,0,1,0,0,1,1,1},
{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1}})
--The following functions can be applied to A, A1, A2
I=ideal(projectivePointsByIntersection(A,R)),M=coker gens gb I
init=ideal(leadTerm gens gb I),degree M
--This function computes the edge biparticity of Petersen graph.
--using the incidence matrix over the rational numbers
genmd1=(d,r)->degree M-max apply(apply(subsets(apply(apply(apply
(toList(set(1,-1))^**(hilbertFunction(d,M))
-(set{0})^**(hilbertFunction(d,M)),toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),r),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==r and not quotient(I,x)==I
then degree(I+x) else 0)
--To compute the r-weight of the dual code
--use genmd(1,r) of the previous procedure:
genmd(1,1),genmd(1,2),genmd(1,3),genmd(1,4),genmd(1,5)
--To compute edge biparticity use: genmd1(1,1)
er=(x)-> if not quotient(init,x)==init then degree ideal(init,x) else 0
```

```
--This is the footprint function
fpr=(d,r)->degree M - max apply(apply(apply(subsets(flatten
entries basis(d,M),r),toSequence),ideal),er)
--To find lower bounds for Hamming weights use the footprint:
fpr(1,1),fpr(1,2),fpr(1,3),fpr(1,4),fpr(1,5),fpr(1,6),fpr(1,7),fpr(1,8)
```

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Chapter 2

Linear codes over signed graphs

Abstract. We give formulas, in terms of graph theoretical invariants, for the generalized Hamming weights of the linear code generated by the rows of the incidence matrix of a signed graph over a finite field, and for those of its dual code. Then we determine the regularity of the ideals of circuits and cocircuits of a signed graph, and prove an algebraic formula in terms of the multiplicity for the frustration index of an unbalanced signed graph.

2.1 Introduction

The generalized Hamming weights (GHWs) of a linear code are parameters of interest in many applications [9, 13, 17, 24, 28, 34, 39, 40, 42] and they have been nicely related to the graded Betti numbers of the ideal of cocircuits of the matroid of a linear code [16, 17], to the nullity function of the dual matroid of a linear code [39], and to the enumerative combinatorics of linear codes [3, 15, 19, 20]. Because of this, their study has attracted considerable attention, but determining them is in general a difficult problem. The notion of generalized Hamming weight was introduced by Helleseth, Kløve and Mykkeltveit in [14] and was first used systematically by Wei in [39]. For convenience we recall this notion. Let $K = \mathbb{F}_q$ be a finite field and let C be a *linear* $[m, k]$ *code* of *length* m and *dimension* k, that is, C is a linear subspace of K^m with $k = \dim_K(C)$. Let $1 \leq r \leq k$ be an integer. Given a linear subspace D of C, the *support* of D, denoted $\mathrm{supp}(D)$, is the set of nonzero positions of D, that is,

$$\mathrm{supp}(D) := \{i : \exists\,(a_1, \ldots, a_m) \in D,\ a_i \neq 0\}.$$

The r-th *generalized Hamming weight* of C, denoted $\delta_r(C)$, is given by

$$\delta_r(C) := \min\{|\mathrm{supp}(D)|\colon D \text{ is a subspace of } C \text{ with } \dim_K(D) = r\}.$$

As usual we call the set $\{\delta_1(C), \dots, \delta_k(C)\}$ the *weight hierarchy* of the linear code C. The 1st Hamming weight of C is the *minimum distance* $\delta(C)$ of C, that is, one has

$$\delta_1(C) = \delta(C) = \min\{\omega(\mathbf{x})\colon \mathbf{x} \in C \setminus \{0\}\},$$

where $\omega(\mathbf{x})$ is the Hamming weight of the vector $\mathbf{x}$, i.e., the number of nonzero entries of $\mathbf{x}$. To determine the minimum distance is essential to find good error-correcting codes [20].

The notion of generalized Hamming weights for linear codes was extended to matroids by Johnsen and Verdure [16, Definition 1], as we now explain.

Let M be a matroid with ground set E, rank function ρ, nullity function η, and let M^* be its dual matroid. The r-th *generalized Hamming weight* of M, denoted $d_r(M)$, is given by

$$d_r(M) := \min\{|X|\colon X \subset E \text{ and } \eta(X) = r\} \quad \text{for } \ 1 \leq r \leq \eta(E).$$

A major result of Johnsen and Verdure [16] shows that the GHWs of a matroid can be read off the minimal graded free resolution of the Stanley-Reisner ideal of the independence complex of the matroid [16, Theorem 2] (see Theorem 2.4.2).

We can associate to a linear $[m, k]$ code C the vector matroid $M[A]$ on the ground set $E = \{1, \dots, m\}$, where A is a generator matrix of C. The rank function (resp. nullity) of $M[A]$ is given by $\rho(X) = \mathrm{rank}(A_X)$ (resp. $\eta(X) = |X| - \rho(X)$) for $X \subset E$, where A_X is the submatrix of A obtained by picking the columns indexed by X. It can be verified that the matroid $M[A]$ does not depend on the generator matrix we choose. We call $M[A]$ the *(generator) matroid* of C. If H is a parity check matrix of C, then $M[A]^* = M[H]$ and $M[A] = M[H]^*$. By Lemma 2.2.4, one has

$$\delta_r(C) = d_r(M[A]^*) \text{ for } 1 \leq r \leq k \ \text{ and } \ \delta_r(C^\perp) = d_r(M[A]) \ \text{ for } \ 1 \leq r \leq m - k.$$

Thus computing GHWs of vector matroids is equivalent to computing those of linear codes. This relationship between the GHWs of linear codes and those of vector matroids is attributed to Wei [39, Theorem 2] (cf. Theorem 2.2.2). In this work we study GHWs of linear codes defined over signed graphs, combining the theory of GHWs of matroids [16, 17] and the combinatorial structure of signed-graphic matroids [43, 44, 45] that we introduce next.

A *signed graph* G_σ is a pair (G, σ) consisting of a multigraph G with vertex set $V(G)$ and edge set $E(G)$ (loops and multiple edges are permitted), and a mapping $\sigma\colon E(G) \to \{\pm\}$, that assigns a sign to each edge. If no loops or multiple edges are permitted, G is called a *simple graph* and G_σ is called a *signed simple graph*. In particular, the signed graph with $\sigma(e) = +$ (resp. $\sigma(e) = -$) for all e, denoted G_+ (resp. G_-), is called the *positive signed graph* (resp. *negative signed graph*) on G. There are more general definitions of signed graphs, where the edge set includes empty loops and half edges, that are essential to represent root systems [43].

Let G_σ be a signed graph. A *cycle* of G_σ is a simple closed path in G. A cycle with an even number of negative edges is called *balanced*. A signed graph is *balanced* if every cycle is balanced.

An isolated vertex is regarded as balanced. A *bowtie* of G_σ is the union of two unbalanced cycles which meet at a single vertex or the union of two vertex-disjoint unbalanced cycles and a simple path which meets one cycle at each end and is otherwise disjoint from them.

A central result of Zaslavsky [43, Theorem 5.1] shows the existence of a matroid $M(G_\sigma)$ with ground set $E(G)$, called the *signed-graphic matroid* of G_σ, whose rank function is

$$\rho(X) = |V(G)| - c_0(X) \quad \text{for} \quad X \subset E(G),$$

where $c_0(X)$ is the number of balanced connected components of the signed subgraph with edge set X and vertex set $V(G)$, the circuits of $M(G_\sigma)$ are the balanced cycles and the bowties of G_σ. The circuits of $M(G_\sigma)$ are called the *circuits* of G_σ. For parts (f) and (g) of [43, Theorem 5.1] the reader is referred to [45, Theorem 2.1].

If $G_\sigma = G_+$, the signed-graphic matroid $M(G_+)$ is the *graphic matroid* $M(G)$ of G whose circuits are the cycles of G [25, 41]. If $G_\sigma = G_-$, the signed-graphic matroid $M(G_-)$ is the *even cycle* matroid [43] whose circuits are the even cycles and the bowties of G_-. The circuits of the matroids $M(G_+)$, $M(G_-)$ and those of their dual matroids, as well as the related notion of an elementary integral vector, occur in coding theory [6, 31], convex analysis [26], the theory of toric ideals of graphs [2, 8, 22, 29, 36, 37], and in matroid theory [25, 30, 43, 46].

The content of this chapter is as follows. In Section 2.2 we briefly introduce matroids and present some well known results about GHWs of matroids and linear codes.

In what follows G_σ denotes a signed graph with s vertices, m edges, c connected components, and c_0 balanced components, and K denotes a finite field $\mathbb{F}_q$ of characteristic p. The *incidence matrix code* of G_σ over the field K, denoted by C, is the linear code generated by the row vectors of the incidence matrix of G_σ (Definition 2.3.6). In Section 2.3 we present our main results on the generalized Hamming weights of incidence matrix codes of signed graphs and those of their dual codes, and describe the GHWs of the signed-graphic matroid of a signed graph and those of its dual matroid, in terms of the combinatorics of a signed graph.

The *frustration index* of G_σ, denoted $\varphi(G_\sigma)$, is the smallest number of edges whose deletion from G_σ leaves a balanced signed graph. The minimum distance of C is bounded from above by $\varphi(G_\sigma)$. We are interested in the following related invariant. The r-th *cogirth* of G_σ, denoted $\upsilon_r(G_\sigma)$, is the minimum number of edges whose removal results in a signed graph with r balanced components. If $r = 1$ and G_σ is connected, $\upsilon_1(G_\sigma)$ is the *cogirth* of $M(G_\sigma)$, that is, the minimum size of a cocircuit of $M(G_\sigma)$ (Lemma 2.3.4). We denote $\upsilon_1(G_\sigma)$ simply by $\upsilon(G_\sigma)$. The r-th *edge connectivity* of G_σ, denoted $\lambda_r(G_\sigma)$ or $\lambda_r(G)$, is the minimum number of edges whose removal results in a signed graph with $r + 1$ connected components. Note that the r-th edge connectivity is a property of the underlying multigraph G, that is, it is independent of σ. If $r = 1$, $\lambda_1(G_\sigma)$ is the edge connectivity of G_σ and is denoted by $\lambda(G_\sigma)$. We will relate these graph invariants to the generalized Hamming weights and the minimum distance of incidence matrix codes.

Our main results on linear codes are the following. First, we give graph theoretical formulas for the generalized Hamming weights of the incidence matrix code of a signed graph.

Theorem 2.3.16 *If C is the incidence matrix code of a connected signed graph G_σ,* *then*

$$\delta_r(C) = \begin{cases} \upsilon_r(G_\sigma), & \text{if } p \neq 2,\ G_\sigma \text{ is unbalanced and } 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } p = 2 \text{ and } 1 \leq r \leq s-1, \\ \lambda_r(G), & \text{if } G_\sigma \text{ is balanced and } 1 \leq r \leq s-1. \end{cases}$$

We show that the formulas of [23, Corollary 2.13] for the generalized Hamming weights of incidence matrix codes of simple graphs can be extended to multigraphs (Corollary 2.3.17). Then we show that the combinatorial formulas of Dankelmann, Key and Rodrigues [6, Theorems 1–3] for the minimum distance of the incidence matrix code of a simple graph can be extended to signed graphs (Corollary 2.3.18).

A family of circuits $\{C_i\}_{i=1}^r$ of a matroid M is called *non-redundant* if $C_i \not\subseteq \bigcup_{j\neq i} C_j$ for $i = 1, \ldots, r$ [16]. Our next result gives graph theoretical formulas for the generalized Hamming weights of the dual code of the incidence matrix code of a signed graph.

Theorem 2.3.19 *Let C be the incidence matrix code of a connected signed graph G_σ.*

(a) If $p = 2$ or G_σ is balanced, and $1 \leq r \leq m - s + 1$ (resp. $1 \leq r \leq s - 1$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant cycles (resp. cocycles) of G.

(b) If $p \neq 2$ and $1 \leq r \leq m - s$ (resp. $1 \leq r \leq s$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant circuits (resp. cocircuits) of G_σ.

If C is the incidence matrix code of a connected digraph $\mathcal{D}$ and G is its underlying multigraph, we show that $\delta_r(C) = \lambda_r(G)$ and give graph theoretical formulas for the generalized Hamming weights of the dual code $C^\perp$ (Corollary 2.3.20). For a connected multigraph, we give formulas for the GHWs of the dual of its incidence matrix code (Corollary 2.3.21).

The main result of Section 2.4 gives explicit formulas for the regularity of the ideals of circuits and cocircuits of the vector matroid of the incidence matrix of a signed graph (Theorem 2.4.7).

Let M be the matroid of C. By Theorems 2.3.16 and 2.3.19, one has graph theoretical formulas for the weight hierarchies of C and $C^\perp$. On the other hand, using *Macaulay*2 [11], the package *Matroids* [5], and the formulas of Johnsen and Verdure (Theorem 2.4.2, Corollary 2.4.3), we can compute the weight hierarchies of C and

$C^\perp$. Hence, our results can be used to compute the r-th *cogirth* $\upsilon_r(G_\sigma)$ of G_σ and the r-th *edge connectivity* $\lambda_r(G_\sigma)$ of G_σ. The main result of Section 2.5 is an algebraic formulation for the frustration index of G_σ, in terms of the degree or multiplicity of graded ideals, that can be used to compute or estimate this number using *Macaulay*2 [11] (Theorem 2.5.4, Example 2.6.6). If G is a graph, the frustration index of G_- is the *edge biparticity* of G, that is, the minimum number of edges whose removal makes the graph bipartite. In Section 2.6 we illustrate how to use our results in practice with some examples.

Our main results and their proofs show that the weight hierarchies of the incidence matrix code C and its dual code $C^\perp$ of a signed graph G_σ can be computed using the field $\mathbb{Q}$ of rational numbers as the ground field. To compute the GHWs of C and $C^\perp$ over a finite field $\mathbb{F}_q$ of characteristic p, we use the incidence matrix of G_σ (resp. G_+) over the field $\mathbb{Q}$ if $p \neq 2$ (resp. $p = 2$). One can also use the rational numbers to compute the cycles, circuits, and cocircuits of a signed graph, as well as its r-th cogirth, frustration index, and r-th edge connectivity. In Section 2.7 we give procedures for *Macaulay*2 [11] that allows us to obtain this information for graphs with a small number of vertices, see the examples of Section 2.6. The package *Matroids* [5] plays an important role here because it computes the circuits and cocircuits of vector matroids over the field of rational numbers, however the problem of computing all circuits of a vector matroid is likely to be NP-hard [18, 35] (cf. [16, p. 76]). The minimum distance of any linear code can be computed using SageMath [27]. For signed simple graphs one can also compute the minimum distance using Proposition 2.5.6 and the algorithms of [9, 21].

2.2 Matroids and linear codes

A *matroid* is a pair $M = (E, \rho)$ where E is a finite set, called the *ground set* of M, and $\rho\colon 2^E \to \mathbb{Z}_+ := \{0, 1, \ldots\}$ is a function, called the *rank function* of M, satisfying:

(R_0) $\rho(\emptyset) = 0$;

(R_1) If $X \subset E$ and $e \in E$, then $\rho(X) \leq \rho(X \cup \{e\}) \leq \rho(X) + 1$;

(R_2) If $X \subset E$ and $Y \subset E$, then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

An *independent set* of a matroid M is subset $X \subset E$ such that $\rho(X) = |X|$. In particular the empty set is always an independent set. A *base* is a maximal independent set. A subset of the ground set which is not independent is called *dependent* and a *circuit* of M is a minimal dependent set. We denote by $\mathcal{C}_M$ the family of all circuits of M. The *rank* of the matroid M, denoted $\rho(M)$, is $\rho(E)$. The *nullity* of $X \subset E$, denoted $\eta(X)$, is defined by

$$\eta(X) := |X| - \rho(X),$$

and the *nullity* of M, denoted $\eta(M)$, is $\eta(E)$. Let $M = (E, \rho)$ be a matroid, its *dual* is the matroid $M^* = (E, \rho^*)$ with the same ground set E and rank function given by

$$\rho^*(X) := |X| - \rho(E) + \rho(E \setminus X) \quad \text{for all } X \subset E,$$

see [25, p. 72]. The nullity function of M^* is denoted by η^*. One can verify that $(M^*)^* = M$.

A family of circuits $\{C_i\}_{i=1}^r$ of a matroid M is called *non-redundant* if $C_i \not\subseteq \bigcup_{j \neq i} C_j$ for $i = 1, \ldots, r$ [16]. Let X be a subset of the ground set E. The *degree* or *non-redundancy* of X is the maximum number of non-redundant circuits contained in X, and it is denoted by $\deg(X)$.

Lemma 2.2.1. [16, Proposition 1] *Let $M = (E, \rho)$ be a matroid, let X be a subset of E, and let η be the nullity function of M. Then* $\deg(X) = \eta(X)$.

Theorem 2.2.2. [39, Theorem 2] *Let C be a linear $[m, k]$ code and let M^* be the dual of the vector matroid of C. Then, the r-th generalized Hamming weight of C is given by*

$$\delta_r(C) = \min\{|X| \colon X \subset E \text{ and } \eta^*(X) \geq r\} \quad \text{for } 1 \leq r \leq k.$$

By Lemma 2.2.1, we can replace the inequality $\eta^*(X) \geq r$ by $\eta^*(X) = r$. This result suggests how to define the generalized Hamming weights of any matroid M.

Definition 2.2.3. [16, Definition 1] Let $M = (E, \rho)$ be a matroid with nullity function η. The *generalized Hamming weights* of M are defined as

$$d_r(M) := \min\{|X| \colon X \subset E \text{ and } \eta(X) = r\} \quad \text{for } 1 \leq r \leq \eta(E).$$

Lemma 2.2.4. *Let C be a linear code of length m and dimension k and let M be the its vector matroid. Then $\delta_r(C) = d_r(M^*)$ for $1 \leq r \leq k$ and $\delta_r(C^\perp) = d_r(M)$ for $1 \leq r \leq m-k$.*

Proof. By Lemma 2.2.1 and Theorem 2.2.2, we obtain $\delta_r(C) = d_r(M^*)$ for $1 \leq r \leq k$. The matroid associated to $C^\perp$ is M^*. Hence $\delta_r(C^\perp) = d_r(M)$ for $1 \leq r \leq m-k$. □

Theorem 2.2.5. ([1, Corollary 1.3], [17, Proposition 6]) *Let $M = (E, \rho)$ be a matroid and let η be its nullity function. The following hold.*

$d_r(M^*) = \min\{|X|\colon X \subset E$ *and* $\rho(E \setminus X) = \rho(E) - r\}$ *for* $1 \leq r \leq \rho(E)$.

$d_r(M) = \min\{|\bigcup_{i=1}^r C_i| : \{C_i\}_{i=1}^r$ *are non-redundant circuits of* $M\}$ *for* $1 \leq r \leq \eta(E)$.

Proof. According to [41, Theorem 2, p. 35], one has $\rho(E \setminus X) = \rho(E) - \eta^*(X)$. Therefore the first equality follows from

$$d_r(M^*) = \min\{|X|\colon X \subset E \text{ and } \eta^*(X) = r\} \text{ for } 1 \leq r \leq \rho(E).$$

On the other hand, recall that by definition of $d_r(M)$, one has

$$d_r(M) = \min\{|X|\colon X \subset E \text{ and } \eta(X) = r\} \text{ for } 1 \leq r \leq \eta(E).$$

Therefore, applying Lemma 2.2.1, the second equality follows. □

Corollary 2.2.6. *Let C be a linear $[m, k]$ code and let $M = (E, \rho)$ be the vector matroid of C. Then the following equalities hold:*

$\delta_r(C) = \min\{|X|\colon X \subset E$ *and* $\rho(E \setminus X) = \rho(E) - r\}$ *for* $1 \leq r \leq k$.

$\delta_r(C^\perp) = \min\{|\bigcup_{i=1}^r C_i| : \{C_i\}_{i=1}^r$ *are non-redundant circuits of* $M\}$ *for* $1 \leq r \leq m-k$.

Proof. By Lemma 2.2.4, we obtain $\delta_r(C) = d_r(M^*)$ for $1 \leq r \leq k$ and $\delta_r(C^\perp) = d_r(M)$ for $1 \leq r \leq m-k$. Thus the result follows from Theorem 2.2.5. □

If C is a linear $[m, k]$ code, then $\delta_1(C) < \cdots < \delta_k(C)$ [15, 39]. The following *duality* theorem of Wei is a classical result in this area.

Theorem 2.2.7. (Wei's duality [39, Theorem 3]) *Let C be a linear $[m, k]$ code. Then*

$$\{\delta_r(C) \,|\, r = 1, \ldots, k\} = \{1, \ldots, m\} \setminus \{m + 1 - \delta_r(C^\perp) \,|\, r = 1, \ldots, m-k\}.$$

This result was generalized by Britz, Johnsen, Mayhew and Shiromoto [4, Theorem 5] from linear codes to arbitrary matroids.

2.3 Generalized weights over signed graphs

In this section we present our main results on linear codes. To avoid repetitions, we continue to employ the notations and definitions used in Sections 2.1 and 2.2.

A multigraph G consists of a finite set of vertices, $V(G)$, and a finite multiset of edges, $E(G)$. Edges of G are of two types. A *link* $e = \{v, w\}$, with two distinct endpoints, v, w in $V(G)$ and a *loop*, $e = \{v, v\}$, with two coincident endpoints. As $E(G)$ is a multiset, multiple edges are allowed. The number of edges of G counted with multiplicity is denoted by $m = |E(G)|$. A multigraph with no loops or multiple edges is called a *simple graph* or a *graph*. Let G be a multigraph. A *cycle* of G is a simple closed path in G. A loop is a cycle of length 1, a pair of parallel links is a cycle of length 2, a triangle is a cycle of length 3, and so on.

Theorem 2.3.1. ([43, Theorem 5.1], [45, Theorem 2.1]) *Let G_σ be a signed graph. Then there exists a matroid $M(G_\sigma)$ on $E(G)$ whose circuits are the balanced cycles and the bowties of G_σ.*

The matroid $M(G_\sigma)$ is called the *signed-graphic matroid* of G_σ. The circuits of $M(G_\sigma)$ are called the *circuits* of G_σ. If G_σ is balanced, then $M(G_\sigma)$ is the graphic matroid $M(G)$ of G.

Definition 2.3.2. Let G_σ be an unbalanced (resp. balanced) signed graph. A *cutset* of G_σ is a set of edges whose removal from G_σ increases the number of balanced connected components (resp. connected components) of G_σ. A *cocircuit* of G_σ is a minimal cutset of G_σ. If G_σ is balanced, a cocircuit of G_σ is called a *cocycle* or *bond* of G.

Lemma 2.3.3. [25, Proposition 2.3.1] *If G_σ is a balanced signed graph, then the cocircuits of G_σ are the cocircuits of the graphic matroid $M(G)$ of G, that is, the circuits of $M(G)^*$.*

Lemma 2.3.4. [43, Theorem 5.1(i)] *If G_σ is a connected unbalanced signed graph and $M(G_\sigma)$ is its signed-graphic matroid, then the cocircuits of G_σ are the cocircuits of $M(G_\sigma)$, that is, the circuits of the dual matroid $M(G_\sigma)^*$.*

Proof. Let ρ be the rank function of $M(G_\sigma)$. We set $V = V(G)$ and $E = E(G)$. Take a cocircuit $X \subset E$ of G_σ. As G_σ is connected, one has, $c_0(E) = 0$ and $c_0(E \setminus X) = 1$

if G_σ is unbalanced, and $c_0(E) = 1$ and $c_0(E \setminus X) = 2$ if G_σ is balanced. Then $c_0(E \setminus X) = c_0(E) + 1$. According to [43, Theorem 5.1(j)], one has

$$\rho(E) = |V| - c_0(E) \quad \text{and} \quad \rho(E \setminus X) = |V| - c_0(E \setminus X). \tag{2.3.1}$$

Therefore $\rho(E \setminus X) = \rho(E) - 1$. Since X is a minimal cutset, it follows that $H := E \setminus X$ is closed, that is, $\rho(H \cup \{e\}) = \rho(H) + 1 = \rho(E)$ for $e \notin H$. Indeed, from the equality $H \cup \{e\} = E \setminus (X \setminus \{e\})$, and the minimality of X, we get $c_0(H \cup \{e\}) = c_0(E)$. Hence, using Eq. (2.3.1), we obtain $\rho(H \cup \{e\}) = \rho(E) = \rho(H) + 1$ for $e \notin H$. As a consequence H is a maximal set of rank $\rho(E) - 1$. Thus, by [41, Lemma 1, p. 38], H is a hyperplane of $M(G_\sigma)$ in the sense of [41], and by [41, Theorem 2, p. 39], X is a cocircuit of $M(G_\sigma)$. Similarly, if X is a cocircuit of $M(G_\sigma)$, it is seen that X is a cocircuit of G_σ. $\square$

Theorem 2.3.5. *Let G_σ be a connected signed graph, let ρ and η be the rank and nullity functions of the signed-graphic matroid $M = M(G_\sigma)$ of G_σ. The following hold.*

(i) If $1 \leq r \leq \eta(M)$, then $d_r(M)$ is equal to the minimum number of edges of G forming a union of r non-redundant circuits of G_σ.

(ii) If $1 \leq r \leq \rho(M)$ and G_σ is unbalanced (resp. balanced), then $d_r(M^*)$ is the r-th cogirth $\upsilon_r(G_\sigma)$ (resp. r-th edge connectivity $\lambda_r(G_\sigma)$) of G_σ.

(iii) If $1 \leq r \leq \rho(M)$, then $d_r(M^*)$ is equal to the minimum number of edges of G_σ forming a union of r non-redundant cocircuits of G_σ.

Proof. (i): By Theorem 2.3.1, the circuits of M are the balanced cycles and the bowties of G_σ. Hence, it suffices to recall the following formula of Theorem 2.2.5:

$$d_r(M) = \min \{|\textstyle\bigcup_{i=1}^r C_i| : \{C_i\}_{i=1}^r \text{ are non-redundant circuits of } M\} \ \text{ for } 1 \leq r \leq \eta(M).$$

(ii): Let E be the edge set of G_σ, which is the ground set of M, and let V be the vertex set of G_σ. According to [43, Theorem 5.1(j)] the rank function of $M(G_\sigma)$ satisfies

$$\rho(E \setminus X) = |V| - c_0(E \setminus X) \quad \text{for} \quad X \subset E, \tag{2.3.2}$$

where $c_0(E\setminus X)$ is the number of balanced connected components of the signed subgraph $G_\sigma\setminus X$ with edge set $E\setminus X$ and vertex set V. Therefore, by Theorem 2.2.5, we obtain

$$\begin{aligned} d_r(M^*) &= \min\{|X| : X\subset E \text{ and } \rho(E\setminus X)=\rho(E)-r\} \\ &= \min\{|X| : X\subset E \text{ and } |V|-c_0(E\setminus X)=\rho(E)-r\} \end{aligned}$$

for $1\leq r\leq \rho(E)$. If G_σ is unbalanced (resp. balanced), then making $X=\emptyset$ in Eq. (2.3.2) we get $\rho(E)=|V|$ (resp. $\rho(E)=|V|-1$). Therefore

$$d_r(M^*)=\begin{cases}\min\{|X|\colon X\subset E \text{ and } c_0(E\setminus X)=r\}=\upsilon_r(G_\sigma), & \text{if } G_\sigma \text{ is unbalanced,}\\ \min\{|X|\colon X\subset E \text{ and } c(E\setminus X)=r+1\}=\lambda_r(G_\sigma), & \text{if } G_\sigma \text{ is balanced,}\end{cases}$$

where $c(E\setminus X)$ is the number of connected components of the signed subgraph $G_\sigma\setminus X$.

(iii): By Lemmas 2.3.3 and 2.3.4, the circuits of the dual matroid M^* of M are the cocircuits of the signed graph G_σ, and by Theorem 2.2.5 we get

$$d_r(M^*)=\min\{|\textstyle\bigcup_{i=1}^r C_i^*| : \{C_i^*\}_{i=1}^r \text{ are non-redundant circuits of } M^*\},\ 1\leq r\leq \eta^*(E).$$

Hence, the required equality follows noticing that $\eta^*(E)=\rho(E)$. $\square$

Definition 2.3.6. Let G_σ be a signed graph with s vertices $t_1,\ldots,t_s$ and m edges, let K be a field, and let $\mathbf{e}_i$ be the i-th unit vector in K^s. The *incidence matrix* of G_σ over the field K is the $s\times m$ matrix A whose column vectors are given by:

(i) $\mathbf{e}_i-\mathbf{e}_j$ (resp. $\mathbf{e}_i+\mathbf{e}_j$) if $e=\{t_i,t_j\}$ is a link with $\sigma(e)=+$ (resp. $\sigma(e)=-$);

(ii) $\mathbf{0}$ (resp. $2\mathbf{e}_i$) if $e=\{t_i,t_i\}$ is a loop with $\sigma(e)=+$ (resp. $\sigma(e)=-$).

Note that the columns of A are defined up to sign, one can pick $\mathbf{e}_i-\mathbf{e}_j$ or $\mathbf{e}_j-\mathbf{e}_i$ if $e=\{t_i,t_j\}$ is a link with $\sigma(e)=+$. To avoid ambiguity we could normalize and pick e_i-e_j if $i>j$. The order of the columns of A and the choice of sign have no significance for the invariants of linear codes, signed graphs, and Stanley-Reisner ideals that we want to study.

If G is a multigraph with vertices $t_1,\ldots,t_s$, the *incidence matrix* of G over a field K is the incidence matrix of the negative signed graph G_-, that is, the matrix whose columns are all vectors $\mathbf{e}_i+\mathbf{e}_j$ such that $\{t_i,t_j\}$ is an edge of G. A *digraph* $\mathcal{D}$ consists of a multigraph G with vertices $t_1, \ldots, t_s$ where all edges of G are directed from one vertex to another. The *edges* or *arrows* of $\mathcal{D}$ are ordered pairs of vertices (t_i, t_j) with $e = \{t_i, t_j\}$ an edge of G, where (t_i, t_j) represents the edge e directed from t_i to t_j. The *incidence matrix* of $\mathcal{D}$ over a field K is the incidence matrix of the positive signed graph G_+, that is, the matrix whose columns are all vectors $\mathbf{e}_i - \mathbf{e}_j$ such that (t_i, t_j) is an edge of $\mathcal{D}$.

Theorem 2.3.7. [43, Theorems 8B.1, 8B.2] *Let G_σ be a signed graph and let A be its incidence matrix over a field of characteristic p. The following hold.*

(a) If $p \neq 2$, then the vector matroid $M[A]$ of A is the signed-graphic matroid $M(G_\sigma)$.

(b) If $p = 2$, then $M[A]$ is the graphic matroid $M(G)$ of G.

Proposition 2.3.8. *Let G_σ be a signed graph with s vertices, c connected components, c_0 balanced connected components, and let A be its incidence matrix over a field K. Then*

$$\operatorname{rank}(A) = \begin{cases} s - c_0, & \text{if } \operatorname{char}(K) \neq 2, \\ s - c, & \text{if } \operatorname{char}(K) = 2 \text{ or } G_\sigma \text{ is balanced.} \end{cases}$$

Proof. Assume $\operatorname{char}(K) \neq 2$. By Theorem 2.3.7 (a), the signed graphic matroid $M(G_\sigma)$ is the vector matroid $M[A]$. According to [43, Theorem 5.1(j)], the rank of $M(G_\sigma)$ is $s - c_0$. Thus in this case $\operatorname{rank}(A) = s - c_0$. Assume $\operatorname{char}(K) = 2$. By Theorem 2.3.7 (b), the graphic matroid $M(G)$ is the vector matroid $M[A]$. If G is connected, then the bases of the matroid $M(G)$ are the spanning trees of G [41, p. 28], and $\operatorname{rank}(A) = s - 1$. As a consequence, if G has c components, one has $\operatorname{rank}(A) = s - c$. If G_σ is balanced, then $c = c_0$, and by the previous two cases $\operatorname{rank}(A) = s - c$, regardless of the characteristic of the field K. $\square$

Corollary 2.3.9. *Let $\mathcal{D}$ be a digraph with s vertices, c connected components, and let A be its incidence matrix over a field K. Then,* $\operatorname{rank}(A) = s - c$.

Proof. Let G be the underlying unoriented simple graph of $\mathcal{D}$. Consider the positive signed graph G_+. Note that G_+ is balanced. Since $\mathcal{D}$ and G_+ have the same incidence matrix, the result follows from Proposition 2.3.8. $\square$

Definition 2.3.10. The *incidence matrix code* of a signed graph G_σ (resp. multigraph G, digraph $\mathcal{D}$), over a finite field $\mathbb{F}_q$, is the linear code C generated by the rows of the incidence matrix of the signed graph G_σ (resp. multigraph G, digraph $\mathcal{D}$).

Corollary 2.3.11. *Let G_σ be a connected signed graph with s vertices and m edges, and let C be the incidence matrix code of G_σ over a finite field of characteristic p. Then*

(a) C (resp. $C^\perp$) is an $[m, s]$ (resp. $[m, m-s]$) code if $p \neq 2$ and G_σ is unbalanced.

(b) C (resp. $C^\perp$) is an $[m, s-1]$ (resp. $[m, m-s+1]$) code if $p = 2$ or G_σ is balanced.

Proof. This follows from Proposition 2.3.8 noticing that $\dim(C) + \dim(C^\perp) = m$. □

Definition 2.3.12. Let G be a multigraph. A *bowtie* of G is the union of two odd cycles which meet at a single vertex or the union of two vertex-disjoint odd cycles and a simple path which meets one cycle at each end and is otherwise disjoint from them.

Corollary 2.3.13. *Let G be a multigraph and let G_+ and G_- be the positive and negative signed graphs, respectively. The following hold.*

(a) The circuits of the signed-graphic matroid $M(G_+)$ are the cycles of G, that is, $M(G_+)$ is the graphic matroid $M(G)$ of G.

(b) The signed-graphic matroid $M(G_+)$ is the vector matroid, over any field K, of the incidence matrix of G_+ whose columns are of the form $\mathbf{e}_i - \mathbf{e}_j$.

(c) The balanced (resp. unbalanced) cycles of G_- are the even (resp. odd) cycles of G. A circuit of $M(G_-)$ is either an even cycle or a bowtie of G.

(d) If G_σ is a balanced signed graph, then $M(G_\sigma)$ is the graphic matroid $M(G)$ of G.

Proof. (a): There are no unbalanced cycles of G_+. Hence, by Theorem 2.3.1, the circuits of $M(G_+)$ are the cycles of G.

(b): Let p be the characteristic of the field K. If $p \neq 2$, by Theorem 2.3.7, $M(G_+)$ is the vector matroid of the incidence matrix A of G_+ and the columns of this matrix have the required form. If $p = 2$, the graphic matroid $M(G)$ of G is the vector matroid

$M[A]$ of the incidence matrix A of G [41, Theorem 3, p. 149]. By part (a), $M(G_+)$ is the graphic matroid of G. Thus $M(G_+)$ is the vector matroid of A. The columns of A have the required form because in this case $1 = -1$.

(c), (d): This follows readily from Theorem 2.3.1. $\square$

Remark 2.3.14. Let G be a multigraph and let A be the incidence matrix of G_+ over the field $K = \mathbb{Q}$ of rational numbers. Since $M[A]$ is the graphic matroid of G, to compute all cycles of G one can use *Macaulay*2 [11] and the package *Matroids* [5].

Corollary 2.3.15. [30, 36, 43] *If A is the incidence matrix of a multigraph G over a field of* $\operatorname{char}(K) \neq 2$*, then the circuits of the vector matroid $M[A]$ are the even cycles and bowties of G.*

Proof. It follows from Theorem 2.3.7(a) and Corollary 2.3.13(c) considering G_-. $\square$

Our main results on linear codes are the following. First, we give graph theoretical formulas for the GHWs for the incidence matrix code of a signed graph.

Theorem 2.3.16. *Let C be the incidence matrix code of a connected signed graph G_σ with s vertices, r-th cogirth $\upsilon_r(G_\sigma)$, r-th edge connectivity $\lambda_r(G)$, over a finite field K of* $\operatorname{char}(K) = p$*. Then, the r-th generalized Hamming weight of C is given by*

$$\delta_r(C) = \begin{cases} \upsilon_r(G_\sigma), & \text{if } p \neq 2,\ G_\sigma \text{ is unbalanced and } 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } p = 2 \text{ and } 1 \leq r \leq s-1, \\ \lambda_r(G), & \text{if } G_\sigma \text{ is balanced and } 1 \leq r \leq s-1. \end{cases}$$

Proof. Let A be the incidence matrix of G_σ and let ρ be the rank function of the vector matroid $M = M[A]$. According to Proposition 2.3.8, $\rho(M) = s$ if $p \neq 2$ and G_σ is unbalanced, and $\rho(M) = s - 1$ if $p = 2$ or G_σ is balanced.

Assume that $p \neq 2$. By Theorem 2.3.7(a), the signed-graphic matroid $M(G_\sigma)$ is the vector matroid $M = M[A]$. Hence, using Lemma 2.2.4 and Theorem 2.3.5(ii), one has

$$\delta_r(C) = d_r(M^*) = \begin{cases} \upsilon_r(G_\sigma), & \text{if } G_\sigma \text{ is unbalanced and } 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } G_\sigma \text{ is balanced and } 1 \leq r \leq s-1. \end{cases}$$

Assume that $p = 2$. By Theorem 2.3.7(b), $M = M[A]$ is the graphic matroid $M(G)$ and, by Corollary 2.3.13(a), $M(G_+)$ is also the graphic matroid $M(G)$. As $M(G_+)$ is balanced, by Lemma 2.2.4 and Theorem 2.3.5(ii), we get $\delta_r(C) = d_r(M^*) = \lambda_r(G_+) = \lambda_r(G)$. $\square$

Let G be a multigraph. The r-th cogirth $\upsilon_r(G_-)$ of G_- is the minimum number of edges whose removal results in a multigraph with r bipartite connected components. If $r = 1$, $\upsilon_1(G_-)$ is denoted $\upsilon(G_-)$. For simple graphs, the following combinatorial formulas for the generalized Hamming weights were shown in [23].

Corollary 2.3.17. *Let C be the incidence matrix code of a connected multigraph G with s vertices over a finite field K of* $\mathrm{char}(K) = p$*. Then*

$$\delta_r(C) = \begin{cases} \upsilon_r(G_-), & \text{if } p \neq 2,\ G \text{ is non-bipartite and } 1 \leq r \leq s, \\ \lambda_r(G), & \text{if } p = 2 \text{ and } 1 \leq r \leq s-1, \\ \lambda_r(G), & \text{if } G \text{ is bipartite and } 1 \leq r \leq s-1. \end{cases}$$

Proof. It follows from Theorem 2.3.16 by considering the negative signed graph G_- and noticing that G_- is balanced if and only if G is bipartite. □

The next result shows that the combinatorial formulas of Dankelmann, Key and Rodrigues [6, Theorems 1–3] for the minimum distance of C can be extended to signed graphs.

Corollary 2.3.18. *Let C be the incidence matrix code of a connected signed graph G_σ with s vertices, cogirth $\upsilon(G_\sigma)$, edge connectivity $\lambda(G_\sigma)$, over a finite field K of* $\mathrm{char}(K) = p$*. Then, the minimum distance $\delta(C)$ of C is given by*

$$\delta(C) = \begin{cases} \upsilon(G_\sigma), & \text{if } p \neq 2,\ G_\sigma \text{ is unbalanced and } 1 \leq r \leq s, \\ \lambda(G_\sigma), & \text{if } p = 2 \text{ and } 1 \leq r \leq s-1, \\ \lambda(G_\sigma), & \text{if } G_\sigma \text{ is balanced and } 1 \leq r \leq s-1. \end{cases}$$

Proof. It follows making $r = 1$ in Theorem 2.3.16. □

Our next result gives graph theoretical formulas for the generalized Hamming weights of the dual code of the incidence matrix code of a signed graph.

Theorem 2.3.19. *Let G_σ be a connected signed graph with s vertices and m edges, and let C be the incidence matrix code of G_σ over a finite field K of characteristic p. The following hold.*

(a) If $p = 2$ or G_σ is balanced, and $1 \leq r \leq m - s + 1$ (resp. $1 \leq r \leq s - 1$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant cycles (resp. cocycles) of G.

(b) If $p \neq 2$ and $1 \leq r \leq m - s$ (resp. $1 \leq r \leq s$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant circuits (resp. cocircuits) of G_σ.

Proof. (a): Assume $p = 2$ and $1 \leq r \leq m-s+1$. Let A be the incidence matrix of G_σ. By Theorem 2.3.7(b), the vector matroid $M = M[A]$ is the graphic matroid $M(G)$. Thus the circuits of $M[A]$ are the cycles of G. Therefore, by Corollary 2.2.6, we get

$$\delta_r(C^\perp) = \min\{|\bigcup_{i=1}^r C_i| : \{C_i\}_{i=1}^r \text{ are non-redundant cycles of } G\}.$$

Assume $p = 2$ and $1 \leq r \leq s - 1$. By the previous part, $M = M[A]$ is the graphic matroid $M(G)$. The circuits of $M^* = M[A]^*$ are the cocycles of G [41, p. 41], that is, these are edge sets X whose removal from G increases the number of connected components of G and are minimal with respect to this property. The dual matroid M^* of M is the vector matroid of $C^\perp$ [41, p. 141]. Therefore, by Corollary 2.2.6 and the equality $C = (C^\perp)^\perp$, we get

$$\begin{aligned} \delta_r(C) &= \min\{|\textstyle\bigcup_{i=1}^r C_i^*| : \{C_i^*\}_{i=1}^r \text{ are non-redundant circuits of } M^*\} \\ &= \min\{|\textstyle\bigcup_{i=1}^r C_i^*| : \{C_i^*\}_{i=1}^r \text{ are non-redundant cocycles of } G\}. \end{aligned}$$

Assume that G_σ is balanced. If $p = 2$, the formulas for $\delta_r(C^\perp)$ and $\delta_r(C)$ follow from the two previous cases. Assume $p \neq 2$. By Theorem 2.3.7(a), the vector matroid $M[A]$ is the signed-graphic matroid $M(G_\sigma)$ and, by Corollary 2.3.13(d), $M(G_\sigma)$ is the graphic matroid $M(G)$. Hence, we can proceed as in the previous cases.

(b): Assume $1 \leq r \leq m - s$. Let A be the incidence matrix of G_σ. By Theorem 2.3.1, the circuits of $M(G_\sigma)$ are the balanced cycles and bowties of G_σ. As $p \neq 2$, by Theorem 2.3.7(a), the vector matroid $M = M[A]$ is $M(G_\sigma)$. Therefore the circuits of $M[A]$ are the balanced cycles and bowties of G_σ, and by Corollary 2.2.6 one has

$$\delta_r(C^\perp) = \min\{|\bigcup_{i=1}^r C_i| : \{C_i\}_{i=1}^r \text{ are non-redundant circuits of } M\}.$$

Assume $1 \leq r \leq s$. As $M = M[A]$ is the signed-graphic matroid $M(G_\sigma)$, the circuits of $M^* = M[A]^*$ are the cocircuits of G_σ by Lemmas 2.3.3 and 2.3.4. The dual matroid M^* of M is the vector matroid of $C^\perp$ [41, p. 141]. Hence, by Corollary 2.2.6 and noticing $C = (C^\perp)^\perp$, we get

$$\begin{aligned}\delta_r(C) &= \min\{|\textstyle\bigcup_{i=1}^r C_i^*| : \{C_i^*\}_{i=1}^r \text{ are non-redundant circuits of } M^*\}\\ &= \min\{|\textstyle\bigcup_{i=1}^r C_i^*| : \{C_i^*\}_{i=1}^r \text{ are non-redundant cocircuits of } G_\sigma\}\,.\end{aligned}$$

This completes the proof of part (b). $\square$

Corollary 2.3.20. *Let C be the incidence matrix code, over a finite field $K = \mathbb{F}_q$, of a connected digraph $\mathcal{D}$ with s vertices and m edges, and let G be its underlying multigraph. Then*

(a) $\delta_r(C) = \lambda_r(G)$ *for* $1 \le r \le s-1$.

(b) If $1 \le r \le m-s+1$ (resp. $1 \le r \le s-1$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant cycles (resp. cocycles) of G.

Proof. Parts (a) and (b) follow from Theorems 2.3.16 and 2.3.19, respectively, by considering the positive signed graph G_+ and noticing that this is a balanced signed graph. $\square$

Corollary 2.3.21. *Let G be a connected multigraph with s vertices and m edges, and let C be its incidence matrix code over a finite field $K = \mathbb{F}_q$ of characteristic p. The following hold.*

(a) If $p = 2$ or G is bipartite, and $1 \le r \le m-s+1$ (resp. $1 \le r \le s-1$), then $\delta_r(C^\perp)$ (resp. $\delta_r(C)$) is the minimum number of edges of G forming a union of r non-redundant cycles (resp. cocycles) of G.

(b) If $p \neq 2$ and $1 \le r \le m-s$, then $\delta_r(C^\perp)$ is the minimum number of edges of G forming a union of r non-redundant even cycles and bowties of G.

Proof. (a): This follows from Theorem 2.3.19(a) noticing that, if G is bipartite, then the circuits of $M(G_+)$ and $M(G_-)$ are the cycles of G.

(b): This part follows from Corollary 2.3.13(c) and Theorem 2.3.19(b), by considering the negative signed graph G_-. $\square$

2.4 The regularity of the ideal of circuits

Let M be a matroid on $E = \{1, \dots, m\}$, let Δ be its independence complex, that is, the faces of Δ are the independent sets of M, and let $R = \mathbb{Q}[x_1, \dots, x_m] = \bigoplus_{d=0}^{\infty} R_d$ be a polynomial ring with the standard grading over the field of rational numbers. It is convenient also to think of E as the set of variables $\{x_1, \dots, x_m\}$. The Stanley-Reisner ideal I_Δ of Δ, in the sense of [33], is the *edge ideal* $I(\mathcal{C}_M)$ of the clutter of circuits $\mathcal{C}_M$ of M, that is, I_Δ is the *ideal of circuits* of M generated by all squarefree monomials $\prod_{i \in X} x_i$ such that X is a circuit of M.

The simplicial complex Δ is pure shellable, in particular the ideal I_Δ is Cohen-Macaulay, and the graded Betti numbers of the Stanley-Reisner ring $K[\Delta] = R/I_\Delta$ are the same if we replace $\mathbb{Q}$ by any other field (see [16, Remark 1, p. 78] and the references therein).

Definition 2.4.1. Let $I \subset R$ be a graded ideal and let $\mathbf{F}$ be the minimal graded free resolution of R/I as an R-module:

$$\mathbf{F}: \quad 0 \to \bigoplus_j R(-j)^{\beta_{g,j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}} \to R \to R/I \to 0.$$

The (i,j)-th *graded Betti number* of R/I, denoted $\beta_{i,j}(R/I)$, is $\beta_{i,j}$, the integer j is a *shift* of the resolution, g is the *projective dimension* of R/I, and the *regularity* of R/I is

$$\operatorname{reg}(R/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

If R/I is Cohen-Macaulay (i.e. $g = \dim(R) - \dim(R/I)$) and there is a unique j such that $\beta_{g,j} \neq 0$, then the ring R/I is called *level*.

An excellent reference for the regularity of graded ideals and Betti numbers is the book of Eisenbud [7]. The shifts and the Betti numbers of the Stanley-Reisner ring of the independence complex Δ of a matroid M were determined by Johnsen and Verdure [16].

The following result shows that one can read the generalized Hamming weights of a matroid M from the minimal graded free resolution of the ideal of circuits of M.

Theorem 2.4.2. [16, Theorem 2] *Let M be a matroid, let R/I_Δ be the Stanley-Reisner ring of the independence complex Δ of M, and let $\beta_{r,j}(M)$ denote the (r,j)-th graded*

Betti number of R/I_Δ. Then the generalized Hamming weights of M are given by

$$d_r(M) = \min\{j : \beta_{r,j}(M) \neq 0\} \quad \textit{for} \quad 1 \leq r \leq \eta(M).$$

Corollary 2.4.3. *Let C be a linear $[m,k]$ code and let R/I_Δ be the Stanley-Reisner ring of the independence complex of the matroid of C. Then*

$$\delta_r(C^\perp) = \min\{j : \beta_{r,j}(R/I_\Delta) \neq 0\} \quad \textit{for} \quad 1 \leq r \leq m-k.$$

Proof. It follows from Lemma 2.2.4 and Theorem 2.4.2. □

The following notion of a non-degenerate code will play a role here.

Definition 2.4.4. If $C \subset K^m$ is a linear code and π_i is the i-th projection map

$$\pi_i : C \to K, \quad (v_1, \ldots, v_m) \mapsto v_i,$$

for $i = 1, \ldots, m$, we say that C is *degenerate* if for some i the image of π_i is zero, otherwise we say that C is *non-degenerate*.

Remark 2.4.5. If $C \subset K^m$ is a non-degenerate linear code, then $\delta_k(C) = m$, where k is the dimension of C. If all columns of a generator matrix of C are nonzero, then $\pi_i \neq 0$ for $i = 1, \ldots, m$ and C is non-degenerate.

Lemma 2.4.6. *Let M be the matroid on E of a linear code C and let Δ (resp. Δ^*) be the independence complex of M (resp. M^*). The following hold.*

(a) If $r = \dim(C^\perp)$, then $\operatorname{reg}(R/I_\Delta) = \delta_r(C^\perp) - \dim(C^\perp)$.

(b) If $r = \dim(C)$, then $\operatorname{reg}(R/I_{\Delta^*}) = \delta_r(C) - \dim(C)$.

(c) If $C^\perp$ is non-degenerate, then $\operatorname{reg}(R/I_\Delta) = \dim(C)$.

(d) If C is non-degenerate, then $\operatorname{reg}(R/I_{\Delta^*}) = \dim(C^\perp)$.

Proof. By [38, Corollary 6.3.5], the Stanley-Reisner ring $K[\Delta] := R/I_\Delta$ has Krull dimension $\dim(\Delta)+1$. As $\dim(\Delta)$ is $\rho(M)-1$ and $\dim(R) = |E|$, one has $|E| - \operatorname{ht}(I_\Delta) = \rho(M)$, where $\operatorname{ht}(I_\Delta)$ is the height of the ideal I_Δ. Therefore

$$\begin{aligned} \eta(M) &= |E| - \rho(M) = \operatorname{ht}(I_\Delta) \\ &= |E| - \dim(C) = \dim(C^\perp) = \rho(M^*). \end{aligned}$$

Thus $\eta(M) = \mathrm{ht}(I_\Delta) = \dim(C^\perp) = \rho(M^*)$. The independence complex Δ is pure shellable [16, Remark 1, p. 78]. Hence I_Δ is Cohen–Macaulay, that is, $\mathrm{ht}(I_\Delta)$ is equal to $\mathrm{pd}_R(R/I_\Delta)$, the projective dimension of R/I_Δ. Let $\beta_{r,j}$ be the (r,j)-th graded Betti number of R/I_Δ, with $r = \eta(M) = \mathrm{pd}_R(R/I_\Delta)$. According to [33, Theorem 3.4], the ring R/I_Δ is level. Therefore, making $r = \dim(C^\perp)$ in Corollary 2.4.3, we get

$$\begin{aligned}\mathrm{reg}(R/I_\Delta) &= \max\{j - r \mid \beta_{r,j} \neq 0\} = \min\{j - r \mid \beta_{r,j} \neq 0\}\\ &= \min\{j \mid \beta_{r,j} \neq 0\} - r = \delta_r(C^\perp) - \dim(C^\perp).\end{aligned}$$

Thus the equality of (a) holds. The equality of (b) follows from (a) using duality. Parts (c) and (d) follow readily from Remark 2.4.5. □

Theorem 2.4.7. *Let G_σ be a signed graph without loops with s vertices, m edges, c connected components, c_0 balanced components, let M be the matroid on E of the incidence matrix code C of G_σ, over a finite field of characteristic p, and let Δ (resp. Δ^*) be the independence complex of M (resp. M^*). The following hold.*

$$\mathrm{reg}(R/I) = \begin{cases} m - s + c_0, & \text{if } I = I_{\Delta^*},\ \ p \neq 2,\\ m - s + c, & \text{if } I = I_{\Delta^*},\ \ p = 2 \ \text{ or } G_\sigma \text{ is balanced},\\ s - c_0, & \text{if } I = I_\Delta,\ p \neq 2,\ \text{and any } i \in E \text{ is in some circuit of } M,\\ s - c, & \text{if } I = I_\Delta,\ p = 2 \text{ or } G_\sigma \text{ is balanced},\ \text{and } G \text{ has no bridges.}\end{cases}$$

Proof. Let A be the incidence matrix of G_σ. As G_σ has no loops, all columns of A are nonzero, that is, C is non-degenerate. Hence, the first two formulas follow at once from Proposition 2.3.8, Lemma 2.4.6(d), and the equality $\dim(C^\perp) = m - \dim(C)$.

Assume that $p \neq 2$ and suppose any $i \in E$ is in some circuit of M. Let H be the parity check matrix of C whose rows correspond to the circuits of C (see the discussion below). The matrix H is a generator matrix for $C^\perp$ and M^* is the vector matroid $M[H]$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the column vectors of A. Take any $i \in E$, then i is in some circuit $X \subset E$ of M. Then $\sum_{j \in X} \lambda_j \mathbf{v}_j = 0$, where $\lambda_j \neq 0$ for $j \in X$. Setting $\lambda_j = 0$ for $j \in E \setminus X$, we get that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a row of H and $\lambda_i \neq 0$. Thus the i-th column of H is nonzero for $i = 1, \ldots, m$, that is, $C^\perp$ is non-degenerate. Therefore, the third formula follows from Proposition 2.3.8 and Lemma 2.4.6(c).

Assume that $p = 2$ or G_σ is balanced, and suppose G has no bridges. Then the vector matroid M is the graphic matroid of G. As G has no bridges, i.e., any edge belongs to a cycle, one has that every edge is in some circuit of M. Hence, by the previous part, $C^{\perp}$ is non-degenerate. Hence, by Proposition 2.3.8 and Lemma 2.4.6(c), the fourth equality follows. $\square$

2.5 An algebraic formula for the frustration index

Let G_σ be a connected signed simple graph with s vertices, m edges, frustration index $\varphi(G_\sigma)$, and let $V(G_\sigma)=\{t_1,\ldots,t_s\}$ be its vertex set. For use below, $\mathbb{X}$ will denote the set of projective points in the projective space $\mathbb{P}^{s-1}$ defined by the column vectors of the incidence matrix of G_σ over a field K of $\mathrm{char}(K)\neq 2$. Consider a polynomial ring $S=K[t_1,\ldots,t_s]=\bigoplus_{d=0}^{\infty}S_d$ over a field K with the standard grading. Given a homogeneous polynomial h in S, that is, $h\in S_d$ for some d, we denote the set of zeros of h in $\mathbb{X}$ by $V_{\mathbb{X}}(h)$. The *vanishing ideal* of $\mathbb{X}$, denoted $I(\mathbb{X})$, is the ideal of S generated by the homogeneous polynomials that vanish at all points of $\mathbb{X}$.

The following characterization of balanced signed graphs is due to Harary [12]. For other characterizations of this property see [43] and the references therein.

Theorem 2.5.1. ([12, Theorem 3], [43, Proposition 2.1]) *A signed simple graph is balanced if and only if its vertex set can be partitioned into two disjoint classes (possible empty), such that an edge is negative if and only if its two endpoints belong to distinct classes.*

Lemma 2.5.2. *Let G_σ be a connected signed simple graph over a field K of* $\mathrm{char}(K)\neq 2$. *Then*

$$\varphi(G_\sigma)=\min\{|\mathbb{X}\setminus V_{\mathbb{X}}(h)|\colon\ h=a_1t_1+\cdots+a_st_s,\ a_i\in\{\pm 1\}\ \textit{for all}\ i\}. \tag{2.5.1}$$

Proof. Let $\mathbf{v}_1,\ldots,\mathbf{v}_m$ be the column vectors of the incidence matrix of G_σ. We set $r=\varphi(G_\sigma)$ and let r_0 be the right-hand side of Eq. (2.5.1). If G_σ is balanced, using Theorem 2.5.1, it is not hard to see that there is a linear polynomial $h=a_1t_1+\cdots+a_st_s$, $a_i\in\{\pm1\}$ for all i, such that $h(\mathbf{v}_i)=0$ for all i, that is, $r_0=0$ and $\varphi(G_\sigma)=r_0$ (see the discussion below). Thus we may assume that G_σ is not balanced. Pick a minimum set of edges $e_1,\ldots,e_r$ such that the signed subgraph $H_\sigma=G_\sigma\setminus\{e_1,\ldots,e_r\}$ is balanced. We may assume that $\{e_1,\ldots,e_r,\ldots,e_m\}$ is the set of edges of G_σ and that e_i corresponds to $\mathbf{v}_i$ for $i = 1, \ldots, m$. We first show the inequality $r \geq r_0$. Note that $V(H_\sigma) = V(G_\sigma)$. According to Theorem 2.5.1, the vertex set of H_σ can be partitioned into two disjoint classes V_1 and V_2 (possible empty) is such a way that an edge of H_σ is negative if and only if its two endpoints belong to distinct classes. We set

$$h := \sum_{t_i \in V_1} t_i - \sum_{t_i \in V_2} t_i.$$

To show the inequality $r \geq r_0$ it suffices to show the equality $V_{\mathbb{X}}(h) = \{\mathbf{v}_i\}_{i=r+1}^m$ because this equality implies $r = |\mathbb{X} \setminus V_{\mathbb{X}}(h)|$, and consequently $r = \varphi(G_\sigma) \geq r_0$.

Case (I): $V_2 = \emptyset$. Therefore, $\sigma(e) = +$ for $e \in E(H_\sigma)$. As $h = \sum_{i=1}^s t_i$, one has the inclusion $\{\mathbf{v}_i\}_{i=r+1}^m \subset V_{\mathbb{X}}(h)$. We claim that $\sigma(e_i) = -$ for $i = 1, \ldots, r$. If $\sigma(e_i) = +$ for some $1 \leq i \leq r$, then $G_\sigma \setminus \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$ is balanced because it is a positive signed graph, a contradiction. As $h = \sum_{i=1}^s t_i$, the inclusion $V_{\mathbb{X}}(h) \subset \{\mathbf{v}_i\}_{i=r+1}^m$ follows because $\operatorname{char}(K) \neq 2$.

Case (II): $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. If $1 \leq i \leq r$ and e_i joins V_1 and V_2, then $\sigma(e_i) = +$ and $h(\mathbf{v}_i) \neq 0$ because $\operatorname{char}(K) \neq 2$. Indeed, if $\sigma(e_i) = -$, then $G_\sigma \setminus \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$ is balanced by Theorem 2.5.1, a contradiction. If $1 \leq i \leq r$ and the two endpoints of e_i are both in V_1 or V_2, then $\sigma(e_i) = -$ and $h(\mathbf{v}_i) \neq 0$ because $\operatorname{char}(K) \neq 2$. Indeed, if $\sigma(e_i) = +$, then $G_\sigma \setminus \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_r\}$ is balanced by Theorem 2.5.1, a contradiction. Thus, one has the inclusion $V_{\mathbb{X}}(h) \subset \{\mathbf{v}_i\}_{i=r+1}^m$. If $i > r$, then $h(\mathbf{v}_i) = 0$, that is, $\{\mathbf{v}_i\}_{i=r+1}^m \subset V_{\mathbb{X}}(h)$. This follows noticing that, for $i > r$, one has $\sigma(e_i) = +$ if the endpoints of e_i are in V_1 or V_2, and $\sigma(e_i) = -$ if e_i joins V_1 and V_2. Therefore the equality $V_{\mathbb{X}}(h) = \{\mathbf{v}_i\}_{i=r+1}^m$ holds.

Now, we show the inequality $r \leq r_0$. Pick $h = a_1 t_1 + \cdots + a_s t_s$, $a_i = \pm 1$ for $i = 1, \ldots, s$, such that $r_0 = |\mathbb{X} \setminus V_{\mathbb{X}}(h)|$. We may assume that the set $\mathbb{X} \setminus V_{\mathbb{X}}(h)$ is equal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r_0}\}$, and we may also assume that $\{e_1, \ldots, e_{r_0}, \ldots, e_m\}$ is the set of edges of G_σ and that e_i corresponds to $\mathbf{v}_i$ for $i = 1, \ldots, m$. It suffices to show that the signed subgraph $H_\sigma = G_\sigma \setminus \{e_1, \ldots, e_{r_0}\}$ is balanced because this implies that $r = \varphi(G_\sigma) \leq r_0$. There are disjoint sets V_1 and V_2 (possibly empty) such that $V(G_\sigma) = \{t_1, \ldots, t_s\} = V_1 \cup V_2$ and

$$h = \sum_{t_i \in V_1} t_i - \sum_{t_i \in V_2} t_i.$$

Note that $h(\mathbf{v}_i) = 0$ if and only if $i > r_0$ and $E(H_\sigma) = \{e_i\}_{i=r_0+1}^m$. If $\sigma(e_i) = -$ for some $i > r_0$, then $h(\mathbf{v}_i) = 0$, and consequently e_i joins V_1 and V_2 because $\operatorname{char}(K) \neq 2$. If $\sigma(e_i) = +$ for some $i > r_0$, then $h(\mathbf{v}_i) = 0$, and consequently the endpoints of e_i are in V_1 or V_2. Therefore, by Theorem 2.5.1, $H_\sigma = G_\sigma \setminus \{e_1, \dots, e_{r_0}\}$ is balanced. $\square$

Let $I \neq (0)$ be a graded ideal of S of Krull dimension k. The *Hilbert function* of S/I is:

$$H_I(d) := \dim_K(S_d/I_d), \quad d = 0, 1, 2, \dots,$$

where $I_d = I \cap S_d$. By a theorem of Hilbert [32, p. 58], there is a unique polynomial $h_I(x) \in \mathbb{Q}[x]$ of degree $k-1$ such that $H_I(d) = h_I(d)$ for $d \gg 0$. The degree of the zero polynomial is -1.

The *degree* or *multiplicity* of S/I, denoted $\deg(S/I)$, is the positive integer given by

$$\deg(S/I) := (k-1)! \lim_{d\to\infty} H_I(d)/d^{k-1} \ \text{ if } \ k \geq 1,$$

and $\deg(S/I) = \dim_K(S/I)$ if $k = 0$. If $f \in S$, the ideal $(I\colon f) = \{g \in S \,|\, gf \in I\}$ is referred to as a *colon ideal*. Note that f is a zero-divisor of S/I if and only if $(I\colon f) \neq I$.

Lemma 2.5.3. [21, Lemma 3.2] *Let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{s-1}$ over a field K and let $I(\mathbb{X}) \subset S$ be its vanishing ideal. If $0 \neq f \in S$ is homogeneous and $(I(\mathbb{X})\colon f) \neq I(\mathbb{X})$, then*

$$|V_{\mathbb{X}}(f)| = \deg(S/(I(\mathbb{X}), f)).$$

The following algebraic formula for the frustration index can be used to compute or estimate this number using *Macaulay*2 [11] (Example 2.6.6).

Theorem 2.5.4. *Let G_σ be a connected unbalanced signed simple graph with frustration index $\varphi(G_\sigma)$ over a field K of $\operatorname{char}(K) \neq 2$, and let $\mathcal{F}$ be the set of linear forms $h = \sum_{i=1}^s a_i t_i$ such that $a_i = \pm 1$ for all i and $(I(\mathbb{X})\colon h) \neq I(\mathbb{X})$. Then*

$$\varphi(G_\sigma) = |\mathbb{X}| - \max\{\deg(S/(I(\mathbb{X}), h))\colon\ h \in \mathcal{F}\}.$$

Proof. The vanishing ideal $I(\mathbb{X})$ does not contains linear forms. This follows noticing that the incidence matrix of G_σ has rank equal to s, the number of vertices of G_σ, because G_σ is unbalanced and connected (see Proposition 2.3.8). Thus $\mathbb{X} \setminus V_{\mathbb{X}}(h) \neq \emptyset$

for any $0 \neq h \in S_1$. If h is a linear form, by [9, Lemma 3.1], $V_{\mathbb{X}}(h) \neq \emptyset$ if and only if $(I(\mathbb{X})\colon h) \neq I(\mathbb{X})$. Therefore, using Lemmas 2.5.2 and 2.5.3, we obtain

$$
\begin{aligned}
\varphi(G_\sigma) &= \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(h)| \colon\ h = \textstyle\sum_{i=1}^{s} a_i t_i,\ a_i = \pm 1 \text{ for all } i\} \\
&= \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(h)| \colon\ h = \textstyle\sum_{i=1}^{s} a_i t_i,\ a_i = \pm 1 \text{ for all } i \text{ and } V_{\mathbb{X}}(h) \neq \emptyset\} \\
&= \min\{|\mathbb{X} \setminus V_{\mathbb{X}}(h)| \colon\ h \in \mathcal{F}\} = |\mathbb{X}| - \max\{|V_{\mathbb{X}}(h)| \colon\ h \in \mathcal{F}\} \\
&= |\mathbb{X}| - \max\{\deg(S/(I(\mathbb{X}), h)) \colon\ h \in \mathcal{F}\}.
\end{aligned}
$$

The second equality follows discarding all h with $V_{\mathbb{X}}(h) = \emptyset$. ☐

Remark 2.5.5. If we allow the coefficients $a_1, \ldots, a_s$ to be in $\{0, \pm 1\}$ such that not all of them are zero, we obtain the minimum distance of the incidence matrix code C of G_σ over any finite field of characteristic $p \neq 2$. This follows from the results of Section 2.3 and Proposition 2.5.6 below.

The following algebraic formula for the minimum distance of an incidence matrix code can be used to compute or estimate this number using *Macaulay*2 [11] and the algorithms of [9, 21].

Proposition 2.5.6. *Let G_σ be a connected signed simple graph and let C be its incidence matrix code over a finite field K. Then the minimum distance of C is given by*

$$
\delta(C) = |\mathbb{X}| - \max\{\deg(S/(I(\mathbb{X}), h)) \colon\ h \in S_1 \setminus I(\mathbb{X}) \text{ and } (I(\mathbb{X})\colon h) \neq I(\mathbb{X})\}.
$$

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the column vectors of the incidence matrix of G_σ and let P_i be the point $[\mathbf{v}_i]$ in $\mathbb{P}^{s-1}$ for $i = 1, \ldots, m$. Thus, $\mathbb{X}$ is the set of points $\{P_1, \ldots, P_m\}$. Note that C is the image of S_1—the vector space of linear forms of S—under the evaluation map

$$
\mathrm{ev}_1 \colon S_1 \to K^m, \quad h \mapsto (h(P_1), \ldots, h(P_m)).
$$

The image of the linear function t_i, under the map ev_1, gives the i-th row of C. This means that C is the Reed-Muller-type code $C_{\mathbb{X}}(1)$ in the sense of [10]. The result now follows readily applying [21, Theorem 4.7]. ☐

2.6 Examples of signed graphs

In this section we illustrate how to use our results in practice with some examples.

Example 2.6.1. Let G_σ be a signed simple graph whose underlying graph G is given in Figure 2.1, let C be the incidence matrix code of G_σ, let A be the incidence matrix of G_σ, and let $M = M[A]$ be the matroid of C. Assume that K is either a field of characteristic 2 or that K is any field and $G_\sigma = +G$. In either case, by Theorem 2.3.7(b) and Corollary 2.3.13(d), M is the cycle matroid of G and, by Proposition 2.3.8, the rank of M is 10.

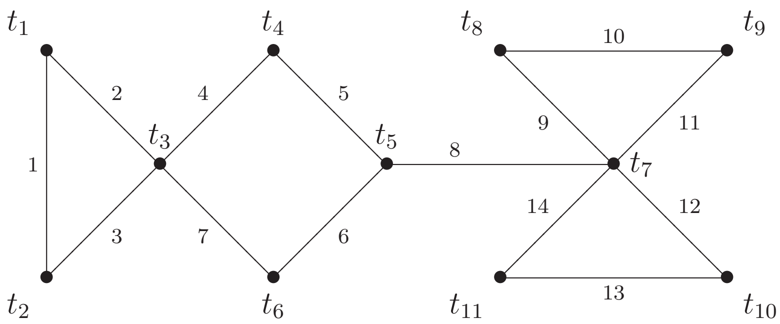

Figure 2.1: Simple graph G with 11 vertices and 14 edges

Therefore, the circuits of M are the cycles of G and they are given by

$$c_1 = \{1,2,3\}, \quad c_2 = \{9,10,11\}, \quad c_3 = \{12,13,14\}, \quad c_4 = \{4,5,6,7\}.$$

Hence, applying Theorem 2.3.19(a), we get the generalized Hamming weights of $C^\perp$:

r	1	2	3	4
$\delta_r(C^\perp)$	3	6	9	13

Concretely, one has $\delta_r(C^\perp) = |c_1 \cup \cdots \cup c_r|$ for $1 \leq r \leq 4$. Let $R = K[x_1, \ldots, x_{14}]$ be a polynomial ring over the field K. The ideal of circuits of M is the squarefree monomial ideal $I = I(\mathcal{C}_M)$ of R generated by all $\prod_{j\in c_i} x_j$ with $i = 1, \ldots, 4$. Using *Macaulay*2 [11], we obtain that the minimal free resolution of R/I is:

$$0 \to R(-13) \to R(-9) \oplus R^3(-10) \to R^3(-6) \oplus R^3(-7) \to R^3(-3) \oplus R(-4) \to R.$$

One can verify the values of the $\delta_r(C^\perp)$'s applying Corollary 2.4.3 to this resolution. By Wei's duality (Theorem 2.2.7), one has

r	1	2	3	4	5	6	7	8	9	10
$\delta_r(C)$	1	3	4	5	7	8	10	11	13	14

.

According to Theorem 2.3.16, $\delta_r(C) = \lambda_r(C)$ for $r = 1, \ldots, 10$. Removing edge 8 from G, we get two connected components. Thus $\delta_1(C) = 1$. To illustrate the equality $\delta_7(C) = 10$, note that removing the ten edges that are not in the square of the graph G results in a subgraph with eight connected components, and $\lambda_7(C) = 10$. The edge biparticity of G is $\varphi(-G) = 3$.

Example 2.6.2. Let G be the graph of Figure 2.1, let K be a field of $\mathrm{char}(K) \neq 2$, and let C be the incidence matrix code of G_-. By Corollary 2.3.13(c), the circuits of the negative signed graph G_-, that is, the circuits of the signed-graphic matroid $M(G_-)$, are the even cycles and the bowties of G:

$$\begin{aligned} &c_1 = \{4,5,6,7\},\ c_2 = \{9,10,11,12,13,14\},\\ &c_3 = \{1,2,3,4,5,8,9,10,11\},\ c_4 = \{1,2,3,4,5,8,12,13,14\},\\ &c_5 = \{1,2,3,6,7,8,9,10,11\},\ c_6 = \{1,2,3,6,7,8,12,13,14\}. \end{aligned}$$

Hence, by Theorem 2.3.19(b), it follows that $\delta_r(C^\perp) = |c_1 \cup \cdots \cup c_r|$ for $1 \leq r \leq 3$, and we obtain the generalized Hamming weights of $C^\perp$:

r	1	2	3
$\delta_r(C^\perp)$	4	10	14

.

Let $R = K[x_1, \ldots, x_{14}]$ be a polynomial ring over the field K and let $I = I(\mathcal{C}_M) \subset R$ be the ideal of circuits of the signed-graphic matroid $M(G_-)$. Using *Macaulay*2 [11], we obtain that the minimal free resolution of R/I is:

$$0 \to R^4(-14) \to R(-10) \oplus R^4(-11) \oplus R^4(-12) \to R(-4) \oplus R(-6) \oplus R^4(-9) \to R.$$

One can verify the values of the $\delta_r(C^\perp)$'s applying Corollary 2.4.3 to this resolution. By Wei's duality (Theorem 2.2.7), we obtain the generalized Hamming weights of C:

r	1	2	3	4	5	6	7	8	9	10	11
$\delta_r(C)$	2	3	4	6	7	8	9	10	12	13	14

.

According to Theorem 2.3.16, $\delta_r(C) = \upsilon_r(G_-)$ for $r = 1, \ldots, 11$. Next we verify these values. Removing edges 2 and 3 from G, we get a graph with a bipartite component. Therefore, by Theorem 2.3.16, $\delta_1(C) = 2$. To check the other values of $\delta_r(C)$ using by Theorem 2.3.16, note that successively removing from the graph G the edges

$$\{1,2\},\ 3,\ 8,\ \{10,13\},\ 9,\ 11,\ 12,\ 14,\ \{4,5\},\ 6,\ 7,$$

we obtain a subgraph with r bipartite connected components at the r-th step. By Theorem 2.4.7, the regularity of R/I is 11. The frustration index of G_- is 3 which is the edge biparticity of G.

Example 2.6.3. Let G_σ be the signed graph of Figure 2.2, let C be the incidence matrix code of G_σ over a finite field of $\mathrm{char}(K) = p \neq 2$, and let $M = M[A]$ be the vector matroid of C, where A is the incidence matrix of G_σ.

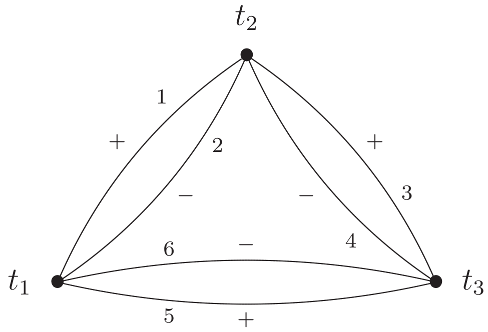

Figure 2.2: Signed graph with 3 vertices and 6 edges

The incidence matrix of the signed graph G_σ is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Using Procedure 2.7.1, we obtain the following information. The ideals of circuits and cocircuits of M are given by

$$\begin{aligned} I &= (x_1x_2x_3x_4,\ x_1x_3x_5,\ x_2x_4x_5,\ x_2x_3x_6,\ x_1x_4x_6,\ x_1x_2x_5x_6,\ x_3x_4x_5x_6), \\ I^* &= (x_2x_4x_6,\ x_1x_3x_6,\ x_1x_4x_5,\ x_2x_3x_5,\ x_3x_4x_5x_6,\ x_1x_2x_5x_6,\ x_1x_2x_3x_4), \end{aligned}$$

and $\mathrm{reg}(R/I) = \mathrm{reg}(R/I^*) = 3$. The generalized Hamming weights of $C^\perp$ and C are

r	1	2	3
$\delta_r(C^\perp)$	3	5	6

r	1	2	3
$\delta_r(C)$	3	5	6

.

Thus, by Theorem 2.3.16, the cogirth of the signed graph G_σ is $\upsilon_1(G_\sigma) = 3$, and one has $\upsilon_2(G_\sigma) = 5$, $\upsilon_3(G_\sigma) = 6$. The frustration index of G_σ is 3.

Example 2.6.4. Let G_+ be the positive signed graph of Figure 2.3, let C be the incidence matrix code of G_+ over a finite field K, and let $M = M[A]$ be the vector matroid of C, where A is the incidence matrix of G_+. By Corollary 2.3.13(b), M is the graphic matroid of the underlying graph G, that is, the circuits and cocircuits of M are the cycles and cocycles of G.

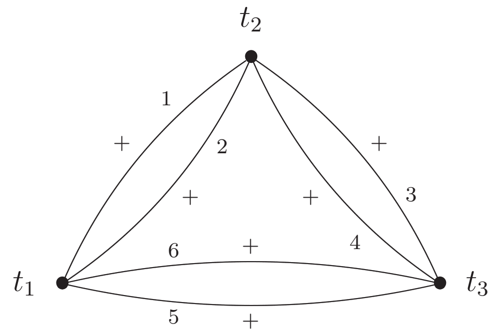

Figure 2.3: Positive signed graph with 3 vertices and 6 edges

The incidence matrix of the positive signed graph G_+ is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Using Procedure 2.7.2, we obtain the following information. The ideals of circuits and cocircuits of M are given by

$$\begin{aligned} I &= (x_5x_6,\ x_3x_4,\ x_1x_2,\ x_2x_4x_6,\ x_1x_4x_6,\ x_2x_3x_6,\ x_1x_3x_6,\ x_2x_4x_5,\ x_1x_4x_5,\ x_2x_3x_5,\ x_1x_3x_5), \\ I^* &= (x_3x_4x_5x_6,\ x_1x_2x_5x_6,\ x_1x_2x_3x_4), \end{aligned}$$

$\mathrm{reg}(R/I) = 2$, and $\mathrm{reg}(R/I^*) = 4$. The generalized Hamming weights of $C^\perp$ and C are

r	1	2	3	4
$\delta_r(C^\perp)$	2	4	5	6

r	1	2
$\delta_r(C)$	4	6

.

Thus, by Theorem 2.3.16, the edge connectivity of G is $\lambda_1(G) = 4$, and $\lambda_2(G) = 6$.

Example 2.6.5. Let G_- be the negative signed graph of Figure 2.4, let C be the incidence matrix code of G_- over a field K of characteristic $p \neq 2$, and let $M = M[A]$ be the vector matroid of C, where A is the incidence matrix of G_-. By Corollary 2.3.15, M is the even cycle matroid of the underlying graph G, that is, the circuits of M are the even cycles and bowties of G.

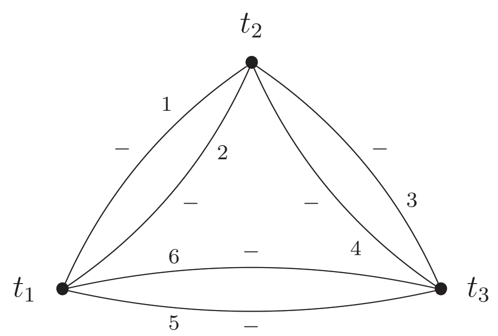

Figure 2.4: Negative signed graph with 3 vertices and 6 edges

The incidence matrix of the negative signed graph G_- is the incidence matrix of G:

$$A = \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Using Procedure 2.7.3, we obtain the following information. The ideals of circuits and cocircuits of M are given by

$$I = (x_1x_2,\, x_3x_4,\, x_5x_6), \quad I^* = (x_1x_2,\, x_3x_4,\, x_5x_6),$$

$\operatorname{reg}(R/I) = \operatorname{reg}(R/I^*) = 3$. The generalized Hamming weights of $C^\perp$ and C are

r	1	2	3
$\delta_r(C^\perp)$	2	4	6

r	1	2	3
$\delta_r(C)$	2	4	6

.

Thus, by Theorem 2.3.16, the cogirth of G_- is $\upsilon_1(G_-) = 2$, and $\upsilon_2(G_-) = 4$, $\upsilon_3(G_-) = 6$.

Example 2.6.6. Let G_σ be the signed graph of Figure 2.5 and let G be its underlying graph. The incidence matrix of G_σ is given in Procedure 2.7.4. Using this procedure we obtain that the frustration index $\varphi(G_\sigma)$ of G_σ is 7 and the frustration index $\varphi(G_-)$ of the negative signed graph G_- is 6. The minimum distance $\delta(C)$ of the incidence matrix code C of G_σ is 4 if $\operatorname{char}(K) \neq 2$ and $\delta(C)$ is 3 if $\operatorname{char}(K) = 2$. In this case $\delta(C) = \delta(C^\perp)$ in any characteristic.

2.7 Procedures for Macaulay2 and Matroids

In this section we give procedures for *Macaulay*2 [11], using the field of rational numbers as the ground field, to compute the generalized Hamming weights of the incidence

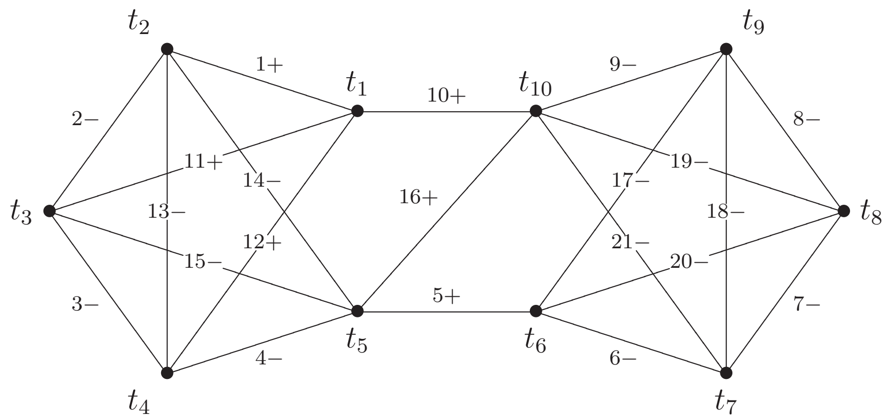

Figure 2.5: Unbalanced signed graph with frustration index 7

matrix code of a signed graph and the corresponding graph theoretical invariants (r-th cogirth, r-th edge connectivity), as well as the ideals of circuits, cocircuits, cycles and cocycles of a signed graph, and their algebraic invariants (Betti numbers, shifts, regularity). We also give a procedure to compute the frustration index of a connected signed simple graph. In all procedures the input is a rational matrix. The package *Matroids* [5] plays an important role here because it computes the circuits and cocircuits of a vector matroid over the field $\mathbb{Q}$ of rational numbers.

Procedure 2.7.1. Given the incidence matrix A of a signed graph G_σ over a field of $\mathrm{char}(K) \neq 2$, the procedure below computes the following:

- The ideal of circuits and the ideal of cocircuits of G_σ, and its regularity.
- The graded Betti numbers of the ideal of circuits and the ideal of cocircuits of G_σ.
- The weight hierarchies of the incidence matrix code C of G_σ and its dual code $C^\perp$.
- The r-th cogirth of G_σ (Theorem 2.3.16).

The next procedure corresponds to Example 2.6.3. To compute other examples just change the incidence matrix A.

```
--Procedure for Macaulay2
loadPackage "Matroids"
loadPackage "BoijSoederberg"
A=transpose matrix{{1,-1,0},{1,1,0},{0,1,-1},{0,1,1},{1,0,-1},{1,0,1}}
MA=matroid(A), I=ideal(MA)
m=matrix{flatten entries gens gb I}
N=coker m, F=res N, B=betti F, regularity N
lowestDegrees B --gives the weight hierarchy of the dual of C
I=ideal(dual(MA))
m=matrix{flatten entries gens gb I}
N=coker m, F=res N, B=betti F, regularity N
lowestDegrees B --gives the weight hierarchy of C
```

Procedure 2.7.2. Using the incidence matrix A of a positive signed graph G_+ over a field K and the Procedure 2.7.1, we can compute the following:

- The ideal of cycles and the ideal of cocycles of G and its regularity.
- The graded Betti numbers of the ideals of cycles and cocycles.
- The weight hierarchies of the incidence matrix code of G_+ and its dual code, and the generalized Hamming weights of the incidence matrix code of a digraph $\mathcal{D}$.
- The r-th edge connectivity of G.

The next incidence matrix corresponds to Example 2.6.4.

```
--Incidence matrix for Macaulay2
A=transpose matrix{{1,-1,0},{1,-1,0},{0,1,-1},{0,1,-1},{1,0,-1},{1,0,-1}}
```

Procedure 2.7.3. Using the incidence matrix A of a negative signed graph G_- over a field K of characteristic $p \neq 2$ and the Procedure 2.7.1, we can compute the following:

- The ideal I of the even cycles and bowties of G and the ideal I^* of cocircuits of G_-.
- The graded Betti numbers of I and I^*, and its regularity.

- The weight hierarchies of the incidence matrix code of G_- and its dual code.
- The r-th cogirth of G_-.

The next incidence matrix corresponds to Example 2.6.5.

```
--Incidence matrix for Macaulay2
A=transpose matrix{{1,1,0},{1,1,0},{0,1,1},{0,1,1},{1,0,1},{1,0,1}}
```

Procedure 2.7.4. One can use Theorem 2.5.4 and *Macaulay*2 [11] to compute the frustration index of a connected unbalanced signed simple graph G_σ. The incidence matrix of the following procedure corresponds to the graph of Figure 2.5 given in Example 2.6.6.

```
--Procedure for Macaulay2
input "points.m2"
R=QQ[t1,t2,t3,t4,t5,t6,t7,t8,t9,t10]
A = transpose matrix{{1,-1,0,0,0,0,0,0,0,0},{0,1,1,0,0,0,0,0,0,0},
{0,0,1,1,0,0,0,0,0,0},{0,0,0,1,1,0,0,0,0,0},{0,0,0,0,1,-1,0,0,0,0},
{0,0,0,0,0,1,1,0,0,0},{0,0,0,0,0,0,1,1,0,0},{0,0,0,0,0,0,0,1,1,0},
{0,0,0,0,0,0,0,0,1,1},{1,0,0,0,0,0,0,0,0,-1},{1,0,-1,0,0,0,0,0,0,0},
{1,0,0,-1,0,0,0,0,0,0},{0,1,0,1,0,0,0,0,0,0},{0,1,0,0,1,0,0,0,0,0},
{0,0,1,0,1,0,0,0,0,0},{0,0,0,0,1,0,0,0,0,-1},{0,0,0,0,0,1,0,0,1,0},
{0,0,0,0,0,0,1,0,1,0},{0,0,0,0,0,0,0,1,0,1},{0,0,0,0,0,1,0,1,0,0},
{0,0,0,0,0,0,1,0,0,1}}
I=ideal(projectivePointsByIntersection(A,R))
M=coker gens gb I, G=gb I
frustration=degree M-max apply(apply(subsets(apply(apply(apply
(toList ((set{1}**(set(1,-1))^**(hilbertFunction(1,M)-1))/splice)-
(set{0})^**(hilbertFunction(1,M)),toList),x->basis(1,M)*vector x),
z->ideal(flatten entries z)),1),ideal),x-> if #set flatten entries
mingens ideal(leadTerm gens x)==1 and not quotient(I,x)==I
then degree(I+x) else 0)--This gives the frustration index
```

Acknowledgments

We thank Thomas Zaslavsky for suggesting to generalize our work on incidence matrix codes of graphs to signed graphs, and for pointing out that the edge biparticity of a graph is a special case of the frustration index of a signed graph. Computations with *Macaulay*2 [11], Matroids [5], and SageMath [27] were important to verifying and computing examples given in this chapter.

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Chapter 3

Hamming polynomial of a demimatroid

Abstract. Following Britz, Johnsen, Mayhew and Shiromoto, we consider demimatroids as a natural generalization of matroids. As they have shown, demimatroids are the appropriate combinatorial objects for studying Wei's duality. Our results here apport further evidence about the trueness of that observation. We define the Hamming polynomial of a demimatroid M, denoted by $W(x, y, t)$, as a generalization of the extended Hamming weight enumerator of a matroid. The polynomial $W(x, y, t)$ is a specialization of the Tutte polynomial of M, and actually is equivalent to it. Guided by work of Johnsen, Roksvold and Verdure for matroids, we prove that Betti numbers of a demimatroid and its elongations determine the Hamming polynomial. Our results may be applied to simplicial complexes since in a canonical way they can be viewed as demimatroids. Furthermore, following work of Brylawski and Gordon, we show how demimatroids may be generalized one step further, to combinatroids. A combinatroid, or Brylawski structure, is an integer valued function ρ, defined over the power set of a finite ground set, satisfying the only condition $\rho(\emptyset) = 0$. Even in this extreme generality, we will show that many concepts and invariants in coding theory can be carried on directly to combinatroids, say, Tutte polynomial, characteristic polynomial, MacWilliams identity, extended Hamming polynomial, and the r-th generalized Hamming polynomial; this last one, at least conjecturelly, guided by the work of Jurrius and Pellikaan for linear codes. All this largely extends the notions of deletion, contraction, duality and codes to non-matroidal structures.

3.1 Introduction

Matroids are combinatorial objects introduced by Whitney in 1935 as a generalization of both graphs and matrices. They capture geometric and combinatorial properties of linear independence over finite structures. Demimatroids (Section 3.3) are a generalization of matroids, and in what follows we will show how demimatroids may be generalized one step further to combinatroids, via the rank function. We will show that combinatroids capture many concepts related with duality in coding theory and matroids. For instance, we define invariants as the Tutte polynomial, the generalized Hamming polynomial and the extended Hamming polynomial; or relationships between them, as deletion, contraction and the MacWilliams identity.

Denote by $\mathcal{C}$ the family of combinatroids defined over the same ground set E, and by $\mathcal{D}$ the smaller subfamily of demimatroids. The four operations: identity, dual, nullity and supplement (Section 3.4), may be seen as duality operators acting on $\mathcal{C}$, actually, these last three operators form a triality, in the sense that the composition of two of them results in the third one. The restriction to $\mathcal{D}$ of these operators behave even better: $\mathcal{D}$ has a natural structure of a bounded distributive lattice, and each demimatroid determines a weight hierarchy and a Duursma zeta function, which is a largely extension of well-known results for linear codes. All these facts show that $\mathcal{D}$ is a mathematical object that merits a further study.

As a final result, by extending work of Johnsen, Roksvold and Verdure for matroids [10], we prove that Betti numbers of a demimatroid and its elongations determine the extended Hamming polynomial of a demimatroid. All these results may be applied to simplicial complexes since in a canonical way they can be viewed as demimatroids. In forthcoming works we will study the k-analogue and q-analogue of a demimatroid. For unexplained notions of graph theory, linear codes and matroids we refer to [6], [8] and [13], respectively.

3.2 Matroids and linear codes

A *matroid* is a pair $M = (E, \rho)$, where E is a finite set called the *ground set* of M, and $\rho : 2^E \to \mathbb{Z}_+ := \{0, 1, \ldots\}$ is a function satisfying:

(R_0) $\rho(\emptyset) = 0$;

(R_1) If $X \subseteq E$ and $x \in E$, then $\rho(X) \le \rho(X \cup \{x\}) \le \rho(X) + 1$;

(R_2) If $X, Y \subseteq E$, then $\rho(X \cup Y) + \rho(X \cap Y) \le \rho(X) + \rho(Y)$.

The function ρ is called the *rank function* of the matroid. Condition (R_2) is known as the *submodularity* condition. An *independent set* of M is a subset $X \subseteq E$ such that $\rho(X) = |X|$, where $|X|$ denotes the cardinality of X; in particular the empty set is always an independent set. A *basis* is an inclusion maximal independent set; one can verify that bases of a matroid are equicardinal. A subset of the ground set which is not independent is called a *dependent set*, and a *circuit* is a minimal dependent set.

Let X be a subset of E. From (R_0) and (R_1), and by a direct induction argument, if follows that $0 \le \rho(X) \le |X|$ for all $X \subseteq E$. The *nullity* of X, denoted by $\eta(X)$, is defined as $\eta(X) := |X| - \rho(X)$. In particular, the nullity of M is defined as $\eta(M) := \eta(E)$. The r*-generalized Hamming weight* of the matroid M is given by

$$d_r(M) := \min\{|X| : \eta(X) = r\}, \qquad 1 \le r \le \eta(E),$$

and the sequence $d_1(M), \ldots, d_{\eta(E)}(M)$ is called the *weight hierarchy* of M.

Let p be a prime, q a positive power of p and $\mathbb{F}_q$ a field with q elements. A *linear* $[n, k]_q$ *code* is a k-dimensional subspace C of $\mathbb{F}_q^n$. In this context the field $\mathbb{F}_q$ is called the *alphabet*, the elements of $\mathbb{F}_q^n$ are the *words* and the elements of C are called *codewords* of the code. We consider $\mathbb{F}_q^n$ provided with its *Hamming distance*, which is the number of coordinates in which two words differ. For $c \in C$ its *weight*, denoted by $w(c)$, is the number of its nonzero coordinates. For a subset X of $\mathbb{F}_q^n$ we define the *support* of X, denoted $\mathrm{supp}(X)$, as the union of all the supports of elements in X, i.e.

$$\mathrm{supp}(X) := \{i : \exists (c_1, \ldots, c_n) \in C \text{ such that } c_i \ne 0\}.$$

Let C be a linear $[n, k]_q$ code. For $1 \le r \le k$, the r-th *generalized Hamming weight* of C is defined as

$$d_r(C) := \min\{|\mathrm{supp}(X)| : X \text{ is a } r\text{-dimensional subspace of } C\}.$$

The number $d_1(C)$ is known as the *minimum distance* of the code and the sequence $d_1(C), \ldots, d_k(C)$ is called the *weight hierarchy* of C.

With each linear code C one associate the vector matroid $M[H]$ on the ground set $E = \{1, \ldots, n\}$, where H is a parity check matrix of C. The rank function of $M[H]$ is given by $\rho(X) := \text{rank}(H_X)$ for $X \subseteq E$, where H_X is the submatrix of H obtained by picking the columns indexed by X. The matroid $M[H]$ does not depend on the parity check matrix we use. We call $M[H]$ the *(parity) matroid* of C. A basic result in this area, relating codes and matroids, is that the weight hierarchies of both the code C and the matroid $M[H]$ coincide [14].

3.3 Demimatroids

A *demimatroid* is a pair $M = (E, \rho)$, where E is a finite set called the *ground set* of M, and $\rho : 2^E \to \mathbb{Z}_+$ is a function such that

(R_0) $\rho(\emptyset) = 0$;

(R_1) If $X \subseteq E$ and $x \in E$, then $\rho(X) \leq \rho(X \cup \{x\}) \leq \rho(X) + 1$;

The function ρ is called the *rank function* of the demimatroid. Clearly matroids are examples of demimatroids. By abuse of notation we will frequently refer to ρ itself as the demimatroid. The *rank* of a demimatroid M is defined as $\rho(M) := \rho(E)$. A straightforward verification shows that $0 \leq \rho(X) \leq |X|$ for all $X \subseteq E$. We define the *nullity* of X as $\eta(X) := |X| - \rho(X)$. The nullity of a demimatroid M is defined as $\eta(M) := \eta(E)$. The *dual* of a demimatroid $M = (E, \rho)$ is the pair $M^* := (E, \rho^*)$, where

$$\rho^*(X) := |X| + \rho(E \backslash X) - \rho(E).$$

Clearly $\rho^*(\emptyset) = 0$. To simplify notation, from here one we will write $X \backslash x$ and $X \cup x$ instead of $X \backslash \{x\}$ and $X \cup \{x\}$, respectively. If $x \in X$, obviously (R_1) is satisfied, and if $x \notin X$, then $\rho^*(X) \leq \rho^*(X \cup x) \leq \rho^*(X) + 1$ if and only if $\rho(E \backslash X) \leq \rho((E \backslash X) \backslash x) + 1 \leq \rho(E \backslash X) + 1$. But each of these last two inequalities readily follows from the properties of ρ. So, in fact, ρ^* is a demimatroid. Moreover, one can verify that $M^{**} = M$; to see this just note that $\rho(E) + \rho^*(E) = |E|$, and then

$$\rho^{**}(X) = |X| + \rho^*(E \backslash X) - \rho^*(E) = |E| + \rho(X) - |E| = \rho(X).$$

As in the case of matroids, we define *independent sets* of a demimatroid as those $X \subseteq E$ such that $\rho(X) = |X|$; and in a similar fashion, one might define bases, dependent sets and circuits. But in this generality we must remark that bases of a demimatroid are not necessarily equicardinal.

Example 3.3.1. Let E be a finite set and $\rho : 2^E \to \mathbb{Z}_+$ given by:

(i) $\rho(X) = 0$ for all $X \subseteq E$. Then ρ is a demimatroid; called the *trivial* demimatroid.

(ii) $\rho(X) = |X|$ for all $X \subseteq E$. Then ρ is a demimatroid; actually it is a matroid.

(iii) $\rho(X) = 0$ if $X \neq E$ and $\rho(E) = 1$. Then ρ is a demimatroid; if E has at least two elements, then ρ is not a matroid.

(iv) $\rho(\emptyset) = 0$ and $\rho(X) = 1$ for all $X \neq \emptyset$. Then $M = (E, \rho)$ is a demimatroid.

Example 3.3.2. Let $M = (E, \rho)$ be a nontrivial dematroid. For $X \subseteq E$ define $\rho^\bullet(X) = \rho(X)$ if $\rho(X) < \rho(E)$ and $\rho^\bullet(X) = \rho(X) - 1$ if $\rho(X) = \rho(E)$. Then $(E, \rho^\bullet)$ is a demimatroid.

A *simplicial complex* Δ on a finite vertex set E is an inclusion closed family of subsets of E, i.e. $\sigma \in \Delta$ and $\tau \subseteq \sigma$ implies $\tau \in \Delta$. Elements of Δ are called *faces* and maximal faces are called *facets*. A face of Δ whose cardinality is $i+1$ is said to be of *dimension* i. The *dimension* of Δ is the maximum dimension of any one of its faces.

Example 3.3.3. Let Δ be a simplicial complex on the vertex set E. We define the demimatroid $\Delta^{\uparrow} := (E, \rho)$, where, for all $X \subseteq E$,

$$\rho(X) := \max\{|\sigma| : \sigma \subseteq X, \sigma \in \Delta\}.$$

Example 3.3.4. A graph may be viewed as a 1-dimensional simplicial complex, and then as a demimatroid. Say the graph has no isolated vertices and let E denote the vertex set. Thus, in this case, the demimatroid in Example 3.3.3 is given by $\rho(\emptyset) = 0$, $\rho(X) = 1$ if X is and independent vertex set of G, and $\rho(X) = 2$ if X is not an independent vertex set of G.

Example 3.3.5. Let $M = (E, \rho)$ be a demimatroid. If $\rho(X) = |X|$ for some $X \subseteq E$, then $\rho(Y) = |Y|$ for all $Y \subseteq X$. Therefore, the set

$$M^{\downarrow} := \{X \subseteq E : \rho(X) = |X|\}$$

is a simplicial complex.

Example 3.3.6. Let Δ be a simplicial complex with vertex set E and $\rho : 2^E \to \mathbb{Z}_+$ given by $\rho(X) = |X|$ if $X \in \Delta$ and $\rho(X) = |X| - 1$ if $X \notin \Delta$. Then $\Delta^{\sharp} := (E, \rho)$ is a demimatroid.

Example 3.3.7. Let $M = (E, \rho)$ be a demimatroid. Since ρ is non-decreasing, it follows that, for all nonnegative integers r, the set $M_{(r)} := \{X \subseteq E : \rho(X) \leq r\}$ is a simplicial complex.

Let E be a finite set. Denote by $\mathcal{S}$ the family of all simplicial complexes with ground set E, and make $\mathcal{S}$ a poset defining $\Delta \leq \Gamma$ when $\Delta \subseteq \Gamma$. Denote by $\mathcal{D}$ the family of all demimatroids with ground set E, and make $\mathcal{D}$ a poset by defining $(E, \rho) \leq (E, \tau)$ when $\rho(X) \leq \tau(X)$ for all $X \subseteq E$. The next lemma is not hard to prove.

Lemma 3.3.8. *(i)* $\Delta \leq \Gamma$ *implies* $\Delta^{\uparrow} \leq \Gamma^{\uparrow}$*;*

(ii) $(E, \rho) \leq (E, \tau)$ *implies* $(E, \rho)^{\downarrow} \leq (E, \tau)^{\downarrow}$*;*

(iii) $(\Delta^{\uparrow})^{\downarrow} = \Delta$.

(iv) $M^{\downarrow} = \Delta$ *implies* $\Delta^{\uparrow} \leq M$*; in particular,* $(M^{\downarrow})^{\uparrow} \leq M$.

(v) $(M^{\downarrow})^{\uparrow} = M$ *if and only if* $M = \Delta^{\uparrow}$ *for some simplicial complex* Δ.

Proof. (iv): Say $M = (E, \rho)$ and $\Delta^{\uparrow} = (E, \tau)$. Take any $X \subseteq E$. $\tau(X) = \max\{|\sigma| : \sigma \subseteq X,\ \rho(\sigma) = |\sigma|\}$. Choose $\sigma \subseteq X$ such that $\tau(X) = |\sigma|$ and $\rho(\sigma) = |\sigma|$. Then $\tau(X) = |\sigma| = \rho(\sigma) \leq \rho(X)$. $\square$

Example 3.3.9. Let Δ be a simplicial complex and M a demimatroid. Then $M^{\downarrow} = \Delta$ if and only if $\Delta^{\uparrow} \leq M \leq \Delta^{\sharp}$.

A *Galois connection* between two posets P and Q is a pair of functions $\alpha : P \to Q$ and $\beta : Q \to P$ with the properties: (1) both α and β are order-inverting; (2) $p \leq \beta(\alpha(p))$ for all $p \in P$ and $q \leq \alpha(\beta(q))$ for all $q \in Q$.

Proposition 3.3.10. *Let $\mathcal{D}$op denote the dual poset of $\mathcal{D}$. The maps $^{\uparrow} : \mathcal{S} \to \mathcal{D}'$, $\Delta \mapsto \Delta^{\uparrow}$ and $^{\downarrow} : \mathcal{D}\mathrm{op} \to \mathcal{S}$, $M \mapsto M^{\downarrow}$ form a Galois connection.*

3.4 Combinatroids

Three important operations on matroids are motivated by graph theory: deletion, contraction and duality. Brylawski realized that it is possible to extend all of these three operations to any finite set E provided with an arbitrary function $r : 2^E \to \mathbb{Z}$, see [4]. Thus we define a *combinatroid* (with values in $\mathbb{Z}$) as a pair $M := (E, \rho)$, where E is a finite set called the *ground set* of M, and $\rho : 2^E \to \mathbb{Z}$ is a function satisfying the only condition $\rho(\emptyset) = 0$. The function ρ is called the *rank function* of the combinatroid. Clearly demimatroids are examples of combinatroids. Another name for a combinatroid is a (normalized) *Brylawski structure*, as is done in [4]. One define the *dual* combinatroid $M^* = (E, \rho^*)$, where ρ^*, called the *dual rank function*, is given by

$$\rho^*(X) = |X| + \rho(E\backslash X) - \rho(E).$$

Then, the *deletion* of $A \subseteq E$, denoted by $M\backslash A$, is defined as the restriction of the rank function ρ to $E\backslash A$, i.e. $\rho_{M\backslash A}(X) := \rho(X)$ for all $X \subseteq E\backslash A$. Moreover, *contraction*, denoted by M/A, is defined using deletion and duality: $M/A := (M^*\backslash A)^*$. Note that both $M\backslash A$ and M/A have the same ground set $E\backslash A$.

Proposition 3.4.1. (Brylawski, Gordon; see [4]) *Let $M = (E, \rho)$ be a combinatroid and $A \subseteq E$.*

(i) $\rho_{M/A}(X) = \rho(X \cup A) - \rho(A)$ *for all* $X \subseteq E\backslash A$;

(ii) $(M^*)^* = M$;

(iii) $(M\backslash A)^* = M^*/A$;

(iv) $(M/A)^* = M^*\backslash A$.

Proposition 3.4.2. *Let $M = (E, \rho)$ be a demimatroid and $A \subseteq E$. Then*

(i) $M\backslash A$ *is a demimatroid;*

(ii) M/A is a demimatroid.

Proof. Let $X \subset E\backslash A$ and $x \in (E\backslash A)\backslash X$.

(i): $\rho_{M\backslash A}(X) \leq \rho_{M\backslash A}(X \cup x) \leq \rho_{M\backslash A}(X) + 1 \Leftrightarrow \rho(X) \leq \rho(X \cup x) \leq \rho(X) + 1$.

(ii): $\rho_{M/A}(X) \leq \rho_{M/A}(X \cup x) \leq \rho_{M/A}(X) + 1 \Leftrightarrow \rho(X \cup A) - \rho(A) \leq \rho(X \cup x \cup A) - \rho(A) \leq \rho(X \cup A) - \rho(A) + 1 \Leftrightarrow \rho(X \cup A) \leq \rho(X \cup A \cup x) \leq \rho(X \cup A) + 1$. □

A *minor* of a demimatroid M is any demimatroid obtainded from M by a sequence of deletions and contractions.

One can also define the *nullity* combinatroid $M^\circ = (E, \rho^\circ)$, where ρ°, called the *nullity function*, is given by

$$\rho^\circ(X) = |X| - \rho(X).$$

Proposition 3.4.3. *Let $M = (E, \rho)$ be a combinatroid. Then*

(i) $\rho^{*\circ}(X) = \rho^{\circ *}(X) = \rho(E) - \rho(E\backslash X)$ for all $X \subseteq E$;

(ii) $(M^\circ)^\circ = M$;

(iii) $(M^*)^\circ = (M^\circ)^*$;

(iv) If M is a demimatroid, then M° is a demimatroid.

Proof. (iv): Obviously $\rho^\circ(\emptyset) = 0$. Let $X \subset E$ and $x \in E\backslash X$. $\rho^\circ(X) \leq \rho^\circ(X \cup x) \leq \rho^\circ(X) + 1$ if and only if $|X| - \rho(X) \leq |X| + 1 - \rho(X \cup x) \leq |X| - \rho(X) + 1$ if and only if $\rho(X) + 1 \geq \rho(X \cup x) \geq \rho(X)$. □

Following [1], we define the *supplement* combinatroid $M^\circledast := (E, \rho^\circledast)$, where $\rho^\circledast$, called the *supplement (or supplementary) function*, is given by

$$\rho^\circledast(X) = \rho(E) - \rho(E\backslash X).$$

Proposition 3.4.4. *Let $M = (E, \rho)$ be a combinatroid. Then*

(i) $(M^\circledast)^\circledast = M$;

(ii) $(M^*)^\circ = (M^\circ)^* = M^\circledast$;

(iii) $(M^\circ)^\circledast = (M^\circledast)^\circ = M^*$;

(iv) $(M^*)^\circledast = (M^\circledast)^* = M^\circ$;

(v) ([1, Thm. 8]) *If M is a demimatroid, then $M^\circledast$ is a demimatroid.*

Proof. (v): Obviously $\rho^\circledast(\emptyset) = 0$. Let $X \subset E$ and $x \in E \backslash X$. $\rho^\circledast(X) \leq \rho^\circledast(X \cup x) \leq \rho^\circledast(X) + 1$ if and only if $\rho(E) - \rho(E\backslash X) \leq \rho(E) - \rho(E\backslash(X \cup x)) \leq \rho(E) - \rho(E\backslash X) + 1$ if and only if $\rho(E\backslash X) \geq \rho((E\backslash X)\backslash x) \geq \rho(E\backslash X) - 1$. But each of these last two inequalities directly follows from the properties of ρ. $\square$

The identity (denoted by "id"), dual, nullity and supplement operations may be viewed as operators acting on the set of combinatroidal structures defined on the same ground set E.

Proposition 3.4.5. *Let $M = (E, \rho)$ be a combinatroid. Then the operators* $\{\mathrm{id}, *, \circ, \circledast\}$ *form an abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$:*

	id	$*$	$\circ$	$\circledast$
id	id	$*$	$\circ$	$\circledast$
$*$	$*$	id	$\circledast$	$\circ$
$\circ$	$\circ$	$\circledast$	id	$*$
$\circledast$	$\circledast$	$\circ$	$*$	id

Remark 3.4.6. Note that the operators $\{*, \circ, \circledast\}$ form a *triality*, in the sense that the composition of two of them gives the third one.

Example 3.4.7. Let E be a finite set and $\rho : 2^E \to \mathbb{Z}_+$, $X \mapsto |X|$. Then $\rho^* \equiv \rho^\circ \equiv 0$ and $\rho^\circledast \equiv \rho$.

Example 3.4.8. Let E be a finite set and $\rho : 2^E \to \mathbb{Z}_+$, $\rho(X) = 0$ if $X \neq E$ and $\rho(E) = 1$. We have that $\rho^*(\emptyset) = 0$ and $\rho^*(X) = |X| - 1$ if $X \neq \emptyset$; $\rho^\circ(X) = |X|$ if $X \neq E$ and $\rho^\circ(E) = |E| - 1$; $\rho^\circledast(\emptyset) = 0$ and $\rho^\circledast(X) = 1$ if $X \neq \emptyset$.

Example 3.4.9. Let $M = (E = \{1,2,3\}, \rho)$ be the matroid whose basis are $\{1,2\}$ and $\{1,3\}$. We have the following table:

X	$\emptyset$	1	2	3	12	13	23	E
ρ	0	1	1	1	2	2	1	2
ρ^*	0	0	1	1	1	1	1	1
ρ°	0	0	0	0	0	0	1	1
$\rho^\circledast$	0	1	0	0	1	1	1	2

Remark 3.4.10. If M is a matroid, then M° and $M^\circledast$ are demimatroids, but they might not be matroids. For instance, in Example 3.4.9, $1 = \rho^\circ(23) + \rho^\circ(\emptyset) \not\leq \rho^\circ(2) + \rho^\circ(3) = 0$ and $1 = \rho^\circledast(23) + \rho^\circledast(\emptyset) \not\leq \rho^\circledast(2) + \rho^\circledast(3) = 0$, show that ρ° and $\rho^\circledast$ do not satisfy the submodularity condition.

Example 3.4.11. Let G be a simple graph with no isolated vertices; we see G as a 1-dimensional simplicial complex. Let E denote the vertex set of G and define $\rho : 2^E \to \mathbb{Z}_+$, $\rho(\emptyset) = 0$, $\rho(X) = 1$ if X is and independent vertex set of G, and $\rho(X) = 2$ if X is not an independent vertex set of G. Then

$$\rho^*(X) = \begin{cases} |X|, & \text{if } X \text{ is not a covering;} \\ |X|-1, & \text{if } X \text{ is a covering;} \\ |X|-2, & \text{if } X = E. \end{cases}$$

$$\rho^\circ(X) = \begin{cases} 0, & \text{if } X = \emptyset; \\ |X|-1, & \text{if } X \text{ is idependent;} \\ |X|-2, & \text{if } X \text{ is not independent.} \end{cases}$$

$$\rho^\circledast(X) = \begin{cases} 0, & \text{if } X \text{ is not a covering;} \\ 1, & \text{if } X \text{ is a covering.} \end{cases}$$

For α and β, combinatroids over E, we define $(\alpha \vee \beta)(X) = \max\{\alpha(X), \beta(X)\}$ and $(\alpha \wedge \beta)(X) = \min\{\alpha(X), \beta(X)\}$ for all $X \subseteq E$.

Lemma 3.4.12. *If α and β are demimatroids, then $\alpha \vee \beta$ and $\alpha \wedge \beta$ are demimatroids.*

Proof. This follows immediately from the fact that for real numbers $a_1 \leq a_2$ and $b_1 \leq b_2$ it holds that $\min\{a_1, b_1\} \leq \min\{a_2, b_2\}$ and $\max\{a_1, b_1\} \leq \max\{a_2, b_2\}$. $\square$

The set of combinatroids on a set E may be partially ordered by defining $\alpha \leq \beta$ if $\alpha(X) \leq \beta(X)$ for all $X \subseteq E$.

Theorem 3.4.13. *The set of demimatroides on a finite set E, with $\vee$ and $\wedge$ defined as above, form a bounded distributive lattice. The maximum demimatroid is $|\cdot| : X \mapsto |X|$ and the minimum demimatroid is $0 : X \mapsto 0$.*

Example 3.4.14. This lattice has only one atom, namely, $\rho : 2^E \to \mathbb{Z}_+$, $\rho(X) = 0$ if $X \neq E$ and $\rho(E) = 1$. And it also has only one coatom, which is the nullity of ρ, i.e. $\rho^\circ(X) = |X|$ for all $X \neq E$ and $\rho^\circ(E) = |E| - 1$.

Let $M = (E, \rho)$ be a nontrivial demimatroid, and set $k := \rho(E) \leq |E|$. Define $\sigma_k(M) := \min\{|X| : \rho(X) = k\}$ and choose $X \subseteq E$ such that $\sigma_k(M) = |X|$. For $x \in X$ we know that $\rho(X \backslash x) < \rho(X) \leq \rho(X \backslash x) + 1$. From this it follows that $\rho(X \backslash x) = k - 1$. Define $\sigma_{k-1}(M) := \min\{|Y| : \rho(Y) = k - 1\}$ and choose $Y \subseteq E$ such that $\sigma_{k-1}(M) = |Y|$. For $y \in Y$ we know that $\rho(Y \backslash y) < \rho(Y) \leq \rho(Y \backslash y) + 1$. From this it follows that $\rho(Y \backslash y) = k - 2$. Continuing this process we obtain that $0 = \sigma_0(M) < \sigma_1(M) < \cdots < \sigma_k(M) \leq |E|$. A subset X of E is said to be of *level* r if $\rho(X) = r$. Thus ρ induce a partition of 2^E by level sets. We put this on record as the following lemma, but first a definition. For $1 \leq r \leq \rho(E)$ we define the r-th *Wei number* of the demimatroid as

$$\sigma_r(M) := \min\{|X| : \rho(X) = r\}. \tag{3.4.1}$$

Lemma 3.4.15. *Let $M = (E, \rho)$ be demimatroid of rank $k := \rho(M)$. Then*

(i) The image of ρ is the set $\{0, 1, \ldots, k\}$;

(ii) $0 < \sigma_1(M) < \cdots < \sigma_k(M) \leq |E|$;

(iii) If $\rho(X) \geq r$, then $|X| \geq \sigma_r(M)$;

(iv) $\min\{|X| : \rho(X) = r\} = \min\{|X| : \rho(X) \geq r\}$.

(v) (Generalized Singleton bound) *For all $0 \leq r \leq k$ it holds that*

$$k + \sigma_r(M) \leq |E| + r.$$

Proof. (iii): Say $\rho(X) = r + s$. Then $|X| \geq \sigma_{r+s}(M) \geq \sigma_r(M)$.

(v): $k + \sigma_k(M) \leq |E| + k$ iff $\sigma_k(M) \leq |E|$, which is true. Suppose the result is true for $r, \ldots, k$. Hence $k + \sigma_{r-1}(M) \leq k + \sigma_r(M) - 1 \leq |E| + r - 1$. □

Example 3.4.16. Let $M = (E = \{1, 2, 3, 4\}, \rho)$ be the matroid whose basis are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{3, 4\}$. We have the following table:

X	$\emptyset$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	E
ρ	0	1	1	1	1	2	2	2	2	1	2	2	2	2	2	2
ρ^*	0	1	1	1	1	2	1	2	2	2	2	2	2	2	2	2
ρ°	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	2
$\rho^\circledast$	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	2

,

	σ_1	σ_2
ρ	2	4
ρ^*	2	4
ρ°	1	2
$\rho^\circledast$	1	2

.

Since $\rho^\circledast(E\backslash X) = \rho^\circledast(E) - \rho(X)$ for all $X \subseteq E$, Eq. (3.4.1) can be rewritten as

$$\sigma_r(M) = \min\{|X| : \rho^\circledast(E\backslash X) = \rho^\circledast(E) - r\}.$$

Thus we may interpret the r-th Wei number $\sigma_r(M)$ as the minimum number of elements that must be removed from E to decrease the rank of $M^\circledast$ by r. A fundamental result is the following.

Theorem 3.4.17. (Wei's duality [1, Thm. 13]) *Let $M = (E, \rho)$ be a demimatroid. Then, with $n = |E|$ and $k = \rho(M)$,*

$$\{\sigma_1(M), \ldots, \sigma_k(M)\} = \{1, \ldots, n\} \setminus \{n + 1 - \sigma_1(M^*), \ldots, n + 1 - \sigma_{n-k}(M^*)\}.$$

Proof. Suppose $\sigma_i(M) = n+1-\sigma_j(M^*)$ for some i, j. Choose $X \subseteq E$ such that $\rho(X) = i$ and $\sigma_i(M) = |X|$. Hence $|E\backslash X| = n - |X| = \sigma_j(M^*) - 1$. By Lemma 3.4.15(iii) we have that $\rho^*(E\backslash X) \leq j - 1$. Similarly, choose $Y \subseteq E$ such that $\rho^*(Y) = j$ and $\sigma_j(M^*) = |Y|$. Hence $\rho(E\backslash Y) \leq i - 1$. But this implies that $i + j - 1 = \rho^*(E\backslash X) + \rho(E\backslash Y) \leq i + j - 2$, which is not possible. $\square$

Remark 3.4.18. In the literature, $\sigma_r(M^\circ)$ is known as the r-th generalized Hamming weight of M, and since $(M^\circ)^\circledast = M^*$, then $\sigma_r(M^\circ)$ is the minimum number of elements that must be removed from E to decrease the rank of M^* by r.

Remark 3.4.19. $\min\{|X| : \eta(X) = r\} + \max\{|Y| : \rho^*(E) - \rho^*(Y) = r\} = |E|$.

Proof of the Remark. Set $a = \min\{|X| : \eta(X) = r\}$ and $b = \max\{|Y| : \rho^*(Y) = \rho^*(E) - r\}$. Choose X such that $a = |X|$. Since $\rho^*(E\backslash X) = \rho(E) - r$, it holds that $|E\backslash X| \leq b$, so $|E| \leq a + b$. To prove the other direction choose Y such that $b = |Y|$. Since $\eta(E\backslash Y) = r$, it holds that $a \leq |E\backslash Y|$, so $a + b \leq |E|$. $\square$

Let $M = (E, \rho)$ be a demimatroid. From the Singleton bound we obtain that $\sigma_1(M) \leq |E| - \rho(E) + 1$. When equality is attained, M is called a *full* demimatroid.

Corollary 3.4.20. *Let $M = (E, \rho)$ be a demimatroid, with $n = |E|$ and $k = \rho(E)$.*

(i) If $k + \sigma_r(M) = n + r$, then $k + \sigma_s(M) = n + s$ for all $s \geq r$.

(ii) If M is full, then M^* is full.

Proof. (i): The result is true for $s = r$. If it is true for $r, \dots, s$, then $k + \sigma_{s+1}(M) \geq k + \sigma_s(M) + 1 = n + s + 1$.

(ii): By (i), $n + 1 - \sigma_r(M) = k - r + 1$. Thus, by Wei's duality, $\sigma_s(M^*) = k + s$ for $1 \leq s \leq n - k$. In particular, $\sigma_1(M^*) = k + 1 = n - (n - k) + 1 = n - \rho^*(E) + 1$. □

Example 3.4.21. Let $M = (E = \{1, 2, 3\}, \rho)$ be the demimatroid, with ρ given by:

X	$\emptyset$	1	2	3	12	13	23	E
ρ	0	0	0	0	1	1	1	2
ρ^*	0	0	0	0	0	0	0	1
ρ°	0	1	1	1	1	1	1	1
$\rho^\circledast$	0	1	1	1	2	2	2	2

,

	σ_1	σ_2
ρ	2	3
ρ^*	3	
ρ°	1	
$\rho^\circledast$	1	2

.

We observe that ρ and ρ^* are full, whereas ρ° and $\rho^\circledast$ are not.

Lemma 3.4.22. *Let $M = (E, \rho)$ be a demimatroid, with $n = |E|$ and $k = \rho(E)$. Then M is full if and only if*

$$\rho(X) = \begin{cases} 0, & \text{if } |X| \leq n - k; \\ r, & \text{if } |X| = n - k + r \text{ and } r \geq 1. \end{cases}$$

Proof. ($\Leftarrow$) Evidently $\sigma_1(M) = \min\{|X| : \rho(X) = 1\} = n - k + 1$.

($\Rightarrow$) By Lemma 3.4.15(v), $\sigma_s(M) = n - k + s$ for all $1 \leq s \leq k$. In particular, $\sigma_k(M) = n$ implies $\rho(E \backslash x) \leq k - 1$ for $x \in E$. Let $X \subseteq E$ with $|X| = n - 1$. Suppose that $\rho(X) \leq k - 2$. Then $\rho(X) \leq \rho(E) \leq \rho(X) + 1 \leq k - 1$, which is not possible. Thus $\rho(X) = k - 1$. Suppose that if $|X| = n - k + r$, then $\rho(X) = s$. Let X such that $|X| = n - k + r - 1$. If $\rho(X) \leq n - k + r - 2$, then $\rho(X) \leq \rho(X \cup x) \leq \rho(X) + 1$, i.e. $r \leq r - 1$, which is not possible. □

Lemma 3.4.23. *Let $M = (E, \rho)$ be a full demimatroid, with $n = |E|$ and $k = \rho(E)$. Then* *(i)* $\rho^*(X) = \begin{cases} 0, & \text{if } |X| \leq k; \\ r, & \text{if } |X| = k + r \text{ and } r \geq 1. \end{cases}$

(ii) $\rho^\circ(X) = \begin{cases} |X|, & \text{if } |X| \leq n - k; \\ n - k, & \text{if } |X| > n - k. \end{cases}$

(iii) $\rho^\circledast(X) = \begin{cases} |X|, & \text{if } |X| \leq k; \\ k, & \text{if } |X| > k. \end{cases}$

We said that a demimatroid M is *uniform* when M° is full.

Corollary 3.4.24. *Let $M = (E, \rho)$ be a full demimatroid, with $n = |E|$ and $k = \rho(E)$. Then M° and $M^\circledast$ are the uniform matroids of rank $n - k$ and k, respectively.*

The Wei numbers $\{\sigma_1(M), \ldots, \sigma_{\rho(M)}(M)\}$ of a demimatroid $M = (E, \rho)$ determine a subset of $\{1, \ldots, |E|\}$. The reciprocal is also true.

Proposition 3.4.25. *Let E be a finite set and $\{\sigma_1 < \ldots < \sigma_k\} \subseteq \{1, \ldots, |E|\}$. Then there exists $\rho : 2^E \to \{0, 1, \ldots, k\}$ such that $M = (E, \rho)$ is a demimatroid, $k = \rho(E)$ and $\sigma_r(M) = \sigma_r$ for all $1 \leq r \leq \rho(E)$.*

Proof. Put $\sigma_0 := 0$, $\sigma_{k+1} := |E|$ and define $\rho(X) = i$ if $\sigma_i \leq |X| < \sigma_{i+1}$. Then $\sigma_r(M) = \min\{|X| : \rho(X) = r\} = \sigma_r$. □

Let $M = (E, \rho)$ be a nontrivial demimatroid, and set $k := \rho(E) \leq |E|$. Define $\sigma^0(M) := \max\{|X| : \rho(X) = 0\}$ and choose $X \subseteq E$ such that $\sigma^0(M) = |X|$. For $x \notin X$ we know that $\rho(X) < \rho(X \cup x) \leq \rho(X) + 1$. From this it follows that $\rho(X \cup x) = 1$. Define $\sigma^1(M) := \max\{|Y| : \rho(Y) = 1\}$ and choose $Y \subseteq E$ such that $\sigma^1(M) = |Y|$. For $y \notin Y$ we know that $\rho(Y) < \rho(Y \cup y) \leq \rho(Y) + 1$. From this it follows that $\rho(Y \cup y) = 2$. Continuing this process we obtain that $0 \leq \sigma^0(M) < \sigma^1(M) < \cdots < \sigma^k(M) = |E|$. We again put this on record as the following lemma, but first a definition. For $1 \leq r \leq \rho(E)$ we define the r-th *upper Wei number* of the demimatroid as

$$\sigma^r(M) := \max\{|X| : \rho(X) = r\}. \tag{3.4.2}$$

Lemma 3.4.26. *Let $M = (E, \rho)$ be demimatroid of rank $k := \rho(M)$. Then*

(i) The image of ρ is the set $\{0, 1, \ldots, k\}$;

(ii) $0 \leq \sigma^0(M) < \cdots < \sigma^k(M) = |E|$;

(iii) *If* $\rho(X) \leq r$, *then* $|X| \leq \sigma^r(M)$;

(iv) $\max\{|X| : \rho(X) = r\} = \max\{|X| : \rho(X) \leq r\}$.

(v) (Generalized upper Singleton bound) *For all* $0 \leq r \leq k$ *it holds that*

$$k + \sigma^r(M) \leq |E| + r.$$

Proof. (iii): Say $\rho(X) = r - s$. Then $|X| \leq \sigma^{r-s}(M) \leq \sigma^r(M)$.

(v) $k + \sigma^k(M) \leq |E| + k$ iff $\sigma^k(M) \leq |E|$, which is true. Suppose the result is true for $r, \ldots, k$. Hence $k + \sigma^{r-1}(M) \leq k + \sigma^r(M) - 1 \leq |E| + r - 1$. □

Theorem 3.4.27. (Upper Wei's duality [1, Thm. 12]) *Let* $M = (E, \rho)$ *be a demimatroid. Then, with* $n = |E|$ *and* $k = \rho(M)$,

$$\{\sigma^0(M) + 1, \ldots, \sigma^{k-1}(M) + 1\} = \{1, \ldots, n\} \setminus \{n - \sigma^0(M^*), \ldots, n - \sigma^{n-k-1}(M^*)\}.$$

Proof. Suppose $\sigma^i(M)+1 = n - \sigma^j(M^*)$ for some i, j. Choose $X \subseteq E$ such that $\rho(X) = i$ and $\sigma^i(M) = |X|$. Hence $|E \backslash X| = n - |X| = \sigma^j(M^*) + 1$. By Lemma 3.4.26(iii) we have that $\rho^*(E \backslash X) \geq j + 1$. Similarly, choose $Y \subseteq E$ such that $\rho^*(Y) = j$ and $\sigma^j(M^*) = |Y|$. Hence $\rho(E \backslash Y) \geq i + 1$. But this implies that $i + j + 1 = \rho^*(E \backslash X) + \rho(E \backslash Y) \geq i + j + 2$, which is not possible. □

3.5 Tutte polynomial

The Tutte polynomial is an important invariant for graphs and matroids. We define the *Tutte polynomial* of a combinatroid $M = (E, \rho)$ as

$$T_M(x, y) := \sum_{A \subseteq E} (x - 1)^{\rho(E) - \rho(A)} (y - 1)^{|A| - \rho(A)}. \tag{3.5.1}$$

Using the classical notation $\eta := \rho^\circ$, this can be rewritten as

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{\eta^*(E \backslash A)} (y - 1)^{\eta(A)}. \tag{3.5.2}$$

Remark 3.5.1. Since a combinatroid ρ may take negative values, we must remark that $T_M(x,y)$, as defined above, could be a rational function; so, a better name would be the Tutte enumerator or the Tutte rational function. However, since we will not use its properties as a rational function, by abuse of language, we will continuous making reference to it as the Tutte polynomial. On the other hand, if ρ is a demimatroid, then $T_M(x,y)$ is in fact a polynomial.

This Tutte polynomial is well-behaved with respect to combinatroidal duality:

Proposition 3.5.2. (Tutte duality) *Let $M=(E,\rho)$ be a combinatroid. Then*

$$T_{M^*}(x,y) = T_M(y,x).$$

Proof. It follows immediately from Eq. (3.5.2) by noticing that $(\rho^*)^\circ = (\rho^\circ)^* = \eta^*$. □

Let $M=(E,\rho)$ be a combinatroid with Tutte polynomial $T_M(x,y)$. We define its *Hamming polynomial* by:

$$W_M(x,y,t) := (x-y)^{\eta(M)} y^{\rho(M)}\, T_M\!\left(\frac{x}{y}, \frac{x+(t-1)y}{x-y}\right). \tag{3.5.3}$$

Example 3.5.3. Let E be a finite set and $\rho : 2^E \to \mathbb{Z}$ given by $\rho(X) = |X|$. Then $\eta(X) = 0$ and $\eta^*(E\backslash X) = |E\backslash X|$ for all $X \subseteq E$. Hence $T(x,y) = W(x,y,t) = x^n$.

Theorem 3.5.4. (MacWilliams identity) *Let $M=(E,\rho)$ be a combinatroid. Then*

$$W_{M^*}(x,y,t) = t^{-\eta(M)}\, W_M(x+(t-1)y, x-y, t).$$

Proof.

$$\begin{aligned}
W_M(x+(t-1)y, x-y, t) &= (ty)^{\eta(E)}(x-y)^{\rho(E)}\, T_M((x+(t-1)y)/(x-y), x/y) \\
&= t^{\eta(E)} \left[y^{\eta(E)}(x-y)^{\rho(E)}(x-y)^{-\eta^*(E)}y^{-\rho^*(E)}\right] \\
&\quad \times (x-y)^{\eta^*(E)}y^{\rho^*(E)}\, T_M((x+(t-1)y)/(x-y), x/y) \\
&= t^{\eta(E)}(1)(x-y)^{\eta^*(E)}y^{\rho^*(E)}\, T_M((x+(t-1)y)/(x-y), x/y) \\
&= t^{\eta(E)}(x-y)^{\eta^*(E)}y^{\rho^*(E)}\, T_{M^*}(x/y, (x+(t-1)y)/(x-y)) \\
&= t^{\eta(E)} W_{M^*}(x,y,t). \qquad \square
\end{aligned}$$

We define the *Whitney generating function*

$$f(M;x,y) := \sum_{A\subseteq E} x^{\eta^*(E\backslash A)} y^{\eta(A)}.$$

Theorem 3.5.5. (Brylawski, Gordon; see [4]) *Let $M=(E,\rho)$ be a combinatroid. Then*

(1) Duality:

$$f(M^*;x,y) = f(M;y,x).$$

(2) Deletion-Contraction: For any $p\in E$,

$$f(M;x,y) = x^{\eta^*(p)} f(M\backslash p;x,y) + y^{1-\rho(p)} f(M/p;x,y).$$

We now proceed to prove a deletion-contraction formula for the Tutte and Hamming polynomials.

Lemma 3.5.6. *Let $M=(E,\rho)$ be a combinatroid. Then*

(a)

$$(x-y)^{\eta(E)} y^{\rho(E)} (x/y-1)^{\rho(E)-\rho(E\backslash p)} = (x-y)(x-y)^{|E\backslash p|-\rho(E\backslash p)} y^{\rho(E\backslash p)}.$$

(b)

$$(x-y)^{\eta(E)} y^{\rho(E)} (x-y)^{\rho(p)-1} = y^{\rho(p)} (x-y)^{|E\backslash p|-\rho(E)+\rho(p)} y^{\rho(E)-\rho(p)}.$$

Proof. (a):

$$\begin{aligned}
(x-y)^{\eta(E)} y^{\rho(E)} (x/y-1)^{\rho(E)-\rho(E\backslash p)} &= (x/y-1)^{\rho(E)-\rho(E\backslash p)} (x-y)^{\eta(E)} y^{\rho(E)} \\
&= (x-y)^{\rho(E)-\rho(E\backslash p)} y^{-\rho(E)+\rho(E\backslash p)} (x-y)^{\eta(E)} y^{\rho(E)} \\
&= (x-y)^{\rho(E)-\rho(E\backslash p)} y^{\rho(E\backslash p)} (x-y)^{\eta(E)} \\
&= (x-y)^{|E|-\rho(E\backslash p)} y^{\rho(E\backslash p)} \\
&= (x-y)(x-y)^{|E\backslash p|-\rho(E\backslash p)} y^{\rho(E\backslash p)}
\end{aligned}$$

(b):

$$\begin{aligned}
(x-y)^{\eta(E)} y^{\rho(E)} &= (x-y)^{1-\rho(p)} (x-y)^{|E\backslash p|-\rho(E)+\rho(p)} y^{\rho(E)-\rho(p)} y^{\rho(p)} \\
&= (x-y)^{1-\rho(p)} y^{\rho(p)} (x-y)^{|E\backslash p|-\rho(E)+\rho(p)} y^{\rho(E)-\rho(p)}. \quad \square
\end{aligned}$$

From the Brylawski recurrence it follows:

Proposition 3.5.7. *Let $M=(E,\rho)$ be a combinatroid. Then*

$$T_M(x,y)=(x-1)^{\eta^*(p)}\,T_{M\backslash p}(x,y)+(y-1)^{1-\rho(p)}\,T_{M/p}(x,y).$$

Example 3.5.8. Let M be the demimatroid in Example 3.4.21, and take $p=3$.

X	$\emptyset$	1	2	3	12	13	23	E
ρ	0	0	0	0	1	1	1	2
ρ^*	0	0	0	0	0	0	0	1
ρ°	0	1	1	1	1	1	1	1
$\rho^\circledast$	0	1	1	1	2	2	2	2

.

$$T_M(x,y)=x-2x^2+y-3xy+3x^2y.$$

Set $\alpha:=\rho_{M\backslash p}$ and $\beta:=\rho_{M/p}$.

X	$\emptyset$	1	2	12
α	0	0	0	1
α^*	0	0	0	1
α°	0	1	1	1
$\alpha^\circledast$	0	1	1	1

,

X	$\emptyset$	1	2	12
β	0	1	1	2
β^*	0	0	0	0
β°	0	0	0	0
$\beta^\circledast$	0	1	1	2

$$T_{M\backslash p}(x,y)=-x-y+2xy;\quad T_{M/p}(x,y)=x^2.$$

$$(x-1)T_{M\backslash p}(x,y)+(y-1)T_{M/p}(x,y)=T_M(x,y).$$

From this we obtain the following recurrence for the Hamming polynomial.

Theorem 3.5.9.

$$W_M(x,y,t)=(x-y)\,W_{M\backslash p}(x,y,t))+t^{1-\rho(p)}y\,W_{M/p}(x,y,t).$$

Proof.

$$\begin{aligned}
W_M(x,y,t) &= (x-y)^{\eta(E)}y^{\rho(E)}\,T_M(x/y,(x+(t-1)y)/(x-y))\\
&= (x-y)^{\eta(E)}y^{\rho(E)}[(x/y-1)^{\rho(E)-\rho(E\backslash p)}\,T_{M\backslash p}(x/y,(x+(t-1)y)/(x-y))\\
&\quad +((x+(t-1)y)/(x-y)-1)^{1-\rho(p)}\,T_{M/p}(x/y,(x+(t-1)y)/(x-y))]\\
&= (x-y)(x-y)^{|E\backslash p|-\rho(E\backslash p)}y^{\rho(E\backslash p)}\,T_{M\backslash p}(x/y,(x+(t-1)y)/(x-y))\\
&\quad +(ty)^{1-\rho(p)}y^{\rho(p)}\\
&\quad \times(x-y)^{|E\backslash p|-\rho(E)+\rho(p)}y^{\rho(E)-\rho(p)}\,T_{M/p}(x/y,(x+(t-1)y)/(x-y))\\
\text{(by 3.5.6)}\quad &= (x-y)\,W_{M\backslash p}(x,y)+t^{1-\rho(p)}y\,W_{M/p}(x,y).\quad\square
\end{aligned}$$

Example 3.5.10. We continuous Example 3.5.8.

$$W_M(x,y,t) = x^3 + 3(t-1)x^2y + 3(1-t)xy^2 + (t-1)y^3.$$

$$W_{M\setminus p}(x,y,t) = x^2 + 2(t-1)xy + (1-t)y^2; \quad W_{M/p}(x,y,t) = x^2.$$

$$(x-y)W_{M\setminus p}(x,y,t) + t\,y\,W_{M/p}(x,y,t) = W_M(x,y,t).$$

3.6 Extended Hamming polynomials

For a combinatroid $M = (E,\rho)$ we define its *characteristic polynomial* as

$$p(M;t) := \sum_{X\subseteq E}(-1)^{|X|}t^{\rho(E)-\rho(X)} = (-1)^{\rho(E)}T_M(1-t,0) = \sum_{X\subseteq E}(-1)^{|E\setminus X|}t^{\eta^*(X)}.$$

Thus the characteristic polynomial of M^* is

$$p(M^*;t) = \sum_{X\subseteq E}(-1)^{|E\setminus X|}t^{\eta(X)}.$$

We generalize $p(M^*;t)$ for every $\sigma \subseteq E$ as: $P_{M,\emptyset}(t) := 1$ and

$$P_{M,\sigma}(t) := \sum_{\gamma\subseteq\sigma}(-1)^{|\sigma\setminus\gamma|}t^{\eta(\gamma)}. \tag{3.6.1}$$

We define the *j-th generalized polynomial* $P_{M,j}(t)$ as $P_{M,0}(t) := 1$ and

$$P_{M,j}(t) := \sum_{|\sigma|=j}P_{M,\sigma}(t) = \sum_{|\sigma|=j}\sum_{\gamma\subseteq\sigma}(-1)^{|\sigma\setminus\gamma|}t^{\eta(\gamma)}, \quad 1 \le j \le n. \tag{3.6.2}$$

Identically as for matroids [10], we define the *Hamming polynomial* of a combinatroid M by

$$W_M(x,y,t) := \sum_{j=0}^{n}P_{M,j}(t)x^{n-j}y^j. \tag{3.6.3}$$

Next, following [10], we will verify that this definition coincides with the one given in Eq. (3.5.3).

Lemma 3.6.1.

$$W_M(x,y,t) = \sum_{\sigma\subseteq E}(x-y)^{|E|-|\sigma|}y^{|\sigma|}\ t^{\eta(\sigma)}.$$

Proof. Set $n = |E|$.

$$\begin{aligned}
\sum_{\sigma} (x-y)^{n-|\sigma|} y^{|\sigma|} t^{\eta(\sigma)} &= \sum_{\sigma} \sum_{i=0}^{n-|\sigma|} \binom{n-|\sigma|}{i} x^i (-y)^{n-|\sigma|-i} y^{|\sigma|}\ t^{\eta(\sigma)} \\
&= \sum_{\sigma} \sum_{\gamma \subseteq E \setminus \sigma} x^{|\gamma|} y^{n-|\gamma|} (-1)^{n-|\sigma|-|\gamma|}\ t^{\eta(\sigma)} \\
&= \sum_{\gamma} x^{|\gamma|} y^{n-|\gamma|} \sum_{\sigma \subseteq E \setminus \gamma} (-1)^{n-|\gamma|-|\sigma|}\ t^{\eta(\sigma)} \\
&= \sum_{\gamma} x^{|\gamma|} y^{n-|\gamma|} P_{M,E\setminus\gamma}(t) \\
&= \sum_{\gamma} x^{n-|\gamma|} y^{|\gamma|} P_{M,\gamma}(t) \\
&= W_M(x,y,t). \qquad \square
\end{aligned}$$

Theorem 3.6.2.

$$W_M(x,y,t) = (x-y)^{\eta(E)} y^{\rho(E)}\, T_M\left(\frac{x}{y}, \frac{x+(t-1)y}{x-y}\right).$$

Proof.

$$\begin{aligned}
T_M\left(\frac{x}{y}, \frac{x+(t-1)y}{x-y}\right) &= \sum_{\sigma} \left(\frac{x}{y} - 1\right)^{\eta^*(E\setminus\sigma)} \left(\frac{x+(t-1)y}{x-y} - 1\right)^{\eta(\sigma)} \\
&= \sum_{\sigma} \frac{(x-y)^{\eta^*(E\setminus\sigma)}}{y^{\eta^*(E\setminus\sigma)}}\ \frac{(ty)^{\eta(\sigma)}}{(x-y)^{\eta(\sigma)}} \\
&= \sum_{\sigma} \frac{(x-y)^{\eta^*(E\setminus\sigma)-\eta(\sigma)}}{y^{\eta^*(E\setminus\sigma)-\eta(\sigma)}}\ t^{\eta(\sigma)} \\
&= \sum_{\sigma} \frac{(x-y)^{\rho(E)-|\sigma|}}{y^{\rho(E)-|\sigma|}}\ t^{\eta(\sigma)} \\
&= \frac{(x-y)^{\rho(E)-n}}{y^{\rho(E)}} \sum_{\sigma} (x-y)^{n-|\sigma|} y^{|\sigma|}\ t^{\eta(\sigma)} \\
&= \frac{(x-y)^{\rho(E)-n}}{y^{\rho(E)}} W_M(x,y,t). \qquad \square
\end{aligned}$$

Theorem 3.6.3.

$$T_M(x,y) = (x-1)^{-\eta(E)} x^{|E|}\, W_M(1, x^{-1}, (x-1)(y-1)).$$

Proof. A straightforward evaluation shows that

$$\begin{aligned}
W_M(1, x^{-1}, (x-1)(y-1)) &= (1-x^{-1})^{n-\rho(E)} x^{-\rho(E)} T_M(x,y) \\
&= (x-1)^{n-\rho(E)} x^{-n} T_M(x,y). \qquad \square
\end{aligned}$$

Example 3.6.4. Let Δ be a simplicial complex of dimension d; so $d+1$ is the largest cardinality of a face. The *f-polynomial* of Δ is defined as

$$f(\Delta,t) := t^{d+1} + c_1 t^{d-1} + \cdots + c_d,$$

where c_i is the number of faces of cardinality i, and its *h-polynomial* is defined as $h(\Delta,t) := f(\Delta,t-1)$. It is well known that $f(\Delta,t) = T(t+1,1)$, where $T(x,y)$ is the Tutte polynomial of Δ. Thus, by Theorem 3.6.3,

$$f(\Delta,t) = (x+1)^{|E|} x^{-\eta(E)} W(1,(x+1)^{-1},0).$$

For instance, let Δ be the simplicial complex with facets $12, 234, 345$, i.e.

$$\Delta = \{\emptyset, 1, 2, 3, 4, 5, 12, 23, 24, 34, 35, 45, 234, 345\}.$$

$$T_{\Delta^{\uparrow}}(x,y) = x - 2x^2 + x^3 + y - 4xy + 4x^2y - y^2 + 2xy^2.$$

$$W_{\Delta^{\uparrow}}(x,y,t) = x^5 + 4(t-1)x^3y^2 + 4(1-t)x^2y^3 + (-1-t+2t^2)xy^4 + (1-t)ty^5.$$

Thus, the f-polynomial of Δ is

$$(x+1)^5 x^{-2} W_{\Delta^{\uparrow}}(1,(x+1)^{-1},0) = x^3 + 5x^2 + 6x + 2.$$

Let $M = (E,\rho)$ be a nontrivial demimatroid, $P_{M,j}(t)$ the polynomial defined in Eq. (3.6.2), and δ the minimum $j > 0$ such that $P_{M,j}(t) \neq 0$.

Proposition 3.6.5. *$\delta = \sigma_1(M^\circ)$ and $P_{M,\delta}(t) = c(t-1)$, where*

$$c = |\{X \subseteq E : |X| = \sigma_1(M^\circ)\}|.$$

Proof. Fix $X \subseteq E$ such that $\eta(X) = 1$ and $|X| = \sigma_1(M^\circ)$. If $\sigma \subseteq E$ and $|\sigma| < |X|$, then by Lemma 3.4.1(i), applied to the restriction of η to σ, it holds that $\eta(\sigma) = 0$. Thus $0 = P_{M,\sigma}(t) := \sum_{\gamma\subseteq\sigma} (-1)^{|\sigma\setminus\gamma|} t^{\eta(\gamma)}$. The same result holds if $|\sigma| = |X|$ and $\eta(\sigma) = 0$. On the other hand, $P_{M,X}(t) = t - 1$. Therefore, we obtain the desired result. $\square$

We call the number $\sigma_1(M^\circ)$ the *formal minimum distance* of M.

Proposition 3.6.6. ([10, Prop. 4.1]) *Let M be a demimatroid, and $P_{M,j}$ as defined in Eq. (3.6.2). Then, for $1 \le r \le \eta(M^\circ)$,*

$$\sigma_r(M^\circ) = \min\{j : \deg P_{M,j} = r\}.$$

Proposition 3.6.7. *Let $M = (E, \rho)$ be a uniform matroid, with $n = |E|$ and $k = \rho(E)$. Then*

$$T_M(x, y) = \sum_{i=0}^{k-1} \binom{n}{i} (x-1)^{k-i} + \binom{n}{k} + \sum_{i=k+1}^{n} \binom{n}{i} (y-1)^{i-k}.$$

Proof. $\rho(X) = |X|$ if $|X| \leq k$ and $\rho(X) = k$ if $|X| > k$. Hence, $\eta(X) = 0$ if $|X| \leq k$ and $\eta(X) = r$ if $|X| = k + r$ with $r > 0$. Moreover, $\eta^*(X) = 0$ if $|E \backslash X| \geq k$, i.e. $|X| \leq n - k$, and $\eta^*(X) = r$ if $|E \backslash X| = k - r$ with $r > 0$, i.e. $|X| = n - k + r$. □

Let $M = (E, \rho)$ be a demimatroid. Set $n = |E|$ and write

$$W_M(x, y, t) = x^n + \sum_{j=\delta}^{n} A_j(t) x^{n-j} y^j.$$

where δ is the formal minimum distance of M.

Proposition 3.6.8. *Let $M = (E, \rho)$ be a uniform matroid, with $n = |E|$, $k = \rho(E)$ and $\delta = \sigma_1(M^\circ)$. Then, for $\delta \leq i \leq n$,*

$$A_i(t) = (t-1) \binom{n}{i} \sum_{j=0}^{i-\delta} (-1)^j \binom{i-1}{j} t^{i-\delta-j}.$$

Proof. The proof readily follows from Proposition 3.6.7. □

Example 3.6.9. Let $M^\circ = (E, \rho^\circ)$ be as in Example 3.4.21. M° is a uniform matroid of rank 1. $T_{M^\circ}(x, y) = x + y + y^2$, $W_{M^\circ}(x, y, t) = x^3 + 3(t-1)xy^2 + (2 - 3t + t^2)y^3$, $\delta = 2$, $A_1(t) = 0$, $A_2(t) = 3(t-1)$, $A_3(t) = 2 - 3t + t^2$.

Example 3.6.10. Let $M^\circ = (E = \{1, 2, 3, 4\}, \rho^\circ)$ be the uniform matroid given by:

X	$\emptyset$	1	2	3	4	12	13	14	23	24	34	123	124	134	234	E
ρ	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	2
ρ^*	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	2
ρ°	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
$\rho^\circledast$	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2

M° is a uniform matroid of rank 2. $T_{M^\circ}(x, y) = 2x + x^2 + 2y + y^2$, $W_{M^\circ}(x, y) = x^4 + 4(t-1)xy^3 + (3 - 4t + t^2)y^4$, $\delta = 3$, $A_1(t) = 0$, $A_2 = 0$, $A_3(t) = 4(t-1)$, $A_4(t) = 3 - 4t + t^2$.

3.7 Elongations

Let $M = (E, \rho)$ be a demimatroid with nullity function η. For $0 \le i \le \eta(M)$ we define the *i-th elongation* of M as the demimatroid $M[i] := (E, \rho^{[i]})$, where

$$\rho^{[i]}(\sigma) := \min\{|\sigma|, \rho(\sigma) + i\},$$

or equivalently,

$$\rho^{[i]}(\sigma) = \begin{cases} |\sigma|, & \eta(\sigma) \le i \\ \rho(\sigma) + i, & \eta(\sigma) > i. \end{cases}$$

Note that $\rho^{[0]} \equiv \rho$, $\rho^{[i]} \equiv (\rho^{[1]})^{[i-1]}$ and $\rho^{[\eta(M)]}(\sigma) = |\sigma|$ for all $\sigma \subseteq E$. When there is no confusion, we will write $M[i]$ instead of $M^{[i]}$.

Proposition 3.7.1. *Let $M = (E, \rho)$ be a demimatroid. Then $M[i]$, as defined above, is a demimatroid.*

Proof. Obviously $\rho^{[i]}(\emptyset) = 0$. Let $X \subseteq E$ and $x \in E$.

If $x \in X$, obviously $\rho^{[i]}(X) \le \rho^{[i]}(X \cup x) \le \rho^{[i]}(X) + 1$, so we may assume $x \notin X$.

If $\rho^{[i]}(X \cup x) = |X| + 1$, thus $\rho^{[i]}(X) = |X|$ and $\rho^{[i]}(X) \le \rho^{[i]}(X \cup x) \le \rho^{[i]}(X) + 1$.

If $\rho^{[i]}(X \cup x) = \rho(X) + i$, thus $\rho^{[i]}(X) = \rho(X) + i$ and $\rho^{[i]}(X) \le \rho^{[i]}(X \cup x) \le \rho^{[i]}(X) + 1$. $\square$

Since $1 \le i \le \eta(M) = |E| - \rho(E)$, it holds that $\rho(E) + i \le |E|$, so $\rho^{[i]}(M[i]) = \rho(M) + i$. If $X \subseteq E$, then the rank function of $M|_X$ is the restriction of ρ to X. We point out that from this it follows that $(M[i])|_X = (M|_X)[i]$.

The nullity function of $M[i]$ is given by

$$\eta^{[i]}(\sigma) = \max\{0, \eta(\sigma) - i\},$$

or equivalently,

$$\eta^{[i]}(\sigma) = \begin{cases} 0, & \eta(\sigma) \le i \\ \eta(\sigma) - i, & \eta(\sigma) > i. \end{cases}$$

An easy verification shows that

$$\eta^{[i]}(\sigma) = 0 \quad \text{if and only if} \quad \eta(\sigma) \le i. \tag{3.7.1}$$

Proposition 3.7.2. *Let $M = (E, \rho)$ be a demimatroid. Then $\sigma_{r+1}(M^\circ) = \sigma_1(M[r]^\circ)$.*

Proof. Choose $X \subseteq E$ such that $\eta(X) = r + 1$ and $|X| = \sigma_{r+1}(M^\circ)$. Hence $\eta^{[r]}(X) = \max\{0, \eta(X) - r\} = 1$, and so $\sigma_1(M[r]^\circ) \leq |X| = \sigma_{r+1}(M^\circ)$. Similarly, choose $Y \subseteq E$ such that $\eta^{[r]}(Y) = 1$ and $|Y| = \sigma_1(M[r]^\circ)$. Hence $\eta(Y) = r + 1$, and so $\sigma_{r+1}(M^\circ) \leq |Y| = \sigma_1(M[r]^\circ)$. □

3.8 Betti numbers

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K. We consider R provided with its canonical $\mathbb{Z}$-grading. Associated with each homogeneous ideal I of R there is a *minimal graded free resolution*

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1j}} \to R \to R/I \to 0,$$

where $R(-j)$ denotes the R-module obtained by shifting the degrees of R by j, i.e $R(-j)_a = R_{a-j}$. The number β_{ij} in the resolution may be interpreted as the minimum number of generators of degree j in the i-th sizygie of R/I; or equivalently

$$\beta_{ij}(R/I) := \beta_{ij} = \dim \operatorname{Tor}_i(R/I, K)_j.$$

These β_{ij}'s are called the *graded Betti numbers* of R/I. We collect all they together by defining the *graded Betti polynomial* of R/I as

$$B(R/I; x, y) := \sum_{i=0}^{p} \sum_j \beta_{ij} x^i y^j.$$

Example 3.8.1. Let $I \subset R = K[x_1, \ldots, x_5]$ be the monomial ideal

$$I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5 \rangle.$$

We have the resolution

$$0 \to R(-5) \to R^3(-3) \oplus R(-4) \to R^4(-2) \to R \to R/I \to 0,$$

so that

$$B(R/I; x, y) = 1 + 4xy^2 + 3x^2y^3 + x^2y^4 + x^3y^5.$$

Let Δ be a simplicial complex; we assume that all the vertices belongs to Δ. It is convenient, abusing notation, to identify $\sigma \subseteq [n]$ with the characteristic vector $\sigma = (\sigma_i) \in \{0,1\}^n$ such that $\sigma_i = 1$ if $i \in \sigma$; and write $|\sigma| := \sigma_1 + \cdots + \sigma_n$. For $\sigma \subseteq [n]$ we denote by Δ_σ the simplicial complex that results from the restriction of Δ to the vertex set σ.

Given a simplicial complex Δ, let I_Δ denote its *Stanley-Reisner ideal* and $K[\Delta]$ its *Stanley-Reisner ring*, i.e. $I_\Delta = \langle x_{i_1} \cdots x_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta \rangle \subset R$ and $K[\Delta] = R/I_\Delta$. Let's also denote by $\widetilde{H}_i(\Delta; K)$ the i-th *reduced homology group* of Δ with coefficients in the field K. We have the fundamental result:

Theorem 3.8.2. (Hochster's Formula [7]) *Let Δ be a simplicial complex with vertex set $[n]$. Then*

$$\beta_{ij}(R/I_\Delta) = \sum_{\sigma \subseteq [n];\, |\sigma| = j} \dim \widetilde{H}_{j-i-1}(\Delta_\sigma).$$

If, instead of the $\mathbb{Z}$-grading, we consider R provided with its $\mathbb{Z}^n$-grading, and for any $\sigma \subseteq [n]$ we define $\beta_{i\sigma}(R/I) := \dim \operatorname{Tor}_i(R/I, K)_\sigma$, then we have

Theorem 3.8.3. (Multigraded Hochster's Formula) *Let Δ be a simplicial complex with vertex set $[n]$. For any $\sigma \subseteq [n]$ we have that*

$$\beta_{i\sigma}(R/I_\Delta) = \dim \widetilde{H}_{|\sigma|-i-1}(\Delta_\sigma).$$

3.9 Hamming polynomial vs Betti numbers

Let Δ be a simplicial complex of dimension d and denote by f_i the number of i-dimensional faces of Δ. The *reduced Euler characteristic* of Δ is defined as

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{d} (-1)^i \dim \widetilde{H}_i(\Delta; K).$$

Lemma 3.9.1. (Euler-Poincaré formula) *The reduced Euler characteristic of a simplicial complex does not depend of the field and*

$$\widetilde{\chi}(\Delta) = -1 + f_0 - \cdots + (-1)^d f_d.$$

Let $M = (E, \rho)$ be a demimatroid with nullity function η, and let $M[i]$ be its i-th elongation. Set $n = |E|$ and denote by $I_{M[i]}$ the Stanley-Reisner ideal of $M[i]$, viewed as a simplicial complex.

Lemma 3.9.2. *For $\sigma \subseteq E$ the coefficient of t^r in $P_{M,\sigma}(t)$ is equal to*

$$\sum_{i=0}^{n}(-1)^i \left(\beta_{i\sigma}(R/I_{M[r]}) - \beta_{i\sigma}(R/I_{M[r-1]})\right).$$

Proof. According to Eq. (3.6.1), the coefficient of t^r is

$$s_{r\sigma} = (-1)^{|\sigma|} \sum_{\gamma\subseteq\sigma;\, \eta(\gamma)=r} (-1)^{|\gamma|}.$$

From Eq. (3.7.1) we have

$$s_{r\sigma} \;=\; (-1)^{|\sigma|}\left(\sum_{\gamma\subseteq\sigma;\, \eta^{[r]}(\gamma)=0} (-1)^{|\gamma|} - \sum_{\gamma\subseteq\sigma;\, \eta^{[r-1]}(\gamma)=0} (-1)^{|\gamma|}\right).$$

By Eq. 3.7.1 and Lemma 3.9.1,

$$\begin{aligned}
(-1)^{|\sigma|} \sum_{\gamma\subseteq\sigma;\, \eta^{[r]}(\gamma)=0} (-1)^{|\gamma|} &= (-1)^{|\sigma|+1}\left(\sum_{\gamma\subseteq\sigma;\, \eta^{[r]}(\gamma)=0} (-1)^{|\gamma|-1}\right)\\
&= (-1)^{|\sigma|+1}\left(\sum_{i=-1}^{\rho^{[r]}(\sigma)} (-1)^i \dim \widetilde{H}_i(M[r]_\sigma; K)\right)\\
&= (-1)^{|\sigma|+1}\left(\sum_{j=\eta^{[r]}(\sigma)-1}^{|\sigma|} (-1)^{|\sigma|-j-1} \dim \widetilde{H}_{|\sigma|-j-1}(M[r]_\sigma; K)\right)\\
&= \sum_{j=\eta^{[r]}(\sigma)-1}^{|\sigma|} (-1)^j \dim \widetilde{H}_{|\sigma|-j-1}(M[r]_\sigma; K)\\
\text{(by 3.8.3)} \quad &= \sum_{j=0}^{|\sigma|}(-1)^j \beta_{j\sigma}(R/I_{M[r]}).
\end{aligned}$$

Similarly,

$$(-1)^{|\sigma|} \sum_{\gamma\subseteq\sigma;\, \eta^{[r-1]}(\gamma)=0} (-1)^{|\gamma|} = \sum_{j=0}^{|\sigma|}(-1)^j \beta_{j\sigma}(R/I_{M[r-1]}). \qquad \square$$

Corollary 3.9.3. *For each $1 \le j \le n$ the coefficient of t^r in $P_{M,j}(t)$ is equal to*

$$\sum_{i=0}^{n}(-1)^i \left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right).$$

Proof. Recall that $P_{M,j}(t) = \sum_{|\sigma|=j} P_{M,\sigma}(t)$ and $\beta_{ij}(R/I_{M[r]}) = \sum_{|\sigma|=j} \beta_{i\sigma}(R/I_{M[r]})$. Hence the coefficient of t^r in $P_{M,j}$ is

$$\sum_{i=0}^{n}(-1)^i \left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right). \quad \square$$

Theorem 3.9.4.

$$W(x,y,t) = x^n \sum_{r=0}^{\eta} \left(B_{M[r]}(-1,y/x) - B_{M[r-1]}(-1,y/x)\right) t^r.$$

Proof. By definition $W(x,y,t) = \sum_{j=0}^{n} P_{M,j}(t) x^{n-j} y^j$. By Corollary 3.9.3,

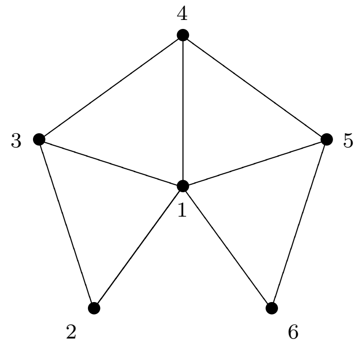
$$\begin{aligned}
W(x,y,t) &= \sum_{j=0}^{n}\left(\sum_{r=0}^{n}\left(\sum_{i=0}^{n}(-1)^i\left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right)\right)t^r\right)x^{n-j}y^j \\
&= \sum_{r=0}^{n}\left(\sum_{j=0}^{n}\left(\sum_{i=0}^{n}(-1)^i\left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right)\right)x^{n-j}y^j\right)t^r \\
&= \sum_{r=0}^{n}\left(\sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{n}\left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right)x^{n-j}y^j\right)\right)t^r \\
&= x^n\sum_{r=0}^{n}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n}\left(\beta_{ij}(R/I_{M[r]}) - \beta_{ij}(R/I_{M[r-1]})\right)(-1)^i(y/x)^j\right)\right)t^r \\
&= x^n\sum_{r=0}^{\eta(E)}\left(B_{M[r]}(-1,y/x) - B_{M[r-1]}(-1,y/x)\right)t^r. \qquad \square
\end{aligned}$$

Remark 3.9.5. (i) $B_{M[-1]}(x,y) = 0$ and $B_{M[\eta(E)]}(x,y) = 1$.

(ii) $W(x,y,0) = x^n B_M(-1,y/x)$.

3.10 Examples

Example 3.10.1. Let G be the graph



and let Δ be the simplicial complex whose facets are the nine edges of this graph. Let us consider Δ provided with its natural structure of demimatroid, i.e. $\rho(\sigma) = \max\{|X| : X \subseteq \sigma \text{ and } X \in \Delta\}$. The circuits of Δ are $\{24, 25, 26, 35, 36, 46, 123, 134, 145, 156\}$; here we have written 24 instead of $\{2, 4\}$, and so on. We have

$$T(x, y) = -x + x^2 - y + 4xy + 2y^2 + xy^2 + 2y^3 + y^4$$

and

$$\begin{aligned} W(x, y, t) &= (x-y)^4 y^2 \; T(\frac{x}{y}, \frac{x + (t-1)y}{x-y}) \\ &= x^6 + 6(-1+t)x^4y^2 + (4 - 5t + t^2)x^3y^3 \\ &\quad + 3(3 - 7t + 4t^2)x^2y^4 + 3(-4 + 11t - 9t^2 + 2t^3)xy^5 \\ &\quad + (4 - 13t + 14t^2 - 6t^3 + t^4)y^6. \end{aligned}$$

The Betti polynomial of the elongations of Δ, over $\mathbb{Q}$, are

$$\begin{aligned} B_0(x, y) &= 1 + 6xy^2 + 4xy^3 + 8x^2y^3 + 12x^2y^4 + 3x^3y^4 + 12x^3y^5 + 4x^4y^6; \\ B_1(x, y) &= 1 + xy^3 + 12xy^4 + 21x^2y^5 + 9x^3y^6; \\ B_2(x, y) &= 1 + 6xy^5 + 5x^2y^6; \\ B_3(x, y) &= 1 + xy^6; \\ B_4(x, y) &= 1. \end{aligned}$$

From this we obtain

$$\begin{aligned} x^6 \sum_{r=0}^{4} \big(B_{M[r]}(-1, y/x) - B_{M[r-1]}(-1, y/x)\big)\, t^r &= x^6 + 6(-1+t)x^4y^2 \\ &\quad + (4-5t+t^2)x^3y^3 + 3(3-7t+4t^2)x^2y^4 \\ &\quad + 3(-4+11t-9t^2+2t^3)xy^5 \\ &\quad + (4-13t+14t^2-6t^3+t^4)y^6. \end{aligned}$$

Example 3.10.2. Let Δ be the simplicial complex whose faces are the independent vertex sets of the graph G in Example 3.10.1, i.e. the facets of Δ are $\{1, 25, 35, 36, 246\}$. The circuits of Δ are all the edges of G. We have

$$T(x, y) = x - 2x^2 + x^3 + y - 2xy + x^2y + y^2 - 5xy^2 + 4x^2y^2 - 2y^3 + 3xy^3$$

and

$$\begin{aligned} W(x, y, t) &= (x-y)^3y^3\; T\Big(\frac{x}{y}, \frac{x+(t-1)y}{x-y}\Big) \\ &= x^6 + 9(-1+t)x^4y^2 + (17-21t+4t^2)x^3y^3 \\ &\quad + 12(-1+t)x^2y^4 + 3(1+t-3t^2+t^3)xy^5 + t(-3+5t-2t^2)y^6. \end{aligned}$$

The Betti polynomial of the elongations of Δ are

$$\begin{aligned} B_0(x, y) &= 1 + 9xy^2 + 17x^2y^3 + x^2y^4 + 13x^3y^4 + 2x^3y^5 + 5x^4y^5 + x^4y^6 + x^5y^6; \\ B_1(x, y) &= 1 + 4xy^3 + 3xy^4 + 3x^2y^4 + 6x^2y^5 + 3x^3y^6; \\ B_2(x, y) &= 1 + 3xy^5 + 2x^2y^6; \\ B_3(x, y) &= 1. \end{aligned}$$

From this we obtain

$$\begin{aligned} x^6 \sum_{r=0}^{3} \big(B_{M[r]}(-1, y/x) - B_{M[r-1]}(-1, y/x)\big)\, t^r &= x^6 + 9(-1+t)x^4y^2 \\ &\quad + (17-21t+4t^2)x^3y^3 + 12(-1+t)x^2y^4 \\ &\quad + 3(1+t-3t^2+t^3)xy^5 + t(-3+5t-2t^2)y^6. \end{aligned}$$

Example 3.10.3. Let C be the Hamming linear $[8,4,4]_2$ code, with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

We have

$$T(x,y) = 6x + 10x^2 + 4x^3 + x^4 + 6y + 14xy + 10y^2 + 4y^3 + y^4$$

and

$$\begin{aligned} W(x,y,t) &= x^8 + 14(-1+t)x^4y^4 + 28(2-3t+t^2)x^2y^6 \\ &\quad + 8(-8+14t-7t^2+t^3)xy^7 + (21-42t+28t^2-8t^3+t^4)y^8. \end{aligned}$$

The Betti polynomial of the elongations of $M[H]$ are

$$\begin{aligned} B_0(x,y) &= 1 + 14xy^4 + 56x^2y^6 + 64x^3y^7 + 21x^4y^8; \\ B_1(x,y) &= 1 + 28xy^6 + 48x^2y^7 + 21x^3y^8; \\ B_2(x,y) &= 1 + 8xy^7 + 7x^2y^8; \\ B_3(x,y) &= 1 + xy^8; \\ B_4(x,y) &= 1. \end{aligned}$$

From this we obtain

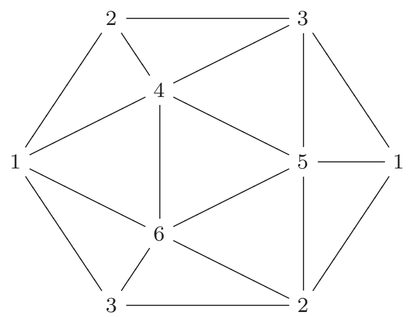
$$x^8 \sum_{r=0}^{4} \big(B_{M[r]}(-1, y/x) - B_{M[r-1]}(-1, y/x)\big)\, t^r = W(x,y,t).$$

Example 3.10.4. Let Δ be the simplicial complex whose facets are the 2-dimensional faces determined by the triangulation of the projective plane

i.e., the facets of Δ are $\{124, 234, 345, 135, 125, 256, 236, 136, 146, 456\}$.

In characteristic 2 the Betti polynomials are

$$\begin{aligned} B_0(x,y) &= 1 + 10xy^3 + 15x^2y^4 + 6x^3y^5 + x^3y^6 + x^4y^6; \\ B_1(x,y) &= 1 + 6xy^5 + 5x^2y^6; \\ B_2(x,y) &= 1 + xy^6; \\ B_3(x,y) &= 1. \end{aligned}$$



In characteristic 3 the Betti polynomials are

$$\begin{aligned} B_0(x,y) &= 1 + 10xy^3 + 15x^2y^4 + 6x^3y^5; \\ B_1(x,y) &= 1 + 6xy^5 + 5x^2y^6; \\ B_2(x,y) &= 1 + xy^6; \\ B_3(x,y) &= 1. \end{aligned}$$

Even though these polynomials do depend of the characteristic of the field, in both cases it results that

$$T(x,y) = -4x + 3x^2 + x^3 - 4y + 10xy + 3y^2 + y^3$$

and

$$W(x,y,t) = x^6 + 10(-1+t)x^3y^3 - 15(-1+t)x^2y^4 + 6(-1+t^2)xy^5 + t(5-6t+t^2)y^6.$$

Note that the coefficient of x^2y^4, i.e. $-15(t-1)$, is negative for any $t > 1$, so $W(x,y,t)$ cannot be the weight enumerator of any code over a finite field.

The Duursma zeta polynomial corresponding to $W(x,y,t)$ is

$$P_q(t) = (1/2)(1 + (1-q)t + qt^2).$$

This polynomial has negative discriminant for $q \in (3-2\sqrt{2}, 3-2\sqrt{2}) \approx (0.17, 5.82)$. For q in this interval, the roots of $P_q(t)$ lie in the circle $(x+1)^2 + y^2 = 2$, moreover all roots have module $1/\sqrt{q}$, so that $P_q(t)$ satisfies the Riemann hypothesis. See [3].

Example 3.10.5. Let M be the Vamos matroid, i.e. the ground set is $E = \{1, \dots, 8\}$ and the bases are all the subsets of E of size 4, except $\{1234, 2356, 1456, 2378, 1478\}$. We have

$$T(x, y) = x^4 + 4x^3 + 10x^2 + 15x + 5xy + 15y + 10y^2 + 4y^3 + y^4$$

and

$$\begin{aligned} W(x, y, t) &= x^8 + 5(-1+t)x^4y^4 + 36(-1+t)x^3y^5 + 2(55 - 69t + 14t^2)x^2y^6 \\ &\quad + 4(-25 + 37t - 14t^2 + 2t^3)xy^7 + (30 - 51t + 28t^2 - 8t^3 + t^4)y^8. \end{aligned}$$

3.11 Generalized Hamming polynomial

For positive integers $j \leq m$ and q an indeterminate, let us define

$$\begin{aligned} [m]_q &:= 1 + q + \cdots + q^{m-1} \\ [m]_q! &:= [1]_q [2]_q \cdots [m]_q \\ \begin{bmatrix} m \\ j \end{bmatrix}_q &:= \frac{[m]_q!}{[j]_q! \, [m-j]_q!} \\ \langle m \rangle_q &:= (q^m - 1)(q^m - q) \cdots (q^m - q^{m-1}). \end{aligned}$$

Since

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \begin{bmatrix} m-1 \\ j \end{bmatrix}_q + q^{m-j} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q,$$

it follows that all these are polynomials in q with integer coefficients.

Let $M = (E, \rho)$ be a combinatroid. Set $n = |E|$ and $k = \rho(E)$. Following [11], for $1 \leq r \leq n$, we define the r-generalized Hamming weight enumerator

$$W^{(r)}(x, y, q) := \frac{1}{\langle r \rangle_q} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix}_q (-1)^{r-j} \, q^{\binom{r-j}{2}} (x-y)^{n-k} \, y^k \, T_M\Big(\frac{x}{y}, \frac{x + (q^j - 1)y}{x - y}\Big).$$

Conjecture 3.11.1. Let $M = (E, \rho)$ be a combinatroid. Set $n = |E|$ and $k = \rho(E)$. Then

$$T_M(x, y) = x^n (x-1)^{k-n} \sum_{r=0}^{n-k} \left(\prod_{j=0}^{r-1} ((x-1)(y-1) - q^j) \right) W^{(r)}(1, 1/x, q). \tag{3.11.1}$$

Remark 3.11.2. When M is the associated matroid to a linear code, via its parity check matrix, this conjecture has been proved by Jurrius [12, Thm. 3.3.5].

Example 3.11.3. Let C be the binary linear $[6,3]$ code with parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(See Example C_1 of Section 5.2 in [12]) The bases of the matroid $M[H]$ are

$$\{145, 146, 156, 245, 246, 256, 345, 346, 356, 456\},$$

and its Tutte polynomial is

$$T(x,y) = x + x^2 + x^3 + y + xy + x^2y + y^2 + xy^2 + x^2y^2 + y^3.$$

$$\begin{aligned} W^{(0)}(x,y,t) &= x^6; \\ W^{(1)}(x,y,t) &= 3x^4y^2 + (-2+t)x^3y^3 + 3x^2y^4 + 3(-2+t)xy^5 + (3-3t+t^2)y^6; \\ W^{(2)}(x,y,t) &= x^3y^3 + 3xy^5 + (-3+t+t^2)y^6; \\ W^{(3)}(x,y,t) &= y^6. \end{aligned}$$

Substituting these $W^{(r)}$'s in Eq. (3.11.1) we recover $T(x,y)$.

Example 3.11.4. Let C be the binary linear $[6,3]$ code with parity check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(See Example C_1 of Section 5.2 in [12]) The bases of the matroid $M[H]$ are

$$\{123, 126, 135, 156, 234, 246, 345, 456\},$$

and its Tutte polynomial is

$$T(x,y) = x^3 + 3x^2y + 3xy^2 + y^3.$$

$$\begin{aligned} W^{(0)}(x,y,t) &= x^6; \\ W^{(1)}(x,y,t) &= 3x^4y^2 + 3(-1+t)x^2y^4 + (-1+t)^2y^6; \\ W^{(2)}(x,y,t) &= 3x^2y^4 + (-2+t+t^2)y^6; \\ W^{(3)}(x,y,t) &= y^6. \end{aligned}$$

Substituting these $W^{(r)}$'s in Eq. (3.11.1) we recover $T(x,y)$.

Example 3.11.5. Let C be the binary Hamming linear $[7,4]$ code with parity check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The Tutte polynomial of $M[H]$ is

$$T(x,y) = 3x + 4x^2 + x^3 + 3y + 7xy + 6y^2 + 3y^3 + y^4.$$

$$\begin{aligned}
W^{(0)}(x,y,t) &= x^7; \\
W^{(1)}(x,y,t) &= 7x^4y^3 + 7x^3y^4 + 21(-2+t)x^2y^5 + 7(6-5t+t^2)xy^6 \\
&\quad +(-13+15t-6t^2+t^3)y^7; \\
W^{(2)}(x,y,t) &= 21x^2y^5 + 7(-5+t+t^2)xy^6 + (15-6t-5t^2+t^3+t^4)y^7; \\
W^{(3)}(x,y,t) &= 7xy^6 + (-6+t+t^2+t^3)y^7; \\
W^{(4)}(x,y,t) &= y^7.
\end{aligned}$$

Substituting these $W^{(r)}$'s in Eq. (3.11.1) we recover $T(x,y)$.

Example 3.11.6. Let M be the demimatroid in Example 3.4.21.

X	$\emptyset$	1	2	3	12	13	23	E
ρ	0	0	0	0	1	1	1	2
ρ^*	0	0	0	0	0	0	0	1
ρ°	0	1	1	1	1	1	1	1
$\rho^\circledast$	0	1	1	1	2	2	2	2

$$T_M(x,y) = x - 2x^2 + y - 3xy + 3x^2y.$$

$$W_M(x,y,t) = x^3 + 3(t-1)x^2y + 3(1-t)xy^2 + (t-1)y^3.$$

$$\begin{aligned}
W^{(0)}(x,y,t) &= (x-y)^3x^3; \\
W^{(1)}(x,y,t) &= (x-y)^3y(3x^2-3xy+y^2); \\
W^{(2)}(x,y,t) &= 0.
\end{aligned}$$

$$x^6(x-y)^{-4}[W^{(0)}(1,1/x,t) + ((x-1)(y-1)-1)W^{(1)}(1,1/x,t)] = T_M(x,y).$$

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