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"On point configurations on grids and crossing numbers"

T H E S I S

Presented by

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Resumen

Preguntas típicas en Geometría Discreta tratan sobre la estructura de conjuntos finitos de objetos geométricos. En particular, conjuntos finitos de puntos en $\mathbb{R}^d$ dan lugar a un gran número de problemas interesantes. En esta tesis estudiamos una variedad de problemas que pertenecen a esta área.

Presentamos las primeras cotas no triviales para el siguiente problema:

> Para una constante $\alpha > 0$, ¿cuál es el número de tipos de orden de n puntos que puede ser representado por coordenadas enteras menores a n^{α}?

Presentamos también mejoras en las cotas superiores de los números de cruce rectilíneo y pseudolineal y estudiamos una variante de el número de cruce rectilíneo, llamado el número de cruce 2-coloreado.

Finalmente, estudiamos trayectorias planas y árboles binarios completos planos en dibujos con aristas ordenadas, la conectividad de la gráfica de giros de las trayectorias Hamiltonianas en la malla y dibujos óptimos de gráficas multipartitas.

Abstract

Typical questions in Discrete Geometry are concerned with the structure of finite sets of geometric objects. In particular, finite sets of points in $\mathbb{R}^d$ give rise to a great number of interesting problems. In this thesis we study a variety of problems that belong to this area.

We give the first non trivial bounds for the following problem:

> For a given constant $\alpha > 0$, what is the number of order types of n points that can be represented by integer coordinates smaller than n^α?

We present improvements on the upper bounds of the rectilinear and pseudolinear crossing numbers and study a variation of the rectilinear crossing number called the 2-colored crossing number.

Finally, we study plane paths and plane binary trees in edge-ordered straight-line drawings, the connectivity of the flip graph of Hamiltonian paths on the grid, and optimal grid drawings of complete multipartite Graphs.

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Contents

Introduction

In this thesis we study a variety of problems that belong to the area of Discrete Geometry. Typical questions in this area of Mathematics are concerned with the structure of finite sets of geometric objects. In particular, finite sets of points in $\mathbb{R}^d$ give rise to a great number of interesting problems. For many of these of problems, the metric properties of the point sets are not relevant. That is, there are sets of points that are combinatorially indistinguishable from one another. An example is convexity: given a set of n points, it does not matter how close to each other they are in order to be in convex position.

There exist several ways to classify sets of points that are combinatorially distinct. One such classification is the *order type*, which consists in considering two point sets as equivalent if there exists a bijection between them that preserves the orientation of every triple of points. This classification is particularly convenient from a computational point of view, because determining the orientation of a triple of points can be done very efficiently.

An aspect about order types that has been studied extensively is their representation. Order types partition the sets of n points into a finite number of equivalence classes; thus, having access to representatives of each equivalence class allows us to perform a complete search over all possible point configurations for any combinatorial property that is invariant under order types.

One possibility for encoding order types is to store the coordinates of one point set of each equivalence class. Although this requires storing only $2n$ coordinates per representative in the plane, storing those coordinates might require an exponential number of bits. There exist point sets whose coordinate representation requires a grid of size at least $2^{2^{cn}}$, where c is a constant, to represent them.

Leaving the size of the coordinates aside, storing representatives for each equivalence class becomes unfeasible very quickly as n increases. Already for $n = 10$ there exist 14,309,547 different order types, and for $n = 11$ the number increases to 2,334,512,907. More generally, if we denote by $f(n)$ the number of different order types in $\mathbb{R}^2$, the following bounds are known:

$$\exp(4(1+O(1/\log n))n\log n) \leq f(n) \leq \exp(4(1+O(1/\log n))n\log n).$$

These results motivate the following question: How many order types can we represent in, say, a polynomial size integer grid? Chapter 1 is dedicated to showing lower bounds for the number of order types that we can represent in grids of sizes n^2 and $n^{2.5}$.

Chapter 2 is devoted to problems related to a well studied invariant under order types: the *rectilinear crossing number*. Let P be a set of n points in the plane. For each pair of points in P, draw a straight-line segment connecting them. This produces a *straight-line drawing* of the complete graph K_n. The *rectilinear crossing number* $\overline{\mathrm{cr}}(D)$ of a straight-line drawing D, is the number of crossings that appear between edges of the drawing. The rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$ of K_n, is the minimum number of crossings that appear in every possible straight-line drawing of K_n.

The asymptotic behavior of $\overline{\mathrm{cr}}(K_n)$ has been studied by several authors. It is known that the limit

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}$$

exists; it is called the *rectilinear crossing constant* and it is denoted by q^*. Currently, the best published upper bound on q^* is:

$$q^* \leq \frac{9363184}{24609375} < 0.380473.$$

There exist many variations of crossing numbers. One such variation comes from representing edges in a drawing with simple curves that can be extended to *pseudolines* that intersect each other exactly once, instead of using straight-line segments. This produces a *pseudolinear drawing* of K_n. As in the rectilinear setting, the *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(K_n)$ of K_n, is the minimum number of crossings that appear in every possible pseudolinear drawing of K_n. There also exists a constant associated with $\widetilde{\mathrm{cr}}(K_n)$, the *pseudolinear crossing constant*, denoted by $\widetilde{q}^*$. Currently, the best published upper bound on $\widetilde{q}^*$ is:

$$\widetilde{q}^* \leq \frac{1437136544749}{3777575768064} < 0.3804388405.$$

In Section 2.1 we present improvements on both of these bounds.

In Section 2.2 we study another variation of crossing numbers: the 2-colored crossing number. In this variation, given a straight-line drawing of

K_n, we color each edge with one of two colors. We are interested in counting the crossings between edges of the same color. Denote by $\overline{\text{cr}}_2(D)$ the minimum number of monochromatic crossings that appear in D over all possible 2-colorings of its edges. The *2-colored crossing number* $\overline{\text{cr}}_2(K_n)$ of K_n, is the minimum of $\overline{\text{cr}}_2(D)$ taken over all possible straight-line drawings of K_n. We present upper and lower bounds on $\overline{\text{cr}}_2(K_n)$ and also study the ratio $\overline{\text{cr}}(D)/\overline{\text{cr}}_2(D)$, where D is a fixed drawing of K_n.

As seen with crossing numbers, different drawings of a graph provide interesting problem variations. Although crossing numbers have a geometric nature to begin with, drawings can also give rise to geometric variations of combinatorial problems; this is the spirit of Chapter 3.

First we study a geometric variation of the *altitude* of a graph. Let G be a graph a with a total order of its edges. A path in G is called *monotone* if its edges are increasing or decreasing with respect to the total order. The altitude of G, denoted by $\alpha(G)$, is the minimum length of the longest monotone path in G under any total order of its edges. We study this parameter in the context of straight-line drawings. Given a straight-line drawing D of G, denote by $\overline{\alpha}(D)$ the length of the longest plane monotone path in D with respect to the order of the edges of G. In Section 3.1 we present bounds on $\overline{\alpha}(K_n)$, the minimum of $\overline{\alpha}(D)$ taken over all possible straight-line drawings of K_n and all possible total orders of the edges of K_n.

In Section 3.2 we turn our attention to Hamiltonian paths in *grid graphs*. Given two positive integers n and m, the grid graph $G_{n,m}$ is the graph with vertex set $\{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Two vertices are adjacent in $G_{n,m}$ if they are at distance one from each other. Let $H(G_{n,m})$ be the set of Hamiltonian paths in $G_{n,m}$. A flip in a Hamiltonian path $H \in H(G_{n,m})$ consists on removing an edge in H and adding an edge of $G_{n,m}$ not in H, such that the resulting graph is also in $H(G_{n,m})$. The number flips needed to transform a Hamiltonian path into another gives a notion of similarity between them. However, it is not obvious that we can transform any two Hamiltonian paths in $H(G_{n,m})$ using flips. We show that this is indeed the case when $m \in \{2,3,4\}$.

Finally, in Section 3.3 we study a relation between straight-line drawings of complete r-partite graphs on d-dimensional integer grids and eigenvalues of such graphs. Given positive integers r, d, and $n_1 \leq \cdots \leq n_r$ such that $\sum n_i = (2M+1)^d$ for some integer M, we consider straight-line drawings of $K_{n_1,\dots,n_r}$ into the d-dimensional integer grid $\left\{(x_1,\dots,x_d) \in \mathbb{Z}^d : -M \leq x_i \leq M\right\}$, where no two vertices of the graph are drawn on the same grid point. These drawings correspond to colorings of the points of the d-dimensional integer grid with r

colors, such that color i appears n_i times. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ be the eigenvalues of the Laplacian matrix of $K_{n_1,\ldots,n_r}$. The drawings that minimize the squared distances between points of different colors approximate the value of λ_2; and the drawings that maximize the squared distances between points of different colors approximate the value of λ_N. We characterize the drawings that best approximate λ_2 and λ_N.

Chapter 1

Order Types in small integer grids

Many problems in combinatorial geometry deal with questions about finite sets of points in $\mathbb{R}^d$. Although there exist infinitely many configurations of n points, not all of them have different combinatorial properties. For example, let S and Q be configurations of 4 points in the plane with no subset of three collinear points such that the convex hull of S is a triangle and the convex hull of Q is a quadrilateral. Any other configuration of 4 points in the plane can be seen as an affine transformation of either S or Q, and it will have the same corresponding combinatorial properties. Thus, there are essentially two combinatorially distinct configurations of 4 points (see Figure 1.1).

This motivates defining an equivalence relation that captures when two point sets are combinatorially the same. One such relation is the *order type*.

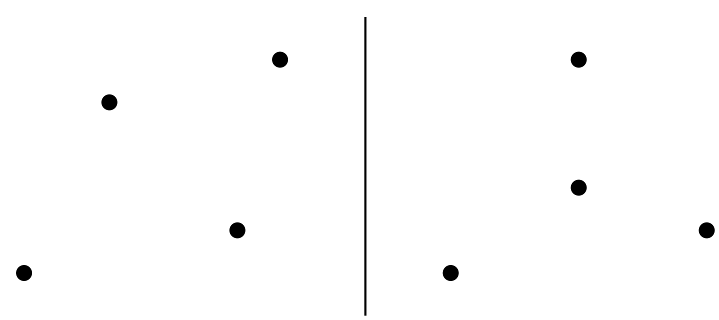


Figure 1.1: Two combinatorially distinct sets of four points.

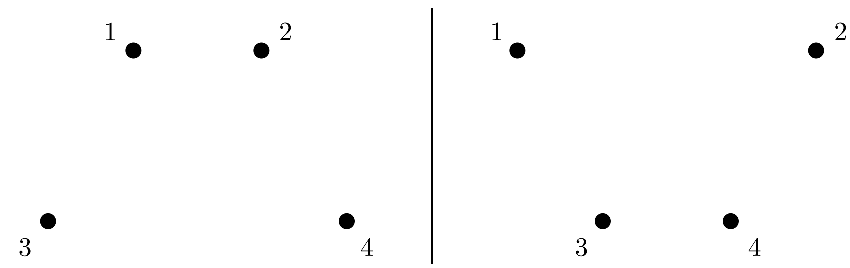


Figure 1.2: Two sets of points with the same order type, but different combinatorial type.

Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be two sets of points in $\mathbb{R}^2$. The order type of P is a mapping that assigns to each ordered triple $1 \leq i < j < k \leq n$ the orientation of the points p_i, p_j, p_k (clockwise, counterclockwise or collinear). The point sets P and Q have the same order type if there exists a bijection $f : P \to Q$ such that p_i, p_j, p_k and $f(p_i), f(p_j), f(p_k)$ have the same orientation for every triple $1 \leq i < j < k \leq n$. Note that the orientation of a triple p, q, r can be determined by computing the determinant

$$\begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix},$$

which is positive when the points p, q, r have clockwise orientation, negative when they have counterclockwise orientation and is equal to zero when the points are collinear. We say that a point configuration is in *general position* if it does not contain a subset of three collinear points.

The order type of a point set P captures many combinatorial properties. Some examples are: number of triangulations, number of non-crossing Hamiltonian cycles, number of k-sets (subsets of that can be separated from P by a straight line), and the number of crossings between the straight line segments defined by pairs of points in P (see [12]). This last property is called the *rectilinear crossing number* of P, and will be the central topic of Chapter 2.

One refinement of the order type equivalence is the *combinatorial type* [49]. This classification associates to any configuration of n labeled points a sequence of permutations of $[1, n]$, and configurations of points that are given the same sequence of permutations are identified. More precisely, project the points in P onto a directed line L. The direction of L determines an order of the projections, which gives us a permutation of $[1, n]$. If we rotate L clockwise, we get a new permutation every time that L passes through a direction orthogonal to a line defined by two points p_i, p_j of P. Note that when this happens, i and j swap places in the new permutation. If we continue to rotate L, we obtain a sequence of permutations called the *circular sequence of permutations* associated to P. Once L completes a full rotation, we end up with the initial permutation. Thus, the circular sequence is periodic and is determined by half a period. The geometric and combinatorial properties of P captured by the order type are also captured by the combinatorial type, but there exist sets of points with the same order type and different combinatorial type (see Figure 1.2).

There are two characteristics that circular sequences satisfy:

- The move from one permutation in the sequence to the next one consists of the swap of disjoint pairs of adjacent elements
- Once two elements swap positions, they never swap again until every other pair of elements has swapped positions

Every cyclic sequence of permutations of $[1, n]$ that satisfies these properties is called an *allowable sequence*. An allowable sequence that comes from a point configuration is called *realizable*. In 1881, Perrin claimed [72] that every allowable sequence was realizable. Goodman and Pollack showed in 1980 [45] that this is not the case by proving that there is exactly one allowable sequence of $[1, 5]$ that is unrealizable.

Allowable sequences are related to *arrangements of pseudolines*, families of topological lines that extend to infinity such that every pair crosses exactly once. A *generalized configuration* is a point configuration together with a pseudoline arrangement that connects every pair of points. Goodman and Pollack proved that every allowable sequence can be realized by a generalized configuration.

Order types correspond to straight line arrangements from *geometric duality*, which assigns to a point (p_x, p_y) the line $y = -p_x x + p_y$ and to each non-vertical line $y = mx + b$ the point $(-m, b)$.

The above definition of order type applies to point sets in $\mathbb{R}^2$, but it can be extended to point sets in $\mathbb{R}^d$ by considering $(d + 1)$ tuples of points. The first d points of the tuple define an oriented hyperplane and the relationship is whether the last point is above, below or on this hyperplane. We say that P is in *general position* if no d points of P lie on a hyperplane.

The notion of order type is also related to sorting. We follow the exposition of Goodman and Pollack [47]. For a set P of distinct points $\{p_1, p_2, \ldots, p_n\}$

on an oriented line, let $\Lambda : \{1, \dots, n\} \to 2^P$ be the function

$$\Lambda(i) = \{p_j \in P | p_j \text{ is to the right of } p_i\},$$

and let $\lambda : \{1, \dots, n\} \to \{0, \dots, |P| - 1\}$ be the function

$$\lambda(i) = |\Lambda(i)|.$$

If either of Λ or λ is known, then the order of the points in P is totally determined: from Λ we can get all the ordered pairs $i < j$ such that $p_i < p_j$ (these are the fundamental atoms of the ordering of the points in P) and from λ we can read the position of each p_i in the ranking $p_{i_1}, \dots, p_{i_n}$. Note that λ can be read from Λ and viceversa.

These functions have a natural counterpart for point configurations in R^d. Let P be a set of n distinct points $\{p_1, \dots, p_n\}$ in R^d. Let $\Lambda_d : \{(i_1, i_2, \dots, i_d) | i_j \neq i_k, 1 \leq i_j \leq n\} \to 2^P$ be the function

$$\Lambda_d(i_1, i_2, \dots, i_d) = \{p_j \in P | p_j \text{ is to the right of the hyperplane defined by } p_{i_1}, \dots, p_{i_d}\},$$

and let $\lambda_d : \{i_1, i_2, \dots, i_d\} \to \{0, \dots, |P| - 1\}$ be the function

$$\lambda_d(i_1, i_2, \dots, i_d) = |\Lambda_d(i_1, i_2, \dots, i_d)|.$$

The function λ_d provides the counterparts in $\mathbb{R}^d$ of the fundamental atoms of sorting in $\mathbb{R}$.

Even for $d = 2$, it is not immediate that Λ_d can be read from λ_d. This case was proved by Goodman and Pollack [48] and the same authors proved in [47] the general case.

Theorem 1.1. *If, for a point configuration $\{P_1, ..., P_n\}$ in $\mathbb{R}^d$ the function λ_d is given, then the function Λ_d is uniquely determined.*

We will abuse of notation and write just λ and Λ whenever the value of d is clear from context.

Since order types in $\mathbb{R}^d$ are completely determined by the function λ, they can be encoded in an $n \times \ldots \times n$ $d-$array. In $\mathbb{R}^2$, this is called the λ-matrix. It is defined as follows:

$$\begin{bmatrix} 0 & \lambda(1,2) & \dots & \lambda(1,n) \\ \lambda(2,1) & 0 & \dots & \lambda(2,n) \\ \vdots & & \ddots & \\ \lambda(n,1) & \lambda(n,2) & \dots & 0 \end{bmatrix}$$

Thus, although order types encode information about a cubic number of triples, they can be stored using $O(n^d \log n)$ bits. It is a natural question to ask how much storage is needed to represent an arbitrary order type. In 1986, Chazelle asked [46] for the number of bits necessary to store an order type. Several authors have studied this question. Recently, Cardinal, Chan, Iacono, Langerman, and Ooms [28] showed an encoding for abstract order types that uses $O(n^2)$ bits in which an orientation query takes $O(\log n)$ time in the word-RAM model. This encoding is space-optimal for abstract order types. They further show how to shorten the encoding to $O(n^2(\log\log n)^2/\log n)$ bits for realizable order types and how to refine their encoding to attain $O(\log n\ \log\log n)$ query time.

One alternative representation of order types is to store the explicit coordinates of a point set. In 1989, Goodman, Pollack, and Sturmfels [44] showed that an exponential number of bits is both sufficient and sometimes necessary to give a coordinate representation of any order type of n points in the plane. More precisely, they proved:

Theorem 1.2. *Let $f(d,n)$ be the smallest integer N such that every configuration of n points in general position in R^d can be realized, up to order type, on the grid $G(d,N) = \{(i_1,\dots,i_d) | -N \leq i_j \leq N\}$. Then there exist constants c_1, c_2 that depend on d such that*

$$2^{2^{c_1 n}} \leq f(d,n) \leq 2^{2^{c_2 n}}.$$

Thus, storing explicitly the coordinates can take an exponential number of bits.

Another natural question is: how many distinct realizable order types exist? Since order types are determined by the λ-matrix, this gives an inmediate upper bound of $\exp(cn^d \log n)$ on the number of order types in $\mathbb{R}^d$. But λ-matrices encode *all* order types, realizable or not. Goodman and Pollack showed in 1986 [50] the following upper bound on the number of different order types:

Theorem 1.3. *Let $f(d,n)$ be the number of labeled order types on n points in R^d. Then*

$$\begin{aligned} f(d,n) &\leq \left(\frac{n}{d}\right)^{d^2 n\left(1+O\left(\frac{1}{\log(n/d)}\right)\right)} \\ &\leq \exp(d^2(1+O(1/\log(n/d)))n\log n). \end{aligned}$$

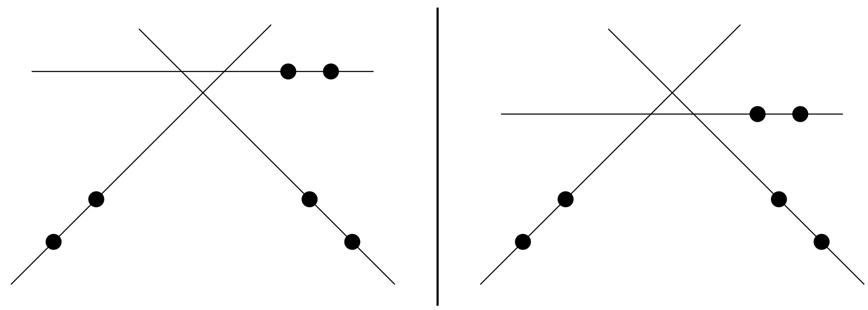


Figure 1.3: Two sets of points with the same order type, but different line arrangements.

The order types in Theorem 1.3 are *labeled* ordered types. That is, two sets $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ have the same labeled order type if p_i, p_j, p_k and q_i, q_j, q_k have the same orientation for every triple $1 \leq i < j < k \leq n$. Thus, a relabeling of a point set might produce a different order type.

Goodman and Pollack also described the following procedure, which gives a lower bound on the number of order types that agrees with the upper bound in the highest order exponent. Let S be a set of n points in general position. This set can be extended to a set of $n+1$ points in general position by adding a point in any cell of the line arrangement determined by P. This arrangement has

$$\binom{\binom{n}{d}}{d} + \binom{\binom{n}{d}}{d-1} + \ldots + \binom{\binom{n}{d}}{0} - n\binom{\binom{n-1}{d-1} - 1}{d}$$

cells [85]. Thus, we get a lower bound of roughly

$$\prod_{i=1}^{n} \frac{i^{d^2}}{(d!)^{(d+1)}} = \frac{(n!)^{d^2}}{(d!)^{(d+1)n}}$$

different order types. Using Stirling's formula, this can be written as

$$\exp(d^2(1 + O(1/\log(n/d)))n \log n).$$

This does not give the exact count of the number of order types because the cells of point sets with the same order type need not be equivalent (see Figure 1.3).

Since we know from Theorem 1.2 that there exist point sets that require coordinates of doubly exponential size to be represented and we also have an upper bound on the number of order types from Theorem 1.3, an interesting problem is determining how many order types can be realized in integer grids of "small" size. In this chapter, we give the first non trivial bounds for the following problem, found in [23]:

Problem 1. *For a given constant $\alpha > 0$, what is the number of order types of n points that can be represented by integer coordinates smaller than n^α?*

For the remaining of this chapter, every point set is labeled and we denote by $g(n, \alpha)$ the number of different labeled order types realizable in an integer grid of size n^α. Our results are the following:

Theorem 1.4. *If $\alpha > 2$ then*

$$g(n, \alpha) \geq \exp\left(2n \log n - O(n \log \log n)\right).$$

Theorem 1.5. *If $\alpha \geq 2.5$ then*

$$g(n, \alpha) \geq \exp(3n \log n - O(n \log \log n)).$$

These results are joint work with Luis E. Caraballo, José-Miguel Díaz-Báñez, Ruy Fabila-Monroy, Jesús Leaños and Amanda Montejano. They have been submitted to publication, the preprint is available at arXiv [27].

1.1 Order Types in Small Integer Grids

One related question to Problem 1 is asking for the smallest integer grid such that at least one order type can be drawn on it. This is the same as asking for the minimum size of an integer grid such that we can place n points in general position. This is known as the *no-three-in-a-line problem* and was introduced by Dudeny [36] in 1917. A construction by Erdős in [75] shows that if p is a prime, then the set

$$Q_p := \{(i, i^2 \mod p) : 0 \leq i < p\}$$

is in general position. See Figure 1.4 for a drawing of Q_{13}. This implies that for any $\epsilon > 0$ and any sufficiently large n, one can place $(1 - \epsilon) \cdot n$ points on a $n \times n$ grid. Hal et. al [52] showed, using an adaptation of Erdős' construction, that for every n there is a subset of the $n \times n$ grid of n points in general position. Therefore, at least one order type can be realized in integer grids of linear size.

Suppose that α is such that at least one order type of n points can be realized in an $n^\alpha \times n^\alpha$ integer grid. For any permutation of the labels of a point set that preserves the order type, the following must hold:

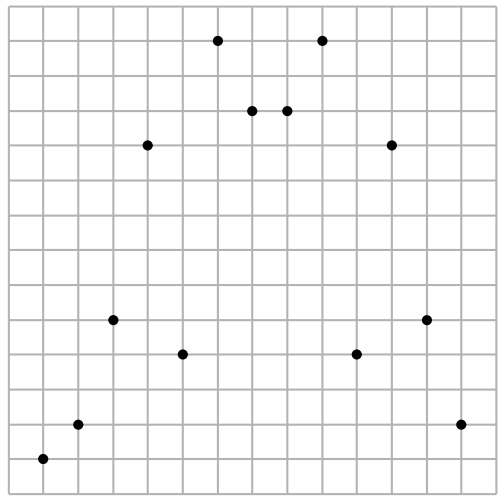


Figure 1.4: Q_{13}

(1) The clockwise cyclic order of the points in the convex hull of S is preserved.

(2) For every point $p \in S$ the clockwise cyclic order by angle of the points of $S \setminus \{p\}$ around p is preserved.

Let H be the size of the convex hull of S. For a given labeling of S, (1) and (2) imply that at most H permutations of the labels of S preserve the order type. Since H is at most n, at least $(n-1)!$ other different order types are realizable in this grid.

By Stirling's approximation, this is at least

$$\exp(n \log n - n + O(\log n)).$$

Thus, we consider a meaningful lower bound for $g(n, \alpha)$ to be of the form

$$\exp(c \cdot n \log n)$$

for some $c > 1$.

We have the following upper bounds. Note that there are at most $n^{2n} = \exp(2n \log n)$ different sets of n points in an $n \times n$ integer grid. Thus

$$g(n, 1) \leq \exp(2n \log n).$$

From Theorem 1.2 there exist point sets of $\log(\alpha \log n)/c_1$ points whose order type cannot be realized with integer coordinates smaller than n^α, let P

be one such set. Consider $P \cup Q$, where Q is any point set of $n - \log(\alpha \log n)/c_1$ points such that $P \cup Q$ is in general position; note that $P \cup Q$ cannot be realized with integer coordinates smaller than n^α. Therefore, for every $\alpha > 0$, there are at least

$$f\left(2, n - \frac{\log(\alpha \log n)}{c_1}\right)$$

realizable order types of n points which are not realizable in integer grids of size n^α.

1.2 Lower Bound Constructions

In this section we show lower bounds for Problem 1. We present two constructions that produce many point sets with different order types in integer grids of size n^α for $\alpha > 2$ and $\alpha \geq 2.5$, respectively. Our approach is similar to the one used to lower bound $f(n)$, that is, iteratively place points and lower bound the number of different available choices that produce different order types. In our case, we have to take care when we place a new point: when we consider a cell in the line arrangement spanned by the straight lines passing through pairs of already placed points, it might not contain a grid point.

To work around this problem, we do the following. We place a portion of our points in a special configuration $\mathcal{C}$; and choose a set of straight lines passing through pairs of points in $\mathcal{C}$. Then, we define a set T of isothethic squares of side length equal to ℓ such that any two squares are separated by one of our chosen lines. Afterwards, we place the remaining points. This is done as follows. At each step we first choose a square from T that

(1) has not been chosen before; and

(2) contains a point p of integer coordinates that does not produce a triple of collinear points with the previously placed points.

We then choose p as our next point.

Our strategy is to lower bound, at each step, the number of squares in T that satisfy (1) and (2). We say that these squares are *alive*; otherwise, we say that they are *dead*. Suppose that a square of T has not been chosen yet. If less than ℓ lines passing through a pair of previously placed points intersect this square, then it is still alive. In what follows, we use this observation extensively.

Theorem 1.4. *If* $\alpha > 2$ *then*

$$g(n,\alpha) \geq \exp\left(2n\log n - O(n\log\log n)\right).$$

Proof. Let n be an arbitrarily large positive integer and let p be the smallest prime greater or equal than $n/(4\log n)$. In this case the configuration $\mathcal{C}$ consists of four sets $\mathcal{U}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$; each set is an affine copy Q'_p of Q_p. $\mathcal{L}$ and $\mathcal{R}$ are rotated by 90° and stretched vertically. $\mathcal{U}$ and $\mathcal{D}$ are stretched horizontally. $\mathcal{L}$ and $\mathcal{R}$ are placed at the same height, with $\mathcal{L}$ to the left of $\mathcal{R}$; $\mathcal{U}$ and $\mathcal{D}$ are placed at the same x-coordinate and between $\mathcal{L}$ and $\mathcal{R}$; $\mathcal{U}$ is above $\mathcal{L}\cup\mathcal{R}\cup\mathcal{D}$ and $\mathcal{D}$ is below $\mathcal{L}\cup\mathcal{R}\cup\mathcal{U}$. Let $k := \lceil\log n\rceil$. The precise definitions are:

$$\begin{aligned}
Q'_p &:= \{(34pk^2\cdot i, 34k^2\cdot(i^2 \mod p)) : 0 \leq i < p\},\\
\mathcal{U} &:= \{(x,y) : (x,y)\in Q'_p\},\\
\mathcal{L} &:= \{(y-136p^2k^2, 2x-238p^2k^2) : (x,y)\in Q'_p\},\\
\mathcal{R} &:= \{(y+153p^2k^2, 2x-238p^2k^2) : (x,y)\in Q'_p\},\\
\mathcal{D} &:= \{(x, y-408p^2k^2) : (x,y)\in Q'_p\} \text{ and}\\
\mathcal{C} &:= \mathcal{U}\cup\mathcal{L}\cup\mathcal{R}\cup\mathcal{D}.
\end{aligned}$$

Every point in $\mathcal{U}$ is joined with a straight line with the point in $\mathcal{D}$ with the same x-coordinate; every point in $\mathcal{L}$ is joined with a straight line with the point in $\mathcal{R}$ with the same y-coordinate. These are our chosen set of lines. See Figure 1.5.

It is not difficult to show that $\mathcal{C}$ is in general position. The set of chosen straight lines form a rectangular grid. In the interior of each of these rectangles place an isothethic square with $32pk^2 \times 32pk^2$ integer grid points. Let T be the set of these squares. Baker, Harman and Pintz [17] showed that, for x sufficiently large, the interval $[x, x+x^{21/40}]$ contains a prime number. Thus, $p = n/(4\log n)+O(n^{21/40})$. Therefore $|\mathcal{C}| = n/\log n + O(n^{21/40})$, $|T| = (p-1)^2$ and $\ell = 32pk^2$ for this construction.

We now iteratively place the remaining $n-4p$ points. At each stage the number of lines passing through a pair of the so far placed points is less than $n^2/2$; each of these lines intersects less than $2p$ squares of T; each square must touched by at least $32pk^2$ straight lines before being dead. Therefore, the number of alive squares at every stage is at least

$$(p-1)^2 - \frac{n^2p}{32pk^2} = p^2 - O(p) - \frac{1}{2}(p - O(n^{21/40}))^2$$

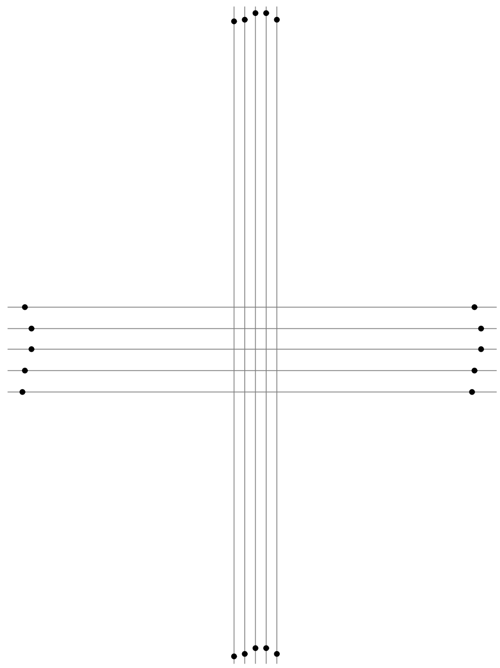


Figure 1.5: The set $\mathcal{C}$ for $p = 5$

$$
\begin{aligned}
&\geq \frac{1}{2}p^2 - O(p) - \frac{1}{2}O(n^{42/40}) \\
&\geq \frac{n^2}{36\log^2 n} + \frac{n \cdot O(n^{21/40})}{4\log n} - O(p) \\
&\geq \frac{n^2}{36\log^2 n}
\end{aligned}
$$

where the last inequality holds for sufficiently large n.

Therefore, we obtain at least

$$\prod_{i=1}^{n-4p} \frac{n^2}{32\log^2 n} = \frac{n^{2(n-4p)}}{(32\log^2 n)^{n-4p}} = \exp\left(2n\log n - O(n\log\log n)\right)$$

different order types with this procedure. Since $\mathcal{C}$ is contained in an integer grid of side length equal to $\Theta(p^2k^2) = \Theta(n^2)$, this proves the theorem. $\square$

The next theorem shows that increasing the value of α slightly lets us place considerably more order types.

Theorem 1.5. *If* $\alpha \geq 2.5$ *then*

$$g(n, \alpha) \geq \exp(3n\log n - O(n\log\log n)).$$

Proof. Let n be an arbitrarily large positive integer; let m be the smallest multiple of 16 larger than $n/\log n$ and let $L := \lceil 64n^2/m^{1/2} \rceil$. Let $\mathcal{C} := \{v_0, \ldots, v_{m-1}\}$ be the vertices, in clockwise order, of a regular polygon P of side length equal to L. These points may not have integer coordinates; their coordinates will be rounded up to the nearest integer later on. Let q^* be the center of this polygon. For $1 \le i \le m$, let $\triangle_i$ be the triangle with vertices v_{i-1}, v_i, and q^*. In what follows we define a set T_i of squares inside $\triangle_i$.

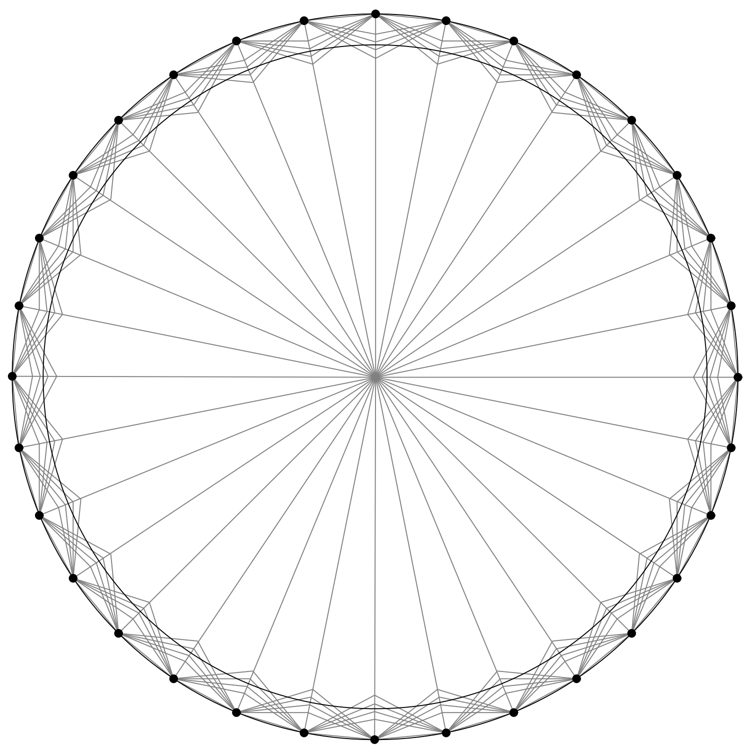


Figure 1.6: The regular polygon construction with $m = 32$.

Starting at the line segment joining v_{i-1} and v_i, let $e_1, \ldots, e_{m-1}$ be the line segments joining v_{i-1} and every other vertex of P, sorted clockwise by angle around v_{i-1}. Starting at the line segment joining v_i and v_{i-1}, let $f_1, \ldots, f_{m-1}$ be the line segments joining v_i and every other vertex of P, sorted counter-clockwise by angle around v_i. Let C_1 be the the circumcircle of P. Since every pair of consecutive vertices of P define a chord of C_1 and these chords have the same length, the angle between any two consecutive e_j and e_{j+1} is the same. Let γ be this angle. Moreover, the angle between any two consecutive f_j and f_{j+1} is also equal to γ; note that

$$\gamma = \frac{\pi}{m}.$$

For indices $2 \leq j \leq m/2$ and $2 \leq k \leq m/2$, let $p_{j,k}$ be the intersection of e_j and f_k; note that $p_{j,k}$ is contained in Δ_i. Let

$$Q := \left\{ p_{j,k} : j, k \text{ even and } \frac{m}{8} \leq j, k < \frac{m}{4} \right\}.$$

Note that

$$|Q| = \frac{m^2}{256}.$$

See Figure 1.6. For each $p_{j,k}$ in Q, place an isothethic square of side length equal to

$$\ell := \frac{L}{m}$$

centered at $p_{j,k}$. Let T_i be the set of these squares. The next lemma shows that the squares in T_i are well separated by the e_j's and f_k's

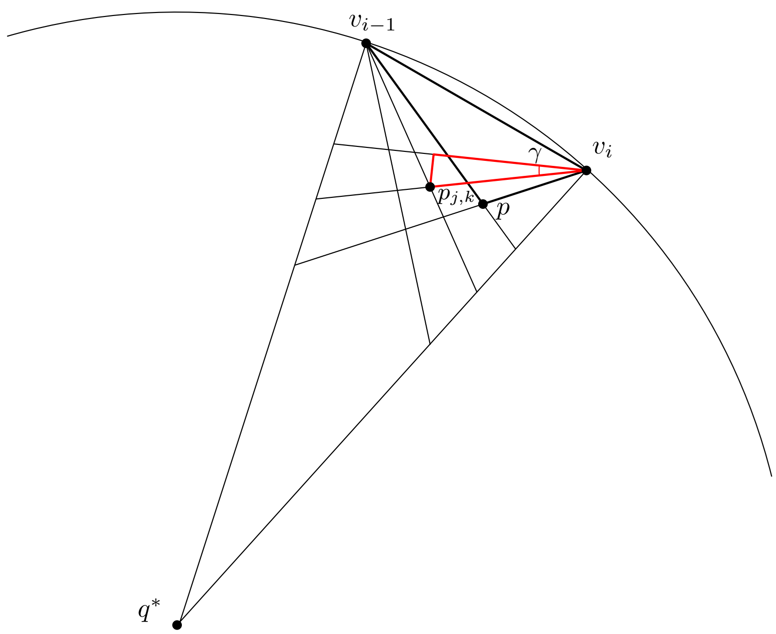


Figure 1.7: The proof of Lemma 1.1

Lemma 1.1. *Let $p_{j,k}$ be a point in Q. Then the distances from $p_{j,k}$ to e_{j-1}, e_{j+1}, f_{k-1} and f_{k+1} are greater than*

$$0.99 \left(\sqrt{2} - 1 \right) \pi \ell$$

Proof. We show that the distances from $p_{j,k}$ to f_{k-1} and f_{k+1} are at least the required value. The proof for e_{j-1} and e_{j+1} is similar.

Let $d = \max\{\mathrm{dist}(p_{j,k}, e_{j-1}), \mathrm{dist}(p_{j,k}, e_{j+1})\}$. Then $d = x\sin(\gamma)$, where x is the distance from $p_{j,k}$ to v_i (see Figure 1.7). Note that among the $p_{j,k}$'s in Q,

$$p := p_{m/8,m/4-1}$$

is the point closest to v_i. Consider the triangle with vertices v_i, v_{i-1} and p (see Figure 1.7). By the law of sines the distance from p to v_i is equal to

$$\begin{aligned}\frac{\sin((m/8)\gamma)}{\sin(\pi-(m/8)\gamma-(m/4-1)\gamma)}L &= \frac{\sin(\pi/8)}{\sin(5\pi/8+\pi/m)}L \\ &> \frac{\sin(\pi/8)}{\sin(5\pi/8)}L = \left(\sqrt{2}-1\right)L.\end{aligned}$$

Therefore, $d > \sin(\gamma)(\sqrt{2}-1)L$. The result follows from the facts that $\sin(x) > 0.99x$ for $x \in [0, \pi/16]$ and $0 < \gamma \leq \pi/16$. □

We are ready to define the set of squares, let

$$T := \bigcup_{i=1}^{m} T_i.$$

Let C_2 be the circle with center q^* and passing through $p_{m/4,m/4}$. Note that T is contained in the annulus A bounded by C_1 and C_2. The following lemma upper bounds the number of squares in T that a given straight line can intersect.

Lemma 1.2. *Every straight line intersects at most*

$$\frac{m^{3/2}}{4}$$

squares of T.

Proof. Let φ be a straight line. Note that φ intersects A in at most two straight line segments. We upper bound the number of squares in T that a straight line segment s can intersect. Each time s intersects a square in T_i, it must intersect an edge e_j or f_k; moreover, only half of these edges define a square in T_i. Therefore, s intersects at most $\frac{m}{8}$ squares in T_i. We upper bound the number of the triangles $\triangle_i$'s that s can intersect. For this we upper bound the length of s.

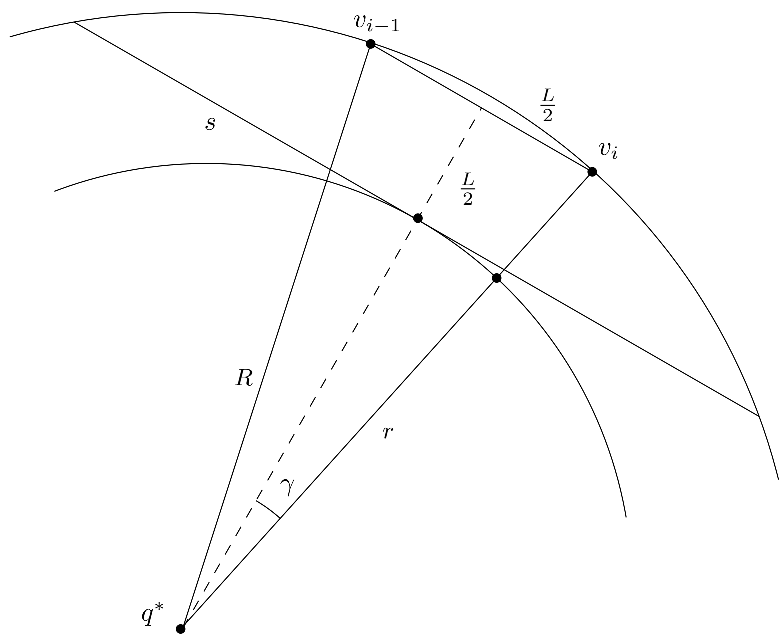


Figure 1.8: The proof of Lemma 1.2

Let R and r be the radii of the circles C_1 and C_2, respectively. Note that s has maximum length when it is tangent to C_2 and its endpoints are in C_1. Therefore,

$$||s|| \leq 2\sqrt{R^2 - r^2}.$$

Let q be the midpoint of the straight line segment from v_{i-1} to v_i. Note that the triangle formed by $p_{m/4,m/4}, q, v_i$ is isosceles, thus the distance from $p_{m/4,m/4}$ to q is equal to $L/2$. This implies that the distance from C_2 to the edge v_i, v_{i-1} is equal to $L/2$.

Since P is a regular polygon $R = \frac{1}{2}L\csc(\gamma)$ and its apotheme is equal to $R\cos(\gamma)$ (see Figure 1.8). This implies that $r = \frac{L}{2}\cot(\gamma) - 1)$. Therefore,

$$||s|| \leq 2\sqrt{R^2 - r^2} = L\sqrt{2\cot(\gamma)} < L\sqrt{\frac{2m}{\pi}};$$

the last inequality follows from the facts that $\tan(x) > x$ for $x \in (0, \pi/16]$ and $0 < \gamma \leq \pi/16$.

Now we lower bound the length of $s \cap \triangle_i$. Note that $s \cap \triangle_i$ has minimum length when s is tangent to C_2 and parallel to the edge v_{i-1}, v_i. Thus,

$$||s \cap \triangle_i|| \geq 2\tan(\gamma)r = (1 - \tan(\gamma))\,L > \sqrt{\frac{2}{\pi}}L.$$

Therefore, s intersects a most

$$\frac{||s||}{\min ||s \cap \triangle_i||} \leq \sqrt{m}$$

of the triangles $\triangle_i$. The result follows. □

To end the construction we round the coordinates of the v_i's to their nearest integer. Redefine the e_j's and f_k's accordingly. By Lemma 1.1, a square in T_i centered at $p_{j,k}$ is separated from edges $e_{j'}$ and $f_{k'}$ different from e_j and f_k by a distance of at least $0.99\left(\sqrt{2}-1\right)\pi\ell$. The endpoints of the new e_j's and f_k's are at a distance of at most one of their original positions. Since $0.99\left(\sqrt{2}-1\right)\pi > 1$, the squares in T_i are still separated by the straight lines containing the e_j's and f_k's.

We now iteratively place the remaining $n-m$ points. At every stage the number of lines passing through every pair of the so far placed points is less than $n^2/2$; each of these lines intersects at most $m^{3/2}/4$ squares of T; each square must be touched by at least $\ell = L/m$ straight lines before being dead. Thus, the number of squares alive at every stage is at least

$$\frac{m^3}{256} - \frac{n^2 m^{5/2}}{8L} \geq \frac{m^3}{512} \geq \frac{n^3}{512 \log^3 n}.$$

Therefore, we obtain at least

$$\prod_{i=1}^{n-m} \frac{n^3}{512 \log^3 n} = \frac{n^{3(n-m)}}{(512 \log n)^{3(n-m)}} = \exp(3n \log n - O(n \log\log n))$$

different order types with this procedure. Recall that $m \leq n/\log n + 16$ and $L \leq 64n^2/m^{1/2} + 1$. Therefore, these point sets lie in an integer grid of side length equal to $L \cdot m = \Theta\left(n^{2.5}/\sqrt{\log n}\right)$. This proves the theorem. □

1.3 Conclusions and Open Problems

In this chapter we have shown the first non-trivial lower bounds for Problem 1. The technique used is, essentially, placing a fraction m of n points in a special configuration and then counting the ways in which the remaining $n-m$ points can be added iteratively to the cells of the line arrangements formed by the so far placed points while having integer coordinates. Thus, it is desirable to have many cells with many integer points.

Some open problems related to the work in this chapter are:

(1) Which configuration of n points in the plane guarantees many cells with many integer points? In particular, let $\text{int}(C)$ be the number of integer points contained in a cell C of a line arrangement. Which configuration of points P minimizes $\text{int}(C_i)/\text{int}(C_j)$ where C_i and C_j are cells in the line arrangement induced by P?

(2) We showed that increasing α from 2 to 2.5 allows us to get from $\exp(2n\log n - O(n\log\log n))$ to at least $\exp(3n\log n - O(n\log\log n))$ different order types. How much can we gain if we increase α to guarantee at least $\exp(4n\log n - O(n\log\log n))$ different order types?

Chapter 2

Crossing Numbers

In this chapter we deal with problems related to *crossing numbers* of graphs. We follow the exposition of Schaefer [76] for an introduction to the basic concepts of crossing numbers.

Let $G = (V, E)$ be a graph on n vertices and m edges. In a *drawing* D of G in the plane, the vertices of G are drawn as different points, and every edge is drawn as a simple curve connecting the two points representing its vertices. We require that in D the edges do not pass through a point representing a vertex other than their endpoints. If two edges intersect finitely often, those intersections are either an endpoint, a *crossing* or a *touching point*. The *crossing number* of D, $\mathrm{cr}(D)$, is the number of crossings in D. The crossing number of G, $\mathrm{cr}(G)$, is the minimum of $\mathrm{cr}(D)$ taken over all the drawings of G.

Straight-line Drawings

Depending on how drawings are defined and on the precise way crossings are counted, there exist a huge number of variants of crossing number problems; see [77] for a comprehensive survey. A widely studied variant is restricting the curves that represent edges to straight-line segments. A drawing where the edges are represented as straight line segments is called a *straight-line drawing* (also called *rectilinear* or *geometric*). Let D be a straight-line drawing of G. As with general drawings, we define the *rectilinear crossing number* of D, $\overline{\mathrm{cr}}(D)$ as the number of crossings in D. The rectilinear crossing number of G, $\overline{\mathrm{cr}}(G)$, is the minimum of $\overline{\mathrm{cr}}(D)$ taken over all the straight-line drawings of G. Thus, if $\overline{\mathrm{cr}}(G) = 0$ then G is planar, and the converse is also true. Given a set P of

n points in general position, we will abuse notation and write $\overline{\mathrm{cr}}(P)$ to denote the rectilinear crossing number of the straight-line drawing of K_n induced by P.

From an algorithmic point of view, the decision variant of the crossing number problem was shown to be NP-complete for general graphs already in the 1980s by Garey and Johnson [43]. The version for straight-line drawings is also known to be NP-hard, and actually, computing the rectilinear crossing number is $\exists\mathbb{R}$-complete [53]. So whenever considering crossing numbers, it is rather likely that one faces computationally difficult problems.

A case of particular interest is the asymptotic behavior of $\overline{\mathrm{cr}}(K_n)$. Let D be a straight-line drawing of K_n such that $\overline{\mathrm{cr}}(D) = \overline{\mathrm{cr}}(K_n)$ and let P be the points in D that represent the vertices of K_n. Note that four points in P are in convex position if and only if they define a pair of edges that cross. On one hand, we have that

$$\sum_{\substack{B \subset P \\ |B|=n-1}} \overline{\mathrm{cr}}(B) \geq n \cdot \overline{\mathrm{cr}}(K_{n-1})$$

and on the other, we have that

$$\begin{aligned} \sum_{\substack{B \subset P \\ |B|=n-1}} \overline{\mathrm{cr}}(B) &= (n-4) \cdot \overline{\mathrm{cr}}(P) \\ &= (n-4) \cdot \overline{\mathrm{cr}}(K_n). \end{aligned}$$

These equalities imply that

$$\frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}} \geq \frac{\overline{\mathrm{cr}}(K_{n-1})}{\binom{n-1}{4}}.$$

Therefore, if we define $a_n = \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$, (a_n) is a monotonically increasing sequence. Note that this sequence is bounded from above by one, thus

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}$$

exists. This limit is called the *rectilinear crossing constant* and is denoted by q^*.

One way to derive upper bounds on q^* is to produce rectilinear drawings of K_n with few crossings for arbitrarily large values of n. The first general construction for such sets was given by Jensen [59] who gave explicit coordinates for the points of K_n. Around the same time Singer [80] proposed another approach: his construction takes a drawing of K_n and produces a drawing of K_{3n}. If the former drawing has few crossings, then so does the latter. Using this drawing of K_{3n} and repeating the process gives a good drawing of K_{9n}, and so on. This approach of iteratively generating larger sets has been successful in improving the upper bound on q^* several times, see [3, 4, 9, 24]. The current best iterative construction is that of Ábrego, Cetina, Fernández-Merchant, Leaños and Salazar ([3] and [4]).

The Construction of [3] and [4]

Let $S := \{p_1, \ldots, p_n\}$ be a set of n points in general position in the plane. A *halving line* of S is a straight line ℓ passing through at least one point of S such that in the two open half-planes defined by ℓ there are the same number of points of S. Note that if n is odd then ℓ passes only through a single point of S, and if n is even then ℓ passes through two points of S.

Let G be the bipartite graph with vertex partition (A, B), where $A := S$ and B is the set of halving lines of S. A pair (p, ℓ) in (A, B) is adjacent in G if and only if ℓ passes through p. A *halving matching* of S is a matching of G, in which every point in A is matched to a halving line in B. If n is odd then a halving matching of S always exists, but it may not exist if n is even.

A rough description of the construction of [3] and [4] is as follows. Let $M = \{(p_1, \ell_1), \ldots, (p, \ell_n)\}$ be a halving matching for S. For every ℓ_i assume that ℓ_i is directed, and let $\vec{v}_i$ be the direction vector of ℓ_i. Let S' be the point set that results by replacing each p_i in S with the pair of points $p_i + \varepsilon\vec{v}_i$ and $p_i - \varepsilon\vec{v}_i$ for a arbitrarily small (but positive) value of ε.

This construction was first described in [4]; furthermore, they showed that if S has an even number of points then S' also has a halving matching. This allows for the iterative construction mentioned above. If S has an odd number of points then it always has a halving matching; however, it is not obvious that S' should also have a halving matching. In [3] they showed that in this case S' also has a halving matching. The following theorem is derived by showing that in every doubling step from S to S' it holds that $\overline{\text{cr}}(S') = 16\overline{\text{cr}}(S) + (n/2)(2n^2 - 7n + 5)$.

Theorem 2.1 ([3, 4]). *Let S be a set of n points in general position in the plane, such that S has a halving matching. Then*

$$q^* \leq \frac{24\overline{\text{cr}}(S) + 3n^3 - 7n^2 + (30/7)n}{n^4}.$$

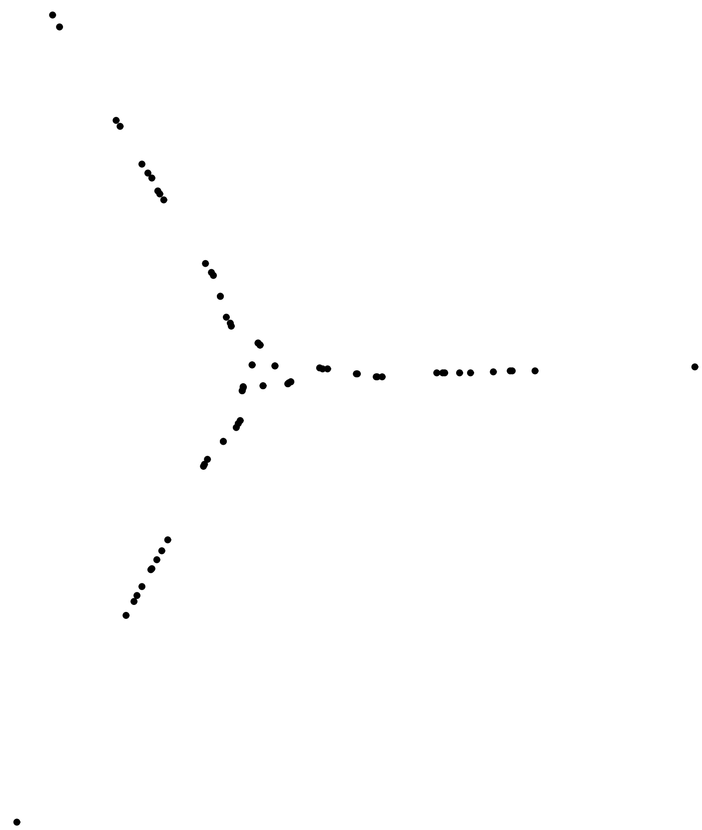


Figure 2.1: A drawing of K_{75} with 450492 crossings

The best published upper bound on q^*, due to Fabila-Monroy and López, comes from using the above theorem with a set of 75 points and 450492 crossings (see Figure 2.1).

Theorem 2.2 ([39]).

$$q^* \leq \frac{9363184}{24609375} < 0.380473$$

Pseudolinear Drawings

Another variation of drawings consists of representing edges with *pseudolines*. A pseudoline is a curve which is isomorphic to a straight line in the plane. A *pseudoline arrangement* is an arrangement of pseudolines such that every pair crosses exactly once. A *pseudolinear drawing* of a graph is a drawing such that its edges can be extended to form a pseudolinear arrangement. Note that pseudolinear drawings are a generalization of rectilinear drawings.

Let D be a pseudolinear drawing of a graph G. Similar as before, the *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(D)$ of D is the number of crossings in D. The *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(G)$ of G is the minimum of $\widetilde{\mathrm{cr}}(D)$ taken over every pseudolinear drawing of G. As with the rectilinear variation, we have that the limit

$$\lim_{n\to\infty} \frac{\widetilde{\mathrm{cr}}(K_n)}{\binom{n}{4}}$$

exists. This constant is denoted by $\widetilde{q}^*$ and is called the *rectilinear crossing constant*.

When bounding q^* and $\widetilde{q}^*$ they are often considered together, see e.g. [5], since $\mathrm{cr}(G) \le \widetilde{\mathrm{cr}}(G) \le \overline{\mathrm{cr}}(G)$ holds for every graph G. It is an open problem whether $\overline{\mathrm{cr}}(K_n) = \widetilde{\mathrm{cr}}(K_n)$ might be true for any n.

Up to and including the work of Fabila-Monroy and López [39] upper bounds for $\widetilde{q}^*$ were simply derived from the upper bounds of q^*, that is, $\widetilde{q}^* \le q^* < 0.380473$. Only recently Balko and Kynčl [18] obtained the first upper bound on $\widetilde{q}^*$ which is below the upper bound of q^*. The method they used is a generalization of [39] to pseudolinear drawings. They give a nice presentation of pseudolinear drawings with n-signatures and show that the construction of [3, 4] can be adopted to the pseudolinear setting. Thus, a similar relation to that of Theorem 2.1 holds when doubling a pseudolinear drawing D to D', namely $\widetilde{\mathrm{cr}}(D') = 16\widetilde{\mathrm{cr}}(D) + 2n\left(\lceil n/2\rceil^2 + \lfloor n/2\rfloor^2\right) - 7n^2/2 + 5n/2$. For even n this is the same bound as for the rectilinear case. Although not explicitly stated in [18] this therefore leads to the following bound for $\widetilde{q}^*$.

Corollary 2.1. *Let D be a pseudolinear drawing of K_n, such that D has a halving matching. Then*

$$\widetilde{q}^* \le \frac{24\widetilde{\mathrm{cr}}(D) + 3n^3 - 7n^2 + (30/7)n}{n^4} \text{ for } n \text{ even; and}$$

$$\widetilde{q}^* \le \frac{24\widetilde{\mathrm{cr}}(D) + 3n^3 - 7n^2 + (81/14)n}{n^4} \text{ for } n \text{ odd.}$$

The best published upper bound on $\widetilde{q}^*$, due to Balko and Kynčl, comes from using the above result with a pseudolinear drawing of K_{216} with 33260204 crossings:

Theorem 2.3 ([18]).

$$\widetilde{q}^* \le \frac{120772213}{317447424} < 0.380448$$

In Section 2.1 we improve the upper bounds of q^* and $\widetilde{q}^*$ using several heuristics and discuss their implementations. The bounds we show are:

Theorem 2.4.

$$q^* \leq \frac{43317373349528}{113858494707069} < 0.3804492011$$

Theorem 2.5.

$$\widetilde{q}^* \leq \frac{1437136544749}{3777575768064} < 0.3804388405.$$

These results are joint work with Oswin Aichholzer, Frank Duque, Ruy Fabila-Monroy and Oscar E. García-Quintero. They have been submitted for publication, the preprint is available in arXiv [10].

In Section 2.2 we study a variation of crossing numbers: the two-colored crossing number $\overline{\text{cr}}_2$. We show that a variation of the duplication method of [4] can be adapted to give bounds on the crossing constant in this setting. We also study the ratio $\overline{\text{cr}}_2/\overline{\text{cr}}$. Our main results are:

Theorem 2.7. *The rectilinear 2-colored crossing constant satisfies*

$$\overline{\text{cr}}_2 \leq \frac{182873519}{1550036250} < 0.11798016.$$

Theorem 2.9. *There exists an integer $n_0 > 0$ and a constant $c > 0$ such that for any straight-line drawing D of K_n on $n \geq n_0$ vertices, $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) < \frac{1}{2} - c$.*

These results are joint work with Oswin Aichholzer, Ruy Fabila-Monroy, Adrian Fuchs, Carlos Hidalgo-Toscano, Irene Parada, Birgit Vogtenhuber and Francisco Zaragoza. They were presented at the 27th International Symposium on Graph Drawing and Network Visualization. It is planned that these results also appear in the thesis of the coauthor Irene Parada.

2.1 Improving Crossing Number Constants

A feature of the iterative constructions in [3] and [4] is that to improve the upper bound on q^* it is sufficient to find for a specific, constant value of n, a sufficiently good rectilinear drawing of K_n.

Our goal is to be able to improve the upper bounds on the crossing constants in a semi-automatic way. We thus used an implementation of the construction of [3, 4] by García-Quintero [42] as well as various heuristics to improve a given rectilinear drawing of K_n. We also implemented its extension to pseudolinear drawings [18] to improve a given pseudolinear drawing of K_n. Both implementations were made in Python. In this section we describe our approach in detail.

2.1.1 Heuristics

We now describe various heuristics that attempt to improve a given drawing of K_n. For a rectilinear drawing the vertices are the points of the point set S and p is a point of S. For a pseudolinear drawing D all triple orientations $v_iv_jv_k$, $1 \leq i < j < k \leq n$ are given.

Moving a Point to a New Random Location

In [39] the following simple heuristic is used for rectilinear drawings. Choose a random point q in a certain neighborhood of p. If $\overline{\mathrm{cr}}(S \setminus \{p\} \cup \{q\}) \leq \overline{\mathrm{cr}}(S)$, replace p with q in S. In [39], $\overline{\mathrm{cr}}(S \setminus \{p\} \cup \{q\})$ is computed in $O(n^2)$ time. If after some time no improvement is found then the size of the neighborhood of p from which q is chosen, is made smaller. We use the amortized faster algorithm of [38]. In that algorithm a set Q of $\Theta(n)$ candidate points for the new position of p is chosen. The set of values

$$\{\overline{\mathrm{cr}}(S \setminus \{p\} \cup \{q\}) : q \in Q\}$$

is computed in $O(n^2)$ time. Note that this is linear per point in Q. We then consider the best point in Q as a possible replacement for p.

For pseudolinear drawings a very similar idea is used. In this setting there are no points to be moved. But observe that moving a point continuously in the rectilinear case corresponds to changing the orientation of point triples one after another (since there are no collinear points), that is, changing the order type of the set step by step. The difficulty in the rectilinear setting is that this is a geometric process, and even if all points have integer coordinates, moving the point on an integer grid might not be sufficient to get all triple changes separately.

To the contrary, changing triple orientations in a pseudolinear drawing given as n-signature is trivial. We choose a random vertex triple $v_iv_jv_k$, $1 \leq i < j < k \leq n$, of the drawing D and invert its orientation. Then we check if this new n-signature is still realizable as a pseudolinear drawing D'. In case D' exists, we keep the new orientation if $\widetilde{\mathrm{cr}}(D') \leq \widetilde{\mathrm{cr}}(D)$ and set $D =$

D'. Otherwise we revert the change. Iterating this process eventually gives pseudolinear drawings with less crossings.

It is interesting to observe that our experiments show that this local optimization heuristic works in average significantly better for pseudolinear drawings than for rectilinear drawings. The main reason might be that for pseudolinear drawings the algorithm has a combinatorial flavour, while for rectilinear drawings the precise geometry of the arrangement of the point set (e.g. the size of cells) plays a role. We therefore present in the next section an approach for rectilinear drawings which avoids this bottleneck.

The Point Set Explorer

Consider the line arrangement, $\mathcal{A}$, spanned by the set of lines passing through every pair of points of $S \setminus \{p\}$. Let C be a cell of $\mathcal{A}$ and let q be a point in C. Note that $\overline{\text{cr}}(S \setminus \{p\} \cup \{q\})$ has the same value regardless of the choice of the precise location of q within C.

All the best known rectilinear drawings of K_n consist of three "arms" with close to $n/3$ points each; the points in each arm are close to being collinear. See Figure 2.1 for an example. This implies that many of the cells in the line arrangements of such sets have area close to zero. Therefore, in the heuristic described in 2.1.1 many of the candidate points fall in the same cells and many cells are never visited.

In [54] the following algorithm is implemented. The algorithm computes the cell of $\mathcal{A}$ that contains p in $O(n^2 \log n)$ time. Afterwards, moving between adjacent cells, and finding a point in the interior of these cells, takes $O(\log^2 n)$ time.

Using this implementation we produce a sequence $C = C_1, \ldots, C_m$ of consecutive adjacent cells. At each C_i we choose a point q_i as a candidate to replace p in S. Since we are moving between adjacent cells, $\overline{\text{cr}}(S \setminus \{p\} \cup \{q_i\})$ can actually be computed in $O(1)$ time.

In this way the size of the cells in the arrangement and thus the precise geometry of our point set does not play a role anymore. We obtain an algorithm with a more combinatorial behaviour than with the previous heuristic.

Finding Good Subdrawings

It is often the case that for $m > n$, drawings of K_m with few crossings contain drawings of K_n with few crossings. We observed that when optimizing drawings of K_m with the heuristics presented in the previous section, they often contain better subdrawings of K_n than before the optimization. In other words, it is possible to improve drawings of K_n by optimizing drawings of K_m. It might sound counter intuitive to optimize the larger set. But observe that there are many subsets of size n in a set of size m. So if just one of these subdrawings is improved, this approach is successful.

Given a drawing of K_m, we use the following heuristics to find drawings of K_n with few crossings.

Removing One Point at a Time

For rectilinear drawings we remove one point of S at a time, to obtain smaller drawings. We use an implementation of an algorithm described in [38] that does the following. In $O(n^2)$ time, it computes the set of values

$$\{\overline{\text{cr}}(S \setminus \{p\} : p) \in S\}.$$

We remove the point p that minimizes $\overline{\text{cr}}(S \setminus \{p\})$, thus finding a rectilinear drawing of K_{m-1}. We proceed iteratively in this way to find rectilinear drawings of K_n with $n = m-1, m-2, \ldots, 27$ points.

For pseudolinear drawings this is done in a similar way.

Removing Two or More Points at a Time

Aichholzer, García, Orden and Ramos [11] showed that point sets minimizing the rectilinear crossing number have a triangular convex hull. Most of them have several layers consisting of three vertices each. Moreover, during the process of generating good examples to get an improved crossing constant we could observe that the cardinality of the best sets seems to be a multiple of 3. So actually it turned out that sometimes it is better to remove more than one point at each step. That means we look at a tuple or triple of points which, when being removed at once, reduces the crossing number the most. Occasionally, this provides rectilinear drawings that cannot be obtained by removing one point at a time as in the previous heuristic. However, the amortized speed up described above is lost, and thus this method needs more computational resources.

To our own surprise good pseudolinear drawings show precisely the same behaviour. There also the best drawings (w.r.t. the obtainable pseudolinear crossing constant) we found so far have a cardinality which is a multiple of 3. We therefore use the same approach of removing a tuple or triple of vertices at the same time also for pseudolinear drawings.

2.1.2 Joining Everything Together

Several computers run all the heuristics described in Section 2.1.1 permanently in the backgroud. Processes are started for the best drawings of K_n for some specific values of n - typically the currently best three or four cardinalities for both types of drawings, rectilinear and pseudolinear. The newly-found drawings are sent daily to a central node. If after for some time, no new improvements are made on the upper bound of either q^* or $\widetilde{q}^*$, we take the corresponding drawing of K_n providing the best upper bound and apply the doubling construction described in Section **The Construction of [3] and [4]** to obtain a drawing of K_{2n} providing the same upper bound on q^* or $\widetilde{q}^*$, respectively. From this set we then compute good subsets by removing points/vertices as described in Section **Finding good subdrawings**. In parallel we localy optimize the new drawing K_{2n} and all the obtained subdrawings by the methods described in Sections **Moving a Point to a New Random Location** and **The Point Set Explorer**. The steps of local optimization and computing subdrawings are interleaved and iterated. Note, however, that the local optimization is applied only for a limited time. The best sets obtained in these iterations serve as new starting sets of our optimization, and the whole process is restarted.

2.1.3 Results for Rectilinear Drawings

With the just described approach we were able to obtain good rectilinear drawings of K_n for up to $n = 3240$ points. The best crossing constant is obtained by using a rectilinear drawing of K_{2643} which has 771218743336 crossings.* The reason why larger sets do not necessarily provide a better constant lies in the before mentioned search for good subsets, which is an essential ingredient of our mixed heuristic. Plugging the values of this drawing into Theorem 2.1 we obtain the following result.

*The interested reader can get a file with the coordinates of the points from `http://www.crossingnumbers.org/projects/crossingconstants/rectilinear.php`.

Theorem 2.4.

$$q^* \leq \frac{43317373349528}{113858494707069} < 0.3804492011$$

2.1.4 Results for Pseudolinear Drawings

For pseudolinear drawings we have been able to get good results for K_n for up to $n = 2502$. The reason why the maximal cardinality is smaller than the one for rectilinear drawings is that here we have to store every triple orientation explicitely, which needs several hundreds MB for each set. The best crossing constant is obtained by using a pseudolinear drawing of K_{2256} which has 409176676250 crossings.† Plugging the values of this drawing into Corollary 2.1 we obtain the following result.

Theorem 2.5.

$$\widetilde{q}^* \leq \frac{1437136544749}{3777575768064} < 0.3804388405.$$

2.2 2-colored Crossing Number

In this section we focus on a version of crossing numbers that combines the *rectilinear crossing number* and the *k-planar crossing number*.

The *k-planar crossing number* $\mathrm{cr}_k(G)$ of a graph G is the minimum of $\mathrm{cr}(G_1) + \cdots + \mathrm{cr}(G_k)$ over all sets of k graphs $\{G_1, \ldots, G_k\}$ whose union is G. For $k = 2$, it was introduced by Owens [70] who called it the *biplanar crossing number*; see [32, 33] for a survey on biplanar crossing numbers. Shahrokhi et al. [78] introduced the generalization to $k \geq 2$.

A *k-edge-coloring* of a drawing D of a graph is an assignment of one of k possible colors to every edge of D. The *rectilinear k-colored crossing number* of a graph G, $\overline{\mathrm{cr}}_k(G)$, is the minimum number of monochromatic crossings (pairs of edges of the same color that cross) in any k-edge-colored straight-line drawing of G. This parameter has been introduced previously and called the *geometric k-planar crossing number* [71]. In the same paper, as well as in [78], also the *rectilinear k-planar crossing number* was considered, which asks for the minimum of $\overline{\mathrm{cr}}(G_1) + \ldots + \overline{\mathrm{cr}}(G_k)$ over all sets of k graphs $\{G_1, \ldots, G_k\}$ whose union is G. We prefer our terminology because the terms geometric

†The interested reader can download the file with the information of the triple orientation from `http://www.crossingnumbers.org/projects/crossingconstants/pseudolinear.php`.

and rectilinear are very often used interchangeably and because the term k-planar is extensively used in graph drawing with a different meaning; see for example [34, 55]. We remark that in graph drawing, *rectilinear* sometimes also refers to orthogonal grid drawings (which is not the case here).

In this section we focus on the case where G is the complete graph K_n, and we prove the following lower and upper bounds on $\overline{\text{cr}}_2(K_n)$:

$$0.03\binom{n}{4} + \Theta(n^3) < \overline{\text{cr}}_2(K_n) < 0.11798016\binom{n}{4} + \Theta(n^3).$$

Our approach is based on theoretical results that guarantee asymptotic bounds from the information of small point sets. Thus, it implies computationally dealing with small sets, both to guarantee a minimum amount of monochromatic crossings (for the lower bound) and to find examples with few monochromatic crossings and some other desired properties (for the upper bound).

On the one hand, we need to optimize the point configuration (order type) to obtain a small number of crossings, which is the original question about the rectilinear crossing number of K_n. On the other hand, we need to determine a coloring of the edges of K_n that minimizes the colored crossing number for a fixed point set.

For the first problem there is not even a conjecture of point configurations that minimize the rectilinear crossing number of K_n for any n. The latter problem corresponds to finding a maximum cut in a segment intersection graph, which in general is NP-complete [13]. Moreover, these two problems are not independent. There exist examples where a point set with a non-minimal number of uncolored crossings allows for a coloring of the edges so that the resulting colored crossing number is smaller than the best colored crossing number obtained from a set minimizing the uncolored crossing number. Thus, the two optimization processes need to interleave if we want to guarantee optimality. But even this combined optimization does not guarantee to yield the best asymptotic result. There are sets of fixed cardinality and with larger 2-colored crossing number which give a better asymptotic constant than the best minimizing sets. This is in contrast to the uncolored setting where for any fixed cardinality, sets with a smaller crossing number always give better asymptotic constants (see Theorem 2.1). Also, it clearly indicates that our extended duplication process for 2-colored crossings differs essentially from the original version.

As mentioned, drawings with few crossings do not necessarily admit a coloring with few monochromatic crossings. This observation motivates the following question: given a fixed straight-line drawing D of K_n, what is the ratio between the number of monochromatic crossings for the best 2-edge-coloring of D and the number of (uncolored) crossings in D? A simple probabilistic argument gives an upper bound: color each edge red with probability $1/2$ and blue with probability $1/2$, a monochromatic crossing appears with probability $1/2$. We improve that bound, showing that for sufficiently large n, it is less than $1/2 - c$ for some positive constant c.

The (rectilinear) 2-colored crossing number of a straight-line drawing D, $\overline{\mathrm{cr}}_2(D)$, is then the minimum of $\overline{\mathrm{cr}}(D_1)+\overline{\mathrm{cr}}(D_2)$, over all pairs of straight-line drawings $\{D_1, D_2\}$ whose union is D. For a given 2-edge-coloring χ of D, we denote with $\overline{\mathrm{cr}}_2(D, \chi)$ the number of monochromatic crossings in D. Thus, $\overline{\mathrm{cr}}_2(D)$ is the minimum of $\overline{\mathrm{cr}}_2(D, \chi)$ over all 2-edge-colorings χ of D.

2.2.1 Upper Bounds on $\overline{\mathrm{cr}}_2(K_n)$

Recall from Section 2.1 that for the rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$, the best upper bound [10] comes from finding examples of straight-line drawings of K_n (for a small value of n) with few crossings which are then used as a seed for the duplication process in [3, 4]. To be able to apply this duplication process, the starting set P with m points has to contain a halving matching. If m is even (odd), a *halving line* of P is a line that passes exactly through two (one) points of P and leaves the same number of points of P to each side. If it is possible to match each point p of P with a halving line of P through this point in such a way that no two points are matched with the same line, P is said to have a *halving matching*. It is then shown in [4] that every point of P can be substituted by a pair of points in its close neighborhood such that the resulting set Q with $2m$ points contains again a halving matching. Iterating this process leads to the mentioned upper bound for $\overline{\mathrm{cr}}(K_n)$, where this bound depends only on m and the number of crossings of the starting set P.

We prove that a significantly more involved but similar approach can be adopted for the 2-colored case. Unlike the original approach, we cannot always get a matching which simultaneously halves both color classes. Moreover, even for sets where such a halving matching exists, it cannot be guaranteed that this property is maintained after the duplication step. We will see below that we need a more involved approach, where the matchings are related to the distribution of the colored edges around a vertex. Consequently, the number of crossings which are obtained in the duplication, and thus, the asymptotic bound we get, not only depends on the 2-colored crossing number of the starting set, but also on the specific distribution of the colors of the edges. In that sense, both the heuristics for small drawings and the duplication process for the 2-colored crossing number differ significantly from the uncolored case.

Throughout this section, P is a set of m points in general position in the plane, where m is even. Let p be a point in P. By slight abuse of notation, in the following we do not distinguish between a point set and the straight-line drawing of K_n it induces. Given a 2-coloring χ of the edges induced by P, we denote by $L(p)$ and $S(p)$ the edges incident to $p \in P$ of the larger and smaller color class at p, respectively. An edge $e = (p, q)$ incident to p is called a *χ-halving edge of p* if the number of edges of $L(p)$ to the right of the line ℓ_e spanned by e (and directed from q to p) and the number of edges of $L(p)$ to the left of ℓ_e differ by at most one. A matching between the points of P and their χ-halving edges is called a *χ-halving matching for P*.

Theorem 2.6. *Let P be a set of m points in general position and let χ be a 2-coloring of the edges induced by P. If P has a χ-halving matching, then the 2-colored rectilinear crossing number of K_n can be bounded by*

$$\overline{\mathrm{cr}}_2(K_n) \leq \frac{24A}{m^4}\binom{n}{4} + \Theta(n^3)$$

where A is a rational number that depends on P, χ, and the χ-halving matching for P.

Proof. First we describe a process to obtain from P a set Q of $2m$ points, a 2-edge-coloring χ' of the edges that Q induces, and a χ'-halving matching for Q. The set Q is constructed as follows. Let p be a point in P and $e = (p, q)$ its χ-halving edge in the matching. We add to Q two points p_1, p_2 placed along the line spanned by e and in a small neighborhood of p such that:

(i) if f is an edge different from e that is incident to p, then p_1 and p_2 lie on different sides of the line spanned by f;

(ii) if f is an edge different from e that is not incident to p, then p_1 and p_2 lie on the same side of the line spanned by f as p; and

(iii) the point p_1 is further away from q than p_2.

The set Q has $2m$ points and the above conditions ensure that they are in general position.

Next, we define a coloring χ' and a χ'-halving matching for Q. For every edge (p, q) of P, we color the four edges $(p_i, q_j), i, j \in \{1, 2\}$ with the same color as (p, q). Hence, the only edges remaining to be colored are the edges (p_1, p_2) between the duplicates of a point $p \in P$. Let ℓ_e be the line spanned by e and directed from q to p. Further, let q_1 and q_2 be the points that originated

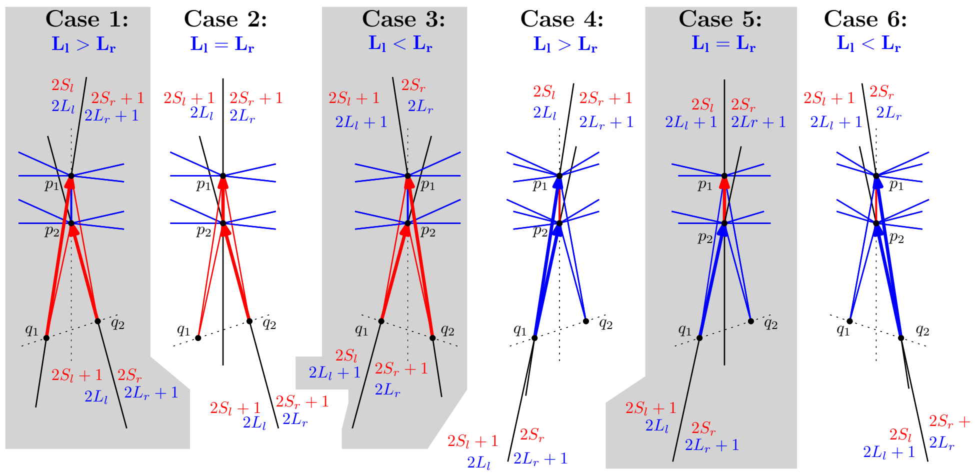


Figure 2.2: The cases in the duplication process of Theorem 2.6 when the larger color class at p is blue. The dotted lines represent the lines spanned by the χ-halving matching edges for P. The numbers of blue (red) edges at p to the left and right of l_e, is denoted with L_l and L_r (S_l and S_r), respectively.

from duplicating q, such that q_1 lies to the left of ℓ_e and q_2 lies to the right of ℓ_e. Denote by $L_l(p)$ and $L_r(p)$ the number of edges in $L(p)$ to the left and right of e, respectively. Analogously, denote by $S_l(p)$ and $S_r(p)$ the number edges in $S(p)$ to the left and right of e. For the following case distinction, we assume that the colors are red and blue and that the larger color class at p is blue.

There are six cases in which p can fall, depending on the color of the edge e and on the relation between the numbers $L_l(p)$ and $L_r(p)$ of blue edges incident to p on the left and the right side of ℓ_e; see Figure 2.2. The edge e of P has color red in the first three cases and color blue in the last three cases. The edge (p_1, p_2) receives color blue in Cases 1 and 3, and color red in the remaining cases. The thick edges in Figure 2.2 represent the matching edges for p_1 and p_2 in Q, where the arrow points to the point it is matched with. For each of p_1 and p_2, the resulting numbers of incident red and blue edges that are to the left and to the right of the line spanned by the matching edge are written next to those lines in the figure. They also show that the matching edges are indeed χ'-halving edges in each case.

Consider a point $p \in P$, its incident edges induced by P, their colors induced by χ, and the χ-halving edge $e = (p, q)$ that is matched with p in the χ-halving matching for P.

There are six cases in which p can fall, depending on the color of e and the numbers $L_l(p)$ and $L_r(p)$ of blue edges incident to p on the left and the right side of the line ℓ_e spanned by e; see again Figure 2.2, where the larger color class is blue. In each case, the color of the edge p_1p_2 and the χ'-halving matching edges for p_1 and p_2 need to be determined.

In the first three cases, e is in the smaller color class $S(p)$ at p while in the last three cases, e is in the larger color class $L(p)$ at p. Note that in all cases, $L_l(p)$ and $L_r(p)$ differ by at most one. Further, as $|P|$ is even, the degree of p is odd and hence $L(p)$ contains at least one more edge than $S(p)$. Thus, no matter how we color the edge (p_1, p_2) in Q, the larger color class at p_1 and p_2 in χ' for Q is the same as the one of p in χ.

Case 1: $e \in S(p)$ and $L_l(p) > L_r(p)$. The edge (p_1, p_2) is colored with the color of $L(p)$. In the matching for Q, p_1 is matched with (p_1, q_1) and p_2 is matched with (p_2, q_2). By this we obtain $L_l(p_i) = 2L_l(p)$ and $L_r(p_i) = 2L_r(p) + 1$ for $i \in \{1, 2\}$, implying that the matched edges are indeed χ'-halving. Further, we have $S_l(p_1) = 2S_l(p)$, $S_r(p_1) = 2S_r(p)+1$, $S_l(p_2) = 2S_l(p) + 1$, and $S_r(p_2) = 2S_r(p)$.

Case 2: $e \in S(p)$ and $L_l(p) = L_r(p)$. The edge (p_1, p_2) is colored with the color of $S(p)$. In the matching for Q, p_1 is matched with (p_1, p_2) and p_2 is matched with (p_2, q_2). By this we obtain $L_l(p_i) = 2L_l(p)$ and $L_r(p_i) = 2L_r(p)$ for $i \in \{1, 2\}$, implying that the matched edges are χ'-halving. Further, $S_l(p_i) = 2S_l(p) + 1$ and $S_r(p_i) = 2S_r(p) + 1$ for $i \in \{1, 2\}$.

Case 3: $e \in S(p)$ and $L_l(p) < L_r(p)$. The edge (p_1, p_2) is colored with the color of $L(p)$. In the matching for Q, p_1 is matched with (p_1, q_2) and p_2 is matched with (p_2, q_1). By this we obtain $L_l(p_i) = 2L_l(p) + 1$ and $L_r(p_i) = 2L_r(p)$ for $i \in \{1, 2\}$, implying that the matched edges are χ'-halving. Further, $S_l(p_1) = 2S_l(p) + 1$, $S_r(p_1) = 2S_r(p)$, $S_l(p_2) = 2S_l(p)$, and $S_r(p_2) = 2S_r(p) + 1$.

Case 4: $e \in L(p)$ and $L_l(p) > L_r(p)$. The edge (p_1, p_2) is colored with the color of $S(p)$. In the matching for Q, p_1 is matched with (p_1, q_1) and p_2 is matched with (p_2, q_1). By this we obtain $L_l(p_i) = 2L_l(p)$ and $L_r(p_i) = 2L_r(p) + 1$ for $i \in \{1, 2\}$, implying that the matched edges are indeed χ'-halving. Further, we have $S_l(p_1) = 2S_l(p)$, $S_r(p_1) = 2S_r(p)+1$, $S_l(p_2) = 2S_l(p) + 1$, and $S_r(p_2) = 2S_r(p)$.

Case 5: $e \in L(p)$ and $L_l(p) = L_r(p)$. The edge (p_1, p_2) is colored with the color of $S(p)$. In the matching for Q, p_1 is matched with (p_1, p_2) and p_2 is matched with (p_2, q_1). By this we obtain $L_l(p_1) = 2L_l(p) + 1$,

$L_r(p_1) = 2L_r(p) + 1$, $L_l(p_2) = 2L_l(p)$, and $L_r(p_2) = 2L_r(p) + 1$. Hence the matched edges are χ'-halving. Further, we have $S_l(p_1) = 2S_l(p)$, $S_r(p_1) = 2S_r(p)$, $S_l(p_2) = 2S_l(p) + 1$, and $S_r(p_2) = 2S_r(p)$.

Case 6: $e \in L(p)$ and $L_l(p) < L_r(p)$. The edge (p_1, p_2) is colored with the color of $S(p)$. In the matching for Q, p_1 is matched with (p_1, q_2) and p_2 is matched with (p_2, q_2). By this we obtain $L_l(p_i) = 2L_l(p) + 1$ and $L_r(p_i) = 2L_r(p)$ for $i \in \{1, 2\}$, implying that the matched edges are indeed χ'-halving. Further, we have $S_l(p_1) = 2S_l(p)+1$, $S_r(p_1) = 2S_r(p)$, $S_l(p_2) = 2S_l(p)$, and $S_r(p_2) = 2S_r(p) + 1$.

Having completed the coloring χ' for the edges induced by Q, we next consider the number of monochromatic crossings in the resulting drawing on Q. We claim the following for $\overline{\text{cr}}_2(Q, \chi')$:

Claim 2.1. *The pair (Q, χ') satisfies*

$$\begin{aligned}\overline{\text{cr}}_2(Q, \chi') = 16\ \overline{\text{cr}}_2(P, \chi) + \binom{m}{2} - m \\ + 4\sum_p \left(\binom{L_l(p)}{2} + \binom{L_r(p)}{2} + \binom{S_l(p)}{2} + \binom{S_r(p)}{2}\right) \\ + 2\sum_p (H_l(p) + H_r(p)).\end{aligned}$$

Proof. We count the crossings in the same way as in the proof of Lemma 3 of [4]. A crossing in Q comes from four points in convex position. We classify the crossings in three types, according to the number of points in P that originated them (see Figure 2.3).

Type I: The points come from two points in P. There are $\binom{m}{2}$ ways of choosing a pair of points in P, and every such pair determines a crossing in Q unless the edge between them is a matching edge. Since we have m matching edges, there are $\binom{m}{2} - m$ crossings of this type.

Type IIa: The points come from three points p, q, and r in P and none of the edges between those points is a matching edge. Without loss of generality, p_1 and p_2 are involved in the crossing. Then q and r lie on the same side of the line spanned by the matching edge e of p and both (p, r) and (p, q) have the same color as e. Any pair (r, q) of points of P that satisfies those conditions with respect to p generates four crossings in Q. Thus, the number of Type IIa crossings for p is $4\left[\binom{L_l(p)}{2} + \binom{L_r(p)}{2} + \binom{S_l(p)}{2} + \binom{S_r(p)}{2}\right]$.

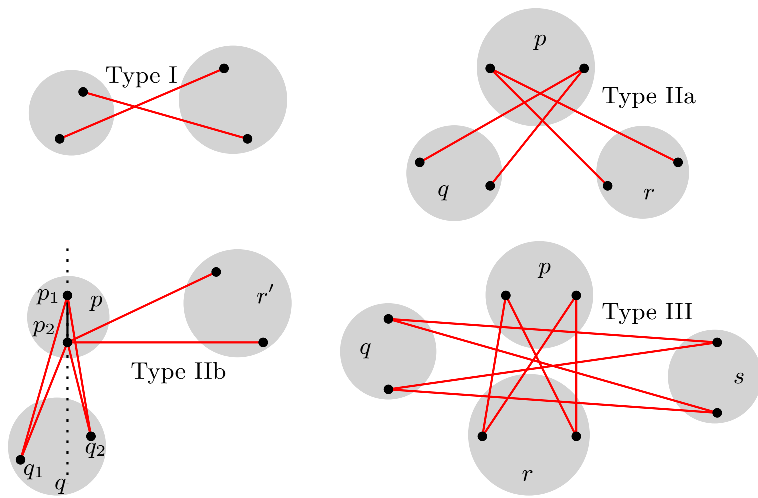


Figure 2.3: Counting the crossings of different types in the duplication process.

Type IIb: The points come from three points p, q, and r in P and one of the edges between those points is a matching edge. Without loss of generality, assume (p, q) is the matching edge of p. Any pair of points that originated from a point $r \in P$ such that (p, r) has the same color as (p, q) generates two crossings with either (p_1, q_1) or (p_1, q_2). Thus, the number of Type IIb crossings for p is $2(H_l(p) + H_r(p))$.

Type III: The points come from four points p, q, r, and s in P. Then those four points generate a crossing in P. There are $\overline{\mathrm{cr}}(P, \chi)$ such quadruples of points, and each one generates 16 crossings in Q. Thus, the number of Type III crossings in Q is $16\overline{\mathrm{cr}}(P, \chi)$.

Summing the Type II crossings over each point p of P and adding them to the crossings of Type III and Type I gives the claimed result. $\square$

We now apply the duplication process multiple times. To this end, consider again the six different cases for a point $p \in P$ when obtaining a coloring and a matching for Q. Note that if one of the Cases 1, 2, 3, 4 and 6 applies for p, then the same case applies for its duplicates $p_1, p_2 \in Q$ (and will apply in all further duplication iterations). If p falls in Case 5, then for p_1 and p_2 we have Case 2 and 4, respectively. As no point in Q falls in Case 5, from now on, we assume that P is such that no point of P falls in Case 5 either.

Let $k \geq 1$ be an integer and let (Q_k, χ_k) be the pair obtained by iterating the duplication process k times, with $(Q_0, \chi_0) = (P, \chi)$. We claim the following on $\overline{\text{cr}}_2(Q_k, \chi_k)$, the number of monochromatic crossings in the 2-edge-colored drawing of K_n induced by Q_k and χ_k:

Claim 2.2. *After k iterations of the duplication process, the following holds*

$$\overline{\text{cr}}_2(Q_k, \chi_k) = A \cdot 2^{4k} + B \cdot 2^{3k} + C \cdot 2^{2k} + D \cdot 2^k$$

where A, B, C, and D are rational numbers that depend on P and its χ-halving matching.

Proof. Let p be a point of P. We iteratively construct a rooted binary tree $T(p)$ of height k containing a vertex for each point q of Q_i that stems from duplicating p in the following way. The root of $T(p)$ contains the tuple $(L_l(p), L_r(p), S_l(p), S_r(p))$ representing p. For vertex v in $T(p)$ that represents a point q of Q_i with $0 \leq i \leq k-1$, its left child contains the tuple $(L_l(q_1),\ L_r(q_1),\ S_l(q_1),\ S_r(q_1))$ and its right child contains the tuple $(L_l(q_2),$ $L_r(q_2),\ S_l(q_2),\ S_r(q_2))$, where $q_1, q_2 \in Q_{i+1}$ are the duplicates of q. In addition, we mark whether the matching edge of p (and hence the ones of all points originating from p) is of the larger or the smaller color class at p.

We next elaborate on the exact content of the tuple stored in the j-th vertex of the i-th level of $T(P)$ with $j \in \{1, \ldots, 2^i\}$, depending on the case to be applied for p in the duplication process.

Cases 1, 3, 4 and 6: Let p be a point in P that falls in Case 1. Then in the i-th level of $T(p)$, the j-th vertex contains the tuple

$$(2^i L_l(p), 2^i L_r(p) + 2^i - 1, 2^i S_l(p) + j - 1, 2^i S_r(p) + 2^i - j).$$

We show this by induction on i. It follows directly from the duplication process that this happens when $i = 1$. Suppose that $i > 1$. From the induction hypothesis, the j-th vertex v of level i contains the tuple $(2^i L_l(p), 2^i L_r(p) + 2^i - 1, 2^i S_l(p) + j - 1, 2^i S_l(p) + 2^i - j)$. Since all the vertices of $T(p)$ represent points that fall in Case 1, the left and right children of v contain the tuples

$$(2^{i+1} L_l(p), 2^{i+1} L_r(p) + 2^{i+1} - 1, 2^{i+1} S_l(p) + (2j-1) - 1, 2^{i+1} S_r(p) + 2^{i+1} - (2j-1))$$

and

$$(2^{i+1} L_l(p), 2^{i+1} L_r(p) + 2^{i+1} - 1, 2^{i+1} S_l(p) + (2j-1), 2^{i+1} S_r(p) + 2^{i+1} - 2j),$$

respectively. These two vertices are precisely the $(2j–1)$-st and the $2j$-th vertex in level $i + 1$.

Note that, if p falls in Case 4, $T(p)$ has the exact same structure as a point of Case 1. Furthermore, if p is a point that falls in Case 3 or Case 6, the structure of $T(p)$ is exactly a mirrored version of the tree from a point that falls in Case 1.

Case 2: Let p be a point in P that falls in Case 2. Then in the i-th level of $T(p)$, the j-th vertex contains the tuple

$$(2^i L_l(p), 2^i L_r(p), 2^i S_l(p) + 2^i - 1, 2^i S_r(p) + 2^i - 1).$$

We again proceed by induction on i. It follows directly from the duplication process that this happens when $i = 1$, so suppose that $i > 1$. From the induction hypothesis, the j-th vertex v of level i contains the tuple $(2^i L_l(p), 2^i L_r(p) + 2^i - 1, 2^i S_l(p) + j, 2^j S_r(p) + 2^j - i)$. Since all the vertices of $T(p)$ represent points that fall in Case 2, the left and right children of v contain the tuple

$$(2^{i+1} L_l(p), 2^{i+1} L_r(p), 2^{i+1} S_l(p) + 2^{i+1} - 1, 2^{i+1} S_r(p) + 2^{i+1} - 1)$$

Note that $T(p)$, together with the information whether the matching edges are of the smaller or the larger color class, contains all the information needed to compute the crossings of Type II in Q_{i+1} that involve points which originate from p.

Using the above observations we can now determine $\overline{\text{cr}}_2(Q_k, \chi_k)$. We will use the following notation: $f_i(x) = \binom{2^i x}{2}, g_i(x) = \binom{2^i x + 2^i - 1}{2}$, $h_{i,j}(x) = \binom{2^i x + j}{2}$, and P_c is the subset of P of points that fall in Case c.

Type III: Each crossing of Type III in P generates 16 crossings in Q. Iterating this process k times, we obtain

$$16^k \ \overline{\text{cr}}_2(P, \chi)$$

crossings in Q_k.

Type I: Every set Q_i has a χ_i-halving matching and $|Q_i| = 2^i m$, thus, there are $\binom{2^i m}{2} - 2^i m$ crossings of Type I in Q_{i+1}. Moreover, each of these crossings becomes a Type III crossing in further duplication steps, that is, it produces 16 crossings per each further duplication step. Hence, adding the crossings of Type I that we get at each iteration and the according crossings of Type III that they generate later, we obtain

$$\sum_{i=0}^{k-1} 16^{k-i-1} \left[\binom{2^i m}{2} - 2^i m \right]$$

crossings in Q_k.

Type II for Case 2: Consider a point $p \in P$ that falls in Case 2, together with all points in Q_i that originate from it (and hence fall in Case 2 as well). Using Claim 2.1, and the information from the i-th level of $T(p)$, we obtain that Q_{i+1} has

$$\begin{aligned} & 4 \cdot 2^i \left[f_i(L_l(p)) + f_i(L_r(p)) + g_i(S_l(p)) + g_i(S_r(p))\right] \\ + \; & 2 \cdot 2^i \left[2^i(H_l(p) + H_r(p)) + 2^{i+1} - 2\right] \end{aligned}$$

crossings of Type II that come from all points in Q_i originating from p. Moreover, each of these crossings becomes a Type III crossing for all further duplication steps. Hence, adding the crossings of Type II that we count for points originating from p at each iteration and the according crossings of Type III that they generate later, we obtain

$$\begin{aligned} & 4 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[f_i(L_l(p)) + f_i(L_r(p)) + g_i(S_l(p)) + g_i(S_r(p))\right] \\ + \; & 2 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[2^i(H_l(p) + H_r(p)) + 2^{i+1} - 2\right] \end{aligned}$$

crossings in Q_k.

Type II for Cases 1 and 3: Consider a point $p \in P$ that falls in Case 1 or 3, and with all points in Q_i that originate from it (and hence fall in Case 1 or Case 3 as well). Using Claim 2.1, and the information from the i-th level of $T(p)$, we obtain that Q_{i+1} has

$$\begin{aligned} & 4 \cdot 2^i \left[f_i(L_l(p)) + g_i(L_r(p))\right] \\ + \; & 4 \sum_{j=0}^{2^i-1} \left[h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))\right] \\ + \; & 2 \sum_{j=0}^{2^i-1} \left[2^i(H_l(p) + H_r(p)) + 2j\right] \end{aligned}$$

crossings of Type II that come from all points in Q_i in the tree $T(p)$. Again, each of these crossings becomes a Type III crossing for all further duplication steps. Hence, adding the crossings of Type II that we we count for points originating from p at each iteration and the according crossings of Type III that they generate later, we obtain

$$
\begin{aligned}
&4\sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[f_i(L_l(p)) + g_i(L_r(p))\right] \\
&+ 4\sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} \left[h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))\right] \\
&+ 2\sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} \left[2^i(H_l(p) + H_r(p)) + 2j\right]
\end{aligned}
$$

crossings in Q_k.

Type II for Cases 4 and 6: Consider a point $p \in P$ that falls in Case 4 or 6, and with all points in Q_i that originate from it (and hence fall in Case 4 or Case 6 as well). Using Claim 2.1, and the information from the i-th level of $T(p)$, we obtain that Q_{i+1} has

$$
\begin{aligned}
&4 \cdot 2^i \left[f_i(L_l(p)) + g_i(L_r(p))\right] \\
&+ 4\sum_{j=0}^{2^i-1} \left[h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))\right] \\
&+ 2 \cdot 2^i \left[2^i(H_l(p) + H_r(p)) + 2^i - 1\right]
\end{aligned}
$$

crossings of Type II that come from all points in Q_i in the tree $T(p)$. Again, each of these crossings becomes a Type III crossing for all further duplication steps. Hence, adding the crossings of Type II that we we count for points originating from p at each iteration and the according crossings of Type III that they generate later, we obtain

$$
\begin{aligned}
&4\sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[f_i(L_l(p)) + g_i(L_r(p))\right] \\
&+ 4\sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} \left[h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))\right] \\
&+ 2\sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[2^i(H_l(p) + H_r(p)) + 2^i - 1\right]
\end{aligned}
$$

crossings in Q_k.

Adding the number of crossings of each type, we obtain the following expression for $\overline{\text{cr}}(Q_k, \chi_k)$, the number of monochromatic crossings in the straight-line 2-edge-colored drawing of K_n induced by Q_k and χ_k:

$$\overline{\text{cr}}_2(Q_k, \chi_k) = 16^k\ \overline{\text{cr}}_2(P, \chi) + \sum_{i=0}^{k-1} 16^{k-i-1} \left[f_i(m) - 2^i m\right] \tag{2.1}$$

$$+ \sum_{p \in P_2} \left[4 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i \left[f_i(L_l(p)) + f_i(L_r(p)) + g_i(S_l(p)) + g_i(S_r(p))\right] \right. \tag{2.2}$$

$$\left. + 2 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i [2^i (H_l(p) + H_r(p)) + 2^{i+1} - 2] \right]$$

$$+ \sum_{p \in P_1 \cup P_3} \left[4 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i [f_i(L_l(p)) + g_i(L_r(p))] \right. \tag{2.3}$$

$$+ 4 \sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} [h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))]$$

$$\left. + 2 \sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} [2^i (H_l(p) + H_r(p)) + 2j] \right]$$

$$+ \sum_{p \in P_4 \cup P_6} \left[4 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i [f_i(L_l(p)) + g_i(L_r(p))] \right. \tag{2.4}$$

$$+ 4 \sum_{i=0}^{k-1} 16^{k-i-1} \sum_{j=0}^{2^i-1} [h_{i,j}(S_l(p)) + h_{i,j}(S_r(p))]$$

$$\left. + 2 \sum_{i=0}^{k-1} 16^{k-i-1} 2^i [2^i (H_l(p) + H_r(p)) + 2^i - 1] \right].$$

What remains to be shown is that this sum can be written as $A \cdot 2^{4k} + B \cdot 2^{3k} + C \cdot 2^{2k} + D \cdot 2^k$, where A, B, C and D depend on $L_l(p), L_r(p), S_l(p), S_r(p), H_l(p)$ and $H_r(p)$ for every point p in P. We will use the following observations:

Observation 1.

$$\sum_{i=0}^{k-1} 16^{k-i-1} 2^i f_i(x) = \frac{2^{4k}}{2^4} \sum_{i=0}^{k-1} \frac{1}{2^{3i}} 2^{i-1} x(2^i x - 1)$$

$$
\begin{aligned}
&= \frac{2^{4k}}{2^4}\sum_{i=0}^{k-1}\frac{x^2}{2\cdot 2^i}-\frac{x}{2\cdot 2^{2i}}\\
&= \frac{2^{4k}}{2^4}\left[\frac{3x^2-2x}{3}-\frac{x^2}{2^k}+\frac{2x}{3\cdot 2^{2k}}\right]\\
&= \frac{3x^2-2x}{48}\cdot 2^{4k}-\frac{x^2}{16}\cdot 2^{3k}+\frac{x}{24}\cdot 2^{2k}.
\end{aligned}
$$

Observation 2.

$$
\begin{aligned}
\sum_{i=0}^{k-1}16^{k-i-1}2^i g_i(x) &= \frac{2^{4k}}{2^4}\sum_{i=0}^{k-1}\frac{1}{2^{3i}}(2^i x+2^i-1)(2^{i-1}x+2^{i-1}-1)\\
&= \frac{2^{4k}}{2^4}\sum_{i=0}^{k-1}\frac{(x+1)^2}{2\cdot 2^i}-\frac{3(x+1)}{2\cdot 2^{2i}}+\frac{1}{2^{3i}}\\
&= \frac{2^{4k}}{2^4}\left[\frac{7x^2+1}{7}-\frac{(x+1)^2}{2^k}+\frac{2(x+1)}{2^{2k}}-\frac{8}{7\cdot 2^{3k}}\right]\\
&= \frac{7x^2+1}{112}\cdot 2^{4k}-\frac{(x+1)^2}{16}\cdot 2^{3k}+\frac{x+1}{8}\cdot 2^{2k}-\frac{1}{14}\cdot 2^k.
\end{aligned}
$$

Observation 3.

$$
\begin{aligned}
\sum_{j=0}^{2^i-1}h_{i,j}(x) &= \frac{1}{2}\sum_{j=0}^{2^i-1}2^{2i}x^2+2^i x(2j-1)+j(j-1)\\
&= \frac{1}{2}\left[2^i\cdot 2^{2i}x^2+2^i x(2^{2i}-2^{i+1})+\frac{2^i(2^i-1)(2^i-2)}{3}\right]\\
&= \frac{3(x^2+x)+1}{6}\cdot 2^{3i}-\frac{2x+1}{2}\cdot 2^{2i}+\frac{1}{3}\cdot 2^i.
\end{aligned}
$$

Observation 4.

$$
\begin{aligned}
&\sum_{i=0}^{k-1}16^{k-i-1}\sum_{j=0}^{2^i-1}h_{i,j}(x)\\
&\quad= \frac{2^{4k}}{2^4}\sum_{i=0}^{k-1}\frac{1}{2^{4i}}\left[\frac{3(x^2+x)+1}{6}\cdot 2^{3i}-\frac{2x+1}{2}\cdot 2^{2i}+\frac{1}{3}\cdot 2^i\right]\\
&\quad= \frac{2^{4k}}{2^4}\sum_{i=0}^{k-1}\frac{1}{2^{4i}}\left[\frac{3(x^2+x)+1}{6\cdot 2^i}-\frac{2x+1}{2\cdot 2^{2i}}+\frac{1}{3\cdot 2^{3i}}\right]\\
&\quad= \frac{2^{4k}}{2^4}\left[\frac{3(x^2+x)+1}{48}(2^{4k}-2^{3k})-\frac{2x+1}{24}(2^{4k}-2^{2k})+\frac{1}{42}(2^{4k}-2^k)\right]\\
&\quad= \frac{21x^2-7x+1}{336}\cdot 2^{4k}-\frac{3(x^2+x)+1}{48}\cdot 2^{3k}+\frac{2x+1}{24}\cdot 2^{2k}-\frac{1}{42}\cdot 2^k.
\end{aligned}
$$

We show that (2.1), (2.2), (2.3) and (2.4) can be written as

$$a \cdot 2^{4k} + b \cdot 2^{3k} + c \cdot 2^{2k} + d \cdot 2^{k}.$$

For (2.1), it follows from Observation 1. For (2.2), it follows from Observations 1 and 2. For (2.3) and (2.4), it follows from Observations 1, 2 and 4. Thus, $\overline{\mathrm{cr}}_2(Q_k, \chi_k)$ can be written as $A \cdot 2^{4k} + B \cdot 2^{3k} + C \cdot 2^{2k} + D \cdot 2^{k}$. □

Applying Claim 2.2 to an initial drawing on m vertices and letting $n = 2^k m$, we get:

$$\overline{\mathrm{cr}}_2(K_n) \leq \overline{\mathrm{cr}}_2(Q_k, \chi_k) = \frac{24A}{m^4}\binom{n}{4} + \Theta(n^3)$$

which completes the proof of Theorem 2.6 when n is of the form $2^k m$. The proof for $2^k m < n < 2^{k+1} m$ then follows from the fact that $\overline{\mathrm{cr}}_2(K_n)$ is an increasing function. □

We remark that the duplication process described in the proof of Theorem 2.6 can also be applied if the initial set P has odd cardinality. However, then it might happen that the resulting matching is not χ'-halving for the resulting set Q. Moreover, a similar process can even be applied with any matching between the points of P and the edges induced by P, where in that situation one needs to specify how the colors for the edges between duplicates of points (and possibly a matching for the resulting set) is chosen.

In the uncolored duplication process for obtaining bounds on $\overline{\mathrm{cr}}(K_n)$, halving matchings always yield the best asymptotic behavior, which only depends on $|P|$ and $\overline{\mathrm{cr}}(P)$. This is not the case for the 2-colored setting, where we ideally would like to achieve simultaneously for every point $p \in P$ that (i) both color classes are of similar size, (ii) both color classes are evenly split by the matching edge, and (iii) $\overline{\mathrm{cr}}_2(P)$ is small. Yet, this is in general not possible. Starting with a χ-halving matching for P we obtain (ii) at least for the larger color class at every point of P. Moreover, this is hereditary by the design of our duplication process.

The results of this section imply that for large cardinality we can obtain straight-line drawings of the complete graph with a reasonably small 2-colored crossing number by starting from *good* sets of constant size. Similar as in [10] we apply a heuristic combining different methods to obtain straight-line drawings of the complete graph with low 2-colored crossing number. Our heuristic iterates three steps of (1) locally improving a set, (2) generating larger good sets, and (3) extracting good subsets, where also after steps (2) and (3) a local optimization is done. The currently best (with respect to the crossing constant, see below) straight-line drawing D with 2-edge coloring χ we found in this way[‡] has $n = 135$ vertices, a 2-colored crossing number of $\overline{\text{cr}}_2(D, \chi) = 1470756$, and contains a χ-halving matching.

Let $\overline{\text{cr}}_2$ be the *rectilinear 2-colored crossing constant*, that is, the constant such that the best straight-line drawing of K_n for large values of n has at most $\overline{\text{cr}}_2\binom{n}{4}$ monochromatic crossings. Its existence follows from the fact that the limit $\lim_{n\to\infty} \overline{\text{cr}}_2(K_n)/\binom{n}{4}$ exists and is a positive number (the proof goes along the same lines as for the uncolored case). Using the above-mentioned currently best straight-line 2-edge colored drawing and plugging it into the machinery developed in the proof of Theorem 2.6 we get

Theorem 2.7. *The rectilinear 2-colored crossing constant satisfies*

$$\overline{\text{cr}}_2 \leq \frac{182873519}{1550036250} < 0.11798016.$$

In [3] a lower bound of $\overline{\text{cr}} \geq \frac{277}{729} > 0.37997267$ has been shown for the rectilinear crossing constant. We can thus give an upper bound on the asymptotic ratio between the best rectilinear 2-colored drawing of K_n and the best rectilinear drawing of K_n of $\overline{\text{cr}}_2/\overline{\text{cr}} \leq 0.31049652$.

2.2.2 Lower Bounds on $\overline{\text{cr}}_2(K_n)$

In this section we consider lower bounds for the 2-colored crossing number and the biplanar crossing number of K_n.

In related work [71], the authors present lower and upper bounds on the $\sup \overline{\text{cr}}_k(G)/\overline{\text{cr}}(G)$ where the supremum is taken over all non-planar graphs. We remark that this lower bound does not yield a lower bound for $\overline{\text{cr}}_2(K_n)$ as their bound is obtained for "midrange" graphs (graphs with a subquadratic but superlinear number of edges). Czabarka et al. mention a lower bound on the biplanar crossing number of general graphs depending on the number of edges [33, Equation 3]. For the complete graph, this yields a lower bound of $\overline{\text{cr}}_2(K_n) \geq 1/1944n^4 - O(n^3)$. A better bound of $\overline{\text{cr}}_2 \geq \frac{24}{29 \cdot 32} = 3/116 > 1/39$ can be obtained from (an improved version of) the crossing lemma [6, 65], which states that for an undirected simple graph with n vertices and e edges with $e > 7n$, the crossing number of the graph is at least $\frac{e^3}{29n^2}$.

Alternatively, the following result shows that from the 2-colored rectilinear crossing number of small sets we can obtain lower bounds for larger sets.

[‡]The interested reader can get a file with the coordinates of the points, the colors of the edges, and a χ-halving matching from `http://www.crossingnumbers.org/projects/monochromatic/sets/n135.php`.

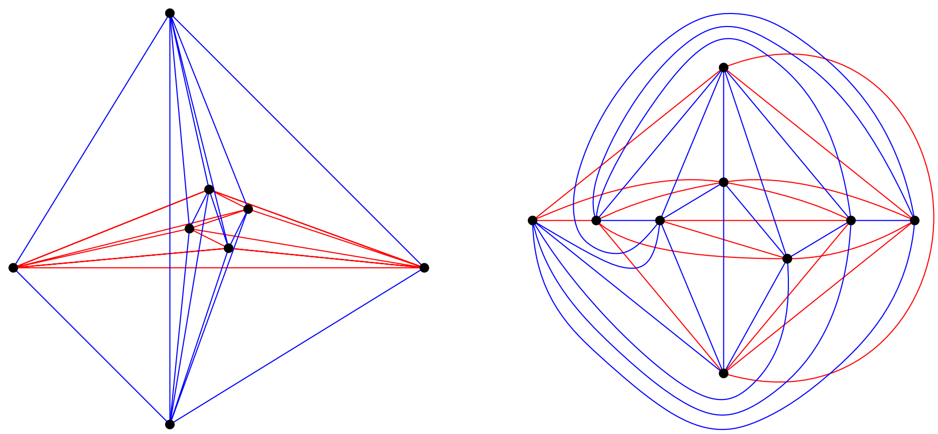


Figure 2.4: Left: a 2-colored rectilinear drawing of K_8 without monochromatic crossings. Right: a 2-colored drawing of K_9 with only one monochromatic (red) crossing.

Lemma 2.1. *Let* $\overline{\text{cr}}_2(m) = \hat{c}$ *for some* $m \geq 4$. *Then for* $n > m$ *we have* $\overline{\text{cr}}_2(K_n) \geq \frac{24\hat{c}}{m(m-1)(m-2)(m-3)}\binom{n}{4}$ *which implies* $\overline{\text{cr}}_2 \geq \frac{24\hat{c}}{m(m-1)(m-2)(m-3)}$.

Proof. Every subset of m points of K_n induces a drawing with at least $\hat{c}$ crossings, and thus we have $\hat{c}\binom{n}{m}$ crossings in total. In this way every crossing is counted $\binom{n-4}{m-4}$ times. This results in a total of $\frac{24\hat{c}}{m(m-1)(m-2)(m-3)}\binom{n}{4}$ crossings. □

As K_8 can be drawn such that $\overline{\text{cr}}_2(K_8) = 0$ (see Figure 2.4 left) we next determine $\overline{\text{cr}}_2(K_9)$. We use the optimization heuristic mentioned from Section 2.2.1 to obtain good colorings for all 158 817 order types of K_9 (which are provided by the order type data base [8]). In this way, it is guaranteed that all (crossing-wise) different straight-line drawings of K_9 (uncolored) are considered.

To prove that the heuristics indeed found the best colorings we consider the intersection graph for each drawing D. In the intersection graph every edge in D is a vertex, and two vertices are connected if their edges in D cross. Note that each odd cycle in the intersection graph of D gives rise to a monochromatic crossing in D. On the other hand, several odd cycles might share a crossing and only one monochromatic crossing is forced by them. We thus set up an integer linear program, where for every crossing of D we have a non-negative variable and for each odd cycle the sum of the variables corresponding to the crossings of the cycle has to be at least 1. The objective function aims to minimize the sum of all variables, which by construction is a lower bound for the number of monochromatic crossings in D.

With that program and some additional methods for speedup (see [41] for details), we have been able to obtain matching lower bounds and hence determine the 2-colored crossing numbers for all order types of K_9 within a few hours. The best drawings we found have 2 monochromatic crossings, and thus $\overline{\text{cr}}_2(K_9) = 2$. Using Lemma 2.1 for $m = 9$ and $\hat{c} = 2$ we get a bound of $\overline{\text{cr}}_2 \geq 1/63$, which is worse than what we obtained from the crossing lemma. Repeating the process of computing lower bounds for sets of small cardinality, we checked all order types of size up to 11 [7] and obtained $\overline{\text{cr}}_2(K_{10}) = 5$ and $\overline{\text{cr}}_2(K_{11}) = 10$. By Lemma 2.1, the latter gives the improved lower bound of $\overline{\text{cr}}_2 \geq 1/33$.

Straight-line versus General Drawings

The best straight-line drawings of K_n with $n \leq 8$ have no monochromatic crossing, see again Figure 2.4 left. In [71, Section 3], the authors state that no graph is known were the k-planar crossing number is strictly smaller than the rectilinear k-planar crossing number for any $k \geq 2$. Moreover, according to personal communication [82], the similar question whether a graph exists where the k-planar crossing number is strictly smaller than the rectilinear k-colored crossing number was open. We next argue that K_9 is such an example. From the previous section we know that $\overline{\text{cr}}_2(K_9) = 2$. Inspecting rotation systems for $n = 9$ [1] which have the minimum number of 36 crossings, we have been able to construct a drawing of K_9 which has only one monochromatic crossing, see Figure 2.4 right. As the graph thickness of K_9 is 3 [20, 83], we cannot draw K_9 with just two colors without monochromatic crossings. Thus, we get the following result.

Observation 5. *The biplanar crossing number for K_9 is 1 and is thus strictly smaller than the rectilinear 2-colored crossing number $\overline{\text{cr}}_2(K_9) = 2$.*

2.2.3 Upper Bounds on the Ratio $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$

In this section we study the extreme values that $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$ can attain for straight-line drawings D of K_n. Using a simple probabilistic argument as in [71], 2-coloring the edges uniformly at random, it can be shown that $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) \leq 1/2$ for every straight-line drawing D, even if the underlying graph is not K_n.

In the following, we show that for K_n this upper bound on $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$ can be improved. To obtain our improved bound, we find subdrawings of D and colorings such that many of the crossings in these drawings are between edges of different colors. To this end, we need to find large subsets of vertices of D with identical geometric properties. We use the following definition and theorem. Let $(Y_1, ..., Y_k)$ be a tuple of finite subsets of points in the plane. A *transversal* of $(Y_1, ..., Y_k)$ is a tuple of points $(y_1, \dots, y_k)$ such that $y_i \in Y_i$ for all i.

Theorem 2.8 (Positive fraction Erdős-Szekeres theorem)**.** *For every integer $k \geq 4$ there is a constant $c_k > 0$ such that every sufficiently large finite point set $X \subset \mathbb{R}^2$ in general position contains k disjoint subsets $Y_1, \dots, Y_k$, of size at least $c_k|X|$ each, such that each transversal of $(Y_1, \dots, Y_k)$ is in convex position.*

The Positive Fraction Erdős-Szekeres theorem was proved by Bárány and Valtr [19], see also Matoušek's book [65]. Although it is not stated in the theorem, every transversal of the Y_i has the same (labelled) order type. Making use of that result we obtained the following theorem.

Theorem 2.9. *There exists an integer $n_0 > 0$ and a constant $c > 0$ such that for any straight-line drawing D of K_n on $n \geq n_0$ vertices, $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) < \frac{1}{2} - c$.*

Proof. Let c_4 be as in Theorem 2.8 and let n_0 be such that Theorem 2.8 holds for $k = 4$ and for point sets with at least n_0 points. Let D be a straight-line drawing of K_n, where $n \geq n_0$.

Our general strategy is as following. We first find subsets of edges of D that can be 2-colored such that many of the crossings between these edges are between pairs of edges of different colors. We remove these edges and search for a subset of edges with the same property. We repeat this process as long as possible. We 2-color the remaining edges so that at most half of the crossings are monochromatic. Afterwards, we put back the edges we removed while 2-coloring them in a convenient way.

We define a sequence of subsets $V = X_0 \supset X_1 \supset \cdots \supset X_m$ of vertices of D, where $V = X_0$ is the set of vertices of D, and tuples $(F_1, F_1'), \dots, (F_m, F_m')$ of sets of edges of D as follows. Suppose that X_i has been defined. If $|X_i| < n_0$, we stop the process. Otherwise we apply Theorem 2.8 to X_i, to obtain a tuple (Y_1, Y_2, Y_3, Y_4) of disjoint subsets of points X_i, each with exactly $\lfloor c_4|X_i| \rfloor$ vertices, such that every transversal (y_1, y_2, y_3, y_4) of (Y_1, Y_2, Y_3, Y_4) is a convex quadrilateral. Without loss of generality we assume that (y_1, y_2, y_3, y_4) appear in clockwise order around this quadrilateral. This implies that the edge (y_1, y_3) crosses the edge (y_2, y_4). Let F_i be the set of edges with an endpoint in Y_1 and an endpoint in Y_3; let F'_i be the set of edges with an endpoint in Y_2 and an endpoint in Y_4; and finally, let $X_{i+1} = X_i \setminus (Y_1 \cup Y_2)$. Note that every edge in F_i crosses every edge in F'_i.

We now consider the remaining edges. Let $\overline{F}$ be the set of edges of D that are not contained in any F_i nor in any F'_i for $1 \leq i \leq m$. Let H be the straight-line drawing with the same vertices as D and with edge set equal to $\overline{F}$. By a probabilistic argument 2-coloring the edges uniformly at random, there is a coloring χ' of the edges of H so that $\overline{\text{cr}}(H)/\overline{\text{cr}}_2(H, \chi') \geq 2$.

We now 2-color the edges in F_i and F'_i. We define a sequence of straight-line drawings $H = D_{m+1}, \subset D_m \subset \cdots \subset D_0 = D$ and a corresponding sequence of 2-edge-colorings $\chi' = \chi_{m+1}, \chi_m, \ldots, \chi_0 = \chi$ that satisfies the following. Each χ_i is a 2-edge-coloring of D_i. Also χ_{i-1} when restricted to D_i equals χ_i. Suppose that D_i and χ_i have been defined and that $0 < i \leq m+1$. Let D_{i-1} be the straight-line drawing with the same vertices as D and with edge set E_{i-1} equal to $E_i \cup F_{i-1} \cup F_{i'-1}$ (where E_i is the edge set of D_i). Since χ_{i-1} coincides with χ_i in the edges of E_i, we only need to specify the colors of F_{i-1} and F'_{i-1}. We color the edges of F_i with the same color and the edges of F'_{i-1} with the other color. There are two options for doing this, and one of them guarantees that at most half of the crossings between an edge of $F_{i-1} \cup F'_{i-1}$ and an edge of D_i are monochromatic. We choose this option to define χ_{i-1}.

In what follows we assume that D has been colored by χ. Let C be the set of pairs of edges of D that cross. Of these, let C_1 be the subset of pairs of edges such that both of them are contained in $F_i \cup F'_i$ for some $1 \leq i \leq m$. Let $C_2 := C \setminus C_1$. Note that, by construction of χ, at most half of the pairs of edges in C_2 are of edges of the same color. For a given i, let E'_i be the subset of pairs of edges in C_1 such that both edges are in $F_i \cup F'_i$. Let (Y_1, Y_2, Y_3, Y_4) be the tuple of disjoint subsets of points X_i used to define F_i and F'_i. Recall that each Y_i consists of $\lfloor c_4 |X_i| \rfloor$ points. Every pair of crossing edges defines a convex quadrilateral and, conversely, every convex quadrilateral defines a unique pair of crossing edges. Therefore, by construction there at most ${c_4}^4 \lfloor |X_i| \rfloor^4/2$ pairs of edges in E'_i such that both edges are of the same color; and there are exactly $\lfloor c_4 |X_i| \rfloor^4$ pairs of edges in E'_i such that the edges are of different color. Thus, at most $\frac{1}{3}$ of the pairs of edges in E'_i are edges of the same color.

Therefore, $\frac{\overline{\text{cr}}_2(D,\chi)}{\overline{\text{cr}}(D)} \leq \frac{\frac{1}{2}|C_1| + \frac{1}{3}|C_2|}{|C_1| + |C_2|}$. This is maximized when C_1 is as large as possible. Since there in total at most $\binom{n}{4}$ pairs of edges that cross, we have

$|C_1| \leq \binom{n}{4} - |C_2|$. Thus,

$$\frac{\overline{\mathrm{cr}}_2(D,\chi)}{\overline{\mathrm{cr}}(D)} \leq \frac{\frac{1}{2}\binom{n}{4} - \frac{1}{6}|C_2|}{\binom{n}{4}}.$$

We now obtain a lower bound for the size of C_2. Note that $|X_0| = n$ and $|X_i| \geq (1-4c_4)|X_{i-1}|$. This implies that $|X_i| \geq (1-4c_4)^i n$ and that $|E_i| \geq {c_4}^4(1-4c_4)^{4i}n^4$. Therefore,

$$|C_2| = \sum_{i=1}^{m} |E_i| \geq \sum_{i=1}^{m} {c_4}^4(1-4c_4)^{4i}n^4 = 24{c_4}^4\left(\frac{1}{1-(1-4c_4)^4} - 1 - o(1)\right)\binom{n}{4},$$

which completes the proof. □

2.2.4 Special Cases: Convex Position and the Double Chain

We consider the ratio $\overline{\mathrm{cr}}_2(D)/\overline{\mathrm{cr}}(D)$ for particular families of drawings D of K_n.

If the vertices of a straight-line drawing D are in convex position then the drawing D is said to be *convex*. For a convex straight-line drawing D of K_n the problem of finding a 2-edge-coloring that minimizes $\overline{\mathrm{cr}}_2(D)$ is equivalent to the problem of finding the 2-page crossing number of the complete graph K_n. In [2], Ábrego et al. proved that the 2-page crossing number of K_n is equal to

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

And, since the number of crossings in a convex straight-line drawing of K_n is $\binom{n}{4}$, we obtain the following theorem.

Theorem 2.10. *If D is a convex straight-line drawing of K_n, then* $\overline{\mathrm{cr}}_2(D)/\overline{\mathrm{cr}}(D) = 3/8 - o(1)$.

The other special case we consider consists of non-complete straight-line drawing whose vertices form a *double-chain*. This configuration is defined as follows. For $n \geq 3$, an (n,n)-*double-chain* consists of two (upper and lower) convex chains of n points each, linearly separable, and facing each other so that (i) two successive points of one chain and two successive points of the other are always in convex position, and (ii) three successive points of one chain and one point of the other are never in convex position.

Theorem 2.11. *Let D be a straight-line drawing of a graph whose vertex set is an (n, n)-double-chain, and in which there exists an edge between two vertices if and only if they belong to different chains. Then* $\overline{\mathrm{cr}}_2(D)/\overline{\mathrm{cr}}(D) \leq 1/3 + o(1)$.

Proof. We label the vertices of the upper chain from left to right as $1, \ldots, n$ and we label the vertices on lower chain from left to right also as $1, \ldots, n$. Let $e = (i, j)$ be an edge of D, with i in the upper chain and j in the lower chain. If $i < j$ then we color e blue; if $i > j$ then we color e red; and if $i = j$ then we color e red or blue.

Let $I = (i, j, k, l)$ be a tuple of indices with $1 \leq i \leq j \leq k \leq l \leq n$, and at most two of them equal. Let S be a set of four vertices of D, whose labels are in $\{i, j, k, l\}$, and such that two vertices are in the upper chain and the other two are in the lower chain. Note that S defines a unique pair of edges of D that cross; and conversely, every pair of edges that cross has two vertices in the upper chain and the other two in the lower chain. There are six possible choices for S (for a given I) and each defines a different pair of crossing edges (except when at least two indices are the same). Of these six pairs of crossing edges, only two are between edges of the same color. Since the number of possible tuples (i, j, k, l) in which at most two indices are equal is $\binom{n}{4} + O(n^3)$, the result follows. $\square$

2.3 Conclusions and Open Problems

In this chapter we have shown upper bounds on the rectilinear and pseudolinear crossing numbers. Our approach of finding good seeds for obtaining improved asymptotic bounds is based on several heuristics. Our observations indicate that none of them would give improved results by their own, but the combination of local optimization, doubling the cardinality, and searching for good subsets is what led to the improvements. We do not have provable results which show that this approach is reasonable good in the sense that we can bound how close we are to the optimal drawings. But this lies in the nature of the crossing number problems. It has been shown that for general graphs computing the rectilinear crossing number is $\exists\mathbb{R}$-complete and computing the pseudolinear crossing number is NP-complete [53].

We have also shown lower and upper bounds on the rectilinear 2-colored crossing number for K_n as well as its relation of the latter to the rectilinear crossing number for fixed drawings of K_n. Besides improving the given bounds, some open problems arise from our work.

(1) How fast can the best edge-coloring of a given straight-line drawing of K_n be computed? This problem is related to the max-cut problem of segment intersection graphs, which has been shown to be NP-complete for general graphs [13]. But for the intersection graph of K_n the algorithmic complexity is still unknown.

(2) What can we say about the structure of 2-colored crossing minimal sets? For the rectilinear crossing number it is known that optimal sets have a triangular convex hull [14]. For $n = 8, 9$ we have optimal sets with 3 and 4 extreme points, but so far all minimal sets for $n \geq 10$ have a triangular convex hull.

(3) We have seen that for convex sets asymptotically, the ratio $\overline{\mathrm{cr}}_2(D)/\overline{\mathrm{cr}}(D)$ approaches $3/8$ from below when $n \to \infty$. It can be observed that among all point sets (order types) of size 10, the convex drawing D of K_{10} is the only one that provides the largest ratio of $\overline{\mathrm{cr}}_2(D)/\overline{\mathrm{cr}}(D) = 2/7$, while the best factor $5/76$ is reached by sets minimizing $\overline{\mathrm{cr}}_2(D)$. Is it true that the convex set has the worst (i.e., largest) factor? And is the best (smallest) factor always achieved by optimizing sets, that is, sets with $\overline{\mathrm{cr}}_2(D) = \overline{\mathrm{cr}}_2(K_n)$?

Chapter 3

Results on Drawings of Graphs

Considering problems under different geometric points of view, often produces interesting variations. An example of this are drawings of graphs, as was shown in Chapter 2: the different geometric properties that the drawings can have give rise to several variations of crossing numbers. While crossing numbers have a geometric origin, there are combinatorial problems that admit a geometric variation; graph related problems tend to be good candidates for this.

An example is the Erdős-Sós conjecture, which states that every simple graph with average degree greater than $k-2$ has every tree on k vertices as a subgraph. A geometric variant of this conjecture was studied by Barba et. al [67]. Given an integer k, the authors ask for the minimum number of edges in a straight-line drawing D of the complete graph K_n necessary to remove in order to forbid at least one tree on k as a plane subgraph. Following this spirit, in Section 3.1 we focus on straight-line drawings of the complete graph K_n. We study a geometric version of the following question: Given a graph G, what is the maximum length of a monotone path that we can guarantee over all total orders of its edges?

This question was posed by Chvátal and Komlós in 1972 [30]. The maximum length that we can guarantee is called the *altitude* of G, and it is denoted by $\alpha(G)$. We study the same question about straight-line drawings of K_n, with the added restriction that the paths we consider must be plane. Given a straight-line drawing D of K_n along with a total order of its edges, let $\overline{\alpha}(D)$ denote the length of the longest monotone plane path in D. Denote by $\overline{\alpha}(K_n)$ the minimum of $\overline{\alpha}(D)$, taken over all drawings of K_n and all possible total orders of their edges. We prove the following bounds:

Theorem 3.1. $\overline{\alpha}(K_n) = \Omega(\log \log n)$.

Theorem 3.2. $\overline{\alpha}(K_n) = O(\log n)$.

We also study an analogue question about complete binary trees. Given a drawing D of K_n along with an ordering of its edges, let $\overline{\tau}(D)$ denote the size of the longest monotone plane complete binary tree in D (that is, a plane complete binary tree such that either every path from its root to a leaf is increasing or decreasing with respect to the total order). Denote by $\overline{\tau}(K_n)$ the minimum of $\overline{\tau}(D)$, taken over all drawings of K_n and all possible total orders of their edges. We prove the following bounds:

Theorem 3.4. $\overline{\tau}(K_n) = \Omega(\log \log \log n)$.

Theorem 3.6. $\overline{\tau}(K_n) = O(\sqrt{n \log n})$.

These results are joint work with Frank Duque, Ruy Fabila-Monroy and Pablo Pérez-Lantero.

In Section 3.2 we turn our attention to grid graphs. Given two positive integers m, n the grid graph $G_{n,m}$ is the graph with vertex set

$$\{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

Two vertices of $G_{n,m}$ are adjacent if they are at distance one from each other. We are interested in relating the Hamiltonian paths in $G_{n,m}$ via *flips*. A flip in a graph consists on removing an edge and adding a different edge, so that the resulting graph remains in the same class as the original one. The number of flips required to transform one graph into another gives a notion of distance and similarity between graphs. A graph that results from taking graphs of a certain class as vertices and making two adjacent if a flip can transform one into the other is called a *flip graph*.

Let $H_{n,m}$ be the flip graph of the Hamiltonian paths of $G_{n,m}$ where two such paths P and Q are adjacent if there exist edges $e \in P$ and $f \notin P$ such that $Q = P - e + f$. We study the problem of determining if $H_{n,m}$ is connected. Our results are as follows:

Theorem 3.7. *$H_{n,2}$ is connected.*

Theorem 3.8. *$H_{n,3}$ is connected.*

Theorem 3.9. *$H_{n,4}$ is connected.*

These results are joint work with Frank Duque, Ruy Fabila-Monroy, David Flores-Peñaloza and Clemens Huemer. They were presented at the XVII Spanish Meeting on Computational Geometry [37].

Finally, in Section 3.3 we focus on drawings of complete multipartite graphs. The drawings we study in this section are rectilinear drawings on d-dimensional integer grids. That is, given positive integers $n_1 \leq \cdots \leq n_r$ such that $\sum n_i = (2M+1)^d$ for some integer M, we consider straight line drawings of the complete r-partite graph $K_{n_1,\ldots,n_r}$ into the d-dimensional integer grid

$$P := \Big\{(x_1,\ldots,x_d) \in \mathbb{Z}^d : -M \leq x_i \leq M\Big\}.$$

One way to see these drawings is as coloring the points of P, such that color i appears n_i times, for $i = 1,\ldots,r$. Our goal is to find colorings that maximize and minimize, respectively, the sum of squared distances between points of different colors. This is motivated by the *Embedding Lemma*, proved by Spielman and Teng [81] in 2007. For a graph G, let λ_2 and λ_N be the second smallest eigenvalue and the largest eigenvalue of the Laplacian matrix of G, respectively.

Lemma 3.7.

$$\lambda_2(G) = \min \frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2},$$

and

$$\lambda_N(G) = \max \frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2},$$

where the minimum, respectively maximum, is taken over all tuples $(\vec{v}_1,\ldots,\vec{v}_N)$ of vectors $\vec{v}_i \in \mathbb{R}^d$ with $\sum_{i=1}^N v_i = \mathbf{0}$, and not all v_i are zero-vectors $\mathbf{0}$.

Let $\mathbf{v} = (\vec{v}_1,\ldots,\vec{v}_N)$ be a tuple of positions defining a drawing of G (vertex i is placed at $\vec{v}_i$). Let

$$\lambda(\mathbf{v}) := \frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2}.$$

Note that $\|\vec{v}_i - \vec{v}_j\|^2$ is equal to squared length of the edge ij in the drawing defined by $\mathbf{v}$. The Embedding Lemma provides a link between the algebraic connectivity of G and its straight-line drawings. Clearly $\lambda_2(G) \leq \lambda(\mathbf{v}) \leq \lambda_N(G)$. Our results characterize the drawings of complete multipartite graphs which minimize/maximize $\lambda\mathbf{v}$ and therefore approximate λ_2 and λ_N.

Theorem 3.10. *Let* $\mathbf{v}$ *be a straight line drawing of* $K_{n_1,\ldots,n_r}$ *that minimizes* $\lambda(\mathbf{v})$. *If* $n_1 = n_2 = \ldots = n_r$, *then* $\mathbf{v}$ *minimizes* $\sum_{i=1}^{r} \left|\left|\sum_{v \in A_i} v\right|\right|^2$; *in particular, if* $\sum_{v \in A_i} v = \mathbf{0}$, *for all* $1 \leq i \leq r$, *then* $\lambda(\mathbf{v}) = N - n_r$. *If* $n_1 < n_2 < \ldots < n_r$, *then* $\mathbf{v}$ *has the following structure: For each* $i = 1, \ldots, r-1$, *the union of the smallest* i *color classes,* $\bigcup_{j=1}^{i} A_j$, *forms a ball centered at* $\mathbf{0}$.

Theorem 3.11. *Let* $\mathbf{v}$ *be a straight-line drawing of* $K_{n_1,\ldots,n_r}$ *on* P *that maximizes* $\lambda(\mathbf{v})$. *If* $n_1 = n_2 = \ldots = n_r$, *then* $\mathbf{v}$ *defines a centroidal Voronoi diagram. If the* n_i *are not all the same, then* $\mathbf{v}$ *defines a multiplicatively weighted centroidal Voronoi diagram.*

These results are joint work with Ruy Fabila-Monroy, Clemens Huemer, Dolores Lara and Dieter Mitsche. They were presented at the 26th International Symposium on Graph Drawing and Network Visualization [68].

3.1 Plane Paths and Plane Binary Trees in Edge-ordered Straight-line Drawings

An *edge-ordering* of a graph is a total order of its edges, an *edge-ordered* graph is a graph with an edge-ordering. A path $P = v_1 v_2 \ldots v_k$ in an edge-ordered graph is called increasing if $(v_i v_{i+1}) < (v_{i+1} v_{i+2})$ for all $i = 1, \ldots, k-2$; it is called decreasing if $(v_i v_{i+1}) > (v_{i+1} v_{i+2})$ for all $i = 1, \ldots, k-2$. We say that P is *monotone* if it is increasing or decreasing. Let G be a graph. Let $\alpha(G)$ be the maximum integer such that G has a monotone path of length $\alpha(G)$ under any edge-ordering. The parameter $\alpha(G)$ is often called the *altitude* of G.

Chvátal and Komlós were the first to pose, in 1972 [30], the problem of estimating the altitude of K_n. This problem has turned out to be challenging and has been studied by several researchers. The first related result is due to Graham and Kleitman [51], who showed in 1973 the following bounds:

$$\frac{1}{2}\left(\sqrt{4n-3} - 1\right) \leq \alpha(K_n) < \frac{3}{4}n.$$

They also conjectured the upper bound to be right order of magnitude of $\alpha(K_n)$. More than ten years after, Calderbank, Chung, and Sturtevant [26] improved the upper bound to

$$\alpha(K_n) \leq \left(\frac{1}{2} + o(1)\right) n.$$

The lower bound by Graham and Kleitman remained the best known for over 40 years, until Milans [66] showed in 2017 that

$$\left(\frac{1}{20} - o(1)\right)\left(\frac{n}{\lg n}\right)^{2/3} \leq \alpha(K_n).$$

Recently, Bucić et al. [25] showed that

$$\alpha(K_n) \geq \frac{n}{2^{O(\sqrt{\log n \log\log n})}} = n^{1-o(1)},$$

almost closing the gap between the upper and lower bounds.

In the meantime, several variations and specific cases of the altitude of a graph have been studied. In 1987 Bialostocki and Roditty [21] showed that a graph G has altitude at least three if and only if G contains as a subgraph an odd cycle of length at least five or one of six fixed graphs.

In 2001 Yuster [84] studied the parameter α_Δ, defined as the maximum of $\alpha(G)$ over all graphs with maximum degree at most Δ. He proved that $\Delta(1-o(1)) \leq \alpha_\Delta \leq \Delta+1$. That same year Roditty, Shoham, and Yuster [74] gave bounds on the altitude of several families of sparse graphs. In particular, if G is a planar graph then $\alpha(G) \leq 9$ and there exist planar graphs with $6 \leq \alpha(G)$. They also proved that if G is a bipartite planar graph then $\alpha(G) \leq 6$ and there exist bipartite planar graphs with $4 \leq \alpha(G)$.

In 2005 Mynhardt, Burger, Clark, Falvai and Henderson [69] characterized cubic graphs with girth at least five and altitude three. They also showed that if G is an r-regular graph ($r \geq 4$) and has girth at least five, then $\alpha(G) \geq 4$.

In 2015 De Silva, Molla, Pfender, Retter and Tait [79] showed that $d/\log d \leq \alpha(Q_d)$, where Q_d is the d-dimensional hypercube.

In 2016 Lavrov and Loh [63] studied the length of a longest monotone path in K_n under a random edge-ordering. They showed that, with probability at least $1/e - o(1)$, K_n contains a monotone Hamiltonian path. They also conjectured that, given a random ordering, K_n contains a monotone Hamiltonian path with probability tending to 1. Shortly after, Martinsson [64] proved this conjecture.

In this section we study $\alpha(G)$ and similar parameters in the context of straight-line drawings of graphs.

Let D be a straight-line drawing of a graph G. Let $\overline{\alpha}(D)$ be the minimum over all edge-orderings of D of the maximum length of a non-crossing monotone path in D. We denote by $\overline{\alpha}(G)$ the minimum of $\overline{\alpha}(D)$ over all straight-line drawings of G.

Now we define parameters similar to $\overline{\alpha}$ related to binary trees instead of paths. From now on, all binary trees are complete and rooted.

A rooted tree in an edge-ordered graph G is increasing (decreasing) if every path from the root to a leaf is increasing (decreasing). It is called monotone if it is increasing or decreasing. Let $\overline{\tau}_+(D)$ ($\overline{\tau}_-(D)$) be the minimum over all edge-orderings of the maximum size of a non-crossing increasing (decreasing) binary tree in D. Let $\overline{\tau}_+(G)$ ($\overline{\tau}_-(G)$) be the minimum of $\overline{\tau}_+(D)$ ($\overline{\tau}_-(D)$) over all straight-line drawings of G.

Similarly, let $\overline{\tau}(D)$ be the minimum over all edge-orderings of the maximum size of a non-crossing monotone binary tree in D and $\overline{\tau}(G)$ the minimum of $\overline{\tau}(D)$ over all straight-line drawings of G.

In this section we prove that $\overline{\alpha}(K_n) = \Omega(\log\log n)$ and $\overline{\alpha}(K_n) = O(\log n)$. For the parameter $\overline{\tau}$ we prove that $\overline{\tau}(K_n) = \Omega(\log\log\log n)$ and $\overline{\tau}(K_n) = O(\sqrt{n\log n})$. As an intermediate result, if we are interested in bounding the size of increasing or decreasing binary trees but not both, we prove that $\overline{\tau}_+(K_n) = O(\log n)$ and $\overline{\tau}_-(K_n) = O(\log n)$.

3.1.1 Monotone Non-crossing Paths

In this section we give bounds for $\overline{\alpha}(K_n)$. Recall that a straight-line drawing D of a graph G is said to be *convex* if its vertices are in convex position.

Observation 6. *Let S be a set of points in convex position and ℓ a straight line that partitions S into two nonempty sets U and V. The maximum length of a non-crossing polygonal chain whose vertices alternate between U and V and whose edges increase in slope is two.*

Proof. Assume that a polygonal chain P of length three with the conditions of the statement exists. Let p, q, r, s denote the vertices of P and assume without loss of generality that q has smaller x coordinate than r. The edges of P have increasing slope, thus both p and s lie to the right of the directed line from q to r. Since the points p, q, r, s are in convex position and the points q and s lie on a different semiplane than the points p and r with respect to ℓ, the edges (pq) and (rs) are the diagonals of a convex quadrilateral. These diagonals intersect; a contradiction the non-crossing property of P. □

Theorem 3.1. $\overline{\alpha}(K_n) = \Omega(\log\log n)$.

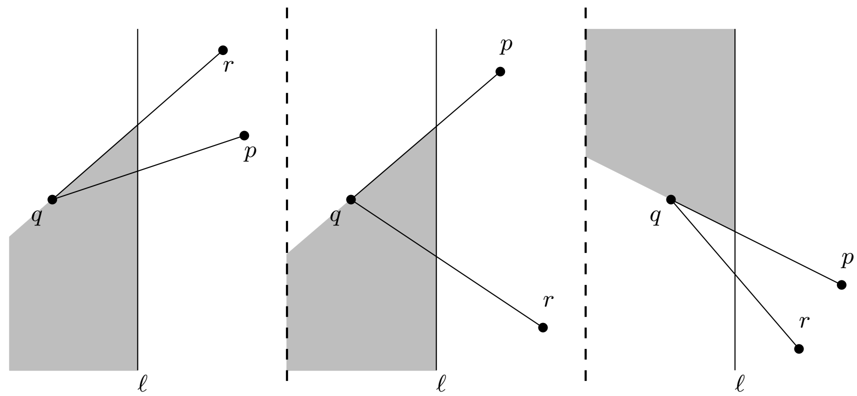


Figure 3.1: The fourth vertex of a polygonal with increasing slope must be in the shaded region.

Proof. Let D be an edge-ordered straight-line drawing of K_n and assume without loss of generality that no two vertices of D have the same x-coordinate. Let $v_1, \dots, v_n$ be the vertices of D ordered by increasing x-coordinate, and let H be the complete 3-uniform hypergraph with same vertex set as D. For $0 \le i < j < k \le n$, color the edge (v_i, v_j, v_k) of H blue if $(v_i v_j) < (v_j v_k)$, otherwise color it red. From Ramsey's Theorem, there exists a complete monochromatic sub-hypergraph K of size $m = \Omega(\log\log n)$ in H with vertices $v_{i_1}, v_{i_2}, \dots, v_{i_m}$. Then $P = v_{i_1}, v_{i_2}, \dots, v_{i_m}$ is a monotone path of length $\Omega(\log\log n)$ in D. Note that P is x-monotone, thus it has no crossings. □

Theorem 3.2. $\overline{\alpha}(K_n) = O(\log n)$.

Proof. Let D be a convex straight-line drawing of K_n. Let ℓ be a vertical line that partitions the vertices of D into two no empty sets U and W, each of size at most $\lceil n/2 \rceil$. Order the edges between U and W so that $e < e'$ if and only if the slope of e is less than the slope of e'. Recursively order the edges of $D[U]$ and $D[W]$ as before. Extend these orders to D by declaring the edges in $D[U] \cup D[W]$ to be less than those between U and W. Let P be a monotone path of maximum length in D. By Observation 6, there are at most two edges of P between sets in the same level of recursion. Moreover, P cannot have edges both in $D[U]$ and in $D[W]$, since there would be a subpath with an edge in $D[U]$, an edge between U and W and an edge in $D[W]$; this path cannot be monotone. Thus, the length $T(n)$ of P satisfies the recursion $T(n) \le 2 + T(n/2)$, which implies that $T(n) = O(\log n)$. □

3.1.2 Monotone Non-crossing Binary Trees

We begin this section with a lower bound for $\overline{\tau}(D)$, where D is a convex straight-line drawing of K_n. We use the same argument used to bound $\overline{\alpha}(K_n)$.

Theorem 3.3. *Let D be an edge-ordered convex straight-line drawing of K_n. Then $\overline{\tau}(D) = \Omega(\log\log n)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of D ordered by increasing x-coordinate and let H be the complete 3-uniform hypergraph with same vertex set as D. For $0 \le i < j < k \le n$, color the edge (v_i, v_j, v_k) of H blue if $(v_i v_j) < (v_j v_k)$, otherwise color it red. From Ramsey's theorem, there exists a complete monochromatic sub-hypergraph K of size $m = \Omega(\log\log n)$ in H. Assume without loss of generality that at least half of the vertices of K belong to the lower convex hull of the vertices of D. Let $v_{i_1}, \ldots, v_{i_m}$ denote those vertices ordered by increasing x-coordinate. Let m' be the largest integer of the form $2^h - 1$, for some integer h, such that $m' \le m$. We embed a binary tree T with vertices $v_{i_1}, \ldots, v_{i_{m'}}$ as follows. Place the root of T at v_{i_1}, and inductively place its left and right subtrees at the vertices $v_{i_2}, \ldots, v_{i_{(m'+1)/2}}$ and $v_{i_{(m'+1)/2+1}}, \ldots, v_{i_{m'}}$ with roots at v_{i_2} and $v_{i_{(m'+1)/2+1}}$, respectively (see Figure 3.2). Note that T is monotone and has no crossings by construction. □

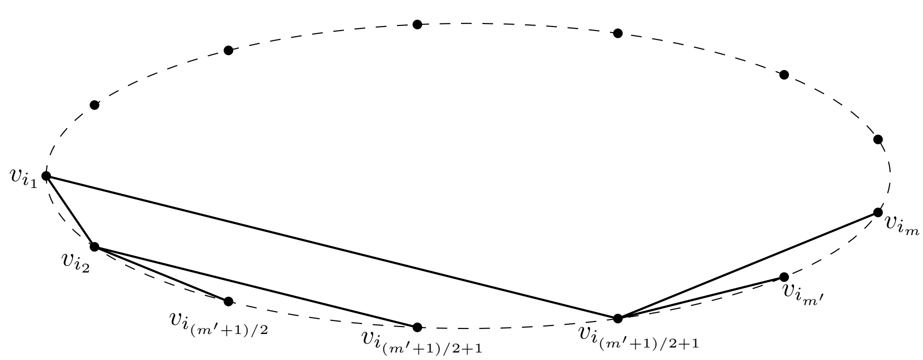


Figure 3.2: A non crossing binary tree in H.

Theorem 3.4. $\overline{\tau}(K_n) = \Omega(\log\log\log n)$.

Proof. Let D be a straight-line drawing of K_n. By the Erdős-Szekeres Theorem, there exists a set of $\Omega(\log n)$ vertices of D in convex position; the bound follows from Theorem 3.3. □

When we search for monotone paths, there is no need to distinguish between increasing and decreasing paths—traversing an increasing path in opposite direction gives us a decreasing path of the same length and vice versa. This is not the case with binary trees. Theorem 3.4 guarantees that we can find a monotone binary tree of size $\Omega(\log\log\log n)$ in any edge-ordered straight-line drawing of K_n, which can be increasing or decreasing.

An upper bound on $\overline{\tau}(K_n)$ must take into account both increasing and decreasing binary trees. We start by bounding $\overline{\tau}_+(K_n)$ and $\overline{\tau}_-(K_n)$.

Theorem 3.5. *$\overline{\tau}_+(K_n) = O(\log n)$ and $\overline{\tau}_-(K_n) = O(\log n)$.*

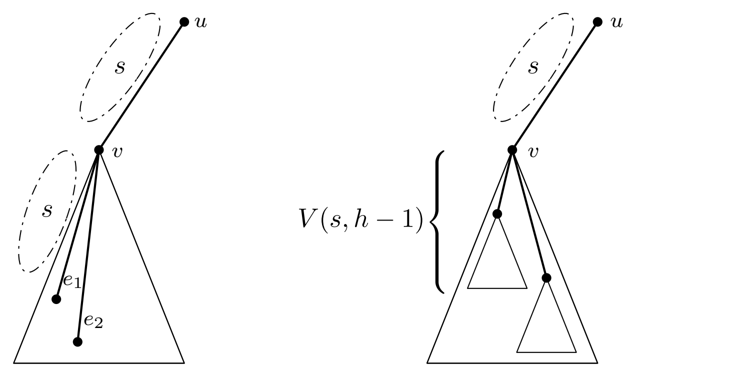
Proof. Let D be a convex straight-line drawing of K_n. We give an edge-ordering of D such that largest non-crossing increasing binary tree has size $O(\log n)$. The proof that $\tau_-(D) = O(\log n)$ is analogous.

The supporting line of every edge $e = (uv)$ of D partitions the vertices of $D \setminus \{u, v\}$ into two sets, let S_e be the smaller one. Construct an edge-ordering of D such that $e < e'$ if $|S_e| < |S_{e'}|$. Let T be a largest non-crossing increasing binary tree in D and denote its root by r.

Let L and R be the convex hulls of the left and right subtrees of T, respectively. We claim that L and R are disjoint. Suppose this is not the case. Order the vertices of K_n counterclockwise starting with r and let $l', v_1, \dots, v_s, r'$ be the vertices of $L \cup R$ as they appear in this order. Let e be the left edge of r. No edge of T can have an endpoint in S_e: any such edge f must intersect e or have both endpoints in S_e, in which case f belongs to the left subtree of T; this is impossible since $|S_f| < |S_e|$. Thus, l' is the root of the left subtree of T and by an analogous argument r' is the root of the right subtree of T. There must be a vertex of R followed by a vertex of L in this order, otherwise L and R are disjoint; let v_i be the first such vertex of R in this order. In the path that joins v_i with r' in T, there exists at least one edge v_jv_k with $j \leq i < k$. The vertices l' and v_{i+1} lie on different sides of the supporting line of v_jv_k, so the path that joins them in T must intersect v_jv_k. This is a contradiction, since T is non-crossing.

Let ℓ be a straight line through r that separates L and R. The line ℓ partitions the vertices of $D \setminus r$ in two parts, one with less than $n/2$ vertices. Let S be this part and let T' be the subtree of T contained in S.

Let h' be the height of T'. Note that for every vertex v of T' with children u and w either the subtree T_u rooted at u is contained in $S_{(vw)}$ or the subtree T_w rooted at w is contained in $S_{(vu)}$. Assume without loss of generality that the first case happens.



$V(s,1) = 2s+4$ $\quad$ $V(s,h) = V(V(s,h-1)-1,h-1)+1$

Figure 3.3: $T_{s,1}$ and $T_{s,h}$.

Let $T_{s,h}$ denote an increasing tree with root u such that:

- The vertex u has only one child v and $|S_{(uv)}| = s$
- The vertex v is the root of a binary tree with height h

Let $V(s,h)$ denote the minimum number of vertices of S needed to embed $T_{s,h}$. If $h=1$, the complete binary tree rooted at v consists of two edges, e_1, e_2; furthermore, $|S_{(uv)}| = s$ implies that $|S_{e_1}| \geq s$ and $|S_{e_2}| \geq s+1$. Thus, $V(s,1) = 2s+4$. (See figure 3.3). Note that if $h > 1$, we need at least $V(s,h-1)$ vertices to embed the left subtree of v, which implies that we need $V(V(s,h-1)-1,h-1)$ vertices to embed the right subtree of v. Thus we obtain the following recurrence:

$$V(s,h) \geq V(V(s,h-1)-1,h-1)+s+1.$$

We show by induction on h that $V(s,h) \geq (s+1)2^{2^{h-1}}$. This certainly holds for the base case, assume it also holds for $h-1$. We have that:

$$\begin{aligned} V(s,h) &\geq V(V(s,h-1)-1,h-1) \\ &\geq V(s,h-1)\cdot 2^{2^{h-2}} \\ &\geq (s+1)\cdot 2^{2^{h-2}}\cdot 2^{2^{h-2}} \\ &\geq (s+1)2^{2^{h-1}} \end{aligned}$$

Note that $h' \geq V(0,h') = \Omega(2^{2^{h'-1}})$, which implies that $h' = O(\log\log n)$. The theorem follows. □

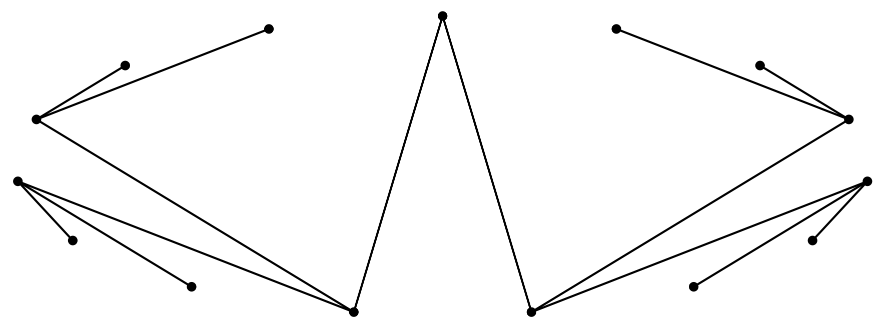


Figure 3.4: A decreasing binary tree of linear size under the edge-ordering used in Theorem 3.5.

The edge-ordering used in the proof of Theorem 3.5 forbids large increasing binary trees, but it is possible to find a decreasing binary tree of linear size (see Figure 3.4). The proof of Theorem 3.6 gives an edge-ordering that forbids both increasing and decreasing large binary trees.

Theorem 3.6. $\overline{\tau}(K_n) = O(\sqrt{n \log n})$.

Proof. Let D be a convex straight-line drawing of K_n. Let $v_1, \ldots, v_n$ denote the vertices of D in counterclockwise order. Let $m = \left\lceil \sqrt{n/\log n} \right\rceil$ and partition the vertices into groups $S_1, S_2, \ldots, S_m$ of consecutive vertices such that each one has size at most $\lceil \sqrt{n \log n} \rceil$. Order the edges that have endpoints within S_i using the same edge-ordering as in Theorem 3.5 so that the largest non-crossing decreasing binary tree contained in S_i has size $O(\log n)$. We refer to those edges as red edges and to the edges that have endpoints in different groups as blue edges. Order the blue edges by increasing slope. Furthermore, order the edges in such a way that every blue edge is greater than every red edge.

Let T be a decreasing binary tree with respect to this edge-ordering and let r be its root. Note that T consists of a possibly empty blue binary tree T_b and a forest of red complete binary trees such that the roots of the red trees are leaves of T_b (Figure 3.5). We claim that the subgraph $T_{i,j}$ of T_b induced by blue the edges between two different groups S_i and S_j has at most one connected component. Suppose for the sake of a contradiction that this does not happen. Choose two connected components such that r is in at most one of them, let e be an edge from the first component and f an edge from the

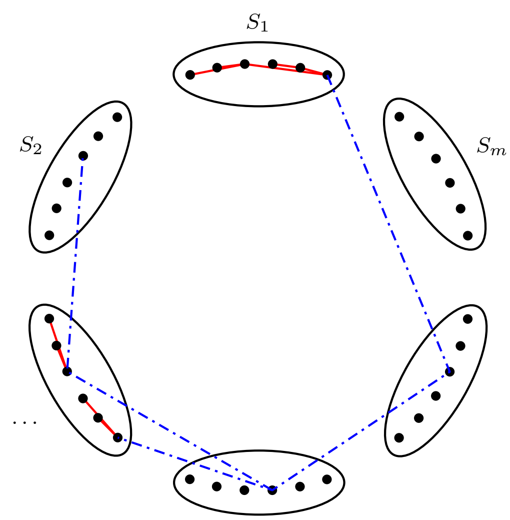


Figure 3.5: A decreasing binary tree with respect to the edge-ordering of Theorem 3.6.

second component. Suppose that one of the support lines (say, the support line of e) leaves f and r in different semiplanes. Then, since the vertices of T_b are in convex position, the path from any endpoint of f to r must intersect e. Therefore, r lies between the support lines of e and f. Without loss of generality, r is not in the connected component where e belongs. Any path from r to an endpoint of e must go first through a vertex in some S_k, where $k \neq i, j$. This path intersects e or f, a contradiction.

Since $T_{i,j}$ has at most one connected component (which by Observation 6 is a tree of height at most two), its number of edges is at most a constant. Let $u_1, \ldots, u_m$ be vertices on a circle ordered counterclockwise. Add an edge between u_i and u_j if and only if there is an edge in T_b between S_i and S_j. Since T_b is non-crossing, the resulting graph is also non-crossing and has $O(\sqrt{n/\log n})$ edges, which implies that T_b has $O(\sqrt{n/\log n})$ edges. There are $O(\sqrt{n/\log n})$ leaves in T_b, thus there are $O(\sqrt{n/\log n})$ red trees and by Theorem 3.5 each has size $O(\log n)$. Thus, T has size $O(\sqrt{n \log n})$.

Now consider an increasing complete binary tree T. Note that T consists of a possibly empty red binary tree T_r and a forest of blue binary trees such that the roots of the blue trees are leaves of T_r (Figure 3.6). The tree T_r is contained in some S_i, thus it has size $O(\sqrt{n \log n})$. Identify all the roots of the blue trees with a vertex of S_i–this produces a non crossing blue tree. There are $O(\sqrt{n \log n})$ blue edges with an endpoint in S_i. We can bound the number of remaining blue edges as before by $O(\sqrt{n/\log n})$. Therefore, T has

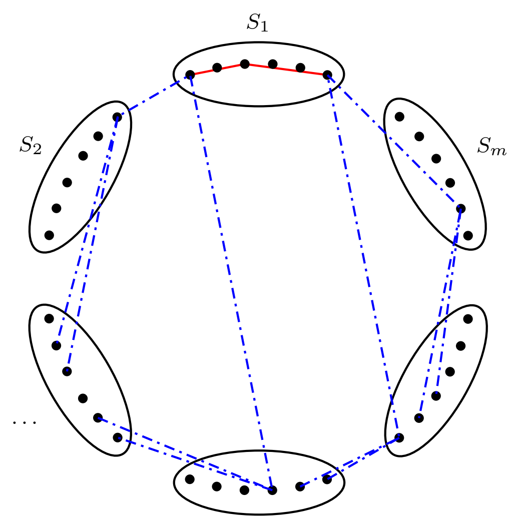


Figure 3.6: An increasing complete binary tree with respect to the edge-ordering of Theorem 3.6.

size $O(\sqrt{n \log n})$. □

3.2 The Flip Graph of Hamiltonian Paths on the Grid

Let m and n be two positive integers, the grid graph $G_{n,m}$ is the graph whose vertex set is the set of points in the plane given by

$$\{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

Two vertices (i,j) and (k,l) of $G_{n,m}$ are adjacent if $i = k$ and $|j - l| = 1$, or $j = l$ and $|i - k| = 1$. Let $H_{n,m}$ be the graph whose vertices are all the Hamiltonian paths of $G_{n,m}$, where two such paths P and Q are adjacent if there exist edges $e \in P$ and $f \notin P$ such that $Q = P - e + f$. We call this graph the *flip graph of Hamiltonian paths of* $G_{n,m}$; the operation of replacing e with f in P is called a *flip*. In this chapter we study the following problem, proposed by Hurtado [56]:

Problem 2. *For which values of n and m is $H_{n,m}$ connected?*

Hamiltonian paths in grid graphs have been studied from different points of view. Itai, Papadimitriou and Szwarcfiter [57] gave necessary and sufficient conditions for a Hamiltonian path to exist between two vertices of $G_{n,m}$. They also showed that the problem of finding a Hamiltonian path (or cycle) between two vertices is NP-complete. Collins and Krompart [5] gave generating functions for counting the number of Hamiltonian paths in $G_{n,m}$ that start in the lower left corner and end in the upper right corner, for $1 \leq m \leq 5$. Jacobsen [58] presented an algorithm for enumerating all the Hamiltonian paths in $G_{n,m}$, and calculated the number of distinct Hamiltonian paths in $G_{n,n}$ for $1 \leq n \leq 17$.

One motivation for studying the connectedness of $H_{n,m}$ is generating random Hamiltonian cycles in $G_{n,m}$: if $H_{n,m}$ is connected, we choose a random Hamiltonian path and then can perform a random walk until we reach a Hamiltonian path whose endpoints are adjacent in $G_{n,m}$. Any such path can be completed into a Hamiltonian cycle.

In this section we show that $H_{n,2}$, $H_{n,3}$ and $H_{n,4}$ are connected.

3.2.1 Preliminaries

We start by introducing some notation. Given a vertex v in $G_{n,m}$, we denote by v_x and v_y its x and y coordinates, respectively. Let R be the boundary rectangle of $G_{n,m}$. That is, R is the subgraph of $G_{n,m}$ induced by the vertices of degree at most three. Consider the subgraphs of R induced by the following sets of vertices:

$$\{(i, 1) : 1 \leq i \leq n\};$$

$$\{(i, m) : 1 \leq i \leq n\};$$

$$\{(1, j) : 1 \leq j \leq m\} \text{ and}$$

$$\{(m, j) : 1 \leq j \leq m\}.$$

We refer to them as the bottom, top, left and right sides of R, respectively.

Given a subgraph H of $G_{n,m}$ with the same vertex set as $G_{n,m}$, denote by $H_{\leftarrow}[i]$ the subgraph of H induced by the vertices with x-coordinate at least i for $1 \leq i \leq n$. Analogously, denote by $H_{\rightarrow}[i]$ the subgraph of H induced by the vertices with x-coordinate at most i for $1 \leq i \leq n$. Let F be another subgraph of $G_{n,m}$ with the same vertex set as $G_{n,m}$. If $H_{\leftarrow}[n] = F_{\leftarrow}[n]$, denote by $\text{same}_{\leftarrow}(H, F)$ the minimum integer i in $[1, n]$ such that $H_{\leftarrow}[i] = F_{\leftarrow}[i]$. If $H_{\rightarrow}[1] = F_{\rightarrow}[1]$, denote by $\text{same}_{\rightarrow}(H, F)$ the maximum integer i in $[1, n]$ such that $H_{\rightarrow}[i] = F_{\rightarrow}[i]$.

A *cyclic Hamiltonian path* of $G_{n,m}$ is a Hamiltonian path of $G_{n,m}$ such that its endpoints are adjacent. Note that adding the edge joining the endpoints of a cyclic Hamiltonian path produces a Hamiltonian cycle of $G_{n,m}$. Given a cyclic Hamiltonian path P, we denote by P_c the associated Hamiltonian cycle.

In the following lemmas it is convenient to assume that the Hamiltonian paths are oriented from one of its endpoints to the other. We refer to them as the first and last vertices, respectively. In what follows let P be such a Hamiltonian path and let s and t be its first and last vertices, respectively.

Lemma 3.1. *Suppose that both endpoints of P lie on R. Let ℓ be a directed vertical line in the plane and let $e_1, \dots, e_k$ be the horizontal edges of P that intersect ℓ, in their order of intersection with ℓ. Then for all $1 \leq i \leq k-1$, the edges e_i and e_{i+1} have opposite directions.*

Proof. Suppose for a contradiction that there exists an index i such that e_i and e_{i+1} have the same direction. Without loss of generality assume that this direction is left-to-right and that the right endpoint of e_i precedes the left endpoint of e_{i+1} in P. Let Q be the subpath of P from the right endpoint of e_i to the left endpoint fo e_{i+1}. Then either the left endpoint of e_i or the right endpoint of e_{i+1} lies in a connected region of the plane bounded by $Q \cup \ell$; this contradicts that s and t lie on R. □

The following observation describes the action of a flip on P, when the new edge added to P is different from (s,t).

Observation 7. *Let $e := (t,x)$ be and edge not in P such that $x \neq s$. Let Q be the Hamiltonian path obtained from P by a flip that adds e. Then the edge removed in the flip is the directed edge (x,y). Assuming that s is the first vertex of Q, then the last vertex of Q is y.*

Given two oriented Hamiltonian paths P and Q of $G_{n,m}$ we say that the endpoints of Q are *monotonically to the right* of P if the three following properties hold.

- The first vertex of Q is not to the left of the first vertex of P
- The last vertex of Q is not to the left of the last vertex of P
- Either the first vertex of Q is to the right of the first vertex of P or the last vertex of Q is to the right of the last vertex of P.

Lemma 3.2. *Suppose that t is not on the right side of R and that the right edge of t is not in P. Then P is adjacent in $H_{n,m}$ to a Hamiltonian path Q whose endpoints are monotonically to the right of P.*

Proof sketch. Let f be the right edge of t and let $Q := P - e + f$ be a Hamiltonian path adjacent to P for some edge e of P. The lemma can be verified by considering possible positions of e. $\square$

3.2.2 $H_{n,2}$ and $H_{n,3}$ are Connected

In this section we prove that $H_{n,2}$ and $H_{n,3}$ are connected for all positive integers n.

Theorem 3.7. *$H_{n,2}$ is connected.*

Proof. Let P be a Hamiltonian path in $G_{n,2}$. Note that all cyclic Hamiltonian paths in $G_{n,2}$ are adjacent. We prove that $H_{n,2}$ is connected by showing that there exists a sequence $Q_1, Q_2, \dots, Q_k$ of Hamiltonian paths of $G_{n,2}$ such that $Q_1 = P$, Q_k is cyclic and Q_i is adjacent to Q_{i+1} in $H_{n,2}$ for all $1 \leq i \leq k-1$.

Let $Q_1 = P$ and suppose that Q_i has been defined. If the endpoints s and t of Q_i have the same x-coordinate then either both are on the left side of R or both are on the right side of R. In both cases Q_i is cyclic. In this case set $Q_k = Q_i$ and we are done.

Suppose without loss of generality that s is to the right of t. Note that the right edge of t is not contained in Q_i; as otherwise, s cannot be to the right of t. Apply Lemma 3.2 to obtain a path Q_{i+1} adjacent to Q_i such that its last vertex is to the right of t and its first vertex is not to the left of s. Note that since the endpoints of the Q_i move monotonically to the right, then eventually both have the same x-coordinate; the result follows. $\square$

Lemma 3.3. *Every Hamiltonian path P of $G_{n,3}$ is connected in $H_{n,3}$ to a Hamiltonian path whose endpoints are on the right side of R.*

Proof. Let s and t be first and last vertices of P, respectively. Assume that at least one endpoint of P is not on the right side of R, as otherwise we are done. We prove the lemma by showing that P is adjacent in $H_{n,3}$ to a Hamiltonian path Q such that the endpoints of Q are monotonically to the right of the endpoints of P.

First suppose an endpoint of P lies in the middle row (it has y-coordinate equal to 2). Without loss of generality suppose that this endpoint is t. Note that t cannot be on the right side of R. If the right edge of t is not in P, then we apply Lemma 3.2 to obtain Q. Assume that the right edge of t is in P and let e be this edge. Let f and g be the edges above and below e in $G_{n,3}$; we claim that one of them appears in P with opposite orientation than e. At least one of f and G must be in P, suppose without loss of generality that it is f. If f has the same orientation than e note that the subpath of P joining f with e must use g, which implies that this edge has opposite direction than e. In any case, we can perform a flip that removes the edge with opposite orientation from P to obtain Q. By Observation 7 the endpoints of Q are monotonically to the right of the endpoints of P.

Now suppose that no endpoint of P lies in the middle row. We may assume that P contains the right edges of both s and t, as otherwise we apply Lemma 3.2 to obtain Q. This implies that s and t cannot have the same y-coordinate. Assume without loss of generality that t is to the left of s and that t lies on the bottom side of R. Let e be the edge of P incident to t and let ℓ be the vertical line through the midpoint of e. Note that since s is to the right of t, P must cross ℓ three times. Therefore, the horizontal edge f above e is in P. By Lemma 3.1 f is directed from left-to-right. Let Q be the Hamiltonian path obtained after performing the flip that removes f; by Observation 7 the endpoints of Q are monotonically to the right of the endpoints of P. The result follows. $\square$

Lemma 3.4. *Let P be a Hamiltonian path of $G_{n,3}$ whose endpoints lie on the right side of R. Then P contains the following sets of edges.*

- $L := \{((1,1),(1,2)),((1,2),(1,3))\}$;
- $B := \{((2i-1,1),(2i,1)) : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$;
- $M := \{((2i,2),(2i+1,2)) : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$; *and*
- $T := \{((2i-1,3),(2i,3)) : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

Proof. Since the endpoints of P are on the right side of R, then P must contain the edges of L. Otherwise, and endpoint of P would be on the left side of R.

For every integer $1 \leq j \leq \lfloor n/2 \rfloor$ let

- $B_j := \{((2i-1,1),(2i,1)) : 1 \leq i \leq j\}$,
- $M_j := \{((2i,2),(2i+1,2)) : 1 \leq i \leq j\}$ and
- $T_j := \{((2i-1,3),(2i,3)) : 1 \leq i \leq j\}$.

We prove by induction on j that P contains B_j, M_j and T_j. Note that since the endpoints of P are on the right side of R, P contains B_1 and T_1.

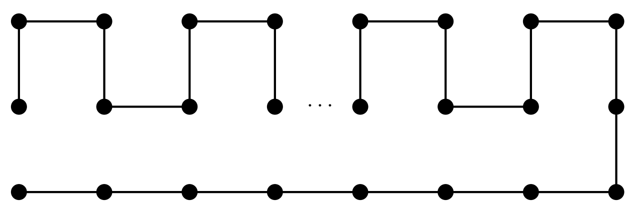


Figure 3.7: The Hamiltonian path Q in Theorem 3.8.

Furthermore, if P does not contain the edge $((2,2),(3,2))$, then $(2,2)$ is an endpoint of P. Therefore, P contains $M_1 = \{((2,2),(3,2))\}$.

Suppose that $1 < j \leq \lfloor n/2 \rfloor$ and that P contains B_i, M_i and T_i for every $1 \leq i < j$. Suppose for a contradiction that P does not contain the last edge $((2j-1,3),(2j,3))$ of T_j. The vertex $(2j-1,3)$ has degree 2 in P, thus its left and down edges are contained in P. Note that since the endpoints of P lie on the right side of R, P does not contain the edge $((2(j-1)-1,1),(2(j-1),1))$. This implies that the edge $((2(j-1),1),(2(j-1),2))$ is in P; thus P cannot pass through any vertex with x-coordinate greater than $2j-1$, a contradiction. Therefore P contains T_j and similar arguments P contains B_j. This implies that P must contain the edge $((2j,2),(2j+1,2))$ and P contains M_j; the result follows. □

Theorem 3.8. *$H_{n,3}$ is connected.*

Proof. Let P be a Hamiltonian path in $G_{n,3}$. By Lemma 3.3, we can assume that the endpoints of P lie on the right side of R. Let Q be the Hamiltonian path in $H_{n,3}$ with set of edges $B \cup T \cup M \cup V \cup ((n,1),(n,2))$, where

$$\begin{aligned}
B &= \{((i,1),(i+1,1)) : 1 \leq i \leq n-1\} \\
T &= \{((n-2i+1,3),(n-2i+2,3)) : 1 \leq i \leq \lfloor n/2 \rfloor\} \\
M &= \{((n-2i,2),(n-2i+1,2)) : 1 \leq i \leq \lfloor n/2 \rfloor\} \\
V &= \{((i,2),(i,3)) : 1 \leq i \leq n\}
\end{aligned}$$

See Figure 3.7.

We prove that $H_{n,3}$ is connected by showing that there exists a sequence of paths $P = P_1, \ldots, P_k = Q$ in $G_{n,3}$ such that P_i and P_{i+1} are adjacent in $H_{n,3}$ and $\mathrm{same}_{\leftarrow}(P_i, Q) \geq \mathrm{same}_{\leftarrow}(P_{i+1}, Q)$ for every $1 \leq i \leq k-1$. We have two cases:

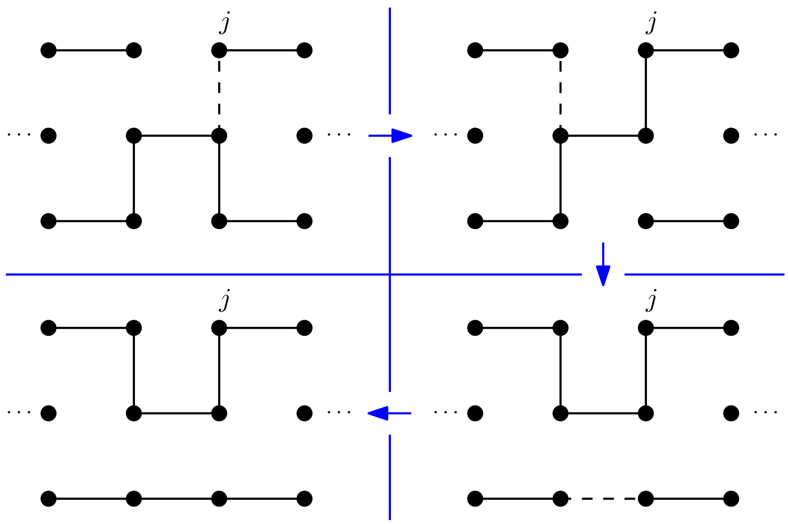


Figure 3.8: The flips for the even case. The flipped edge in each step is drawn with dashes.

- *n is even.* If n is even, P is a cyclic Hamiltonian path in $G_{n,3}$. By Lemma 3.4, the set of edges $D = \{P_c \setminus \{((1,1),(1,2))\}\} \setminus Q$ consists only of edges from T and V. Furthermore, D can be partitioned in sets of the form $\{((2i,3),(2i+1,3)),((2i,1),(2i,2)),((2i+1,1),(2i+1,2))\}$, where $1 \leq i \leq n$.

  Let $j := \text{same}_{\leftarrow}(P_c, Q)$ and let $P_2 := P_c \setminus \{((j-1,3),(j,3))\}$. Figure 3.8 shows a sequence of flips that leads us to a path P_6 such that $\text{same}_{\leftarrow}(P_6, Q) < j$. We can repeat this process as many times as necessary until we arrive at Q.

- *n is odd.* If n is odd, there are two possible configurations for the endpoints of P, depicted in Figure 3.9. The same figure shows a sequence of flips that transforms configuration a into configuration b, thus we can assume without loss of generality that the endpoints of P have configuration b. Furthermore, one more flip leads us to a path P_2 whose endpoints are $(n-1,2)$ and $(n,3)$, and such that $\text{same}_{\leftarrow}(P_2, Q) = n$. In general, given a path S such that $\text{same}_{\leftarrow}(S, Q) = n-i+1$ and whose endpoints have coordinates of the form $(n-i,2)$ and $(n-i+1,3)$, there are only two possible configurations for the edges whose endpoints have x-coordinates in $\{n-i-2, n-i-1\}$. Figure 3.10 shows sequences of flips for each configuration that take S to a path S' whose endpoints have coordinates $(n-i-2,2)$ and $(n-i-1,3)$. Furthermore, $\text{same}_{\leftarrow}(S', Q) = n-i-1$. Thus, applying these sequences of flips repeatedly yields a sequence of

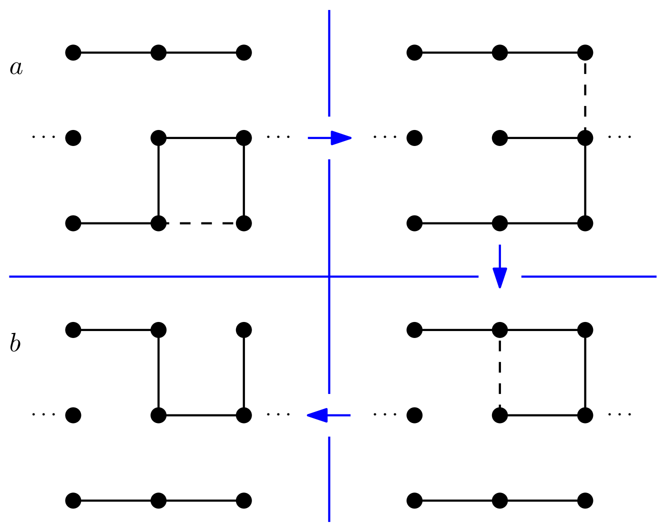


Figure 3.9: The two possible configurations for the endpoints in the odd case and the flips that take one into the other.

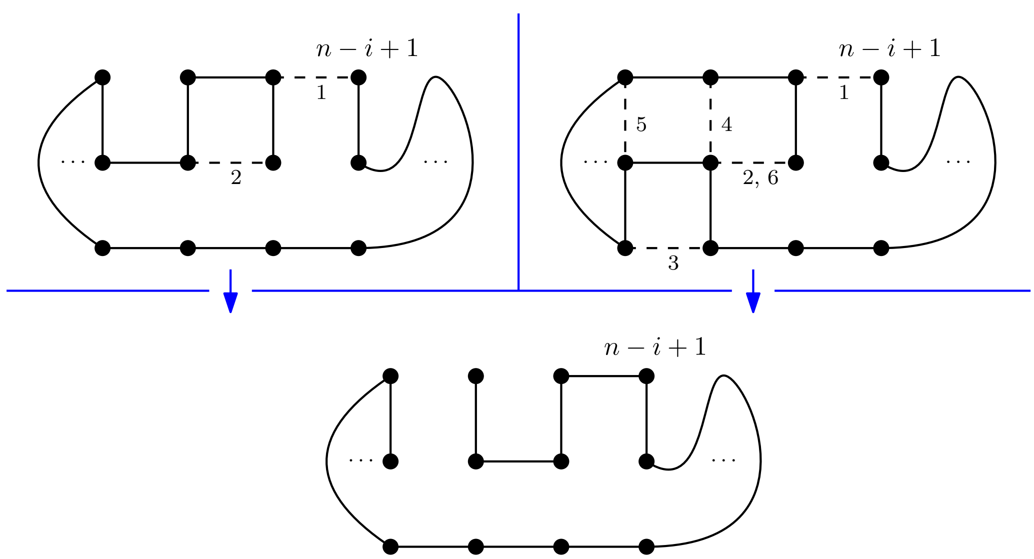


Figure 3.10: The flips for the odd case. The numbers indicate the order in which the edges are flipped.

paths that ends with Q. □

3.2.3 $H_{n,4}$ is Connected

In this section we prove that $H_{n,4}$ is connected. Throughout this section, assume $G_{n,4}$ has a 2-coloration of its vertices.

Observation 8. *Suppose that each vertex of $G_{n,4}$ is colored with one of two colors in such a way that adjacent vertices receive different colors. Then the endpoints of every Hamiltonian path of $G_{n,4}$ have different colors.*

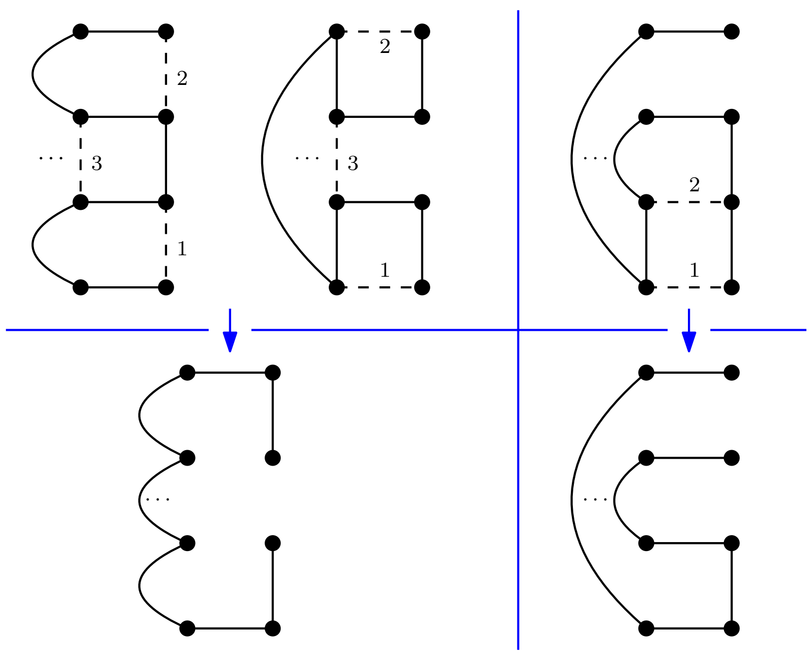


Figure 3.11: On top, the three possible configurations for the endpoints that do not correspond to a cyclic path. On the bottom, the cyclic paths that the indicated flips yield.

Lemma 3.5. *Let P be a Hamiltonian path of $G_{n,4}$. Then there exists a sequence $P = P_1, \ldots, P_t$ of Hamiltonian paths of $G_{n,4}$ such that for every $1 \leq i < t$ the following three properties hold.*

1) P_i and P_{i+1} are adjacent in $H_{n,4}$.

2) Either the first vertices of P_i and P_{i+1} are equal and the last vertex of P_{i+1} is to the right of the last vertex of P_i, or the last vertices of P_i and

P_{i+1} are equal and the first vertex of P_{i+1} is to the right of the first vertex of P_i.

3) Let j be the smallest x-coordinate of the endpoints of P_i and P_{i+1}. Then $\text{same}_{\rightarrow}(P_i, P_{i+1}) \geq j$.

Furthermore, P_t is cyclic and its endpoints lie on the right side of R.

Proof. We proceed by induction on i. Let $P_1 := P$ and assume that $i \geq 1$ and that $P_1, \ldots, P_i$ have been defined. We construct P_{i+1}. Let s and t be the first and last vertices of P_i, respectively. Assume that the right edges of both s and t are in P_i, otherwise Lemma 3.2 gives a path P_{i+1} that satisfies all three conditions. We consider the following cases:

- $s_x = t_x$
  Let k be the x-coordinate of s and t. If s and t are adjacent we are done: remove any edge from P_{ic} whose endpoint with smallest x-coordinate is greater than k to obtain P_{i+1}. Thus, assume that s and t lie on R and have y-coordinates 1 and 4, respectively, and let $u = (k, 2)$, $v = (k, 3)$. The right edges of u and v are in P_i. Moreover, the right edge of u has opposite direction than the right edge of s and the right edge of v has opposite direction than the right edge of t. Therefore we can perform a flip that deletes either the right edge of u or the right edge of v, which by Observation 7 gives us P_{i+1}.

From now on, we assume that $t_x < s_x$.

- $t \notin R$
  Assume without loss of generality that $t_y = 3$ and denote the right edge of t by e. The right edge of $(t_x, 4)$ is in P_i, if it has different orientation than e, we perform a flip where we add the edge $(t, (t_x, 4))$ and we are done. Suppose that it has the same orientation. If the right edge of $(t_x, 2)$ is in P_i, it has different orientation than e, thus we can perform a flip where we add the edge $(t, (t_x, 2))$ and we are done. Suppose the right edge of $(t_x, 2)$ does not belong to P_i. Then P_i contains the edges $((t_x - 1, 4), (t_x, 4))$ and $((t_x - 1, 2), (t_x, 2))$. The subgraph of P_i induced by the vertices with x-coordinate less than t_x is a Hamiltonian path on $G_{t_x-1,4}$ with endpoints $(t_x - 1, 4)$ and $(t_x - 1, 2)$. These vertices have the same color, a contradiction to Observation 8.

- $s \notin R$
  Assume without loss of generality that $s_y = 3$ and denote the right edge

of s by e. The right edge of $(s_x, 4)$ is in P_i, if it has different orientation than e, we perform a flip where we add the edge $(s, (s_x, 4))$ and we are done. Suppose that it has the same orientation as e and let ℓ be a vertical line that passes through the middle point of e. Since $t_x < s_x$, P_i must cross ℓ two times from right to left. This implies that the edge $((s_x, 2), (s_x + 1, 2)$ has different orientation than e. Thus we can make a flip where we add the edge $(s, (s_x, 2))$ to obtain P_{i+1}.

- $\{s, t\} \subset R$
  Assume without loss of generality that $t_y = 4$ and denote the right edge of t by e. If the edge $((t_x, 3), (t_x+1, 3))$ is in P_i, it has different orientation than e by Lemma 3.1, thus we can get P_{i+1} by performing a flip where we add the edge $(t, (t_x, 3)$. Assume $((t_x, 3), (t_x + 1, 3))$ is not in P_i and let ℓ be a vertical line that passes through the middle point of e. Since $t_x < s_x$, P_i must intersect ℓ two more times. The subgraph of P_i induced by the vertices with x-coordinate less than t_x is a Hamiltonian path on $G_{t_x-1,4}$ with endpoints $(t_x, 2)$ and $(t_x, 1)$. These vertices the same color, a contradiction to Observation 8.

It follows that P_t is a Hamiltonian path whose endpoints lie on the right side of R. There is only left to prove that P_t is cyclic. There are three possible configurations for the endpoints of P_t that do not correspond to a cyclic path, depicted in Figure 3.11. The same figure shows sequences of flips that take each of the configurations to the configuration of a cyclic path. $\square$

In the following, let Q be the Hamiltonian cycle in $G_{n,4}$ with set of edges $H_1 \cup H_2 \cup V_1 \cup V_2$, where

$$\begin{aligned}
H_1 &= \{((i, j), (i + 1, j)) : 1 \le i \le n - 2, 1 \le j \le 4\} \\
H_2 &= \{((n - 1, 1), (n, 1)), ((n - 1, 4), (n, 4))\} \\
V_1 &= \{((n, j), (n, j + 1)) : 1 \le j \le 3\} \\
V_2 &= \{((1, 1), (1, 2)), ((1, 3), (1, 4)), ((n - 1, 2), (n - 1, 3))\}
\end{aligned}$$

See Figure 3.12.

Lemma 3.6. *Let S be a cyclic Hamiltonian path in $G_{n,4}$ whose endpoints lie on the right side of R. Then S is connected in $H_{n,4}$ to a cyclic Hamiltonian path S' whose endpoints lie on the right side of R such that:*

- *If* $S_{c\to}[1] \neq Q[1]$, *then* $\mathrm{same}_{\to}(S'_c, Q) \ge 1$

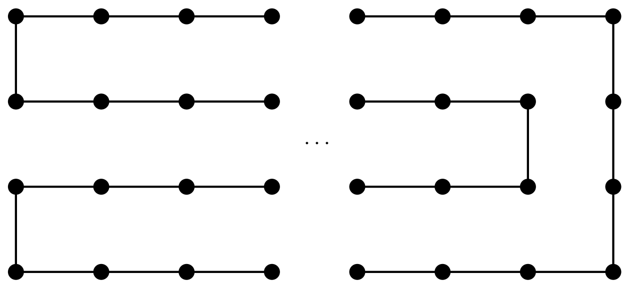


Figure 3.12: The Hamiltonian path Q in Theorem 3.6.

- *If* $S_{c\rightarrow}[1] = Q_{\rightarrow}[1]$, *then* $\mathrm{same}_{\rightarrow}(S'_c, Q) \geq \mathrm{same}_{\rightarrow}(S_c, Q)$

Proof. If $S_{c\rightarrow}[1] \neq Q_{\rightarrow}[1]$, S_c must contain the left side of R. Let $S_1 = S_c \setminus \{((1,2),(1,3))\}$. Figure 3.13 shows two flips that transform S_1 into a Hamiltonian path S_3 such that $\mathrm{same}_{\leftarrow}(S_3, Q) \geq 1$. By Lemma 3.5, we can transform S_3 into a cyclic Hamiltonian path S' such that $\mathrm{same}_{\rightarrow}(S'_c, Q) \geq 1$.

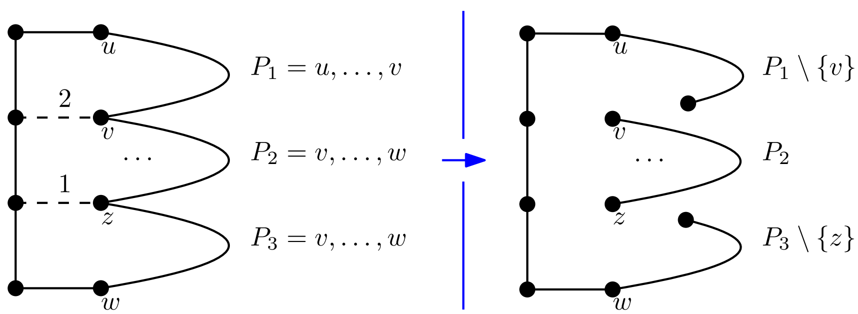


Figure 3.13: The flips that take S to S_3 when $S_c[1] \neq Q[1]$.

Suppose that $S_{c\rightarrow}[1] = Q[1]$ and let $m := \mathrm{same}_{\rightarrow}(S_c, Q)$. Let H be the set of horizontal edges of S whose left endpoint has x-coordinate m. We claim that H consists exactly of two edges of R. There cannot be 4 edges in H, otherwise m would be bigger. Furthermore, since the endpoints of S lie on the right side of R, there cannot be 3 edges in H. Thus, there are exactly two edges in H. The y-coordinates of the edges in H cannot be $\{2,3\}$, as such a configuration cannot be completed to a Hamiltonian path which is cyclic (see Figure 3.14a). Similarly, the y-coordinates cannot be $\{1,2\}$ or $\{3,4\}$, as this configuration cannot be completed to a path (see Figure 3.14b).

The positions of the edges in H imply that $((m,2),(m,3)) \in S$. Let $S_1 = S_c \setminus \{((m,2),(m,3))\}$. Figure 3.15 shows two flips that transform S_1 into a Hamiltonian path S_3 such that $\mathrm{same}_{\rightarrow}(S_3, Q) > m$. By Lemma 3.5, we can

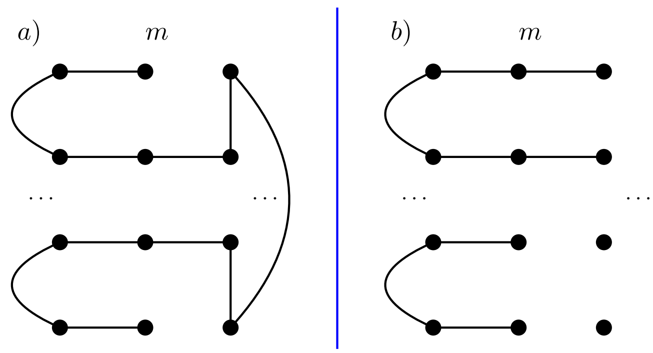


Figure 3.14: Forbidden configurations of horizontal edges.

transform S_3 into a cyclic Hamiltonian path S' such that $\text{same}_{\rightarrow}({S'}_c, Q) > m$. $\square$

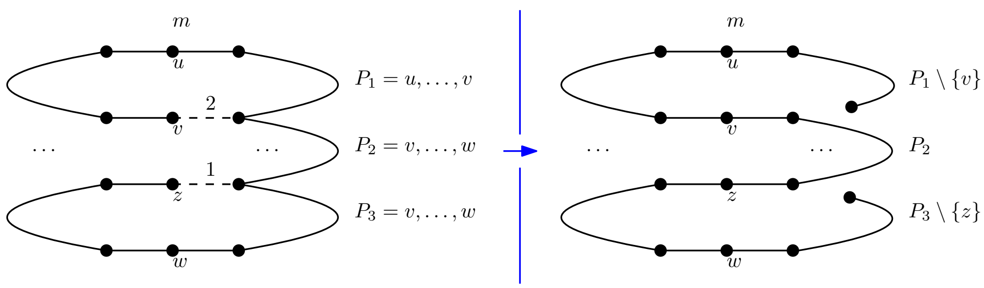


Figure 3.15: The flips that take S to S_3 when $S_{c\rightarrow}[1] = Q_{\rightarrow}[1]$.

Theorem 3.9. *$H_{n,4}$ is connected.*

Proof. Let P be a Hamiltonian path in $G_{n,4}$. By Lemma 3.5, we can assume that the endpoints of P lie on the right side of R. The result follows from a repeated application of Lemma 3.6. $\square$

3.3 Optimal Grid Drawings of Complete Multipartite Graphs

Let r, d be positive integers. Let $n_1 \leq \cdots \leq n_r$ be positive integers such that $\sum n_i = (2M+1)^d$ for some integer M. We consider straight-line drawings of the complete r-partite graph $K_{n_1,\ldots,n_r}$ into the d-dimensional integer grid

$$P := \Big\{(x_1,\ldots,x_d) \in \mathbb{Z}^d : -M \le x_i \le M\Big\}.$$

No two vertices of the graph are drawn on the same grid point. Note that such a drawing corresponds to a coloring of the points of P with r colors, such that color i appears n_i times, for $i = 1,\ldots,r$. The goal is to find the assignment of colors to the points of P such that the sum of squared distances between points of different colors is (i) minimized or (ii) maximized. The motivation for this problem stems from the following relation between drawings of a graph and spectral theory:

Let $G = (V, E)$ be a graph with vertex set $V = \{1,\ldots,N\}$, and let $deg(i)$ denote the degree of vertex i. The *Laplacian* matrix of G is the $N \times N$ matrix, $L = L(G)$, whose entries are

$$L_{i,j} = \begin{cases} \deg(i), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } ij \in E, \\ 0, & \text{if } i \neq j \text{ and } ij \notin E. \end{cases}$$

Let $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_N(G)$ be the eigenvalues of L. The *algebraic connectivity* (also known as the Fiedler value [40]) of G is the value of $\lambda_2(G)$. It is related to many graph invariants (see [40]), and in particular to the size of the separator of a graph, giving rise to partitioning techniques using the associated eigenvector (see [81]). Spielman and Teng [81] proved the following lemma:

Lemma 3.7.

$$\lambda_2(G) = \min \frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2},$$

and

$$\lambda_N(G) = \max \frac{\sum_{ij\in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i\in V} \|\vec{v}_i\|^2},$$

where the minimum, respectively maximum, is taken over all tuples $(\vec{v}_1,\ldots,\vec{v}_N)$ of vectors $\vec{v}_i \in \mathbb{R}^d$ with $\sum_{i=1}^N v_i = \mathbf{0}$, and not all v_i are zero-vectors $\mathbf{0}$.

In fact, Spielman and Teng [81] proved the Embedding Lemma for $\lambda_2(G)$, but the result for $\lambda_N(G)$ follows by very similar arguments; when adapting the proof of [81] we have to replace the last inequality given there by the inequality $\sum_i x_i / \sum_i y_i \le \max_i \frac{x_i}{y_i}$, for $x_i, y_i > 0$.

Let $\mathbf{v} = (\vec{v}_1, \ldots, \vec{v}_N)$ be a tuple of positions defining a drawing of G (vertex i is placed at $\vec{v}_i$). Let

$$\lambda(\mathbf{v}) := \frac{\sum_{ij \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}. \tag{3.1}$$

Note that $\|\vec{v}_i - \vec{v}_j\|^2$ is equal to squared length of the edge ij in the drawing defined by $\mathbf{v}$. Lemma 3.7 provides a link between the algebraic connectivity of G and its straight line drawings. Clearly

$$\lambda_2(G) \leq \lambda(\mathbf{v}) \leq \lambda_N(G).$$

We remark that in dimension $d = 1$, optimal drawings $\mathbf{v}$ are eigenvectors of $L(G)$ and $\lambda(\mathbf{v})$ is the well known Rayleigh quotient.

In this section we study how well we can approximate $\lambda_2(G)$ and $\lambda_N(G)$ with drawings with certain restrictions. First, we restrict ourselves to drawings in which the vertices are placed at points with integer coordinates and no two vertices are placed at the same point. Since $\lambda(\alpha\mathbf{v}) = \lambda(\mathbf{v})$ for $\alpha \in \mathbb{R} \setminus \{0\}$, we have that $\lambda_2(G)$ and $\lambda_N(G)$ can be approximated arbitrarily closely with straight line drawings with integer coordinates of sufficiently large absolute value. We therefore bound the absolute value of such drawings and consider only drawings in the bounded d-dimensional integer grid P. Juvan and Mohar [60, 61] already studied drawings of graphs with integer coordinates for $d = 1$. More precisely, the authors consider the *minimum-p-sum*-problem: for $0 < p < \infty$, a graph G and a bijective mapping Ψ from V to $\{1, \ldots, N\}$, define $\sigma_p(G, \Psi) = \left(\sum_{uv \in E(G)} |\Psi(u) - \Psi(v)|^p\right)^{1/p}$, and for $p = \infty$, let $\sigma_p(G, \Psi) = \max_{uv \in E(G)} |\Psi(u) - \Psi(v)|$. The quantity $\sigma_p(G) = \min_\Psi \sigma_p(G, \Psi)$ (where the minimum is taken over all bijective mappings) is then called the minimum-p-sum of G, and if $p = \infty$, it is also called the bandwidth of G. In [61] relations between the min-p-sum and $\lambda_2(G)$ and $\lambda_N(G)$ are analyzed, and also polynomial-time approximations of the minimum-p-sum based on the drawing suggested by the eigenvector corresponding to $\lambda_2(G)$ are given. In [60] the minimum-p-sums and its relations to $\lambda_2(G)$ and $\lambda_N(G)$ are studied for the cases of random graphs, random regular graphs, and Kneser graphs. For a survey on the history of these problems, see [29] and [31].
The use of eigenvectors in graph drawing has been studied for instance in [62], and we also mention [73] as a recent work on spectral bisection.

In the following we characterize the optimal drawings $\mathbf{v}$ for complete multipartite graphs $K_{n_1,\ldots,n_r}$ which minimize/maximize $\lambda(\mathbf{v})$. The assumption

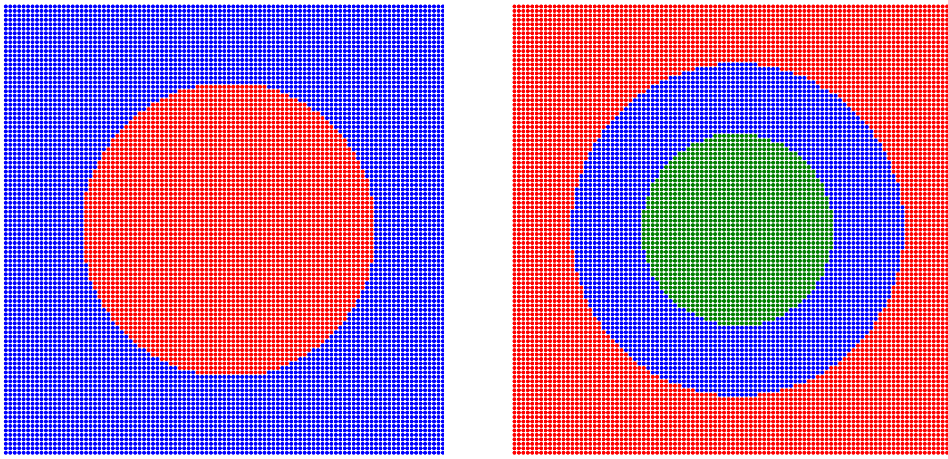


Figure 3.16: Approximations to optimal drawings that minimize the sum of squared distances between points of different colors on a 101×101 integer grid. Left: for $r = 2$ colors, with 1/3 of the points in red and 2/3 of the points in blue. Right: for $r = 3$ colors, with 4/7 red points, 2/7 blue points, and 1/7 green points.

$N = \sum_{i=1}^{r} n_i = (2M+1)^d$ made in the beginning is to ensure that every drawing satisfies the condition $\sum_{i=1}^{N} \vec{v}_i = \mathbf{0}$. The Laplacian eigenvalues of $K_{n_1,\ldots,n_r}$ are known to be, see [22],

$$0^1, (N-n_r)^{n_r-1}, (N-n_{r-1})^{n_{r-1}-1}, \ldots, (N-n_2)^{n_2-1}, (N-n_1)^{n_1-1}, N^{r-1}$$

where the superindexes denote the multiplicities of the eigenvalues. Therefore, $N - n_r \leq \lambda(\mathbf{v}) \leq N$.

Two examples of optimal drawings in dimension $d = 2$ which minimize $\lambda(\mathbf{v})$ are given in Figure 3.16. Figure 3.17 and Figure 3.18 show examples which maximize $\lambda(\mathbf{v})$. We mention that we obtained all these drawings with computer simulations, using simulated annealing. The solution for minimizing $\lambda(\mathbf{v})$ shown in Figure 3.16 consists of concentric rings and applies to the case when all color classes have different size. While this solution is unique, we will show that if the color classes have the same size, then there are exponentially many drawings that minimize $\lambda(\mathbf{v})$. In that case, the solutions are characterized as those drawings where for each color class all its points sum up to $\mathbf{0}$.

As can be observed in Figures 3.17 and Figure 3.18 (and proved in Section 3.3.2), the solution for maximizing $\lambda(\mathbf{v})$ is given by (weighted) centroidal Voronoi diagrams, which are related to clustering [35]. Let us give the defini-

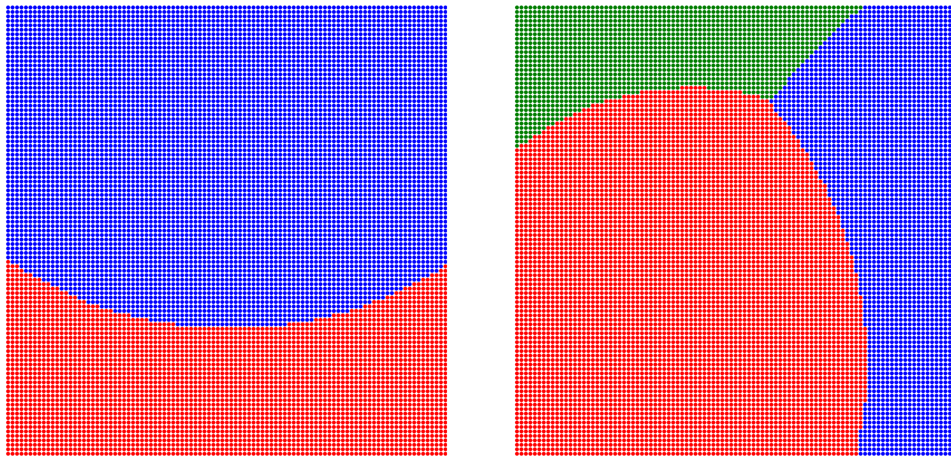


Figure 3.17: Approximations to optimal drawings that maximize the sum of squared distances between points of different colors on a 101×101 integer grid. Left: for $r = 2$ colors, with 2/3 of the points in blue and 1/3 of the points in red. Right: for $r = 3$ colors, with 4/7 red points, 2/7 blue points, and 1/7 green points.

tion of a centroidal Voronoi tessellation, according to [35]. Given an open set $\Omega \subseteq \mathbb{R}^d$, the set $\{V_i\}_{i=1}^r$ is called a *tessellation* of Ω if $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^r \overline{V_i} = \overline{\Omega}$. Given a set of points $\{c_i\}_{i=1}^r$ belonging to $\overline{\Omega}$, the Voronoi region $\hat{V}_i$ corresponding to the point c_i is defined by

$$\hat{V}_i = \{x \in \Omega \mid ||x - c_i|| < ||x - c_j|| \text{ for } j = 1, \dots, r, j \neq i\}.$$

The points $\{c_i\}_{i=1}^r$ are called *generators* or *sites*. The set $\{\hat{V}_i\}_{i=1}^r$ is a *Voronoi tessellation* or *Voronoi diagram*. A Voronoi diagram is *multiplicatively weighted*, see [16], if each generator c_i has an associated weight $w_i > 0$ and the weighted Voronoi region of c_i is

$$\hat{V}_i = \{x \in \Omega \mid ||x - c_i|| w_j < ||x - c_j|| w_i \text{ for } j = 1, \dots, r, j \neq i\}.$$

A Voronoi tessellation is *centroidal* if the generators are the centroids for each Voronoi region. Voronoi diagrams have also been defined for discrete sets P instead of regions Ω [35].

3.3.1 Optimal Drawings Minimizing $\lambda(\mathbf{v})$

In the following we give bounds on $\lambda(\mathbf{v})$. Note that in Equation (3.1), the term $\sum_{i \in V} \|\vec{v}_i\|^2$ is the same for all drawings on P. Let $S := \sum_{v \in P} \|v\|^2$. We first calculate the value of S which we need later on.

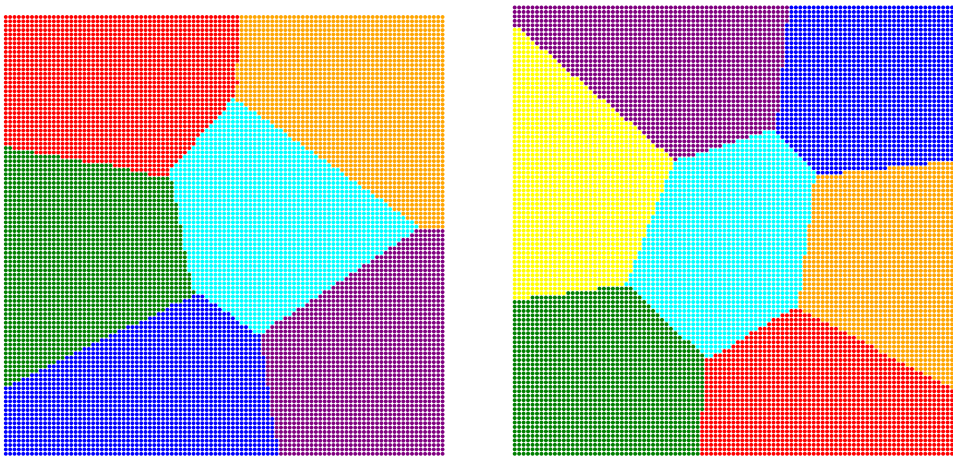


Figure 3.18: Approximations to optimal drawings that maximize the sum of squared distances between points of different colors on a 101×101 integer grid. Left: for $r = 6$ colors, with $1/6$ points of each color. Right: for $r = 7$ colors, with $1/7$ points of each color.

Proposition 3.1.

$$S = 2d(2M+1)^{d-1}\frac{M(M+1)(2M+1)}{6}.$$

Proof. We have that

$$S = \sum_{v \in P} \|v\|^2 = \sum_{(x_1,\dots,x_d) \in P} \left(x_1^2 + \cdots + x_d^2\right) = 2d(2M+1)^{d-1}\sum_{i=1}^{M} i^2.$$

The last equation comes from counting the number of appearances of the term i^2 in the left hand side of the equation. This is equivalent to counting the number of times i^2 is equal to the squared coordinate of a vector v of P. Such a coordinate may be a i or a $-i$, this gives a factor of two; it has d possibilities to appear as a coordinate of v, this gives a factor of d; once the sign and position are fixed, the other $(d-1)$ coordinates can take any one of $(2M+1)$ possible values; this gives a total of $2d(2M+1)^{d-1}$ vectors. Finally,

$$2d(2M+1)^{d-1}\sum_{i=1}^{M} i^2 = 2d(2M+1)^{d-1}\frac{M(M+1)(2M+1)}{6}.$$

□

Let $A_1, \dots, A_r$ be the partition classes of $K_{n_1,\dots,n_r}$ with $|A_i| = n_i$ (for $1 \le i \le r$). Let $N = (2M+1)^d = \sum_{i=1}^{r} n_i$. Let $\mathbf{v}$ be a fixed straight line drawing of $K_{n_1,\ldots,n_r}$. In what follows we abuse notation and say that a point $v \in P$ is in A_i if a vertex of A_i is mapped to v. We also use A_i to refer to the image of A_i under $\mathbf{v}$.

Let A and B be two finite subsets of $\mathbb{R}^d$. We define

$$A \cdot B := \sum_{\substack{v \in A \\ w \in B}} v \cdot w,$$

where $\cdot$ is the dot product. We will need the following property:

Proposition 3.2. *Let $A_1, \ldots, A_r$ be $r \geq 2$ finite subsets of $\mathbb{R}^d$ such that*

$$\sum_{i=1}^{r} \sum_{v \in A_i} v = \vec{0}.$$

Then

$$\sum_{i=1}^{r-1} \sum_{j=i+1}^{r} A_i \cdot A_j = -\frac{1}{2} \sum_{i=1}^{r} \left\| \sum_{v \in A_i} v \right\|^2 .$$

Proof. For the case $r = 2$,

$$\begin{aligned} A_1 \cdot A_2 &= \sum_{\substack{v \in A_1 \\ w \in A_2}} v \cdot w \\ &= \left(\sum_{v \in A_1} v \right) \cdot \left(\sum_{w \in A_2} w \right) \\ &= \left(\sum_{v \in A_1} v \right) \cdot \left(- \sum_{v \in A_1} v \right) \\ &= - \left\| \sum_{v \in A_1} v \right\|^2 \leq 0. \end{aligned}$$

In the same way,

$$A_1 \cdot A_2 = \left(\sum_{w \in A_2} w \right) \cdot \left(- \sum_{w \in A_2} w \right)$$

$$= -\left\| \sum_{w \in A_2} w \right\|^2 \leq 0.$$

Summing the two equations, the result follows for $r = 2$. Let then $r > 2$. For each $i \in \{1, \ldots, r\}$ we have

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq i}}^{r} A_i \cdot A_j &= \sum_{\substack{j=1 \\ j \neq i}}^{r} \sum_{\substack{v \in A_i \\ w \in A_j}} v \cdot w \\
&= \left(\sum_{v \in A_i} v \right) \cdot \left(\sum_{\substack{w \in \cup_{j=1}^{r} A_j \\ j \neq i}} w \right) \\
&= -\left\| \sum_{v \in A_i} v \right\|^2 \leq 0,
\end{aligned}$$

where we applied the result for $r = 2$. Then, when summing these r equations (summing over all i), each term $A_i \cdot A_j$ appears exactly twice in the sum. The result follows. □

Lemma 3.8. *Let* $\mathbf{v}$ *be a fixed straight line drawing of* $G = (V, E) = K_{n_1, \ldots, n_r}$. *Then*

$$\lambda(\mathbf{v}) = N + \frac{1}{S} \sum_{i=1}^{r} \left(-n_i \sum_{v \in A_i} \|v\|^2 \right) + \frac{1}{S} \sum_{i=1}^{r} \left\| \sum_{v \in A_i} v \right\|^2 .$$

Proof.

$$\lambda(\mathbf{v}) = \frac{1}{S} \sum_{(v,w) \in E} \|v - w\|^2 = \frac{1}{S} \sum_{(v,w) \in E} \left(\|v\|^2 + \|w\|^2 - 2v \cdot w \right).$$

Since in the complete multipartite graph each $v \in A_i$ is adjacent to all vertices but the n_i vertices of its class A_i, this further equals

$$\lambda(\mathbf{v}) = \frac{1}{S} \sum_{i=1}^{r} \left((N - n_i) \sum_{v \in A_i} \|v\|^2 \right) - \frac{2}{S} \sum_{i \neq j} A_i \cdot A_j$$

$$= N + \frac{1}{S}\sum_{i=1}^{r}\left(-n_i \sum_{v\in A_i}\|v\|^2\right) + \frac{1}{S}\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2 .$$

□

The following theorem provides best possible drawings whenever one can draw $K_{n_1,\ldots,n_r}$ on P such that for each class A_i we have $\sum_{v\in A_i} v = \vec{0}$. This can be achieved for instance if $|A_i|$ is even for all but one of the classes, and for each point $v \in A_i$ in the drawing, also $-v \in A_i$, and the remaining vertex is drawn at $\mathbf{0}$. If all $|A_i|$ are even, then the theorem also holds under the assumption that no vertex is drawn at $\mathbf{0}$ (recall that $|P|$ is odd). Otherwise, the best drawings are such that $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2$ is minimized, and the drawing in the second case of the theorem only gives an approximation.

Theorem 3.10. *Let* $\mathbf{v}$ *be a straight line drawing of* $K_{n_1,\ldots,n_r}$ *that minimizes* $\lambda(\mathbf{v})$. *If* $n_1 = n_2 = \ldots = n_r$, *then* $\mathbf{v}$ *minimizes* $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2$; *in particular, if* $\sum_{v\in A_i} v = \mathbf{0}$, *for all* $1 \le i \le r$, *then* $\lambda(\mathbf{v}) = N - n_r$. *If* $n_1 < n_2 < \ldots < n_r$, *then* $\mathbf{v}$ *has the following structure: For each* $i = 1, \ldots, r-1$, *the union of the smallest* i *color classes,* $\bigcup_{j=1}^{i} A_j$, *forms a ball centered at* $\mathbf{0}$.

Proof. Consider first the case when all classes A_i have the same number of points $n = n_i$. Take a drawing $\mathbf{v}$. By Lemma 3.8,

$$\begin{aligned}\lambda(\mathbf{v}) &= N + \frac{1}{S}\sum_{i=1}^{r}\left(-n_i \sum_{v\in A_i}\|v\|^2\right) + \frac{1}{S}\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2 \\ &= N - n + \frac{1}{S}\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2 .\end{aligned}$$

Then $\lambda(\mathbf{v})$ is minimized if $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2$ is minimized. If there are drawings $\mathbf{v}$ such that $\sum_{v\in A_i} v = \mathbf{0}$ for each class, then $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2 = 0$. Since the algebraic connectivity of $K_{n,n\ldots,n}$ (with $N = r\cdot n$) is $N-n$, such a drawing is best possible. Consider then the case $n_1 < n_2 < \ldots < n_r$. By Lemma 3.8, $\mathbf{v}$ minimizes $\lambda(\mathbf{v})$ if $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2$ is minimized and $\sum_{i=1}^{r}\left(n_i\sum_{v\in A_i}\|v\|^2\right)$ is as large as possible. Both conditions can be guaranteed at the same time. $\sum_{i=1}^{r}\left\|\sum_{v\in A_i} v\right\|^2$ can be kept small (or equal to 0) when drawing each A_i

in a symmetric way around the origin. The quantity $\sum_{i=1}^{r}\left(n_i \sum_{v\in A_i}\|v\|^2\right)$ is maximized when the smallest class A_1 is drawn such that $\sum_{v\in A_1}\|v\|^2$ is as small as possible, which is the case when the vertices of A_1 are drawn as close as possible to the origin; then, in an optimal drawing the vertices of the second smallest class A_2 are drawn as close as possible to the origin on grid points which are not occupied by A_1. In the same way, iteratively, for the i-th smallest class all grid points closest to the origin, that are not yet occupied by smaller classes, are selected. This results in a drawing with concentric rings around the origin. □

Remark 3.1. *If some of the classes have the same number of elements, then the optimal solutions are given by a combination of the two cases of Theorem 3.10. That is, several classes with the same number of elements can form one of the concentric rings in the drawing which satisfies $\sum_{v\in A_i} v = \mathbf{0}$ for all color classes.*

We show next that the number of optimal drawings of $K_{n_1,\dots,n_r}$ that minimize $\lambda(\mathbf{v})$ can be exponential if some classes have the same number of elements. For the sake of simplicity of the exposition, we show this only for the case $K_{1,2m,2m}$ and dimension $d = 1$. The argument can be adapted to the general case.

Proposition 3.3. *Let $d = 1$, and $P = \{-2m, -2m+1, \dots, 2m-1, 2m\}$. There exists a constant $c > 0$ such that the number $\mathcal{N}$ of straight line drawings $\mathbf{v}$ of $K_{1,2m,2m}$ on P which minimize $\lambda(\mathbf{v})$ satisfies $c16^m/m^5 < \mathcal{N} < 16^m$.*

Proof. Let A_1, A_2, A_3 be the classes of $K_{1,2m,2m}$, with $n_1 = 1$ and $n_2 = n_3 = 2m$. Theorem 3.10 characterizes the optimal drawings as all drawings that satisfy $\sum_{v\in A_i} v = 0$. Then the only vertex of class A_1 is drawn at position 0 in any optimal drawing. For the upper bound, the number of such drawings is at most $\binom{4m}{2m} < 16^m$, since there are at most $\binom{4m}{2m}$ choices for mapping the vertices of A_2 to $P\setminus\{0\}$, and then the positions of the vertices in A_3 are already determined. Regarding the lower bound, in order to have $\sum_{v\in A_2} v = 0$, we must have $\sum_{v\in A_2, v<0} -v = \sum_{v\in A_2, v>0} v$. We may thus consider only drawings with exactly m elements v of A_2 with $v > 0$. There are at most $\sum_{i=1}^{2m} i = 2m^2+m$ different sums that can be obtained by $\sum_{v\in A_2, v>0} v$, and the same holds for $\sum_{v\in A_2, v<0} -v$. Thus, one of these sums, call it s, appears in at least $\frac{\binom{2m}{m}}{2m^2+m}$ of all the drawings of $\{v \in A_2, v > 0\}$, and by symmetry, the same sum s appears also at least $\frac{\binom{2m}{m}}{2m^2+m}$ times when considering $\sum_{v\in A, v<0} -v$. Any drawing for which at the same time we have $\sum_{v\in A_2, v>0} v = s$ and $\sum_{v\in A_2, v<0} -v = s$ is an optimal drawing. There are at least $\left(\frac{\binom{2m}{m}}{2m^2+m}\right)^2 = \Omega\left(\frac{16^m}{m^5}\right)$ such drawings, where we use the asymptotic estimate $\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}}$. Hence the lower bound follows. $\square$

3.3.2 Optimal Drawings Maximizing $\lambda(\mathbf{v})$

We now study drawings of $K_{n_1,\ldots,n_r}$ that maximize $\lambda(\mathbf{v})$. The following solution as a Voronoi diagram has to be considered as an approximation, due to the discrete setting and due to the given bounding box. However, the bigger the numbers n_i, the better the approximation to the boundary curves between adjacent Voronoi regions.

Theorem 3.11. *Let $\mathbf{v}$ be a straight-line drawing of $K_{n_1,\ldots,n_r}$ on P that maximizes $\lambda(\mathbf{v})$. If $n_1 = n_2 = \ldots = n_r$, then $\mathbf{v}$ defines a centroidal Voronoi diagram. If the n_i are not all the same, then $\mathbf{v}$ defines a multiplicatively weighted centroidal Voronoi diagram.*

Proof. We make use of the following fact: let Q be an arbitrary set of n points $p_1, \ldots, p_n$ in $\mathbb{R}^d$. Let c be the centroid of Q, $c = \frac{1}{n}\sum_{i=1}^n p_i$. Then, see [15],

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} ||p_i - p_j||^2 = n\sum_{i=1}^{n} ||p_i - c||^2. \tag{3.2}$$

In the case of our theorem, let $\mathbf{v}$ be a drawing of $K_{n_1,\ldots,n_r}$ drawn on

$$P = \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : -M \le x_i \le M\right\}.$$

Denote by $c_{A_1}, \ldots, c_{A_r}$ the centroids of the classes $A_1, \ldots, A_r$, respectively. Then, from Equation (3.1) and $S = \sum_{v\in P} \|v\|^2$ we get

$$\begin{aligned}
\lambda(\mathbf{v})|S| &= \sum_{(v,w)\in E} \|v-w\|^2 \\
&= \sum_{i<j} \sum_{\substack{v\in A_i\\ w\in A_j}} ||v-w||^2 \\
&= \sum_{v,w\in P} ||v-w||^2 - \sum_{i=1}^{r} \sum_{v,w\in A_i} ||v-w||^2 \\
&= \sum_{v,w\in P} ||v-w||^2 - \sum_{i=1}^{r} n_i \sum_{v\in A_i} ||v-c_{A_i}||^2,
\end{aligned}$$

where in the last equation we use (3.2). The quantity $\sum_{v,w\in P}||v-w||^2$ is the same for each drawing of $K_{n_1,\ldots,n_r}$, and $\sum_{i=1}^{r} n_i \sum_{v\in A_i}||v-c_{A_i}||^2$ is minimized if for each class A_i, its vertices are drawn as close as possible to its centroid c_{A_i}. Then the union of the r regions defined by $A_1,\ldots,A_r$ forms a centroidal Voronoi tessellation, see [35]. Note that when the $n_i's$ are different, then this is a multiplicatively weighted Voronoi diagram, see [16]. $\square$

3.4 Conclusions and Open Problems

In this chapter we proved results related to drawings of graphs. In Section 3.1 we presented results on a geometric variation of the well known problem of estimating the altitude of a graph. We also introduced a related parameter, the size of the largest plane complete binary tree. Future work for the problems in this section includes:

1. Closing the gap between our bounds of $\overline{\alpha}(K_n)$ and $\overline{\tau}(K_n)$
2. Studying other graphs from this point of view, besides monotone paths and monotone complete binary trees

In Section 3.2 we studied the connectivity of the flip graph of Hamiltonian paths on the grid of size $n\times m$. We proved that this graph is connected for $m\in\{2,3,4\}$. It is still an open problem to decide if this graph is connected for every value of n and m. We conjecture that it is indeed connected.

Finally, in Section 3.3, we studied straight-line drawings of complete multipartite graphs, where the vertices of the drawings are restricted to points in an integer grid. These drawings are connected to the eigenvalues of the graphs they represent. We characterized the drawings that approximate the second smallest eigenvalue and the largest eigenvalue of the graph.

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