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"Switching control problems and reflected backward stochastic differential equations with càdlàg data"

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Abstract

This work is about the study of optimal control problems and dynamic games and its relation with backward stochastic differential equations (BSDE). We analyze the existence and uniqueness of solutions of several types of BSDE with a general degree of generality for the cases of zero, one and two barriers. We are going to base ourselves on these equations in order to analyze the existence of optimal values related to the aforementioned control and game problems.

The first part of this thesis surveys very well-known results on BSDE, Reflected BSDE (RBSDE) and systems of RBSDE with continuous barriers that later are applied to find optimality associated to optimal stopping times and switching control problems. Later, the second part constitutes the originality of this work. In this sense, we provide new results on the existence and uniqueness and/or characterizations of systems of interconnected RBSDE with either one and two barriers of càdlàg type (i.e., right continuous with left limits).

Our original contributions can be summarized as follows:

(1) We prove the existence and uniqueness of interconnected systems of RBSDE for one càdlàg barriers.

(2) Assuming the existence of a Markov process as a driver of the randomness of the model, we characterize the solutions of the systems above mentioned in item (1) as solutions of systems of interconnected PDE's with discontinuous obstacles. This connection is possible when the former system is evaluated at the initial condition.

(3) We ensure the existence of the interconnected PDE's of one discontinuous barrier in a weak viscosity sense.

(4) The optimality of a switching control problem with càdlàg switching cost is guaranteed through the theory of Snell envelopes.

(5) The above switching control problem can be also characterized as a solution of special cases of both systems of (RBSDEs and PDEs) with one càdlàg barrier above mentioned in (2).

(6) The same property given in (5) applies for switching games when the systems of (BRSDEs and PDEs) are càdlàg with double barriers.

Resumen

Esta tesis trata sobre el estudio de problemas de control óptimo y juegos dinámicos así como su relación con las ecuaciones diferenciales estocásticas hacia atrás (BSDE). Analizamos la existencia y unicidad de soluciones de varios tipos de BSDE especialmente aquellas que tienen asociadas una o dos restricciones o barreras. Usando estas ecuaciones analizamos la existencia de valores óptimos relacionados con los problemas de control y juegos antes mencionados.

La primera parte de esta tesis revisa resultados muy conocidos sobre BSDE, BSDE Reflejado (RBSDE) y sistemas de RBSDE con barreras continuas que luego se aplican para encontrar las optimalidades asociadas a tiempos de parada óptimos y problemas de control tipo switching. Posteriormente, la segunda parte constituye la originalidad de este trabajo. En este sentido, brindamos nuevos resultados sobre la existencia y singularidad y/o caracterizaciones de sistemas de RBSDE interconectados con una o dos barreras de tipo càdlàg (i.e., continuas por la derecha con límites por la izquierda).

Nuestras contribuciones originales se pueden resumir de la siguiente manera:

(1) Demostramos la existencia y unicidad para sistemas de ecuaciones interconectadas de RBSDE con una barrera tipo càdlàg.

(2) Suponiendo que la aleatoriedad del modelo proviene un proceso de Markov, caracterizamos las soluciones de los sistemas arriba mencionados en el ítem (1) como soluciones de sistemas de PDEs interconectadas y con obstáculos discontinuos. Esta conexión es posible cuando el sistema anterior se evalúa en la condición inicial.

(3) Aseguramos la existencia de las PDE interconectadas de una barrera discontinua en un sentido de viscosidad débil.

(4) A través de la teoría de las envolturas de Snell se proporciona un teorema de verificación para el problema de control tipo switching con costos càdlàg.

(5) El problema de control tipo switching también se puede caracterizar en terminos de las soluciones de casos especiales de ambos sistemas de (RBSDE y PDE) con una barrera càdl àg mencionada anteriormente en (2). item [(6)] La misma propiedad dada en (5) se aplica para los juegos tipo switching cuando los sistemas de (BRSDEs y PDE) son càdlàg con barreras dobles.

1

Introduction

We can say that the theory of optimal control is essentially a technique of mathematical modeling designed to optimize, along time, the employment of limited resources. This theory is quite vast and the advances on this interesting field have still successfully prevailed during decades due to its huge applicability in many fields of natural and social sciences, such as economy, engineering, finance, biology, among others. At the same time this theory has been also of great interest from the mathematical point of view as it studies a wide spectrum of topics among "pure" and "applied" mathematics. The classification of optimal control problems can be given according to the controlled system itself (for instance, we can have deterministic or stochastic systems) and/or by its performance criterion (for instance, finite- or infinite-horizon problems, with random horizon, or with a discounted or an ergodic criterion).

Among the family of optimal control problems we can highlight those whose control is applied on the discontinuities to the dynamic. A special type of these problems is the so-called *optimal multiple switching problems* consisting in configuring the state of system according to doing changes of *regimes* (a.k.a. *configurations*) allowed for the controller. The times on which these changes are triggered are also part of the control, so the controller needs to apply a sequence, say (τ_n, ξ_n) such that at time τ_n, he/she changes the state from the regime ξ_{n-1} to ξ_n, $n \geq 1$. The objective for the controller is to find an optimal sequence like the one above that optimizes a certain total payoff.

On the other hand, in a general setting, a *game* can be regarded as a mathematical model of conflict or bargaining between players (a.k.a. agents, or controllers). A first classification in game theory consists when a game is played either one or several times. As for the former type, players apply one decision only in order to optimize a single payoff and the game is over; this type of games is known in the literature as *static games*. On the other hand, when the game is sequentially played in time, it is repeated a finite or an infinite number of times. Sometimes these types of games involve a *state of the system* driven by a given *dynamics*, and thus, based on the state of the system, players apply decisions/actions from time to time in order to optimize a (more sophisticated) payoff function. This kind of games is well-known as *dynamic games*. Similar to the optimal control theory, dynamic games can be classified according to the dynamical system (for instance, we can have deterministic or stochastic systems) and/or by their performance criteria (for instance, with a deterministic finite- or infinite-horizon, with random horizon, of discounted or ergodic type, etc...). But also games can be classified according to their rules. Among this classification we have cooperative and noncooperative games; the latter category, can in turn be classified as zero and nonzero sum games. In this work we deal with a special case of dynamic games so-called zero-sum switching games that can somehow be regarded as a natural extension of the above mentioned switching control problems when the number or controllers are $n \geq 2$.

There are several methods to analyze optimality associated to the theory of both (1) switching control problems and (2) switching games. One of these methods consists to regard the optimal values of either problems (1) and (2) as solutions of systems of reflected backward stochastic differential equations (RBSDEs) with one barrier for the case (1), and the same type of systems but with two barriers (rather than one) for the case (2).

We can say that a system of RBSDEs is a generalization of the theory of backward stochastic differential equations (BSDEs) when there are more than one equation, interconnected in some way, and with one extra element that plays the role to not allow the process to exit from certain region. These types of systems will be defined with details in Chapters 4 and 5.

1.1 Related literature.

The class of switching control problems has been studied in the literature by several authors. For instance, Carmona and Ludkovski [9] study this kind of problems in order to find management optimal strategies with the purpose to release a power plant that converts natural gas into electricity and hence to sell this commodity in the market. Doucet and Ristic [18] apply the switching control theory to problems of target tracking that are commonly used in aerospace and electronic systems. Trigeorgis [51, 52] relates this type of problems to real option theory. Perhaps the most studied switching control problem is when only two-modes are considered. Several authors have put attention on this type of problems (see e.g., Brekke and Oksendal [5, 6], Hamadène and Jeanblanc [30], Duckworth and Zervos [19], among others).

During the last decade, the switching control problem has been extensively studied by several authors including [9, 10, 16, 30, 50, 35, 36], etc. (see also the references therein).

However all the aforementioned papers consider the cases where the switching costs are continuous. To the best of our knowledge the case where the switching costs are discontinuous has not been considered yet. This is one of the main objectives of this work. In summary, Chapter 4 (see also reference [29]) is somehow the extension of the references Djehiche et. al. [16] and Hamadène and Morlais [33] when the switching costs are of càdlàg type.

Switching games, on the other hand, have been studied in a Markovian framework in Djehiche et. al. [15] where the authors provide solutions of a system of partial differential equations (PDEs) with bilateral interconnected obstacles of min-max and max-min types. It is also justified how the solutions of such PDEs are related to the solution of a system of double RBSDE (DRBSDE) with bilateral interconnected barriers. The authors conclude the analysis by showing that the solution of these systems coincides with the value of the switching game in the case when the coefficient and the terminal conditions satisfy a certain separability condition. On the other hand, the case when the switching costs are continuous but their coefficients and terminal condition are not separated was tackled in Hamadène et. al. [32]. In fact, this last reference was our departure point for the development of Chapter 5.

1.2 Summary of the following chapters.

The material in this thesis is organized as follows. After this introductory part, Chapter 2 provides a series of properties related to the theory of BSDEs as well as some key miscellaneous results of probability theory. Namely, in Sections 2.1 and 2.2 there are established some known results on Snell envelopes, martingale theory and some famous inequalities. The concept of BSDE and more complex extensions such as reflected backward stochastic differential equations (RBSDE) and double RBSDE, are stated in Section 2.3. The key results on the existence and uniqueness of all these equations as well as their corresponding comparison theorems are also discussed with details, and finally, in order to illustrate the theory of BSDEs, we have included an example regarding to mathematical finance. Chapter 3 is devoted to the preliminary theory of our later original results. Firstly, we introduce the concept of a switching control problem with continuous costs. Later, in Section 3.2, and based on the theory of Snell envelopes, we provide a verification theorem that is very common in control theory. This theorem provides, among other things, the characterizations of switching optimal control strategies. We then present the existence and uniqueness of solutions for a system of RBSDEs with interconnected continuous barriers in Section 3.3, whereas in Section 3.4 we show that under a Markovian framework i.e. the dynamic of the system is also governed by a underlying diffusion process, our unique solution, obtained in the previous section provides a viscosity solution for a system of PDEs with interconnected obstacles. On the other hand, in Chapter 4, we present a similar theory than its predecessor chapter. The difference lies in that the switching costs associated to the switching control problem are not longer continuous but càdlàg with respect to the time variable. This new assumption produces substantial changes throughout the entire analysis due that we need to deal with weaker concepts. For instance, the related verification theorem is valid also in the context of ε-optimal control strategies. Another substantial change is the treatment on the existence and uniqueness of systems of RBSDEs with càdlàg barriers, since both the barriers and the generator of such system are assumed to be dependent on the unknown solutions, making the problem more general but at the same time more challenging; this analysis is covered in Section 4.3. The later part of this chapter, namely, Section 4.4, is similar to the material of Section 3.4, but in this new case, the solution of the systems of PDE's is not necessarily continuous. This setting forces us to deal with the concept of weak viscosity solutions. Existence of solutions of systems of PDEs with the aforementioned characteristics in this later sense was also proved within Section 4.4. In Chapter 5, we give a probabilistic verification theorem for an optimal switching game with càdlàg switching costs in a general setting. This theorem is formulated in terms of a solution of a system of DRBSDEs with càdlàg barriers. By the end of the chapter, we analyze two special cases that ensure the existence of solutions to this system, so through the use of our previously proved verification theorem, it will follow the existence of the value of the switching game associated to those instances. We conclude the thesis with Chapter 6 by presenting our general remarks and some possible extensions of our work.

1.3 Notation and terminology

Throughout this thesis we consider a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ on which a standard d-dimensional Brownian motion $B = (B_t)_{t\leq T}$ is defined, with $(\mathcal{F}_t)_{t\leq T}$ being the natural filtration of B_t which is completed with the $\mathbb{P}$-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{t\leq T}$ satisfies the usual conditions, i.e., it is right continuous and complete. Associated to $\mathbb{P}$, we denote by $\mathbb{E}$ its respective expectation.

Next let us consider the following elements:

- $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^l$, for some appropriate $l \in \mathbb{N}$.
- We denote by $\mathbf{1}_A$ the indicator function of a given set A
- Given $\theta \in [0,T]$, $L^2_l(\mathcal{F}_\theta)$ is the set of random variables ξ, $\mathcal{F}_\theta$-measurable, and $\mathbb{R}^l$-valued such that $\mathbb{E}\Big[|\xi|^2\Big] < \infty$.
- $\mathcal{P}$ denotes the σ-algebra on $[0,T] \times \Omega$ of $\mathbb{F}$-progressively measurable sets.
- $\mathcal{H}^{2,l}$ the set of $\mathcal{P}$-measurable processes $w = (w_t)_{t\leq T}$, $\mathbb{R}^l$-valued such that $\|w\|_{\mathcal{H}^{2,l}} := \mathbb{E}[\int_0^T |w_s|^2\, ds]^{\frac{1}{2}} < \infty$.
- $\mathcal{S}^{2,l}$ (resp. $\mathcal{S}^{2,l}_c$) stands for the set of $\mathcal{P}$-measurable, càdlàg (resp. continuous), $\mathbb{R}^l$-valued processes $w = (w_t)_{t\leq T}$ such that $\|w\|_{\mathcal{S}^{2,l}} = \{\mathbb{E}[\sup_{t\leq T} |w_t|^2]\}^{\frac{1}{2}} < \infty$.
- If $l = 1$, then we will simply write $\mathcal{H}^{2,1}$, $\mathcal{S}^{2,1}$, and $L^{2,1}$ by $\mathcal{H}^2$, $\mathcal{S}^2$, and L^2, respectively.
- A random variable τ defined on Ω and valued in $\mathbb{R}_+ \cup \{+\infty\}$ is called a stopping time with respect to the filtration $\mathbb{F}$, or simply an $\mathcal{F}_t$-stopping time, if for all $t \in \mathbb{R}_+$, $\{\omega|\tau(\omega) \leq t\} \in \mathcal{F}_t$. For a given stopping time τ, define $\mathcal{T}_\tau$ the set of all stopping times θ such that $\tau \leq \theta \leq T$, $\mathbb{P}$-a.s.
- $\mathcal{I} = \{1, \ldots, q\}$ denotes the set of indices so-called set of configurations (a.k.a. modes or regimes), while the notation $\mathcal{I}^{-i}$ means $\mathcal{I} - \{i\}$.
- The notation $D^2_{xx}\phi$ and $D_x\phi$ means the Hessian matrix and the gradient vector of the function ϕ, respectively.
- Given A and B given metric spaces, we denote by $C^{p,q}(A\times B, \mathbb{R})$ the set of continuous functions $f : A\times B \mapsto \mathbb{R}$ such that f is of class C^p on the interior of A, and of class C^q on the interior of B.

- Let $\Gamma^k = \{1, \ldots, m_k\}$, $k \in \{1, 2\}$, be a finite, discrete set representing the operating modes that player k can choose, while the notation $(\Gamma^k)^{-l}$ means $\Gamma^k - \{l\}$. Let $\Gamma = \Gamma^1 \times \Gamma^2$ denote the product space of operating modes.
- The notation $\mathbb{P}-$ a.s. means almost surely with respect to a (generic) probability measure $\mathbb{P}$.
- The space of function $C^{1,2}([0, T] \times \mathbb{R}^k)$ represents all the continuous functions $h : [0, T] \times \mathbb{R}^k \mapsto \mathbb{R}$, that are of class C^1 in $(0, T)$ and of class C^2 in $\mathbb{R}^k$.

2

Preliminaries

In this chapter we present a survey of some key results of Snell envelopes and backward stochastic differential equations (BSDEs) with some extensions. We also establish a well-known relation of BSDEs with the theory of partial differential equations (PDE's) when the randomness is carried out by means of a given Markov process. We illustrate the aforementioned theories through an application on mathematical finance.

The material of this chapter is thought to introduce the reader to this interesting area but at the same time most of the results posed in here are the basis of a more sophisticated theory provided in later chapters, which are in fact the novelty of this work.

2.1 Some results of stochastic analysis

In this section we extract useful miscellaneous results of probability theory. The details can be seen consulted in [40, 49].

Theorem 2.1. *(Martingale representation theorem) Let $B = (B^1, \ldots, B^d)$ be a d-dimensional Brownian motion defined on the probability space in Section 1.3. Then, every local martingale $M = (M_t)_{0 \leq t \leq T}$, adapted to the filtration $(\mathcal{F}_t)_{t \leq T}$, can be expressed as*

$$M_t = M_0 + \sum_{i=1}^{d} \int_0^t Z_s^i \, dB_s^i$$

for predictable processes Z^i satisfying $\int_0^t (Z_s^i)^2 \, ds < \infty$, almost surely, for each $0 \leq t \leq T$.

Theorem 2.2. *(Doob-Meyer decomposition) Any local supermartingale X, adapted to the filtration $(\mathcal{F}_t)_{t \leq T}$, has a unique decomposition*

$$X = M - A$$

where M is a local martingale and A is a predictable increasing process starting from zero.

Theorem 2.3. *(Burkholder-Davis-Gundy) For any $p \in [1, \infty)$ there exist positive constants c_p, C_p such that, for all local martingales X, adapted to the filtration $(\mathcal{F}_t)_{t \leq T}$, with $X_0 = 0$ and stopping times τ, the following inequality holds.*

$$c_p \mathbb{E}\left[[X]_\tau^{p/2} \right] \leq \mathbb{E}\left[(\sup_{0 \leq s \leq \tau} |X_s|)^p \right] \leq C_p \mathbb{E}\left[[X]_\tau^{p/2} \right].$$

where $[X]_\tau$ denotes the quadratic variation of a process X. Furthermore, for continuous local martingales, this statement holds for all $p \in (0, \infty)$.

The next result will be useful in later sections for obtaining a priori estimates.

Lemma 2.4 (The Generalized Bellman-Gronwall Inequality)**.** *Assume that $f(t)$, $g(t)$ and $y(t)$ are non-negative integrable functions in $[0,T]$ verifying the integral inequality*

$$y(t) \leq g(t) + C \int_0^t y(u)du, \qquad t \in [0,T].$$

Then we have

$$y(t) \leq g(t) + C \int_0^t g(u)e^{C(t-u)}du, \qquad t \in [0,T].$$

2.2 The Snell envelope

The notion of Snell envelopes is very useful to deal with the optimal properties of optimal stopping and switching problems. We begin this section by showing a proposition that summarizes some key results concerning Snell envelopes. All these properties have been borrowed from Proposition 2 in [16].

Recall that for any right-continuous process $X = \{X_t,\, t \in [0,T]\}$ we define its *jump size* ΔX_t at time t as the difference $\Delta X_t = X_t - X_{t-}$ for $t \in [0,T]$, and $X_{0-} = 0$, where X_{t-} denotes the left-hand limit of X at time t.

Proposition 2.5. *Let $Z = (Z_t)_{0\leq t\leq T}$ be an $\mathbb{F}$-adapted $\mathbb{R}$-valued càdlàg process that belongs to the class $[D]$; i.e., the family of random variables $\{Z_\tau, \tau \in \mathcal{T}_0\}$ is uniformly integrable. Then there exists an $\mathbb{F}$-adapted and $\mathbb{R}$-valued càdlàg process $Y := (Y_t)_{0\leq t\leq T}$ of class [D] such that Y is the smallest supermartingale which dominates Z; i.e., if $(\bar{Y}_t)_{0\leq t\leq T}$ is another càdlàg supermartingale of class $[D]$ such that for all $0 \leq t \leq T$, $\bar{Y}_t \geq Z_t$, then $\bar{Y}_t \geq Y_t$ for any $0 \leq t \leq T$. The process Y is called the Snell envelope of Z.*

Moreover it satisfies the following properties:

(i) For any $\mathbb{F}$-stopping time τ we have

$$Y_\tau = \operatorname*{ess\,sup}_{\theta \in \mathcal{T}_\tau} \mathbb{E}[Z_\theta | \mathcal{F}_\tau].$$

In particular, $Y_0 = \sup_{\theta \in \mathcal{T}_0} \mathbb{E}[Z_\theta]$.

(ii) The Doob-Meyer decomposition of Y implies the existence of a martingale $(M_t)_{0\leq t\leq T}$ and two nondecreasing processes $(A_t)_{0\leq t\leq T}$ and $(B_t)_{0\leq t\leq T}$ which are, respectively, continuous and purely discontinuous predictable such that for all $0 \leq t \leq T$,

$$Y_t = M_t - A_t - B_t \quad (with \quad A_0 = B_0 = 0).$$

Moreover, for any $0 \leq t \leq T$, $\{\Delta B_t > 0\} \subset \{\Delta Z_t < 0\} \cap \{Y_{t-} = Z_{t-}\}$.

(iii) If Z has only positive jumps, then Y is a continuous process. Furthermore, if τ is an $\mathbb{F}$-stopping time and $\theta^*_\tau = \inf\{s \geq \tau, Y_s = Z_s\} \wedge T$, then θ^*_τ is optimal after τ, i.e.,

$$Y_\tau = \mathbb{E}[Y_{\theta^*_\tau} | \mathcal{F}_\tau] = \mathbb{E}[Z_{\theta^*_\tau} | \mathcal{F}_\tau] = \operatorname*{ess\,sup}_{\theta \geq \tau} \mathbb{E}[Z_\theta | \mathcal{F}_\tau].$$

In particular, $Y_0 = \sup_{\theta \in \mathcal{T}_0} \mathbb{E}[Z_\theta] = \mathbb{E}[Z_{\theta^*_0}]$.

(iv) If $(Z_n)_{n\geq 0}$ and Z are càdlàg and of class $[D]$ and such that the sequence $(Z_n)_{n\geq 0}$ converges increasingly and pointwisely to Z, then the sequence of Snell envelopes associated to the former sequence, denoted by $(Y^{Z_n})_{n\geq 0}$, converges increasingly and pointwisely to the corresponding Snell envelope Y^Z associated to Z. Finally, if Z belongs to $\mathcal{S}^2_c$, then Y^Z belongs to $\mathcal{S}^2_c$.

2.3 Backward stochastic differential equations (BSDEs)

Unlike the theory of deterministic ordinary differential equations where one can formulate the model under study by using indistinctly the initial or final condition, in the stochastic framework there is a huge difference on the use of such initial and final conditions. Indeed, let us consider the following forward stochastic differential equation (FSDE) with an initial condition,

$$\begin{cases} dX_t = 0 \cdot dB_t, \quad t \in [0,T] \\ X_0 = \xi. \end{cases} \tag{2.1}$$

By virtue of the theory of FSDE and Itô's calculus, it is well known that this equation has a unique *adapted* solution. Furthermore due that the filtration is generated by the Brownian motion $(B_t)_{0 \le t \le T}$, it is very easy to know that $X_0 = \xi$ is $\mathcal{F}_0$-measurable, which implies that ξ is constant. Hence, $X_t = \xi$ is the unique *adapted* solution of (2.1).

Now, let us change the initial condition to a final one, and use the next model

$$\begin{cases} dY_t = 0 \cdot dB_t, \quad t \in [0,T] \\ Y_T = \xi. \end{cases} \tag{2.2}$$

with ξ being a $\mathcal{F}_T$-measurable random variable. From the dynamics in (2.2), we see that the process $\{Y_t\}_{0 \le t \le T}$ is equal to the random variable ξ for all $t \in [0,T]$. But from the theory of FSDE and Itô's calculus, the only *adapted* solution of this equation is when ξ is a constant (not random), that would make the problem idle.

In order to overcome this lack of adaptability, it is natural to think on the conditional expectation $Y = \mathbb{E}[\xi|\mathcal{F}_t]$ rather than the relation $Y_t = \xi$. If we assume that $\xi \in L^2(\mathcal{F}_T)$, then by the Representation Martingale Theorem (Theorem 2.1), there exist a unique $Z \in \mathcal{H}^2$ such that $\mathbb{P}$-a.s.

$$Y_t = Y_0 + \int_0^t Z_t dB_t \quad \text{for all } t \le T.$$

Taking into account that $Y_T = \xi$, and writing the above integral in differential form we have

$$\begin{cases} dY_t = Z_t dB_t, \quad t \in [0,T] \\ Y_T = \xi. \end{cases} \tag{2.3}$$

Note from the above arguments that the processes Y and Z are both *adapted*. Thus, if we now look for a pair (Y,Z) instead of a single Y (as in (2.2)), we are able to find adapted processes that verify (2.3).

The previous example is the simplest one to introduce the theory of BSDE. Another example appears in Bismut [3], in which the author uses linear BSDE as a tool to solve both stochastic control problems and financial problems. As for the attribute "linear" we mean a BSDE of type

$$\begin{cases} dY_t = (a + bY_t + cZ_t)dt + Z_t \cdot dB_t, \quad t \in [0,T] \\ Y_T = \xi. \end{cases} \tag{2.4}$$

The nonlinear case was tackled by Pardoux and Peng in [44], where they showed the well-posedness of these equations with reasonable integrability and Lipschitz conditions on the data (f,ξ) —see Definition 2.1 below. More precisely, the authors consider the following assumptions on the data (f,ξ),

(A_1) The terminal value $\xi \in L^2_l(\mathcal{F}_T)$.

(A_2) The coefficient (a.k.a. drift or generator) $f : \Omega \times [0,T] \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \to \mathbb{R}^l$ satisfies

- (a) $f(\cdot,0,0) \in \mathcal{H}^{2,l}$
- (b) $f(\cdot,y,z)$ is progressively measurable for each y,z in $\mathbb{R}^l \times \mathbb{R}^{l \times d}$.

(c) f is Lipschitz continuous with respect to (y, z) uniformly in (t, ω); this implies that, $\mathbb{P}$-a.s., for all $t \in [0, T]$, (y_1, z_1) and (y_2, z_2), we have

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|)$$

where C is a fixed constant.

In the above conditions for f, we have omitted the variable $\omega \in \Omega$ for simplicity.

Definition 2.1. A solution of a BSDE with coefficient f and terminal condition ξ is pair $(Y, Z) \in \mathcal{S}_c^{2,\,l} \times \mathcal{H}^{2,\,l\times d}$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \quad \text{ for all } \ t \in [0, T], \ \text{ a.s.} \tag{2.5}$$

where $Y_T = \xi$ a.s.

Remark 2.1. For a sake of simplicity, hereafter we analyze just the case $l = 1$.

The existence and uniqueness theorem for BSDE is as follows. For further details see [44, 48].

Theorem 2.6 (Pardoux and Peng)**.** *If the data* (ξ, f) *satisfy* (A_1) *and* (A_2)*, then the BSDE* (2.5) *has a unique solution* $(Y, Z) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,\,d}$.

The proof of the theorem uses three results:

- Martingale representation theorem.
- Fixed point method.
- Itô's formula.

Sketch of the proof. The idea of the proof is to define a convenient mapping and show the existence of a (unique) fixed point that will match desired solution. Indeed, the convenient mapping is the next one:

$$\begin{aligned} \Phi: \quad \mathcal{S}_c^2 \times \mathcal{H}^{2,\,d} \quad &\longrightarrow \quad \mathcal{S}_c^2 \times \mathcal{H}^{2,\,d} \\ (U, V) \quad &\mapsto \quad (Y, Z), \end{aligned}$$

where, given $(U, V) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,\,d}$, the pair (Y, Z) is obtaining as follows:

- Since ξ and f satisfy assumptions (A_1) and (A_2) then

$$M_t = \mathbb{E}\Big[\xi + \int_0^T f(s, U_s, V_s)ds \Big| \mathcal{F}_t\Big]$$

  is a square integrable martingale and thus, by Theorem 2.1, there exist a unique $Z \in \mathcal{H}^{2,\,d}$ such that

$$M_t = M_0 + \int_0^t Z_s dB_s \qquad t \in [0, T]. \tag{2.6}$$

- The process Y is defined by

$$Y_t = \mathbb{E}\Big[\xi + \int_t^T f(s, U_s, V_s)ds \Big| \mathcal{F}_t\Big] = M_t - \int_0^t f(s, U_s, V_s)ds, \quad 0 \leq t \leq T. \tag{2.7}$$

From (2.6) and (2.7), it follows easily that Y and Z hold

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)ds - \int_t^T Z_s dB_s \quad \text{for all } t \in [0, T], \text{ a.s.} \tag{2.8}$$

Thus, the mapping Φ is defined by $\Phi(U, V) = (Y, Z)$ such that (2.8) holds. The following step is to show that Φ is a contraction on the Banach space $\mathcal{S}_c^2 \times \mathcal{H}^{2,d}$, when this space is endowed with the norm

$$\|(Y, Z)\|_\beta := \left\{ \mathbb{E}\left[\int_0^T e^{\beta s} \left(|Y_s|^2 + |Z_s|^2 \right) ds \right] \right\}^{\frac{1}{2}}.$$

We begin by applying Itô's formula to the process $e^{\beta s}\left|Y_s^1 - Y_s^2\right|^2$, where $(Y^1, Z^1) = \Phi(U^1, V^1)$ and $(Y^2, Z^2) = \Phi(U^2, V^2)$ and hence applying expectation, we get

$$\mathbb{E}\left[\left|Y_0^1 - Y_0^2\right|^2 + \int_0^T e^{\beta s} \left(\left|Y_s^1 - Y_s^2\right|^2 + \left|Z_s^1 - Z_s^2\right|^2 \right) ds \right]$$
$$= 2\mathbb{E}\left[\int_0^T e^{\beta s} \left|Y_s^1 - Y_s^2\right| . \left|f(s, U_s^1, V_s^1) - f(s, U_s^2, V_s^2)\right| ds \right].$$

Using the Lipschitz condition of f and the inequality $ab \le 2a^2 + \frac{b^2}{4}$ we have

$$\mathbb{E}\left[\left|Y_0^1 - Y_0^2\right|^2 + \int_0^T e^{\beta s} \left(\left|Y_s^1 - Y_s^2\right|^2 + \left|Z_s^1 - Z_s^2\right|^2 \right) ds \right] \le 4C^2 \mathbb{E}\left[e^{\beta s} |Y_s^1 - Y_s^2|^2 \right]$$
$$+ \mathbb{E}\left[\frac{1}{2} \int_0^T e^{\beta s} \left(\left|U_s^1 - U_s^1\right|^2 + \left|V_s^1 - V_s^2\right|^2 \right) ds \right].$$

Thus, taking $\beta = 1 + 4C^2$, we get

$$\mathbb{E}\left[\int_0^T e^{\beta s} \left(\left|Y_s^1 - Y_s^2\right|^2 + \left|Z_s^1 - Z_s^2\right|^2 \right) ds \right] \le \frac{1}{2} \mathbb{E}\left[\int_0^T e^{\beta s} \left(\left|U_s^1 - U_s^1\right|^2 + \left|V_s^1 - V_s^2\right|^2 \right) ds \right],$$

which proves that Φ is a contraction and hence has fixed point, i.e., the BSDE (2.5) has a unique solution. □

One of the most important tools within the theory of BSDEs, is the following comparison theorem, which is a powerful tool in many applications. For further details on the proof of this theorem, we refer the interested reader to El Karoui *et al* [23], page 23 or Pham [48], page 142. For self-contained purposes, we only provide a sketch of the proof.

Theorem 2.7. *Let (ξ^1, f^1) and (ξ^2, f^2) be two pairs of data satisfying assumptions (A_1) and (A_2). Suppose in addition that*

(i) $\xi^1 \le \xi^2$ $\mathbb{P}$- a.s.

(ii) $f^1(t, y, z) \le f^2(t, y, z)$ $dt \otimes d\mathbb{P}$- a.e., for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

Then $Y_t^1 \le Y_t^2$, $0 \le t \le T$, $\mathbb{P}$- a.s. Moreover, if $Y_0^2 \le Y_0^1$, then $Y_t^1 = Y_t^2$, $0 \le t \le T$. In particular, if $\xi^1 < \xi^2$ $\mathbb{P}$- a.s. or $f^1(t, y, z) < f^2(t, y, z)$ $dt \otimes d\mathbb{P}$- a.e., then $Y_0^1 < Y_0^2$.

Before we write the sketch, we give a proposition which establishes an explicit solution when the coefficient f is linear in y and z. The proof can be consulted in references [23, 48].

Proposition 2.8. *Consider the so-called linear BSDE given by*

$$Y_t = \xi + \int_t^T (\alpha_s + \beta_s Y_s + \gamma_s Z_s) ds - \int_t^T Z_s dB_s$$

where β and γ are bounded progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, and $\alpha \in \mathcal{H}^2$. Then the unique solution (Y, Z) of this equation is given by

$$Y_t = \mathbb{E}\Big[\Gamma_t^T \xi + \int_t^T \Gamma_t^s \alpha_s ds | \mathcal{F}_t\Big]$$

with

$$\Gamma_t^s = \exp\Big[\int_t^s \beta_r dr - \frac{1}{2}\int_t^s |\gamma_r|^2 dr + \int_t^s \gamma_r dB_r\Big].$$

Sketch of the proof of Theorem 2.7 Let us set $\bar{Y} = Y^2 - Y^1$, $\bar{Z} = Z^2 - Z^1$, then $(\bar{Y}, \bar{Z})$ is solution of the BSDE

$$d\bar{Y}_t = -\Big[\alpha_t \bar{Y}_t + \beta_t \bar{Z}_t + \gamma_t\Big] + \bar{Z}_t dB_t, \quad \bar{Y}_T = \xi^2 - \xi^1$$

with

$$\begin{aligned} \alpha_t &= \tfrac{f^2(t,Y_t^2,Z_t^2) - f^2(t,Y_t^1,Z_t^2)}{Y_t^2 - Y_t^1} \mathbf{1}_{[Y_t^2 - Y_t^1 \neq 0]} \\ \beta_t &= \tfrac{f^2(t,Y_t^1,Z_t^2) - f^2(t,Y_t^1,Z_t^1)}{Z_t^2 - Z_t^1} \mathbf{1}_{[Z_t^2 - Z_t^1 \neq 0]} \\ \gamma_t &= f^2(t, Y_t^1, Z^1) - f^1(t, Y_t^1, Z_t^1). \end{aligned}$$

Since the drift (or generator) f^2 is uniformly Lipschitz in y and z, the processes α and β are bounded. Moreover, γ is a process in $\mathcal{H}^2$. From Proposition 2.8, $\overline{Y} = Y^2 - Y^1$ is given by

$$Y_t^2 - Y_t^1 = \mathbb{E}\Big[\Gamma_t^T(\xi^2 - \xi^1) + \int_t^T \Gamma_t^s (f^2(t, Y_s^1, Z_s^1) - f^1(t, Y_s^1, Z_s^1)) ds | \mathcal{F}_t\Big].$$

Note also that assumptions (i) and (ii) imply that this expectation is positive since Γ is strictly positive, i.e., $Y_t^2 \leq Y_t^1$, $0 \leq t \leq T$, a.s. □

2.3.1 Reflected backward stochastic differential equations (RBSDEs)

A reflected backward stochastic differential equation (RBSDE) is a BSDE where the element Y of the solution is conditioned to stay either above or below of a given process $(S_t)_{0 \leq t \leq T}$ called barrier or obstacle. To fix ideas, throughout this work we will choose a 'lower' barrier since the 'upper' case is similar. More precisely, given an initial data (f, ξ) satisfying (A_1) and (A_2), together with a barrier process $(S_t)_{0 \leq t \leq T}$ in $\mathcal{S}_c^2$, we want to find a solution (Y, Z) such that $Y_t \geq S_t$, $0 \leq t \leq T$. In order to guarantee that the element Y is always above S, an increasing process K is necessary to push Y upwards once this later process reaches S. In other words, in this new scenario the solution of a RBSDE is a triplet $(Y, Z, K) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,d} \times \mathcal{S}_c^2$ such that

$$\begin{aligned} &Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s \quad \text{for all} \quad 0 \leq t \leq T, \\ &Y_t \geq S_t \quad \text{for all} \quad 0 \leq t \leq T, \end{aligned} \tag{2.9}$$

and

$$\int_0^T (Y_s - S_s) dK_s = 0, \tag{2.10}$$

where the new process K is a finite variation process which is increasing and whose "target" is to keep Y above S. Moreover, the condition (2.10) is imposed in order to guarantee that the process K acts just when $Y_t = S_t$

for some $0 \leq t \leq T$. Summarizing, the system (2.9) and (2.10) is so-called a RBSDE that was introduced by El Karoui *et al* in [25].

The following theorem ensures that under the conditions (A_1) and (A_2) as well as some additional conditions on the barrier, the above RBSDE has indeed a unique solution.

Theorem 2.9. *Given $(\xi, f, S) \in L^2(\mathcal{F}_T) \times \mathcal{H}^2 \times \mathcal{S}_c^2$ satisfying (A_1), (A_2) and $S_T \leq \xi$ a.s. Then, there exists a unique solution $(Y, Z, K) \in \mathcal{S}_c^2 \times \mathcal{H}^2 \times \mathcal{S}_c^2$ of the RBSDE (2.9)-(2.10).*

There exist at least two approaches to demonstrate this theorem:

- fixed point method
- Penalization method.

The proof via fixed point method is very similar to that used in Theorem 2.6, but by doing suitable changes (see El Karoui *et al* [25], Theorem 5.2). Therefore, we will only provide a sketch of the proof by using the penalization method. This method consists in transforming a constrained optimization problem to a family of parametric unconstrained problems whose solutions converge to the solution of the original constrained problem. In this case, the solution (Y, Z, K) is found by using a sequence $(Y^n, Z^n)_{n\geq 0}$ of unconstrained solutions of BSDEs (i.e. BSDEs without barriers). We refer the reader to El Karoui *et al* [25], page 719, for further details of the proof (see also Pham [48], page 154).

Sketch of the proof. The idea is to consider for each $n \geq 0$, the sequence $(Y^n, Z^n)_{n\geq 0}$ of solutions of unconstrained standard BSDEs.

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + n\int_t^T (Y_s^n - S_s)^- ds - \int_t^T Z_s^n dB_s. \tag{2.11}$$

Besides, we shall define for any $t \leq T$

$$K_t^n := n\int_t^T (Y_s^n - S_s)^- ds, \quad \text{(penalized term)}.$$

Therefore, the proof consists in showing that under the assumptions on the data (f, ξ, S), the sequence $(Y^n, Z^n, K^n)_{n\geq 0}$ converges (in the norm of their corresponding spaces) to the unique solution (Y, K, Z) of the RBSDE. Note that for any n, the solution (Y^n, Z^n) does exist by Theorem 2.5, since

$$f_n(s, y, z) = f(s, y, z) + n(y - S_s)^- \tag{2.12}$$

satisfies (A_2) and $\xi \in L^2(\mathcal{F}_T)$.

More precisely, the proof can be divided into three main steps:

Step 1. The sequence $(Y^n, Z^n, K^n)_{n\geq 0}$ satisfies the following a priori estimate (uniformly in n),

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} \left|Y_t^n\right|^2 + \int_0^T \left|Z_t^n\right|^2 dt + (K_T^n)^2\Big] \leq C, \quad n \in \mathbb{N}. \tag{2.13}$$

Indeed, by applying Itô's formula to $|Y_t^n|^2$ and taking expectation we have

$$\mathbb{E}[|Y_t^n|^2] + \mathbb{E}[\int_t^T |Z_s^n|^2 ds] = \mathbb{E}[\xi^2] + 2\mathbb{E}[\int_t^T f(s, Y_s^n, Z_s^n)Y_s^n ds] + 2\mathbb{E}[\int_t^T Y_s^n dK_s^n], \tag{2.14}$$

This last equality together with the Lipschitz assumption on f along with the inequalities $2ab \leq \beta a^2 + \frac{1}{\beta}b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$, with β properly taken, lead to

$$\begin{aligned}\mathbb{E}[|Y_t^n|^2] + \mathbb{E}[\int_t^T |Z_s^n|^2 ds] \quad &\leq \quad C\Big(1 + \mathbb{E}[\int_t^T |Y_s^n|^2 ds]\Big) + \frac{1}{3}\mathbb{E}[\int_t^T |Z_s^n|^2 ds] \\ &\quad + \frac{1}{\beta}\mathbb{E}[\sup_{0\leq t\leq T} (S_t^+)^2] + \beta\mathbb{E}(K_T^n - K_t^n)^2.\end{aligned} \tag{2.15}$$

Note that from (2.11) one can see that the last term in the right hand side of (2.15) satisfies

$$\mathbb{E}[(K_T^n - K_t^n)^2] \le C\Big(\mathbb{E}[|Y_t^n|^2] + \mathbb{E}|\xi|^2 + 1 + \int_t^T (|Y_s^n|^2 + |Z_s^n|^2)ds\Big). \tag{2.16}$$

Combining this last expression with (2.15), and taking $\beta = \frac{1}{3}C$, we see that

$$\frac{2}{3}\mathbb{E}[|Y_t^n|^2] + \frac{1}{3}\mathbb{E}[\int_t^T |Z_s^n|^2 ds] \le C\Big(1 + \mathbb{E}[\int_t^T |Y_s^n|^2 ds]\Big)$$

and thus by Gronwall's Lemma (Lemma 2.4), and taking into account (2.16) along with that $K_0 = 0$, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|Y_t^n\big|^2 + \int_0^T \big|Z_t^n\big|^2 dt + (K_T^n)^2\Big] \le C, \quad n \in \mathbb{N}.$$

Finally, using this last inequality along with (2.11) and Burkholder-Davis-Gundy inequality (Theorem 2.3), we obtain the a priori estimate (2.13).

Step 2. The sequence $(Y^n)_{n\ge 0}$ converges uniformly to Y in $\mathcal{S}_c^2$. Indeed, from (2.12) it is easily seen that

$$f_n(t,y,z) \le f_{n+1}(t,y,z)$$

and thus by Comparison Theorem (Theorem 2.7), we get that $Y^n \le Y^{n+1}$, a.s., which implies that there exist a process $(Y_t)_{0\le t\le T}$ such that

$$Y_t^n \nearrow Y_t, \quad 0 \le t \le T, \quad \text{a.s.}. \tag{2.17}$$

Furthermore, from the a priori estimate (2.13) the sequences $(Y_n)_{n\ge 0}$ is uniform bounded in $\mathcal{S}_c^2$ and hence $\mathbb{E}(\sup_{0\le t\le T}|Y_t|^2) < \infty$. Moreover, this implies that

$$\mathbb{E}\Big[\int_0^T (Y_t - Y_t^n)^2 dt\Big] \to 0 \quad \text{as} \quad n \to \infty \tag{2.18}$$

by the Dominated Convergence Theorem.

On the other hand, using condition (A_1) and (A_2), we apply Itô's formula to $|Y_t^n - Y_t^p|^2$, for some $n, p \in \mathbb{N}$, and then take expectation to obtain

$$\begin{aligned}\mathbb{E}\big[|Y_t^n - Y_t^p|^2\big] + \mathbb{E}\int_t^T |Z_s^n - Z_s^p|^2 ds \quad &\le \quad 2C\mathbb{E}\Big[\int_t^T \Big(|Y_s^n - Y_s^p|^2 + |Y_s^n - Y_s^p)| \cdot |Z_s^n - Z_s^p|\Big)ds\Big] \\ &\quad +2\mathbb{E}\Big[\int_t^T (Y_s^n - S_s)^- dK_s^p\Big] + 2\mathbb{E}\Big[\int_t^T (Y_s^p - S_s)^- dK_s^n\Big]\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{E}\int_t^T |Z_s^n - Z_s^p|^2 ds \quad &\le \quad C\mathbb{E}\Big[\int_t^T |Y_s^n - Y_s^p|^2 ds\Big] + 2\mathbb{E}\Big[\int_t^T (Y_s^n - S_s)^- dK_s^p\Big] \\ &\quad +4\mathbb{E}\Big[\int_t^T (Y_s^p - S_s)^- dK_s^n\Big].\end{aligned} \tag{2.19}$$

The key step of the proof is to show that the last two term in the right side of (2.19) go to zero as $n \to 0$, which is achieved if we assume that

$$\mathbb{E}\Big[\sup_{0\le t\le T}|(Y_t^n - S_t)^-|^2\Big] \to 0 \quad \text{as} \quad n \to \infty. \tag{2.20}$$

We omit the proof of this last claim and we refer to the reader to [25] Lemma 6.1. Then, this result along with the convergence in (2.18) yields

$$\mathbb{E}\int_t^T (|Y_t^n - Y_t^p|^2 + |Z_s^n - Z_s^p|^2)ds \to 0 \quad \text{as} \quad n,p \to \infty. \tag{2.21}$$

Finally, applying Itô's formula to $|Y^n - Y^p|^2$ and hence taking the supremum over $[0,T]$, we get

$$\begin{aligned}\sup_{0\le t\le T} |Y_t^n - Y_t^p|^2 \;&\le\; 2\int_0^T |f(s,Y_s^n,Z_s^n) - f(s,Y_s^p,Z_s^p)|\cdot|Y_s^n - Y_s^p)|ds + 2\int_0^T (Y_s^n - S_s)^- dK_s^p \\ &+2\int_0^T (Y_s^p - S_s)^- dK_s^n + 2\sup_{0\le t\le T}\Big|\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)dB_s\Big|.\end{aligned}$$

Taking expectations to the last expression and hence applying Burkholder-Davis-Gundy Theorem (Theorem 2.3) to the last term as well as the Lipschitz condition on f, we get

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T} |Y_t^n - Y_t^p|^2] \;&\le\; C\mathbb{E}\Big[\int_0^T (|Y_s^n - Y_s^p|^2 - |Z_s^n - Z_s^p)|^2)\Big]ds + 2\mathbb{E}\Big[\int_0^T (Y_s^n - S_s)^- dK_s^p\Big] \\ &+2\mathbb{E}\Big[\int_0^T (Y_s^p - S_s)^- dK_s^n\Big] + \tfrac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T} |Y_s^n - Y_s^p|^2\Big] + C\mathbb{E}\Big[\int_0^T |Z_s^n - Z_s^p|^2 ds\Big],\end{aligned}$$

and thus, from the previous estimations, it is easy to verify that $\mathbb{E}(\sup_{0\le t\le T} |Y_t^n - Y_t^p|^2) \to 0$ as $n,p\to\infty$. From this last property, it is not difficult to prove that

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|K_t^n - K_t^p\big|^2\Big] \to 0.$$

Given that $\mathcal{S}_c^2$ and $\mathcal{H}^2$ are Banach spaces, then there exists a pair (Z,K) of progressively measurable processes such that the triplet (Y,Z,K) verifies the first equality in (2.9), but also, by (2.20), this pair satisfies also the barrier condition in the second part of (2.9).

Step 3. This part is about the verification of (2.10). Since K_t is increasing and (Y^n, K^n) converges to (Y,K) in $\mathcal{S}_c^2$, the measure dK^n converges to dK in probability, i.e.,

$$\int_0^T (Y_t^n - S_t)dK_t^n \to \int_0^T (Y_t - S_t)dK_t.$$

Moreover, we have

$$\int_0^T (Y_t^n - S_t)dK_t^n \le 0, \quad \text{for all } n\in\mathbb{N},$$

thus we have

$$\int_0^T (Y_t - S_t)dK_t \le 0.$$

On the other hand, from the proof of Lemma 6.1 in [25], we obtain

$$\int_0^T (Y_t - S_t)dK_t \ge 0, \quad n\in\mathbb{N}.$$

The proof is finished. □

In the proof of Theorem 2.9, we saw how useful was to compare solutions of standard BSDEs allowing, among other things, to obtain a monotone sequences of processes $(Y^n)_{n\ge 0}$. In the case of RBSDE it is also possible to obtain a similar result. For more details we quote the reference [25].

Theorem 2.10 (Comparison theorem for RBSDE). *Let (ξ^1, f^1, S^1) and (ξ^2, f^2, S^2) be two pairs of data, both satisfying (A_1), (A_2). Moreover suppose that the following conditions hold:*

(i) $\xi^1 \le \xi^2$ *a.s.*

(ii) $f^1(t, y, z) \le f^2(t, y, z)$ $d\mathbb{P} \otimes dt$ *a.e., for all* $t \ge 0$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

(iii) $S_t^1 \le S_t^2$, $0 \le t \le T$, *a.s.*

Then, if (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) are the respective solutions of the RBSDE, then we have

$$Y_t^1 \le Y_t^2 \qquad 0 \le t \le T, \text{ a.s.}$$

Sketch of the proof. First note that it is enough to show that $(Y_t^1 - Y_t^2)^+ = 0$, for all $0 \le t \le T$. Namely, applying Itô's formula to $|(Y_t^1 - Y_t^2)^+|^2$ and then taking expectation, we derive

$$\begin{aligned}\mathbb{E}\big[|(Y_t^1 - Y_t^2)^+|^2\big] \le 2\mathbb{E}\Big[\int_t^T \Big((Y_s^1 - Y_s^2)^+\big[f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\big] - |Z_s^1 - Z_s^2|^2 \mathbf{1}_{\{Y^1 > Y^2\}}\Big) ds\Big] \\ +2\mathbb{E}\Big[\int_t^T \big(Y_s^1 - Y_s^2\big)^+ \cdot \big(dK_s^1 - dK_s^2\big)\Big].\end{aligned} \tag{2.22}$$

Since on the event $\{Y_t^1 > Y_t^2\}$, the next relations hold $Y_t^1 > Y_t^2 \ge S_t^2 \ge S_t^1$, so $dK_t^1 = 0$, and hence

$$\int_t^T \big(Y_s^1 - Y_s^2\big)^+ \cdot \big(dK_s^1 - dK_s^2\big) = -\int_t^T \big(Y_s^1 - Y_s^2\big)^+ dK_s^2 \le 0. \tag{2.23}$$

Therefore, using Lipschitz condition on f^1, assumption (ii) of this theorem and inequality (2.23), we get from (2.22) that

$$\mathbb{E}\big[\big|(Y_t^1 - Y_t^2)^+\big|^2\big] \le 2C\mathbb{E}\Big[\int_t^T \Big(\big(Y_s^1 - Y_s^2\big)^+ \big(|Y_s^1 - Y_s^2| + |Z_s^1 - Z_s^2|)\big] - \big|Z_s^1 - Z_s^2\big|^2 \mathbf{1}_{\{Y^1 > Y^2\}}\Big) ds\Big], \tag{2.24}$$

and using the inequalities $2ab \le \beta a^2 + \frac{1}{\beta} b^2$ and $(a + b)^2 \le 2a^2 + 2b^2$, with β properly taken, in the right side of (2.24), we have

$$\mathbb{E}\big[|(Y_t^1 - Y_t^2)^+|^2\big] \le 2C\mathbb{E}\Big[\int_t^T \Big(|\big(Y_s^1 - Y_s^2\big)^+|^2 + |Z_s^1 - Z_s^2|^2 \mathbf{1}_{\{Y^1 > Y^2\}} - |Z_s^1 - Z_s^2|^2 \mathbf{1}_{\{Y^1 > Y^2\}}\Big) ds\Big],$$

for some suitable constant C, and thus

$$\mathbb{E}\big[|(Y_t^1 - Y_t^2)^+|^2\big] \le 2CK\mathbb{E}\Big[\int_t^T |\big(Y_s^1 - Y_s^2\big)^+|^2 ds\Big].$$

Finally, a simple use of Gronwall's inequailty (Lemma 2.4), we obtain $(Y_t^1 - Y_t^2)^+ = 0$, $0 \le t \le T$. □

Remark 2.2. Note that in (2.23) we have only applied the condition (2.10) to the process Y^1. Therefore, given two solutions Y^1 and Y^2 of a RBSDE with the same data where Y^1 satisfies the condition (2.10) but Y^2 does not, then applying this comparison result, we can show that $Y^1 \le Y^2$. That is why, the relation (2.10) is so-named *minimal condition*.

2.3.2 Double reflected backward stochastic differential equations (DRBSDEs)

A DRBSDE is a BSDE which is forced to stay between two given processes L and U called lower and upper barrier, respectively.

Definition 2.2. A solution for DRBSDE with lower barrier L, upper barrier U, coefficient f and terminal value ξ, is a quadruple of processes $(Y, Z, K^+, K^-) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,d} \times \mathcal{S}_c^2 \times \mathcal{S}_c^2$ such that a.s. and for all $0 \le t \le T$,

$$\begin{aligned} & Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s \\ & L_t \le Y_t \le U_t \\ & \int_0^T (Y_s - L_s) dK_s^+ = \int_0^T (U_s - Y_s) dK_s^- = 0. \end{aligned} \tag{2.25}$$

where K^+ and K^- are the increasing processes with $K_0^+ = K_0^- = 0$.

Existence and uniqueness results for (2.25) can be proven under any of the following conditions:

- Mokobodski's condition: There exist two non-negative supermartingales such that the difference of these two last processes is always between L and U,

or

- Regularity on the barrier: Either L or U is a semimartingale.

This type of equations was introduced by Cvitanic and Karatzas in [12] and inspired the work of El Karoui *et al* [25]. The following result ensures the existence of a unique solution to this type of equations. We omit the proof, for it can be found in [12].

Theorem 2.11. *Given $(\xi, f, L, U) \in L^2(\mathcal{F}_T) \times \mathcal{H}^2 \times \mathcal{S}_c^2 \times \mathcal{S}_c^2$ with f and ξ satisfying (A_1) and (A_2), and L and U satisfying Mokobosdski's condition with $L_T \le \xi \le U_T$. Then, there exists a unique solution $(Y, Z, K^+, K^-) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,d} \times \mathcal{S}_c^2 \times \mathcal{S}_c^2$ of the DRBSDE (2.25).*

We just point out that this theorem can be also proved either by a fixed point argument or penalization argument as in the cases of BSDE and RBSDE.

2.3.3 An example of application of BSDEs in financial derivatives

In this section we illustrate how the theory of BSDE can be applied to financial mathematics, more precisely, to the theory of European options and American contingent claims, in which their corresponding fair values (pricing option) can be formulated in terms of solutions of BSDE and RBSDE respectively. We will base our analysis from the examples in [8, 23, 43]

European options. We consider a standard complete market consisting of one non-risky asset and d risky assets, whose prices are given by the following system of forward stochastic differential equations

$$\begin{aligned} dP_t^0 &= P_t^0 r_t dt && (Bond) \\ dP_t^i &= P_t^i(\mu_t^i dt + \sigma_t^i dB_t) \quad i = 1, \dots, d && (Stocks) \end{aligned} \tag{2.26}$$

for $0 \le t \le T$ with $P_0^i > 0$ for $i = 0, 1, \dots, d$. The processes $r \in \mathbb{R}$, $\mu^i \in \mathbb{R}$, and $\sigma^i \in \mathbb{R}^d$ are supposed to be progressively measurable, bounded and stand for the short rate, the i-th stock appreciation rate and the i-th volatility vector, respectively. Moreover, the volatility matrix $\sigma = (\sigma^{i,j})$ is invertible and its inverse σ^{-1} is bounded. We will refer to as *underlying securities or underlying market* a subset of prices in (2.26) that will be using along this example.

In a general setting, an option is a contract between two parts: a seller and a holder. Based on the value of the underlying securities, the holder of the option has the right but not the obligation to either sell or buy a stock contained in the underlying market at determined prescribed time. Besides, an option depends on the following elements:

(1) a predetermined price q (strike price)

(2) a terminal time T (maturity date).

(3) an exercise time.

Let us focus now on the European options. These financial instruments are contracts in which the exercise time coincides with the terminal time T. For instance, let us consider the following situation: (1) assume that the underlying market is just P_t^1, and (2) assume that an investor is interested to buy a stock associated whose prices is P_t^1 at a future time T (maturity time), at a specified price q (strike price) on that future time. For this reason he/she can engage on an European call option in order to have the right (but not the obligation) of buying the desired stock at time T at price q. Obviously, if $P_T^1 < q$ then he/she does not exercise the option, otherwise, if $P_T^1 > q$ then she/he exercises it. Note in any case the buyer's payoff at time T is given by $\xi = (P_T^1 - q)^+$. Observe also in this example that the payoff is written explicitly as a function of P_T^1. In general, when the option price at terminal T becomes an explicit function of the price, then the contract is called option, otherwise, it is called a *contingent claim*.

On the other hand, there is the problem of calculating the price at which the seller trades the option, i.e., the so-called option pricing problem. Let us then assume, in order to get a *fair price*, that both parts of the contract decide to adopt the following principle: if the sale price is y_0 of the option is reinvested in the $q+1$ assets, then the value of the portfolio at time T must be sufficient to guarantee the amount $(P_T - q)^+$. More precisely, let Y_t be the total wealth at time t, once he/she has invested, at time 0, the amount y_0. Therefore, denoting by $\pi = (\pi_t^0, \pi_t^1, \ldots, \pi_t^d)_{0\le t\le T}$, the so-called replicating strategy or portfolio, where π_t^0 and π_t^i represent the amounts invested in both, the bond and the i-th stock, respectively. Then we have that at time t,

$$Y_t = \pi_t^0 + \pi_t^1 + \cdots + \pi_t^d,$$

where $Y_0 = y_0$. Here we assume that the agent cannot withdraw his wealth at any time t.

Definition 2.3. A portfolio π is said to be:

(a) *admissible* if its components are progressively measurable and

$$\int_0^T |\pi_t^0| dt < +\infty, \quad \int_0^T |\pi_t^i \sigma_t^i|^2 dt < +\infty \quad \text{and} \quad Y_t^\pi \ge 0, \quad 0 \le t \le T \tag{2.27}$$

for each $i = 1, \ldots, q$ $\mathbb{P}$-a.s. where Y^π is the value associated to the portfolio π.

(b) *self-financing* if

$$dY_t^\pi = \pi_t^0 \frac{dP_t^0}{P_t^0} + \pi_t^1 \frac{dP_t^1}{P_t^1} + \cdots \pi_t^q \frac{dP_t^q}{P_t^q}, \quad \text{for all } 0 \le t \le T. \tag{2.28}$$

We denote by $\mathcal{A}$ the set of self-financing admissible portfolios. Observe that by combining (2.28) with (2.26), we get

$$dY_t^\pi = r_t Y_t^\pi dt + \sum_{i=1}^q [\pi_t^i(\mu_t^i - r_t)dt + \pi_t^i \sigma_t^i dB_t].$$

Therefore, rewriting this last expression and using a vector notation we obtain

$$dY_t^\pi = [r_t Y_t^\pi + Z_t \theta_t]dt + Z_t dB_t.$$

where $Z_t = \pi_t \sigma_t$ and $\theta = \sigma_t^{-1}[\mu_t - r_t \mathbf{1}_d]$, with $\mathbf{1}_d$ the vector of ones of dimension d. This last process is the so-called the *risk premium process*.

The problem to tackle is to find a fair price for the European option but at the same time to also find an admissible replicating strategy. Rigorously speaking, we are looking for a value y_0 satisfying

$$y_0 = \inf\{y \ge 0 : Y_0^\pi = y \quad \text{and} \quad Y_T^\pi = \xi, \text{ for } \pi \in \mathcal{A}\}.$$

The next theorem provides the solution of our problem. Details can be seen in [8, 23].

Theorem 2.12. *Assume that the matrix $\sigma_t = [\sigma_t^1, \dots, \sigma_t^d]$ is invertible, and that the risk premium vector $\theta_t = [\sigma_t]^{-1}(\mu_t - r_t \mathbf{1}_d)$ is bounded. Given a nonnegative random variable $\xi \in L^2(\mathcal{F}_T)$, if $(Y_t, Z_t)_{0\le t\le T}$ is the unique solution of the standard BSDE*

$$\begin{cases} dY_t = (r_t Y_t + Z_t\theta_t)dt + Z_t dB_t, \\ Y_T = \xi, \end{cases} \tag{2.29}$$

then $y_0 := Y_0$ is a fair price of the claim ξ and $\pi_t := Z_t[\sigma_t^{-1}]$ is a replicating self-financing portfolio.

Sketch of the proof. Since $\xi \in L^2(\mathcal{F}_T)$, then the solution (Y, Z) of (2.29) does exist by a simple use of Theorem 2.5. Moreover, since (2.29) is a linear BSDE, then by Proposition 2.8, we have

$$\Gamma_0^t Y_t = \mathbb{E}\Big[\Gamma_0^T \xi | \mathcal{F}_t\Big] \tag{2.30}$$

with

$$\Gamma_t^s = \exp\Big[\int_t^s r_r dr - \frac{1}{2}\int_t^s |\theta_r|^2 dr + \int_t^s \theta_r dB_r\Big].$$

Therefore, $Y_t \geq 0$ because ξ and Γ_t^T are nonnegative. Moreover, as $\mathcal{F}_0$ is the trivial σ-algebra, the value Y_0 is deterministic and thus $y_0 := Y_0$ is well defined. On the other hand, we have that $\int_0^T |Z_t|^2 < +\infty$ $\mathbb{P}$ a.s., which combined with the assumption that $[\sigma]^{-1}$ is bounded, all together yield that $\pi^* := Z.[\sigma]^{-1}$ satisfies both, the self-financing property, and

$$\int_0^T |\pi_t^{i,*}\sigma_t^i|^2 dt < +\infty \quad \text{a.s.,} \quad \text{for each } i = 1, \dots, q,$$

and this along with the non-negativeness of Y and a simple algebra to get the first property of (2.27), gives that $\pi^* \in \mathcal{A}$, and thus this last set is nonempty. Let us define $Y^{\pi^*} := Y$.

Now, if $\pi \in \mathcal{A}$ is another admissible replicating portfolio, (i.e., such that $Y_T^\pi = \xi$), using Itô's formula for the product $(\Gamma_0^t Y_t^\pi)_{0\le t\le T}$ and taking into account (2.29) for Y^π, we have that this product is a nonnegative local martingale (hence a supermartingale) with terminal value $\Gamma_0^T Y_T^\pi = \Gamma_0^T\xi = \Gamma_T Y_T^{\pi^*}$, which implies that for any $t \le T$,

$$\Gamma_0^t Y_t^\pi \geq \mathbb{E}[\Gamma_0^T \xi|\mathcal{F}_t] = \mathbb{E}[\Gamma_0^T Y_T^{\pi^*}|\mathcal{F}_t] = \Gamma_0^t Y_t^{\pi^*},$$

where the last equality is due to (2.30). Hence, for $t = 0$, we conclude that $Y_0^\pi \geq Y_0^{\pi^*}$, which proves the result. □

American contingent claims. The key difference of an American contingent claim with respect to a European contingent claim is that the buyer can exercise the contract at any time t that might be less or equal to the terminal time T.

Recall that $\mathcal{T}_t$ is the set of all the stopping times on $[t, T]$. Using this definition, let us take $\tau \in \mathcal{T}_t$ that is the time when the buyer exercises the claim after time t. Hence, if $\tau < T$, then the payoff is S_τ, whereas if $\tau = T$, then the payoff is ξ. The former and later expressions are also known as *running* and *terminal* payoffs, respectively. These two payoffs in turn define the *actual payoff* of the contingent claim defined by follows:

$$\tilde{S}_\tau = S_\tau \mathbf{1}_{[\tau<T]} + \xi\mathbf{1}_{[\tau=T]}.$$

Next, in order to find the fair price of the American contingent claim, we assume that for any fixed t, the buyer decides to execute his/her right of claiming the contract at time $\tau \in \mathcal{T}_t$, and therefore by Theorem 2.12, there exist a unique solution (Y^τ, Z^τ) which replicates $\tilde{S}_\tau$ that satisfies

$$\begin{cases} dY_s^\tau = f(s, Y_s^\tau, Z_s^\tau)ds + Z_s^\tau dB_s \quad t \le s \le \tau \\ Y_\tau^\tau = \tilde{S}_\tau, \end{cases} \tag{2.31}$$

where $f(s, y, z)$ is the linear coefficient in (2.29). In this later argument, we are assuming implicitly that both the running payoff $(S_t)_{0 \le t \le T}$ belongs to $\mathcal{S}_c^2$ and the terminal payoff as a random variable in $L^2(\mathcal{F}_T)$.

As is established in [8], the fair price for the American contingent claim with payoff $(\tilde{S}_s)_{t \le s \le T}$ is given by

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} Y_t^\tau. \tag{2.32}$$

In other words, the fair price of an American contingent claim can be seen as an optimal stopping problem associated to a family of solutions of BSDEs, whose elements represent the price of a European contingent claim. Moreover, the price (2.32) can be characterized in terms of a solution of a RBSDE as is established in the following proposition. For more details, we refer the reader to [8, 41].

Theorem 2.13. *There exist a $Z \in \mathcal{H}^{2,d}$ and a nondecreasing continuous process K such that for any $t \le T$, the fair price of the American contingent claim Y_t satisfies*

$$\begin{cases} Y_t = \xi - \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \\ Y_t \ge S_t, \\ \int_0^T (Y_s - S_s) dK_s = 0. \end{cases} \tag{2.33}$$

Furthermore, the stopping time

$$\tau^* = \inf\{s \ge t : Y_s = S_s\} \wedge T$$

is an optimal exercise time after t and $Y_t = Y_t^{\tau^*}$.

Sketch of the proof. The existence of a solution (Y, Z, K) of (2.33) is given by Theorem 2.9. Now, if $\tau \in \mathcal{T}_t$, then by Theorem 2.12 we have

$$Y_s^\tau = \tilde{S}_\tau - \int_s^\tau f(r, Y_r^\tau, Z_r^\tau) dr - \int_s^\tau Z_r^\tau dB_r \quad t \le s \le \tau \tag{2.34}$$

(recall equation (2.31)). But since

$$Y_s = Y_\tau - \int_s^\tau f(r, Y_r, Z_r) dr + K_\tau - K_s - \int_s^\tau Z_r dB_r \quad t \le s \le \tau, \tag{2.35}$$

then comparing equations (2.34) and (2.35) and using the fact that $\xi \ge S_T$, we deduce that $Y_\tau \ge \tilde{S}_\tau$. Furthermore, as the process K is nonnegative and increasing, then it is straightforward to see that the generator of (2.35) is greater than the generator of (2.34), and hence, a simple use of the Comparison Theorem (Theorem 2.7) , yields that $Y_s \ge Y_s^\tau$, for every $t \le s \le \tau$. Therefore, we get

$$Y_t \ge \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} Y_t^\tau.$$

On the other hand, by definition of τ^* and K, this later process is zero on $[t, \tau^*]$. This last assertion, together with equation (2.35) evaluated at τ^* gives that

$$Y_s = Y_{\tau^*} - \int_s^{\tau^*} f(r, Y_r, Z_r) dr - \int_s^{\tau^*} Z_r dB_r, \quad t \le s \le \tau^*.$$

Furthermore, $Y_{\tau^*} = \tilde{S}_{\tau^*}$. Finally by the uniqueness of the BSDEs, we conclude $Y_t = Y_t^{\tau^*}$, and so $Y_t \le \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} Y_t^\tau$. □

2.4 Markovian framework and partial differential equations (PDEs)

In this section we shall describe the link between BSDEs and second order semilinear PDEs. It turns out that solutions of BSDEs are naturally related with viscosity solutions of PDEs when the randomness of the data (f, ξ) comes from a Markov-diffusion process. More precisely, consider the following system of forward stochastic differential equations (FSDE) and BSDE

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s \quad \text{for all} \quad t \le s \le T; \\ X_s^{t,x} = x \quad \text{for all} \quad 0 \le s \le t, \end{cases} . \tag{2.36}$$

and

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dB_r \quad \text{for all} \quad t \le s \le T, \tag{2.37}$$

where

(B_1) the functions $b : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0,T] \times \mathbb{R}^k \to \mathbb{R}^{k\times d}$ are jointly continuous in (t,x) and Lipschitz continuous with respect to x uniformly in t, i.e., there exists a constant K such that, for all $t \ge 0$, $x_1, x_2 \in \mathbb{R}^k$,

$$|b(t,x_1) - b(t,x_2)| + |\sigma(t,x_1) - \sigma(t,x_2)| \le K|x_1 - x_2|,$$

(B_2) the functions $f : [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$ satisfy

(a) g is continuous and has polynomial growth.

(b) f is jointly continuous and there exist constants non-negative real constants K and γ such that

$$|f(t,x,0,0)| \le K(1 + |x|^\gamma), \qquad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^k. \tag{2.38}$$

(c) f is Lipschitz continuous with respect to (y,z) uniformly in (t,x); this implies that for any $(t,x) \in [0,T] \times \mathbb{R}^k$, and for any (y_1,z_1), $(y_2,z_2) \in \mathbb{R} \times \mathbb{R}^d$, we have

$$|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|)$$

where C is a fixed constant.

BSDEs vs PDEs. Let us consider the parabolic second order semilinear PDE

$$\begin{cases} -\dfrac{\partial v}{\partial t}(t,x) - \mathcal{L}v(t,x) - f(t,x,v,\sigma^{\mathsf{T}} D_x v) = 0 \quad (t,x) \in [0,T) \times \mathbb{R}^k, \\ \\ v(T,x) = g(x) \quad x \in \mathbb{R}^k, \end{cases} \tag{2.39}$$

where $\mathcal{L}$ is the infinitesimal generator associated to the diffusion in (2.36), whose nature is as follows

$$\mathcal{L}\varphi(t,x) = \frac{1}{2} Tr \left[\left(\sigma.\sigma^{\mathsf{T}}\right)(t,x) D_{xx}^2 \varphi(t,x) \right] + b(t,x)^{\mathsf{T}} D_x \varphi(t,x) \quad \forall\ \varphi \in C^{1,2}([0,T] \times \mathbb{R}^k). \tag{2.40}$$

In the expression above, $Tr(.)$ represents the trace of a square matrix and, $\mathtt{A}^{\mathsf{T}}$ is the transpose of a matrix $\mathtt{A}$. For notational convenience, sometimes we write $(\partial_t + \mathcal{L})\varphi(t,x)$ instead of $\partial_t \varphi(t,x) + \mathcal{L}\varphi(t,x)$.

Next, given a solution $v \in C^{1,2}([0,T) \times \mathbb{R}^k)$ of the PDE (2.39), which is continuous on $[0,T] \times \mathbb{R}^k$, then the process $(Y_s^{t,x}, Z_s^{t,x})_{t \le s \le T}$, defined by

$$Y_s^{t,x} := v(s, X_s^{t,x}) \quad \text{and} \quad Z_s^{t,x} := \sigma^{\mathsf{T}} D_x v(s, X_s^{t,x}),$$

provides a solution for a BSDE with data $(f, g(X_T))$. This assertion is straightforward by just applying Itô's formula to $v(s, X_s^{t,x})$, and using the fact that v is solution of (2.39). Further details can be seen in [45].

We now focus on the converse; that is, a solution of the BSDE with data $(f, g(X_T))$ provides a solution of PDE (2.39). For this purpose, first note that for a fixed $t \in [0,T]$ we can consider the unique solution $(X_s^{t,x})_{t\leq s\leq T}$ of the FSDE (2.36) —that in fact do exist because of our former assumptions—, then it is not difficult to obtain a unique solution $(Y^{t,x}, Z^{t,x})$ of the BSDE with data $(f(s, X_s^{t,x}, y, z),\, g(X_T^{t,x}))$. Moreover, in virtue of the Markovian framework of the forward dynamics, it can be proved that the solution $(Y^{t,x}, Z^{t,x})$ is adapted to the filtration generated by $(B_s - B_t)_{t\leq s\leq T}$ and thus $Y_t^{t,x}$ is deterministic (see, Proposition 4.2 in [23]). This fact allows us to define the deterministic function

$$u(t,x) := Y_t^{t,x}. \tag{2.41}$$

We point out that under the assumption imposed to f, this latter function is continuous, but not necessarily twice continuously differentiable and thus it could not be a classical solution of (2.39). The following definition clarifies in which sense the function u can be a solution of (2.39).

Definition 2.4. Let $v \in C([0,T]\times\mathbb{R}^k)$ be a continuous function satisfying $u(T,x) = g(x)$, $x\in\mathbb{R}^k$. Then v is called a *viscosity subsolution* (resp. *viscosity supersolution*) of PDE (2.39) if, for any $(t_0,x_0)\in[0,T]\times\mathbb{R}^k$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^k)$, under which (t_0,x_0) is a minimum (resp. maximum) of $\varphi - v$, it verifies

$$-\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) - f(t_0,x_0,\varphi(t_0,x_0),\left(\sigma^{\mathrm{T}}D_x\right)\varphi(t_0,x_0)) \leq 0$$

(resp.

$$-\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) - f(t_0,x_0,\varphi(t_0,x_0),\left(\sigma^{\mathrm{T}}D_x\right)\varphi(t_0,x_0)) \geq 0).$$

Moreover, v is said to be a viscosity solution of (2.39), if it is both, a viscosity subsolution and supersolution.

The link from BSDE to PDE is then as follows.

Theorem 2.14. *Under the Assumptions (B_1) and (B_2), the function $u(t,x)$ defined in (2.41) belongs to $C([0,T]\times\mathbb{R}^k)$ and it is viscosity solution of (2.39). If we suppose in addition that, for each $R>0$, there exists a continuous function $m_R:\mathbb{R}^+\to\mathbb{R}^+$ such that $m_R(0)=0$ and*

$$|f(t,x_1,y,z) - f(t,x_2,y,z)| \leq m_R(|x_1-x_2|(1+|z|)),$$

for all $t\in[0,T]$, $|x_1|,|x_2|\leq R$, $|z|\leq R$, $z\in\mathbb{R}^d$, then u is the unique viscosity solution of (2.39) in the space of continuous functions of polynomial growth.

Sketch of the proof. The arguments used to prove the continuity property can be seen in [48], whereas the uniqueness property is proved by methods from viscosity solutions, see for instance, [11]. Then we focus just on the existence inspired by references [25, 48]. More specifically, we shall briefly show that $u(t,x) = Y_t^{t,x}$ is a viscosity subsolution of (2.39). The viscosity supersolution property is analogous, so we will omit it from our analysis. Indeed, take $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^k)$ and $(t_0,x_0)\in[0,T)\times\mathbb{R}^k$ such that $\varphi-u$ has a minimum at (t_0,x_0) with $\varphi(t_0,x_0) = u(t_0,x_0)$. Suppose by contradiction that

$$-\partial_t\varphi(t_0,x_0) - \mathcal{L}\varphi(t_0,x_0) - f(t_0,x_0,\varphi(t_0,x_0),\left(\sigma^{\mathrm{T}}D_x\right)\varphi(t_0,x_0)) > 0.$$

Since (t_0,x_0) is a local minimum of $\varphi-u$ and in virtue of the continuity of f, φ, and the derivatives of this latter function, there exist both $h>0$ and $\varepsilon>0$ such that

$$u(t,x)\leq\varphi(t,x) \quad\text{and}\quad -\partial_t\varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,\varphi(t,x),\left(\sigma^{\mathrm{T}}D_x\right)\varphi(t,x)) > 0 \tag{2.42}$$

for all $(t,x)\in[t_0,t_0+h]\times[x_0-\varepsilon,x_0+\varepsilon]$. We now consider the stopping time

$$\theta := \inf\left\{s\geq t_0 : \left|X_s^{t_0,x_0} - x_0\right| \geq \varepsilon\right\} \wedge (t_0+h).$$

Applying Itô's formula to $\varphi(s, X_s^{t_0,x_0})$, it can be seen that

$$\big(\varphi(s, X_s^{t_0,x_0}), \big(\sigma^{\mathrm{T}} D_x\big)\, \varphi(s, X_s^{t_0,x_0});\ \ t_0 \le s \le \theta\big)$$

is the solution of the BSDE with coefficient $-\left(\partial_s + \mathcal{L}\right)\varphi(s,x)$, terminal time θ and terminal value $\varphi(\theta, X_\theta^{t_0,x_0})$

The next step is to compare this BSDE with the solution $(Y_s^{t_0,x_0})_{t_0 \le s \le \theta}$ of the BSDE with coefficient f and terminal condition $Y_\theta^{t_0,x_0} = u(\theta, X_\theta^{t_0,x_0})$. Note that by definition of θ and from inequalities in (2.42), we have

$$u(\theta, X_\theta^{t_0,x_0}) \le \varphi(\theta, X_\theta^{t_0,x_0}) \ \text{ and } \ f(t, X_t^{t_0,x_0}, \varphi(t, X_t^{t_0,x_0}), (\sigma^{\mathrm{T}} D_x)\varphi(t, X_t^{t_0,x_0})) < -(\partial_t + \mathcal{L})\varphi(t, X_t^{t_0,x_0})$$

thus by (strict) comparison Theorem 2.7, we get $u(t_0, x_0) < \varphi(t_0, x_0)$, that is a contradiction. Hence, u turns out a subsolution of the PDE (2.39). □

RBSDEs vs PDEs. In the same way, the solutions of a RBSDE in this Markovian framework can be linked to the following variational inequality (VI),

$$\begin{cases} \min\Big\{v(t,x) - h(t,x);\ -\partial_t v(t,x) - \mathcal{L}v(t,x) - f(t,x,v(t,x),\sigma^{\mathrm{T}} D_x v(t,x))\Big\} = 0; \\ \\ v(T,x) = g(x), \end{cases} \tag{2.43}$$

where $\mathcal{L}$ is the infinitesimal generator in (2.40) associated to $(X_s^{t,x})_{t \le s \le T}$ and $h : [0,T] \times \mathbb{R}^k \to \mathbb{R}$ is a jointly continuous function satisfying polynomial growth, i.e., there exist two non-negative real constants K and γ such that $|h(t,x)| \le K(1 + |x|^\gamma)$, for all $(t,x) \in [0,T] \times \mathbb{R}^k$. Moreover, we assume that $h(T,x) \le g(x)$, for $x \in \mathbb{R}^k$.

Next, given a solution v of (2.43), the processes defined by

$$\begin{aligned} &Y_s := v(s, X_s), \\ &Z_s := \sigma^{\mathrm{T}}(s, X_s) D_x v(s, X_s), \quad \text{and} \\ &K_s := \int_0^s -\partial_s v(r, X_r) - \mathcal{L}v(r, X_r) - f(r, X_r, v(r, X_r), \sigma^{\mathrm{T}} D_x v(r, X_r))\, dr. \end{aligned} \tag{2.44}$$

are solution of a RBSDE (2.9) with data $(f(t, X_t, y, z), g(X_T), h(t, X_t))$. (In this direction we can consider $t = 0$ and omit the superscripts t and x in X.) Indeed, plugging (2.44) into the equality in (2.9), simplifying and then using Itô's formula to $v(s, X_s)$, with terminal condition $g(X_T)$, it is straightforward that the equality is satisfied. The barrier condition $Y_s \ge h(s, X_s)$ is obtained directly from expression of (2.43). Finally, note that $(K_s)_{0 \le s \le T}$ is an increasing process and $K_0 = 0$. Moreover, considering the stopping time

$$\tau = \inf\{s \ge 0 : Y_s = h(s, X_s)\} \wedge T,$$

we have $Y_s > h(s, X_s)$, for $s \in [0, \tau]$, which implies that

$$-\partial_s v(s, X_s) - \mathcal{L}v(s, X_s) - f(s, X_s, v(s, X_s), \sigma^{\mathrm{T}} D_x v(s, X_s)) = 0,$$

in other words, $K_s = 0$, for $s \in [0, \tau]$, and then

$$\int_0^\tau (Y_t - h(t, X_t)) dK_t = 0.$$

We now focus on the converse, that is, the solution of a RBSDE provides a solution of a VI of type (2.43). As before, we fix t and consider the unique solution $X_{t \le s \le T}$ of (2.36) with b and σ satisfying (B_1). The assumption on f and g imply, by Theorem 2.9, that the RBSDE (2.37) has a unique solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$.

Therefore, this Markovian framework implies that $Y_t^{t,x}$ is deterministic which allows to define the deterministic function

$$u(t,x) := Y_t^{t,x}. \tag{2.45}$$

In this section we will define an alternative definition of viscosity solution related to the VI (2.43) that uses the concept of subjet and superjet. Before we give such alternative definition, we define the latter concepts.

Definition 2.5. Let $u \in C([0,T] \times \mathbb{R}^k)$ and $(t_0, x_0) \in [0,T) \times \mathbb{R}^k$. We denote by $\mathcal{P}^{2,+}u(t_0,x_0)$ (the "parabolic superjet" of u at (t_0,x_0)) the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^k \times S(k)$, where $S(k)$ is the set of symmetric $k \times k$ matrices, such that

$$u(s,y) \leq u(t_0,x_0) + p(s-t_0) + q \cdot (y-x_0) + \frac{1}{2}(y-x_0)^{\intercal} X (y-x_0) + o(|s-t_0| + |y-x_0|^2).$$

Similarly, we denote by $\mathcal{P}^{2,-}u(t_0,x_0)$ (the "parabolic subjet" of u at (t_0,x_0)) the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^k \times S(k)$ such that

$$u(s,y) \geq u(t_0,x_0) + p(s-t_0) + q \cdot (y-x_0) + \frac{1}{2}(y-x_0)^{\intercal} X (y-x_0) + o(|s-t_0| + |y-x_0|^2)$$

We have arrived to the definition of viscosity solution of the VI (2.43) that is given in terms of both superjet and subjet.

Definition 2.6. (a) We say that $v \in C([0,T] \times \mathbb{R}^k)$ is a viscosity subsolution of (2.43) if $v(T,x) \leq g(x)$, $x \in \mathbb{R}^k$, and at any point $(t_0,x_0) \in [0,T) \times \mathbb{R}^k$, for any $(p,q,X) \in \mathcal{P}^{2,+}v(t_0,x_0)$,

$$\min\Big\{ v(t_0,x_0) - h(t_0,x_0);\ -p - \frac{1}{2}Tr(\sigma\sigma^{\intercal}X) - b \cdot q - f(t_0,x_0,v(t_0,x_0),\sigma^{\intercal}(t_0,x_0)q) \Big\} \leq 0.$$

In other words at any point (t_0,x_0) where $v(t_0,x_0) > h(t_0,x_0)$,

$$-p - \frac{1}{2}Tr(\sigma\sigma^{\intercal}X) - b \cdot q - f(t_0,x_0,v(t_0,x_0),\sigma^{\intercal}(t_0,x_0)q) \leq 0.$$

(b) We say that $v \in C([0,T] \times \mathbb{R}^k)$ is a viscosity supersolution of (2.43) if $v(T,x) \geq g(x)$, $x \in \mathbb{R}^k$, and at any point $(t_0,x_0) \in [0,T) \times \mathbb{R}^k$, for any $(p,q,X) \in \mathcal{P}^{2,-}v(t_0,x_0)$,

$$\min\Big\{ v(t_0,x_0) - h(t_0,x_0);\ -p - \frac{1}{2}Tr(\sigma\sigma^{\intercal}X) - b \cdot q - f(t_0,x_0,v(t_0,x_0),\sigma^{\intercal}(t_0,x_0)q) \Big\} \geq 0$$

In other words at any point (t_0,x_0) where $v(t_0,x_0) \geq h(t_0,x_0)$ and

$$-p - \frac{1}{2}Tr(\sigma\sigma^{\intercal}X) - b \cdot q - f(t_0,x_0,v(t_0,x_0),\sigma^{\intercal}(t_0,x_0)q) \geq 0.$$

c) $v \in C([0,T] \times \mathbb{R}^k)$ is said to be a viscosity solution of (2.43) if it is both a viscosity sub- and supersolution.

The following theorem is borrowed from [25].

Theorem 2.15. *Consider the assumptions (B_1), (B_2) and that $h(T,x) \leq g(x)$, for $x \in \mathbb{R}^k$. Then, the function $u(t,x) = Y_t^{t,x}$, defined in (2.45), is a viscosity solution of the obstacle problem (2.43). Moreover, if for each $R > 0$, there exists a continuous function $m_R : \mathbb{R}^+ \to \mathbb{R}^+$ such that $m_R(0) = 0$ and*

$$|f(t,x_1,y,z) - f(t,x_2,y,z)| \leq m_R(|x_1 - x_2|(1+|z|)),$$

for all $t \in [0,T]$, $|x_1|, |x_2| \leq R$, $|y| \leq R$, $z \in \mathbb{R}^d$, then u is the unique viscosity solution of (2.43) in the class of continuous functions of polynomial growth.

Sketch of the proof. As in the case of standard BSDEs we show only the existence of sub- and supersolution property. As for the continuity of the function u and the uniqueness property, we refer the reader to [25] and [11], respectively. Moreover, from the continuity of u and the terminal condition of the BSDE, it is clear that $u(T,x) = g(x)$, for $x \in \mathbb{R}^k$.

Let us then prove the existence of a solution: To this end, we are going to use the penalized method for the RBSDE, which was introduced in the proof of Theorem 2.9. Namely, for each $(t,x) \in [0,T] \times \mathbb{R}^k$ and $n \in \mathbb{N}$, we let (Y^n, Z^n) be the solution of the standard BSDE

$$Y_s^{n,t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) + n(Y_r^{n,t,x} - S_r)^- dr - \int_s^T Z_r^{n,t,x} dB_r,$$

which exists since $f_n(t,x,y,z) := f(t,x,y,z) + n(y - h(t,x))^-$ and $g(X_T^{t,x})$ satisfies (A_1) and (A_2), respectively. Therefore, by Theorem 2.14 we have that

$$u^n(t,x) = Y_t^{n,t,x}, \quad 0 \le t \le T,\, x \in \mathbb{R}^k$$

is continuous and is a viscosity solution of the parabolic PDE

$$\begin{cases} -\partial_t u^n(t,x) - \mathcal{L}u^n(t,x) - f_n(t,x,u^n(t,x), \sigma^\intercal D_x u^n(t,x)) = 0, \quad 0 \le t \le T,\, x \in \mathbb{R}^k \\ \\ u^n(T,x) = g(x), \quad x \in \mathbb{R}^k. \end{cases}$$

Moreover, as in the convergence (2.17) in the proof of Theorem 2.9, we can see that, for each $0 \le t \le T$, $x \in \mathbb{R}^k$,

$$u^n(t,x) \nearrow u(t,x) \quad \text{as } n \to \infty. \tag{2.46}$$

Since u_n and u are continuous, it follows from Dini's Theorem that the above convergence is uniform on compact sets of $[0,T] \times \mathbb{R}^k$.

We now show that u is a subsolution of (2.43). Let us choose a point (t,x) for which $u(t,x) > h(t,x)$, and let $(p,q,M) \in \mathcal{P}^{2,+}u(t,x)$. From Lemma 6.1 in [11], there exist sequences

$$n_j \to \infty, \quad (t_j, x_j) \to (t,x), \quad (p_j, q_j, M_j) \in \mathcal{P}^{2,+}u_{n_j}(t_j, x_j),$$

such that

$$(p_j, q_j, M_j) \to (p,q,M).$$

But for any j,

$$-p_j - \frac{1}{2} Tr(\sigma\sigma^\intercal M_j) - b \cdot q_j - f(t_j, x_j, u^{n_j}(t_j,x_j), \sigma^\intercal(t_j,x_j) q_j) - n_j(u^{n_j}(t_j,x_j) - h(t_j,x_j))^- \le 0.$$

From the assumption that $u(t,x) > h(t,x)$ and the uniform convergence of u_n, for j large enough, it follows that $u^{n_j}(t_j,x_j) > h(t_j,x_j)$; hence, taking the limit as $j \to \infty$ in the above inequality we get

$$-p - \frac{1}{2} Tr(\sigma\sigma^\intercal M) - b \cdot q - f(t,x,u(t,x), \sigma^\intercal(t,x) q) \le 0.$$

and we have proved that u is a subsolution of (2.43).

We conclude the proof by showing that u is a supersolution of (2.43). Let (t,x) be an arbitrary point in $[0,T] \times \mathbb{R}^k$, and $(p,q,M) \in \mathcal{P}^{2,-}u(t,x)$. We already know that $u(t,x) \ge h(t,x)$. By the same argument as above, there exist sequences:

$$n_j \to \infty, \quad (t_j, x_j) \to (t,x), \quad (p_j, q_j, M_j) \in \mathcal{P}^{2,-}u^{n_j}(t_j, x_j),$$

such that

$$(p_j, q_j, M_j) \to (p,q,M).$$

However, for any j,

$$-p_j - \frac{1}{2}Tr(\sigma\sigma^{\intercal}M_j) - b\cdot q_j - f(t_j, x_j, u^{n_j}(t_j,x_j), \sigma^{\intercal}(t_j,x_j)q_j) - n_j(u^{n_j}(t_j,x_j) - h(t_j,x_j))^- \geq 0.$$

Hence

$$-p_j - \frac{1}{2}Tr(\sigma\sigma^{\intercal}M_j) - b\cdot q_j - f(t_j, x_j, u^{n_j}(t_j,x_j), \sigma^{\intercal}(t_j,x_j)q_j) \geq 0,$$

and taking the limit as $j \to \infty$, the above inequality yields

$$-p - \frac{1}{2}Tr(\sigma\sigma^{\intercal}M) - b\cdot q - f(t,x,u(t,x),\sigma^{\intercal}(t,x)q) \geq 0.$$

and we have proved that u is a supersolution of (2.43). □

DRBSDEs vs PDEs. Finally, let us see the relation of the solution of a DRBSDE and a min-max PDE of the type

$$\begin{cases} \min\left\{v(t,x) - L(t,x), \max\left[v(t,x) - U(t,x), -(\partial_t + \mathcal{L})v(t,x) \right.\right. \\ \qquad\qquad\qquad \left.\left. -f(t,x,v(t,x),\sigma^{\intercal}D_x v(x,t))\right]\right\} = 0, \\ v(T,x) = g(x), \end{cases} \tag{2.47}$$

where L and U are given functions, which are called the lower and upper obstacles, respectively. In this Markovian framework, consider the unique solution $(Y^{t,x}, Z^{t,x}, K^{t,x,+}, K^{t,x,-})$ of a DRBSDE associated to both, the process $(X_s^{t,x})_{0\leq s\leq T}$, where the data $(f, U, L, g(X_T^{t,x}))$, with f and g satisfying assumptions (B_1) and (B_2), and L and U are assumed to be completely separated, i.e., $L_t < U_t$, for all $t \in [0,T]$. The existence and uniqueness of this aforementioned solution is given by Theorem 3.7 in Hamadène and Hassani [28].

As in previous analysis, the part Y of this DRBSDE defines a deterministic function

$$u(t,x) := Y_t^{t,x}, \tag{2.48}$$

which provides a viscosity solution for (2.47), as will be established in Theorem 2.16. The definition of viscosity solution for the system (2.47) is as follows.

Definition 2.7. Let v be a function in $C([0,T]\times\mathbb{R}^k)$. This function is called a viscosity:

(i) subsolution of (2.47) if $v(T,x) \leq g(x)$ and for any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^k)$ and any minimum point $(t_0,x_0) \in (0,T)\times\mathbb{R}^k$ of $\varphi - v$, we have

$$\min\Big\{v(t_0,x_0) - L(t_0,x_0),\ \max\Big[v(t_0,x_0) - U(t_0,x_0),\ -(\partial_t + \mathcal{L})\varphi(t_0,x_0) - f(t_0,x_0,(t_0,x_0),\sigma^{\intercal}D_x\varphi(t_0,x_0))\Big]\Big\} \leq 0$$

(ii) supersolution of (2.47) if $v(T,x) \geq g(x)$, and for any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^k)$ and any maximum point $(t_0,x_0) \in (0,T)\times\mathbb{R}^k$ of $\varphi - v$, we have

$$\min\Big\{v(t_0,x_0) - L(t_0,x_0),\ \max\Big[v(t_0,x_0) - U(t_0,x_0),\ -(\partial_t + \mathcal{L})\varphi(t_0,x_0) - f(t_0,x_0,v(t_0,x_0),\sigma^{\intercal}D_x\varphi(t_0,x_0))\Big]\Big\} \geq 0$$

(iii) solution of (2.47) if it is both a viscosity subsolution and supersolution

The following result ensures the existence and uniqueness of viscosity solutions associated to (2.47). We will only give the main ideas of the proof. For further details, the reader is referred to the paper [28].

Theorem 2.16. *Under assumptions (B_1), (B_2) and $L_T < g(X_T^{t,x}) < U_T$, the function $u(t,x) = Y_t^{t,x}$, defined in (2.48), is a continuous viscosity solution of the obstacle problem (2.47). Moreover, if for each $R > 0$, there exists a continuous function $m_R : \mathbb{R}^+ \to \mathbb{R}^+$ such that $m_R(0) = 0$ and*

$$|f(t,x_1,y,z) - f(t,x_2,y,z)| \leq m_R(|x_1 - x_2|(1+|z|)),$$

for all $t \in [0,T]$, $|x_1|, |x_2| \leq R$, $|y| \leq R$, $z \in \mathbb{R}^d$, then u is the unique viscosity solution of (2.47) in the space of continuous functions of polynomial growth.

Sketch of the proof. The part ensuring the continuity of $u(t,x) = Y_t^{t,x}$ on $[0,T]\times\mathbb{R}^k$ follows by showing that u is the limit of both an increasing sequence of continuous functions and decreasing sequences of continuous functions. Indeed, let $(\underline{Y}^{n,t,x}, \underline{Z}^{n,t,x}, \underline{K}^{n,t,x})$ (resp. $(\bar{Y}^{n,t,x}, \bar{Z}^{n,t,x}, \bar{K}^{n,t,x})$) be the solution of one lower (resp. upper) barrier penalized RBSDE, i.e., for all $t \leq s \leq T$,

$$\underline{Y}_s^n = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \underline{Y}_r^{n,t,x}, \underline{Z}_r^{n,t,x})dr - n\int_s^T (U(r,X_r^{t,x}) - \underline{Y}_s^{n,t,x})^- dr + (\underline{K}_T^{n,t,x} - \underline{K}_s^{n,t,x}) - \int_t^T \underline{Z}_s^{n,t,x} dB_r,$$

$$\underline{Y}_s^{n,t,x} \geq L(s, X_s^{t,x}),$$

$$\int_t^T (\underline{Y}_r^{n,t,x} - L_r) d\underline{K}_r^{n,t,x} = 0.$$

(resp.

$$\bar{Y}_s^n = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \bar{Y}_r^{n,t,x}, \bar{Z}_r^{n,t,x})dr + n\int_s^T (L(r,X_r^{t,x}) - \bar{Y}_s^{n,t,x})^+ dr + (\bar{K}_T^{n,t,x} - \bar{K}_s^{n,t,x}) - \int_t^T \bar{Z}_s^{n,t,x} dB_r$$

$$\bar{Y}_s^{n,t,x} \leq U(s, X_s^{t,x})$$

$$\int_t^T (\bar{Y}_r^{n,t,x} - U_r) d\bar{K}_r^{n,t,x} = 0.)$$

Therefore, by Theorem 2.15, the deterministic functions $\underline{u}^n(t,x) := \underline{Y}_t^{n,t,x}$ (resp. $\bar{u}^n(t,x) := \bar{Y}_t^{n,t,x}$) is a continuous viscosity solution of the variational inequality

$$\begin{cases} \min\Big\{v(t,x) - L(t,x);\ -(\partial_t + \mathcal{L})v(t,x) - f(t,x,v(t,x),\sigma^\intercal D_x v(t,x)) + n(U(t,x) - v(t,x))^-\Big\} = 0; \\ v(T,x) = g(x), \end{cases} \tag{2.49}$$

(resp.

$$\begin{cases} \max\Big\{v(t,x) - U(t,x);\ -(\partial_t + \mathcal{L})v(t,x) - f(t,x,v(t,x),\sigma^\intercal D_x v(t,x)) - n(L(t,x) - v(t,x))^+\Big\} = 0; \\ v(T,x) = g(x)). \end{cases}$$

Moreover, by comparison theorem for RBSDEs (Theorem 2.10), the sequence $\overline{Y}^{n,t,x}$ (resp. $\underline{Y}^{n,t,x}$) is decreasing (resp. increasing) and moreover it converges in $\mathcal{S}_c^2$ to $Y^{t,x}$. In particular, for any $(t,x) \in [0,T]\times\mathbb{R}^k$,

$$\underline{u}^n(t,x) \nearrow u(t,x) \quad \text{as } n \to \infty \qquad (\text{resp.} \quad \bar{u}^n(t,x) \searrow u(t,x) \quad \text{as } n \to \infty),$$

and thus u is both lower and upper semicontinuous, which yields the continuity of u, yielding also that the convergence of $(\overline{u}^n)_{n\geq 0}$ and $(\underline{u}^n)$ is uniform on compact subsets of $[0,T]\times\mathbb{R}^k$.

It remains to prove that the function u is a viscosity subsolution of (2.47). Namely, since $u(T,x) = g(x)$ and $L(t,x) \leq u(t,x) \leq U(t,x)$, it is sufficient to prove that for any $\varphi \in C^{1,2}([0,T]) \times \mathbb{R}^k)$ and for any minimum point $(t,x) \in (0,T) \times \mathbb{R}^k$ of $\varphi - u$ such that $u(t,x) > L(t,x)$, we have

$$-\partial_t \varphi(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,v(t,x), \sigma^{\mathsf{T}} D_x \varphi(t,x)) \leq 0.$$

Let (t_n, x_n) be a sequence of local minimum points of $\varphi - u^n$ such that (t_n, x_n) converges to (t,x) (the existence of such a sequence follows from the uniform convergence of u^n to u (see e.g. [39], pp.117)). Note that for n large enough we have $\overline{u}^n(t_n,x_n) > L(t_n,x_n)$ then, using the fact that $\overline{u}^n$ is a viscosity solution of (2.49) we have,

$$-\partial_t \varphi(t_n,x_n) - \mathcal{L}\varphi(t_n,x_n) - f(t_n,x_n,u^n(t_n,x_n), \sigma^{\mathsf{T}} D_x \varphi(t_n,x_n)) \leq -n(U(t_n,x_n) - u^n(t_n,x_n))^- \leq 0.$$

Now the continuity of the functions and the uniform convergence yields the desired result. In a similar way we can show that u is also a viscosity supersolution. □

3

Switching control problems and systems of RBSDEs: continuous costs

In this chapter we give a review of some known results related to the theory of optimal switching problems when the switching costs are continuous. From a probabilistic point of view, these optimal problems can be studied by combining both, martingale approach (via Snell envelopes), and the theory of BSDEs. In the Markovian framework, we also review the link between switching problems and systems of variational inequalities with interconnected obstacles or the so-called Hamilton-Jacobi-Bellman equations for switching problems.

3.1 Optimal switching problem formulation

Let $\mathcal{I} = \{1, \cdots, q\}$ be a finite set, and consider the stochastic processes $\psi^i \in \mathcal{H}^2$, $i \in \mathcal{I}$, and $g^{ik} \in \mathcal{S}^2_c$, $i \in \mathcal{I}$ and $k \in \mathcal{I}^{-i}$, together with a sequence

$$\mathcal{S} = \big((\tau_n, \xi_n)\big)_{n\geq 0} \tag{3.1}$$

of non-decreasing stopping times (with respect to the filtration $\mathbb{F}$) τ_n, and random variables ξ_n, which are $\mathcal{F}_{\tau_n}$-measurable with values in $\mathcal{I} = \{1, \ldots, q\}$, such that $\tau_0 = t$, $\xi_0 = i$ for some initial state $(t, i) \in [0, T] \times \mathcal{I}$.

Let us fix an initial state $(t, i) \in [0, T] \times \mathcal{I}$ and define the next payoff function

$$J^i_t(\mathcal{S}) = \mathbb{E}\left[\sum_{n=0}^{\infty} \int_{\tau_n}^{\tau_{n+1}} \psi^{\xi_n}_s\, ds - \sum_{n=1}^{\infty} g^{\xi_{n-1}\xi_n}_{\tau_n} \mathbf{1}_{[\tau_n < T]} \Big| \mathcal{F}_t\right], \tag{3.2}$$

with $\psi^{\xi_n} := \psi^j$, when $\xi_n = j$; and the same reasoning applies to g^{ξ_{n-1},ξ_n}, i.e., $g^{\xi_{n-1},\xi_n} = g^{i,k}$ if $\xi_{n-1} = i$ and $\xi_n = k$. The processes ψ^i and g^{ik} are usually called the payoff rate per unit of the time and the switching cost, respectively.

Definition 3.1. A sequence $\mathcal{S} = ((\tau_n, \xi_n))_{n\geq 0}$ defined as in (3.1) is called a strategy or switching control policy for the controller. Furthermore, we say that a strategy $\mathcal{S}$ is *admissible* if it satisfies the following condition:

$$\mathbb{P}\left[\tau_n < T,\ \forall n \geq 0\right] = 0.$$

For each $i \in \mathcal{I}$, we denote by $\mathcal{A}^i_t$, the set of admissible strategies with the property, of $\tau_0 = t$ and $\xi_0 = i$.

Given an *admissible* strategy $((\tau_n, \xi_n))_{n\geq 0}$, we define the indicator process $(u_t)_{0\leq t\leq T}$ of the system at time t by

$$u(t) = \xi_0 \mathbf{1}_{[0,\tau_1)}(t) + \sum_{n=1}^{\infty} \xi_n \mathbf{1}_{[\tau_n,\tau_{n+1})}(t), \quad \text{for all} \quad t \in [0, T]. \tag{3.3}$$

It is clear that we can use indistinctly either a strategy

$$((\tau_n, \xi_n))_{n\geq 0}$$

or $(u(t))_{0\leq t\leq T}$. Thus, when we refer to an admissible policy, we shall use sometimes one or the other. Thus, using the indicator u, the payoff (3.2) is written as

$$J_t^i(u) = \mathbb{E}\left[\int_0^T \psi_s^{u(s)} ds - \sum_{n=1}^{\infty} g_{\tau_n}^{u(\tau_{n-1}),u(\tau_n)} \mathbf{1}_{[\tau_n<T]} \Big| \mathcal{F}_t\right]. \tag{3.4}$$

We will impose a condition on the switching cost processes g^{ik}, $i \in \mathcal{I}$, $k \in \mathcal{I}^{-i}$ that will be considered throughout this chapter.

(C) There exists a constant $\gamma > 0$ such that for each $i, j \in \mathcal{I}$, $g_t^{ik} \geq \gamma$ $\mathbb{P}$-a.s. for any $0 \leq t \leq T$.

Switching control problem: Find an admissible sequence $\mathcal{S}^* = ((\tau_n^*, \xi_n^*))_{n\geq 0}$ in $\mathcal{A}_t^i$ such that

$$J_t^i(\mathcal{S}^*) = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}_t^i} J_t^i(\mathcal{S}), \tag{3.5}$$

where J is the functional defined in (3.2). The expression in the right-hand side of (3.5) is called the value function of the switching problem, and $\mathcal{S}^*$, when it exists, it is referred to as an optimal control or strategy. In the particular case when $t = 0$, we have

$$J^i(\mathcal{S}^*) = \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}), \tag{3.6}$$

where $J^i := J_0^i$ and $\mathcal{A}^i := \mathcal{A}_0^i$.

The next results have been studied previously in the literature. For self-contained purposes they are mentioned with their respective rigorous proofs. The reader is referred for more details to [4, 9, 16, 21, 33, 36].

3.2 Verification theorem and existence results

Let us introduce the so-named *probabilistic verification theorem* for the switching problem that consists to characterize the value function as a process that satisfy a given system of interconnected Snell envelopes. Such a result uses as ancillary source of the martingale approach via Snell envelope theory (recall Proposition 2.5). For further details, we quote Theorem 1 in [16].

Theorem 3.1. *Assume that there exist q processes $(Y_t^i := (Y_t^i)_{0\leq t\leq T})_{i=1,\dots,q}$ in $\mathcal{S}_c^2$ such that for all $i \in \mathcal{I}$, $t \leq T$ and $\mathbb{P}$-a.s.,*

$$Y_t^i = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{ik}\right) \mathbf{1}_{[\tau<T]} \Big| \mathcal{F}_t\right]. \tag{3.7}$$

Then, for every initial state $(t, i) \in [0, T] \times \mathcal{I}$,

$$Y_t^i = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}_t^i} J_t^i(\mathcal{S}), \ a.s. \tag{3.8}$$

In particular, when $t = 0$ we have

$$Y_0^i = \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}).$$

Furthermore the strategy $\mathcal{S}_t^{i,*} = ((\tau_n^*, \xi_n^*))_{n\geq 0}$ defined as follows:

(i) $\tau_0^* := t, \quad \tau_1^* := \inf\left\{s \geq t : Y_s^i = \max_{k\in\mathcal{I}^{-i}} \left(Y_s^k - g_s^{i,k}\right)\right\} \wedge T$

and, for $n \geq 2$,

(ii) $\tau_n^* := \inf\left\{ s \geq \tau_{n-1}^* : Y_s^{\xi_{n-1}} = \max_{k\in\mathcal{I}^{-\xi_{n-1}}} \left(Y_s^k - g_s^{\xi_{n-1},k}\right)\right\} \wedge T$

where

- $\xi_0^* := i, \quad \xi_1^* = \arg\max_{k\in\mathcal{I}^{-i}} \left\{ Y_{\tau_1^*}^k - g_{\tau_1^*}^{i,k}\right\}$,

and for $n \geq 2$,

- $\xi_n^* = \arg\max_{k\in\mathcal{I}^{-\xi_{n-1}}} \left\{ Y_{\tau_n^*}^k - g_{\tau_n^*}^{\xi_{n-1},k}\right\}$,

is optimal.

Sketch of the proof. It can be shown using assumption (C) that strategy $((\tau_n^*, \xi_n^*))_{n\geq 0}$ is admissible (see Proposition 5.3 in [33]). We will show first that, by iterating the expression in (3.7) along with the strategy $((\tau_n^*, \xi_n^*))_{n\geq 0}$ and using repeatedly the Proposition 2.5 (iii), the expression in (3.7) will take the form of (3.4). Indeed, at time t the expression Y^i can be rewritten as, for any $t \leq s \leq T$,

$$Y_t^i + \int_0^t \psi_s^i ds = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_0^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{ik}\right) \mathbf{1}_{[\tau<T]} \Big| \mathcal{F}_t \right].$$

By Proposition 2.5 (iii), τ_1^* is optimal after t and thus

$$Y_t^i + \int_0^t \psi_s^i ds = \mathbb{E}\left[\int_0^{\tau_1^*} \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_{\tau_1^*}^k - g_{\tau_1^*}^{i,k}\right) \mathbf{1}_{[\tau_1^*<T]} \Big| \mathcal{F}_t \right] = \mathbb{E}\left[\int_0^{\tau_1^*} \psi_s^i ds + \left(Y_{\tau_1^*}^{\xi_1^*} - g_{\tau_1^*}^{i,\xi_1^*}\right) \mathbf{1}_{[\tau_1^*<T]} \Big| \mathcal{F}_t \right]$$

which yields

$$Y_t^i = \mathbb{E}\left[\int_t^{\tau_1^*} \psi_s^i ds + \left(Y_{\tau_1^*}^{\xi_1^*} - g_{\tau_1^*}^{i,\xi_1^*}\right) \mathbf{1}_{[\tau_1^*<T]} \Big| \mathcal{F}_t \right]. \tag{3.9}$$

Next, we point out that $\mathbb{P}$-a.s. for every $\tau_1^* \leq s \leq T$,

$$Y_s^{\xi_1^*} = \operatorname*{ess\,sup}_{\tau\geq s} \mathbb{E}\left[\int_s^\tau \psi_s^{\xi_1^*} ds + \max_{k\in\mathcal{I}^{-\xi_1^*}} \left(Y_\tau^k - g_\tau^{\xi_1^*,k}\right) \mathbf{1}_{[\tau<T]} \Big| \mathcal{F}_s \right]. \tag{3.10}$$

We do not prove this claim, but we refer the interested reader to [16] (page 2757). Therefore, from (3.10), the definition of τ_2^*, and again Proposition 2.5 (iii), we have

$$Y_{\tau_1^*}^{\xi_1^*} = \mathbb{E}\left[\int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds + \max_{k\in\mathcal{I}^{-\xi_1^*}} \left(Y_{\tau_2^*}^k - g_{\tau_2^*}^{\xi_1^*,k}\right) \mathbf{1}_{[\tau_2^*<T]} \Big| \mathcal{F}_{\tau_1^*} \right] = \mathbb{E}\left[\int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds + \left(Y_{\tau_2^*}^{\xi_2^*} - g_{\tau_2^*}^{\xi_1^*,\xi_2^*}\right) \mathbf{1}_{[\tau_2^*<T]} \Big| \mathcal{F}_{\tau_1^*} \right].$$

Inserting this back into (3.9) and noting both $\mathbf{1}_{[\tau_1^*<T]}$ is $\mathcal{F}_{\tau_1^*}$-measurable and $[\tau_2^* < T] \subset [\tau_1^* < T]$, it follows that

$$Y_t^i = \mathbb{E}\left[\int_t^{\tau_1^*} \psi_s^i ds + \int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds - g_{\tau_1^*}^{i,\xi_1^*} \mathbf{1}_{[\tau_1^*<T]} - g_{\tau_2^*}^{\xi_1^*,\xi_2^*} \mathbf{1}_{[\tau_2^*<T]} + Y_{\tau_2^*}^{\xi_2^*} \mathbf{1}_{[\tau_2^*<T]} \Big| \mathcal{F}_t \right].$$

Repeating this procedure n times, we have

$$Y_t^i = \mathbb{E}\left[\int_t^{\tau_n^*} \psi_s^{u^*(s)} ds - \sum_{k=1}^n g_{\tau_k^*}^{\xi_{k-1}^*,\xi_k^*} \mathbf{1}_{[\tau_k^*<T]} + Y_{\tau_n^*}^{\xi_n^*} \mathbf{1}_{[\tau_n^*<T]} \Big| \mathcal{F}_t \right],$$

where u^* is the indicator process of $((\tau_n^*, \xi_n^*)_{n\geq 0})$. Since the $(\tau_n^*)_{n\geq 0}$ is admissible control, then letting $n \to \infty$

$$Y_t^i = \mathbb{E}\left[\int_t^T \psi_s^{u^*(s)} ds - \sum_{k=1}^\infty g_{\tau_k^*}^{\xi_{k-1}^*,\xi_k^*} \mathbf{1}_{[\tau_k^*<T]} \Big| \mathcal{F}_t \right]. \tag{3.11}$$

It remains to show that $u^* = (\tau_n^*, \xi_n^*)$ is optimal, i.e., $J_t^i(u^*) \geq J_t^i(u)$, for all $u \in \mathcal{A}_t^i$. Indeed, let $u = ((\tau_n, \xi_n))_{n\geq 0}$ in $\mathcal{A}_t^i$ arbitrary. Since

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\bigg[\int_t^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{ik}\right) \mathbf{1}_{[\tau<T]}\Big|\mathcal{F}_t\bigg],$$

then iterating over u as we previously did with u^*, with inequality instead of equality, we have that after n steps we obtain

$$Y_t^i \geq \mathbb{E}\bigg[\int_t^{\tau_n} \psi_s^{u(s)} ds - \sum_{k=1}^n g_{\tau_k}^{\xi_{k-1},\xi_k} \mathbf{1}_{[\tau_k<T]} + Y_{\tau_n}^{\xi_n} \mathbf{1}_{[\tau_n<T]}\Big|\mathcal{F}_t\bigg].$$

Finally, using the Dominated Convergence Theorem, we can take limit as $n \to \infty$, to obtain

$$Y_t^i \geq \mathbb{E}\bigg[\int_t^{\tau_n} \psi_s^{u(s)} ds - \sum_{k=1}^n g_{\tau_k}^{\xi_{k-1},\xi_k} \mathbf{1}_{[\tau_k<T]}\Big|\mathcal{F}_t\bigg]. \tag{3.12}$$

From (3.11) and (3.12) we have that $J_t^i(u^*) \geq J_t^i(u)$, i.e., u^* is optimal and Y_t^i is the value function of the switching problem at the initial state (t, i). In particular, for $t = 0$, we deduce that $J^i(u^*) \geq J^i(u)$, for all $u \in \mathcal{A}^i$. □

In the previous theorem the existence of the processes $(Y^i)_{i\in\mathcal{I}}$ is assumed. Now we are going to briefly show how these processes can be constructed. The idea is to prove that such a processes can be obtained as limit of the following recursive sequence: for $i \in \mathcal{I}$, and any $0 \leq t \leq T$, define $(Y_t^{i,n})_{n\geq 0}$ as

$$Y_t^{i,0} = \mathbb{E}\bigg[\int_0^T \psi_s^i ds|\mathcal{F}_t\bigg] - \int_0^t \psi_s^i ds, \tag{3.13}$$

and for $n \geq 1$,

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\bigg[\int_0^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^{k,n-1} - g_\tau^{i,k})\right) \mathbf{1}_{[\tau<T]}\Big|\mathcal{F}_t\bigg] - \int_0^t \psi_s^i ds. \tag{3.14}$$

The next result was borrowed from Djehiche *et al* [16] and it is presented next.

Proposition 3.2. *Assume that for each $i, k \in \mathcal{I}$, $\psi^i \in \mathcal{H}^2$ and $g^{i,k} \in \mathcal{S}_c^2$. Then,*

(i) Increasing and bounded sequences: for each $n \geq 0$, the processes $Y^{1,n}, \ldots, Y^{q,n}$ belong to $\mathcal{S}_c^2$, and verify for each $i \in \mathcal{I}$ and $t \leq T$,

$$Y_t^{i,n} \leq Y_t^{i,n+1} \leq \mathbb{E}\Big[\int_t^T \max_{k\in\mathcal{I}} |\psi_s^k| ds|\mathcal{F}_t\Big].$$

(ii) Limit processes: there exist q processes $Y^1, \ldots, Y^q$ in $\mathcal{S}_c^2$ such that for any $i \in \mathcal{I}$ and $t \leq T$,

(a) The sequence converges: for all $n \geq 0$,

$$\lim_{n\to\infty} Y_t^{i,n} \nearrow Y_t^i \quad \text{and} \quad \mathbb{E}\Big[\sup_{s\leq T} |Y_s^{i,n} - Y_s^i|^2\Big] \to 0 \quad \text{as } n \to +\infty.$$

(b) The limit processes satisfy the verification theorem: the q processes $Y^1, \ldots, Y^q$ hold

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\bigg[\int_t^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{i,k}\right) \mathbf{1}_{[\tau<T]}|\mathcal{F}_t\bigg].$$

Sketch of the proof. (i) We proceed by induction on n to show that for any $i \in \mathcal{I}$, $Y^{i,n} \in \mathcal{S}_c^2$, for all $n \in \mathbb{N}$. If $n = 0$, then we can see that $Y^{i,0}$ is the sum of a continuous martingales w.r.t the Brownian filtration minus a continuous process of finite variation. Moreover, since $\psi_s^i \in \mathcal{H}^2$, then by using Doob's inequality we have that $Y^{i,0} \in \mathcal{S}_c^2$. Now assume that for $n \geq 0$, $Y^{i,n} \in \mathcal{S}_c^2$. Note that

$$Y_t^{i,n+1} = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\Big[\int_0^\tau \psi_s^i ds + \max_{k \in \mathcal{I}^{-i}} \big(Y_\tau^{k,n} - g_\tau^{i,k}\big)\big) \mathbf{1}_{[\tau < T]} | \mathcal{F}_t\Big] - \int_0^t \psi_s^i ds. \tag{3.15}$$

Therefore, let us analyze the continuity of $[\max_{k \in \mathcal{I}^{-i}} \big(Y_s^{k,n} - g_s^{i,k})\big) \mathbf{1}_{[s<T]}]_{0 \leq s \leq T}$ in $[0,T)$ and its jump at T. By the induction hypothesis $Y^{k,n}$ is continuous on $[0,T]$ and $Y_T^{k,n} = 0$. On the other hand, since the processes $g^{i,k}$ are continuous on $[0,T]$ and that there exist a constant $\gamma > 0$ such that $g^{i,k} > \gamma$, then it is clear that the process $[\max_{k \in \mathcal{I}^{-i}} \big(Y_s^{k,n} - g_s^{i,k})\big) \mathbf{1}_{[s<T]}]_{0 \leq s \leq T}$ is continuous on $[0,T)$ and has a positive jump at T, which implies by Proposition 2.5 (iii) that the first part in the right side on (3.15) is continuous, and it is clear that the second integral is continuous too. Hence using again Doob's inequality we can deduce $Y^{i,n+1} \in \mathcal{S}_c^2$. Let us now see that the sequence $(Y^{i,n})_{n \geq 0}$ is increasing. Note that each iteration in the sequence produces a switching, i.e., if $\mathcal{A}_t^{i,n} = \{u \in \mathcal{A}_t^i : u_0 = i,\ \tau_1 \geq t, \quad \text{and} \quad \tau_{n+1} = T\}$, then we have that

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^{i,n}} \mathbb{E}\Big[\int_t^{\tau_n} \psi_s^{u(s)} ds - \sum_{k=1}^n g_{\tau_k}^{\xi_{k-1},\xi_k} \mathbf{1}_{[\tau_k < T]} \Big| \mathcal{F}_t\Big]. \tag{3.16}$$

Therefore, since from step n to step $n+1$ one switching arises, then $\mathcal{A}_t^{i,n} \subset \mathcal{A}^{i,n+1}$ and thus by continuity of $Y^{i,n}$, we have that $\mathbb{P}$-a.s. for all $t \in [0,T]$, $Y_t^{i,n} \leq Y_t^{i,n+1}$. Moreover, from representation (3.16) and taking into account that for $i,j \in \mathcal{I}$, $g^{i,j} \geq \gamma > 0$, we have that for any $i \in \mathcal{I}$, $t \in [0,T]$,

$$Y_t^{i,n} \leq Y_t^{i,n+1} \leq \mathbb{E}\Big[\int_t^T \max_{k \in \mathcal{I}} |\psi_s^k| ds \Big| \mathcal{F}_t\Big]. \tag{3.17}$$

(ii) The last inequality implies that for any $i \in \mathcal{I}$, the sequences $(Y^{i,n})_{n \geq 0}$ converges. Letting $Y_t^i := \lim_{n \to \infty} Y_t^{i,n}$ for $t \leq T$, inequality (3.17) implies that Y^i satisfies for $t \leq T$

$$Y_t^{i,0} \leq Y_t^i \leq \mathbb{E}\Big[\int_t^T \max_{k \in \mathcal{I}} |\psi_s^k| ds \Big| \mathcal{F}_t\Big].$$

Note that for any $i \in \mathcal{I}$, $(Y_t^i)_{0 \leq t \leq T}$ is a càdlàg supermartingale as it is the limit of the next increasing sequence of the continuous supermartingale $(Y_t^{i,n} + \int_0^t \psi_s^i ds)_{0 \leq t \leq T}$ (see e.g. [17], page 473). Actually, the continuity of the switching costs along with the hypothesis (C), yield for any $i \in \mathcal{I}$, $(Y_t^i)_{0 \leq t \leq T}$ are continuous —for further details of this last assertion, we refer the reader to Theorem 2 in [16]. Hence, by Dini's Theorem and Lebesgue Dominated Convergence Theorem, the convergence is also uniformly, i.e.,

$$\mathbb{E}\Big[\sup_{s \leq T} |Y_s^{i,n} - Y_s^i|^2\Big] \to 0 \quad \text{as } n \to \infty.$$

The part (ii)-(b) follows from Proposition 2.5 (iv). □

3.3 Systems of RBSDEs with continuous barriers

In this section we introduce the theory of systems of RBSDEs that are a powerful tool to give optimality solutions for switching control problems. As we will see later, under suitable conditions, a solution that verifies this type of systems, will coincide with the value function associated to a switching control problems.

The theory illustrated in here will be studied with a high degree of generality. As a special case we will be able to link this theory with optimal switching control problems.

The rest of this section is devoted to the study of existence and uniqueness of the system of RBSDEs (3.18). For self-containedness purposes, we will include the main ideas of the proofs same that were inspired from references [10, 33].

Consider the following system of RBSDEs, for all $i \in \mathcal{I}$ and all $0 \le s \le T$,

$$\begin{cases} \text{Find } \left(Y^i_\cdot, Z^i_\cdot, K^i_\cdot\right) \in \mathcal{S}^2_c \times \mathcal{H}^{2,d} \times \mathcal{S}^2_c \text{ such that :} \\ Y^i_s = h^i(X_T) + \displaystyle\int_s^T f^i(r, X_r, Y^1_r, \ldots, Y^q_r, Z^i_r)dr + K^i_T - K^i_s - \int_s^T Z^i_r dB_r; \\ Y^i_s \ge \displaystyle\max_{k \in \mathcal{I}^{-i}} \left\{ Y^k_s - g^{i,k}(s, X_s) \right\}; \\ \displaystyle\int_0^T \Big(Y^i_r - \max_{k \in \mathcal{I}^{-i}} \left\{ Y^k_r - g^{i,k}(r, X_r) \right\} \Big) dK^i_r = 0, \end{cases} \tag{3.18}$$

where $K^i_\cdot$ is non-decreasing and $K^i_0 = 0$.

Note that in this system the coefficient f^i depends on the other solutions $(Y^k)_{k \in \mathcal{I}^{-i}}$. Existence and uniqueness of solutions of this type of systems are guaranteed under the next set of assumptions.

Assumption H.

(H1) : The stochastic process $(X_t)_{t \ge 0}$ belongs to $\mathcal{S}^{2,k}_c$

(H2) : For any $i \in \mathcal{I}$, the function $f^i : [0,T] \times \mathbb{R}^k \times \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R}$ verifies:

(i) $(t, x) \mapsto f^i\left(t, x, y^1, \ldots, y^q, z\right)$ is continuous uniformly with respect to $(y^1, \ldots, y^q, z)$;

(ii) f^i is Lipschitz continuous with respect to $(y^1, \ldots, y^q, z)$ uniformly on $[0,T] \times \mathbb{R}^k$, i.e., for some $C \ge 0$,

$$\left| f^i(t, x, y^1, \ldots, y^q, z) - f^i(t, x, \bar{y}^1, \ldots, \bar{y}^q, \bar{z}) \right| \le C \left(\left| y^1 - \bar{y}^1 \right| + \cdots + |y^q - \bar{y}^q| + |z - \bar{z}| \right).$$

(iii) The mapping $(t, x) \mapsto f^i(t, x, 0, \ldots, 0)$ is of polynomial growth.

(iv) *monotonicity*: For all $i \in \mathcal{I}$, for all $k \in \mathcal{I}^{-i}$, the mapping $y_k \mapsto f^i(t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_q, z)$ is non-decreasing whenever the other components $(t, x, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_q, z)$ are fixed.

(H3) :

(i) For each $i, j \in \mathcal{I}$, the function $g^{ij} : [0,T] \times \mathbb{R}^k \to \mathbb{R}$ is bounded from below; i.e. there exists a real constant $\gamma > 0$ such that, $g^{ij} \ge \gamma$. Furthermore it is jointly continuous and of polynomial growth in x.

(ii) *Non-free loop property*: For any loop of length $N-1$, i.e., a sequence $\{i_1, \ldots, i_N\}$ in $\mathcal{I}$ with $N-1$ distinct elements such that $i_1 \ne i_2, \ldots, i_{N-1} \ne i_N, i_N = i_1$, we have for all $(t, x) \in [0,T] \times \mathbb{R}^k$,

$$\sum_{j=1}^{N-1} g^{j,j+1}(t, x) > 0 \quad \mathbb{P}\text{-a.s.}$$

(H4) : *Consistency*: For each $i \in \mathcal{I}$, the function $h^i : \mathbb{R}^k \to \mathbb{R}$ is continuous of polynomial growth and satisfies

$$\forall x \in \mathbb{R}^k \quad h^i(x) \ge \max_{j \in \mathcal{I}^{-i}} \left(h^j(x) - g^{ij}(T, x) \right).$$

We begin by introducing an optimal control switching problem associated with a particular BSDEs. This allows us to give a switching representation for a solution (Y, Z, K) of the system (3.18).

Given a state $(t,i)\in[0,T]\times\mathcal{I}$, let $\mathcal{D}_t^i$ be the following set of strategies:

$$\mathcal{D}_t^i := \Big\{u=(\theta_n,\kappa_n)_{n\geq 0} : \theta_0 = t, \kappa_0 = i \text{ and } \mathbb{E}[(\mathtt{C}_T^u)^2]<\infty\Big\}$$

where $\mathtt{C}_r^u$, $r\leq T$, is the following cumulative costs up to time r, i.e.,

$$\mathtt{C}_r^u := \sum_{n=1}^{\infty} g^{\kappa_{n-1},\kappa_n}(\theta_n, X_{\theta_n})\mathbf{1}_{[\theta_n\leq r]} \quad \text{for } r<T \text{ and } \mathtt{C}_T^u = \lim_{r\to T}\mathtt{C}_r^u, \ \mathbb{P}\text{-a.s.}$$

Therefore for any admissible strategy $u\in\mathcal{D}_t^i$ we have:

$$\mathtt{C}_T^u = \sum_{n=1}^{\infty} g^{\kappa_{n-1},\kappa_n}(\theta_n, X_{\theta_n})\mathbf{1}_{[\theta_n< T]}. \tag{3.19}$$

Given a solution $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ of (3.18), we denote $\mathbf{Y}=(Y^i,\dots,Y^q)$. For any $u=(\theta_n,\kappa_n)_{n\geq0}\in\mathcal{D}_t^i$, let $(P_s^u,Q_s^u)_{s\leq T}$ be the solution of the following BSDE

$$\begin{cases} P_\cdot^u \text{ is càdlàg and } \mathbb{E}\big[\sup_{s\leq T}|P_s^u|^2\big]<\infty, \ \ Q_\cdot^u\in\mathcal{H}^{2,d}; \\ P_s^{u(s)} = h^{u(T)}(X_T) + \int_s^T f^{u(r)}(r,X_r,(\mathbf{Y}^{-u(r)},P_r^{u(r)}),Q_r^{u(r)})dr - (\mathtt{C}_T^u-\mathtt{C}_s^u) - \int_s^T Q_r^{u(s)}dB_r,\ s\leq T, \end{cases} \tag{3.20}$$

where $(\mathbf{Y}^{-i},P^i) := (Y^1,\cdots,Y^{i-1},P^i,Y^{i+1},\cdots,Y^q)$, and

$$h^{u(T)}(x) = h^{\kappa_n}(x)\mathbf{1}_{[\theta_n<T\leq\theta_{n+1}]} \text{ and}$$

$$f^{u(r)}(r,x,v_1,\dots,v_q,z) := \sum_{n=0}^{\infty} f^{\kappa_n}(r,x,v_1,\dots,v_q,z)\mathbf{1}_{[\theta_n\leq r<\theta_{n+1})}.$$

Making the change of variable $\bar{P}_\cdot^u := P_\cdot^u - \mathtt{C}_\cdot^u$, the equation in (3.20) is transformed in a standard BSDE. Since $\mathtt{C}^u$ is adapted and $\mathbb{E}[(\mathtt{C}_T^u)^2]<\infty$, we easily deduce the existence and uniqueness of the process $(P_\cdot^u,Q_\cdot^u)$.

Now we are in conditions to present the following characterization of the solutions of the system (3.18). Details of the proof of this result, can be consulted in [33].

Theorem 3.3. *Suppose there exists a solution $(Y^j,Z^j,K^j)_{j\in\mathcal{I}}$ to the system of RBSDEs (3.18). Then, for every initial state $(t,i)\in[0,T]\times\mathcal{I}$,*

$$Y_t^i = \operatorname*{ess\,sup}_{u(\cdot)\in\mathcal{D}_t^i}[P_t^{u(\cdot)} - \mathtt{C}_t^{u(\cdot)}],$$

where $\mathtt{C}_\cdot^u$ and $P_\cdot^u$ are defined in the previous lines.

Sketch of the proof. Let us denote $\mathbf{Y}=(Y^1,\dots,Y^q)$. Now, given an initial state $(t,i)\in[0,T]\times\mathcal{I}$, we know that

$$Y_t^i = h^i(X_T) + \int_t^T f^i(s,X_s,\mathbf{Y}_s,Z_s^i)ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s.$$

Given $u\in\mathcal{D}_t^i$, consider the first switching τ_1 and the associated state $u(\tau_1)$ of this strategy, then we get

$$\begin{aligned} Y_t^i &= h^i(X_T) + \int_t^{\tau_1} f^i(s,X_s,\mathbf{Y}_s,Z_s^i)ds + \int_{\tau_1}^T f^i(s,X_s,\mathbf{Y}_s,Z_s^i)ds + K_T^i - K_{\tau_1}^i + K_{\tau_1}^i - K_t^i \\ &\qquad - \int_t^{\tau_1} Z_s^i dB_s - \int_{\tau_1}^T Z_s^i dB_s. \\ &= Y_{\tau_1}^i + \int_t^{\tau_1} f^i(s,X_s,\mathbf{Y}_s,Z_s^i)ds + K_{\tau_1}^i - K_t^i - \int_t^{\tau_1} Z_s^i dB_s \\ &\geq Y_{\tau_1}^{u(\tau_1)} - g_{\tau_1}^{i,u(\tau_1)} + \int_t^{\tau_1} f^i(s,X_s,\mathbf{Y}_s,Z_s^i)ds - \int_t^{\tau_1} Z_s^i dB_s, \end{aligned} \tag{3.21}$$

where in the last inequality we use both that the process K^i is non-decreasing, and $Y^i \geq \max_{j\in\mathcal{I}^{-i}}(Y^{i,j} - g^{i,j})$. Next, note that

$$Y^{u(\tau_1)}_{\tau_1} = h^{u(\tau_1)}(X_T) + \int_{\tau_1}^T f^{u(\tau_1)}(s, X_s, \mathbf{Y}_s, Z^{u(\tau_1)}_s)ds + K^{u(\tau_1)}_T - K^{u(\tau_1)}_{\tau_1} - \int_{\tau_1}^T Z^{u(\tau_1)}_s dB_s.$$

and this considering the next switching $(\tau_2, u(\tau_2))$ and writing an analogous procedure as in (3.21), we have that

$$Y^{u(\tau_1)}_{\tau_1} \geq Y^{u(\tau_2)}_{\tau_2} - g^{u(\tau_1),u(\tau_2)}_{\tau_2} + \int_{\tau_1}^{\tau_2} f^{u(\tau_1)}(s, X_s, \mathbf{Y}_s, Z^{u(\tau_1)}_s)ds - \int_{\tau_1}^{\tau_2} Z^{u(\tau_1)}_s dB_s$$

Inserting this back into (3.21) we see that

$$Y^i_t \geq Y^{u(\tau_2)}_{\tau_2} - \sum_{n=1}^{2} g^{u(\tau_{n-1}),u(\tau_n)}_{\tau_n} + \sum_{n=1}^{2}\int_{\tau_{n-1}}^{\tau_n} f^{u(\tau_{n-1})}(s, X_s, \mathbf{Y}_s, Z^{u(\tau_{n-1})}_s)ds - \sum_{n=1}^{2}\int_{\tau_{n-1}}^{\tau_n} Z^{u(\tau_{n-1})}_s dB_s.$$

Iterating N times, we obtain

$$Y^i_t \geq Y^{u(\tau_N)}_{\tau_N} + \sum_{n=1}^{N} g^{u(\tau_{n-1}),u(\tau_n)}_{\tau_n} + \sum_{n=1}^{N}\int_{\tau_{n-1}}^{\tau_n} f^{u(\tau_{n-1})}(s, X_s, \mathbf{Y}_s, Z^{u(\tau_{n-1})}_s)ds - \sum_{n=1}^{N}\int_{\tau_{n-1}}^{\tau_n} Z^{u(\tau_{n-1})}_s dB_s.$$

Since the control strategy u is finite, we can let $N \to \infty$ to obtain

$$Y^i_t \geq Y^{u(T)}_{T} + \sum_{n=1}^{\infty} g^{u(\tau_{n-1}),u(\tau_n)}_{\tau_n} + \sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n} f^{u(\tau_{n-1})}(s, X_s, \mathbf{Y}_s, Z^{u(\tau_{n-1})}_s)ds - \sum_{n=1}^{\infty}\int_{\tau_{n-1}}^{\tau_n} Z^{u(\tau_{n-1})}_s dB_s.$$

Taking into account that $Y^{u(T)}_T = h^{u(T)}(X_T)$ as well as (3.19), the last expression can be rewritten as

$$Y^i_t \geq h^{u(T)}(X_T) + \mathtt{C}_T + \int_t^T f^{u(s)}(s, X_s, \mathbf{Y}_s, Z^{u(s)}_s)ds - \int_t^T Z^{u(s)}_s dB_s.$$

We denote the right side of this last inequality as Y^u. Note that Y^u and $P^u - \mathtt{C}^u$ coincide since they have the same data $h^{u(T)}$ and f^u. Therefore, we have

$$Y^i_t \geq P^u_t - \mathtt{C}^u_t,$$

and this implies

$$Y^i_t \geq \operatorname*{ess\,sup}_{u\in\mathcal{D}^i_t}(P^u_t - \mathtt{C}^u_t).$$

To show the equality, we define the following strategy $u^* = (\tau^*_n, \xi^*_n)_{n\geq 0} \in \mathcal{D}^i_t$ given by

$$\tau^*_n := \inf\left\{s \geq \tau^*_{n-1} : Y^{\xi^*_{n-1}}_s = \max_{k\in\mathcal{I}^{-\xi^*_{n-1}}}\left(Y^k_s - g^{\xi^*_{n-1},k}_s\right)\right\} \wedge T, \qquad \text{for } n \geq 1,$$

and

$$\xi^*_0 := i, \quad \xi^*_n = \arg\max_{k\in\mathcal{I}^{-\xi_{n-1}}}\left\{Y^k_{\tau^*_n} - g^{\xi_{n-1},k}_{\tau^*_n}\right\}, \quad \text{for } n \geq 1.$$

Proceeding as above and taking into account that $K^{\xi_{n-1}}_{\tau^*_n} - K^{\xi_{n-1}}_s = 0$ for $\tau^*_{n-1} \leq s < \tau^*_n$, we can deduce

$$Y^i_t = P^{u^*}_t - \mathtt{C}^{u^*}_t$$

Hence u^* is optimal, which proves the desired result. □

We now give an existence result of the solution of the system (3.18), whose extended proof can be seen in [33].

Theorem 3.4. *Assume that for any $i, j \in \mathcal{I}$, the data f^i, $g^{i,j}$ and $h^{i,j}$ satisfy the Assumption H. Then, there exists $((Y^i, Z^i, K^i))_{i=1,\dots,q}$ a unique solution of the system (3.18).*

Sketch of the Proof. The idea is to construct a monotone sequence of stochastic processes $\{(Y^{1,n}, Y^{2,n}, \cdots, Y^{q,n})\}_{n\geq 0}$ such that its limit denoted by $(Y^1, Y^2, \cdots, Y^q)$ coincides with of the solution of (3.18). As for the existence for the remaining part (Z, K), the arguments are based on the so-named Peng's Monotonic Limit Theorem (see [46]).

Indeed, let us define

$$f_{\max}(s,y,z) := \max_{i\in\mathcal{I}}[f^i](s,X_s,y,\dots,y,z) \quad \text{and} \quad f_{\min}(s,y,z) := \min_{i\in\mathcal{I}}[f^i](s,X_s,y,\dots,y,z)$$

and

$$h_{\max} := \max_{i\in\mathcal{I}}[h^i] \quad \text{and} \quad h_{\min} := \min_{i\in\mathcal{I}}[h^i]$$

Therefore, since the data f^i, and h^i hold Assumption H, then the data $(f_{\max}, h_{\max})$ and $(f_{\min}, h_{\min})$ in turn satisfy assumption (A_1) and (A_2), then a simple application of Theorem 2.6 ensures the existence of solutions $(\overline{Y}, \overline{Z})$ and $(\underline{Y}, \underline{Z})$ of the standard BSDEs

$$\overline{Y}_t = h_{\max}(X_T) + \int_t^T f_{\max}(s, X_s, \overline{Y}_s, \overline{Z}_s)ds - \int_t^T \overline{Z}_s dB_s \quad \text{for all } t\in[0,T], \text{ a.s.}$$

and

$$\underline{Y}_t = h_{\min}(X_T) + \int_t^T f_{\min}(s, X_s, \underline{Y}_s, \underline{Z}_s)ds - \int_t^T \underline{Z}_s dB_s \quad \text{for all } t\in[0,T], \text{ a.s.}$$

Next, consider the following sequences of RBSDEs defined recursively by : for any $i \in \mathcal{I}$, $Y^{i,0} = \underline{Y}$ and for $n \geq 1$ and $s \leq T$,

$$\begin{cases} Y^{i,n}_{\cdot}, K^{i,n}_{\cdot} \in \mathcal{S}^2 \quad \text{and} \quad Z^{i,n}_{\cdot} \in \mathcal{H}^{2,d};\ K^{i,n}_{\cdot} \text{ is non-decreasing with } \quad K^i_0 = 0, \\ Y^{i,n}_s = h^i + \int_s^T f^i(r, X_r, (\mathbf{Y}^{-i,n-1}_r, Y^{i,n}_r), Z^{i,n}_r)dr + K^{i,n}_T - K^{i,n}_s - \int_s^T Z^{i,n}_r dB_r \\ Y^{i,n}_s \geq \max_{k\in\mathcal{I}^{-i}} \left\{Y^{k,n-1}_s - g^{ik}(s,X_s)\right\}, \quad \text{for all } 0\leq s\leq T, \\ \int_0^T \left(Y^{i,n}_r - \max_{k\in\mathcal{I}^{-i}} \left\{Y^{k,n-1}_r - g^{ik}(r,X_r)\right\}\right) dK^{i,n}_r = 0, \end{cases} \tag{3.22}$$

where $(\mathbf{Y}^{-i,n-1}, Y^{i,n})$ denote the situation when all the components other than i are fixed at $n-1$ and the i-th component is considering at the iteration n. Therefore, by Theorem 2.9 and an induction argument, for any $n \geq 0$, there exist $(Y^{i,n}.Z^{i,n}, K^{i,n})$ solution of (3.22). Thus by using the comparison theorem for standard BSDEs, Theorem 2.7, we have that $Y^{i,0} \leq Y^{i,1}$, for any $i \in \mathcal{I}$. On the other hand, since f^i satisfies the monotonicity property and combined with the comparison theorem for RBSDEs, Theorem 2.10, we get by induction that for $n \geq 0$ and $i \in \mathcal{I}$

$$Y^{i,n} \leq Y^{i,n+1}.$$

On the other hand, the q processes $((\overline{Y}, \overline{Z}, \overline{K} := 0))_{i\in\mathcal{I}}$ are solutions of the system of RBSDEs with data $(f_{\max}, h_{max}, \max_{k\in\mathcal{I}^{-i}}(\overline{Y} - g^{i,k}))$. Then, by the monotonicity property of f^i and the comparison Theorem 2.10, we have for all for any $i, j \in \mathcal{I}$,

$$Y^{i,n} \leq \overline{Y} \quad \text{for all } n\geq 0.$$

Hence, for each $i \in \mathcal{I}$, the sequence $(Y^{i,n})_{n\geq 0}$ is bounded and monotone and thus a simple use of Peng's Monotonic Limit Theorem [46] yield that there exist $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{S}^2$ such that $(Y^{i,n}) \nearrow Y^i$ pointwisely, $Z^{i,n} \to Z^i$ weakly in $\mathcal{H}^{2,d}$ and $(K^{i,n}) \to K^i$ is a càdlàg square-integrable increasing process.

Finally, to prove that the limit processes $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ verifies (3.18), we refer the reader to Theorem 3.2 in [35]. □

3.4 Markovian framework and systems of PDE's with continuous obstacles

In this section we illustrate some results about the connection of special systems of RBSDEs related with other systems of PDE's. As a by-product, we will also be able to link this latter relation with a certain switching optimal control problems. The most part of the analysis is based on the references [16, 21].

Let us then consider the following special cases of some mathematical objects introduced in previous sections. For any $i, j \in \mathcal{I}$, we will assume that f^i does not depend on Y^i, $g^{i,j}$ does not depend on X and $h^i(X_T) = 0$. Other type of data as well as other assumptions on the data have been also studied. We encourage the reader to check the references [10, 26, 33, 35, 36] for further details.

As we mentioned in Chapter 2, when randomness comes from a diffusion process $(X_s^{t,x})_{t\le s\le T}$ as in (2.36), solutions of RBSDEs provide a deterministic functions which in turn they become solutions of PDE's (3.23); the same happens when we want to extend this property for interconnected systems as we will show later on. Another important feature is that the switching problem is related with the viscosity solution of the following system of PDE's or variational inequalities (VI)

$$\begin{cases} \min\left\{v^i(t,x) - \max\limits_{k\in\mathcal{I}^{-i}}\left(v^k(t,x) - g^{ik}(t)\right);\ -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - f^i(t,x)\right\} = 0; \\ \\ v^i(T,x) = 0 \qquad \text{for all } i \in 1,\cdots,q. \end{cases} \tag{3.23}$$

Let us first recall the following estimates of the process $(X^{t,x})_{t\le s\le T}$ defined in (2.36), whose proof is given in [49, 40].

Proposition 3.5. *(i) There exist a constant $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{0\le s\le T}|X_s^{t,x}|^2\Big] \le C(1+|x|^2)$$

(ii) There exists a constant $C > 0$ such that for all $(t,x),(t',x')$ in $[0,T]\times\mathbb{R}^k$

$$\mathbb{E}\Big[\sup_{0\le s\le T}|X_s^{t,x} - X_s^{t',x'}|^2\Big] \le C(1+|x|^2)(|x-x'|^2 + |t-t'|).$$

We now focus on showing that there exist deterministic functions $(v^i)_{i\in\mathcal{I}}$ such that for each $i \in \mathcal{I}$ and $t \le s \le T$,

$$Y_s^{i,t,x} = v^i(s, X_s^{t,x}), \tag{3.24}$$

where $Y^{i,t,x}$ is the solution of the RBSDE with data $(f^i(s,X_s^{t,x}), 0, \max_{j\in\mathcal{I}^{-i}}\{Y_s^{i,t,x} - g^{ij}(s)\})$. Note that for any $i \in \mathcal{I}$, the barriers (resp. obstacle) of the system of RBSDEs (3.18) (resp. PDE (3.23)), depend on the processes $(Y^j)_{j\in\mathcal{I}^{-i}}$ (resp. on the functions $(v^j)_{j\in\mathcal{I}^{-i}}$), i.e., both systems are interconnected just in the barrier (resp. in the obstacle). As a consequence, we cannot use the same argument provided in Chapter 2 to determine the relation (3.24).

Let $(Y^{i,n})_{n\ge0}$ be the sequence defined in (3.13)–(3.14), with $f^i(s,x)$ rather than ψ. Note that for $n = 0$, we can repeat the same argument in the proof of Theorem 2.6 to show the existence of a process $Z^{i,0,t,x}$ such that $(Y^{i,0,t,x}, Z^{i,0,t,x})$ is solution of a standard BSDE with data $(f^i, 0)$. Therefore there exist a deterministic function $v^{i,0}$ (see (2.41) and Theorem 2.14), such that

$$Y_s^{i,0,t,x} = v^{i,0}(s, X_s^{t,x}).$$

Now, for $n \ge 1$, we have from Proposition 5.1 in [25] that there exist unique $(Z^{i,n,t,x}, K^{i,n,t,x})$ such that $(Y^{i,n,t,x}, Z^{i,n,t,x}, K^{i,n,t,x})$ is solution of the RBSDE with data $(f^i, 0, \max_{k\in\mathcal{I}^{-i}}(Y^{k,n-1,t,x} - g^{i,k}))$ and thus in this Markovian framework there exist deterministic function $v^{i,n}$ (see (2.45) and Theorem 2.15), such that

$$Y_s^{i,n,t,x} = v^{i,n}(s, X_s^{t,x}).$$

Therefore, by virtue of Proposition 3.2, the sequence $Y^{i,n,t,x}$ is non decreasing in n and bounded above by its limit $Y^{i,n,t,x}$. As a consequence, the sequence $v^{i,n}$ is also non-decreasing and bounded above by $Y^{i,n,t,x}$ too. Thus, evaluating at the inital point t, we obtain

$$v^{i,n}(t,x) \leq Y_t^{i,t,x} \leq \mathbb{E}\Big[\int_t^T \max_{j\in\mathcal{I}} |f^j(s,X_s^{t,x})|ds\Big|\mathcal{F}_t\Big].$$

Taking expectation in both sides of the above expression, and taking into account that $v^{i,n}(t,x)$ is deterministic as well as the iterated conditional expectation property, we get

$$v^{i,n}(t,x) \leq \mathbb{E}\Big[\int_t^T \max_{j\in\mathcal{I}} |f^j(s,X_s^{t,x})|ds\Big].$$

Therefore, for any $i \in \mathcal{I}$, the sequence $(v^{i,n})_{n\geq 0}$ converges pointwisely to a deterministic function v^i which has also the polynomial growth property because f^i has the same property at the variable x and by using Proposition 3.5 -(i). As a consequence this limit is not infinite.

Furthermore, by uniqueness of the limit, for any $(t,x) \in [0,T]\times\mathbb{R}^k$ we have

$$Y_s^{i,t,x} = v^i(s,X_s^{t,x}), \quad t \leq s \leq T, \tag{3.25}$$

in particular $Y_t^{i,t,x} = v^i(t,x)$. Note that by Proposition 3.2 (ii)-(b) we have that

$$\begin{aligned} Y_t^{i,t,x} = v^i(t,x) &= \sup_{\tau\geq t}\mathbb{E}\left[\int_t^\tau f^i(s,X_s^{t,x})ds + \max_{k\in\mathcal{I}^{-i}}\left(Y_\tau^{k,t,x} - g_\tau^{i,k}\right)\mathbf{1}_{[\tau<T]}\right]. \\ &= \sup_{u\in\mathcal{A}_t^i}\mathbb{E}\left[\sum_{n=0}^\infty\int_{\tau_n}^{\tau_{n+1}} f^{\xi_n}(s,X_s^{t,x})ds - \sum_{n=1}^\infty g_{\tau_n}^{\xi_{n-1}\xi_n}\mathbf{1}_{[\tau_n<T]}\right], \end{aligned} \tag{3.26}$$

where the last equality is by verification Theorem 3.1.

Remark 3.1. (a) The relation (3.26) shows a bridge between systems of RBSDEs, systems of PDE's, and switching control problems.

(b) Note also that the variable t in this analysis has a different meaning to that t in the verification Theorem 3.1. Indeed, the one used here represents the current time, whereas that in the aforementioned theorem represents a future stage.

It only remains to show that $(v^1,\cdots,v^q)$ is a viscosity solution of the system (3.23). To this end, let us give the definition of viscosity solution of the system (3.23).

Definition 3.2. (a) A continuous function $(v^1,\ldots,v^q) : [0,T]\times\mathbb{R}^k \to \mathbb{R}^q$ is said to be a viscosity subsolution of (3.23) if for any $i\in\mathcal{I}$, for any $(t_0,x_0)\in[0,T)\times\mathbb{R}^k$ and any $(\phi^1,\ldots,\phi^q)\in(C^{1,2}([0,T]\times\mathbb{R}^k))^q$, with $\phi^i(t_0,x_0) = v^i(t_0,x_0)$ and $\phi^i - v^i$ attaining its minimum at (t_0,x_0), then we have

$$\begin{cases} \min\left\{v^i(t_0,x_0) - \max\limits_{k\in\mathcal{I}^{-i}}(v^k(t_0,x_0) - g^{ik}(t_0)); \quad -\partial_t\phi^i(t_0,x_0) - \mathcal{L}\phi^i(t_0,x_0) - f^i\big(t_0,x_0\big)\right\} \leq 0. \\ \\ v^i(T,x_0) = 0 \end{cases}$$

In this case, if $v^i(t_0,x_0) > \max_{k\in\mathcal{I}^{-i}}\left(v^k(t_0,x_0) - g^{ik}(t_0)\right)$, then

$$-\partial_t\phi^i(t_0,x_0) - \mathcal{L}\phi^i(t_0,x_0) - f^i\big(t_0,x_0\big) \leq 0.$$

(b) A continuous function $(v^1,\dots,v^q):[0,T]\times\mathbb{R}^k\to\mathbb{R}^q$ is said to be a viscosity super-solution of (3.23) if for all $(t_0,x_0)\in[0,T)\times\mathbb{R}^k$, $i\in\mathcal{I}$ and any $(\phi^1,\dots,\phi^q)\in(C^{1,2}([0,T]\times\mathbb{R}^k))^q$ with $\phi^i(t_0,x_0)=v^i(t_0,x_0)$ and ϕ^i-v^i attaining its maximum at (t_0,x_0), then we have

$$\begin{cases}\min\big\{v^i(t_0,x_0)-\max\limits_{k\in\mathcal{I}^{-i}}\big(v^k(t_0,x_0)-g^{ik}(t_0)\big);\ -\partial_t\phi^i(t_0,x_0)-\mathcal{L}\phi^i(t_0,x_0)-f^i\big(t_0,x_0\big)\big\}\geq 0\\ \\ v^i(T,x_0)=0.\end{cases}$$

Consequently, we have both $v^i(t_0,x_0)\geq\max_{k\in\mathcal{I}^{-i}}\big(v^k(t_0,x_0)-g^{ik}(t_0)\big)$ and

$$-\partial_t\phi^i(t_0,x_0)-\mathcal{L}\phi^i(t_0,x_0)-f^i(t_0,x_0)\geq 0.$$

(c) We say that $(v^1,\dots,v^q)$ is viscosity solution of (3.23) if it is both a viscosity sub- and super-solution of (3.23).

Theorem 3.6. *Under the Assumption H, the deterministic functions v^i, $i=1,\cdots,q$ which satisfy (3.25) are continuous viscosity solutions of the system of PDE's (3.23).*

Sketch of the proof. The part concerned to the continuity of $(t,x)\mapsto v(t,x)$ was rigorously analyzed in [16]. The key idea is to show that the following convergence is true

$$\mathbb{E}\Big[\sup_{0\leq r\leq T}|Y_r^{i,t,x}-Y_r^{i,t',x'}|^2\Big]\to 0\quad\text{as}\quad(t,x)\to(t',x')\quad\text{for any }\ i\in\mathcal{I}.\tag{3.27}$$

where $Y_s^{i,t,x}$ comes from the verification theorem (Theorem 3.1), for any $i\in\mathcal{I}$, $s\in[0,T]$, and $(t,x)\in[0,T]\times\mathbb{R}^k$; i.e.,

$$Y_s^{i;t,x}=\operatorname*{ess\,sup}_{u\in\mathcal{A}_s^i}\mathbb{E}\Big[\int_s^T f^{u(r)}(r,X_r^{t,x})dr-\sum_{j\geq 1}g^{u(\tau_{j-1}),u(\tau_j)}(\tau_j)\mathbf{1}_{[\tau_j<T]}\Big|\mathcal{F}_s\Big],$$

with $\mathcal{A}_s^i$ being the set of finite strategies such that $\tau_1\geq s$, $\mathbb{P}$-a.s., and $u(0)=i$.

In virtue of the inequality

$$|Y_{s'}^{i,t',x'}-Y_s^{i,t,x}|^2\leq 2|Y_{s'}^{i;t',x'}-Y_{s'}^{i,t,x}|^2+2|Y_{s'}^{i,t,x}-Y_s^{i,t,x}|^2\leq 2\sup_{0\leq r\leq T}\Big(|Y_r^{i,t',x'}-Y_r^{i;t,x}|^2\Big)+|Y_{s'}^{i;t,x}-Y_s^{i;t,x}|^2,$$

it follows that, for any $i\in\mathcal{I}$, the function $(s,t,x)\to Y_s^{i;t,x}$ from $[0,T]^2\times\mathbb{R}^k$ into $L^2(\Omega)$ is continuous in the mean square after taking expectations in both sides of last expression and using the convergence in (3.27). Finally, doing $s=t$, the continuity of $(t,t,x)\to Y_t^{i;t,x}$ is straightforward, which shows that v^i is also continuous in (t,x).

The argument that shows that the functions $v^1,\dots,v^q$ are viscosity solutions of the system of PDEs (3.23) follows by Theorem 2.15, because the barriers are already known and due that the functions v^i, $i\in\mathcal{I}$ are continuous of polynomial growth. □

4

Switching control problems and systems of RBSDEs: discontinuous costs

This chapter is inspired from the content of Chapter 3. Namely, we begin this material by providing a probabilistic verification theorem for a switching problem when the switching cost is discontinuous in time. Next, the analysis is focused to show a general result that guaranties the existence and uniqueness of a system of RBSDEs with interconnected coefficients and barriers and càdlàg barriers. This degree of generality forces us to use other tools different to those used in Chapter 3. The last part of the chapter deals with the relation of systems of RBSDEs with càdlàg barriers and systems of PDE's with interconnected discontinuous obstacles. This latter system is proved to contain a weak viscosity solution because the obstacles are not necessarily continuous. All this material has been also analyzed in reference [29].

4.1 The statement of the problem

Use again the set $\mathcal{I} = \{1, \cdots, q\}$, and consider the stochastic processes $\psi^i \in \mathcal{H}^2$, $i \in \mathcal{I}$ but, in contrast to Chapter 3, assume now that the switching costs g^{ik} are in $\mathcal{S}^2$, $i \in \mathcal{I}$ and $k \in \mathcal{I}^{-i}$, and satisfy condition (C). Now assume the existence of the sequence

$$\mathcal{S} = (\tau_n, \xi_n)_{n \geq 0} \tag{4.1}$$

of non-decreasing stopping times (with respect to filtration $\mathbb{F}$) τ_n, and random variables ξ_n which are $\mathcal{F}_{\tau_n}$-measurable with values in $\mathcal{I}$, such that $\tau_0 = 0$, $\xi_0 = i$ for some $i \in \mathcal{I}$.

Define the functional J by

$$J^i(\mathcal{S}) = \mathbb{E}\left[\sum_{n=0}^{\infty} \int_{\tau_n}^{\tau_{n+1}} \psi_s^{\xi_n} ds - \sum_{n=1}^{\infty} g_{\tau_n}^{\xi_{n-1}\xi_n} \mathbf{1}_{[\tau_n < T]}\right], \tag{4.2}$$

with $\psi^{\xi_n} := \psi^j$, when $\xi_n = j$; similarly, $g^{\xi_{n-1}\xi_n}$ is such that $g^{\xi_{n-1}\xi_n} = g^{ik}$ if $\xi_{n-1} = i$ and $\xi_n = k$.

Remark 4.1. For simplicity, the statement of this switching problem is posed for the case when the initial time is $t = 0$ (compare to the switching problem in Chapter 3). However the proofs below are given in a general framework that also cover the cases when the initial time is $t \geq 0$.

Let us define now the concept of admissibility of controls or strategies that were also defined in a similar way in Chapter 3.

Definition 4.1. A sequence $\mathcal{S} = (\tau_n, \xi_n)_{n \geq 0}$ defined as in (4.1) is called a strategy or switching control policy for the controller. Furthermore, we say that a strategy $\mathcal{S}$ is *admissible* if it satisfies the following condition:

$$\mathbb{P}\left[\tau_n < T,\ \forall n \geq 0\right] = 0.$$

Denote by $\mathcal{A}^i$ the set of admissible strategies with the property of $\tau_0 = 0$ and $\xi_0 = i$, for $i = 1, \dots, q$.

As in the previous chapter, the processes ψ^i and g^{ik} are called the payoff rate per unit of the time and the switching cost, respectively.

Switching control problem: A finite horizon switching control problem with q-modes and initial configuration $\xi_0 = i$ for $i \in \mathcal{I}$, consists in finding an admissible sequence $S^* = (\tau_n^*, \xi_n^*)_{n\geq 0} \in \mathcal{A}^i$ such that

$$J^i(\mathcal{S}^*) = \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}), \tag{4.3}$$

where J is the functional defined in (4.2).

There is also a weaker formulation of what we understand for optimal strategy, namely, we say that $\mathcal{S}^* \in \mathcal{A}^i$ is ε-optimal strategy if for all $\varepsilon > 0$, we have

$$J^i(\mathcal{S}^*) \geq \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}) - \varepsilon.$$

4.2 Verification theorem and existence of results

In this section we prove the existence of certain processes that are connected with the cost function (4.2) and its optimal value. To this end, we first provide an existence result of q-interconnected processes, which will be useful later on.

Theorem 4.1. *Consider q processes $\psi^i \in \mathcal{H}^2$, $i \in \mathcal{I}$ and $q(q-1)$ processes $g^{ik} \in \mathcal{S}^2$, $i \in \mathcal{I}$, $k \in \mathcal{I}^{-i}$. Then, under Assumption (C), there exist q $\mathbb{R}-$valued càdlàg processes $\left(Y^i_\cdot := (Y^i_t)_{t\leq T}, i = 1, \dots, q\right) \in (\mathcal{S}^2)^q$ satisfying: for all $i \in \mathcal{I}$*

$$Y_t^i = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{ik}\right)\mathbf{1}_{[\tau<T]}\Big|\mathcal{F}_t\right] \quad \mathbb{P}-a.s.,\ 0\leq t\leq T. \tag{4.4}$$

Proof. For $i \in \mathcal{I}$, and any $0 \leq t \leq T$, use the sequence $(Y_t^{i,n})_{n\geq 0}$ defined by:

$$Y_t^{i,0} = \mathbb{E}\left[\int_t^T \psi_s^i ds\Big|\mathcal{F}_t\right],$$

and for $n \geq 1$,

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^{k,n-1} - g_\tau^{ik}\right)\mathbf{1}_{[\tau<T]}\Big|\mathcal{F}_t\right]. \tag{4.5}$$

First note that the process $(Y_t^{i,0})_{t\leq T}$ is continuous for all $i \in \mathcal{I}$. Next since the process g^{ik} is càdlàg, $(Y_t^{i,1})_{t\leq T}$ is also a càdlàg process, and thus by an induction procedure we have that for all $n \geq 1$, $Y_t^{i,n}$ is càdlàg too.

Let us prove now that, for $i \in \mathcal{I}$, the sequence $(Y_\cdot^{i,n})_{n\geq 0}$ converges increasingly and pointwisely $\mathbb{P}$-a.s. for any $0 \leq t \leq T$ and in the norm $\mathcal{H}^2$ to a càdlàg process $Y_\cdot^i$. To begin with, for any $n \geq 1$ let us define $\mathcal{A}_t^{i,n} = \{\mathcal{S} = (\tau_m, \xi_m)_{m\geq 0} : \xi_0 = i, \tau_0 = t \ \text{ and } \ \tau_{n+1} = T\}$, and let us prove that for N fixed, $Y_\cdot^{i,N}$ can be characterized by

$$Y_t^{i,N} = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}_t^{i,N}} \mathbb{E}\left[\sum_{j=0}^{N}\int_{\tau_j}^{\tau_{j+1}} \psi_s^{\xi_j} ds - \sum_{j=0}^{N-1} g_{\tau_{j+1}}^{\xi_j\xi_{j+1}} \mathbf{1}_{[\tau_{j+1}<T]}\Big|\mathcal{F}_t\right]. \tag{4.6}$$

Since the processes g^{ik} for $i, k \in \mathcal{I}$ are càdlàg, it is not obvious to use the same procedure as given in Djehiche. et al. [16] Proposition 3-(ii) (see also Chapter 3). In contrast, we shall consider the sequence of ε-stopping times $(\tau_n^\varepsilon)_{n\geq 0}$ given by follows: $\tau_0^\varepsilon := t$,

$$\tau_1^\varepsilon := \inf\left\{s \geq t : Y_s^{i,N} \leq \max_{k\in\mathcal{I}^{-i}} \left(Y_s^{k,N-1} - g_s^{ik}\right) + \frac{\varepsilon}{2}\right\} \wedge T$$

and for $2 \le n \le N$,

$$\tau_n^\varepsilon := \inf\Big\{ s \ge \tau_{n-1}^\varepsilon : Y_s^{\hat{\xi}_{n-1},N-n+1} \le \max_{k\in\mathcal{I}^{-\hat{\xi}_{n-1}}} \big(Y_s^{k,N-n} - g_s^{\hat{\xi}_{n-1}k}\big) + \tfrac{\varepsilon}{2^n} \Big\} \wedge T.$$

$$\tau_{N+1}^\varepsilon := T,$$

where

- $\hat{\xi}_0 := i, \quad \hat{\xi}_1 := \arg\max_{k\in\mathcal{I}^{-i}} \Big\{ Y_{\tau_1^\varepsilon}^{k,N-1} - g_{\tau_1^\varepsilon}^{ik} \Big\},$

and for $n \ge 2$,

- $\hat{\xi}_n = \arg\max_{k\in\mathcal{I}^{-\hat{\xi}_{n-1}}} \Big\{ Y_{\tau_n^\varepsilon}^{k,N-n} - g_{\tau_n^\varepsilon}^{\hat{\xi}_{n-1}k} \Big\}.$

Note that by (4.5) the process $(Y_s^{i,N} + \int_t^s \psi_r^i dr)_{t\le s\le \tau_1^\varepsilon}$ is a super-martingale. Hence, its Doob-Meyer decomposition is given by $(M_s - K_s)_{t\le s\le\tau_1^\varepsilon}$ (recall that M is a martingale and K a non-decreasing process), then by definition of τ_1^ε, we have that $K_s = 0$ on $s\in[t,\tau_1^\varepsilon]$, i.e., $(Y_s^{i,N} + \int_t^s \psi_r^i dr)_{t\le s\le\tau_1^\varepsilon}$ is a martingale. Therefore,

$$\begin{aligned}
Y_t^{i,N} &= \mathbb{E}\bigg[Y_{\tau_1^\varepsilon}^{i,N} + \int_t^{\tau_1^\varepsilon} \psi_r^i dr \Big| \mathcal{F}_t \bigg] \\
&\le \mathbb{E}\bigg[\max_{k\in\mathcal{I}^{-i}} \Big(Y_{\tau_1^\varepsilon}^{k,N-1} - g_{\tau_1^\varepsilon}^{ik} \Big) \mathbf{1}_{[\tau_1^\varepsilon<T]} + \tfrac{\varepsilon}{2} + \int_t^{\tau_1^\varepsilon} \psi_r^i dr \Big| \mathcal{F}_t \bigg] \\
&= \mathbb{E}\bigg[\Big(Y_{\tau_1^\varepsilon}^{\hat{\xi}_1,N-1} - g_{\tau_1^\varepsilon}^{i\hat{\xi}_1} \Big) \mathbf{1}_{[\tau_1^\varepsilon<T]} + \tfrac{\varepsilon}{2} + \int_t^{\tau_1^\varepsilon} \psi_r^i dr \Big| \mathcal{F}_t \bigg] \\
&= \mathbb{E}\bigg[\int_t^{\tau_1^\varepsilon} \psi_r^i dr - g_{\tau_1^\varepsilon}^{i\hat{\xi}_1} \mathbf{1}_{[\tau_1^\varepsilon<T]} + Y_{\tau_1^\varepsilon}^{\hat{\xi}_1,N-1} \mathbf{1}_{[\tau_1^\varepsilon<T]} \Big| \mathcal{F}_t \bigg] + \frac{\varepsilon}{2}.
\end{aligned} \tag{4.7}$$

Analogously, taking:

$$\tau_2^\varepsilon = \inf\Big\{ s \ge \tau_1^\varepsilon, Y_s^{\hat{\xi}_1,N-1} \le \max_{k\in\mathcal{I}^{-\hat{\xi}_1}} \big(Y_s^{k,N-2} - g_s^{\hat{\xi}_1 k}\big) + \frac{\varepsilon}{4} \Big\} \wedge T$$

we have again that $(Y_s^{\hat{\xi}_1,N-1} + \int_{\tau_1^\varepsilon}^s \psi_r^{\hat{\xi}_1} dr)_{\tau_1^\varepsilon\le s\le\tau_2^\varepsilon}$ is a martingale. Arguing similarly as above, we have

$$\begin{aligned}
Y_{\tau_1^\varepsilon}^{\hat{\xi}_1,N-1} &= \mathbb{E}\bigg[Y_{\tau_2^\varepsilon}^{\hat{\xi}_1,N-1} + \int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_r^{\hat{\xi}_1} dr \Big| \mathcal{F}_{\tau_1^\varepsilon} \bigg] \\
&\le \mathbb{E}\bigg[\max_{k\in\mathcal{I}^{-\hat{\xi}_1}} \Big(Y_{\tau_2^\varepsilon}^{k,N-2} - g_{\tau_2^\varepsilon}^{\hat{\xi}_1 k} \Big) \mathbf{1}_{[\tau_2^\varepsilon<T]} + \tfrac{\varepsilon}{4} + \int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_r^{\hat{\xi}_1} dr \Big| \mathcal{F}_{\tau_1^\varepsilon} \bigg] \\
&= \mathbb{E}\bigg[\Big(Y_{\tau_2^\varepsilon}^{\hat{\xi}_2,N-2} - g_{\tau_2^\varepsilon}^{\hat{\xi}_1\hat{\xi}_2} \Big) \mathbf{1}_{[\tau_2^\varepsilon<T]} + \tfrac{\varepsilon}{4} + \int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_r^{\hat{\xi}_1} dr \Big| \mathcal{F}_{\tau_1^\varepsilon} \bigg] \\
&= \mathbb{E}\bigg[\int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_r^{\hat{\xi}_1} dr - g_{\tau_2^\varepsilon}^{\hat{\xi}_1\hat{\xi}_2} \mathbf{1}_{[\tau_2^\varepsilon<T]} + Y_{\tau_2^\varepsilon}^{\hat{\xi}_2,N-2} \mathbf{1}_{[\tau_2^\varepsilon<T]} \Big| \mathcal{F}_{\tau_1^\varepsilon} \bigg] + \frac{\varepsilon}{4}.
\end{aligned} \tag{4.8}$$

Plugging (4.8) into (4.7), rearranging terms and since that $[\tau_2^\varepsilon < T] \subset [\tau_1^\varepsilon < T]$, we see that

$$Y_t^{i,N} \le \mathbb{E}\bigg[\sum_{j=0}^{1} \Big(\int_{\tau_j^\varepsilon}^{\tau_{j+1}^\varepsilon} \psi_r^{\hat{\xi}_j} dr - g_{\tau_{j+1}^\varepsilon}^{\hat{\xi}_j\hat{\xi}_{j+1}} \mathbf{1}_{[\tau_{j+1}^\varepsilon<T]} \Big) + Y_{\tau_2^\varepsilon}^{\hat{\xi}_2,N-2} \mathbf{1}_{[\tau_2^\varepsilon<T]} \Big| \mathcal{F}_t \bigg] + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}.$$

Repeating this procedure N times, we obtain

$$Y_t^{i,N} \leq \mathbb{E}\bigg[\sum_{j=0}^{N-1}\bigg(\int_{\tau_j^\varepsilon}^{\tau_{j+1}^\varepsilon}\psi_s^{\hat{\xi}_j}\,ds - g_{\tau_{j+1}^\varepsilon}^{\hat{\xi}_j\hat{\xi}_{j+1}}\mathbf{1}_{[\tau_{j+1}^\varepsilon<T]}\bigg) + Y_{\tau_N^\varepsilon}^{\hat{\xi}_N,0}\mathbf{1}_{[\tau_N^\varepsilon<T]}\bigg|\mathcal{F}_t\bigg] + \varepsilon\left(\sum_{i=1}^{N}\frac{1}{2^i}\right). \tag{4.9}$$

But

$$Y_{\tau_N^\varepsilon}^{\hat{\xi}_N,0} = \mathbb{E}\bigg[\int_{\tau_N^\varepsilon}^{T}\psi_s^{\hat{\xi}_N}\,ds\bigg|\mathcal{F}_{\tau_N}\bigg]. \tag{4.10}$$

Plugging (4.10) into (4.9), and noting that $\left(\sum_{i=1}^{n}\frac{1}{2^i}\right) < 1$, we deduce

$$Y_t^{i,N} \leq \mathbb{E}\bigg[\sum_{j=0}^{N}\int_{\tau_j^\varepsilon}^{\tau_{j+1}^\varepsilon}\psi_s^{\hat{\xi}_j}\,ds - \sum_{j=0}^{N-1} g_{\tau_{j+1}^\varepsilon}^{\hat{\xi}_j\hat{\xi}_{j+1}}\mathbf{1}_{[\tau_{j+1}^\varepsilon<T]}\bigg|\mathcal{F}_t\bigg] + \varepsilon \quad \text{for all } \varepsilon > 0.$$

Since $(\tau_n^\varepsilon, \hat{\xi}_n)_{0\leq n\leq N+1}$ belongs to $\mathcal{A}_t^{i,N}$, we can take the essential supremum over $\mathcal{S} \in \mathcal{A}_t^{i,N}$ and then let $\varepsilon \to 0$ to obtain

$$Y_t^{i,N} \leq \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}_t^{i,N}} \mathbb{E}\bigg[\sum_{j=0}^{N}\int_{\tau_j}^{\tau_{j+1}}\psi_s^{\hat{\xi}_j}\,ds - \sum_{j=0}^{N-1} g_{\tau_{j+1}}^{\hat{\xi}_j\hat{\xi}_{j+1}}\mathbf{1}_{[\tau_{j+1}<T]}\bigg|\mathcal{F}_t\bigg]. \tag{4.11}$$

Now we derive the inverse inequality. Let $\mathcal{S} = (\tau_n, \xi_n) \in \mathcal{A}_t^{i,N}$ be an arbitrary strategy. Since $\tau_1 \geq t$, $\mathbb{P}$-a.s., and $\xi_0 = i$, then from (4.5) we have

$$\begin{aligned} Y_t^{i,N} &= \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\bigg[\int_t^{\tau}\psi_s^i ds + \max_{k\in\mathcal{I}^{-i}}\left(Y_\tau^{k,N-1} - g_\tau^{ik}\right)\mathbf{1}_{[\tau<T]}\bigg|\mathcal{F}_t\bigg] \\ &\geq \mathbb{E}\bigg[\int_t^{\tau_1}\psi_s^i ds + \max_{k\in\mathcal{I}^{-i}}\left(Y_{\tau_1}^{k,N-1} - g_{\tau_1}^{ik}\right)\mathbf{1}_{[\tau_1<T]}\bigg|\mathcal{F}_t\bigg] \\ &\geq \mathbb{E}\bigg[\int_t^{\tau_1}\psi_s^i ds + \left(Y_{\tau_1}^{\xi_1,N-1} - g_{\tau_1}^{i\xi_1}\right)\mathbf{1}_{[\tau_1<T]}\bigg|\mathcal{F}_t\bigg]. \end{aligned} \tag{4.12}$$

In the same way, since $\tau_2 \geq \tau_1$ and τ_1 is also $\mathcal{F}_{\tau_2}$- measurable, then

$$\begin{aligned} Y_{\tau_1}^{\xi_1,N-1} &= \operatorname*{ess\,sup}_{\tau\geq\tau_1} \mathbb{E}\bigg[\int_{\tau_1}^{\tau}\psi_s^{\xi_1} ds + \max_{k\in\mathcal{I}^{-\xi_1}}\left(Y_\tau^{k,N-2} - g_\tau^{\xi_1 k}\right)\mathbf{1}_{[\tau<T]}\bigg|\mathcal{F}_{\tau_1}\bigg] \\ &\geq \mathbb{E}\bigg[\int_{\tau_1}^{\tau_2}\psi_s^{\xi_1} ds + \max_{k\in\mathcal{I}^{-\xi_1}}\left(Y_{\tau_2}^{k,N-2} - g_{\tau_2}^{\xi_1 k}\right)\mathbf{1}_{[\tau_2<T]}\bigg|\mathcal{F}_{\tau_1}\bigg] \\ &\geq \mathbb{E}\bigg[\int_{\tau_1}^{\tau_2}\psi_s^{\xi_1} ds + \left(Y_{\tau_2}^{\xi_2,N-2} - g_{\tau_2}^{\xi_1\xi_2}\right)\mathbf{1}_{[\tau_2<T]}\bigg|\mathcal{F}_{\tau_1}\bigg]. \end{aligned}$$

Plugging this last inequality into (4.12), rearranging terms and using that $[\tau_2 < T] \subset [\tau_1 < T] \in \mathcal{F}_{\tau_1}$, we see that

$$Y_t^{i,N} \geq \mathbb{E}\bigg[\sum_{j=0}^{1}\bigg(\int_{\tau_j}^{\tau_{j+1}}\psi_s^{\xi_j}\,ds - g_{\tau_j}^{\xi_j\xi_{j+1}}\mathbf{1}_{[\tau_{j+1}<T]}\bigg) + Y_{\tau_2}^{\xi_2,N-2}\mathbf{1}_{[\tau_2<T]}\bigg|\mathcal{F}_t\bigg].$$

Iterating this procedure, we have

$$Y_t^{i,N} \geq \mathbb{E}\bigg[\sum_{j=0}^{N-1}\bigg(\int_{\tau_j}^{\tau_{j+1}}\psi_s^{\xi_j}\,ds - g_{\tau_{j+1}}^{\xi_j\xi_{j+1}}\mathbf{1}_{[\tau_{j+1}<T]}\bigg) + Y_{\tau_N}^{\xi_N,0}\mathbf{1}_{[\tau_N<T]}\bigg|\mathcal{F}_t\bigg].$$

But again, since $Y^{\xi_N,0}_{\tau_N} = \mathbb{E}\big[\int_{\tau_N}^T \psi^{\xi_n}_s ds \big| \mathcal{F}_{\tau_N}\big]$, we get

$$Y^{i,N}_t \geq \mathbb{E}\left[\sum_{j=0}^{N}\int_{\tau_j}^{\tau_{j+1}} \psi^{\xi_j}_s ds - \sum_{j=0}^{N-1} g^{\xi_j\xi_{j+1}}_{\tau_{j+1}} \mathbf{1}_{[\tau_{j+1}<T]} \middle| \mathcal{F}_t\right] \quad \text{for all } \mathcal{S} \in \mathcal{A}^{i,N}_t.$$

Thus, taking the essential supremum on $\mathcal{A}^{i,N}_t$, we get

$$Y^{i,N}_t \geq \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}^{i,N}_t} \mathbb{E}\left[\sum_{j=0}^{N}\int_{\tau_j}^{\tau_{j+1}} \psi^{\xi_j}_s ds - \sum_{j=0}^{N-1} g^{\xi_j\xi_{j+1}}_{\tau_{j+1}} \mathbf{1}_{[\tau_{j+1}<T]} \middle| \mathcal{F}_t\right].$$

This last inequality together with (4.11), yield the characterization (4.6).

Since $\mathcal{A}^{i,n}_t \subset \mathcal{A}^{i,n+1}_t$, we have $Y^{i,n}_t \leq Y^{i,n+1}_t$, $\mathbb{P}$-a.s. for all $t \in [0,T]$. On the other hand, by assumption (C), we obtain for each $i \in \mathcal{I}$,

$$Y^{i,n}_t \leq \mathbb{E}\left[\int_t^T \max_{[i=1,\ldots,q]} \left|\psi^i_s\right| ds \middle| \mathcal{F}_t\right] \qquad \text{for all } t \leq T \text{ and } n \geq 0$$

and hence the sequence $(Y^{i,n}_t)_{n\geq 1}$ is convergent. We now let $Y^i_t := \lim_{n\to\infty} Y^{i,n}_t$ for $t \leq T$. Note that the process $Y^i_\cdot$ satisfies

$$Y^{i,0}_t \leq Y^i_t \leq \mathbb{E}\left[\int_t^T \max_{[i=1,\ldots,q]} \left|\psi^i_s\right| ds \middle| \mathcal{F}_t\right] \quad \text{for all } t \leq T. \tag{4.13}$$

Besides, $Y^i_\cdot$ is also càdlàg process. Indeed, from (4.5) the process $(Y^{i,n}_t + \int_0^t \psi^i_s ds)_{0\leq t\leq T}$ is a càdlàg super-martingale for all $i \in \mathcal{I}$ and $n \geq 1$. Thus its limit process $(Y^i_t + \int_0^t \psi^i_s ds)_{0\leq t\leq T}$ is càdlàg, as it is a limit of increasing sequence of càdlàg super-martingales (see Dellacherie and Meyer [[14], p. 86]), which gives the desired càdlàg property of $Y^i_\cdot$. Moreover, from (4.13), the L^2-properties of ψ^i and by Doob's maximal inequality, for each $i \in \mathcal{I}$, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|Y^i_t\right|^2\right] < \infty,$$

and hence by the Lebesgue Dominated Convergence Theorem, the sequence $(Y^{i,n}_\cdot)_{n\geq 0}$ converges to $Y^i_\cdot$ in $\mathcal{H}^2$. Thus, by Snell envelope properties (see Proposition 2.5-(iv)), the càdlàg processes $Y^1_\cdot, \ldots, Y^q_\cdot$ satisfy (4.4). □

Let us show now some properties of the ε-strategy introduced in Theorem 4.1.

Proposition 4.2. *The ε-strategy $\mathcal{S}^\varepsilon = (\tau^\varepsilon_n, \xi^\varepsilon_n)_{n\geq 0}$ defined as follows:*

- $\tau^\varepsilon_0 := 0, \quad \tau^\varepsilon_1 := \inf\left\{s \geq 0 : Y^i_s \leq \max_{k\in\mathcal{I}^{-i}} \left(Y^k_s - g^{ik}_s\right) + \frac{\varepsilon}{2}\right\} \wedge T,$

and, for $n \geq 2$,

- $\tau^\varepsilon_n := \inf\left\{s \geq \tau^\varepsilon_{n-1} : Y^{\xi^\varepsilon_{n-1}}_s \leq \max_{k\in\mathcal{I}^{-\xi^\varepsilon_{n-1}}} \left(Y^k_s - g^{\xi^\varepsilon_{n-1}k}_s\right) + \frac{\varepsilon}{2^n}\right\} \wedge T,$

and the sequence (ξ^ε_n) given by

- $\xi^\varepsilon_0 := i, \quad \xi^\varepsilon_1 = \arg\max_{k\in\mathcal{I}^{-i}} \left\{Y^k_{\tau^\varepsilon_1} - g^{ik}_{\tau^\varepsilon_1}\right\},$

and for $n \geq 2$,

- $\xi^\varepsilon_n = \arg\max_{k\in\mathcal{I}^{-\xi_{n-1}}} \left\{Y^k_{\tau^\varepsilon_n} - g^{\hat{\xi}_{n-1}k}_{\tau^\varepsilon_n}\right\},$

is admissible.

Proof. We proceed by a reductio ad absurdum argument. Suppose that $\mathcal{S}^\varepsilon$ is not admissible, that is, $\mathbb{P}[\tau_n^\varepsilon < T, \text{for all } n \geq 1] > 0$. Then, by definition of τ_n^ε we have

$$\mathbb{P}\left[Y_{\tau_n^\varepsilon}^{\xi_{n-1}^\varepsilon} \leq Y_{\tau_n^\varepsilon}^{\xi_n^\varepsilon} - g_{\tau_n^\varepsilon}^{\xi_{n-1}^\varepsilon \xi_n^\varepsilon} + \frac{\varepsilon}{2^n}, \;\; \xi_n^\varepsilon \in \mathcal{I}^{-\xi_{n-1}^\varepsilon}, \forall n \geq 1\right] > 0.$$

If the event $B = \{\omega \in \Omega : \tau_n^\varepsilon(\omega) < T, \forall n \geq 1\}$ has positive probability, then there is a state $i_1 \in \mathcal{I}$, a loop $i_1, i_2 \ldots, i_k$ (with $i_1 = i_k$) of elements of $\mathcal{I}$ (recall that $\mathcal{I}$ is a finite set), and a subsequence $\tau_n^\varepsilon, \ldots, \tau_{n+k}^\varepsilon$ corresponding to this configuration such that

$$\mathbb{P}\left[Y_{\tau_{n+l}^\varepsilon}^{i_l} \leq Y_{\tau_{n+l}^\varepsilon}^{i_{l+1}} - g_{\tau_{n+l}^\varepsilon}^{i_l, i_{l+1}} + \frac{\varepsilon}{2^n}, l = 1, \ldots k-1, (i_k = i_1), \forall n \geq 0\right] > 0. \tag{4.14}$$

Since $(\tau_n^\varepsilon)_{n\geq 1}$ is monotone and bounded, then we can define $\tau := \lim_{n\to\infty} \tau_n^\varepsilon$. Taking the limit with respect to n in (4.14), we obtain

$$\mathbb{P}\left[Y_{\tau-}^{i_l} \leq Y_{\tau-}^{i_{l+1}} - g_{\tau-}^{i_l, i_{l+1}}, l = 1, \ldots k-1, (i_k = i_1)\right] > 0. \tag{4.15}$$

It is easy to verify that

$$\left\{Y_{\tau-}^{i_l} \leq Y_{\tau-}^{i_{l+1}} - g_{\tau-}^{i_l, i_{l+1}}, l = 1, \ldots k-1, (i_k = i_1)\right\} \subseteq \left\{g_{\tau-}^{i_1, i_2} + \cdots + g_{\tau-}^{i_{k-1}, i_1} \leq 0\right\},$$

then from (4.15) we have

$$\mathbb{P}\left[g_{\tau-}^{i_1, i_2} + \cdots + g_{\tau-}^{i_k, i_1} \leq 0\right] > 0.$$

Since $g^{ij} \geq \gamma > 0$ $\mathbb{P}$-a.s., we have a contradiction. Therefore, $\mathcal{S}^\varepsilon$ is admissible. □

Our next result has to do with a so-called verification theorem for the switching problem (4.3) in the context of càdlàg cost functions.

Theorem 4.3. *The q $\mathcal{S}^2$-processes $(Y_\cdot^i := \left(Y_t^i\right)_{t\leq T}, i = 1, \ldots, q)$ in Theorem 4.1 are unique and they have the following relation with the switching problem (4.3):*

(i) For each $i \in \mathcal{I}$,

$$Y_0^i = \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}). \tag{4.16}$$

(ii) The ε-strategy $\mathcal{S}^\varepsilon$ defined in Proposition 4.2 forms an ε-optimal strategy, i.e., for $\mathcal{S}^\varepsilon = (\tau_n^\varepsilon, \xi_n^\varepsilon)_{n\geq 0}$,

$$J^i\left(\mathcal{S}^\varepsilon\right) \geq \sup_{\mathcal{S}\in\mathcal{A}^i} J^i(\mathcal{S}) - \varepsilon. \tag{4.17}$$

Proof.

(i) Assuming that at time $t = 0$ the system is in mode i, it follows by (4.4) that, for any $0 \leq t \leq T$,

$$Y_t^i + \int_0^t \psi^i(s)ds = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\bigg[\int_0^\tau \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_\tau^k - g_\tau^{ik}\right) \mathbf{1}_{[\tau<T]}\bigg|\mathcal{F}_t\bigg].$$

Since Y_0^i is $\mathcal{F}_0$-measurable, it is a $\mathbb{P}$-a.s. constant, that is, $Y_0^i = \mathbb{E}\left[Y_0^i\right]$. Now take $\mathcal{S}^\varepsilon$ defined in Proposition 4.2. Arguing similarly to Theorem 4.1, we can deduce

$$\begin{aligned} Y_0^i &\leq \mathbb{E}\bigg[\int_0^{\tau_1^\varepsilon} \psi_s^i ds + \max_{k\in\mathcal{I}^{-i}} \left(Y_{\tau_1^\varepsilon}^k - g_{\tau_1^\varepsilon}^{ik}\right) \mathbf{1}_{[\tau_1^\varepsilon<T]}\bigg] + \frac{\varepsilon}{2} \\ &= \mathbb{E}\bigg[\int_0^{\tau_1^\varepsilon} \psi_s^i ds + \left(Y_{\tau_1^\varepsilon}^{\xi_1^\varepsilon} - g_{\tau_1^\varepsilon}^{i\xi_1^\varepsilon}\right) \mathbf{1}_{[\tau_1^\varepsilon<T]}\bigg] + \frac{\varepsilon}{2}. \end{aligned} \tag{4.18}$$

The rest of the proof uses the same arguments as in the proof of Theorem 4.1. Namely, for every $\tau_1^\varepsilon \le t \le T$, we can deduce

$$Y_t^{\xi_1^\varepsilon} = \operatorname*{ess\,sup}_{\tau \ge t} \mathbb{E}\Big[\int_t^\tau \psi_s^{\xi_1^\varepsilon} ds + \max_{j \in \mathcal{I}^{-\xi_1^\varepsilon}} \left(Y_\tau^j - g_\tau^{\xi_1^\varepsilon j} \right) \mathbf{1}_{[\tau < T]} \Big| \mathcal{F}_t \Big].$$

Then, from the definition of τ_2^ε and since $(Y_t^{\xi_1^\varepsilon} + \int_{\tau_1^\varepsilon}^t \psi_s^{\xi_1^\varepsilon} ds)_{\tau_1^\varepsilon \le t \le \tau_2^\varepsilon}$ is a martingale, we get

$$\begin{aligned} Y_{\tau_1^\varepsilon}^{\xi_1^\varepsilon} &\le \mathbb{E}\Big[\int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_s^{\xi_1^\varepsilon} ds + \max_{j \in \mathcal{I}^{-\xi_1^\varepsilon}} \left(Y_{\tau_2^\varepsilon}^j - g_{\tau_2^\varepsilon}^{\xi_1^\varepsilon j} \right) \mathbf{1}_{[\tau_2^\varepsilon < T]} \Big| \mathcal{F}_{\tau_1^\varepsilon} \Big] + \frac{\varepsilon}{4} \\ &= \mathbb{E}\Big[\int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_s^{\xi_1^\varepsilon} ds + \left(Y_{\tau_2^\varepsilon}^{\xi_2^\varepsilon} - g_{\tau_2^\varepsilon}^{\xi_1^\varepsilon \xi_2^\varepsilon} \right) \mathbf{1}_{[\tau_2^\varepsilon < T]} \Big| \mathcal{F}_{\tau_1^\varepsilon} \Big] + \frac{\varepsilon}{4}. \end{aligned} \tag{4.19}$$

Plugging (4.19) into (4.18) and noting that $\mathbf{1}_{[\tau_1^\varepsilon < T]}$ is $\mathcal{F}_{\tau_1^\varepsilon}$-measurable, it follows that:

$$\begin{aligned} Y_0^i &\le \mathbb{E}\Big[\int_0^{\tau_1^\varepsilon} \psi_s^i ds - g_{\tau_1^\varepsilon}^{i\xi_1^\varepsilon} \mathbf{1}_{[\tau_1^\varepsilon < T]} \Big] \\ &\qquad + \mathbb{E}\Big[\int_{\tau_1^\varepsilon}^{\tau_2^\varepsilon} \psi_s^{\xi_1^\varepsilon} ds + \left(Y_{\tau_2^\varepsilon}^{\xi_2^\varepsilon} - g_{\tau_2^\varepsilon}^{\xi_1^\varepsilon \xi_2^\varepsilon} \right) \mathbf{1}_{[\tau_2^\varepsilon < T]} \Big] + \varepsilon \left(\frac{1}{2} + \frac{1}{4} \right). \\ &= \mathbb{E}\Bigg[\sum_{j=0}^{1} \left(\int_{\tau_j^\varepsilon}^{\tau_{j+1}^\varepsilon} \psi_s^{\xi_j^\varepsilon} ds - g_{\tau_j^\varepsilon}^{\xi_j^\varepsilon \xi_{j+1}^\varepsilon} \mathbf{1}_{[\tau_{j+1}^\varepsilon < T]} \right) + Y_{\tau_2^\varepsilon}^{\xi_2^\varepsilon} \mathbf{1}_{[\tau_2^\varepsilon < T]} \Bigg] + \varepsilon \left(\frac{1}{2} + \frac{1}{4} \right) \end{aligned}$$

since $[\tau_2^\varepsilon < T] \subset [\tau_1^\varepsilon < T]$. Repeating this procedure n times, we obtain

$$Y_0^i \le \mathbb{E}\Bigg[\sum_{j=0}^{n-1} \left(\int_{\tau_j^\varepsilon}^{\tau_{j+1}^\varepsilon} \psi_s^{\xi_j^\varepsilon} ds - g_{\tau_{j+1}^\varepsilon}^{\xi_j^\varepsilon \xi_{j+1}^\varepsilon} \mathbf{1}_{[\tau_{j+1}^\varepsilon < T]} \right) + Y_{\tau_n^\varepsilon}^{\xi_n^\varepsilon} \mathbf{1}_{[\tau_n^\varepsilon < T]} \Bigg] + \varepsilon \left(\frac{1}{2} + \cdots + \frac{1}{2^n} \right).$$

Taking liminf as $n \to \infty$ we obtain

$$Y_0^i \le \mathbb{E}\Bigg[\sum_{j=0}^{\infty} \left(\int_{\tau_j^\varepsilon \wedge T}^{\tau_{j+1}^\varepsilon \wedge T} \psi_s^{\xi_j^\varepsilon} ds - g_{\tau_{j+1}^\varepsilon}^{\xi_j^\varepsilon \xi_{j+1}^\varepsilon} \mathbf{1}_{[\tau_{j+1}^\varepsilon < T]} \right) \Bigg] + \varepsilon. \tag{4.20}$$

By Proposition 4.2 we can take supremum over all admissible strategies $\mathcal{A}^i$, to obtain

$$\begin{aligned} Y_0^i &\le \sup_{\mathcal{S} \in \mathcal{A}^i} \mathbb{E}\Bigg[\sum_{j=0}^{\infty} \left(\int_{\tau_j \wedge T}^{\tau_{j+1} \wedge T} \psi_s^{\xi_j} ds - g_{\tau_{j+1}}^{\xi_j \xi_{j+1}} \mathbf{1}_{[\tau_{j+1} < T]} \right) \Bigg] + \varepsilon \\ &= \sup_{\mathcal{S} \in \mathcal{A}^i} J^i(\mathcal{S}) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, it follows that $Y_0^i \le \sup_{\mathcal{S} \in \mathcal{A}^i} J^i(\mathcal{S})$. The inverse inequality is analogous to the previous Theorem 4.1. Hence, the result follows.

(ii) From part (i), specifically, (4.16) and inequality (4.20), we deduce

$$\sup_{\mathcal{S} \in \mathcal{A}^i} J^i(\mathcal{S}) - \varepsilon \le J^i(\mathcal{S}^\varepsilon) \le \sup_{\mathcal{S} \in \mathcal{A}^i} J^i(\mathcal{S}),$$

which proves (ii). □

4.3 Systems of RBSDEs with càdlàg barriers

In this section we will provide the existence as well as uniqueness of the solution of the system of reflected backward stochastic differential equations (RBSDEs) of type

$$\begin{cases} \forall i \in \mathcal{I}, \ \text{ find } \ \left(Y^i_\cdot, Z^i_\cdot, K^i_\cdot\right) \ \text{ such that :} \\ Y^i_\cdot, K^i_\cdot \in \mathcal{S}^2 \quad \text{and} \quad Z^i_\cdot \in \mathcal{H}^{2,d}; \ K^i_\cdot \ \text{ is non-decreasing and } \quad K^i_0 = 0, \\ Y^i_s = h^i(X_T) + \displaystyle\int_s^T f^i(r, X_r, Y^1_r, \ldots, Y^q_r, Z^i_r)dr + K^i_T - K^i_s - \int_s^T Z^i_r dB_r \quad \text{for all } 0 \le s \le T, \\ Y^i_s \ge \displaystyle\max_{k \in \mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\} \qquad \text{for all } 0 \le s \le T, \\ \text{and if } \ K^i_\cdot = K^{i,c}_\cdot + K^{i,d}_\cdot, \ \text{where } \ K^{i,c}_\cdot \ (\text{resp. } \ K^{i,d}_\cdot) \ \text{ is the continuous} \\ (\text{resp. purely discontinuous) part of } \ K^i_\cdot, \ \text{ then:} \\ \displaystyle\int_0^T \Big(Y^i_r - \max_{k \in \mathcal{I}^{-i}} \left\{Y^k_r - \gamma^{ik}(r, X_r)\right\}\Big) dK^{i,c}_r = 0. \\ \Delta Y_s := Y_s - Y_{s^-} = -\Big(\displaystyle\max_{k \in \mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\} - Y^i_s\Big)^+ \qquad \text{for all } 0 \le s \le T, \end{cases} \tag{4.21}$$

in which the associated barriers are *càdlàg* processes. This system is connected with the previous switching problem. Actually when $(f^i)_{i\in\mathcal{I}}$ do not depend on $(Y^i)_{i\in\mathcal{I}}$, the system (4.21) is exactly the translation of the verification Theorem 4.1 in terms of reflected BSDEs as it is well-known that the Snell envelope can be expressed through reflected BSDEs (see e.g. El Karoui [22] or Hamadène [27]). On the other hand, this form of system (4.21) allows to consider switching problems when the cost functions are of risk sensitive type (utility functions) —see El Karoui and Hamadène [24].

To begin with our analysis, we will first introduce the following assumptions related to the items involved in (4.21): These hypotheses are the same ones as those established in Chapter 3, but here the switching cost are no longer continuous, just càdlàg, and we repeat them here for easy of reference.

Remark 4.2. We warn the reader that the index k plays a different role here as it is used as a switching index, whereas in Chapter 3 it was used to denote the dimension of the forward Markov process X. Actually in this Chapter the dimension of this later Markov process will be denoted by $\mathbb{R}^r$.

Assumption H.

(H1) : The stochastic process $(X_t)_{t\ge0}$ is in $\mathcal{H}^{2,r}$ for any $r \in \mathbb{N}$.

(H2) : For any $i \in \mathcal{I}$, the function $f^i : [0,T] \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R}$ satisfies:

- (i) $(t,x) \mapsto f^i\left(t, x, y^1, \ldots, y^q, z\right)$ is continuous uniformly with respect to $(y^1, \ldots, y^q, z)$;
- (ii) f^i is uniformly Lipschitz continuous in the following sense: for some $C \ge 0$,

$$\left|f^i(t,x,y^1,\ldots,y^q,z) - f^i(t,x,\bar{y}^1,\ldots,\bar{y}^q,\bar{z})\right| \le C\left(\left|y^1 - \bar{y}^1\right| + \cdots + |y^q - \bar{y}^q| + |z - \bar{z}|\right).$$

- (iii) the mapping $(t,x) \mapsto f^i(t,x,0,\ldots,0)$ is of polynomial growth.
- (iv) *monotonicity*: For all $i \in \mathcal{I}$, for all $k \in \mathcal{I}^{-i}$, the mapping $y_k \mapsto f^i(t,x,y^1,\ldots,y^{k-1},y^k,y^{k+1},\ldots,y^q,z)$ is non-decreasing whenever the other components $(t,x,y^1,\ldots,y^{k-1},y^{k+1},\ldots,y^q,z)$ are fixed.

(H3) : For each $i,k \in \mathcal{I}$, the function $\gamma^{ik} : [0,T] \times \mathbb{R}^r \to \mathbb{R}$ is bounded from below, i.e. there exists a real constant $\gamma > 0$ such that, $\gamma^{ik} \ge \gamma$. Furthermore it is càdlàg in t, continuous and of polynomial growth in x.

(H4) : For each $i \in \mathcal{I}$, the function $h^i : \mathbb{R}^r \to \mathbb{R}$ is continuous with polynomial growth and satisfies

$$\forall x \in \mathbb{R}^r, \quad h^i(x) \ge \max_{k \in \mathcal{I}^{-i}} \left(h^k(x) - \gamma^{ik}(T,x)\right).$$

Note that in (4.21), the process X does not play a specific role. In later sections, this process will have a specific dynamic and thus the system (4.21) can be associated to a system PDEs.

Proposition 4.4. *Under Assumption H, the system of RBSDEs (4.21) has a solution* $(Y^i_\cdot, Z^i_\cdot, K^i_\cdot)_{i=1,\dots,q}$.

Proof. To begin with, we first consider the following standard BSDEs:

$$\begin{cases}(\overline{Y}_\cdot, \overline{Z}_\cdot) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \quad ; \\ \overline{Y}_s = \max\limits_{i=1,\dots,q} h^i(X_T) + \displaystyle\int_s^T \big[\max\limits_{i=1,\dots,q} f^i\big](r, X_r, \overline{Y}_r, \dots, \overline{Y}_r, \overline{Z}_r)dr - \int_s^T \overline{Z}_r dB_r, \quad \text{for all } s \le T,\end{cases} \tag{4.22}$$

and

$$\begin{cases}(\underline{Y}_\cdot, \underline{Z}_\cdot) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \quad ; \\ \underline{Y}_s = \min\limits_{i=1,\dots,q} h^i(X_T) + \displaystyle\int_s^T \big[\min\limits_{i=1,\dots,q} f^i\big](r, X_r, \underline{Y}_r, \dots, \underline{Y}_r, \underline{Z}_r)dr - \int_s^T \underline{Z}_r dB_r, \quad \text{for all } s \le T.\end{cases} \tag{4.23}$$

It is easy to verify that under Assumption H, the data of (4.22) and (4.23) satisfy the conditions of Pardoux and Peng's Theorem 2.6, then there exist unique solutions of both (4.22) and (4.23). To solve the system (4.21), we shall use an iterative method and regard (4.21) as a limit system. To this end, for any $i \in \mathcal{I}$, we set $Y^{i,0}_\cdot := \underline{Y}_\cdot$, and for $n \ge 1$, we seek a triplet $(Y^{i,n}_\cdot, Z^{i,n}_\cdot, K^{i,n}_\cdot)$ such that, for $n \ge 1$:

$$\begin{cases} Y^{i,n}_\cdot, K^{i,n}_\cdot \in \mathcal{S}^2 \quad \text{and} \quad Z^{i,n}_\cdot \in \mathcal{H}^{2,d};\ K^{i,n}_\cdot \ \text{ is non-decreasing with } \ K^{i,n}_0 = 0, \\ Y^{i,n}_s = h^i(X_T) + \displaystyle\int_s^T f^i(r, X_r, Y^{1,n-1}_r, \dots, Y^{i-1,n-1}_r, Y^{i,n}_r, Y^{i+1,n-1}_r, \dots Y^{q,n-1}_r, Z^{i,n}_r)dr \\ \qquad +K^{i,n}_T - K^{i,n}_s - \int_s^T Z^{i,n}_r dB_r, \quad \text{for all } 0 \le s \le T; \\ Y^{i,n}_s \ge \max\limits_{k \in \mathcal{I}^{-i}} \left\{Y^{k,n-1}_s - \gamma^{ik}(s, X_s)\right\}, \quad \text{for all } 0 \le s \le T, \\ \text{and if } \ K^{i,n}_\cdot = K^{i,n,c}_\cdot + K^{i,n,d}_\cdot, \text{ where } \ K^{i,n,c}_\cdot \text{ (resp. } \ K^{i,n,d}_\cdot) \text{ is the continuous} \\ \text{(resp. purely discontinuous) part of } \ K^{i,n}_\cdot, \ \text{ then:} \\ \displaystyle\int_0^T \left(Y^{i,n}_r - \max\limits_{k \in \mathcal{I}^{-i}} \left\{Y^{k,n-1}_r - \gamma^{ik}(r, X_r)\right\}\right) dK^{i,n,c}_r = 0\,; \\ \Delta Y_s := Y^{i,n}_s - Y^{i,n}_{s^-} = -\Big(\max\limits_{k \in \mathcal{I}^{-i}} \left\{Y^{k,n-1}_s - \gamma^{ik}(s, X_s)\right\} - Y^{i,n}_s\Big)^+, \quad \text{for all} \quad 0 \le s \le T.\end{cases}$$

Note that for each $k \in \mathcal{I}$ the process $Y^{k,0}_\cdot$ is given. Then, by letting

$$\tilde{f}^i(s, Y^{i,1}_s, Z^{i,1}_s) := f^i(s, X_s, Y^{1,0}_s, \dots, Y^{i-1,0}_s, Y^{i,1}_s, Y^{i+1,0}_s, \dots Y^{q,0}_s, Z^{i,1}_s)$$

for $i \in \mathcal{I}$, the data of the RBSDE associated with $(Y^{i,1}_\cdot, Z^{i,1}_\cdot, K^{i,1}_\cdot)$ satisfy the assumptions in Hamadène [27], Theorem 1.4, and hence the processes $(Y^{i,1}_\cdot, Z^{i,1}_\cdot, K^{i,1}_\cdot)$ do exist. Next, using the comparison theorem of solutions of BSDEs (see Theorem 2.7), we deduce that for any $i \in \mathcal{I}$, $Y^{i,0}_\cdot \le Y^{i,1}_\cdot$. Besides, as f^i satisfies the monotonicity property (H2)-(iv) and using again the comparison of solutions of RBSDEs with càdlàg barrier (see Theorem 1.5 in Hamadène [27]), we obtain by induction that:

$$\text{for all } \ n \ge 1 \ \text{ and } \ i \in \mathcal{I}, \quad Y^{i,n}_\cdot \le Y^{i,n+1}_\cdot. \tag{4.24}$$

On the other hand, the process $(\overline{Y}_\cdot, \overline{Z}_\cdot)$ in (4.22), can be regarded as the triplet $((\overline{Y}^i_\cdot, \overline{Z}^i_\cdot, 0))_{i \in \mathcal{I}}$ (i.e. $K^i_\cdot = 0$), which is a solution for the system of RBSDEs with data

$$([\max_{i=1,\dots,q} f^i](t, X_t, y^1, \dots, y^q, z), \gamma^{ik}(t, X_t), \max_{i=1,\dots,q} h^i(X_T)), \quad i, k \in \mathcal{I}.$$

Note that

$$\begin{aligned} f^i(t, \overline{Y}_t, \overline{Z}_t) \quad &:= f^i(t, X_t, Y^{1,0}_t, \dots, \overline{Y}_t, \dots, Y^{q,0}_t, \overline{Z}_t) \\ &\le \max\nolimits_{i=1,\dots,q} f^i(t, X_t, \overline{Y}_t, \dots, \overline{Y}_t, \dots, \overline{Y}_t, \overline{Z}_t) \\ &:= \max\nolimits_{i=1,\dots,q} f^i(t, \overline{Y}_t, \overline{Z}_t), \end{aligned}$$

where the above inequalities follow since f^i satisfies the monotonicity property (H2)-(iv) and due that, for each $k \in \mathcal{I}^{-i}$ (the fixed processes), $Y^{k,0}_\cdot = \underline{Y}_\cdot \leq \overline{Y}_\cdot$. Therefore, by comparison Theorem 1.5 in Hamadène [27], we get that $Y^{i,1}_\cdot \leq \overline{Y}_\cdot$. In general, through an induction procedure, we can obtain for all $n \geq 0$ and $i \in \mathcal{I}$, $Y^{i,n}_\cdot \leq \overline{Y}_\cdot$ and hence

$$\underline{Y}_\cdot = Y^{i,0}_\cdot \leq Y^{i,n}_\cdot \leq Y^{i,n+1}_\cdot \leq \overline{Y}_\cdot. \tag{4.25}$$

Arguing as in the last part of Theorem 4.1, we can see that there exists $Y^i_\cdot$ such that $Y^{i,n}_\cdot \nearrow Y^i_\cdot$ and $\mathbb{E}\big[\sup_{0\leq t\leq T} \left|Y^i_t\right|^2\big] < \infty$. Therefore, using Peng's Monotonic Limit Theorem (see Theorem 2.1 and Theorem 3.6 in Peng [46]), we deduce that for any $i \in \mathcal{I}$, the limit process $Y^i_\cdot$ is a càdlàg process and that there exists $(Z^i_\cdot, K^i_\cdot) \in \mathcal{H}^{2,d} \times \mathcal{S}^2$ with K^i non-decreasing process and $K^i_0 = 0$ such that: $\forall s \leq T$,

$$\begin{cases} Y^i_s = h^i(X_T) + \displaystyle\int_s^T f^i(r, X_r, Y^1_r, \ldots, Y^i_r, \ldots, Y^q_r, Z^i_r)dr + K^i_T - K^i_s - \int_s^T Z^i_r dB_r\,; \\ Y^i_s \geq \displaystyle\max_{k\in\mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\}. \end{cases}$$

Now we claim that $\left(Y^i, Z^i, K^i\right)_{i=1,\ldots,q}$ is, in fact, the desired solution of (4.21). Indeed, consider the RBSDEs at the i-th variable and the other variables $Y^1, \ldots, Y^{i-1}, Y^{i+1}, \ldots, Y^q$ fixed, that is to say

$$\begin{cases} \forall i \in \mathcal{I}, \ \text{ find } \ \left(\tilde{Y}^i_\cdot, \tilde{Z}^i_\cdot, \tilde{K}^i_\cdot\right) \ \text{ such that :} \\ \tilde{Y}^i_\cdot, \tilde{K}^i_\cdot \in \mathcal{S}^2 \quad \text{and} \quad \tilde{Z}^i_\cdot \in \mathcal{H}^{2,d}; \tilde{K}^i_\cdot \ \text{ is non-decreasing and } \ \tilde{K}^i_0 = 0 \\ \tilde{Y}^i_s = h^i(X_T) + \displaystyle\int_s^T f^i(r, X_r, Y^1_r, \ldots, Y^{i-1}_r, \tilde{Y}^i_r, Y^{i+1}_r, \ldots Y^q_r, Z^i_r)dr + \tilde{K}^i_T - \tilde{K}^i_s - \int_s^T \tilde{Z}^i_r dB_r, \ \text{for all } 0 \leq s \leq T; \\ \tilde{Y}^i_s \geq \displaystyle\max_{k\in\mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\} \qquad \text{for all } 0 \leq s \leq T \\ \text{and if } \ \tilde{K}^i_\cdot = \tilde{K}^{i,c}_\cdot + \tilde{K}^{i,d}_\cdot, \ \text{where } \ \tilde{K}^{i,c}_\cdot \ (\text{resp.} \tilde{K}^{i,d}_\cdot) \ \text{ is the continuous} \\ \text{(resp. purely discontinuous) part of } \ \tilde{K}^i, \ \text{ then:} \\ \displaystyle\int_0^T \Big(\tilde{Y}^i_r - \max_{k\in\mathcal{I}^{-i}} \left\{Y^k_r - \gamma^{ik}(r, X_r)\right\}\Big) d\tilde{K}^{i,c}_r = 0. \\ \Delta\tilde{Y}_s := \tilde{Y}^i_s - \tilde{Y}^i_{s^-} = -\Big(\displaystyle\max_{k\in\mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\} - \tilde{Y}^i_s\Big)^+ \qquad \text{for all } 0 \leq s \leq T. \end{cases} \tag{4.26}$$

The solution of (4.26) do exist by using again Theorem 1.4 in Hamadène [27]. Such a solution $\tilde{Y}^i_\cdot$ becomes the smallest f^i-supermartingale that dominates $\max_{k\in\mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\}$ (for more details on this last assertion, see Peng and Xu [47]). But, the limit process Y^i is also a f^i-supermartingale dominating the same barrier and thus $\tilde{Y}^i_t \leq Y^i_t$. On the other hand, since $Y^{i,n-1}_t \leq Y^i_t$ for any $i \in \mathcal{I}$ and $n \geq 1$, we get

$$\max_{k\in\mathcal{I}^{-i}} \left\{Y^{k,n-1}_s - \gamma^{ik}(s, X_s)\right\} \leq \max_{k\in\mathcal{I}^{-i}} \left\{Y^k_s - \gamma^{ik}(s, X_s)\right\}.$$

Also observe that assumptions (H2)-(iv) yields that

$$f^i(t, x, Y^{1,n-1}_t, \ldots, \tilde{Y}^i_t, \ldots, Y^{q,n-1}_t, Z^i_t) \leq f^i(t, x, Y^1_t, \ldots, \tilde{Y}^i_t, \ldots, Y^q_t, Z^i_t).$$

Then using again the comparison theorem for RBSDEs given in Theorem 1.5 in Hamadène [27], we have $Y^{i,n}_t \leq \tilde{Y}^i_t$. This implies that $Y^i_t \leq \tilde{Y}^i_t$ after taking limit $n \to \infty$, and hence $\tilde{Y}^i_t = Y^i_t$. Moreover, this also implies that $\tilde{Z}^i_t = Z^i_t$ and $\tilde{K}^i_t = K^i_t$ for any $0 \leq t \leq T$, $\mathbb{P}$-a.s. This proves the existence of solution for (4.21). □

We now provide a representation result for the solutions of system (4.21) and, as a by-product, we obtain the uniqueness. For later use, let us fix $\mathbf{u}_\cdot := \left(u^1_\cdot, \ldots, u^q_\cdot\right)$ in $\mathcal{H}^{2,q}$ and let us consider the following system of

RBSDEs:

$$\begin{cases} \forall i \in \mathcal{I}, \text{ find } \left(Y_\cdot^{\mathbf{u},i}, Z_\cdot^{\mathbf{u},i}, K_\cdot^{\mathbf{u},i}\right) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^{2,d} \text{ such that :} \\ Y_s^{\mathbf{u},i} = h^i(X_T) + \int_s^T f^i(r, X_r, \mathbf{u}_r, Z_r^{\mathbf{u},i})dr + K_T^{\mathbf{u},i} - K_s^{\mathbf{u},i} - \int_s^T Z_r^{\mathbf{u},i} dB_r \quad \text{for all } 0 \le s \le T; \\ Y_s^{\mathbf{u},i} \ge \max_{k \in \mathcal{I}^{-i}} \left\{Y_s^{\mathbf{u},k} - \gamma^{ik}(s, X_s)\right\} \quad \text{for all } 0 \le s \le T. \\ \text{and if } K_\cdot^{\mathbf{u},i} = K_\cdot^{\mathbf{u},i,c} + K_\cdot^{\mathbf{u},i,d}, \text{ where } K_\cdot^{i,\mathbf{u},c} \text{ (resp.} K_\cdot^{i,\mathbf{u},d}) \text{ is the continuous} \\ \text{(resp. purely discontinuous) part of } K_\cdot^{\mathbf{u},i}, \text{ then:} \\ \int_0^T \left(Y_r^{\mathbf{u},i} - \max_{k \in \mathcal{I}^{-i}} \left\{Y_r^{\mathbf{u},k} - \gamma^{ik}(r, X_r)\right\}\right) dK_r^{i,\mathbf{u},c} = 0. \\ \Delta Y_s^{\mathbf{u}} := Y_s^{\mathbf{u},i} - Y_{s^-}^{\mathbf{u},i} = -\left(\max_{k \in \mathcal{I}^{-i}} \left\{Y_s^{\mathbf{u},k} - \gamma^{ik}(s, X_s)\right\} - Y_s^{\mathbf{u},i}\right)^+ \quad \text{for all } 0 \le s \le T. \end{cases} \tag{4.27}$$

Observe that f^i does not depend on $Y^1, \dots, Y^q$. Let $s \le T$ be fixed, $i \in \mathcal{I}$ and let $\mathcal{D}_s^i$ be the following set of strategies as in Definition 4.1, such that:

$$\mathcal{D}_s^i := \left\{\alpha = (\theta_n, \kappa_n)_{n \ge 0} : \theta_0 = s, \kappa_0 = i \text{ and } \mathbb{E}[(\mathtt{C}_T^\alpha)^2] < \infty\right\}$$

where $\mathtt{C}_r^\alpha$, $r \le T$, is the following cumulative costs up to time r, i.e.,

$$\mathtt{C}_r^\alpha := \sum_{n=1}^\infty \gamma^{\kappa_{n-1},\kappa_n}(\theta_n, X_{\theta_n})\mathbf{1}_{[\theta_n \le r]} \quad \text{for } r < T \text{ and } \mathtt{C}_T^\alpha = \lim_{r \to T} \mathtt{C}_r^\alpha, \ \mathbb{P}\text{-a.s.}$$

Therefore and for any admissible strategy $\alpha \in \mathcal{D}_s^i$ we have:

$$\mathtt{C}_T^\alpha = \sum_{n=1}^\infty \gamma^{\kappa_{n-1},\kappa_n}(\theta_n, X_{\theta_n})\mathbf{1}_{[\theta_n < T]}.$$

Consider a strategy $\alpha = (\theta_n, \kappa_n)_{n \ge 0} \in \mathcal{D}_s^i$ and let $(P_\cdot^\alpha, Q_\cdot^\alpha) := (P_s^\alpha, Q_s^\alpha)_{s \le T}$ be the solution of the following BSDE

$$\begin{cases} P_\cdot^\alpha \text{ is càdlàg and } \mathbb{E}\big[\sup_{s \le T} |P_s^\alpha|^2\big] < \infty, \ \ Q_\cdot^\alpha \in \mathcal{H}^{2,d}; \\ P_s^\alpha = h^\alpha(X_T) + \int_s^T f^\alpha(r, X_r, \mathbf{u}_r, Q_r^\alpha)dr - (\mathtt{C}_T^\alpha - \mathtt{C}_s^\alpha) - \int_s^T Q_r^\alpha dB_r, \ s \le T, \end{cases} \tag{4.28}$$

with

$$h^\alpha(x) = h^{\kappa_n}(x)\mathbf{1}_{[\theta_n < T \le \theta_{n+1}]} \text{ and}$$
$$f^\alpha(r, x, v_1, \dots, v_q, z) := \sum_{n=0}^\infty f^{\kappa_n}(r, x, v_1, \dots, v_q, z)\mathbf{1}_{[\theta_n \le r < \theta_{n+1})}. \tag{4.29}$$

Making the change of variable $\bar{P}_\cdot^\alpha := P_\cdot^\alpha - \mathtt{C}_\cdot^\alpha$, the equation in (4.28) is transformed in a standard BSDE. Since $\mathtt{C}^\alpha$ is adapted and $\mathbb{E}[(\mathtt{C}_T^\alpha)^2] < \infty$, we easily deduce the existence and uniqueness of the process $(P_\cdot^\alpha, Q_\cdot^\alpha)$. We then have the following representation for the solution of (4.27).

Proposition 4.5. *Assume that for any $i, k \in \mathcal{I}$:*

(i) f^i satisfies (H2)-(ii),(iii);

(ii) γ^{ik} (resp. h^i) satisfies (H3) (resp. (H4)).

Then the solution of system of RBSDEs (4.27) exists, it is unique and satisfies:

$$Y_s^{\mathbf{u},i} = \operatorname*{ess\,sup}_{\alpha \in \mathcal{D}_s^i} \{P_s^\alpha - \mathtt{C}_s^\alpha\} \quad \forall s \le T, \quad \forall i \in \mathcal{I}. \tag{4.30}$$

Proof. Since f^i does not depend on variables $Y^1_\cdot, \dots, Y^q_\cdot$, then, it trivially satisfies (H2)-(iv). Then, by hypothesis *(i)* and *(ii)*, and Proposition 4.4, the solution $(Y^{\mathbf{u},i}_\cdot, Z^{\mathbf{u},i}_\cdot, K^{\mathbf{u},i}_\cdot)$ of the system (4.27) exists. Therefore, proceeding as in the proof of Theorem 3.3, we can plug an arbitrary strategy $\alpha \in \mathcal{D}^i_s$ in (4.27) to obtain

$$Y^{\mathbf{u},i}_s \geq h^\alpha(X_T) + \int_s^T f^\alpha(r, X_r, \mathbf{u}_r, Z^\alpha_r)dr + \tilde{K}^\alpha_T - \mathtt{C}^\alpha_T - \int_s^T Z^\alpha_r dB_r. \tag{4.31}$$

with h^α and f^α as in (4.29), and,

$$\tilde{K}^\alpha_T = (K^{\mathbf{u},i}_{\theta_1} - K^{\mathbf{u},i}_s) + \sum_{n=1}^\infty (K^{\mathbf{u},\kappa_n}_{\theta_{n+1}} - K^{\mathbf{u},\kappa_n}_{\theta_n}) \text{ and } Z^\alpha_r = \sum_{n=0}^\infty Z^{\mathbf{u},\kappa_n}_r \mathbf{1}_{[\theta_n \leq r < \theta_{n+1})}, \forall r \leq T. \tag{4.32}$$

Adding $\mathtt{C}^\alpha_s$ from both sides of (4.31) and taking into account that $\tilde{K}^\alpha_T \geq 0$, we have

$$\begin{aligned} Y^{\mathbf{u},i}_s + \mathtt{C}^\alpha_s &\geq h^\alpha(X_T) + \int_s^T f^\alpha(r, X_r, \mathbf{u}_r, Z^\alpha_r)dr - (\mathtt{C}^\alpha_T - \mathtt{C}^\alpha_s) - \int_s^T Z^\alpha_r dB_r \\ &= P^\alpha_s. \end{aligned}$$

Therefore, we have

$$Y^{\mathbf{u},i}_s \geq \operatorname*{ess\,sup}_{\alpha \in \mathcal{D}^i_s} \{P^\alpha_s - \mathtt{C}^\alpha_s\}, \quad \forall \alpha \in \mathcal{D}^i_s. \tag{4.33}$$

Next let $\alpha^\varepsilon = (\theta^\varepsilon_n, \kappa^\varepsilon_n)_{n\geq 0}$ be the strategy defined recursively as follows (compare to the ε-strategy $\mathcal{S}^\varepsilon$ in Proposition 4.2): $\theta^\varepsilon_0 = 0, \kappa^\varepsilon_0 = i$ and for $n \geq 0$,

$$\theta^\varepsilon_{n+1} = \inf\left\{ s \geq \theta^\varepsilon_n : Y^{\mathbf{u},\kappa^\varepsilon_n}_s \leq \max_{k \in \mathcal{I}^{-\kappa^\varepsilon_n}} \left(Y^{\mathbf{u},k}_s - \gamma^{\kappa^\varepsilon_n,k}(s, X_s)\right) + \frac{\varepsilon}{2^{n+1}} \right\} \wedge T$$

and

$$\kappa^\varepsilon_{n+1} = \arg \max_{k \in \mathcal{I}^{-\kappa^\varepsilon_n}} \left\{ Y^{\mathbf{u},k}_{\theta^\varepsilon_{n+1}} - \gamma^{\kappa^\varepsilon_n,k}(\theta^\varepsilon_{n+1}, X_{\theta^\varepsilon_{n+1}})\right\}.$$

In a similar manner as in the proof of Proposition 4.2, we can ensure that $\alpha^\varepsilon \in \mathcal{D}^i_s$ satisfies that $P[\theta^\varepsilon_n < T, \forall n \geq 0] = 0$. Let us prove now that $\mathbb{E}\big[(\mathtt{C}^{\alpha^\varepsilon}_T)^2\big] < \infty$ and that α^ε is ε-optimal in $\mathcal{D}^i_s$ for the problem (4.30). Following the strategy α^ε and since $(Y^{\mathbf{u},i})_{i\in\mathcal{I}}$ solves the RBSDE (4.27), it turns out that,

$$Y^{\mathbf{u},i}_s \leq Y^{\mathbf{u},\kappa^\varepsilon_n}_{\theta^\varepsilon_n} + \int_s^{\theta^\varepsilon_n} f^{\alpha^\varepsilon}(r, X_r, \mathbf{u}_r, Z^{\alpha^\varepsilon}_r)dr - \mathtt{C}^{\alpha^\varepsilon}_{\theta^\varepsilon_n} - \int_s^{\theta^\varepsilon_n} Z^{\alpha^\varepsilon}_r dB_r + \varepsilon \sum_{i=1}^n \frac{1}{2^i}, \quad \forall n \geq 1 \tag{4.34}$$

since $K^{\mathbf{u},\kappa^\varepsilon_n}_r - K^{\mathbf{u},\kappa^\varepsilon_n}_{\theta^\varepsilon_n} = 0$ for $\theta^\varepsilon_n \leq r < \theta^\varepsilon_{n+1}$. Taking now the limit with respect to n in (4.34) we get:

$$Y^{\mathbf{u},i}_s \leq h^{\alpha^\varepsilon}(X_T) + \int_s^T f^{\alpha^\varepsilon}(r, X_r, \mathbf{u}, Z^{\alpha^\varepsilon}_r)dr - \mathtt{C}^{\alpha^\varepsilon}_T - \int_s^T Z^{\alpha^\varepsilon}_r dB_r + \varepsilon, \tag{4.35}$$

and thus

$$\mathtt{C}^{\alpha^\varepsilon}_T \leq |Y^{\mathbf{u},i}| + |h^{\alpha^\varepsilon}(X_T)| + \int_s^T |f^{\alpha^\varepsilon}(r, X_r, \mathbf{u}, Z^{\alpha^\varepsilon}_r)|dr + \int_s^T |Z^{\alpha^\varepsilon}_r| dB_r + \varepsilon. \tag{4.36}$$

Using the assumptions (H4) and (H2)-(ii),(iii) applied for h^i and f^i respectively, and since $\mathbf{u} \in \mathcal{H}^{2,q}$, $Z^{\alpha^\varepsilon} \in \mathcal{H}^{2,d}$, and $(Y^{1,\mathbf{u}}_\cdot, \dots, Y^{q,\mathbf{u}}_\cdot) \in (\mathcal{S}^2)^q$, we deduce from (4.36) that $\mathbb{E}[(\mathtt{C}^{\alpha^\varepsilon}_T)^2] < \infty$. The last assertion together with the admisibility of α^ε show that $\alpha^\varepsilon \in \mathcal{D}^i_s$. On the other hand, from (4.35), we have

$$\begin{aligned} Y^{\mathbf{u},i}_s &\leq h^{\alpha^\varepsilon}(X_T) + \int_s^T f^{\alpha^\varepsilon}(r, X_r, \mathbf{u}, Z^{\alpha^\varepsilon}_r)dr - \mathtt{C}^{\alpha^\varepsilon}_T - \int_s^T Z^{\alpha^\varepsilon}_r dB_r + \varepsilon \\ &= P^{\alpha^\varepsilon}_s - \mathtt{C}^{\alpha^\varepsilon}_s + \varepsilon. \end{aligned} \tag{4.37}$$

Taking supremum over all $\alpha \in \mathcal{D}_s^i$, and next letting $\varepsilon \to 0$ we deduce that $Y_s^{\mathbf{u},i} \le \operatorname{ess\,sup}_{\alpha\in\mathcal{D}_s^i} \{P_s^\alpha - \mathtt{C}_s^\alpha\}$. This last fact together with (4.33) yield (4.30). Therefore, we obtain that the solution of (4.27) is unique. $\square$

Next for $\mathbf{u} := (u^1, \ldots, u^q) \in \mathcal{H}^{2,q}$ let us define

$$\Phi(\mathbf{u}) := (Y_\cdot^{\mathbf{u},1}, \ldots, Y_\cdot^{\mathbf{u},q}),$$

where $(Y_\cdot^{\mathbf{u},i}, Z_\cdot^{\mathbf{u},i}, K_\cdot^{\mathbf{u},i})_{i=1,\ldots,q}$ is the solution of system (4.27) which exists and is unique under the assumptions of Proposition 4.5. Note that the processes $(Y_\cdot^{\mathbf{u},i}, \ldots, Y_\cdot^{\mathbf{u},q})$ belong to $(\mathcal{S}^2)^q \subseteq \mathcal{H}^{2,q}$. Hence, Φ is a mapping from $\mathcal{H}^{2,q}$ to $\mathcal{H}^{2,q}$.

We introduce the norm defined on $\mathcal{H}^{2,q}$ by

$$\left\|(u^1, \ldots, u^q)\right\|_\beta^2 := \mathbb{E}\left[\int_0^T e^{\beta s}\left(\sum_{i=1}^q \left|u_s^i\right|^2\right) ds\right].$$

Note that $\|w\|_{\mathcal{H}^{2,q}} \le \|w\|_\beta \le e^{\beta T}\|w\|_{\mathcal{H}^{2,q}}$, for all $w \in \mathcal{H}^{2,q}$, implies that these norms are equivalent. For sake of completeness, we present the following result, established in Chassagneux et al. [10], which ensures that Φ is a contraction on the Banach space $(\mathcal{H}^{2,q}, \|\cdot\|_\beta)$.

Proposition 4.6. *Assume that for any $i, j \in \mathcal{I}$ the following hypotheses are in force:*

*(i) f^i verifies **(H2)-(ii),(iii)**;*

*(ii) γ^{ij} (resp. h^i) verifies **(H3)** (resp. **(H4)**).*

Then, there exists $\beta_0 \in \mathbb{R}$ such that the mapping Φ is a contraction operator on $(\mathcal{H}^{2,q}, \|\cdot\|_{\beta_0})$. Therefore Φ has a fixed point $(Y_\cdot^1, \ldots, Y_\cdot^q)$ which belongs to $(\mathcal{S}^2)^q$ and which provides a unique solution for system (4.21).

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathcal{H}^{2,q}$ and consider the respective images under Φ, $Y_\cdot^{\mathbf{u},i} := \Phi(\mathbf{u})$ and $Y_\cdot^{\mathbf{v},i} := \Phi(\mathbf{v})$. Besides, let us introduce the following "auxiliary dominating" RBSDE, for $i \in \mathcal{I}$:

$$\begin{cases} \check{Y}_s^i = h^i(X_T) + \int_s^T \check{f}^i(r, X_r, \check{Z}_r^i)dr + \check{K}_T^i - \check{K}_s^i - \int_s^T \check{Z}_r^i dB_r \quad \text{for all } 0 \le s \le T, \\ \check{Y}_s^i \ge \max\limits_{k\in\mathcal{I}^{-i}} \left\{\check{Y}_s^k - \gamma^{ik}(s, X_s)\right\} \quad \text{for all } 0 \le s \le T, \\ \int_0^T \left(\check{Y}_s^i - \max\limits_{k\in\mathcal{I}^{-i}} \left\{\check{Y}_s^k - \gamma^{ik}(s, X_s)\right\}\right) d\check{K}_s^{i,c} = 0. \\ \Delta_s \check{Y}_\cdot^i := \check{Y}_s^i - \check{Y}_{s^-}^i = -\left(\max\limits_{k\in\mathcal{I}^{-i}} \left\{\check{Y}_s^k - \gamma^{ik}(s, X_s)\right\} - \check{Y}_s^i\right)^+ \quad \text{for all } 0 \le s \le T \end{cases} \tag{4.38}$$

where $\check{f}^i(s, X_s, z^i) = \max\{f^i(s, X_s, \mathbf{u}_r, z^i), f^i(s, X_s, \mathbf{v}_r, z^i)\}$, and $\check{K}_\cdot^{i,c}$ and $\check{K}_\cdot^{i,d}$ are the continuous and discontinuous parts of $\check{K}^i$. Note that by Proposition 4.5 a unique solution $(\check{Y}_\cdot^i, \check{Z}_\cdot^i, \check{K}_\cdot^i)$ exists for (4.38).
For $s \in [0, T]$ fixed, and for any $\alpha \in \mathcal{D}_s^i$, denote by $(U_\cdot^\alpha, Z_\cdot^\alpha)$,$(\bar{U}_\cdot^\alpha, \bar{Z}_\cdot^\alpha)$ and $(\check{U}_\cdot^\alpha, \check{Z}_\cdot^\alpha)$ the respective solutions of the following one-dimensional BSDEs: $\forall s \le T$,

$$U_s^\alpha = h^\alpha(X_T) + \int_s^T f^\alpha(r, X_r, \mathbf{u}_r, Z_r^\alpha)dr - (\mathtt{C}_T^\alpha - \mathtt{C}_s^\alpha) - \int_s^T Z_r^\alpha dB_r,$$

$$\bar{U}_s^\alpha = h^\alpha(X_T) + \int_s^T f^\alpha(r, X_r, \mathbf{v}_r, \bar{Z}_r^\alpha)dr - (\mathtt{C}_T^\alpha - \mathtt{C}_s^\alpha) - \int_s^T \bar{Z}_r^\alpha dB_r,$$

$$\check{U}_s^\alpha = h^\alpha(X_T) + \int_s^T \check{f}^\alpha(r, X_r, \check{Z}_r^\alpha)dr - (\mathtt{C}_T^\alpha - \mathtt{C}_s^\alpha) - \int_s^T \check{Z}_r^\alpha dB_r.$$

We deduce from Proposition 4.5 that

$$Y_s^{\mathbf{u},i} = \operatorname*{ess\,sup}_{\alpha\in\mathcal{D}_s^i}\{U_s^\alpha - \mathtt{C}_s^\alpha\}, \quad Y_s^{\mathbf{v},i} = \operatorname*{ess\,sup}_{\alpha\in\mathcal{D}_s^i}\{\bar{U}_s^\alpha - \mathtt{C}_s^\alpha\}, \quad \check{Y}_s^i = \operatorname*{ess\,sup}_{\alpha\in\mathcal{D}_s^i}\{\check{U}_s^\alpha - \mathtt{C}_s^\alpha\}. \tag{4.39}$$

Besides, note that for an ε-optimal strategy $\alpha^\varepsilon \in \mathcal{D}_s^i$, we have

$$\check{Y}_s^i \le \check{U}_s^{\alpha^\varepsilon} - \mathtt{C}_s^{\alpha^\varepsilon} + \varepsilon. \tag{4.40}$$

Using a comparison argument, we easily check that $\check{U}^{\alpha}_{\cdot} \geq U^{\alpha}_{\cdot} \vee \bar{U}^{\alpha}_{\cdot}$ for any strategy $\alpha \in \mathcal{D}^i_s$, and hence, by (4.39) we get that $\check{Y}^i_s \geq Y^{\mathbf{u},i}_s \vee Y^{\mathbf{v},i}_s$. Therefore, taking into account the last two inequalities and (4.40), we get that

$$U^{\alpha^\varepsilon}_s - \mathsf{C}^{\alpha^\varepsilon}_s \leq Y^{\mathbf{u},i}_s \leq \check{Y}^i_s \leq \check{U}^{\alpha^\varepsilon}_s - \mathsf{C}^{\alpha^\varepsilon}_s + \varepsilon \quad \text{and} \quad \bar{U}^{\alpha^\varepsilon}_s - \mathsf{C}^{\alpha^\varepsilon}_s \leq Y^{\mathbf{v},i}_s \leq \check{Y}^i_s \leq \check{U}^{\alpha^\varepsilon}_s - \mathsf{C}^{\alpha^\varepsilon}_s + \varepsilon.$$

This implies

$$\left|Y^{\mathbf{u},i}_s - Y^{\mathbf{v},i}_s\right| \leq \left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right| + \left|\check{U}^{\alpha^\varepsilon}_s - \bar{U}^{\alpha^\varepsilon}_s\right| + 2\varepsilon,$$

and by using the inequality $(a+b+c)^2 \leq 4a^2 + 4b^2 + 2c^2$, we have

$$\left|Y^{\mathbf{u},i}_s - Y^{\mathbf{v},i}_s\right|^2 \leq 4\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2 + 4\left|\check{U}^{\alpha^\varepsilon}_s - \bar{U}^{\alpha^\varepsilon}_s\right|^2 + 4\varepsilon^2. \tag{4.41}$$

Now, applying Itô's formula to $e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2$, using the inequality $|x \vee y - y| \leq |x - y|$ and the fact that f^{α_ε} is Lipschitz, taking expectation, to obtain: $\forall s \leq T$,

$$\mathbb{E}\Big[e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2 + \int_s^T e^{\beta r}\left|\check{Z}^{\alpha^\varepsilon}_r - Z^{\alpha^\varepsilon}_r\right|^2 dr\Big] \leq -\mathbb{E}\Big[\int_s^T \beta e^{\beta r}\left|\check{U}^{\alpha^\varepsilon}_r - U^{\alpha^\varepsilon}_r\right|^2 dr\Big]$$

$$+2C\mathbb{E}\Big[\int_s^T e^{\beta r}\left|\check{U}^{\alpha^\varepsilon}_r - U^{\alpha^\varepsilon}_r\right|\left(|\mathbf{v}_r - \mathbf{u}_r| + \left|\check{Z}^{\alpha^\varepsilon}_r - Z^{\alpha^\varepsilon}_r\right|\right) dr\Big].$$

The inequalities $2ab \leq \beta a^2 + \frac{1}{\beta}b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$ also imply

$$\mathbb{E}\Big[e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2\Big] + \mathbb{E}\Big[\int_s^T e^{\beta r}\left|\check{Z}^{\alpha^\varepsilon}_r - Z^{\alpha^\varepsilon}_r\right|^2 dr\Big] \leq -\mathbb{E}\Big[\int_s^T \beta e^{\beta r}\left|\check{U}^{\alpha^\varepsilon}_r - U^{\alpha^\varepsilon}_r\right|^2 dr\Big]$$

$$+\mathbb{E}\Big[\int_s^T \{\beta e^{\beta r}\left|\check{U}^{\alpha^\varepsilon}_r - U^{\alpha^\varepsilon}_r\right|^2 + \tfrac{2C^2}{\beta}e^{\beta r}|\mathbf{v}_r - \mathbf{u}_r|^2 + \tfrac{2C^2}{\beta}e^{\beta r}\left|\check{Z}^{\alpha^\varepsilon}_r - Z^{\alpha^\varepsilon}_r\right|^2\}dr\Big].$$

Rearranging terms, we obtain:

$$\mathbb{E}\Big[e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2\Big] + \Big(1 - \frac{2C^2}{\beta}\Big)\mathbb{E}\Big[\int_s^T e^{\beta r}\left|\check{Z}^{\alpha^\varepsilon}_r - Z^{\alpha^\varepsilon}_r\right|^2 dr\Big] \leq \frac{2C^2}{\beta}\mathbb{E}\Big[\int_s^T e^{\beta r}|\mathbf{v}_r - \mathbf{u}_r|^2 dr\Big].$$

Taking $\beta \geq 2C^2$, we get

$$\mathbb{E}\Big[e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - U^{\alpha^\varepsilon}_s\right|^2\Big] \leq \frac{2C^2}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta r}|\mathbf{v}_r - \mathbf{u}_r|^2 dr\Big].$$

Now, an analogous procedure to $e^{\beta s}\left|\check{U}^{\alpha_\varepsilon}_s - \bar{U}^{\alpha_\varepsilon}_s\right|^2$ lead to similar result, namely

$$\mathbb{E}\Big[e^{\beta s}\left|\check{U}^{\alpha^\varepsilon}_s - \bar{U}^{\alpha^\varepsilon}_s\right|^2\Big] \leq \frac{2C^2}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta r}|\mathbf{v}_r - \mathbf{u}_r|^2 dr\Big].$$

Combining these two inequalities with (4.41), we deduce

$$\mathbb{E}\Big[e^{\beta s}\left|Y^{\mathbf{u},i}_s - Y^{\mathbf{v},i}_s\right|^2\Big] \leq \frac{16C^2}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta r}|\mathbf{v}_r - \mathbf{u}_r|^2 dr\Big] + 4\varepsilon^2 e^{\beta T}.$$

By integrating with respect to s and summing up over i both sides of the last inequality and taking into account the fact that such inequality holds true for any $i = 1, \ldots, q$ and for all $s \in [0, T]$, we get

$$\|\Phi(Y^{\mathbf{u}}) - \Phi(Y^{\mathbf{v}})\|_\beta \leq 4C\sqrt{Tq\beta^{-1}}\,\|\mathbf{u} - \mathbf{v}\|_\beta + 2\varepsilon\sqrt{Tqe^{\beta T}}.$$

Finally, choosing $\beta_0 > \max\left(16C^2Tq, 2C^2\right)$ and taking $\varepsilon \to 0$, we see that this mapping is a contraction. This contraction property also gives the existence and uniqueness of the system of RBSDEs (4.21). □

4.4 Markovian framework and systems of PDE's with càdlàg obstacles

In this section we will provide more specifications to the process $X.$ treated in previous sections. Namely, we will assume now that this process has a Markovian evolution described by means of a stochastic differential equation (diffusion process) as in (4.44) below. Under this framework our previous analysis can be reduced to study a system of partial differential equations with obstacles (quasi-variational system). Among the main results of this section we can highlight the characterization of both the optimal function (4.3) and the solution of the system of RBSDEs (4.21) as a viscosity solution in a weak sense (see Theorem 4.12 below). We will start to introduce the following functions:

$$\begin{cases} b : (t,x) \in [0,T] \times \mathbb{R}^r \mapsto b(t,x) \in \mathbb{R}^r; \\ \sigma : (t,x) \in [0,T] \times \mathbb{R}^r \mapsto \sigma(t,x) \in \mathbb{R}^{r\times d}, \end{cases}$$

satisfying the following hypotheses:

The functions b and σ are jointly continuous and Lipschitz continuous with respect to x uniformly in t, that is, there exists a constant $C \geq 0$ such that for any $t \in [0,T]$ and $x,\, x' \in \mathbb{R}^r$

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq C\,|x - x'|\,. \tag{4.42}$$

Note that continuity and (4.42) imply that b and σ are of linear growth, i.e., there exists a constant C such that:

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|),\ \forall (t,x). \tag{4.43}$$

It is well known that under (4.42)-(4.43), there exists a unique Markov process $(X_s^{t,x})_{s\leq T}$, for $(t,x) \in [0,T]\times\mathbb{R}^r$, that is a (strong) solution of the following standard stochastic differential equation:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s \quad \text{ for all } t \leq s \leq T; \\ X_s^{t,x} = x \quad \text{ for all } 0 \leq s \leq t, \end{cases} \tag{4.44}$$

satisfying the following estimates: For any $p \geq 2$, $x, x' \in \mathbb{R}^r$ and $s \geq t$

$$\mathbb{E}\big[\sup_{s\leq T} \big|X_s^{t,x}\big|^p\,\big] \leq C\,(1 + |x|^p)\,, \quad \mathbb{E}[\sup_{r\in[t,s]} |X_r^{t,x} - x|^p] \leq M_p(s-t)(1 + |x|^p)] \text{ and} \tag{4.45}$$
$$\mathbb{E}[\sup_{r\in[t,s]} |X_r^{t,x} - X_r^{t,x'} - (x - x')|^p] \leq M_p(s-t)|x - x'|^p$$

for some constant M_p (one can refer to Karatzas and Shreve [40] or Revuz and Yor [49], for more details).

Recall that the associated infinitesimal generator to $(X_s^{t,x})_{s\leq T}$ is given by :

$$\mathcal{L}\phi(t,x) = \frac{1}{2} Tr\left[\left(\sigma.\sigma^{\mathtt{T}}\right)(t,x) D_{xx}^2\phi(t,x)\right] + b(t,x)^{\mathtt{T}} D_x\phi(t,x)$$

for ϕ in $C^{1,2}([0,T]\times\mathbb{R}^r)$ ($Tr(.)$ is the trace of a square matrix and, $\mathtt{A}^{\mathtt{T}}$ is the transpose of a matrix $\mathtt{A}$).

Now let $(t,x) \in [0,T]\times\mathbb{R}^r$ be fixed and let $((Y_s^{i,t,x}, Z_s^{i,t,x}, K_s^{i,t,x})_{t\leq s\leq T})_{i=1,\ldots,q}$ be the unique solution of system (4.21) when the process X is taken to be equal to $X^{t,x}$ of (4.44), i.e., the solution associated with $(f^i(s, X_s^{t,x}, y^1, \ldots, y^i, \ldots y^q, z^i), h^i(X_T^{t,x}), g^{ik}(s, X_s^{t,x}))$ (g^{ik} are the switching costs and they satisfy the same assumptions as γ^{ik} in Assumption H) with $y^i \in \mathbb{R}$ and $z^i \in \mathbb{R}^d$.

Assume now that Assumption H is satisfied. Since we are in the Markovian framework then there exist deterministic functions u^i, $i \in \mathcal{I}$, with polynomial growth such that for any (t,x)

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ i \in \mathcal{I},\ \mathbb{P}-a.s.,\ \forall s \in [t,T].$$

Note that we also have

$$u^i(t,x) = Y_t^{i,t,x}, \ \text{ for } \ (t,x) \in [0,T]\times\mathbb{R}^r \ \text{ and } \ i \in \mathcal{I}. \tag{4.46}$$

On the other hand the polynomial growth of u^i stems from the polynomial growths of the data assumed in Assumption H and the BSDEs (4.22), (4.23) as well.

Notation: For a sake of simplicity of notation, hereafter we sometimes denote by $(\psi)_{k=1,\dots,q} := (\psi_1,\dots,\psi_q)$, for some generic function or vector ψ.

Remark 4.3. From now on we will assume that f^i is non-decreasing w.r.t y^k for any $k = 1,...,q$ and not only w.r.t $y^1,...,y^{i-1},y^{i+1},...,y^q$ (as precised in (H2)-(iv)). This assumption is not really restrictive since by considering the system of RBSDEs verified by $(e^{\alpha t}Y^i_t)_{t\le T}$, we obtain new generators f^i given by

$$f^i(t,y^1,...,y^m,z^i) := e^{\alpha t} f^i(t,x,e^{-\alpha t}y^1,...,e^{-\alpha t}y^m,e^{-\alpha t}z^i) - \alpha y^i$$

which have the same properties as $(f^i)_{i=1,\dots,q}$. Moreover, with an appropriate choice of α, those new generators are non-decreasing w.r.t y^k for any $k = 1,\dots,q$, i.e., they fulfill the property we are requiring for $(f^i)_{i=1,\dots,q}$ (one can see Hamadène and Morlais [33], for more details on this transform).

Our main interest will be to show that the function $(u^i)_{i=1,\dots,q} : (t,x) \in [0,T]\times\mathbb{R}^r \mapsto (u^i(t,x))_{i=1,\dots,q} \in \mathbb{R}^q$ is a solution in a weak viscosity sense for the Hamilton-Jacobi-Bellman system of PDEs associated with the switching problem. This type of systems is also regarded as a system of quasi-variational inequalities (QVIs). In the case when the functions g^{ij} and h^i, $i,j\in\mathcal{I}$, are continuous, this system reads as: for all $i\in\mathcal{I}$,

$$\begin{cases} \min\{v^i(t,x) - \max\limits_{k\in\mathcal{I}^{-i}} \left(v^k(t,x) - g^{ik}(t,x)\right);\ -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - \\ \qquad\qquad -f^i(t,x,\left(v^1,\dots,v^i,\dots,v^q\right)(t,x),\sigma^{\mathrm{T}}(t,x)D_x v^i(t,x))\} = 0; \\ v^i(T,x) = h^i(x). \end{cases} \tag{4.47}$$

and it is shown that $(u^i)_{i=1,\dots,q}$ is the unique viscosity solution of system (4.47) —see Hamadène and Morlais [33]. But in our framework the functions g^{ij}, $i,j\in\mathcal{I}$, are no longer continuous w.r.t t, therefore the definition should be adapted. We are going to show that $(u^i)_{i=1,\dots,q}$ is a viscosity solution in a weak sense for the HJB system of PDEs (4.47), associated with the switching problem. This definition is inspired by Ishii's works [37, 38], and also by the paper of Barles and Perthame [1].

To this end, we recall that for a locally bounded $\mathbb{R}$-valued function $v(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^r$ $(r\ge 1)$, the lower (resp. upper) semi-continuous envelope v_* (resp. v^*) of v is next: For any $(t,x)\in[0,T]\times\mathbb{R}^r$,

$$v^*(t,x) := \limsup_{(t',x')\to(t,x),\ t'<T} v(t',x') \quad (\text{resp.} \quad v_*(t,x) := \liminf_{(t',x')\to(t,x),\ t'<T} v(t',x')).$$

Note that the function v^* can also be seen as the smallest upper semi-continuous (usc) function which is greater than v. Similarly, the function v_* can also be seen as the largest lower semi-continuous (lsc) function which is smaller than v. On the other hand, the following properties of the semi-continuous envelopes of functions will be useful later.

Lemma 4.7. *Let $(t,x)\in[0,T]\times\mathbb{R}^r$ and $\varphi^i(t,x)$, $i=1,2$, be two locally bounded $\mathbb{R}$-valued functions. We then have:*

(i) If φ^1 is continuous then $(\varphi^1+\varphi^2)_* = \varphi^1 + (\varphi^2)_*$ and $(\varphi^1+\varphi^2)^* = \varphi^1+(\varphi^2)^*$.

(ii) $(-\varphi^1)_* = -(\varphi^1)^*$.

(iii) $(\varphi^1\wedge\varphi^2)_* = (\varphi^1)_*\wedge(\varphi^2)_*$ and $(\varphi^1\vee\varphi^2)^* = (\varphi^1)^*\vee(\varphi^2)^*$.

(iv) If φ^1 is continuous then $(\varphi^1\wedge\varphi^2)^* = \varphi^1\wedge(\varphi^2)^*$ and $(\varphi^1\vee\varphi^2)_* = \varphi^1\vee(\varphi^2)^*$.

Proof. (i) Obviously we have $\varphi^1+\varphi^2 \ge \varphi^1+(\varphi^2)_*$ and then $(\varphi^1+\varphi^2)_* \ge \varphi^1+(\varphi^2)_*$ since this latter is lsc. On the other hand $(\varphi^1+\varphi^2)_* - \varphi^1 \le \varphi^2$ and then $(\varphi^1+\varphi^2)_* - \varphi^1 \le (\varphi^2)_*$ since $(\varphi^1+\varphi^2)_*-\varphi^1$ is lsc. This completes the proof of the claim as the other property can be obtained similarly.

Parts (ii) and (iii) are rather obvious.

(iv) First note that $(\varphi^1 \wedge \varphi^2)^* \leq \varphi^1 \wedge (\varphi^2)^*$. Next let $((t_n, x_n))_n$ be a sequence such that $(\varphi^2(t_n, x_n))_n \to (\varphi^2)^*(t, x)$ as $n \to \infty$. As φ^1 is continuous then $\varphi^1(t_n, x_n) \wedge \varphi^2(t_n, x_n) \to \varphi^1(t, x) \wedge (\varphi^2)^*(t, x)$ as $n \to \infty$. Therefore, by definition of the usc envelope, $(\varphi^1 \wedge \varphi^2)^*(t, x) \geq \varphi^1(t, x) \wedge (\varphi^2)^*(t, x)$ which completes the proof of the first claim. The proof of the other one is similar. □

Next for $i = 1, ..., q$, let us denote by f^i the non-linearity which defines the i-th equation in (4.47), i.e.,

$$f^i(t, x, (y^j)_{j=1,\dots,q}, r, p, X) = \min\left\{y^i - \max_{k\in\mathcal{I}^{-i}}(y^k - g^{ik}(t, x));\ G^i(t, x, (y^j)_{j=1,\dots,q}, r, p, X)\right\} \tag{4.48}$$

where

$$G^i(t, x, (y^j)_{j=1,\dots,q}, r_i, p_i, X_i) = -r_i - \frac{1}{2}Tr(\sigma\sigma^\intercal X_i) - b^\intercal p_i - f^i(t, x, (y^j)_{j=1,\dots,q}, \sigma^\intercal p_i). \tag{4.49}$$

Note that by Assumption H on f^i and (4.42), the function G^i is jointly continuous in its arguments. Therefore, taking into account the results of Lemma 4.7, for any $i = 1, \dots, q$, the semi-continuous envelopes of f^i (in all arguments) are given by:

$$(f^i)^*(t, x, (y^j)_{j=1,\dots,q}, r, p, X) = \min\left\{y^i - (\max_{k\in\mathcal{I}^{-i}}(y^k - g^{ik}(t, x)))_*;\ G^i(t, x, (y^j)_{j=1,\dots,q}, r, p, X)\right\}$$

and

$$(f^i)_*(t, x, (y^j)_{j=1,\dots,q}, r, p, X) = \min\left\{y^i - (\max_{k\in\mathcal{I}^{-i}}(y^k - g^{ik}(t, x)))^*;\ G^i(t, x, (y^j)_{j=1,\dots,q}, r, p, X)\right\}.$$

We are now ready to precise the definition of the viscosity solution of HJB system associated with the switching problem. As noticed previously it is inspired by the papers [1, 37, 38]. On the other hand, the discontinuities of the functions $(u^i)_{i=1,\dots,q}$ generated by the ones of $(g^{ij})_{i,j\in\mathcal{I}}$ make that the terminal condition at time $t = T$ is not the same as in (4.47), but should be adapted as well to this weak sense (see e.g. [4]).

Definition 4.2. Let $\mathbf{v} := (v^1, \dots, v^q)$ be a locally bounded function from $[0, T] \times \mathbb{R}^r$ into $\mathbb{R}^q$.

(1) We say that $\mathbf{v}$ is a viscosity subsolution of (4.47) if for any $i \in \mathcal{I}$, and $x_0 \in \mathbb{R}^r$,

(a) v^{i*} verifies the following inequality at point (T, x_0):

$$\min\left\{v^{i*}(T, x_0) - h^i(x_0);\ u^{i*}(T, x_0) - \max_{j\in\mathcal{I}^{-i}}\left(v^{j*} - g^{ij}\right)^*(T, x_0)\right\} \leq 0. \tag{4.50}$$

(b) Moreover, at $(t_0, x_0) \in [0, T) \times \mathbb{R}^r$, the function v^i is such that, for any $\phi \in C^{1,2}([0, T] \times \mathbb{R}^r)$ with $\phi(t_0, x_0) = v^{i*}(t_0, x_0)$ and $\phi - v^{i*}$ attaining its minimum at (t_0, x_0), we have

$$\begin{aligned}(F^i)_*(t_0, x_0, (v^{j*}(t_0, x_0))_{j=1,\dots,q}, \partial_t\phi(t_0, x_0), D_x\phi(t_0, x_0), D^2_{xx}\phi(t_0, x_0))\\ = \min\Bigg\{v^{i*}(t_0, x_0) - \max_{k\in\mathcal{I}^{-i}}\left(v^{k*} - g^{ik}\right)^*(t_0, x_0);\\ -(\partial_t + \mathcal{L})\,\phi(t_0, x_0) - f^i(t_0, x_0, (v^{j*}(t_0, x_0))_{j=1,\dots,q}, (\sigma^\intercal D_x)\,\phi(t_0, x_0))\Bigg\} \leq 0.\end{aligned}$$

(2) In the same manner, $\mathbf{v}$ is said to be a viscosity supersolution of (4.47) if for any $i \in \mathcal{I}$, and $x_0 \in \mathbb{R}^r$,

(a) v^i_* verifies at (T, x_0) the following:

$$\min\left\{v^i_*(T, x_0) - h^i(x_0);\ v^i_*(T, x_0) - \left(\max_{j\in\mathcal{I}^{-i}}\left(v^j_* - g^{ij}\right)\right)_*(T, x_0)\right\} \geq 0. \tag{4.51}$$

(b) Similarly, at $(t_0, x_0) \in [0, T) \times \mathbb{R}^r$, v^i satisfies the next: for any $\phi \in C^{1,2}([0, T] \times \mathbb{R}^r)$ with $\phi(t_0, x_0) = v^i_*(t_0, x_0)$ and $\phi - v^i_*$ attaining its maximum at (t_0, x_0), we have

$$\begin{aligned}(F^i)^*(t_0, x_0, (v^j_*(t_0, x_0))_{j=1,\dots,q}, \partial_t\phi(t_0, x_0), D_x\phi(t_0, x_0), D^2_{xx}\phi(t_0, x_0))\\ = \min\Big\{ v^i_*(t_0, x_0) - (\max_{k\in\mathcal{I}^{-i}} (v^k_* - g^{ik}))_*(t_0, x_0);\\ -(\partial_t + \mathcal{L})\,\phi(t_0, x_0) - f^i(t_0, x_0, (v^j_*(t_0, x_0))_{j=1,\dots,q}, (\sigma^{\mathsf{T}} D_x)\,\phi(t_0, x_0))\Big\} \geq 0.\end{aligned}$$

(3) We say that $\mathbf{v}$ is viscosity solution of (4.47) if it is both a viscosity sub- and supersolution.

To proceed we are going to show that the functions $(u^i)_{i=1,\dots,q}$ defined in (4.46) are a viscosity solution of the system (4.47) in a weak sense, i.e., according to Definition 4.2. However we need some preliminary results which we give as lemmas hereafter. From now $B_\eta(t_0, x_0)$ is the open ball of radius η and center (t_0, x_0).

Lemma 4.8. *Under the Assumption (H2), the mapping*

$$(t, x) \longmapsto f^i\big(t, x, \big(v^{1*}, \dots, v^{q*}\big)(t, x), \big(\sigma^{\mathsf{T}} D_x\big)\,\phi(t, x)\big)$$

is usc. for any $\phi \in C^{1,2}([0, T] \times \mathbb{R}^r)$.

Proof. Let $(t_0, x_0) \in [0, T] \times \mathbb{R}^r$. Since v^{k*} is *usc* for $k = 1, \dots, q$, then for all $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for all (t, x), satisfying $\|(t, x) - (t_0, x_0)\| < \eta_\varepsilon$, we have

$$v^{k*}(t_0, x_0) \geq v^{k*}(t, x) - \varepsilon \qquad \text{for all} \quad k = 1, \dots, q.$$

Next, by monotonicity and Lipschitz properties of f^i, for all $(t, x) \in B_{\eta_\varepsilon}(t_0, x_0)$ we get

$$\begin{aligned}&f^i\big(t_0, x_0, \big(v^{k*}\big)_{k=1,\dots,q}(t_0, x_0), \big(\sigma^{\mathsf{T}} D_x\big)\,\phi(t_0, x_0)\big) \geq f^i\big(t_0, x_0, \big(v^{k*}(t, x) - \varepsilon\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\phi(t_0, x_0)\big)\\ &\geq f^i\big(t_0, x_0, \big(v^{k*}(t, x)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\phi(t_0, x_0)\big) - C\varepsilon = f^i\big(t, x, \big(v^{k*}(t, x)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\,\phi(t, x)\big) - C\varepsilon +\\ &+ \Big\{ f^i\big(t_0, x_0, \big(v^{k*}(t, x)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\phi(t_0, x_0)\big) - f^i\big(t, x, \big(v^{k*}(t, x)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\phi(t, x)\big)\Big\},\end{aligned}$$

where C is the Lipschitz constant of f^i. By continuity of f^i with respect to (t, x) and Lipschitz in z^i (Assumptions (H2)-(i)-(ii)), the quantity inside the brackets goes to zero as $(t, x) \to (t_0, x_0)$. Therefore, taking a suitable $\eta_\varepsilon > 0$, we obtain

$$f^i\big(t_0, x_0, \big(v^{k*}(t_0, x_0)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\,\phi(t_0, x_0)\big) \geq f^i\big(t, x, \big(v^{k*}(t, x)\big)_{k=1,\dots,q}, \big(\sigma^{\mathsf{T}} D_x\big)\,\phi(t, x)\big) - C'\varepsilon$$

for all $(t, x) \in B_{\eta_\varepsilon}(t_0, x_0)$ and for some other constant C' and the claim follows. □

Lemma 4.9. *Let* $\phi \in C^{1,2}([0, T] \times \mathbb{R}^r)$, $(t_0, x_0) \in [0, T) \times \mathbb{R}^r$ *and* $\phi(t_0, x_0) = v^{i*}(t_0, x_0)$. *If*

$$\phi(t_0, x_0) = v^{i*}(t_0, x_0) > \max_{k\in\mathcal{I}^{-i}} \big(v^{k*} - g^{ik}\big)^*(t_0, x_0) \tag{4.52}$$

and

$$-(\partial_t + \mathcal{L})\,\phi(t_0, x_0) > f^i(t_0, x_0, (v^{k*})_{k=1,\dots,q}(t_0, x_0), (\sigma^{\mathsf{T}} D_x)\phi(t_0, x_0)), \tag{4.53}$$

then there exist ε *and a ball* $B_{\eta_\varepsilon}(t_0, x_0)$ *such that for all* $(t, x) \in B_{\eta_\varepsilon}(t_0, x_0)$ *we have:*

$$\phi(t, x) \geq \max_{k\in\mathcal{I}^{-i}} \big(v^{k*}(t, x) - g^{ik}(t, x)\big) + \varepsilon \tag{4.54}$$

and

$$-(\partial_t + \mathcal{L})\,\phi(t, x) \geq f^i\big(t, x, (v^{k*})_{k=1,\dots,q}(t, x), (\sigma^{\mathsf{T}} D_x)\phi(t, x)\big) + \varepsilon. \tag{4.55}$$

Proof: By (4.52) and the continuity of ϕ there exist ε and a ball $B_{\eta_\varepsilon}(t_0, x_0)$ such that

$$\phi(t,x) \geq \max_{k\in\mathcal{I}^{-i}} \left(v^{k*}(t_0,x_0) - g^{ik}(t_0,x_0)\right)^* + 2\varepsilon \tag{4.56}$$

for all $(t,x) \in B_{\eta_\varepsilon}(t_0,x_0)$. Next, by the *usc* property, there exists $\eta_\varepsilon^{'}$ such that for all $(t,x) \in B_{\eta_\varepsilon^{'}}(t_0,x_0)$ we have

$$\begin{aligned}\max_{k\in\mathcal{I}^{-i}} \left(v^{k*}(t_0,x_0) - g^{ik}(t_0,x_0)\right)^* &\geq \max_{k\in\mathcal{I}^{-i}} \left(v^{k*}(t,x) - g^{ik}(t,x)\right)^* - \varepsilon \\ &\geq \max_{k\in\mathcal{I}^{-i}} \left(v^{k*}(t,x) - g^{ik}(t,x)\right) - \varepsilon\end{aligned} \tag{4.57}$$

where in the last inequality we use that the usc envelope of a function is greater or equal to the function itself. Therefore, from (4.56), (4.57) and assuming , without loss of generality, that $\eta_\varepsilon \leq \eta_\varepsilon^{'}$ we have

$$\phi(t,x) \geq \max_{k\in\mathcal{I}^{-i}} \left(v^{k*}(t,x) - g^{ik}(t,x)\right) + \varepsilon \tag{4.58}$$

for all $(t,x) \in B_{\eta_\varepsilon}(t_0,x_0)$. ☐

As for the second inequality we can do a similar procedure since $(\partial_t + \mathcal{L})\phi$ is continuous and $(t,x) \mapsto f^i(t,x,(v^{k*})_{k=1,\dots,q}(t,x),(\sigma^\intercal D_x)\phi(t,x))$ is *usc*. Namely, there exist $\varepsilon^{'}$ and $\eta_\varepsilon^{''}$ such that for each $(t,x) \in B_{\eta_\varepsilon^{''}}(t_0,x_0)$ we have

$$-\left(\partial_t + \mathcal{L}\right)\phi(t,x) \geq f^i\left(t,x,(v^{k*})_{k=1,\dots,q}(t,x),(\sigma^\intercal D_x)\phi(t,x)\right) + \varepsilon'. \tag{4.59}$$

Now, supposing, without loss of generality, that $\varepsilon \leq \varepsilon'$ and $\eta_\varepsilon \leq \eta_\varepsilon^{''}$, we have that inequalities (4.54) and (4.55) hold true for all $(t,x) \in B_{\eta_\varepsilon}$.

Remark 4.4. In a similar manner, it is possible to obtain a parallel result as in Lemmas 4.8 and 4.9 for v_*^i in lieu of v^{i*}. Namely, it can be proved that under Assumption (H2) the mapping

$$(t,x) \longmapsto f^i\left(t,x,(v_*^k)_{k=1,\dots,q}(t,x),\left(\sigma^\intercal D_x\right)\phi(t,x)\right)$$

is lsc, and if

$$-\left(\partial_t + \mathcal{L}\right)\phi(t_0,x_0) < f^i(t_0,x_0,(v_*^k)_{k=1,\dots,q}(t_0,x_0),(\sigma^\intercal D_x)\phi(t_0,x_0)), \tag{4.60}$$

then there exists $\varepsilon > 0$ and η_ε such that for all $(t,x) \in B_{\eta_\varepsilon}(t_0,x_0)$:

$$-\left(\partial_t + \mathcal{L}\right)\phi(t,x) \leq f^i\left(t,x,(v_*^k)_{k=1,\dots,q}(t,x),\left(\sigma^\intercal D_x\right)\phi(t,x)\right) - \varepsilon.$$

The proofs are very similar as the proofs given in the aforementioned lemmas, so shall omit them.

Finally, we recall two comparison results for BSDEs and RBSDEs that we have borrowed from Lemma 19 and Proposition 20, in Dumitrescu et al. [20].

Lemma 4.10. *Fix $t_0 \in [0,T]$ and let θ be a stopping time with values in $[t_0,T]$. Consider two random variables ξ^1 and $\xi^2 \in L^2(\mathcal{F}_\theta)$ and two drivers (a.k.a generators) f^1, f^2 such that f^2 satisfies (H2) with Lipschitz constant $C > 0$. For $i = 1,2$, let (Y_t^i, Z_t^i) be the solution in $\mathcal{S}^2 \times \mathcal{H}^2$ of the BSDE with associated data (f^i, ξ^i), and terminal time θ. In this case, f^i and ξ^i represent the driver and terminal condition, respectively. Suppose that for some $\epsilon > 0$ we have*

$$\mathbf{1}_{\{t_0\leq t\leq\theta\}}(t) f^1(t,Y_t^1,Z_t^1) \geq \mathbf{1}_{\{t_0\leq t\leq\theta\}}(t) f^2(t,Y_t^1,Z_t^1),\ dt\otimes d\mathbb{P}-a.e. \quad \text{and} \quad \xi_1 \geq \xi_2 + \varepsilon,\ \mathbb{P}-a.s.$$

Then we have $Y_t^1 \geq Y_t^2 + \varepsilon e^{-CT}$, $\mathbb{P}$-a.s. for each $t \in [t_0,\theta]$.

Lemma 4.11 (A comparison result between a BSDE and a RBSDE)**.** *Fix $t_0 \in [0,T]$ and let θ be a stopping time on $[t_0,T]$. Consider the random variable $\xi^1 \in L^2(\mathcal{F}_\theta)$ and a driver f^1. Let (Y_t^1, Z_t^1) be the associated BSDE solution with driver f^1, terminal time θ and terminal condition ξ_1. Consider also $g^2(\cdot) \in \mathcal{S}^2$ and let f^2 be a driver satisfying (H2) with Lipschitz constant $C > 0$. Assume the existence of the solution Y_t^2 of the associated RBSDE with driver f^2, terminal time θ and obstacle g^2, and assume that*

$$\mathbf{1}_{\{t_0 \le t \le \theta\}}(t) f^1(t, Y_t^1, Z_t^1) \ge \mathbf{1}_{\{t_0 \le t \le \theta\}}(t) f^2(t, Y_t^1, Z_t^1),\ dt \otimes d\mathbb{P} - a.e.$$

and

$$\mathbf{1}_{\{t_0 \le t \le \theta\}}(t) Y_t^1 \ge \mathbf{1}_{\{t_0 \le t \le \theta\}}(t)(g^2(t) + \varepsilon), \forall t \ge 0, \mathbb{P} - a.s.$$

where ε is a positive constant. Then, we have $Y_t^1 \ge Y_t^2 + \varepsilon e^{-CT}$, $\mathbb{P} - a.s.$, for each $t \in [t_0, \theta]$.

We now give the main result of this section.

Theorem 4.12. *The function $\mathbf{u} := (u^1, \dots, u^q)$, where for each $i = 1, \dots, q$, u^i is defined as in (4.46), is a weak viscosity solution of the system (4.47).*

Proof. Step 1: Viscosity sub-solution property on $[0,T) \times \mathbb{R}^r$.

Let $\phi \in C^{1,2}([0,T] \times \mathbb{R}^r)$ and $(t_0, x_0) \in [0,T) \times \mathbb{R}^r$ be such that $\phi(t,x) \ge u^{i*}(t,x)$, for all $(t,x) \in [0,T) \times \mathbb{R}^r$ and $\phi(t_0,x_0) = u^{i*}(t_0,x_0)$. Without loss of generality, we can assume that the minimum of $\phi - u^{i*}$ attained at (t_0,x_0) is strict. We need to show that if

$$\phi(t_0,x_0) = u^{i*}(t_0,x_0) > \max_{k \in \mathcal{I}^{-i}} \left(u^{k*}(t_0,x_0) - g^{ik}(t_0,x_0) \right)^* \tag{4.61}$$

then

$$-\left(\partial_t + \mathcal{L}\right) \phi(t_0,x_0) - f^i\left(t_0, x_0, (u^{k*})_{k=1,\dots,q}(t_0,x_0), (\sigma^{\intercal} D_x)\phi(t_0,x_0)\right) \le 0. \tag{4.62}$$

We proceed by contradiction; i.e. we shall assume

$$-\left(\partial_t + \mathcal{L}\right) \phi(t_0,x_0) - f^i\left(t_0, x_0, (u^{k*})_{k=1,\dots,q}(t_0,x_0), (\sigma^{\intercal} D_x)\phi(t_0,x_0)\right) > 0,$$

then by Lemma 4.9 there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that, for all $(t,x) \in B_{\eta_\varepsilon}(t_0,x_0)$, we have both

$$\phi(t,x) \ge \max_{k \in \mathcal{I}^{-i}} \left(u^{k*}(t,x) - g^{ik}(t,x) \right) + \varepsilon \ge \max_{k \in \mathcal{I}^{-i}} \left(u^{k}(t,x) - g^{ik}(t,x) \right) + \varepsilon, \tag{4.63}$$

since $u^{k*} \ge u^k$, and

$$-\left(\partial_t + \mathcal{L}\right) \phi(t,x) - f^i(t, x, (u^{k*})_{k=1,\dots,q}(t,x), \left(\sigma^{\intercal} D_x\right) \phi(t,x)) \ge \varepsilon. \tag{4.64}$$

By definition of u^{i*}, there exists a sequence $(t_m, x_m)_{m \ge 0}$ in $B_{\eta_\varepsilon}(t_0,x_0)$, such that $(t_m,x_m) \to (t_0,x_0)$ and $u^i(t_m,x_m) \to u^{i*}(t_0,x_0)$. Now let us fix m and take the associated state process X^{t_m,x_m} defined in (4.44) and define the stopping time θ^m as

$$\theta^m := (t_0 + \eta_\varepsilon) \wedge \inf\left\{ s \ge t_m : \left| X_s^{t_m,x_m} - x_0 \right| \ge \eta_\varepsilon \right\}. \tag{4.65}$$

Applying Itô's lemma to $\phi(s, X_s^{t_m,x_m})$, it can be seen that

$$\left(\phi(s, X_s^{t_m,x_m}), \left(\sigma^{\intercal} D_x\right) \phi(s, X_s^{t_m,x_m});\ \ t_m \le s \le \theta^m \right)$$

is the solution of the BSDE with coefficient $-\left(\partial_s + \mathcal{L}\right) \phi(s,x)$, terminal time θ^m and terminal value $\phi(\theta^m, X_{\theta^m}^{t_m,x_m})$. The idea is to compare this BSDE with the solution $(\underline{Y}_s^{i,t_m,x_m})_{t_m \le s \le \theta^m}$ of the RBSDE with coefficient f^i, barrier $\max_{k\in\mathcal{I}^{-i}}\{u^k - g^{ik}\}$ and terminal condition $u^{i*}(\theta^m, X_{\theta^m}^{t_m,x_m})$. Note that by definition of θ^m and inequality (4.64), we have

$$\begin{aligned} -(\partial_s + \mathcal{L})\,\phi\big(s, X_s^{t_m,x_m}\big) &\geq f^i\left(s, X_s^{t_m,x_m}, (u^{k*})_{k=1,\dots,q}(s, X_s^{t_m,x_m}), (\sigma^{\mathsf{T}} D_x)\,\phi(s, X_s^{t_m,x_m})\right) + \varepsilon \\ &\geq f^i\left(s, X_s^{t_m,x_m}, (u^{k})_{k=1,\dots,q}(s, X_s^{t_m,x_m}), (\sigma^{\mathsf{T}} D_x)\,\phi(s, X_s^{t_m,x_m})\right) + \varepsilon \end{aligned}$$

for each $t_m \leq s \leq \theta^m$, where to reach the last inequality we use that $u^* \geq u$, the monotonicity property (H2)-(iv) and the Remark 4.3. It remains to compare the solution $\phi(s, X_s^{t_m,x_m})$ of the BSDE with the barrier $\max_{k\in\mathcal{I}^{-i}}\{u^k(s, X_s^{t_m,x_m}) - g^{ik}(s, X_s^{t_m,x_m})\}\mathbf{1}_{[s<\theta^m]} + u^{i*}(s, X_s^{t_m,x_m})\mathbf{1}_{[s=\theta^m]}$ of the RBSDE for $t_m \leq s \leq \theta^m$. From inequality (4.63) and definition of θ^m we derive that

$$\begin{aligned} \phi(s, X_s^{t_m,x_m}) &\geq \max_{k\in\mathcal{I}^{-i}}\left(u^{k*}(s, X_s^{t_m,x_m}) - g^{ik}(s, X_s^{t_m,x_m})\right) + \varepsilon \\ &\geq \max_{k\in\mathcal{I}^{-i}}\left(u^{k}(s, X_s^{t_m,x_m}) - g^{ik}(s, X_s^{t_m,x_m})\right) + \varepsilon \quad \text{for} \quad t_m \leq s < \theta^m. \end{aligned} \tag{4.66}$$

On the other hand, to show that the inequality holds at θ^m, we recall that the minimum (t_0, x_0) is strict and hence there exists a constant γ_ε such that

$$\phi(t,x) - u^{i*}(t,x) \geq \gamma_\varepsilon \quad \text{on } [0,T]\times\mathbb{R}^r \setminus B_{\eta_\varepsilon}(t_0,x_0).$$

In particular, we have

$$\phi(\theta^m, X_{\theta^m}^{t_m,x_m}) \geq u^{i*}(\theta^m, X_{\theta^m}^{t_m,x_m}) + \gamma_\varepsilon. \tag{4.67}$$

Therefore, from (4.66), (4.67) and letting $\delta_\varepsilon := \min(\varepsilon, \gamma_\varepsilon)$, we get

$$\phi(s, X_s^{t_m,x_m}) \geq \max_{k\in\mathcal{I}^{-i}}\left(u^k(s, X_s^{t_m,x_m}) - g^{ik}(s, X_s^{t_m,x_m}) + \delta_\varepsilon\right)\mathbf{1}_{[s<\theta^m]} + \left(u^{i*}(\theta^m, X_{\theta^m}^{t_m,x_m}) + \delta_\varepsilon\right)\mathbf{1}_{[s=\theta^m]}$$

for $t_m \leq s \leq \theta^m$ a.s.. Thus, by the comparison result in Lemma 4.11, we have

$$\phi(s, X_s^{t_m,x_m}) \geq \underline{Y}_s^{i,t_m,x_m} + \delta_\varepsilon K \qquad \text{for} \quad t_m \leq s \leq \theta^m$$

where K is a positive constant which only depends on T and the Lipschitz constant of f^i. In particular, for $t = t_m$, we have

$$\phi(t_m, x_m) \geq \underline{Y}_{t_m}^{i,t_m,x_m} + \delta_\varepsilon K.$$

Now, since $u^i(t_m, x_m) \to u^{i*}(t_0, x_0)$ and ϕ is continuous with $\phi(t_0, x_0) = u^{i*}(t_0, x_0)$, for m sufficiently large we have both

$$|u^i(t_m, x_m) - u^{i*}(t_0, x_0)| \leq \frac{1}{4}\delta_\varepsilon K \tag{4.68}$$

and

$$|u^{i*}(t_0, x_0) - \phi(t_m, x_m)| \leq \frac{\delta_\varepsilon K}{4}, \tag{4.69}$$

whence $|\phi(t_m, x_m) - u^i(t_m, x_m)| \leq \frac{1}{2}\delta_\varepsilon K$, and hence

$$u^i(t_m, x_m) \geq \underline{Y}_{t_m}^{i,t_m,x_m} + \frac{1}{4}\delta_\varepsilon K. \tag{4.70}$$

But $u^{i*}(\theta^m, X_{\theta^m}^{t_m,x_m}) \geq u^i(\theta^m, X_{\theta^m}^{t_m,x_m})$, then by comparison theorem $u^i(s, X_s^{t_m,x_m}) = Y_s^{i,t_m,x_m} \leq \underline{Y}_s^{i,t_m,x_m}$ for $t_m \leq s \leq \theta^m$. Thus, for $s = t_m$, we get $u^i(t_m, x_m) \leq \underline{Y}_{t_m}^{i,t_m,x_m}$ that produces a contradiction with (4.70). Therefore (4.62) holds true and then also the viscosity subsolution property in $[0,T)\times\mathbb{R}^r$.

Step 2: Viscosity super-solution property on $[0,T)\times\mathbb{R}^r$.

Let $(t_0, x_0) \in [0,T)\times\mathbb{R}^r$ and $\phi \in C^{1,2}([0,T]\times\mathbb{R}^r)$ be such that $\phi(t_0, x_0) = u^i_*(t_0, x_0)$ and $\phi(t,x) \leq u^i_*(t,x)$, for all $(t,x) \in [0,T]\times\mathbb{R}^r$. As stated above, we can suppose that the maximum is strict in (t_0, x_0). Since by construction $u^i \geq \max_{k\in\mathcal{I}^{-i}}\left(u^k - g^{ik}\right)$, then it is easy to see that $u^i_*(t_0,x_0) \geq (\max_{k\in\mathcal{I}^{-i}}(u^k_* - g^{ik}))_*(t_0,x_0)$. Now, we show that

$$-\left(\partial_t + \mathcal{L}\right)\phi(t_0,x_0) - f^i\left(t_0,x_0,(u^k_*)_{k=1,\dots,q}(t_0,x_0),\left(\sigma^{\mathrm{T}}D_x\right)\phi(t_0,x_0)\right) \geq 0.$$

Similar to the subsolution case, we shall proceed by contradiction, namely, suppose that

$$-\left(\partial_t + \mathcal{L}\right)\phi(t_0,x_0) - f^i\left(t_0,x_0,(u^k_*)_{k=1,\dots,q}(t_0,x_0),\left(\sigma^{\mathrm{T}}D_x\right)\phi(t_0,x_0)\right) < 0,$$

then by Remark 4.4 there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that, for all $(t,x) \in B_{\eta_\varepsilon}(t_0,x_0)$, we have

$$-\left(\partial_t + \mathcal{L}\right)\phi(t,x) - f^i\left(t,x,(u^k_*)_{k=1,\dots,q}(t,x),\left(\sigma^{\mathrm{T}}D_x\right)\phi(t,x)\right) \leq -\varepsilon. \tag{4.71}$$

Let $(t_m,x_m)_{m\geq 1}$ be a sequence in $B_{\eta_\varepsilon}(t_0,x_0)$ such that $(t_m,x_m)\to(t_0,x_0)$ and $u^i(t_m,x_m)\to u^i_*(t_0,x_0)$. We introduce the state process X^{t_m,x_m} and define the stopping time θ^m as in (4.65). Next, we apply Itô's formula to $\phi(s,X^{t_m,x_m}_s)$ in order to obtain

$$\left(\phi(s,X^{t_m,x_m}_s),\left(\sigma^{\mathrm{T}}D_x\right)\phi(s,X^{t_m,x_m}_s); t_m \leq s \leq \theta^m\right)$$

is the solution of the BSDE associated with terminal time θ^m, terminal value $\phi(\theta^m,X^{t_m,x_m}_{\theta^m})$ and driver $(-\left(\partial_t + \mathcal{L}\right)\phi(s,X^{t_m,x_m}_s))_{s\in[t_m,\theta_m]}$. Then by definition of θ^m and inequality (4.71), we get

$$\begin{aligned}-\left(\partial_t + \mathcal{L}\right)\phi\left(s,X^{t_m,x_m}_s\right) &\leq f^i\left(s,X^{t_m,x_m}_s,(u^k_*)_{k=1,\dots,q}(s,X^{t_m,x_m}_s),(\sigma^{\mathrm{T}}D_x)\,\phi(s,X^{t_m,x_m}_s)\right) - \varepsilon\\ &\leq f^i\left(s,X^{t_m,x_m}_s,(u^k)_{k=1,\dots,q}(s,X^{t_m,x_m}_s),(\sigma^{\mathrm{T}}D_x)\,\phi(s,X^{t_m,x_m}_s)\right) - \varepsilon\end{aligned} \tag{4.72}$$

for $t_m \leq s \leq \theta^m$ a.s., where to reach the last inequality we use the monotonicity property (H2)-(iv) and Remark 4.3 and that $u^j \geq u^j_*$ for $j = 1,\dots,q$. It remains to compare the terminal conditions of the BSDEs with coefficients $-\left(\partial_t + \mathcal{L}\right)\phi$ and f^i respectively. Since the maximum (t_0,x_0) is strict, there exists γ_ε (which depends on η_ε) such that $u^i_*(t,x) \geq \phi(t,x) + \gamma_\varepsilon$ on $[0,T]\times\mathbb{R}^r \setminus B_{\eta_\varepsilon}(t_0,x_0)$, which implies

$$\phi(\theta_m,X^{t_m,x_m}_{\theta^m}) \leq u^i_*(\theta^m,X^{t_m,x_m}_{\theta^m}) - \gamma_\varepsilon.$$

Thus using inequality (4.72) and the comparison result for BSDEs, Lemma 4.10, we derive that

$$\phi(s,X^{t_m,x_m}_s) \leq \bar{Y}^{i,t_m,x_m}_s, \quad \text{for } \; t_m \leq s \leq \theta^m$$

and therefore, in $s = t_m$, we have $\phi(t_m,x_m) \leq \bar{Y}^{i,t_m,x_m}_{t_m}$. As above mentioned, we can assume that m is sufficient large so that $|\phi(t_m,x_m) - u^i(t_m,x_m)| \leq \frac{\delta_\varepsilon K}{2}$. We thus get

$$u^i(t_m,x_m) - \frac{\gamma_\varepsilon K}{2} \leq \phi(t_m,x_m) \leq \bar{Y}^{i,t_m,x_m}_{t_m}$$

and hence

$$u^i(t_m,x_m) < \bar{Y}^{i,t_m,x_m}_{t_m}. \tag{4.73}$$

But $u^i_*(\theta^m,X^{t_m,x_m}_{\theta^m}) \leq u^i(\theta^m,X^{t_m,x_m}_{\theta^m})$, then by Lemma 4.10 we get $\bar{Y}^{i,t_m,x_m}_s \leq Y^{i,t_m,x_m}_s = u^i(s,X^{t_m,x_m}_s)$ for $t_m \leq s \leq \theta^m$, and thus $\bar{Y}^{i,t_m,x_m}_{t_m} \leq u^i(t_m,x_m)$, which is a contradiction with (4.73). Therefore the viscosity supersolution property in $[0,T)\times\mathbb{R}^r$ holds true.

Step 3: Subsolution property at (T,x).

We now show that for any $i = 1,\dots,m$,

$$\min\left\{u^{i*}(T,x_0) - h^i(x_0);\;\; u^{i*}(T,x_0) - \max_{j\in\mathcal{I}^{-i}}\left(u^{j*} - g^{ij}\right)^*(T,x_0)\right\} \leq 0.$$

We follow here the same idea as in Bouchard [4] (see also Theorem 1 in Hamadène and Morlais [33]). We reason by contradiction, namely, we assume that

$$\min\left\{u^{i*}(T,x_0)-h^i(x_0);\;\; u^{i*}(T,x_0)-\max_{j\in\mathcal{I}^{-i}}\left(u^{j*}-g^{ij}\right)^*(T,x_0)\right\}=2\varepsilon>0. \tag{4.74}$$

Let (t_k,x_k) be a sequence in $[0,T)\times\mathbb{R}^r$ such that

$$(t_k,x_k)\to(T,x_0)\quad\text{and}\quad u^i(t_k,x_k)\to u^{i*}(T,x_0)\quad\text{as}\;\; k\to\infty. \tag{4.75}$$

Since u^{i*} is *usc* and of polynomial growth, we can find a sequence $(\varphi^n)_{n\geq 0}$ of functions of $C^{1,2}([0,T]\times\mathbb{R}^r)$ and neighborhood B_n of (T,x_0) such that $\varphi^n\to u^{i*}$, and hence from the inequality (4.74) we have

$$\min\left\{\varphi^n(t,x)-h^i(x);\;\; \varphi^n(t,x)-\max_{j\in\mathcal{I}^{-i}}\left(u^{j*}-g^{ij}\right)^*(t,x)\right\}\geq\varepsilon\quad\text{for all}\;\;(t,x)\in B_n, \tag{4.76}$$

for n large enough. On the other hand, after possibly passing to a sub-sequence of $(t_k,x_k)_{k\geq 1}$ we can assume that the previous inequality holds on $B_n^k:=[t_k,T]\times B(x_k,\delta_n^k)$ for some $\delta_n^k\in(0,1)$ small enough in such a way that $B_n^k\subset B_n$. Since u^{i*} is locally bounded (recall it has polynomial growth), there exists $\zeta>0$ such that $\left|u^{i*}\right|\leq\zeta$ on B_n. We can then assume that $\varphi^n\geq-2\zeta$ on B_n. Next we define

$$\tilde{\varphi}_k^n(t,x):=\varphi^n(t,x)+\frac{4\zeta\left|x-x_k\right|^2}{(\delta_n^k)^2}+\sqrt{T-t}.$$

Note that $\tilde{\varphi}_k^n\geq\varphi^n$ and

$$\left(u^{i*}-\tilde{\varphi}_k^n\right)(t,x)\leq-\zeta\quad\text{for}\;\;(t,x)\in[t_k,T]\times\partial B(x_k,\delta_n^k). \tag{4.77}$$

Since $\partial_t(\sqrt{T-t})\to-\infty$ as $t\to T$, we can choose t_k close enough to T to ensure that

$$-\left(\partial_t+\mathcal{L}\right)\tilde{\varphi}_k^n(t,x)\geq 0\quad\text{on}\quad B_n^k. \tag{4.78}$$

Next let us consider the following stopping times

$$\theta_n^k:=\inf\{s\geq t_k:(s,X_s^{t_k,x_k})\in B_n^{k^c}\}\wedge T \tag{4.79}$$

and

$$\vartheta_k^\varepsilon:=\inf\{s\geq t_k,u^i(s,X_s^{t_k,x_k})\leq\max_{j\in\mathcal{I}^{-i}}(u^j(s,X_s^{t_k,x_k})-g^{ij}(s,X_s^{t_k,x_k}))+\frac{\varepsilon}{4}\}\wedge T \tag{4.80}$$

where $B_n^{k^c}$ is the complement of B_n^k.

First note that for a subsequence $\{k\}$, $\mathbb{P}[\vartheta_k^\varepsilon>t_k]=1$. Actually from (4.74), we have

$$\begin{aligned}u^{i*}(T,x_0)&\geq\max_{j\in\mathcal{I}^{-i}}\left(u^{j*}-g^{ij}\right)^*(T,x_0)+2\varepsilon\\&\geq\max_{j\in\mathcal{I}^{-i}}\left(u^{j}-g^{ij}\right)^*(T,x_0)+2\varepsilon.\end{aligned}$$

Therefore taking into account of (4.75), at least for a subsequence, for any $k\geq 1$,

$$u^{i*}(t_k,x_k)\geq\max_{j\in\mathcal{I}^{-i}}\left(u^j-g^{ij}\right)(t_k,x_k)+\varepsilon.$$

Now let us stick to this subsequence. If $\mathbb{P}[\vartheta_k^\varepsilon=t_k]>0$, then by the càdlàg property of the processes which define ϑ_k^ε we have $u^i(t_k,x_k)\leq\max_{j\in\mathcal{I}^{-i}}(u^j(t_k,x_k)-g^{ij}(t_k,x_k))+\frac{\varepsilon}{4}$, which contradicts the previous inequality and then the claim is valid.

On the other hand the property which characterizes the jumps of Y^i in the definition (4.21), implies that on $[t_k, \vartheta_k^\varepsilon]$ the process Y^i is continuous and $dK_s^i = 0$ for $s \in [t_k, \vartheta_k^\varepsilon]$. Applying now Itô's formula to the process $(\tilde{\varphi}_k^n(s, X_s))_{s\in[t_k,\theta_n^k\wedge\vartheta_k^\varepsilon]}$ and taking expectation, we obtain

$$
\begin{aligned}
\tilde{\varphi}_k^n(t_k,x_k) &= \mathbb{E}\Big[\tilde{\varphi}_k^n(\theta_n^k\wedge\vartheta_k^\varepsilon, X^{t_k,x_k}_{\theta_n^k\wedge\vartheta_k^\varepsilon}) - \int_{t_k}^{\theta_n^k\wedge\vartheta_k^\varepsilon}(\partial_t+\mathcal{L})\,\tilde{\varphi}_k^n(s,X_s^{t_k,x_k})ds\Big]\\
&\geq \mathbb{E}\big[\tilde{\varphi}_k^n(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})\mathbf{1}_{[\theta_n^k\leq\vartheta_k^\varepsilon]} + \tilde{\varphi}_k^n(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})\mathbf{1}_{[\theta_n^k>\vartheta_k^\varepsilon]}\big] \qquad \text{by (4.78)}\\
&= \mathbb{E}\big[\big\{\tilde{\varphi}_k^n(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})\mathbf{1}_{[\theta_n^k<T]} + \tilde{\varphi}_k^n(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})\mathbf{1}_{[\theta_n^k=T]}\big\}\mathbf{1}_{[\theta_n^k\leq\vartheta_k^\varepsilon]} + \tilde{\varphi}_k^n(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})\mathbf{1}_{[\theta_n^k>\vartheta_k^\varepsilon]}\big]\\
&\geq \mathbb{E}\Big[\big\{\big(u^{i*}(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})+\zeta\big)\mathbf{1}_{[\theta_n^k<T]} + \big(h^i(T,X_T^{t_k,x_k})+\varepsilon\big)\mathbf{1}_{[\theta_n^k=T]}\big\}\mathbf{1}_{[\theta_n^k\leq\vartheta_k^\varepsilon]}\\
&\qquad + \Big\{\max_{j\in\mathcal{I}^{-i}}\big(u^{j*}-g^{ij}\big)^*(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})+\varepsilon\Big\}\mathbf{1}_{[\theta_n^k>\vartheta_k^\varepsilon]}\Big] \qquad \text{by (4.77) and (4.76)}\\
&\geq \mathbb{E}\Big[\big\{\big(u^{i}(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})+\zeta\big)\mathbf{1}_{[\theta_n^k<T]} + \big(h^i(T,X_T^{t_k,x_k})+\varepsilon\big)\mathbf{1}_{[\theta_n^k=T]}\big\}\mathbf{1}_{[\theta_n^k\leq\vartheta_k^\varepsilon]}\\
&\qquad + \Big\{\max_{j\in\mathcal{I}^{-i}}\big(u^{j}(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})-g^{ij}(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})\big)+\varepsilon\Big\}\mathbf{1}_{[\theta_n^k>\vartheta_k^\varepsilon]}\Big]\\
&\geq \mathbb{E}\Big[\big\{\big(u^{i}(\theta_n^k, X^{t_k,x_k}_{\theta_n^k})+\zeta\big)\mathbf{1}_{[\theta_n^k<T]} + \big(h^i(T,X_T^{t_k,x_k})+\varepsilon\big)\mathbf{1}_{[\theta_n^k=T]}\big\}\mathbf{1}_{[\theta_n^k\leq\vartheta_k^\varepsilon]}\\
&\qquad + \big\{u^i(\vartheta_k^\varepsilon, X^{t_k,x_k}_{\vartheta_k^\varepsilon})+\tfrac{3\varepsilon}{4}\big\}\mathbf{1}_{[\theta_n^k>\vartheta_k^\varepsilon]}\Big] \qquad \text{by (4.80)}\\
&\geq \mathbb{E}\big[u^i(\theta_n^k\wedge\vartheta_k^\varepsilon, X^{t_k,x_k}_{\theta_n^k\wedge\vartheta_k^\varepsilon})\big] + \big(\zeta\wedge\tfrac{3\varepsilon}{4}\big)\\
&= \mathbb{E}\big[u^i(t_k,x_k)\big] - \mathbb{E}\Big[\int_{t_k}^{\theta_n^k\wedge\vartheta_k^\varepsilon} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big] + \Big(\zeta\wedge\frac{3\varepsilon}{4}\Big)
\end{aligned} \tag{4.81}
$$

where the last equality is due to the fact that the process $Y^i_\cdot = u^i(\cdot, X_\cdot)$, stopped at time $\theta_n^k\wedge\vartheta_k$, solves a RBSDE system of the type (4.21) with data given by $((f^i)_{i\in\mathcal{I}}, (h^i)_{i\in\mathcal{I}}, (g^{ij})_{i\in\mathcal{I}})$, and the last inequality is obtained by monotonicity property of f^i and since $u^{j*}\geq u^j$ for $j\in\mathcal{I}^{-i}$. Besides, note that by definition of $\theta_n^k\wedge\vartheta_k$ we have $dK^{i,t,x}=0$ on $[t_k,\vartheta_k^\varepsilon]$. Next, we have that both $(u^j)_{j=1,\dots,m}$ and $(t,x)\to\left\|Z^{i,t,x}_\cdot\right\|_{\mathcal{H}^{2,d}}(t,x)$ are of polynomial growth. Thus by Assumption (H2)-(i),(iii) and inequality (4.45) we deduce that

$$
\lim_{k\to\infty}\mathbb{E}\Big[\int_{t_k}^{\theta_n^k\wedge\vartheta_k^\varepsilon} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big] = 0, \tag{4.82}
$$

and hence taking the limit in both hand sides of the inequality (4.81) as $k\to\infty$ yields

$$
\begin{aligned}
\varphi^n(T,x_0) &= \lim_{k\to\infty}\big[\varphi^n(t_k,x_k)+\sqrt{T-t_k}\big] = \lim_{k\to\infty}\tilde{\varphi}_k^n(t_k,x_k)\\
&\geq \lim_{k\to\infty} u^i(t_k,x_k) + \big(\zeta\wedge\tfrac{3\varepsilon}{4}\big) = u^{i*}(T,x_0) + \big(\zeta\wedge\tfrac{3\varepsilon}{4}\big).
\end{aligned} \tag{4.83}
$$

Therefore, taking n large enough and recalling that $\varphi^n\to u^{i*}$ pointwisely, we get a contradiction. Thus for any $x\in\mathbb{R}^r$ and $i\in\mathcal{I}$ we have

$$
\min\Big\{u^{i*}(T,x)-h^i(x);\;\; u^{i*}(T,x)-\max_{j\in\mathcal{I}^{-i}}\big(u^{j*}-g^{ij}\big)^*(T,x)\Big\}\leq 0. \tag{4.84}
$$

which is the claim.

Step 4: Supersolution property at (T, x_0).

We are going to show that

$$\min\Big\{u_*^i(T,x_0)-h^i(x_0);\ \ u_*^i(T,x_0)-\big(\max_{j\in\mathcal{I}^{-i}}\big(u_*^j(T,x_0)-g^{ij}(T,x_0)\big)\big)_*\Big\}\geq 0. \tag{4.85}$$

Let $(t_k, x_k)_{k\geq 1}$ be a sequence in $[0,T)\times\mathbb{R}^r$ such that

$$(t_k,x_k)\to(T,x_0)\ \text{ and }\ u^i(t_k,x_k)\to u_*^i(T,x_0)\quad\text{as}\quad k\to\infty. \tag{4.86}$$

Since $u^i(t,x)$ is deterministic, we have from the definition of u^i that

$$\begin{aligned} u^i(t_k,x_k)&=\mathbb{E}\Big[h^i(X_T^{t_k,x_k})+\int_{t_k}^T f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds+K_T^i-K_{t_k}^i\Big]\\ &\geq\mathbb{E}\Big[h^i(X_T^{t_k,x_k})+\int_{t_k}^T f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\end{aligned} \tag{4.87}$$

where we have used that $dK^{i,t,x}\geq 0$ on $[t_k,T]$. Next taking the limit in both hand sides as $k\to\infty$, using that h^i is continuous and arguing similarly to (4.82) we have

$$\begin{aligned} u_*^i(T,x_0)&\geq\lim_{k\to\infty}\mathbb{E}\Big[h^i(X_T^{t_k,x_k})+\int_{t_k}^T f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\\ &=\mathbb{E}\big[h^i(X_T^{T,x_0})\big]=h^i(x_0),\end{aligned}$$

that is, $u_*^i(T,x_0)\geq h^i(x_0)$. On the other hand, setting $\tau_k=(T+t_k)/2$, considering the RBSDE (4.21) on $[t_k,\tau_k]$, taking expectation to obtain

$$\begin{aligned} u^i(t_k,x_k)&\geq\mathbb{E}\Big[u^i(\tau_k,X_{\tau_k}^{t_k,x_k})+\int_{t_k}^{\tau_k} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\\ &\geq\mathbb{E}\Big[\max_{j\in\mathcal{I}^{-i}}\big(u^j(\tau_k,X_{\tau_k}^{t_k,x_k})-g^{ij}(\tau_k,X_{\tau_k}^{t_k,x_k})\big)+\int_{t_k}^{\tau_k} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\\ &\geq\mathbb{E}\Big[\big(\max_{j\in\mathcal{I}^{-i}}\big(u_*^j-g^{ij}\big)\big)_*(\tau_k,X_{\tau_k}^{t_k,x_k})+\int_{t_k}^{\tau_k} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\end{aligned} \tag{4.88}$$

since $dK^{i,t,x}\geq 0$ and $u^i(\tau_k,X_{\tau_k}^{t_k,x_k})\geq\max_{j\in\mathcal{I}^{-i}}\big(u^j(\tau_k,X_{\tau_k}^{t_k,x_k})-g^{ij}(\tau_k,X_{\tau_k}^{t_k,x_k})\big)$. It implies that

$$\begin{aligned}\lim_{k\to\infty}u^i(t_k,x_k)&\geq\liminf_{k\to\infty}\mathbb{E}\Big[\big(\max_{j\in\mathcal{I}^{-i}}\big(u_*^j-g^{ij}\big)\big)_*(\tau_k,X_{\tau_k}^{t_k,x_k})\\ &\qquad+\int_{t_k}^{\tau_k} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\\ &\geq\mathbb{E}\Big[\liminf_k\big(\max_{j\in\mathcal{I}^{-i}}\big(u_*^j-g^{ij}\big)\big)_*(\tau_k,X_{\tau_k}^{t_k,x_k})\\ &\qquad+\int_{t_k}^{\tau_k} f^i(s,X_s^{t_k,x_k},(u^k)_{k=1,\dots,q}(s,X_s^{t_k,x_k}),Z_s^{i,t_k,x_k})ds\Big]\\ &\geq\big(\max_{j\in\mathcal{I}^{-i}}\big(u_*^j-g^{ij}\big)\big)_*(t_k,x_k).\end{aligned}$$

The second inequality stems from Fatou's Lemma while the third one is due to the fact that $\big(\max_{j\in\mathcal{I}^{-i}}\big(u^j_* - g^{ij}\big)\big)_*$ is lower semicontinuous and by (4.86), at least for a subsequence, $((\tau_k, X^{t_k,x_k}_{\tau_k}))_k \to (T,x)$ $\mathbb{P}-a.s.$. Thus

$$\min\Big\{u^i_*(T,x_0) - h^i(x_0);\;\; u^i_*(T,x_0) - \big(\max_{j\in\mathcal{I}^{-i}}\big(u^j_*(T,x_0) - g^{ij}(T,x_0)\big)\big)_*\Big\} \geq 0 \tag{4.89}$$

which is the claim. The proof is now complete. $\square$

Remark 4.5. If the switching costs g^{ij} are continuous, conditions (4.51) and (4.50), read respectively as:

$$\min\big\{v^{i*}(T,x_0) - h^i(x_0);\;\; u^{i*}(T,x_0) - \max_{j\in\mathcal{I}^{-i}}\big(v^{j*} - g^{ij}\big)(T,x_0)\big\} \leq 0$$

and

$$\min\Big\{v^i_*(T,x_0) - h^i(x_0);\;\; v^i_*(T,x_0) - \big(\max_{j\in\mathcal{I}^{-i}}\big(v^j_* - g^{ij}\big)\big)(T,x_0)\Big\} \geq 0.$$

Therefore $v^i_*(T,x_0) \geq h^i(x_0)$ and by the non free-loop property one deduces that $v^{i*}(T,x_0) \leq h^i(x_0)$ which implies that $v^i(T,x_0) = h^i(x_0)$. For more details one can see e.g. Hamadène and Morlais [33]. $\square$

5

Switching Games

This chapter is about zero-sum switching games whose switching costs associated to each player satisfy a càdlàg property. We can highlight as the main material the characterization of the value of the game viewed as a solution of a system of RBSDEs with double càdlàg barriers (verification theorem). We also deduce the existence of ε-optimal saddle points that turn out a special case of the well-known Nash equilibria in the context of zero-sum games.

We warn the reader that the attributes "control" and "strategy" used in this chapter have a special meaning and wont represent the usual meaning that in many works they represent the same thing.

5.1 The game model and main assumptions

Throughout this chapter we shall assume the existence of two players. Player k ($k \in \{1,2\}$) chooses his/her actions over the set $\Gamma^k = \{1, \ldots, m_k\}$, $k \in \{1,2\}$, same that represent his/her operating modes. Gathering these two sets, we let $\Gamma = \Gamma^1 \times \Gamma^2$ be the product space of operating modes of both players.

Let us define now the type *controls* for each player.

Definition 5.1. (Individual switching controls). A control for Player 1 is a sequence $\alpha = (\sigma_n, \xi_n)_{n\geq 0}$ such that,

1. For all $n \geq 0$, $\sigma_n \in \mathcal{T}_0$ and is such that $\sigma_n \leq \sigma_{n+1}$, $\mathbb{P}$-a.s., and $\mathbb{P}(\{\sigma_n < T, \forall n \geq 0\}) = 0$.
2. For all $n \geq 0$, ξ_n is an $\mathcal{F}_{\sigma_n}$-measurable Γ^1-valued random variable.
3. For $n \geq 1$, on $\{\sigma_n < T\}$ we have $\sigma_n < \sigma_{n+1}$ and $\xi_n \neq \xi_{n-1}$, while on $\{\sigma_n = T\}$ we have $\xi_n = \xi_{n-1}$.

We will denote by A the set of controls for Player 1. The set B of controls $\beta = (\tau_n, \zeta_n)_{n\geq 0}$ for Player 2, where the ζ_n are Γ^2-valued, is defined analogously as for the case of Player 1. For each $i \in \Gamma^1$ (resp. $j \in \Gamma^2$) we also denote by A_t^i (resp. B_t^j), the set of controls with the property of $\sigma_0 = t$, $\xi_0 = i$ (resp. $\tau_0 = t$ and $\zeta_n = j$).

Given two controls $\alpha \in \mathsf{A}$ and $\beta \in \mathsf{B}$, let us define the *coupled control* of α and β under the following assumption: if both players decide to switch at the same instant, then Player 1's switch is implemented first.

Definition 5.2 (Coupling of controls)**.** Given controls $\alpha \in \mathsf{A}$ and $\beta \in \mathsf{B}$, define the coupling control $\gamma(\alpha, \beta) = (\rho_n, \gamma_n)_{n\geq 0}$ where $\rho_n \in \mathcal{T}_0$ is defined by,

$$\rho_n = \sigma_{r_n} \wedge \tau_{s_n},$$

with $r_0 = s_0 = 0$, $r_1 = s_1 = 1$ and for $n \geq 2$,

$$r_n = r_{n-1} + 1_{\{\sigma_{r_{n-1}} \leq \tau_{s_{n-1}}\}}, \qquad s_n = s_{n-1} + 1_{\{\tau_{s_{n-1}} < \sigma_{r_{n-1}}\}},$$

and γ_n is a Γ-valued random variable such that $\gamma_0 = (\xi_0, \zeta_0)$ and for $n \geq 1$,

$$\gamma_n = \begin{cases} (\xi_{r_n}, \gamma^{(2)}_{n-1}), & on \quad \{\sigma_{r_n} \le \tau_{s_n}, \sigma_{r_n} < T\} \\ (\gamma^{(1)}_{n-1}, \zeta_{s_n}), & on \quad \{\tau_{s_n} < \sigma_{r_n}\} \\ \gamma_{n-1}, & on \quad \{\tau_{s_n} = \sigma_{r_n} = T\}. \end{cases}$$

Define the indicator process $(u_t)_{0\le t\le T}$ of the system at time t, for all $t \in [0,T]$, by

$$u_t = \gamma_0 \mathbf{1}_{[\rho_0,\rho_1]}(t) + \sum_{n\ge 1} \gamma_n \mathbf{1}_{(\rho_n,\rho_{n+1}]}(t). \tag{5.1}$$

Note that the coupling $\gamma = (\alpha, \beta) = (\rho_n, \gamma_n)_{n\ge 0}$ of the controls $\alpha \in \mathsf{A}^i_s$ and $\beta \in \mathsf{B}^j_s$ has the following properties:

- $\rho_0 = s$ and for all $n \ge 0$ we have $\rho_n \in \mathcal{T}_s$ and $\rho_n \le \rho_{n+1}$ $\mathbb{P}$-a.s., and $\mathbb{P}(\{\rho_n < T, \forall n \ge 0\}) = 0$;
- $\gamma_0 = (i, j)$ and for all $n \ge 0$ the random variable γ_n is $\mathcal{F}_{\rho_n}$-measurable, Γ-valued and $\gamma_{n+1} \ne \gamma_n$ on $\{\gamma_{n+1} < T\}$.

Next assume the following: For each $(i, j) \in \Gamma$, $i_1, i_2 \in \Gamma^1$, and $j_1, j_2 \in \Gamma^2$, there exist processes satisfying $f^{i,j} \in \mathcal{H}^2$ and $h^{i,j} \in L^2(\mathcal{F}_T)$ and $\hat{g}^{i_1,i_2}, \check{g}^{j_1,j_2} \in \mathcal{S}^2$, where:

- $f^{i,j}$ stands for the running reward paid by player 2 to player 1 and $h^{i,j}$ the terminal reward paid by player 2 to player 1, when player 1's (resp. player 2's) active mode is i (resp. j).
- $\hat{g}^{i_1,i_2}$ stands for the non-negative payment from player 1 to player 2 when the former switches from mode i_1 to mode i_2.
- $\check{g}^{j_1,j_2}$ stands for the non-negative payment from player 2 to player 1 when the former switches from mode j_1 to mode j_2.
- For all $(i, j) \in \Gamma$ and $t \in [0, T]$ we set $\hat{g}^{i,i}_t = \check{g}^{j,j}_t = 0$.

With the above ingredients, we then introduce the players' payoff, with initial condition $(s, i, j) \in [0, T] \times \Gamma$ by

$$J^{i,j}_s(\gamma(\alpha,\beta)) = \mathbb{E}\Big[\int_s^T f^{u_t}_t\, dt - \sum_{n=1}^{\infty}\Big[\hat{g}^{\gamma^{(1)}_{n-1},\gamma^{(1)}_n}_{\rho_n} - \check{g}^{\gamma^{(2)}_{n-1},\gamma^{(2)}_n}_{\rho_n}\Big] + h^{u_T}\Big|\mathcal{F}_s\Big] \quad \alpha \in \mathsf{A}^i_s,\ \beta \in \mathsf{B}^j_s. \tag{5.2}$$

Let us now define the admissibility of a control in terms of its interaction with the switching costs. Given $\alpha \in \mathsf{A}$ (resp. $\beta \in \mathsf{B}$), we denote by C^α_N (resp. C^β_N) the cost of the first $N \ge 1$ switchings,

$$C^\alpha_N := \sum_{n=1}^N \hat{g}^{\xi_{n-1},\xi_n}_{\sigma_n}, \quad (\text{resp. } C^\beta_N := \sum_{n=1}^N \check{g}^{\zeta_{n-1},\zeta_n}_{\tau_n}).$$

Note that the limit $\lim_{N\to\infty} C^\alpha_N$ (resp. $\lim_{N\to\infty} C^\beta_N$) is well defined.

Definition 5.3. A control $\alpha \in \mathsf{A}$ for Player 1 is said to be square-integrable if,

$$\mathbb{E}\Big[|\lim_{N\to\infty} C^\alpha_N|^2\Big] < \infty.$$

Let $\mathcal{A}$ denote the set of such controls. Similarly, the set $\mathcal{B}$ of square-integrable controls for Player 2 consists of those $\beta \in \mathsf{B}$ satisfying,

$$\mathbb{E}\Big[|\lim_{N\to\infty} C^\beta_N|^2\Big] < \infty.$$

On the other hand, we denote by $C^{\gamma(\alpha,\beta)}_N$ the joint cumulative cost of the first N switches,

$$C^{\gamma(\alpha,\beta)}_N = \sum_{n=1}^N \Big[\hat{g}^{\gamma^{(1)}_{n-1},\gamma^{(1)}_n}_{\rho_n} - \check{g}^{\gamma^{(2)}_{n-1},\gamma^{(2)}_n}_{\rho_n}\Big], \qquad N \ge 1.$$

Definition 5.4. The coupling $\gamma(\alpha,\beta) = (\rho_n, \gamma_n)_{n\geq 0}$ of controls $\alpha \in \mathsf{A}_s^i$ and $\beta \in \mathsf{B}_s^j$ is said to be *admissible*, if

$$\mathbb{E}\Big[\sup_{N\geq 1} |C_N^{\gamma(\alpha,\beta)}|^2\Big] < \infty.$$

We denote by $\mathcal{G}_s^{i,j}$ the set of all these controls.

Note that for every $\alpha \in \mathsf{A}$ and $\beta \in \mathsf{B}$ have $\lim_{N\to\infty} C_N^{\gamma(\alpha,\beta)} = \lim_{N\to\infty} C_N^{\alpha} - \lim_{N\to\infty} C_N^{\beta}$. Using the triangle inequality, we see that every pair of square-integrable controls (α,β), $\alpha \in \mathcal{A}_s^i$ and $\beta \in \mathcal{B}_s^j$, satisfies $\gamma(\alpha,\beta) \in \mathcal{G}_s^{i,j}$.

Similar to the definition of controls, we next introduce the notion of *strategies* which is crucial in this chapter.

Definition 5.5. Fix $s \in [0,T]$ and $\tau \in \mathcal{T}_s$.

(a) Two controls $\alpha^1, \alpha^2 \in \mathsf{A}$ with $\alpha^1 = (\sigma_n^1, \xi_n^1)_{n\geq 0}$ and $\alpha^2 = (\sigma_n^2, \xi_n^2)_{n\geq 0}$ are said to be equivalent on $[s,\tau]$, and we denote it as $\alpha^1 \equiv \alpha^2$, if we have $\mathbb{P}$–a.s.,

$$\xi_0^1 \mathbf{1}_{[\sigma_0^1,\sigma_1^1]}(t) + \sum_{n=1}^{\infty} \xi_n^1 \mathbf{1}_{(\sigma_n^1,\sigma_{n+1}^1]}(t) = \xi_0^2 \mathbf{1}_{[\sigma_0^2,\sigma_1^2]}(t) + \sum_{n=1}^{\infty} \xi_n^2 \mathbf{1}_{(\sigma_n^2,\sigma_{n+1}^2]}(t), \quad s \leq t \leq \tau.$$

(b) A non-anticipative strategy for Player 1 is a mapping $\overline{\alpha} : \mathsf{B} \to \mathsf{A}$ such that:

(b.1) Non-anticipativity property: for any $s \in [0,T]$, $\tau \in \mathcal{T}_s$, and $\beta^1, \beta^2 \in \mathsf{B}$ such that $\beta^1 \equiv \beta^2$ on $[s,\tau]$, we have $\overline{\alpha}(\beta^1) \equiv \overline{\alpha}(\beta^2)$ on $[s,\tau]$.

(b.2) Square-integrability: for any $\beta \in \mathcal{B}$ we have $\overline{\alpha}(\beta) \in \mathcal{A}$.

Let $\mathscr{A}$ and $\mathscr{B}$ denote the set of non-anticipative strategies for players 1 and 2 respectively.

Roughly speaking, non-anticipative strategies mean that Player 2 cannot respond differently to equivalent controls of Player 1. In a similar manner we define non-anticipative strategies for Player 2.

We conclude this section by setting the value functions of the zero-sum switching game under study.

Definition 5.6. The lower and upper values for the switching game related to the payoff (5.2), denoted respectively by $\check{V}_s^{i,j}$ and $\hat{V}_s^{i,j}$, are defined as follows:

$$\begin{cases} \check{V}_s^{i,j} := \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_s^i} \operatorname*{ess\,inf}_{\beta\in\mathcal{B}_s^j} J_s^{i,j}(\gamma(\alpha,\beta)) \\ \hat{V}_s^{i,j} := \operatorname*{ess\,inf}_{\beta\in\mathcal{B}_s^j} \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_s^i} J_s^{i,j}(\gamma(\alpha,\beta)). \end{cases}$$

The game is said to have a value at (s,i,j), denoted by $V_s^{i,j}$, if

$$V_s^{i,j} = \check{V}_s^{i,j} = \hat{V}_s^{i,j} \qquad \text{a.s.} \tag{5.3}$$

Note from the above definition that $\check{V}_s^{i,j} \leq \hat{V}_s^{i,j}$ a.s. The common value $V_s^{i,j}$, when it exists, is referred to as the *value of the game* (a.k.a. *game's solution*) at (s,i,j). Finally, when $s = T$ we formally set $\check{V}_T^{i,j} = \hat{V}_T^{i,j} = h^{i,j}$.

5.2 Verification theorem

In this section we shall introduce the main tool for showing the existence of the value (5.3) of the zero-sum optimal switching game. Such existence will in turn be ensured by assuming the existence of solutions related to the next system of double RBSDEs, introduced in the next definition:

Definition 5.7. The processes $(Y^{i,j}, Z^{i,j}, K^{i,j,+}, K^{i,j,-})_{(i,j)\in\Gamma}$ are solutions of the system of DRBSDEs with terminal value $h^{i,j} \in L^2(\mathcal{F}_T)$, coefficient $f^{i,j} \in \mathcal{H}^2$ and switching costs $\hat{g}^{i,i_1}$ and $\check{g}^{j,j_1}$ in $\mathcal{S}^2$ if, for any $(i,j) \in \Gamma$ and all $0 \le s \le T$,

$$\begin{aligned}
&(i) \quad (Y^{i,j}, Z^{i,j}, K^{i,j,+}, K^{i,j,-}) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{S}^2 \times \mathcal{S}^2;\\
&(ii) \quad Y_s^{i,j} = h^{i,j} + \int_s^T f_t^{i,j} dt + K_T^{i,j,+} - K_s^{i,j,+} - (K_T^{i,j,-} - K_s^{i,j,-}) - \int_s^T Z_t^{i,j} dB_t;\\
&(iii) \quad \max_{i_1 \in (\Gamma^1)^{-i}} \{Y_s^{i_1,j} - \hat{g}_s^{i,i_1}\} \le Y_s^{i,j} \quad \text{and} \quad Y_s^{i,j} \le \min_{j_1 \in (\Gamma^2)^{-j}} \{Y_s^{i,j_1} + \check{g}_s^{j,j_1}\};\\
&(iv) \quad \int_s^T \Big(Y_{t-}^{i,j} - \min_{j_1 \in (\Gamma^2)^{-j}} \{Y_{t-}^{i,j_1} + \check{g}_{t-}^{j,j_1}\}\Big) dK_t^{i,j,-} = \int_s^T \Big(\max_{i_1 \in (\Gamma^1)^{-i}} \{Y_{t-}^{i_1,j} - \hat{g}_{t-}^{i,i_1}\} - Y_{t-}^{i,j}\Big) dK_t^{i,j,+} = 0,
\end{aligned} \tag{5.4}$$

where $K^{i,j,+}$ and $K^{i,j,-}$ are the nondecreasing processes with $K_0^{i,j,+} = K_0^{i,j,-} = 0$. Note that in (iv) we are applying left limit at t on the integrands.

We now give a definition and some assumptions on the switching costs $\hat{g}$ and $\check{g}$.

Definition 5.8. For $N \ge 2$ a loop in Γ of length $N-1$ is a sequence $\{(i_1, j_1), \ldots, (i_N, j_N)\}$ of elements in Γ with $N-1$ distinct members such that $(i_N, j_N) = (i_1, j_1)$ and either $i_{q+1} = i_q$ or $j_{q+1} = j_q$ for all $q = 1, \ldots, N-1$.

We are going to consider the following standard assumptions on the switching and terminal costs.

Assumption G.

(G1) : Non-negativity: $\min_{i_1 \in \Gamma^1} \hat{g}^{i,i_1} \ge 0$ and $\min_{j_1 \in \Gamma^2} \check{g}^{j,j_1} \ge 0$ for all $i \in \Gamma^1$, $j \in \Gamma^2$.

(G2) : Consistency:

(i) For all sequences $\{i_1, i_2, i_3\} \in \Gamma^1$ and $\{j_1, j_2, j_3\} \in \Gamma^2$ with $i_1 \ne i_2$, $i_2 \ne i_3$ and $j_1 \ne j_2$, $j_2 \ne j_3$, we have for all $t \in [0,T]$,

$$\hat{g}_t^{i_1,i_3} < \hat{g}_t^{i_1,i_2} + \hat{g}_t^{i_2,i_3} \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad \check{g}_t^{j_1,j_3} < \check{g}_t^{j_1,j_2} + \check{g}_t^{j_2,j_3} \quad \mathbb{P}\text{-a.s.} \tag{5.5}$$

(ii) For all $(i,j) \in \Gamma$ we have,

$$\max_{i_1 \in (\Gamma^1)^{-i}} \{h^{i_1,j} - \hat{g}_T^{i,i_1}\} \le h^{i,j} \le \min_{j_1 \in (\Gamma^2)^{-j}} \{h^{i,j_1} + \check{g}_T^{j,j_1}\} \quad \mathbb{P}\text{-a.s.} \tag{5.6}$$

(G3) : Non-free loop property: For any loop $\{(i_1, j_1), \ldots, (i_N, j_N)\}$ in Γ we have for all $t \in [0,T]$,

$$\sum_{q=1}^{N-1} \varphi_t^{q,q+1} \ne 0 \quad \mathbb{P}\text{-a.s.} \tag{5.7}$$

where $\varphi_t^{q,q+1} = -\hat{g}_t^{i_q,i_{q+1}} 1_{\{i_q \ne i_{q+1}\}} + \check{g}_t^{j_q,j_{q+1}} 1_{\{j_q \ne j_{q+1}\}}$.

The main goal in this section is to prove the following theorem when the switching costs are càdlàg.

Theorem 5.1. *Let the Assumption G holds true. Suppose that there exists a solution $(Y^{i,j}, Z^{i,j}, K^{i,j})_{(i,j)\in\Gamma}$ to the system of DRBSDEs (5.4). For every initial state $(s,i,j) \in [0,T] \times \Gamma$,*

(i) Existence of ε-optimal controls: there exists a pair of ε-controls $(\alpha^\varepsilon, \beta^\varepsilon) \in \mathsf{A}_s^i \times \mathsf{B}_s^j$ such that the coupling control associated $\gamma(\alpha^\varepsilon, \beta^\varepsilon)$ belongs to $\mathcal{G}_s^{i,j}$ and

$$J_s^{i,j}(\gamma(\alpha, \beta^\varepsilon)) - \varepsilon \le Y_s^{i,j} \le J_s^{i,j}(\gamma(\alpha^\varepsilon, \beta)) + \varepsilon \quad \textit{for all } \alpha \in \mathcal{A}_s^i \textit{ and } \beta \in \mathcal{B}_s^j$$

(ii) Existence of value: the switching game has a value with,

$$Y_s^{i,j} = V_s^{i,j} \qquad a.s. \tag{5.8}$$

Before we give the proof of this theorem, we provide a proposition which links a solution of a DRBSDE and Dynkin's games. Next, we prove two lemmas which show, on the one hand, that a coupling control related to two ε-optimal switching controls is admissible, and on the other hand, that the strategies associated to ε- optimal controls are admissible too. Finally, following the idea in [32], we will be in conditions to prove Theorem 5.1.

For some generic index $m \in \mathbb{N}$, let us define the lower and upper switching operators $L^{i,j} : \mathcal{S}^{2,m} \to \mathcal{S}^2$ and $U^{i,j} : \mathcal{S}^{2,m} \to \mathcal{S}^2$ as follows: For $\mathbf{Y} \in \mathcal{S}^{2,m}$,

$$\begin{cases} L^{i,j}(\mathbf{Y}) = \max\limits_{i_1 \in (\Gamma^1)^{-i}} \{Y^{i_1,j} - \hat{g}^{\,i,i_1}\}, \\ U^{i,j}(\mathbf{Y}) = \min\limits_{j_1 \in (\Gamma^2)^{-j}} \{Y^{\,i,j_1} + \check{g}^{j,j_1}\}. \end{cases}$$

Proposition 5.2. *Suppose there exists a solution $(Y^{i,j}, Z^{i,j}, K^{i,j})_{(i,j)\in\Gamma}$ to the DRBSDE (5.4). Then for all initial state $(s,i,j) \in [0,T] \times \Gamma$ a.s., we have*

(a) The part $Y^{i,j}$ of the solution can be represented as follows:

$$Y_s^{i,j} = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_s} \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_s} \mathcal{J}_s^{i,j}(\sigma,\tau) = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_s} \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_s} \mathcal{J}_s^{i,j}(\sigma,\tau),$$

where,

$$\mathcal{J}_s^{i,j}(\sigma,\tau) := \mathbb{E}\bigg[\int_s^{\sigma\wedge\tau} f_t^{i,j} dt + \mathbf{1}_{\{\tau<\sigma\}} U_\tau^{i,j}(\mathbf{Y}) + \mathbf{1}_{\{\sigma\le\tau,\sigma<T\}} L_\sigma^{i,j}(\mathbf{Y}) + h^{i,j}\mathbf{1}_{\{\sigma=\tau=T\}}\Big|\mathcal{F}_s\bigg],$$

and $h^{i,j}$, $f^{i,j}$, $L^{i,j}(\mathbf{Y})$ and $U^{i,j}(\mathbf{Y})$ are the data for the DRBSDE.

(b) Consider the ε-stopping times $\sigma_s^{i,j,\varepsilon} \in \mathcal{T}_s$ and $\tau_s^{i,j,\varepsilon} \in \mathcal{T}_s$ defined by

$$\begin{cases} \sigma_s^{i,j,\varepsilon} = \inf\{s \le t \le T : Y_t^{i,j} \le L_t^{i,j}(\mathbf{Y}) + \varepsilon\} \wedge T, \\ \\ \tau_s^{i,j,\varepsilon} = \inf\{s \le t \le T : Y_t^{i,j} \ge U_t^{i,j}(\mathbf{Y}) - \varepsilon\} \wedge T. \end{cases} \tag{5.9}$$

Then, we have that

$$\mathcal{J}_s^{i,j}(\sigma,\tau_s^{i,j,\varepsilon}) - \varepsilon \le \mathcal{J}_s^{i,j}(\sigma_s^{i,j,\varepsilon},\tau_s^{i,j,\varepsilon}) \le \mathcal{J}_s^{i,j}(\sigma_s^{i,j,\varepsilon},\tau) + \varepsilon \qquad \forall \sigma \in \mathcal{T}_s \ \text{ and } \ \tau \in \mathcal{T}_s. \tag{5.10}$$

That is, $(\sigma_s^{i,j,\varepsilon},\tau_s^{i,j,\varepsilon})$ is a ε-Nash equilibrium for a Dynkin game.

Proof. Analogously to [32], the result follows from Proposition 3.1 in [42]. □

Let us define now the next sequence $(\rho_n^\varepsilon, \gamma_n^\varepsilon)_{n\ge0}$ so-called ε-coupling switching controls

$$\rho_0^\varepsilon = s, \quad \gamma_0^\varepsilon = (i,j) \quad \text{and for} \quad n \ge 1, \tag{5.11}$$

$$\rho_n^\varepsilon = \sigma_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon}, \wedge \tau_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon}, \quad \gamma_n^\varepsilon = \begin{cases} \Big(\mathcal{L}_{\rho_n^\varepsilon}^{\gamma_{n-1}^\varepsilon}(\mathbf{Y}) \ , \ \gamma_{n-1}^{(2),\varepsilon}\Big), & on \quad \{\sigma_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon} \le \tau_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon}, \sigma_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon} < T\} \\ \\ \Big(\gamma_{n-1}^{(1),\varepsilon} \ , \ \mathcal{U}_{\rho_n^\varepsilon}^{\gamma_{n-1}^\varepsilon}(\mathbf{Y})\Big), & on \quad \{\tau_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon} < \sigma_{\rho_{n-1}^\varepsilon}^{\gamma_{n-1}^\varepsilon}\} \\ \\ \gamma_{n-1}^\varepsilon, & otherwise, \end{cases} \tag{5.12}$$

where $\sigma_{\rho^\varepsilon_{n-1}}^{\gamma^\varepsilon_{n-1}}$ and $\tau_{\rho^\varepsilon_{n-1}}^{\gamma^\varepsilon_{n-1}}$ are defined using (5.9) above, and the operators $\mathcal{L}_{\rho^\varepsilon_n}^{\gamma^\varepsilon_{n-1}}(\mathbf{Y})$ and $\mathcal{U}_{\rho^\varepsilon_n}^{\gamma^\varepsilon_{n-1}}(\mathbf{Y})$ are obtained from the switching selectors,

$$\begin{cases} \mathcal{L}_t^{i,j}(\mathbf{Y}) \in \underset{i_1\in(\Gamma^1)^{-i}}{\arg\max}\{Y_t^{i_1,j} - \hat{g}_t^{i,i_1}\} \\ \\ \mathcal{U}_t^{i,j}(\mathbf{Y}) \in \underset{j_1\in(\Gamma^2)^{-j}}{\arg\min}\{Y_t^{i,j_1} + \check{g}_t^{j,j_1}\}. \end{cases} \tag{5.13}$$

The following lemma provides an admissible ε-coupling switching control.

Lemma 5.3. *Let $(Y^{i,j}, Z^{i,j}, K^{i,j,+}, K^{i,j,-})_{(i,j)\in\Gamma}$ be a solution of the system (5.4). Under Assumption G, for any initial state $(s,i,j) \in [0,T]\times\Gamma$, we have that $\gamma(\alpha^\varepsilon, \beta^\varepsilon) \in \mathcal{G}_s^{i,j}$, where $\alpha^\varepsilon = (\sigma_n^\varepsilon, \xi_n^\varepsilon)_{n\geq 0}$ and $\beta^\varepsilon = (\tau_n^\varepsilon, \zeta_n^\varepsilon)_{n\geq 0}$ are defined as:*

$$\begin{cases} \sigma_0^\varepsilon := s, \ \ \textit{and for } n\geq 1,\ \sigma_n^\varepsilon := \inf\left\{ s \geq \sigma_{n-1}^\varepsilon : Y_s^{\xi^\varepsilon_{n-1},\gamma^{(2)}_{n-1}} \leq \underset{i_1\in(\Gamma^1)^{-\xi^\varepsilon_{n-1}}}{\max}\left(Y_s^{i_1,\gamma^{(2)}_{n-1}} - \hat{g}_s^{\xi^\varepsilon_{n-1},i_1}\right) + \frac{\varepsilon}{2^{n+1}}\right\} \wedge T \\ \tau_0^\varepsilon := s, \ \ \textit{and for } n\geq 1,\ \tau_n^\varepsilon := \inf\left\{ s \geq \tau_{n-1}^\varepsilon : Y_s^{\gamma^{(1)}_{n-1},\zeta^\varepsilon_{n-1}} \geq \underset{j_1\in(\Gamma^2)^{-\zeta^\varepsilon_{n-1}}}{\min}\left(Y_s^{\gamma^{(1)}_{n-1},j_1} + \check{g}_s^{\zeta^\varepsilon_{n-1},j_1}\right) - \frac{\varepsilon}{2^{n+1}}\right\} \wedge T \end{cases}$$

and

$$\begin{cases} \xi_0^\varepsilon := i, \quad \xi_n^\varepsilon = \underset{i_1\in(\Gamma^1)^{-\xi_{n-1}}}{\arg\max}\left\{Y_{\sigma_n^\varepsilon}^{i_1,\gamma^{(2)}_{n-1}} - \hat{g}_{\sigma_n^\varepsilon}^{\xi_{n-1},i_1}\right\}, \\ \zeta_0^\varepsilon := j, \quad \zeta_n^\varepsilon = \underset{j_1\in(\Gamma^2)^{-\zeta_{n-1}}}{\arg\min}\left\{Y_{\tau_n^\varepsilon}^{\gamma^{(1)}_{n-1},j_1} + \check{g}_{\tau_n^\varepsilon}^{\zeta_{n-1},j_1}\right\}. \end{cases}$$

Proof. For easy of notation, we shall omit the variable ε to the coupled control (ρ,γ) associated to the individual controls $(\sigma_n^\varepsilon, \xi_n^\varepsilon)$ and $(\tau_n^\varepsilon, \zeta_n^\varepsilon)$. This proof is divided in two steps:

Step 1. Let us show that $\alpha^\varepsilon \in \mathsf{A}_s^i$ and $\beta^\varepsilon \in \mathsf{B}_s^j$. Indeed, by Assumption G:

- Since the switching costs satisfy the non-free loop property (G3), then by proceeding as in Proposition 4.2, we have that $\mathbb{P}(\{\rho_n < T, \text{for all } n\geq 0\}) = 0$. Thus, $\mathbb{P}(\{\sigma_n^\varepsilon < T, \text{for all } n\geq 0\}) = 0$ since $\sigma_n^\varepsilon \geq \rho_n^\varepsilon$ for $n \geq 0$.
- The consistency property (G2)-(i) implies that $\sigma_n^\varepsilon < \sigma_{n+1}^\varepsilon$ on $\{\sigma_n^\varepsilon < T\}$ for $n\geq 1$, since it is not optimal to switch more than once.

On the other hand, since that the filtration $\mathbb{F}$ is right continuous, then ξ_n^ε is $\mathcal{F}_{\sigma_n^\varepsilon}$-measurable. Hence, $\alpha^\varepsilon \in \mathsf{A}_s^i$ (see Definition 5.1). Similarly, $\beta^\varepsilon \in \mathsf{B}_s^j$.

Step 2. Let us show that $\gamma(\alpha^\varepsilon,\beta^\varepsilon) \in \mathcal{G}_s^{i,j}$. Note that by construction of ρ_1 and since $(Y^{i,j}, Z^{i,j}, K^{i,j,+}, K^{i,j,-})$ is solution of (5.4) we have $\mathbb{P}$-a.s.,

$$\begin{aligned} Y_s^{i,j} &= Y_{\rho_1}^{i,j}\mathbf{1}_{\{\rho_1<T\}} + h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + \int_s^{\rho_1} f_t^{i,j}dt + \int_s^{\rho_1} dK_t^{i,j,+} - \int_s^{\rho_1} dK_t^{i,j,-} - \int_s^{\rho_1} Z_t^{i,j}dB_t \\ &= Y_{\rho_1}^{i,j}\mathbf{1}_{\{\rho_1<T\}} + h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + \int_s^{\rho_1} f_t^{i,j}dt - \int_s^{\rho_1} Z_t^{i,j}dB_t, \end{aligned} \tag{5.14}$$

where in the last equality we use that $K_t^{i,j,+} = K_t^{i,j,-} = 0$, for $t\in[s,\rho_1]$. On the other hand, on the event $\{\rho_1 = \sigma_1^\varepsilon\}$ we have $Y_{\rho_1}^{i,j} = Y_{\sigma_1^\varepsilon}^{i,j}$ and thus by definition of σ_1^ε we obtain $Y_{\rho_1}^{i,j} \leq Y_{\rho_1}^{\gamma_1} - \hat{g}_{\rho_1}^{\gamma_0^{(1)},\gamma_1^{(1)}} + \frac{\varepsilon}{4}$. Moreover, on the event $\{\rho_1 = \tau_s^\varepsilon\}$ we have trivially that $Y_{\rho_1}^{i,j} \leq Y_{\rho_1}^{\gamma_1} + \check{g}_{\rho_1}^{\gamma_0^{(2)},\gamma_1^{(2)}}$ (see Definition 5.7-(iii)). Therefore, by considering the first switch ρ_1 for any of the two player we have from (5.14) that

$$
\begin{aligned}
Y_s^{i,j} &\leq h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + \int_s^{\rho_1} f_t^{i,j}dt + (Y_{\sigma_s^\varepsilon}^{\gamma_1^{(1)},j} - \hat{g}_{\sigma_s^\varepsilon}^{i,\gamma_1^{(1)}})\mathbf{1}_{\{\sigma_s^\varepsilon<T\}}\mathbf{1}_{\{\sigma_s^\varepsilon\leq\tau_s^\varepsilon\}} + (Y_{\tau_s^\varepsilon}^{i,\gamma_1^{(2)}} + \check{g}_{\tau_s^\varepsilon}^{j,\gamma_1^{(2)}})\mathbf{1}_{\{\tau_s^\varepsilon<\sigma_s^\varepsilon\}} \\
&\quad - \int_s^{\rho_1} Z_t^{i,j}dB_t + \frac{\varepsilon}{4} \qquad (5.15)\\
&= h^{\gamma_0}\mathbf{1}_{\{\rho_1=T\}} + \int_s^{\rho_1} f_t^{u_t}dt + Y_{\rho_1}^{\gamma_1}\mathbf{1}_{\{\rho_1<T\}} - \left[\hat{g}_{\rho_1}^{\gamma_0^{(1)},\gamma_1^{(1)}} - \check{g}_{\rho_1}^{\gamma_0^{(2)},\gamma_1^{(2)}}\right] - \int_s^{\rho_1} Z_t^{u_t}dB_t + \frac{\varepsilon}{4}.
\end{aligned}
$$

Proceeding iteratively for $n = 1, \ldots, N$ we get

$$
\begin{aligned}
Y_s^{i,j} \leq \sum_{n=1}^{N} h^{\gamma_{n-1}}\mathbf{1}_{\{\rho_n=T,\rho_{n-1}<T\}} + \int_s^{\rho_N} f_t^{u_t}dt + Y_{\rho_N}^{\gamma_N}\mathbf{1}_{\{\rho_N<T\}} - \sum_{n=1}^{N}\left[\hat{g}_{\rho_n}^{\gamma_{n-1}^{(1)},\gamma_n^{(1)}} - \check{g}_{\rho_n}^{\gamma_{n-1}^{(2)},\gamma_n^{(2)}}\right] \\
- \int_s^{\rho_N} Z_t^{u_t}dB_t + \sum_{n=2}^{N+1}\frac{\varepsilon}{2^n}.
\end{aligned}
\tag{5.16}
$$

An analogous procedure, by changing the roles of σ's and τ's, yields

$$
\begin{aligned}
Y_s^{i,j} \geq \sum_{n=1}^{N} h^{\gamma_{n-1}}\mathbf{1}_{\{\rho_n=T,\rho_{n-1}<T\}} + \int_s^{\rho_N} f_t^{u_t}dt + Y_{\rho_N}^{\gamma_N}\mathbf{1}_{\{\rho_N<T\}} - \sum_{n=1}^{N}\left[\hat{g}_{\rho_n}^{\gamma_{n-1}^{(1)},\gamma_n^{(1)}} - \check{g}_{\rho_n}^{\gamma_{n-1}^{(2)},\gamma_n^{(2)}}\right] \\
- \int_s^{\rho_N} Z_t^{u_t}dB_t - \sum_{n=2}^{N+1}\frac{\varepsilon}{2^n}.
\end{aligned}
\tag{5.17}
$$

Therefore, by combining (5.16) and (5.17), and using the triangle inequality, we have

$$
\left|C_N^{\gamma(\alpha^\varepsilon,\beta^\varepsilon)}\right| \leq \left|Y_s^{i,j} + \sum_{n=1}^{N} h^{\gamma_{n-1}}\mathbf{1}_{\{\rho_n=T,\rho_{n-1}<T\}} + \int_s^{\rho_N} f_t^{u_t}dt + Y_{\rho_N}^{\gamma_N}\mathbf{1}_{\{\rho_N<T\}} - \int_s^{\rho_N} Z_t^{u_t}dB_t\right| + \sum_{n=2}^{N+1}\frac{\varepsilon}{2^n}
\tag{5.18}
$$

and thus

$$
\sup_{N\geq 1}\left|C_N^{\gamma(\alpha^\varepsilon,\beta^\varepsilon)}\right| \leq \left|Y_s^{i,j}\right| + \max_{(i,j)\in\Gamma}\left|h^{i,j}\right| + \int_s^T \left|f_t^{u_t}\right|dt + \max_{(i,j)\in\Gamma}\sup_{s\leq t\leq T}\left|Y_t^{i,j}\right| + \sup_{s\leq t\leq T}\left|\int_s^t Z_t^{u_t}dB_t\right| + \varepsilon. \tag{5.19}
$$

Hence, since $f^{i,j} \in \mathcal{H}^2$, $h^{i,j} \in L^2(\mathcal{F}_T)$ and $(Y^{i,j}, Z^{i,j}) \in \mathcal{S}^2 \times \mathcal{H}^{2,d}$, we conclude that the right-hand side of (5.19) is a square-integrable random variable, which proves that $\gamma(\alpha^\varepsilon, \beta^\varepsilon) \in \mathcal{G}_s^{i,j}$ □

Let us define an ε- strategy $\overline{\beta^\varepsilon} : \mathsf{A}_s^i \to \mathsf{B}_s^j$, for Player 2, in the following manner: given $\alpha = (\sigma_n, \xi_n)_{n\geq 0} \in \mathsf{A}_s^i$, we define $\overline{\beta^\varepsilon}(\alpha) = (\tau_n^\varepsilon, \zeta_n^\varepsilon)_{n\geq 0}$ in B_s^j, such that the coupling control associated to this two controls is given by

$$
\rho_0 = s \quad \text{and} \quad \gamma_0 = (i, j) \tag{5.20}
$$

and for $n \geq 1$,

$$
\rho_n = \sigma_{r_n} \wedge \tau_n^\varepsilon, \qquad \gamma_n = \begin{cases} (\xi_{r_n}, \gamma_{n-1}^{(2)}), & \text{on} \quad \{\sigma_{r_n} \leq \tau_n^\varepsilon,\, \sigma_{r_n} < T\} \\ (\gamma_{n-1}^{(1)}, \zeta_n^\varepsilon), & \text{on} \quad \{\tau_n^\varepsilon < \sigma_{r_n}\} \\ \gamma_{n-1}, & \text{otherwise.} \end{cases}
$$

where $\zeta_n^\varepsilon = \mathcal{U}_{\rho_n}^{\gamma_{n-1}}$ is obtained from (5.13), $\tau_n^\varepsilon = \tau_{\rho_{n-1}}^{\gamma_{n-1}}$ for $n \geq 1$, $\{r_n\}_{n\geq 0}$ is defined iteratively by $r_0 = 0$, $r_1 = 1$ and for $n \geq 2$,

$$
r_n = r_{n-1} + \mathbf{1}_{\{\sigma_{r_{n-1}}\leq\tau_{n-1}^\varepsilon\}}. \tag{5.21}
$$

Similarly, for each $\beta \in \mathsf{B}_s^j$ we define the control $\overline{\alpha^\varepsilon}(\beta) \in \mathsf{A}_s^i$, for player 1, using the switching selector $\mathcal{L}(Y)$ instead of $\mathcal{U}(Y)$ (see (5.13)).

Lemma 5.4. *We have $\overline{\alpha^\varepsilon} \in \mathscr{A}_s^i$ and $\overline{\beta^\varepsilon} \in \mathscr{B}_s^j$.*

Proof. Let us only show that $\overline{\beta^\varepsilon} \in \mathscr{B}_s^i$, since $\overline{\alpha^\varepsilon} \in \mathscr{A}_s^i$ is similar. Note that given a control $\alpha \in \mathsf{A}_s^i$, then the ε-optimal control $\overline{\beta^\varepsilon}(\alpha) = \beta^\varepsilon = (\tau_n^\varepsilon, \zeta_n^\varepsilon)_{n\geq 0}$ belongs to B_s^j (see **Step 1** in Lemma 5.3). Hence, the mapping $\overline{\beta^\varepsilon}$ is well defined. Moreover, by construction of $\overline{\beta^\varepsilon}$ satisfies the non anticipative property.

On the other hand, given $\alpha \in \mathcal{A}_s^i$ we want to see that $\overline{\beta^\varepsilon}(\alpha) = \beta^\varepsilon \in \mathcal{B}_s^i$. To this end, let $\gamma(\alpha, \beta^\varepsilon)$ the coupling strategy associated to α and β^ε. For notational convenience, we delete the superscript ε but keep in mind that the control β of player 2 is ε-optimal and the control α of player 1 is arbitrary. Therefore, by proceeding as in Lemma 5.3 we obtain from (5.14) that $\mathbb{P}$–a.s.,

$$\begin{aligned} Y_s^{i,j} &= \int_s^{\rho_1} f_t^{i,j} dt + h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + Y_{\rho_1}^{i,j}\mathbf{1}_{\{\rho_1<T\}} + \int_s^{\rho_1} dK_t^{i,j,+} - \int_s^{\rho_1} dK_t^{i,j,-} - \int_s^{\rho_1} Z_t^{i,j} dB_t, \\ &\geq \int_s^{\rho_1} f_t^{i,j} dt + h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + Y_{\rho_1}^{i,j}\mathbf{1}_{\{\rho_1<T\}} - \int_s^{\rho_1} Z_t^{i,j} dB_t. \end{aligned}$$

In the last inequality, since α is arbitrary then $\gamma^{(1)}$ is not necessarily ε-optimal at time ρ_1, then the non-negative term $K_t^{i,j,+}$ may not be zero on $[s, \rho_1]$, but we still have $K_t^{i,j,-} = 0$ for $s \leq t \leq \rho_1$, by the definition of $\overline{\beta^\varepsilon}$. Besides, as in Lemma 5.3 we have $Y_{\rho_1}^{i,j} \geq L_{\rho_1}^{i,j}(\mathbf{Y})$ (when $\rho_1 = \sigma_1$) or $Y_{\rho_1}^{i,j} \geq Y_{\rho_1}^{\gamma_1} + g_{\rho_1}^{j,\gamma_1^{(2)}} - \frac{\varepsilon}{4}$ (when $\rho_1 = \tau_1^\varepsilon$), and thus

$$Y_s^{i,j} \geq h^{i,j}\mathbf{1}_{\{\rho_1=T\}} + \int_s^{\rho_1} f_t^{u_t} dt + Y_{\rho_1}^{\gamma_1}\mathbf{1}_{\{\rho_1<T\}} - \left[\hat{g}_{\rho_1}^{i,\gamma_1^{(1)}} - g_{\rho_1}^{j,\gamma_1^{(2)}}\right] - \int_s^{\rho_1} Z_t^{u_t} dB_t - \frac{\varepsilon}{4}.$$

and iterating for $1, \ldots, N$ it follows

$$\begin{aligned} Y_s^{i,j} \geq \sum_{n=1}^N h^{\gamma_{n-1}}\mathbf{1}_{\{\rho_n=T,\rho_{n-1}<T\}} + \int_s^{\rho_N} f_t^{u_t} dt + Y_{\rho_N}^{\gamma_N}\mathbf{1}_{\{\rho_N<T\}} - \sum_{n=1}^N \left[\hat{g}_{\rho_n}^{\gamma_{n-1}^{(1)},\gamma_n^{(1)}} - \check{g}_{\rho_n}^{\gamma_{n-1}^{(2)},\gamma_n^{(2)}}\right] \\ - \int_s^{\rho_N} Z_t^{u_t} dB_t - \sum_{n=2}^{N+1} \frac{\varepsilon}{2^n}, \end{aligned} \tag{5.22}$$

and thus we have

$$\begin{aligned} \sum_{n=1}^N \check{g}_{\rho_n}^{\gamma_{n-1}^{(2)},\gamma_n^{(2)}} \leq Y_s^{i,j} - \sum_{n=1}^N h^{\gamma_{n-1}}\mathbf{1}_{\{\rho_n=T,\rho_{n-1}<T\}} - \int_s^{\rho_N} f_t^{u_t} dt - Y_{\rho_N}^{\gamma_N}\mathbf{1}_{\{\rho_N<T\}} + \sum_{n=1}^N \hat{g}_{\rho_n}^{\gamma_{n-1}^{(1)},\gamma_n^{(1)}} \\ + \int_s^{\rho_N} Z_t^{u_t} dB_t + \sum_{n=2}^{N+1} \frac{\varepsilon}{2^n}. \end{aligned} \tag{5.23}$$

Since $\mathbb{P}(\{\rho_N < T \ \ \forall N \geq 1\}) = 0$, the limits as $N \to \infty$ on both sides of (5.23) are well defined. As the switching costs are non-negative we have

$$0 \leq \sum_{n=0}^\infty \check{g}_{\tau_n}^{\zeta_{n-1}^\varepsilon,\zeta_n^\varepsilon} \leq Y_s^{i,j} - h^{u_T} - \int_s^T f_t^{u_t} dt + \sum_{n=0}^\infty \hat{g}_{\sigma_n}^{\xi_{n-1},\xi_n} + \int_s^T Z_t^{u_t} dB_t + \varepsilon. \tag{5.24}$$

Hence, since $\alpha \in \mathcal{A}_s^i$, $(Y^{i,j}, Z^{i,j}) \in \mathcal{S}^2 \times \mathcal{H}^{2,d}$, $h^{ij} \in L^2(\mathcal{F}_T)$, and $f^{i,j}$ belongs to $\mathcal{H}^2$ for all $(i,j) \in \Gamma$, the random variable on the right-hand side of (5.24) belongs to L^2 and we conclude that the control β is square-integrable. □

5.3 Proof of Theorem 5.1.

(i) We first show that for any arbitrary $\alpha \in \mathcal{A}_s^i$ we have,

$$J_s^{i,j}(\gamma(\alpha, \overline{\beta^\varepsilon}(\alpha))) - \varepsilon \leq Y_s^{i,j}, \tag{5.25}$$

where $\overline{\beta^\varepsilon}(\alpha)$ was introduced in Lemma 5.4. Namely, conditional expectations in (5.22) above we get,

$$Y_s^{i,j} \geq \mathbb{E}\Big[\sum_{n=1}^{N} h^{\gamma_{n-1}} \mathbf{1}_{\{\rho_n = T,\, \rho_{n-1}<T\}} + \int_s^{\rho_N} f_t^{u_t}\,dt + Y_{\rho_N}^{\gamma_N} \mathbf{1}_{\{\rho_N<T\}} - \sum_{n=1}^{N}\Big[\hat{g}_{\rho_n}^{\gamma_{n-1}^{(1)},\gamma_n^{(1)}} - \check{g}_{\rho_n}^{\gamma_{n-1}^{(2)},\gamma_n^{(2)}}\Big]\Big|\mathcal{F}_s\Big] - \varepsilon. \tag{5.26}$$

By Lemma 5.4, we have $\gamma(\alpha, \overline{\beta^\varepsilon}(\alpha)) \in \mathcal{G}_s^{i,j}$, then taking the limit $N \to \infty$ in (5.26) proves the inequality (5.25). Similarly, we get

$$Y_s^{i,j} \leq J_s^{i,j}(\gamma(\overline{\alpha^\varepsilon}(\beta), \beta)) + \varepsilon \tag{5.27}$$

for each $\beta \in \mathcal{B}_s^j$. Therefore, the desired result follows from (5.25) and (5.27).

(ii) Since $\alpha \in \mathcal{A}_s^i$ is arbitrary, by taking *ess sup* over $\mathcal{A}_s^i$ in (5.25), we have that

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_s^i} J_s^{i,j}(\gamma(\alpha, \overline{\beta^\varepsilon}(\alpha)) - \varepsilon \leq Y_s^{i,j}. \tag{5.28}$$

Moreover, since $\overline{\beta}^\varepsilon(\alpha) \in \mathcal{B}_s^j$, for each $\alpha \in \mathcal{A}_s^i$, then

$$\operatorname*{ess\,inf}_{\beta \in \mathcal{B}_s^j} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_s^i} J_s^{i,j}(\gamma(\alpha, \beta)) - \varepsilon \leq \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_s^i} J_s^{i,j}(\gamma(\alpha, \overline{\beta^\varepsilon}(\alpha)) - \varepsilon \leq Y_s^{i,j}.$$

and letting $\varepsilon \to 0$ we get

$$\operatorname*{ess\,inf}_{\beta \in \mathcal{B}_s^j} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_s^i} J_s^{i,j}(\gamma(\alpha, \beta)) \leq Y_s^{i,j}.$$

Similarly, given $\beta \in \mathcal{B}_s^j$ and defining $\overline{\alpha}^\varepsilon(\beta) = (\sigma_n^\varepsilon, \xi_n^\varepsilon)$, we obtain that

$$Y_s^{i,j} \leq \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_s^i} \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_s^j} J_s^{i,j}(\gamma(\alpha, \beta)).$$

Thus, combining the last two inequalities we get $\hat{V}_s \leq Y_s^{i,j} \leq \check{V}_s$ a.s., which together with $\hat{V} \geq \check{V}$ yields to the equality

$$\hat{V}_s = Y_s^{i,j} = \check{V}_s$$

and so the game has a value. □

5.4 *Special cases*

In previous sections we are assuming the existence of systems of DRBSDEs (5.4). In this section we show that under a special structure such systems have indeed a solution.

Case 1. Suppose that for each $(i, j) \in \Gamma$, the switching costs $(\hat{g}^{i,j})$ and $(\check{g}^{i,j})$ are càdlàg. Besides, for each $(i, j) \in \Gamma$, the functions $f^{i,j}$ and $h^{i,j}$ satisfy

$$f^{i,j} := f^i + f^j \quad \text{and} \quad h^{i,j} := h^i + h^j. \tag{5.29}$$

Consider the following two systems of RBSDEs with one interconnected lower (resp. upper) barriers associated with data $((f^i)_{i\in\Gamma^1}, (h^i)_{i\in\Gamma^1}, (\hat{g}^{ik})_{i,k\in\Gamma^1})$ (resp. $((f^j)_{j\in\Gamma^2}, (h^j)_{j\in\Gamma^2}, (\check{g}^{j\ell})_{j,\ell\in\Gamma^2})$): for any $i \in \Gamma$ and $s \leq T$

(resp. for any $j \in \Gamma$ and $s \le T$),

$$
\begin{aligned}
&(i) \quad \overline{Y}^i_s = h^i + \int_s^T f^i_t dt + \overline{K}^i_T - \overline{K}^i_s - \int_s^T \overline{Z}^i_t dB_t;\\
&(ii) \quad \overline{Y}^i_s \ge \max_{k\in(\Gamma^1)^{-i}} \{\overline{Y}^k_s - \hat{g}^{i,k}_s\};\\
&(iii) \quad \int_0^T (\overline{Y}^i_{t-} - \max_{k\in(\Gamma^1)^{-i}} \{\overline{Y}^k_{t-} - \hat{g}^{i,k}_{t-}\}) d\overline{K}^i_t = 0.
\end{aligned}
\tag{5.30}
$$

(resp.

$$
\begin{aligned}
&(i) \quad \overline{Y}^j_s = h^j + \int_s^T f^j_t dt - (\overline{K}^j_T - \overline{K}^j_s) - \int_s^T \overline{Z}^j_t dB_t;\\
&(ii) \quad \overline{Y}^j_s \le \min_{\ell\in(\Gamma^2)^{-j}} \{\overline{Y}^\ell_s + \breve{g}^{j,\ell}_s\};\\
&(iii) \quad \int_0^T (\overline{Y}^j_{t-} - \min_{\ell\in(\Gamma^2)^{-j}} \{\overline{Y}^\ell_{t-} + \breve{g}^{j,\ell}_{t-}\}) d\overline{K}^j_t = 0.)
\end{aligned}
\tag{5.31}
$$

Since for any $i, i_1, i_2 \in \Gamma^1$ and $j, j_1, j_2 \in \Gamma^2$, $f^i.f^j \in \mathcal{H}^2$, $h^i, h^j \in L^2(\mathcal{F}_T)$, $\hat{g}^{i_1,i_2}, \breve{g}^{j_1,j_2} \in \mathcal{S}^2$ and along with Assumption G, then there exist unique solutions $(\overline{Y}^i, \overline{Z}^i, \overline{K}^i)_{i\in\Gamma^1}$ and $(\overline{Y}^j, \overline{Z}^j, \overline{K}^j)_{j\in\Gamma^2}$ of the equations (5.30) and (5.31), respectively (see Proposition 4.6). By using these solutions, we define the processes $(\overline{Y}^{i,j}, \overline{Z}^{i,j}, \overline{K}^{i,j,+}, \overline{K}^{i,j,-})$ by the formula: for all $(i,j) \in \Gamma$

- $\overline{Y}^{i,j} := \overline{Y}^i + \overline{Y}^j$.
- $\overline{Z}^{i,j} := \overline{Z}^i + \overline{Z}^j$.
- $\overline{K}^{i,j,+} := \overline{K}^i$.
- $\overline{K}^{i,j,-} := \overline{K}^j$.

Then, it is easily seen that the quadruple $(\overline{Y}^{i,j}, \overline{Z}^{i,j,+}, \overline{K}^{i,j}, \overline{K}^{i,j,-})$ is solution of the following system: for all $(i,j) \in \Gamma$ and $s \le T$,

$$
\begin{aligned}
&\text{(i)} \quad \overline{Y}^{i,j}_s = h^{i,j} + \int_s^T f^{i,j}_t dt + \overline{K}^{i,j,+}_T - \overline{K}^{i,j,+}_s - (\overline{K}^{i,j,-}_T - \overline{K}^{i,j,-}_s) - \int_s^T \overline{Z}^{i,j}_t dB_t;\\
&\text{(ii)} \quad \max_{k\in(\Gamma^1)^{-i}} \{\overline{Y}^{k,j}_s - \hat{g}^{i,k}_s\} \le \overline{Y}^{i,j}_s \le \min_{\ell\in(\Gamma^2)^{-j}} \{\overline{Y}^{i,\ell}_s + \breve{g}^{j,\ell}_s\};\\
&\text{(iii)} \quad \int_0^T (\overline{Y}^{i,j}_{t-} - U^{i,j}_{t-}(\overline{\mathbf{Y}})) d\overline{K}^{i,j,-}_t = \int_0^T (\overline{Y}^{i,j}_{t-} - L^{i,j}_{t-}(\overline{\mathbf{Y}})) d\overline{K}^{i,j,+}_t = 0.
\end{aligned}
$$

Indeed, the equality in (i) is clear. On the other hand, to show part (ii), note that for $(i,j) \in \Gamma$, $s \le T$,

$$
\max_{k\in(\Gamma^1)^{-i}} \{\overline{Y}^k_s - \hat{g}^{i,k}_s\} \le Y^i_s,
$$

and thus adding Y^j in both side of this inequality and using the definition of $Y^{i,j}$ we get for $s \le T$,

$$
\max_{k\in(\Gamma^1)^{-i}} \{\overline{Y}^{k,j}_s - \hat{g}^{i,k}_s\} \le Y^{i,j}_s.
$$

Similarly the other inequality in (ii) is obtained in the same manner. Finally, item (iii) is directly obtained from the definition of $Y^{i,j}$, $K^{i,j,+}$ and $K^{i,j,-}$ for each $(i,j) \in \Gamma$.

Case 2. We now assume that $T = 1$, $\Gamma^1 = \{1, 2\}$ and $\Gamma^2 = \{3, 4\}$, and let us consider the following lower switching costs, for $t \in [0, 1]$,

$$\begin{aligned} \hat{g}^{12}(t) &= 1 \cdot \mathbf{1}_{[0,1/2)}(t) + 2 \cdot \mathbf{1}_{[1/2,1]}(t) \\ \hat{g}^{21}(t) &= 4 \cdot \mathbf{1}_{[0,1/2)}(t) + 5 \cdot \mathbf{1}_{[1/2,1]}(t) \end{aligned},$$

and the upper switching costs $\check{g}^{3,4}$ and $\check{g}^{4,3}$ are assumed semimartingales (e.g. constants).

In order to find processes that are solutions of the system (5.4) with these given switching costs, we start by considering the following penalization scheme: for $n \geq 0$, $(i, j) \in \Gamma$ and $s \in [0, 1]$,

$$\begin{cases} \tilde{Y}_s^{ij,n} = \xi^{ij} + \int_s^1 f_t^{ij} dt + \tilde{K}_1^{ij,n} - \tilde{K}_s^{ij,n} - n \int_s^1 (\tilde{Y}_t^{ij,n} - (\tilde{Y}_t^{il,n} + \check{g}_t^{jl}))^+ dt - \int_s^1 \tilde{Z}_t^{ij,n} dB_t \\ \tilde{Y}_s^{ij,n} \geq \tilde{Y}_s^{kj,n} - \hat{g}_s^{ik} \\ \int_0^1 \left(\tilde{Y}_{t-}^{ij,n} - (\tilde{Y}_{t-}^{kj,n} - \hat{g}_{t-}^{ik})\right) d\tilde{K}_t^{ij,n} = 0. \end{cases} \tag{5.32}$$

Note that for any $n \geq 0$ and $(i, j) \in \Gamma$, the solution $(\tilde{Y}^{ij,n}, \tilde{Z}^{ij,n}, \tilde{K}^{ij,n}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2$ exists by Proposition 4.6 (see also Proposition 3.1 in [29]). Since the lower switching costs are càdlàg at $t = 1/2$, then for each $n \geq 0$ and $(i, j) \in \Gamma$, $\tilde{Y}^{ij,n}$ may also have a negative jump at $t = 1/2$. Moreover, by comparison theorem we have that $\mathbb{P}$- a.s.,

$$\tilde{Y}^{ij,n} \geq \tilde{Y}^{ij,n+1} \geq \widehat{Y}^{ij}$$

where $(\widehat{Y}^{ij}, \widehat{Z}^{ij}, \widehat{K}^{ij})_{(i,j)\in\Gamma}$ is the unique solution of system of reflected BSDEs with upper barriers associated to the data $((f^{ij})_{(i,j)\in\Gamma}, (\xi^{ij})_{(i,j)\in\Gamma}, (\check{g}_{jl})_{j,l\in\Gamma^2})$ (see the same arguments in the proof of Proposition 3.3. in [34]). Therefore, taking limit we get $\tilde{Y}^{ij} := \lim_n \tilde{Y}^{ij,n}$ with $\tilde{Y}^{ij}$ right upper semicontinuous. On the other hand, by using a similar procedure as in step 2 in Proposition 3.3. in [34], we have that the penalized terms satisfy the following estimate: for any $t \in [0, 1]$, $(i, j) \in \Gamma$ and $n \geq 0$,

$$\mathbb{E}\Big[n^2((\tilde{Y}_t^{ij,n} - (\tilde{Y}_t^{il,n} + \check{g}_t^{jl}))^+)^2\Big] \leq C \tag{5.33}$$

where C is a constant independent of n (unlike Proposition 3.3. in [34], this constant considers the only possible jump of the processes $\tilde{Y}^{ij,n}$ and $\tilde{Y}^{il,n}$ in (5.33)). Therefore, there exists a subsequence which we still denote by $\{n\}$ such that for some $\phi^{ij} \in \mathcal{H}^2$ the following weak convergence follows

$$n(\tilde{Y}^{ij,n} - (\tilde{Y}^{il,n} + \check{g}^{jl}))^+ \rightharpoonup \phi^{ij}.$$

Thus the process defined as $\tilde{k}_t^{ij,-} := \int_0^t \phi_s^{ij} ds$ is continuous, non decreasing and $\mathbb{E}[(\tilde{k}_1^{ij,-})^2] < \infty$. Moreover, we have

$$n \int_t^1 (\tilde{Y}^{ij,n} - (\tilde{Y}^{il,n} + \check{g}^{jl}))^+ ds \rightharpoonup \tilde{k}_1^{ij,-} - \tilde{k}_t^{ij,-}.$$

On the other hand, by applying Itô's formula to $(\tilde{Y}^{ij,n})^2$ and taking into account (5.33), we have that sequences $(\tilde{Z}^{ij,n})_{n\geq 0}$ and $(\tilde{K}^{ij,n})_{n\geq 0}$ are bounded in $\mathcal{H}^{2,d}$ and $\mathcal{H}^2$, respectively and thus they converge weakly to a processes $\tilde{Z}^{ij}$ and $\tilde{K}^{ij}$ in $\mathcal{H}^{2,d}$ and $\mathcal{H}^2$, respectively. Hence, it follows that

$$\tilde{Y}_s^{ij} = \xi^{ij} + \int_s^1 f_t^{ij} dt + \tilde{K}_1^{ij,+} - \tilde{K}_s^{ij,+} - \int_s^1 \phi_t^{ij} dt - \int_s^1 \tilde{Z}_t^{ij} dB_t. \tag{5.34}$$

Since $(\tilde{K}_t^{ij,+})_{0\leq t\leq 1}$ is non decreasing process, then for any t, it has both left and right limits. Thus from (5.34), we also have that the process $(\tilde{Y}_t^{ij})_{0\leq t\leq 1}$ has both left and right limit since the other terms of the right hand side of (5.34) are continuous.

Other other hand, in the interval $[1/2,1]$ we have continuity of both lower and upper switching costs and thus, by Theorem 3.6 in [34], there exist unique processes $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})$ such that for any $s\in[1/2,1]$,

$$\begin{aligned}
&(i)\quad Y_s^{ij}=\xi^{ij}+\int_s^1 f_t^{ij}dt+K_1^{ij,+}-K_s^{ij,+}-(K_1^{ij,-}-K_s^{ij,-})-\int_s^1 Z_t^{ij}dB_t\\
&(ii)\quad Y_s^{kj}-\hat{g}_s^{ik}\le Y_s^{ij}\le Y_s^{il}+\check{g}_s^{jl}\\
&(iii)\quad \int_s^1 (Y_t^{ij}-(Y_t^{kj}-\hat{g}_t^{ik}))dK_t^{ij,+}=\int_s^1 (Y_t^{il}+\check{g}_t^{jl}-Y_t^{ij})dK_t^{ij,-}=0,
\end{aligned}$$

where $K^{ik,+}$ and $K^{ij,-}$ are non decreasing processes with $K_{1/2}^{ij,+}=K_{1/2}^{ij,-}=0$.

Next for the interval $[0,1/2]$, we have again by Theorem 3.6 in [34] that there exist $(\overline{Y}^{ij}, \overline{Z}^{ij}, \overline{K}^{ij,+}, \overline{K}^{ij,-})$ such that, for any $s\in[0,1/2]$,

$$\begin{aligned}
&(i)\quad \overline{Y}_s^{ij}=\theta^{ij}+\int_s^{1/2} f_t^{ij}dt+\overline{K}_1^{ij,+}-\overline{K}_s^{ij,+}-(\overline{K}_1^{ij,-}-\overline{K}_s^{ij,-})-\int_s^{1/2} \overline{Z}_t^{ij}dB_t\\
&(ii)\quad \overline{Y}_s^{kj}-\hat{g}_s^{ik}\le \overline{Y}_s^{ij}\le \overline{Y}_s^{il}+\check{g}_s^{jl}\\
&(iii)\quad \int_s^{1/2} (\overline{Y}_{t-}^{ij}-(\overline{Y}_{t-}^{kj}-\hat{g}_{t-}^{ik}))dK_t^{ij,+}=\int_s^{1/2} (\overline{Y}_{t-}^{il}+\check{g}_t^{jl}-\overline{Y}_{t-}^{ij})d\overline{K}_t^{ij,-}=0
\end{aligned}$$

where θ^{ij} is defined by

$$\theta^{ij}:=Y_{1/2}^{ij}\mathbf{1}_{[\tilde{Y}_{1/2-}^{ij}=Y_{1/2}^{ij}]}+(\tilde{Y}_{1/2-}^{kj}-\hat{g}_{1/2-}^{ik})\mathbf{1}_{[\tilde{Y}_{1/2-}^{ij}>Y_{1/2}^{ij}]} \tag{5.35}$$

Finally, we proceed to concatenate to obtain the desired solution. Indeed, let us define the concatenated solution $(Y^{ij,c}, Z^{ij,c}, K^{ij,c}, K^{ij,c})$ as follows, for any $s\in[0,1]$,

(c1) $Y_s^{ij,\,c}:=\overline{Y}_s^{ij}\mathbf{1}_{[s<1/2]}+Y_s^{ij}\mathbf{1}_{[1/2\le s\le 1]}$,

(c2) $Z_s^{ij,\,c}:=\overline{Z}_s^{ij}\mathbf{1}_{[s<1/2]}+Z_s^{ij}\mathbf{1}_{[1/2\le s\le 1]}$,

(c3) $K_s^{ij,+,\,c}:=\overline{K}_s^{ij,+}\mathbf{1}_{[s<1/2]}+(K_s^{ij,+}+(\tilde{Y}_{1/2^-}^{kj}-\hat{g}_{1/2^-}^{ik}-Y_{1/2}^{ij})^++\overline{K}_{1/2}^{ij,+})\mathbf{1}_{[1/2\le s\le 1]}$,

(c4) $K_s^{ij,-,\,c}:=\overline{K}_s^{ij,-}\mathbf{1}_{[s\le 1/2]}+(K_s^{ij,-}+\overline{K}_{1/2}^{ij,-})\mathbf{1}_{[1/2<s\le 1]}$.

It is straightforward to show that these processes $(Y^{ij,c}, Z^{ij,c}, K^{ij,c}, K^{ij,c})$ satisfy for each $(i,j)\in\Gamma$ and $s\in[0,1]$,

$$\begin{aligned}
&(i)\quad Y_s^{ij,c}=\xi^{ij}+\int_s^1 f_t^{ij}dt+K_1^{ij,+,c}-K_s^{ij,+,c}-(K_1^{ij,-,c}-K_s^{ij,-,c})-\int_s^1 Z_t^{ij,c}dB_t\\
&(ii)\quad Y_s^{kj,c}-\hat{g}_s^{ik}\le Y_s^{ij,c}\le Y_s^{il,c}+\check{g}_s^{jl}\\
&(iii)\quad \int_s^1 (Y_{t-}^{ij,c}-(Y_{t-}^{kj,c}-\hat{g}_{t-}^{ik}))dK_t^{ij,+,c}=\int_s^1 (Y_{t-}^{il,c}+\check{g}_t^{jl}-Y_{t-}^{ij,c})dK_t^{ij,-,c}=0.
\end{aligned} \tag{5.36}$$

Hence the system (5.4) has solution to this special case.

Remark 5.1. In a similar manner we can extend the lower barrier from one to a finite, say $0 < t_1 < t_2 <, \cdots, < t_k < 1$, $k \geq 2$ number of discontinuities. The idea is to do the analysis in the following backward reasoning: choose the intervals $[t_{k-1}, t_k]$ and $[t_k, 1]$. Then apply the above analysis for the discontinuity t_k, in particular to get a θ_k as in (5.35) and thus to obtain a solution Y^{ij} on $[t_{k-1}, 1]$. Next, let us construct the solution on the intervals $[t_{k-2}, t_{k-1}]$ by giving a terminal condition θ_{k-1} that in turn depends on both values $Y^{ij}_{t_{k-1}}$ and $\tilde{Y}^{ij}_{t_{k-1}-}$ —recall that $\tilde{Y}^{ij}$ was defined in (5.34) on the whole interval $[0, 1]$. In this way, we can deduce a solution on $[t_{k-1}, 1]$ by concatenating according to the procedures (c1)–(c4). Proceeding in this way, we can obtain the solution in the whole interval $[0, 1]$.

Remark 5.2. Another interesting example is the one that includes discontinuities affecting both the lower and the upper switching costs. We think that the above results can be applied only for the case when there is not a point $x \in [0, 1]$ that produces a discontinuity to both costs. This new challenging example deserves more attention and will be tackled in the near future.

6

Conclusions and future work

In this thesis we have studied optimal switching control problems and switching games for a general class of switching cost functions. Our leading motivation was the acquired knowledge of the well-known theory of BSDEs, specifically the area of systems of inteconnected RBSDEs of one and two càdlàg barriers. We have given conditions ensuring the existence of optimal control switching strategies by taking advantage of the richness of the theory of BSDE and the connection with Snell envelopes and the PDEs theory.

More specifically, the first two chapters were the basis for the development of the new original results proposed in the later two chapters. Among the topics of these two preliminary chapters, we can highlight results on existence and uniqueness as well as comparison theorems of (1) BSDE, (2) RBSDE, (3) Systems of inteconected RBSDE with continuous barriers. In turn, these results provide links between the solutions of these equations and certain systems of PDE's; the existence of the later systems is also analyzed through the theory of viscosity solutions. In the meantime, there has been also stipulated a bridge of the aforementioned systems with the optimal elements of switching control problems for the case of the systems of RBSDE and stopping times for the case of RSDE.

Next, the idea was then to extend the results of these preliminaries chapters for the cases when (a) a switching control problem has a switching cost of càdlàg type and (b) a switching game with the same characteristics on the switching costs for each player. All these extensions were posed and solved in Chapters 4 and 5.

Nevertheless, we want to mention that the results studied in here leave open other interesting lines of study. For instance:

- The study of switching control problems and games when the time horizon is infinite (i.e., $T = \infty$). These later type of problem are very common in applications related to pollution problems, population growth models, among others.
- Non zero-sum switching because many economic problems need not to have the property that the gain of one player is the loss of the other. So, the treatment of these type of models combined with our theory would be so interesting.
- Mixing control problems when the controller (resp. the player) not only controls the switching but also needs to control the dynamics of the system at every instant of time between switchings. The usefulness of these problems is vast and generalizes of course our results.

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