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Acerca del cálculo de los números de Betti de ideales de aristas

TESIS

Que presenta

David Camilo Molano Valbuena

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Dedication and Aknowledgments

As always, this work is dedicated to my cat, Minush, but there are many people to aknowledge. The effort put by my advisor Carlos Valencia to help me get through all the challenges in this master program, for example. With him, I thank the two reviewers of this work, the professors Jesús González and Rafael Villarreal. In particular, Jesús González was the one who gave me my first course in algebraic topology. This knowledge was essential for me to be able to make this work.

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Last, but not least, I thank my family. All their support during my life made me the person who I can be proud to say I am today.

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Abstract

In this work we'll use Mayer-Vietoris sequences to compute the multigraded Betti numbers of squarefree monomial ideals in a polynomial ring. This also allows us to compute the Betti numbers of non-squarefree monomial ideals, by methods like polarization, or others, without computing the entire free resolutions of such ideals.

Mayer-Vietoris sequences also allow us, when adding some combinatorial tools to the mix, to give algorithms and simple formulas for the computations of the Betti numbers of the edge ideals of some graphs, like forests and cycles.

Resumen

En este trabajo usaremos sucesiones de Mayer-Vietoris para calcular los números de Betti multigraduados de ideales monomiales libres de cuadrados en un anillo de polinomios. Esto también nos permite calcular los números de Betti de ideales monomiales no libres de cuadrados, por métodos como polarización y otros, sin calcular completamente las resoluciones libres minimales de dichos ideales.

Las sucesiones de Mayer-Vietoris también nos permiten, al añadir algunas herramientas combinatoriales a la mezcla, dar algoritmos y fórmulas simples para el cálculo de los números de Betti de los ideales de aristas de algunos grafos, como bosques y ciclos.

Contents

Introduction

A broad interest for us in this work is to compute the multigraded Betti numbers of monomial ideals. In particular, we're interested in finding a recursive way to compute the multigraded Betti numbers of some families of monomial ideals. Computing these are a first step in computing minimal free resolutions, which give us important information about the ideal, and about its associated variety.

We'll be using algebraic-topological methods to compute these, using Hochster's formula as a first step with the upper Koszul complex, and then we'll deepen using other tools, like all the Mayer-Vietoris sequences we can associate to each monomial ideal, using the fact that the upper Koszul complex behaves appropiately with respect to the lattice structure of the monomial ideals. There are some methods which allow us to consider only squarefree ideals. A well known method for this is called polarization, but we won't need it: For a monomial ideal we'll be able to compute the upper Koszul complex X, and by taking the complements of the facets of such complex we'll get a squarefree ideal having X as its upper Koszul complex, therefore sharing one Betti number with the original ideal.

We'll use these tools to simplify the computations of the Betti numbers of the edge ideals of some families of graphs, going through stars, complete bipartite graphs, cones, graphs with isolated edges, forests, paths and cycles in that order. We'll also be talking about Alexander duality (The lower Koszul complex) since some homologies can be computed easier in the Alexander dual complex of a simplicial complex.

Lastly, we'll conclude talking about how we can find other simplicial complexes which homology also give us the Betti numbers of G. The methods used in this work will be used to get some preliminary results about these complexes.

Objectives

The objectives for this work are, at short term, to compute the Betti numbers and similar invariants of a broad family of squarefree monomial ideals, and at large term, to use these results, and other combinatorial properties of these ideals to be able to compute minimal free resolutions of them.

Chapter 1

Preliminaries

In this chapter we're concerned about giving an introduction to some topics needed later. We assume the reader has a basic knowledge in basic algebra (Groups, rings and modules) and general topology. We'll start with Categories and Functors (This might not be absolutely necessary but since there are a lot of categories and functors later, it's a good way to start).

1.1 Categories and Functors

In this section we give the basics of category theory needed for this work. You can use [3], [8], or any other book on homological or commutative algebra, to get deepen in those topics.

Let **Sets** denote the class of all sets (Later in this section we'll redefine it).

Definition 1.1.1. A Category $\mathcal{C}$ consists in:

1. A class $\mathrm{ob}\,\mathcal{C}$ of sets called objects.

2. A function $\hom_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to$ **Sets** denoted by hom. The elements of the images of hom are called morphisms, and $f \in \hom(A, B)$ will be denoted by $f : A \to B$ or $A \xrightarrow{f} B$.

3. For objects A, B, C of $\mathcal{C}$, a function

$$\hom(A, B) \times \hom(B, C) \to \hom(A, C)$$

denoted by $(f, g) \mapsto gf$.

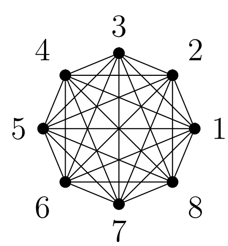
such that

1. For $(A, B) \neq (C, D) \in \mathcal{C} \times \mathcal{C}$, $\hom(A, B) \cap \hom(C, D) = \varnothing$.

2. For $A \in \mathcal{C}$, there exists an element $1_A \in \hom(A, A)$. such that $f 1_A = f$ and $1_B f = f$ for every $f : A \to B$.

3. If we have
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$
then $(hg)f = h(gf)$.

If, for $A, B \in$ **Sets** we define $\hom_{\mathbf{Sets}}(A, B) = B^A$ then **Sets** becomes a category. So are the categories **Groups** , **Rings** ,$k-$**Vect** , $R-$**Mod** , of groups, rings, $k-$vector spaces and $R-$modules, where k is a field and R is a ring. We can also talk about the category **Graph** of all graphs: A graph is a pair (V, E), where V is called the vertex set of G and $E \subseteq \wp(V)$, called the edge set of G is a set such that $|e| = 2$ for any $e \in E$. A graph can be shown with a drawing:



A morphism $f : G \to H$ between graphs is just a map $f : V_G \to V_H$ between their vertex sets such that $f(e) \in E_H$ for any $e \in E_G$.

When there is risk of ambiguity, as just now, we'll use $\hom_{\mathcal{C}}$ instead of just hom, and for the category of $R-$modules and other algebraic structures we'll use Hom. The morphisms will also be called just *maps*, so, for example, a map of $R-$modules will be a homomorphism, instead of just a function.

Definition 1.1.2. For categories $\mathcal{C}, \mathcal{D}$, a covariant functor $F : \mathcal{C} \to \mathcal{D}$ consists on:

1. A map $F : \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$.

2. For $A, B \in \operatorname{ob}\mathcal{C}$ a map $F : \hom(A, B) \to \hom(FA, FB)$

such that

1. $F(gf) = F(g)F(f)$ for $f : A \to B, g : B \to C$ and $A, B, C \in \operatorname{ob}\mathcal{C}$.
2. $F(1_A) = 1_{FA}$ for $A \in \operatorname{ob} C$.

Definition 1.1.3. For categories $\mathcal{C}, \mathcal{D}$, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ consists on:

1. A map $F : \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$.
2. For $A, B \in \operatorname{ob}\mathcal{C}$ a map $F : \hom(A, B) \to \hom(FB, FA)$

such that

1. $F(gf) = F(f)F(g)$ for $f : A \to B, g : B \to C$ and $A, B, C \in \operatorname{ob}\mathcal{C}$.
2. $F(1_A) = 1_{FA}$ for $A \in \operatorname{ob} C$.

The induced morphisms $F(f)$ will often by denoted by f^* or f_*.

1.2 Hom and Tensor

Here we'll talk about some results about the functors Hom and $\otimes$ which we'll be using. These two functors have some nice properties, but calculations involving both of them can become cumbersome, which makes, among other things, the Betti numbers of a module hard to compute normally. This is an issue we want to be addressing in this work.

Let R be a commutative ring.

Definition 1.2.1. For a (possibly finite) sequence (C, ϕ) of $R-$modules C_i and homomorphisms ϕ_i:

$$C : \cdots \overset{\phi_{-1}}{\leftarrow} C_{-1} \overset{\phi_0}{\leftarrow} C_0 \overset{\phi_1}{\leftarrow} C_1 \overset{\phi_2}{\leftarrow} \cdots$$

we say that C is a chain complex if $\phi_i \circ \phi_{i+1} = 0$ for all i. If, in fact, $\ker \phi_i = \operatorname{im} \phi_{i-1}$ for some i, then C is said to be exact at the homological degree i. If it's exact at all the homological degrees, then we say it's an exact sequence. An exact sequence

$$0 \to L \to M \to N \to 0$$

is said to be a short exact sequence. In this case, this is equivalent to the fact that $L \to M$ is a monomorphism, $M \to N$ is an epimorphism and $\ker(M \to N) = \mathrm{im}(L \to M)$. Sequences $0 \to L \to M \to 0$ and $0 \to L \to 0$ are also called short exact sequences.

We'll use the notations $C_\bullet$, (C, ϕ) or simply C for a chain complex, depending on the need to differentiate two of them, or state the maps ϕ_i.

Proposition 1.2.1. For a fixed $R-$module A,

$$\mathrm{Hom}(A, -) : R - \mathbf{Mod} \to R - \mathbf{Mod}$$

is a covariant functor and

$$\mathrm{Hom}(-, A) : R - \mathbf{Mod} \to R - \mathbf{Mod}$$

is a contravariant functor.

Proof. For $\phi \in \mathrm{Hom}(M, N)$ there is an induced morphism

$$\phi_* \in \mathrm{Hom}(\mathrm{Hom}(A, M), \mathrm{Hom}(A, N))$$

defined by $\phi_*(f) = \phi f$. For $1 \in \mathrm{Hom}(M, M)$, we have that

$$1_* \in \mathrm{Hom}(\mathrm{Hom}(A, M), \mathrm{Hom}(A, M))$$

satisfies $1_*(f) = 1f = f$ we have 1_* is the identity map of $\mathrm{Hom}(A, M)$ onto itself. Also $(\phi\psi)_*(f) = \phi\psi f = \phi(\psi f) = \phi_*\psi_*(f)$. So $\mathrm{Hom}(A, -)$ is a covariant functor. Similarly when, for $\phi : M \to N$, we define the transpose map $\phi^* : \mathrm{Hom}(N, A) \to \mathrm{Hom}(M, A)$ by $\phi^*(f) = f\phi$, we get that $\mathrm{Hom}(-, A)$ is a contravariant functor. ■

Proposition 1.2.2. If (C, ϕ) is a chain complex of $R-$modules, then for any $R-$module J, the sequence $(\mathrm{Hom}(C, J), \phi^*)$ is also a chain complex. In particular, the dual (C^*, ϕ^*) is a chain complex, where $C^* = \mathrm{Hom}(C, R)$.

It also holds that $(\mathrm{Hom}(J, C), \phi_*)$ is a chain complex.

Proof. To prove the first part, note that if $0 : M \to N$ is such that $0(m) = 0$ for all $m \in M$, then $0^* : \mathrm{Hom}(N, J) \to \mathrm{Hom}(M, J)$ satisfies $0^*(f)(x) = f0(x) = 0$ for any $f \in N^*$ and $x \in N$. The result follows from this and the definition of chain complex.

The second part is similar. The map $0_* : \mathrm{Hom}(J, M) \to \mathrm{Hom}(J, N)$ satisfies $0_*(f) = 0f = 0$, so the result follows. ■

Proposition 1.2.3. If $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is a short exact sequence of $R-$modules, and J is an $R-$module, then the sequence

$$0 \to \operatorname{Hom}(J, L) \xrightarrow{\alpha_*} \operatorname{Hom}(J, M) \xrightarrow{\beta_*} \operatorname{Hom}(J, N)$$

is also exact.

Similarly, if $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is exact then

$$0 \to \operatorname{Hom}(N, J) \xrightarrow{\beta^*} \operatorname{Hom}(M, J) \xrightarrow{\alpha^*} \operatorname{Hom}(L, J).$$

Proof. First consider the sequence $0 \to L \to M \to N$. We first have to proof that α_* is injective. Indeed, suppose $f : J \to L$ is such that $\alpha_*(f) = \alpha f = 0$. Then for every $j \in J$, $\alpha f(j) = 0$. Since α is injective, we have $f(j) = 0$ for every j, so $f = 0$. Now we have to prove that $\ker \beta_* = \operatorname{im} \alpha_*$. By the proposition 1.2.2 we have $\operatorname{im} \alpha \subseteq \ker \beta$, so take $g : J \to M$ such that $\beta g(j) = 0$ for every $j \in J$. Then, for each j there is $l_j \in L$ such that $\alpha(l_j) = g(j)$. Define $f : J \to L$ as $f(j) = l_j$. See that f is a homomorphism. Indeed, for $j = j' \in J$, $g(j) = g(j')$ and since $\alpha(l_j) = \alpha(l_{j'}) = g(j)$ then $l_j = l_{j'}$. Also for any $j, j' \in J$ we have $f(j + j') = l_{j+j'} = l_j + l_{j'}$ since both $l = l_j + l_{j'}$ and $l' = l_{j+j'}$ satisfy $\alpha(l) = \alpha(l') = g(j + j') = g(j) + g(j')$. Furthermore for any $j \in J$, $\alpha f(j) = g(j)$ by construction, so indeed $g = \alpha_*(f)$. Therefore $\ker \beta_* = \operatorname{im} \alpha_*$.

Now, consider an exact sequence $L \to M \to N \to 0$. First we prove β^* is injective. Let $f : N \to J$ be such that $\beta^*(f) = f\beta = 0$. Then $f\beta(m) = 0$ for every $m \in M$. Since β is surjective, this means that $f = 0$. So β^* is injective. Now we want to prove that $\ker \alpha^* = \operatorname{im} \beta^*$. Let $f : M \to J$ be such that $\alpha^*(f) = f\alpha = 0$. Then, for any $l \in L$, $f\alpha(l) = 0$. Let $g : N \to J$ be defined by $g(n) = f(m_n)$ where $\beta(m_n) = n$. This is a homomorphism. Indeed, if $n = n' \in N$ then for any choice of $m_n, m_{n'}$ we have $m_n - m_{n'} \xrightarrow{\beta} 0$. Thus, there exists $l \in L$ such that $\alpha(l) = m_n - m_{n'}$. So $f(m_n) - f(m_{n'}) = f\alpha(l) = 0$, therefore $g(n) = f(m_n) = f(m_{n'}) = g(n')$. Also for any $n, n' \in N$ we have $m_n + m_{n'} \xrightarrow{\beta} n + n'$ and

$$g(n + n') = f(m_n + m_{n'}) = f(m_n) + f(m_{n'}) = g(n) + g(n').$$

And by construction $f = \beta^*(g) = g\beta$. Therefore $\ker \alpha^* = \operatorname{im} \beta^*$, from which the result follows. ■

There is a sort of converse:

Proposition 1.2.4. If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is such that

$$\mathrm{Hom}(N, J) \xrightarrow{\beta^*} \mathrm{Hom}(M, J) \xrightarrow{\alpha^*} \mathrm{Hom}(L, J)$$

is exact for every $R-$module J then $L \to M \to N$ is exact.

Proof. The trick lies in taking appropiate choices for J. First take $J = N$. Then the sequence is

$$\mathrm{Hom}(N, N) \xrightarrow{\beta^*} \mathrm{Hom}(M, J) \xrightarrow{\alpha^*} \mathrm{Hom}(L, J).$$

For the identity map 1_N we have $0 = \alpha^*\beta^*(1_N) = \beta\alpha 1_N = \beta\alpha$, so $L \to M \to N$ is a chain complex.

To finish proving the exactness, take $J = \operatorname{coker}\alpha = M/\operatorname{im}\alpha$. Then, for $\pi : M \to \operatorname{coker}\alpha$ the canonical map, we have $\alpha^*(\pi) = \pi\alpha = 0$, so there exists $g : N \to J$ such that $\beta^*(g) = g\beta = \pi$. Therefore $\ker\beta \subseteq \ker\pi = \operatorname{im}\alpha$, from which the result follows. ■

Definition 1.2.2. A tensor product of $R-$modules M, N is a pair $(T, \odot)$ where

1. $T(M, N)$ is an $R-$module.
2. $\odot : M \times N \to T(M, N)$ is bilinear, i.e. both restrictions $\odot|_{M\times\{n\}} : M \to T(M, N), \odot|_{\{m\}\times N} : N \to T(M, N)$ are homomorphisms for any $m \in M, n \in N$.
3. For any $R-$module U and any bilinear map $\beta : M \times N \to U$, there exists a unique homomorphism $S_\beta : T(M, N) \to U$ such that $\beta = S\odot$.

The property 3. in the definition is called the universal property of the tensor product.

Proposition 1.2.5. If two tensor products $(T_1, \odot), (T_2, \otimes)$ of M, N exist, then there exists an isomorphism $\phi : T_1 \to T_2$ such that $\phi\odot = \otimes$.

Proof. Since $\otimes : M \times N \to T_2$ is bilinear, then there exists a unique homomorphism $\phi : T_1 \to T_2$ such that $\otimes = \phi\odot$. Also there exists a unique isomorphism $\psi : T_2 \to T_1$ such that $\odot = \psi\otimes$. Therefore $\psi\phi : T_1 \to T_1$ is such that $\psi\phi\odot = \odot$. Since 1_{T_1} also satisfies $1_{T_1}\odot = \odot$, by uniqueness of such map in the definition, $\psi\phi = 1_{T_1}$. Analogously, $\phi\psi = 1_{T_2}$. Therefore, ϕ is an isomorphism such that $\phi\odot = \otimes$. ■

Proposition 1.2.6. For $R-$modules M, N there exists a tensor product $(M \otimes_R N, \otimes)$. When there is no risk of ambiguity, such tensor product will be denoted by $M \otimes N$.

Proof. Consider the free $R-$module $F = R^{M\times N}$, and Z the submodule generated by all the elements of the form:

$$\begin{aligned}
&(m + m', n) - (m, n) - (m', n),\\
&(m, n + n') - (m, n) - (m, n'),\\
&(rm, n) - r(m, n)\\
&(m, rn) - r(m, n)
\end{aligned}$$

for $m, m' \in M, n, n' \in N, r \in R$. Let $M \otimes N = F/Z$ and, for $(m, n) \in M \times N$, $m \otimes n = (m, n) + Z$. Then $M \otimes N$ is a tensor product, the tensor product of M and N. Indeed, $M \otimes N$ is an $R-$module, $\otimes$ is bilinear by construction, and every bilinear map $\beta : M \times N \to U$ extends uniquely to an homomorphism $F \to U$ which factors through Z to the desired homomorphism $S_\beta : M \otimes N \to U$. ∎

There are many natural properties of tensor products: For example, for any $R-$modules L, M, N we have $M \otimes N \cong N \otimes M$, $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$. There is a first special one:

Proposition 1.2.7. Tensor product is a bit more than a functor, i.e. for $R-$module M, M', N, N', and homomorphisms $\alpha M \to M', \beta : N \to N'$, there exists an induced map $\alpha \otimes \beta : M \otimes N \to M' \otimes N'$ such that:

1. For $\beta = 1_N : N \to N$, and $\gamma : M' \to M''$ we have $(\beta\alpha) \otimes 1_N = (\beta \otimes 1_N)(\alpha \otimes 1_N)$. Similarly $1_N \otimes (\beta\alpha) = (1_N \otimes \beta)(1_N \otimes \alpha)$.

2. $1_M \otimes 1_N = 1_{M\otimes N}$.

Proof. It follows from the definition: The induced map $\alpha \otimes \beta$ is given by $\alpha \otimes \beta(m \otimes n) = \alpha(m) \otimes \beta(n)$ and it's a well defined homomorphism since the map $\alpha \times \beta : M \times N \to M' \otimes N'$ is bilinear. ∎

Lemma 1.2.8. For $R-$modules M, N, J we have

$$\mathrm{Hom}(M \otimes N, J) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, J)).$$

Proof. It's not hard to prove that the map

$$\phi : \mathrm{Hom}(M \otimes N, J) \to \mathrm{Hom}(M, \mathrm{Hom}(N, J))$$

given by $f \mapsto \tilde{f}$ is an isomorphism, where $\tilde{f}(m)(n) = f(m \otimes n)$ for $(m, n) \in M \times N$. The linearity of $\otimes$ in N implies that $\tilde{f}(m) \in \mathrm{Hom}(N, J)$ for every m and the linearity of $\otimes$ in M implies that $\tilde{f} \in \mathrm{Hom}(M, \mathrm{Hom}(N, J))$. Also

$$\widetilde{f+g}(m)(n) = (f+g)(m \otimes n) = f(m \otimes n) + g(m \otimes n) = \tilde{f}(m)(n) + \tilde{g}(m)(n),$$

so ϕ is a homomorphism. The fact that every map $f : M \to \mathrm{Hom}(N, J)$ can be seen as a bilinear map $M \times N \to J$ means that ϕ is surjective. If $\tilde{f} = 0$ this means that $\tilde{f}(m)(n) = 0$ for every m, n, thus $f(m \otimes n) = 0$ for every m, n, so $f = 0$. Therefore ϕ is an isomorphism. ∎

The isomorphism from the previous lemma is natural in the sense that for $R-$modules M, M', N, J and a map $\alpha : M \to M'$ it makes the following diagram commute:

$$\begin{array}{ccc}
\mathrm{Hom}(M' \otimes J, N) & \xrightarrow{(\alpha \otimes 1)^*} & \mathrm{Hom}(M \otimes J, N) \\
\phi_{M'} \downarrow & & \downarrow \phi_M \\
\mathrm{Hom}(M', \mathrm{Hom}(J, N)) & \xrightarrow[\alpha^*]{} & \mathrm{Hom}(M, \mathrm{Hom}(J, N))
\end{array}$$

Indeed, for $f : M' \otimes J \to N$, we have

$$\phi_M((\alpha \otimes 1)^*(f)) = \phi_M(f(\alpha \otimes 1)).$$

Furthermore,

$$\alpha^*(\phi_{M'}(f)) = \phi_{M'}(f)\alpha.$$

For $m \in M, j \in J$, we have

$$\phi_M(f(\alpha \otimes 1)(m)(j)) = f(\alpha \otimes 1)(m \otimes j) = f(\alpha(m) \otimes j)$$

and

$$(\phi_{M'}(f)\alpha)(m)(j) = \phi_{M'}(f)(\alpha(m))(j) = \tilde{f}(\alpha(m))(j) = f(\alpha(m) \otimes j).$$

Proposition 1.2.9. The tensor product functor is an right-exact functor. This is, given an exact sequence

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

and a module J, the induced sequence

$$L \otimes J \xrightarrow{\alpha \otimes 1} M \otimes J \xrightarrow{\beta \otimes 1} N \otimes J \to 0$$

is also exact.

Proof. It's easy to prove that $\beta \otimes 1$ is surjective, every element of $N \otimes J$ is a sum of elements of the form $n \otimes j, n \in N, j \in J$ which themselves can bee seen as $\beta \otimes 1(m \otimes j)$ for $m \in M$ such that $\beta(m) = n$. It's also easy to prove that the induced sequence is a chain complex, but we'll take a different approach, since the exactness is somewhat problematic.

So, we start with the sequence

$$L \to M \to N \to 0$$

and we get, for any $R-$module U, an induced exact sequence

$$0 \to \mathrm{Hom}(N, \mathrm{Hom}(J, U)) \to \mathrm{Hom}(M, \mathrm{Hom}(J, U)) \to \mathrm{Hom}(L, \mathrm{Hom}(J, U)).$$

By the natural isomorphism we get the exact sequence:

$$0 \to \mathrm{Hom}(N \otimes J, U) \to \mathrm{Hom}(M \otimes J, U) \to \mathrm{Hom}(L \otimes J, U)$$

for any U. Therefore, the sequence

$$L \otimes J \to M \otimes J \to N \otimes J$$

is exact. Since we already proved that $\beta \otimes 1$ is surjective, the sequence

$$L \otimes J \to M \otimes J \to N \otimes J \to 0$$

is exact. ∎

1.3 Simplicial Homology and Cohomology

Here we'll cover a topological invariant for simplicial complexes: Their homology. Some of the results here can be generalized adding some small technical hypothesis.

Definition 1.3.1. An $n-$simplex is just a topological space homeomorphic to

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} t_i = 1\}.$$

We define a face $\sigma \subseteq \Delta^n$ by taking a subset $A_\sigma \subseteq \{0, \dots, n\}$ and defining

$$\sigma = \{(t_0, \dots, t_n) \in \Delta^n \mid (\forall i \in A_\sigma), t_i = 0\}.$$

We then say that σ is a $n - |A_\sigma|-$face of Δ^n. In this case, it's also clear that σ is a $n - |A_\sigma|-$simplex for $|A_\sigma| \leq n + 1$.

We define $\Delta^{-1} = \varnothing$.

Definition 1.3.2 (Simplicial Complex)**.** A simplicial complex Δ is a topological space, with a family of simplices $\mathcal{S} \subseteq \{\sigma \subseteq \Delta\}$ such that

$$\Delta = \bigcup \mathcal{S}$$

and also for every $\sigma, \tau \in \mathcal{S}$ we have that $\sigma \cap \tau$ is a face of both σ and τ. We say that ρ is a face of Δ if it's a face of some simplex of Δ.

Remark. By the previous definition we have that every simplex $\sigma \in \mathcal{S}$ is determined by its $0-$faces, i.e. vertices. Otherwise, if there were $\sigma \neq \tau$ both with the same vertices, this would mean $\sigma \cap \tau$ is a face of σ (resp. τ), having all its vertices, which can only be σ (since it would correspond to the face of the simplex Δ^n corresponding to the set $A = \varnothing$,which is Δ^n), thus $\sigma = \tau \cap \sigma = \tau$.

Definition 1.3.3. For a $n-$simplex σ, we define its dimension by $\dim \sigma = n$. For a simplicial complex Δ we define its dimension by $\dim \Delta = \max\{\dim \sigma : \sigma \in \mathcal{S}\}$.

Definition 1.3.4. For a simplicial complex Δ we can define its face poset $F(\Delta)$ as the set of all its faces ordered by inclusion. It has a *graded* structure, by taking

$$F(\Delta) = F_0 \sqcup \dots \sqcup F_{\dim \Delta}.$$

where $F_i = \{\sigma \in F(\Delta) : \dim \sigma = i\}$.

Remark. A simplicial complex Δ remains unchanged if we replace $\mathcal{S}$ with $F(\Delta)$. So we can add an extra condition for $\mathcal{S}$: That every $\tau \subseteq \sigma$ is also in $\mathcal{S}$ for any $\sigma \in \mathcal{S}$. This means that a all the faces of a simplicial complex are determined by its maximal faces; its facets. This also gives us a recipe for the next definition.

Definition 1.3.5 (Abstract Simplicial Complex)**.** An abstract simplicial complex Δ on the vertex set $V = \Delta_0$ is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. The $i-$faces of Δ are the elements of Δ with $i+1$ elements. This way we define the dimension of both the faces of Δ and Δ as above. We define Δ_i as the set of all the $i-$faces of Δ.

Remark. For a simplicial complex Δ, the face poset $F(\Delta)$ is an abstract simplicial complex on the vertex set F_0. Conversely, if $\Delta \neq \varnothing$ is instead an abstract simplicial complex, we get a simplicial complex $T(\Delta)$ by replacing all $i-$faces of Δ by $i-$simplices, in a way they intersect according to their intersections in Δ.

This way, the empty space $\varnothing$ corresponds to the *irrelevant* (abstract) simplicial complex $\{\varnothing\}$. The empty abstract simplicial complex $\varnothing$ does not correspond to a simplicial complex.

Given an $i-$face $\sigma = \{v_{j_0}, \ldots, v_{j_i}\} \in \Delta$ such that $j_l \leq j_m$ for $l \leq m$, we shall denote it as $\sigma = [v_{j_0}, \ldots, v_{j_i}]$.

Definition 1.3.6 (Skeleton)**.** Let Δ be a simplicial complex. The simplicial complex given by all the faces of dimension $\leq i \in \mathbb{N}$ of Δ is called the $i-$skeleton of Δ.

Definition 1.3.7 (Euler Characteristic)**.** For a simplicial complex Δ, define the Euler characteristic of $\chi(\Delta)$ as $\chi(\Delta) = \rho - \iota - 1$ where ρ is the number of faces of even dimension and ι is the number of faces of odd dimension (counting the empty $-1-$face).

Definition 1.3.8 (Chain complex of a simplicial complex and its homology)**.** For an abstract simplicial complex Δ on a vertex set V(It's analogous for a simplicial complex using its face poset) and a commutative ring with unity R, we define, for every $-1 \leq i \leq \dim \Delta$, $C_i(\Delta; R) = R^{\Delta_i}$ formally, i.e. the free $R-$module generated by Δ_i, or more explicitly, the $R-$module of all the formal sums of the form $\sum_{\sigma \in \Delta_i} \alpha_\sigma \sigma$. We also define boundary maps

$$\partial_i : C_i(\Delta; R) \to C_{i-1}(\Delta; R)$$

by

$$\partial_i([v_{j_0},\dots,v_{j_i}]) = \sum_{l=0}^{i}(-1)^l([v_{j_0},\dots,\widehat{v_{j_l}},\dots,v_{j_i}])$$

where $\widehat{v_{j_l}}$ means that component is removed. If we have some matrix

$$(a_{\sigma\sigma'})_{\sigma\in\Delta_i,\sigma'\in\Delta_{i-1}}$$

representing ∂, this means that

$$\partial(\sigma) = \sum_{\sigma'\in\Delta_{i-1},\sigma'\subseteq\sigma} a_{\sigma\sigma'}\sigma'.$$

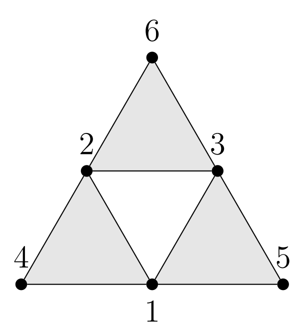
Since $\sigma - \sigma'$ consists in a single vertex k, we'll define $\operatorname{sgn}(k,\sigma)$ as $a_{\sigma\sigma'}$. This also means that for $\sigma = [v_{j_0},\dots,v_{j_i}]$ and $k = v_{j_l} \in \sigma$ we have $\operatorname{sgn}(k,\sigma) = (-1)^l$. We define the simplicial homology of Δ with coefficients in R as $\widetilde{H}_i(\Delta;R) = \ker\partial_i/\operatorname{im}\partial_{i+1}$. The elements of $Z_i(\Delta;R) = \ker\partial_i$ are called cycles, and the elements of $B_i(\Delta;R) = \operatorname{im}\partial_{i+1}$ are called boundaries.

When there is no risk of ambiguity we'll omit R in the notation, so $C_\bullet(\Delta;R)$ will be $C_\bullet(\Delta)$ and every module is an $R-$module.

Example 1.3.1. A simplicial complex (abstract or not) can be represented by a drawing. For example, consider the abstract simplicial complex Δ defined on the vertex set $\Delta_0 = \{1,\dots,6\}$ determined by the facets

$$\{1,2,4\},\{1,3,5\},\{2,3,6\}.$$

This simplicial complex has the drawing:



Computing its homology is not hard, but the calculations are a bit cumbersome. We have to compute the chain modules:

$$\begin{aligned}
C_2(\Delta; R) &= R[1,2,4] \oplus R[1,3,5] \oplus R[2,3,6], \\
C_1(\Delta; R) &= R[1,2] \oplus R[1,3] \oplus R[2,3] \oplus R[1,4] \oplus R[2,4] \oplus R[1,5] \oplus R[3,5] \\
&\quad \oplus R[2,6] \oplus R[3,6], \\
C_0(\Delta; R) &= \bigoplus_{i=1}^{6} R[i], \\
C_{-1}(\Delta; R) &= R\varnothing.
\end{aligned}$$

The boundary maps are also given by the following matrices (with respect to the given bases and considering the vectors of C_i as columns):

$$\partial_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \partial_1 = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\partial_0 = \mathbf{1}$ where $\mathbf{1}$ is the 1×6 vector with $1s$ as its entries. now we have to compute the kernel and image of the three maps. We'll then have that $\tilde{H}_2(\Delta; R) \cong \ker \partial_2, \tilde{H}_1(\Delta; R) \cong \ker \partial_1 / \operatorname{im} \partial_2$ and $\tilde{H}_0(\Delta; R) \cong \ker \partial_0 / \operatorname{im} \partial_1$. To solve the system

$$\partial_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we observe that, in particular, by computing the products with the first three rows, that $x = y = z = 0$, so $\tilde{H}_2(\Delta; R) \cong \ker \partial_2 = 0$. To compute $\operatorname{im} \partial_2$ we see that for $x, y, z \in R$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -x \\ x \\ -y \\ y \\ -z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so $\operatorname{im} \partial_2$ is generated by those three vectors which are clearly a basis for it.

Now we solve the system

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which in particular, implies that $x_8 = -x_9, x_6 = -x_7$ and $x_4 = -x_5$. We also get the relations

$$x_2 + x_3 - x_7 - x_9 = x_1 - x_3 - x_5 - x_8 = -(x_1 + x_2 + x_4 + x_6) = 0$$

which translate into

$$x_2 + x_3 + x_6 - x_9 = x_1 - x_3 + x_4 + x_9 = x_1 + x_2 + x_4 + x_6 = 0,$$

which are associated to the kernel of the 3×6 matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}$$

which can be reduced further by (carefully made) elementary row operations:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $x_1 = x_3 - x_4 - x_9, x_2 = -x_3 - x_6 + x_9$. Therefore any vector u in the kernel must have the form

$$u = \begin{bmatrix} x_3 - x_4 - x_9 \\ -x_3 - x_6 + x_9 \\ x_3 \\ x_4 \\ -x_4 \\ x_6 \\ -x_6 \\ -x_9 \\ x_9 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

These four vectors u_1, u_2, u_3, u_4 make a basis, so this submodule of R^9 is isomorphic to R^4. We also see that u_2, u_3 and $u_4 + u_1$ are in $\operatorname{im} \partial_2$, which means that $u_2 + \operatorname{im} \partial_2 = u_3 + \operatorname{im} \partial_2 = 0$ and $u_1 + \operatorname{im} \partial_2 = -u_4 + \operatorname{im} \partial_2$. So $\tilde{H}_1(\Delta; R)$ is generated by $u_1 + \operatorname{im} \partial_2$. Since no multiple of u_1 is in $\operatorname{im} \partial_2$, we get $\tilde{H}_1(\Delta; R) \cong R$.

Lastly, $\ker \partial_0 : R^6 \to R$ consists on all the vectors

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$, which means that

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ -x_1 - x_2 - x_3 - x_4 - x_5 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

so $\ker \partial_0$ is generated by those five vectors v_1, v_2, v_3, v_4, v_5. The image of ∂_1 is just the submodule of all vectors v of the form

$$v = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 - x_2 - x_4 - x_6 \\ x_1 - x_3 - x_5 - x_8 \\ x_2 + x_3 - x_7 - x_9 \\ x_4 + x_5 \\ x_6 + x_7 \\ x_8 + x_9 \end{bmatrix}$$

Each of the v_i's is in there, it can be checked making the following matrix products for each vector, so that we can see that $\tilde{H}_0(\Delta; R) = 0$:

$$\partial_1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

and lastly

$$\partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

The calculations above are very cumbersome, even when R is a field (in which case they're still simplified a lot because of theorems like rank-nullity and other properties of the dimension of finite dimensional vector spaces). But there are other tools to help us make these calculations in an easier way; two of them are the long exact sequence of homology and the Mayer-Vietoris sequence.

Proposition 1.3.2. The sequence

$$C_\bullet(\Delta; R) : 0 \leftarrow C_{-1}(\Delta; R) \overset{\partial_0}{\leftarrow} C_0(\Delta; R) \overset{\partial_1}{\leftarrow} \cdots$$

is indeed a chain complex.

Proof. It's very straighforward. Let $\sigma = [v_0, \dots, v_n] \in C_n(\Delta)$. Then

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma - v_i)$$

and

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) + \sum_{j>i} (-1)^{j-1} (-1)^i (\sigma - v_i - v_j) \\ &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) + \sum_{j<i} (-1)^{i-1} (-1)^j (\sigma - v_i - v_j) \\ &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) - \sum_{j<i} (-1)^i (-1)^j (\sigma - v_i - v_j) \\ &= 0 \end{aligned}$$

$\blacksquare$

This can be generalized a bit.

Definition 1.3.9 (Homology of chain complex)**.** For a chain complex

$$C : \cdots \xleftarrow{\partial_{i-1}} C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1} \xleftarrow{\partial_{i+2}} \cdots$$

we define the homology of C, denoted $H_i(C)$ as $\ker \partial_i / \operatorname{im} \partial_{i+1}$. When the chain complex comes from a simplicial complex, we'll denote $H_i(\Delta; R)$, if we take $C_{-1}(\Delta; R) = 0$ or $\tilde{H}_i(\Delta; R)$ if we consider $C_{-1}(\Delta; R) = R$. In the second case the notation C will be replaced by $\tilde{C}$ and $\tilde{H}$ will be called reduced homology. The elements of the set $Z_i(C) = \ker \partial_i$ will be called cycles and the elements of the set $B_i(C) = \operatorname{im} \partial_{i+1}$ will be called boundaries.

Definition 1.3.10. A homomorphism $\phi : C \to D$ of two chain complexes $(C, \partial), (D, \delta)$ is just a sequence of maps $\phi = (\phi_i : C_i \to D_i)_i$ such that the following diagram commutes (i.e. the rectangle in the diagram commutes for every i):

$$\begin{tikzcd}
\cdots & C_{i-1} \arrow[l, "\partial_{i-1}"'] \arrow[d, "\phi_{i-1}"'] & C_i \arrow[l, "\partial_i"'] \arrow[d, "\phi_i"] & \cdots \arrow[l, "\partial_{i+1}"'] \\
\cdots & D_{i-1} \arrow[l, "\delta_{i-1}"'] & D_i \arrow[l, "\delta_i"'] & \cdots \arrow[l, "\delta_{i+1}"']
\end{tikzcd}$$

Definition 1.3.11. A chain complex (resp. exact sequence) of chain complexes is just a sequence $(A_i, \phi_i)_{i \in \mathbb{Z}}$:

$$\cdots \xleftarrow{\phi_{i-1}} A_{i-1} \xleftarrow{\phi_i} A_i \xleftarrow{\phi_{i+1}} A_{i+1} \xleftarrow{\phi_{i+2}} \cdots$$

such that for every $i \in \mathbb{Z}$, A_i is a chain complex, ϕ_i is a homomorphism of chain complexes and the sequence:

$$\cdots \xleftarrow{\phi_{i-1,j}} A_{i-1,j} \xleftarrow{\phi_{i,j}} A_{i,j} \xleftarrow{\phi_{i+1,j}} A_{i+1,j} \xleftarrow{\phi_{i+2,j}} \cdots$$

is a chain complex (resp. exact) for every j. I In particular,

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of chain complexes if

$$0 \to A_j \to B_j \to C_j \to 0$$

is a short exact sequence of modules for every j.

Theorem 1.3.3 (Long exact sequence of homology)**.** If there is a short exact sequence of chain complexes:

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

for chain complexes $(A,\gamma),(B,\delta),(C,\epsilon)$then there is a long exact sequence:

$$\cdots \overset{\beta^*_{i+1}}{\to} H_{i+1}(C) \overset{\partial_{i+1}}{\to} H_i(A) \overset{\alpha^*_i}{\to} H_i(B) \overset{\beta^*_i}{\to} H_i(C) \overset{\partial_i}{\to} H_{i-1}(A) \overset{\alpha^*_{i-1}}{\to} \cdots$$

where the homomorphisms $\alpha^*_i, \beta^*_i, \partial_i$ are to be defined in the proof.

Proof. We have the following commutative mesh with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "\gamma_{i+1}"'] & B_{i+1} \arrow[r, "\beta_{i+1}"] \arrow[d, "\delta_{i+1}"'] & C_{i+1} \arrow[r] \arrow[d, "\epsilon_{i+1}"'] & 0 \\
0 \arrow[r] & A_i \arrow[r, "\alpha_i"] \arrow[d, "\gamma_i"'] & B_i \arrow[r, "\beta_i"] \arrow[d, "\delta_i"'] & C_i \arrow[r] \arrow[d, "\epsilon_i"'] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d] & B_{i-1} \arrow[r, "\beta_{i-1}"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

First we want to define the maps $\alpha^*, \beta^*, \partial$ of the sequence. To define α^*_i consider $\alpha_i : A_i \to B_i$. Since α is a chain complex homomorphism then $\alpha_i : A_i \to B_i$ maps $\ker \gamma_i$ into $\ker \delta_i$ and $\operatorname{im} \gamma_{i+1}$ into $\operatorname{im} \delta_{i+1}$. So first we can take $\alpha'_i = \alpha_i|_{\ker \gamma_i} : \ker \gamma_i \to \ker \delta_i$. Thus, there is a unique map $\alpha^*_i : H_i(A) \to H_i(B)$ such that the following diagram commutes:

$$\begin{tikzcd}
\ker \gamma_i \arrow[r, "\alpha'_i"] \arrow[d, "\pi_{A_i}"'] & \ker \delta_i \arrow[d, "\pi_{B_i}"] \\
H_i(A) \arrow[r, "\alpha^*_i"'] & H_i(B)
\end{tikzcd}$$

where π_{A_i}, π_{B_i} are the canonical maps. So that's how α^* and β^* are defined. About how ∂ is defined, we'll focus on the following part of the mesh:

$$\begin{tikzcd} & B_i \arrow[r, "\beta_i"] \arrow[d, "\delta_i"'] & C_i \arrow[r] \arrow[d, "\epsilon_i"'] & 0 \\ 0 \arrow[r] & A_{i-1} \arrow[r, "\alpha_{i-1}"] & B_{i-1} \arrow[r, "\beta_{i-1}"] & C_{i-1} \arrow[r] & 0 \end{tikzcd}$$

So, take $[z] \in H_i(C)$ for $z \in \ker \epsilon_i$ (a cycle) so that $\epsilon_i(z) = 0$. Since β_i is surjective, then there exists some $b \in B_i$ such that $z = \beta_i(b)$. But since $\epsilon_i\beta_i = \beta_{i-1}\delta_i$, then $\delta_i(b) \in \ker \beta_{i-1} = \operatorname{im} \alpha_{i-1}$. So there is $a \in A_{i-1}$ such that $\alpha_{i-1}(a) = \delta_i(b)$. But also $\alpha_{i-2}\gamma_{i-1} = \delta_{i-1}\alpha_{i-1}$, so since $\delta_{i-1}\alpha_{i-1}(a) = \delta_{i-1}\delta_i(b) = 0$, we get $\alpha_{i-2}\gamma_{i-1}(a) = 0$, and since α_{i-1} is injective this means $\gamma_{i-1}(a) = 0$ so $a \in \ker \gamma_{i-1}$ and $[a] \in H_{i-1}(A)$. So just define $\partial_i([z]) = [a]$. So we have just gone backwards through the mesh

$$\begin{array}{ccccc} & & B_i & \xrightarrow{\beta_i} & C_i \\ & & \Big\downarrow{\scriptstyle \delta_i} \quad b & \longleftarrow\!\shortmid & z \\ & & \qquad \Big\downarrow & & \\ A_{i-1} & \xrightarrow{\alpha_{i-1}} & B_{i-1} & & \\ a & \longleftarrow\!\shortmid & \delta_i(b) & & \end{array}$$

just to find a. This choice of a might a priori be dependent on the choice of the representative z of $[z]$ and on the choice of b, so we must prove these choices don't affect the homology class of a, by showing that ∂_i is a well defined map.

To do that, suppose $[z] = [z']$ so that $z - z' \in \operatorname{im} \epsilon_{i+1}$, and we'll do the process of finding a, a' for both of the representatives z, z'. Take $b, b' \in B$ such that $\beta_i(b) = z, \beta_i(b') = z'$, so $\beta_i(b - b') = z - z'$. Also, since $z - z' \in \operatorname{im} \epsilon_{i+1}$, take c such that $\epsilon_{i+1}(c) = z - z'$. Since β_{i+1} is surjective there exists $b'' \in B_{i+1}$ such that $\beta_{i+1}(b'') = c$, so, since the diagram

$$\begin{array}{ccc} B_{i+1} & \xrightarrow{\beta_{i+1}} & C_{i+1} \\ {\scriptstyle \delta_{i+1}}\downarrow & & \downarrow{\scriptstyle \epsilon_{i+1}} \\ B_i & \xrightarrow[\beta_i]{} & C_i \end{array}$$

commutes, we have $\epsilon_{i+1}\beta_{i+1}(b'') = \beta_i\delta_{i+1}(b'') = z - z'$. Thus

$$\beta_i(b - b' - \delta_{i+1}(b'')) = 0.$$

This means that there is $a'' \in A_i$ such that $\alpha_i(a'') = b - b' - \delta_{i+1}(b'')$. Now,

$$\delta_i(b - b' - \delta_{i+1}(b'')) = \delta_i(b) - \delta_i(b') = \delta_i\delta_{i+1}(b'') = \delta(b) - \delta(b').$$

If we take a, a' such that $\alpha_{i-1}(a) = \delta_i(b), \alpha_{i-1}(a') = \delta_i(b')$ then

$$\alpha_{i-1}(a - a') = \delta_i(b) - \delta_i(b') = \delta_i(b - b' - \delta_{i+1}(b'')) = \delta_i\alpha_i(a'').$$

So, since the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & B_i \\ {\scriptstyle \gamma_i}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ A_{i-1} & \xrightarrow[\alpha_{i-1}]{} & B_{i-1} \end{array}$$

commutes, we have

$$\alpha_{i-1}\gamma_i(a'') = \delta_i\alpha_i(a'') = \alpha_{i-1}(a - a'),$$

so since α_{i-1} is a monomorphism, we get get $\gamma_i(a'') = a - a'$. So $a - a'$ is a boundary (lies in the image of γ_i) and $[a - a'] = [a] - [a'] = 0$, thus $[a] = [a']$, therefore $\partial_i([z]) = \partial_i([w])$, so ∂_i is indeed a well defined function.

To check it's a homomorphism, note that if you take cycles $z, w \in C_i$, and b, b' such that $\beta_i(b) = z, \beta_i(b') = w$, and then a, a' such that $\alpha_{i-1}(a) = \delta_i(b), \alpha_{i-1}(a') = \delta_i(b')$ then $\alpha_{i-1}(a + a') = \delta_i(b + b')$ and $\beta_i(b + b') = z + w$. So $\partial([z + w]) = [a + a'] = [a] + [a'] = \partial([z]) + \partial([w])$.

Now we only have to prove that the new sequence is exact. Start with a cycle a such that $\alpha_i^*([a]) = 0$ so that $\alpha_i(a)$ is a boundary in B_i. So $\beta_i\alpha_i(a)$ is a boundary in C_i, therefore $[\beta_i \alpha_i(a)] = \beta_i^* \alpha_i^*([a]) = 0$. Conversely suppose that $[b]$ is such that $\beta_i^*([b]) = 0$, so that $c = \beta_i(b)$ is a boundary in C_i. So there is some $c' \in C_{i+1}$ such that $\epsilon_{i+1}(c') = c$. Also since β_{i+1} is surjective there is $b'' \in B_{i+1}$ such that $\beta_{i+1}(b'') = c'$. Set $b' = \delta_{i+1}(b'')$. Then $\beta_i(b') = \beta_i(b) = c$. Thus $\beta_i(b - b') = 0$, so there is some $a \in A_i$ such that $\alpha_i(a) = b - b'$. But b is a cycle and b' is a boundary, so since the diagram above commutes we have

$$\alpha_{i-1}\gamma_i(a) = \delta_i \alpha_i(a) = \delta_i(b - b') = 0,$$

but since α_{i-1} is injective, this means that $\gamma_i(a) = 0$, and a is a cycle. So $\alpha_i^*([a]) = [b - b'] = [b]$, therefore $\ker \beta_i^* = \operatorname{im} \alpha_i^*$.

Now, we need to prove that $\ker \partial_i = \operatorname{im} \beta_i^*$. Take a cycle z of C_i such that $\partial_i([z]) = 0$. This means that, for $b \in B_i$ such that $\beta_i(b) = z$ we have $0 = \delta_i(b)$, so b is a cycle in B_i and $\beta_i(b) = z$, which means that $\beta_i^*([b]) = [z]$. Conversely if we take a cycle b then $\partial_i(\beta_i^*([b])) = 0$ since $\delta_i(b) = 0$ and α_{i-1} is injective.

Lastly, to prove that $\ker \alpha_i^* = \operatorname{im} \partial_{i+1}$, suppose first that we take a cycle a such that $\alpha_i^*([a]) = 0$. Then $\alpha_i(a)$ is a boundary in B_i and there is $b \in B_{i+1}$ such that $\delta_{i+1}(b) = \alpha_i(a)$. Let $z = \beta_{i+1}(b)$. Then z is a cycle in C_{i+1} since $\epsilon_{i+1}(z) = \beta_i(\alpha_i(a)) = 0$. Also, by the construction of ∂ we have $\partial_{i+1}([z]) = [a]$. Conversely, if we take cycles $z \in C_{i+1}$ and $a \in A_{i+1}$ such that $\partial_{i+1}([z]) = [a]$ then $\alpha_i(a)$ is a boundary, thus $\alpha_i^* \partial_{i+1}([z]) = 0$. Therefore, the sequence is exact, which is the desired result. ■

We can learn two important things from this long and convoluted, but easy proof. First we'll talk about relative homology.

Definition 1.3.12 (Relative homology)**.** For a simplicial complex Δ and a subcomplex $A \subseteq \Delta$ (A subspace of Δ which is a simplicial complex with all its simplices being faces of Δ), define $C_i(\Delta, A) = C_i(\Delta)/C_i(A)$ and $\tilde{H}_i(\Delta, A) = H_i(C(\Delta, A))$.

This states implicitly that $C_i(\Delta, A)$ is a chain complex. We must wonder how are the boundary maps defined, but this is not a problem:

$$\delta_i : C_i(\Delta, A) \to C_{i-1}(\Delta, A)$$

is the only map making the following diagram commute

$$\begin{tikzcd} C_{i-1}(\Delta) \arrow[d, "\pi_{i-1}"'] & C_i(\Delta) \arrow[l, "\partial_i"'] \arrow[d, "\pi_i"] \\ C_{i-1}(\Delta, A) & C_i(\Delta, A) \arrow[l, "\delta_i"] \end{tikzcd}$$

This map is well defined since ∂_i sends $C_i(A)$ into $C_{i-1}(A)$. This can easily be made explicit: A basis for $C_i(\Delta, A)$ is given by all the faces $\sigma = [v_0, \dots, v_i] \in \Delta_i - A_i$, so

$$\delta_i(\sigma) = \sum_{j=0}^{i} u_{\sigma,j}(-1)^j(\sigma - \{v_j\}).$$

where $u_{\sigma,j} = 1$ iff $\sigma - \{v_j\} \notin A$, otherwise $u_{\sigma,j} = 0$. This can be written as follows:

$$\delta_i(\sigma) = \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin A}} \mathrm{sgn}(v,\sigma)(\sigma - \{v\}).$$

Proposition 1.3.4. There is a long exact sequence:

$$\cdots \overset{\beta^*_{i+1}}{\to} \tilde{H}_{i+1}(\Delta, A) \overset{\partial_{i+1}}{\to} \tilde{H}_i(A) \overset{\alpha^*_i}{\to} \tilde{H}_i(\Delta) \overset{\beta^*_i}{\to} \tilde{H}_i(\Delta, A) \overset{\partial_i}{\to} \tilde{H}_{i-1}(A) \overset{\alpha^*_{i-1}}{\to} \cdots$$

Proof. There is a short exact sequence

$$0 \to C(A) \overset{\alpha}{\to} C(\Delta) \overset{\beta}{\to} C(\Delta, A) \to 0$$

where α is induced by the direct image of the inclusion $A \hookrightarrow \Delta$ on its faces (So it becomes the inclusion $C(A) \hookrightarrow C(\Delta)$), and β is the canonical projection. ■

Proposition 1.3.5 (Mayer-Vietoris sequence)**.** For a simplicial complex X, and A, B subcomplexes such that $X = A \cup B$, there is a long exact sequence:

$$\cdots \overset{\partial_{i+1}}{\to} \tilde{H}_i(A\cap B) \overset{\alpha^*_i}{\to} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \overset{\beta^*_i}{\to} \tilde{H}_i(X) \overset{\partial_i}{\to} \tilde{H}_{i-1}(A\cap B) \overset{\alpha^*_{i-1}}{\to} \cdots$$

Proof. There is a natural short exact sequence:

$$0 \to C(A\cap B) \overset{\alpha}{\to} C(A) \oplus C(B) \overset{\beta}{\to} C(X) \to 0,$$

for $\alpha = (\iota_A, -\iota_B), \beta = \iota'_A + \iota'_B$ where $\iota_A, \iota_B, \iota'_A, \iota'_B$ are induced by the inclusions given by the following diagram:

$$\begin{tikzcd}
& X & \\
B \arrow[ur, "\iota'_J"] & & A \arrow[ul, "\iota'_I"'] \\
& A \cap B \arrow[ul, "\iota_J"] \arrow[ur, "\iota_I"'] &
\end{tikzcd}$$

$\blacksquare$

For the Mayer-Vietoris sequence, it's really easy to make the homomorphism $\partial : \tilde{H}_i(X) \to \tilde{H}_i(A \cap B)$ explicit. Take a cycle

$$z = \sum_{\sigma \in A_i} r_\sigma \sigma + \sum_{\sigma \in B_i - A_i} r_\sigma \sigma \in Z_i(X).$$

Since it's a cycle, its boundary is 0, and if we take

$$x = \sum_{\sigma \in A_i} r_\sigma \sigma \in C_i(A), y = \sum_{\sigma \in B_i - A_i} r_\sigma \sigma \in C_i(B),$$

we get

$$\partial(x) = -\partial(y),$$

so $\partial(x) \in C_{i-1}(A \cap B)$. So, just define $\partial([z]) = [\partial(x)]$, since $\alpha(\partial(x)) = (\partial(x), -\partial(x)) = (\partial(x), \partial(y))$.[i]

Proposition 1.3.6 (Euler Characteristic)**.** For a simplicial complex of dimension n,

$$\chi(\Delta) = \sum_{i=0}^{n} \operatorname{rank} H_i(\Delta, \mathbb{Z}).$$

By the fundamental theorem for finitely generated abelian groups every abelian group Q can be decomposed as

$$Q = \mathbb{Z}^n \oplus T$$

[i]Here we're abusing the notation by letting lots of ∂s have the same name, making it tacit that when we say $\partial([z])$ we're talking about $\partial_i : \tilde{H}_i(X) \to \tilde{H}_i(A \cap B)$, where the i is determined by the location of $[z]$, and when we say $\partial(x)$ we're talking about $\partial_i : C_i(A) \to C_{i-1}(A)$ where A, i are determined by the location of x.

where T is a torsion subgroup. As used in the previous proposition, the rank of the abelian group Q will be defined as n. Similarly, for a free $R-$module R^n over a ring R, its rank will be n. A similar definition can be given more generally over principal ideal domains.

Proof of 1.3.6. Consider a chain complex (C, ϕ) denote $H_i(C)$ by H_i and by Z_i, B_i the cycles and boundaries of C respectively.

Then, for each i, we have an exact sequence

$$0 \to B_i \to Z_i \to H_i \to 0,$$

and one

$$0 \to Z_i \to C_i \to B_{i-1} \to 0.$$

Then it's easy to check that

$$\operatorname{rank} Z_i = \operatorname{rank} B_i + \operatorname{rank} H_i$$

and

$$\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}.$$

So, $\operatorname{rank} C_i = \operatorname{rank} B_i + \operatorname{rank} H_i + \operatorname{rank} B_{i-1}$, which means that

$$\sum_i (-1)^i \operatorname{rank} C_i = \sum_i (-1)^i (\operatorname{rank} B_i + \operatorname{rank} B_{i-1}) + \sum_i (-1)^i \operatorname{rank} H_i.$$

Since $\sum_i (-1)^i (\operatorname{rank} B_i + \operatorname{rank} B_{i-1}) = 0$, we get

$$\sum_i (-1)^i \operatorname{rank} C_i = \sum_i (-1)^i \operatorname{rank} H_i.$$

Applying it to the chain complex $C = C(\Delta; \mathbb{Z})$, and since $\operatorname{rank} C_i = |\Delta_i|$ and $(-1)^i = 1$ iff i is even and -1 otherwise, we have the result. ■

Definition 1.3.13 (Chain homotopy)**.** We say a map $\alpha : C \to D$ are chain-null-homotopic if there is a sequence of maps $h_i : C_i \to D_{i+1}$ such that given the following diagram:

$$\begin{tikzcd}
\cdots & C_{i-1} \arrow[l, "\partial_{i-1}"'] \arrow[d, "\alpha_{i-1}"'] \arrow[dr, "h_{i-1}"] & C_i \arrow[l, "\partial_i"'] \arrow[d, "\alpha_i"] \arrow[dr, "h_i"] & C_{i+1} \arrow[l, "\partial_{i+1}"'] \arrow[d, "\alpha_{i+1}"] & \cdots \arrow[l, "\partial_{i+2}"'] \\
\cdots & D_{i-1} \arrow[l, "\delta_{i-1}"] & D_i \arrow[l, "\delta_i"] & D_{i+1} \arrow[l, "\delta_{i+1}"] & \cdots \arrow[l, "\delta_{i+2}"]
\end{tikzcd}$$

we have $h_{i-1}\partial_i + \delta_{i+1}h_i = \alpha_i$ for every i. We say two maps $\alpha, \beta : C \to D$ are chain-homotopic if $\alpha - \beta$ is chain-null-homotopic. The map h is called a chain homotopy, and we say $\alpha \simeq \beta$ or $\alpha \simeq_h \beta$ when the map h is relevant.

Proposition 1.3.7. If a map $\alpha : C \to D$ is chain-null-homotopic then the induced homomorphisms in homology are 0.

Proof. As in the proof of the long exact sequence of homology, α maps cycles into cycles and boundaries into boundaries so it induces maps $\alpha_i^* : H_i(C) \to H_i(D)$ by $\alpha_i^*([z]) = [\alpha_i(z)]$ for $z \in Z_i(C)$. Since α is chain-null-homotopic we have

$$\alpha_i(z) = h_{i-1}\partial_i(z) + \delta_{i+1}h_i(z) = \delta_{i+1}(h_i(z))$$

which is clearly a boundary of D so its homology class is 0. Therefore $\alpha_i^* = 0$ for every i. ■

Corollary 1.3.8. If $\alpha, \beta : C \to D$ are chain-homotopic then they induce the same homomorphisms in homology.

Definition 1.3.14. Given two chain complexes C, D, then we say they are chain-homotopically equivalent if there are maps $\alpha : C \to D$ and $\beta : D \to C$ such that $\beta\alpha \simeq 1_C$ and $\alpha\beta \simeq 1_D$.

Proposition 1.3.9. If two chain complexes C, D are chain-homotopically equivalent then they have the same homology.

Proof. Since there are maps $\alpha : C \to D$ and $\beta : D \to C$ such that $\beta\alpha \simeq 1_C$ and $\alpha\beta \simeq 1_D$ then $(\beta\alpha)_i^* = \beta_i^*\alpha_i^* : H_i(C) \to H_i(C)$ and $(\alpha\beta)_i^* = \alpha_i^*\beta_i^* : H_i(D) \to H_i(D)$ are identity maps, therefore α_i^*, β_i^* are isomorphisms for every i. ■

We'll use these results to give a convoluted proof of an elementary fact: That a simplex has trivial homology. This can be proven using the fact that a simplex is contractible, and that homology remains invariant under deformations. We haven't introduced deformations, but we can use the argument behind their validity to make it work here.

Example 1.3.10. Let Δ be the full simplex on $V = \{1, \ldots, n\}$, i.e. $\Delta = \wp(V)$. We'll show that the complexes $C(\Delta)$ and $C(1)$ are homotopically equivalent. The candidate map $c : C(1) \to C(\Delta)$ is just the induced map $1 \mapsto k$ for a fixed $k \in V$. This one induces an inclusion $C_0(1) \to C_0(\Delta)$, the identity map $C_{-1}(1) \to C_{-1}(\Delta)$ and the zero map everywhere else. For its homotopical inverse we'll take the map induced by sending every $x \in V$ into 1. This map induces the $C_{-1}(1) \leftarrow C_{-1}(\Delta)$, a kind of projection $C_0(1) \leftarrow C_0(\Delta)$ and the zero map everywhere else. By a composition of these two maps and a difference with the identity we have a new map $f : C(\Delta) \to C(\Delta)$, which fits into the following diagram:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_{-1}(\Delta) & \xleftarrow{\partial_0} & C_0(\Delta) & \xleftarrow{\partial_1} & C_1(\Delta) & \xleftarrow{\partial_2} & \cdots \\
 & & \downarrow f_{-1} & & \downarrow f_0 & & \downarrow f_1 & & \\
0 & \longleftarrow & C_{-1}(\Delta) & \xleftarrow[\partial_0]{} & C_0(\Delta) & \xleftarrow[\partial_1]{} & C_1(\Delta) & \xleftarrow[\partial_2]{} & \cdots
\end{array}$$

There, f_0 is just the map induced by $v \mapsto v - 1$ (This difference is formal) for every $v \in V$, $f_{-1} = 0$ and $f_i = 1_{C_i(\Delta)}$ for $i > 0$. We want to prove that f is chain-null-homotopic. A chain-null-homotopy h is given by

$$h_i(\sigma) = \sigma \cup \{1\}$$

if $1 \notin \sigma$ and 0 if $1 \in \sigma$, for $i \geq 0$ and $\sigma \in \Delta_i$. For $i = -1$ just define $h_{-1} = 0$. Now we prove that, for every i,

$$\partial_{i+1} h_i + h_{i-1} \partial_i = f_i.$$

First suppose that $i = -1$. Then since $h_{-1}(\varnothing) = 0$ we get that $\partial_0 h_{-1} = 0 = f_{-1}$. For $i = 0$, and a vertex $v \in V$ we have $h_0(v) = [1, v]$ and $\partial_1 h_0(v) = v - 1 = f_0(v)$. Since $h_{-1} = 0$ then the result holds. Now, for $i > 0$, consider $\sigma \in \Delta_i$. If $1 \in \sigma$ then $h_i(\sigma) = 0$, so that $\partial_{i+1} h_i(\sigma) = 0$. Furthermore $\partial_i(\sigma)$ is a linear combination of faces σ' such that exactly one of them does not contain 1. The coefficient in such σ' is $(-1)^0 = 1$, so $h_{i-1}(\partial_i(\sigma)) = h_{i-1}(\sigma') = \sigma' \cup \{1\} = \sigma$. If, otherwise, $1 \notin \sigma = \{v_0, \cdots, v_i\}$, then not a single such face σ' contains 1, so

$$h_{i-1}(\partial_i(\sigma)) = \sum_{l=0}^{i} (-1)^l ((\sigma - \{v_l\}) \cup \{1\})$$

Also $h_i(\sigma) = \sigma \cup \{1\}$ and

$$\begin{aligned}
\partial_{i+1}(h_i(\sigma)) &= (-1)^0\sigma + \sum_{l=1}^{i+1}(-1)^l((\sigma \cup \{1\}) - \{v_{l-1}\}) \\
&= \sigma + \sum_{l=1}^{i+1}(-1)^l((\sigma - \{v_{l-1}\}) \cup \{1\}) \\
&= \sigma + \sum_{l=0}^{i}(-1)^{l+1}((\sigma - \{v_l\}) \cup \{1\}) \\
&= \sigma - \sum_{l=0}^{i}(-1)^{l}((\sigma - \{v_l\}) \cup \{1\}) \\
&= \sigma - h_{i-1}(\partial_i(\sigma))
\end{aligned}$$

Therefore, as expected, we have

$$\partial_{i+1}h_i + h_{i-1}\partial_i = f_i,$$

for every i. The composition $g : C(1) \to C(1)$ in the remaining order is just the identity map, so there is nothing to prove here. This means the chain complex of a $n-$simplex is chain-homotopically-equivalent to the chain complex of a $0-$simplex, which has trivial homology. We're thus done.

Definition 1.3.15. We say a simplicial complex Δ is connected if and only if for any two vertices x, y, there is a $x-y-$path, i.e. a path in the $1-$skeleton of Δ connecting x and y.

This means that Δ is connected iff its $1-$skeleton is connected.

Proposition 1.3.11. If Δ is a connected simplicial complex over the vertex set $V = \{v_1, \cdots v_n\}$ then $\tilde{H}_0(\Delta) = 0$.

Proof. We need to prove that $\ker \partial_0 = \operatorname{im} \partial_1$. We already have that $\operatorname{im} \partial_1 \subseteq \ker \partial_0$. So let

$$s = \sum_{i=1}^{n} \alpha_i v_i \in \ker \partial_0$$

so that

$$\sum_{i=1}^{n} \alpha_i = 0$$

Fix a vertex $v \in V$ and take a $v - v_i-$path $\tau_i = \{e_{1i}, \ldots, e_{r_i i}\}$ for each i. Consider τ_i as the $1-$chain

$$\tau_i = \sum_{j=1}^{r_i} e_{ji}$$

so that by a telescopic property, $\partial_1(\tau_i) = v_i - v$. Let

$$t = \sum_{i=1}^{n} \alpha_i \tau_i.$$

Then

$$\partial_1(t) = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \alpha_i v = \sum_{i=1}^{n} \alpha_i v_i = s$$

since

$$\sum_{i=1}^{n} \alpha_i v = v \sum_{i=1}^{n} \alpha_i = 0.$$

Therefore, $\ker \partial_0 = \operatorname{im} \partial_1$ and $\tilde{H}_0(\Delta) = 0$. ■

With all this we can compute lots of homologies!

Proposition 1.3.12. Let $X = A \sqcup B$ for simplicial complexes A, B. Then, for all $i \geq 1$,

$$\tilde{H}_i(X) \cong \tilde{H}_i(A) \oplus \tilde{H}_i(B).$$

For $i = 0$ we have $\tilde{H}_i(X) \cong \tilde{H}_i(A) \oplus \tilde{H}_i(B) \oplus R$.

Proof. In the Mayer-Vietoris sequence for $X = A \cup B$, we have $A \cap B = \varnothing$, so for $i \geq 0$, $\tilde{H}_i(A \cap B) = 0$. But $\tilde{H}_{-1}(A \cap B) = R$. Therefore the sequence divides into the short exact sequences:

$$0 \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(X) \to 0$$

for each $i \geq 1$ and

$$0 \to \tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(X) \to R \to 0.$$

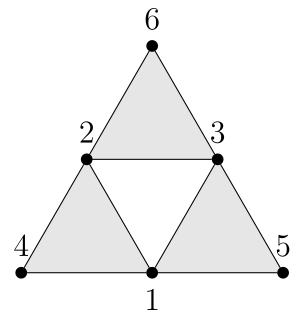
But a short exact sequence of this kind always splits: since $\tilde{H}_0(X) \to R$ is surjective finding a section is just finding a preimage of $1 \in R$. Therefore $\tilde{H}_0(X) \cong \tilde{H}_0(A) \oplus h_0(B) \oplus R$. ■

Proposition 1.3.13. Let $X = A \vee B$, for simplicial complexes A, B. This is, $A \cap B$ consists in a single vertex. Then $\tilde{H}_i(X) \cong \tilde{H}_i(A) \oplus \tilde{H}_i(B)$ for **all** i.

Proof. In the Mayer-Vietoris sequence for $A \cup B = X$, $\tilde{H}_i(A \cap B) = 0$ for all i. ■

The space $A \vee B$ is called the wedge sum of A, B.

Example 1.3.14. Remember the simplicial complex of example 1.3.1:



If A, B, C are the simplices $\{1,2,4\}, \{2,3,6\}, \{1,3,5\}$, then $X = (A \cup B) \cup C$. The Mayer-Vietoris sequence for this union is

$$\cdots \to \tilde{H}_i(1 \sqcup 2) \to \tilde{H}_i(A \cup B) \oplus \tilde{H}_i(C) \to \tilde{H}_i(X) \to \cdots .$$

Since $A \cup B$ is the wedge of A and B, and A, B, C are simplices we have, $\tilde{H}_i(A \cup B) = \tilde{H}_i(C) = 0$ for all i. Thus, the Mayer-Vietoris sequence separates in sequences

$$0 \to \tilde{H}_i(X) \to 0$$

for $i > 1$,

$$0 \to \tilde{H}_1(X) \to \tilde{H}_0(1 \sqcup 2) \to 0$$

and $0 \to \tilde{H}_0(X) \to 0$. For $i < 0$, everything is already 0.

Therefore, $\tilde{H}_i(X) = 0$ for $i \neq 1$ and $\tilde{H}_1(X) \cong \tilde{H}_0(1 \sqcup 2) = R$. With all our theory these calculations became a lot simpler.

We're interested in one more result from algebraic topology. It is a particular case of the Universal Coefficient Theorem for cohomology.

For a simplicial complex Δ and its associated chain complex $C_\bullet(\Delta; R)$ we can define its dual complex $C^\bullet$; its homology will be called the cohomology of Δ with coefficients in R.

Definition 1.3.16. For a simplicial complex Δ define $C^\bullet(\Delta; R)$ as

$$C_\bullet(\Delta; R)^* = \mathrm{Hom}(C_\bullet(\Delta; R), R).$$

The homology $\ker \partial^*_{i+1} / \operatorname{im} \partial^*_i$ of $C^\bullet(\Delta; R)$ will be called the cohomology of Δ with coefficients in R and denoted by $H^i(\Delta; R)$. The elements of the set $Z^i(\Delta; R) = \ker \partial^*_{i+1}$ will be called cocycles, and the elements of the set $B^i(\Delta, R) = \operatorname{im} \partial_i$ will be called coboundaries.

The boundary maps of the cochain complex $C^\bullet$, called coboundary maps can easily be made explicit: If we take $\phi \in C^j(\Delta; R)$ then, for

$$\tau = [v_0, \cdots, v_{n+1}] \in C_{j+1}(\Delta; R)$$

we have

$$\partial^*_{j+1}(\phi)(\tau) = \phi \circ \partial_{j+1}(\tau) = \sum_{i=0}^{j+1} (-1)^i \phi(\tau - \{v_i\}).$$

In particular, for the basis Δ_j of $C_j(\Delta; R)$ and $\sigma \in \Delta_j$ if we take the maps $f_\sigma : C_j(\Delta; R) \to R$ (These maps form a basis for $C^j(\Delta)$) determined by

$$f_\sigma(\sigma') = \delta_{\sigma\sigma'}$$

where δ is the Kronecker delta, then this means that

$$\partial^*_{j+1}(f_\sigma)(\tau) = \sum_{i=0}^{j+1} (-1)^i f_\sigma(\tau - \{v_i\})$$

so that

$$\partial^*_{j+1}(f_\sigma) = \sum_{\substack{v \notin \sigma \\ \sigma \cup v \in \Delta}} \mathrm{sgn}(v, \sigma \cup \{v\}) f_{\sigma \cup \{v\}}$$

since $f_\sigma(\tau - \{v\}) \neq 0$ if and only if $\tau - \{v\} = \sigma$, i.e. when $\tau = \sigma \cup \{v\}$, in which case $f_\sigma(\tau - \{v\}) = 1$; and since v_i is the $i-$th element of τ.

We were unexpectedly ready to prove the following weak version of the Universal Coefficient theorem for cohomology, with just the definition of cohomology (And a bit of linear algebra):

Theorem 1.3.15 (Universal Coefficient Theorem for Cohomology)**.** Let k be a field. Then there exists an isomorphism:

$$h : H^i(\Delta; k) \to \mathrm{Hom}_k(H_i(\Delta; k), k).$$

Proof. Take $[f] \in H^i(\Delta; k)$ for a cocycle f, i.e. a linear map $f : C_i(\Delta; k) \to k$ such that $\partial^*_{i+1}(f) = f\partial_{i+1} = 0$. We can restrict f to $Z_i(\Delta; k)$, and since $f\partial_{i+1} = 0$ this means that $f|_{B_i(\Delta;k)} = 0$. Thus, there exists a unique linear map $\tilde{f} : H_i(\Delta; k) \to k$ such that the following triangle commutes

$$\begin{tikzcd} Z_i(\Delta; k) \arrow[r, "f"] \arrow[d, "\pi"'] & k \\ H_i(\Delta; k) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where π is the canonical map. So just define $h([f]) = \tilde{f}$. Since this choice might depend on the choice of representative $[f]$ we must first prove that h is a well defined function. Let f, f' be cocycles such that $[f] = [f']$, i.e. $f - f' \in B^i(\Delta; k)$, so that there exists g such that $\partial^*_i(g) = g\partial_i = f - f'$. Then, for $x \in Z_i(\Delta; k)$ we have

$$(f - f')(x) = g\partial_i(x) = g(0) = 0.$$

So, while f and f' may be different maps, their restrictions to $Z_i(\Delta; k)$ are the same: The induced maps $\tilde{f}, \tilde{f}'$ are the same one. So this is indeed a well defined function. It's clearly surjective: If you take a linear map $q : H_i(\Delta; k) \to k$, then, for $\pi : C_i(\Delta, k) \to H_i(\Delta, k)$, the map $f = q\pi$ is a well defined linear functional on $C_i(\Delta; k)$ such that for all

$$\partial^*_{i+1} f(c) = f\partial_{i+1}(c) = q\pi\partial_{i+1}(c) = q(0) = 0,$$

thus f is a cocycle and $h([f]) = q$. Therefore, h is indeed surjective.

It's also a homomorphism. Indeed, if we take f, f' representatives of $[f], [f'] \in H^i(\Delta; k)$ then, for any cycle $z \in Z_i(\Delta; k)$ we have

$$\widetilde{f + f'}([z]) = (f + f')(z) = f(z) + f'(z) = \tilde{f}([z]) + \tilde{f}'([z]),$$

therefore $h([f] + [f']) = h([f]) + h([f'])$ and h is a homomorphism. Now we must prove that h is injective. Suppose f is such that $h([f]) = 0$. This means that $f(z) = 0$ for every $z \in Z_i(\Delta, k)$. We must build some $g \in C^{i-1}(\Delta; k)$ such that $\partial^*_i(g) = g\partial_i = f$. Since k is a field this is easy. Take a basis $\mathcal{B}$ of $\partial_i(Z_i(\Delta; k))$, a basis $\mathcal{E}$ of $B_{i-1}(\Delta; k)$ extending $\mathcal{B}$ and a basis $\mathcal{F}$ of $C_{i-1}(\Delta; k)$ extending $\mathcal{E}$. For every $b \in \mathcal{B}$ define $g(b) = 0$. For every $b = \partial_i(c) \in \mathcal{E} - \mathcal{B}$ define $g(b) = f(c)$. For every $b \in \mathcal{F} - \mathcal{E}$ define $g(b)$ arbitrarily. If we prove that the value of $g(b)$ does not depend on the choice of the preimage c then we can extend g by linearity to all $C_{i-1}(\Delta; k)$. So consider c, c' such that $\partial_i(c) = \partial_i(c') = b$. Then $\partial_i(c - c') = 0$ so $c - c'$ is a cycle and $f(c - c') = f(c) - f(c') = 0$, so $f(c) = f(c')$. Thus g becomes a well defined linear map $C_{i-1}(\Delta; k) \to k$, and by construction $f(x) = g(\partial_i(x))$ for every x, i.e. $f = \partial_i^* g$. This means that, f is a coboundary and $[f] = 0$. Therefore h is injective, and an isomorphism. ■

The strategy for the proof is purely linear algebra, but it's not a coincidence. We didn't use the fact that k is a field until we wanted to prove the injectiveness of h. We didn't even use much the ring structure of k as the codomain of the cocycles and maps $H_i(\Delta; k) \to k$. In the more general case, for a Principal Ideal Domain R and an $R-$module N, the proof shows $h : H^i(\Delta; N) \to \mathrm{Hom}_R(H_i(\Delta; R), N)$ is an epimorphism, so there is an exact sequence of the form

$$H^i(\Delta; N) \to \mathrm{Hom}_R(H_i(\Delta; R), N) \to 0.$$

The idea of the complete form of the Universal Coefficient Theorem is to discover the structure of the module $M = \ker h$, which fits into the following short exact sequence:

$$0 \to M \to H^i(\Delta; N) \to \mathrm{Hom}_R(H_i(\Delta; R), N) \to 0.$$

In our case, this module will be 0 thanks to the fact that every $k-$vector space V has a basis, and every linearly independent set of V is contained in a basis. In the Principal Ideal Domain case, we won't have a longer exact sequence because free resolutions are short enough (submodules of free modules are free). We'll talk about free resolutions soon.

This also means that when a simplicial complex has a finite number of simplices of each dimension, the homology and cohomology groups will be (non-naturally) isomorphic.

1.4 Homological Algebra

Here we'll talk about the fundamentals of homological algebra needed to give a proof of Hochster's formula. Once we do that, surprisingly, we won't need many of the results given here; using Hochster's formula and Mayer-Vietoris sequences we'll be able to prove many things in a more arithmetical and combinatorial way.

A chain complex

$$\mathcal{M}_\bullet : 0 \leftarrow M_0 \overset{\phi_1}{\leftarrow} M_1 \overset{\phi_2}{\leftarrow} M_2 \leftarrow \cdots$$

of $R-$modules is said to be a free resolution of an $R-$module M if $\mathcal{M}$ is exact everywhere except in the homological degree 0, and $M \cong \operatorname{coker} \phi_1 = M_0/\operatorname{im} \phi_1$. Analogously, $\mathcal{M}$ is a free resolution if the sequence

$$\mathcal{M}'_\bullet : 0 \leftarrow M \overset{\phi_0}{\leftarrow} M_0 \overset{\phi_1}{\leftarrow} M_1 \overset{\phi_2}{\leftarrow} M_2 \leftarrow \cdots$$

is exact everywhere, where ϕ_0 is the projection (since $M \cong \operatorname{coker} \phi_1$). We'll name both $\mathcal{M}$ and $\mathcal{M}'$ as $\mathcal{M}$ and use one or the other according to the context.

The following proposition can be generalized to projective resolutions, but since we'll be working over a polynomial ring we'll have enough with free ones.

Proposition 1.4.1. Any two free resolutions $(\mathcal{M}, \phi), (\mathcal{N}, \psi)$ of a $R-$module M are chain-homotopically equivalent.

Proof. We'll construct the maps $\alpha : \mathcal{M} \to \mathcal{N}, \beta : \mathcal{N} \to \mathcal{M}$ inductively. Then we'll similarly build the appropiate chain homotopies. First, we start with the following commutative tree with the directed paths exact:

$$\begin{array}{ccc} M_0 & & N_0 \\ & \phi_0 \searrow \quad \swarrow \psi_0 & \\ & M & \\ & \downarrow & \\ & 0 & \end{array}$$

Take a basis $\mathcal{B}$ of M_0. Then $\phi_0(\mathcal{B}) \subseteq M$, and for each $b \in \mathcal{B}_0$ there is $c \in N_0$ such that $\psi_0(c) = \phi_0(b)$. Make some choice c_b of such c and define $\alpha_0(b) = c_b$.

Now we have the following diagram

$$\begin{array}{ccccc} M_1 & & & & N_1 \\ \phi_1 \downarrow & & & & \downarrow \psi_1 \\ M_0 & & \xrightarrow{\alpha_0} & & N_0 \\ & \phi_0 \searrow & & \swarrow \psi_0 & \\ & & M & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

and by exactness of the free resolutions (After augmenting them with M), the boundaries in M_0 are sent into boundaries in N_0 by α_0.

Inductively, we want to build α_i with the assumptions that we have been able to build $\alpha_{i-1}, \alpha_{i-2}$ (For $i = 1$ we say $\alpha_{-1} = 1_M$) making the diagram

$$\begin{array}{ccc} M_{i-1} & \xrightarrow{\alpha_{i-1}} & N_{i-1} \\ \phi_{i-1} \downarrow & & \downarrow \psi_{i-1} \\ M_{i-2} & \xrightarrow[\alpha_{i-2}]{} & N_{i-2} \end{array}$$

commute. So, as before, we take a basis $\mathcal{B}$ of M_i, which is sent into M_{i-1} by ϕ_i and then into N_{i-1} by α_{i-1}. Because of the diagram above, adn since the images of $\mathcal{B}$ in M_{i-1} are mapped onto 0 in M_{i-2} by ϕ_{i-1}, $\alpha_{i-1}(\phi_i(\mathcal{B}))$ is also mapped onto 0 in N_{i-2} by ψ_{i-1}, so by exactness $\mathcal{U} = \alpha_{i-1}(\phi_i(\mathcal{B})) \subseteq \operatorname{im} \psi_i$. Make some choice of preimages of $\mathcal{U}$ in N_i and define α_i by mapping $\mathcal{B}$ into this choice. By construction, α_i makes the diagram

$$\begin{array}{ccc} M_i & \xrightarrow{\alpha_i} & N_i \\ \phi_i \downarrow & & \downarrow \psi_i \\ M_{i-1} & \xrightarrow[\alpha_{i-1}]{} & N_{i-1} \end{array}$$

commute. Define $\beta : \mathcal{N} \to \mathcal{M}$ the same way. Now, to prove that $\beta\alpha \simeq 1_{\mathcal{M}}$ we take $\gamma : \mathcal{M} \to \mathcal{M} = \beta\alpha - 1_{\mathcal{M}}$ and we'll prove γ is chain-null-homotopic. First, $\gamma_{-1} = 0$, so we can define, for $i \leq -1$, $h_i : M_i \to M_{i+1}$ as 0.

Now, γ_0 makes the diagram:

$$\begin{array}{ccc} M_0 & \xrightarrow{\gamma_0} & M_0 \\ \phi_0 \downarrow & & \downarrow \phi_0 \\ M & \xrightarrow[0]{} & M \end{array}$$

commute. Since $0\phi_i = \phi_i\gamma_0 = 0$, then $\operatorname{im}\gamma_0 \subseteq \operatorname{im}\phi_1$. Choose a basis $\mathcal{B}$ for M_0, and make a choice of preimages in M_1 of $\gamma_0(\mathcal{B})$. Define $h_0 : M_0 \to M_1$ to map $\mathcal{B}$ into this choice. Then $\phi_1 h_0 = \gamma_0$ and since $h_{-1} = 0$ then $\gamma_0 = \phi_1 h_0 + h_{-1}\phi_0$.

So, now suppose we've been able to define the part h_j of the chain-homotopy for $-1 \leq j < i$ and we want to define h_i.

In the diagram

$$\begin{tikzcd}
M_i \arrow[r, "\gamma_i"] \arrow[d, "\phi_i"'] & M_i \arrow[d, "\phi_i"] \\
M_{i-1} \arrow[r, "\gamma_{i-1}"] \arrow[ur, "h_{i-1}"] \arrow[d, "\phi_{i-1}"'] & M_{i-1} \arrow[d, "\phi_{i-1}"] \\
M_{i-2} \arrow[r, "\gamma_{i-2}"'] \arrow[ur, "h_{i-2}"] & M_{i-2}
\end{tikzcd}$$

we have $\gamma_{i-1} = \phi_i h_{i-1} + h_{i-2}\phi_{i-1}$. We want to define $h_i : M_i \to M_{i+1}$ such that $\gamma_i = h_{i-1}\phi_i + \phi_{i+1}h_i$, or which is the same, that $\gamma_i - h_{i-1}\phi_i = \phi_{i+1}h_i$.

Note that

$$\phi_i\gamma_i = \gamma_{i-1}\phi_i = \phi_i h_{i-1}\phi_i + h_{i-2}\phi_{i-1}\phi_i = \phi_i h_{i-1}\phi_i$$

so

$$\phi_i(\gamma_i - h_{i-1}\phi_i) = 0.$$

Therefore, by exactness $\operatorname{im}(\gamma_i - h_{i-1}\phi_i) \subseteq \operatorname{im}\phi_{i+1}$. So, taking a basis $\mathcal{B}$ of M_i, we can make a choice of preimages in M_{i+1} of $(\gamma_i - h_{i-1}\phi_i)(\mathcal{B})$, and define h_i to map $\mathcal{B}$ into this choice. This assures us that $\phi_{i+1}h_i = \gamma_i - h_{i-1}\phi_i$, as desired. Therefore $\gamma = \beta\alpha - 1_{\mathcal{M}}$ is chain-null-homotopic and $\beta\alpha \simeq 1_{\mathcal{M}}$. Analogously $\alpha\beta \simeq 1_{\mathcal{N}}$. Therefore $\mathcal{M} \simeq \mathcal{N}$. ■

Now we're interested in giving a definition of Tor suitable to our purposes. This is, our ring is $S = k[x_1, \ldots, x_n$, every module is a $S-$module, and we will only be working with free resolutions instead of projective ones. This, as many things in the previous sections, can be generalized a lot, nonetheless, and many proofs are very similar. For the sake of keeping it as simple as possible, we won't generalize it. In the previous sections we proved that $\otimes$ is a right-exact functor, i.e. a sequence

$$L \to M \to N \to 0$$

of $S-$modules gives rise to an induced sequence

$$L \otimes J \to M \otimes J \to N \otimes J \to 0$$

for any $S-$module J. We want to find some functorial construction of $S-$modules $F_i(-, J)$ such that we have a long exact sequence

$$\begin{aligned} \cdots \to F_i(L, J) \to F_i(M, J) \to F_i(N, J) \to \cdots \\ \to F_1(N, J) \to L \otimes J \to M \otimes J \to N \otimes J \to 0. \end{aligned}$$

Definition 1.4.1 (Tor)**.** We define $\operatorname{Tor}(M, J) = H_i(\mathcal{M} \otimes J)$ where $\mathcal{M}$ is a free resolution of M.

Proposition 1.4.2. The definition above makes sense. This is, $\operatorname{Tor}(M, N)$ does not depend on the choice of the free resolution.

Proof. Consider free resolutions $\mathcal{M}, \mathcal{N}$ of M. Then $\mathcal{M}, \mathcal{N}$ are homotopically equivalent, i.e. there exist $\alpha : \mathcal{M} \to \mathcal{N}$ and $\beta : \mathcal{N} \to \mathcal{M}$ such that $\beta\alpha \simeq 1_{\mathcal{M}}, \alpha\beta \simeq 1_{\mathcal{N}}$. Let $h : M_i \to M_{i+1}$ the homotopy between $\beta\alpha$ and $1_{\mathcal{M}}$.

Now, when tensoring with J we have the maps $\beta\alpha \otimes 1_J$ and $h \otimes 1_J$. We shall prove that $h \otimes 1_J$ is a chain-homotopy between $\beta\alpha \otimes 1_J$ and $1_{\mathcal{M}} \otimes 1_J = 1_{\mathcal{M}\otimes J}$. This is purely arithmetical.

Check that for an element of the form $m \otimes j \in M_i \otimes J$ we have

$$(\beta_i\alpha_i \otimes 1_J - 1_{\mathcal{M}} \otimes 1_J)(m \otimes j) = \beta_i\alpha_i(m) \otimes j - m \otimes j = (\beta_i\alpha_i(m) - m) \otimes j.$$

so for any i $\beta_i\alpha_i \otimes 1_J - 1_{\mathcal{M}} \otimes 1_J = (\beta_i\alpha_i - 1_{\mathcal{M}}) \otimes 1_J$. Since $\beta_i\alpha - 1_{\mathcal{M}} = h_{i-1}\phi_i + \phi_{i+1}h_i$ and as above,

$$(h_{i-1}\phi_i + \phi_{i+1}h_i) \otimes 1_J = (h_{i-1} \otimes 1_J)(\phi_i \otimes 1_J) + (\phi_{i+1} \otimes 1_J)(h_i \otimes 1_J),$$

we have that

$$(\beta_i\alpha_i - 1_{\mathcal{M}}) \otimes 1_J = (h_{i-1} \otimes 1_J)(\phi_i \otimes 1_J) + (\phi_{i+1} \otimes 1_J)(h_i \otimes 1_J).$$

This means that $(\beta_i\alpha_i) \otimes 1_J \simeq 1_{\mathcal{M}} \otimes 1_J$, and by similar means we get that $\alpha_i\beta_i \otimes 1_J \simeq 1_{\mathcal{N}} \otimes 1_J$, therefore $\alpha \otimes 1_J : \mathcal{M} \otimes J \to \mathcal{N} \otimes J$ is a chain-homotopy equivalence, so $\mathcal{M} \otimes J$ and $\mathcal{N} \otimes J$ have the same homology. ■

Now we observe that $\operatorname{Tor}(M, J)$ can also be computed by using a free resolution of J. An elementary proof can be found in [8].

Proposition 1.4.3 (Theorem 6.32, [8])**.** Let $\mathcal{J}$ be a free resolution of J. Then $\operatorname{Tor}(M, J) \cong H_i(M \otimes \mathcal{J})$.

1.5 Hypergraphs and Their Ideals

The ideals described here will be the ones we'll be mostly working on. There is a lot of theory about these ideals given in [10] or [4], but most of it won't be needed for our work here.

Definition 1.5.1. A hypergraph G is a pair (V, E) where $V = V(G)$ is a set (called vertex set) and $E = E(G)$ is a set of nonempty subsets of V (called edge set).

Note that when every $e \in E$ satisfies $|e| = 2$, then G is a graph. When for any $e \in E$ and $f \subseteq e$ we have $f \in E$ then G is a simplicial complex.

Definition 1.5.2. Let k be a field and G be a hypergraph on a vertex set $V = \{v_1, \ldots, v_n\}$. For a set of indeterminates (Which may as well be V) $\{x_1, \ldots, x_n\}$ define $x(v_i) = x_i$ for all i. The ideal

$$I = I_G = \left(\prod_{v \in e} x(v) \mid e \in E \right) \subseteq k[x_1, \ldots, x_n]$$

is called the edge ideal of G.

If G is a hypergraph with vertex set $\{1, \ldots, n\}$ this can be rewritten as

$$I_G = \left(\prod_{i \in e} x_i \mid e \in E \right).$$

Furthermore, if G is a graph this can be rewritten further as

$$I_G = (x_i x_j \mid \{i, j\} \in E).$$

For a graph we'll use the notation vw for the edge $\{v, w\}$. A subgraph H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an induced subgraph if $E(H) = \{e \in E(G) \mid e \subseteq V(H)\}$; this definition can be generalized to define induced subhypergraph or induced simplicial subcomplex.

We say a cycle C in a graph G is a subgraph such that if $V(C) = \{v_1, \ldots, v_r\}$ then $E(C) = \{v_1v_2, v_2v_3, \ldots, v_{r-1}v_r, v_1v_r\}$. We also define the cycle and path C_n, P_n: C_n is the graph with n vertices which is an induced cycle in itself, and P_n is the graph resulting from C_n after removing an edge. A path in a graph G is said to be a subgraph isomorphic to P_n for some n. A graph G is said to be connected if between any two vertices of G there is a path containing both. A graph without cycles is called a forest. A connected graph without cycles is called a tree.

Chapter 2

Hochster's Formula

Here we'll talk about Hochster's formula. This formula shows that the Betti numbers of a monomial ideal can be computed by instead computing the homology of a family of simplicial complexes. Its proof is very convoluted, but it's fair enough, since it allows us to avoid giving even more convoluted proofs of many other results. It's fundamental for this work.

2.1 Minimal Free Resolutions

Our first ingredient for computing the Betti numbers of an ideal is a minimal free resolution of it.

Definition 2.1.1 (Minimal free resolution)**.** Let $S = k[x_1, \dots, x_n]$ be the polynomial ring in n variables,

$$\mathfrak{m} = (x_1, \dots, x_n),$$

M a $S-$module, and $\mathcal{M} = (M_i, \phi_i)$ a free resolution of M. We say that $\mathcal{M}$ is minimal if $\operatorname{im} \phi_i \subseteq \mathfrak{m} M_{i-1}$ for all $i \geq 1$. The same definition will do when $(S, \mathfrak{m})$ is instead a local ring.

Remark. Equivalently, $\mathcal{M}$ is minimal iff $\mathcal{M} \otimes_S k$ is a complex with null morphisms, where $k = S/\mathfrak{m}$.

Lemma 2.1.1. Let M be a $S-$module and $\mathfrak{m}$ as before. Then there is an isomorphism

$$\psi : M \otimes_S k \to M/\mathfrak{m}M.$$

Proof. Consider the exact sequence:

$$0 \to \mathfrak{m} \to S \to k \to 0,$$

where the two nonzero morphisms are, from left to right, the inclusion and canonical projection. By the exactness of the tensor product the induced sequence:

$$\mathfrak{m} \otimes_S M \to S \otimes_S M \to k \otimes_S M \to 0$$

is exact. Furthermore, $S \otimes M \cong M$ by the isomorphism $m \mapsto 1 \otimes m$. By this isomorphism, the submodule $\mathfrak{m}M$ is given by $\mathfrak{m} \otimes_S M$, therefore

$$M/\mathfrak{m}M \cong S \otimes_S M/\mathfrak{m}(S \otimes_S M) \cong k \otimes_S M.$$

■

Proof of the Remark. By the lemma, for every i,

$$M_i/\mathfrak{m}M_i \cong k \otimes_S M_i$$

by the map $m + \mathfrak{m}M_i \mapsto 1 \otimes m$, with inverse $r \otimes m \mapsto rm$ for $r \in S$. So, $\mathcal{M}$ is minimal if for every i, $\phi_i \otimes 1 = 0$ iff, for every i, m, $\phi_i(m) \otimes 1 = 0$, iff for every i, m, $\phi_i(m) + \mathfrak{m}M_{i-1} = 0$, iff for every i, m, $\phi_i(m) \in \mathfrak{m}M_{i-1}$, iff for every i, $\operatorname{im} \phi_i \subseteq \mathfrak{m}M_{i-1}$. ■

This means that if all the free modules of the free resolution are finitely generated (as in Hilbert's Syzygy Theorem), any matrix representing some ϕ_i has all its entries in M.

The following about free resolutions can generalized to projective resolutions over any commutative ring with unity.

Now, suppose that everything is $\mathbb{N}^n-$graded, so that a free graded module M of finite rank has a direct sum decomposition

$$M = S(-\mathbf{a}_1) \oplus \cdots \oplus S(-\mathbf{a}_r)$$

for some $\mathbf{a}_i \in \mathbb{N}^n$, with grading given by $S(-\mathbf{a}_i)_\mathbf{s} \cong S_{\mathbf{s}-\mathbf{a}_i}$ for $\mathbf{s} \in \mathbb{N}^n$ such that $\mathbf{s} - \mathbf{a}_i \in \mathbb{N}^n$. Also a graded free resolution $\mathcal{M}$ will be a free resolution given by free graded modules with boundary maps of degree 0, i.e. $\phi(M_\mathbf{s}) = \phi(M)_\mathbf{s}$ for a boundary map ϕ of $\mathcal{M}$ and a module M in the resolution.

Definition 2.1.2. A monomial matrix is a matrix $M \in M_{n\times n}(k)$ with labels $\mathbf{a}_p, \mathbf{a}_q$ in its pth column and qth row, and entries λ_{qp} such that $\lambda_{qp} = 0$ unless $\mathbf{a}_p - \mathbf{a}_q \in \mathbb{N}^n$.

The previously stated condition is equivalent to $\frac{\mathbf{x}^{\mathbf{a}_p}}{\mathbf{x}^{\mathbf{a}_q}}$ being a well-defined polynomial or $\mathbf{x}^{\mathbf{a}_q} \mid \mathbf{x}^{\mathbf{a}_p}$.

Remark. The $S-$homomorphism

$$\bigoplus_p S(-\mathbf{a}_p) \to \bigoplus_q S(-\mathbf{a}_q)$$

determined by the monomial matrix $(\lambda_{qp})_{q,p}$ is just given by matricial product with the matrix $(\lambda_{qp}\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q})_{q,p}$.

Definition 2.1.3. A monomial matrix is minimal if $\lambda_{qp} = 0$ for $\mathbf{a}_p = \mathbf{a}_q$.

Remark. Given a free resolution of M given by minimal monomial matrices, then the free resolution is in fact minimal, since all the entries of the matrices are in $\mathfrak{m}$. Conversely, if a minimal free resolution has a representation of its morphisms by monomial matrices, they must be minimal, because all of its entries must lie in $\mathfrak{m}$.

In a free graded module $\bigoplus_p S(-\mathbf{a}_p)$, each $\mathbf{a}_p$ may appear more than once, in such case we can instead use the notation

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{\mathbf{a}}}.$$

In a minimal free resolution (M_i, ϕ_i) of a module M, with

$$M_i = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$$

each $\beta_{i,\mathbf{a}}$ is called the i^{th} Betti number of M in degree $\mathbf{a}$.

We can also define the Betti numbers by computing the derived functors $\mathrm{Tor}^S(-,k)$ of $-\otimes_S k$[(i)]. If we do so, it'll follow that the Betti numbers can, in theory, be computed without having a *minimal* free resolution, so that we can be content with any free resolution.

Remark. In the graded case, we have an isomorphism

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{\mathbf{a}}} \otimes k \cong \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{\mathbf{a}}}.$$

[(i)] The functors Tor are computed by taking any projective resolution, in particular, free ones, tensoring with k and computing homology.

Lemma 2.1.2. The i^{th} Betti number of an $\mathbb{N}^n-$graded module M in degree $\mathbf{a}$ equals the vector space dimension $\dim_k \operatorname{Tor}_i^S(k, M)_{\mathbf{a}}$.

Proof. Take a minimal free resolution

$$\mathcal{M} : 0 \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i+1,\mathbf{a}}} \leftarrow \cdots \leftarrow 0.$$

When tensoring with k, we get the complex

$$\mathcal{M}\otimes k : 0 \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{i+1,\mathbf{a}}} \leftarrow \cdots \leftarrow 0,$$

with all maps zero. Thus,

$$\tilde{H}_i(\mathcal{M}\otimes k) \cong \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$$

and

$$\beta_{i,\mathbf{a}} = \dim_k \operatorname{Tor}_i^S(k, M)_{\mathbf{a}}.$$

■

2.2 Hochster's Formula

Here every ideal is supposed to be monomial, i.e. generated by monomials.

Definition 2.2.1 (Koszul Complex)**.** Consider the simplex Δ^{n-1} consisting on all the subsets of $\{x_1, \ldots, x_n\}$. In every matrix $(a_{ij})_{i,j}$ of the exact sequence $\tilde{C}_\bullet(\Delta^n, S)$, consider it as a monomial matrix, with labels in rows and columns given by χ_σ for faces σ generating the free summands of $S^{\binom{n}{i}}$ in homological degree $i+1$. Then, renumber the homological degrees so that the empty face $\varnothing$ is in homological degree 0. The resulting complex $K_\bullet$ is called the Koszul complex.

This is just done by taking the reduced chain complex of Δ^n and adjusting the morphisms so that they become graded.

Proposition 2.2.1. The Koszul complex is a minimal free resolution of k.

Proof. Since the boundary maps of the Koszul complex are given by considering the matrices of the boundary maps of $\tilde{C}_\bullet(\Delta^n, S)$ as monomial matrices, then every one of the maps in the Koszul complex are monomial. Also since every entry of every one of these matrices is zero unless the face associated to the row is strictly contained in the face associated to the column, the maps of the Koszul complex are in fact minimal.

To prove the Koszul complex is a resolution, we must prove it's exact everywhere except in homological degree 0. So, restricting the Koszul complex to some degree $\mathbf{d} \in \mathbb{N}^n$, we get a subcomplex of $k-$vector spaces, which we want to prove is (isomorphic to) the reduced complex $\tilde{C}(\Delta^{\dim \mathbf{d}}, k)$ where $\dim \mathbf{d}$ is just the number of variables of $\mathbf{d}$ minus one. For degree $\mathbf{d} = 0$ we get the complex $0 \to k \to 0$, which is not exact. Otherwise, since a summand of each homological degree of the Koszul complex is $\neq 0$ in degree $\mathbf{d}$ iff it's has the form $S(-\mathbf{d}') = S\mathbf{x}^{\mathbf{d}'}$ for $\mathbf{d}' \leq \mathbf{d}$, we get the complex $0 \leftarrow k \leftarrow k^s \leftarrow k^{\binom{s}{2}} \leftarrow \cdots \leftarrow k^{\binom{s}{s-2}} \leftarrow k^s \leftarrow k \leftarrow 0$, where, in each homological degree i, $k^{\binom{s}{i}}$ is considered as the direct sum

$$\bigoplus_{\mathbf{d}' \leq \mathbf{d}, \dim \mathbf{d}' = i-1} k\mathbf{x}^{\mathbf{d}}$$

and $s = \dim \mathbf{d} + 1$. And given that,

$$\partial(\mathbf{x}^{\mathbf{d}} e_{\mathbf{d}'}) = \sum_{\mathbf{d}'' \leq \mathbf{d}', \dim \mathbf{d}'' = i-2} \alpha_{\mathbf{d}'\mathbf{d}''} \mathbf{x}^{\mathbf{d}} e_{\mathbf{d}''}$$

where $\alpha_{\mathbf{d}'\mathbf{d}''}$ is the $\mathbf{d}'\mathbf{d}''$ entry of the matrix of the map ∂ in $\tilde{C}(\Delta^{\dim \mathbf{d}}, k)$. This means than when we restrict to degree $\mathbf{d}$ in $K_\bullet$, the resulting map is just the boundary map of the reduced chain complex of $\Delta^{\dim \mathbf{d}}$ with coefficients in k, which is what we expected.

Then, since every non-irrelevant simplex is contractible, this complex has null homology for $\mathbf{d} \neq 0$, from which it follows that $K_\bullet$ also has null homology except in the homology degree 0, on which the homology is k. It follows that $K_\bullet$ is a free resolution of k. ■

Definition 2.2.2 (Upper Koszul Complex)**.** For a monomial ideal I and a degree $\mathbf{b} \in \mathbb{N}^n$, define

$$K^{\mathbf{b}}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{b}-\tau} \in I\}$$

to be the (upper) Koszul simplicial complex of I in degree $\mathbf{b}$.

Theorem 2.2.2 (Hochster's Formula)**.** Given a vector $\mathbf{b} \in \mathbb{N}^n$, the Betti numbers of I in degree $\mathbf{b}$ can be expressed as

$$\beta_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I); k).$$

From now on, when we talk about the homology of a simplicial complex X, it will be the homology over k, unless stated otherwise.

Proof. The proof is similar to the proof of the minimality of the Koszul complex as a resolution of k. First, to compute $\beta_{i,\mathbf{b}}(I)$, we can do it by taking a free resolution of I, tensoring with k and computing homologies. But we can also compute it by taking a minimal free resolution of k, i.e. the Koszul complex, tensoring it with I and computing homologies. We'll do that.

Let $K_\bullet$ be the Koszul complex and take the chain complex $K_\bullet(I) = K_\bullet \otimes_S I$. We'll restrict the complex to the degree $\mathbf{b}$, then we'll show that $K_\bullet(I)_\mathbf{b}$ is basically the reduced chain complex of $K^{\mathbf{b}}(I)$ over k. Remember that $(K_\bullet)_\mathbf{b}$ is the chain complex of a simplex $\Delta = \Delta^{\dim \mathbf{b}}$ with $\dim \mathbf{b} + 1$ vertices, over k. So, since $K_\bullet(I)_\mathbf{b}$ is a subcomplex of $(K_\bullet)_\mathbf{b}$, it will be the chain complex of a subcomplex of Δ. So we only have to see which faces of Δ contribute to nonzero vectors in $K_\bullet(I)_\mathbf{b}$.

Taking a vector $\tau \in \{0,1\}^n$, the summand of $K_\bullet$ associated to τ is a cyclic $S-$module $S(-\tau)$. Thus $I \otimes_S S(-\tau)$ is $I(-\tau)$, and since

$$I(-\tau)_\mathbf{b} = I_{\mathbf{b}-\tau}$$

, then $I(-\tau)_\mathbf{b} \neq 0$ if and only if $I_{\mathbf{b}-\tau}$ is nonzero i.e. $\mathbf{x}^{\mathbf{b}-\tau} \in I$.

Therefore, the faces of the subcomplex of Δ having as chain complex $K_\bullet(I)_\mathbf{b}$ are precisely the ones in $K^{\mathbf{b}}(I)$. We're done. ■

We can define another simplicial complex:

$$K_\mathbf{b}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau} \notin I\},$$

and an interesting result, which will be presented in the section 2.4, is that the dimensions in its homology (In fact, the cohomology, but because of the universal coefficient theorem they're the same) also give the Betti numbers of I. More precisely:

$$\beta_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{n-i-2}(K_\mathbf{b}(I)).$$

2.3 Properties of $K^{\mathbf{b}}(I)$.

The operator $K^{\mathbf{b}}$ has some nice properties with respect to the order structure of the monomial ideals S. It's known that the monomial ideals of S form a lattice, ordered by inclusion, with $+$ and $\cap$ being the lattice operations. It will be proven that $K^{\mathbf{b}}$ preserves this structure.

Here $I = (m_1, \ldots, m_r)$, $J = (m'_1, \ldots, m'_s)$ are monomial ideals of S.

Proposition 2.3.1. $K^{\mathbf{b}}(I + J) = K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(J)$.

Proof. We have

$$\begin{aligned}
K^{\mathbf{b}}(I + J) &= \{\tau \in \{0,1\}^n : \mathbf{x}^{b-\tau} \in I + J\} \\
&= \{\tau \in \{0,1\}^n : (\exists i \in [r]); m_i \mid \mathbf{x}^{b-\tau} \text{ o } (\exists j \in [s]); m'_j \mid \mathbf{x}^{b-\tau}\} \\
&= \{\tau \in \{0,1\}^n : (\exists i \in [r]); m_i \mid \mathbf{x}^{b-\tau}\} \\
&\cup \{\tau \in \{0,1\}^n : (\exists j \in [s]); m'_j \mid \mathbf{x}^{b-\tau}\} \\
&= K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(J).
\end{aligned}$$

■

Proposition 2.3.2. $K^{\mathbf{b}}(I \cap J) = K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(J)$.

Proof. Using the previous proposition, and the fact that

$$\begin{aligned}
K^{\mathbf{b}}(m_i) \cap K^{\mathbf{b}}(m'_j) &= \{\tau \in \{0,1\}^n : m_i \mid \mathbf{x}^{b-\tau}\} \cap \{\tau \in \{0,1\}^n : m'_j \mid \mathbf{x}^{b-\tau}\} \\
&= \{\tau \in \{0,1\}^n : m_i \mid \mathbf{x}^{b-\tau} \text{ y } m'_j \mid \mathbf{x}^{b-\tau}\} \\
&= \{\tau \in \{0,1\}^n : \operatorname{lcm}(m_i, m'_j) \mid \mathbf{x}^{b-\tau}\} \\
&= K^{\mathbf{b}}(\operatorname{lcm}(m_i, m'_j)),
\end{aligned}$$

we have

$$\begin{aligned}
K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(J) &= \left(\bigcup_{i \in [r]} K^{\mathbf{b}}(m_i)\right) \cap \left(\bigcup_{i \in [r]} K^{\mathbf{b}}(m_i)\right) \\
&= \bigcup_{(i,j) \in [r] \times [s]} K^{\mathbf{b}}(m_i) \cap K^{\mathbf{b}}(m'_j) \\
&= \bigcup_{(i,j) \in [r] \times [s]} K^{\mathbf{b}}(\operatorname{lcm}(m_i, m'_j)) \\
&= K^{\mathbf{b}}((\operatorname{lcm}(m_i, m'_j) : (i,j) \in [r] \times [s])) \\
&= K^{\mathbf{b}}(I \cap J).
\end{aligned}$$

$\blacksquare$

2.4 Duality

Here we'll talk about some dual objects of simplicial complexes which inherit the properties of the original objects. The homology of the dual of a simplicial complex will be, up to some permutation of the indexes, the same as the homology of the original complex.

Definition 2.4.1. Let Δ be the simplex with vertex set $\{1, \ldots, n\}$. Then we can define a duality map $\iota : \Delta \to \Delta$ given by $\sigma \mapsto \sigma^c$ for every $\sigma \in \Delta$. This map is just set complementation and is of course an involution, i.e. $\iota^2 = 1_\Delta$.

So we can define the *Alexander dual* of a simplicial complex:

Definition 2.4.2. For a simplicial complex X with vertex set $\{1, \ldots, n\}$ we define the Alexander dual $X^\vee$ of X as:

$$X^\vee = \{\iota(\sigma) : \sigma \notin X\} = \{\sigma : \iota(\sigma) \notin X\}.$$

Since the set of all the non-faces of X has a dual property to the one of the simplicial complexes (Every set containing a non-face is a non-face) and complementation reverses inclusions this makes clear that $X^\vee$ is a simplicial complex. We also have the following fact:

Proposition 2.4.1. For a simplicial complex X we have $X^{\vee\vee} = X$.

Proof. A face σ is in $X^{\vee\vee}$ if and only if $\iota(\sigma) \notin X^\vee$, which happens if and only if $\iota^2(\sigma) = \sigma \in X$. $\blacksquare$

We, then, have the following four related structures:

$$\begin{tikzcd}
X \arrow[r, leftrightarrow, "\nu"] \arrow[d, leftrightarrow, "\vee"'] \arrow[dr, leftrightarrow, "\iota"] & \nu(X) \arrow[d, leftrightarrow, "\vee"] \arrow[dl, leftrightarrow] \\
X^\vee \arrow[r, leftrightarrow, "\nu"'] & \iota(X)
\end{tikzcd}$$

where $\nu(X)$ is the simplicial *cocomplex* (A fancy name for an upper set under the inclusion relation) given by $\iota(X^\vee)$; the set of all the nonfaces of X.

The following proof of the Simplicial Alexander duality theorem is due to Anders Björner and Martin Tancer in [1]. For every face σ of a simplicial complex X with vertex set $V = \{1, \ldots, n\}$, define

$$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}.$$

Lemma 2.4.2 (Lemma 2.1,[1])**.** Let $k \in \sigma \subset \{1, \ldots, n\}$. Then

$$\operatorname{sgn}(k, \sigma) p(\sigma - \{k\}) = \operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) p(\sigma).$$

Proof. We have that $\operatorname{sgn}(k, \sigma) = (-1)^j$ where k is the $j-$th element of σ, i.e.

$$\operatorname{sgn}(k, \sigma) = \prod_{\substack{i \in \sigma \\ i<k}} (-1).$$

Also

$$\operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) = \prod_{\substack{i \notin \sigma \\ i<k}} (-1)$$

and

$$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}$$

Therefore, we have

$$p(\sigma) p(\sigma - \{k\}) = (-1)^{k-1} \prod_{i \in \sigma - \{k\}} (-1)^{i-1} \prod_{i \in \sigma - \{k\}} (-1)^{i-1} = (-1)^{k-1}$$

and

$$\operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) \operatorname{sgn}(k, \sigma) = \prod_{\substack{i \notin \sigma \\ i<k}} (-1) \prod_{\substack{i \in \sigma \\ i<k}} (-1) = \prod_{i<k} (-1) = (-1)^{k-1}.$$

Therefore

$$p(\sigma) p(\sigma - \{k\}) = \operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) \operatorname{sgn}(k, \sigma)$$

Since all the factors in the expressions above are in $\{-1, 1\}$, they are involutive in R, therefore multiplying at both sides by $p(\sigma) \operatorname{sgn}(k, \sigma)$ we have

$$\operatorname{sgn}(k, \sigma) p(\sigma - \{k\}) = \operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) p(\sigma)$$

which is the desired result. ∎

Theorem 2.4.3 (Alexander Duality)**.** Let X be a simplicial complex over the vertex set $V = \{1, \dots, n\}$. Then

$$\tilde{H}_i(X) \cong \tilde{H}^{n-i-3}(X^\vee).$$

Proof. First, if we take the simplex Δ on V, by the long exact sequence of homology we have an exact sequence

$$\cdots \to \tilde{H}_{i+1}(\Delta) \to \tilde{H}_{i+1}(\Delta, X) \to \tilde{H}_i(X) \to \tilde{H}_i(\Delta) \to \tilde{H}_i(\Delta, X) \to \cdots,$$

but since a simplex has no homology, this sequence breaks into short sequences of the form

$$0 \to \tilde{H}_{i+1}(\Delta, X) \to \tilde{H}_i(X) \to 0$$

for every i, which means the maps

$$\tilde{H}_{i+1}(\Delta, X) \to \tilde{H}_i(X)$$

are isomorphisms.

Now we must stablish an isomorphism

$$\tilde{H}_{i+1}(\Delta, X) \to \tilde{H}^{n-i-3}(X^\vee).$$

We'll do that by establishing an appropiate chain complex isomorphism between the complexes $C_\bullet(\Delta, X)$ and $C^\bullet(X^\vee)$. Define $\phi_i : C_i(\Delta, X) \to C^{n-i-2}(X^\vee)$ by

$$\phi_i(\sigma) = p(\sigma) f_{\iota(\sigma)}$$

for $\sigma \in \Delta_i - X_i$. This is a well defined map, since if $\sigma \in \Delta_i - X_i$ then $\iota(\sigma) \in X^\vee_{n-i-2}$, thus $f_{\iota(\sigma)} \in C^{n-i-2}(X^\vee)$. This also sends the standard basis of $C_i(\Delta, X)$ into the (up to signs) standard basis of $C^{n-i-2}(X^\vee)$, so it's an isomorphism at each dimension. The only thing left to prove is that it's a chain complex isomorphism, so that it makes the following diagram commute for each i:

$$\begin{tikzcd}
C_{i-1}(\Delta, X) \arrow[d, "\phi_{i-1}"'] & C_i(\Delta, X) \arrow[l, "\delta_i"'] \arrow[d, "\phi_i"] \\
C_{n-i-1}(\Delta, X) & C^{n-i-2}(\Delta, X) \arrow[l, "\partial^*_{n-i-1}"]
\end{tikzcd}$$

But this is straighforward, for $\sigma \in \Delta_i - X_i$:

$$\begin{aligned}
\phi_{i-1}\delta_i(\sigma) &= \phi_{j-1}\left(\sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} \operatorname{sgn}(k,\sigma)(\sigma-\{v\})\right) \\
&= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} \operatorname{sgn}(k,\sigma)p(\sigma-\{v\})f_{\iota(\sigma-\{v\})} \\
&= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} p(\sigma)\operatorname{sgn}(k,\iota(\sigma)\cup\{v\})f_{\iota(\sigma-\{v\})} \\
&= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} p(\sigma)\operatorname{sgn}(k,\iota(\sigma)\cup\{v\})f_{\iota(\sigma)\cup\{v\}} \\
&= \partial^*_{n-i-1}(p(\sigma)f_{\iota(\sigma)}) \\
&= \partial^*_{n-i-1}\phi_i(\sigma)
\end{aligned}$$

which is the desired result. Therefore, since the chain complexes above are isomorphic, they have the same homology. Therefore, for each i,

$$\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Delta, X) \cong \tilde{H}^{n-i-3}(X^\vee).$$

■

Corollary 2.4.4 (Alexander Duality)**.** Let X be a simplicial complex over the vertex set $V = \{1,\dots,n\}$ and k a field. Then

$$\dim_k \tilde{H}_i(X;k) = \dim_k \tilde{H}_{n-i-3}(X^\vee;k).$$

Proof. It follows from the universal coefficient theorem and the fact that X and $X^\vee$ have a finite number of faces of each dimension: Since in each (topological) dimension, the cohomology is the dual of the homology, and the homology is a finite dimensional $k-$vector space, they have the same dimension. ■

Definition 2.4.3 (Lower Koszul complex)**.** Let $\mathbf{b} \in \mathbb{N}^n$ and I a monomial ideal of the polynomial ring S in n variables. Define the lower Koszul complex of I as

$$K_{\mathbf{b}}(I) = \{\tau \in \{0,1\}^n \mid \tau \leq \mathbf{b}, \mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau} \notin I\}$$

where $\mathbf{1} = \operatorname{Supp}\mathbf{b}$.

Theorem 2.4.5 (Hochster's formula, dual version)**.** Given a vector $\mathbf{b} \in \mathbb{N}^n$ such that $\mathbf{x}^{\mathbf{b}} \in I$, the Betti numbers of I in degree $\mathbf{b}$ can be expressed as

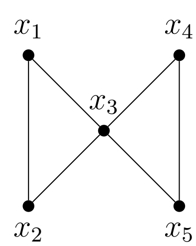
$$\beta_{i,\mathbf{b}}(I) = \dim_k \tilde{H}_{n-i-2}(K_{\mathbf{b}}(I); k).$$

Proof. A face τ is in $K_{\mathbf{b}}(I)$ iff $\tau \leq \mathbf{b}$ and $\mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau} \notin I$. This means that no generating monomial of I divides $\mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau}$. So, a nonface τ of $K_{\mathbf{b}}(I)$ with respect to the simplex $\{\tau \in \{0,1\}^n \mid \tau \leq \mathbf{b}\}$ is a squarefree vector such that $\mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau} \in I$ which is the same as saying $\mathbf{x}^{\mathbf{b}-(\mathbf{1}-\tau)} \in I$, which means that $\mathbf{1} - \tau \in K^{\mathbf{1}}(I)$. Since the complement of τ is $\mathbf{1} - \tau$, this means that $K^{\mathbf{1}}(I) = K^{\vee}_{\mathbf{1}}(I)$. The result follows from Alexander's duality. ∎

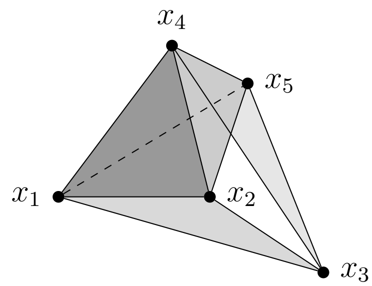
Chapter 3

Using Hochster's Formula Recursively

Hochster's formula is made to compute Betti numbers of ideals. For example, consider the edge ideal I_G of the graph



i.e. the ideal $I_G = (x_1x_2, x_1x_3, x_2x_3, x_3x_5, x_3x_4, x_4x_5)$. We'll start by computing $K^{\mathbf{1}}(I_G)$. Since a vector $abcde := (a,b,c,d,e) \in \{0,1\}^5 \in K^{\mathbf{1}}(I_G)$ iff $\mathbf{x}^{\mathbf{1}-abcde} \in I$, we'll clearly have the vectors $00111, 01011, 10011, 11010, 11001$ and 11100 in $K^{\mathbf{1}}(I_G)$ since they are the complements of the exponents of the generators of I_G. They can also be written as $x_3x_4x_5$, $x_2x_4x_5$, $x_1x_4x_5$, $x_1x_2x_4$, $x_1x_2x_5$ and $x_1x_2x_3$. A quick check shows these are all the posible $2-$faces of $K^{\mathbf{1}}(I_G)$. Also we can see there are no $i-$faces for $i \geq 3$ since for such a face τ we'd have that $\mathbf{x}^{\mathbf{1}-\tau} \in I_G$ but $\mathbf{x}^{\mathbf{1}-\tau}$ has less than 1 variables so it cannot be in I_G. We can also see by any method (Like checking the subsets of the $2-$faces) that all the posible $1-$faces for the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$ are in $K^{\mathbf{1}}(I_G)$ so its $1-$skeleton is isomorphic to K_5. Drawing the faces we get the following:



Considering $K^{\mathbf{1}}(I_G)$ as a $CW-$complex, and using the proposition 0.17 from [5], we get, by computing $X_i = X_{i-1}/A_{i-1}$ sequentially, first for $X_0 = K^{\mathbf{1}}(_G)$ and $A_0 = x_1x_2x_3$, then $A_1 = x_4x_5$, and then $A_2 = x_{123}x_{45}$ (The strip resulting from contracting $x_1x_2x_3$ and then x_4x_5 to points x_{123}, x_{45}). So, $X_3 \simeq K^{\mathbf{1}}(I_G)$ but $X_3 \cong S^2/\{a, -a\}$ for some point $a \in S^2$. This is a sphere with two points identified, and using the long exact sequence for homology we get $H_2(X_3) \cong H_2(S^2) \cong k$ and $H_1(X_3) \cong H_0(S^0) = k$ (where S^0 is the $0-$dimensional sphere, a two-point space). Being a bit more careful we can check that $K^{\mathbf{1}}(I_G)$ has the homotopy type of $S^2 \vee S^1$ which gives the same result.

Of course these tools used are strongly topological. We're interested in finding tools to compute Betti numbers which don't rely so much in the topology of $K^{\mathbf{b}}$, but in the algebra and combinatorics behind it. The Mayer-Vietoris sequences which arise here are a first example of such a tool.

3.1 Mayer-Vietoris Sequences

Since $K^{\mathbf{b}}$ preserves intersections, and transforms $+$ into $\cup$, it would be a shame not to make use of this to build Mayer-Vietoris sequences. We have the inclusions:

$$\begin{tikzcd}
 & K^{\mathbf{b}}(I+J) & \\
K^{\mathbf{b}}(J) \arrow[ur, "\iota'_J"] & & K^{\mathbf{b}}(I) \arrow[ul, "\iota'_I"'] \\
 & K^{\mathbf{b}}(I\cap J) \arrow[ul, "\iota_J"] \arrow[ur, "\iota_I"'] &
\end{tikzcd}$$

so, if $\mathbf{b} \in \mathbb{N}^n$ is such that $\mathbf{x}^\mathbf{b} \in K^\mathbf{b}(I \cap J)$, then there is a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \tilde{H}_i(K^\mathbf{b}_{I\cap J}) \arrow[r, "\alpha_*"] & \tilde{H}_i(K^\mathbf{b}_I) \oplus \tilde{H}_i(K^\mathbf{b}_J) \arrow[r, "\beta_*"] & \tilde{H}_i(K^\mathbf{b}_{I+J}) \arrow[dll, "\partial"'] & \\
& \tilde{H}_{i-1}(K^\mathbf{b}_{I\cap J}) \arrow[r, "\alpha_*"] & \tilde{H}_{i-1}(K^\mathbf{b}_I) \oplus \tilde{H}_{i-1}(K^\mathbf{b}_J) \arrow[r, "\beta_*"] & \tilde{H}_{i-1}(K^\mathbf{b}_{I+J}) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

where $K^\mathbf{b}_* = K^\mathbf{b}(*)$, $\alpha = (\iota_I, -\iota_J)$, $\beta = \iota'_I + \iota'_J$ and the maps α_*, β_* are the induced maps. In particular, if we take $I' = (m_1, \ldots, m_{r-1})$ and $I'' = (\mathrm{lcm}(m_1, m_r), \ldots, \mathrm{lcm}(m_{r-1}, m_r))$, then for $\mathbf{b}$ such that $\mathbf{x}^\mathbf{b} \in I''$ there is a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \tilde{H}_i(K^\mathbf{b}_{I''}) \arrow[r] & \tilde{H}_i(K^\mathbf{b}_{I'}) \oplus \tilde{H}_i(K^\mathbf{b}_{m_r}) \arrow[r] & \tilde{H}_i(K^\mathbf{b}_I) \arrow[dll, "\partial"'] & \\
& \tilde{H}_{i-1}(K^\mathbf{b}_{I''}) \arrow[r] & \tilde{H}_{i-1}(K^\mathbf{b}_{I'}) \oplus \tilde{H}_{i-1}(K^\mathbf{b}_{m_r}) \arrow[r] & \tilde{H}_{i-1}(K^\mathbf{b}_I) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

so if we can compute $\tilde{H}_i(K^\mathbf{b}(I'')), \tilde{H}_i(K^\mathbf{b}(I')), \tilde{H}_i(K^\mathbf{b}(m_r))$ then we'll have a lot of information of $\tilde{H}_i(K^\mathbf{b}(I))$. In particular, maybe we'll have in this information the numbers $\dim_k \tilde{H}_i(K^\mathbf{b}(I))$, from which we'd be able to compute the Betti numbers of I recursively. This is a reason this will be called a recursive Mayer-Vietoris sequence of I (A sequence gotten from decomposing the ideal into the sum of a principal ideal and of another ideal). These sequences are used extensively in [9]. By an application of a recursive Mayer-Vietoris sequence we have the following result:

Proposition 3.1.1. Let $I = (m_1, \ldots, m_r)$ and $m = \mathrm{lcm}(m_1, \ldots, m_r) = \mathbf{x}^a$. Then $\tilde{H}_i(K^\mathbf{b}(I)) = 0$ for all i and $\mathbf{a} < \mathbf{b}$.

Proof. By induction on r. For such b, we can take Mayer-Vietoris sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \tilde{H}_i(K^\mathbf{b}_{I''}) \arrow[r] & \tilde{H}_i(K^\mathbf{b}_{I'}) \oplus \tilde{H}_i(K^\mathbf{b}_{m_r}) \arrow[r] & \tilde{H}_i(K^\mathbf{b}_I) \arrow[dll, "\partial"'] & \\
& \tilde{H}_{i-1}(K^\mathbf{b}_{I''}) \arrow[r] & \tilde{H}_{i-1}(K^\mathbf{b}_{I'}) \oplus \tilde{H}_{i-1}(K^\mathbf{b}_{m_r}) \arrow[r] & \tilde{H}_{i-1}(K^\mathbf{b}_I) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

By induction hypothesis, and the base case in which $K^{\mathbf{b}}(m_r)$ will be a simplex, $\tilde{H}_i(K^{\mathbf{b}}(I')), \tilde{H}_i(K^{\mathbf{b}}(m_r)), \tilde{H}_{i-1}(K^{\mathbf{b}}(I''))$ are all 0, so $\tilde{H}_i(K^{\mathbf{b}}(I))$ is 0. The proof will only be complete if we prove the base case, where I is principal.

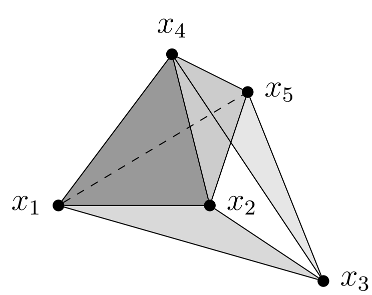
If $I = (\mathbf{x}^{\mathbf{a}})$ is principal then I is a free $S-$module, so the minimal free resolution of I is just $0 \to S\mathbf{x}^{\mathbf{a}} \to 0$, so the only Betti number is $\beta_{0,\mathbf{a}} = 1 = \dim_k \tilde{H}_{-1}(K^{\mathbf{a}}(\mathbf{x}^{\mathbf{a}}))$. Therefore, for $\mathbf{b} > \mathbf{a}$, $\tilde{H}_i(K^{\mathbf{b}}(\mathbf{x}^{\mathbf{a}})) = 0$. ■

Using recursive Mayer-Vietoris sequences is a natural way to do induction proofs on the number of generating monomials, since both the ideals I'' and I' have less generating monomials than I. But the underlying decomposition of I is not unique, and it's not always the most useful we can use. An example of this is in order.

Example 3.1.2. Remember the ideal

$$I = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_4x_5).$$

We wanted to compute $H_i(K^{\mathbf{1}}(I))$. Also remember that $K^{\mathbf{1}}(B)$ has a topological representation as:



Another method to compute its homology is to use (by a topological reasoning) a Mayer-Vietoris sequence for $X = K^{\mathbf{1}}(I) = A \cup B$ where

$$A \cong \partial\Delta^3 \cong S^2 = \{x_1x_2x_4, x_1x_2x_5, x_1x_4x_5, x_2x_4x_5\}$$

and

$$B \simeq \{x_3\} = \{x_1x_2x_3, x_3x_4x_5\}.$$

Also note that $A \cap B \simeq \{x_1, x_4\} = \{x_1x_2, x_4x_5\}$, in the Mayer-Vietoris sequence we have the same sequences we got before

$$0 \to \tilde{H}_1(X) \to \tilde{H}_0(A \cap B) \to 0$$

and

$$0 \to \tilde{H}_2(A) \to \tilde{H}_2(X) \to 0$$

which mean that $\tilde{H}_2(X) \cong \tilde{H}_2(\partial\Delta^3) \cong k$ and $\tilde{H}_1(X) \cong \tilde{H}_0(A \cap B) \cong k$. The rest of (reduced) homologies of X are zero. But we can do this without using so many *homotopical equivalence* arguments.

First, note that $K^{\mathbf{1}}(x_1x_2) = \{x_3x_4x_5\}$ and $K^{\mathbf{1}}(x_4x_5) = \{x_1x_2x_3\}$, so $K^{\mathbf{1}}(x_1x_2, x_4x_5) = B$. Also we know that

$$\tilde{H}_i(K^{\mathbf{1}}(x_1x_2, x_4x_5)) = 0$$

for every i since $\mathrm{lcm}(x_1x_2, x_4x_5) = x_1x_2x_4x_5 \neq x_1x_2x_3x_4x_5$. Also

$$K^{\mathbf{1}}(x_1x_3, x_2x_3, x_3x_4, x_3x_5) = A,$$

and

$$A \cap B = K^{\mathbf{1}}(x_1x_2x_3, x_3x_4x_5),$$

so $\tilde{H}_0(A \cap B) = k$ and 0 elsewhere. Now, $A = K^{\mathbf{1}}(x_1, x_2, x_4, x_5) \cap K_1(x_3)$.

So we can now refer to the Mayer-Vietoris sequence of the ideal $\mathfrak{m}$. In this sequence everything is 0 except $h_3(K^{\mathbf{1}}(\mathfrak{m})) \cong k$ since a minimal free resolution of $\mathfrak{m}$ is just the Koszul complex snipping the first S. So

$$\tilde{H}_2(A) \cong \tilde{H}_3(K^{\mathbf{1}}(\mathfrak{m})) \cong k.$$

While this decomposition works, there are some more general decompositions of I which we're interested in. The following lemma can be proven by other means but we'll use Mayer-Vietoris sequences to prove it.

Lemma 3.1.3. Let $I = (m_1, \ldots, m_r) \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal and $I' = (m_1, \ldots, m_r) \subseteq k[x_1, \ldots, x_n, y]$. Then, for a degree $\mathbf{b} \in \{0,1\}^{n+1}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$ we have $\beta_{i,\mathbf{b}}(I') = 0$ for all i. Additionally, for $\mathbf{b} \in \{0,1\}^{n+1}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$, $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I)$, where $\mathbf{b}' = (\mathbf{b}_i)_{i=1}^{n}$.

Proof. We'll start with the last part. We know that

$$\beta_{i,\mathbf{b}}(I') = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I'))$$

and

$$\beta_{i,\mathbf{b}'}(I) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}'}(I)).$$

But for $\mathbf{b}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$ and $\tau \in \{0,1\}^{n+1}$, we have $\mathbf{x}^{\mathbf{b}-\tau} \in I'$ iff $\mathbf{x}^{\mathbf{b}'-\tau'} \in I$. Thus, the inclusion $\tau' \mapsto \tau$ induces an isomorphism $K^{\mathbf{b}}(I') \cong K^{\mathbf{b}'}(I)$ of simplicial complexes (They're just the same except for a last coordinate in each face which is 0). For $\mathbf{b}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$ we'll prove $\beta_{i,\mathbf{b}}(I') = 0$ by induction on r. If $\mathbf{x}^{\mathbf{b}} \notin I$ there is nothing to prove. So we suppose $\mathbf{x}^{\mathbf{b}} \in I$.

By an induction argument, suppose $r = 1$, then since $m_1 \mid \mathbf{x}^{\mathbf{b}}$ but $m_1 \neq \mathbf{x}^{\mathbf{b}}$ we have that $K^{\mathbf{b}}(I')$ is a nontrivial simplex and therefore has trivial reduced homology. Now suppose $r > 1$ and that for every ideal I'' generated by less than r monomials in the variables $x_1, \ldots, x_n$ and $\mathbf{b}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$, we have $\beta_{i,\mathbf{b}}(I'') = 0$.

If I' is such that there is only one generating monomial, say, m_r such that $m_r \mid \mathbf{x}^{\mathbf{b}}$ then $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(m_r)$ which is already known to have trivial reduced homology. So we can suppose there are at least two monomials m_r, m_{r-1} such that $m_r, m_{r-1} \mid \mathbf{x}^{\mathbf{b}}$. So, for $I'' = (m_1, \ldots, m_{r-1})$ and $I''' = (m_1 m_r, \ldots, m_{r-1} m_r)$ there is a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to \tilde{H}_i(K^{\mathbf{b}}(I''')) \to \tilde{H}_i(K^{\mathbf{b}}(I'')) \oplus \tilde{H}_i(K^{\mathbf{b}}(m_r)) \\ \to \tilde{H}_i(K^{\mathbf{b}}(I')) \to \tilde{H}_{i-1}(K^{\mathbf{b}}(I''')) \to \cdots \end{aligned}$$

By induction hypothesis, all the homologies in the exact sequence, except for $\tilde{H}_i(K^{\mathbf{b}})(I')$ are zero, so by exactness $\tilde{H}_i(K^{\mathbf{b}})(I')$ is also zero for every i. We're, thus, done. ■

From this, it follows that if we have the edge ideal I_G of a graph G, then for a squarefree degree $\mathbf{b} \neq \mathbf{1}$ and every i, $\tilde{H}_i(K^{\mathbf{b}}(I_G)) \cong \tilde{H}_i(K^{\mathbf{1}'}(I_{G'}))$ where G' is the subgraph induced by the vertices which characteristic vector is $\mathbf{b}$ and $\mathbf{1}'$ is the vector filled with 1s in $\{0,1\}^{|}G'|$ and $I_{G'} \subseteq k[V(G')]$. It also follows that this number does not depend on the number of the variables (as long as there are as many as ones in $\mathbf{b}$).

Proposition 3.1.4. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal of the ring $k[x_1, \ldots, x_n, y]$, and suppose that for all the monomials m_i such that $y \mid m_i$, there is some variable $x_j \in \operatorname{Supp}(m_1, \ldots, m_r)$ such that $x_j \nmid m_i$, and that $I' = (m_i : y \mid m_i) \neq I$. Then, for $I'' = (m_i : y \nmid m_i)$, $I''' = I' \cap I''$, and any $\mathbf{b} \in \mathbb{N}^{n+1}$ such that $\mathbf{x}^{\mathbf{b}} \in I'''$ there is an isomorphism:

$$\tilde{H}_i(K^{\mathbf{b}}(I)) \to \tilde{H}_{i-1}(K^{\mathbf{b}}(I''')).$$

For $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \in I' - I''$ there is an isomorphism

$$\tilde{H}_i(K^{\mathbf{b}}(I')) \to \tilde{H}_i(K^{\mathbf{b}}(I)).$$

For $\mathbf{b}$ such that $\mathbf{x^b} \in I'' - I'$ there is an isomorphism

$$\tilde{H}_i(K^{\mathbf{b}}(I'')) \to \tilde{H}_i(K^{\mathbf{b}}(I)).$$

Proof. For $\mathbf{b}$ such that $\mathbf{x^b} \notin I'''$ we have $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x^b})$. If $\mathbf{x^b} \in I' - I''$ this means that the monomials m_i such that $m_i \mid \mathbf{x^b}$ are in the generating set of I', so

$$K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x^b}) = K^{\mathbf{b}}(I').$$

If, otherwise $\mathbf{x^b} \in I'' - I'$ then similarly $K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x^b}) = K^{\mathbf{b}}(I'')$. So, suppose that $\mathbf{x^b} \in I'''$. This means that, in particular, $\mathbf{x^b}$ is divisible by two generating monomials $m_i, m_{i'}$ of I, such that $y \mid m_i, y \nmid m_{i'}$, which means that $y \mid \mathbf{x^b}$. Also there is some other variable x_j such that $x_j \mid \mathbf{x^b}$ and $x_j \notin \operatorname{Supp} I'$. Therefore for all i, $\tilde{H}_i(K^{\mathbf{b}}(I')) \cong \tilde{H}_i(K^{\mathbf{b}}(I'')) \cong 0$ and in the Mayer-Vietoris sequence associated to this decomposition we'll find the desired isomorphism. ■

The conditions for the proposition 3.1.4 seem to be harsh, but are not so much. This Mayer-Vietoris sequence will be called an *incomplete star* Mayer-Vietoris sequence. This is because when we have an edge ideal of a graph, the condition is equivalent to the graph having a vertex not being adyacent to every other vertex of G.

3.2 Stars, Complete Bipartite Graphs, Cones

Here we'll study some families of ideals for which the Betti numbers can be computed rather easily. The first one of them is the family of the edge ideals of the complete bipartite graphs. Let $G = (V, E) = K_{n,m}$ be the complete bipartite graph with bipartition $H = \{x_1, \ldots, x_n\}, K = \{y_1, \ldots, y_n\}$ and

$$I = I_G \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

its edge ideal. For a fixed vertex, which we can, by a permutation of A, B or $y_1, \ldots, y_m$ declare to be y_m, we can say $I = I_m + I'_m$ where $I_m = (e \in E : y_m \in e)$ and $I'_m = (e \in E : y_m \notin e)$. Given this, for $m = 1$ we can use a trick to compute the Betti numbers of I.

Proposition 3.2.1. If $G = K_{n,1}$ is a star with vertex set $\{x_1, \dots, x_n, y_1\}$, if $\mathbf{b} \in \{0,1\}^{n+1}$ is such that $\mathbf{x^b} \in I_G$, we have

$$\beta_{i,\mathbf{b}}(I_G) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}})(I_G) = \dim_k \tilde{H}_i(K^{\mathbf{b}}(\mathfrak{m})) = \beta_{i+1,\mathbf{b}}(\mathfrak{m}),$$

for $\mathfrak{m} = (x_1, \dots, x_n, y_1)$.

Proof. Take $y = y_1$ and $I = I_G$. Since $x^{\mathbf{b}} \in I$ then we can assume by a permutation $\sigma \in S_n$ that $x_n y \mid x^{\mathbf{b}}$. So, since $I = (x_1, \dots, x_n) \cap (y)$, we have, for $I' = (x_1, \dots, x_n)$, a Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \to \tilde{H}_i(K^{\mathbf{b}}(I)) \to \tilde{H}_i(K^{\mathbf{b}}(I')) \oplus \tilde{H}_i(K^{\mathbf{b}}(y)) \\ \to \tilde{H}_i(K^{\mathbf{b}}(\mathfrak{m})) \to \tilde{H}_{i-1}(K^{\mathbf{b}}(I)) \to \cdots \end{aligned}$$

Because of the lemma 3.1.3 we have that $\tilde{H}_i(K^{\mathbf{b}}(I')), \tilde{H}_i(K^{\mathbf{b}}(y))$ are both 0 for all i, therefore $\tilde{H}_i(K^{\mathbf{b}}(\mathfrak{m})) \cong \tilde{H}_{i-1}(K^{\mathbf{b}}(I))$ for all i. The result follows. ■

From the previous proposition we can easily compute the Betti numbers of the star $K_{n,1}$. We know that the Koszul complex $K_\bullet$ is a minimal free resolution of $k = S/\mathfrak{m}$, so when we remove the S corresponding to the empty face from it, it becomes a minimal free resolution $K'_\bullet$ of $\mathfrak{m}$. Suppose that $\mathbf{b}$ is squarefree (otherwise its Betti number would be 0) and take $r = |\mathbf{b}|$. Then $\mathbf{x^b}$ appears exactly once as a generator in homological degree $r-1$, and doesn't appear elsewhere in $K'_\bullet$. So

$$\beta_{i,\mathbf{b}}(I_G) = \beta_{i+1,\mathbf{b}}(\mathfrak{m}) = \begin{cases} 1 & \text{if } i = r-2 \\ 0 & \text{otherwise.} \end{cases}$$

It's no surprise if we can then compute the Betti numbers of the ideal I_G of the complete bipartite graph $G = K_{n,m}$.

Proposition 3.2.2. Let $G = K_{n,m}$ and $I = I_G \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$. Then, for $\mathbf{b}$ such that $\mathbf{x^b} \in I$,

$$\beta_{\mathbf{b},i}(I) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I)) = \begin{cases} 1 & \text{if } i = n_{\mathbf{b}} + m_{\mathbf{b}} - 2 \\ 0 & \text{otherwise.} \end{cases}$$

where $n_{\mathbf{b}} = |\{x_i : x_i \mid \mathbf{x^b}\}|, m_{\mathbf{b}} = |\{y_j : y_j \mid \mathbf{x^b}\}|$.

Proof. For $\mathbf{b} < \mathbf{1}$ we have $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(I_{K_{n_{\mathbf{b}},m_{\mathbf{b}}}})$ so we only have to prove the result for $\mathbf{b} = \mathbf{1}$. If m or n equals 1 then we're done: our graph is a star. So suppose that $m, n > 1$. Since no vertex is adjacent to every other vertex, the incomplete star Mayer-Vietoris sequence with respect to the the vertex y_m gives an isomorphism:

$$\tilde{H}_i(K^{\mathbf{1}}(I)) \cong \tilde{H}_{i-1}(K^{\mathbf{1}}(I_1))$$

where

$$I_1 = (x_i y_j y_m : 1 \leq i \leq n, 1 \leq j \leq m) = I_{G-y_n} \cap I_{\Sigma(y_m)}$$

where $\Sigma(y_m)$ is the star of y_m. We can repeat the process, since not every generating monomial is divisible by y_{m-1}. So, this way we find a decreasing sequence $I_s, s \in \mathbb{N}$ of ideals such that $I_0 = I, I_{n+m-2} = (x_1 \dots x_n y_1 \dots y_m)$, and

$$\tilde{H}_i(I_s) \cong \tilde{H}_{i-1}(I_{s+1}).$$

Therefore, $\tilde{H}_{n+m-3}(I) \cong \tilde{H}_{-1}(I_{n+m-2}) = k$. ■

For these cases, Alexander duality is specially powerful. The Lower Koszul Complex $K_{\mathbf{1}}(I_G)$ is just the disjoint union of two simplexes, one corresponding to the stable set of the variables x, and other corresponding to the stable set of the variables y. It's reduced homology is clearly 0 everywhere besides at dimension 0, on which we have $\tilde{H}_0(K_{\mathbf{1}}(I_G)) = k$. This is exactly the same result. One can go even further. For a graph G which is the join of two graphs H, K (The graph resulting from taking the disjoint union of H and K, and adding all the possible $H-K-$edges), the Lower Koszul Complex $K_{\mathbf{1}}(I_G)$ is the disjoint union $K_{\mathbf{1}'}(I_H) \sqcup K_{\mathbf{1}''}(I_K)$, where $\mathbf{1}' = \chi(V(H)), \mathbf{1}'' = \chi(V(K))$. Thus, for every dimension but 0, the homology of $K_{\mathbf{1}}(G)$ is the direct sum of the homologies of $K_{\mathbf{1}'}(H)$ and $K_{\mathbf{1}''}(K)$. In dimension 0, it's the direct sum of $K_{\mathbf{1}'}(H), K_{\mathbf{1}''}(K)$ and k.

Now we're interested in generalizing some of the ideas in the proof above. One of them is, given a monomial ideal

$$I = (m_1, \dots, m_r) \subseteq k[x_1, \dots, x_n, y]$$

such that for all $i \in \{1, \dots, r\}$, $y \nmid m_i$, computing the Betti numbers of

$$I' = (m_1, \dots, m_r, y).$$

Using a recursive Mayer-Vietoris sequence this is equivalent to computing the Betti numbers of $I'' = (m_1 y, \dots, m_r y)$. Ideals like I'' appear often when using incomplete star Mayer-Vietoris sequences, so it would be useful if we could compute these Betti numbers in terms on the Betti numbers of I. These ideals appear in the sequence of ideals in the proof above, for example.

Other ideas are to compute the Betti numbers of the cone of a graph, i.e. a graph with a vertex adyacent to every other one, and to compute the Betti numbers of the graph resulting from duplicating a vertex of a given graph.

Proposition 3.2.3. Let $I = (m_1, \dots, m_r)$ be an ideal of $k[x_1, \dots, x_n, y]$ not using the variable y. Then, for $I' = I + (y)$ and $\mathbf{b} \in \mathbb{N}^{n+1}$ such that $y \mid \mathbf{x}^{\mathbf{b}} \in I$ we have

$$\beta_{i+1,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I).$$

Proof. For any other $\mathbf{b}$, we have

$$K^{\mathbf{b}}(I') = K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(y).$$

We can characterize $K^{\mathbf{b}}(y)$ as the simplex of all the faces $\tau \leq \mathbf{b}$ such that $y \nmid \mathbf{x}^{\tau}$. Also since I does not use the variable y, we see that

$$\begin{aligned} K^{\mathbf{b}}(I) =& \{\tau = (t_1, \dots, t_n, q) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \\ =& \{\tau = (t_1, \dots, t_n, 0) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \\ & \cup \{\tau = (t_1, \dots, t_n, 1) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \end{aligned}$$

so $K^{\mathbf{b}}(I)$ is just the simplicial complex of all the subfaces of the faces of

$$T_1 = \{\tau = (t_0, \dots, t_n, 1) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\}.$$

This means that

$$\begin{aligned} K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(y) &= \{\tau = (t_1, \dots, t_n, q) \in K^{\mathbf{b}}(I) : q = 0\} \\ &= K^{(\mathbf{b}',0)}(I) \end{aligned}$$

where $\mathbf{b}' \in \mathbb{N}^n$ is such that $\mathbf{b} = (\mathbf{b}', 1)$. Therefore, by a recursive Mayer-Vietoris sequence,

$$\tilde{H}_i(K^{\mathbf{b}}(I')) \cong \tilde{H}_{i-1}(K^{\mathbf{b}}(I \cap (y))) \cong \tilde{H}_{i-1}(K^{(\mathbf{b}',0)}(I))$$

as desired. ∎

We can also compute all the remaining Betti numbers of the ideal in the previous proposition:

For $\mathbf{b}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$ we have $K^{\mathbf{b}}(I') = K^{\mathbf{b}}(I)$ so $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}}(I)$. Also if $\mathbf{x}^{\mathbf{b}} \notin I$ we have $K^{\mathbf{b}}(I') = K^{\mathbf{b}}(y)$ so $\beta_i(K^{\mathbf{b}}(I')) = \delta_{i0}$ where δ_{ij} is the Kronecker delta.

We can also compute the Betti numbers of the cone of a graph:

Proposition 3.2.4. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$, and $\kappa = C(G, y)$ the cone of G with y being the new vertex. Then

$$\beta_{n-1,\mathbf{1}}(I_\kappa) = \beta_{n-2,\mathbf{1}'}(I_G) + 1,$$

where $\mathbf{1} = (\mathbf{1}', 0)$. Furthermore $\beta_{i,\mathbf{1}}(I_\kappa) = \beta_{i,\mathbf{1}'}(I_G)$ for every $i \neq n-1$

Proof. Here we take the Mayer-Vietoris sequence relative to the *complete* star $\Sigma(y)$ of y. We know that $\tilde{H}_i(K^{\mathbf{1}}(I_{\Sigma(y)})) \cong k$ iff $i = n-2$. Also $\tilde{H}_i(K^{\mathbf{1}}(I_G)) = 0$ for every i since G does not use the variable y and $\tilde{H}_{n-2}(K^{\mathbf{1}}(I')) = 0$ where $I' = I_G \cap I_{\Sigma(y)}$. So, we've got the following short exact sequence:

$$0 \to \tilde{H}_{n-2}(K^{\mathbf{1}}(I_{\Sigma(y)})) \to \tilde{H}_{n-2}(I_\kappa) \to \tilde{H}_{n-3}(K^{\mathbf{1}}(I')) \to 0$$

and isomorphisms $\tilde{H}_i(K^{\mathbf{1}}(I)) \to \tilde{H}_{i-1}(I')$ for every $i \neq n-2$. Therefore

$$\begin{aligned}\dim \tilde{H}_{n-2}(K^{\mathbf{1}}(I_\kappa)) &= \dim \tilde{H}_{n-2}(K^{\mathbf{1}}(I_{\Sigma(y)})) + \dim \tilde{H}_{n-3}(K^{\mathbf{1}}(I')) \\ &= 1 + \dim \tilde{H}_{n-3}(K^{\mathbf{1}}(I')).\end{aligned}$$

The trick lies, then, in disclosing the identity of $K^{\mathbf{1}}(I')$. Since $I' \neq 1$ then $\mathbf{1} \notin K^{\mathbf{1}}(I')$. Also $K^{\mathbf{1}}(I_{\Sigma(y)})$ is the $n-2-$skeleton of the simplex with vertices $x_1, \ldots, x_n$: The set of all the faces with $n-1$ vertices. So, with the same reasoning as the one in the proposition 3.2.3, the only thing we're doing when intersecting is removing the faces of $K^{\mathbf{1}}(I_G)$ containing y. As in the analysis of the proposition 3.2.3, we end up with $K^{(\mathbf{1}',0)}(I)$. So, replacing it in the equation above and replacing it all with Betti, we get

$$\beta_{n-1,\mathbf{1}}(I_\kappa) = 1 + \beta_{n-2,\mathbf{1}'}(I_G),$$

which is what we wanted to prove. ∎

Of course, we're also able to compute the remaining Betti numbers of κ, given. For $y \mid \mathbf{x}^{\mathbf{b}}$ we have the Betti numbers of the cone of an induced subgraph of G, from which we can apply the previous proposition, and for $y \nmid \mathbf{x}^{\mathbf{b}}$ we have the same Betti numbers as G. This, in particular, gives us another way to compute all the Betti numbers of any complete graph, since it's the consecutive application of the cone operator to an edge.

The following proposition is clearly related to the proposition 3.2.3; it's a sort of generalization of it and it can be generalized further, but for the sake of simplicity we'll state it as follows.

Proposition 3.2.5. Let $I = (m_1, \ldots, m_{r-1})$ be a monomial ideal of $S = k[x_1, \ldots, x_n]$ and suppose $m_r = x_a x_b, a \neq b \in \{1, \ldots, n\}$ such that

$$\operatorname{Supp}(m_r) \cap \operatorname{Supp}(m_1, \ldots, m_{r-1}) = \varnothing^{(\mathrm{i})}$$

Let $I' = I + (m_r)$. Then

$$\beta_{i+1,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I)$$

where $\mathbf{b}'_i = (1-\delta_{ia})(1-\delta_{ib})\mathbf{b}_i$, i.e. $\mathbf{b}'$ is the result of letting the entries of $\mathbf{b}$ corresponding to x_a, x_b be zero.

Proof. By a permutation of the variables suppose $m_r = x_{n-1}x_n$. As before, if $\mathbf{b}$ is such that $\mathbf{x}^{\mathbf{b}}$ is not divisible by m_r we have $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}}(I)$. If $\mathbf{x}^{\mathbf{b}}$ is divisible only by m_r then $\beta_{i,\mathbf{b}}(I) = \beta_{i,\mathbf{b}}(m_r)$. So, suppose $\mathbf{x}^{\mathbf{b}}$ is divisible by both m_r and some other m_i. Then

$$\begin{aligned} K^{\mathbf{b}}(I) =& \{\tau \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} \\ =& \{\tau = (t_1, \ldots, t_{n-2}, 0, 0) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} \\ & \cup \{\tau = (t_1, \ldots, t_{n-2}, t_{n-1}, t_n) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I, (t_{n-1}, t_n) \neq 0\}. \end{aligned}$$

Also $K^{\mathbf{b}}(m_r)$ is the simplex on the vertex set $x_1, \ldots, x_{n-2}$. Therefore

$$K^{\mathbf{b}}(I \cap (m_r)) = \{\tau = (t_1, \ldots, t_{n-2}, 0, 0) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} = K^{\mathbf{b}'}(I).$$

Using the isomorphism arising from the recursive Mayer-Vietoris sequence of I' with respect to m_r we get the result. ■

(i) If $I = I_G$, this means that G is disconnected being the edge $m_r = x_a x_b$ one of the connected components.

3.3 Forests, Paths and Cycles

With the results of the previous section, we're able to compute the Betti numbers of a few more families of ideals. For example, edge ideals of forests. This has already been done in [6], but the approach used there is different, and the results are given to compute their graded Betti numbers. In counterpart, our result gets a simpler formula and algorithm to compute the multigraded Betti numbers. It also uses less terminology so it has a simpler proof.

Lemma 3.3.1. Let T be a forest without isolated vertices. Let x_n be a leaf of T with x_{n-1} being its only neighbor. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_{i-d_T(x_{n-1})}(K^{\mathbf{1}'}(I_{T-N_T[x_{n-1}]}))$$

for $\mathbf{1}' = \chi(V(T - N_T[x_{n-1}]))$.

Proof. If T is a tree with vertex set $V = \{x_1, \ldots, x_n\}$, then we can suppose by a permutation of V that x_n is a leaf, and that $x_n x_{n-1} \in E(T)$, so that $I_T = (m_1, \ldots, m_u, x_n x_{n-1})$. Then, since x_n is only adjacent to x_{n-1}, there is an isomorphism

$$\tilde{H}_i(K^{\mathbf{1}}(I_T)) \to \tilde{H}_{i-1}(K^{\mathbf{1}}(I'))$$

where $I' = I_{T-x_n} \cap (x_n x_{n-1})$.

We can split $E(T - x_n)$ in two sets E_1, E_2, where E_1 consists in all the edges in $T - x_n$ incident to x_{n-1} or its neighbors, and E_2 consists in the remaining edges. By a permutation of the m_i, suppose that there is some $k \in \{0, \ldots, u\}$ such that $E_2 = \{m_1, \ldots, m_k\}$ and $E_1 = \{m_{k+1}, \ldots, m_u\}$. We can deal with the edges in E_1 as follows:

$$I_{E_1} = (\operatorname{lcm}(m_i, x_{n-1}x_n) : x_{n-1} \mid m_i)$$

since every other edge in E_1 has the form $x_r x_s$ where x_s is adjacent to x_{n-1}, so $\operatorname{lcm}(x_r x_s, x_{n-1}x_n) = x_r x_s x_{n-1} x_n$ which is divisible by $x_s x_{n-1} x_n = \operatorname{lcm}(x_s x_{n-1}, x_{n-1} x_n)$. We can then reorder the edges in E_1 in such a way that there is some $l \in \{k+1, u\}$ such that $x_{n-1} \mid m_i$ for $k+1 \leq i \leq l$ and $x_{n-1} \nmid m_i$ for $l < i \leq u$. So, with this,

$$I' = (m_1 x_{n-1} x_n, \ldots, m_k x_{n-1} x_n, m_{k+1} x_n, \ldots, m_l x_n)$$

After this we can go back and see that

$$\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_i(K^{\mathbf{1}}(I'_T))$$

where $I'_T = (m_1, \ldots, m_k, x_{m_{k+1}}, \ldots, x_{m_l}, x_{n-1}x_n)$ for $x_{m_i} = \frac{m_i}{x_{n-1}}$. Now, since no generating monomial of I'_T besides $x_{n-1}x_n$ is divisible by x_{n-1} or x_n, we can use the proposition 3.2.5:

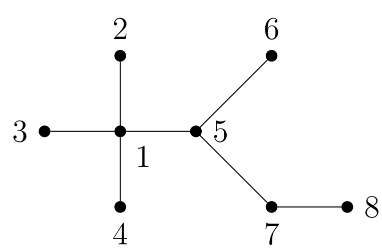
$$\tilde{H}_i(K^{\mathbf{1}}(I'_T)) \cong \tilde{H}_{i-1}(K^{\mathbf{1}'}(I''))$$

where $I'' = (m_1, \ldots, m_k, x_{m_{k+1}}, \ldots, x_{m_l})$ and $\mathbf{1}' = (1, \ldots, 1, 0, 0)$. Now, there is no generating monomial of I'' such that there is some variable in $\{x_{m_{k+1}}, \ldots, x_{m_l}\}$ dividing it, because E_2 consists in the edges of T not adjacent to x_{n-1} or any of its neighbors (The variables $x_{m_{k+1}}, \ldots, x_{m_l}$ are the neighbors of x_{n-1}). So, for all $i \in \{1, \ldots, k\}$, m_i is not divisible by any of the variables $x_{m_{k+1}}, \ldots, x_{m_l}$. Then, by an iterative application of the proposition 3.2.3 we have

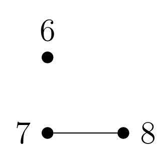
$$\tilde{H}_{i-1}(K^{\mathbf{1}'}(I'')) \cong \tilde{H}_{i-d_T(x_{n-1})}(K^{\mathbf{1}''}(\tilde{I}))$$

where $\mathbf{1}''$ results from $\mathbf{1}'$ by making all the entries of $\mathbf{1}'$ corresponding to the variables $x_{m_{k+1}}, \ldots, x_{m_l}$ zero and $\tilde{I} = (m_1, \ldots, m_k)$. But $\tilde{I}$ is the edge ideal of a subforest T' of T, with less vertices than T. The identity of the subforest T' is clear: Its edges are just the edges not incident to x_{n-1} or any of its neighbors, i.e. it's $(T - N_T(x_{n-1})) - x_{n-1}$. The result follows. ∎

Example 3.3.2. Let T be the following graph:



Then by the previous lemma $\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_{i-4}(K^{\mathbf{1}'}(I_{T'}))$, where T' is as follows:



An isolated vertex means the ideal $I_{T'}$ doesn't use the variable associated to 6, which means $\tilde{H}_i(K^{\mathbf{1}'}(I_{T'})) = 0$ for all i. Therefore $\tilde{H}_i(K^{\mathbf{1}}(I_T))$ is also 0 for all i. This can be verified. $K^{\mathbf{1}}(T)$ is generated by the facets

$$\{3,4,5,6,7,8\}$$
$$\{2,3,4,6,7,8\}$$
$$\{1,2,3,4,6,8\}$$
$$\{2,3,5,6,7,8\}$$
$$\{2,4,5,6,7,8\}$$
$$\{1,2,3,4,7,8\}$$
$$\{1,2,3,4,5,6\}$$

which can be saved as a .sim file from a text editor, e.g. UpperKoszul.sim, and running the command *homsimpl UpperKoszul.sim* with CHOMP, we get the following result:

```
HOMSIMPL, ver. 0.01, 11/09/04. Copyright (C) 1997-2013
by Pawel Pilarczyk.
This is free software. No warranty. Consult 'license.txt'
for details.
[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'UpperKoszul.sim'... 7 simplices
read.
Collapsing faces in X... ..... 184 removed, 1 left.
Note: The dimension of X decreased from 5 to 0.
Creating the chain complex of X... Done.
Time used so far: 0.01 sec (0.000 min).
Computing the homology of X over the ring of integers...
H_0 = Z
Total time used: 0.02 sec (0.000 min).
Thank you for using this software. We appreciate your
business.
```

since this is an unreduced homology, it's the same result we got.

This can also be computed in Macaulay, using the command (without the line breaks)

```
load "SimplicialComplexes.m2";
```

```
R = ZZ[a..e];

X = simplicialComplex{c*d*e*f*g*h,b*c*d*f*g*h,a*b*c*d*f*h,

b*c*e*f*g*h,b*d*e*f*g*h,a*b*c*d*g*h,a*b*c*d*e*f};

C = chainComplex X

prune HH X
```

which gives us the reduced homology of the simplicial complex:

```
i1 : load "SimplicialComplexes.m2";

i2 :
     R = ZZ[a..h];

i3 :
     X = simplicialComplex{c*d*e*f*g*h,b*c*d*f*g*h,
     a*b*c*d*f*h,b*c*e*f*g*h,b*d*e*f*g*h,a*b*c*d*g*h,
     a*b*c*d*e*f};

i4 :
     C = chainComplex X

       1       8       28       53       57       32       7
o4 = ZZ  <-- ZZ  <-- ZZ   <-- ZZ   <-- ZZ   <-- ZZ   <-- ZZ

     -1      0       1        2        3        4        5

o4 : ChainComplex

i5 :
     prune HH X

o5 = -1 : 0

      0 : 0
```

```
     1 : 0

     2 : 0

     3 : 0

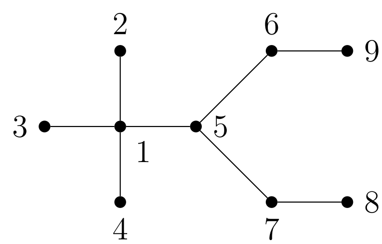
     4 : 0

     5 : 0

o5 : GradedModule
```

The following example has nonzero homology:

Example 3.3.3. Let T be the following graph:



With the same process as before, $\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_{i-4}(K^{\mathbf{1}}(I_{T'}))$ where T' is the following graph:

Then, using the proposition 3.2.5, we get that $\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_{i-5}(K^{\mathbf{1}}(I_{T''}))$ where T'' is just an edge, so $\tilde{H}_i(K^{\mathbf{1}}(I_{T''})) \cong k^{\delta_{i,-1}}$, i.e. it has nonzero homology only at dimension -1, and that homology is k. Therefore $K^{\mathbf{1}}(I_T)$ has nonzero homology only at dimension 4. Again, this can be verified computationally: The Upper Koszul complex is just given by

$$
\begin{gathered}
\{2,4,5,6,7,8,9\} \\
\{1,2,3,4,6,8,9\} \\
\{1,2,3,4,7,8,9\} \\
\{3,4,5,6,7,8,9\} \\
\{2,3,4,6,7,8,9\} \\
\{2,3,5,6,7,8,9\} \\
\{1,2,3,4,5,6,9\} \\
\{1,2,3,4,5,7,8\}
\end{gathered}
$$

CHOMP returns

```
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for details.
[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'UpperKoszul.sim'...
8 simplices read.
Collapsing faces in X... ...... 234 removed, 164 left.
Note: The dimension of X decreased from 6 to 4.
Creating the chain complex of X... .... Done.
Time used so far: 0.02 sec (0.000 min).
Computing the homology of X over the ring of
integers...
Reducing D_4: 0 + 21 reductions made.
Reducing D_3: 32 + 0 reductions made.
Reducing D_2: 21 + 0 reductions made.
Reducing D_1: 3 + 4 reductions made.
H_0 = Z
H_1 = 0
H_2 = 0
H_3 = 0
H_4 = Z
Total time used: 0.03 sec (0.001 min).
```

```
Thank you for using this software.
We appreciate your business.
```

After a while, Macaulay also returns:

```
i5 :
     prune HH X

o5 = -1 : 0

      0 : 0

      1 : 0

      2 : 0

      3 : 0

            1
      4 : ZZ

      5 : 0

      6 : 0

o5 : GradedModule
```

since the calculations become a bit complex computationally to do them directly using the chain complex (Since CHOMP collapses lots of faces before creating the chain complex it's a more efficient software for calculations of homology).

Definition 3.3.1. Let T be a forest. We say that T is sequentially star-coverable of class r if there is a sequence of subgraphs $\Sigma_0, \ldots, \Sigma_r \subseteq T$ such that

1. Σ_j is a star for every $j \in \{0, \ldots, r\}$.
2. For $T_0 = T$ and $T_j = T_{j-1} - V(\Sigma_{j-1})$ (for $j \in \{1, \ldots, r\}$), Σ_j is induced in T_j and Σ_j contains at least one leaf of T_j different from its center.

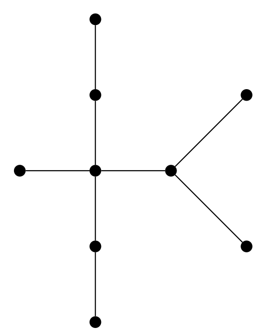
3. $V(\Sigma_i) \cap V(\Sigma_j) = \varnothing$ for $i \neq j$.
4. $V(T) = \bigcup_{j=1}^{r} V(\Sigma_j)$.

Proposition 3.3.4. Let T be a forest. If T is sequentially star-coverable of class r then

$$\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \begin{cases} k & \text{if } i = n - r - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise $\tilde{H}_i(K^{\mathbf{1}}(I_T)) = 0$ for every i.

Proof. A repeated use of the previous lemma shows that if T is sequentially star-coverable of class r then by a counting argument we get the result after the second-last graph in the process is a star (The last one is the empty one). Otherwise when we end up with a graph with no edges, independently of the choice of stars, we'll end up with a graph made up of isolated vertices. If the second to last graph already had isolated vertices, then we're done. Otherwise the second to last graph is a tree, of height 3 on which the root has a neighbor of degree 1, similar to the one in the following picture:



In this case we can change our choice of stars. Removing one of the stars of a neighbor of the root which does not have degree 1 (If every neighbor had degree 1 the graph would be a star and T would be sequentially star-coverable; a contradiction) we isolate the vertex of degree 1, making all the homologies $\tilde{H}_i(K^{\mathbf{1}}(I_T))$ become 0. ■

The proposition above also shows that the class r of T only depends on the homology of $K^{\mathbf{1}}(I_T)$. So, independently of the choice of stars for a sequentially star-coverable graph, the number of stars remains constant.

This allows us to increase the number of graphs for which we can compute their Betti numbers.

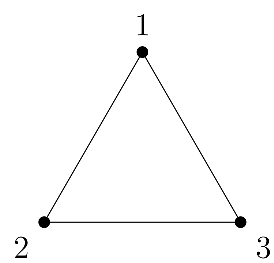
Proposition 3.3.5. Let P_n be the path with n vertices. Then P_n is sequentially star-coverable iff $n \not\equiv 1 \bmod 3$. If $n \equiv 0 \bmod 3$ then its class is $r = \frac{n}{3}$, and if $n \equiv 2 \bmod 3$ then its class is $r = \frac{n+1}{3}$.

Corollary 3.3.6. Let P_n be the path with n vertices. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_{P_n})) \cong \begin{cases} k & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-6}{3}, \\ k & \text{if } n \equiv 2 \bmod 3 \text{ and } i = \frac{2n-7}{3}, \\ 0 & \text{if } n \equiv 1 \bmod 3. \end{cases}$$

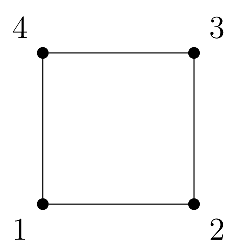
We'll actually be able to use this to compute the Betti numbers of a something which is totally not a forest. First we'll show two examples:

Example 3.3.7. Let $G = C_3$, i.e. the graph given by the following figure:



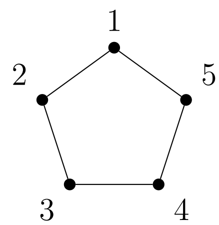
The ideal I_G is (x_1x_2, x_2x_3, x_1x_3). The upper Koszul complex $K^{\mathbf{1}}(I_G)$ is given by the facet set $\{\{1\}, \{2\}, \{3\}\}$. This simplicial complex is a discrete set and its homology $\tilde{H}_i$ is k^2 iff $i = 0$ and 0 otherwise.

Example 3.3.8. Let $G = C_4$, i.e. the graph given by the following figure:



The ideal I_G is $(x_1x_2, x_2x_3, x_3x_4, x_1x_4)$. The upper Koszul complex $K^{\mathbf{1}}(I_G)$ is given by the facet set $\{\{3,4\}, \{1,4\}, \{1,2\}, \{2,3\}\}$. This simplicial complex is also C_4 and its homology $\tilde{H}_i$ is k iff $i = 1$ and 0 otherwise.

Example 3.3.9. Let $G = C_5$, i.e. the graph given by the following figure:



The ideal I_G is just

$$(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5).$$

The upper Koszul complex $K^{\mathbf{1}}(I_G)$ is given by the facet set

$$\{\{3,4,5\},\{1,4,5\},\{1,2,5\},\{1,2,3\},\{2,3,4\}\}.$$

This simplicial complex can be seen as $A \cup B$ where

$$A = \{\{1,2,3\},\{1,2,5\},\{1,4,5\}\}$$

and

$$B = \{\{2,3,4\},\{3,4,5\}\}.$$

Both A and B have all homologies 0 and

$$A \cap B = \{\{2,3\},\{4,5\}\},$$

so that $\tilde{H}_i(A \cap B) \cong k$ if $i = 0$ and 0 otherwise. Therefore $\tilde{H}_i(K^{\mathbf{1}}(I_G)) \cong k$ iff $i = 1$ and 0 otherwise.

Proposition 3.3.10. Let C_n be the cycle with n vertices $x_1, \ldots, x_n$. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \cong \begin{cases} k^2 & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-6}{3}, \\ k & \text{if } n \equiv 2 \bmod 3 \text{ and } i = \frac{2n-7}{3}, \\ k & \text{if } n \equiv 1 \bmod 3 \text{ and } i = \frac{2n-5}{3}. \end{cases}$$

Proof. First suppose that $n > 5$. We take a recursive Mayer-Vietoris sequence as follows:

$$\cdots \to \tilde{H}_i(K^{\mathbf{1}}(I')) \to \tilde{H}_i(K^{\mathbf{1}}(I_{P_n})) \oplus \tilde{H}_i(K^{\mathbf{1}}(x_1x_n)) \to \tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \to \cdots$$

where

$$\begin{aligned}
I' &= I_{P_n} \cap (x_1x_n) \\
&= (x_1x_2, x_2x_3, \ldots, x_{n-2}x_{n-1}, x_{n-1}x_n) \cap (x_1x_n) \\
&= (x_1x_2x_n, x_1x_2x_3x_n, \ldots, x_1x_{n-2}x_{n-1}x_n, x_1x_{n-1}x_n) \\
&= (x_1x_2x_n, x_1x_3x_4x_n, \ldots, x_1x_{n-3}x_{n-2}x_n, x_1x_{n-1}x_n) \\
&= (x_2, x_3x_4, \ldots, x_{n-3}x_{n-2}, x_{n-1}) \cap (x_1x_n).
\end{aligned}$$

We know that for all i,

$$\tilde{H}_i(K^{\mathbf{1}}(x_1x_n)) = 0$$

and,

$$\tilde{H}_i(K^{\mathbf{1}}(P_n)) \cong \begin{cases} k & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-6}{3}, \\ k & \text{if } n \equiv 2 \bmod 3 \text{ and } i = \frac{2n-7}{3}, \\ 0 & \text{if } n \equiv 1 \bmod 3, \end{cases}$$

so the result will also depend on the remainder of n modulo 3. Furthermore for $h_i(K^{\mathbf{1}}(x_2, x_3x_4, \ldots, x_{n-3}x_{n-2}, x_{n-1})) = 0$ for every i, since $x_1, x_n \notin \mathrm{Supp}(x_2, x_3x_4, \ldots, x_{n-3}x_{n-2}, x_{n-1})$. So we have a second recursive Mayer-Vietoris sequence:

$$\cdots \to \tilde{H}_{i+1}(K^{\mathbf{1}}(I'')) \to \tilde{H}_i(K^{\mathbf{1}}(I')) \to 0 \to \tilde{H}_i(K^{\mathbf{1}}(I'')) \to \cdots$$

where $I'' = (x_2, x_3x_4, \ldots, x_{n-3}x_{n-2}, x_{n-1}, x_1x_n)$. For a suitable $\mathbf{1}'$, we have

$$\begin{aligned}
\tilde{H}_i(K^{\mathbf{1}}(I'')) &\cong \tilde{H}_{i-3}(K^{\mathbf{1}'}(P_{n-4})) \\
&\cong \begin{cases} k & \text{if } n \equiv 1 \bmod 3 \text{ and } i = \frac{2n-5}{3}, \\ k & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-6}{3}, \\ 0 & \text{if } n \equiv 2 \bmod 3. \end{cases}
\end{aligned}$$

Therefore, for all i,

$$\tilde{H}_i(K^{\mathbf{1}}(I')) \cong \tilde{H}_{i+1}(K^{\mathbf{1}}(I'')) \cong \begin{cases} k & \text{if } n \equiv 1 \bmod 3 \text{ and } i = \frac{2n-8}{3}, \\ k & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-9}{3}, \\ 0 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

So, in the first Mayer-Vietoris sequence, we know all the homologies, except for $\tilde{H}_i(K^{\mathbf{1}}(I_{C_n}))$. Suppose first that $n \cong 2 \bmod 3$. In this case the Mayer-Vietoris sequence divides into sequences:

$$0 \to \tilde{H}_i(K^{\mathbf{1}}(I_{P_n})) \to \tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \to 0$$

In this case
$$\tilde{H}_i(K^{\mathbf{1}}(I_{P_n})) \cong \tilde{H}_i(K^{\mathbf{1}}(I_{C_n}))$$
for every i.

Suppose, then that $n \cong 1 \bmod 3$. In this case, we have sequences of the form:
$$0 \to \tilde{H}_{i+1}(K^{\mathbf{1}}(C_n)) \to \tilde{H}_i(K^{\mathbf{1}}(I')) \to 0$$
which means that
$$\tilde{H}_{i+1}(K^{\mathbf{1}}(C_n)) \cong \tilde{H}_i(K^{\mathbf{1}}(I'))$$
for every i.

Lastly, suppose that $n \equiv 0 \bmod 3$. Then, the sequence
$$\cdots \to \tilde{H}_i(K^{\mathbf{1}}(I')) \to \tilde{H}_i(K^{\mathbf{1}}(I_{P_n})) \to \tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \to \cdots$$
divides into the sequences
$$0 \to \tilde{H}_s(K^{\mathbf{1}}(I_{P_n})) \to \tilde{H}_s(K^{\mathbf{1}}(I_{C_n})) \to \tilde{H}_{s-1}(K^{\mathbf{1}}(I')) \to 0$$
for $s = \frac{2n-6}{3}$, and
$$0 \to \tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \to 0$$
for $i \neq s$. Then we have the short exact sequence:
$$0 \to k \to \tilde{H}_s(K^{\mathbf{1}}(I_{C_n})) \to k \to 0$$
which makes
$$\tilde{H}_s(K^{\mathbf{1}}(I_{C_n})) \cong k^2.$$
Therefore
$$\tilde{H}_i(K^{\mathbf{1}}(I_{C_n})) \cong \begin{cases} k^2 & \text{if } n \equiv 0 \bmod 3 \text{ and } i = \frac{2n-6}{3}, \\ k & \text{if } n \equiv 2 \bmod 3 \text{ and } i = \frac{2n-7}{3}, \\ k & \text{if } n \equiv 1 \bmod 3 \text{ and } i = \frac{2n-5}{3}. \end{cases}$$
Since the formula above also holds for $n < 5$, then we're done. ∎

Chapter 4

Conclusion - Future Work

There is another simplicial complex which can, heuristically (At least for now), be used to compute the Betti numbers of an edge ideal. So, let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and $I = I_G$ its associated ideal. The upper Koszul complex can be seen as the set of all sets $\sigma \subseteq V$ of vertices which are not vertex-covers in G, i.e. there are edges in G non-incident with any vertex of σ. Indeed, if $\sigma \in X = K^{\mathbf{1}}(I)$ then $\mathbf{x}^{\mathbf{1}-\sigma} \in I$ means that there is some monomial in I dividing $\mathbf{x}^{\mathbf{1}-\sigma}$, which is the same as the vertices of the edge associated to that monomial not being vertices in σ.

We also have the complex $K_{\mathbf{1}}(I)$ of all the independent sets of G, which is the Alexander dual of $K^{\mathbf{1}}(I)$, so it's homology also gives (after a shifting and a translation in dimension) the Betti numbers of I.

But analogously we have another simplicial complex which also can be used for that: The complex $\mathcal{K}(I)$ of all the non edge-covers of the graph. Intuitively, there is no reason why this complex should be able to compute the Betti numbers of the graph, at least not directly, and even if it did, since it almost has a higher dimension than $K^{\mathbf{1}}(I)$ and $K_{\mathbf{1}}(I)$, it should be expected to have the homologies in different dimensions. This is surprisingly not the case. In the case of forests, since $|E| < |V|$, the dimension of this complex is usually lower, though.

A first easy result is that $\mathcal{K}(G)$ has the same Euler characteristic of $K^{\mathbf{1}}(I)$. Next we'll use three examples which show they have the same homology (In these cases they're even homotopically equivalent). Then we'll show an example which shows that it also seems to hold in more general squarefree ideals. The non edge-covers of hypergraphs are still contained in complements of stars, and the complements of stars are still non edge-covers, so the maximal non edge-covers are still along the complements of stars. One example we'll use is the hypergraph which upper Koszul complex is the projective plane (since the previous examples lack torsion).

This is normally not a more efficient way to compute Betti numbers, since it has almost always more vertices than the Koszul complexes (Since its vertices are the edges of the graph, instead of the vertices, as in the Koszul complexes). It would be a new result, nonetheless: This complex does not appear in the literature of complexes associated to graphs.

In our search for some related simplicial complexes we also came upon the simplicial complex of the matchings of the graph associated to I. This one does appear in the literature (See [7]), and its rational homology is known, but unfortunately does not coincide with the homology of $K^{\mathbf{1}}(I)$. It is to be seen if there is a relationship between them and if some of its topological information can be translated into the algebraical information of I. We can also look into the homology of $K^{\mathbf{1}}(I_{L(G)})$, where $L(G)$ is the line graph of G, i.e. a graph with E as a vertex set and edges ee' such that $e \cap e' \neq \varnothing$.

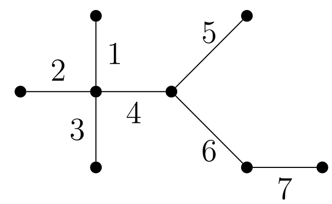
4.1 Complex of non edge-covers

In this section $\mathcal{K}(G)$ is the simplicial complex of all the edge sets of G which don't cover V, where $G = (V, E)$ is a graph with vertex set $V = \{1, \ldots, n\}$. Since the complement of a non edge-cover contains a star, and the maximal non edge-covers are all complements of stars, then

$$\mathcal{K}(G) = K^{\mathbf{1}}(I_{\Sigma(G)})$$

where $\Sigma(G)$ is the hypergraph of all the stars of G, i.e. the hypergraph with vertex set $E(G)$ and edge set $\{\sigma_1, \ldots, \sigma_n\}$ for $\sigma_i = \{e \in E \mid i \in e\}$. There are no issues about leaving in $\Sigma(G)$ the stars corresponding to not maximal non edge-covers of G, since for stars σ_i, σ_j, if we have that $E - \sigma_j \subseteq E - \sigma_i$ then we also have $\sigma_i \subseteq \sigma_j$ and $m_i \mid m_j$, where for each i, m_i is the monomial in $k[E]$ associated to σ_i.

Example 4.1.1. Consider the following tree T:



The complex $\mathcal{K}(T)$ is given by the facets:

$$\{2,3,4,5,6,7\},$$
$$\{1,3,4,5,6,7\},$$
$$\{1,2,4,5,6,7\},$$
$$\{1,2,3,4,6,7\},$$
$$\{1,2,3,4,5,6\}.$$

The three remaining complements of stars were just $\{5,6,7\}$, $\{1,2,3,7\}$ and $\{1,2,3,4,5\}$. According to CHOMP, the homology of this simplicial complex is given by

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for details.
[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'Edgenoncover.sim'... 1
simplices read.
Collapsing faces in X... ..... 62 removed, 1 left.
Note: The dimension of X decreased from 5 to 0.
Creating the chain complex of X... Done.
Time used so far: 0.02 sec (0.000 min).
Computing the homology of X over the ring of integers...
H_0 = Z
Total time used: 0.02 sec (0.000 min).
Thank you for using this software. We appreciate
your business.
```

We'll develop a method for computing these homologies which will show it agrees with the homology of $K^{\mathbf{1}}(I_T)$, at least for forests. We'll start showing that the Euler characteristic of both complexes is the same. The way to prove this is not so direct, so we'll need some results to do it.

Definition 4.1.1. For a monomial ideal $I \subseteq S$, define

$$\mu(I) = \sum_{m \in \operatorname{Mon} I} m \in k[[x_1, \ldots, x_n]]$$

as the sum of all monomials of I, where the sum above is a formal sum.

We can have some more compact expresions for $\mu(I)$ in $k((x_1, \ldots, x_n))$, for example, for $I = (1)$ we have

$$\mu(I) = \prod_{i=1}^{n} \frac{1}{1 - x_i}.$$

For a principal ideal (m), we have that

$$\mu((m)) = m \prod_{i=1}^{n} \frac{1}{1 - x_i}.$$

Definition 4.1.2. For a subset $\sigma \subseteq N = \{1, \ldots, n\}$ define

$$P_\sigma = \prod_{i \in \sigma} \frac{1}{1 - x_i} \in k((x_1, \ldots, x_n)).$$

The expression above is really in $k(x_1, \ldots, x_n)$ but its meaning as a series can only be grasped in $k((x_1, \ldots, x_n))$.

Definition 4.1.3. For a monomial ideal $I = (m_1, \ldots, m_r)$ and a subset $\sigma \subseteq \{1, \ldots, r\}$ define

$$m_\sigma = \operatorname{lcm}(m_i \mid i \in \sigma)$$

Definition 4.1.4. For a monomial ideal $I = (m_1, \ldots, m_r)$, we'll define its lcm-poset as the poset

$$L_I = \{m_\sigma \mid \sigma \subseteq \{1, \ldots, r\}\},$$

ordered by divisibility. For a monomial $q \in L_I$ define

$$W_I(q) = \{J \subseteq \{1, \ldots, n\} \mid m_J = q\}.$$

Then define

$$\omega_I(q) = \sum_{c \in W_I(q)} (-1)^{|c|-1}$$

and $Q_i = \{q \in L_I \mid \omega_I(q) \neq 0\}$.

Lemma 4.1.2. For a monomial ideal I generated minimally by

$$M = \{m_1, \ldots, m_r\}$$

we have

$$\mu(I) = \sum_{q \in Q_I} \omega_I(q) q P_N.$$

Proof. For $\sigma \subseteq \{1, \ldots, r\}$ let $A_\sigma = \mathrm{Mon}((m_\sigma))$, i.e. the set of all the monomials divisible by m_σ. So, since

$$\bigcap_{i \in \sigma} A_i = A_\sigma$$

we have, by the inclusion-exclusion principle applied to the sets A_j, that

$$\mu(I) = \mu\left(\bigcup_{i \in N} A_i\right) = \sum_{k=1}^{r} (-1)^{k-1} \sum_{\sigma \in N_k} \mu\left(\bigcap_{i \in \sigma} A_i\right) = \sum_{k=1}^{r} (-1)^{k-1} \sum_{\sigma \in N_k} \mu(A_J)$$

$$= \sum_{q \in Q_I} \mu((q)) \left(\sum_{c \in W_I(q)} (-1)^{|c|-1} \right) = \sum_{q \in Q_i} \omega_I(q) \mu((q)) = \sum_{q \in Q_I} \omega(q) q P_N.$$

where N_k is the set of all the subsets of N with k elements. ■

Now, for a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ and $I = I_G$, consider the set $\mathcal{C}(G)$ as the set of all edge-covers of G. For an induced subgraph H of G it's clear that

$$\omega_G(H) := \omega_I(q(H)) = \sum_{c \in \mathcal{C}(H)} (-1)^{|c|-1},$$

since $W_I(q(H)) = \mathcal{C}(H)$, where $q(H) = \mathrm{lcm}(e | e \in E(H))$.

Therefore,

$$\mu(G) := \mu(I_G) = \sum_{q \in Q_I} \omega(H) q(H) P_N.$$

Using the previous remark and a Möbius inversion formula it can be proven that

$$\omega(G) = (-1)^n \sum_{k=0}^{\alpha(G)} (-1)^{k-1} u_k = (-1)^n \chi(K_{\mathbf{1}}(G)).$$

This can be rewritten as:

Proposition 4.1.3. For a graph G with n vertices,

$$\chi(K_{\mathbf{1}}(I_G)) = (-1)^{n-1}\chi(\mathcal{K}(G)).$$

There is also a relationship between $\chi(\Delta)$ and $\chi(\Delta^\vee)$ for any simplicial complex Δ. First, it's clear that

$$\sum_{\sigma\in\Delta}(-1)^{|\sigma|-1} + \sum_{\sigma\notin\Delta}(-1)^{|\sigma|-1} = \chi(\Delta^{n-1}) = 0.$$

Also

$$\begin{aligned}
\sum_{\sigma\notin\Delta}(-1)^{|\sigma|-1} &= (-1)^n(-1)^n\sum_{\sigma\notin\Delta}(-1)^{|\sigma|-1} \\
&= (-1)^n\sum_{\sigma\notin\Delta}(-1)^{n-|\sigma|-1} \\
&= (-1)^n\sum_{\sigma\notin\Delta}(-1)^{|\Delta^n-\sigma|-1} \\
&= (-1)^n\sum_{\sigma\in\Delta^\vee}(-1)^{|\sigma|-1}. \\
&= (-1)^n\chi(\Delta^\vee).
\end{aligned}$$

Therefore, $\chi(\Delta) = (-1)^{n-1}\chi(\Delta^\vee)$. So it follows that

Corollary 4.1.4. For a graph G with n vertices,

$$\chi(\mathcal{K}(G)) = \chi(K^{\mathbf{1}}(I_G)).$$

We can go further, getting the following results, ironically, in the opposite order to the way we got them in the previous chapter.

Theorem 4.1.5. Let $G = C_n$ be the cycle with n vertices. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_G)) \cong \tilde{H}_i(\mathcal{K}(G))$$

for all i.

Proof. In this case, if $V(G) = \{v_1,\ldots,v_n\}$ and $E(G) = \{e_1,e_2,\ldots,e_n\}$, where $e_i = v_iv_{i+1}$ for $i<n$ and $e_n = v_1v_n$, the star ideal of G is

$$I_{\Sigma(G)} = (e_1e_2, e_2e_3, \ldots, e_{n-1}e_n, e_ne_1),$$

which coincides to the edge ideal of G (up to a change of variables). Therefore, $\mathcal{K}(G) = K^{\mathbf{1}}(I_\Sigma(G)) = K^{\mathbf{1}}(I_G)$. The result follows. ■

Theorem 4.1.6. Let $G = P_n$ be the path with n vertices. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_G)) \cong \tilde{H}_i(\mathcal{K}(G))$$

for all i.

Proof. In this case, if $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_{n-1}\}$, where for $i < n$ $e_i = v_i v_{i+1}$, the star ideal of G is

$$I = I_{\Sigma(G)} = (e_1, e_1e_2, e_2e_3, \ldots, e_{n-2}e_{n-1}, e_{n-1}).$$

Since $e_1 \mid e_1e_2$ and $e_n \mid e_{n-1}e_n$ is not generated minimally, and

$$I = (e_1, e_2e_3, \ldots, e_{n-2}e_{n-1}, e_n).$$

Since e_1, e_n are independent from the remaining monomials generating I, we have, for all i, that

$$\tilde{H}_i(K^{\mathbf{1}}(I)) \cong \tilde{H}_{i-2}(K^{\mathbf{1}'}(I'))$$

where

$$I' = (e_2e_3, \ldots, e_{n-2}e_{n-1})$$

and $\mathbf{1}'$ results from removing the entries of $\mathbf{1}$ associated to e_1 and e_n. This is, up to a change of variables, the edge ideal of P_{n-3}. Therefore

$$\tilde{H}_i(\mathcal{K}(G)) \cong \tilde{H}_{i-2}(K^{\mathbf{1}'})(I_{P_{n-3}}).$$

Since we already know, by applying the lemma 3.3.1, that

$$\tilde{H}_i(K^{\mathbf{1}}(G)) \cong \tilde{H}_{i-2}(K^{\mathbf{1}'})(I_{P_{n-3}}).$$

the result follows. ∎

Theorem 4.1.7. Let T be a forest with n vertices and m edges. Then

$$\tilde{H}_i(K^{\mathbf{1}}(I_T)) \cong \tilde{H}_i(\mathcal{K}(T))$$

for all i.

Proof. First, if T is a star, then $\mathcal{K}(T)$ consists of all the edge-sets except for $\{e_1, \ldots, e_m\}$. Therefore $\mathcal{K}(T)$ is the boundary of the simplex with vertex set $e_1, \ldots, e_m$, i.e. the sphere S^{m-2}. The homology of $\mathcal{K}(T)$ can be computed directly. The chain complex of $\Delta = \Delta^{m-1}$ is an exact sequence of $k-$vector spaces

$$0 \leftarrow C_{-1}(\Delta) \leftarrow C_0(\Delta) \leftarrow C_1(\Delta) \leftarrow \cdots \overset{\psi}{\leftarrow} C_{m-2}(\Delta) \overset{\phi}{\leftarrow} C_{m-1}(\Delta) \leftarrow 0$$

where $C_{m-1}(\Delta) \cong k$ is generated by the full face $\{e_1, \ldots, e_m\}$. Note that $\ker \psi = \operatorname{im} \phi \cong k$. Now, the chain complex of $\mathcal{K}(T)$ is

$$0 \leftarrow C_{-1}(\Delta) \leftarrow C_0(\Delta) \leftarrow C_1(\Delta) \leftarrow \cdots \overset{\psi}{\leftarrow} C_{m-2}(\Delta) \leftarrow 0$$

which is exact everywhere except in $C_{m-2}(\Delta)$. So, we have

$$\tilde{H}_{m-2}(\mathcal{K}(T)) \cong \ker \psi \cong k.^{(\mathrm{i})}$$

Since stars are connected, then $m-2 = n-3$ and the result follows: When T is a star we have

$$\tilde{H}_i(\mathcal{K}(T)) \cong \begin{cases} k & \text{if } i = m-2 = n-3 \\ 0 & \text{otherwise.} \end{cases}$$

So the second case is next: If T is any graph which has an isolated vertex then $\mathcal{K}(T)$ is the full simplex with E as a vertex set. Therefore $\tilde{H}_i(\mathcal{K}(T)) = 0$ for all i. For the last case, consider T as any forest with edges and no isolated vertices. Let v be a leaf of T, e be the only edge incident to v, and S the star of the only vertex w adjacent to v. Let $S_1, \ldots, S_r$ the stars of T which don't share edges with S and $S_{r+1}, \ldots, S_{n-2}, S, e$ the remaining stars. Then

$$I = I_{\Sigma(T)} = (S_1, \ldots, S_r, S_{r+1}, \ldots, S_{n-2}, S, e) = (S_1, \ldots, S_r, S_{r+1}, \ldots, S_{n-2}, e)$$

since $I_{\Sigma(T)}$ is not generated minimally. Now, since e is independent from the remaining monomials generating I, we have

$$\tilde{H}_i(\mathcal{K}(T)) \cong \tilde{H}_{i-1}(K^{\mathbf{1}'}(I'))$$

where $I' = (S_1, \ldots, S_r, S_{r+1}, \ldots, S_{n-2})$ and $\mathbf{1}'$ results from removing from $\mathbf{1}$ the entry corresponding to e. Since T is a forest, the stars $S_{r+1}, \ldots, S_{n-2}$ are all independent from each other (Otherwise two of them would share an edge, making a triangle with their two edges incident to w). Also each of these stars has at least one edge not being on any of the stars $S_1, \ldots, S_r$

(i) This holds for any ring, not only fields.

(The edge incident to w). If there is only one remaining star (i.e. $r = 0$ and $n = 3$) then $T = P_3$ and we're done. If, furthermore, we have that $r = 0$ then T is a star, and we're also done here. So we can suppose this is not the case, so that there is an edge in $S_1, \ldots, S_r$ not being in any of the stars $S_{r+1}, \ldots, S_{n-2}$. With this, and a recursive Mayer-Vietoris sequence we have that

$$\tilde{H}_i(K^{\mathbf{1}'}(I')) \cong \tilde{H}_{i-1}K^{\mathbf{1}'}(I_1)$$

where

$$I_1 = (\operatorname{lcm}(S_1, S_{n-2}), \ldots, \operatorname{lcm}(S_r, S_{n-2}), S_{r+1}S_{n-2}, \ldots, S_{n-3}S_{n-2}).$$

Reversing the process as in the lemma 3.3.1, we have the ideal

$$I^{(1)} = (\operatorname{lcm}(S_1, S_{n-2})/S_{n-2}, \ldots, \operatorname{lcm}(S_r, S_{n-2})/S_{n-2}, S_{r+1}, \ldots, S_{n-3})$$

which appears in a Mayer-Vietoris sequence:

$$\cdots \to \tilde{H}_i(K^{\mathbf{1}'}(I_1)) \to \tilde{H}_i(K^{\mathbf{1}'}(I^{(1)})) \to \tilde{H}_i(K^{\mathbf{1}'}(I^{(1)} + (S_{n-2}))) \to \cdots$$

If one of the stars is in fact an edge incident to S_{n-2} then

$$I^{(1)} = (1) = I^{(1)} + (S_{n-2})$$

so, since (1) is principal, $\tilde{H}_i(K^{\mathbf{1}'}(1))$ is k iff $i = -1$ and is 0 otherwise. This sequence turns into the short exact sequences

$$0 \to \tilde{H}_i(K^{\mathbf{1}'}(I_1)) \to 0$$

for $i \geq 0$ and

$$0 \to \tilde{H}_0(K^{\mathbf{1}'}(I_1)) \to k \to k \to 0.$$

Both of these make $\tilde{H}_i(K^{\mathbf{1}'}(I_1)) = 0$ for all i.

So, analogously, if, for some $i > r$ we have that S_i contains a leaf, this will imply that $\tilde{H}_i(K^{\mathbf{1}}(I)) = 0$ for all i. In the context of the lemma 3.3.1, this means that removing $N_T[w]$ leaves an isolated vertex which would make $\tilde{H}_i(K^{\mathbf{1}}(I_T)) = 0$ too. So in this case, we're done.

Suppose next that this is not the case, i.e. no star S_i contains a leaf for $i > r$. Then for each star S_i with $i > r$, every star S_j for $j \leq r$ contains an edge e_j not being in S_i. Furthermore e_j is not in S_i for any other $i > r$, because of the following:

1. If, for every $i > r$, S_j does not share edges with S_i, then no edge of S_j is S_i for any $i > r$.

2. If S_j shares edges with S_i for exactly one i, then, since S_j must have another edge, this edge is not in $S_{i'}$ for any $i' > r$.

3. S_j cannot share edges with S_i for more than one $i > r$, otherwise the edges
$$e \in E(S_j) \cap E(S_i), e' \in E(S_j) \cap E(S_{i'})$$
and
$$e'' \in E(S_i) \cap E(S), e''' \in E(S_{i'}) \cap E(S)$$
make a cycle.

So, in this case, the Mayer-Vietoris sequence above gives an isomorphism $\tilde{H}_i(K^{\mathbf{1}'}(I')) \cong \tilde{H}_i(K^{\mathbf{1}'})(I^{(1)})$. We can then repeat the process to build ideals $I^{(2)}, \dots, I^{(d_T(w)-1)}$, with the same homology in $K^{\mathbf{1}'}$, and where

$$I'' = I^{(d_T(w)-1)} = (S'_1, S'_2, \dots, S'_r, S_{r+1}, \dots, S_{n-2})$$

and, for each $j \leq r$, S'_j is a monomial not sharing any edge variable with any S_i for $i > r$. Now we can safely remove the stars $S_i, i > r$ from I'' since these monomials are independent from each other monomial in I''. So,

$$\tilde{H}_i(K^{\mathbf{1}'}(I')) \cong \tilde{H}_{i-d_T(w)+1}(K^{\mathbf{1}''}(I''))$$

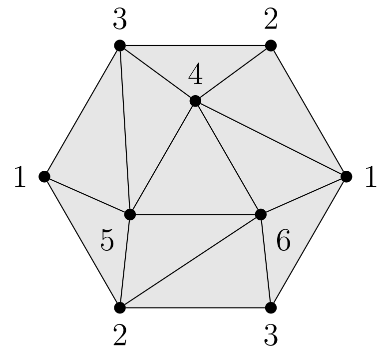
where $\mathbf{1}''$ results from removing the entries of $\mathbf{1}'$ associated to the edges of the removed stars. Therefore

$$\tilde{H}_i(K^{\mathbf{1}}(I)) \cong \tilde{H}_{i-d_T(w)}K^{\mathbf{1}''}(I'').$$

But $K^{\mathbf{1}''}(I'') = \mathcal{K}(I_{T-N_T[w]})$ and I'' is the star ideal of $T - N_T[w]$, which means that the homology of $\mathcal{K}(T)$ and $K^{\mathbf{1}}(I_T)$ satisfy the same recurrence relation, and they also satisfy the base cases of such recurrence (stars and forests with isolated vertices). The result follows. ■

With such a strong evidence, what's left to do is to prove that this result holds for any graph, and even more, for any hypergraph. All our examples until now are independent of the characteristic of k. The following is an example in which the homology of $K^{\mathbf{1}}$ does depend on the characteristic of k.

Example 4.1.8. Consider the following simplicial structure of $\mathbb{R}P^2$:



The integer homology of this simplicial complex is given by

$$\tilde{H}_i(\mathbb{R}P^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This complex is associated to the ideal

$$I = (x_1x_2x_3, x_1x_5x_6, x_1x_2x_6, x_1x_3x_4, x_1x_4x_5, x_3x_5x_6, x_2x_3x_5, \\ x_3x_4x_6, x_2x_4x_6, x_2x_4x_5).$$

so that $K^{\mathbf{1}}(I) = \mathbb{R}P^2$. The complex $\mathcal{K}(I)$ has as facets:

$$\{1,3,5,7,10\},$$
$$\{1,2,4,5,6\},$$
$$\{2,4,7,9,10\},$$
$$\{1,2,3,8,9\},$$
$$\{3,4,6,7,8\},$$
$$\{5,6,8,9,10\}.$$

According to CHOMP, its homology is also $\mathbb{Z}_2$.

```
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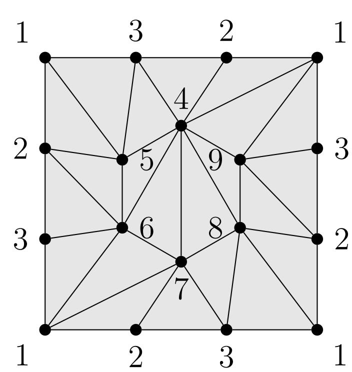
[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'edgenoncover.sim'...
```

```
6 simplices read.
Collapsing faces in X... .... 96 removed, 55 left.
Note: The dimension of X decreased from 4 to 2.
Creating the chain complex of X... .. Done.
Time used so far: 0.01 sec (0.000 min).
Computing the homology of X over the ring of integers...
Reducing D_2: 0 + 18 reductions made.
Reducing D_1: 3 + 6 reductions made.
H_0 = Z
H_1 = Z_2
Total time used: 0.02 sec (0.000 min).
Thank you for using this software. We appreciate your business.
```

When we wonder about the problem of proving that $K^{\mathbf{1}}(I_G)$ and $\mathcal{K}(G)$ have the same homology, we only need to prove the homologies agree when the coefficients lie on a field. But they might have the same homology over every field and still have different homology over some other ring. This is not the case in the examples above, but this is not surprising, since most of them have no torsion, and when there is no torsion the homology over any field of characteristic zero is enough to compute the integer homology. The example above is also not surprising, since when the integer homology has simple torsion we still can build the integer homology from the homology over a few fields. The following example is one when the integer homology of $K^{\mathbf{1}}$ and $\mathcal{K}$ is still the same even if this homology is not recovered completely from their homology over every field.

Example 4.1.9. Consider the following simplicial complex X:



This is a simplicial structure of a CW-complex generalizing the projective plane having as integer homology $\tilde{H}_1(X) \cong \mathbb{Z}_4$ (It's built by taking a circle and gluing a $2-$cell by a map going around the circle four times) and zero everywhere else. Its homology over $\mathbb{Z}_2$ is $\tilde{H}_1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\tilde{H}_2(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$. It's zero everywhere else. The associated ideal has many generators:

$$\begin{aligned} I = (&x_1x_3x_4x_7x_8x_9, x_2x_4x_5x_6x_7x_8, x_1x_2x_6x_7x_8x_9, x_1x_4x_5x_6x_8x_9,\\ &x_2x_4x_5x_6x_7x_9, x_1x_4x_5x_7x_8x_9, x_3x_4x_5x_6x_7x_9, x_2x_4x_5x_7x_8x_9,\\ &x_3x_4x_5x_6x_8x_9, x_1x_5x_6x_7x_8x_9, x_2x_4x_6x_7x_8x_9, x_3x_4x_6x_7x_8x_9,\\ &x_3x_5x_6x_7x_8x_9, x_1x_2x_3x_5x_6x_7, x_1x_3x_4x_5x_6x_7, x_1x_2x_3x_5x_6x_9,\\ &x_1x_2x_4x_5x_6x_9, x_1x_2x_3x_5x_8x_9, x_1x_2x_3x_7x_8x_9, x_2x_3x_4x_5x_8x_9,\\ &x_1x_4x_5x_6x_7x_8, x_2x_3x_5x_6x_7x_8). \end{aligned}$$

The complex $\mathcal{K}(I)$ has the facets:

$$\begin{aligned} &\{4,9,16,17,18,20\},\\ &\{3,10,13,14,16,18,19,22\},\\ &\{1,3,11,12,19\},\\ &\{2,3,4,5,6,8,10,11,17,21\},\\ &\{1,4,6,7,9,10,12,13,15,21\},\\ &\{1,6,8,18,19,20\},\\ &\{2,5,7,8,9,11,12,13,20,22\},\\ &\{5,7,14,15,16,17\},\\ &\{2,14,15,21,22\} \end{aligned}$$

and its homology over $\mathbb{Z}$ and $\mathbb{Z}_2$ are the same as the ones of X[ii].

Over $\mathbb{Z}$:

```
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[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'edgenoncover.sim'...
9 simplices read.
```

[ii] These spaces are called the Eilenberg-Maclane spaces.

```
Collapsing faces in X... ......... 3268 removed, 203 left.
Note: The dimension of X decreased from 9 to 2.
Creating the chain complex of X... .. Done.
Time used so far: 0.03 sec (0.000 min).
Computing the homology of X over the ring of integers...
Reducing D_2: 0 + 80 reductions made.
Reducing D_1: 21 + 0 reductions made.
H_0 = Z
H_1 = Z_4
Total time used: 0.04 sec (0.001 min).
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```

Over $\mathbb{Z}_2$:

```
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[Tech info: simpl 4, chain 12, addr 4, intgr 2.]
Reading simplices to X from 'edgenoncover.sim'...
9 simplices read.
Collapsing faces in X... ......... 3268 removed, 203 left.
Note: The dimension of X decreased from 9 to 2.
Creating the chain complex of X... .. Done.
Time used so far: 0.03 sec (0.001 min).
Computing the homology of X over Z modulo 2...
Reducing D_2: 0 + 79 reductions made.
Reducing D_1: 18 + 3 reductions made.
H_0 = Z_2
H_1 = Z_2
H_2 = Z_2
Total time used: 0.04 sec (0.001 min).
Thank you for using this software. We appreciate your business.
```

4.2 Combinatorial Hochster's Formula

For edge ideals of a graph G we're interested in making a combinatorial construction which helps us compute, not only the Betti numbers of I_G, but an entire minimal free resolution of it. An idea of such construction goes as follows.

Consider a minimal vertex-cover c of the graph. Then the vertex set associated to it is not a face of the simplicial complex $K^{\mathbf{1}}(I_G)$, but every proper subset of it is a face. So the boundary of c is a potential element of $C_\bullet(K^{\mathbf{1}}(I_G))$ with nonzero homology. Now consider the subgraph resulting from G after removing the edges with both end of c, and call it G_c. The set of all G_c generates an abelian group with symmetric difference as its sum. This is expected to have as its dimension at least the combined dimension of $\tilde{H}(K^{\mathbf{1}}(I_G))$. Furthermore it's also expected that this correspondence preserves the operations of the homology group.

Chapter 5

Appendix

5.1 Java Code

The following code was used to compute the Koszul homology of a graph given its incidence matrix. The code just computes the facets of both the edge non-cover complex, and the upper Koszul complex of an edge ideal, given its incidence matrix. Then it runs the `homsimpl` command of CHOMP (Computational Homology Project) to compute both homologies, printing the output in a JFrame.

```java
package Hedgeideal;

import java.awt.Font;
import java.util.*;
import java.io.*;
import javax.swing.*;

/**
 *
 * @author dcmol
 */

public class Hedgeideal {

    public static <T> Set<Set<T>> powerSet(Set<T> aSet) {
        Set<Set<T>> sets = new HashSet<Set<T>>();
        if (aSet.isEmpty()) {
```

```
            sets.add(new HashSet<T>());
            return sets;
        }
        List<T> list = new ArrayList<T>(aSet);
        T head = list.get(0);
        Set<T> rest = new HashSet<T>(
                list.subList(1, list.size()));
        for (Set<T> set : powerSet(rest)) {
            Set<T> eachSet = new HashSet<T>();
            eachSet.add(head);
            eachSet.addAll(set);
            sets.add(eachSet);
            sets.add(set);
        }
        return sets;
    }
    public static String Edgenotation(Set edgeset, int v) {
        String Ret="";
        for (Object e : edgeset) {
            Ret=Ret+","+tag((Set)e,v);
        }
        Ret=Ret.substring(1);
        Ret="{"+Ret+"}";
        return Ret;
    }
    public static String nnotation(Set edgeset, int v) {
        String Ret="";
        for (Object e : edgeset) {
            Ret=Ret+","+tag((Set)e,v);
        }
        return Ret.substring(1);
    }
    public static String Nnotation(Set set) {
        String Ret="";
        String S=set.toString();
        String E=S.substring(1, S.length()-1);
        Ret="{"+E+"}";
        return Ret;
    }
    public static String tomonomial(Set<Integer> edge) {
```

```
        String Ret="";
        Integer[] EDGE=edge.toArray(new Integer[edge.size()]);
        Ret=Ret+"x"+Integer.toString(EDGE[0])+"*"
            +"x"+Integer.toString(EDGE[1]);
        return Ret;
    }
    public static String edgeideal(Set edgeset) {
        String Ret="";
        for (Object edge: edgeset) {
            Ret=Ret+","+tomonomial((Set)edge);
        }
        Ret=Ret.substring(1);
        Ret="("+Ret+")";
        return Ret;
    }
    public static String tag(Set<Integer> edge, int v) {
        String Ret="";
        Set E=new HashSet();
        for (int i = 1; i < v; i++) {
            for (int j = i+1; j < v+1; j++) {
            Set e=new HashSet<Integer>();
            e.add(i);
            e.add(j);
            E.add(e);

            }
        }
        Object[] EE=E.toArray();
        for (int i = 0; i < EE.length; i++) {
            if (EE[i].equals(edge)) {
                Ret=Ret+Integer.toString(i+1);
            }
        }
        return Ret;
    }
    public static void main(String[] args)
            throws IOException {
    int e=0;
    int v=0;
    Set V=new HashSet();
```

```
Set E=new HashSet();
List textread=new ArrayList();
BufferedReader reader;
  try {
     reader = new BufferedReader(new FileReader(
            "IncidenceMatrix.txt"));
            /*
            You can write the incidence matrix in a text file
            removing all brackets, commas and spaces. For
            example, you can use the array
            1100
            1010
            1001
            0110
            0101
            0011
            for the complete graph with four vertices.
            */
     String line = reader.readLine();
     while (line != null) {
            e++;
            v=line.length();
            textread.add(line);
        line = reader.readLine();

     }
     reader.close();
  } catch (IOException exception) {
        System.out.println("Error reading the file.");
  }
    for (int i = 1; i < v+1; i++) {
        V.add(i);
    }
    for (Object l : textread) {
        String ll=(String)l;
        Set m=new HashSet<Integer>();
        for (int i = 0; i < ll.length(); i++) {
            String lu=ll.substring(i, i+1);
            int mu=Integer.parseInt(lu);
            if (mu!=0) {
```

```
                m.add(i+1);
            }
        }
        E.add(m);
    }
    System.out.println("Vertex set:"+V);
    System.out.println("Edge set:"+E);
    Set Simp=new HashSet();
    for (Object i : V) {
        Set notcov=new HashSet();
        for (Object q : E) {
            Set f=(Set)q;
            if (!f.contains(i)) {
                notcov.add(f);
            }
        }
        Simp.add(notcov);
    }
    Set Facets=new HashSet();
    for (Object B : Simp) {
        Set C=(Set)B;
        boolean esmax=true;
        for (Object A : Simp) {
            Set D=(Set)A;
            if (D.containsAll(C)
                &&!C.containsAll(D)) {
            esmax=false;
            }
        }
        if (esmax) {
            Facets.add(C);
        }
    }
    System.out.println("Simp"+Facets);
    /*The following is to fix the notation of the complex, to use
    it as input for CHOMP*/
    Set NFacets=new HashSet();
    for (Object o : Facets) {
        NFacets.add(Edgenotation((Set)o,v));
    }
```

```
System.out.println(NFacets);
String noncoversimp="";
PrintWriter Cod = new PrintWriter("Edgenoncover.sim");
for (Object f : NFacets) {
    Cod.println(f);
    noncoversimp=noncoversimp+f+","+"<br/>";
}
noncoversimp="{"
    +noncoversimp.substring(
        0,noncoversimp.length()-6)+"}";
Cod.close();
/*Here we run CHOMP from the command line to compute
    homology*/
Process p = Runtime.getRuntime().exec("cmd /c \"homsimpl "
    + "Edgenoncover.sim -g Edgenoncover.txt\"");
BufferedReader stdInput = new BufferedReader(new
InputStreamReader(p.getInputStream()));
BufferedReader stdError = new BufferedReader(new
InputStreamReader(p.getErrorStream()));
PrintWriter Result = new PrintWriter("Edgehomology.txt");
/*Here we create a frame for the output*/
String s = null;
JFrame f = new JFrame();
JPanel pa = new JPanel();
JLabel la = new JLabel();
f.setSize(500, 500);
f.setDefaultCloseOperation(JFrame.EXIT_ON_CLOSE);
f.add(pa);
String Outp="The Simplicial Complex of edge non-covers "
    + "is<br/>X="+noncoversimp;
JScrollPane pane = new JScrollPane(pa);
pa.add(la);
f.setContentPane(pane);
f.setVisible(true);
la.setFont(new Font("Arial", Font.PLAIN, 12));
while ((s = stdInput.readLine()) != null) {
    System.out.println(s);
    Result.println(s);
    if (s.contains("Reducing")) {
        s=s.substring(0, 13)+s.substring(s.length()-29);
```

```
    }
    if (s.contains("Collapsing")) {
        s=s.substring(0, 21)+s.substring(s.length()-29);
    }
    if (s.contains("Press")) {
        s="";
    }
    if (s.contains("Thank")) {
        s="";
    }
    if (s.contains("Copyright")) {
        s="";
    }
    if (s.contains("free")) {
        s="";
    }
    if (s.contains("Tech")) {
        s="";
    }
    Outp=Outp+"<br/>"+s;
    la.setText("<html>"+Outp+"<br/>CopyRight (C) 1997-2013 "
        + "by Pawel Pilarczyk"+"</html>");
}
Result.close();
while ((s = stdError.readLine()) != null) {
    System.out.println(s);
}
Set K=new HashSet();
for (Object q : E) {
    Set w=new HashSet<Integer>();
    w.addAll(V);
    w.removeAll((Set)q);
    K.add(w);
}
Set NK=new HashSet();
for (Object k : K) {
    NK.add(Nnotation((Set)k));
}
System.out.println("The Upper Koszul Complex is");
Outp=Outp
```

```
    +"<br/>The Upper Koszul Complex is<br/> X={";
System.out.println(Nnotation(NK));
PrintWriter Kos = new PrintWriter("UpperKoszul.sim");
for (Object k : NK) {
    Outp=Outp+k+",<br/>";
    Kos.println(k);
}
Outp=Outp.substring(0, Outp.length()-6)+"}";
la.setText("<html>"+Outp+"<br/>CopyRight (C) 1997-2013 "
    + "by Pawel Pilarczyk"+"</html>");
Kos.close();
Process p2 = Runtime.getRuntime().exec("cmd /c \"homsimpl "
    + "UpperKoszul.sim -g UpperKoszul.txt\"");
BufferedReader stdInput2 = new BufferedReader(new
InputStreamReader(p2.getInputStream()));
BufferedReader stdError2 = new BufferedReader(new
InputStreamReader(p2.getErrorStream()));
PrintWriter Result2 = new PrintWriter("KoszulHomology.txt");
while ((s = stdInput2.readLine()) != null) {
    System.out.println(s);
    Result.println(s);
    if (s.contains("Reducing")) {
        s=s.substring(0, 13)
            +s.substring(s.length()-29);
    }
    if (s.contains("Collapsing")) {
        s=s.substring(0, 21)
            +s.substring(s.length()-29);
    }
    if (s.contains("Press")) {
        s="";
    }
    if (s.contains("Thank")) {
        s="";
    }
    if (s.contains("Copyright")) {
        s="";
    }
    if (s.contains("free")) {
        s="";
```

```
            }
            if (s.contains("Tech")) {
                s="";
            }
            Outp=Outp+"<br/>"+s;
            la.setText("<html>"+Outp+"<br/>CopyRight (C) 1997-2013 "
                + "by Pawel Pilarczyk"+"</html>");
        }
        Result2.close();
        while ((s = stdError.readLine()) != null) {
            System.out.println(s);
        }
    }
}
```

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