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Local Zeta Functions for Graphs

A dissertation presented by

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Resumen

En esta tesis, definimos y estudiamos la función zeta local asociada a un gráfico simple-finito. Este es un nuevo objeto matemático que se construye en el marco de funciones zeta multivariadas, definidas por Loeser en [33]. La continuación meromórfica de la función zeta local de un gráfico como función racional se desprende del trabajo de Loeser. Aquí nos centramos en métodos explícitos para calcular esta función zeta. Nuestro resultado principal es un algoritmo recursivo para el cálculo de dichos objetos. Los resultados de este trabajo se utilizaron en el artículo. [49], escrito en colaboración con mis asesores, el Dr. Wilson Zúñiga Galindo y el Dr. Edwin León Cardenal. En este artículo, estudiamos las transiciones de fase para gases log-Coulomb usando funciones zeta locales. Por otro lado, toda la programación en Python de los algoritmos presentados en esta tesis fue realizada únicamente por el autor.

Abstract

In this thesis, we define and study the local zeta function attached to a finite, simple graph. This is a new mathematical object that is constructed in the framework of multivariate zeta functions, defined by Loeser in [33]. The meromorphic continuation of the local zeta function of a graph as a rational function follows from the work of Loeser. Here we focus on explicit methods of computing this zeta function. Our main result is a recursive algorithm for the computation of such objects. The results of this work were used in the article [49], which was written in collaboration with my advisors, Dr. Wilson Zúñiga Galindo and Dr. Edwin León Cardenal. In this article, we study phase transitions for log-Coulomb gases using local zeta functions. On the other hand, all the programming in Python of the algorithms presented in this thesis was carried out solely by the author.

Overview

Let $\mathbb{K}$ be a local field, for instance $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, and let $f(x) \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-constant polynomial and let $\phi(x)$ be a test function. The local zeta function attached to the pair $(f(x), \phi(x))$ is defined as

$$Z_\phi(s, f) = \int_{\mathbb{K}^n \setminus f^{-1}(0)} \phi(x) \, |f(x)|_{\mathbb{K}}^s \, d^n x, \quad Re(s) > 0,$$

where $|\cdot|_{\mathbb{K}}$ denotes the absolute value of $\mathbb{K}$, $s \in \mathbb{C}$, and $d^n x$ denotes a normalized Haar measure of the topological group $(\mathbb{K}^n, +)$. These integrals give rise to holomorphic functions of s in the half-plane $Re(s) > 0$.

The main motivation to study local zeta functions was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. It is important to mention here, that in the p-adic framework, the existence of fundamental solutions for pseudodifferential operators is also a consequence of the fact that the Igusa local zeta functions, i.e. when $\mathbb{K} = \mathbb{Q}_p$, admit a meromorphic continuation (see e.g. [47, Theorem 5.5.1]).

In the middle 60s, Weil initiated the study of local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [44]. In the 70s, Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero [20], [22]. Later, Loeser introduced in [33] the multivariate zeta functions, which constitute a generalization of the $Z_\phi(s, f)$.

In the last thirty-five years there has been a strong interest on p-adic models of quantum field theory, which is motivated by the fact that these models are exactly solvable. There is a large list of p-adic type Feynman and string amplitudes that are related with local zeta functions of Igusa-type, and it is interesting to mention that it seems that the mathematical community working on local zeta functions is not aware of this fact (see e.g. [30], [47], [48], [29], and the references therein).

Let $(\mathbb{Q}_p, |\cdot|_p)$ denote the field of p-adic numbers. A test function $\varphi(x)$ on $\mathbb{Q}_p^n$ is a locally constant function with compact support, see Section 1.2. Given G a finite, simple graph and $\varphi(x)$ a test function, the local zeta function attached to G, $\varphi(x)$ is defined as

$$Z_\varphi(\boldsymbol{s}; G) = \int_{\mathbb{Q}_p^{|V(G)|}} \varphi(\boldsymbol{x}) \prod_{\substack{u,v \in V(G) \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{v \in V(G)} dx_v,$$

where $\boldsymbol{s} = (s(u,v))$ for $u, v \in V(G)$ for $u \sim v$, $s(u,v)$ is a complex variable attached to the edge connecting the vertices u and v, and $\prod_{v \in V(G)} dx_v$ is a Haar measure of the locally compact group $(\mathbb{Q}_p^{|V(G)|}, +)$. The integral converges for $\mathrm{Re}(s(u,v)) > 0$ for any (u,v).
The integral $Z_\varphi(\boldsymbol{s}; G)$ is a particular case of a multivariate local zeta function, and it is known that they admit meromorphic continuations as rational function to the whole space $\mathbb{C}^{|V(G)|}$, see [33, Theorem 1.1.4].

There exists a large family of zeta functions attached to finite graphs, which can be considered as discrete analogues of the Riemann zeta function, see [42] and the references therein. There are also zeta functions attached to infinite graphs, see e.g. [5], [14], [16], and attached to hypergraphs [23].

The graph zeta functions that are studied in this work are related to the p-adic Feynman integrals, worked out by Lerner and Missarov in [31]. In [49], we use the integrals $Z_{\varphi}(\boldsymbol{s}; G)$ to study log-gases on a network (given by a graph G) confined in a bounded subset of a local field (i.e. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$ the field of p-adic numbers). In this gas, a log-Coulomb interaction between two charged particles occurs only when the sites of the particles are connected by an edge of the network. The partition functions of such gases turn out to be a particular class of multivariate local zeta functions attached to the network and a test function which is determined by the confining potential. The methods and results of the theory of local zeta functions allow us to establish that the partition functions admit meromorphic continuations in the parameters.

In the case of p-adic fields the meromorphic continuations of the partition functions are rational functions in the parameters. We give an Algorithm *ZetaFunctionGraph* for computing such rational functions, see Section 3.3.

From a physical perspective, the study of models over ultrametric spaces started in the 80s with the works of Frauenfelder, Parisi, Stein, among others, see e.g. [11], [13], [38], see also [24], [25], [47], and the references therein. The Ising models over ultrametric spaces have been studied intensively, see e.g. [12], [15], [26], [32], [36], [35], [37], and the references therein.

We now describe in detail the results and contributions presented in this thesis. In Chapter 1 we present the essential aspects of p-adic analysis, including the definitions of the field of p-adic numbers and the Bruhat-Schwartz space. We also define the integration over $\mathbb{Q}_p^n$ and the change of variables theorem. Chapter 2 contains an introduction to local zeta functions, including a sketch of the proof of its meromorphic continuation. We also show a relation between local zeta functions and Poincaré series. Finally, we present the definition of multivariate zeta functions, which is the main object of this thesis.

Chapter 3 is the core of this work. We present a review of graph theory in Section 3.1, we also define vertex coloring, see Definition 14, and chromatic function, see Definition 18. They are essential to computing $Z(\boldsymbol{s}; G)$. Vertex coloring, notion introduced here, is different from classical colorings for graph, since we use the first one to get some subgraphs of a fixed graph G while, the second is used to find independent subsets of $V(G)$. Despite this, they are related by Proposition 1. Additionally, we define the set $Indgraphs(G)$, see Definition 17, which provide a relation between poles of $Z(\boldsymbol{s}; G)$ and subgraphs of G. We define the local zeta function for a graph G in Section 3.2, where we show that this zeta function is invariant by isomorphism of graphs, i.e., if G are H are isomorphic graphs, then $Z(\boldsymbol{s}; G) = Z(\boldsymbol{s}; H)$, see Lemma 3. We also show some examples where we exhibit the use of vertex coloring and chromatic function to calculate $Z(\boldsymbol{s}; G)$ for some graphs. Finally, Section 3.3 contains the main results, which are Theorem 7 and Proposition 2. Theorem 7 contains a recursive formula to calculate $Z(\boldsymbol{s}; G)$ for connected graphs. This result also provide a list of possible set of poles $Z(\boldsymbol{s};G)$. Proposition 2 presents a relation between poles of $Z(\boldsymbol{s};G)$ and subgraphs H in $Indgraphs(G)$. Along this chapter, we present examples of the new objects introduced to clarify these notions.

Chapter 4 contains closed formulas of $Z(\boldsymbol{s};G)$ for some graphs. In Section 4.3 we use Algorithm *ZetaFunctionGraph* to get the irreducible factors of the denominator of $Z(\boldsymbol{s};G)$ for some well-known graphs.

In order to facilitate the understanding of Algorithm *ZetaFunctionGraph*, we have included after the theoretical developments, the corresponding subroutines. These subroutines are then incorporated in *ZetaFunctionGraph*.

In appendix B, we present the implementation *ZetaFunctionGraph.py* of Algorithm *ZetaFunctionGraph*.

The development of the implementation *ZetaFunctionGraph.py*, in the Python language, has taken advantage of the potential given by the classes and methods of the SymPy library, which were very useful in the development of the program since symbolic calculations were already implemented.

SymPy (`https://www.sympy.org/en/index.html`) is a Python library for symbolic mathematics. It aims to become a full-featured computer algebra system (CAS) while keeping the code as simple as possible in order to be comprehensible and easily extensible. SymPy is written entirely in Python.

Contents

Chapter 1

Essential aspects of p-adic analysis

In this document, p will be a fixed prime number. In this chapter, we present some important results about of p-adic numbers. For an in-depth review of p-adic analysis, the reader may consult [28], [1], and the references therein.

1.1 The field of p-adic numbers

Given x a nonzero rational number, we can represent x as $p^r a/b$ where p does not divide a neither b. We define $r = ord_p(x)$,to be the p-adic order of x. We also set $ord_p(0) := \infty$.

The p-adic norm, define in $\mathbb{Q}$ is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-ord_p(x)} & \text{if} \quad x \neq 0. \end{cases}$$

Th norm $|\cdot|_p$ is non-archimedean, i.e., for every x and y, rational numbers, $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. This implies, for instance, that for any integer number a, $|a|_p \leq 1$.

The metric space $(\mathbb{Q}, |\cdot|_p)$ is not a complete space, its completion $(\mathbb{Q}_p, |\cdot|_p)$ is called the field of p-adic numbers.

Any nonzero p-adic number x has a representation as a power series of the form

$$p^{ord_p(x)} \sum_{i=0}^{\infty} x_i p^i,$$

where $x_i \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$.

The p-adic numbers satisfying $|x|_p \leq 1$ are called *p-adic integers*. The set of p-adic integers is denoted as $\mathbb{Z}_p$. Furthermore, $\mathbb{Z}_p$ is a discrete valuation ring with maximal ideal $p\mathbb{Z}_p$. The group of units $\mathbb{Z}_p^\times$ consists of the p-adic numbers x with $|x|_p = 1$. *The residue field* $\mathbb{Z}_p/p\mathbb{Z}_p$ of $\mathbb{Z}_p$ is the field with p elements, $\mathbb{F}_p$.

1.1.1 The topology of $\mathbb{Q}_p$

We present some general definitions and results about the topology of $\mathbb{Q}_p$. For further details, the reader may consult [43, Section 1.3] and [1, Section 1.8].

We define *the ball with center a and radius p^r* as

$$B_r(a) = \{x \in \mathbb{Q}_p;\ |x-a|_p \leq p^r\},$$

and *the sphere with center a and radius p^r* as

$$S_r(a) = \{x \in \mathbb{Q}_p;\ |x-a|_p = p^r\}.$$

When the center of a ball or a sphere is 0, we use B_r and S_r instead of $B_r(0)$ and $S_r(0)$, respectively. Note that $B_0 = \mathbb{Z}_p$ and $S_0 = \mathbb{Z}_p^{\times}$.

The following assertions provide the main properties of the topology of $\mathbb{Q}_p$.

1. $\left(\mathbb{Q}_p, |\cdot|_p\right)$ is a separable, complete, ultrametric space, with dense sub-space $\mathbb{Q}$.
2. For every $a \in \mathbb{Q}_p$ and $r \in \mathbb{Z}$, $B_r(a)$ is a compact set of $\left(\mathbb{Q}_p, |\cdot|_p\right)$. This implies that a sub-set A of $\mathbb{Q}_p$ is compact if and only if A is closed and bounded in $\mathbb{Q}_p$. Thus $\left(\mathbb{Q}_p, |\cdot|_p\right)$ is a locally compact topological space.
3. $\mathbb{Q}_p$ is a totally disconnected space, i.e., a subset A of $\mathbb{Q}_p$ is connected if and only if $A = \{x\}$ for some $x \in \mathbb{Q}_p$ or $A = \emptyset$.
4. For every $a, a' \in \mathbb{Q}_p$ and $r, r' \in \mathbb{Z}$, either $B_r(a) \cap B_{r'}(a') = \emptyset$ or one of the balls is contained in the other one.
5. For every $a \in \mathbb{Q}_p$ and $l \in \mathbb{Z}$, $B_l(a) = a + p^{-l}\mathbb{Z}_p$ and $S_l(a) = a + p^{-l}\mathbb{Z}_p^{\times}$.
6. $\mathbb{Z}_p = \bigsqcup_{i=0}^{p-1} (i + p\mathbb{Z}_p)$.

1.1.2 The topology of $\mathbb{Q}_p^n$

We will review some topological properties of $\mathbb{Q}_p^n$.

For this, we extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$\|(x_1, \ldots, x_n)\|_p = \max\{|x_1|_p, \ldots, |x_n|_p\}.$$

Moreover, this norm is also a non-archimedean norm.

We denote the *ball with center a and radius p^r* as

$$B_r^n(a) = \{x \in \mathbb{Q}_p^n;\ \|x-a\|_p \leq p^r\},$$

and *the sphere with center a and radius p^r* as

$$S_r^n(a) = \{x \in \mathbb{Q}_p^n;\ \|x-a\|_p = p^r\}.$$

We set, $B_r^n(0) := B_r$ and $S_r^n(0) := S_r^n$.

Properties (1-4) of Section 1.1.1 also hold in $\mathbb{Q}_p^n$. For any $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$ we have $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, this implies that the product topology of $\mathbb{Q}_p^n$ is equal to the topology induced by the norm $\|\cdot\|_p$.

The reader may find a proof of these assertions in [1, Section 1.8] and [43, Section 1.3].

1.2 The Bruhat-Schwartz space

A complex-valuated function $\phi : \mathbb{Q}_p^n \to \mathbb{C}$ is called *locally constant*, if for any $x \in \mathbb{Q}_p^n$ there exists $l(x) \in \mathbb{Z}$ such that:

$$\phi(y) = \phi(x); \quad y \in B_{l(x)}^n(x).$$

A *Bruhat-Schwartz function* or *test function* $\phi : \mathbb{Q}_p^n \to \mathbb{C}$ is a locally constant function with compact support. Since ϕ has compact support there exists $l \in \mathbb{Z}$ such that for any $x \in \mathbb{Q}_p^n$,

$$\phi(y) = \phi(x), \quad \text{for any } y \in B_l^n(x).$$

We define the set of test functions as $D(\mathbb{Q}_p^n)$.

Example 1. *The characteristic function $1_A(x)$ of a open and compact subset $A \subseteq \mathbb{Q}_p^n$ is test function.*

1.3 Integration over $\mathbb{Q}_p^n$

We now review Haar's theorem for locally compact topological groups, which allow us to develop an integration theory over $\mathbb{Q}_p^n$. For further details, the reader may consult [43, Chapter 4] and [1, Chapter 3].

Theorem 1. *([18, Thm B. Sec.58]) Let $(G, +)$ be a locally compact topological group. There exists a Borel measure dx, unique up to multiplication by a positive constant, such that $\int_U dx > 0$ for every non empty Borel open set U, and $\int_{x+E} dx = \int_E dx$, for every Borel set E.*

The measure dx is called a Haar measure of G. Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, by Theorem 1, it has a Haar measure dx. We normalize this measure using the condition $\int_{\mathbb{Z}_p} dx = 1$.

In the n-dimensional case, we denote by $d^n x$ the product measure $\underbrace{dx \cdots dx}_{n-times}$. This measure satisfies that $d^n(x + a) = d^n x$, for $a \in \mathbb{Q}_p^n$, and $\int_{\mathbb{Z}_p^n} d^n x = 1$.

Example 2. *1. $\int_{p\mathbb{Z}_p} dx = p^{-1}$. Indeed,*

$$1 = \int_{\mathbb{Z}_p} dx = \int_{\bigsqcup_{i=0}^{p-1} i+p\mathbb{Z}_p} dx = \sum_{i=0}^{p-1} \int_{i+p\mathbb{Z}_p} dx = \sum_{i=0}^{p-1} \int_{p\mathbb{Z}_p} dx = p \int_{\mathbb{Z}_p} dx.$$

2. $\int_{B^n_{-1}} d^n x = p^{-n}$.

3. $\int_{S^n_{-1}} d^n x = (1 - p^{-n})$, *this formula is obtained as follows:*

$$\int_{S^n_{-1}} d^n x = \int_{\mathbb{Z}_p^n \setminus B^n_{-1}} d^n x = \int_{\mathbb{Z}_p^n} d^n x - \int_{B^n_{-1}} d^n x = 1 - p^{-n}.$$

4. $\int_{B^n_r(a)} d^n x = p^{rn}$.

5. $\int_{S^n_r(a)} d^n x = p^r(1 - p^{-n})$.

6. *Set* $s \in \mathbb{C}$ *with* $Re(s) > 0$, *then*

$$\int_{a+p^l\mathbb{Z}_p} |x|_p^{s-1}\, dx = \begin{cases} p^{-ls}\left(\frac{1-p^{-1}}{1-p^{-s}}\right) & \text{if} \quad a \in p^l\mathbb{Z}_p \\ p^{-l}\,|a|_p^{s-1} & \text{if} \quad a \notin p^l\mathbb{Z}_p. \end{cases} \tag{1.3.1}$$

Indeed,

if $a \notin p^l\mathbb{Z}_p$, *by changing variables as* $x \mapsto y - a$ *we have*

$$\int_{a+p^l\mathbb{Z}_p} |x|_p^{s-1}\, dx = \int_{p^l\mathbb{Z}_p} |-a+y|_p^{s-1}\, dy = |a|_p^{s-1} \int_{p^l\mathbb{Z}_p} dy = p^{-l}\,|a|_p^{s-1}.$$

If $a \in p^l\mathbb{Z}_p$ *then* $a + p^l\mathbb{Z}_p = p^l\mathbb{Z}_p$, *by changing variables as* $x \mapsto yp^l$ *we have*

$$\begin{aligned} \int_{a+p^l\mathbb{Z}_p} |x|_p^{s-1}\, dx &= p^{-l} \int_{\mathbb{Z}_p} \left|p^l y\right|_p^{s-1} dy = p^{-ls} \int_{\mathbb{Z}_p} |y|_p^{s-1}\, dy \\ &= p^{-ls} \sum_{k=0}^{\infty} \int_{S_k} |y|_p^{s-1}\, dy = p^{-ls} \sum_{k=0}^{\infty} p^{-k(s-1)} p^{-k}(1 - p^{-1}) \\ &= p^{-ls}(1 - p^{-1}) \sum_{k=0}^{\infty} p^{-ks} = p^{-ls} \frac{1 - p^{-1}}{1 - p^{-s}}. \end{aligned}$$

1.3.1 Change of variables

In order to establish the change of variables theorem for p-adic integrals we first introduce the notion of analytic function, following Igusa' book [22, Section 2.4].

For $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we set $x^i = x_1^{i_1} \cdots x_n^{i_n}$.

Definition 1. 1. *Let* $f : U \to \mathbb{Q}_p$ *be a function in* U, *an open subset of* $\mathbb{Q}_p^n$. *We say that* f *is an analytic function, if for all* $a \in U$ *there exists* $l \in \mathbb{Z}$ *and a convergent power series* $\sum_{i \in \mathbb{N}^n} b_i x^i$ *in* $B^n_l(a) \subseteq U$, *such that* $f(x) = \sum_{i \in \mathbb{N}^n} b_i x^i$ *for all* $x \in B^n_l(a)$. *In this case,* $\frac{\partial}{\partial x_j} f(x) = \sum_{i \in \mathbb{N}^n} b_i \frac{\partial}{\partial x_j} x^i$.

2. *A function* $F : U \subseteq \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ *is called an analytic function, if every* f_i *in* $F = (f_1, \ldots, f_m)$ *is an analytic function. If* $n = m$, *we denote* $\frac{\partial F(x)}{\partial x_1 \cdots \partial x_n}$ *the determinant of the Jacobian matrix,* $\left(\frac{\partial f_i(x)}{\partial x_j}\right)_{1 \le i,j \le n}$, $x \in U$.

Theorem 2. *([22, Proposition 7.4.1]) Let K_1, $K_2 \subseteq \mathbb{Q}_p^n$ be compact open subsets, and let $F = (f_1, \ldots, f_n) : K_2 \to K_1$ a bi-analytic map such that*

$$\frac{\partial F(y)}{\partial y_1 \cdots \partial y_n} \neq 0, \quad \textit{for any } y \in K_2.$$

If ϕ is a continuous function on K_1, then

$$\int_{K_1} \phi(x)\, d^n x = \int_{K_2} \phi(F(y)) \left| \frac{\partial F(y)}{\partial y_1 \cdots \partial y_n} \right|_p d^n y, \quad (x = F(y))\,.$$

Chapter 2

Local zeta functions

In this chapter, we review some well-known results about local zeta functions, the Poincaré series attached to a polynomial with p-adic integer coefficients. For an in-depth discussion of the classical aspects of the local zeta functions, we recommend [10], [33], [22], [29].

Definition 2. *Let $f(x)$ be a non-constant polynomial in $\mathbb{Q}_p[x_1,\ldots,x_n]$ and let ϕ be a test function. The Igusa local zeta function attached to the pair (f,ϕ) is*

$$Z_\phi(s,f)=\int_{\mathbb{Q}_p^n\setminus f^{-1}(0)}\phi(x)\,|f(x)|_p^s\,d^nx,$$

for $s\in\mathbb{C}$ and $Re(s)>0$[1]*. In the case $\phi=1_{\mathbb{Z}_p^n}$, we use the notation $Z(s,f)$ instead of $Z_\phi(s,f)$.*

Remark 1. *By using the fact that for $Re(s)>0$ the function $\phi(x)\,|f(x)|_p^s$ is continuous with compact support, and the fact that the Haar measure of any compact set is finite, one immediately obtains that the integral $Z_\phi(s,f)$ converges for $Re(s)>0$.*

Lemma 1. *([22, Lemma 5.3.1]) Let (X,μ) denote a measure space, U a nonempty open subset of $\mathbb{C}$, and f a $\mathbb{C}-$valued measurable function on $X\times U$. Assume that the following properties hold:*

1. *If C is any compact subset of U, there exists an integrable function $\phi_C\geq 0$ on X, satisfying*
$$|f(x,s)|\leq\phi_C(x),$$
for all $(x,s)\in X\times C$.
2. *$f(x,\cdot)$ is a holomorphic function on U for every $x\in X$.*

Then

$$F(s)=\int_X f(x,s)d\mu(x),$$

defines a holomorphic function F on U.

[1] For $a>0$ and $s\in\mathbb{C}$, we set $a^s=e^{s\ln(a)}$.

Corollary 1. *The integral $Z_\phi(s,f)$ is a holomorphic function in s in the half-plane $Re(s) > 0$.*

Proof. Let $X = \mathbb{Q}_p^n$ and $g(x,s) = \phi(x)\,|f(x)|_p^s$ which is a continuous function on $X \times U$ where $U = \{s \in \mathbb{C} \mid Re(s) > 0\}$. Take $C \subseteq U$ a compact set, note that $g(x,s)$ has compact support on $X \times C$, so there exists $A_C \in \mathbb{R}^+$ such that $|g(x,s)|_p \leq A_C$ for all $(x,s) \in X \times C$. We take $\phi_C(x) = A_C 1_{Supp(\phi)}(x)$. By using that $g(x,s)$ is a holomorphic function for any fixed $x \in X$, it follows from Lemma 1 that $Z_\phi(s,f)$ is a holomorphic function in s in the half-plane $Re(s) > 0$. □

2.1 Meromorphic continuation of $Z_\phi(s,f)$

In the middle of the seventies, Igusa proved that for every non-constant polynomial $f(x) \in Q_p[x_1,\dots,x_n]$and any test function $\phi(x_1,\dots,x_n)$, the local zeta function $Z_\phi(s,f)$ attached to (f,ϕ) has a meromorphic continuation to the whole complex plane as a rational function of p^{-s}, see [22, Theorem 5.4.1].

The proof given by Igusa depends on Hironaka's resolution of singularities theorem, which is a profound result in algebraic geometry. For further details the reader may consult [46], [19], [20], [22].

The following definitions and results are based on [22, Section 2].

Definition 3. *Let X be a Hausdorff space and n a fixed non-negative integer. A pair (U,ϕ_U), where U is a nonempty open subset of X and $\phi_U : U \to \phi_U(U) \subseteq \mathbb{Q}_p^n$ is a homeomorphism, is called a chart. For a variable point $x \in U$, the local coordinates of x are $\phi_U(x) = (x_1,\dots,x_n)$.*

A set of charts $\{(U,\phi_U)\}$ is called an atlas, if X equals the union of all U and for every U, U' with $U \cap U' \neq \emptyset$ the map

$$\phi_{U'} \circ \phi_U^{-1} : \phi_U(U \cap U') \to \phi_{U'}(U \cap U')$$

is an analytic function.

Two atlases are considered equivalent if their union is also an atlas. Any equivalence class is called a n-dimensional p-adic analytic structure on X. If $\{(U,\phi_U)\}$ is an atlas in the equivalence class, we say that X is a n-dimensional p-adic analytic manifold, and we write $n = dim(X)$.

Theorem 3. *(Implicit Function Theorem, [22, Theorem 2.1.1])*

Take $F(x,y) = (F_1(x,y),\dots,F_m(x,y))$ an analytic function, such that $F_i(0,0) = 0$ for all i, and

$$\frac{\partial F(0,0)}{\partial x_1 \cdots \partial x_n} \neq 0.$$

Then there exists a unique $f(x) = (f_1(x),\dots,f_m(x))$, analytic function, satisfying $F_i(x,f(x)) = 0$ for all i. Furthermore, if a is near 0 in $\mathbb{Q}_p^n$, then $f(a)$ is near 0 in $\mathbb{Q}_p^m$; and if (a,b) is near $(0,0)$ in $\mathbb{Q}_p^n \times \mathbb{Q}_p^m$ and $F(a,b) = 0$, then $b = f(a)$.

Definition 4. *Let X be a p-adic manifold of dimension n, with defining atlas (U, ϕ_U). Take m a positive integer with $0 < m \le n$. Suppose further that Y is a nonempty closed subset of X and that (U, ϕ_U) can be chosen with the following properties:*

If $\phi_U(x) = (x_1, \dots, x_n)$ and $U' = Y \cap U \ne \emptyset$, then there exists a p-adic analytic function $F = (F_1, \dots, F_m)$ on U such that firstly U' becomes the set of all x in U satisfying $F_1(x) = \cdots = F_m(x) = 0$, and secondly,

$$\frac{\partial F(a)}{\partial x_1 \cdots \partial x_m} \ne 0 \ \text{ at every } a \text{ in } U'.$$

Then by Theorem 3 the mapping $x \mapsto (F_1(x), \dots, F_m(x), x_{m+1}, \dots, x_n)$ is a bi-analytic mapping from a neighborhood of a in U to its image in $\mathbb{Q}_p^n$. If we denote by V the intersection of such neighborhood of a and Y, and put $\psi_V(x) = (x_{m+1}, \dots, x_n)$ for every x in V, then $\{(V, \psi_V)\}$ gives an atlas on Y. Therefore Y becomes a p-adic analytic manifold with $dim(Y) = n - m$. We call Y a closed submanifold of X of codimension m.

Definition 5. *Let X be a p-adic analytic manifold with an atlas $\{(U, \phi_U)\}$.*

1. *Let V be a open subset of X and $F : V \to \mathbb{Q}_p^n$. We say F is p-adic analytic, if $F \circ \phi_U^{-1} : \phi_U(V \cap U) \to \mathbb{Q}_p^n$ is analytic for all U with $U \cap V \ne \emptyset$.*
2. *Fix $a \in X$. If V, V' are neighborhoods of a, and f, g are p-adic analytic functions respectively on V, V' such that $f|_W = g|_W$ for some neighborhood $W \subseteq V \cap V'$ of a, then we say that f and g are equivalent at a.*

   An equivalence class is said to be a germ of analytic functions at a. The set of germs of analytic functions at a form a local ring denoted by $\mathcal{O}_a$.

Definition 6. *Take X a p-adic analytic manifold with atlas $\{(U, \phi_U)\}$. Set α a differential form of degree $n = dim(X)$ on X; then $\alpha|_U$ has an expression of the form*

$$\alpha(x) = f_U(x) dx_1 \wedge \cdots \wedge dx_n,$$

where f_U is an analytic function on U. We denote by μ_n the normalize n-dimensional Haar measure of $\mathbb{Q}_p^n$, this meas that

$$\mu_n(B) = \int_B d^n x$$

for a Borel subset $B \subseteq \mathbb{Q}_p^n$. For A an open and compact subset of X contained in U, we define

$$\mu_\alpha(A) = \int_A d\mu_n(\phi_U(x)) = \sum_{e \in \mathbb{Z}} p^{-e} \mu_n \left(\phi_U \left(f_U^{-1}(p^e \mathbb{Z}_p^\times \cap A) \right) \right).$$

The above series is convergent because $f_U(A)$ is a compact subset of $\mathbb{Q}_p$. The measure μ_α is independent of the chosen chart. Notice that if $X = U \subseteq \mathbb{Q}_p^n$ is an open subset, and $\alpha = dx_1 \wedge \cdots \wedge dx_n$, then $\mu_\alpha = \mu_n$.

Theorem 4. *(Hironaka, [22, Theorem 3.2.1])*

Take $f(x)$ a nonconstant polynomial in $\mathbb{Q}_p[x_1, \ldots, x_n]$, and put $X = \mathbb{Q}_p^n$. Then there exists an n-dimensional p-adic analytic manifold Y, a finite set $T = \{E\}$ of closed submanifolds of Y of co-dimension 1 with a pair of positive integers (N_E, v_E) assigned to each $E \in T$, and a p-adic analytic proper mapping $h : Y \to X$ satisfying the following conditions:

1. *h is the composition of a finite number of monoidal transformations each one with a smooth center[2].*
2. *$(f \circ h)^{-1}(0) = \bigcup_{E \in T} E$ and h induces a p-adic bi-analytic map $Y \setminus h^{-1}(Sing_f(\mathbb{Q}_p)) \to \mathbb{Q}_p^n \setminus Sing_f(\mathbb{Q}_p)$. Where $Sing_f(\mathbb{Q}_p)$ is the set of singular points of f on $\mathbb{Q}_p$.*
3. *At every point $b \in Y$, if $E_1, \ldots, E_m$ are all $E \in T$ containing b with local equations $y_1, \ldots, y_m$ around b and $(N_i, v_i) = (N_E, v_E)$ for $E = E_i$, there exist local coordinates of Y around b which has the form $(y_1, \ldots, y_n)$ such that*

$$(f \circ h)(y) = \epsilon(y) \prod_{i=1}^{m} y_i^{N_{E_i}}, \quad h^* \left(\bigwedge_{i=1}^{n} dx_i \right) = \eta(y) \left(\prod_{i=1}^{m} y_i^{v_{E_i} - 1} \right) \bigwedge_{i=1}^{n} dy_i,$$

on some neighborhood of b, in which $\epsilon(y), \eta(y)$ are units in the local ring $\mathcal{O}_b$ of b in Y[3].

Theorem 5. *(Igusa, [22, Theorem 5.4.1])*

Let $f(x)$ be a non-constant polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$, $s \in \mathbb{C}$ with $Re(s) > 0$ and $\phi(x_1, \ldots, x_n)$ be a test function. Then exist a finite number of pairs $(N_E, v_E) \in \mathbb{N} \setminus \{0\} \times \cdots \times \mathbb{N} \setminus \{0\}$, $E \in T$, such that

$$Z_\phi(s, f) = \frac{M(p^{-s})}{\prod_{E \in T} (1 - p^{v_E - N_E s})},$$

where $M(p^{-s}) \in \mathbb{Q}_p[p^{-s}]$.

Sketch of the proof. We will denote by $|\bigwedge_{i=1}^n dx_i|$ the measure induced by the differential form $\bigwedge_{i=1}^n dx_i$. Let h be the p-adic analytic proper mapping given by Hironaka's theorem. By Theorem 1.3.1 we have that

$$Z_\phi(s, f) = \int_{Y \setminus h^{-1}(f^{-1}(0))} \phi(h(y)) \, |f(h(y))|_p^s \, |h^* \left(\wedge_{i=1}^n dx_i \right)(y)| \, .$$

Since $\phi(x)$ is a test function and $h(x)$ is proper, $\phi \circ h$ has compact support. We take a covering $\{U_i\}_{i \in \mathbb{N}}$ of $h^{-1}(Supp\phi)$, such that each U_i is contained in a chart (U, ψ_U) and all the formulas in the third part of Theorem 4 are valid. In addition, we may assume that each U_i is sufficiently small so that

1. There exists a finite number of U_i in the chart. Moreover, their union cover $h^{-1}(Supp(\phi))$.
2. $(\phi \circ h)|_{U_i} = \phi(h(b))$, $|\epsilon(y)|_p |_{U_i} = |\epsilon(b)|_p$, and $|\eta(y)|_p |_{U_i} = |\eta(b)|_p$ for some $b \in U_i$.

[2]A *monomial transformation or blow up* can be interpreted as a special change of variables between p-adic analytic manifolds. For a formal definition, the reader may consult [22, Section 3.1].

[3]h^* is the pull-back of the differential form $dx_1 \wedge \cdots \wedge dx_n$ through h. For a formal definition consult [22, Section 2.4].

3. $\phi_{U_i}(U_i) = a + p^m\mathbb{Z}_p^n$, for some $a \in \mathbb{Q}_p^n$.

In consequence, we have

$$Z_\phi(s,f) = \sum_i \phi(h(b_i)) \left|\epsilon(b_i)\right|_p \left|\eta(b_i)\right|_p \prod_{j=1}^{n} \int_{a_j+p^m\mathbb{Z}_p} |y_i|_p^{N_is+v_i-1}\, dy_i.$$

Now the Theorem follows by using Formula 1.3.1. □

2.2 Poincaré series

Definition 7. *Let $f(x) \in \mathbb{Z}_p[x_1,\ldots,x_n]$ and $f(x) \not\equiv 0 \pmod p$. The Poincaré series of $f(x)$ is defined as*

$$P_f(t) = \sum_{i=0}^{\infty} N_i p^{-in} t^i,$$

where $N_0 = 1$, $N_i = |\{(a_1,\ldots,a_n) \in \mathbb{Z}/p^i\mathbb{Z};\ f(a_1,\ldots,a_n) \equiv 0 \pmod{p^i}\}|$, for $i =\geq 1$, and $|t| < 1$.

Notation 1. *For a finite subset A, we denote by $|A|$ its cardinality.*

Lemma 2. *Let $f(x) \in \mathbb{Z}_p[x_1,\ldots,x_n]$. Then*

$$Z(s,f) = P_f(t) - t^{-1}(P_f(t) - 1),$$

where $t = p^{-s}$.

Proof. Let $A_i = \{(x_1,\ldots,x_n) \in \mathbb{Z}_p;\ |f(x_1,\ldots,x_n)|_p \leq p^{-m}\}$, we have that

$$\begin{aligned} Z(s,f) &= \sum_{i=0}^{\infty} \int_{A_i\setminus A_{i-1}} |f(x)|_p^s\, d^nx = \sum_{i=0}^{\infty} p^{-is}\left(\int_{A_i} d^nx - \int_{A_{i-1}} d^nx\right) \\ &= \sum_{i=0}^{\infty} p^{-is}\left(N_ip^{-in} - N_{i-1}p^{(-i-1)n}\right) = P_f(t) - t^{-1}\left(P_f(t) - 1\right). \end{aligned}$$

□

In [4] Borevich and Shafarevich conjectured $P_f(t)$ is a rational function of t. Igusa proved this conjecture as a corollary of the meromorphic continuation of $Z_\phi(s,f)$.

2.3 Multivariate Igusa zeta function

The multivariate local zeta functions are generalizations of the Igusa zeta functions introduced by F. Loeser in [33]. As we will see in the rest of this document, they constitute the main object of study of the present dissertation.

Definition 8. *Let $f_1, \ldots, f_l$ be nonconstant polynomials in $\mathbb{Q}_p[x_1, \ldots, x_n] \backslash \mathbb{Q}_p$ and $\phi(x_1, \ldots, x_n)$ a test function. The multivariate Igusa zeta function attached to $(f_1, \ldots, f_l, \phi)$ is defined as the integral*

$$Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l) = \int_{\mathbb{Q}_p^n \backslash \cup_{i=1}^l f_i^{-1}(0)} \phi(x) \prod_{i=1}^{l} |f_i(x)|_p^{s_i} \, d^n x,$$

for $(s_1, \ldots, s_l) \in \mathbb{C}^l$ and $Re(s_i) > 0$ for $i = 1, \ldots, l$.

Remark 2. *This integral defines a holomophic function in the subspace of $\mathbb{C}^l$ defined by $Re(s_i) > 0$, $i = 1, \ldots, l$.*

Theorem 6. *(F. Loeser, [33, Theorem 1.1.4])*

The multivariate zeta function $Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l)$ attached to $(f_1, \ldots, f_l, \phi)$ admits a meromorphic continuation to $\mathbb{C}^l$ as a rational function in the variables p^{-s_i}, $i = 1, \ldots, l$, more precisely,

$$Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l) = \frac{P_\phi(s_1, \ldots, s_l)}{\prod_{i \in T} \left(1 - p^{-N_0 - \sum_{i=1}^{l} N_i s_i}\right)}, \tag{2.3.1}$$

where T is a finite set, the N_0, N_i are non-negative integers, and $P_\phi(s_1, \ldots, s_l)$ is a polynomial in the variables $\{p^{-s_i}\}$.

Remark 3. *Theorem 5 and Theorem 6 yield in general to very big list of candidate poles. However, due to intrincated cancellations, usually many of these candidates are not poles. Thus identifying the poles of $Z_\phi(s, f)$ and $Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l)$ is a difficult open problem, for an in-depth discussion the reader may consult [10] and [33].*

Chapter 3

Zeta functions for graphs

In this Chapter, we introduce the main object of study of the present thesis, which is a multivariate zeta function attached to a finite simple graph. We give its definition in Section 3.2 and study some of its properties in the subsequent sections. But before this, we give a quick review of graph theory. We also provide in Section 3.3 the pseudocode of an algorithm to compute the zeta function of a finite and simple graph. This algorithm has been implemented in Python and the full code is available in Appendix B.

3.1 Graphs

We review quickly some basic aspects of graph theory, including the chromatic polynomial and chromatic function. For details the reader may consult [2].

Definition 9. *A graph is an ordered triple $G = (V(G), E(G), i_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and i_G is an incidence relation that associates each element of $E(G)$ an unordered pair of distinct elements of $V(G)$. The elements of $V(G)$ are called the vertices of G, and the elements of $E(G)$ are called the edges of G.*

Let G be a graph. Given $l \in E(G)$, we use the notation $i_G(l) = \{u, v\}$ or the notation $u \sim v$, where $u, v \in V(G)$ are the vertices of the edge l. From now on we only work with finite, simple graphs G, i.e. graphs with no loops and no multiple edges, see e.g. [2, Definition 1.2.4].

Remark 4. *Due to technical reasons, we consider the empty set as a graph and we will denote it as $\emptyset$.*

Example 3. 1. ***Complete graph*** *K_n. $V(K_n) = \{v_1, \cdots, v_n\}$ and $E(K_n) = \{\{v_i, v_j\}\,;\ 1 \leq i < j \leq n\}$.*

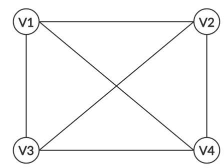

Figure 3.1: Graph K_4.

2. ***Path graph*** A_n. $V(A_n) = \{v_1, \cdots, v_n\}$ *and* $E(A_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}\}$

Figure 3.2: Graph A_4.

3. ***Circle graph*** C_n. $V(C_n) = \{v_1, \cdots, v_n\}$ *and* $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$

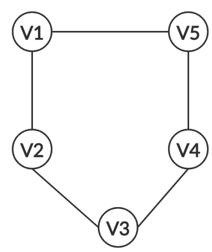

Figure 3.3: Graph C_5.

4. ***Star graph.*** S_n. $V(S_n) = \{v_1, \cdots, v_n\}$ *and* $E(S_n) = \{\{v_1, v_2\}, \{v_1, v_3\}, \cdots, \{v_1, v_n\}\}$

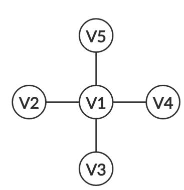

Figure 3.4: Graph S_5.

We recall that a graph H is called a *subgraph* of G if $V(H) \subset V(G)$, $E(H) \subset E(G)$. If $E(H) \neq \emptyset$, i_H is the restriction of i_G to $E(H)$. If $E(H) = \emptyset$, H consists of a subset of vertices of G without edges, and thus i_H is the empty function.

Definition 10. *Let G be a graph and I be a non-empty subset of $V(G)$. We denote by G_I (or $G[I]$) the subgraph induced by I, which is the subgraph defined as $V(G_I) = I$,*

$$E(G_I) = \{l \in E(G); i_G(l) = \{v, v'\} \text{ for some } v, v' \in I\},$$

and $i_{G_I} = i_G|_{E(G_I)}$. If $I = \emptyset$, by definition $G_I = \emptyset$.

Example 4. *Take $G = K_4$ as in Figure 3.1, and set $I = \{v_1, v_2, v_3\}$. Then $K_4[I]$ is the red graph in Figure 3.5. Note that $K_4[I]$ is isomorphic to K_3.*

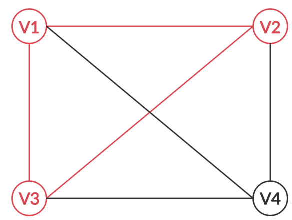

Figure 3.5: Graph $K_4[I]$.

Definition 11. *Let G be a graph. Two vertices u and v of G are said to be connected if there is a path subgraph A_n in G with $u, v \in V(A_n)$. The relation "connected" is an equivalence relation in $V(G)$. Let $V_1, \ldots, V_r$ be the equivalence classes. The subgraphs $G[V_1], \ldots, G[V_r]$ are called the components of G. If $r = 1$, the graph G is connected; otherwise, the graph G is disconnected.*

There is a simple algorithm to determine if a given graph G is connected or not. The algorithm is based on the BSF algorithm which is described in Appendix A, see [9, Section 22].

Algorithm 1: Connected graph.

Function `IsConnected`(G)**:**
- **input** : Let G be a graph.
- **output: True** if G is a connected graph. **False** otherwise.
- Let $s \in V(G)$
- **BFS**(G,s) `// See Algorithm 10 for BSF function.`
- **if** $V(Tree(s)) \neq V(G)$ **then**
  - **return False**
- **end**
- **else**
  - **return True**
- **end**

Example 5. 1. *For all $N \in \mathbb{N}$, the graphs K_N, A_N, C_N, and S_N, in Example 3, are connected graphs.*

2. *The graph G in Figure 3.7 is not a connected graph.*

3. ***Tree graph.*** *A tree is a connected graph without cycles.*

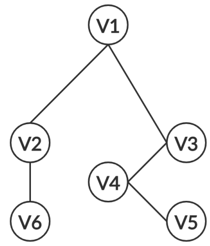

Figure 3.6: Tree with 6 vertices.

Remark 5. *Let G be a graph. We use the notation $G = G_1\#\cdots\#G_k$ to mean that $G_1, \dots, G_k$ are all the distinct connected components of G.*

Algorithm 2 below gives a list of the connected components of a graph G. This algorithm uses the BFS algorithm 10 to find the connected equivalence classes, Definition 11, of the set $V(G)$.

Algorithm 2: Connected components.

Function `GetComponents(`G`):`
  input : Let G be a graph.
  output: A list of connected components of G.
  $Ans = [\,]$ the empty list
  $A = \emptyset$
  while $A \neq V(G)$ **do**
    $v \in V(H) \setminus A$
    BFS(G,v) `// See Algorithm 10 for BSF function.`
    $A = A \cup V(Tree(v))$
    $G' = G[V(Tree(v))]$
    add G' to Ans
  end
  return Ans

Example 6. *Let G be the following graph:*

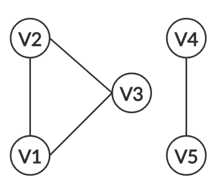

Figure 3.7: Graph G.

The graph G is not connected. Its connected components are $G_1 = G[\{v_1, v_2, v_3\}]$ and $G_2 = G[\{v_4, v_5\}]$.

Definition 12. *Let G be a graph and let I be a nonempty subset of $V(G)$. Suppose that $G_I = G_I^{(1)} \# \cdots \# G_I^{(m)}$. If $G_I^{(j)} = \{v\}$, we say that v is an isolated vertex of G_I. We denote by G_I^{iso} the set of all the isolated vertices of G_I. Then*

$$G_I = G_I^{red} \bigsqcup G_I^{iso},$$

where $G_I^{red} := G_I^{(i_1)} \# \cdots \# G_I^{(i_l)}$ and $\left| G_I^{(i_k)} \right| > 1$ for $k = 1, \dots, l$. We call G_I^{red} the reduced subgraph of G_I. We adopt the convention that if $I = \varnothing$, then $G_I^{red} = G_I^{iso} = \varnothing$.

Example 7. *Take G the graph depicted in Figure 3.7 and put $I = \{v_1, v_2, v_3, v_4\}$, then G_I is the following graph*

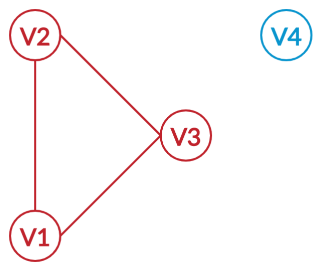

Figure 3.8: Graph G_I.

In this case, G_I^{red} is the graph $G[\{v_1, v_2, v_3\}]$, which is colored in red in Figure 3.8. We also have that G^{iso} is the graph $G[v_4] = v4$, which is colored in blue in Figure 3.8.

Definition 13. *Let G be a graph and let $v \in V(G)$. The number of incident edges at v in G is called the degree of the vertex v in G and is denoted by $d_G(v)$, or $d(v)$ when no confusion can arise.*

3.1.1 Vertex Colorings and Chromatic Functions

In this section, we will introduce the definitions of a *vertex coloring* and *chromatic functions*. For this section we fix a simple and finite connected graph G.

Vertex colorings

We color graphs using p colors, more precisely, we attach to every element of $\{0, 1, \dots, p-1\}$ (which we identify with an element of $\mathbb{F}_p$) a color.

Definition 14. *A vertex coloring of G is a mapping $C : V(G) \to \mathbb{F}_p$. If v is a vertex of G, then $C(v)$ is its color. We denote by $Colors(G)$, the set of all possible vertex-colorings of G.*

Notice that any coloring C is given by a vector $\boldsymbol{a} = (a_v)_{v \in V(G)} \in \mathbb{F}_p^{|V(G)|}$ with $C(v) = a_v$ for $v \in V$. We will identify C with $\boldsymbol{a}$. Our notion of vertex coloring is completely different from the classical one which requires that adjacent vertices of G receive distinct colors of $\mathbb{F}_p$, see e.g. [2, Section 7.2].

Definition 15. *Given a pair* (G, C)*, we attach to it a colored graph* G^C *defined as follows:* $V(G^C) = V(G)$*,*

$$E(G^C) = \{l \in E(G); C(u) = C(v) \text{ where } i_G(l) = \{u, v\}\}$$

and $i_{G^C} = i_G \mid_{E(G^C)}$*.*

We note that if $G_1^C, \cdots, G_r^C$, with $r = r(C)$, are all the connected components of G^C, then $C \mid_{G_k^C}$ is constant for $k = 1, \ldots, r$. If C is identified with $\boldsymbol{a}$ we use the notation $G^{\boldsymbol{a}}$. Definition 15 tell us how to color the edges of a graph if we have already assigned colors to the vertices of the graph. To an edge having its two vertices colored with the same color we assign the color of its vertices, in other case, we discard the edge.

Example 8. *Take* $G = K_4$*, see Figure 3.1, and* $p = 3$*. Define the vertex coloring* C *of* G *as* $C(v_1) = 0$*,* $C(v_2) = 1$*,* $C(v_3) = C(v_4) = 2$*. Then* G^C *is the graph*

Figure 3.9: Graph K_4^C.
We represent 0, 1, and 2 with black, yellow, and red, respectively.

In this case, $r(C) = 3$ *and* $G_1^C = \{v_1\}$*,* $G_2^C = \{v_2\}$*, and* $V(G_3^C) = \{v_3, v_4\}$ *and* $E(G_3^C) = \{v_3 \sim v_4\}$*.*

Definition 16. *We set* $Colored(G) := \left\{G^C; C \in Colors(G)\right\}$*, and* $Subgraphs(G, |G|)$ *to be the set of all graphs* H *such that* $V(H) = V(G)$*,* $E(H) \subset E(G)$*, and if* $E(H) \neq \emptyset$*,* i_H *is the restriction of* i_G *to* $E(H)$*. We define*

$$\mathfrak{F} : Colored(G) \to Subgraph(G, |G|)$$

as follows: $\mathfrak{F}\left(G^C\right) = H$ *if and only if* $V(H) = V(G^C)$*,* $E(H) = E(G^C)$ *and* $i_H = i_{G^C}$*. We set* $Subgraph_{\mathfrak{F}}(G, |G|) = \mathfrak{F}\,(Colored(G))$*.*

The family $Colored(G)$ is formed by all the possible colored versions of G, the operation 'forgetting the coloring' $\mathfrak{F}$ assigns to an element of $Colored(G)$ a subgraph of G having the same vertices as G. Any graph in $Subgraphs(G, |G|)$ is obtained from G by deleting one or more edges, 'but keeping' the corresponding vertices. The pseudocode for computing $Subgraph_{\mathfrak{F}}(G, |G|)$ is given below in Algorithm 3.

This algorithm uses the fact that each subgraph H in $Subgraphs(G, |G|)$ is obtained by deleting edges from G, and it is not connected. To determine whether or not H is connected, we use Algorithm 1.

Algorithm 3: Colored graphs of G.

Function `Coloredgraphs(`G`):`

  input : Let G be a connected graph.
  output: A set A such that $Subgraph_{\mathfrak{F}}(G, |G|) \setminus \{G\} \subseteq A$.
  $Ans = \emptyset$
  for $A \subseteq E(G)$ **do**
    $H = (V(G), E(G) \setminus A)$ the graph obtained from G deleting the edges in A.
    if $IsConnected(H) = False$ **then**
      Add H to Ans
    end
  end
  return Ans

Example 9. *Put $G = K_3$ and assume that $p \geq 3$. Then $Subgraphs(G, |G|)$ is the set of graphs:*

$\{G, G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$ given in Figure 3.10. Then $Subgraph_{\mathfrak{F}}(G, |G|)$ is the collection $\{G_4, G_5, G_6, G_7, G\}$, where

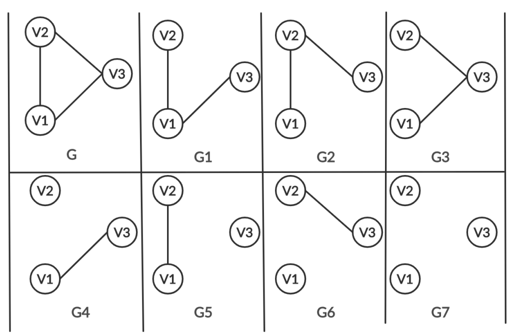

Figure 3.10: $Subgraphs(G, |G|)$.

Definition 17. *We define $Indgraphs(G)$ to be the set of all connected graphs H such that there exists a coloring C, with $G^C = G_1^C \# \cdots \# G_r^C$, and $H = G_i^C$ for exactly one index i.*

By Definition 10, we have

$$Indgraphs(G) = \{G\,[I]\,; \varnothing \neq I \subset V(G) \text{ and } G\,[I] \text{ is connected}\},$$

where $G\,[I]$ denotes the subgraph induced by I.

The pseudocode for computing $Indgraphs(G)$ is given in Algorithm 4. This algorithm uses the fact that each subgraph H in $Indgraphs(G)$ is connected and has the form $G[I]$ for some

I, nonempty subset of $V(G)$.

Algorithm 4: The set $Indgraphs(G)$.

Function `GetIndgraphs(`G`)`**:**
input **:** Let G be a graph.
output: The set $Indgraphs(G)$.
$Ans = \emptyset$
for $\emptyset \neq A \subseteq V(G)$ **do**
  if $IsConnected(G[A]) = True$ **then**
    Add $G[A]$ to Ans
  end
end
return Ans

Example 10. *Take $G = K_3$. In this case, we have that $Indgraphs(G)$ consists of the following graphs:*

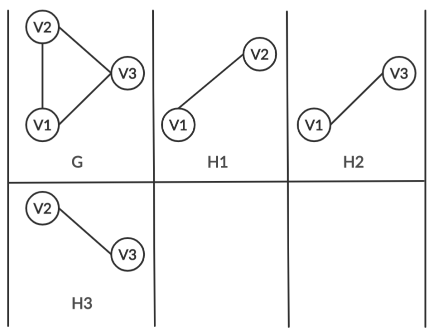

Figure 3.11: $Indgraphs(K_3)$.

Chromatic Functions

We will introduce the chromatic function, which is part of the zeta function attached to a fixed simple connected graph G, and its relation with the chromatic polynomial.

Definition 18. *Given H in $Subgraphs(G, |G|)$, we define its chromatic function as*

$$\mathcal{C}(p; H) = \left|\left\{G^C \in Colored(G); \mathfrak{F}\left(G^C\right) = H\right\}\right|.$$

Notice that if G is connected, then $\mathcal{C}(p; G) = p$. Indeed, if we use at least two colors then G^C has at least two connected components, and thus $\mathfrak{F}(G^C) \neq G$. So we can use only constant colorings to have $\mathfrak{F}(G^C) = G$.

Definition 19. *Given u, $v \in V(G)$, we denote by $d(u, v)$ the length of the shortest path in G joining u and v. Given H, W subgraphs of G, we set*

$$d(H, W) = \min_{u \in V(H),\ v \in V(W)} d(u, v) \in \mathbb{N}.$$

Let H, W be subgraphs of a fixed graph G. The pseudocode for computing the distance between H and W is given by Algorithm 5. This algorithm uses the BFS algorithm 10 to find the distance between vertices u and v of H and W, respectively, and returns the minimum of these distances.

Algorithm 5: Distance of graphs.

```
Function disGraph(G,H,W):
    input  : Let G be a graph and H,W sub-graphs of G.
    output: The distances from H to W in G, d(H,W).
    ans = [ ] The empty list
    for u ∈ V(H) do
        BSF(G,u) // See Algorithm 10 for BSF function.
        for v ∈ V(W) do
            Add d(u,v) to ans
        end
    end
    return min_{a∈ans}(a)
```

Example 11. *In this example, we compute the chromatic function $\mathcal{C}(p; H)$, where H is in $Subgraphs(G, |G|)$, with G and H as in Figure 4.1.*

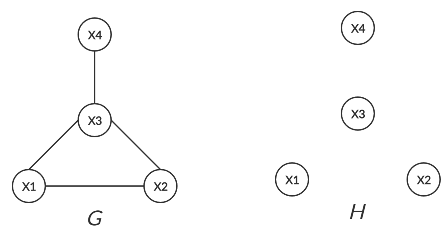

Figure 3.12: Graphs G and H.

In this case $H = H_1 \# \cdots \# H_4$, where $H_i = \{x_i\}$ is the vertex x_i, for $i = 1, 2, 3, 4$. Set $C(H_i) = a_i$, for $i = 1, 2, 3, 4$. There are three different types of conditions (colorings) coming from $\mathfrak{F}(G^C) = H$:

$$\begin{cases} a_1 \neq a_2,\ a_1 \neq a_3,\ a_2 \neq a_3,\ a_3 \neq a_4; \\ \\ a_1 \neq a_4,\ a_2 \neq a_4; \end{cases} \tag{3.1.1}$$

$$\begin{cases} a_1 \neq a_2,\ a_1 \neq a_3,\ a_2 \neq a_3,\ a_3 \neq a_4; \\ \\ a_1 = a_4. \end{cases} \tag{3.1.2}$$

$$\begin{cases} a_1 \neq a_2,\ a_1 \neq a_3,\ a_2 \neq a_3,\ a_3 \neq a_4; \\ \\ a_2 = a_4. \end{cases} \tag{3.1.3}$$

Consequently

$$C(p,H) = p(p-1)(p-2)(p-3) + 2p(p-1)(p-2),$$

for any prime number p.

Remark 6. *Suppose that* $H = H_1 \# \cdots \# H_r$. *The condition* $\mathfrak{F}\left(G^C\right) = H$ *implies that* $C|_{H_i} = a_i \in \mathbb{F}_p$ *for* $i = 1, \ldots, l$. *Now if* $d(H_i, H_j) = 1$, *then* $a_i \neq a_j$. *If* $d(H_i, H_j) \geq 2$, *the colors* a_i, a_j *may be equal. We now define*

$$D_1(H) := D_1 = \left\{\{H_i, H_j\}\,; H_i, H_j \text{ are connected components of } H,\, d(H_i, H_j) = 1\right\},$$

and

$$D_2(H) := D_2 = \left\{\{H_i, H_j\}\,; H_i, H_j \text{ are connected components of } H,\, d(H_i, H_j) \geq 2\right\}.$$

We set $\Pi_1 : A \times B \to A$, *respectively* $\Pi_2 : A \times B \to B$, *for the canonical projections, and define* $\widetilde{D} = \Pi_1 D_2 \cup \Pi_2 D_2$. *Any coloring* C *satisfying* $\mathfrak{F}\left(G^C\right) = H$ *is determined by a set conditions of the following form. There exists a partition* $\mathcal{P}\left(\widetilde{D}\right) = \left\{\widetilde{D}_1, \ldots, \widetilde{D}_k\right\}$, *with* $\left|\widetilde{D}_i\right| \geq 1$ *for* $i = 1, \ldots, k$, *such that*

$$\{C(H_i) \neq C(H_j) \text{ for } d\left(H_i, H_j\right) = 1; \tag{3.1.4}$$

$$\left\{\begin{array}{l} C(H_i) = C(H_j) = b_l \in \mathbb{F}_p, \text{ for any } \{H_i, H_j\} \in \widetilde{D}_i, \\ \text{with } b_l \neq b_m \text{ if } l \neq m, \text{ for } l, m \in \{1, \ldots, k\}. \end{array}\right. \tag{3.1.5}$$

The set of conditions (3.1.4)-(3.1.5) defines a relative closed subset of the affine space $\mathbb{F}_p^M$, *for a suitable* M, *and the solution set of these conditions corresponds to the colorings defined by conditions (3.1.4)-(3.1.5).*

Definition 20. *Let* G *be a graph and let* k *be a positive integer. A proper* k*-coloring of the vertices of* G *is a function* $f : V(G) \to \{0, \ldots, k-1\}$ *such that* $f^{-1}(j)$ *is an independent set, i.e. for any* u,$v \in f^{-1}(j)$ *there is no edge in* $E(G)$ *joining them.*

Let $\mathcal{P}(k; G)$ *denotes the number of proper* k*-colorings of* G. $\mathcal{P}(k; G)$ *is called the chromatic polynomial of* G.

There is a simple algorithm to determine the chromatic polynomial of a graph G. Algorithm

6 is based in [2, Theorem 7.9.2].

Algorithm 6: Chromatic Polynomial of G.

Function `ChromaticPoly(`G,x`)`**:**

  input : Let G be a graph and $x \in \mathbb{N}$.
  output: The Chromatic Polynomial $\mathcal{P}(G, x)$.
  Put $e \in E(G)$
  $G - e$ is the graph obtained from G deleting the edge e.
  $G \circ e$ is the graph obtained from G contracting the edge e.
  if $E(G) = \emptyset$ **then**
    return $x^{|V(G)|}$
  end
  if $|V(G)| + |E(G)| = 3$ **then**
    return $x(x-1)$
  end
  else
    return $ChromaticPoly(G - e, x) - ChromaticPoly(G \circ e, x)$
  end

Remark 7. *There exists a polynomial $\mathcal{P}(x; G)$, with integer coefficients, satisfying*

$\mathcal{P}(x; G)\mid_{x=k} = \mathcal{P}(k; G)$ for any positive integer k, see e.g. [2, Theorem 7.9.2].

In the following examples we present closed formulae for the chromatic polynomials of some families of graphs. See [2, Section 7.9] and [3, Section 5.1] for the proofs of these results.

Example 12. *1.* $\mathcal{P}(x; K_N) = x(x-1)\cdots(x-(N-1))$.

2. $\mathcal{P}(x; T_N) = x(x-1)^{N-1}$.

3. $\mathcal{P}(x; C_N) = (x-1)^N + (-1)^N(x-1)$.

Definition 21. *Let H be a subgraph in $Subgraphs(G, |G|)$, such that $H = H_1\#\cdots\#H_r$, where the H_is are the different connected components of H. We attach to H the graph G_H^* defined as follows:*

$$V(G_H^*) = \{H_1, \cdots, H_r\}, \quad E(G_H^*) = \{\{H_i, H_j\}\, ; d\,(H_i, H_j) = 1\},$$

and

$$i_{G_H^*}(\{H_i, H_j\}) = \{H_i, H_j\}; \quad \forall \{H_i, H_j\} \in E(G_H^*). \tag{3.1.6}$$

Let H be a subgraph of a fixed graph G with connected components $H_1, \ldots, H_r$. There is a simple algorithm to obtain the graph G_H^*. Algorithm 7 uses Algorithm 5 to find the distances between the graphs H_i and H_j for $1 \le i < j \le r$ and get the edges of the graph

G_H^*.

Algorithm 7: The graph G_H.

```
Function GetGraphGH(G,H):
    input : Let G be a graph and H a sub-graph of G.
    output: The graph G_H.
    Ans=the empty graph
    Components=GetComponents(H)
    for A ∈ Components do
        Add A to V(Ans)
        for B ∈ Components do
            if disGraph(G, A, B) = 1 then
                Add {A, B} to E(Ans)
            end
        end
    end
    return ans
```

Proposition 1. *For any H in $Subgraphs(G, |G|)$, $\mathcal{C}(p; H) = \mathcal{P}(x; G_H^*) \mid_{x=p}$.*

Proof. We assume that $H = H_1 \# \cdots \# H_r$ as in Definition 21. The result follows by establishing a bijection between the following two sets:

$$A\left(G^C, H\right) := \left\{C \in Colors(G); \mathfrak{F}\left(G^C\right) = H\right\},$$

$$B\left(G_H^*\right) := \left\{p\text{-colorings of } G_H^*\right\}.$$

Given a coloring $C \in A\left(G^C, H\right)$, we define

$$\begin{array}{rlcl} C^* : & V(G_H^*) & \rightarrow & \{0, \ldots, p-1\} \\ & H_i & \rightarrow & C\left(H_i\right). \end{array}$$

Now, if $C_1,\ C_2 \in A\left(G^C, H\right)$ and $C_1 \neq C_2$, then there exists $j \in \{1, \ldots, r\}$ such that $C_1|_{H_j} \neq C_2|_{H_j}$ which implies that $C_1^* \neq C_2^*$.

Given a p-coloring C^* of G_H^*, we define

$$\begin{array}{rlcl} C : & V(G) & \rightarrow & \{0, \ldots, p-1\} \\ & v & \rightarrow & C^*(H_i), \end{array}$$

for any $v \in H_i$. Then $C \in A\left(G^C, H\right)$. Indeed, by the definition of C, $G^C = H_1 \# \cdots \# H_r = H$, with $C|_{H_i} = a_i \in \mathbb{F}_p$ for $i = 1, \ldots, r$. Then $V(G^C) = V(H)$. Additionally, an edge $l \in E\left(G^C\right)$ is and edge of G, say $i_G(l) = \{u, v\}$, satisfying $C(u) = C(v)$. Then $u, v \in V(H_i)$, and $l \in E(H_i)$, i.e. $E(G^C) \subset E(H)$. Conversely, given $l \in E(H_i)$, with $i_H(l) = \{u, v\}$, we have $C(u) = C(v) = C^*(H_i)$, and thus $l \in V(G^C)$. $\square$

Example 13. *Using the notation given in Example 9, we obtain that* $G^*_{G_4}$*,* $G^*_{G_5}$*, and* $G^*_{G_6}$ *are isomorphic to the complete graph* K_2*. And* $G^*_{G_7}$ *is isomorphic to the complete graph* K_3*. Using Proposition 1 and Example 12, we have*

$$\mathcal{C}(p;G_4) = \mathcal{C}(p;G_5) = \mathcal{C}(p;G_6) = p(p-1),$$

and

$$\mathcal{C}(p;G_7) = p(p-1)(p-2).$$

3.2 Zeta function for a graph

This section is dedicated to the study of the function $Z_\varphi(\boldsymbol{s};G)$, see Definition 23. This function admits a meromorphic continuation as a rational function in the variables $p^{-s(u,v)}$, see Corollary 2.

Definition 22. *Let* G *be a graph. To each vertex* $v \in V$ *we attach a* p*-adic variable* x_v*, and to each edge* $l \in E$ *we attach a complex variable* $s\left(l\right)$*. We also use the notation* $s\left(u,v\right)$ *if* $u \sim v$*. We set* $\boldsymbol{x} := \left\{x_v\right\}_{v\in V}$*,* $\boldsymbol{s} := \left\{s\left(l\right)\right\}_{l\in E}$*.*

Given $l \in E$*, with* $i_G\left(l\right) = \{u, v\}$*, we set*

$$F_l\left(x_u, x_v, s\left(l\right)\right) := \left|x_u - x_v\right|_p^{s(l)}$$

and

$$F_G\left(\boldsymbol{x},\boldsymbol{s}\right) := \prod_{l\in E} F_l\left(x_u, x_v, s\left(l\right)\right) = \prod_{\substack{u,v\in V\\ u\sim v}} \left|x_u - x_v\right|_p^{s(u,v)}. \tag{3.2.1}$$

Remark 8. *(i) If* $V(G) \neq \emptyset$ *and* $E(G) = \emptyset$*, then* G *consists of a finite set of vertices without edges connecting them, thus incidence relation is the empty function. In this case we set* $F_G\left(\boldsymbol{x},\boldsymbol{s}\right) := 1$*. Due to technical reasons, we set* $F_\varnothing\left(\boldsymbol{x},\boldsymbol{s}\right) := 1$*.*

Example 14. 1. *For the graph* A_4 *of Figure 3.2, its function* $F_{A_4}\left(\boldsymbol{x},\boldsymbol{s}\right)$ *is*

$$\left|x_{v_1} - x_{v_2}\right|_p^{s(v_1,v_2)} \left|x_{v_2} - x_{v_3}\right|_p^{s(v_2,v_3)} \left|x_{v_3} - x_{v_4}\right|_p^{s(v_3,v_4)}.$$

2. *For* S_5*, Figure 3.4, its function* $F_{S_5}\left(\boldsymbol{x},\boldsymbol{s}\right)$ *is*

$$\left|x_{v_1} - x_{v_2}\right|_p^{s(v_1,v_2)} \left|x_{v_1} - x_{v_3}\right|_p^{s(v_1,v_3)} \left|x_{v_1} - x_{v_4}\right|_p^{s(v_1,v_4)} \left|x_{v_1} - x_{v_5}\right|_p^{s(v_1,v_5)}.$$

Notation 2. *We denote by* $\mathcal{D}_{sym}\left(\mathbb{Q}_p^N\right)$ *the* $\mathbb{C}$*-vector space of symmetric test functions, i.e. all the complex-valued test functions satisfying* $\varphi\left(x_1,\ldots,x_N\right) = \varphi\left(x_{\pi(1)},\ldots,x_{\pi(N)}\right)$ *for any permutation* π *of* $\{1,2,\ldots,N\}$*.*

Let G and H be graphs. By a graph isomorphism $\sigma : G \to H$, we mean a pair of mappings $\{\sigma_E, \sigma_V\}$, where $\sigma_V : V(G) \to V(H)$, $\sigma_E : E(G) \to E(H)$ are bijections, with the property that $i_G\left(l\right) = \{u, v\}$ if and only if $i_H\left(\sigma_E\left(l\right)\right) = \left\{\sigma_V\left(u\right), \sigma_V\left(v\right)\right\}$. In the case of simple graphs, σ_E is completely determined by σ_V. For the sake of simplicity, we will denote the pair $\{\sigma_E, \sigma_V\}$ as σ, see e.g. [2, Sections 1.2.9, 1.2.10].

We denote by $\mathrm{Aut}(G)$ the automorphism group of G. Let $\sigma : G \to H$ be a graph isomorphism. Assume that the cardinality of $|V(G)| = |V(H)| = N$. Let x_u, $u \in V(G)$, be p-adic variables as before. Then the mapping

$$\begin{array}{rccc} \sigma^* : & \mathbb{Q}_p^N & \to & \mathbb{Q}_p^N \\ & x_v & \to & x_{\sigma(v)} \end{array} \tag{3.2.2}$$

is a p-adic analytic isomorphism that preserves the Haar measure of $\mathbb{Q}_p^N$, see (1.3.1).

Definition 23. *Given $\varphi \in \mathcal{D}_{sym}\left(\mathbb{Q}_p^{|V(G)|}\right)$, the p-adic zeta function attached to (G, φ) is defined as*

$$Z_\varphi(\boldsymbol{s}; G) = \int\limits_{\mathbb{Q}_p^{|V(G)|}} \varphi\left(\boldsymbol{x}\right) F_G\left(\boldsymbol{x}, \boldsymbol{s}\right) \prod_{v \in V(G)} dx_v, \tag{3.2.3}$$

for $\mathrm{Re}(s\left(l\right)) > 0$ for every $l \in E$, where $\prod_{v \in V(G)} dx_v$ denotes the normalized Haar measure on $\left(\mathbb{Q}_p^{|V(G)|}, +\right)$. If φ is the characteristic function of $\mathbb{Z}_p^{|V(G)|}$, we use the notation $Z(\boldsymbol{s}; G)$.

Lemma 3. *Let G and H be graphs. If $\sigma : G \to H$ is a graph isomorphism, then*

$$Z_\varphi(\{s\left(l\right)\}_{l \in E(G)}; G) = Z_\varphi(\{s\left(l\right)\}_{l \in E(H)}; H).$$

Furthermore, for any $\sigma = (\sigma_V, \sigma_E) \in Aut(G)$, it holds true that

$$Z(\{s\left(l\right)\}_{l \in E(G)}; G) = Z(\{s\left(\sigma_E\left(l\right)\right)\}_{l \in E(G)}; G), \tag{3.2.4}$$

where the integrals exist.

Proof. By using that

$$Z_\varphi(\boldsymbol{s}; G) = \int\limits_{\mathbb{Q}_p^{|V(G)|}} \varphi\left(\{x_v\}_{v \in V(G)}\right) \prod_{\substack{u,v \in V(G) \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{v \in V(G)} dx_v,$$

and changing variables as $\sigma^* : \mathbb{Q}_p^N \to \mathbb{Q}_p^N$, $x_v \mapsto x_{\sigma(v)}$, see (3.2.2), we have

$$\varphi\left(\{x_v\}_{v \in V(G)}\right) = \varphi\left(\{x_{\sigma(v)}\}_{v \in V(G)}\right) = \varphi\left(\{x_{v'}\}_{v' \in V(H)}\right),$$

because the list $\{x_{v'}\}_{v' \in V(H)}$ is a permutation of the list $\{x_v\}_{v \in V(G)}$. In addition,

$$\begin{aligned} \prod_{\substack{u,v \in V(G) \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} &= \prod_{\substack{\sigma(u),\sigma(v) \\ u,v \in V(G) \\ u \sim v}} \left|x_{\sigma(u)} - x_{\sigma(v)}\right|_p^{s(\sigma(u),\sigma(v))} \\ &= \prod_{\substack{u',v' \in V(H) \\ u' \sim v'}} |x_{u'} - x_{v'}|_p^{s(u',v')}, \end{aligned}$$

and by using that σ^* preserves the Haar measure,

$$\prod_{v\in V(G)} dx_v = \prod_{v\in V(G)} dx_{\sigma(v)} = \prod_{v'\in V(H)} dx_{v'}.$$

Consequently $Z_\varphi(\{s\,(l)\}_{l\in E(G)}\,;G) = Z_\varphi(\{s\,(l)\}_{l\in E(H)}\,;H)$. □

Example 15. *1. The p-adic zeta function, $Z(\boldsymbol{s};A_4)$, attached to $\left(A_4, 1_{\mathbb{Z}_p^4}\right)$, see Figure 3.2, is*

$$\int_{\mathbb{Z}_p^4} |x_{v_1}-x_{v_2}|_p^{s(v_1,v_2)}\,|x_{v_2}-x_{v_3}|_p^{s(v_2,v_3)}\,|x_{v_3}-x_{v_4}|_p^{s(v_3,v_4)}\,dx_{v_1}dx_{v_2}dx_{v_3}dx_{v_4}.$$

2. The p-adic zeta function, $Z(\boldsymbol{s};S_5)$, attached to $\left(S_5, 1_{\mathbb{Z}_p^5}\right)$, see Figure 3.4, is

$$\int_{\mathbb{Z}_p^5} \left|x_{v_1}-x_{v_2}\right|_p^{s(v_1,v_2)}\left|x_{v_1}-x_{v_3}\right|_p^{s(v_1,v_3)}\left|x_{v_1}-x_{v_4}\right|_p^{s(v_1,v_4)}\left|x_{v_1}-x_{v_5}\right|_p^{s(v_1,v_5)}dx_{v_1}dx_{v_2}dx_{v_3}dx_{v_4}dx_{v_5}.$$

Corollary 2. *The zeta function $Z_\varphi(\boldsymbol{s};G)$ admits a meromorphic continuation to $\mathbb{C}^{|E(G)|}$ as a rational function in the variables $p^{-s(l)}$, $l\in E\,(G)$, more precisely,*

$$Z_\varphi(\boldsymbol{s};G) = \frac{P_\varphi(\boldsymbol{s})}{\prod_{i\in T}\left(1-p^{-N_0^i-\sum_{l\in E(G)} N_l^i s(l)}\right)}, \tag{3.2.5}$$

where T is a finite set, the N_0^i, N_l^i are non-negative integers, and $P_\varphi(\boldsymbol{s})$ is a polynomial in the variables $\left\{p^{-s(l)}\right\}_{l\in E(G)}$.

Proof. The results follows from Theorem 6. □

Example 16. *We use all the notation introduced in Example 11. We now explain the connection between chromatic functions and the computation of certain p-adic integrals. Set*

$$F_G\left(\boldsymbol{x},\boldsymbol{s}\right) = |x_1-x_2|_p^{s_{12}}\,|x_1-x_3|_p^{s_{13}}\,|x_2-x_3|_p^{s_{23}}\,|x_3-x_4|_p^{s_{24}},$$

and

$$I(\boldsymbol{s},\boldsymbol{a}) = \int\limits_{\boldsymbol{a}+p\mathbb{Z}_p^4} F_G(\boldsymbol{x},\boldsymbol{s})\prod_{i=1}^4 dx_i,$$

where $\boldsymbol{a}=(a_1,a_2,a_3,a_4)\in\mathbb{F}_p^4$. Assume that $\boldsymbol{a}$ is a coloring of one the types (3.1.1)-(3.1.3), i.e. $\boldsymbol{a}$ is a solution of exactly one of the conditions systems (3.1.1)-(3.1.3), then by using that

$$|a_1-a_2-p\,(x_1-x_2)|_p^{s_{12}}\,|a_1-a_3-p\,(x_1-x_3)|_p^{s_{13}}\,|a_2-a_3-p\,(x_2-x_3)|_p^{s_{23}}\times$$
$$|a_3-a_4-p\,(x_3-x_4)|_p^{s_{24}}=1,\ \textit{for any } x_1,x_2,x_3,x_4,$$

we have $I(\boldsymbol{s},\boldsymbol{a}) = p^{-4}$. *Now notice that*

$$\left|\left\{\boldsymbol{a} \in \mathbb{F}_p^4; I(\boldsymbol{s},\boldsymbol{a}) = p^{-4}\right\}\right| = C(p,H) \text{ for any prime number } p.$$

So $Z(\boldsymbol{s};G)$ *can now be calculated directly using Lemma 6 or Lemma 7 below.*

Corollary 3. *The following functional equations hold true:*

$$\frac{P_\varphi(\{s\,(l)\}_{l\in E(G)})}{\prod_{i\in T}\left(1-p^{-N_0^i-\sum_{l\in E(G)}N_l^i s(l)}\right)} = \frac{P_\varphi(\{s\,(\sigma_E\,(l))\}_{l\in E(G)})}{\prod_{i\in T}\left(1-p^{-N_0^i-\sum_{\sigma_E(l)\in E(G)}N_{\sigma_E(l)}^i s(\sigma_E(l))}\right)},$$

for any $\sigma = (\sigma_V, \sigma_E) \in Aut(G)$.

Proof. The results follows from (3.2.4) by using the fact (3.2.5) gives an equality between functions in an open set containing $\{\mathrm{Re}(s(l)) > 0; l \in E(G)\}$. $\square$

Example 17. *Let* K_2 *be the complete graph with two vertices,* v_0, v_1. *We denote the corresponding edge as* l. *Then* $F_{K_2}(\boldsymbol{x},\boldsymbol{s}) = |x_{v_0} - x_{v_1}|_p^{s(l)}$ *and*

$$Z(\boldsymbol{s};K_2) = \int_{\mathbb{Z}_p^2} |x_{v_0} - x_{v_1}|_p^{s(l)}\, dx_{v_0} dx_{v_1} = \int_{\mathbb{Z}_p}\left\{\int_{\mathbb{Z}_p} |x_{v_0} - x_{v_1}|_p^{s(l)}\, dx_{v_0}\right\} dx_{v_1}.$$

By changing variables as $y = x_{v_0} - x_{v_1}$, $z = x_{v_1}$, *we have*

$$Z(\boldsymbol{s};K_2) = \int_{\mathbb{Z}_p}\left\{\int_{\mathbb{Z}_p} |y|_p^{s(l)}\, dy\right\} dz = \int_{\mathbb{Z}_p} |y|_p^{s(l)}\, dy = \frac{1-p^{-1}}{1-p^{-1-s(l)}}.$$

Remark 9. *If* $G = G_1\#\cdots\#G_k$, *then* $F_G(\boldsymbol{x},\boldsymbol{s}) = \prod_{i=1}^k F_{G_i}(\boldsymbol{x},\boldsymbol{s})$ *and*

$$Z(\boldsymbol{s};G) = \prod_{i=1}^k Z(\boldsymbol{s};G_i).$$

Notice that $Z(\boldsymbol{s};G_i) = 1$, *if* G_i *consists of only one vertex.*

3.3 Rationality and recursive formulas

We provide a recursive algorithm for computing $Z(\boldsymbol{s};G)$. The algorithm uses vertex colorings and chromatic polynomials. This algorithm allows us to describe the possible poles of $Z(\boldsymbol{s};G)$ in terms of the subgraphs of G, see Theorem 7 and Corollary 4.

Theorem 7. *Let* G *be a connected graph. Then, for any prime number* p, $Z(\boldsymbol{s};G)$ *satisfies:*

(i)

$$Z(\boldsymbol{s};G) = \frac{\sum_{\substack{H\in Subgraphs_{\mathcal{F}}(G,|G|)\\ H\neq G}} p^{-|V(G)|-\sum_{l\in E(H)}s(l)}\mathcal{C}(p;H)Z(\boldsymbol{s};H)}{1-p^{1-|V(G)|-\sum_{l\in E(G)}s(l)}}. \tag{3.3.1}$$

(ii) $Z(\boldsymbol{s};G)$ *admits a meromorphic continuation to* $\mathbb{C}^{|E(G)|}$ *as a rational function of* $\left\{p^{-s(l)}; l \in E(G)\right\}$*. More precisely,*

$$Z(\boldsymbol{s};G) = \frac{M\left(\left\{p^{-s(l)}; l \in E(G)\right\}\right)}{\prod\limits_{\substack{H \in Indgraphs(G) \\ |V(H)| \geq 2}} \left(1 - p^{1-|V(H)|-\sum_{l\in E(H)} s(l)}\right)}, \tag{3.3.2}$$

where $M\left(\left\{p^{-s(l)}; l \in E(G)\right\}\right)$ *denotes a polynomial with rational coefficients in the variables* $\left\{p^{-s(l)}\right\}_{l\in E(G)}$.

Proof. (i) We attach to $\boldsymbol{a} = \{a_v\}_{v\in V(G)} \in \mathbb{F}_p^{|V(G)|}$ a color C defined as $C(v) = a_v$, for $v \in V(G)$. We set

$$I(s;\boldsymbol{a}) := \int\limits_{\boldsymbol{a}+p\mathbb{Z}_p^{|V(G)|}} F_G(\boldsymbol{x},\boldsymbol{s}) \prod_{v\in V(G)} dx_v,$$

then

$$Z(\boldsymbol{s};G) = \sum_{\boldsymbol{a}\in\mathbb{F}_p^{|V(G)|}} I(s;\boldsymbol{a}).$$

Now

$$I(s;\boldsymbol{a}) = p^{-|V(G)|} \int\limits_{\mathbb{Z}_p^{|V(G)|}} F_G(\boldsymbol{a}+p\boldsymbol{x},\boldsymbol{s}) \prod_{v\in V(G)} dx_v,$$

where

$$\begin{aligned} F_G(\boldsymbol{a}+p\boldsymbol{x},\boldsymbol{s}) &= \prod_{\substack{l\in E(G) \\ i_G(l)=\{v,u\}}} |a_v - a_u + px_v - px_u|_p^{s(l)} \\ &= \prod_{\substack{l\in E(G) \\ i_G(l)=\{v,u\}}} \begin{cases} 1 & \text{if} \quad C(v) \neq C(u) \\ \\ p^{-s(l)} |x_v - x_u|_p^{s(l)} & \text{if} \quad C(v) = C(u). \end{cases} \end{aligned}$$

By attaching to $I(s;\boldsymbol{a})$ the colored graph $G^C = \left(G^C\right)_{\text{red}} \# \left(G^C\right)^{\text{iso}}$, and using $G^C_{\text{red}} = \left(G^C\right)_{\text{red}}$ by simplicity, we have

$$F_G(\boldsymbol{a}+p\boldsymbol{x},\boldsymbol{s}) = p^{-\sum_{l\in E\left(G^C_{\text{red}}\right)} s(l)} \prod_{\substack{l\in E\left(G^C_{\text{red}}\right) \\ i_G(l)=\{v,u\}}} |x_v - x_u|_p^{s(l)},$$

and

$$I(\boldsymbol{s};\boldsymbol{a}) = p^{-|V(G)|-\sum_{l\in E\left(G^C_{\text{red}}\right)} s(l)} Z\left(\{s(l)\}_{l\in E\left(G^C_{\text{red}}\right)}, \{x_v\}_{v\in V\left(G^C_{\text{red}}\right)}\right).$$

Therefore

$$Z(\boldsymbol{s};G) = \sum_{G^C,\ C\in Colors(G)} p^{-|V(G)|-\sum_{l\in E\left(G^C_{\text{red}}\right)} s(l)} Z\left(\boldsymbol{s};G^C_{\text{red}}\right).$$

By fixing a graph H in $Subgraphs_{\mathcal{F}}(G,|G|)$, we have

$$\sum_{\mathcal{F}(G^C)=H} p^{-|V(G)|-\sum_{l\in E\left(G^C_{\mathrm{red}}\right)} s(l)} Z\left(\boldsymbol{s};G^C_{\mathrm{red}}\right) = p^{-|V(G)|-\sum_{l\in E(H)} s(l)} \mathcal{C}(p;H) Z\left(\boldsymbol{s};H\right), \tag{3.3.3}$$

and consequently

$$Z\left(\boldsymbol{s};G\right) = \sum_{H\in Subgraphs_{\mathcal{F}}(G,|G|)} p^{-|V(G)|-\sum_{l\in E(H)} s(l)} \mathcal{C}(p;H) Z\left(\boldsymbol{s};H\right) \tag{3.3.4}$$

By taking $H=G$, $\mathcal{C}(p;H)=p$, in (3.3.3), we get

$$\sum_{\mathcal{F}(G^C)=G} p^{-|V(G)|-\sum_{l\in E(G)} s(l)} Z\left(s;G^C\right) = p^{1-|V(G)|-\sum_{l\in E(G)} s(l)} Z\left(\boldsymbol{s};G\right)$$

and thus from (3.3.4),

$$Z(\boldsymbol{s};G) = \frac{\displaystyle\sum_{\substack{H\in Subgraphs_{\mathcal{F}}(G,|G|)\\ H\neq G}} p^{-|V(G)|-\sum_{l\in E(H)} s(l)} \mathcal{C}(p;H) Z\left(\boldsymbol{s};H\right)}{1-p^{1-|V(G)|-\sum_{l\in E(G)} s(l)}}. \tag{3.3.5}$$

Now, taking $H=H_1\#\cdots\#H_{r(H)}\#H^{\mathrm{iso}}$, where the H_is are different graphs in $Indgraphs(H)$, we have

$$Z\left(\boldsymbol{s};H\right) = \prod_{j=1}^{r(H)} Z(\boldsymbol{s};H_j). \tag{3.3.6}$$

By using recursively (3.3.5)-(3.3.6), and the formula for $Z(\boldsymbol{s};K_2)$, we obtain (3.3.2). Notice that at the beginning of any iteration of the formulas (3.3.5)-(3.3.6), with $|H_j|\geq 2$ for $j=1,\ldots,r(H)$, we have

$$\prod_{j=1}^{r(H)} Z(\boldsymbol{s};H_j) = \frac{A(\boldsymbol{s};H_1,\ldots,H_{r(H)})}{\prod_{j=1}^{r(H)}\left(1-p^{1-|V(H_j)|-\sum_{l\in E\left(H_j\right)} s(l)}\right)},$$

where all the factors in the denominator are different since $H_j\cap H_i=\varnothing$ if $j\neq i$. $\square$

Algorithm 8 computes the local zeta function of a connected graph using Formula 3.3.1. This algorithm returns a list with Formula 3.3.1, the possible set of poles, and a dictionary[1] with

[1] A dictionary in Python can be understood as a set of key/value pairs, see [17, Section 5.5].

$\left(H, p^{-|V(G)|-\sum_{l\in E(H)} s(l)}\mathcal{C}(p;H)Z(\boldsymbol{s};H)\right)$ pairs, for every $H \in Subgraphs_{\mathcal{F}}(G,|G|)$, $H \neq G$.

Algorithm 8: The Zeta Function $Z(\mathbf{s}, G)$.

Function `ZetaFunctionGraph(`G`):`

  input : Let G be a connected graph.

  output: A list with components $Z(\mathbf{s}, G)$, $\prod_{T\in Indgraphs(G)} 1 - p^{1-|V(T)|-\sum_{l\in E(T)} s(l)}$, a set with the possible poles of $Z(\mathbf{s}, G)$, and a dictionary *AddedGraphs* with key word H and item $p^{-|V(G)|-\sum_{l\in E(H)} s(l)}\mathcal{C}(p,H)Z(\mathbf{s},H)$ for every $H \in Subgraphs_{\mathcal{F}}(G,|G|)$ and $H \neq G$.

  Begin

    Put $N = |V(G)|$
    Put $Z_G = 0$
    Put *AddedGraphs* the empty dictionary
    for $l \in E(G)$ **do**
      Put $p^{-s(l)}$ as variable
    end
    if $N = 1$ **then**
      return $[1, \emptyset, AddedGraphs]$
    end
    if $N = 2$ **then**
      Put $AddedGraphs\,[\{u\}\#\{v\}] := 1 - p^{-1}$ `// Where` u `and` v `are the vertices of` G
      return $\left[1 - p^{-1}/1 - p^{-1-s(l)}, \{1 - p^{-1-s(l)}\}, AddedGraphs\right]$
    end
    else
      for $H \in Coloredgraphs(G)$`// See Algorithm 3`
       do
        Put $G_H^* = GetGraph(G, H)$ `// See Algorithm 7`
        Put $\mathcal{C}(p, H) = ChromaticPoly(G_H^*, p)$ `// See Algorithm 6`
        Put $Fact_H = p^{-N}$
        for $l \in E(H)$ **do**
          $Fact_H = Fact_H \times p^{-s(l)}$
        end
        Put $Z_H = 1$
        for $H' \in GetComponents(H)$`// See Algorithm 2`
         do
          $Z_H = Z_H \times ZetaFunctionGraph(H')$
        end
        $Z_G = Z_G + (Fact_H \times \mathcal{C}(p,H) \times Z_H)$
        $AddedGraphs[H] := Fact_H \times \mathcal{C}(p,H) \times Z_H$
      end
    end

Put $D_G = p^{1-N}$;
Put $PossiblePoles = \emptyset$;
for $l \in E(G)$ **do**
  $D_G = D_G \times p^{-s(l)}$
Put $D_G = 1 - D_G$;
for $T \in GetIndgraphs(G)$`// See Algorithm 4`
 do
  Put $D_T = p^{1-|V(T)|}$;
  for $l \in E(T)$ **do**
   $D_T = D_T \times p^{-s(l)}$
  Put $D_T = 1 - D_T$;
  Add D_T to $PossiblePoles$
return $[Z_G/D_G, PossiblePoles, AddedGraphs]$

Proposition 2. *(i) Set* $s\,(l) = \gamma \in \mathbb{C}$ *for any* $l \in E(G)$*, and define* $\mathcal{Z}_{G,p}\,(\gamma) := Z\,(\boldsymbol{s};G)|_{s(l)=\gamma}$*. Then the integral* $\mathcal{Z}_{G,p}\,(\gamma)$ *converges for*

$$\mathrm{Re}(\gamma) \geq \max_{\substack{H \in Indgraphs(G)\\ |V(H)| \geq 2}} \frac{1 - |V(H)|}{|E(H)|} =: \gamma_0.$$

More generally, for G *and* p *fixed,* $\mathcal{Z}_{G,p}\,(\gamma)$ *is an analytic function in* γ *for* $\mathrm{Re}(\gamma) \geq \gamma_0$*.*

(ii) Let $G = K_N$ *be the complete graph with* N *vertices. Then* $\mathcal{Z}_{G,p}\,(\gamma)$ *is an analytic function in* γ *for* $\mathrm{Re}(\gamma) \geq \frac{-2}{N}$*.*

(iii) Let $M\left(\left\{p^{-s(l)}; l \in E\,(G)\right\}\right)$ *be the polynomial defined in (3.3.2). Then the following functional equations hold true:*

$$M\left(\left\{p^{-s(l)}; l \in E\,(G)\right\}\right) = M\left(\left\{p^{-s(\sigma_E(l))}; l \in E\,(G)\right\}\right)$$

for any $\sigma = (\sigma_V, \sigma_E) \in$*Aut*$(G)$*.*

Proof. (i) It follows directly from Theorem 7-(ii), by using the properties of the geometric series. (ii) It follows from the fact that any induced subgraph H of K_N is complete, say $H = K_l$, $|V(H)| = l$, $|E(H)| = \frac{l(l-1)}{2}$ for $l = 2, \ldots, N$. Then

$$\gamma_0 = \max_{2 \leq l \leq N} \frac{-2}{l} = \frac{-2}{N}.$$

(iii) It follows from Theorem 7-(ii) and Corollary 3 by using the fact that any isomorphism of G induces a permutation on the set $\{H \in Indgraphs(G); |V\,(H)| \geq 2\}$. □

Corollary 4. *(i) Let* G_I *be an Indgraph of* G *generated by* $I \subset V(G)$*. Then*

$$Z(\boldsymbol{s}; G_I) = Z(\boldsymbol{s}; G)\mid_{\substack{s(l)=0\\ l \notin E(G_I)}}.$$

(ii) If $\lim_{s(l)\to a_l} Z(\boldsymbol{s}; G_I) = \infty$*, then*

$$\lim_{\substack{s(l)\to a_l\\ l\in E(G_I)}} \lim_{\substack{s(l)\to 0\\ l\notin E(G_I)}} Z(\boldsymbol{s}; G) = \infty.$$

(iii) Let $l_0 \in E(G)$ *and let* K_2 *be the corresponding induced graph. Then*

$$\lim_{\substack{s(l)\to 0\\ l\in E(G_I)\smallsetminus\{l_0\}}} \lim_{s(l_0)\to\ -1} Z(\boldsymbol{s}; G) = \infty.$$

Proof. (i) It follows from Theorem 7-(i). (ii) It follows from (i). (iii) It follows from (ii) by using the formula for $Z(s, K_2)$. □

3.4 Universal zeta functions for graphs

Let G be a graph as before. Let $\boldsymbol{L}$, $\boldsymbol{X}_l$ for $l \in E(G)$ be indeterminates. We set $\boldsymbol{X} = \{\boldsymbol{X}_l\}_{l\in E(G)}$. We denote by $\mathbb{Z}_{\text{loc}}[\boldsymbol{L}]$ the localization of $\mathbb{Z}[\boldsymbol{L}]$ with respect to the multiplicative system $\left\{\boldsymbol{L}^k; k \in \mathbb{N}\right\}$. We denote by $\mathbb{Z}_{\text{loc}}[\boldsymbol{L}]\left(\{\boldsymbol{X}_l\}_{l\in E(G)}\right)$ the field of rational function in the indeterminates $\{\boldsymbol{X}_l\}_{l\in E(G)}$ with coefficients in $\mathbb{Z}_{\text{loc}}[\boldsymbol{L}]$.

Definition 24. *We define the universal zeta function of G recursively as follows: if* $|E(G)| \geq 2$*,*

$$Z(\boldsymbol{X}; G) = \frac{\sum\limits_{\substack{H\in Subgraphs_{\mathcal{F}}(G)\\ H\neq G}} \boldsymbol{L}^{-|V(G)|}\mathcal{C}(\boldsymbol{L}; H)\left(\prod_{l\in E(H)} \boldsymbol{X}_l\right) Z(\boldsymbol{X}; H)}{1 - \boldsymbol{L}^{1-|V(G)|}\prod_{l\in E(G)} \boldsymbol{X}_l}.$$

If $|E(G)| = 1$*, i.e.* $G = K_2$*, then* $Z(\boldsymbol{X}; K_2) = \frac{1-\boldsymbol{L}^{-1}}{1-\boldsymbol{L}^{-1}\boldsymbol{X}}$*. When* G *is a graph with just one vertex, then* $Z(\boldsymbol{X}; G) = 1$*.*

Theorem 8. *Let* G*,* W *be simple, finite, connected graphs. Then the function* $Z(\boldsymbol{X}; G)$ *satisfies the following:*

(i) $Z(\boldsymbol{X}; G) \in \mathbb{Z}_{loc}[\boldsymbol{L}]\left(\{\boldsymbol{X}_l\}_{l\in E(G)}\right)$*;*

(ii) for $p = p(G)$ *sufficiently large,*

$$Z(\boldsymbol{X}; G)\mid_{L=p,\ X_l=p^{-s(l)}} = Z(\{s(l)\}_{l\in E(G)}; G);$$

Proof. (i) It follows from the recursive definition of $Z(\boldsymbol{X}; G)$ by using the reasoning given in the proof of Theorem 7-(i) and Proposition 2.

(ii) It follows from Theorem 7-(i) and Proposition 2, by using the definition of $Z(\boldsymbol{X}; G)$. □

Chapter 4

Local zeta functions of some specific graphs

In this chapter, we compute the local zeta functions for some specific graphs. In all the graphs considered here, we determine the actual poles of the corresponding zeta functions. As we already mentioned, in general, this is a difficult task.

4.1 Zeta function for tree and tree-like graphs

4.1.1 Star graphs

Lemma 4. *Let S_N be a star graph. Denote by $x_{v_1} = x_1$, $x_{v_i} = x_i$, and $s_i = s(v_1, v_i)$ for $i = 2, \ldots, N$. Then*

$$Z\left(\boldsymbol{s}, S_N\right) = \frac{\left(1-p^{-1}\right)^{N-1}}{\prod_{i=2}^{N} 1 - p^{-1-s_i}}.$$

Proof. Note that $F_{S_N}(\mathbf{x}, \mathbf{s}) = \prod_{i=2}^{N} |x_1 - x_i|_p^{s_i}$. By changing variables as $z_1 = x_1$, $z_i = x_1 - x_i$ for $i = 2, \cdots, N$, we obtain that

$$\begin{aligned} Z\left(\mathbf{s}, S_N\right) &= \int_{\mathbb{Z}_p^N} \prod_{i=2}^{N} |x_1 - x_i|_p^{s_i} \prod_{i=1}^{N} dx_i = \int_{\mathbb{Z}_p} \left(\int_{\mathbb{Z}_p^{N-1}} \prod_{i=2}^{N} |x_1 - x_i|_p^{s_i} \prod_{i=2}^{N} dx_i \right) dx_1 \\ &= \left(\int_{\mathbb{Z}_p} dz_1 \right) \left(\int_{\mathbb{Z}_p^{N-1}} \prod_{i=2}^{N} |z_i|_p^{s_i} \prod_{i=2}^{N} dz_i \right) = \prod_{i=2}^{N} \frac{1-p^{-1}}{1-p^{-1-s_i}} = \frac{\left(1-p^{-1}\right)^{N-1}}{\prod_{i=2}^{N} 1 - p^{-1-s_i}}. \end{aligned}$$

□

4.1.2 Path graphs

Lemma 5. *Let A_N be a path graph, we denote by $s_i = s(v_i, v_{i+1})$ and $x_{v_i} = x_i$ for $i = 1, \cdots, N$. Then*

$$Z(\boldsymbol{s}, A_N) = \frac{(1-p^{-1})^{N-1}}{\prod_{i=1}^{N-1} 1 - p^{-1-s_i}}.$$

Proof. We have that $F_{A_N}(\mathbf{x}, \mathbf{s}) = \prod_{i=1}^{N-1} |x_i - x_{i+1}|_p^{s_i}$. By using the change of variables $z_1 = x_1$ and $z_i = x_i - x_{i+1}$ for $i = 1, \cdots, N-1$ we obtain the result. □

Remark 10. *With these two examples, we can see that the reciprocal of Lemma 3 is not true since S_N and A_N are not isomorphic for $N > 2$ but its zeta functions are the same.*

4.1.3 Tree graphs

Lemma 6. *Let T_N any tree graph as in Definition 3-(3) with $M = |E(T_N)|$. Then*

$$Z(\boldsymbol{s}, T_N) = \frac{(1-p^{-1})^{M}}{\prod_{l \in E(T_N)} 1 - p^{-1-s(l)}}.$$

Proof. The proof follows by induction over N, the number of vertices of T_N. If $N = 2$ note that $T_2 = A_2$ and

$$Z(\mathbf{s}, T_2) = \frac{(1-p^{-1})}{1 - p^{-1-s(l)}},$$

so the assertion is true. Now, suppose that

$$Z(\mathbf{s}, T_j) = \frac{(1-p^{-1})^{|E(T_j)|}}{\prod_{l \in E(T_j)} 1 - p^{-1-s(l)}}, \text{ for any } j < N.$$

Define D_1 as the set of vertices v in G with degree 1, see Definition 13. Take T' to be the subgraph of G induced by the set $V(G) \setminus D_1$, see Definition 10. Note that T' is a tree, since G is a tree. By the induction hypothesis, we have

$$\int_{\mathbb{Z}_p^{|V(T_N)\setminus D_1|}} \prod_{\substack{u,v \in V(T_N)\setminus D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{v \in V(T_N)\setminus D_1} dx_v = \frac{(1-p^{-1})^{M-|D_1|}}{\prod_{l \in E(T')} 1 - p^{-1-s(l)}}. \tag{4.1.1}$$

On the other hand,

$$F_{T_N}(\boldsymbol{x}, \boldsymbol{s}) = \left(\prod_{\substack{u,v \in V(T_N)\setminus D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \right) \left(\prod_{\substack{v \in V(T_N)\setminus D_1;\, u \in D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \right) \tag{4.1.2}$$

and

$$Z(\mathbf{s}, T_N) = \int_{\mathbb{Z}_p^N} \prod_{\substack{u,v \in V(T_N) \setminus D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{\substack{v \in V(T_N) \setminus D_1;\, u \in D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{v \in V(T_N)} dx_v.$$

we may assume without loss of generality that $D_1 = \{1, \ldots, k\}$. Denote by $x_1, \ldots, x_k$, the variables associated to the elements of D_1. By definition of D_1 and the fact that T_N is connected, there is only one vertex v_k in $V(T_N) \setminus D_1$ connected with each element of D_1. Denote by $x_{v_1}, \ldots, x_{v_k}$ the variables corresponding to these v_k. Then

$$Z(\mathbf{s}, T_N) = \int_{\mathbb{Z}_p^N} \prod_{\substack{u,v \in V(T_N) \setminus D_1 \\ u \sim v}} |x_u - x_v|_p^{s(u,v)} \prod_{i=1}^{k} |x_i - x_{v_1}|_p^{s(i,v_i)} \prod_{v \in V(T_N)} dx_v.$$

By changing variables as

$$\begin{cases} z_i = x_i - x_{v_i} & \text{for} \quad i = 1, \ldots, k \\ z_i = x_i & \text{for} \quad i = k+1, \ldots, N, \end{cases} \tag{4.1.3}$$

we get that

$$Z(\mathbf{s}, T_N) = \int_{\mathbb{Z}_p^N} \prod_{\substack{u,v \in V(T_N) \setminus D_1 \\ u \sim v}} |z_u - z_v|_p^{s(u,v)} \prod_{i=1}^{k} |z_i|_p^{s(i,v_i)} \prod_{v \in V(T_N)} dz_v. \tag{4.1.4}$$

Note that the Jacobian matrix of the change of variables (4.1.3) has the form

$$\begin{bmatrix} I_k & * \\ 0 & I_{N-k} \end{bmatrix}, \tag{4.1.5}$$

since T_N is connected and it has no cycles $x_{v_j} \neq x_i$ for every $j = 1, \ldots, k$ and any $i = 1, \ldots, k$.

By using 4.1.4 and 1.3.1, we have

$$Z(\mathbf{s}, T_N) =$$
$$\left(\prod_{i=1}^{k} \int_{\mathbb{Z}_p} |z_i|_p^{s(i,v_i)}\, dz_i \right) \int_{\mathbb{Z}_p^{|V(T_N) \setminus D_1|}} \prod_{\substack{u,v \in V(T_N) \setminus D_1 \\ u \sim v}} |z_u - z_v|_p^{s(u,v)} \prod_{v \in V(T_N) \setminus D_1} dz_v =$$
$$\left(\prod_{i=1}^{k} \frac{1 - p^{-1}}{1 - p^{-1-s(i,v_i)}} \right) \frac{\left(1 - p^{-1}\right)^{M - |D_1|}}{\prod_{l \in E(T')} 1 - p^{-1-s(l)}}.$$

Finally, since every $u \in D_1$ has degree 1, we have

$$\prod_{i=1}^{k} 1 - p^{-1-s(i,v_i)} \prod_{l \in E(T')} 1 - p^{-1-s(l)} = \prod_{l \in E(T_N)} 1 - p^{-1-s(l)}.$$

$\square$

Remark 11. *Note that Lemma 4 and Lemma 5 are particular cases of the Lemma 6, since S_N and A_N are trees.*

4.2 Graph K_3

Lemma 7. *If we denote the graph K_3 as in the figure*

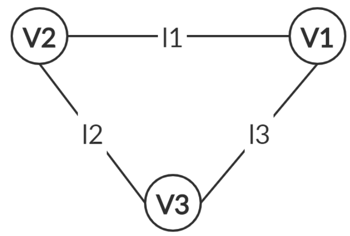

Figure 4.1: Graph K_3.

Then

$$Z(\boldsymbol{s}, K_3) = \frac{p^{-2}(p-1)\left(L_1(\boldsymbol{s}) + L_2(\boldsymbol{s}) + L_3(\boldsymbol{s}) + L_4(\boldsymbol{s})\right)}{\left(1-p^{-1-s(l_1)}\right)\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_2)}\right)\left(1-p^{-2-s(l_1)-s(l_3)-s(l_2)}\right)},$$

where

$$\begin{aligned}
L_1(\boldsymbol{s}) &= p^{-s(l_1)}\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_2)}\right), \\
L_2(\boldsymbol{s}) &= p^{-s(l_3)}\left(1-p^{-1-s(l_1)}\right)\left(1-p^{-1-s(l_2)}\right), \\
L_3(\boldsymbol{s}) &= p^{-s(l_2)}\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_1)}\right), \\
L_4(\boldsymbol{s}) &= (p-2)\left(1-p^{-1-s(l_1)}\right)\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_2)}\right).
\end{aligned}$$

Moreover, the polynomials $\left(1-p^{-1-s(l_1)}\right)$, $\left(1-p^{-1-s(l_3)}\right)$, $\left(1-p^{-1-s(l_2)}\right)$, and $\left(1-p^{-2-s(l_1)-s(l_3)-s(l_2)}\right)$ do not divide $L_1(\boldsymbol{s}) + L_2(\boldsymbol{s}) + L_3(\boldsymbol{s}) + L_4(\boldsymbol{s})$.

Proof. The announced formula for $Z(\mathbf{s}, K_3)$ follows from Examples 11, 9, and 10, and 3.3.1. Now, We only have to see that each factor of its denominator does not divide the numerator. First, note that

$$L_1(\mathbf{s})+L_2(\mathbf{s})+L_3(\mathbf{s})+L_4(\mathbf{s}) \equiv p^{-s(l_1)}\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_2)}\right) \not\equiv 0 \left(\mod 1-p^{-1-s(l_1)}\right).$$

So, without loss of generality, $\left(1-p^{-1-s(l_1)}\right)\left(1-p^{-1-s(l_3)}\right)\left(1-p^{-1-s(l_2)}\right)$ does not divide the numerator of $Z(\mathbf{s}, K_3)$, since $\left(1-p^{-1-s(l_1)}\right)$, $\left(1-p^{-1-s(l_3)}\right)$, and $\left(1-p^{-1-s(l_2)}\right)$ are irreducible polynomials in $\mathbb{Q}[p^{-s(l_1)}, p^{-s(l_2)}, p^{-s(l_3)}]$.

Second, note that the coefficient of $p^{-s(l_1)-s(l_2)-s(l_3)}$ in the numerator of $Z(\mathbf{s}, K_3)$ is

$$\left(3p^{-2} - p^{-3}(p-2)\right)(p-1)p^{-2}$$

and its constant coefficient is $p^{-2}(p-1)(p-2)$. Then, if $\left(1-p^{-2-s(l_1)-s(l_3)-s(l_2)}\right)$ divides the numerator of $Z(\mathbf{s},K_3)$, it has the form $M\left(1-p^{-2-s(l_1)(l_1)-s(l_3)-s(l_2)}\right)$ with $M\in\mathbb{Q}$, this implies that $M=2p^{-2}(p^{-1}+1)$ and $M=p^{-2}(p-1)(p-2)$. But this is impossible, consequently $\left(1-p^{-2-s(l_1)-s(l_3)-s(l_2)}\right)$ do not divide the numerator of $Z(\mathbf{s},K_3)$.

$\square$

4.3 More examples

We now compute the irreducible factors of the denominator of $Z(\mathbf{s},G)$ for some well known small graphs, see [45] for a list of small graphs. The calculation in this section were made using Program *ZetaFunctionGraph.py* in Appendix B. We denote $p^{-s(v_i,v_j)}$ as zij.

1. **Barbell Graph:**

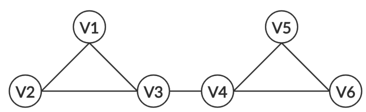

Figure 4.2: Barbell graph with 6 vertices.

Irreducible factors:

$$1-\frac{z12}{p},\ 1-\frac{z13}{p},\ 1-\frac{z23}{p},\ 1-\frac{z34}{p},\ 1-\frac{z45}{p},\ 1-\frac{z46}{p},\ 1-\frac{z56}{p},$$
$$1-\frac{z12z13z23}{p^2},\ 1-\frac{z45z46z56}{p^2},\ 1-\frac{z12z13z23z34z45z46z56}{p^5}.$$

2. **Book Graph:**

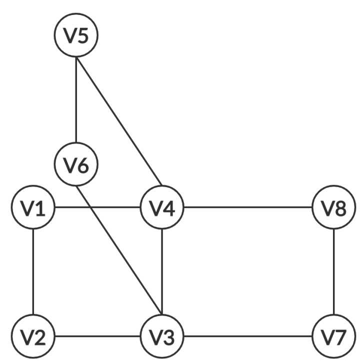

Figure 4.3: Book graph with 8 vertices.

Irreducible factors:

$$1-\frac{z_{12}}{p},\ 1-\frac{z_{14}}{p},\ 1-\frac{z_{23}}{p},\ 1-\frac{z_{34}}{p},\ 1-\frac{z_{36}}{p},\ 1-\frac{z_{37}}{p},\ 1-\frac{z_{45}}{p},$$
$$1-\frac{z_{48}}{p},\ 1-\frac{z_{56}}{p},\ 1-\frac{z_{78}}{p},\ 1-\frac{z_{12}z_{14}z_{23}z_{34}}{p^3},\ 1-\frac{z_{34}z_{36}z_{45}z_{56}}{p^3},$$
$$1-\frac{z_{34}z_{37}z_{48}z_{78}}{p^3},\ 1-\frac{z_{12}z_{14}z_{23}z_{36}z_{45}z_{56}}{p^5},\ 1-\frac{z_{12}z_{14}z_{23}z_{37}z_{48}z_{78}}{p^5},$$
$$1-\frac{z_{36}z_{37}z_{45}z_{48}z_{56}z_{78}}{p^5},\ 1-\frac{z_{12}z_{14}z_{23}z_{34}z_{36}z_{45}z_{56}}{p^5},$$
$$1-\frac{z_{12}z_{14}z_{23}z_{34}z_{37}z_{48}z_{78}}{p^5},\ 1-\frac{z_{34}z_{36}z_{37}z_{45}z_{48}z_{56}z_{78}}{p^5},$$
$$1-\frac{z_{12}z_{14}z_{23}z_{34}z_{36}z_{37}z_{45}z_{48}z_{56}z_{78}}{p^7}.$$

3. **Gear Graph:**

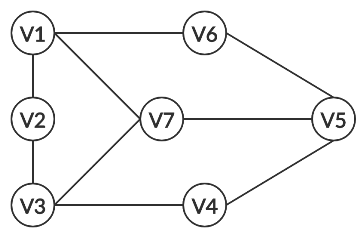

Figure 4.4: Gear graph with 7 vertices.

Irreducible factors:

$$1-\frac{z_{12}}{p},\ 1-\frac{z_{17}}{p},\ 1-\frac{z_{23}}{p},\ 1-\frac{z_{34}}{p},\ 1-\frac{z_{37}}{p},\ 1-\frac{z_{45}}{p},\ 1-\frac{z_{56}}{p},\ 1-\frac{z_{57}}{p},\ 1-\frac{z_{12}z_{17}z_{23}z_{37}}{p^3},\ 1-\frac{z_{34}z_{37}z_{45}z_{57}}{p^3},\ 1-\frac{z_{12}z_{17}z_{23}z_{34}z_{37}z_{45}z_{57}}{p^5}.$$

4. **Prism Graph**

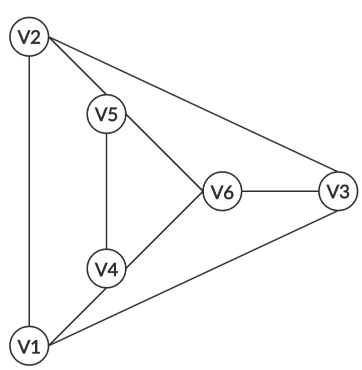

Figure 4.5: Prism graph with 6 vertices.

Irreducible factors:

$1-\frac{z_{12}}{p}$, $1-\frac{z_{13}}{p}$, $1-\frac{z_{14}}{p}$, $1-\frac{z_{23}}{p}$, $1-\frac{z_{25}}{p}$, $1-\frac{z_{36}}{p}$, $1-\frac{z_{45}}{p}$, $1-\frac{z_{46}}{p}$, $1-\frac{z_{56}}{p}$, $1-\frac{z_{12}z_{13}z_{23}}{p^2}$, $1-\frac{z_{45}z_{46}z_{56}}{p^2}$, $1-\frac{z_{12}z_{14}z_{25}z_{45}}{p^3}$, $1-\frac{z_{13}z_{14}z_{36}z_{46}}{p^3}$, $1-\frac{z_{23}z_{25}z_{36}z_{56}}{p^3}$, $1-\frac{z_{12}z_{13}z_{25}z_{36}z_{56}}{p^4}$, $1-\frac{z_{12}z_{14}z_{23}z_{36}z_{46}}{p^4}$, $1-\frac{z_{12}z_{14}z_{25}z_{46}z_{56}}{p^4}$, $1-\frac{z_{13}z_{14}z_{23}z_{25}z_{45}}{p^4}$, $1-\frac{z_{13}z_{14}z_{36}z_{45}z_{56}}{p^4}$, $1-\frac{z_{23}z_{25}z_{36}z_{45}z_{46}}{p^4}$, $1-\frac{z_{12}z_{13}z_{14}z_{23}z_{25}z_{45}}{p^4}$, $1-\frac{z_{12}z_{13}z_{14}z_{23}z_{36}z_{46}}{p^4}$, $1-\frac{z_{12}z_{13}z_{23}z_{25}z_{36}z_{56}}{p^4}$, $1-\frac{z_{12}z_{14}z_{25}z_{45}z_{46}z_{56}}{p^4}$, $1-\frac{z_{13}z_{14}z_{36}z_{45}z_{46}z_{56}}{p^4}$, $1-\frac{z_{23}z_{25}z_{36}z_{45}z_{46}z_{56}}{p^4}$, $1-\frac{z_{12}z_{13}z_{14}z_{23}z_{25}z_{36}z_{45}z_{46}z_{56}}{p^5}$.

5. **Wheel Graph**

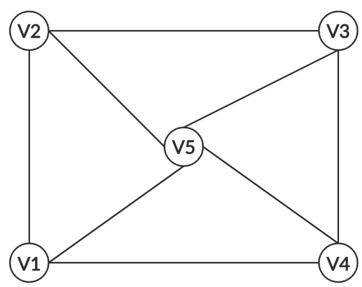

Figure 4.6: Wheel graph with 5 vertices.

Irreducible factors:

$1-\frac{z_{12}}{p}$, $1-\frac{z_{14}}{p}$, $1-\frac{z_{15}}{p}$, $1-\frac{z_{23}}{p}$, $1-\frac{z_{25}}{p}$, $1-\frac{z_{34}}{p}$, $1-\frac{z_{35}}{p}$, $1-\frac{z_{45}}{p}$, $1-\frac{z_{12}z_{15}z_{25}}{p^2}$, $1-\frac{z_{14}z_{15}z_{45}}{p^2}$, $1-\frac{z_{23}z_{25}z_{35}}{p^2}$, $1-\frac{z_{34}z_{35}z_{45}}{p^2}$, $1-\frac{z_{12}z_{14}z_{23}z_{34}}{p^3}$, $1-\frac{z_{12}z_{14}z_{25}z_{45}}{p^3}$, $1-\frac{z_{12}z_{15}z_{23}z_{35}}{p^3}$, $1-\frac{z_{14}z_{15}z_{34}z_{35}}{p^3}$, $1-\frac{z_{23}z_{25}z_{34}z_{45}}{p^3}$, $1-\frac{z_{12}z_{14}z_{15}z_{25}z_{45}}{p^3}$, $1-\frac{z_{12}z_{15}z_{23}z_{25}z_{35}}{p^3}$, $1-\frac{z_{14}z_{15}z_{34}z_{35}z_{45}}{p^3}$, $1-\frac{z_{23}z_{25}z_{34}z_{35}z_{45}}{p^3}$, $1-\frac{z_{12}z_{14}z_{15}z_{23}z_{25}z_{34}z_{35}z_{45}}{p^4}$.

Chapter 5

Conclusions

Since this is a mathematical thesis, we usually do not have conclusions from our work.

Appendix A

Breadth first search algorithm

In this section, we discuss the implementation of the Breadth First Search algorithm (*BFS*) in Python. For an in-depth review of this algorithm, the reader may consult [9, Section 22.2].

Definition 25. *A Queue is a dynamic set in which the element removed from the set by Delete operation is prespecified. In this case, the element deleted is always the one that has been in the set for the longest time. We call the INSERT operation on a queue ENQUEUE, and we call the DELETE operation DEQUEUE.*

Definition 26. *Let G be a graph and u a vertex of G. A subgraph G_u of G is a breadth-first tree if $V(G_u)$ consists of the vertices reachable from u and, for all $v \in V(G_u)$, the subgraph G_u contains a unique path from u to v that is also a shortest path from u to v in G.*

Algorithm 9: Breadth First Search.

```
Function BFS(G,s):
    input  : Let G be a graph and s ∈ V(G).
    output: A list with the distances from s to each vertex v ∈ V(G) and the
            breadth-first tree G_s.
    for u ∈ V(G); u ≠ s do
        color(u) = white
        d(u) = ∞
        π(u) = Nan
    end
    color(s) = gray
    d(s) = 0
    π(s) = Nan
    Enqueue(A, s)
    while A ≠ ∅ do
        u = Dequeue(A)
        for v ∈ Γ(u) do
            if color(v) = white then
                color(v) = gray
                d(v) = d(u) + 1
                π(v) = u
                Enqueue(A,v)
            end
        end
        color(u) = black
    end
```

Remark 12. *To keep track of progress, breadth-first search colors each vertex withe, gray, or black. All vertices star out white and may later become gray and then black. A vertex is* ***discovered*** *the first time it is encountered during the search, at which time it becomes nonwhite. Gray and black vertices, therefore, have been discovered, but breadth-first search distinguishes between them to ensure that the search proceeds in a breadth-first manner. If* $\{u, v\} \in E(G)$ *and vertex u is black, then vertex v is either gray or black; that is, all vertices adjacent to black vertices have been discovered. Gray vertices may have some adjacent white vertices; they represent the frontier between discovered and undiscovered vertices.*

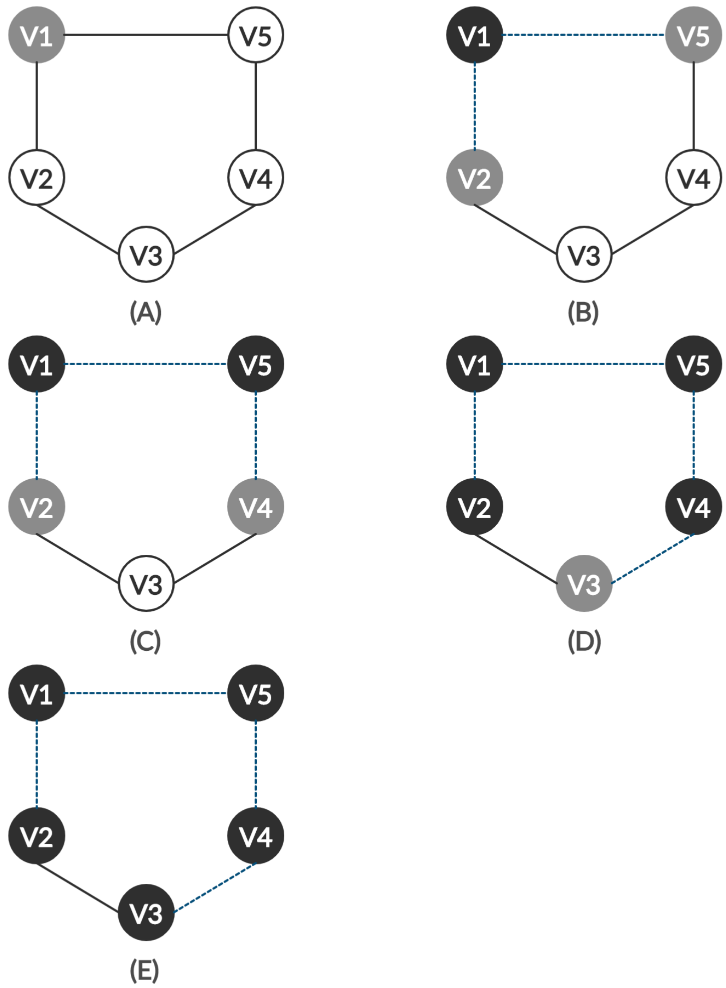

Figure A.1: The operation of BFS on the graph C_5. Tree edges are shown with the dotted lines as they are produced by BFS.

Appendix B

Implementation

In this section, we present an implementation of Algorithm *ZetaFunctionGraph* in Python language. For this, we will use the following conventions. First, for all graphs G, we set $V(G) = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. Second, we set $p^{-s(i,j)} = zij$ if $i \sim j$.

B.1 Class graph

In the class graph, we have implemented the graph's properties which we use to calculate the zeta function, using in Theorem 7.

```
from itertools import chain, combinations
def powerset(iterable):

    s = list(iterable)
    return chain.from_iterable(combinations(s, r) for r in ↩
       range(len(s)+1))
class Graph(object):
    def __init__(self, Vertices, Edges):

        self.Vertices=Vertices
        self.Edges=Edges
    def getVertices(self):
        return self.Vertices.copy()
    def getEdges(self):
        return self.Edges[:]
    def getNeighborhood(self,v):

         A=[]
         for u in self.getVertices():
              if {u,v} in self.getEdges():
                   A.append(u)
```

```
        return set(A)
    def addVertex(self,x):
        V=self.getVertices()
        V.add(x)
        return Graph(V,self.getEdges())
    def addEdge(self,x):
        E=self.getEdges()
        if type(x)!=set:
            raise("Invalit␣value.")

        else:

            if x.issubset(self.Vertices):
               if not(x in self.Edges):
                   E.append(x)
               return Graph(self.getVertices(),E)

    def DeletingEdge(self,e):
           E=self.getEdges()
           if e in self.Edges:
               E.remove(e)
           return Graph(self.getVertices(),E)
    def DeletingVertex(self,v):
           V=self.getVertices()
           if v in self.getVertices():
               V.discard(v)
               E=self.getEdges()
               for e in self.getEdges():
                   if v in e:
                      E.remove(e)
               return Graph(V,E)

    def getDegree(self,v):
        return len( self.getNeighborhood(v))

    def Induced(self,S):

          if S.issubset(self.getVertices()):
            E_S=[]
            for a in self.getEdges():
                  if a.issubset(S):
                      E_S.append(a)
            return Graph(S,E_S)
          else: raise("Invalit␣value")
```

```
    def joindVertices(self,x,y):

        if {x,y} in self.getEdges():
            v1=min(x,y)
            v2=max(x,y)
            J_Edges=self.getEdges()
            J_Vertices=self.getVertices()
            J_Vertices.discard(v2)
            for u in self.getNeighborhood(v2):
                J_Edges.remove({u,v2})
                if u!=v1 and not({u,v1} in J_Edges):
                    J_Edges.append({u,v1})

            return Graph(J_Vertices,J_Edges)
        else:
            return Graph(self.getVertices(),self.getEdges())
    def joindEdge(self,e):
        E=list(e)
        return self.joindVertices(E[0],E[1])
    def BFS(self,s):

        V=self.getVertices()

        color={}
        dis={}
        pred={}
        for u in V:
            if u!=s:
                color[u]='white'
                dis[u]=0
                pred[u]=None
        color[s]='gray'
        dis[s]=0
        pred[s]=None
        A=[s]
        while A!=[]:
            u=A.pop(0)
            for v in self.getNeighborhood(u):
                if color[v]=='white':
                    color[v]='gray'
                    dis[v]=dis[u]+1
                    pred[v]=u
                    A.append(v)
            color[u]='black'
        E_S=[]
```

```
        V_S={s}
        for u in V:
            if dis[u]>0:
               V_S.add(u)
               E_S.append({u,pred[u]})
        return (Graph(V_S,E_S),dis)
    def disVertices(self,x,y):
        if {x,y} in self.getEdges():
            return 1
        else:
            tree_x, dis=self.BFS(x)
            if y in tree_x.getVertices():
                return dis[y]
            else:
                return None
    def disGraph(self,H,W):

        ans=[]
        for u in H.getVertices():
            for v in W.getVertices():
                if self.disVertices(u,v)==None:
                    return None
                else:
                   ans.append(self.disVertices(u,v) )
        return min(ans)
    def IsConnected(self):
       if self.getVertices()==set():
           return True
       else:
        s=self.getVertices().pop()
        tree=self.BFS(s)[0]
        if self.getVertices()==tree.getVertices():
            return True
        else:
            return False
    def getComponents(self):
        ans=[]
        A=set()
        V=self.getVertices()
        while A != V:
            s=(V-A).pop()
            Tree_s=self.BFS(s)[0]
            A=A|Tree_s.getVertices()
            G1=self.Induced(Tree_s.getVertices())
            ans.append(G1)
```

```
        return ans
    def DeletingsetEdges(self,A):

        G=self
        for e in A:
            G=G.DeletingEdge(e)
        return G
    def ColoredGraphs(self):
        ans=[]
        E=self.getEdges()
        for A in powerset(E):
          if not(A==()):
            A1=list(A)
            H=self.DeletingsetEdges(A1)
            if not(H.IsConnected()):
                ans.append(H)
        return ans
    def getGraphGH(self,H):
        V_GH=set()
        E_GH=[]
        Com_H=H.getComponents()
        l=len(Com_H)

        for i in range(l):
            V_GH.add(i+1)
            for j in range(i+1,l):
                if self.disGraph(Com_H[i],Com_H[j])==1:
                   E_GH.append({i+1,j+1})
        return Graph(V_GH,E_GH)

    def ChromaticPoly(self,x):

       V=self.getVertices()
       E=self.getEdges()
       if E==[]:
           return x**len(V)
       elif not(self.IsConnected()):
           Poly=1
           for H in self.getComponents():
               Poly=Poly*H.ChromaticPoly(x)
           return Poly
       elif len(V)+len(E)==3:
           return x*(x-1)
       else:
           s=E.pop()
```

```
            return self.DeletingEdge(s).ChromaticPoly(x) - ↩
               self.joindEdge(s).ChromaticPoly(x)

    def getIndgraphs(self):
        ans=[]
        V1=list(self.getVertices())
        for A in powerset(V1):
          if len(A)>1:
            A1=set(A)
            H=self.Induced(A1)
            if H.IsConnected():
                ans.append(H)
        return ans

    def __eq__(self,other):
        return (self.getVertices()==other.getVertices()) and ↩
           (self.getEdges()==other.getEdges())
    def __str__(self):
        return 'V='+ str(self.Vertices) +',␣'+ '␣E='+str(self↩
           .Edges)
```

Listing B.1: Class Graph.

B.2 Zeta function computation

Here we give an implementation of the recursive procedure given in Theorem 7 for computing the zeta function for a connected graph. We use the SymPy library to do symbolical calculations.

```
import sympy
from Class_Graph import *
p=sympy.symbols('p', integer=True)
def getVariables(G):

     Var=[]
     E=G.getEdges()
     for e in E:
         e1=list(e)
         e1.sort()
         z='z'+str(e1[0])+str(e1[1])
         w=sympy.symbols(z)
         Var.append(w)
     return Var
def ZetaFunctionGraph(G):
```

```python
    N=len(G.getVertices())
    Z_G=0
    AddedGraphs={}
    Variables_G=getVariables(G)
    if N==1:
        return [1,set(),{}]
    elif N==2:
        l=G.getEdges()[0]
        H=G.DeletingEdge(l)
        z=Variables_G[0]
        AddedGraphs[H.__str__()]=1-p**(-1)
        return [(1-p**(-1))/(1-p**(-1)*z),{1-p**(-1)*z},AddedGraphs]
    else:
        for H in G.ColoredGraphs():
            G_H=G.getGraphGH(H)
            C_H=G_H.ChromaticPoly(p)
            Fact_H=p**(-N)
            for z in getVariables(H):
                Fact_H=Fact_H*z
            Z_H=1
            for T in H.getComponents():
                Z_H=Z_H*ZetaFunctionGraph(T)[0]
            Z_G+=Fact_H*C_H*Z_H
            AddedGraphs[H.__str__()]=Fact_H*C_H*Z_H
        D_G=p**(1-N)
        PossiblePoles=set()
        for z in Variables_G:
            D_G=D_G*z
        D_G=1-D_G
        PossiblePoles.add(D_G)
        for T in G.getIndgraphs():
            D_T=p**(1-len(T.getVertices()))
            for z in getVariables(T):
                D_T=D_T*z
            D_T=1-D_T
            PossiblePoles.add(D_T)
        return [Z_G/D_G,PossiblePoles,AddedGraphs]
```

Listing B.2: Zeta Function Computation.

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