Cinvestav

Centro de Investigación y de Estudios Avanzados del I.P.N.

A Survey of Potential Differential Games: Optimal Control and Stability Techniques

Submitted in partial fulfillment of the requirements of the degree of

Master of Science in Mathematics

By

ALFONSO SÁNCHEZ AGUILAR

Supervisor:

Dr. Onésimo Hernández Lerma.

Mexico city December 7,2020

Centro de Investigación y de Estudios Avanzados del I.P.N.

Estudio de Juegos Diferenciales Potenciales :Técnicas de Control Óptimo y Estabilidad

Que para obtener el grado de

Maestro en Ciencias con Especialidad en Matemáticas

PRESENTA

ALFONSO SÁNCHEZ AGUILAR

Director de Tesis:

Dr. Onésimo Hernández Lerma

Ciudad de México Diciembre 7 del 2020

Acknowledgements

To my parents and my siblings for always being with me and for their immense support to complete my master's degree and this work. Because despite being physically far away, their love and affection have always been with me.

To Dr. Onésimo Hernández-Lerma for his eminent guidance in this work, for his time and his excellent corrections because thanks to them I improved the final version of my thesis.

I would also like to thank the Centro de Investigación y de Estudios Avanzados del IPN (CINVESTAV) for the extraordinary academic preparation and the Consejo Nacional de Ciencia y Tecnología (CONACYT) for the scholarship granted throughout my master's degree.

Agradecimientos.

A mis padres y a mis hermanos por estar siempre conmigo y por su inmenso apoyo para concluir mi maestría y este trabajo. Porque a pesar de estar lejos físicamente su cariño y amor siempre me han acompañado.

Al Dr. Onésimo Hernández-Lerma por su eminente dirección en este trabajo, por su tiempo y sus excelentes correcciones ya que gracias a ellas mejore la versión final de mi tesis.

De igual forma quiero agradecer al Centro de Investigación y de Estudios Avanzados del IPN (CINVESTAV) por la extraordinaria preparación académica y al Consejo Nacional de Ciencia y Tecnología (CONACYT) por la beca otorgada a lo largo de mi maestría.

Contents

Abstract. **3**

1 Introduction. **7**

1.1 Static and Dynamic Games. 7

1.2 Differential Games. 9

2 Optimal Control theory . **13**

2.1 Optimal Control Problems(OCP). 13

2.2 OCPs with Finite Time Horizon. 14

2.2.1 The HJB equation. 15

2.2.2 Maximum principle. 17

2.3 OCPs with Infinite Time Horizon. 20

2.3.1 The discounted case. 20

2.3.2 Other optimality criteria. 20

3 Potential Differential Games (PDG). **25**

3.1 PDGs: the exact potential approach 26

3.1.1 PDGs Over an Infinite Horizon. 26

3.1.2 PDGs Over a Finite Horizon. 29

3.2 PDGs the Fictitious-Potential Approach. 30

3.2.1 PDGs Over an Infinite Horizon. 33

3.2.2 PDGs Over a Finite Horizon. 36

4 Stability . **39**

4.1 Stability of Optimal Control. 39
4.2 Stability of Zero-Sum Games. 44
4.2.1 Asymptotic Stability:Basic concepts and results. 47
4.2.2 A Result about Stability of Zero-Sum Games. 49

5 Applications **53**

Conclusions. **59**

Bibliography and references. **61**

Abstract.

This work concerns the concept of a noncooperative potential differential game (PDG). Briefly put, a PDG is a game to which we can associate an optimal control problem (OCP) whose optimal solutions are Nash equilibria for the original game. We consider three important problem areas on PDGs. First: how to identify them. Second: how to solve them. Third: the stability analysis of the corresponding Nash equilibria.

In fact, one of our main objectives is to illustrate the stabilization problem for noncooperative differential games, that is, to find Nash equilibria that, in addition, stabilize the game's state process.

Our study is largely restricted to open-loop strategies. However, in the case of infinite-horizon zero-sum games we consider closed-loop strategies. Most of our results are illustrated by examples of theoretical and practical interest.

We begin in Chapter 1 with a brief introduction to game theory, including basic concepts such as static and dynamic games, Nash equilibrium, and so forth.

Chapter 2 summarizes some ideas from optimal control theory. First, we consider finite- and infinite-horizon optimal control problems (OCPs), and introduce two important techniques to analyze them, namely, dynamic programming and the maximum principle. The former is based on the so-called Bellman or Hamilton-Jacobi-Bellman (HJB) equation, and it gives sufficient conditions for optimality. In contrast, the maximum principle gives necessary conditions for optimality.

In Chapter 3 we introduce PDGs, and two approaches to identify them, that is, conditions ensuring that a given differential game is a PDG. The first approach, which we call the exact potential method, is based on the game's primitive data, that is, the payoff functions and the game dynamics. The second approach assumes the existence of a smooth concave

function that together with the primitive data and suitable conditions ensures that we indeed have a PDG. The latter approach is called the fictitious-potential method.

Chapter 4 is about stability issues. More precisely, given a PDG with a certain Nash equilibrium, is the corresponding state process stable?. First, we state two results on the stability of OCPs with an autonomous system. Next, we apply these satbility results to a class of zero-sum games.

In the final Chapter 5, we introduce some examples that illustrate our results.

Resumen.

Esta obra se refiere al concepto de juego diferencial potencial (JDP) no cooperativo.De manera breve, un JDP es un juego al cual podemos asociar un problema de control óptimo cuyas soluciones óptimas son equilibrios de Nash para el juego original. Consideramos tres áreas de estudio importantes sobre juegos diferenciales potenciales. Primero: cómo identificarlos. Segundo: cómo resolverlos. Tercero: el análisis de la estabilidad de los correspondientes equilibrios de Nash.

En efecto, uno de nuestros principales objetivos es ilustrar el problema de estabilización para juegos diferenciales no cooperativos, esto es, encontrar un equilibrio de Nash que, adicionalmente, estabilice la trayectoria del juego.

Nuestro estudio es mayormente restringido a open-loop estrategias. Sin embargo, en el caso de juegos de suma cero con tiempo horizonte finito e infinito, consideramos closed-loop estrategias. La mayoría de nuestros resultados son ilustrados con ejemplos de interés teórico y práctico.

Comenzamos, en el Capítulo 1, con una breve introducción a la teoría de juegos, incluyendo conceptos básicos tales como juegos estáticos, dinámicos ,equilibrio de Nash, etc.

El Capítulo 2 resume algunas ideas de teoría de control óptimo. Primero, consideramos problemas de control óptimo con horizonte finito e infinito, e introducimos dos técnicas importantes para analizarlos, mś precisamente, programación dinámica y el principio del máximo. El primero está basado en la ecuación de Hamilton-Jacobi-Bellman, está primera técnica da condiciones suficientes de optimalidad. En contraste, el principio del máximo da condiciones necesarias de optimalidad.

En el Capítulo 3 introducimos el concepto de juego diferencial potencial, así como dos métodos para identificarlos, esto es, condiciones que aseguran que un juego diferencial dado es

un juego diferencial potencial. El primer método, el cual es llamado el método del potencial exacto, está basado en los datos iniciales del juego, cómo lo son las funciones de pago del juego y dinámica del juego. El segundo método asume la existencia de una función suave y cóncava que en conjunto con los datos iniciales y condiciones adecuadas aseguran, que en efecto tenemos un JDP. Esté último método es llamado el método del potencial ficticio.

El capítulo 4 aborda aspectos de estabilidad. Más precisamente, dado un JDP con un equilibrio de Nash dado, ¿es estable la correspondiente trayectoria?. Primero, establecemos dos resultados sobre problemas de control óptimo con sistema autónomo. Después, aplicamos estos resultados a un juego de suma cero.

En nuestro capítulo final, el Capítulo 5, damos algunos ejemplos que ilustran nuestros resultados.

Chapter 1

Introduction.

Game theory is a useful tool to analyze many important applications in engineering, economics, and natural resources, among many other fields. These applications include capital accumulation games [12], marketing games[12] and games in natural resources [12, 35], among many other applications. Moreover, game theory provide us rich mathematical structures to study: zero-sum games [2, 27, 29], pursuit evasion games [2], linear quadratic games [12, 35], linear state games [12], exponential games [12, 35] ,and potential games [15, 31] are some examples. The latter class of games is the main issue in our study.

In this chapter we introduce some basic concepts of game theory, namely, dynamic games (Section 1.1) and the particular case of differential games (Section 1.2). We also introduce some notation and terminology that will be used throughout the following.

The material in this chapter is quite standard; it appears in any texbook on dynamic games. (See, for instance, [2, 12, 16]).

1.1 Static and Dynamic Games.

For future reference, in this section we briefly introduce some general terminology and notation used in game theory.

1.1 Definition. *A static (or one-shot) game in normal form $\mathcal{G}$ can be expressed as*

$\mathcal{G} := (\mathcal{N}, U_1, ..., U_N, J_1, ..., J_N)$

and it consists of

1. *a set $\mathcal{N} = \{1, ..., N\}$ of players, for some integer $N \geq 2$; and, for each $i \in \mathcal{N}$,*

2. *a set U_i of feasible actions (or strategies) for player i, and*

3. *a payoff function $J_i : U \to \mathbb{R}$ for player i, where*

$$U := U_1 \times \cdots \times U_N. \tag{1.1}$$

A game in normal form is also known as a game in *strategic form.* An element $u = (u_1, ..., u_N)$ of U in (1.1) is called a *strategy profile.*

This work concerns *noncooperative* games, which means that collusions or teams between players are not allowed. In this case, the most common solution concept is a *Nash equilibrium* defined below. First we introduce some notation.

1.1 Remark. *Let $U^{-i} := \prod_{j \neq i} U_j$. Given a strategy profile $u = (u_1, ..., u_N) \in U$, we define*

$$u^{-i} := (u_1, ..., u_{i-1}, u_{i+1}, ..., u_N) \in U^{-i}.$$

Moreover, by an abuse of notation, we also write u as (u_i, u^{-i}). In general, for each $i \in \mathcal{N}$ and $a_i \in U_i$, we write

$$(a_i, u^{-i}) := (u_1, ..., u_{i-1}, a_i, u_{i+1}, ..., u_N). \qquad \diamondsuit$$

1.2 Definition. *Consider the game $\mathcal{G}$ in Definition 1.1. A strategy profile $\hat{u} = (\hat{u_1}, ..., \hat{u_N}) \in U$ is called a Nash equilibrium (NE) for $\mathcal{G}$ if, for every $i \in \mathcal{N}$,*

$$J_i(\hat{u}_i, \hat{u}^{-i}) = \max_{u_i \in U_i} J(u_i, \hat{u}^{-i}), \tag{1.2}$$

that is, for every i,

$$J_i(\hat{u}) \geq J_i(u_i, \hat{u}^{-i}) \;\; \forall \;\; u_i \in U_i.$$

The NE $\hat{u}$ in (1.2) is said to be *pure* (or *nonrandomized*) to distinguish it from a *mixed* (or *randomized*) NE, which we do not consider in this work. Also note, on the other hand, that (1.2) defines N *coupled* optimization problems.

1.3 Definition. *Consider a game in normal form $\mathcal{G}$ with payoff functions $J_i : U \to \mathbb{R}$, for $i = 1, ..., N$. Then $\mathcal{G}$ is said to be a potential game if there exists a function $P : U \to \mathbb{R}$ with the following property: If $\hat{u} \in U$ maximizes P, then $\hat{u}$ is a Nash equilibrium for $\mathcal{G}$. In this case, P is called a potential function for $\mathcal{G}$.*

An elementary example of a potential game is a *team game*, that is, a game $\mathcal{G}$ for which, for some real-valued function p on U, the payoff $J_i \equiv p$ for every $i = 1, ..., N$. In this case, $\mathcal{G}$ is a potential game with potential function $P = p$. (In optimal control theory, a team game is called a *decentralized control problem*).

1.4 Example. Mallozzi [33] considered a game with two players, action sets $U_1 = U_2 = [0, 1]$ and payoff functions

$$J_1(u_1, u_2) = J_2(u_1, u_2) = u_1 u_2 - 1.$$

This is a team game with potential function $P(u_1, u_2) = u_1 u_2 - 1$. Thus, this is an example of potential game.

1.5 Example. Consider two firms producing a certain homogeneous product which is sold in the same market. Each firm (i=1,2) decides its quantity u_i to produce in the action set $U_1 = U_2 = [0, \infty)$ having a cost function $C_i(u_i)$. If firms face an inverse demand function $p(Q)$, with $Q := u_1 + u_2$, then each firm wants to maximize its profit function (or payoff function)

$$J_i(u_1, u_2) := p(u_1 + u_2)u_i - C_i(u_i),\ i = 1, 2.$$

Suppose that the functions p and C_i are given by

$$p(Q) := \begin{cases} \alpha - Q, & \text{if } 0 \leq Q \leq \alpha, \\ 0, & \text{if } Q > \alpha \end{cases}$$

and $C_i(u_i) = cu_i$, where $\alpha > c$, respectively. Assuming that $\alpha - c - u_i \geq 0$ for each i ,it can be shown that $u_1^* = u_2^* = \frac{\alpha - c}{3}$ is the unique Nash equilibrium for this game. A direct calculation shows that $F(u_1, u_2) := (\alpha - c)(u_1 + u_2) - u_1 u_2 - u_1^2 - u_2^2$ is a potential function for our game. Therefore, this game is a potential game.

1.2 Differential Games.

In this work we are mainly interested in a class of dynamic games called *differential games*. In this class, the state of the game $t \to x(t)$ evolves as the solution of an ordinary differential equation, as (1.3), below. (In a later chapter we also consider *stochastic* differential games.) Strictly speaking we have the following.

1.6 Definition. *Consider :*

1. *A set of players* $\mathcal{N} = \{1, \cdots, N\}$, $N \in \mathbb{N}$, $N \geq 2$.

2. *$T := [0,h]$ with $h \leq \infty$, h is called the game's time horizon.*

3. *A set $X \subset \mathbb{R}^l$ called the game's **state space**, and $l = l_1 + \cdots + l_N$.*

4. *For each player $i \in \mathcal{N}$, a set of **feasible controls** $U_i \subseteq \mathbb{R}^{m_i}$, with $m_i \in \mathbb{N}$. We define $U := U_1 \times \cdots \times U_N \subseteq \mathbb{R}^m$, with $m = m_1 + \cdots + m_N$.*

5. *Define for each i the set of functions*

$$\mathbf{U_i} = \{\mathbf{u_i} : T \to U_i | \ \mathbf{u_i} \text{ is Borel-measurable } \},$$

   *which is called the **open-loop strategy space** and*

$$\mathbf{U} := \mathbf{U_1} \times \cdots \times \mathbf{U_N},$$

   *called the **space of open-loop multistrategies.***

6. *Consider $f : T \times X \times U - > X$ a measurable function. For each $u \in U$, we say that the function $x : T - > X$ is an admissible state path for the game corresponding to the multistrategy $\mathbf{u}$, if $\mathbf{x}$ is the unique solution to the system*

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \ \ \mathbf{x}(0) = x_0 \in X. \tag{1.3}$$

7. *For each $i \in \mathcal{N}$, $L^i : T \times X \times U \to \mathbb{R}$ denotes the current (or instantaneous) payoff ,and $S^i : X \to \mathbb{R}$ is the final payoff function, with $S_i(\cdot) \equiv 0$ if $h = \infty$. The payoff function for player i defined for each $\mathbf{u} \in \mathbf{U}$ with feasible state path $\mathbf{x}$*

$$J_h^i(\mathbf{u}) := \begin{cases} \int_0^h L^i(t, \mathbf{x}(t), \mathbf{u}(t))dt + S^i(\mathbf{x}(h)) & \text{if } h < \infty, \\ \int_0^h e^{-\beta t} L^i(t, \mathbf{x}(t), \mathbf{u}(t))dt & \text{if } h = \infty, \end{cases}$$

   where $\beta > 0$ is the intertemporal discount rate, considered the same for every player.

A differential game consists of the components 1-7 in Definition 1.6. In a compact form we can express this game as

$$\Gamma_{x_0}^h := [N, \{\mathbf{U_i}\}_{i \in \mathcal{N}}, \{J_h^i\}_{i \in \mathcal{N}}, f], h \leq \infty. \tag{1.4}$$

1.2 Remark. *If there is no risk of confusion we write J^i instead of J_h^i in Definition 1.6.*

1.7 Definition. *A strategy profile* $\mathbf{u}^* = (\mathbf{u}_1^*, ..., \mathbf{u}_N^*)$ *is called an open-loop Nash equilibrium (OLNE) for the differential game (1.4), if, for each* $i \in \mathcal{N}$, *the condition*

$$J^i(\mathbf{u}_1^*, ..., \mathbf{u}_N^*) \geq J^i(\mathbf{u}_1^*, ..., \mathbf{u}_{i-1}^*, \mathbf{u}_i, \mathbf{u}_{i+1}^*, ..., \mathbf{u}_N^*) \tag{1.5}$$

holds for all $\mathbf{u}_i \in \mathbf{U}_i$.

In a later chapter we will give conditions for a differential games to have a Nash equilibrium.

Chapter 2

Optimal Control theory .

We already have the concept of a Nash equilibrium, which is the cornerstone of game theory. Finding Nash equilibria, however, can be a formidable task. Hence, the main goal of this work is to introduce a class of differential games to which we can associate *optimal control problems* (OCPs) whose optimal solutions are Nash equilibria for the original games. To this end, in this chapter we summarize some important concepts and results from optimal control theory. We consider finite- and infinite-horizon OCPs. To analyze these problems we introduce two well-known techniques, dynamic programming and the maximum principle. The former gives sufficient conditions to have an optimal control, and it is based in the so-called *dynamic programming equation*, also known as the Hamilton-Jacobi-Bellman (HJB) equation. In contrast, the maximum principle gives necessary conditions. The material in this chapter is mainly borrowed from [8,9,11,24,30].

2.1 Optimal Control Problems(OCP).

2.1 Definition. *Consider* $X \subseteq \mathbb{R}^l$*, and* $U \subseteq \mathbb{R}^m$*, three functions* $F : T \times X \times U \to \mathbb{R}$*,* $f : T \times X \times U \to \mathbb{R}^l$ *and* $S : X \to \mathbb{R}$*, where* $T = [0, h]$*,* $h \leq \infty$*,* X *is called the state space and* U *is the set of feasible controls. Define the open-loop strategy space*

$\mathbf{U} = \{\mathbf{u} : T \to U | \mathbf{u} \,\text{is Borel} - \text{measurable}\}$*, and consider the system (2.1)*

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \;\; \mathbf{x}(0) = x_0 \in X. \tag{2.1}$$

Again $\mathbf{x}$ *is an admissible state path for the strategy* $\mathbf{u} \in \mathbf{U}$*, if* $\mathbf{x}$ *is the unique solution* *of equation (2.1). The optimal control problem (OCP) is to maximize the payoff function* $J_h : \mathbf{U} \to \mathbb{R}$ *defined by*

$$J_h(\mathbf{u}) := \begin{cases} \int_0^h F(t, \mathbf{x}(t), \mathbf{u}(t))dt + S(\mathbf{x}(h)) & \text{if } h < \infty \\ \int_0^h e^{-\beta t} F(t, \mathbf{x}(t), \mathbf{u}(t))dt & \text{if } h = \infty, \end{cases}$$

over all $u \in U$, *subject to (2.1),* S *is called the scrap value function .*

2.2 Example. Consider a market for a durable good consisting of many consumers on the demand side and a single firm on the supply side, in a finite interval of time $T = [0, h]$.

Let the total market potential be constant and equal to M and denote by $x(t) \in X = [0, 100]$ the percentage of the market potential which has bought the product from the monopolist by time t. Furthermore, denote the advertising rate of the firm at time t by $u(t) \in U = [0, \infty)$ and assume that advertising costs are given by the quadratic function $\frac{1}{2}u(t)^2$. Thus, we obtain the next differential system

$$\dot{x}(t) = u(t)M[1 - x(t)], x(0) = 0.$$

The goal is to maximize the market penetration by time h minus the advertising cost incurred up to time h, that is,maximize the objective functional

$$J(u(\cdot)) = -\frac{1}{2}\int_0^h u(t)^2 dt + x(h).$$

restricted to the previous differential system.

2.3 Definition. *A function* $\mathbf{u}^* \in \mathbf{U}$ *that solves (maximizing) the OCP in Definition 2.1 is called an* **open − loop optimal control** *or* **optimal solution**.

We denote by $U(x, t)$ the set of feasible controls $u \in U$ given that at time $t \geq 0$ the state is $x \in X$.

2.2 OCPs with Finite Time Horizon.

In this section we give two useful theorems to solve OCPs with finite time horizon and smooth functions, they are based in the Hamilton-Jacobi-Bellman (HJB) equation and the maximum principle.

2.2.1 The HJB equation.

In one of his books, Bellman wrote "In place of determining the optimal sequence of decisions from the fixed state of the system, we wish to determine the optimal decision to be made at any state of the system. Only if we know the latter, we do understand the intrinsic structure of the solution". This idea (known today as "the principle of optimality") would set the foundations of dynamic programming theory. Eventually, this leaded to the HJB equation (2.2) below, which gives sufficient conditions for optimality.

In the following, for $a, b \in \mathbb{R}^l$ we denote $\langle a, b\rangle_l$ the inner product in $\mathbb{R}^l$.

2.1 Theorem. *Consider an OCP as in Definition 2.1.Let $V : X \times [0, h] \to \mathbb{R}$ be a continuously differentiable function which satisfies the HJB equation*

$$\beta V(x,t) - V_t(x,t) = \max\{F(x,u,t) + \langle V_x(x,t), f(x,u,t)\rangle_l \,|u \in U(x,t)\}, \tag{2.2}$$

for all $(x,t) \in X \times [0,h)$, and the terminal condition

$$V(x,h) = S(x). \tag{2.3}$$

Let $\Phi(x,t)$ denote the set of controls $u \in U(x,t)$ maximizing the right-hand side of Eq. (2.2). If $\mathbf{u}(\cdot)$ is a feasible control path with corresponding state trajectory $\mathbf{x}(\cdot)$ and if $\mathbf{u}(t) \in \Phi(\mathbf{x}(t),t)$ holds for almost all $t \in [0,h]$, then $\mathbf{u}(\cdot)$ is an optimal control path for our OCP.

Proof: Let the pair $(\mathbf{u}(\cdot), \mathbf{x}(\cdot))$ be as in the theorem. We wish to verify that $\mathbf{u}(\cdot)$ is indeed an optimal control.

Let $v(\cdot)$ be any feasible control with corresponding state trajectory $y(\cdot)$, for $t \in [0,h]$ we have that, in general the inequality

$$\begin{aligned}\beta V(y(t),t) - V_t(y(t),t) &= \max\{F(y(t),w,t) + \langle V_x(y(t),t), f(y(t),w,t)\rangle_l \,|w \in U(y(t),t)\} \\ &\geq F(y(t),v(t),t) + \langle V_x(y(t),t), f(y(t),v(t),t)\rangle_l\end{aligned}$$

holds.The first equality results because of the definition of the HJB equation applied to $y(t)$ and t, and the inequality because of the definition of "$\underline{\max}$" and the feasibility of $v(\cdot)$, thus we have

$$\beta V(y(t),t) - V_t(y(t),t) - \langle V_x(y(t),t), f(y(t),v(t),t)\rangle_l \geq F(y(t),v(t),t).$$

Multiplying by $e^{-\beta t}$ the previous inequation, we obtain the last inequality, of the next equation

$$(*)\quad -\tfrac{d[e^{-\beta t}V(y(t),t)]}{dt} = -[-\beta e^{-\beta t}V(y(t),t) + e^{-\beta t}\{\tfrac{dV(y(t),t)}{dt}\tfrac{dt}{dt} + \left\langle \tfrac{dV(y(t),t)}{dy}, \tfrac{dy(t)}{dt}\right\rangle_l\}]$$
$$= e^{-\beta t}\beta V(y(t),t) - e^{-\beta t}V_t(y(t),t) - e^{-\beta t}\langle V_y(y(t),t), \dot{y}(t)\rangle_l$$
$$= e^{-\beta t}\beta V(y(t),t) - e^{-\beta t}V_t(y(t),t) - e^{-\beta t}\langle V_x(y(t),t), f(y(t),v(t),t)\rangle_l$$
$$(**) \geq e^{-\beta t}F(y(t),v(t),t),$$

where the first and second equality holds doing some calculations and the third equality is obtained from

$$\dot{y}(t) = f(y(t),v(t),t),$$

which implies

$$-\frac{d[e^{-\beta t}V(y(t),t)]}{dt} \geq e^{-\beta t}F(y(t),v(t),t). \tag{2.4}$$

In a similar way , for $\mathbf{x}$ and $\mathbf{u}$ we obtain

$$-\frac{d[e^{-\beta t}V(\mathbf{x}(t),t)]}{dt} = e^{-\beta t}F(\mathbf{x}(t),\mathbf{u}(t),t), \tag{2.5}$$

where the equality holds instead of the inequality (**) due to $\mathbf{u}(\cdot)$ maximizes the right side of Eq. (2.2).

Substituting Eqs. (2.4) and (2.5) in the payoff function and using Eq. (2.3) we obtain

$$(\Diamond) J(\mathbf{u}(\cdot)) - J(v(\cdot))$$
$$= \int_0^h e^{-\beta t}[F(\mathbf{x}(t),\mathbf{u}(t),t) - F(y(t),v(t),t)]dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$$
$$\geq \int_0^h \tfrac{d}{dt}\{e^{-\beta t}[V(y(t),t) - V(\mathbf{x}(t),t)]\}dt + e^{-\beta h}[V(\mathbf{x}(h),h) - V(y(h),h)]$$
$$= -e^{-\beta t}[V(\mathbf{x}(t),t) - V(y(t),t)]|_0^h + e^{-\beta h}[V(\mathbf{x}(h),h) - V(y(h),h)]$$
$$= -e^{-\beta h}[V(\mathbf{x}(h),h) - V(y(h),h)] + e^{-r0}[V(\mathbf{x}(0),0) - V(y(0),0)] +$$
$$+ e^{-\beta h}[V(\mathbf{x}(h),h) - V(y(h),h)]$$
$$= V(\mathbf{x}(0),0) - V(y(0),0)$$
$$= V(x_0,0) - V(x_0,0) = 0.$$

The first equality holds because of the linearity of the integral, the first inequality holds substituting Eqs. (2.4) and (2.5) in the payoff function J, the second and third equality hols using the fundamental calculus theorem and operating , finally the fifth equality holds because $\mathbf{x}(\cdot)$ and $y(\cdot)$ are solution of the system dynamic Eq. (2.1), thus $\mathbf{x}(0) = x_0 = y(0)$.

This shows $J(\mathbf{u}(\cdot)) - J(v(\cdot)) \geq 0$, hence $J(\mathbf{u}(\cdot)) \geq J(v(\cdot))$. This concludes the proof.■

It is important to know that Eq (2.2) is obtained as a necessary condition assuming $\mathbf{u}(\cdot)$ is an optimal control path (for an heuristic but illustrating derivation see [12]), using the maximum principle [32], similarly we can derive different kinds of HJB equations depending on the conditions of our problem for instance we have in [2], the derivation of the Isaacs equations from a physic framework, using a HJB equation, which helps to solve numerically the pursuit-evader game [24].

2.2.2 Maximum principle.

The Maximum Principle is a first order condition for smooth problems. In our framework, if an OCP satisfies some curvatures properties then any control path which satisfies The Maximum Principle is optimal. For a formal derivation of The Maximum Principle we have [38] and for a more illustrating but heuristic derivation [12].

2.4 Definition. *For an optimal control problem (see Definition 2.1) we define its (current value)Hamiltonian function* $H : D_H = \{(x, u, \lambda, t) | x \in X, u \in U(x, t), \lambda \in \mathbb{R}^l, t \in [0, h]\} \to \mathbb{R}$ *by*

$$H(x, u, \lambda, t) = F(x, u, t) + \langle \lambda, f(x, u, t) \rangle_l \ \forall\, (x, u, \lambda, t) \in D_H.$$

2.5 Definition. *For an optimal control problem we define the maximized Hamiltonian function* $H^* : D_{H^*} = X \times \mathbb{R}^n \times [0, h] \to \mathbb{R}$ *by*

$$H^*(x, \lambda, t) = \max\{H(x, u, \lambda, t) | u \in U(x, t)\} \ \forall\, (x, \lambda, t) \in D_{H^*}. \tag{2.6}$$

2.2 Theorem. *Consider the optimal control problem of Definition 2.1 and define the Hamiltonian function H and the maximized Hamiltonian function* H^* *as above.Assume that the state space X is a convex set and that the scrap value function S is continuously differentiable and concave.Let* $\mathbf{u}(\cdot)$ *be a feasible control path with corresponding state trajectory* $\mathbf{x}(\cdot)$*. If there exists an absolutely continuous function* $\lambda : [0, T] \to \mathbb{R}^n$ *such that the maximum condition*

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = H^*(\mathbf{x}(t), \lambda(t), t), \tag{2.7}$$

the adjoint equation

$$\dot{\lambda}(t) = \beta\lambda(t) - H^*_x(\mathbf{x}(t), \lambda(t), t), \tag{2.8}$$

and the transversality condition

$$\lambda(h) = S'(\mathbf{x}(h)) \tag{2.9}$$

are satisfied ,and such that the function $x \to H^*(x, \lambda(t), t)$ *is concave and continuously differentiable with respect to x for all* $t \in [0, h]$, *then* $\mathbf{u}(\cdot)$ *is an optimal path.*

Proof : Let us check the condition of optimality for $\mathbf{u}(\cdot)$. Let $v(\cdot)$ and arbitrary feasible control path with corresponding state path $y(\cdot)$, consider

$$J(\mathbf{u}(\cdot)) - J(v(\cdot)) =$$

$$= \int_0^h e^{-\beta t} F(\mathbf{x}(t), \mathbf{u}(t), t)dt - \int_0^h e^{-\beta t} F(\mathbf{y}(t), v(t), t)dt + +e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$$

$$= \int_0^h e^{-\beta t}[F(\mathbf{x}(t), \mathbf{u}(t), t) + \langle\lambda(t), f(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)\rangle_l - \langle\lambda(t), f(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)\rangle_l]dt - \tag{2.10}$$

$$- \int_0^h e^{-\beta t}[F(y(t), v(t), t) + \langle\lambda(t), f(y(t), v(t), \lambda(t), t)\rangle_l - \langle\lambda(t), f(y(t), v(t), \lambda(t), t)\rangle_l]dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$$

$$= \int_0^h e^{-\beta t}[H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) - \langle\lambda(t), \dot{\mathbf{x}}(t)\rangle_l]dt -$$

$$- \int_0^h e^{-\beta t}[H(y(t), v(t), \lambda(t), t) - \langle\lambda(t), \dot{y}(t)\rangle_l]dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))].$$

Where the last equality holds using the Definitions 2.4 of H and the feasibility of $y(\cdot)$ and $\mathbf{x}(\cdot)$, furthermore, from the definition of " max" and the feasibility of $y(\cdot)$ we have

$$H^*(y(t), \lambda(t), t) = \max\{H(y(t), u, \lambda(t), t) | u \in U(y(t), t)\} \geq H(y(t), v(t), \lambda(t), t) \tag{2.11}$$

From Eqs. (2.7) and (2.11) we have inequality (2.12)

$$J(\mathbf{u}(\cdot)) - J(v(\cdot)) =$$

$$= \int_0^h e^{-\beta t}[H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) - \langle\lambda(t), \dot{\mathbf{x}}(t)\rangle_l]dt - \int_0^h e^{-\beta t}[H(y(t), v(t), \lambda(t), t) - \langle\lambda(t), \dot{y}(t)\rangle_l]dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$$

$$\geq \int_0^h e^{-\beta t}[H^*(\mathbf{x}(t), \lambda(t), t) - \langle\lambda(t), \dot{\mathbf{x}}(t)\rangle_l]dt - \int_0^h e^{-\beta t}[H^*(y(t), \lambda(t), t) - \langle\lambda(t), \dot{y}(t)\rangle_l]dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))] \tag{2.12}$$

The differentiability and concavity of the function $x \to H^*(x, \lambda(t), t)$ imply (see [43]

Theorem 17.7 pag.629)

$$H^*(\mathbf{x}(t),\lambda(t),t) - H^*(y(t),\lambda(t),t) \geq \langle H_x^*(\mathbf{x}(t),\lambda(t),t),[\mathbf{x}(t)-y(t)]\rangle_l \tag{2.13}$$

Using Eq. (2.8), we can write Eq. (2.13) in the next form

$$H^*(\mathbf{x}(t),\lambda(t),t) - H^*(y(t),\lambda(t),t) \geq \left\langle [\beta\lambda(t)-\dot{\lambda}(t)],[\mathbf{x}(t)-y(t)]\right\rangle_l \tag{2.14}$$

Using Eqs. (2.12) and (2.14) we obtain

$$J(\mathbf{u}(\cdot)) - J(v(\cdot)) \geq \tag{2.15}$$

$\geq \int_0^h e^{-\beta t}[H^*(\mathbf{x}(t),\lambda(t),t) - \langle\lambda(t),\dot{\mathbf{x}}(t)\rangle_l]dt - \int_0^h e^{-\beta t}[H^*(y(t),\lambda(t),t) - \langle\lambda(t),\dot{y}(t)\rangle_l]dt$
$+e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$
$= \int_0^h e^{-\beta t}[H^*(\mathbf{x}(t),\lambda(t),t) - H^*(y(t),\lambda(t),t) - \langle\lambda(t),[\dot{\mathbf{x}}(t)-\dot{y}(t)]\rangle_l]dt$
$+e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$
$\geq \int_0^h e^{-\beta t}\{\left\langle [\beta\lambda(t)-\dot{\lambda}(t)],[\mathbf{x}(t)-y(t)]\right\rangle_l - \langle\lambda(t),[\dot{\mathbf{x}}(t)-\dot{y}(t)]\rangle_l\}dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$
$= \int_0^h \frac{d}{dt}\{e^{-\beta t}\langle\lambda(t),[y(t)-\mathbf{x}(t)]\rangle_l\}dt + e^{-\beta h}[S(\mathbf{x}(h)) - S(y(h))]$
$= e^{-\beta h}\{\langle\lambda(h),[y(h)-\mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h))\} - \langle\lambda(0),[\mathbf{x}(0)-y(0)]\rangle_l$

where the first equality holds because of Eq. (2.12), the second inequality holds due to 2.14, and the third and fourth equality holds due to some operations and the fundamental theorem of calculus, we conclude.

$$J(\mathbf{u}(\cdot)) - J(v(\cdot)) \geq e^{-\beta h}\{\langle\lambda(h),[y(h)-\mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h))\} - \langle\lambda(0),[\mathbf{x}(0)-y(0)]\rangle_l \tag{2.16}$$

Now, observe that $\mathbf{x}(0) - y(0) = 0$ due to $\mathbf{x}$ and y are solutions of 2.1 ($\mathbf{x}(0) = x_0 = y(0)$), then

$$\langle\lambda(0),[\mathbf{x}(0)-y(0)]\rangle_l = 0 \tag{2.17}$$

and, from Theorem 17.7 of [43] and Eq. (2.9)

$$S(\mathbf{x}(h)) - S(y(h)) \geq \langle S'(\mathbf{x}(h)),[\mathbf{x}(h)-y(h)]\rangle_l = \langle\lambda(h),[\mathbf{x}(h)-y(h)]\rangle_l \tag{2.18}$$

because of the differentiability and concavity of S. From (2.18) we obtain

$$\langle\lambda(h),[y(h)-\mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h)) \geq 0.$$

Thus

$$e^{-\beta h}\{\langle\lambda(h),[y(h)-\mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h))\} \geq 0.$$

This last inequality, (2.16) and (2.17) imply

$J(\mathbf{u}(\cdot)) - J(v(\cdot)) \geq e^{-\beta h}\{\langle\lambda(h), [y(h) - \mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h))\} - \langle\lambda(0), [\mathbf{x}(0) - y(0)]\rangle_l = e^{-\beta h}\{\langle\lambda(h), [y(h) - \mathbf{x}(h)]\rangle_l + S(\mathbf{x}(h)) - S(y(h))\} \geq 0$

Thus, $J(\mathbf{u}(\cdot)) \geq J(v(\cdot))$ which concludes the result.■

2.3 OCPs with Infinite Time Horizon.

In this section, we consider infinite-horizon OCPs and several standard optimality concepts in the infinite-horizon case. First, we introduce the discounted cost criterion (Section 2.3.1), and then, for future reference, we introduce three other criteria that are quite common in applications to economics (Section 2.3.2).

2.3.1 The discounted case.

For ease of reference, we repeat part of Definition 2.1 as follows.

2.6 Definition. ***(Optimal discounted case).***

The infinite-horizon discounted cost OCP is to maximize

$$J_\infty(\mathbf{u}) := \int_0^\infty e^{-\beta t} F(t, \mathbf{x}(t), \mathbf{u}(t)) dt \tag{2.19}$$

over all $\mathbf{u} \in \mathbf{U}$, *subject to (2.1).*

The functions f and F in (2.1) and (2.19), respectively, are supposed to satisfy conditions ensuring that (2.1) has a unique solution $\mathbf{x}(\cdot)$ and, in addition, $J_\infty(\cdot)$ is a well-defined, finite-valued mapping from $\mathbf{U}$ to $\mathbb{R}$.To this end, there are many well-known conditions. See, for instance, Dmitruk and Kuz'kina [11], Gaitsgory, Quincampoix [19], and Sydsaeter, Seierstad [44].

2.3.2 Other optimality criteria.

Consider an OCP as in Definition 2.1 with $h = \infty$. For $T^* \geq 0$ the <u>T^*-truncation</u> $J_{T^*}(\mathbf{u}(\cdot))$ of the objective functional J_∞ is defined by

$$J_{T^*}(\mathbf{u}(\cdot)) = \int_0^{T^*} e^{-\beta t} F(\mathbf{x}(t), \mathbf{u}(t), t) dt. \tag{2.20}$$

2.7 Definition. *A feasible control path* $\mathbf{u}(\cdot)$ *is called*

i. *overtaking optimal if for every feasible control path* $v(\cdot)$ *there exists* $\tau = \tau(\mathbf{u}(.), v(.)) > 0$ *such that*

$$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) \geq 0, \forall\ T^* \in [\tau, \infty),$$

ii. *catching up optimal if*

$$\liminf_{T^* \to \infty}[J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot))] \geq 0,$$

iii. *sporadically catching up optimal if*

$$\limsup_{T^* \to \infty}[J_{T^*}((\mathbf{u}(\cdot))) - J_{T^*}((v(\cdot)))] \geq 0.$$

2.3 Theorem. *Consider the discounted cost OCP in Definition 2.6.*

I. *If optimality is understood in the sense of Definition 2.7.i, then Theorem 2.1 is valid, replacing equation (2.3) by the assumption that for every feasible control path* $v(\cdot)$ *there exists a finite number* ρ*, such that*

$$V(y(T^*), T^*) - V(\mathbf{x}(T^*), T^*) \geq 0\ \ \forall\ \ T^* \geq \rho. \tag{2.21}$$

Similarly, Theorem 2.2 remains valid if the transversality condition 2.9 is replaced by the condition that for every feasible control path $v(\cdot)$ *there exists a finite number* ρ *such that its feasible state path holds*

$$\langle \lambda(T^*), [y(T^*) - \mathbf{x}(T^*)] \rangle_l \geq 0\ \ \forall\ \ T^* \geq \rho. \tag{2.22}$$

II. *If optimality is understood in the sense of Definition 2.7.ii, then Theorem 2.1 is valid, replacing equation (2.3) by the assumption that for every feasible control path* $v(\cdot)$*, the corresponding state path satisfies that*

$$\liminf_{T^* \to \infty} e^{-\beta T^*}[V(y(T^*), T^*) - V(\mathbf{x}(T^*), T^*)] \geq 0. \tag{2.23}$$

Similarly, Theorem 2.2 remains valid if the transversality condition 2.9 is replaced by the condition

$$\liminf_{T^* \to \infty} e^{-\beta T^*} \langle \lambda(T^*), [y(T^*) - \mathbf{x}(T^*)] \rangle_l \geq 0. \tag{2.24}$$

III. *If optimality is understood in the sense of Definition 2.8.iii, then Theorem 2.1 is valid, replacing equation (2.3) by the assumption that for every feasible control path* $v(\cdot)$*, the corresponding state path satisfies that*

$$\limsup_{T^*\to\infty} e^{-\beta T^*}[V(y(T^*),T^*) - V(\mathbf{x}(T^*),T^*)] \geq 0. \tag{2.25}$$

Similarly, Theorem 2.2 remains valid if the transversality condition 2.9 is replaced by the condition

$$\limsup_{T^*\to\infty} e^{-\beta T^*} \langle \lambda(T^*), [y(T^*) - \mathbf{x}(T^*)]\rangle_l \geq 0. \tag{2.26}$$

Proof :

I. Given $T^* \geq 0$, consider the T^*-truncation $J_{T^*}(\cdot)$. Following an analogous arguments as in ($\Diamond$) of Theorem 2.1

$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) =$

$$\begin{aligned}
&= \int_0^{T^*} e^{-\beta t}[F(\mathbf{x}(t),\mathbf{u}(t),t) - F(y(t),v(t),t)]dt \\
&\geq \int_0^{T^*} \tfrac{d}{dt}\{e^{-\beta t}[V(y(t),t) - V(\mathbf{x}(t),t)]\}dt \\
&= -e^{-\beta t}[V(\mathbf{x}(t),t) - V(y(t),t)]|_0^{T^*} \\
&= -e^{-\beta T^*}[V(\mathbf{x}(T^*),T^*) - V(y(T^*),T^*)] + [V(\mathbf{x}(0),0) - V(y(0),0)] \\
&= e^{-\beta T^*}(V(y(T^*),T^*) - V(\mathbf{x}(T^*),T^*)).
\end{aligned}$$

As before, the first inequality holds due to, Eqs. (2.4) and (2.5) hold again for every $t \in [0, T^*]$, the fourth equality holds due to $\mathbf{x}, y$ are solutions of the dynamic system (2.1).

Hence

$$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) \geq e^{-\beta T^*}(V(y(T^*),T^*) - V(\mathbf{x}(T^*),T^*)) \tag{2.27}$$

Similarly, Eqs. (2.10)-(2.17) of Theorem 2.2 remain valid, if we remove $e^{-rh}[S(\mathbf{x}(h)) - S(y(h))]$ wherever it appears, thus, we can write

$$\begin{aligned}
J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) &= e^{-\beta T^*}\{\langle \lambda(T^*), [y(T^*) - \mathbf{x}(T^*)]\rangle_l\} - \langle \lambda(0), [\mathbf{x}(0) - y(0)]\rangle_l = \\
&= e^{-\beta T^*}\{\langle \lambda(T^*), [y(T^*) - \mathbf{x}(T^*)]\rangle_l\}.
\end{aligned}$$

Hence

$$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) = e^{-\beta T^*}\{\langle\lambda(T^*), [y(T^*) - \mathbf{x}(T^*)]\rangle_l\}. \tag{2.28}$$

If we assume (2.21), from (2.27) we conclude

$$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) \geq 0 \ \ \forall \ T^* \geq \tau. \tag{2.29}$$

If we assume (2.22), from (2.28) we conclude

$$J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot)) \geq 0 \ \ \forall \ T^* \geq \tau. \tag{2.30}$$

From (2.29) and (2.30) we conclude the result I.

II. In the same way to the item I, we obtain (2.27) and (2.28). If we assume (2.23), from (2.27) we conclude

$$\liminf_{T^*\to\infty}[J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot))] \geq 0. \tag{2.31}$$

If we assume (2.24), from (2.28) we conclude

$$\liminf_{T^*\to\infty}[J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot))] \geq 0. \tag{2.32}$$

From (2.31) and (2.32) we conclude the result II.

III. In the same way to the item I, we obtain (2.27) and (2.28). If we assume (2.25), from (2.27) we conclude

$$\limsup_{T^*\to\infty}[J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot))] \geq 0. \tag{2.33}$$

If we assume (2.26), from (2.28) we conclude

$$\limsup_{T^*\to\infty}[J_{T^*}(\mathbf{u}(\cdot)) - J_{T^*}(v(\cdot))] \geq 0. \tag{2.34}$$

From (2.33) and (2.34) we conclude the result III.

Chapter 3

Potential Differential Games (PDG).

In general, finding a Nash equilibrium for a differential game is a very difficult task. In some cases, under certain structural assumptions, we can derive characterizations of open-loop and Markov perfect Nash equilibrium, using optimization techniques [12].However, there is an important class of games called potential games [13, 14, 15, 31] ,whose Nash equilibrium study can be reduced to an optimal control problem, whose solution is a Nash equilibrium of our initial differential game. In this section we give basic concepts and identifying theorems for the class of potential games. We use two different approaches. The first one is based on the game's primitive data, and the second one assumes the existence of a smooth concave functions that together with the primitive data give sufficient conditions to ensure the existence of a PDG. From here on, we will use the discounted case as long as we treat OCPs. In this chapter we mainly follow [15] .

3.1 Definition. *A differential game* $\Gamma^h_{x_0}, h \leq \infty$*, as in* **Definition1.6** *is an open-loop potential differential game, if there exists an OCP such that an optimal solution of this OCP is an OLNE for* $\Gamma^h_{x_0}$.

3.1 Example A game $\Gamma^h_{x_0} := [N, \{\mathbf{U_i}\}_{i\in\mathcal{N}}, \{J^i_h\}_{i\in\mathcal{N}}, f], h \leq \infty$, where h is either finite or infinite, is said to be a team game if there is a (payoff) function $J_h : \mathbf{U} \to \mathbb{R}$ such that $J^i_h = J_h \;\; , \forall i \in \mathcal{N}$.

It is easily seen that a team game is an OL-PDG.Indeed, let $\mathbf{u}^* = (\mathbf{u}^*_1, ..., \mathbf{u}^*_n) \in \mathbf{U}$ be a multistrategy that maximizes J_h subject to $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \;\; \mathbf{x}(0) = x_0 \in X$. Then,by definition of optimality, for every $i \in \mathcal{N}$, we have

$$J_h(\mathbf{u}^*) \geq J_h(\mathbf{u}_i, \mathbf{u}^*_{-i}), \;\; \forall\, \mathbf{u_i} \in \mathbf{U_i}.$$

Hence, by Definition 3.1, $\mathbf{u}^*$ is an OLNE. For example ,consider a N-Game $\Gamma^h_{x_0} := [N, \{\mathbf{U_i}\}_{i\in\mathcal{N}}, \{J^i_h\}_{i\in\mathcal{N}}, f], h = \infty$, of extraction of exhaustible resources. The payoff function of player i to maximize is

$$J^i_h(\mathbf{u}) = \sum_{j=1}^{N} \int_0^{\infty} e^{-\beta t}[\mathbf{u}_j(t)]^{\frac{1}{2}} dt, \;\; i = 1, ..., N,$$

subject to

$$\dot{x}(t) = -\mathbf{u}_i(t) - \sum_{j\neq i} \mathbf{u}_j(t),$$

with $\mathbf{u}_i \geq 0$, $\lim_{t\to\infty} x(t) \geq 0$,$x(0) = x_0 > 0$, and β is the discount rate. This is an example of team game (the players have the same payoff function), thus, a team game, hence, an OL-PDG.

3.2 Definition. *Taking into account a N-Game (as in Definition 1.6), given $i \in \mathcal{N}$ we define*

$$\mathbf{U_{-i}} = (\mathbf{U}_1, \ldots, \mathbf{U}_{i-1}, \mathbf{U}_{i+1}, \ldots, \mathbf{U}_N), \tag{3.1}$$

$$X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N). \tag{3.2}$$

Similarly, given $\mathbf{v} \in \mathbf{U_i}$, $\mathbf{u} = (\mathbf{u_1}, ..., \mathbf{u_N}) \in \mathbf{U}$ and $\mathbf{x} = (\mathbf{x_1}, ..., \mathbf{x_N})$, we define

$$\mathbf{u_{-i}} = (\mathbf{u_1}, ., \mathbf{u_{i-1}}, \mathbf{u_{i+1}}, .., \mathbf{u_N}) \in \mathbf{U_{-i}}, \tag{3.3}$$

$$(\mathbf{v}, \mathbf{u_{-i}}) = (\mathbf{u_1}, ., \mathbf{u_{i-1}}, \mathbf{v}, \mathbf{u_{i+1}}, .., \mathbf{u_N}) \in \mathbf{U}\;, \textit{and} \tag{3.4}$$

$$\mathbf{x_{-i}} = (\mathbf{x_1}, ., \mathbf{x_{i-1}}, \mathbf{x_{i+1}}, .., \mathbf{x_N}). \tag{3.5}$$

3.1 PDGs: the exact potential approach .

3.1.1 PDGs Over an Infinite Horizon.

3.1 Theorem. *Let $\Gamma^{\infty}_{x_0}$ be a differential game as in Definition 1.6, and $p : T \times X \times U \to \mathbb{R}$ a certain function. We assume that one of the following conditions holds for every $i \in \mathcal{N}$:*

(a) There exists a function $c^i : T \times U_{-i} \to \mathbb{R}$ such that

$$L^i(s,x,u) = p(s,x,u) + c^i(s,u_{-i}). \tag{3.6}$$

(b)There exist functions $c^i : T \times X \times U_{-i} \to \mathbb{R}$ and $g^i : T \times X \to X_i$ such that

$$L^i(s,x,u) = p(s,x,u) + c^i(s,x,u_{-i}), \quad \textit{and} \tag{3.7}$$

$$f^i(s,x,u) = g^i(s,x). \tag{3.8}$$

(c) There exist functions $c^i : T \times X_{-i} \times U_{-1} \to \mathbb{R}$ and $g^i : T \times X_i \times U_i \to X_i$ such that

$$L^i(s,x,u) = p(s,x,u) + c^i(s,x_{-i},u_{-i}), \quad \textit{and} \tag{3.9}$$

$$f^i(s,x,u) = g^i(s,x_i,u_i). \tag{3.10}$$

Then $\Gamma^\infty_{x_0}$ is an $OL-PDG$ with potential function p.

Proof : Let us consider the OCP in Definition 2.1 from section 2.1 with $F := p$, $h = \infty$. We shall prove that this OCP and $\Gamma^\infty_{x_0}$ satisfy Definition 3.1's conditions in each case a),b),c).

Suppose that $\mathbf{u}^* = (\mathbf{u_1}^*, ..., \mathbf{u_N}^*)$ is an open-loop optimal solution of our OCP, and $\mathbf{x}^* = (\mathbf{x_1}^*, ..., \mathbf{x_N}^*)$ is the corresponding feasible path. Fix an arbitrary $i \in N$ and let $\mathbf{u}_i \neq \mathbf{u}_i^*$ be an open-loop strategy for player i. Let $\mathbf{x} = (\mathbf{x_1}, ..., \mathbf{x_N})$ be the new state trajectory of (1.3), corresponding to $\mathbf{u} = (\mathbf{u}_i, \mathbf{u_{-i}}^*)$.

Because $\mathbf{u}^*, \mathbf{x}^*$ are optimal for the OCP we have

$$\int_0^\infty e^{-\beta s} p(s,\mathbf{x}(s),\mathbf{u}(s)))ds \leq \int_0^\infty e^{-\beta s} p(s,\mathbf{x}^*(s),\mathbf{u}^*(s)))ds. \tag{3.11}$$

For case a), adding the constant

$$\textstyle\int_0^\infty e^{-\beta s} c^i(s,\mathbf{u_{-i}^*})ds,$$

in both sides of (3.11) we obtain

$$\begin{aligned} R1 = \int_0^\infty e^{-\beta s} p(s,\mathbf{x}^*(s),\mathbf{u}^*(s)))ds + \int_0^\infty e^{-\beta s} c^i(s,\mathbf{u}_{-i})ds \geq \\ \geq \textstyle\int_0^\infty e^{-\beta s} p(s,\mathbf{x}(s),\mathbf{u}(s)))ds + \int_0^\infty e^{-\beta s} c^i(s,\mathbf{u}_{-i})ds = R2, \end{aligned} \tag{3.12}$$

which implies

$$\begin{aligned} J^i(\mathbf{u}^*) = \int_0^\infty e^{-\beta s} L^i(s,\mathbf{x}^*(s),\mathbf{u}^*(s))ds = R1 \geq R2 = \int_0^\infty e^{-\beta s} L^i(s,\mathbf{x}(s),\mathbf{u}(s))ds, \\ = J^i(\mathbf{u}) \end{aligned}$$

with u_i arbitrary in $\mathbf{U}_i$, from which we have the result for a).

For case b), observe that condition (3.8), implies $\mathbf{x}$ is solution of

$$(\dot{x_1}(s),\ldots,\dot{x_N}(s)) = (f^1(s,\mathbf{x}(s),\mathbf{u}(s)),\ldots,f^N(s,\mathbf{x}(s),\mathbf{u}(s)))$$
$$= (g^1(s,\mathbf{x}(s)),\ldots,g^N(s,\mathbf{x}(s))), \quad \mathbf{x}(0)=\mathbf{x}_0,$$

and $\mathbf{x}^*$ is solution of

$$(\dot{x_1}(s),\ldots,\dot{x_N}(s)) = (f^1(s,\mathbf{x}(s),\mathbf{u}^*(s)),\ldots,f^N(s,\mathbf{x}(s),\mathbf{u}^*(s)))$$
$$= (g^1(s,\mathbf{x}(s)),\ldots,g^N(s,\mathbf{x}(s))), \quad \mathbf{x}(0)=\mathbf{x}_0.$$

Thus, $\mathbf{x}$ and $\mathbf{x}^*$ are solutions of

$$(\dot{x_1}(s),\ldots,\dot{x_N}(s)) = (g^1(s,\mathbf{x}(s)),\ldots,g^N(s,\mathbf{x}(s))).$$

From the uniqueness of the solution of the last system we have $\mathbf{x}=\mathbf{x}^*$ and $\mathbf{u}_{-i}={\mathbf{u}^*}_{-i}$, therefore

$$\int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}^*,\mathbf{u}^*_{-\mathbf{i}})ds = \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x},\mathbf{u}_{-\mathbf{i}})ds.$$

Adding this constant to (3.11) , we obtain

$$J^i(\mathbf{u}^*) = \int_0^\infty e^{-\beta s}L^i(s,\mathbf{x}^*(s),\mathbf{u}^*(s))ds \tag{3.13}$$
$$= \int_0^\infty e^{-\beta s}p(s,\mathbf{x}^*(s),\mathbf{u}^*(s)))ds + \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}^*,\mathbf{u}^*_{-\mathbf{i}})ds$$
$$\geq \int_0^\infty e^{-\beta s}p(s,\mathbf{x}(s),\mathbf{u}(s)))ds + \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x},\mathbf{u}_{-\mathbf{i}})ds$$
$$= \int_0^\infty e^{-\beta s}L^i(s,\mathbf{x}(\mathbf{s}),\mathbf{u}(\mathbf{s}))ds = J^i(\mathbf{u}).$$

Finally, for c), we have that for any $j\neq i$, $\mathbf{u}_j={\mathbf{u}^*}_j$ the uniqueness of the solution of the system

$$\dot{x}_j(s) = f^j(s,x(s),\mathbf{u}^*(s))$$
$$= g^j(s,x_j(s),(\mathbf{u}^*)_j)$$
$$= g^j(s,x_j(s),(\mathbf{u})_j) = f^j(s,x(s),\mathbf{u}(s)), \quad x_j(0)=(x_0)_j,$$

implies $\mathbf{x}^*_j=\mathbf{x}_j$ $\forall j\neq i$ ($\mathbf{x}^*_j,\mathbf{x}_j$ are solutions of the last system) thus ${\mathbf{x}^*}_{-i}=\mathbf{x}_{-i}$, and

$$\int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}^*_{-\mathbf{i}},\mathbf{u}^*_{-\mathbf{i}})ds = \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}_{-\mathbf{i}},\mathbf{u}_{-\mathbf{i}})ds.$$

Hence, adding the last constant to Eq.(3.11) we obtain

$$J^i(\mathbf{u}^*) = \int_0^\infty e^{-\beta s}L^i(s,\mathbf{x}^*(s),\mathbf{u}^*(s))ds \tag{3.14}$$
$$= \int_0^\infty e^{-\beta s}p(s,\mathbf{x}^*(s),\mathbf{u}^*(s)))ds + \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}^*_{-\mathbf{i}},\mathbf{u}^*_{-\mathbf{i}})ds$$
$$\geq \int_0^\infty e^{-\beta s}p(s,\mathbf{x}(s),\mathbf{u}(s)))ds + \int_0^\infty e^{-\beta s}c^i(s,\mathbf{x}_{-\mathbf{i}},\mathbf{u}_{-\mathbf{i}})ds$$
$$= \int_0^\infty e^{-\beta s}L^i(s,\mathbf{x}(\mathbf{s}),\mathbf{u}(\mathbf{s}))ds = J^i(\mathbf{u})$$

From (3.12)-(3.14) we obtain the desired result ■

3.1.2 PDGs Over a Finite Horizon.

3.2 Theorem. *Let $\Gamma_{x_0}^h$ be a differential game as in (Definition 1.6), $h < \infty$ and $\bar{s} : X \to \mathbb{R}$ a certain function. Let us assume that one of the following conditions holds for every $i \in \mathcal{N}$*

a')The functions f^i and L^i satisfy part a) in Theorem 3.1 and the terminal payoff function is independent of i, that is

$$S^i(x) = \bar{s}(x) \;\; \forall \;\; i \in \mathcal{N}, \; x \in X$$

b')The functions f^i and L^i satisfy part b) in Theorem 3.1, the final payoff functions S^i for player i have no restrictions.

c')The functions f^i and L^i satisfy part c) in Theorem 3.1, and there exists a function $k^i : X_{-i} \to \mathbb{R}$ such that

$$S^i(x) = \bar{s}(x) + k^i(x_{-i})$$

Then the differential game $\Gamma_{x_0}^h$ is an OL-PDG.The potential function is p or $p + \sum_{j=1}^n c^j$, moreover, the potential terminal payoff function is $\bar{s}$ for a'), identically zero for b') and $\bar{s}$ for c').

Proof :We reason in a similar way as in the proof of Theorem 3.1, let us consider the OCP in Definition 2.1 from section 2.1 in chapter 2, with $F := p$, $h < \infty$.We shall prove that this OCP and $\Gamma_{x_0}^h$ satisfy Definition 3.1's conditions in each case a'),b'),c').

Suppose that $\mathbf{u}^* = (\mathbf{u_1}^*, ..., \mathbf{u_n}^*)$ is an open-loop optimal solution of our OCP, and $\mathbf{x}^* = (\mathbf{x_1}^*, ..., \mathbf{x_N}^*)$ is the corresponding feasible path. Fix an arbitrary $i \in N$, and let $\mathbf{u}_i \neq \mathbf{u}_i^*$ be an open-loop strategy for player i. Let $\mathbf{x} = (\mathbf{x_1}, ..., \mathbf{x_N})$ be the new state trajectory of (1.3), corresponding to $\mathbf{u} = (\mathbf{u_1}^*, \ldots, , \mathbf{u_{i-1}}^*, \mathbf{u_i}, \mathbf{u_{i+1}}^*, \ldots, \mathbf{u_N}^*)$.

Because of $\mathbf{u}^*, \mathbf{x}^*$ are optimal for the OCP we have

$$\int_0^h p(s, \mathbf{x}(s), \mathbf{u}(s)))ds + \bar{s}(\mathbf{x}(h)) \leq \int_0^h p(s, \mathbf{x}^*(s), \mathbf{u}^*(s)))ds + \bar{s}(\mathbf{x}^*(h)) \tag{3.15}$$

For case a), adding the constant

$$\int_0^h c^i(s, \mathbf{u}_{-\mathbf{i}}^*)ds = \int_0^h c^i(s, \mathbf{u}_{-\mathbf{i}})ds$$

in both sides of (3.15) we obtain

$$\int_0^h p(s, \mathbf{x}^*(s), \mathbf{u}^*(s)))ds + \bar{s}(\mathbf{x}^*(h)) + \int_0^h c^i(s, \mathbf{u}_{-\mathbf{i}}^*)ds \geq$$
$$\geq \int_0^h p(s, \mathbf{x}(s), \mathbf{u}(s)))ds + \bar{s}(\mathbf{x}(h)) + \int_0^h c^i(s, \mathbf{u}_{-\mathbf{i}})ds.$$

Hence

(3.16)
$$J^i(\mathbf{u}^*) = \int_0^h L^i(s, \mathbf{x}^*(s), \mathbf{u}^*(s))ds + \bar{s}(\mathbf{x}^*(h)) \geq \int_0^h L^i(s, \mathbf{x}(s), \mathbf{u}(s))ds + \bar{s}(\mathbf{x}(h)) = J^i(\mathbf{u})$$

For b') and c') using an analogous process to the proofs b) and c) in Theorem 3.1, the inequalities (3.13) and (3.14) remain valid, modifying the functions J^i used in Definition 2.1 with $h < \infty$. In each case we conclude

$$J(\mathbf{u}^*) \geq J(\mathbf{u}).$$

Thus, we conclude the result.■

3.2 PDGs the Fictitious-Potential Approach.

Now we give the definitions, assumptions and conditions that characterize our second approach, called The Fictitious-Potential Approach. In the following we consider every notation in Definition 1.6 of a differential game.

Assumption 1 For each $i \in \mathcal{N}$,

(a) the sets U_i and X_i are open and convex,

(b) the function L_i is in $C^2(X \times U)$,

(c) the function f^i is in $C^2(X \times U)$,

Denote for each $i \in \mathcal{N}$, the gradient of the function L^i with respect to the vector u_i by

$$\nabla_{u_i} L^i = \left(\frac{\partial L^i}{\partial u_1^i}, \ldots, \frac{\partial L^i}{\partial u_{m_i}^i} \right) \tag{3.17}$$

and ,for each fixed $(t, \bar{u}_{-i}) \in T \times U_{-i}$, the Hessian matrix of L^i with respect to the vector (x, u_i) is denoted by

$$\text{Hess}[L^i(t, x, (u_i, \bar{u}_{-i}))] = \begin{pmatrix} \frac{\partial^2 L^i}{\partial x_1^1 \partial x_1^1} & \cdots & \frac{\partial^2 L^i}{\partial u_{m_i}^i \partial x_1^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L^i}{\partial x_1^1 \partial u_{m_i}^i} & \cdots & \frac{\partial^2 L^i}{\partial u_{m_i}^i \partial u_{m_i}^i} \end{pmatrix} \tag{3.18}$$

Analogously, for a function $P : T \times X \times U \to \mathbb{R}$, we define the gradients $\nabla_{x_k} L^i, \nabla_{u_i} P$ and $\nabla_{x_k} P$ with their respective dimensions, and for each fixed $(t, \bar{u}_{-i}) \in T \times U_{-i}$, $\text{Hess}[P(t, x, (u_i, \bar{u}_{-i}))]$ denotes the Hessian matrix of the function P with respect to the vector (x, u_i), $i \in \mathcal{N}$, assuming that P has second-order partial derivatives on $X \times U$.

Takink into account a game and notations as in Definition 1.6, let $R \subseteq \mathcal{N}$ be the subsets of indices k such that $l_k > 0$ (for some k we may have $l_k = 0$ see Definition 1.6), this set is non-empty because $l > 0$.

Assumption 2 The functions $L^1, ..., L^N$ satisfy that

$$\nabla_{x_k} L^1 = \cdots = \nabla_{x_k} L^n \ \ \forall k \in R \tag{3.19}$$

Assumption 3 Let r be an index in R.

(a)There is at least another index $k \in R \setminus \{r\}$.

(b)For every $j \in R, l_j = l_r$.

Assumption 4 Let $r \in R$ be as in Assumption 3. For each $i \in \mathcal{N}$, the function L^i satisfies that

$$\nabla_{x_r} L^i = \nabla_{x_j} L^i \ \ \forall j \in R. \tag{3.20}$$

Condition 1(Sufficient conditions)The function $P : T \times X \times U \to \mathbb{R}$ is in $C^2(X \times U)$, is concave in (x, u), and for every $i \in \mathcal{N}$ satisfies

$$\nabla_{u_i} P = \nabla_{u_i} L^i \tag{3.21}$$

$$\nabla_{x_i} P = \nabla_{x_i} L^i. \tag{3.22}$$

3.3 Remark. (See [10] Chapter 22,[45],or [47] Chapter 3) *Let us assume the existence of a function P as in condition 1 and, in addition consider the OCP in Definition 2.1 described by P with $h = \infty$. If $\mathbf{u}^*$ is an open-loop solution for this OCP, and $\mathbf{x}^*$ is the state path corresponding to $\mathbf{u}^*$, then there exists a vector of Lagrange multipliers $\lambda^* : T \to \mathbb{R}^l$ defined by the rule $s \to \lambda^*(s) = (\lambda^{1*}(s), ..., \lambda^{k*}(s)) \in \mathbb{R}^l$, such that using the notation $(\dagger) := (s, \mathbf{x}^*(s), \mathbf{u}^*(s)), s \in T$:*

I. for $k \in R$, each coordinate λ^{k*} of λ^* is defined as the function $\lambda^{k*} : T \to \mathbb{R}^{l_k}$ that is the solution to the linear adjoint system

$$\dot{\lambda}^{k*}(s) = \beta\lambda^{k*}(s) - \nabla_{x_k} P(\dagger) - \nabla_{x_k} \langle f(\dagger), \lambda^*(s)\rangle_l \tag{3.23}$$

that satisfies the transversality conditions

$$\lim_{s\to\infty} e^{-\beta \cdot s}\lambda^{k*}(s) = 0; \ \ \textit{and} \tag{3.24}$$

II. for almost every $s \in T$, the following maximality condition holds

$$H(s, \mathbf{x}^*(s), \mathbf{u}^*(s), \lambda^*(s)) = \max_{u \in U} H(s, \mathbf{x}^*(s), u, \lambda^*(s)), \tag{3.25}$$

where $H : X \times U \times \mathbb{R}^l \times T \to \mathbb{R}$ is the current value Hamiltonian associated to P (see Definition 2.4).

Condition 2(Sufficient conditions) Let P be a function as in Condition 1. We assume that for each Lagrange multiplier λ^* as in Remark 3.3 the function

$$(x, u) \to H(t, x, u, \lambda^*(t))$$

is concave in (x, u).

3.4 Lemma. *Consider a game as in Definition 1.6, under Assumption 1, a function P satisfying Eqs. (3.21) and (3.22) in Condition 1. Suppose that one of the following condition holds:*

(a)Assumption 2;

(b)Assumption 4 and Assumption 3

Then, for each $i \in \mathcal{N}$,and each point $(s, \bar{u}_{-i}) \in T \times U_{-i}$, P is concave in (x, u_i) if and only if L^i is concave in (x, u_i).

Proof For each $i \in \mathcal{N}$, and each fixed point $(s, \bar{u}_{-i})$ in $T \times U_{-i}$ we will show that

$$\text{Hess}[P(t, x, (u_i, \bar{u}_{-i}))] = \text{Hess}[L^i(t, x, (u_i, \bar{u}_{-i}))] \tag{3.26}$$

First observe that, by Eqs. (3.21) and (3.22), we have for each $i, k \in \mathcal{N}$, and $q = 1, \ldots, l_k; v, w = 1, \cdots, m_i$ that

$$\frac{\partial P}{\partial x_q^k u_v^i} = \frac{\partial L^i}{\partial x_q^k u_v^i} \tag{3.27}$$

$$\frac{\partial P}{\partial u_v^i u_w^i} = \frac{\partial L^i}{\partial u_v^i u_w^i} \tag{3.28}$$

On the other hand, considering Assumption 2, and using Eq. (3.22), for each $i, k, j \in \mathcal{N}$, and $q = 1, \cdots, l_k$,$r = 1, \cdots, l_j$,

$$\frac{\partial P}{\partial x_q^k \partial x_r^j} = \frac{\partial L^j}{\partial x_q^k \partial x_r^j} = \frac{\partial L^i}{\partial x_q^k \partial x_r^j} \tag{3.29}$$

Similarly, if we consider Assumptions 3, 4, and use Eq. (3.22), then for each $i, k, j \in \mathcal{N}$, and $q = 1, \ldots, l_k$,$r = 1, \cdots, l_j$

$$\frac{\partial P}{\partial x_q^k \partial x_r^j} = \frac{\partial L^k}{\partial x_q^k \partial x_r^j} = \frac{\partial L^k}{\partial x_q^k \partial x_r^i} = \frac{\partial P}{\partial x_q^k \partial x_r^i} = \frac{\partial L^i}{\partial x_q^k \partial x_r^i} = \frac{\partial L^i}{\partial x_q^k \partial x_r^j} \tag{3.30}$$

Hence, Eqs.(3.27)-(3.29) imply (3.26), and also Eqs. (3.27), (3.28) and (3.30) imply Eq.(3.26). Therefore, for each $i \in \mathcal{N}$, we have that P is concave in (x, u_i), if and only if (3.26) is negative-semidefinite on $X \times U_i$, if and only if, the function L^i is concave in (x, u_i). (See [34], Chapter 6 ,Section 3,Theorem 2).■

3.2.1 PDGs Over an Infinite Horizon.

3.5 Theorem. *Suppose that a differential game $\Gamma_{x_0}^{\infty}$ (as in Definition 1.6) satisfies Assumptions 1 and 2. If there exists a function P satisfying Conditions 1 and 2, then $\Gamma_{x_0}^{\infty}$ is an OL-PDG with potential function P.*

Proof : Consider the OCP in Definition 2.1 described by P, with $h = \infty$. Let $\mathbf{u}^* = (\mathbf{u_1}^*, ..., \mathbf{u_N}^*)$ be an open-loop optimal solution of our OCP, and $\mathbf{x}^* = (\mathbf{x_1}^*, ..., \mathbf{x_N}^*)$ is the corresponding feasible state path. We will show that $\mathbf{u}^*$ is an OLNE for $\Gamma_{x_0}^{\infty}$.

Again we use the notation $(\dagger) := (s, \mathbf{x}^*(s), \mathbf{u}^*(s)), s \in T$, then there exists a vector Lagrange multipliers $\lambda^* : T \to \mathbb{R}^l$ that satisfies the conditions in Remark 3.3.

Now, consider $H = P + \langle f, \lambda^* \rangle_l$, using the fact from [42], Chapter 7, Theorem 7.15 and conditions in Remark 3.3, we obtain for every $i \in R$.

$$\nabla_{u_i} H = 0 \tag{3.31}$$

and using Eq. (3.21) of Condition 1, we obtain

$$\nabla_{u_i} H = \nabla_{u_i} P(\dagger) + \sum_{k \in R} \nabla_{u_i} \left\langle f^k(\dagger), \lambda^{k*}(\dagger) \right\rangle_l = \nabla_{u_i} L^i(\dagger) + \sum_{k \in R} \nabla_{u_i} \left\langle f^k(\dagger), \lambda^{k*}(\dagger) \right\rangle_l = 0 \tag{3.32}$$

Moreover, applying (3.23) and using Assumption 2 we obtain

$$\dot{\lambda}^{k*}(s) = \beta \lambda^{k*}(s) - \nabla_{x_k} P(\dagger) - \nabla_{x_k} \left\langle f(\dagger), \lambda^*(s) \right\rangle_l = \tag{3.33}$$
$$= \beta \lambda^{k*}(s) - \nabla_{x_k} L^i(\dagger) - \nabla_{x_k} \left\langle f(\dagger), \lambda^*(s) \right\rangle_l, \ k \in R.$$

Define for each $i \in \mathcal{N}$ a Lagrange multiplier $p^{i*} : T \to \mathbb{R}^l$, by the rule $t \mapsto (p_k^{i*}(t))_{k \in R} \in \mathbb{R}^l$ with

$$p_k^{i*} = \lambda^{k*} \tag{3.34}$$

i.e., $p^{i*} = \lambda^*$ for every $i \in \mathcal{N}$. Observe that $p_k^{i*} : T \to \mathbb{R}^{l_k}$,

$$(p_{k1}^{i*}(s), \ldots, p_{kl_k}^{i*}(s)) \in \mathbb{R}^{l_k} \tag{3.35}$$

and for each $i \in \mathcal{N}$, $p_{kj}^{i*} = \lambda_j^{k*}$,$j = 1, ..., l_k$, $k \in R$.

By equation (3.33) we obtain that, for each $i \in \mathcal{N}$, the lagrange multipliers, $p_k^{i*}, k \in R$, solve the system

$$\dot{p}_k^{i*}(s) = \beta p_k^{i*}(s) - \nabla_{x_k} L^i(\dagger) - \nabla_{x_k} \left\langle f(\dagger), p^{i*}(s) \right\rangle_l \tag{3.36}$$

under the transversality conditions

$$\lim_{s \to \infty} e^{-\beta \cdot s} p_k^{i*}(s) = 0 \tag{3.37}$$

Now consider the Hamiltonian function $H^i : T \times X \times U \times \mathbb{R}^l \to \mathbb{R}$ for every player i ,which is defined as

$$H^i(s, x, u, p^i) = L^i(s, x, u) + \left\langle f(s, x, u), p^i \right\rangle_l \tag{3.38}$$

Hence, by Assumptions 1, 2 and Condition 1 and 2, from Lemma 3.4(a) and Eq.(3.34) , for each $i \in \mathcal{N}$, the function

$$(x, u_i) \to H^i(t, x, (u_i, \mathbf{u}_{-i}^*), p^{i*}(t)) \tag{3.39}$$

is concave given that

$$\text{Hess}[H(t, x, (u_i, \mathbf{u}_{-i}^*), \lambda^*(t))] = \text{Hess}[H^i(t, x, (u_i, \mathbf{u}_{-i}^*), p^{i*}(t))] \tag{3.40}$$

In fact observe that Lemma 3.4(a) implies, for each $i, k, j \in \mathcal{N}$, and $q = 1, \cdots, l_k$,$r = 1, \cdots, l_j$, that

$$\text{Hess}[P(t, x, (u_i, \mathbf{u}^*{}_{-i}))] = \text{Hess}[L^i(t, x, (u_i, \mathbf{u}^*{}_{-i}))] \tag{3.41}$$

and $\langle f, \lambda^* \rangle_l = \langle f, p^{i*} \rangle_l$, then

$$\text{Hess}[P(t, x, (u_i, \mathbf{u}^*{}_{-i})) + \langle f, \lambda^* \rangle_l] = \text{Hess}[L^i(t, x, (u_i, \mathbf{u}^*{}_{-i})) + \left\langle f, p^{i*} \right\rangle_l] \tag{3.42}$$

so (3.40) holds.

Thus from Eq. (3.31) and Theorem 7.15 from [25], chapter 7, we obtain for every $i \in \mathcal{N}$

$$H^i(s, \mathbf{x}^*(s), \mathbf{u}^*(s), p^{i*}(s)) = \max_{u_i \in U_i} H^i(s, \mathbf{x}^*(s), (u_i, \mathbf{u}_{-i}^*(s)), p^{i*}(s)) \tag{3.43}$$

Moreover, since (3.39) is concave in (x, u_i) the function

$$x \mapsto \max_{u_i \in U_i} H^i(t, x, (u_i, \mathbf{u}_{-i}^*(t)), p^{i*}(t)) \tag{3.44}$$

is also concave.

From our hypothesis on $\mathbf{x}^*, \mathbf{u}^*$ and the concavity of function (3.44) we conclude that $\mathbf{u}^*$ is an OLNE (see [25], Chapter 7; [37] or [47], Chapter 3).■

3.6 Theorem. *Suppose that the differential game $\Gamma_{x_0}^{\infty}$ (as in Definition 1.6), the set R, and the index $r \in R$ satisfy Assumptions 1 and 4. Moreover, suppose that for $j \in R$, there exist functions $g^j : T \times X_j \times U_j \to \mathbb{R}$ such that*

$$f^j(t,x,u) = g^j(t,x_j,u_j) \tag{3.45}$$

and such that

$$\nabla_{x_r} g^r = \nabla_{x_j} g^j \ \ \forall j \in R \tag{3.46}$$

If there exists a function P that satisfies the Conditions 1 and 2, then $\Gamma_{x_0}^{\infty}$ is an OL-PDG with potential function P

Proof :We use similar arguments as in the proof of Theorem 3.5, so, again there exists a vector λ^* such that Eq. (3.31) remains valid in this case. Now instead of (3.34), for each $i \in \mathcal{N}$, we set

$$p_j^i := \lambda^{i*}, \forall j \in R \tag{3.47}$$

in other words, we have for each $i \in \mathcal{N}$

$$p_{jk}^i := \lambda_k^{i*},\ k = 1, \cdots, l_i\ ,\ j \in R \tag{3.48}$$

the definition of λ^* leads, for every $k' \in \mathcal{N}$

$$\dot{\lambda}^{k'*}(s) = \beta\lambda^{k'*}(s) - \nabla_{x_{k'}} P(\dagger) - \nabla_{x_{k'}} \langle f(\dagger), \lambda^*(s)\rangle_l \tag{3.49}$$

From (3.22) of Condition 1 and Assumption 4 we have for every $j \in R$

$$\nabla_{x_{k'}} P = \nabla_{x_{k'}} L^{k'} = \nabla_{x_r} L^{k'} = \nabla_{x_j} L^{k'} \tag{3.50}$$

Using(3.45), (3.46) and (3.48) in (3.49), we have for every $k' \in N$ and $j \in R$

$$\dot{p}_j^{k'*}(s) = \beta p_j^{k'*}(s) - \nabla_{x_j} L^{k'}(\dagger) - \nabla_{x_j} \left\langle f(\dagger), p^{k'*}(s)\right\rangle_l \tag{3.51}$$

Under the transversality condition

$$\lim_{s\to\infty} e^{-\beta\cdot s} p_j^{k'*}(s) = 0 \tag{3.52}$$

In the same form we write (3.38), thus, we have

$$\nabla_{u_i} H^i = 0 \tag{3.53}$$

and using Assumptions 1 and 4, Conditions 1, and 2, we obtain the conclusion in Lemma 3.4(b). From Lemma 3.4(b) and Eq. (3.45) we obtain Eq. (3.42) again, note that from (3.45) the entries of

$$\mathrm{Hess}[P(t,x,(u_i,\mathbf{u}^*{}_{-i}))+\langle f,\lambda^*\rangle_l]$$

are equal to the entries of

$$\mathrm{Hess}[L^i(t,x,(u_i,\mathbf{u}^*{}_{-i}))+\langle f,p^{i*}\rangle_l]$$

Finally Eqs. (3.43), (3.44) remain valid from Theorem 7.15 chapter 7 of [42], so using the same arguments as in the last part of the proof of Theorem 3.5 we conclude, the multistrategy $\mathbf{u}^*$ is an OLNE with state variable $\mathbf{x}^*$ for the game with $h=\infty$.■

3.2.2 PDGs Over a Finite Horizon.

3.7 Theorem. *Consider a differential game as in Definition 1.6, with $h<\infty$. Assume that one of the following conditions holds:*

a)The set R and the functions L^i and f^i satisfy the hypotheses in Theorem 3.5. Moreover, the terminal payoff function S^i, $i\in\mathcal{N}$, satisfy that

(3.54) $$\nabla_{x_k}S^1=\cdots=\nabla_{x_k}S^N\ \ \forall k\in R$$

b)The set R and the functions L^i and f^i satisfy the hypotheses in Theorem 3.6, whereas the terminal payoff function S^i satisfies that

(3.55) $$\nabla_{x_1}S^i=\cdots=\nabla_{x_N}S^i\ \ \forall i\in\mathcal{N}$$

In addition, suppose that there are functions $P:T\times X\times U\to\mathbb{R}$ and $S:X\to\mathbb{R}$ such that P satisfies Conditions 1 and 2, and S is convex in x and satisfies that

(3.56) $$\nabla_{x_i}S=\nabla_{x_i}S^i\ \ \forall i\in R$$

Then the differential game $\Gamma_{x_0}^h$ is an OL-PDG with potential function P and potential terminal payoff function S.

Proof :Consider the OCP in Definition 2.1 related to P,S and $h<\infty$. Let $\mathbf{u}^*$ be an optimal solution to this OCP and $\mathbf{x}^*$ the corresponding admissible path.

Assume Condition 1. Adapting Remark 3.3 in the case $h<\infty$, there exists a vector of Lagrange multipliers $\lambda^*:T\to\mathbb{R}^l$ such that,with $\beta=0$, for each index $k\in R$ we have

Eq.(3.23) and the final condition

$$\lambda^{k*} = \nabla_{x_k} S(\mathbf{x}^*(h)) \tag{3.57}$$

and, for almost every $s \in T$, the maximality condition (3.25) holds.

a) Considering Eqs.(3.21) and (3.22) on the functions P and L^i, Eqs.(3.54) and (3.56) on the functions S^i and S, we have that for each i and k in R, the Lagrange multiplier $p_k^i : T \to \mathbb{R}^{l_k}$ defined as in (3.34) satisfies the linear differential equation (3.36) with $\beta = 0$ and the final conditions

$$p_k^{i*}(h) = \nabla_{x_k} S^i(\mathbf{x}^*(h)).$$

Since (3.54) and (3.56) hold, S is convex in x if and only if S^i, $i \in \mathcal{N}$, is convex in x. Thus Lemma 3.4(a) and the concavity of the functions (3.39) imply the maximality condition (3.43) holds,which combined with the concavity of function (3.44) imply the multistrategy $\mathbf{u}^*$ is an OLNE.

b)Similarly considering Eqs. (3.21) and (3.22) on the functions P and L^i, Eqs. (3.55)and (3.56) on the functions S^i and S, we have that for each i and k in R, the Lagrange multiplier $p_k^i : T \to \mathbb{R}^{l_k}$ defined as in (3.47) satisfies the linear differential equation (3.51) with $\beta = 0$ and final conditions $p_k^i(h) = \lambda^{i*}(h) = \nabla_{x_i} S(\mathbf{x}(h))$. Since Eqs. (3.55) and (3.56) hold, S is convex in x if and only if S^i,$i \in \mathcal{N}$, is convex in x. Thus from Lemma 3.4(b) and the concavity of the functions (3.39) (due to (3.40) holds again) imply the maximality condition (3.43) holds, as the same time (3.43) combined with the concavity of function (3.44) (due to (3.39) is again concave),imply the multistrategy $\mathbf{u}^*$ is an OLNE. ■

Chapter 4

Stability .

To complete the theoretical results of our survey, we present results on stability of OCPs and pursuer-evader game problems.In both cases we consider the infinite time horizon case. The first topic is very useful, in order to fulfill our study on PDGs (in this case we face an OCP). These results are obtained by means of Lyapunov functions and some curvature assumptions on the maximized Hamiltonian H^* and the system (4.3) . On the other hand we give a result about stability of zero-sum games which is obtained by means of Lyapunov functions , and a result on asymptotic stability of the system given by Eqs. (4.29) and (4.30) (Theorem 4.4). This result is illustrating and give us useful and interesting applications as we shall see. Some results for the finite time horizon case can be derived from the infinite horizon case.

This chapter follows principally papers [5, 7, 39] for the first part, and papers [29, 30] for the second part.

4.1 Stability of Optimal Control.

Consider again $X \subseteq \mathbb{R}^l$, and $U \subseteq \mathbb{R}^m$ with $m, l \in \mathbb{N}$, two functions $F : X \times U \to \mathbb{R}$ $f : X \times U \to \mathbb{R}^l$, $T = [0, \infty)$, and the system

(4.1) $$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \;\; \mathbf{x}(0) = x_0 \in X$$

Again $\mathbf{x}$ is an admissible state path for the strategy $\mathbf{u} \in \mathbf{U}$, if $\mathbf{x}$ is the unique solution of equation (4.1). These functions and the set $\mathbf{U}$ define an optimal control problem (OCP) in which one player wants to maximize the payoff function $J : \mathbf{U} \to \mathbb{R}$ defined by

$$J(\mathbf{u}) := \int_0^\infty e^{-\beta t} F(\mathbf{x}(t), \mathbf{u}(t))dt \tag{4.2}$$

over the elements of the set $\mathbf{U}$ which have an admissible state path, where $\beta \geq 0$.

4.1 Example. Consider the optimal control problem defined by $X = U = \mathbb{R}$, restricted to

$$\dot{x(t)} = \alpha x(t) + \mathbf{u}(t), \quad x(0) = x_0.$$

The goal is to maximize

$$J(\mathbf{u}) = \int_0^\infty e^{-\beta t}[-\frac{\mathbf{u}^2(t)}{2} - \frac{\gamma x^2(t)}{2}]dt.$$

Where α and γ are real constants.

We consider $H : \{(x, u, \lambda) | x \in X, u \in U, \lambda \in \mathbb{R}^l\} \to \mathbb{R}$ the current Hamiltonian function

$$H(x, u, \lambda) = F(x, u) + \langle \lambda, f(x, u) \rangle_l$$

and the maximized Hamiltonian function $H^* : X \times \mathbb{R}^n \to \mathbb{R}$ by

$$H^*(x, \lambda) = \max\{H(x, u, \lambda) | u \in U\}.$$

Let $\mathbf{u}^*(\cdot)$ be a control that maximizes (4.2) and $\mathbf{x}(\cdot)$ the corresponding state path. Then Arrow and Kurz [1] showed that there exist lagrange multipliers $\lambda(\cdot) = (\lambda(\cdot)_1, ..., \lambda(\cdot)_l)$, such that on each interval of continuity of $\mathbf{u}^*(\cdot)$

$$\begin{cases} \dot{\lambda}(t) = \beta\lambda(t) - \frac{\partial}{\partial x} H^*(x(t), \lambda(t)) \\ \dot{x}(t) = \frac{\partial}{\partial \lambda} H^*(x(t), \lambda(t)) \\ x(0) = x_0 \end{cases} \tag{4.3}$$

where

$$\begin{aligned} H^*(x(t), \lambda(t)) &= H(x(t), \mathbf{u}^*(\mathbf{t}), \lambda(t)) \\ &= \max\{H(x(t), u(t), \lambda(t)) | u(t) \in U(x(t), t)\} \; \forall \, t \geq 0. \end{aligned}$$

A steady solution of system 4.3, is a solution that is a constant over the variable t. Now we are ready to give the first two results related to the stability of the system (4.1).

4.1 Definition. *For $y \in \mathbb{R}^l$,we write $y \geq 0$ if $y_i \geq 0$ for all $i = 1, ..., l$.*

4.1 Theorem. *Assume that the function $(x, \lambda) \to H^*(x, \lambda)$ defined over the set*

$\{(x,\lambda)|x,\lambda\in\mathbb{R}^l, x,\lambda\geq 0\}$*is strictly concave in the first variable, strictly convex in the second variable, and the function* $(x,\lambda)\to H^*(x,\lambda)-\beta\langle\lambda,x\rangle_l$ *defined on* $\{(x,\lambda)|x,\lambda\in\mathbb{R}^l, x,\lambda\geq 0\}$ *is strictly concave in the first variable. Moreover, let* $\bar{x},\bar{\lambda}$ *be a steady solution of (4.3). Assume: for every* $\epsilon>0$*,there is a* $\delta>0$ *such that* $\|x-\bar{x}\|>\epsilon$ *implies*

$$\left\langle(\lambda-\bar{\lambda}),\frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l-\left\langle\frac{\partial}{\partial x}H^*(x,\lambda),(x-\bar{x})\right\rangle_l+\beta\left\langle\bar{\lambda},(x-\bar{x})\right\rangle_l> \\ >-\beta\left\langle(\lambda-\bar{\lambda}),(x-\bar{x})\right\rangle_l+\delta \tag{4.4}$$

Then if $\lambda(\cdot),x(\cdot)$ *are continuously differentiable functions that solves (4.3) with* x *uniformly continuous and additionally*

$$\lim_{t\to\infty}\lambda(t)x(t)e^{-\beta t}=0, \tag{4.5}$$

then $\lim_{t\to\infty}\|x(t)-\bar{x}\|=0$.

Proof :Define $V(\lambda,x)=\left\langle\lambda-\bar{\lambda},x-\bar{x}\right\rangle_l$. Note that

$$\begin{aligned}\dot{V}(\lambda,x)&=\left\langle\lambda-\bar{\lambda},\dot{x}\right\rangle_l+\left\langle\dot{\lambda},x-\bar{x}\right\rangle_l\\ &=\left\langle\lambda-\bar{\lambda},\tfrac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l+\left\langle(\beta\lambda-\tfrac{\partial}{\partial x}H^*(x,\lambda)),x-\bar{x}\right\rangle_l\\ &=\left\langle\lambda-\bar{\lambda},\tfrac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l-\left\langle\tfrac{\partial}{\partial x}H^*(x,\lambda)-\beta\lambda,x-\bar{x}\right\rangle_l\\ &=\left\langle\lambda-\bar{\lambda},\tfrac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l-\left\langle\tfrac{\partial}{\partial x}H^*(x,\lambda)-\beta\bar{\lambda},x-\bar{x}\right\rangle_l+\left\langle\beta(\lambda-\bar{\lambda}),x-\bar{x}\right\rangle_l\end{aligned}$$

Note that if $x(t)\neq\bar{x}$, then from the last equation and (4.4), we have $\dot{V}>\delta>0$ for some $\delta>0$, if $x(t)=\bar{x}$, if we can find $\{r_i\}$ such that $\lim_{i\to\infty}r_i=t$, $x(r_i)\neq\bar{x}$ then $0\leq\lim_{i\to\infty}\dot{V}(\lambda(r_i),x(r_i))=\dot{V}(\lambda(t),x(t))$ if we can not find such sequence it follows that V is constant in a neighbourhood of t then it has derivative 0.

Thus

$$\dot{V}\geq 0 \tag{4.6}$$

On the other hand

$\frac{d}{dt}Ve^{-\beta t}=\dot{V}e^{-\beta t}-\beta e^{-\beta t}V=e^{-\beta t}[\dot{V}-\beta V]$

$=e^{-\beta t}[\left\langle\lambda-\bar{\lambda},\frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l-\left\langle\frac{\partial}{\partial x}H^*(x,\lambda)-\beta\bar{\lambda},x-\bar{x}\right\rangle_l+\left\langle\beta(\lambda-\bar{\lambda}),x-\bar{x}\right\rangle_l-\left\langle\beta(\lambda-\bar{\lambda})^T,x-\bar{x}\right\rangle_l]$

$$=e^{-\beta t}[\left\langle\lambda-\bar{\lambda},\frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l-\left\langle\frac{\partial}{\partial x}H^*(x,\lambda),x-\bar{x}\right\rangle_l+\left\langle\beta\bar{\lambda},x-\bar{x}\right\rangle_l] \tag{4.7}$$

Observe that from the concavity-convexity of H^*

$$\left\langle(\bar{x}-x),\frac{\partial}{\partial x}H^*(x,\lambda)\right\rangle_l=\left\langle-(x-\bar{x}),\frac{\partial}{\partial x}H^*(x,\lambda)\right\rangle l\geq H^*(\bar{x},\lambda)-H^*(x,\lambda) \tag{4.8}$$

$\left\langle \bar{\lambda}-\lambda, \frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l \leq H^*(x,\bar{\lambda}) - H^*(x,\lambda)$

note that the last inequality implies

$$\left\langle \lambda-\bar{\lambda}, \frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l \geq H^*(x,\lambda) - H^*(x,\bar{\lambda}) \tag{4.9}$$

Similarly concavity-convexity of $(x,\lambda) \to H(x,\lambda)$ and concavity of $(x,\lambda) \to H^*(x,\lambda) - \beta\langle x,\lambda\rangle$ implies

$$H^*(x,\bar{\lambda}) - \beta\left\langle \bar{\lambda}, x\right\rangle \leq H^*(\bar{x},\bar{\lambda}) - \beta\left\langle \bar{\lambda}, \bar{x}\right\rangle \leq H^*(\bar{x},\lambda) - \beta\left\langle \bar{\lambda}, \bar{x}\right\rangle,\ \forall x,\lambda \tag{4.10}$$

thus

$$H^*(\bar{x},\lambda) - H^*(x,\bar{\lambda}) + \beta\left\langle \bar{\lambda}, x-\bar{x}\right\rangle_l \geq 0\ \forall x,\lambda \tag{4.11}$$

Using (4.8), (4.9) and (4.11) in (4.7) we obtain

$$\frac{d}{dt}Ve^{-\beta t} = e^{-\beta t}[\left\langle \lambda-\bar{\lambda}, \frac{\partial}{\partial\lambda}H^*(x,\lambda)\right\rangle_l - \left\langle \frac{\partial}{\partial x}H^*(x,\lambda), x-\bar{x}\right\rangle_l + \beta\left\langle \bar{\lambda}, x-\bar{x}\right\rangle_l] \tag{4.12}$$

$= e^{-\beta t}[H^*(\bar{x},\lambda) - H^*(x,\lambda) + H^*(x,\lambda) - H^*(x,\bar{\lambda}) + \beta\left\langle \bar{\lambda}, x-\bar{x}\right\rangle_l]$

$= e^{-\beta t}[H^*(\bar{x},\lambda) - H^*(x,\bar{\lambda}) + \beta\left\langle \bar{\lambda}, x-\bar{x}\right\rangle_l] \geq 0$

Furthermore

$$\lim_{t\to\infty} Ve^{-\beta t} = \lim_{t\to\infty} e^{-\beta t}\left\langle \lambda-\bar{\lambda}, x-\bar{x}\right\rangle_l = \lim_{t\to\infty} e^{-\beta t}[\langle \lambda, x\rangle_l - \left\langle \bar{\lambda}, x\right\rangle_l - \langle \lambda, \bar{x}\rangle_l + \left\langle \bar{\lambda}, \bar{x}\right\rangle_l] \leq \tag{4.13}$$

$$\leq \lim_{t\to\infty} e^{-\beta t}\langle \lambda, x\rangle_l + e^{-\beta t}\left\langle \bar{\lambda}, \bar{x}_l\right\rangle = \begin{cases} 0 & \text{if } \beta > 0 \\ 2\left\langle \bar{\lambda}, \bar{x}\right\rangle_l & \text{if } \beta = 0 \end{cases}$$

From (4.12) and (4.13) we have that $Ve^{-\beta t}$ is increasing and nonpositive for $\beta > 0$(bounded for $\beta = 0$), then there exists a constant V^* such that, $0 \leq V^*$,and

$$V(\lambda(t), x(t)) \leq V^*\ \forall t \in [0,\infty) \tag{4.14}$$

In what follows we write $V(t) = V(\lambda(t), x(t))$. Now observe that from (4.6), V is increasing and from (4.14), $\lim_{t\to\infty} V = V^\infty$ exists and

$$\lim_{t\to\infty} V = V^\infty \leq V^* < \infty \tag{4.15}$$

Suposse that $\lim_{t\to\infty} x(t) = \bar{x}$ were not true,i.e, that for some $\epsilon > 0$ there were a sequence of points $\{t_i\}$ such that $\|x(t_i) - \bar{x}\| > 2\epsilon$ we can ask $t_{i+1} - t_i > 1$, from the uniform continuity of x we have that for $\epsilon > 0$ exists $1 > \eta > 0$ such that ,defining $\bar{t}_i = t_i + \eta$, for all $t \in [t_i, \bar{t}_i]$,$0 < \|x(t_i) - x(t)\| < \epsilon$, so $\|x(t) - \bar{x}\| > \epsilon$. Thus from (4.4) there is a $\delta > 0$ such that

$$\dot{V}(t) > \delta, \ \forall\, t \in [t_i, \bar{t}_i] \tag{4.16}$$

Hence using (4.15), for sufficiently large t' we have $V^{\infty} - \delta \leq V(t) \leq V^{\infty}$ for every $t \geq t'$ so

$$0 \leq V(t) - V(t') \leq \delta \tag{4.17}$$

Now, since V is increasing due to (4.6), let be $I = \min\{j | t' \leq t_j\}$,choose J such that $(J - I)\eta > 1$ and $t > \bar{t}_J \geq t_J$ then since V is increasing due to (4.6), we have

$$V(t) - V(t') \geq \sum_{i=I}^{J} [V(\bar{t}_i) - V(t_i)] \geq \sum_{i=I, t_i \leq t \leq \bar{t}_i}^{J} \eta \dot{V}(t) = (J - I + 1)\eta\delta > \delta \tag{4.18}$$

From (4.17) and (4.18) we obtain

$$\delta < V(t) - V(t') \leq \delta \tag{4.19}$$

This contradiction concludes our result.■

For a matrix $A \in \mathbb{R}^{n \times m}$, we denote $A' \in \mathbb{R}^{m \times n}$ the transpose of A.

4.2 Theorem. *Let*

$$B(\lambda, x) = \begin{pmatrix} \frac{\partial^2}{\partial \lambda^2} H^*(x, \lambda) & \frac{\rho}{2} I_l \\ \frac{\rho}{2} I_l & -\frac{\partial^2}{\partial x^2} H^*(x, \lambda) \end{pmatrix} \tag{4.20}$$

where I_l is the identity matrix with $l \times l$ entries. Let us assume that the steady solutions of system (4.3) are isolated.

Let $(x(\cdot), \lambda(\cdot))$ be a bounded solution of system (4.3) independently of t, define $z : [0, \infty) \to \mathbb{R}^l \times \mathbb{R}^l$ defined by the rule $t \to (\lambda(t), x(t))$ and $G : D = \{(\lambda(t), x(t)) | t \geq 0)\} \to \mathbb{R}^l \times \mathbb{R}^l$ defined by the rule $(\lambda(t), x(t)) \mapsto [\beta\lambda(t) - \frac{\partial}{\partial x} H^*(x(t), \lambda(t)), \frac{\partial}{\partial \lambda} H^*(x(t), \lambda(t))] = [\dot{\lambda}(t), \dot{x}(t)] = \dot{z}(t)$. If $G(\lambda, x)' B(\lambda, x) G(\lambda, x) > 0$ for all $G(\lambda, x) \neq 0$, then there is a steady solution $(\bar{x}, \bar{\lambda})$ of system (4.3) such that $(x(t), \lambda(t)) \to (\bar{x}, \bar{\lambda})$ when $t \to \infty$.

Proof :

Let $\gamma^+ = \{z \in \mathbb{R}^l \times \mathbb{R}^l | z = (\lambda(t), x(t))$, for some $t \geq 0\}$ and

$\Omega(\gamma^+) = \{z \in \mathbb{R}^l \times \mathbb{R}^l |$there exists increasing sequence $\{t_n\}$ such that, $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty}(\lambda(t_n), x(t_n)) = z\}$,

note that the closure of γ^+ is compact on $D = \{(\lambda(t), x(t)) | t \geq 0)\}$, because of it is bounded (D is bounded) so its closure is bounded, and its closed by definition,so, using Theorem 1.1, [22], p.145, $\Omega(\gamma^+)$ is nonempty, compact and connected.

Let P_1, P_2 be the projections over the first and second variable, respectively. Define $W(z) = P_1(z)'P_2(z)$.Using (4.3) we have;

$$\begin{aligned} \dot{W}(z) &= \dot{P}_1(G(z))'P_2(G(z)) + P_1(G(z))'\dot{P}_2(G(z)) \\ &= \beta P_1(G(z))'P_2(G(z)) + P_1'H_{11}(G(z))P_1(G(z)) - P_2'H_{22}(G(z))P_2(G(z)) \\ &= G(z)'B(z)G(z) \geq 0 \end{aligned} \tag{4.21}$$

By Lemma 2.1, of [5], p. 172 and Eq. (4.21), if $(\bar{\lambda}, \bar{x}) = \bar{z} \in \Omega(\gamma^+)$ ($\bar{z}$ is a limit point of the solution $(\lambda(\cdot), x(\cdot)))$, then $\dot{W}(\bar{z}) = G(\bar{z})'B(\bar{z})G(\bar{z}) = 0$, and, hence, $G(\bar{z}) = 0$; i.e., $\bar{z}$ is a steady solution of 4.3. Since the steady solutions of (4.3) are isolated and $\Omega(\gamma^+)$ is connected, then $\Omega(\gamma^+) = \{\bar{z}\}$.Hence $\lim_{t\to\infty}(\lambda(t), x(t)) = (\bar{\lambda}, \bar{x})$.■

There are some other interesting geometrical conditions that ensure stability of autonomous OCPs, for instance we have [39], [5] and [18], which we recommend to the interested reader.

4.2 Stability of Zero-Sum Games.

In this section we follow [29, 30]. Firstly we address the two-player zero-sum differential game problem for non-linear dynamical systems over the infinite time horizon.We seek conditions on state-feedback control laws, which guarantee partial-state asymptotic stability of the closed-loop system.These results are obtained by means of Lyapunov functions.

Let us consider a two-player game,with infinite time horizon $h = \infty$, $\mathcal{N} = \{1, 2\}, T = [0, \infty)$. In what follows consider $X_1 \subseteq \mathbb{R}^{l_1}, X_2 = \mathbb{R}^{l_2}$, and the corresponding notation (see Definition 1.6).

Observe that we only have two current payoff functions L^1, L^2. In addition, we consider an autonomous system , i.e.

$$f : X_1 \times X_2 \times U_1 \times U_2 \to \mathbb{R}^l = \mathbb{R}^{l_1} \times \mathbb{R}^{l_2}, \;\; x_0 = (x_{10}, x_{20}) \tag{4.22}$$

with $f = (f_1', f_2')'$, and

$$f_1 : X_1 \times X_2 \times U_1 \times U_2 \to \mathbb{R}^{l_1}, f_2 : X_1 \times X_2 \times U_1 \times U_2 \to \mathbb{R}^{l_2}. \tag{4.23}$$

Thus the system is

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), u_1(t), u_2(t)), \;\; x_1(0) = x_{10} \tag{4.24}$$

(4.25) $$\dot{x}_2(t) = f_2(x_1(t), x_2(t), u_1(t), u_2(t)), \;\; x_2(0) = x_{20} \qquad t \geq 0,$$

where X_1 is an open set with $0 \in X_1$ and $0 \in U_2$. Moreover, f_1, f_2 are Lipschitz continuous in x_1, x_2, u_1 and u_2, and $f_1(0, x_2, 0, 0) = 0$ for every $x_2 \in X_2$

and

$$L^1 : X_1 \times X_2 \times U_1 \times U_2 \to \mathbb{R}, L^2 : X_1 \times X_2 \times U_1 \times U_2 \to \mathbb{R}$$

Furthermore we ask $L^1 + L^2 = 0$, the game obtained is a zero-sum game. So, a zero-sum game can be defined taking into account just one function let us say

(4.26) $$L := L^1,$$

so $L^2 = -L$.

4.1 Example.

Consider the zero-sum game defined by equations of motion of a spacecraft given by [46]

$$\dot{w}_1(t) = I_{23} w_2(t) + u_1(t), \quad w_1(0) = w_{10},$$
$$\dot{w}_2(t) = I_{31} w_3(t) w_1(t) + u_2(t), \quad w_2(0) = w_{20},$$
$$\dot{w}_3(t) = I_{12} w_1(t) w_2(t), \quad w_3(0) = w_{30},$$

and the function

$$J(u_1(\cdot), u_2(\cdot)) = \int_0^\infty [I_{31}^2 w_1^2(t) - I_{23}^2 w_2^2(t) + 4 I_{23} w_2(t) u_2(t) + u_1^2(t) - u_2^2(t)] dt,$$

where $I_{23} = \frac{I_2 - I_3}{I_1}$, $I_{31} = \frac{I_3 - I_1}{I_2}$, $I_{12} = \frac{I_1 - I_2}{I_3}$, and I_1, I_2, I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 < I_2 < I_3$, $[w_1, w_2, w_3]' : [0, \infty) \to \mathbb{R}^3$ denote the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and $u_1 : [0, \infty) \to \mathbb{R}$ and $u_2 : [0, \infty) \to \mathbb{R}$ are the spacecraft control moments.

If we write $x_1(t) = (w_1(t), w_2(t))$, $x_2(t) = w_3(t)$ and $x_0 = (x_{10}, x_{20}) = (w_{10}, w_{20}, w_{30})$, then, our game can be written in the form

$$\begin{aligned} \dot{x}(t) &= (\dot{x}_1(t), \dot{x}_2(t))' \\ &= f(x(t), u_1(t), u_2(t)) \\ &= f(x_1(t), x_2(t), u_1(t), u_2(t)), \qquad x(0) = x_0, t \geq 0, \end{aligned}$$

and

$$L = I_{31}^2 P_1(x_1(t))^2(t) - I_{23}^2 P_2(x_1(t))^2(t) + 4 I_{23} P_2(x_1(t)) u_2(t) + u_1^2(t) - u_2^2(t),$$

$J(u_1(\cdot), u_2(\cdot)) = \int_0^\infty [I_{31}^2 P_1(x_1(t))^2(t) - I_{23}^2 P_2(x_1(t))^2(t) + 4I_{23}P_2(x_1(t))u_2(t) + u_1^2(t) - u_2^2(t)]dt$,

where P_1, P_2 be the projections over the first and second variable, respectively. We will tackle this example later to discuss more details about it.

In order to give the main results of this section, we give the following definitions.

4.2 Definition. *Two continuous functions $\phi : X_1 \times X_2 \to U_1$ and $\psi : X_1 \times X_2 \to U_2$ which satisfy*

$$\phi(0, x_2) = 0, \psi(0, x_2) = 0, \ \ \forall x_2 \in X_2$$

are called control laws.

Considering the zero sum game whose goal is to maximize

$$J(u_1, u_2) = \int_0^\infty L(u_1, u_2, x_1, x_2)dt$$

restricted to (4.24) and (4.25.)If

$$u_1(t) = \phi(x_1(t), x_2(t)), u_2(t) = \psi(x_1(t), x_2(t)), \ \ \forall t \geq 0,$$

ϕ, ψ are control laws and x_1, x_2 satisfy (4.24)-(4.25), then we call u_1 and u_2 ,feedback control laws. In this case we write (4.24) and (4.25) in the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \ \ x_1(0) = x_{10} \tag{4.27}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t)), \psi(x_1(t), x_2(t))), \ \ x_2(0) = x_{20} \qquad t \geq 0. \tag{4.28}$$

Eqs (4.27) and (4.28) define a close-loop system related to (4.24) and (4.25).

4.2 Example Considering the 4.1 Example, set

$$\phi(x_1, x_2) = -I_{31}P_1(x_1) = \phi(w_1, w_2, w_3) = -I_{31}w_1,$$
$$\psi(x_1, x_2) = \psi(w_1, w_2, w_3) = I_{23}P_2(x_1) = I_{23}w_2.$$

It is easy to see that ϕ, ψ are control laws, and they define feedback control laws, setting $u_1(t) = \phi(x_1(t), x_2(t)), u_2(t) = \psi(x_1(t), x_2(t))$(see 5.7 Example of Chapter 5), with x_1, x_2 solutions of the system $(4.24) - (4.25)$. In this case we can write the system $(4.24) - (4.25)$ in the form $(4.27) - (4.28)$.

4.2.1 Asymptotic Stability:Basic concepts and results.

Let $\hat{f} : X_1 \times X_2 \to \mathbb{R}^{l_1} \times \mathbb{R}^{l_2}$, with component functions $\hat{f}_1 : X_1 \times X_2 \to \mathbb{R}^{l_1}$, $\hat{f}_2 : X_1 \times X_2 \to \mathbb{R}^{l_2}$,$(\hat{f} = (\hat{f}_1', \hat{f}_2')')$. Consider the nonlinear autonomous dynamical system

$$\dot{x}_1(t) = \hat{f}_1(x_1(t), x_2(t)), \;\; x_1(0) = x_{10} \tag{4.29}$$

$$\dot{x}_2(t) = \hat{f}_2(x_1(t), x_2(t)), \;\; x_2(0) = x_{20} \qquad t \geq 0 \tag{4.30}$$

where X_1 is an open set with $0 \in X_1$, for every $x_2 \in X_2$, $\hat{f}_1(0, x_2) = 0$, $f_1(\cdot, x_2)$ is locally Lipschitz continuos and for every $x_1 \in X_1$, $f_1(x_1, \cdot)$ is locally Lipschitz continuos.

4.3 Definition. *The nonlinear dynamical system given by Eqs (4.29) and (4.30)*

i. *is Lyapunov stable with respect to x_1 uniformly in x_{20} if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_{10}\| \leq \delta$ implies $\|x_1(t)\| < \epsilon$ for all $t \geq 0$ and for all $x_{20} \in \mathbb{R}^{l_2}$,*

ii. *is asymptotically stable with respect to x_1 uniformly in x_{20}, if it is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{20} \in X_2$,*

iii. *is globally asymptotically stable with respect to x_1 uniformly in x_{20}, if it is Lyapunov stable with respect to x_1 uniformly in x_{20} and $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{10} \in \mathbb{R}^{l_1}$ and $x_{20} \in \mathbb{R}^{l_2}$.*

4.3 Example. Using a result that we give later in this section, we will show that, the OCP defined in 4.1 Example, is globally asymptotically stable with respect to x_1 uniformly in x_{20}, with the feedback controls u_1 and u_2 defined in 4.2 Example.

4.4 Definition. *Suppose $a \leq \infty$.*

A continuous function $\alpha : [0, a) \to [0, \infty)$ is a function of class K , if it is strictly increasing and $\alpha(0) = 0$. If in addition we have $a = \infty$,and $\lim_{t\to\infty} \alpha(t) = \infty$, then we call α a function of class K_∞.

Notation. Let $V : X_1 \times \mathbb{R}^{l_2} \to \mathbb{R}$, dV is the Frechet derivative of V (see [40] pp. 37-38).

4.3 Theorem. ([21],Theorem 4.1.) *Consider the nonlinear dynamical system given by (4.29) and (4.30). Then the following statements hold:*

i. *If there exists a continuously differentiable function* $V : X_1 \times \mathbb{R}^{l_2} \to \mathbb{R}$ *and class* K *functions* $\alpha(\cdot), \eta(\cdot)$ *and* $\theta(\cdot)$ *such that*

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \eta(\|x_1\|), \quad (x_1, x_2) \in X_1 \times \mathbb{R}^{l_2} \tag{4.31}$$

$$dV(x_1, x_2)' \hat{f}(x_1, x_2) \le -\theta(\|x_1\|), \quad (x_1, x_2) \in X_1 \times \mathbb{R}^{l_2} \tag{4.32}$$

then the nonlinear dynamical system given by Eqs. (4.29) and (4.30) is asymptotically stable with respect to x_1 *uniformly in* x_{20}.

ii. *If there exist a continuously differentiable function* $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, *a class* K *function* $\theta(\cdot)$ *, and class* K_∞ *functions* $\theta(\cdot)$*and* $\eta(\cdot)$ *satisfying Eqs. (4.31) and (4.32), then the nonlinear dynamical system given by Eqs. (4.29) and (4.30) is globally asymptotically stable with respect to* x_1 *uniformly in* x_{20}.

4.5 Definition. *We define the set of regulation controllers related to the closed-loop system defined by (4.24) and (4.25), as*

$$S(x_0) = S(x_{10}, x_{20}) := \{(u_1(\cdot), u_2(\cdot)) | u_1(\cdot), u_2(\cdot) \text{ are admissible controls and } x_1(\cdot) \text{ solution } of \text{ (4.24) satisfies } x_1(t) \to 0 \text{ as } t \to \infty\} \tag{4.33}$$

In addition given the control law ψ

$S_\psi(x_0) = S_\psi(x_{10}, x_{20}) = \{u_1(\cdot) | (u_1(\cdot), \psi(x_1(\cdot), x_2(\cdot))) \in S(x_{10}, x_{20})\}$

and given the control law ϕ, let

$S_\phi(x_0) = S_\phi(x_{10}, x_{20}) = \{u_2(\cdot) | (\phi(x_1(\cdot), x_2(\cdot)), u_2(\cdot)) \in S(x_{10}, x_{20})\}$

4.6 Definition. *Given* $F : X \times Y \to \mathbb{R}$, *where* $X \subseteq \mathbb{R}^{l_1}$,$Y \subseteq \mathbb{R}^{l_2}$, *we define*

$$\text{argminmax}_{(x,y) \in X \times Y} F(x, y) = \{(\bar{x}, \bar{y}) \in (X, Y) | F(x, \bar{y}) \ge F(\bar{x}, \bar{y}) \ge F(\bar{x}, y), \forall x \in X, y \in Y\} \tag{4.34}$$

and

$$\text{minmax}_{(x,y) \in X \times Y} F(x, y) = F(\bar{x}, \bar{y}), (\bar{x}, \bar{y}) \in \text{argminmax}_{(x,y) \in X \times Y} F(x, y) \tag{4.35}$$

We are ready to give the main results of this section, related to the stability of the system given by (4.27), (4.28)

4.2.2 A Result about Stability of Zero-Sum Games.

4.4 Theorem. *Consider the controlled nonlinear dynamical system given by (4.24) and (4.25) with*

(4.36) $$J(u(\cdot),w(\cdot)) = \int_0^\infty L(x_1(t),x_2(t),u(t),w(t))dt,$$

where $u(\cdot)$ and $w(\cdot)$ are admissible controls. Assume that there exist a continuously differentiable function $V : X_1 \times \mathbb{R}^{l_2} \to \mathbb{R}$,class K functions $\alpha(\cdot), \eta(\cdot)$,and $\theta(\cdot)$, and control laws $\phi : X_1 \times \mathbb{R}^{n_2} \to U_1$ and $\psi : X_1 \times \mathbb{R}^{l_2} \to U_2$, such that

(4.37) $$\alpha(\|x_1\|) \le V(x_1,x_2) \le \eta(\|x_1\|), \quad (x_1,x_2) \in X_1 \times \mathbb{R}^{l_2}$$

(4.38) $$dV(x_1,x_2)'f(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2)) \le -\theta(\|x_1\|), \;\; (x_1,x_2) \in X_1 \times \mathbb{R}^{l_2}$$

(4.39) $$\phi(0,x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2},$$

(4.40) $$\psi(0,x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2},$$

(4.41) $$L(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2)) + dV(x_1,x_2)'f(x_1,x_2,\phi(x_1,x_2),\psi(x_1,x_2)) = 0,$$
$$(x_1,x_2) \in X_1 \times \mathbb{R}^{l_2}$$

(4.42) $$L(x_1,x_2,u,\psi(x_1,x_2)) + dV(x_1,x_2)'f(x_1,x_2,u,\psi(x_1,x_2)) \ge 0,$$
$$(x_1,x_2,u) \in X_1 \times \mathbb{R}^{l_2} \times U_1$$

(4.43) $$L(x_1,x_2,\phi(x_1,x_2),w) + dV(x_1,x_2)'f(x_1,x_2,\phi(x_1,x_2),w) \le 0,$$
$$(x_1,x_2,w) \in X_1 \times \mathbb{R}^{l_2} \times U_2$$

Then with the feedback controls $u_1 = \phi(x_1,x_2)$ and $u_2 = \psi(x_1,x_2)$ ($(x_1(\cdot),x_2(\cdot))$ the feasible control path), the closed-loop system given by Eqs.(4.27) and (4.28) is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq X_1$ of $x_1 = 0$ such that

(4.44) $$J(\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot))) = V(x_{10},x_{20}), \quad (x_{10},x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{l_2}$$

In addition, if $(x_{10},x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{l_2}$, then

(4.45)
$$J(\phi(x_1(\cdot),x_2(\cdot)),\psi(x_1(\cdot),x_2(\cdot))) = \mathrm{minmax}_{(u_1(\cdot),u_2(\cdot)) \in S_\psi(x_{10},x_{20}) \times S_\phi(x_{10},x_{20})} J(u_1(\cdot),u_2(\cdot))$$

and

$$J(\phi(x_1(\cdot), x_2(\cdot)), \psi(x_1(\cdot), x_2(\cdot))) \leq V(x_{10}, x_{20}), \;\; w(\cdot) \in S_\phi(x_{10}, x_{20}) \tag{4.46}$$

Finally, if $X_1 = \mathbb{R}^{l_1}$,$U_1 = \mathbb{R}^{m_1}$,$U_2 = \mathbb{R}^{m_2}$,and the functions $\alpha(\cdot)$ and $\eta(\cdot)$ satisfying Eq. (4.37) are class K_∞, then the closed-loop system given by Eqs. (4.27) and (4.28) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Proof It follows from Eqs (4.37), (4.38) and (i) of Theorem 4.4, that the closed-loop system is asymptotically stable with respect to x_1 uniformly in x_{20}.Consequently, $x_1(t) \to 0$ as $t \to \infty$ for all initial $(x_{10}, x_{20}) \in \mathcal{D}_0 \subseteq X_1$ of 0 for some neighborhood $\mathcal{D}_0 \subseteq X_1$ of 0. Now, because

$$\dot{V}(x_1(t), x_2(t)) = dV(x_1(t), x_2(t))' f(x_1(t), x_2(t), u_1(t), u_2(t)), t \geq 0$$

it follows

$$0 = -\dot{V}(x_1(t), x_2(t)) + dV(x_1(t), x_2(t))' f(x_1(t), x_2(t), u_1(t), u_2(t)), t \geq 0 \tag{4.47}$$

.

Using Eqs. (4.41) and (4.47)

$$\begin{aligned} L(x_1(t), x_2(t), u_1(t), u_2(t)) &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t), u_1(t), u_2(t)) + \\ &\quad + dV(x_1(t), x_2(t))' f(x_1(t), x_2(t), u_1(t), u_2(t)) \\ &= -\dot{V}(x_1(t), x_2(t)), t \geq 0 \end{aligned} \tag{4.48}$$

Integrating (4.48) over $[0, \tau]$, we have

$$\begin{aligned} \int_0^\tau L(x_1(t), x_2(t), u_1(t), u_2(t)) dt &= \int_0^\tau -\dot{V}(x_1(t), x_2(t)) dt \\ &= V(x_{10}, x_{20}) - V(x_1(\tau), x_2(\tau)) \end{aligned} \tag{4.49}$$

Using (4.37) and letting $\tau \to \infty$ in (4.49), we have

$$\begin{aligned} V(x_{10}, x_{20}) - \alpha(\lim_{\tau \to \infty)} \|x_1(\tau)\|) &\geq \int_0^\infty L(x_1(t), x_2(t), u_1(t), u_2(t)) dt \\ &\geq V(x_{10}, x_{20}) - \beta(\lim_{\tau \to \infty)} \|x_1(\tau)\|) \end{aligned} \tag{4.50}$$

From the fact that α and β are class K functions we obtain from (4.50)

$$V(x_{10}, x_{20}) \geq \int_0^\infty L(x_1(t), x_2(t), u_1(t), u_2(t)) dt \geq V(x_{10}, x_{20}) \tag{4.51}$$

then (4.51) implies (4.44)

Next let $x_0 = (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{l_2}$, u, w admissible controls with (y_1, y_2) feasible path, then

$$0 = -\dot{V}(y_1(t), y_2(t)) + dV(y_1(t), y_2(t))' f(y_1(t), y_2(t), u(t), w(t)), t \geq 0 \tag{4.52}$$

Hence,

$$L(y_1(t), y_2(t), u(t), w(t)) = -\dot{V}(y_1(t), y_2(t)) + L(y_1(t), y_2(t), u(t), w(t)) + \\ + dV(y_1(t), y_2(t))' f(y_1(t), y_2(t), u(t), w(t)), t \geq 0 \tag{4.53}$$

Per definition, if $u(\cdot) \in S_\psi(x_0)$, then $y_1(t) \to 0$ as $t \to \infty$. Thus it follows from Eq (4.37) that

$$0 = \lim_{t\to\infty} \alpha(\|y_1(t)\|) \leq \lim_{t\to\infty} V(y_1(t), y_2(t)) \leq \lim_{t\to\infty} \eta(\|y_1(t)\|) = 0 \tag{4.54}$$

for every $u(\cdot) \in S_\psi(x_0)$. Consequently ,Eqs (4.53), (4.54) and (4.42) imply that

$$\begin{aligned} J(u(\cdot), \psi(y_1(\cdot), y_2(\cdot))) &= \int_0^\infty L(y_1(t), y_2(t), u(t), \psi(y_1(\cdot), y_2(\cdot)))dt \\ &= \int_0^\infty -\dot{V}(y_1(t), y_2(t))dt + \int_0^\infty [L(y_1(t), y_2(t), u(t), \psi(y_1(\cdot), y_2(\cdot))) + \\ &\quad + dV(y_1(t), y_2(t))' f(y_1(t), y_2(t), u(t), \psi(y_1(\cdot), y_2(\cdot)))]dt \\ &\geq \int_0^\infty -\dot{V}(y_1(t), y_2(t))dt \\ &= -[\lim_{t\to\infty} V(y_1(t), y_2(t)) - V(x_{10}, x_{20})] \\ &= J(\phi(y_1(\cdot), y_2(\cdot)), \psi(y_1(\cdot), y_2(\cdot))) \end{aligned} \tag{4.55}$$

Similarly, $w(\cdot) \in S_\phi(x_0)$, it follows from Eqs (4.53),(4.54) and (4.43) that for every $w(\cdot) \in S_\phi(x_0)$ we have

$$\begin{aligned} J(\phi(y_1(\cdot), y_2(\cdot)), w(\cdot)) &= \int_0^\infty L(y_1(t), y_2(t), \phi(y_1(\cdot), y_2(\cdot)), w(t))dt \\ &\leq \int_0^\infty -\dot{V}(y_1(t), y_2(t)) \\ &= -\lim_{t\to\infty} V(y_1(t), y_2(t)) + V(x_{10}, x_{20}) \\ &= J(\phi(y_1(\cdot), y_2(\cdot)), \psi(y_1(\cdot), y_2(\cdot))) \end{aligned} \tag{4.56}$$

From Eqs. (4.55) and (4.56) we obtain Eq. (4.45).Furthermore from Eq. (4.45) (ϕ, ψ) is a saddle point for (4.36) on $S_\psi(x_{10}, x_{20}) \times S_\phi(x_{10}, x_{20})$,$(x_{10}, x_{20} \in \mathcal{D}_0 \times \mathbb{R}^{l_2})$,it implies by definition and Eq. (4.44) that

$$\begin{aligned} J(\phi(y_1(\cdot), y_2(\cdot)), w(\cdot)) &\leq J(\phi(y_1(\cdot), y_2(\cdot)), \psi(y_1(\cdot), y_2(\cdot))) \\ &= V(x_{10}, x_{20}), \;\; w \in S_\phi(x_{10}, x_{20}) \end{aligned} \tag{4.57}$$

and Eq. (4.46) holds.

Finally, if $X_1 = \mathbb{R}^{l_1}$,$U_1 = \mathbb{R}^{m_1}$,$U_2 = \mathbb{R}^{m_2}$, (4.40) hold with $\alpha(\cdot)$ and $\beta(\cdot)$ are class K_∞, then from Eqs. (4.37), (4.38) Theorem 4.4 (ii) we obtain the global asymptotic stability of the system.■

Chapter 5

Applications

In order to illustrate some of our results we give some examples of applications.

First ,we apply Theorem 2.1 and Theorem 2.2 to solve an optimal control problem. Consider the OCP [12] defined by the objective functional

$$J(u(\cdot)) = \int_0^h e^{-\beta t}[-x(t) - \frac{\alpha}{2}u(t)^2]dt \tag{5.1}$$

subject to

$$\dot{x}(t) = \eta(t) - u(t)\sqrt{x(t)}, \quad u(t) \geq 0, \quad x(0) = x_0 \tag{5.2}$$

where α, β, h and x_0 are positive constants and $\eta : [0, h] \to \mathbb{R}$ is a positive-valued function and $u(t), x(t)$ are real numbers for all $t \in [0, h]$. One may interpret this control problem as one of finding an optimal maintenance policy for a machine,building,or piece of equipment which is subject to continuous deterioration(see [12],pag.53). The goal is to minimize the present value of the sum of the costs over the finite time interval $[0, h]$.

5.1 Example. Let us apply Theorem 2.2. In this case ,the Hamiltonian function is given by

$$H(x, u, \lambda, t) = -x - \frac{\alpha}{2}u^2 + \lambda[\eta(t) - u\sqrt{x}] \tag{5.3}$$

According to condition (2.7) of Theorem 2.2, this function should be maximized with respect to u ,taking into account (5.2). This yields $u^* = \max\{0, \frac{-\lambda\sqrt{x}}{\alpha}\}$, as a candidate. Note that we cannot have $u^*(t) = 0$ except possibly at $t = h$, in fact, from the nature of the problem we can see that λ must be negative for all $t \leq h$, so

$$u^*(t) = \frac{-\lambda(t)\sqrt{x(t)}}{\alpha} \tag{5.4}$$

From (5.4) we have

$$H^*(x, \lambda, t) = -x + \frac{1}{2\alpha}\lambda^2 x + \eta(t)\lambda \tag{5.5}$$

The adjoint equation (2.8) and the transversality condition (2.9) of Theorem 2.2 yield the terminal value problem

$$\dot{\lambda}(t) = 1 + \beta\lambda(t) - \frac{1}{2\alpha}\lambda^2(t), \qquad \lambda(h) = 0 \tag{5.6}$$

Equation (5.6) is a differential equation of the Riccati type. The unique solution is

$$\lambda(t) = \frac{2[1 - e^{C(t-T)}]}{(\beta - C)e^{C(t-T)} - (\beta + C)} \tag{5.7}$$

where $C = \sqrt{\frac{\beta^2+2}{\alpha}}$.Since $C > \beta > 0$ it is easily seen that $\lambda(t) \leq 0$ holds for all $t \in [0, h]$. It follows that $u^*(t)$ defined in (5.4) is nonnegative ;hence it maximizes the Hamiltonian $H(x(t), u, \lambda(t), t)$ over all feasible u, with $u(t) \geq 0$,for all $t \in [0, h]$.

To summarize, we have obtained a unique candidate $u(\cdot)$ for an optimal control path.Now observe that the maximized Hamiltonian function H^* is linear(and hence concave) with respect to the state variable and the scrap value function is identically equal to 0 so that Theorem 2.2 applies.

5.2 Example. Now, let us see how we can solve the OCP defined by (5.1) and (5.2) applying Theorem 2.1.In this case equation (2.2) of Theorem 2.1 is

$$\beta V(x,t) - V_t(x,t) = \max\{-x - \frac{\alpha}{2}u^2 + V_x(x,t)[\eta(t) - u\sqrt{x}] | u(t) \geq 0, t \in [0, h]\} \tag{5.8}$$

Carrying out the maximization on the right-hand and using some extra argument similar to the previous application yields

$$u = \frac{-V_x(x,t)\sqrt{x}}{\alpha} \tag{5.9}$$

Substituting (5.9) in (5.8), we obtain

$$\beta V(x,t) - V_t(x,t) = -x + \frac{1}{2\alpha}xV_x(x,t)^2 + \eta(t)V_x(x,t) \tag{5.10}$$

In order to solve(5.10) we assume $V(x,t) = A(t)x + B(t)$. The boundary condition (2.3) implies $A(h)x + B(h) = 0$ for all $x \geq 0$. This conditions can be satisfied if and only if

$$A(h) = B(h) = 0. \tag{5.11}$$

On the other hand, after some rearranging this implies from (5.10)

$$x[1+\beta A(t)-\frac{1}{2\alpha}A(t)^2-\dot{A}(t)]+\beta B(t)-\eta(t)A(t)-\dot{B}(t)=0 \tag{5.12}$$

A necessary and sufficient conditions for this equation to hold for all $x \geq 0$ and all $t \in [0,h]$ is that $A(\cdot)$ and $B(\cdot)$ satisfy the two differential equations

$$\dot{A}(t)=1+\beta A(t)-\frac{1}{2\alpha}A(t)^2, \dot{B}(t)=\beta B(t)-\eta A(t) \tag{5.13}$$

The first equation together with the boundary condition $A(h)=0$ from (5.11) coincides with (5.6), so the unique solution is $A=\lambda$. The differential equation for $B(t)$, with $A(t)=\lambda(t)$ and boundary condition $B(h)=0$, is a nonautonomous linear equation which unique solution $B(t)=\int_t^h e^{-\beta(s-t)}\nu(s)\lambda(s)ds$, so

$$V(x,t)=\lambda(t)x+\int_t^h e^{-\beta(s-t)}\eta(s)\lambda(s)ds \tag{5.14}$$

Since $A(t)=\lambda(t)\geq 0$ for all $t\in[0,h]$,the maximum of the right hand side of the HJB equation is indeed given by $u^*=\frac{-\lambda(t)\sqrt{x(t)}}{\alpha}$,so all conditions of Theorem 2.1 are satisfied and u is the solution of the OCP given by (5.1) and (5.2).

5.3 Example. Using Theorem 3.2, we can show that the following example is an OL-PDG.

$$L^1=-x-\tfrac{\alpha}{2}u_2+u_2, \quad \alpha>0 \text{ and, } L^2=-x+u_2$$
$$f=-u_1\sqrt{x}+u_2+1$$

5.4 Example. Using Theorem 3.5, we can show that the following example is an OL-PDG.

$$L^i=-x-\tfrac{1}{2}u_i^2, \quad i\in\mathcal{N}$$
$$f=-\sqrt{x}(\textstyle\sum_{i\in\mathcal{N}}u_i)$$

5.5 Example. Using Theorem 3.7, we can show that the following example is an OL-PDG.

$$L^i=x-K_i(u_i), \quad S^i=W_i(x), \; i=1,2$$
$$f=u_1+u_2-\alpha x, \; \alpha>0$$

where, for each $i=1,2$,K_i is a convex function, and W_i a concave function.

5.6 Example. Using Theorem 4.4, setting $m_1=m$ and $m_2=0$, Eqs. (4.24) and (4.25) reduce to the conditions of Theorem 3.2 of [30]. At the same time, Theorem 3.2 has Theorem 4.1, [30] as a consequence, which leads to the applications 6.1 and 6.2 of [30].

Setting $l_2=0, m_1=m$, and $m_2=0$, the nonlinear controlled dynamical system given by Eqs. (4.24) and (4.25) reduces to

$$\dot{x}(t)=f(x(t),u(t)), \quad x(0)=x_0, \quad t\geq 0$$

In this case, the conditions of Theorem 4.5 reduce to the conditions of Theorem 3.1 of [3].

5.7 Example. Now we provide a numerical example to highlight the theoretical framework developed in the section "Stability of Zero-Sum Games". Consider the equations of motion of a spacecraft given by [46]

$$\dot{w}_1(t) = I_{23}w_2(t) + u_1(t), \quad w_1(0) = w_{10}, \tag{5.15}$$

$$\dot{w}_2(t) = I_{31}w_3(t)w_1(t) + u_2(t), \quad w_2(0) = w_{20}, \tag{5.16}$$

$$\dot{w}_3(t) = I_{12}w_1(t)w_2(t), \quad w_3(0) = w_{30}. \tag{5.17}$$

Where $I_{23} = \frac{I_2-I_3}{I_1}$,$I_{31} = \frac{I_3-I_1}{I_2}$,$I_{12} = \frac{I_1-I_2}{I_3}$, and I_1, I_2, I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 < I_2 < I_3$, $[w_1, w_2, w_3]' : [0,\infty) \to \mathbb{R}^3$ denote the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and $u_1 : [0,\infty) \to \mathbb{R}$ and $u_2 : [0,\infty) \to \mathbb{R}$ are the spacecraft control moments.For this example we proof that two given state feedback controllers $u_1 = \phi(x_1, x_2))$, $u_2 = \psi(x_1, x_2))$,$x_1 = [w_1, w_2]'$, $x_2 = w_3$, guarantee that the dynamical system (5.15)-(5.17) is globally asymptotically stable with respect to x_1 uniformly and (4.45) holds with

$$J(u_1(\cdot), u_2(\cdot)) = \int_0^\infty [I_{31}^2 w_1^2(t) - I_{23}^2 w_2^2(t) + 4I_{23}w_2(t)u_2(t) + u_1^2(t) - u_2^2(t)]dt \tag{5.18}$$

where $x_{10} = [w_{10}, w_{20}]'$a and $x_{20} = w_{30}$, satisfies $(x_{10}, x_{20}) \in \mathbb{R}^2 \times \mathbb{R}^1$

Note that Eqs. (5.15)-(5.17) with objective functional (5.18) is in the form of Eqs.(4.24) and (4.25) and objective functional (4.36). In this case Theorem 4.5 can be applied with $l_1 = 2$,$l_2 = m_1 = m_2 = 1$,$f(x_1, x_2, u_1, x_2) = K(x_1, x_2) + u_1G_1(x_1, x_2) + u_2G_2(x_1, x_2)$, where

$K(x_1, x_2) = [I_{23}w_3w_2, I_{31}w_3w_1, I_{12}w_3w_1]'$,$G_1(x_1, x_2) = [1, 0, 0]'$, $G_2(x_1, x_2) = [0, 1, 0]'$

Let

$$V(w_1, w_2, w_3) = V(x_1, x_2) = x_1' \begin{pmatrix} I_{31} & 0 \\ 0 & -I_{23} \end{pmatrix} x_1 = I_{31}w_1^2 - I_{23}w_2^2, \quad (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^1 \tag{5.19}$$

$$\phi(w_1, w_2, w_3) = \phi(x_1, x_2) = -\frac{1}{2}[G_1(x_1, x_2)'dV(x_1, x_2)] = -I_{31}w_1, \tag{5.20}$$

$$\psi(w_1, w_2, w_3) = \psi(x_1, x_2) = \frac{1}{2}[G_2(x_1, x_2)'dV(x_1, x_2) + 4I_{23}w_2] = I_{23}w_2, \text{ and} \tag{5.21}$$

(5.22) $$\alpha(\|x_1\|) = \alpha_1\|x_1\|^2, \eta(\|x_1\|) = \eta_1\|x_1\|^2, \theta(\|x_1\|) = 2\alpha_1^2\|x_1\|^2$$

where $\alpha_1 = \min\{I_{31}, -I_{23}\}$, $\eta_1 = \max\{I_{31}, -I_{23}\}$.

Since $\alpha_1 w_1^2 \leq I_{31}w_1^2 \leq \eta_1 w_1^2$ and $\alpha_1 w_2^2 \leq -I_{23}w_2^2 \leq \eta_1 w_2^2$ we have

(5.23) $$\alpha(\|x_1\|) = \alpha_1 w_1^2 + \alpha_1 w_2^2 \leq I_{31}w_1^2 - I_{23}w_2^2 \leq \eta_1 w_1^2 + \eta_1 w_2^2 = \eta(\|x_2\|)$$

From (5.19), we obtain $dV = (2I_{23}w_1, -2I_{23}w_2, 0)'$, so doing some calculations we have

(5.24) $$dV'f = (2I_{23}w_1, -2I_{23}w_2, 0)(I_{23}w_3w_2 - I_{23}w_1, I_{31}w_3w_1 + I_{23}w_2, I_{12}w_3w_1)' =$$

$$= -2I_{31}^2w_1^2 - 2I_{23}^2w_2^2 \leq -2\alpha^2\|x_1\|^2$$

In addition

(5.25) $$\phi(0, x_2) = -\frac{1}{2}[G_1(0, x_2)'dV(0, x_2)] = -I_{31}0 = 0 = I_{23}0$$

$$= \tfrac{1}{2}[G_2(0, x_2)'dV(0, x_2) + 4I_{23}0] = \psi(0, x_2)$$

Note that (5.23)-(5.25) imply conditions (4.37)-(4.40) of Theorem 4.4, furthermore observe that $u_1 = \phi(x_1, x_2), u_2 = \psi(x_1, x_2)$ are solution of

(5.26) $$\frac{d}{du_1}[L(x_1, x_2, u_1, u_2) + dV(x_1, x_2)'K(x_1, x_2) + dV(x_1, x_2)'G_1(x_1, x_2)u_1$$
$$+dV(x_1, x_2)'G_2(x_1, x_2)u_2] = 0$$

(5.27) $$\frac{d}{du_2}[L(x_1, x_2, u_1, u_2) + dV(x_1, x_2)'K(x_1, x_2) + dV(x_1, x_2)'G_1(x_1, x_2)u_1$$
$$+dV(x_1, x_2)'G_2(x_1, x_2)u_2] = 0$$

It follows

(5.28)
$$[\phi(x_1, x_2)', \psi(x_1, x_2)']' \in \text{argminmax}_{(u_1,u_2)\in S_\psi(x_{10},x_{20})\times S_\phi(x_{10},x_{20})} H(x_1, x_2, dV(x_1, x_2), u_1, u_2)$$

Now observe that

(5.29) $$L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) + dV(x_1, x_2)'f(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) =$$

$$= I^2{}_{31}w^2{}_1 - I^2{}_{23}w^2{}_2 + 4I_{23}w_2I_{23}w_2 + I_{31}^2w_1^2 - I_{23}^2w_2^2 - 2I_{31}^2w_1^2 - 2I_{23}^2w_2^2 = 0$$

And (5.29) implies (4.41) of Theorem 4.5 .Lastly, from (5.27) we have

(5.30) $$L(x_1, x_2, u_1, \psi(x_1, x_2)) + dV(x_1, x_2)'f(x_1, x_2, u_1, \psi(x_1, x_2)) =$$

$L(x_1, x_2, u_1, \psi(x_1, x_2)) + dV(x_1, x_2)'f(x_1, x_2, u_1, \psi(x_1, x_2)) - [L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) + dV(x_1, x_2)'f(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2))] \geq 0$, and

(5.31) $$L(x_1, x_2, \phi(x_1, x_2), u_2) + dV(x_1, x_2)'f(x_1, x_2, \phi(x_1, x_2), u_2) =$$

$L(x_1, x_2, \phi(x_1, x_2), u_2) + dV(x_1, x_2)' f(x_1, x_2, \phi(x_1, x_2), u_2) - [L(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2)) + dV(x_1, x_2)' f(x_1, x_2, \phi(x_1, x_2), \psi(x_1, x_2))] \leq 0,$

where (5.30) and (5.31) imply (4.42) and (4.43) of Theorem 4.5. Then Theorem 4.5 holds and ϕ,ψ guarantee globally asymptotic stability with respect to x_1 uniformly in x_{20} of the system given by (5.15), (5.16) and (5.17). Moreover

$$J(\phi, \psi) = I_{31} w_1^2 - I_{23} w_2^2, \;\; \forall \, (x_{10}, x_{20}) \in \mathbb{R}^2 \times \mathbb{R}^1.$$

Conclusions.

The main goal of this work is to present a brief survey on the *stabilization problem* for noncooperative differential games, that is, the problem of finding Nash equilibria that, in addition, stabilize the game's state process in some sense (usually, but not necessarily, in the sense of Lyapunov). To this end, and to simplify the presentation, we concentrate on *potential differential games*, for which there is an optimal control problem (OCP) whose optimal solutions are Nash equilibria for the original game. Accordingly, our work includes presentations on potential games, and the two main techniques to analyze OCPs, namely, dynamic programming and the maximum principle. Under additional conditions we can also obtain stability results. These results are illustrated by means of some interesting applications.

Bibliography and references

[1] Arrow K, Kurz M(1970)Public Investment,The Rate of Return ,and Optimal Fisical Policy.The Johns Hopkins Press,Baltimore

[2] Başar T, Olsder GJ (1995) Dynamic Noncooperative Game Theory, 2nd edn. Academic Press, London

[3] Bernstein DS(1993) Nonquadratic cost and nonlinear feedback control, Int. J. Robust nonlinear control, 3 (3): 211–229.

[4] Brock WA,Malliaris AG (1989) Differential Equations Stability and Chaos in Dynamics Economics.Elsevier Science B.V.,Amsterdam, The Netherlands

[5] Brock WA , Scheinkman JA (1976) Global asymptotic stability of optimal control systems with applications to the theory of economic growth.Journal of Economic Theory, 12:164-190.

[6] Carlson DA, Haurie A (1991) Infinite Horizon Optimal Control.Theory and Applications.Springer-Verlag, Berlin

[7] Cass D , Shell K (1976a) The structure and stability of competitive dynamical systems.Journal of Economic Theory, 12:31-70.

[8] Chiang AC (1992) Elements of Dynamic Optimization.McGraw-Hill,New York

[9] Clarke FH (2013) Functional Analysis,Calculus of Variations and Optimal Control. Springer, Berlin

[10] Clarke FH (1983) Optimization and Nonsmooth Analysis.John Wiley and Sons, Inc., New York

[11] Dmitruk AV, Kuz'kina NV (2005) Existence theorem in the optimal control problem on an infinite time interval. Mathematical Notes 78, 466-480

[12] Dockner EJ, Jørgensen S, Long NV , Sorger G (2000) Differential Games in Economics and Management Science. Cambridge University Press, Cambridge

[13] Dragone D, Lambertini L, Leitmann G, Palestini A (2009) Hamiltonian potential functions for differential games. IFAC Proc 42: 1–8

[14] Dragone D, Lambertini L, Leitmann G, Palestini A (2015) Hamiltonian potential functions for differential games. Automatica 62: 134–138

[15] Fonseca-Morales A , Hernández-Lerma O (2018) Potential differential games. Dynamic Games and Applications : 254–279

[16] Friedman A (1971) Differential Games. Wiley, New York

[17] González-Sánchez D, Hernández-Lerma O (2016) A survey of static and dynamic potential games. Sci China Math 59:2075–2102

[18] Gaitsgory V,Grune L, Thatcher N(2015)Stabilization with discounted optimal control.Syst. Contr. Lett., 82:91–98

[19] Gaitsgory V, Quincampoix M (2009)Linear programming approach to deterministic infinite horizon optimal control problems with discounting. SIAM J. Control Optim. 48 .

[20] Gopalakrishnan R, Marden JR, Wierman A (2014) Potential games are necessary to ensure pure Nash equilibria in cost sharing games. Math Oper Res 39: 1252–1296

[21] Haddad WM, Chellaboina V (2008) Nonlinear dynamical systems and control: A Lyapunov-based approach. Princeton University Press, Princeton, New Jersey

[22] Hartman P (1964) Ordinary Differential Equations.John Wiley and Sons, New York

[23] Rilwan J,Kuman P,Hernández-Lerma O ()Stabilization of capital accumulation games

[24] Johnson A (2009) Numerical Solution Methods for Differential Game Problems(Master of Science in Aeronautics and Astronautics).Massachusetts Institute of Technology,Massachusetts

[25] Jorgensen S, Zaccour G (2012) Differential games in marketing, vol 15. Springer, Berlin

[26] Kirk DE (1970) Optimal Control Theory ,An Introduction.Dover Publications Inc., New York.

[27] L'Afflitto A (2017) Differential games,continuous Lyapunov functions,and stabilisation of non-linear dynamical systems.IET ,Control Theory Applications, Volume: 11 :2486 - 2496.

[28] L'Afflitto A (2016) Differential games, asymptotic stabilization, and robust optimal control of nonlinear systems. IEEE Conf. Decision and Control, vol. 2: 1933–1938

[29] L'Afflitto A (2017) Differential games, partial-state stabilization, and model reference adaptive control. J. Franklin Inst., 354(1):456–478

[30] L'Afflitto A , Haddad WM, Bakolas E (2016) Partial-state stabilization and optimal feedback control. Int. J. Robust Nonlinear Control 26 (5) : 1026–1050.

[31] La QD, Chew YH, Soong BH (2016) Potential game theory: applications in radio resource allocation.Springer, Berlin

[32] Liberzon D (2012) Calculus of Variations and Optimal Control Theory.A Consice Introduction.Princeton University Press,New Jersey

[33] Mallozzi L. An application of optimization theory to the study of equilibria for games: A survey. Cent Eur J Oper Res, 2013, 21: 523–539

[34] Mangasarian OL(1969)Nonlinear Programing.McGraw-Hill,New York.

[35] Mehlmann A (1988) Applied Differential Games.Springer Science+Business Media, New York

[36] Monderer D, Shapley LS (1996) Potential games. Game Econ Behav 14: 124–143

[37] Mou L, Yong J (2007) A variational formula for stochastic controls and some applications. Pure Appl Math Q 3:539–567

[38] Pontryagin LS, Boltyanskij VG, Gamkrelidze RV, and Mishchenko EF(1961),The Mathematical Theory of Optimal Processes.Fizmatgiz, Moscow

[39] Rockafellar RT (1976) Saddle points of Hamiltonian systems in convex Lagrange problems having a nonzero discount rate.Journal of Economic Theory 12:71-113

[40] Coleman R (2012)Calculus on Normed Vector Spaces.Springer-Verlag,New York

[41] Rosenthal RW (1973) A class of games possessing pure-strategy Nash equilibria. Int J Game Theory 2:65–67

[42] Sundaram RK (1996) A First Course in Optimization Theory. Cambridge University Press, Cambridge

[43] Sydsaeter K, Hammond PJ (1995) Mathematics for Economic Analysis. Prentice Hall, Upper Saddle River, New Jersey

[44] Sydsaeter K, Seierstad A (1987) Optimal Control Theory with Economic Applications.Elsevier Science B.V.,North-Holland, Amsterdam

[45] Tauchnitz N (2015) The Pontryagin maximum principle for nonlinear optimal control problems with infinite horizon. J Optim Theory Appl 167:27–48

[46] Wie B (1998) Space Vehicle Dynamics and Control.American Institute of Aeronautics and Astronautics, Reston, VA

[47] Yong J, Zhou XY (1999) Stochastic Controls,Hamiltonian Systems and HJB Equations, vol 43. Springer, Berlin