Valores propios de matrices tridiagonales hermitianas de Toeplitz con ciertas perturbaciones en las esquinas

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Eigenvalues of tridiagonal Hermitian Toeplitz matrices with certain perturbations in the corners

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Abstract

In this work we analyze the individual behavior of the eigenvalues of two families of large tridiagonal Hermitian Toeplitz matrices with certain perturbations in the corners.

The first family are $n \times n$ Hermitian Toeplitz matrices with entries 2, -1, 0, …, 0, $-\alpha$ in the first column. We prove that, if $|\alpha| \leq 1$, then the eigenvalues belong to $[0, 4]$ and are asymptotically distributed as the function $g(x) = 4\sin^2(x/2)$ on $[0, \pi]$. The situation changes drastically when $|\alpha| > 1$ and n tends to infinity; we prove that, in this case, the two extreme eigenvalues (the minimal and the maximal one) lay out of $[0, 4]$ and converge exponentially to certain limits determined by the value of α, whilst all others belong to $[0, 4]$ and are asymptotically distributed as g.

The second family are $n \times n$ Laplacian matrices of the cyclic graph with one weighted edge, whose weight we denote by α. We prove that, if $0 \leq \alpha \leq 1$, then the eigenvalues belong to $[0, 4]$ and are asymptotically distributed as the function $g(x) = 4\sin^2(x/2)$ on $[0, \pi]$. For the case $\alpha < 0$, the extreme minimal eigenvalue lies out $[0, 4]$ and converge exponentially to certain limit determined by the value α, whilst all others belong to $[0, 4]$ and are asymptotically distributed as g. In the case $\alpha > 1$, the outlier is the extreme maximal eigenvalue.

In the analysis of both matrix families, we localize the eigenvalues in disjoint intervals, then transform the characteristic equation to a form convenient to solve by numerical methods, and derive asymptotic formulas for the eigenvalues, as n tends to infinity.

Resumen

En este trabajo analizamos el comportamiento individual de los valores propios de dos familias de matrices grandes hermitianas tridiagonales de Toeplitz con ciertas perturbaciones en las esquinas.

La primer familia son las matrices hermitianas de Toeplitz de tamaño $n \times n$ con componentes 2, -1, 0, …, 0, $-\alpha$ en la primera columna. Demostramos que, si $|\alpha| < 1$, entonces los valores propios pertenecen a $[0, 4]$ y están asintóticamente distribuidos como la función $g(x) = 4 \sin^2(x/2)$ sobre $[0, \pi]$. La situación cambia drásticamente cuando $|\alpha| > 1$ y n tiende a infinito; para este caso probamos que, los dos valores propios extremos (el mínimo y el máximo) se encuentran fuera de $[0, 4]$ y convergen exponencialmente a ciertos límites determinados por el valor de α, mientras que todos los demás pertenecen a $[0, 4]$ y se encuentran asintóticamente distribuidos como la función g.

La segunda familia son las matrices laplacianas de tamaño $n \times n$ del grafo cíclico con peso en un vértice, dicho peso se denota por α. Demostramos que, si $0 \leq \alpha \leq 1$, entonces los valores propios pertenecen a $[0, 4]$ y están asintóticamente distribuidos como la función $g(x) = 4 \sin^2(x/2)$ sobre $[0, \pi]$. Para el caso $\alpha < 0$, el valor propio mínimo se encuentra afuera de $[0, 4]$ y converge exponencialmente a cierto límite determinado por el valor de α, mientras que todos los otros pertenecen a $[0, 4]$ y están asintóticamente distribuidos como la función g. En el caso $\alpha > 1$, el valor propio máximo es el aislado.

En el análisis de ambas familias de matrices, localizamos los valores propios en intervalos disjuntos, luego transformamos la ecuación característica a una forma conveniente para resolver por medio de métodos numéricos, y derivamos fórmulas asintóticas para los valores propios, cuando n tiende a infinito.

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Contents

Introduction

Objects of study and historical background

In this work we investigate the individual behaviour of the eigenvalues of $n \times n$ matrices $A_{\alpha,n}$ and $L_{\alpha,n}$ of the following form:

$$A_{\alpha,6} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad L_{\alpha,6} = \begin{bmatrix} 1+\overline{\alpha} & -1 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix},$$

notice these matrices depend on the complex parameter α.

Matrices of the form $A_{\alpha,n}$ may appear in the study of one-dimensional shift-invariant models on a finite interval, with some special interactions between the extremes of the interval. This matrix family belongs simultaneously to three matrix classes:

· periodic Jacobi matrices (in this case, with constant diagonals);

· Hermitian Toeplitz matrices; notice that the generating symbol of $A_{\alpha,n}$ is the following Laurent polynomial depending on the parameters α and n:

$$-\overline{\alpha}t^{-n+1} - t^{-1} + 2 - t - \alpha t^{n-1}; \tag{1}$$

· tridiagonal Toeplitz matrices, with values $-1, 2, -1$ on the diagonals and perturbations in the off-diagonal corners $(n, 1)$ and $(1, n)$.

If α is real, then the matrices $L_{\alpha,n}$ are the Laplacian matrices of the cyclic graph with one weighted edge. These matrices may appear in the study of electric flow, network dynamics, and many other physical phenomena. This matrix family belongs simultaneously to the matrix classes of tridiagonal Toeplitz matrices with values $-1,\ 2,\ -1$ on the diagonals and perturbations in the four corners, and periodic Jacobi matrices.

Periodic Jacobi matrices were studied by Ferguson, da Fonseca, and other authors [15, 16]. In particular, they found formulas for the characteristic polynomial and analyzed the inverse eigenvalue problem.

Laplacian matrices are matrix representation of graphs; general theory of this subject is explained in [24]. Recently, the characteristic polynomial, spectrum localization, and eigenvalues of particular cases of Laplacian matrices have been studied by Arenas Velilla and Rocha Hernández in their M.Sc. thesis [2, 27].

Toeplitz matrices are naturally associated to discrete truncated shift-invariant models. The general theory of such matrices is explained in the books and reviews [8, 9, 10, 14, 20, 21]. Explicit formulas for the determinants of banded symmetric Toeplitz matrices were found in [30]. The determinants, minors, cofactors, and components of eigenvectors of banded Toeplitz matrices were recently expressed in terms of skew Schur polynomials, see [1, 22]. These formulas are not needed in the present work, but may be useful in the study of pentadiagonal Toeplitz matrices with perturbations on the corners.

The individual behavior of the eigenvalues of Hermitian Toeplitz matrices was investigated in [3, 5, 6, 7, 13].

Determinants of non-singular Toeplitz matrices with low-rank perturbations were studied in [8]. The eigenvalues and eigenvectors of tridiagonal Toeplitz matrices with some special perturbations on the diagonal corners are computed in [11, Section 1.1] and [17, 26]. The determinants and inverses of a family of non-symmetric tridiagonal Toeplitz matrices with perturbed corners are computed in [32].

The theorem of Szegő has many implications on the localization and asymptotic distribution of eigenvalues of Toeplitz matrices, see [19, 29, 31, 33] for results in this direction; although some of these works apply to this thesis, we obtain, by elementary but not easy computations, the localization and asymptotic distribution of the eigenvalues.

The localization of the eigenvalues of a family of non-Hermitian Jacobi matrices (which can be viewed as a family of tridiagonal Toeplitz matrices with perturbation in the position $(n, n-1)$) was studied in [18].

Yueh and Cheng [34] considered the tridiagonal Toeplitz matrices with four perturbed corners. Using the techniques of finite differences they derived the characteristic equation into a trigonometric form and formulas for the eigenvectors in terms of the eigenvalues. Unlike the present work, [34] deals with arbitrary complex coefficients, but does not contain the analysis of the localization of the eigenvalues nor approximate formulas for the eigenvalues.

For $\alpha = 0$, the matrix $A_{\alpha,n}$ is the well studied tridiagonal Toeplitz matrix with the symbol

$$g(x) \coloneqq 4\sin^2(x/2).$$

The characteristic polynomial of $A_{0,n}$ is $\det(\lambda I_n - A_{0,n}) = U_n((\lambda - 2)/2)$, where U_n is the nth Chebyshev polynomial of the second type. Directly from the trigonometric form of the Chebyshev polynomials we obtain that the eigenvalues of $A_{0,n}$ are $g(j\pi/(n+1))$, $1 \le j \le n$.

Objectives of the thesis

The objective of this work is the individual study of eigenvalues of matrices $A_{\alpha,n}$ and $L_{\alpha,n}$, including the localization of eigenvalues in disjoint intervals, an equation derived from the characteristic equation comfortable to solve by numerical methods, and asymptotic expansions for the eigenvalues. Most of the results in this direction seems to be new, although they are based in well-known ideas.

In a certain way, in this thesis we try to generalize for $A_{\alpha,n}$ and $L_{\alpha,n}$ the procedure described above for $A_{0,n}$.

Structure of the thesis

In Chapter 1 we study the characteristic polynomial and the eigenvectors of tridiagonal symmetric Toeplitz matrices with values -1, 2, -1 on the diagonals and arbitrary perturbations in the corners $(1,1)$, $(n,1)$, $(1,n)$ and (n,n). We denote these matrices by S_n. We first review some properties of the Chebyshev polynomials, then expand by cofactors $\det(\lambda I_n - S_n)$ and obtain an expression of the characteristic polynomial in terms of Chebyshev polynomials (Proposition 1.13). Using standard methods for solving linear recurrences we find in terms of Chebyshev polynomials formulas for the eigenvectors (Proposition 1.16). This approach is different and relatively simpler than in [34], where the authors used techniques of the ring of sequences [12]. The characteristic polynomial of S_n can also be found with methods from [16]. We conclude this chapter by proving in Proposition 1.21 that, given a tridiagonal Toeplitz matrix with arbitrary values on the diagonals and arbitrary perturbations in the four corners, then this matrix is similar to a matrix of the form S_n. Due to Proposition 1.21, there is no loss of generality in studying only the eigenvalues of matrices S_n.

Recall that the eigenvalues of the tridiagonal Toeplitz matrices $A_{0,n}$ belong to the interval $[0, 4]$. The same holds for $A_{\alpha,n}$ and $L_{\alpha,n}$ for some values of α; in this situation we speak about "weak perturbations". For other values of α and n large enough, some of the eigenvalues do not belong to $[0, 4]$, and we refer this case as "strong perturbations".

In Chapter 2 we study the individual behaviour of the eigenvalues of matrices $A_{\alpha,n}$. By applying the general formula for the characteristic polynomial (Proposition 1.13), we write the characteristic polynomial of $A_{\alpha,n}$ in terms of Chebyshev polynomials of the second kind (Proposition 2.1). We find that, if $|\alpha| < 1$ ("weak perturbation"), then the eigenvalues belong to the interval $[0, 4]$ and are asmyptotically distributed as the function g (Theorem 2.6). If $1 < |\alpha|$ ("strong perturbation") and n is sufficiently big, the first and last eigenvalues lay out of $[0, 4]$, while the intermediate eigenvalues behave as in the case $|\alpha| < 1$ (Theorem 2.7). Then we apply the trigonometric and hyperbolic representation of the Chebyshev polynomials to the characteristic equation, transforming it to a form that can be solved by the fixed point iteration (Theorems 2.11 and 2.18). This leads to the main result of the chapter, the asymptotic expansions of the eigenvalues (theorems 2.15 and 2.22). The ideas in the asymptotic expansions in this and in next chapter are based on the paper [3]. We have found explicit formulas for the eigenvalues of $A_{\alpha,n}$ only for $|\alpha| = 1$.

In Chapter 3 we study the individual behaviour of the eigenvalues of matrices $L_{\alpha,n}$. By applying the Proposition 1.13, we write the characteristic polynomial of $L_{\alpha,n}$ in terms of Chebyshev polynomials (Proposition 3.1). We realized that the characteristic polynomial of $L_{\alpha,n}$ does not depend on the imaginary part of α; this is the reason why we consider $\mathrm{Im}(\alpha) = 0$ troughout the chapter. The analysis of the characteristic equation threw that, if $0 < \alpha < 1$ ("weak perturbation"), then the eigenvalues belong to the interval $[0, 4]$ and are asymptotically distributed as the function g (Theorem 3.7). If $\alpha < 0$ ("left strong perturbation") or $1 < \alpha$ ("strong perturbation") and n is sufficiently big, then the first or, respectively, the last eigenvalue lay out of $[0, 4]$, whilst the $n - 1$ remaining eigenvalues behave as in the case of weak perturbations (Theorems 3.8 and 3.9). Similarly to Chapter 2, we transform the characteristic equation to a form that can be solved by the fixed point iteration (Theorems 3.16, 3.23 and 3.28), and obtain asymptotic expansions for the eigenvalues of $L_{\alpha,n}$ (Theorems 3.20, 3.26 and 3.34). We have found explicitly the eigenvalues of $L_{\alpha,n}$ only for $\alpha = 0$ and $\alpha = 1$.

In Chapters 2 and 3 we prove that the eigenvalues of $A_{\alpha,n}$ and $L_{\alpha,n}$ can be obtained by solving some equation via the fixed iteration point, or by computing their asymptotic expansions. We programmed in Sagemath these numerical approximations; in the Appendices B and C we write the main sections of the programs.

Chapter 1

Chebyshev polynomials and tridiagonal Toeplitz matrices with corner perturbations

In this chapter we study some relations between Chebyshev polynomials and the characteristic equation and eigenvectors of tridiagonal Toeplitz matrices with corner perturbations.

In Section 1.1 we review the definition and some properties of Chebyshev polynomials.

In Section 1.2 we study the characteristic polynomials of tridiagonal Toeplitz matrices S_n with values -1, 2, -1 on the diagonals and arbitrary complex perturbations in the corners $(1, 1)$, $(n, 1)$, $(1, n)$ and (n, n). We express the characteristic equation in terms of Chebyshev polynomials.

In Section 1.3, with the aid of Chebyshev polynomials, we develop some formulas for the eigenvectors of matrices of the form S_n.

Finally in Section 1.4, we give a method of transforming an arbitrary tridiagonal Toeplitz matrix with corner perturbations to a matrix of the form S_n.

1.1 Chebyshev polynomials

The incoming definitions and propositions can be found in the book [23].

Definition 1.1 (Chebyshev polynomials of the first kind)**.** *The Chebyshev polynomials of* *the first kind T_n are defined by the following recurrence relation with two initial conditions:*

$$
\begin{aligned}
T_0(t) &:= 1, \\
T_1(t) &:= t, \\
T_n(t) &:= 2tT_{n-1}(t) - T_{n-2}(t) \qquad (n \geq 2).
\end{aligned}
$$

Definition 1.2 (Chebyshev polynomials of the second kind)**.** *The Chebyshev polynomials of the second kind U_n are defined by the following recurrence relation with two initial conditions:*

$$
\begin{aligned}
U_0(t) &:= 1, \\
U_1(t) &:= 2t, \\
U_n(t) &:= 2tU_{n-1}(t) - U_{n-2}(t) \qquad (n \geq 2).
\end{aligned}
$$

Definition 1.3 (Chebyshev polynomials of the third kind)**.** *The Chebyshev polynomials of the third kind V_n are defined by the following recurrence relation with two initial conditions:*

$$
\begin{aligned}
V_0(t) &:= 1, \\
V_1(t) &:= 2t - 1, \\
V_n(t) &:= 2tV_{n-1}(t) - V_{n-2}(t) \qquad (n \geq 2).
\end{aligned}
$$

Definition 1.4 (Chebyshev polynomials of the fourth kind)**.** *The Chebyshev polynomials W_n of the fourth kind are defined by the following recurrence relation with two initial conditions:*

$$
\begin{aligned}
W_0(t) &:= 1, \\
W_1(t) &:= 2t + 1, \\
W_n(t) &:= 2tW_{n-1}(t) - W_{n-2}(t) \qquad (n \geq 2).
\end{aligned}
$$

Proposition 1.5. *For every n in $\mathbb{N}_0$ and every x in $(0,\pi)$,*

$$\begin{aligned} T_n(\cos(x)) &= \cos(nx), & T_n(-\cos(x)) &= (-1)^n T_n(\cos(x)),\\ U_n(\cos(x)) &= \frac{\sin((n+1)x)}{\sin(x)}, & U_n(-\cos(x)) &= (-1)^n U_n(\cos(x)),\\ V_n(\cos(x)) &= \frac{\cos\left(\left(n+\frac{1}{2}\right)x\right)}{\cos\frac{x}{2}}, & V_n(-\cos(x)) &= (-1)^n W_n(\cos(x)),\\ W_n(\cos(x)) &= \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}}, & W_n(-\cos(x)) &= (-1)^n V_n(\cos(x)). \end{aligned}$$

Proof. We give a proof by induction for the assertion related to the Chebyshev polynomials of the second kind, the remaining can be demonstrated in similar way.

Fix x in $(0,\pi)$. For $n=0$,

$$\frac{\sin(x)}{\sin(x)} = 1 = U_0(\cos(x)).$$

For $n=1$,

$$\frac{\sin(2x)}{\sin(x)} = \frac{2\sin(x)\cos(x)}{\sin(x)} = 2\cos(x) = U_1(\cos(x)).$$

For $n=2$,

$$\begin{aligned} \frac{\sin(3x)}{\sin(x)} &= \frac{\sin(2x)\cos(x)+\cos(2x)\sin(x)}{\sin(x)}\\ &= 2\cos(x)2\cos(x) - 1. \end{aligned}$$

By Definition 1.2 this expression equals $U_2(\cos(x))$. Suppose the conclusion for $n-1$, then

$$\begin{aligned} \frac{\sin((n+1)x)}{\sin(x)} &= \frac{\sin(nx)\cos(x)+\sin(x)\cos(nx)}{\sin(x)}\\ &= 2\cos(x)\frac{\sin(nx)}{\sin(x)} + \frac{\sin(x)\cos(nx)-\sin(nx)\cos(x)}{\sin(x)}\\ &= 2\cos(x)\frac{\sin(nx)}{\sin(x)} - \frac{\sin((n-1)x)}{\sin(x)}\\ &= 2\cos(x)U_{n-1}(\cos(x)) - U_{n-2}(\cos(x)). \end{aligned}$$

The conclusion follows from the recurrence relation of Definition 1.2. $\square$

Corollary 1.6. *For every n in $\mathbb{N}_0$,*

$$\begin{aligned} T_n(1) &= 1, & T_n(-1) &= (-1)^n 1, \\ U_n(1) &= n+1, & U_n(-1) &= (-1)^n (n+1), \\ V_n(1) &= 1, & V_n(-1) &= (-1)^n (2n+1), \\ W_n(1) &= 2n+1, & W_n(-1) &= (-1)^n. \end{aligned}$$

By de Moivre's formula, the Proposition 1.5 is generalized to the complex plane, then the following propositions hold.

Proposition 1.7. *For every n in $\mathbb{N}_0$ and x in $\mathbb{R}$,*

$$\begin{aligned} T_n(\cosh(x)) &= \cosh(nx), & T_n(-\cosh(x)) &= (-1)^n T_n(\cosh(x)), \\ U_n(\cosh(x)) &= \frac{\sinh((n+1)x)}{\sinh(x)}, & U_n(-\cosh(x)) &= (-1)^n U_n(\cosh(x)), \\ V_n(\cosh(x)) &= \frac{\cosh\left(\left(n+\frac{1}{2}\right)x\right)}{\cosh\frac{x}{2}}, & V_n(-\cosh(x)) &= (-1)^n W_n(\cosh(x)), \\ W_n(\cosh(x)) &= \frac{\sinh\left(\left(n+\frac{1}{2}\right)x\right)}{\sinh\frac{x}{2}}, & W_n(-\cosh(x)) &= (-1)^n V_n(\cosh(x)). \end{aligned}$$

Proposition 1.8. *For every n in $\mathbb{N}_0$ and z in $\mathbb{C}$,*

$$\begin{aligned} T_n\left(\frac{z+z^{-1}}{2}\right) &= \frac{z^n+z^{-n}}{2}, \\ U_n\left(\frac{z+z^{-1}}{2}\right) &= \frac{z^{n+1}-z^{-n-1}}{z-z^{-1}}, \\ V_n\left(\frac{z^2+z^{-2}}{2}\right) &= \frac{z^{2n+1}+z^{-2n-1}}{z+z^{-1}}, \\ W_n\left(\frac{z^2+z^{-2}}{2}\right) &= \frac{z^{2n+1}-z^{-2n-1}}{z-z^{-1}}. \end{aligned}$$

We conclude the section listing some properties of the Chebyshev polynomials.

Proposition 1.9. *For every n in $\mathbb{N}_0$, the following statements hold.*

1) $U_n(-t) = (-1)^n U_n(t)$.

2) $2U_n(t)T_{n+1}(t) = U_{2n+1}(t)$.

3) $2T_n^2(t) = 1 + T_{2n}(t)$.

4) $2(1-t^2)U_{n-1}^2(t) = 1 - T_{2n}(t)$.

5) $U_{2n}(t) = tU_{2n-1}(t) + T_{2n}(t)$.

6) $U_{2n}(t) = 2tU_{n-1}(t)T_n(t) + 2T_n^2(t) - 1$.

7) $U_{2n}(t) = 2tU_{n-1}(t)T_n(t) + 1 - 2(1-t^2)U_{n-1}^2(t)$.

8) $W_n(t) = (-1)^n V_n(t)$.

9) $U_{2n}(t) = W_n(t)V_n(t)$.

10) $T_{2n+1}(t) = (1+t)V_n^2(t) - 1$.

11) $U_{2n+1}(t) = tW_n(t)V_n(t) + T_{2n+1}(t)$.

12) $U_{2n+1}(t) = tW_n(t)V_n(t) + (1+t)V_n^2(t) - 1$.

13) $tU_{n-1}(t) - T_n(t) = U_{n-2}(t)$.

14) $U_{n-1}(t)T_{n-1}(t) - T_n(t)U_{n-2}(t) = 1$.

15) $T_m(t)U_{n-m}(t) - T_{n+1-m}(t)U_{m-1}(t) = U_{n-2m}(t)$, $0 < 2m < n+1$.

1.2 Characteristic polynomial of S_n

A Toeplitz matrix is a $n \times n$ matrix of the form:

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{1.1}$$

that is to say, these matrices are characterized by the property of being constant along the parallels to the main diagonal.

Let $a\colon \mathbb{T} \to \mathbb{C}$ be the function defined by

$$a(t) \coloneqq \sum_{k=-n}^{n} a_n t^n \quad (t \in \mathbb{T}), \tag{1.2}$$

where the coefficients a_k's are the entries of the Toeplitz matrix (1.1). Then we can say that the Toeplitz matrix (1.1) is generated by a, we denoted this matrix by $M_n(a)$ and a is referred to be the symbol of the matrix.

In this thesis we are concern with corner perturbed tridiagonal Toeplitz matrices of order n, with values -1, 2 , -1 on the diagonals and complex numbers $-\alpha$, $-\beta$, $-\gamma$, $-\delta$, respectively in the corners $(1,1)$, $(1,n)$, $(n,1)$, (n,n). Denote by S_n the matrix defined by these rules. For example, if $n=6$,

$$S_6 \coloneqq \begin{bmatrix} -\alpha & -1 & 0 & 0 & 0 & -\beta \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\gamma & 0 & 0 & 0 & -1 & -\delta \end{bmatrix}. \tag{1.3}$$

If $\alpha = \delta = -2$ and $\beta = \gamma = 0$, we have the case of tridiagonal Toeplitz matrix, whose generating symbol is the function

$$a(\theta) \coloneqq -e^{-i\theta} + 2 - e^{i\theta} = 2 - 2\cos(\theta) \quad (\theta \in [0, 2\pi]). \tag{1.4}$$

Fix $n \geq 3$ and p, q in $\{1, \ldots, n\}$, let $E_{p,q}$ be the $n \times n$ matrix with 1 in the position (p, q) and 0 elsewhere. Observe that we can express the matrix S_n as follows

$$S_n = M_n(a) - (2+\alpha)E_{1,1} - \beta E_{1,n} - \gamma E_{n,1} - (2+\delta)E_{n,n}.$$

Denote by $D_n(\lambda)$ the characteristic polynomial of S_n, i.e., $D_n(\lambda) \coloneqq \det(\lambda I_n - S_n)$.

Recall that U_n is the Chebyshev polynomial of the second kind (Definition 1.2).

Proposition 1.10. *Let $n \geq 1$, then the characteristic polynomial of the Toeplitz matrix $M_n(a)$ generated by the symbol (1.4) is*

$$\det(\lambda I_n - M_n(a)) = U_n\left(\frac{\lambda - 2}{2}\right). \tag{1.5}$$

Proof. We prove (1.5) by induction.

Put $\lambda = 2t + 2$. For $n = 1$, $D_1(2t + 2) = 2t$, and by Definition 1.2 this equals $U_1(t)$. For $n = 2$, we easily obtain $D_2(2t + 2) = 4t^2 - 1$, and again by Definition 1.2 this equals $U_1(t)$. Let $n > 2$ and suppose (1.5) for all natural lesser than n. Expand D_n by cofactors along the first row

$$D_n(2t+2) = 2t\begin{vmatrix} 2t & 1 & 0 & 0 & \cdots \\ 1 & 2t & 1 & 0 & \cdots \\ 0 & 1 & 2t & 1 & \cdots \\ 0 & 0 & 1 & 2t & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n-1} - \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 2t & 1 & 0 & \cdots \\ 0 & 1 & 2t & 1 & \cdots \\ 0 & 0 & 1 & 2t & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n-1}.$$

The first determinant on the right side of last equality is D_{n-1}. Expand by cofactors the second determinant along the first column and get $D_{n-2}(2t+2)$. Then

$$D_n(2t+2) = 2tD_{n-1}(2t+2) - D_{n-2}(2t+2).$$

By the induction hypothesis $D_{n-1}(2t+2) = U_{n-1}(t)$ and $D_{n-2}(2t+2) = U_{n-2}(t)$; and so, by Definition 1.2 we get (1.5). □

Corollary 1.11. *Let $n \geq 1$. Then the eigenvalues of $M_n(a)$ are*

$$\lambda_j = 2 - 2\cos\frac{j\pi}{n+1} \qquad (1 \leq j \leq n). \tag{1.6}$$

Proof. From Proposition 1.10,

$$\det(\lambda I_n - M_n(a)) = U_n\left(\frac{\lambda - 2}{2}\right).$$

We do the change of variable $\lambda = 2 - 2\cos(x)$, $x \in [0, \pi]$. Then by Proposition 1.5,

$$\det((2 - 2\cos(x))I_n - M_n(a)) = U_n\left(-\cos(x)\right) = (-1)^{n+1}\frac{\sin((n+1)x)}{\sin(x)}.$$

Note that $x = 0$ and $x = \pi$ are not zeros of the last expression, and the only zeros on $[0, \pi]$ are the numbers $j\pi/(n+1)$, with $1 \leq j \leq n$. Hence, the correspondent numbers λ_j (1.6) are n different eigenvalues of $M_n(a)$. Since the order of the matrix $M_n(a)$ is n, we have found all the eigenvalues. □

Lemma 1.12. *Let $\alpha = -2$, $\beta = \gamma = 0$ and δ be an arbitrary complex number, and let*

$n \geq 2$. *Then*

$$D_n(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) + (2+\delta)U_{n-1}\left(\frac{\lambda-2}{2}\right). \tag{1.7}$$

Proof. Put $\lambda = 2 + 2t$. Expand $D_n(2+2t)$ by cofactors along the last row

$$D_n(2+2t) = (2+2t+\delta)\begin{vmatrix} 2t & 1 & 0 & 0 & \cdots \\ 1 & 2t & 1 & 0 & \cdots \\ 0 & 1 & 2t & 1 & \cdots \\ 0 & 0 & 1 & 2t & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n-1} - \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 2t & 1 & 0 & 0 \\ \cdots & 1 & 2t & 1 & 0 \\ \cdots & 0 & 1 & 2t & 0 \\ \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1}.$$

By (1.5) the first determinant on the right of last equality is $U_{n-1}(t)$. Expand by cofactors the second determinant along the last column

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 2t & 1 & 0 & 0 \\ \cdots & 1 & 2t & 1 & 0 \\ \cdots & 0 & 1 & 2t & 0 \\ \cdots & 0 & 0 & 1 & 1 \end{vmatrix}_{n-1} = 1 \times \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 2t & 1 & 0 \\ \cdots & 1 & 2t & 1 \\ \cdots & 0 & 1 & 2t \end{vmatrix}_{n-2}.$$

By (1.5) this is $U_{n-2}(t)$. Hence

$$D_n(2t+2) = (2+2t+\delta)U_{n-1}(t) - U_{n-2}(t).$$

In order to obtain (1.7) we just apply to the last expression the Definition 1.2. □

Proposition 1.13. *Let* α, β, γ *and* δ *in* $\mathbb{C}$, *and* $n \geq 3$. *Then*

$$\begin{aligned} D_n(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) &+ (4+\alpha+\delta)U_{n-1}\left(\frac{\lambda-2}{2}\right) \\ &+ (4+2(\alpha+\delta)+\alpha\delta-\beta\gamma)U_{n-2}\left(\frac{\lambda-2}{2}\right) + (-1)^{n+1}(\beta+\gamma). \end{aligned} \tag{1.8}$$

Proof. Put $\lambda = 2 + 2t$. Expand $D_n(2 + 2x)$ by cofactors along the first row

$$D_n(2+2t) = (2+2t+\alpha)\begin{vmatrix} 2t & 1 & \cdots & 0 & 0 \\ 1 & 2t & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2t & 1 \\ 0 & 0 & \cdots & 1 & 2+2t+\delta \end{vmatrix}_{n-1}$$

$$- \begin{vmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 2t & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2t & 1 \\ \gamma & 0 & \cdots & 1 & 2+2t+\delta \end{vmatrix}_{n-1} + (-1)^{n+1}\beta \begin{vmatrix} 1 & 2t & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 2t \\ \gamma & 0 & \cdots & 0 & 1 \end{vmatrix}_{n-1}.$$

In last expression apply (1.7) to the first determinant on the right, for the second and third determinants expand by cofactors along the first column, use (1.5) and (1.7), then

$$\begin{aligned} D_n(2+2t) = {} & (2+2t+\alpha)\left(U_{n-1}(t) + (2+\delta)U_{n-2}(t)\right) \\ & - \left(U_{n-2}(t) + (2+\delta)U_{n-3}(t)\right) + (-1)^{n+1}\gamma \\ & + (-1)^{n+1}\beta - \beta\gamma U_{n-2}(t). \end{aligned}$$

By the recurrence relation $2tU_{n-2}(t) - U_{n-3}(t) = U_{n-1}(t)$,

$$\begin{aligned} D_n(2+2t) = {} & (4+2t+\alpha+\delta)U_{n-1}(t) - U_{n-2}(t) \\ & + (4+2(\alpha+\delta)+\alpha\delta-\beta\gamma)U_{n-2}(t) + (-1)^{n+1}(\beta+\gamma). \end{aligned} \tag{1.9}$$

Finally, (1.8) follows by applying $2tU_{n-1}(t) - U_{n-2}(t) = U_n(t)$. □

Observe that, the expression for the determinant (1.8) has three Chebyshev polynomials of distinct grades, for some purposes the formula (1.9), written below in terms of λ

$$\begin{aligned} D_n(\lambda) = {} & (2+\lambda+\alpha+\delta)U_{n-1}\left(\frac{\lambda-2}{2}\right) \\ & + (4+2(\alpha+\delta)+\alpha\delta-\beta\gamma)U_{n-2}\left(\frac{\lambda-2}{2}\right) + (-1)^{n+1}(\beta+\gamma), \end{aligned}$$

could be more convenient, since it has only two Chebyshev polynomials.

1.3 Eigenvectors of S_n

Over all this section S_n is the $n \times n$ matrix defined in (1.3), where α, β, γ and δ are fixed complex numbers and $n \geq 3$. Also λ will denote an eigenvalue of S_n and $v = [v_k]_{k=1}^n$ the correspondent eigenvector.

Let z be a solution of $\omega^2 + (\lambda - 2)\omega + 1 = 0$, i.e., z satisfies

$$\lambda = -z^{-1} + 2 - z. \tag{1.10}$$

The aim of this section is to prove that the eigenvector v can be constructed by the rule

$$v_k = C_1 z^k + C_2 z^{-k} \qquad (1 \leq k \leq n), \tag{1.11}$$

where C_1 and C_2 are some complex constants.

From now on we suppose that λ satisfies (1.10) and that the components of v satisfies (1.11).

Denote $\lambda I_n - S_n$ by $B_{n,\lambda}$, then for every w in $\mathbb{C}^n$

$$\begin{aligned}(B_{n,\lambda}w)_1 &= (\lambda + \alpha)w_1 + w_2 + \beta w_n, \\ (B_{n,\lambda}w)_k &= w_{k-1} + (\lambda - 2)w_k + w_{k+1} \qquad (2 \leq k \leq n-1), \\ (B_{n,\lambda}w)_n &= \gamma w_1 + w_{n-1} + (\lambda + \delta)w_n.\end{aligned} \tag{1.12}$$

Note that for every eigenvector associate to λ, every equation in (1.12) equals zero.

Proposition 1.14. *Let C_1 and C_2 in $\mathbb{C}$ and $n \geq 3$. Then $(B_{n,\lambda}v)_k = 0$ for every $2 \leq k \leq n-1$, where the components of v are given by (1.11).*

Proof. By (1.11) and (1.12),

$$\begin{aligned}(B_{n,\lambda}v)_k &= v_{k-1} + (\lambda - 2)v_k + v_{k+1} \\ &= C_1 z^{k-1} + C_2 z^{-(k-1)} + C_1(\lambda - 2)z^k + C_2(\lambda - 2)z^{-k} + C_1 z^{k+1} + C_2 z^{-(k+1)} \\ &= \left(C_1 z^k + C_2 z^{-k}\right)\left(z^{-1} + (\lambda - 2) + z\right) \\ &= 0.\end{aligned}$$

□

Now, we proceed to determine C_1 and C_2 such that $(B_{n,\lambda}v)_1 = 0$ and $(B_{n,\lambda}v)_n = 0$. Denote $(C_1 + C_2)/2$ and $(C_1 - C_2)/2$ by x and y respectively, then $(B_{n,\lambda}v)_1 = 0$ and

$(B_{n,\lambda}v)_n = 0$ transform into

$$\begin{aligned}
&\big((\lambda+\alpha)(z+z^{-1}) + z^2 + z^{-2} + \beta(z^n + z^{-n})\big)\, x + \\
&\qquad + \big((\lambda+\alpha)(z-z^{-1}) + z^2 - z^{-2} + \beta(z^n - z^{-n})\big)\, y = 0, \\
&\big(\gamma(z+z^{-1}) + z^{n-1} + z^{-(n-1)} + (\lambda+\delta)(z^n + z^{-n})\big)\, x + \\
&\quad + \big(\gamma(z-z^{-1}) + z^{n-1} - z^{-(n-1)} + (\lambda+\delta)(z^n - z^{-n})\big)\, y = 0.
\end{aligned}$$

We use the representation of the Chebyshev polynomials given in Proposition 1.8 on previous equations, converting them in

$$\begin{aligned}
\mathrm{a}_{\alpha,\beta,n} x + \mathrm{b}_{\alpha,\beta,n} y = 0, \\
\mathrm{c}_{\gamma,\delta,n} x + \mathrm{d}_{\gamma,\delta,n} y = 0,
\end{aligned} \tag{1.13}$$

where

$$\begin{aligned}
\mathrm{a}_{\alpha,\beta,n} &\coloneqq (2+\alpha)(z+z^{-1}) - 2 + 2\beta T_n\left(\frac{z+z^{-1}}{2}\right), \\
\mathrm{b}_{\alpha,\beta,n} &\coloneqq (z-z^{-1})\left(2+\alpha+\beta U_{n-1}\left(\frac{z+z^{-1}}{2}\right)\right), \\
\mathrm{c}_{\gamma,\delta,n} &\coloneqq \gamma(z+z^{-1}) + 2(2+\delta)T_n\left(\frac{z+z^{-1}}{2}\right) - 2T_{n+1}\left(\frac{z+z^{-1}}{2}\right), \\
\mathrm{d}_{\gamma,\delta,n} &\coloneqq (z-z^{-1})\left(\gamma + (2+\delta)U_{n-1}\left(\frac{z+z^{-1}}{2}\right) - U_n\left(\frac{z+z^{-1}}{2}\right)\right).
\end{aligned} \tag{1.14}$$

Suppose (1.14) are not all zero simultaneously, then in order to obtain non trivial solutions of (1.13) we must have

$$\begin{vmatrix} \mathrm{a}_{\alpha,\beta,n} & \mathrm{b}_{\alpha,\beta,n} \\ \mathrm{c}_{\gamma,\delta,n} & \mathrm{d}_{\gamma,\delta,n} \end{vmatrix} = 0. \tag{1.15}$$

Recall that $\det(\lambda I_n - S_n)$ is denoted by $D_n(\lambda)$.

Lemma 1.15. *Let α, β, γ, $\delta \in \mathbb{C}$ and $n \geq 3$. Then*

$$(-1)^n \frac{\mathrm{a}_{\alpha,\beta,n}\mathrm{d}_{\gamma,\delta,n} - \mathrm{b}_{\alpha,\beta,n}\mathrm{c}_{\gamma,\delta,n}}{2(z-z^{-1})} = D_n(2-(z+z^{-1})). \tag{1.16}$$

Proof. By the statements of Proposition 1.9,

$$\frac{\mathrm{ad}-\mathrm{bc}}{2(z-z^{-1})} = U_n\left(\frac{z+z^{-1}}{2}\right) - (4+\alpha+\delta)U_{n-1}\left(\frac{z+z^{-1}}{2}\right) + (4+2(\alpha+\delta)+\alpha\delta-\beta\gamma)U_{n-2}\left(\frac{z+z^{-1}}{2}\right) - (\beta+\gamma).$$

From Proposition 1.9, for every m in $\mathbb{N}_0$, $U_m(-t) = (-1)^m U_m(t)$. Use this fact after substitute $z + z^{-1} = -(\lambda-2)/2$ in last equation. Then multiply by $(-1)^n$. The result now equals the characteristic polynomial (1.8). □

In next proposition we use the convention that $U_{-1}(t) \coloneqq 0$ and the fact that every eigenvalue of S_n can be written in the form (1.10).

Proposition 1.16. *Let α, β, γ, $\delta \in \mathbb{C}$, $n \geq 3$ and λ an eigenvalue of S_n. If $\mathrm{a}_{\alpha,\beta,n} \neq 0$ or $\mathrm{b}_{\alpha,\beta,n} \neq 0$, then the vector $v = [v_k]_{k=1}^n$ with components*

$$v_k \coloneqq U_{k-1}\left(\frac{2-\lambda}{2}\right) - (2+\alpha)U_{k-2}\left(\frac{2-\lambda}{2}\right) + \beta U_{n-k-1}\left(\frac{2-\lambda}{2}\right) \quad (1 \leq k \leq n) \tag{1.17}$$

is an eigenvector of S_n associated to λ. If $\mathrm{c}_{\gamma,\delta,n} \neq 0$ or $\mathrm{d}_{\gamma,\delta,n} \neq 0$, then the vector $v = [v_k]_{k=1}^n$ with components

$$v_k \coloneqq \gamma T_{k-1}\left(\frac{\lambda-2}{2}\right) + (2+\delta)T_{n-k}\left(\frac{\lambda-2}{2}\right) - T_{n+1-k}\left(\frac{\lambda-2}{2}\right) \quad (1 \leq k \leq n) \tag{1.18}$$

is an eigenvector of S_n associated to λ.

Proof. Since λ is an eigenvalue of S_n, then $D_n(\lambda) = 0$, moreover formula (1.10) holds. By Lemma 1.15 the equation (1.15) holds. So, if at least one of the coefficients (1.14) is not zero, then the system (1.13) has non trivial solutions for x and y. This implies that the components of v can be constructed by the formula (1.11) with $C_1 = x + y$ and $C_2 = x - y$.

If $\mathrm{a}_{\alpha,\beta} \neq 0$ or $\mathrm{b}_{\alpha,\beta} \neq 0$, put $x = \mathrm{b}_{\alpha,\beta}/2$ and $y = -\mathrm{a}_{\alpha,\beta}/2$, then

$$C_1 = \frac{\mathrm{b}_{\alpha,\beta,n} - \mathrm{a}_{\alpha,\beta,n}}{2} = 1 - (2+\alpha)z^{-1} - \beta z^{-n},$$
$$C_2 = \frac{\mathrm{b}_{\alpha,\beta,n} + \mathrm{a}_{\alpha,\beta,n}}{2} = -1 + (2+\alpha)z + \beta z^{n}.$$

Now, for every $1 \leq k \leq n$, (1.11) converts in

$$v'_k \coloneqq z^k - z^{-k} - (2+\alpha)(z^{k-1} - z^{-(k-1)}) + \beta(z^{n-k} - z^{-(n-k)}). \tag{1.19}$$

Hence the vector with components v'_k $(1 \leq k \leq n)$, turns to be an eigenvector associated to λ. Any constant multiple of v is again an eigenvector, so, divide (1.19) by $z - z^{-1}$ and from the definition of the Chebyshev polynomials of the second kind (Definition 1.2) this is (1.17).

If $\mathrm{c}_{\gamma,\delta,n} \neq 0$ or $\mathrm{d}_{\gamma,\delta,n} \neq 0$, put $x = \mathrm{c}_{\gamma,\delta,n}/2$ and $y = -\mathrm{d}_{\gamma,\delta,n}/2$- We obtain (1.18) by proceeding in the same manner as in the previous case.

If $\mathrm{a}_{\gamma,\delta,n} = \mathrm{b}_{\gamma,\delta,n} = \mathrm{c}_{\gamma,\delta,n} = \mathrm{d}_{\gamma,\delta,n} = 0$, then we can take any non zero value on x and any other number on y or vice versa, in other words, we can take any non zero value on C_1 and any other number on C_2 or vice versa, and so, construct an eigenvector associated to λ similar to (1.17) or (1.18). □

In next chapters we use Proposition 1.17 for describing the eigenvectors of the matrices we study in this thesis. For now, we apply this result for the tridiagonal Toeplitz matrix $M_n(a)$ generated by the symbol (1.4).

Corollary 1.17. *Let $n \geq 1$ and $M_n(a)$ be the Toeplitz matrix generated by the symbol (1.4). Then for every $1 \leq j \leq n$, the vector $v_j = [v_j^k]_{k=1}^n$ with components*

$$v_j^k = \sin\frac{kj\pi}{n+1} \quad (1 \leq k \leq n) \tag{1.20}$$

is an eigenvector associated to the eigenvalue λ_j given by equation (1.6).

Proof. The matrix $M_n(a)$ is a particular case of the family matrices of the form (1.3) with $\alpha = \delta = -2$ and $\beta = \gamma = 0$. Hence, we can apply Proposition 1.17.

By Corollary 1.11, every eigenvalue of $M_n(a)$ is of the form (1.6). So, for $1 \leq j \leq n$, the correspondent eigenvalue λ_j can be written as in (1.10), thus

$$\lambda_j = 2 - e^{ij\pi/(n+1)} - e^{-ij\pi/(n+1)}.$$

Hence

$$\mathrm{a}_{-2,0,n} = -4\cos^2\frac{j\pi}{n+1} + 2\cos\frac{2j\pi}{n+1} = -2 \neq 0.$$

So, by Proposition 1.16 for every k with $1 \leq k \leq n$ we can use the formula (1.17). Multiply (1.17) by $\sin(j\pi/(n+1))$,

$$v_k = 2\cos\frac{j\pi}{n}\sin\frac{(k-1)j\pi}{n} - \sin\frac{(k-2)j\pi}{n}.$$

Apply to v_k the trigonometric identities of the sum of the arguments, then (1.20) follows. □

1.4 Schmidt-Spitzer matrix transformation

In this section we show that, given an arbitrary tridiagonal Toeplitz matrix with arbitrary perturbations in corners, then it is possible to transform it to a matrix of the form (1.3). See [9] for the procedure of the transformation explained here. We start by showing that this is true for tridiagonal Toeplitz matrices without corner perturbations. Let $b_{-1}, b_0, b_1 \in \mathbb{C}$ and $n \geq 3$. Denote by $M_n(b)$ the $n \times n$ tridiagonal Toeplitz generated by the symbol

$$b(\vartheta) := b_{-1}e^{-i\vartheta} + b_0 + b_1 e^{i\vartheta}. \tag{1.21}$$

For example if $n = 4$

$$M_4(b) = \begin{bmatrix} b_0 & b_{-1} & 0 & 0 \\ b_1 & b_0 & b_{-1} & 0 \\ 0 & b_1 & b_0 & b_{-1} \\ 0 & 0 & b_1 & b_0 \end{bmatrix}. \tag{1.22}$$

For every $\rho \in \mathbb{C}^n$ define the $n \times n$ matrix $\operatorname{diag}(\rho)$ by the diagonal matrix with entries $(\operatorname{diag}(\rho))_{j,j} = \rho_j$ and $(\operatorname{diag}(\rho))_{j,k} = 0$ if $j \neq k$. For example if $n = 4$

$$\operatorname{diag}(\rho) := \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}. \tag{1.23}$$

Note that, if $\mathsf{\rho}_j \neq 0$ for every $j = 1, \ldots, n$, then $\operatorname{diag}(\mathsf{\rho})^{-1}$ exist, moreover, $(\operatorname{diag}(\mathsf{\rho})^{-1})_{j,j} = \mathsf{\rho}_j^{-1}$ and $(\operatorname{diag}(\mathsf{\rho})^{-1})_{j,k} = 0$ if $j \neq k$. For example if $n = 4$

$$\operatorname{diag}(\mathsf{\rho})^{-1} \coloneqq \begin{bmatrix} \mathsf{\rho}_1^{-1} & 0 & 0 & 0 \\ 0 & \mathsf{\rho}_2^{-1} & 0 & 0 \\ 0 & 0 & \mathsf{\rho}_3^{-1} & 0 \\ 0 & 0 & 0 & \mathsf{\rho}_4^{-1} \end{bmatrix}. \tag{1.24}$$

Proposition 1.18. *Let $n \geq 3$, $b_{-1}, b_0, b_1 \in \mathbb{C} \setminus \{0\}$ and $\rho \coloneqq \sqrt{b_{-1}/b_1}$. Then*

$$\operatorname{diag}(\mathsf{\rho}) M_n(b) \operatorname{diag}(\mathsf{\rho})^{-1} = \begin{bmatrix} b_0 & \sqrt{b_{-1}b_1} & 0 & \cdots \\ \sqrt{b_{-1}b_1} & b_0 & \sqrt{b_{-1}b_1} & \cdots \\ 0 & \sqrt{b_{-1}b_1} & b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{1.25}$$

where $\mathsf{\rho} \coloneqq [\rho, \rho^2, \ldots, \rho^n]$ and b is defined by (1.21).

Proof. We first compute $\operatorname{diag}(\mathsf{\rho}) M_n(b)$, thus

$$\begin{aligned} \operatorname{diag}(\mathsf{\rho}) M_n(b) &= \begin{bmatrix} \rho & 0 & 0 & \cdots \\ 0 & \rho^2 & 0 & \cdots \\ 0 & 0 & \rho^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} b_0 & b_{-1} & 0 & \cdots \\ b_1 & b_0 & b_{-1} & \cdots \\ 0 & b_1 & b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\ &= \begin{bmatrix} \rho b_0 & \rho b_{-1} & 0 & \cdots \\ \rho^2 b_1 & \rho^2 b_0 & \rho^2 b_{-1} & \cdots \\ 0 & \rho^3 b_1 & \rho^3 b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \end{aligned}$$

Now

$$\operatorname{diag}(\mathsf{\rho})M_n(b)\operatorname{diag}(\mathsf{\rho})^{-1} = \begin{bmatrix} \rho b_0 & \rho b_{-1} & 0 & \cdots \\ \rho^2 b_1 & \rho^2 b_0 & \rho^2 b_{-1} & \cdots \\ 0 & \rho^3 b_1 & \rho^3 b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \rho^{-1} & 0 & 0 & \cdots \\ 0 & \rho^{-2} & 0 & \cdots \\ 0 & 0 & \rho^{-3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$
$$= \begin{bmatrix} b_0 & \rho^{-1} b_{-1} & 0 & \cdots \\ \rho b_1 & b_0 & \rho^{-1} b_{-1} & \cdots \\ 0 & \rho b_1 & b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

By the assumption on ρ

$$\rho b_1 = \sqrt{\frac{b_{-1}}{b_1}} b_1 = \sqrt{b_{-1} b_1} = \sqrt{b_1/b_{-1}} b_{-1} = \rho^{-1} b_{-1}.$$

Then (1.25) follows. □

Proposition 1.19. *Let $n \geq 3$, and $M_n(a)$, $M_n(b)$ the Toeplitz matrices generated by (1.4) and (1.22), respectively. Then*

$$M_n(a) = -\frac{1}{\sqrt{b_1 b_{-1}}} \operatorname{diag}(\mathsf{\rho}) \left(M_n(b) - \left(2\sqrt{b_1 b_{-1}} + b_0 \right) I_n \right) \operatorname{diag}(\mathsf{\rho})^{-1}, \tag{1.26}$$

or equivalently

$$M_n(b) = \operatorname{diag}(\mathsf{\rho})^{-1} \left(-\sqrt{b_1 b_{-1}} M_n(a) + \left(2\sqrt{b_1 b_{-1}} + b_0 \right) I_n \right) \operatorname{diag}(\mathsf{\rho}), \tag{1.27}$$

where $\rho := \sqrt{b_{-1}/b_1}$ and $\mathsf{\rho} := [\rho, \rho^2, \ldots, \rho^n]$.

Proof. By the fact that $\operatorname{diag}(\mathsf{\rho})\operatorname{diag}(\mathsf{\rho})^{-1} = I_n$,

$$\begin{aligned}\operatorname{diag}(\mathsf{\rho}) \left(M_n(b) - \left(2\sqrt{b_1 b_{-1}} + b_0 \right) I_n \right) \operatorname{diag}(\mathsf{\rho})^{-1} &= \operatorname{diag}(\mathsf{\rho}) M_n(b) \operatorname{diag}(\mathsf{\rho})^{-1} \\ &\quad - \operatorname{diag}(\mathsf{\rho})(2\sqrt{b_1 b_{-1}} + b_0) I_n \operatorname{diag}(\mathsf{\rho})^{-1} \\ &= \operatorname{diag}(\mathsf{\rho}) M_n(b) \operatorname{diag}(\mathsf{\rho})^{-1} \\ &\quad - \left(2\sqrt{b_1 b_{-1}} + b_0 \right) I_n.\end{aligned}$$

Now, (1.26) follows by applying (1.25). □

Corollary 1.20. *Let $n \geq 3$, and $M_n(a)$, $M_n(b)$ the Toeplitz matrices generated by* (1.4) *and* (1.22)*, respectively. Then for every eigenvalue λ of $M_n(a)$, $\sqrt{b_1 b_{-1}}(2-\lambda) - b_0$ is eigenvalue of $M_n(b)$.*

Proof. The conclusion follows from (1.27). □

Now, we consider the $n \times n$ tridiagonal Toeplitz matrix M_n generated by the symbol (1.21) with complex numbers $-\phi$, $-\chi$, $-\psi$, $-\omega$, respectively in the corners $(1,1)$, $(1,n)$, $(n,1)$, (n,n). For example if $n = 6$

$$M_6 \coloneqq \begin{bmatrix} -\phi & b_1 & 0 & 0 & 0 & -\chi \\ b_1 & b_0 & b_{-1} & 0 & 0 & 0 \\ 0 & b_1 & b_0 & b_{-1} & 0 & 0 \\ 0 & 0 & b_1 & b_0 & b_{-1} & 0 \\ 0 & 0 & 0 & b_1 & b_0 & b_{-1} \\ -\psi & 0 & 0 & 0 & b_1 & -\omega \end{bmatrix}. \tag{1.28}$$

Proposition 1.21. *Let $b_{-1}, b_0, b_1, \phi, \chi, \psi, \omega \in \mathbb{C}$, $b_{-1}, b_1 \neq 0$ $\rho \coloneqq \sqrt{b_{-1}/b_1}$, $n \geq 3$ and let M_n be the matrix of the form* (1.28)*. Let S_n be the matrix defined by* (1.3) *where $\alpha = -2 - (\phi + b_0)/\sqrt{b_1 b_{-1}}$, $\beta = -\rho^{-(n-1)}\chi/\sqrt{b_1 b_{-1}}$, $\gamma = -\rho^{n-1}\psi/\sqrt{b_1 b_{-1}}$ and $\delta = -2 - (\omega + b_0)/\sqrt{b_1 b_{-1}}$. Then*

$$S_n = -\frac{1}{\sqrt{b_1 b_{-1}}} \operatorname{diag}(\boldsymbol{\rho}) \left(M_n - (2\sqrt{b_1 b_{-1}} + b_0) I_n \right) \operatorname{diag}(\boldsymbol{\rho})^{-1}; \tag{1.29}$$

equivalently

$$M_n = \operatorname{diag}(\boldsymbol{\rho})^{-1} \left(-\sqrt{b_1 b_{-1}} S_n + \left(2\sqrt{b_1 b_{-1}} + b_0 \right) I_n \right) \operatorname{diag}(\boldsymbol{\rho}) \tag{1.30}$$

where $\boldsymbol{\rho} \coloneqq [\rho, \rho^2, \ldots, \rho^n]$.

Proof. The proof is similar to the proof of Proposition 1.19. □

Corollary 1.22. *Under the assumptions of Proposition 1.21, for every eigenvalue* λ *of* S_n, $\sqrt{b_1 b_{-1}}(2-\lambda) - b_0$ *is eigenvalue of* M_n.

Proof. The conclusion follows from (1.30). □

Chapter 2

Tridiagonal Toeplitz matrices with perturbations in the off diagonal corners

In this Chapter we study the individual behaviour of the eigenvalues of the Toeplitz matrix generated by the function

$$a_{\alpha,n}(t) = -\overline{\alpha}t^{-n+1} - t^{-1} + 2 - t - \alpha t^{n-1},$$

where α is a complex number and n is a natural number greater than 3, this matrix will be denoted over all the chapter by $A_{\alpha,n}$. For example, if $n = 6$,

$$A_{\alpha,6} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

This family matrices are a particular case of the matrix (1.3), in the notation of Section 1.2 $\upalpha = \updelta = -2$, $\upbeta = -\overline{\alpha}$ and $\upgamma = -\alpha$.

The matrices $A_{\alpha,n}$ are Hermitian, hence their eigenvalues are real and we enumerate them in the ascending order:

$$\lambda_{\alpha,n,1} \leq \lambda_{\alpha,n,2} \leq \cdots \leq \lambda_{\alpha,n,n}.$$

In section 2.1 we study the characteristic polynomial of $A_{\alpha,n}$. We obtain that for weak perturbations ($|\alpha| < 1$) the eigenvalues are on the interval $[0, 4]$, and for strong perturbations ($1 < |\alpha|$) with n sufficiently big $\lambda_{\alpha,n,1}$ is negative and $\lambda_{\alpha,n,n}$ is greater than 4, whereas the intermediate eigenvalues lie on the interval $[0, 4]$.

In Section 2.2 we do an analytic analysis of the characteristic equation of $A_{\alpha,n}$ with weak perturbations. We show that it is possible to determine numerically by iteration of fixed point the eigenvalues. We give asymptotic expansions of the eigenvalues.

In Section 2.3 we focus on the extreme eigenvalues $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$ of $A_{\alpha,n}$ with strong perturbations. We show that as n tends to infinity they converge exponentially to determined values depending only on α, and prove that we can find them by iteration of fixed point.

We write also formulas of eigenvectors for both types of perturbations.

Finally in Section 2.4, we give exact formulas for the eigenvalues and eigenvectors of $A_{\alpha,n}$ with $|\alpha| = 1$.

Observe that if $\alpha = 0$, then $A_{0,n}$ is the tridiagonal Toeplitz matrix without corner perturbation. The eigenvalues and eigenvectors of this matrix are given Corollaries 1.11 and 1.17, respectively. We omit this case through the chapter.

2.1 Characteristic equation and eigenvalues localization of $A_{\alpha,n}$

Through all the chapter $D_{\alpha,n}(\lambda)$ represents $\det(\lambda I_n - A_{\alpha,n})$ and U_n the Chebyshev polynomial of the second kind (Definition 1.2).

Proposition 2.1. *For every α, λ in $\mathbb{C}$ and $n \geq 3$,*

$$D_{\alpha,n}(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) - |\alpha|^2 U_{n-2}\left(\frac{\lambda-2}{2}\right) - 2(-1)^n \operatorname{Re}(\alpha). \qquad (2.1)$$

Proof. In the notation of Proposition 1.13 we have for the matrix $A_{\alpha,n}$ that $\boldsymbol{\alpha} = \boldsymbol{\delta} = -2$, $\boldsymbol{\beta} = \overline{\alpha}$ and $\boldsymbol{\gamma} = \alpha$. Then equation (1.8) transforms directly in (2.1). □

In order to solve the characteristic equation for λ, we are going to do some change of variables, namely we are choosing for λ to be one of the functions:

$$g(x) \coloneqq 4\sin^2\frac{x}{2}, \quad g_-(x) \coloneqq -4\sinh^2\frac{x}{2}, \quad g_+(x) \coloneqq 4 + 4\sinh^2\frac{x}{2}. \tag{2.2}$$

Later we see that the choice of the change of variables depends on whether α is or not in the unitary disc.

Proposition 2.2. *Let $\alpha \in \mathbb{C}$ and $n \geq 3$. Then*

$$D_{\alpha,n}(0) = (-1)^n\big(n(1-|\alpha|^2) + |\alpha-1|^2\big), \tag{2.3}$$

$$D_{\alpha,n}(4) = n(1-|\alpha|^2) + |1-(-1)^n\alpha|^2, \tag{2.4}$$

and for $1 \leq j \leq n-1$,

$$D_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = (-1)^{n+j}\left|\alpha - (-1)^j\right|^2. \tag{2.5}$$

Furthermore,

$$\lim_{\lambda\to-\infty}((-1)^n D_{\alpha,n}(\lambda)) = +\infty, \qquad \lim_{\lambda\to+\infty} D_{\alpha,n}(\lambda) = +\infty. \tag{2.6}$$

Proof. Formulas (2.6) are obvious since the leading term of the polynomial $D_{\alpha,n}(\lambda)$ is λ^n. Formulas (2.3) and (2.4) follow easily from (2.1) using the values of the polynomial U_n at the points 1 and -1 given in Corollary 1.6.

Let us verify (2.5):

$$\begin{aligned} D_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) &= \frac{(-1)^n}{\sin\frac{j\pi}{n}}\left(\sin\left(j\pi + \frac{j\pi}{n}\right) - |\alpha|^2\sin\left(j\pi - \frac{j\pi}{n}\right) - 2\operatorname{Re}(\alpha)\sin\frac{j\pi}{n}\right) \\ &= (-1)^n\big((-1)^j + |\alpha|^2(-1)^j - 2\operatorname{Re}(\alpha)\big) = (-1)^{n+j}\left|\alpha-(-1)^j\right|^2. \end{aligned}$$

□

Proposition 2.2 yields the following conclusions about the localization of the eigenvalues.

1. Let $|\alpha| \leq 1$, $\alpha \neq \pm 1$, and $1 \leq j \leq n$. Then the characteristic polynomial changes its sign in the interval

$$I_{n,j} \coloneqq \left(\frac{(j-1)\pi}{n}, \frac{j\pi}{n}\right). \tag{2.7}$$

Hence, $A_{\alpha,n}$ has an eigenvalue in this interval.

2. Let $|\alpha| < 1$, then 0 and 4 are not eigenvalues of $A_{\alpha,n}$.
3. Let $\alpha \in \mathbb{C}$, $\alpha \notin \{-1, 1\}$, then the points $g(j\pi/n)$, $1 \le j \le n-1$, are not eigenvalues of $A_{\alpha,n}$.
4. Let $|\alpha| > 1$ and $2 \le j \le n-1$. Then the characteristic polynomial changes its sign in the interval $I_{n,j}$, and $A_{\alpha,n}$ has an eigenvalue in this interval.
5. Let $|\alpha| > 1$. Then

$$\lim_{\lambda \to -\infty} ((-1)^n D_{\alpha,n}(\lambda)) = +\infty, \qquad (-1)^n D_{\alpha,n}\left(g\left(\frac{\pi}{n}\right)\right) < 0.$$

So, there is an eigenvalue in $(-\infty, g(\pi/n))$.

6. Let $|\alpha| > 1$. Then

$$D_{\alpha,n}\left(g\left(\frac{(n-1)\pi}{n}\right)\right) < 0, \qquad \lim_{\lambda \to +\infty} D_{\alpha,n}(\lambda) = +\infty.$$

So, there is an eigenvalue in $(g((n-1)\pi/n), +\infty)$.

For $|\alpha| > 1$, the conclusions are sufficient to enumerate the eigenvalues, but it is not clear if $\lambda_{\alpha,n,1}$ and $\lambda_{\lambda,n,n}$ belong to $[0, 4]$ or not. We are going to provide the correspondent criteria. For every α in $\mathbb{C}$ with $|\alpha| \neq 1$, we define

$$N_1(\alpha) := \frac{|\alpha - 1|^2}{||\alpha|^2 - 1|}, \quad N_2(\alpha) := \frac{|\alpha + 1|^2}{||\alpha|^2 - 1|}, \quad N_3(\alpha) := \max\{N_1(\alpha), N_2(\alpha)\}. \tag{2.8}$$

Proposition 2.3 (Criteria for first eigenvalue)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, $n \ge 3$.*

1) If $n < N_1(\alpha)$, then $0 < \lambda_{\alpha,n,1} < g(\pi/n)$.
2) If $n = N_1(\alpha)$, then $\lambda_{\alpha,n,1} = 0$.
3) If $n > N_1(\alpha)$, then $\lambda_{\alpha,n,1} < 0$.

Proof. We determine the sign of $(-1)^n D_{\alpha,n}(0)$ using (2.3).

1. If $n < N_1(\alpha)$, then

$$(-1)^n D_{\alpha,n}(0) > 0, \qquad (-1)^n D_{\alpha,n}(g(\pi/n)) < 0,$$

and $(-1)^n D_{\alpha,n}(\lambda)$ changes its sign on $(0, g(\pi/n))$.

2. If $n = N_1(\alpha)$, then $(-1)^n D_{\alpha,n}(\lambda)(0) = 0$.
3. If $n > N_1(\alpha)$, then

$$(-1)^n D_{\alpha,n}(0) < 0, \qquad \lim_{\lambda \to -\infty} (-1)^n D_{\alpha,n}(\lambda) = +\infty,$$

   and $(-1)^n D_{\alpha,n}$ changes its sign on $(-\infty, 0)$.

Hence, the sentences *1)*, *2)* and *3)* hold. □

Proposition 2.4 (Criteria for last eigenvalue n even)**.** *Let* $\alpha \in \mathbb{C}$, $|\alpha| > 1$, $n \geq 3$.

1) If $n < N_1(\alpha)$, *then* $g\left(\frac{(n-1)\pi}{n}\right) < \lambda_{\alpha,n,n} < 4$.
2) If $n = N_1(\alpha)$, *then* $\lambda_{\alpha,n,n} = 0$.
3) If $n > N_1(\alpha)$, *then* $4 < \lambda_{\alpha,n,n}$.

Proof. We determine the sign of $(-1)^n D_{\alpha,n}(4)$ using (2.4).

1. If $n < N_1(\alpha)$, then

$$(-1)^n D_{\alpha,n}(4) > 0, \qquad (-1)^n D_{\alpha,n}\left(g\left(\frac{(n-1)\pi}{n}\right)\right) < 0,$$

   and $(-1)^n D_{\alpha,n}$ changes its sign on $\left(g\left(\frac{(n-1)\pi}{n}\right), 4\right)$.
2. If $n = N_1(\alpha)$, then $(-1)^n D_{\alpha,n}(4) = 0$.
3. If $n > N_1(\alpha)$, then

$$(-1)^n D_{\alpha,n}(4) < 0, \qquad \lim_{\lambda \to \infty} (-1)^n D_{\alpha,n}(\lambda) = +\infty,$$

   and $(-1)^n D_{\alpha,n}$ changes its sign on $(-\infty, 0)$.

Hence, the sentences *1)*, *2)* and *3)* hold. □

Proposition 2.5 (Criteria for last eigenvalue n odd)**.** *Let* $\alpha \in \mathbb{C}$, $|\alpha| > 1$, $n \geq 3$.

1) If $n < N_2(\alpha)$, *then* $g\left(\frac{(n-1)\pi}{n}\right) < \lambda_{\alpha,n,n} < 4$.
2) If $n = N_2(\alpha)$, *then* $\lambda_{\alpha,n,n} = 0$.
3) If $n > N_2(\alpha)$, *then* $4 < \lambda_{\alpha,n,n}$.

Proof. The proof is similar to the proof of Proposition 2.4. □

We now prove the main results of this section, the eigenvalues localization.

Theorem 2.6 (localization of the eigenvalues for weak perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| \le 1$, $\alpha \notin \{-1, 1\}$, and $n \ge 3$. Then the matrix $A_{\alpha,n}$ has n different eigenvalues belonging to $(0, 4)$. More precisely, for every $1 \le j \le n$,*

$$g\left(\frac{(j-1)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\frac{j\pi}{n}\right). \tag{2.9}$$

Proof. Proposition 2.2 implies that for $|\alpha| < 1$ and $n \ge 3$, the numbers $D_{\alpha,n}(g(j\pi/n))$, $0 \le j \le n$ are nonzero, and their signs alternate. By the intermediate value theorem, this yields the result (2.9). □

Theorem 2.7 (localization of the eigenvalues for strong perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and $n \ge 3$.*

1) If $n > N_1(\alpha)$, then $\lambda_{\alpha,n,1} < 0$, else $0 \le \lambda_{\alpha,n,1} < g(\pi/n)$.

2) If n is odd and $n > N_1(\alpha)$, or n is even and $n > N_2(\alpha)$, then $\lambda_{\alpha,n,n} > 4$. In the other case, $g((n-1)\pi/n) < \lambda_{\alpha,n,n} \le 4$.

3) For $2 \le j \le n-1$, the eigenvalues $\lambda_{\alpha,n,j}$ belong to $(0, 4)$ and satisfy (2.9).

Proof. Proposition 2.3 implies 1, Propositions 2.4 and 2.5 implies 2, and Proposition 2.2 implies 3 (this case is similar as Theorem 2.6). □

Theorem 2.6 implies immediately that for every $|\alpha| < 1$ and for every v in $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{\#\{j \in \{1, \dots, n\} \colon\ \lambda_{\alpha,n,j} \le v\}}{n} = \frac{\mu\left(\{x \in [0, \pi] \colon g(x) \le v\}\right)}{\pi}, \tag{2.10}$$

i.e., the eigenvalues of $A_{\alpha,n}$ are asymptotically distributed as the function g on $[0, \pi]$. For every $|\alpha| > 1$, Theorem 2.7 implies a similar conclusion for all the eigenvalues except for the extreme ones. We see in Section 2.3 that $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$ converge to determined values as n tends to infinity, hence there is a determined gap between the sets $\{\lambda_{\alpha,n,1}\}$, $\{\lambda_{\alpha,n,n}\}$ and $\{\lambda_{\alpha,n,j} \colon\ j \in \{2, \dots, n-1\}\}$ as n tends to infinity.

Now we give a trigonometric expression for the characteristic equation, using for this the change of variable $\lambda = g(x)$.

Define

$$k_\alpha := \frac{1-|\alpha|^2}{|1+\alpha|^2}, \qquad l_\alpha := \frac{|1-\alpha|}{|1+\alpha|}. \tag{2.11}$$

Proposition 2.8. *Let $\alpha \in \mathbb{C}$, $\alpha \notin \{-1,1\}$, $n \geq 3$, $x \in (0,\pi)$. Then*

$$D_{\alpha,n}(g(x)) = \frac{(-1)^{n+1}|\alpha+1|^2 \left(\tan^2 \frac{nx}{2} - 2k_\alpha \cot(x) \tan \frac{nx}{2} - l_\alpha^2\right)}{1+\tan^2 \frac{nx}{2}}. \tag{2.12}$$

Proof. We start with (2.1), write λ as $2-2\cos(x)$, use the parity or imparity of U_n and the trigonometric formula in 1.5. Then

$$D_{\alpha,n}(g(x)) = \frac{(-1)^n}{\sin(x)} \left(\sin((n+1)x) - |\alpha|^2 \sin((n-1)x) - 2\operatorname{Re}(\alpha)\sin(x)\right). \tag{2.13}$$

Applying the trigonometric identities

$$\sin((n \pm 1)x) = \sin(nx)\cos(x) \pm \cos(nx)\sin(x),$$

$$\sin(nx) = \frac{2\tan\frac{nx}{2}}{1+\tan^2\frac{nx}{2}}, \qquad \cos(nx) = \frac{1-\tan^2\frac{nx}{2}}{1+\tan^2\frac{nx}{2}}, \tag{2.14}$$

and regrouping the summands, we get

$$\begin{aligned} D_{\alpha,n}(g(x)) = &\frac{(-1)^{n+1}}{1+\tan^2\frac{nx}{2}} \times \\ &\times \left(|\alpha+1|^2 \tan^2\frac{nx}{2} - 2(1-|\alpha|^2)\cot(x)\tan\frac{nx}{2} - |\alpha-1|^2\right), \end{aligned} \tag{2.15}$$

which is equivalent to (2.12). □

If $|\alpha| < 1$, the Theorem 2.6 and Proposition 2.8 motivates the change of variable $\lambda = g(x)$. In next section we write asymptotic formulas for the eigenvalues of $A_{\alpha,n}$ by finding the roots of (2.12). If $|\alpha| > 1$ and n is sufficiently big, by Theorem 2.7 there are only $n-2$ eigenvalues in $(0,4)$, hence it is convenient to do the same change of variable $\lambda = g(x)$. However we prove in Section 2.3 that for the remaining eigenvalues, namely

$\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$, we require to do $\lambda = g_-(x)$ and $g_+(x)$, respectively. Knowing this, we show below the characteristic polynomial after doing these changes of variable.

Observe that for x in $(-\infty, \infty)$, we have the relations

$$g(ix) = g_-(x), \qquad g(\pi + ix) = g_+(x). \tag{2.16}$$

It is evident that for $x = 0$ these relations turn into $g(0)$ and $g(\pi)$ respectively, both values $D_{\alpha,n}(g(0))$ and $D_{\alpha,n}(g(\pi))$ were already analyzed in 2.2. Furthermore the functions $g_\pm$ are even, so, we just study only the case when x is in the open ray $(0, \infty)$.

Proposition 2.9. *Let $\alpha \in \mathbb{C}$, $\alpha \notin \{-1, 1\}$, $n \geq 3$, $x > 0$. Then*

$$\begin{aligned} D_{\alpha,n}(g_-(x)) = \frac{(-1)^n}{1 - \tanh^2 \frac{nx}{2}} \Bigg(& |\alpha + 1|^2 \tanh^2 \frac{nx}{2} \\ & - 2(|\alpha|^2 - 1) \tanh \frac{nx}{2} \coth(x) + |\alpha - 1|^2 \Bigg), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} D_{\alpha,n}(g_+(x)) = \frac{1}{1 - \tanh^2 \frac{nx}{2}} \Bigg(& |\alpha + (-1)^n|^2 \tanh^2 \frac{nx}{2} \\ & - 2(|\alpha|^2 - 1) \tanh \frac{nx}{2} \coth(x) + |\alpha - (-1)^n|^2 \Bigg). \end{aligned} \tag{2.18}$$

Proof. The proof is similar to the proof of Proposition 2.8. □

2.2 Eigenvalues and eigenvectors of $A_{\alpha,n}$ with weak perturbations $|\alpha| < 1$

Let $|\alpha| < 1$ and $n \geq 3$. In this section we are interested in solving (2.12). Motivated by (2.9), we use the function g (2.2) as a change of variable in the characteristic equation and put $\vartheta_{\alpha,n,j} \coloneqq g^{-1}(\lambda_{\alpha,n,j})$. Note that inequality (2.9) is equivalent to

$$\frac{(j-1)\pi}{n} < \vartheta_{\alpha,n,j} < \frac{j\pi}{n} \qquad (1 \leq j \leq n). \tag{2.19}$$

For $|\alpha| \neq 1$ and j in $\mathbb{Z}$, we define $u_{\alpha,j} \colon (0, \pi) \to \mathbb{R}$ by

$$u_{\alpha,j}(x) \coloneqq k_\alpha \cot(x) + (-1)^{j+1} \sqrt{k_\alpha^2 \cot^2(x) + l_\alpha^2}. \tag{2.20}$$

The first derivative of $u_{\alpha,j}$ is

$$u'_{\alpha,j}(x) = -\frac{k_\alpha}{\sin^2(x)} \left(1 + \frac{(-1)^{j+1} k_\alpha \cot(x)}{\sqrt{k_\alpha^2 \cot^2(x) + l_\alpha^2}} \right), \tag{2.21}$$

if $|\alpha| < 1$, $u'_{\alpha,j}$ is negative and positive if $|\alpha| > 1$. For x in $(0, \pi)$, different from the points $j\pi/n$, the expression $\frac{(-1)^{n+1}|\alpha+1|^2}{1+\tan^2 \frac{nx}{2}}$ takes finite nonzero values. Omitting this factor, we consider the right-hand side of (2.12) as a quadratic polynomial in $\tan \frac{nx}{2}$, with coefficients depending on α and x. The roots of this quadratic polynomial are $u_{\alpha,1}(x)$ and $u_{\alpha,2}(x)$. So, the characteristic equation $D_{\alpha,n}(g(x)) = 0$ is equivalent to the disjunction of the equations

$$\tan \frac{nx}{2} = u_{\alpha,1}(x), \qquad \tan \frac{nx}{2} = u_{\alpha,2}(x). \tag{2.22}$$

Recall that $I_{n,j}$ is the interval defined by (2.7).

Proposition 2.10. *Let $|\alpha| < 1$, $n \geq 3$, and $1 \leq j \leq n$. Then the equation*

$$\tan \frac{nx}{2} = u_{\alpha,j}(x) \tag{2.23}$$

has a unique solution in $I_{n,j}$ that coincides with $\vartheta_{\alpha,n,j}$ and the correspondent value $g(\vartheta_{\alpha,n,j})$ is $\lambda_{\alpha,n,j}$.

Proof. By the intermediate value theorem the first of the equations in (2.22) has solution in each interval $I_{n,j}$ with j odd, and the second one has solution in each interval $I_{n,j}$ with j even. The uniqueness of the solutions follows from Theorem 2.6, but it can also be verified directly, since the first derivative of $u_{\alpha,j}$ is negative. □

See in Figure 2.1 a graph of both sides of (2.23). This figure confirm that, if $|\alpha| < 1$, then for every $1 \leq j \leq n$ there exist a solution of (2.23) in the interval $I_{n,j}$; and if $|\alpha| > 1$ and n is sufficiently big, then (2.23) does not have solution in $I_{n,1}$ nor in $I_{n,n}$.

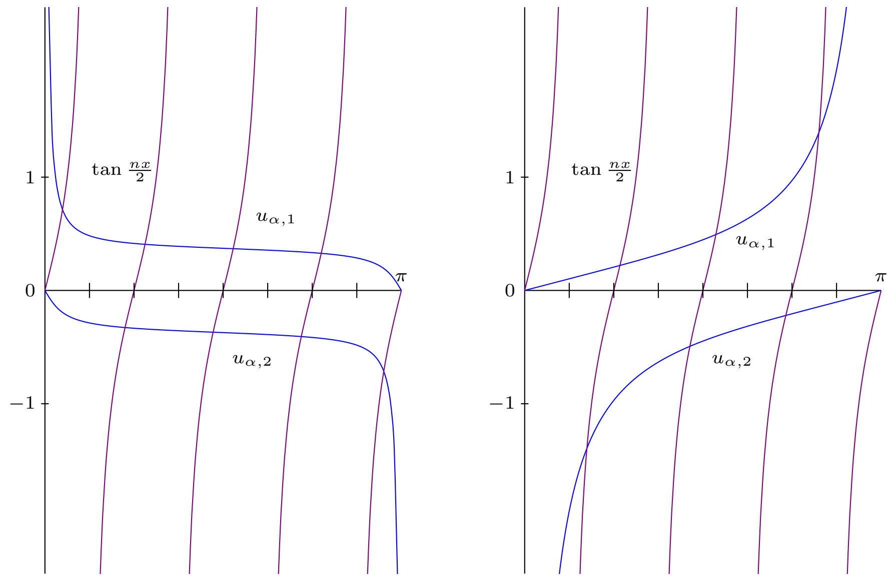


Figure 2.1: Functions of (2.23) for $\alpha = 0.7 + i0.6$ and $n = 8$ (left) and for $\alpha = 2 + i$ and $n = 8$ (right).

Theorem 2.11 (characteristic equation for weak perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$, $\alpha \notin \{-1, 1\}$, $n \geq 3$, and $1 \leq j \leq n$. Then the number $\vartheta_{\alpha,n,j}$ satisfies*

$$\vartheta_{\alpha,n,j} = -\frac{2\arctan\left(\left((-1)^{j+1}k_\alpha \cot(\vartheta_{\alpha,n,j}) + \sqrt{k_\alpha^2 \cot^2(\vartheta_{\alpha,n,j}) + l_\alpha^2}\right)^{(-1)^j}\right)}{n} + \frac{j\pi}{n}. \quad (2.24)$$

Proof. Equality (2.24) is equivalent to (2.23). □

Motivated by (2.24), for every α in $\mathbb{C}$ with $\alpha \notin \{-1, 1\}$ and every integer j we define the function $\eta_{\alpha,j} \colon [0, \pi] \to \mathbb{R}$ by

$$\eta_{\alpha,j}(x) \coloneqq -2\arctan\left(\left((-1)^{j+1}k_\alpha \cot(x) + \sqrt{k_\alpha^2 \cot^2(x) + l_\alpha^2}\right)^{(-1)^j}\right), \quad (2.25)$$

where the constants are defined by (2.11) In fact, $\eta_{\alpha,j}$ depends only on α and on the parity of j. Thus, for every α there are only two different functions: $\eta_{\alpha,1}$ and $\eta_{\alpha,2}$. These functions take values in $[-\pi, 0]$. See a couple of examples in Figures 2.2 and 2.3 for some $|\alpha| < 1$ and $|\alpha| > 1$.

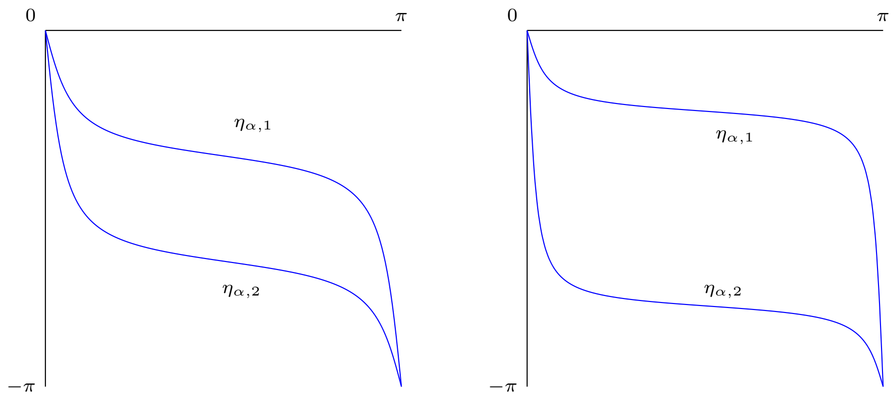


Figure 2.2: Function (2.25) for $\alpha = -0.3 + 0.5i$ (left) and $\alpha = 0.7 + i0.6$ (right).

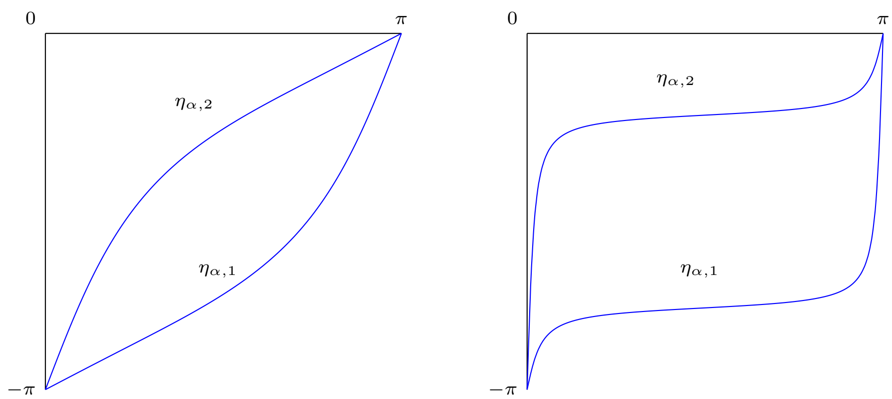


Figure 2.3: Function (2.25) for $\alpha = 2 + i$ (left) and $\alpha = 0.8 - i0.7$ (right).

A straightforward computation yields

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1 + \frac{(-1)^{j+1}k_\alpha\cot(x)}{\sqrt{k_\alpha^2\cot^2(x)+l_\alpha^2}}\right)(1+\cot^2(x))}{1 + \left((-1)^{j+1}k_\alpha\cot(x) + \sqrt{k_\alpha^2\cot^2(x) + l_\alpha^2}\right)^2}. \tag{2.26}$$

Equivalently,

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1 + \frac{(-1)^{j+1}k_\alpha\operatorname{sign}(\tan(x))}{\sqrt{k_\alpha^2+l_\alpha^2\tan^2(x)}}\right)(1+\tan^2(x))}{\tan^2(x) + \left((-1)^{j+1}k_\alpha\operatorname{sign}(\tan(x)) + \sqrt{k_\alpha^2 + l_\alpha^2\tan^2(x)}\right)^2}. \tag{2.27}$$

Recall that the constants $N_1(\alpha)$, $N_2(\alpha)$ and $N_3(\alpha)$ are defined by (2.8).

Proposition 2.12. *Let $\alpha \in \mathbb{C}$, $|\alpha| \neq 1$, $j \in \mathbb{Z}$. Then each derivative of $\eta_{\alpha,j}$ is a bounded function on $(0,\pi)$. In particular,*

$$\sup_{0<x<\pi} |\eta'_{\alpha,j}(x)| = N_3(\alpha). \tag{2.28}$$

Proof. Case I: $(-1)^{j+1}k_\alpha \tan(x) > 0$, i.e., $(-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x)) = |k_\alpha|$. Let us denote $\tan^2(x)$ by t. Then (2.27) simplifies to

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha\left(1+t+\frac{|k_\alpha|\,(1+t)}{\sqrt{k_\alpha^2+l_\alpha^2 t}}\right)}{2k_\alpha^2+(1+l_\alpha^2)t+2|k_\alpha|\sqrt{k_\alpha^2+l_\alpha^2 t}}. \tag{2.29}$$

Since $l_\alpha \geq |k_\alpha|$, $k_\alpha^2 + l_\alpha^2 t \geq k_\alpha^2(1+t)$, and $1 + l_\alpha^2 \geq 2k_\alpha^2$,

$$|\eta'_{\alpha,j}(x)| \leq \frac{2|k_\alpha|\,(1+t+\sqrt{1+t})}{2k_\alpha^2\,(1+t+\sqrt{1+t})} = \frac{1}{|k_\alpha|} = N_1(\alpha).$$

Case II: $(-1)^{j+1}k_\alpha \tan(x) < 0$, i.e. $(-1)^{j+1}k_\alpha \operatorname{sign}(\tan(x)) = -|k_\alpha|$. Denote again $\tan^2(x)$ by t. Then, using the identity

$$\left(\sqrt{k_\alpha^2+l_\alpha^2 t}-|k_\alpha|\right)\left(\sqrt{k_\alpha^2+l_\alpha^2 t}+|k_\alpha|\right) = l_\alpha^2 t,$$

we transform (2.27) to

$$\eta'_{\alpha,j}(x) = -\frac{2k_\alpha l_\alpha^2(1+t)}{\left(1+l_\alpha^2+\frac{(1-l_\alpha^2)|k_\alpha|}{\sqrt{k_\alpha^2+l_\alpha^2 t}}\right)(k_\alpha^2+l_\alpha^2 t)}. \tag{2.30}$$

Since $k_\alpha^2 + l_\alpha^2 t \geq k_\alpha^2(1+t)$,

$$|\eta'_{\alpha,j}(x)| \leq \frac{2l_\alpha^2}{|k_\alpha|}\cdot\frac{1}{1+l_\alpha^2+\frac{(1-l_\alpha^2)|k_\alpha|}{\sqrt{k_\alpha^2+l_\alpha^2 t}}}. \tag{2.31}$$

Denote by $R(t)$ the expression in the right-hand side of (2.31). If $l_\alpha \geq 1$, then R decreases, and

$$\sup_{0<t<+\infty} R(t) = \lim_{t\to 0^+} R(t) = \frac{2l_\alpha^2}{|k_\alpha|}\cdot\frac{1}{2} = N_2(\alpha).$$

If $l_\alpha < 1$, then R increases, and

$$\sup_{0<t<+\infty} R(t) = \lim_{t\to+\infty} R(t) \le \frac{2l_\alpha^2}{|k_\alpha|} \cdot \frac{1}{1+l_\alpha^2} < \frac{1}{|k_\alpha|} = N_1(\alpha).$$

In both cases I and II, the inequality $\le$ in (2.28) is proven.

The limit values of $|\eta'_{\alpha,j}|$ at the points 0 and π can be computed by applying (2.29) and (2.30), and coincide with $N_1(\alpha)$ and $N_2(\alpha)$, or vice versa, depending on the sign of $(-1)^{j+1}k_\alpha$. This implies the inverse inequality $\ge$ in (2.28).

For the higher derivatives of $\eta_{\alpha,j}$, the explicit estimates are too tedious, and we propose the following argument. By (2.26), $\eta'_{\alpha,j}$ is analytic in a neighborhood of x, for any x in $(0,\pi)$. Moreover, formulas (2.29) and (2.30) show that $\eta'_{\alpha,j}$ has an analytic extension in some neighborhoods of the points 0 and π. Hence, $\eta'_{\alpha,j}$ has an analytic extension to a certain open set in the complex plane containing the segment $[0,\pi]$. Therefore, this function and all their derivatives are bounded on $(0,\pi)$. □

For every α in $\mathbb{C}$ with $\alpha \notin \{-1,1\}$, $n \ge 3$ and $1 \le j \le n$ we define the function $f_{\alpha,n,j}$ on $[0,\pi]$ by

$$f_{\alpha,n,j}(x) := \frac{j\pi + \eta_{\alpha,j}(x)}{n}. \tag{2.32}$$

Proposition 2.13. *Let $|\alpha| < 1$, $n > N_3(\alpha)$, and $1 \le j \le n$. Then $f_{\alpha,n,j}$ defined by (2.32) is contractive on $[0,\pi]$, and its fixed point belongs to $I_{n,j}$, evenmore, coincides with $\vartheta_{\alpha,n,j}$.*

Proof. Since the function $\eta_{\alpha,j}$ takes values in $[-\pi,0]$ and its derivative is bounded by (2.28), it is easy to see that $f_{\alpha,n,j}(x) \in [0,\pi]$ for every x in $[0,\pi]$, and

$$|f'_{\alpha,n,j}(x)| \le \frac{N_3(\alpha)}{n} < 1.$$

Moreover, the assumption $|\alpha| < 1$ implies that $\eta_{\alpha,j}(0) = 0$ and $\eta_{\alpha,j}(\pi) = -\pi$. Therefore

$$f_{\alpha,n,j}(0) > 0, \qquad f_{\alpha,n,j}(\pi) < \pi.$$

We have proved that $f_{\alpha,n,j}$ is contractive on $[0,\pi]$. By Theorem A.4 this implies that $f_{\alpha,n,j}$ has a unique fixed point, by Theorem 2.11 it coincides with $\vartheta_{\alpha,n,j}$. □

In order to compute the eigenvalues, the previous proposition justifies the implementation of the fixed point iteration method.

We use the notation $\lambda^{\text{gen}}_{\alpha,n,j}$, for the eigenvalues computed in Sagemath by general algorithms, with double-precision arithmetic; and $\lambda^{\text{fp}}_{\alpha,n,j}$ denote the eigenvalues computed by formulas of Theorem 2.11, i.e. solving the equation (2.24) by the fixed point iteration; these computations are performed in high-precision arithmetic with 3322 binary digits.

We have constructed a large series of examples with random values of α and n, see the Codes B.2 and B.3 for the main parts of the developed program. For example with $\alpha = 1/3$ and $n = 64$ we run the Code B.3 in the command window and get the Code 2.1. In all these examples, we have obtained

$$\max_{1\leq j\leq n} |\lambda^{\text{gen}}_{\alpha,n,j} - \lambda^{\text{fp}}_{\alpha,n,j}| < 2 \cdot 10^{-13}. \tag{2.33}$$

Code 2.1: Test eigenvalues approximation by fixed point iteration

```
sage: load('Off_weak_test_eigenvalues_by_fixed_point.sage')
sage: max_error_eigenvalues_gen_minus_fp(1/3, 64, 3322)
7.993605777301127e−15
```

In [3] Barrera, Bötcher, Grudsky and Maximenko develop asymptotic expansions of the eigenvalues of certain symmetric pentadiagonal Toeplitz matrix, we use these ideas in next proposition and theorem.

Proposition 2.14. *Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$. Then there exists $C_1(\alpha) > 0$ such that for n large enough and $1 \leq j \leq n$,*

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)\eta'_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n^2} + r_{\alpha,n,j}, \tag{2.34}$$

where $|r_{\alpha,n,j}| \leq \frac{C_1(\alpha)}{n^3}$.

Proof. Theorem 2.6 assures the initial approximation $\vartheta_{\alpha,n,j} = j\pi/n + O(1/n)$. Substitute it into the right-hand side of (2.24) and expand $\eta_{\alpha,j}$ by Taylor's formula around $j\pi/n$

$$\vartheta_{\alpha,n,j} = \frac{j\pi + \eta_{\alpha,j}\left(\frac{j\pi}{n} + O\left(\frac{1}{n}\right)\right)}{n} = \frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + O_\alpha\left(\frac{1}{n^2}\right).$$

Iterate once again in (2.24),

$$\vartheta_{\alpha,n,j} = \frac{j\pi + \eta_{\alpha,j}\left(\frac{j\pi}{n} + \frac{\eta_{\alpha,j}\left(\frac{j\pi}{n}\right)}{n} + O_\alpha\left(\frac{1}{n^2}\right)\right)}{n}.$$

Expanding $\eta_{\alpha,j}$ around $j\pi/n$ with two exact terms and estimating the residue term with Proposition 2.12 we obtain the desired result. □

For every α in $\mathbb{C}$ with $|\alpha| \neq 1$ and $1 \leq j \leq n$, define on $[0, \pi]$ the function

$$\Lambda_{\alpha,n,j}(x) := g(x) + \frac{g'(x)\eta_{\alpha,j}(x)}{n} + \frac{g'(x)\eta_{\alpha,j}(x)\eta'_{\alpha,j}(x) + \frac{1}{2}g''(x)\eta^2_{\alpha,j}(x)}{n^2}. \tag{2.35}$$

For every $1 \leq j \leq n$, define $\lambda^{\text{asympt}}_{\alpha,n,j}$ by

$$\lambda^{\text{asympt}}_{\alpha,n,j} := \Lambda_{\alpha,n,j}\left(\frac{j\pi}{n}\right). \tag{2.36}$$

Note that this asymptotic expansion

Theorem 2.15 (asymptotic expansion of the eigenvalues for weak perturbations)**.** *Let* $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$. *Then there exists* $C_2(\alpha) > 0$ *such that for* n *large enough and* $1 \leq j \leq n$,

$$|\lambda_{\alpha,n,j} - \lambda^{\text{asympt}}_{\alpha,n,j}| \leq \frac{C_2(\alpha)}{n^3}. \tag{2.37}$$

Proof. The conclusion follows from Proposition 2.14: we just evaluate g at the expression (2.34) and expand it by Taylor's formula around $j\pi/n$. □

For Theorem 2.15 we have computed the errors

$$R_{\alpha,n,j} := \lambda^{\text{asympt}}_{\alpha,n,j} - \lambda^{\text{fp}}_{\alpha,n,j},$$

and their maximums $\|R_{\alpha,n}\|_\infty = \max_{1\leq j\leq n}|R_{\alpha,n,j}|$, with both $\lambda^{\text{asympt}}_{\alpha,n,j}$ and $\lambda^{\text{fp}}_{\alpha,n,j}$ computed in high-precision arithmetic with 3322 binary digits, see the Codes B.4 and B.5 for the main part of the written program. For example for $\alpha = 1/3 + i1/5$ and $n = 64$ we execute the Code B.5 in the command window and get the Code 2.2.

Code 2.2: Test eigenvalues approximation by asymptotic expansion

```
sage: load('Off_weak_test_eigenvalues_by_asymp.sage')
sage: max_error_eigenvalues_asymp_minus_fp(1/3+i*1/5, 64, 3322)
test eigenvalues by asymptotic expansion
n = 64
al = 1/5*I + 1/3
maximal error = 1.538e-04
normalized error = 4.032e+01
```

Tables 2.1 show that these errors indeed can be bounded by $C_2(\alpha)/n^3$, and $C_2(\alpha)$ has to take bigger values when $|\alpha|$ is close to 1.

Table 2.1: Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some $|\alpha| < 1$.

$\alpha = -0.3 + 0.5i, \quad \|\alpha\| \approx 0.58$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	4.89×10^{-3}	20.05
32	1.05×10^{-3}	34.48
64	1.76×10^{-4}	46.05
128	2.49×10^{-5}	52.13
256	3.29×10^{-6}	55.12
512	4.22×10^{-7}	56.58
1024	5.34×10^{-8}	57.31
2048	6.71×10^{-9}	57.67
4096	8.42×10^{-10}	57.84
8192	1.05×10^{-10}	57.93

$\alpha = 0.7 + 0.6i, \quad \|\alpha\| \approx 0.92$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	2.83×10^{-2}	116.25
32	5.72×10^{-3}	187.54
64	1.02×10^{-3}	266.71
128	1.59×10^{-4}	333.02
256	2.24×10^{-5}	376.61
512	2.99×10^{-6}	401.28
1024	3.86×10^{-7}	414.29
2048	4.90×10^{-8}	420.94
4096	6.17×10^{-9}	424.30
8192	7.75×10^{-10}	425.99

Let $n \geq 3$. For every α in $\mathbb{C}$ and $1 \leq j \leq n$ define the next functions on $[0, \pi]$

$$\begin{aligned} d_0(x) &:= g(x), \\ d_{1,\alpha,n,j}(x) &:= \frac{g'(x)\eta_{\alpha,j}(x)}{n}, \\ d_{2,\alpha,n,j}(x) &:= \frac{g'(x)\eta_{\alpha,j}(x)\eta'_{\alpha,j}(x) + \frac{1}{2}g''(x)\eta^2_{\alpha,j}(x)}{n^2}. \end{aligned} \tag{2.38}$$

It is evident that

$$\Lambda_{\alpha,n,j}(x) = d_0(x) + d_{1,\alpha,n,j}(x) + d_{2,\alpha,n,j}(x).$$

See in Figure 2.4 the behaviour of the asymptotic expansion (2.35). There the dots in blue and red are respectively the pairs $(j\pi/n, \lambda_{\alpha,n,j})$ and $(j\pi/n, \lambda^{\text{asympt}}_{\alpha,n,j})$; the first pair of numbers were computed with general algorithms and the last ones computed with one, two and three terms of the asymptotic expansion, respectively to each figure. The curves are the functions d_0, $d_0 + d_{1,\alpha,j}$ and $d_0 + d_{1,\alpha,n,j} + d_{2,\alpha,n,j}$, respectively to each figure. These figures confirm that, if we take more terms of the asymptotic expansion, then we obtain a better approximation for the eigenvalues.

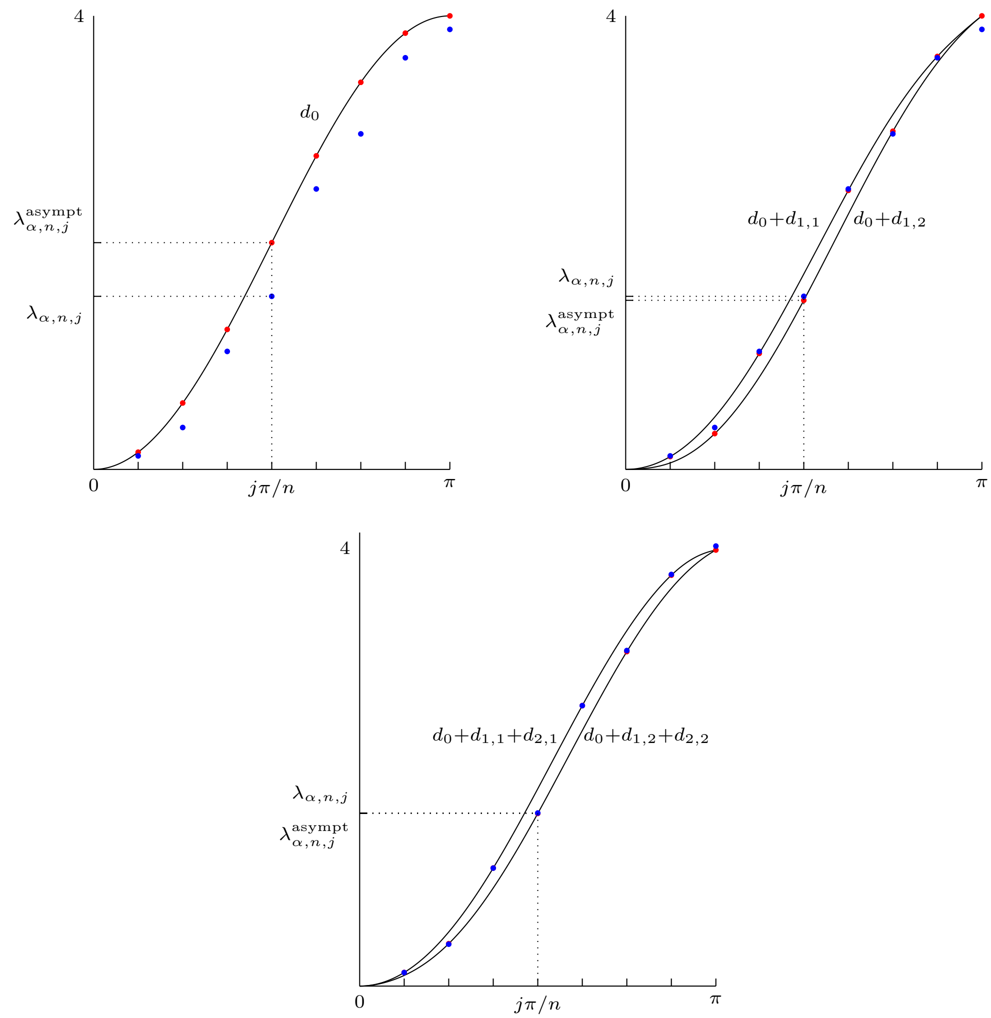


Figure 2.4: One, two and three terms of the function (2.35) for $\alpha = -0.3 + 0.5i$ and $n = 8$ (we omitted the subscripts α and n in the functions).

Proposition 2.16 (the eigenvectors for weak perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$, $n \geq 3$,*

$1 \le j \le n$. *Then the vector* $v_{\alpha,n,j} = \left[v_{\alpha,n,j,k}\right]_{k=1}^{n}$ *with components*

$$v_{\alpha,n,j,k} := \sin(k\vartheta_{\alpha,n,j}) + \overline{\alpha}\sin((n-k)\vartheta_{\alpha,n,j}) \qquad (1 \le k \le n) \tag{2.39}$$

is an eigenvector of the matrix $A_{\alpha,n}$ *associated to the eigenvalue* $\lambda_{\alpha,n,j}$.

Proof. Let $1 \le j \le n$, and let $\mathrm{b}_{-2,-\overline{\alpha},n}$ be the constant defined here (1.14), there $z = e^{i\vartheta_{\alpha,n,j}}$. Then

$$\mathrm{b}_{-2,-\overline{\alpha},n} = -2i\overline{\alpha}\sin(n\vartheta_{\alpha,n,j}).$$

Since $\vartheta_{\alpha,n,j} \ne j\pi/n$, then $\mathrm{b}_{-2,-\overline{\alpha},n}(\vartheta_{\alpha,n,j}) \ne 0$. So, by Proposition 1.16 the vector v' with components

$$v'_k = \frac{\sin(k\vartheta_{\alpha,n,j}) + \overline{\alpha}\sin((n-k)\vartheta_{\alpha,n,j})}{\sin(\vartheta_{\alpha,n,j})} \quad (1 \le k \le n)$$

is an eigenvector associated to $\lambda_{\alpha,n,j}$. Since every constant multiple of v' is also an eigenvector, then formula (2.39) follows by multiplying v'_k by $\sin(\vartheta_{\alpha,n,j})$. □

Motivated by (2.39), for every $|\alpha| < 1$, $n \ge 3$ and $1 \le j \le n$, we define for every x in $[0, 16]$ the function:

$$w_{\alpha,n,j}(x) := \sin(x\vartheta_{\alpha,n,j}) + \overline{\alpha}\sin((n-x)\vartheta_{\alpha,n,j}). \tag{2.40}$$

Observe that for every $1 \le j \le n$ and for every $1 \le k \le n$, $w_{\alpha,n,j}(k)$ equals (2.39). Observe that if $-1 < \alpha < 1$, then (2.40) can be written as

$$w_{\alpha,n,j}(x) = P_{\alpha,n,j}(\sin(\vartheta_{\alpha,n,j}x + \omega_{\alpha,n,j})) \tag{2.41}$$

where

$$P_{\alpha,n,j} := \sqrt{1 - 2\alpha\cos(n\vartheta_j) + \alpha^2}, \quad \omega_{\alpha,n,j} := \arctan\frac{\alpha\sin(n\vartheta_{\alpha,n,j})}{1 - \alpha\cos(n\vartheta_{\alpha,n,j})}.$$

Hence $w_{\alpha,n,j}$ is a sinusoidal function, where the amplitude $P_{\alpha,n,j}$, the angular frequency $\vartheta_{\alpha,n,j}$ and the phase shift $\omega_{\alpha,n,j}$ depend on α and n and do not depend on x. Let $\alpha = -0.3$ and $n = 16$. See in Figure 2.5 the function $w_{\alpha,n,j}$ for some values of j. The dots in red are the pairs $(k, w_{\alpha,n,j}(k))$.

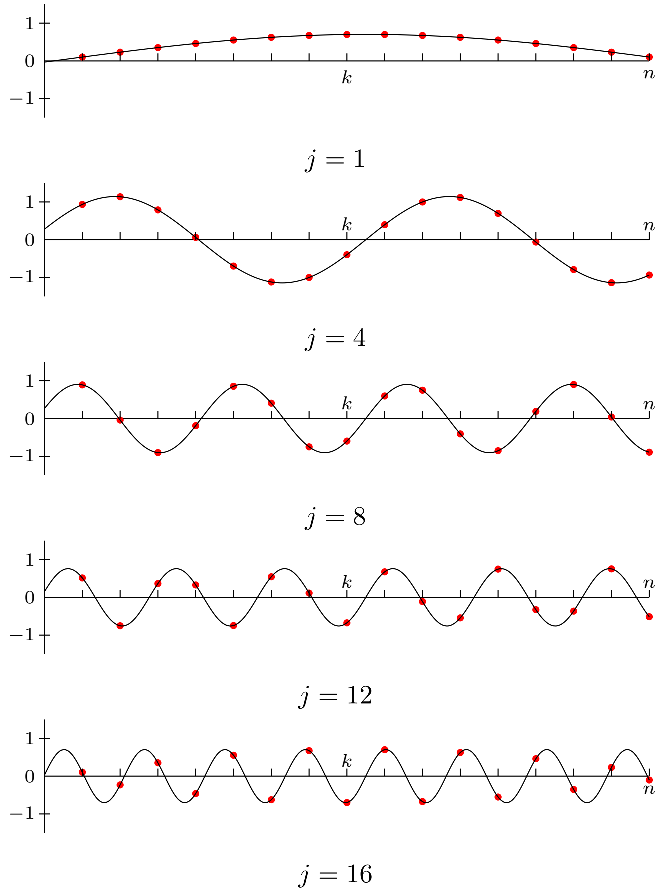


Figure 2.5: Function (2.40) for $\alpha = -0.3$, $n = 16$ and different values of j.

2.3 Eigenvalues and eigenvectors of $A_{\alpha,n}$ with strong perturbations $|\alpha| > 1$

Let $|\alpha| > 1$ and $n \geq 3$. By Theorem 2.7, for $2 \leq j \leq n-1$, the eigenvalues $\lambda_{\alpha,n,j}$ of $A_{\alpha,n}$ behave as in the case of weak perturbations and can be approximated by the methods of Section 2.2. Hence, in this section we are only interested in solving (2.17) and (2.18).

Recall some notation. The functions g, g_+ and g_- are defined by (2.2). The function $D_{\alpha,n}(\lambda)$ is the characteristic polynomial of $A_{\alpha,n}$. The constant $N_3(\alpha)$ is defined by (2.8). Define the constant $N_4(\alpha)$ by

$$N_4(\alpha) := \frac{20\log(|\alpha|+1) - 4\log(\log(|\alpha|))}{\log|\alpha|}. \tag{2.42}$$

Proposition 2.17. *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$. For $x > 0$ and $\lambda = g_-(x)$, the equation $D_{\alpha,n}(g_-(x)) = 0$ is equivalent to*

$$\tanh(x) = \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + 1|^2 \tanh^2\frac{nx}{2} + |\alpha - 1|^2}. \tag{2.43}$$

For $x > 0$ and $\lambda = g_+(x)$, the equation $D_{\alpha,n}(g_+(x)) = 0$ is equivalent to

$$\tanh(x) = \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + (-1)^n|^2 \tanh^2\frac{nx}{2} + |\alpha - (-1)^n|^2}. \tag{2.44}$$

Proof. The expression (2.17) for the characteristic polynomial yields (2.43). The proof of (2.44) is analogous. □

Theorem 2.18 (characteristic equations for strong perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$, and $n > \max\{N_3(\alpha), N_4(\alpha)\}$. Then*

$$\lambda_{\alpha,n,1} = g_-(\vartheta_{\alpha,n,1}), \qquad \lambda_{\alpha,n,n} = g_+(\vartheta_{\alpha,n,n}),$$

where $\vartheta_{\alpha,n,1}$ is the unique positive solution of the equation

$$x = \operatorname{arctanh} \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + 1|^2 \tanh^2\frac{nx}{2} + |\alpha - 1|^2}, \tag{2.45}$$

and $\vartheta_{\alpha,n,n}$ is the unique positive solution of the equation

$$x = \operatorname{arctanh} \frac{2(|\alpha|^2 - 1)\tanh\frac{nx}{2}}{|\alpha + (-1)^n|^2 \tanh^2\frac{nx}{2} + |\alpha - (-1)^n|^2}. \tag{2.46}$$

For $2 \le j \le n-1$, $\lambda_{\alpha,n,j} = g(\vartheta_{\alpha,n,j})$, where $\vartheta_{\alpha,n,j}$ can be found as in Theorem 2.11.

Proof. For $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$ the conclusion follows directly from Proposition 2.17, for $\lambda_{\alpha,n,j}$ with $2 \le j \le n-1$ the conclusion follows similar to proof of Proposition 2.13. □

In what follows, we restrict ourselves to the analysis of the equation (2.43), because (2.44) is similar. Remark, in formula (2.15), $\tan\frac{nx}{2}$ is rapidly oscillating and $\cot(x)$ is much slower, therefore we solve (2.15) for $\tan\frac{nx}{2}$. The situation in (2.17) is different: if

x is separated from zero and n is large enough, then $\tanh \frac{nx}{2}$ is almost a constant, and we prefer to solve (2.17) for $\tanh(x)$.

Define $\psi_\alpha \colon [0,1] \to [0,+\infty)$ by

$$\psi_\alpha(t) := \frac{2(|\alpha|^2-1)t}{|\alpha+1|^2t^2+|\alpha-1|^2}. \tag{2.47}$$

Notice that

$$\psi_\alpha(1) = \frac{2(|\alpha|^2-1)}{|\alpha+1|^2+|\alpha-1|^2} = \frac{|\alpha|^2-1}{|\alpha|^2+1} = \tanh(\log|\alpha|). \tag{2.48}$$

See in Figure 2.6 a graph of both sides of (2.43), there we see that, from a certain point the function $\psi_\alpha(\tanh(nx/2))$ behaves as a constant, hence we are going to construct explicitly a left neighborhood of 1 where the values of ψ_α are close enough to $\tanh(\log|\alpha|)$.

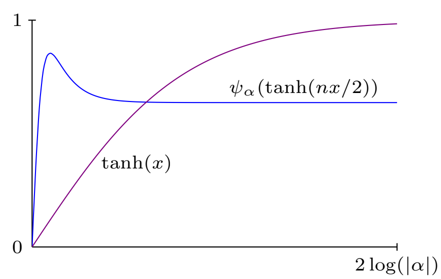


Figure 2.6: Function (2.43) for $\alpha = 2+i$ and $n = 8$.

Lemma 2.19. *Let $|\alpha| > 1$. Then for every t with*

$$1 - \frac{|\alpha|-1}{(|\alpha|+1)^3} \le t \le 1, \tag{2.49}$$

the following inequalities hold:

$$|\psi'_\alpha(t)| \le 1, \tag{2.50}$$

$$\tanh\frac{\log|\alpha|}{2} \le \psi_\alpha(t) \le \tanh\frac{3\log|\alpha|}{2}, \tag{2.51}$$

$$1 - \psi_\alpha^2(t) \ge \frac{2}{(|\alpha|+1)^3}. \tag{2.52}$$

Proof. Assumption (2.49) implies that

$$1-t^2 \le \frac{2(|\alpha|-1)}{(|\alpha|+1)^3}, \qquad |\alpha+1|^2 t^2 + |\alpha-1|^2 \ge \frac{2(|\alpha|^3+|\alpha|^2+2)}{|\alpha|+1}.$$

With these estimates we obtain (2.50). After that, the mean value theorem and (2.48) provide (2.51). Inequality (2.52) follows from (2.51). □

Define the segment

$$S_\alpha := \left[\frac{\log|\alpha|}{2}, \frac{3\log|\alpha|}{2}\right]. \tag{2.53}$$

We define $\varphi_{\alpha,n}$ on S_α as the right-hand side of (2.45),

$$\varphi_{\alpha,n}(x) := \operatorname{arctanh} \frac{2(|\alpha|^2-1)\tanh\frac{nx}{2}}{|\alpha+1|^2\tanh^2\frac{nx}{2} + |\alpha-1|^2}.$$

Proposition 2.20. *Let $\alpha\in\mathbb{C}$, $|\alpha|>1$, and $n>\max\{N_3(\alpha),N_4(\alpha)\}$. Then $\varphi_{\alpha,n}$ is contractive on S_α, and its fixed point coincides with $\vartheta_{\alpha,n,1}$.*

Proof. We represent $\varphi_{\alpha,n}$ as the following composition,

$$\varphi_{\alpha,n}(x) = \operatorname{arctanh}\left(\psi_\alpha\left(\tanh\frac{nx}{2}\right)\right).$$

For x in S_α, denote $\tanh\frac{nx}{2}$ by t. Then

$$1-t \le 2e^{-nx} \le 2e^{-N_2(\alpha)\frac{\log|\alpha|}{2}} < \frac{|\alpha|-1}{(|\alpha|+1)^3}.$$

Therefore, by (2.52) we have $\psi_\alpha(\tanh\frac{nx}{2})<1$, and the definition of $\varphi_{\alpha,n}$ makes sense. By (2.51), $\varphi_{\alpha,n}$ takes values in S_α. Estimate from above the derivative of $\varphi_{\alpha,n}$ using (2.52), (2.50), and the elementary inequality $ue^{-u}\le 1/e$:

$$\begin{aligned}|\varphi'_{\alpha,n}(x)| &\le \frac{|\psi'_\alpha(t)|}{1-\psi_\alpha^2(t)}\cdot\frac{n}{2\cosh^2\frac{nx}{2}} \le (|\alpha|+1)^3 n e^{-nx}\\ &\le (|\alpha|+1)^3 n e^{-\frac{n\log|\alpha|}{2}} = (|\alpha|+1)^3 n e^{-\frac{n\log|\alpha|}{4}} e^{-\frac{n\log|\alpha|}{4}}\\ &\le (|\alpha|+1)^3\cdot\frac{4}{\log|\alpha|}\cdot\frac{\log|\alpha|}{(|\alpha|+1)^5} = \frac{4}{(|\alpha|+1)^2} < 1.\end{aligned}$$

We have proved that $\varphi_{\alpha,n}$ is a contraction on S_α. By Theorem A.4 this implies that $\varphi_{\alpha,n}$ has a unique fixed point, and by Theorem 2.18 it coincides with $\vartheta_{\alpha,n,1}$. □

For Proposition 2.20 we have constructed a large series of examples with random values of α and n, denote by $\lambda^{\mathrm{fp}}_{\alpha,n,1}$ and $\lambda^{\mathrm{fp}}_{\alpha,n,n}$ the eigenvalues obtained by iteration of fixed point with high-precision arithmetic with 3322 binary digits,$\lambda^{\mathrm{gen}}_{\alpha,n,1}$ and $\lambda^{\mathrm{gen}}_{\alpha,n,n}$ the eigenvalues obtained by general algorithms with double-precision arithmetic. We also tested the eigenvalues approximation by fixed point iteration for the $2 \le j \le n-1$. For all these numerical experiments we obtained

$$\max_{j=1,\dots,n} |\lambda^{\mathrm{gen}}_{\alpha,n,j} - \lambda^{\mathrm{fp}}_{\alpha,n,j}| < 2 \cdot 10^{-13}. \tag{2.54}$$

See the Codes B.6 and B.7 for the main part of the written program for the extreme eigenvalues computations, for example in Code 2.3 we tested this program with parameters $\alpha = 2$ and $n = 64$.

Code 2.3: Test extreme eigenvalues computation by fixed point iteration

```
sage: load('Off_strong_test_extreme_eigvalues_by_fixed_point.sage')
sage: max_error_extreme_eigenvalues_gen_minus_fp(2,64,3322)
test extreme eigenvalues by fixed point iteration
n = 64
al = 2
maximal error = 3.553e-15
```

Proposition 2.21. *Let $|\alpha| > 1$, then exist $C_3(\alpha)$ positive constant such that for all $n > \max\{N_3(\alpha), N_4(\alpha)\}$*

$$\left|\vartheta_{\alpha,n,1} - \log|\alpha|\right| \le \frac{C_3(\alpha)}{|\alpha|^n}, \qquad |\vartheta_{\alpha,n,n} - \log|\alpha|| \le C_3(\alpha)/|\alpha|^n. \tag{2.55}$$

Proof. For brevity, put $x = \vartheta_{\alpha,n,1}$. Apply the mean value theorem to ψ_α, taking into account (2.50):

$$\left|\tanh(x) - \tanh(\log|\alpha|)\right| = \left|\psi_\alpha\left(\tan\frac{nx}{2}\right) - \psi_\alpha(1)\right| \le 1 - \tan\frac{nx}{2} \le 2e^{-nx}.$$

On the other hand, apply the mean value theorem to tanh:

$$|\tanh(x) - \tanh(\log|\alpha|)| \geq \frac{\big|x - \log|\alpha|\big|}{\cosh^2 \frac{3\log|\alpha|}{2}} \geq \frac{2}{|\alpha|^3}\big|x - \log|\alpha|\big|.$$

From this chain of inequalities,

$$\big|x - \log|\alpha|\big| \leq |\alpha|^3 e^{-nx}. \tag{2.56}$$

We already know from Proposition 2.20 that $x \geq \frac{\log|\alpha|}{2}$. Thus,

$$x \geq \log|\alpha| - |\alpha|^3 e^{-\frac{n\log|\alpha|}{2}}.$$

Using the elementary inequality $ue^{-u} \leq 1/e$ we get

$$nx \geq n\log|\alpha| - |\alpha|^3\, ne^{-\frac{n\log|\alpha|}{2}} \geq n\log|\alpha| - \frac{|\alpha|^3}{\log|\alpha|}. \tag{2.57}$$

By (2.56) and (2.57), inequality (2.55) holds. In a similar manner we prove the second equation in (2.55). □

In order to describe the asymptotic behavior of the extreme eigenvalues $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$, we introduce the following notation:

$$s_\alpha := |\alpha| - 2 + \frac{1}{|\alpha|}, \quad \text{i.e.} \quad s_\alpha = \frac{(|\alpha|-1)^2}{|\alpha|} = \left(\sqrt{|\alpha|} - \frac{1}{\sqrt{|\alpha|}}\right)^2. \tag{2.58}$$

For every $|\alpha| > 1$ and $n \geq 3$ we define:

$$\lambda_{\alpha,n,1}^{\text{asympt}} := -s_\alpha, \qquad \lambda_{\alpha,n,n}^{\text{asympt}} := 4 + s_\alpha. \tag{2.59}$$

Theorem 2.22 (asymptotic expansion of the eigenvalues for strong perturbations)**.** *Let $\alpha \in \mathbb{C}$, $|\alpha| > 1$. As n tends to infinity, the extreme eigenvalues of $A_{\alpha,n}$ converge exponentially* *to* $\lambda_{\alpha,n,1}^{\text{asympt}}$ *and* $\lambda_{\alpha,n,n}^{\text{asympt}}$*, respectively:*

$$|\lambda_{\alpha,n,1} - \lambda_{\alpha,n,1}^{\text{asympt}}| \le \frac{C_4(\alpha)}{|\alpha|^n}, \tag{2.60}$$

$$|\lambda_{\alpha,n,n} - \lambda_{\alpha,n,n}^{\text{asympt}}| \le \frac{C_4(\alpha)}{|\alpha|^n}. \tag{2.61}$$

Here $C_4(\alpha)$ *is a positive constant depending only on* α*. For* $2 \le j \le n-1$*, the eigenvalues* $\lambda_{\alpha,n,j}$ *satisfy the asymptotic formulas* (2.37)*.*

Proof. Since the derivatives of g_- and g_+ are bounded on $\left[0, \frac{3}{2}\log|\alpha|\right]$, by the mean value theorem we get

$$|\lambda_{\alpha,n,1} + s_\alpha| = |g_-(\vartheta_{\alpha,n,1}) - g_-(\log|\alpha|)| \le \left| g_-'\left(\frac{3}{2}\log|\alpha|\right)\right| |\vartheta_{\alpha,n,1} - \log|\alpha||\,,$$

using the first equation in (2.55) we obtain (2.60) with $C_4(\alpha) \coloneqq C_3(\alpha) g_-'(3\log(|\alpha|)/2)$. In a similar manner we prove (2.61). □

For Theorem 2.22, we have computed the errors

$$R_{\alpha,n,j} \coloneqq \lambda_{\alpha,n,j}^{\text{asympt}} - \lambda_{\alpha,n,j}^{\text{fp}} \qquad (2 \le j \le n-1),$$

and their maximums $\|R_{\alpha,n}\|_\infty = \max_{1\le j\le n} |R_{\alpha,n,j}|$, with both $\lambda_{\alpha,n,j}^{\text{asympt}}$ and $\lambda_{\alpha,n,j}^{\text{fp}}$ where computed in high-precision arithmetic with 3322 binary digits, the main parts of the code we used are similar to the Codes B.4 and B.5. Tables 2.2 show that these errors indeed can be bounded by $C_2(\alpha)/n^3$, and $C_2(\alpha)$ has to take bigger values when $|\alpha|$ is close to 1. We also tested (2.60) and (2.61). As n grows, $|\alpha|^n|R_{\alpha,n,1}|$ and $|\alpha|^n|R_{\alpha,n,n}|$ approach rapidly the same limit value depending on α. For example, we execute the Code B.9 and get

for $\alpha = 2 + i$, $\qquad \lim_{n\to\infty} (|\alpha|^n|R_{\alpha,n,1}|) \approx 2.86,$

for $\alpha = 0.8 - 0.7i$; $\qquad \lim_{n\to\infty} (|\alpha|^n|R_{\alpha,n,1}|) \approx 1.12 \cdot 10^{-2}.$

Proposition 2.23 (the eigenvectors for strong perturbations)**.** *Let* $\alpha \in \mathbb{C}$, $|\alpha| > 1$, $n > \max\{N_3(\alpha), N_4(\alpha)\}$*. Then the vectors* $v_{\alpha,n,1} \coloneqq [v_{\alpha,n,1,k}]_{k=1}^n$ *and* $v_{\alpha,n,n} \coloneqq [v_{\alpha,n,n,k}]_{k=1}^n$

Table 2.2: Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some $|\alpha| > 1$.

$\alpha = 2 + i,\quad |\alpha| \approx 2.23$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
64	1.55×10^{-4}	40.59
128	2.15×10^{-5}	45.18
256	2.82×10^{-6}	47.33
512	3.60×10^{-7}	48.36
1024	4.55×10^{-8}	48.86
2048	5.72×10^{-9}	49.10
4096	7.16×10^{-10}	49.22
8192	8.97×10^{-11}	49.29

$\alpha = 0.8 - 0.7i,\quad |\alpha| \approx 1.06$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
64	2.19×10^{-4}	57.51
128	2.19×10^{-5}	45.90
256	1.40×10^{-5}	235.36
512	2.99×10^{-6}	401.90
1024	4.55×10^{-7}	488.25
2048	6.16×10^{-8}	528.84
4096	7.98×10^{-9}	548.04
8192	1.01×10^{-9}	557.32

with components

$$v_{\alpha,1,n,k} = \sinh(k\vartheta_{\alpha,n,1}) + \overline{\alpha}\sinh((n-k)\vartheta_{\alpha,n,1}) \quad (1 \le k \le n), \tag{2.62}$$
$$v_{\alpha,n,n,k} = (-1)^k \sinh(k\vartheta_{\alpha,n,n})) + (-1)^{k+n}\,\overline{\alpha}\sinh((n-k)\vartheta_{\alpha,n,n}) \quad (1 \le k \le n), \tag{2.63}$$

are the eigenvectors of the matrix $A_{\alpha,n}$ associated to the eigenvalues $\lambda_{\alpha,1,n}$ and $\lambda_{\alpha,n,n}$, respectively. For $2 \le j \le n-1$, the vector $v_{\alpha,n,j}$ defined by (2.39) is an eigenvector of $A_{\alpha,n}$ associated to the eigenvalue $\lambda_{\alpha,n,j}$.

Proof. From Theorem 2.18 we have

$$\lambda_{\alpha,n,1} = 2 - (e^{\vartheta_{\alpha,n,1}} + e^{-\vartheta_{\alpha,n,1}}).$$

Let $\mathrm{b}_{-2,-\overline{\alpha},n}$ be the constant defined here (1.14), there $z = e^{\vartheta_{\alpha,n,1}}$, so

$$\mathrm{b}_{-2,-\overline{\alpha},n} = -2\overline{\alpha}\sinh(n\vartheta_{\alpha,n,1}).$$

Since $\vartheta_{\alpha,n,1} \ne 0$, then $\mathrm{b}_{-2,-\overline{\alpha},n} \ne 0$. So by Proposition 1.16 the vector v' with components

$$v'_k = \frac{\sinh(k\vartheta_{\alpha,n,1}) + \overline{\alpha}\sinh((n-k)\vartheta_{\alpha,n,1})}{\sinh(\vartheta_{\alpha,n,1})} \quad (1 \le k \le n),$$

is an eigenvector associated to $\lambda_{\alpha,n,j}$. Since every constant multiple of v' is also an eigenvector, then formula (2.62) follows by multiplying v'_k by $\sinh(\vartheta_{\alpha,n,1})$. In similar manner (2.63) is proven. For $2 \le j \le n-1$ the proof is the same as in 2.16. □

Motivated by (2.62), for every $|\alpha| > 1$, $n > \max\{N_3(\alpha), N_4(\alpha)\}$, we define for every x in $[0, 16]$ the function:

$$w_{\alpha,n,1}(x) \coloneqq \sinh(x\vartheta_{\alpha,n,1}) + \overline{\alpha}\sinh((n-x)\vartheta_{\alpha,n,1}). \tag{2.64}$$

Let $\alpha = 3/2$ and $n = 16$. See in Figure 2.7 the function $w_{\alpha,n,1}$ normalized by $\|v_{\alpha,n,1}\|$. The dots in red are the pairs $(k, w_{\alpha,n,1}(k)/\|v_{\alpha,n,1}\|)$.

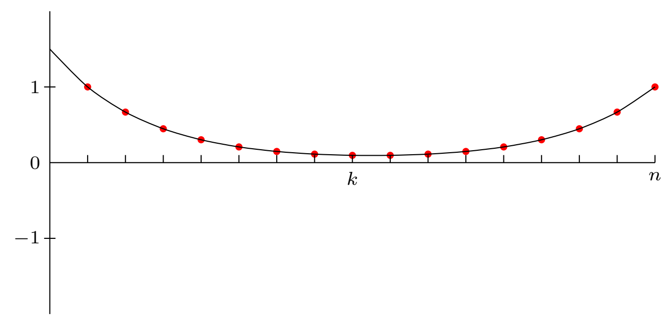


Figure 2.7: Function (2.64) normalized by $\|v_{\alpha,n,1}\|$, for $\alpha = 3/2$ 1 $n = 16$, scale 0.5 (x axis) to 1.0 (y axis).

2.4 Eigenvalues and eigenvectors of $A_{\alpha,n}$ with $|\alpha| = 1$

Finally we present the computations for the eigenpairs of $A_{\alpha,n}$ with $|\alpha| = 1$. Unlike the cases of weak and strong perturbations, we give exact formulas for the eigenvalues of $A_{\alpha,n}$ that are easily derived from the characteristic equation.

Proposition 2.24. *Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\alpha \neq \pm 1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{\alpha,n,j} = g(\vartheta_{\alpha,n,j})$ is an eigenvalue of $A_{\alpha,n}$, where*

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} - \frac{2}{n}\arctan\left(l_\alpha^{(-1)^j}\right). \tag{2.65}$$

Furthermore, $\vartheta_{\alpha,n,j} \in I_{n,j}$, and the vector with components (2.39) is an eigenvector of $A_{\alpha,n}$ associated to $\lambda_{\alpha,n,j}$.

Proof. The condition about α implies that $k_\alpha = 0$. In this case, the functions $\eta_{\alpha,1}$ and $\eta_{\alpha,2}$ are just constants

$$\eta_{\alpha,j}(x) = -2\arctan\left(l_\alpha^{(-1)^j}\right),$$

and the characteristic equation (2.24) simplifies to the direct formula (2.65).

Since $\alpha \neq \pm 1$ then $l_\alpha \neq 0$ nor $l_\alpha \neq \infty$, so $\vartheta_{\alpha,n,j} \neq j\pi/n$. Then the same proof of Proposition 2.16 applies for the eigenvectors of this case. □

Proposition 2.25. *Let $\alpha = 1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{1,n,j} = g(\vartheta_{1,n,j})$, where*

$$\vartheta_{1,n,j} = \left(j - \frac{1 - (-1)^j}{2} \right) \frac{\pi}{n} = \begin{cases} \dfrac{2q\pi}{n}, & j = 2q, \\ \dfrac{2q\pi}{n}, & j = 2q + 1. \end{cases} \tag{2.66}$$

The vector $v_{1,n,j} = [v_{1,n,j,k}]_{k=1}^n$ with components

$$v_{1,n,j,k} \coloneqq \sin\left(k\vartheta_{1,n,j} + \frac{(1 - (-1)^j)\pi}{4} \right) = \begin{cases} \sin \dfrac{2kq\pi}{n}, & j = 2q, \\ \cos \dfrac{2kq\pi}{n}, & j = 2q + 1, \end{cases} \tag{2.67}$$

is an eigenvector of $A_{1,n}$ associated to $\lambda_{1,n,j}$.

Proof. The numbers $\vartheta_{1,n,j}$ can be found by passing to the limit $\alpha \to 1^-$ in (2.24). The equalities $A_{1,n}v_{1,n,j} = \lambda_{1,n,j}v_{1,n,j}$ are easy to verify directly. □

Proposition 2.26. *Let $\alpha = -1$, $n \geq 3$, and $1 \leq j \leq n$. Then $\lambda_{-1,n,j} = g(\vartheta_{-1,n,j})$, where*

$$\vartheta_{-1,n,j} = \left(j - \frac{1 + (-1)^j}{2} \right) \frac{\pi}{n} = \begin{cases} \dfrac{(2q-1)\pi}{n}, & j = 2q - 1, \\ \dfrac{(2q-1)\pi}{n}, & j = 2q. \end{cases} \tag{2.68}$$

The vector $v_{-1,n,j} = [v_{-1,n,j,k}]_{k=1}^n$ with components

$$v_{-1,n,j,k} \coloneqq \sin\left(k\vartheta_{-1,n,j} + \frac{(1 + (-1)^j)\pi}{4} \right) = \begin{cases} \sin \dfrac{k(2q-1)\pi}{n}, & j = 2q - 1, \\ \cos \dfrac{k(2q-1)\pi}{n}, & j = 2q, \end{cases} \tag{2.69}$$

is an eigenvector of $A_{-1,n}$ associated to $\lambda_{-1,n,j}$.

Proof. Similar to the proof of Proposition 2.25. □

Chapter 3

Laplacian matrix of the cyclic graph with one weighted edge

In this chapter we study the individual behaviour of the eigenvalues of $n \times n$ matrices $L_{\alpha,n}$, depending on the real parameter α, of the following form:

$$L_{\alpha,7} = \begin{bmatrix} 1+\alpha & -1 & 0 & 0 & 0 & 0 & -\alpha \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix}. \tag{3.1}$$

This family matrices are a particular case of the matrix (1.3), in the notation of Section 1.2 $\mathsf{a} = \mathsf{d} = -(1+\alpha)$, $\mathsf{b} = \mathsf{c} = -\alpha$.

The matrix $L_{\alpha,n}$ is Hermitian, their eigenvalues are real, and we enumerate them in the ascending order:

$$\lambda_{\alpha,n,1} \le \lambda_{\alpha,n,2} \le \cdots \le \lambda_{\alpha,n,n}.$$

By definition of the Laplacian matrix of a simple undirected graph, the entries (j,k) and (k,j), for $j \neq k$, are equal to the weight of the edge $\{j,k\}$ with the opposite sign. The diagonal entry (j,j) is the sum of the weights of the edges $\{j,k\}$. It follows from this definition that the Laplacian matrix is symmetric, and the sum of the entries in each row

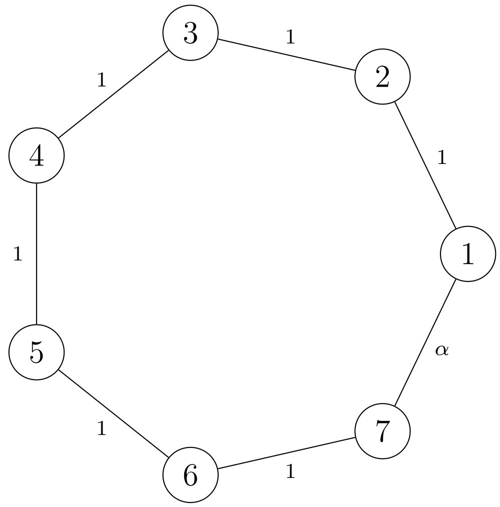


Figure 3.1: Cycle with one weighted edge

is 0. In other words, every Laplacian matrix has eigenvalue 0 associated to the eigenvector $[1, \ldots, 1]^\top$, see [24] for more information of this matrices.

Denote by $G_{\alpha,n}$ the cyclic graph of order n, where one of the edge has weight α real, and all others have weight 1, from previous definition is clear that, $L_{\alpha,n}$ is the Laplacian matrix of the graph in Figure 3.1.

In Section 3.1 we study the characteristic polynomial of $L_{\alpha,n}$, we extract from it some properties that allow us to determine that, for weak perturbations ($0 < \alpha < 1$) the eigenvalues are on the interval $[0, 4]$, and for strong left or right perturbations ($\alpha < 0$ or $1 < \alpha$), respectively the eigenvalue $\lambda_{\alpha,n,1}$ or $\lambda_{\alpha,n,n}$ lies outside $[0, 4]$.

Forward in Section 3.2, we prove that the eigenvalues of $L_{\alpha,n}$ with weak perturbations can be computed by iteration of fixed point, moreover we write asymptotic expansions for them.

In Sections 3.3 and 3.4 we study the extreme eigenvalues of $L_{\alpha,n}$ with strong perturbations, we show that these eigenvalues can be obtained by the fixed point iteration; moreover we show that as n tends to infinity they converge to some values determined only on α.

Finally in Section 3.5 we give extact formulas for the eigenvalues of the particular matrix $L_{0,n}$.

For all cases of perturbations we give formulas for eigenvectors of $L_{\alpha,n}$.

Remark. Let α in $\mathbb{C}$, we define $L_{\alpha,n}$ as before but with values $(1 + \overline{\alpha})$ and $-\overline{\alpha}$, respectively in the entries $(1,1)$ and $(n,1)$. In Proposition 3.1 we prove that the characteristic equation only depends of $\mathrm{Re}(\alpha)$, this implies that even for α complex, all the eigenvalues of $L_{\alpha,n}$ are real. This is the reason why in almost every result and reasoning concerning to the eigenvalues we think α as a real number, and if we want to extend the forward analysis to the complex plane we just need to replace $\mathrm{Re}(\alpha)$ instead of α where needed.

Remark. The formulas we provide for the eigenvectors of $L_{\alpha,n}$ in Propositions 3.22, 3.27 and 3.35 are considered with α real; this is because, in the analysis performed for the eigenvalues we consider only α real. However, these propositions can easily be generalized to the case of α complex, by just replacing $\overline{\alpha}$ in the correspondent formula.

3.1 Characteristic equation and eigenvalues localization of $L_{\alpha,n}$

Through all the chapter $D_{\alpha,n}(\lambda)$ represents $\det(\lambda I_n - L_{\alpha,n})$ and T_n, U_n, V_n and W_n the Chebyshev polynomials defined in Section 1.1.

Recall that if α is a complex number, then we define $L_{\alpha,n}$ as before, but with values $(1+\overline{\alpha})$ and $-\overline{\alpha}$, respectively in the entries $(1,1)$ and $(n,1)$. For example if $n=7$

$$L_{\alpha,7} = \begin{bmatrix} 1+\overline{\alpha} & -1 & 0 & 0 & 0 & 0 & -\overline{\alpha} \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -\alpha & 0 & 0 & 0 & 0 & -1 & 1+\alpha \end{bmatrix}.$$

Proposition 3.1. *For every α, λ in $\mathbb{C}$ and $n \geq 3$,*

$$\begin{aligned} D_{\alpha,n}(\lambda) = (\lambda - 2\,\mathrm{Re}(\alpha))U_{n-1}\left(\frac{\lambda-2}{2}\right) - 2\,\mathrm{Re}(\alpha)U_{n-2}\left(\frac{\lambda-2}{2}\right) \\ + 2(-1)^{n+1}\,\mathrm{Re}(\alpha). \end{aligned} \tag{3.2}$$

Proof. Under the assumptions of Proposition 1.13 we have for the matrix $L_{\alpha,n}$ that

$\alpha = -1 - \overline{\alpha}$, $\delta = -1 - \alpha$, $\beta = \overline{\alpha}$ and $\gamma = \alpha$, then equation (1.8) transforms directly in

$$D_{\alpha,n}(\lambda) = U_n\left(\frac{\lambda-2}{2}\right) + 2(1-\operatorname{Re}(\alpha))U_{n-1}\left(\frac{\lambda-2}{2}\right) \\ + (1-2\operatorname{Re}(\alpha))U_{n-2}\left(\frac{\lambda-2}{2}\right) + 2(-1)^{n+1}\operatorname{Re}(\alpha).$$

Substitute in last expression the recurrence relation

$$U_n\left(\frac{\lambda-2}{2}\right) = (\lambda-2)U_{n-1}\left(\frac{\lambda-2}{2}\right) - U_{n-2}\left(\frac{\lambda-2}{2}\right),$$

then the conclusion follows. $\square$

From now onward we consider α to be only a real number.

For every α in $\mathbb{R}$ and every $n \geq 3$ we define on $\mathbb{R}$ the functions:

$$E_{\alpha,n}(\lambda) \coloneqq \begin{cases} 2\lambda U_{\frac{n}{2}-1}\left(\frac{\lambda-2}{2}\right), & \text{if } n \text{ is even,} \\ \lambda V_{\frac{n-1}{2}}\left(\frac{\lambda-2}{2}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.3}$$

$$F_{\alpha,n}(\lambda) \coloneqq \begin{cases} (1-\alpha)T_{\frac{n}{2}}\left(\frac{\lambda-2}{2}\right) - \alpha\frac{4-\lambda}{2}U_{\frac{n}{2}-1}\left(\frac{\lambda-2}{2}\right), & \text{if } n \text{ is even,} \\ (1-\alpha)W_{\frac{n-1}{2}}\left(\frac{\lambda-2}{2}\right) + \alpha V_{\frac{n-1}{2}}\left(\frac{\lambda-2}{2}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.4}$$

Proposition 3.2. *Let* $\alpha,\ \lambda \in \mathbb{R}$ *and* $n \geq 3$*. Then*

$$D_{\alpha,n}(\lambda) = E_{\alpha,n}(\lambda)F_{\alpha,n}(\lambda). \tag{3.5}$$

Proof. For simplicity put $\lambda = 2 + 2t$. Suppose $n = 2m$. Apply the recurrence relation $U_{2m-2}(t) = -U_{2m}(t) + 2tU_{2m-1}(t)$ to (3.2),

$$D_{\alpha,2m}(2t+2) = 2\left[\alpha U_{2m}(t) + (t+1-\alpha-2\alpha t)U_{2m-1}(t) - \alpha\right].$$

Substitute in last expression the relations:

$$\begin{aligned} U_{2m-1}(t) &= 2U_{m-1}(t)T_m(t), \\ U_{2m}(t) &= 2tU_{m-1}(t)T_m(t) + 2T_m^2(t) - 1, \end{aligned}$$

then

$$D_{\alpha,2m}(2t+2) = 4\big[(t+1)(1-\alpha)U_{m-1}(t)T_m(t) + \alpha(T_m^2 - 1)\big].$$

Equation (3.5) follows after applying $T_m^2(t) - 1 = (t^2-1)U_{m-1}^2(t)$.

Now suppose $n = 2m-1$. Use the recurrence relation $U_{2m-3}(t) = -U_{2m-1}(t) + 2tU_{2m-2}(t)$ on (3.2),

$$D_{\alpha,2m-1}(2+2t) = 2\big[(t+1-\alpha-2\alpha t)U_{2m-2}(t) + \alpha U_{2m-1}(t) + \alpha\big].$$

Equation (3.5) follows after substitute the relations:

$$\begin{aligned} U_{2m-2}(t) &= W_{m-1}(t)V_{m-1}(t), \\ U_{2m-1}(t) &= tW_{m-1}(t)V_{m-1}(t) + (1+t)V_{m-1}^2(t) - 1. \end{aligned}$$

□

Now, the problem of obtaining the roots of $D_{\alpha,n}$ transforms in obtaining in independently fashion the roots of $E_{\alpha,n}$ and $F_{\alpha,n}$.

Recall that g, g_- and g_+ are the functions defined by (2.2).

Proposition 3.3. *Let α in $\mathbb{R}$ and $n \geq 3$, then*

$$E_{\alpha,n}(0) = 0, \tag{3.6}$$

$$E_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = 0 \quad (1 \leq j \leq n-1,\ \textit{even}). \tag{3.7}$$

Proof. Notice that, independently of the parity of n, λ is a factor of $E_{\alpha,n}$, then (3.6) follows.

Using the trigonometric formulas in 1.5 we can see that $j\pi/n$, with j even, are zeros of $U_{\frac{n}{2}-1}(-\cos(x))$ and of $V_{\frac{n-1}{2}}(-\cos(x))$, respectively if n is even or odd. Then (3.7) follows. □

Corollary 3.4. *Let α in $\mathbb{R}$, $n \geq 3$ and $1 \leq j \leq n-1$, where j is even. Then the numbers 0 and $g(j\pi/n)$ are eigenvalues of $L_{\alpha,n}$.*

Proof. By Proposition 3.3, the numbers of the hypothesis are zeros of $D_{\alpha,n,}$, hence eigenvalues of $L_{\alpha,n}$. □

Unlike $E_{\alpha,n}$, the zeros of the polynomial $F_{\alpha,n}$ are not trivial. In order to obtain these zeros in later sections we study the method of fixed iteration point on functions that are a slightly variation of the polynomial $F_{\alpha,n}$. For now, we can derive the following propositions.

Proposition 3.5. *Let $\alpha \in \mathbb{R}$ and $n \geq 3$. If n is even and $1 \leq j \leq n-1$, then*

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = (1-\alpha)(-1)^{\frac{n+j}{2}} \quad (j \text{ even}), \tag{3.8}$$

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = \frac{\alpha\left(1+\cos\frac{j\pi}{n}\right)}{\sin\frac{j\pi}{n}}(-1)^{\frac{n+j-1}{2}} \quad (j \text{ odd}), \tag{3.9}$$

$$(-1)^{n/2}F_{\alpha,n}(0) = (1-\alpha)+\alpha n, \quad F_{\alpha,n}(4) = 1-\alpha. \tag{3.10}$$

Proof. Let $1 \leq j \leq n-1$. If we evaluate $F_{\alpha,n}$ at the points $g(j\pi/n)$, then one of the terms in (3.4) is always zero. So,

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = (1-\alpha)T_{\frac{n}{2}}\left(-\cos\frac{j\pi}{n}\right) \quad (j \text{ even}),$$
$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = -\alpha\left(1+\cos\frac{j\pi}{n}\right)U_{\frac{n}{2}-1}\left(-\cos\frac{j\pi}{n}\right) \quad (j \text{ odd}).$$

Hence, (3.8) and (3.9) follows by Proposition 1.5. Moreover,

$$F_{\alpha,n}(0) = (1-\alpha)T_{\frac{n}{2}}(-1) - 2\alpha U_{\frac{n}{2}-1}(-1),$$
$$F_{\alpha,n}(4) = (1-\alpha)T_{\frac{n}{2}}(1).$$

Therefore, (3.10) follows from Corollary 1.6. □

Proposition 3.6. *Let $\alpha \in \mathbb{R}$ and $n \geq 3$. If n is odd and $1 \leq j \leq n-1$, then*

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = \frac{1-\alpha}{\cos\frac{j\pi}{2n}}(-1)^{\frac{n+j-1}{2}} \quad (j \text{ even}), \tag{3.11}$$

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = \frac{\alpha}{\sin\frac{j\pi}{2n}}(-1)^{\frac{n+j-2}{2}} \quad (j \text{ odd}), \tag{3.12}$$

$$(-1)^{(n-1)/2}F_{\alpha,n}(0) = 1-\alpha+\alpha n, \quad F_{\alpha,n}(4) = (1-\alpha)n+\alpha. \tag{3.13}$$

Proof. Recall that for every t and m, $W_m(t) = (-1)^m V_m(-t)$. Let $1 \leq j \leq n-1$, then

$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = (1-\alpha)W_{\frac{n-1}{2}}\left(-\cos\frac{j\pi}{n}\right) \quad (j \text{ even}),$$
$$F_{\alpha,n}\left(g\left(\frac{j\pi}{n}\right)\right) = \alpha V_{\frac{n-1}{2}}\left(-\cos\frac{j\pi}{n}\right) \quad (j \text{ odd}).$$

So, (3.11) and (3.12) follows by Proposition 1.5. Moreover,

$$(-1)^{(n-1)/2}F_{\alpha,n}(0) = (1-\alpha)W_{\frac{n-1}{2}}(-1) + \alpha V_{\frac{n-1}{2}}(-1),$$
$$F_{\alpha,n}(4) = (1-\alpha)W_{\frac{n-1}{2}}(1) + \alpha V_{\frac{n-1}{2}}(1).$$

Hence, (3.13) follows from Corollary 1.6. □

Let $I_{n,j}$ be the interval defined by (2.7), and define for every $\alpha \neq 0$,

$$\varkappa_\alpha := \frac{1-\alpha}{\alpha}. \tag{3.14}$$

We have the relations

$$\begin{aligned} 0 < \varkappa_\alpha < \infty \quad &\text{if} \quad 0 < \alpha < 1, \\ -\infty < \varkappa_\alpha < -1 \quad &\text{if} \quad \alpha < 0, \\ -1 < \varkappa_\alpha < 0 \quad &\text{if} \quad 1 < \alpha. \end{aligned} \tag{3.15}$$

We refer to these cases as weak, left strong and right strong perturbations, respectively.

Theorem 3.7 (eigenvalues localization for weak perturbations)**.** *Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, and $n \geq 3$. Then the matrix $L_{\alpha,n}$ has n different eigenvalues belonging to $[0,4]$. More* *precisely, for every j with $1 \leq j \leq n$,*

$$\lambda_{\alpha,n,j} = g\left(\frac{(j-1)\pi}{n}\right) \quad (j \text{ odd}), \tag{3.16}$$

$$g\left(\frac{(j-1)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\frac{j\pi}{n}\right) \quad (j \text{ even}). \tag{3.17}$$

Proof. Let $1 \leq j \leq n$. If j is odd, then (3.16) follows by Corollary 3.4.

From propositions 3.5 and 3.6 we obtain that, $F_{\alpha,n}$ change of sign in the intervals $I_{n,j}$, $1 \leq j \leq n-1$ even. Therefore, if j is even, by the intermediate value theorem we have (3.17). □

Theorem 3.8 (localization of eigenvalues for strong left perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha < 0$, and $n \geq 3$.*

1) If $n < |\kappa_\alpha|$, then $\lambda_{\alpha,n,1} = 0$ and $0 < \lambda_{\alpha,n,2} < g\left(\frac{\pi}{n}\right)$.
2) If $n = |\kappa_\alpha|$, then $\lambda_{\alpha,n,1} = \lambda_{\alpha,n,2} = 0$.
3) If $n > |\kappa_\alpha|$, then $\lambda_{\alpha,n,1} < 0$ and $\lambda_{\alpha,n,2} = 0$.

Furthermore, for every j such that $3 \leq j \leq n$,

$$\lambda_{\alpha,n,j} = g\left(\frac{(j-1)\pi}{n}\right) \qquad (j \text{ odd}), \tag{3.18}$$

$$g\left(\frac{(j-2)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\frac{(j-1)\pi}{n}\right) \qquad (j \text{ even}). \tag{3.19}$$

Proof. From propositions 3.5 and 3.6 we obtain the following sentences.

1. If $n < |(1-\alpha)/\alpha|$, then $F_{\alpha,n}$ change of sign in the interval $I_{n,1}$
2. If n is even and satisfies $n = |(1-\alpha)/\alpha|$, then $F_{\alpha,n}(-1) = 0$.
3. If $|(1-\alpha)/\alpha| < n$ then change of sign in the interval $(-\infty, -1)$.

Hence, the sentences *1)*, *2)* and *3)* hold.

If $3 \leq j \leq n$, the conclusions follows as in the proof of Theorem 3.7 □

Theorem 3.9 (Localization of eigenvalues for strong right perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha > 1$, $n \geq 3$.*

1) If $n < 1/|\kappa_\alpha|$ odd, then $g\left(\frac{(n-1)\pi}{n}\right) < \lambda_{\alpha,n,n} < g(\pi) = 4$.
2) If $n = 1/|\kappa_\alpha|$ odd, then $\lambda_{\alpha,n,n} = 1/|\kappa_\alpha|$.
3) If n even or if odd and $n > 1/|\kappa_\alpha|$, then $4 < \lambda_{\alpha,n,n}$.

Furthermore $\lambda_{n,1} = 0$, and for every j such that $2 \le j \le n-1$,

$$, \lambda_{\alpha,n,j} = g\left(\frac{j\pi}{n}\right) \qquad (j \text{ even}), \tag{3.20}$$

$$g\left(\frac{(j-1)\pi}{n}\right) < \lambda_{\alpha,n,j} < g\left(\frac{j\pi}{n}\right) \qquad (j \text{ odd}). \tag{3.21}$$

Proof. From propositions 3.5 and 3.6 we obtain the following sentences.

1. If n is odd and satisfies $n < |\alpha/(1-\alpha)|$, then $F_{\alpha,n}$ change of sign in $I_{n,n}$.
2. If $|\alpha/(1-\alpha)| < n$, then $F_{\alpha,n}$ change of sign in the interval $(1,\infty)$.
3. If n is odd and satisfies $n = |(1-\alpha)/\alpha|$, then $F_{\alpha,n}(1) = 0$, hence $\lambda = 4$ is an eigenvalue of $L_{\alpha,n}$.

Hence, the sentences *1)*, *2)* and *3)* hold.

If $1 \le j \le n-1$, the conclusion follows as in the proof of Theorem 3.7. □

Theorem 3.7 implies immediately that for every $0 < \alpha < 1$ and for every v in $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{\#\{j \in \{1,\dots,n\}\colon\ \lambda_{\alpha,n,j} \le v\}}{n} = \frac{\mu\left(\{x \in [0,\pi]\colon g(x) \le v\}\right)}{\pi}, \tag{3.22}$$

i.e., the eigenvalues of $L_{\alpha,n}$ are asymptotically distributed as the function g on $[0,\pi]$. For every $\alpha < 0$ ($\alpha > 1$), Theorem 3.8 (Theorem 3.9) implies a similar conclusion for all the eigenvalues except for the minimal one $\lambda_{\alpha,n,1}$ (maximal $\lambda_{\alpha,n,n}$). We see in Section 3.3 (Section 3.4) that $\lambda_{\alpha,n,1}$ ($\lambda_{\alpha,n,n}$) converge to a determined value as n tends to infinity, hence, there is a determined gap between $\{\lambda_{\alpha,n,1}\}$ ($\{\lambda_{\alpha,n,n}\}$) and $\{\lambda_{\alpha,n,j}\colon\ j \in \{2,\dots,n-1\}\}$ as n tends to infinity.

Motivated by (3.17), (3.19) and (3.21), if $\lambda_{\alpha,n,j}$ is in $[0,4]$, then we use the function g as a change of variable in the characteristic equation and put $\vartheta_{\alpha,n,j} := g^{-1}(\lambda_{\alpha,n,j})$. For example, inequalities (3.17) and (3.16), respectively, are equivalent to

$$\vartheta_{\alpha,n,j} = \frac{(j-1)\pi}{n} \qquad (j \text{ odd}), \tag{3.23}$$

$$\frac{(j-1)\pi}{n} < \vartheta_{\alpha,n,j} < \frac{j\pi}{n} \qquad (j \text{ even}). \tag{3.24}$$

In similar fashion, by Theorem 3.8, if $\alpha < 0$ and $\lambda_{\alpha,n,1} < 0$, then we put $\vartheta_{\alpha,n,1} \coloneqq g_{-}^{-1}(\lambda_{\alpha_n,1})$. Analogously, by Theorem 3.9, if $\alpha > 1$ and $\lambda_{\alpha,n,n} > 4$, then we put $\vartheta_{\alpha,n,n} \coloneqq g_{+}^{-1}(\lambda_{\alpha_n,n})$.

By Corollary 3.4, the function $E_{\alpha,n}$ yields the "half" of the eigenvalues of $L_{\alpha,n}$. The other half are going to be obtained by solving $F_{\alpha,n}$. With this in mind, we now make the change of variable $\lambda = g(x)$.

Proposition 3.10. *Let $\alpha \in \mathbb{R}$, $n \geq 3$ and $x \in (0,\pi)$. Then*

$$\begin{aligned} F_{\alpha,n}(g(x)) &= (-1)^{n/2} \cot\frac{x}{2} \cos\frac{nx}{2} \left(\alpha \tan\frac{nx}{2} - (\alpha-1)\tan\frac{x}{2}\right) \quad \text{if } n \text{ is even}, \\ F_{\alpha,n}(g(x)) &= \frac{(-1)^{(n-1)/2}\cos\frac{nx}{2}}{\sin\frac{x}{2}} \left(\alpha \tan\frac{nx}{2} - (\alpha-1)\tan\frac{x}{2}\right) \quad \text{if } n \text{ is odd}. \end{aligned} \tag{3.25}$$

Proof. Write λ as $g(x) = 2 - 2\cos(x)$. Then, by Propositions 1.5 and 1.9 the polynomial $F_{\alpha,n}$ (3.4) transforms in (3.25). □

Proposition 3.11. *Let $\alpha \in \mathbb{R}$, $n \geq 3$, and $x \in (0,\pi)$. Then*

$$D_{\alpha,n}(g(x)) = \frac{4(-1)^{n+1}\tan\frac{nx}{2}}{1+\tan^2\frac{nx}{2}} \left(\alpha \tan\frac{nx}{2} - (\alpha-1)\tan\frac{x}{2}\right), \tag{3.26}$$

and

$$D_{\alpha,n}(g(x)) = \frac{2(-1)^n}{\sin(x)} \Big(\sin(nx)(1-\cos(x))(\alpha-1) - \alpha\sin(x)(1-\cos(nx))\Big). \tag{3.27}$$

Proof. Write λ as $g(x) = 2 - 2\cos(x)$. By (3.5)

$$D_{\alpha,n}(g(x)) = E_{\alpha,n}(g(x))F_{\alpha,n}(g(x)).$$

Supoose n is even. From Propositions 1.5, 1.9 and 3.10

$$D_{\alpha,n}(g(x)) = 2(-1)^{n+1}\sin(nx)\left(\alpha\tan\frac{nx}{2} - (\alpha-1)\tan\frac{x}{2}\right). \tag{3.28}$$

Hence, (3.26) follows by applying the identity

$$\sin(nx) = \frac{2\tan\frac{nx}{2}}{1+\tan^2\frac{nx}{2}}.$$

If n is odd the proof is similar.

Finally, (3.27) follows by applying to (3.28) the identities

$$\tan\frac{nx}{2} = \frac{1-\cos(nx)}{\sin(nx)}, \quad \tan\frac{x}{2} = \frac{1-\cos(x)}{\sin(x)}. \qquad \square$$

Due to Proposition 3.10, equation $F_{\alpha,n}(g(x)) = 0$ is equivalent to

$$\tan\frac{nx}{2} = -\kappa_\alpha\tan\frac{x}{2}. \tag{3.29}$$

See in Figure 3.2 a graph of both sides of (3.29) for some α in $(0,1)$. From this figure we confirm that the Theorem 3.7 really yields the intervals where the intersections take place. On the other hand in the graphs of Figure 3.3 where α is not in $(0,1)$, is evident that the sign and direction of $\kappa_\alpha\tan(x/2)$ has changed, and as the theorems 3.8 and 3.9 pointed out, we see that there is no solution of (3.29) in the extreme intervals $I_{n,1}$, $I_{n,n}$, respectively.

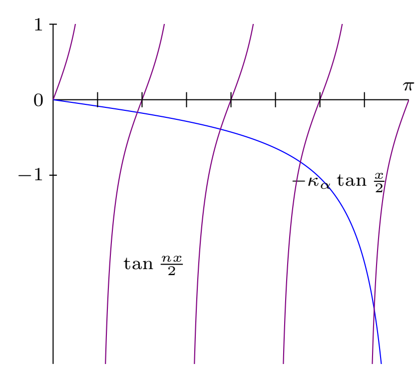


Figure 3.2: Functions in (3.29) for $\alpha = 0.7$ and $n = 8$, scale 1.5 (x axis) to 1(y axis).

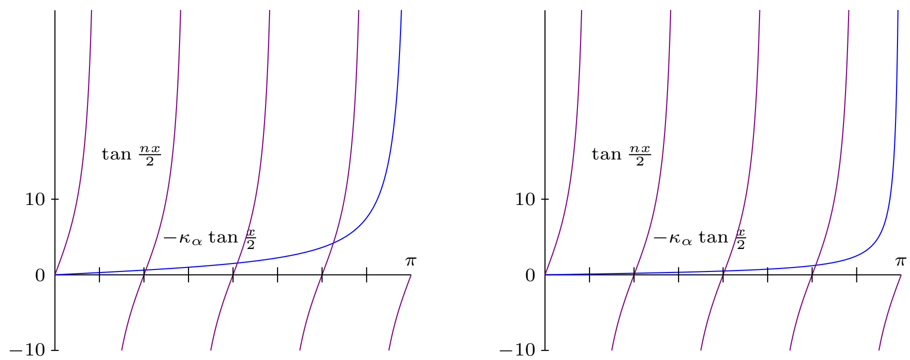


Figure 3.3: Functions in (3.29) for $\alpha = -2$ and $n = 8$ (left) and $\alpha = 2$ and $n = 8$ (right), scale 1.5 (x axis) to 0.1 (y axis).

If $\alpha < 0$ or $\alpha > 1$, then Theorem 3.8, respectively Theorem 3.9, say that, for n sufficiently big, using the change of variable $\lambda = g(x)$ is not enough to find all eigenvalues of $L_{\alpha,n}$. We now make the changes of variable $\lambda = g_-(x)$ and $\lambda = g_+(x)$, where x is in $(0, \infty)$.

Proposition 3.12. *Let $\alpha \in \mathbb{R}$, $n \geq 3$, $x \in (0, \infty)$. Then*

$$\begin{aligned} F_{\alpha,n}(g_-(x)) &= \frac{(-1)^{n/2}\cosh\frac{nx}{2}}{\tanh\frac{x}{2}}\left(\alpha\tanh\frac{nx}{2} - (\alpha-1)\tanh\frac{x}{2}\right) \quad (n \text{ even}),\\ F_{\alpha,n}(g_-(x)) &= \frac{(-1)^{(n-1)/2}\cosh\frac{nx}{2}}{\sinh\frac{x}{2}}\left(\alpha\tanh\frac{nx}{2} - (\alpha-1)\tanh\frac{x}{2}\right) \quad (n \text{ odd}), \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} F_{\alpha,n}(g_+(x)) &= \cosh\frac{nx}{2}\tanh\frac{x}{2}\left(\alpha\tanh\frac{nx}{2} + (1-\alpha)\coth\frac{x}{2}\right) \quad (n \text{ even}),\\ F_{\alpha,n}(g_+(x)) &= \frac{\cosh\frac{nx}{2}}{\sinh\frac{x}{2}}\left((1-\alpha)\tanh\frac{nx}{2} + \alpha\tanh\frac{x}{2}\right) \quad (n \text{ odd}). \end{aligned} \tag{3.31}$$

Proof. The proof is similar to proof of Proposition 3.10. □

Proposition 3.13. *Let $\alpha \in \mathbb{R}$, $n \geq 3$ and $x > 0$. Then*

$$D_{\alpha,n}(g_-(x)) = \frac{4(-1)^n\tanh\frac{nx}{2}}{1-\tanh^2\frac{nx}{2}}\left(\alpha\tanh\frac{nx}{2} + (1-\alpha)\tanh\frac{x}{2}\right), \tag{3.32}$$

and

$$
\begin{aligned}
D_{\alpha,n}(g_+(x)) &= \frac{4\tanh\frac{nx}{2}}{1-\tanh^2\frac{nx}{2}}\left(\alpha\tanh\frac{nx}{2} + (1-\alpha)\coth\frac{x}{2}\right) \quad (n \ \textit{even}),\\
D_{\alpha,n}(g_+(x)) &= 4\coth\frac{x}{2}\cosh^2\frac{nx}{2}\left((1-\alpha)\tanh\frac{nx}{2} + \alpha\tanh\frac{x}{2}\right) \quad (n \ \textit{odd}).
\end{aligned} \tag{3.33}
$$

Proof. The proof is similar to proof of Proposition 3.11. □

By (3.30), we note that finding the zero of $F_{\alpha,n}(g_-(x))$ is a problem of finding the number x for which the next equation have solution

$$
\tanh\frac{nx}{2} = -\kappa_\alpha \tanh\frac{x}{2}. \tag{3.34}
$$

Analogously, to get the zero of $F_{\alpha,n}(g_+(x))$, by (3.31), we just need to look for the number x such that one of the next equations hold

$$
\tanh\frac{nx}{2} = -\kappa_\alpha \coth\frac{x}{2}, \qquad \tanh\frac{nx}{2} = -\frac{1}{\kappa_\alpha}\tanh\frac{x}{2}, \tag{3.35}
$$

respectively if n is even or odd.

For every α in $\mathbb{R}\setminus[0,1]$, define

$$
\rho_\alpha := \log|2\alpha - 1|. \tag{3.36}
$$

See in Figure 3.4 a graph of both sides of (3.34) for some $\alpha < 0$, and in Figure 3.5 the graphs of both sides of the two equations in (3.35) for $\alpha > 1$. In these figures the drawn curves intersect just once, even more, the point x that satisfies (3.34) or (3.35) is close to ρ_α. This is not a mere coincidence since we prove later that as n is bigger the point x gets closer to ρ_α.

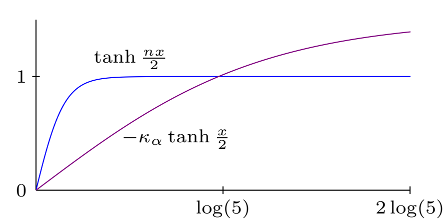


Figure 3.4: Left and right functions of (3.34) for $\alpha = -2$ and $n = 8$.

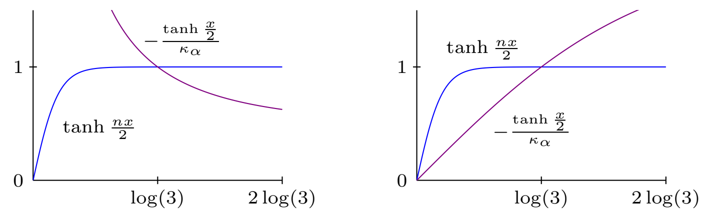


Figure 3.5: Left and right functions of (3.35) for $\alpha = 2$ and $n = 8$ (left), $n = 9$ (right).

The localization given in Theorems 3.8 and 3.9 for the extreme eigenvalues that are outside $[0, 4]$, is kind of coarse. To have a more precise localization, we start with the idea that $\tanh(nx/2)$ increases faster from 0 to 1 than $\tanh(x/2)$, then we can in certain way omit this factor in (3.34) and (3.35), i.e., replace $\tanh(nx/2)$ by 1, and then solve for x the equations

$$\tanh\frac{x}{2} = \frac{1}{|\kappa_\alpha|} \quad (\alpha < 0), \quad \tanh\frac{x}{2} = |\kappa_\alpha| \quad (1 < \alpha). \tag{3.37}$$

Taking into consideration the relations given of κ_α (3.15), direct computations shows that the solutions for (3.37) are respectively ρ_α (3.36).

Proposition 3.14. *Let $\alpha < 0$ and $n > |\kappa_\alpha|$. Then $g_-(\rho_\alpha) < \lambda_{\alpha,n,1} < 0$.*

Proof. Since ρ_α is solution of (3.37), then for every $x > \rho_\alpha$

$$\tanh\frac{nx}{2} < 1 = -\kappa_\alpha \tanh\frac{\rho_\alpha}{2} < -\kappa_\alpha \tanh\frac{x}{2}.$$

By Theorem 3.8, $\vartheta_{\alpha,n,1}$ is solution of (3.34). Hence, by last inequality we must have $\vartheta_{\alpha,n,1} < \rho_\alpha$. □

Proposition 3.15. *Let $\alpha > 1$ and $n \geq 3$. If n is even, then $g_+(\rho_\alpha) < \lambda_{\alpha,n,n}$ and, if n is odd and satisfies $n > 1/|\kappa_\alpha|$, then $4 < \lambda_{\alpha,n,n} < g_+(\rho_\alpha)$.*

Proof. Let n be even. Since ρ_α is solution of (3.37), then for every $x < \rho_\alpha$

$$\tanh\frac{nx}{2} < 1 = -\kappa_\alpha \coth\frac{\rho_\alpha}{2} < -\kappa_\alpha \coth\frac{x}{2}.$$

By Theorem 3.9, $\vartheta_{\alpha,n,n}$ is solution of the first equation in (3.35). Hence, by last inequality we must have $\rho_\alpha < \vartheta_{\alpha,n,n}$.

Let n be odd and $n > 1/|\kappa_\alpha|$. Since ρ_α is solution of (3.37), then for every $x > \rho_\alpha$

$$-\kappa_\alpha \coth \frac{x}{2} < -\kappa_\alpha \coth \frac{\rho_\alpha}{2} = 1 < \coth \frac{nx}{2}.$$

By Theorem 3.9, $\vartheta_{\alpha,n,n}$ is solution of the second equation in (3.35), hence, by last inequality we must have $\vartheta_{\alpha,n,n} < \rho_\alpha$ □

For every α in $\mathbb{R} \setminus [0, 1]$, define

$$s_\alpha := \frac{4\alpha^2}{|2\alpha - 1|}, \tag{3.38}$$

then we have

$$\begin{aligned} g_-(\rho_\alpha) &= -s_\alpha \qquad (\alpha < 0), \\ g_+(\rho_\alpha) &= s_\alpha \qquad (\alpha > 1). \end{aligned} \tag{3.39}$$

3.2 Eigenvalues and eigenvectors of $L_{\alpha,n}$ with weak perturbations $0 < \alpha < 1$

Let $0 < \alpha < 1$ and $n \geq 3$. As before, define $I_{n,j}$ by (2.7) and κ_α by (3.14). In this section we are interested in solutions of (3.29). By the Theorem of localization 3.7 we know that, if j is even and satisfies $1 \leq j \leq n$, then there exist $\vartheta_{\alpha,n,j}$ (3.24) in $I_{n,j}$ solution to (3.29).

Theorem 3.16 (characteristic equation for weak perturbations)**.** *Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, $n \geq 3$, and $1 \leq j \leq n$, where j is even. Then the number $\vartheta_{\alpha,n,j}$ satisfies*

$$\vartheta_{\alpha,n,j} = -\frac{2}{n} \arctan\left(\kappa_\alpha \tan \frac{\vartheta_{\alpha,n,j}}{2}\right) + \frac{j\pi}{n}. \tag{3.40}$$

Proof. The localization theorem for weak perturbations (Theorem 3.7) assures the existence of $\vartheta_{\alpha,n,j}$ in $I_{n,j}$. By Proposition 3.10 $\vartheta_{\alpha,n,j}$ is solution to (3.29); this is equivalent to (3.40). □

Motivated by (3.40) we define the next functions. For every α in $\mathbb{R} \setminus \{0\}$, we define the function $\zeta_\alpha \colon [0, \pi] \to \mathbb{R}$ by

$$\zeta_\alpha(x) := -2 \arctan\left(\kappa_\alpha \tan \frac{x}{2}\right). \tag{3.41}$$

For every α in $\mathbb{R} \setminus \{0\}$, $n \geq 3$ and $1 \leq j \leq n$, we define the function $f_{\alpha,n,j} \colon [0, \pi] \to \mathbb{R}$ by

$$f_{\alpha,n,j}(x) \coloneqq \frac{j\pi + \zeta_\alpha(x)}{n}. \tag{3.42}$$

The function ζ_α takes values in $[-\pi, 0]$ or in $[0, \pi]$, depending on the sign of κ_α. Therefore, $f_{\alpha,n,j}$ takes values in $((j-1)\pi/n, j\pi/n)$ or in $(j\pi/n, ((j+1)\pi/n))$, respectively. See the graphs of ζ_α in Figure 3.6 for two different values of α.

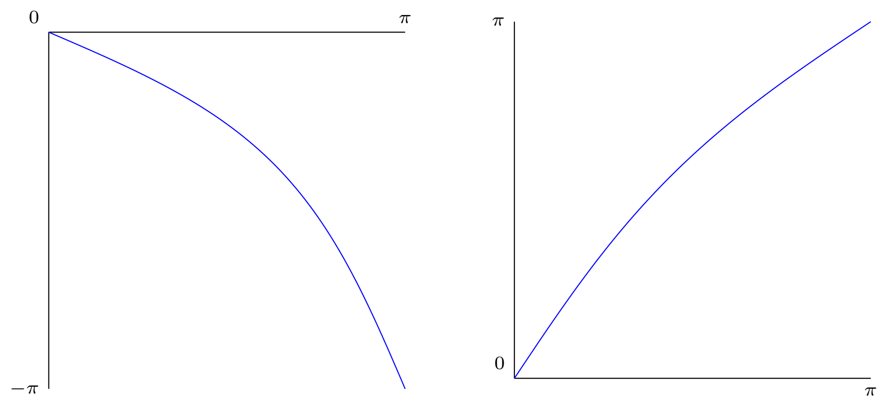


Figure 3.6: Function ζ_α (3.41) for $\alpha = 0.7$ (left) and for $\alpha = -2$ (right).

A straightforward computation yields the derivative of ζ_α:

$$\zeta_\alpha'(x) = -\frac{\kappa_\alpha(1 + \tan^2 \frac{x}{2})}{1 + \kappa_\alpha^2 \tan^2 \frac{x}{2}} = -\frac{\kappa_\alpha(1 + \cot^2 \frac{x}{2})}{\kappa_\alpha^2 + \cot^2 \frac{x}{2}}, \tag{3.43}$$

hence

$$f_{\alpha,n,j}'(x) = \frac{\zeta_\alpha'(x)}{n}. \tag{3.44}$$

For every α in $\mathbb{R}$, put

$$K(\alpha) \coloneqq \max\left\{|\kappa_\alpha|, \frac{1}{|\kappa_\alpha|}\right\}. \tag{3.45}$$

Proposition 3.17. *Let $\alpha \in \mathbb{R}$. Then each derivative of ζ_α is a bounded function on $(0, \pi)$. In particular,*

$$\|\zeta_\alpha'\|_\infty = K(\alpha). \tag{3.46}$$

Proof. In order to prove (3.46), we start by expressing (3.43) as follows

$$\zeta_\alpha'(x) = -\frac{1}{\kappa_\alpha}\left(1 + \frac{\kappa_\alpha^2 - 1}{1 + \kappa_\alpha^2 \tan^2 \frac{x}{2}}\right). \tag{3.47}$$

Since $\tan^2(x/2)$ increases from 0 to ∞ as x goes from 0 to π, then (3.47) increases or decreases from $\zeta'_\alpha(0) = -\kappa_\alpha$ to $\zeta'_\alpha(\pi) = -1/\kappa_\alpha$ depending on whether $\kappa_\alpha > 1/\kappa_\alpha$ or $\kappa_\alpha < 1/\kappa_\alpha$.

For the higher derivatives of $\zeta_{\alpha,j}$, the explicit estimates are too tedious, and we purpose the following argument. By (3.43), ζ'_α is analytic in a neighborhood of x, for any x in $(0, \pi)$. Even more, ζ'_α has an analytic extension in some neighborhoods of the points 0 and π. Hence, $\zeta'_{\alpha,j}$ has an analytic extension to a certain open set in the complex plane containing the segment $[0, \pi]$. Therefore, this function and all their derivatives are bounded on $(0, \pi)$. □

Observe that (3.40) can be written as $\vartheta_{\alpha,n,j} = f_{\alpha,n,j}(\vartheta_{\alpha,n,j})$.

Proposition 3.18. *Let $\alpha \in \mathbb{R}$, with $0 < \alpha < 1$, $n \geq 3$ with $n > K(\alpha)$, and $1 \leq j \leq n$, where j is even. Then $f_{\alpha,n,j}$ is contractive in $\operatorname{clos}(I_{n,j})$. Its fixed point belongs to $I_{n,j}$ and coincides with $\vartheta_{\alpha,n,j}$.*

Proof. Since the function ζ_α takes values in $[-\pi, 0]$, for every x in $\operatorname{clos}(I_{n,j})$

$$\frac{(j-1)\pi}{n} \leq \frac{j\pi + \zeta_\alpha(x)}{n} \leq \frac{j\pi}{n},$$

i.e. $f_{\alpha,n,j}(x)$ lies in $\operatorname{clos}(I_{n,j})$. Since ζ'_α is bounded by $K(\alpha)$, then by Proposition 3.17

$$|f_{\alpha,n,j}(x)| \leq \frac{K(\alpha)}{n} < 1.$$

We have proved that $f_{\alpha,n,j}$ is a contractive function on $\operatorname{clos}(I_{n,j})$. By Theorem A.4, this implies that $f_{\alpha,n,j}$ has a unique fixed point, and by Theorem 3.16 it coincides with $\vartheta_{\alpha,n,j}$ and belongs to $I_{n,j}$. □

We use the notation $\lambda^{\text{gen}}_{\alpha,n,j}$, for the eigenvalues computed by general algorithms, with double-precision arithmetic; and $\lambda^{\text{fp}}_{\alpha,n,j}$ denote the eigenvalues computed by formulas of Theorem 3.16, i.e., solving the equation (3.40) by the fixed point iteration; these computations are performed in high-precision arithmetic with 3322 binary digits.

We have constructed a large series of examples with random values of α and n, see the Codes C.1, C.2 and C.3 for the main part of the program, for example we ran the program C.3 with parameters $\alpha = 7/8$ and $n = 64$ and got Code 3.1. In all these numerical tests we obtained

$$\max_{1\leq j\leq n} |\lambda^{\text{gen}}_{\alpha,n,j} - \lambda^{\text{fp}}_{\alpha,n,j}| < 10^{-12}.$$

Code 3.1: Test eigenvalues approximation by fixed point iteration

```
sage: load('L_weak_test_eigenvalues_by_fixed_point.sage')
sage: max_error_eigenvalues_gen_minus_fp(7/8, 64, 3322)
test eigenvalues by fixed point iteration
n = 64
al = 7/8
maximal error = 1.554e−14
```

Proposition 3.19. *Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. Then there exists $C_3(\alpha) > 0$ such that for n large enough and $1 \leq j \leq n$ with j even,*

$$\vartheta_{\alpha,n,j} = \frac{j\pi}{n} + \frac{\zeta_\alpha\left(\frac{j\pi}{n}\right)}{n} + \frac{\zeta_\alpha\left(\frac{j\pi}{n}\right)\zeta'_\alpha\left(\frac{j\pi}{n}\right)}{n^2} + r_{\alpha,n,j}, \tag{3.48}$$

where $|r_{\alpha,n,j}| \leq \frac{C_3(\alpha)}{n^3}$.

Proof. The proof is similar to the proof of Proposition 2.14. □

For every α in $\mathbb{R} \setminus \{0\}$, define on $[0, \pi]$ the function

$$\Lambda_{\alpha,n}(x) \coloneqq g(x) + \frac{g'(x)\zeta_\alpha(x)}{n} + \frac{g'(x)\zeta_\alpha(x)\zeta'_\alpha(x) + \frac{1}{2}g''(x)\zeta^2_\alpha(x)}{n^2}. \tag{3.49}$$

For every α in $\mathbb{R}$ with $0 < \alpha < 1$, and $1 \leq j \leq n$ even, define $\lambda^{\text{asympt}}_{\alpha,n,j}$ by

$$\lambda^{\text{asympt}}_{\alpha,n,j} \coloneqq \Lambda_{\alpha,n}\left(\frac{j\pi}{n}\right). \tag{3.50}$$

Remark that $\Lambda_{\alpha,n}$ does not depend on the parameter j ($1 \leq j \leq n$), however we use this function only for the case where j is even. The idea behind this reasoning is that, for j odd the values $g((j-1)\pi/n)$ are eigenvalues of $L_{\alpha,n}$ (Theorem 3.7), so, in this case there is no need to find an expression like (3.50).

Theorem 3.20 (asymptotic expansion of the eigenvalues for weak perturbations)**.** *Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. Then there exists $C_1(\alpha) > 0$ such that for n large enough and j even with $1 \le j \le n$,*

$$|\lambda_{\alpha,n,j} - \lambda_{\alpha,n,j}^{\text{asympt}}| \le \frac{C_1(\alpha)}{n^3}. \tag{3.51}$$

Proof. The proof is similar to the proof of Theorem 2.15. □

For Theorem 3.20 we have computed the errors

$$R_{\alpha,n,j} \coloneqq \lambda_{\alpha,n,j}^{\text{asympt}} - \lambda_{\alpha,n,j}^{\text{fp}}$$

and their maximums $\|R_{\alpha,n}\|_\infty = \max_{1\le j\le n} |R_{\alpha,n,j}|$, these computations where performed in high-precision arithmetic with 3322 binary digits, see the Codes C.4 and C.5 for the main part of the written program, for example we ran in the command window the Code C.5 with parameters $\alpha = 7/9$ and $n = 64$ and got the Code 3.2. Tables 3.1 show that these errors indeed can be bounded by $C_1(\alpha)/n^3$.

Table 3.1: Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some α with $0 < \alpha < 1$.

$\alpha = 1/3$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	2.67×10^{-3}	10.93
32	3.37×10^{-4}	11.04
64	4.23×10^{-5}	11.10
128	5.31×10^{-6}	11.14
256	6.65×10^{-7}	11.15
512	8.32×10^{-8}	11.16
1024	1.04×10^{-8}	11.16
2048	1.30×10^{-9}	11.17
4096	1.63×10^{-10}	11.17
8192	2.03×10^{-11}	11.17

$\alpha = 0.9$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	2.17×10^{-2}	88.71
32	3.67×10^{-3}	120.36
64	5.52×10^{-4}	144.77
128	7.63×10^{-5}	160.07
256	1.00×10^{-5}	168.58
512	1.29×10^{-6}	173.05
1024	1.63×10^{-7}	175.33
2048	2.05×10^{-8}	176.49
4096	2.58×10^{-9}	177.07
8192	3.23×10^{-10}	177.36

Let $n \ge 3$. For every α in $\mathbb{C}$ define the next functions on $[0, \pi]$

$$\begin{aligned} d_0(x) &\coloneqq g(x), \\ d_{1,\alpha,n}(x) &\coloneqq \frac{g'(x)\zeta_\alpha(x)}{n}, \\ d_{2,\alpha,n}(x) &\coloneqq \frac{g'(x)\eta_{\alpha,j}(x)\zeta'_\alpha(x) + \frac{1}{2}g''(x)\zeta_\alpha^2(x)}{n^2}. \end{aligned} \tag{3.52}$$

It is evident that

$$\Lambda_{\alpha,n,j}(x) = d_0(x) + d_{1,\alpha,n}(x) + d_{2,\alpha,n}(x).$$

See in Figure 3.7 the behaviour of the asymptotic expansion (3.49). There the dots in blue and red are respectively the pairs $(j\pi/n, \lambda_{\alpha,n,j})$ and $(j\pi/n, \lambda_{\alpha,n,j}^{\text{asympt}})$ for j even; the first pair of numbers were computed with general algorithms and the last ones computed with one, two and three terms of the asymptotic expansion (3.50), respectively to each figure. The curves are the functions d_0, $d_0 + d_{1,\alpha}$ and $d_0 + d_{1,\alpha,n} + d_{2,\alpha,n}$, respectively to each figure. These figures confirm that, if we take more terms of the asymptotic expansion, then we obtain a better approximation for the eigenvalues.

Code 3.2: Test eigenvalues approximation by fixed asymptotic expansion

```
sage: load('L_weak_test_eigenvalues_by_asymp.sage')
sage: max_error_eigenvalues_gen_minus_fp(7/9, 64, 3322)
test eigenvalues by asymptotic expansion
n = 64
al = 7/9
maximal error = 2.433e−04
normalized error = 6.379e+01
```

Proposition 3.21. *Let $\alpha \in \mathbb{R}$ and $n \geq 3$. Then the vector $[1, \ldots, 1]^\top$ is an eigenvector of the matrix $L_{\alpha,n}$ associated to the eigenvalue $\lambda = 0$.*

Proof. By Corollary 3.4, $\lambda = 0$ is an eigenvalue of $L_{\alpha,n}$. The conclusion follows by the fact that for every $1 \leq j \leq n$ the sum of the entries of the j-th row of $L_{\alpha,n}$ equals zero. □

Proposition 3.22 (eigenvectors of $L_{\alpha,n}$ with weak perturbations)**.** *Let $0 \leq \alpha \leq 1$ and $n \geq 3$. Then the vector $[1, \ldots, 1]^\top$ is an eigenvector of the matrix $L_{\alpha,n}$ associated to the eigenvalue $\lambda_{\alpha,n,1} = 0$; and for every j with $2 \leq j \leq n$, the vector $v_{\alpha,n,j} = [v_{\alpha,n,j,k}]_{k=1}^n$ with components*

$$v_{\alpha,n,j,k} := \sin(k\vartheta_{\alpha,n,j}) - (1-\alpha)\sin((k-1)\vartheta_{\alpha,n,j}) + \alpha\sin((n-k)\vartheta_{\alpha,n,j}) \quad (1 \leq k \leq n), \tag{3.53}$$

is an eigenvector of $L_{\alpha,n}$ associated to $\lambda_{\alpha,n,j}$.

Proof. The first conclusion follows by Proposition 3.21.

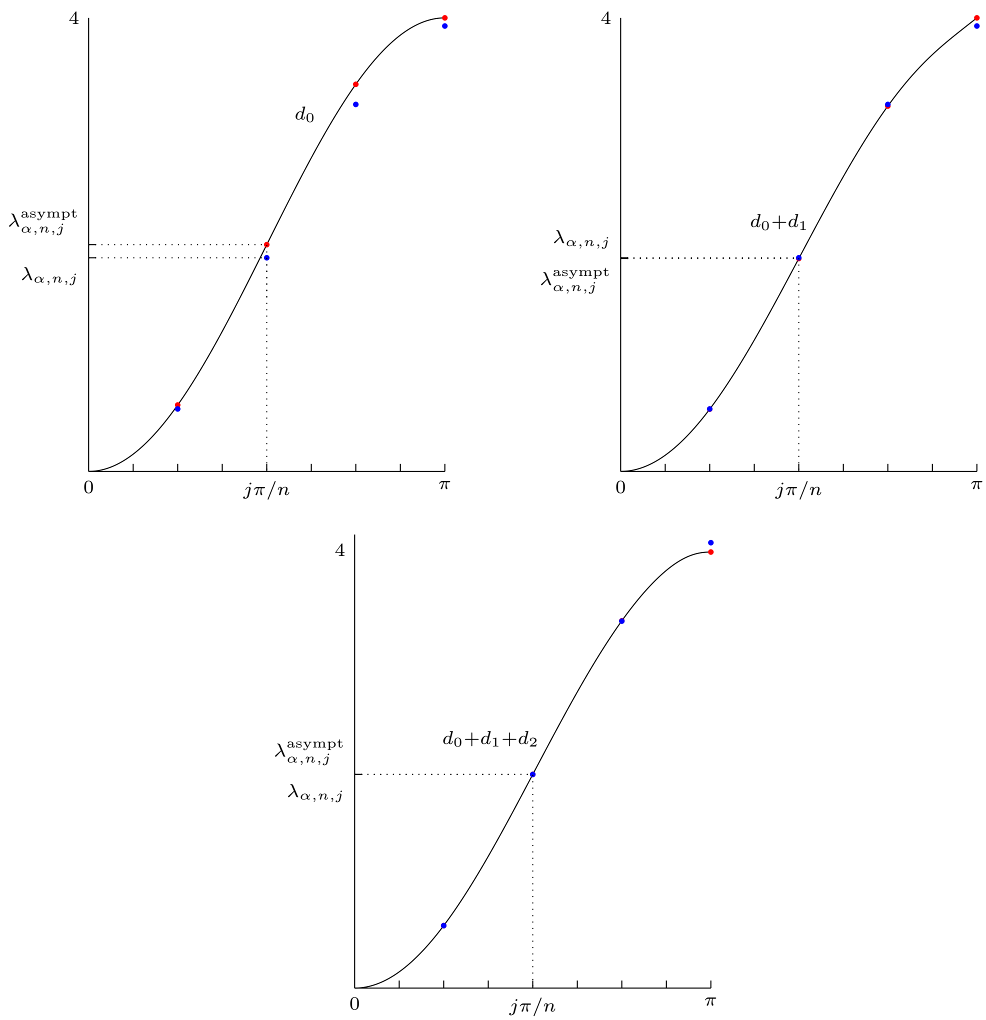


Figure 3.7: One, two and three terms of the function (3.49) for $\alpha = 0.8$ and $n = 8$ (we omitted the subscripts α and n in the functions).

By (3.16) for $j \geq 3$ odd, the number $\lambda_{\alpha,n,j} = g(\vartheta_{\alpha,n,j})$, with $\vartheta_{\alpha,n,j} = (j-1)\pi/n$, is an eigenvalue of $L_{\alpha,n}$. Let $\mathrm{b}_{-1-\alpha,\alpha,n}$ be the constant defined in (1.14), there $z = e^{i\vartheta_{\alpha,n,j}}$, so

$$\mathrm{b}_{-1-\alpha,\alpha,n} = 2i\sin(\vartheta_{\alpha,n,j})(1-\alpha) \neq 0.$$

Hence, by Proposition 1.16 the vector v' with components

$$v'_k = \frac{\sin(k\vartheta_{\alpha,n,j}) - (1-\alpha)\sin((k-1)\vartheta_{\alpha,n,j}) + \alpha\sin((n-k)\vartheta_{\alpha,n,j})}{\sin(\vartheta_{\alpha,n,j})} \quad (1 \leq k \leq n)$$

is an eigenvector associated to $\lambda_{\alpha,n,j}$. Since every constant multiple of v' is also an eigenvector associated to $\lambda_{\alpha,n,j}$, in order to obtain (3.53) just multiply v'_k by $\sin(\vartheta_{\alpha,n,j})$.

By (3.17), for $j \geq 2$ even, the correspondent $\vartheta_{\alpha,n,j}$ such that $\lambda_{\alpha,n,j} = g(\vartheta_{\alpha,n,j})$ is an eigenvalue, lies in the open interval $((j-1)\pi/n, j\pi/n)$. Therefore, if $\mathrm{a}_{-1-\alpha,\alpha,n} = 0$ and $\mathrm{b}_{-1-\alpha,\alpha,n} = 0$, the constants defined in (1.14), then

$$(1-\alpha)\cos(\vartheta_{\alpha,n,j}) = 1 - \alpha\cos(n\vartheta_{\alpha,n,j}) \quad \& \quad (\alpha-1)\sin(\vartheta_{\alpha,n,j}) = \alpha\sin(n\vartheta_{\alpha,n,j}). \tag{3.54}$$

The left equation in (3.54) implies

$$(\alpha-1)(1-\cos(\vartheta_{\alpha,n,j})) = \alpha(1-\cos(n\vartheta_{\alpha,n,j})).$$

So, (3.27) transforms in

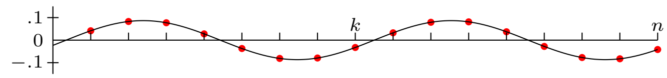
$$D_{\alpha,n}(g(x)) = \frac{2(-1)^n}{\sin(\vartheta_{\alpha,n,j})}\alpha(1-\cos(n\vartheta_{\alpha,n,j}))(\sin(n\vartheta_{\alpha,n,j}) - \sin(\vartheta_{\alpha,n,j})),$$

since $\vartheta_{\alpha,n,j} \neq j\pi/n$, then necessarily $\sin(n\vartheta_{\alpha,n,j}) - \sin(\vartheta_{\alpha,n,j}) = 0$. The right equation in (3.54) implies $\alpha - 1 = \alpha$, but this cannot occur. Hence, at least one of $\mathrm{a}_{-1-\alpha,\alpha,n}$ and $\mathrm{b}_{-1-\alpha,\alpha,n}$ is different from zero. Then we can proceed in the same way as before to obtain (3.53). □

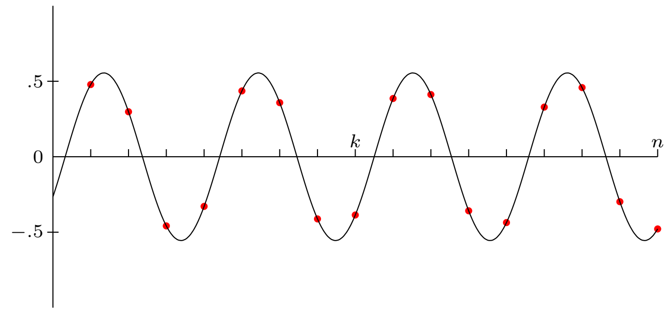
Motivated by (3.53), for every $0 < \alpha < 1$, $n \geq 3$ and $2 \leq j \leq n$, we define for every x in $[0, 16]$ the function:

$$w_{\alpha,n,j}(x) \coloneqq \sin(x\vartheta_{\alpha,n,j}) - (1-\alpha)\sin((x-1)\vartheta_{\alpha,n,j}) + \alpha\sin((n-x)\vartheta_{\alpha,n,j}). \tag{3.55}$$

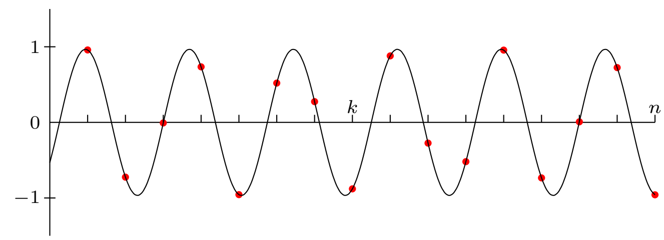
Observe that for every $2 \leq j \leq n$ and for every $1 \leq k \leq n$, $w_{\alpha,n,j}(k)$ equals (2.39). This function can be expressed in a similar manner as (2.41). Let $\alpha = 7/9$ and $n = 16$. See in Figure 3.8 the function $w_{\alpha,n,j}$ for some values of j. The dots in red are the pairs $(k, w_{\alpha,n,j}(k))$.



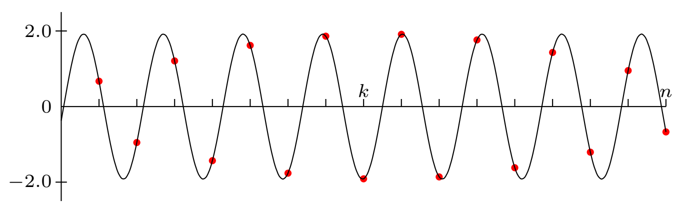
$j = 4$, scale 0.5 (x axis) to 3.0 (y axis)



$j = 8$, scale 0.5 (x axis) to 2.0 (y axis)



$j = 12$, scale 0.5 (x axis) to 1.0 (y axis)



$j = 16$

Figure 3.8: Function (3.55) for $\alpha = 7/9$, $n = 16$ and different values of j.

3.3 Eigenvalues and eigenvectors of $L_{\alpha,n}$ with strong left perturbations $\alpha < 0$

Let $\alpha < 0$ and $n \geq 3$. By Theorem 3.8, for $2 \leq j \leq n$, the eigenvalues $\lambda_{\alpha,n,j}$ of $L_{\alpha,n}$ behave as in the case of weak perturbations and can be approximated by the methods of Section 3.2. Hence, in this section we are only interested in solving (3.34).

Recall some notation. Functions g and g_- are defined by (2.2). Constants κ_α, ρ_α and

$K(\alpha)$ are defined respectively by (3.14), (3.36) and (3.45); in specific $\rho_\alpha = \log(2|\alpha|+1)$ and $K(\alpha) = |\kappa_\alpha|$.

Theorem 3.23 (characteristic equations for strong left perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha < 0$, and $n > |\kappa_\alpha|$. Then*

$$\lambda_{\alpha,n,1} = g_-(\vartheta_{\alpha,n,1}),$$

where $\vartheta_{\alpha,n,1}$ is the unique positive solution of the equation

$$x = 2\,\mathrm{arctanh}\left(\frac{\tanh\frac{nx}{2}}{|\kappa_\alpha|}\right). \tag{3.56}$$

For j even with $4 \le j \le n$, the number $\vartheta_{\alpha,n,j} = g^{-1}(\lambda_{\alpha,n,j})$ satisfies

$$\vartheta_{\alpha,n,j} = \frac{(j-2)\pi + \zeta_\alpha(\vartheta_{\alpha,n,j})}{n}. \tag{3.57}$$

Proof. Under the asumptions of Theorem 3.8 we can conclude that there is a unique solution in $(0,\infty)$, to (3.34), namely $\vartheta_{\alpha,n,1}$, rephrasing, $\vartheta_{\alpha,n,1}$ satisfies (3.56).

The proof of (3.57) is similar to the proof of (3.40). □

Recall that by Proposition 3.14 we have that $\vartheta_{\alpha,n,1}$ belongs to the interval $(0,\rho_\alpha)$. Motivated by (3.56) , for every $\alpha < 0$ and every $n \ge \max\{3, K(\alpha)+1\}$ we define the mapping $\varphi_{\alpha,n}\colon [0,\infty) \to \mathbb{R}$ by

$$\varphi_{\alpha,n}(x) \coloneqq 2\,\mathrm{arctanh}\left(\frac{\tanh\frac{nx}{2}}{|\kappa_\alpha|}\right) = 2\,\mathrm{arctanh}\left(\frac{\tanh\frac{nx}{2}}{K(\alpha)}\right). \tag{3.58}$$

A straightforward computation yields

$$\varphi'_{\alpha,n}(x) = \frac{n|\kappa_\alpha|}{(|\kappa_\alpha|^2 - \tanh^2\frac{nx}{2})\cosh^2\frac{nx}{2}} = \frac{n|\alpha|(1+|\alpha|)}{\alpha^2 + (1+2|\alpha|)\cosh^2\frac{nx}{2}}. \tag{3.59}$$

Proposition 3.24. *Let $\alpha < 0$ and $n \ge \max\{3, K(\alpha)+1\}$. Then $\varphi_{\alpha,n}$ is a contraction on $[\vartheta_{\alpha,n,1}, \rho_\alpha]$ and its fixed point coincides with $\vartheta_{\alpha,n,1}$.*

Proof. Since $\vartheta_{\alpha,n,1}$ is a fixed point of $\varphi_{\alpha,n}$ (see Theorem 3.23) and $\varphi_{\alpha,n}$ is an increasing function (by (3.59)), therefore for every x in $[\vartheta_{\alpha,n,1}, \rho_\alpha]$

$$\vartheta_{\alpha,n,1} = \varphi_{\alpha,n}(\vartheta_{\alpha,n,1}) \leq \varphi_{\alpha,n}(x) \leq 2\operatorname{arctanh}\left(\frac{\tanh\frac{n\rho_\alpha}{2}}{|\kappa_\alpha|}\right) < 2\operatorname{arctanh}\left(\frac{1}{|\kappa_\alpha|}\right) = \rho_\alpha,$$

so $\varphi_{\alpha,n}([\vartheta_{\alpha,n,1}, \rho_\alpha]) \subset [\vartheta_{\alpha,n,1}, \rho_\alpha]$.

By the mean value theorem there exists $\xi_{\alpha,n}$ in $(0, \vartheta_{\alpha,n,1})$ that satisfies $\varphi'_{\alpha,n}(\xi_{\alpha,n}) = 1$. Since $\varphi'_{\alpha,n}$ decreases, for every x in $[\vartheta_{\alpha,n,1}, \rho_\alpha]$

$$\varphi'_{\alpha,n}(x) \leq \varphi'_{\alpha,n}(\vartheta_{\alpha,n,1}) < \varphi'_{\alpha,n}(\xi_{\alpha,n}) = 1.$$

By Theorem A.4 this implies that $\varphi_{\alpha,n}$ has a unique fixed point, so, it coincides with $\vartheta_{\alpha,n,1}$. □

For Proposition 3.24 we have constructed a large series of examples with random values of α and n, denote by $\lambda^{\mathrm{fp}}_{\alpha,n,1}$ the eigenvalue obtained by iteration of fixed point with high-precision arithmetic with 3322 binary digits,$\lambda^{\mathrm{gen}}_{\alpha,n,1}$ the eigenvalue obtained by general algorithms with double-precision arithmetic, see the Codes C.6 and C.7 for the main part of the program; for example, we executed this code with parameters $\alpha = -3/2$ and $n = 64$, obtaining the Code 3.3. We also tested the eigenvalues approximation by fixed point iteration for the $2 \leq j \leq n$ by running similar codes to the ones written in Section C.1. For all these numerical experiments we obtained

$$\max_{j=1,n} |\lambda^{\mathrm{gen}}_{\alpha,n,j} - \lambda^{\mathrm{fp}}_{\alpha,n,j}| < 2 \cdot 10^{-13}.$$

Code 3.3: Test eigenvalues approximation by fixed point iteration

```
sage: load('L_strong_test_extreme_eigenvalues_by_fixed_point.sage')
sage: max_error_eigenvalues_gen_minus_fp(-3/2,64,3322)
test eigenvalues by fixed point iteration
n = 64
al = -3/2
maximal error = 1.332e-15
```

We remark that ρ_α is likely the best choice for the evaluation in the fixed iteration method, this because $\vartheta_{\alpha,n,1}$ converges exponentially to it as n tends to ∞. We prove this convergence in the proposition below.

Proposition 3.25. *Let $\alpha < 0$ and n be large enough. Then*

$$|\vartheta_{\alpha,n,1} - \rho_\alpha| \leq \frac{4\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|} e^{\frac{8\cosh^2\frac{\rho_\alpha}{2}}{\rho_\alpha|\kappa_\alpha|}} e^{-n\rho_\alpha}. \tag{3.60}$$

Proof. We apply the mean value theorem to $\tanh(x/2)$ on $[\vartheta_{\alpha,n,1}, \rho_\alpha]$, and that $\vartheta_{\alpha,n,1}$ is the fixed point for $\varphi_{\alpha,n}$, thus

$$\begin{aligned}\rho_\alpha - \vartheta_{\alpha,n,1} &\leq 2\cosh^2\frac{\rho_\alpha}{2}\left(\tanh\frac{\rho_\alpha}{2} - \tanh\frac{\vartheta_{\alpha,n,1}}{2}\right)\\ &= \frac{2\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|}\left(1 - \tanh\frac{n\vartheta_{\alpha,n,1}}{2}\right)\\ &\leq \frac{4\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|} e^{-n\vartheta_{\alpha,n,1}}.\end{aligned}$$

By Lemma A.5, $\vartheta_{\alpha,m,1} < \vartheta_{\alpha,n,1}$ if $m < n$. Then for $n > n_0 := [|\kappa_\alpha|] + 1$

$$\frac{4\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|} e^{-n\vartheta_{\alpha,n,1}} \leq \frac{4\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|} e^{-n\vartheta_{\alpha,n_0,1}}.$$

So, $\vartheta_{\alpha,n,1} > \rho_\alpha/2$ for n large enough, hence

$$\vartheta_{\alpha,n,1} \geq \rho_\alpha - \frac{4\cosh^2\frac{\rho_\alpha}{2}}{|\kappa_\alpha|} e^{-n\rho_\alpha/2}.$$

Using the elementary inequality $ue^{-u} \leq 1/e < 1$ we get

$$n\vartheta_{\alpha,n,1} \geq n\rho_\alpha - \frac{8\cosh^2\frac{\rho_\alpha}{2}}{\rho_\alpha|\kappa_\alpha|}\frac{n\rho_\alpha}{2} e^{-n\rho_\alpha/2} \geq n\rho_\alpha - \frac{8\cosh^2\frac{\rho_\alpha}{2}}{\rho_\alpha|\kappa_\alpha|},$$

the conclusion now follows. ☐

Recall that s_α was defined here (3.38), define now

$$\lambda_{\alpha,n,1}^{\text{asympt}} := -s_\alpha. \tag{3.61}$$

Theorem 3.26 (asymptotic expansion of the eigenvalues for left strong perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha < 0$. As n tends to infinity, the extreme eigenvalue $\lambda_{\alpha,n,1}$ of $L_{\alpha,n}$ converges* *exponentially to* $\lambda_{\alpha,n,1}^{asympt}$,

$$|\lambda_{\alpha,n,1} - \lambda_{\alpha,n,1}^{asympt}| \leq C_3(\alpha) e^{-n\rho_\alpha}, \tag{3.62}$$

here $C_3(\alpha)$ *is a positive constant depending only on* α. *For even* j *with* $4 \leq j \leq n$, *the eigenvalue* $\lambda_{\alpha,n,j}$ *of* $L_{\alpha,n}$ *satisfies the asymptotic formula* (3.49), *but with*

$$\lambda_{\alpha,n,j}^{\text{asympt}} := \Lambda_{\alpha,n}\left(\frac{(j-2)\pi}{n}\right). \tag{3.63}$$

Proof. (3.62) follows by Proposition 3.25 and the mean value theorem applied to g_-. The proof of (3.63) is similar to the proof of Theorem 3.20. □

For Theorem 3.26 we have computed the errors

$$R_{\alpha,n,j} := \lambda_{\alpha,n,j}^{\text{asympt}} - \lambda_{\alpha,n,j}^{\text{fp}}$$

and their maximums $\|R_{\alpha,n}\|_\infty = \max_{1\leq j\leq n} |R_{\alpha,n,j}|$, both where computed in high-precision arithmetic with 3322 binary digits, for $2 \leq j \leq n$ the written code is similar to the ones written in the Section C.1. Tables 3.2 show that these errors indeed can be bounded by $C_3(\alpha)/n^3$.

Table 3.2: Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some α with $\alpha < 0$.

$\alpha = -2$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	3.23×10^{-3}	13.25
32	4.11×10^{-4}	13.45
64	5.10×10^{-5}	13.37
128	6.35×10^{-6}	13.32
256	7.93×10^{-7}	13.30
512	9.90×10^{-8}	13.29
1024	1.24×10^{-8}	13.28
2048	1.55×10^{-9}	13.28
4096	1.93×10^{-10}	13.28
8192	2.41×10^{-11}	13.28

$\alpha = -1/3$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	3.00×10^{-3}	12.27
32	3.67×10^{-4}	12.02
64	4.52×10^{-5}	11.85
128	5.60×10^{-6}	11.75
256	6.97×10^{-7}	11.70
512	8.70×10^{-8}	11.67
1024	1.09×10^{-8}	11.66
2048	1.36×10^{-9}	11.65
4096	1.70×10^{-10}	11.65
8192	2.12×10^{-11}	11.65

We have also tested (3.62), see the Codes C.8 and C.9 for the main section of the program used. As n grows, $|2\alpha - 1|^n |R_{\alpha,n,1}|$ approach rapidly to a value depending on α.

For example,

$$\text{for } \alpha = -2, \quad \lim_{n\to\infty} (|2\alpha - 1|^n |R_{\alpha,n,1}|) \approx 23.04;$$
$$\text{for } \alpha = -1/3, \quad \lim_{n\to\infty} (|2\alpha - 1|^n |R_{\alpha,n,1}|) \approx 1.14.$$

Proposition 3.27 (eigenvectors of $L_{\alpha,n}$ with strong left perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha < 0$ and $n > |\kappa_\alpha|$. Then the vector $[1,\ldots,1]^\top$ is an eigenvector associated to the eigenvalue $\lambda_{\alpha,n,2} = 0$; and the vector $v_{\alpha,n,1} = [v_{\alpha,n,1,k}]_{k=1}^n$ with components*

$$\begin{aligned} v_{\alpha,n,1,k} &:= \sinh(k\vartheta_{\alpha,n,1}) \\ &\quad - (1-\alpha)\sinh((k-1)\vartheta_{\alpha,n,1}) + \alpha \sinh((n-k)\vartheta_{\alpha,n,1}) \quad (1 \le k \le n), \end{aligned} \tag{3.64}$$

is an eigenvector associated to $\lambda_{\alpha,n,1}$, and for every $3 \le j \le n$ the vector $v_{\alpha,n,j} = [v_{\alpha,n,j,k}]_{k=1}^n$ with components (3.53) is an eigenvector associated to $\lambda_{\alpha,n,j}$.

Proof. By Theorem 3.8 we know that $\lambda_{\alpha,n,1} = g_-(\vartheta_{\alpha,n,1})$ is an eigenvalue of $L_{\alpha,n}$, where $0 < \vartheta_{\alpha,n,1}$. Let $\mathrm{b}_{-1-\alpha,\alpha,n}$ be the constant defined in (1.14), there $z = e^{\vartheta_{\alpha,n,1}}$. So

$$\mathrm{b}_{-1-\alpha,\alpha,n} = 2[(1-\alpha)\sinh(\vartheta_{\alpha,n,1}) + \alpha \sinh(n\vartheta_{\alpha,n,1})] > 0.$$

Therefore, by Proposition 1.16 the vector v' with components

$$v'_k = \frac{\sinh(k\vartheta_{\alpha,n,1})}{\sinh(\vartheta_{\alpha,n,1})} - (1-\alpha)\frac{\sinh((k-1)\vartheta_{\alpha,n,1})}{\sinh(\vartheta_{\alpha,n,1})} + \alpha \frac{\sinh((n-k)\vartheta_{\alpha,n,1})}{\sinh(\vartheta_{\alpha,n,1})} \qquad (1 \le k \le n),$$

is an eigenvector associated to $\lambda_{\alpha,n,1}$. Since every constant multiple of v' is also an eigenvector, in order to obtain (3.53) just multiply v'_k by $\sinh(\vartheta_{\alpha,n,1})$.

For the case $j = 2$, the conclusion follows from Proposition 3.21. For $3 \le j \le n$, the proof is similar to the proof given in Proposition 3.22. □

3.4 Eigenvalues and eigenvectors of $L_{\alpha,n}$ with strong right perturbations $\alpha > 1$

Let $\alpha > 1$ and $n \ge 3$. By Theorem 3.9, for $1 \le j \le n-1$, the eigenvalues $\lambda_{\alpha,n,j}$ of $L_{\alpha,n}$ behave as in the case of weak perturbations and can be approximated by the methods of Section 3.2. Hence, in this section we are only interested in solving (3.35).

Recall some notation. Functions g and g_- are defined by (2.2). Constants κ_α, ρ_α and $K(\alpha)$ are defined respectively by (3.14), (3.36) and (3.45); in specific $K(\alpha) = 1/|\kappa_\alpha|$ and $\rho_\alpha = \log(2\alpha - 1)$.

Theorem 3.28 (characteristic equations for strong right perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha > 1$, and $n > 1/|\kappa_\alpha|$. Then*

$$\lambda_{\alpha,n,n} = g_+(\vartheta_{\alpha,n,n}),$$

where $\vartheta_{\alpha,n,n}$ is the unique positive solution of the equation

$$x = 2\operatorname{arctanh}\left(|\kappa_\alpha| \left[\tanh\frac{nx}{2}\right]^{(-1)^{n+1}}\right). \tag{3.65}$$

For each odd j with $3 \le j \le n-1$, the number $\vartheta_{\alpha,n,j} = g^{-1}(\lambda_{\alpha,n,j})$ satisfies

$$\vartheta_{\alpha,n,j} = \frac{(j-1)\pi + \zeta_\alpha(x)}{n}. \tag{3.66}$$

Proof. Under the assumptions of Theorem 3.9, we can conclude that there is a unique solution of (3.35) in $(0, \infty)$, and (3.35) is equivalent to (3.65). The proof of (3.66) is similar to the proof of (3.40). □

Motivated by (3.65), for every $\alpha > 1$ and $n \ge \max\{3, K(\alpha)+1\}$ we define the function:

$$\varphi_{\alpha,n}(x) := \begin{cases} 2\operatorname{arctanh}\left(|\kappa_\alpha| \coth\frac{nx}{2}\right), & x \in [\rho_\alpha, \infty) \text{ and } n \text{ even},\\ 2\operatorname{arctanh}\left(|\kappa_\alpha| \tanh\frac{nx}{2}\right), & x \in (0, \rho_\alpha] \text{ and } n \text{ odd}. \end{cases} \tag{3.67}$$

In both cases (n even or odd), the domain of $\varphi_{\alpha,n}$ could be larger than the proposed intervals, but by Proposition 3.15 these are sufficient because they contain the fixed point.

Proposition 3.29. *Let $\alpha > 1$, $n \ge \max\{3, K(\alpha)+1\}$ odd. Then $\lambda_{\alpha,n,n} = 4 - \lambda_{1-\alpha,n,1}$.*

Proof. Indeed, for these values of α and n, $K(\alpha) = K(1-\alpha)$ and $\rho_\alpha = \rho_{1-\alpha}$. By the assumption that n is odd,

$$\varphi_{\alpha,n}(x) = 2\operatorname{arctanh}\left(\frac{\tanh\frac{nx}{2}}{K(\alpha)}\right).$$

The last expression agrees with (3.58) , so, $\varphi_{\alpha,n} = \varphi_{1-\alpha,n}$. Recall that the last eigenvalue is computed as $g_+(\vartheta_{\alpha,n,n})$ where $\vartheta_{\alpha,n,n}$ is solution of (3.65). Then $\lambda_{\alpha,n,n} = g_+(\vartheta_{\alpha,n,n}) = 4 - g_-(\vartheta_{1-\alpha,n,1}) = 4 - \lambda_{1-\alpha,n,1}$. □

The subsequent Propositions 3.30 and 3.31 can be viewed as corollaries of Propositions 3.24 and 3.25, respectively.

Proposition 3.30. *Let $\alpha > 1$ and $n \geq \max\{K(\alpha)+1, 3\}$ odd. Then $\varphi_{\alpha,n}$ is a contraction on $[\vartheta_{\alpha,n,n}, \rho_\alpha]$ and its fixed point that coincides with $\vartheta_{\alpha,n,n}$.*

Proposition 3.31. *Let $\alpha > 1$ and n large enough and odd. Then*

$$|\vartheta_{\alpha,n,n} - \rho_\alpha| \leq 4|\kappa_\alpha| \cosh^2 \frac{\rho_\alpha}{2} e^{\frac{8|\kappa_\alpha| \cosh^2 \frac{\rho_\alpha}{2}}{\rho_\alpha}} e^{-n\rho_\alpha}. \tag{3.68}$$

Now we consider that n is even. A straightforward computation gives

$$\varphi'_{\alpha,n}(x) = -\frac{n(\alpha-1)\alpha}{(2\alpha-1)\cosh^2 \frac{nx}{2} - \alpha^2}. \tag{3.69}$$

It is easy to verify that $\varphi'_{\alpha,n}$ is negative on $[\rho_\alpha, \infty)$. Hence, $\varphi_{\alpha,n}$ decreases and

$$\rho_\alpha < \varphi_{\alpha,n}(x) < \infty \qquad (x > \rho_\alpha). \tag{3.70}$$

For every $\alpha > 1$ we define

$$M(\alpha) := \max\left\{\frac{2\log(2\alpha/\rho_\alpha)}{\rho_\alpha},\ \frac{2\log(2\alpha)}{\rho_\alpha},\ K(\alpha)\right\}. \tag{3.71}$$

For every $\alpha > 1$ and $n > M(\alpha)$ even, we put $\mu_{\alpha,n} := \varphi_{\alpha,n}(\rho_\alpha)$. By (3.70) $\rho_\alpha < \mu_{\alpha,n}$.

Proposition 3.32. *Let $\alpha > 1$, $n > M(\alpha)$ even. Then $\varphi_{\alpha,n}$ is a contraction on $[\rho_\alpha, \mu_{\alpha,n}]$ and its fixed point coincides with $\vartheta_{\alpha,n,n}$.*

Proof. That $\vartheta_{\alpha,n,n}$ is a fixed point follows from Theorem 3.9. Since $\rho_\alpha < \vartheta_{\alpha,n,n}$ and $\varphi_{\alpha,n}$ decreases,

$$\rho_\alpha < \vartheta_{\alpha,n,n} = \varphi_{\alpha,n}(\vartheta_{\alpha,n,n}) < \varphi_{\alpha,n}(\rho_\alpha) = \mu_{\alpha,n},$$

i.e., $\vartheta_{\alpha,n,n}$ belongs to $[\rho_\alpha, \mu_{\alpha,n}]$. Even more, for every x in this interval,

$$\rho_\alpha < \varphi_{\alpha,n}(\mu_\alpha) \le \varphi_{\alpha,n}(x) \le \varphi_{\alpha,n}(\rho_\alpha) = \mu_{\alpha,n},$$

so, $\varphi_{\alpha,n}([\rho_\alpha, \mu_{\alpha,n}]) \subseteq [\rho_\alpha, \mu_{\alpha,n}]$.

From (3.69) we know that $\varphi'_{\alpha,n}$ is negative and $|\varphi'_{\alpha,n}|$ decreases in $[\rho_\alpha, \mu_\alpha]$. Hence, $|\varphi'_{\alpha,n}(x)| \le |\varphi'_{\alpha,n}(\rho_\alpha)|$ for every x in $[\rho_\alpha, \mu_{\alpha,n}]$.

By the assumptions, $n > M(\alpha) \ge 2\log(2\alpha)/\rho_\alpha$. Then $4\alpha^2 e^{-n\rho_\alpha} < 1$. So,

$$\begin{aligned}|\varphi'_{\alpha,n}(\rho_\alpha)| &= \frac{4n(\alpha-1)\alpha e^{-n\rho_\alpha}}{(2\alpha-1)(1+e^{-n\rho_\alpha})^2 - 4\alpha^2 e^{-n\rho_\alpha}} \\ &\le \frac{4n(\alpha-1)\alpha e^{-n\rho_\alpha}}{(2\alpha-1)\cdot 1 - 1} = 2n\alpha e^{-n\rho_\alpha}.\end{aligned}$$

Using the inequality $ue^{-u} \le 1/e$ we get

$$|\varphi'_{\alpha,n}(\rho_\alpha)| \le 2n\alpha e^{-n\rho_\alpha/2} e^{-n\rho_\alpha/2} \le \frac{4\alpha e^{-n\rho_\alpha/2}}{e\rho_\alpha} \le \frac{2\alpha e^{-n\rho_\alpha/2}}{\rho_\alpha}.$$

Now we apply the assumption $n > 2\log(2\alpha/\rho_\alpha)/\rho_\alpha$ and conclude that $|\varphi'_{\alpha,n}(\rho_\alpha)| < 1$. So, we have proved that $\varphi_{\alpha,n}$ is a contraction on $[\rho_\alpha, \mu_{\alpha,n}]$. By Theorem A.4, this implies that $\varphi_{\alpha,n}$ has a unique fixed point in $[\rho_\alpha, \mu_{\alpha,n}]$, and by Theorem 3.28, it coincides with $\vartheta_{\alpha,n,n}$. □

We define $\mu_\alpha := \mu_{\alpha,[M(\alpha)]+1} = \varphi_{\alpha,[M(\alpha)]+1}(\rho_\alpha)$.

Next proposition says that ρ_α is indeed a good first approximation for $\vartheta_{\alpha,n,n}$.

Proposition 3.33. *Let $\alpha > 1$ and $n > M(\alpha)$ even. Then*

$$|\vartheta_{\alpha,n,n} - \rho_\alpha| \le \frac{16(\alpha-1)\alpha\cosh^2\frac{\mu_\alpha}{2}}{4\alpha^2-1} e^{-n\rho_\alpha}. \tag{3.72}$$

Proof. First notice that $\varphi_{\alpha,n}(x) \le \varphi_{\alpha,[M(\alpha)]+1}(x)$ for every $n \ge [M(\alpha)]+1$ and $x \ge \rho_\alpha$. In particular this holds for $x = \rho_\alpha$, then $\mu_{\alpha,n} \le \mu_\alpha$, hence $[\rho_\alpha, \mu_{\alpha,n}] \subset [\rho_\alpha, \mu_\alpha]$.

Apply the mean value theorem to $\tanh(x/2)$ on $[\rho_\alpha, \mu_\alpha]$, and the fact that $\vartheta_{\alpha,n,n}$ is the fixed point for $\varphi_{\alpha,n}$, thus

$$\begin{aligned}\vartheta_{\alpha,n,n} - \rho_\alpha &\le 2\cosh^{\frac{\mu_\alpha}{2}}\left(\tanh\frac{\vartheta_{\alpha,n,n}}{2} - \tanh\frac{\rho_\alpha}{2}\right)\\ &= 2\cosh^2\frac{\mu_\alpha}{2}\left(|\kappa_\alpha|\coth\frac{n\vartheta_{\alpha,n,n}}{2} - |\kappa_\alpha|\right)\\ &= 2|\kappa_\alpha|\cosh^2\frac{\mu_\alpha}{2}\left(\coth\frac{n\vartheta_{\alpha,n,n}}{2} - 1\right)\\ &= 4|\kappa_\alpha|\cosh^2\frac{\mu_\alpha}{2}\left(\frac{e^{-n\rho_\alpha}}{1-e^{-n\rho_\alpha}}\right).\end{aligned}$$

Now, (3.72) follows by applying the inequality $n > M(\alpha) \ge 2\log(2\alpha)/\rho_\alpha$ in the denominator. □

Recall that s_α is defined by (3.38). Define now

$$\lambda^{\mathrm{asympt}}_{\alpha,n,n} := s_\alpha. \tag{3.73}$$

Theorem 3.34 (asymptotic expansion of the eigenvalues for right strong perturbations)**.** *Let $\alpha \in \mathbb{R}$, $\alpha > 1$. As n tends to infinity, the extreme eigenvalue $\lambda_{\alpha,n,n}$ of $A_{\alpha,n}$ converges exponentially to $\lambda^{asympt}_{\alpha,n,n}$ thus*

$$|\lambda_{\alpha,n,n} - \lambda^{asympt}_{\alpha,n,n}| \le C_4(\alpha)e^{-n\rho_\alpha}, \tag{3.74}$$

here $C_4(\alpha)$ is a positive constant depending only on α. For $1 \le j \le n-1$, the eigenvalue $\lambda_{\alpha,n,j}$ satisfies the asymptotic formula (3.49), but with

$$\lambda^{\mathrm{asympt}}_{\alpha,n,j} := \Lambda_{\alpha,n}\left(\frac{(j-2)\pi}{n}\right). \tag{3.75}$$

Proof. Inequality (3.74) follows from Propositions 3.31 and 3.33. The proof of (3.63) is similar to the proof of Theorem 3.20. □

We have constructed a large series of examples with random values of α and n, denote by $\lambda^{\mathrm{fp}}_{\alpha,n,n}$ the eigenvalue obtained by iteration of fixed point with high-precision arithmetic with 3322 binary digits,$\lambda^{\mathrm{gen}}_{\alpha,n,n}$ the eigenvalue obtained by general algorithms with double-precision arithmetic. We also tested the eigenvalues approximation by fixed point iteration for the $1 \le j \le n-1$. For all these numerical experiments we obtained

$$\max_{j=1,n} |\lambda_{\alpha,n,j}^{\text{gen}} - \lambda_{\alpha,n,j}^{\text{fp}}| < 2 \cdot 10^{-13}.$$

The main parts of the program used are similar to the codes written in the Section C.1.

For Theorem 3.34 we have computed the errors

$$R_{\alpha,n,j} := \lambda_{\alpha,n,j}^{\text{asympt}} - \lambda_{\alpha,n,j}^{\text{fp}}$$

and their maximums $\|R_{\alpha,n}\|_\infty = \max_{1\le j\le n} |R_{\alpha,n,j}|$, these computations where performed in high-precision arithmetic with 3322 binary digits. The main pieces of the program used are similar to the codes written in the Section C.2. Tables 3.3 show that these errors indeed can be bounded by $C_4(\alpha)/n^3$.

Table 3.3: Values of $\|R_{\alpha,n}\|_\infty$ and $n^3\|R_{\alpha,n}\|_\infty$ for some α with $1 < \alpha$.

$\alpha = 2$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	4.06×10^{-3}	16.65
32	8.68×10^{-4}	28.43
64	1.31×10^{-4}	34.30
128	1.76×10^{-5}	37.01
256	2.28×10^{-6}	38.28
512	2.90×10^{-7}	38.89
1024	3.65×10^{-8}	39.18
2048	4.58×10^{-9}	39.33
4096	5.73×10^{-10}	39.41
8192	7.17×10^{-11}	39.44

$\alpha = 3/2$		
n	$\|R_{\alpha,n}\|_\infty$	$n^3\|R_{\alpha,n}\|_\infty$
16	2.54×10^{-3}	10.42
32	9.89×10^{-4}	32.42
64	1.77×10^{-4}	46.27
128	2.53×10^{-5}	53.03
256	3.35×10^{-6}	56.21
512	4.30×10^{-7}	57.74
1024	5.45×10^{-8}	58.49
2048	6.85×10^{-9}	58.85
4096	8.59×10^{-10}	59.04
8192	1.08×10^{-10}	59.13

We have also tested (3.62), see the Code C.9 for a similar piece of program used. As n grows, $|2\alpha-1|^n|R_{\alpha,n,1}|$ approach rapidly to a value depending on α. For example,

$$\text{for } \alpha = 3/2, \quad \lim_{n\to\infty} (|2\alpha-1|^n|R_{\alpha,n,1}|) \approx 2.25;$$

$$\text{for } \alpha = 2, \quad \lim_{n\to\infty} (|2\alpha-1|^n|R_{\alpha,n,1}|) \approx 7.11.$$

Proposition 3.35 (eigenvectors of $L_{\alpha,n}$ with strong right perturbations)**.** *Let* $\alpha \in \mathbb{R}$, $\alpha > 1$, $n > 1/|\kappa_\alpha|$. *Then the vector* $[1,\ldots,1]^\top$ *is an eigenvector of the matrix* $L_{\alpha,n}$

associated to the eigenvalue $\lambda_{\alpha,n,1} = 0$; and the vector $v_{\alpha,n,n} = [v_{\alpha,n,n,k}]_{k=1}^{n}$ with components

$$\begin{aligned} v_{\alpha,n,n,k} &\coloneqq (-1)^{k-1}\,[(-1)^n \alpha \sinh((n-k)\vartheta_{\alpha,n,n}) \\ &\quad +(1-\alpha)\sinh((k-1)\vartheta_{\alpha,n,n}) + \sinh(k\vartheta_{\alpha,n,n})] \qquad (1 \le k \le n), \end{aligned} \tag{3.76}$$

is an eigenvector associated to $\lambda_{\alpha,n,n}$, and for every $2 \le j \le n-1$ the vector $v_{\alpha,n,j} = [v_{\alpha,n,j,k}]_{k=1}^{n}$ with components (3.53) is an eigenvector of $L_{\alpha,n}$ associated to $\lambda_{\alpha,n,j}$.

Proof. The first conclusion follows from Proposition 3.21.

By Theorem 3.28 we know that $\lambda_{\alpha,n,n} = g_+(\vartheta_{\alpha,n,n})$ is an eigenvalue of $L_{\alpha,n}$, where $0 < \vartheta_{\alpha,n,n},$. Let $\mathrm{b}_{-1-\alpha,\alpha,n}$ be the constant defined in (1.14), there $z = -e^{\vartheta_{\alpha,n,n}}$. So

$$\mathrm{b}_{-1-\alpha,\alpha,n} = -2[(1-\alpha)\sinh(\vartheta_{\alpha,n,n}) + (-1)^n \alpha \sinh(n\vartheta_{\alpha,n,n})] \neq 0. \tag{3.77}$$

Therefore, by Proposition 1.16 the vector v' with components

$$\begin{aligned} v'_k = \frac{(-1)^{k-1}\sinh(k\vartheta_{\alpha,n,n})}{\sinh(\vartheta_{\alpha,n,n})} - (1-\alpha)(-1)^{k-2}\frac{\sinh((k-1)\vartheta_{\alpha,n,n})}{(\vartheta_{\alpha,n,n})} \\ + \alpha(-1)^{n-k-1}\frac{\sinh((n-k)\vartheta_{\alpha,n,n})}{\sinh(\vartheta_{\alpha,n,n})} \qquad (1 \le k \le n), \end{aligned}$$

is an eigenvector associated to $\lambda_{\alpha,n,n}$. Since every constant multiple of v' is also an eigenvector, in order to obtain (3.76) just multiply v'_k by $\sinh(\vartheta_{\alpha,n,n})$.

For the remaining eigenvectors the proof is similar to the proof of Proposition 3.22 □

3.5 Eigenvalues and eigenvectors of $L_{\alpha,n}$ with $\alpha = 0$

Recall that $D_{\alpha,n}$ denote the characteristic polynomial of $L_{\alpha,n}$, and U_n the n-th Chebyshev polynomial of the second kind.

Remark that for $\alpha = 1$ it is clear that $L_{1,n}$ is the particular case $A_{1,n}$ studied in Section 2.4. So, we now focus on the analysis of the eigenvalues and eigenvectors of $L_{0,n}$.

Proposition 3.36. *For α, λ in $\mathbb{R}$ with $\alpha = 0$ and $n \ge 3$,*

$$D_{0,n}(\lambda) = \lambda U_{n-1}\left(\frac{\lambda-2}{2}\right). \tag{3.78}$$

Proof. Substitute $\alpha = 0$ in (3.2), then (3.78) follows. □

Recall that g is the function defined in (2.2).

Proposition 3.37. *Let $\alpha = 0$ and $n \geq 3$. Then the eigenvalues of $L_{0,n}$ are*

$$\lambda_{0,n,j} = g\left(\frac{(j-1)\pi}{n}\right) \quad (1 \leq j \leq n). \tag{3.79}$$

Proof. It is evident that $\lambda = 0$ is a zero of (3.78). And after the change of variable $\lambda = g(x)$ we get from 1.5 that the numbers $(j-1)\pi/n$, where $2 \leq j \leq n$, are zeros of U_{n-1}, so these are zeros of (3.78). We have found n different zeros of (3.78), then the conclusion follows. □

Proposition 3.38 (eigenvectors of $L_{\alpha,n}$ with $\alpha = 0$)**.** *Let $\alpha = 0$ and $n \geq 3$. Then the vector $[1,\ldots,1]^\top$ is an eigenvector of the matrix $L_{0,n}$ associated to the eigenvalue $\lambda_{0,n,1} = 0$; and for every $2 \leq j \leq n$, the vector $v_{0,n,j} = [v_{\alpha,n,j,k}]_{k=1}^n$ with components*

$$v_{0,n,j,k} \coloneqq \cos\left(\left(k - \frac{1}{2}\right)\frac{(j-1)\pi}{2n}\right) \qquad (1 \leq k \leq n), \tag{3.80}$$

is an eigenvector of $L_{0,n}$ associated to $\lambda_{0,n,j}$.

Proof. That $[1,\ldots,1]^\top$ is an eigenvector associated to the eigenvalue $\lambda_{0,n,1} = 0$ follows from Proposition 3.21.

For $j \geq 2$, by (3.79), $\lambda_{0,n,j} = 2 - e^{i\vartheta_{0,n,j}} - e^{-i\vartheta_{0,n,j}}$ with $\vartheta_{0,n,j} = (j-1)\pi/n$. Recall that $\mathrm{b}_{0,n}$ is defined in (1.14), there $z = e^{i\vartheta_{0,n,j}}$, so

$$\mathrm{b}_{0,n} = 2i\sin(\vartheta_{0,n,j}) \neq 0.$$

Similar to the proof of Proposition 3.22, the conclusion follows by Proposition 1.16. □

Conclusion

In this thesis we considered the problem of finding asymptotic expansions for the eigenvalues of tridiagonal toeplitz matrices with corner perturbations, and solved this problem for two particular families of matrices, $A_{\alpha,n}$ and $L_{\alpha,n}$.

For each of these two families, we understood the localization of the eigenvalues of these matrices, i.e., we divided the real line into n disjoint intervals containing exactly one eigenvalue, and successfully developed asymptotic expansions for the eigenvalues of $A_{\alpha,n}$ and $L_{\alpha,n}$, as α is fixed and n tends to infinity. However, these asymptotic expansions are not uniform with respect to the parameter α.

We realized that the Chebyshev polynomials play an important role in the study of the spectrum of the matrices $A_{\alpha,n}$ and $L_{\alpha,n}$. In particular, the characteristic polynomial and eigenvectors are expressed in terms of these polynomials. Moreover, after writing the Chebyshev polynomials in terms of trigonometric or hyperbolic functions, we transformed the characteristic equation to a form convenient to solve by the fixed point iteration method.

We also showed that the eigenvectors of tridiagonal Toeplitz matrices with arbitrary corner perturbations can be found as linear combinations of geometric progressions; this approach is different from the methods found in literature.

Here are some natural goals for future investigations on this theme:

· prove that the characteristic equation, written in an appropriate form, can be solved by the Newton method, providing precise sufficient conditions and simple upper estimates for the convergence;

· understand the localization and find asymptotic expansions of the eigenvalues of tridiagonal Toeplitz matrices with four arbitrary perturbed corners or with block corner perturbations.

Appendix A

Some classical theory

The following propositions are part of the classic theory in real analysis, cf. [4, 25, 28].

Theorem A.1. *Let T a contraction from a metric space in itself,* $0 \leq \tau < 1$

$$d(T(x), T(y)) \leq \tau d(x, y)$$

then, T has a unique fixed point. Moreover if x_0 is any point in the space, and it is defined the following sequence (x_n) by $x_n = T(x_{n-1})$, then

$$\lim x_n = \bar{x}$$

and

$$d(\bar{x}, x_n) \leq \frac{\tau}{1-\tau} d(x_{n-1}, x_n) \leq \frac{\tau^n}{1-\tau} d(x_0, x_1)$$

Theorem A.2 (Intermediate Value Theorem)**.** *If f is continuous on $[a, b]$ and $f(a) > c > f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$.*

Theorem A.3 (Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proposition A.4 (fixed point for differentiable functions)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), with $f([a, b]) \subset [a, b]$ and $\|f'\|_\infty < 1$, then f is a contraction on $[a, b]$ and has a unique fixed point.*

Proof. From the Mean Value Theorem A.3 we have that, for every $x < y$ in $[a, b]$ there exists z in (x, y), such that the following holds

$$|f(y) - f(x)| = |f'(z)||y - x| \leq \|f'\|_\infty < |y - x|.$$

This implies that f is a contraction on $[a, b]$, and by the Theorem A.1 there exists a unique fixed point for f. □

Lemma A.5. *Let f_1 and f_2 be contractive functions on $[a, b]$ such that $f_1(x) < f_2(x)$ for every x in $[a, b]$. If x_1 and x_2 are the correspondent fixed points of f_1 and f_2, then $x_1 < x_2$.*

Proof. Let $0 \leq L_2 < 1$ be the Lipschitz constant of f_2. Since $f_1(x) < f_2(x)$ for every x in $[a, b]$, then $x_1 \neq x_2$. Suppose $x_1 > x_2$, then

$$L_2(x_1 - x_2) \geq f_2(x_1) - f_2(x_2) > f_1(x_1) - x_2 = x_1 - x_2.$$

This is a contradiction, so $x_1 < x_2$. □

Appendix B

Programs of numerical experiments for the eigenvalues approximations of $A_{\alpha,n}$

In this appendix we write the main codes developed for the numerical experiments on the eigenvalues approximations of the matrix $A_{\alpha,n}$ studied in Chapter 2.

Note that the codes in Sections B.1 and B.2 should belong to the same directory. We remark some notations and issues on the codes:

· the variable "al" denote the perturbation α of the matrix $A_{\alpha,n}$;
· the variable 'n' denote the dimension of the matrix $A_{\alpha,n}$;
· if x approaches π from the left, then $\cot(x)$ decreases rapidly to $-\infty$, this could imply computation overflow, so we decided to compute $\eta_{\alpha,j}$ (2.25) in terms of tan when x is greater than $\pi/2$;
· for the same overflow reasons we compute $\tanh \frac{nx}{2}$ as $1 - 2e^{-nx}/(1 + e^{-nx})$,
· the parameter 'prec' denotes the precision in bits used for computations;
· at the top of each code we type the name of the file.

B.1 Eigenvalues approximation of $A_{\alpha,n}$ with weak perturbations

Over all this section we suppose $|\alpha| < 1$.

The code B.1 has the next functions:

- "my_matrix" returns the matrix $A_{\alpha,n}$ in double-precision complex field;
- "eigenvalues_general_alg" returns the eigenvalues of $A_{\alpha,n}$, sorted in ascending order and computed by general algorithm with double-precision arithmetic.

Code B.1: Eigenvalues of $A_{\alpha,n}$ computed by general algorithm.

```
# File name: Off_eigenvalues_by_general_algorithm.sage

def my_matrix(al, n):
    v = vector(CDF, n)
    w = vector(CDF, n-1)
    v[0] = 2
    v[1] = -1
    v[n-1] = CDF(-al)
    w[0] = -1
    w[n-2] = CDF(-conjugate(al))
    return matrix.toeplitz(v, w, CDF)

def eigenvalues_general_alg(al, n):
    A = my_matrix(al, n)
    eigvals = A.eigenvalues()
    eigvals1 = sorted(eigvals)
    return vector(CDF, n, eigvals1)
```

The code B.2 has the next functions:

- "g" returns the function g defined in (2.2);
- "myparams" returns a list with entries α and the constants (2.11);
- "eta" returns the function $\eta_{\alpha,j}$ (2.25);
- "eta_fixed_point" returns the value $\vartheta_{\alpha,n,j}$ that satisfies (2.24) computed with precision determined by the precision of α;
- "la_fixed_point" returns the j-th eigenvalue of $A_{\alpha,n}$ computed by fixed iteration point, this algorithm is valid due to Proposition 2.13;
- "eigenvalues_fixed_point" returns the vector with components the eigenvalues of $A_{\alpha,n}$ computed by fixed iteration point.

Code B.2: Eigenvalues of $A_{\alpha,n}$ computed by iteration fixed point.

```
# File: Off_weak_eigenvalues_by_fixed_point.sage

def g(x):
    return 4*(sin(x/2))**2

def myparams(al):
    abs_al = abs(al)
    abs_al_plus_1 = abs(al+1)
    abs_al_min_1 = abs(al-1)
    coef_k = (1-abs_al**2)/abs_al_plus_1**2
    coef_l = abs_al_min_1/abs_al_plus_1
    return [al, coef_k, coef_l]

def eta(params,j,x):
    [al, coef_k, coef_l] = params
    if x > al.parent()(pi/2):
        coef_k_tan_x = tan(x)/coef_k
        sgnj = (-1)**(j)
        expr = sqrt(1+(coef_l*coef_k_tan_x)**2)
        expr1 = (-(sgnj+expr)/coef_k_tan_x)**sgnj
        return -2*arctan(expr1)
    else:
        coef_k_cotx = coef_k * cot(x)
        sgnj = (-1)**(j)
        expr = sqrt(coef_k_cotx**2+coef_l**2)-sgnj*coef_k_cotx
        expr1 = expr**sgnj
        return -2*arctan(expr1)

def tht_fixed_point(params, n, j):
    F = params[0].parent()
    tht0 = F(j*pi/n)
    thttemp = tht0
    er = F.epsilon()*16
    while (er>F.epsilon()*8) :
        thtprev = thttemp
        thttemp = F(tht0+eta(params,j,thttemp)/n )
        er = abs(thttemp-thtprev)
    return thttemp
```

```
def la_fixed_point(params,n, j):
    return g(tht_fixed_point(params, n, j))

def eigenvalues_fixed_point(al, n, prec):
    al = ComplexField(prec)(al)
    params = myparams(al)
    eigvals = [la_fixed_point(params,n, j+1) for j in range(n)]
    return vector(al.parent(), n, eigvals)
```

The code B.3 has the function "max_error_eigenvalues_gen_minus_fp" that tests Proposition 2.13, it prints on the command window the maximal error of the difference between the eigenvalues computed by general algorithms and the fixed iteration point.

Code B.3: Test of the eigenvalues of $A_{\alpha,n}$ computed by iteration fixed point.

```
# File name: Off_weak_test_eigenvalues_by_fixed_point.sage

load('Off_weak_eigenvalues_by_fixed_point.sage')
load('Off_eigenvalues_by_general_algorithm.sage')

def max_error_eigenvalues_gen_minus_fp(al, n, prec):
    eigvals_gen = eigenvalues_general_alg(al, n)
    temp_eigvals_fp = eigenvalues_fixed_point(al, n, prec)
    eigvals_fp = vector(CDF,temp_eigvals_fp)
    eigvals_dif = eigvals_gen-eigvals_fp
    er = eigvals_dif.norm(Infinity)
    str_er = '%.3e' % er
    print('test eigenvalues by fixed point iteration')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er + '\n')
```

The Code B.4 has the next functions:

- "gder1" and "gder1" return the first and second derivative of the function g (2.2);
- "etader" returns the derivative $\eta'_{\alpha,j}$ (2.26);
- "la_asymp" returns the j-th eigenvalue of $A_{\alpha,n}$ computed by asymptotic expansion (2.36);
- "eigenvalues_asymp" return a vector with components the eigenvalues of $A_{\alpha,n}$ computed by asymptotic expansion.

Code B.4: Eigenvalues of $A_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: Off_weak_eigenvalues_by_asmyp.sage

load('Off_weak_eigenvalues_by_fixed_point.sage')

def gder1(x):
    return 2*sin(x)

def gder2(x):
    return 2*cos(x)

def etader(params,j,x):
    [al, coef_k, coef_l] = params
    if x>al.parent()(pi/2):
        coef_l_tan_x = coef_l*tan(x)
        raiz = sqrt(coef_k**2+coef_l_tan_x**2)
        sgnj = (-1)**(j+1)
        num = -2*coef_k*(1-sgnj*coef_k/raiz)*(1+tan(x)**2)
        den = tan(x)**2+(-sgnj*coef_k+raiz )**2
        return num/den
    else:
        raiz = sqrt((coef_k*cot(x))**2+coef_l**2)
        sgnj = (-1)**(j+1)
        num = -sgnj*(2/(sin(x)**2))*(sgnj*coef_k+coef_k**2*cot(x)/raiz)
        den = 1+(sgnj*coef_k*cot(x)+raiz)**2
        return num/den

def la_asymp(params, n, j):
    F = params[0].parent()
    thtj = F(j*pi/n)
    c1 = eta(params, j, thtj)
    c2 = c1*etader(params, j, thtj)
    g1 = gder1(thtj)
    g2 = gder2(thtj)
    d0 = g(thtj)
    d1 = g1*c1/n
    d2 = (g1*c2+(1/2)*g2*c1**2)/(n**2)
    return d0+d1+d2

def eigenvalues_asymp(al, n, prec):
```

```
    al = ComplexField(prec)(al)
    params = myparams(al)
    return vector(al.parent(), [la_asymp(params,n,j+1) for j in range(n)])
```

The Code B.5 has the function "max_error_eigenvalues_asymp_minus_fp" that tests the Theorem 2.15, it prints on the command window the maximal and normalized errors of the difference between the eigenvalues of $A_{\alpha,n}$ computed by fixed point iteration and asymptotic expansion.

Code B.5: Test of the eigenvalues of $A_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: Off_weak_test_eigenvalues_by_asymp.sage

load('Off_weak_eigenvalues_by_asmyp.sage')

def max_error_eigenvalues_asymp_minus_fp(al, n, prec):
    eigvals_fp = eigenvalues_fixed_point(al, n, prec)
    eigvals_asymp = eigenvalues_asymp(al, n, prec)
    eigvals_dif = eigvals_fp-eigvals_asymp
    er = eigvals_dif.norm(infinity)
    er_norm = er*n**3
    str_er = '%.3e' % er
    str_norm_er = '%.3e' % er_norm
    print('test eigenvalues by asymptotic expansion')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er)
    print('normalized error = ' + str_norm_er + '\n')
```

B.2 Eigenvalues approximation of $A_{\alpha,n}$ with strong perturbations

Over all this section we suppose $|\alpha| > 1$.

The Code B.6 has the next functions:

- "g_minus" and "g_plus" respectively return the functions g_- and g_+ defined in (2.2);
- "phi_1" and "phi_n" respectively return the functions on the right of (2.45) and (2.46);

- "tht_fixed_point_1" and "tht_fixed_point_n" return respectively $\vartheta_{\alpha,n,1}$ and $\vartheta_{\alpha,n,n}$, the numbers that satisfies respectively (2.45) and (2.46), computed by fixed point iteration;
- "la_fixed_point_1" and "la_fixed_point_n" respectively return the eigenvalues $\lambda_{\alpha,n,1}$ and $\lambda_{\alpha,n,n}$ of $A_{\alpha,n}$ computed by fixed point iteration, this algorithm is valid due to Proposition 2.20;
- "extreme_eigenvalues_fixed_point" return the extreme eigenvalues of $A_{\alpha,n}$ computed by fixed point iteration.

Code B.6: Extreme eigenvalues of $A_{\alpha,n}$ computed by fixed point iteration.

```
# File name: Off_strong_extreme_eigvalues_by_fixed_point.sage

load('Off_eigenvalues_by_general_algorithm.sage')
load('Off_weak_eigenvalues_by_fixed_point.sage')

def g_minus(x):
    return -4*(sinh(x/2))**2

def g_plus(x):
    return 4+4*(sinh(x/2))**2

def phi_1(params, n, x):
    [al, coef_k, coef_l] = params
    e_min_nx = e**(-n*x)
    tanh_x = 1-2*e_min_nx/(1+e_min_nx)
    num = -2*coef_k*tanh_x
    den = tanh_x**2+coef_l**2
    return arctanh(num/den)

def phi_n(params, n, x):
    [al, coef_k, coef_l] = params
    e_min_nx = e**(-n*x)
    tanh_x = 1-2*e_min_nx/(1+e_min_nx)
    sgn_n = (-1)**n
    abs_al_plus_sign = abs(al+sgn_n)
    abs_al_minus_sign = abs(al-sgn_n)
    coef_k_signed = (1-abs(al)**2)/abs_al_plus_sign**2
    coef_l_signed = abs_al_minus_sign/abs_al_plus_sign
```

```
    num = −2*coef_k_signed*tanh_x
    den = tanh_x**2+coef_l_signed**2
    return arctanh(num/den)

def tht_fixed_point_1(params, n):
    al = params[0]
    F = al.parent()
    thttemp = F(log(abs(al)))
    er = F.epsilon()*16
    while (er>F.epsilon()*8):
        thtprev = thttemp
        thttemp = F(phi_1(params, n, thttemp))
        er = abs(thttemp−thtprev)
    return thttemp

def la_fixed_point_1(params, n):
    return g_minus(real(tht_fixed_point_1(params, n)))

def tht_fixed_point_n(params, n):
    al = params[0]
    F = al.parent()
    thttemp = F(log(abs(al)))
    er = F.epsilon()*16
    while (er>F.epsilon()*8):
        thtprev = thttemp
        thttemp = F(phi_n(params, n, thttemp))
        er = abs(thttemp−thtprev)
    return thttemp

def la_fixed_point_n(params,n):
    return g_plus(real(tht_fixed_point_n(params, n)))

def extreme_eigenvalues_fixed_point(al, n, prec):
        al = ComplexField(prec)(al)
        params = myparams(al)
        la_1 = la_fixed_point_1(params, n)
        la_n = la_fixed_point_n(params, n)
        return vector(al.parent(), 2, [la_1,la_n])
```

The Code B.7 has the function "max_error_extreme_eigenvalues_gen_minus_fp" that tests the Proposition 2.20, it prints on the command window the maximal error of the difference between the extreme eigenvalues of $A_{\alpha,n}$ computed by general algorithm and fixed point iteration.

Code B.7: Test of the extreme eigenvalues of $A_{\alpha,n}$ computed by fixed point iteration.

```
# File name: Off_strong_test_extreme_eigvalues_by_fixed_point.sage

load('Off_strong_extreme_eigvalues_by_fixed_point.sage')

def max_error_extreme_eigenvalues_gen_minus_fp(al, n, prec):
    eigvals_gen = eigenvalues_general_alg(al, n)
    extreme_eigvals_gen = vector(CDF, [eigvals_gen[0], eigvals_gen[n-1]])
    temp_extreme_eigvals_fp = extreme_eigenvalues_fixed_point(al, n, prec)
    extreme_eigvals_fp = vector(CDF, temp_extreme_eigvals_fp)
    eigvals_dif = extreme_eigvals_gen-extreme_eigvals_fp
    er = eigvals_dif.norm(Infinity)
    str_er = '%.3e' % er
    print('test extreme eigenvalues by fixed point iteration')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er + '\n')
```

The Code B.8 has the function:

- "extreme_eigenvalues_asymp" return the extreme eigenvalues of $A_{\alpha,n}$ computed by the formulas $-s_\alpha$ and $4 + s_\alpha$ shown in (2.60) and (2.61), respectively.

Code B.8: Extreme eigenvalues of $A_{\alpha,n}$ computed by fixed asymptotic expansion.

```
# File name: Off_strong_extreme_eigvalues_by_asymp.sage

load('Off_strong_extreme_eigvalues_by_fixed_point.sage')

def extreme_eigenvalues_asymp(al, n, prec):
    al = ComplexField(prec)(al)
    abs_al = abs(al)
    x0 = log(abs_al)
    la_1 = g_minus(x0)
    la_n = g_plus(x0)
    return vector(al.parent(), 2, [la_1,la_n])
```

The Code B.9 has the functions "max_error_extreme_eigenvalues_asymp_minus _fp" that tests Theorem 2.22, it prints on the command window the maximal error of the difference between the extreme eigenvalues of $A_{\alpha,n}$ computed by iteration of fixed point and asymptotic expansion.

Code B.9: Test of the extreme eigenvalues of $A_{\alpha,n}$ computed by fixed asymptotic expansion.

```
# File name: Off_strong_test_extreme_eigvalues_by_asymp.sage

load('Off_strong_extreme_eigvalues_by_asymp.sage')

def max_error_extreme_eigenvalues_asymp_minus_fp(al, n, prec):
    extreme_eigvals_fp = extreme_eigenvalues_fixed_point(al, n, prec)
    extreme_eigvals_asymp = extreme_eigenvalues_asymp(al, n, prec)
    eigvals_dif = extreme_eigvals_asymp-extreme_eigvals_fp
    er = eigvals_dif.norm(Infinity)
    norm_er = er*abs(al)**n
    str_er = '%.3e' % er
    str_norm_er = '%.3e' % norm_er
    print('test extreme eigenvalues by asymptotic expansion')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er + '\n')
    print('normalized error = ' + str_norm_er + '\n')
```

Appendix C

Programs of numerical experiments for the eigenvalues approximations of $L_{\alpha,n}$

In this appendix we write the main codes developed for the numerical experiments on eigenvalues approximations of the matrix $L_{\alpha,n}$ studied in Chapter 3.

Note that the codes in Sections C.1 and C.2 should belong to the same directory. We remark some notations and issues on the codes:

- the variable "al" denote the perturbation α of the matrix $A_{\alpha,n}$;
- the variable 'n' denote the dimension of the matrix $A_{\alpha,n}$
- since $\tanh \frac{nx}{2}$ increases rapidly to 1, in some cases this could imply computation overflow, so we decided to transform it in the codes to the form $1 - 2e^{-nx}/(1 + e^{-nx})$.
- the parameter 'prec' denotes the precision in bits used for computations;
- at the top of each code we type the name of the file.

C.1 Eigenvalues approximation of $L_{\alpha,n}$ with weak perturbations

Over all the section we suppose $0 < \alpha < 1$.

The Code C.1 has the next functions:

- "my_matrix" returns the matrix $L_{\alpha,n}$;
- "eigenvalues_general_alg" returns a vector with components the eigenvalues of $L_{\alpha,n}$ computed by general algorithm.

Code C.1: Eigenvalues of $L_{\alpha,n}$ computed by general algorithm.

```
# File name: L_eigenvalues_by_general_algorithm.sage

def my_matrix(al, n):
    v = vector(CDF, n)
    w = vector(CDF, n-1)
    v[0] = 2
    v[1] = -1
    v[n-1] = CDF(-al)
    w[0] = -1
    w[n-2] = CDF(-conjugate(al))
    T = matrix.toeplitz(v, w, CDF)
    T[0,0] = CDF(1 + conjugate(al))
    T[n-1,n-1] = CDF(1+al)
    return T

def eigenvalues_general_alg(al, n):
    A = my_matrix(al, n)
    eigvals = A.eigenvalues()
    eigvals1 = sorted(eigvals)
    return vector(CDF, n, eigvals1)
```

The Code C.2 has the next functions:

- "g" returns the function g defined in (2.2);
- "myparams" returns a list with components, α, $\mathrm{Re}(\alpha)$ and κ_α (3.14);
- "zeta" returns the function $\zeta_{\alpha,n}$ (3.41);
- "tht_fixed_point" returns the number $\vartheta_{\alpha,n,j}$ that satisfies (3.40) computed by fixed point iteration;
- "la_fixed_point" returns the j-th eigenvalue of $L_{\alpha,n}$ computed by fixed point iteration, this algorithm is valid due to Proposition 3.18;
- "eigenvalues_fixed_point" returns a vector with components the eigenvalues of $A_{\alpha,n}$ computed by fixed point iteration.

Code C.2: Eigenvalues of $L_{\alpha,n}$ computed by fixed point iteration.

```
# File name: L_weak_eigenvalues_by_fixed_point.sage

def g(x):
    return 4*(sin(x/2))**2

def myparams(al):
    al_real = real(al)
    coef_kappa = (1-al_real)/al_real
    return [al, al_real, coef_kappa]

def zeta(params, j, x):
    [al, al_real, coef_kappa] = params
    if mod(j,2) == 1:
        return 0
    else:
        if x>al.parent()(pi):
            coef_kappa_cotx = cot(x/2)/coef_kappa
            return -pi+2*arctan(coef_kappa_cotx)
        else:
            coef_kappa_tanx = coef_kappa*tan(x/2)
            return -2*arctan(coef_kappa_tanx)

def tht_fixed_point(params, n, j):
    F = params[0].parent()
    if mod(j,2) == 1:
        return F((j-1)*pi/n)
    else:
        tht0 = F(j*pi/n)
        thttemp = tht0
        er = F.epsilon()*16
        while (er>F.epsilon()*8) :
            thtprev = thttemp
            thttemp = F(tht0+zeta(params,j,thttemp)/n)
            er = abs(thttemp-thtprev)
            return thttemp

def la_fixed_point(params, n, j):
    return g(tht_fixed_point(params, n, j))
```

```
def eigenvalues_fixed_point(al, n, prec):
    al = ComplexField(prec)(al)
    params = myparams(al)
    eigvals = [la_fixed_point(params,n, j+1) for j in range(n)]
    return vector(al.parent(), n, eigvals)
```

The Code C.3 has the next function "max_error_eigenvalues_gen_minus_fp" that tests Proposition 3.18, it prints on the command window the maximal error of the difference between the eigenvalues of $L_{\alpha,n}$ computed by general algorithm and fixed point iteration.

Code C.3: Test of the eigenvalues of $L_{\alpha,n}$ computed by fixed point iteration.

```
# File name: L_weak_test_eigenvalues_by_fixed_point.sage

load('L_eigenvalues_by_general_algorithm.sage')
load('L_weak_eigenvalues_by_fixed_point.sage')

def max_error_eigenvalues_gen_minus_fp(al, n, prec):
    eigvals_gen = eigenvalues_general_alg(al, n)
    temp_eigvals_fp = eigenvalues_fixed_point(al, n, prec)
    eigvals_fp = vector(CDF, temp_eigvals_fp)
    eigvals_dif = eigvals_gen-eigvals_fp
    er = eigvals_dif.norm(Infinity)
    str_er = '%.3e' % er
    print('test eigenvalues by fixed point iteration')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er + '\n')
```

The Code C.4 has the next functions:

- "gder1" and "gder2" return respectively the first and second derivatives of the function g (2.2);
- "zetader" returns the derivative $\zeta'_{\alpha,n}$ (3.43);
- "la_asymp" returns the j-th eigenvalue of $L_{\alpha,n}$ computed by asymptotic expansion (3.50);
- "eigvalues_asymp" returns a vector with components the eigenvalues of $L_{\alpha,n}$ computed by asymptotic expansion.

Code C.4: Eigenvalues of $L_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: L_weak_eigenvalues_by_asymp.sage

load('L_weak_eigenvalues_by_fixed_point.sage')

def gder1(x):
    return 2*sin(x)

def gder2(x):
    return 2*cos(x)

def zetader(params, j, x):
    [al, al_real, coef_kappa] = params
    return -coef_kappa/(cos(x/2)**2+(coef_kappa*sin(x/2))**2)

def la_asymp(params, n, j):
    [al, abs_al, coef_k] = params
    F = params[0].parent()
    if mod(j,2) == 1:
        return F(g((j-1)*pi/n))
    else:
        thtj = F(j*pi/n)
        zeta0 = zeta(params, j, thtj)
        zeta1 = zetader(params, j, thtj)
        c1 = F(g(thtj))
        c2 = F(gder1(thtj)*zeta0/n)
        c3 = F(gder1(thtj)*zeta0*zeta1/(n**2))
        c4 = F((1/2)*gder2(thtj)*zeta0**2/(n**2))
        return c1+c2+c3+c4

def eigvalues_asymp(al, n, prec):
    al = ComplexField(prec)(al)
    params = myparams(al)
    eigvals = [la_asymp(params, n, j+1) for j in range(n)]
    return vector(al.parent(), eigvals)
```

The Code C.5 has the next function "max_error_eigenvalues_asymp_min_fp" that tests Theorem 3.20, it prints on the command window the maximal and normalized errors of the difference between the eigenvalues of $L_{\alpha,n}$ computed by asymptotic expansion and fixed iteration point.

Code C.5: Test of the eigenvalues of $L_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: L_weak_test_eigenvalues_by_asymp.sage

load('L_weak_eigenvalues_by_asymp.sage')

def max_error_eigenvalues_asymp_minus_fp(al, n, prec):
    eigvals_fp = eigenvalues_fixed_point(al, n, prec)
    eigvals_asymp = eigvalues_asymp(al, n, prec)
    eigvals_dif = eigvals_asymp-eigvals_fp
    er = eigvals_dif.norm(Infinity)
    er_norm = er*n**3
    str_er = '%.3e' % er
    str_norm_er = '%.3e' % er_norm
    print('test eigenvalues by asymptotic expansion')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er)
    print('normalized error = ' + str_norm_er + '\n')
```

C.2 Eigenvalues approximation of $L_{\alpha,n}$ with strong perturbations

On this section we consider $\alpha < 0$.

The Code C.6 has the next functions:

- "gmin" returns the function g_- defined in (2.2);
- "phi_1" returns $\varphi_{\alpha,n}$ (3.58);
- "tht_fp_extreme" returns the number $\vartheta_{\alpha,n}$ that satisfies (3.56) computed by fixed point iteration;
- "la_extreme_1_fixed_point" returns the first eigenvalue computed by iteration of fixed point, this algorithm is valid due to Proposition 3.24.

Code C.6: Extreme eigenvalue $\lambda_{\alpha,n,1}$ of $L_{\alpha,n}$ computed by fixed point iteration.

```
# File name: L_strong_extreme_eigenvalues_by_fixed_point.sage
```

```
load('L_weak_eigenvalues_by_fixed_point.sage')

def gmin(x):
    return 2-2*cosh(x)

def phi_1(params, n, x):
    F = params[0].parent()
    [al, al_real, coef_kappa] = params
    abs_kappa = abs(coef_kappa)
    e_min_nx = e**(-n*x)
    tanh_x = 1-2*e_nx/(1+e_min_nx)
    return 2*arctanh(tanh_x/abs_kappa)

def tht_fp_extreme(params, n):
    F = params[0].parent()
    [al, al_real,coef_kappa] = params
    thttemp = F(log(abs(2*al_real-1)))
    cnt = 0
    er = F.epsilon()*16
    while (er>F.epsilon()*8) :
        thtprev = thttemp
        thttemp = F(phi_1(params, n, thttemp))
        er = abs(thttemp-thtprev)
    return thttemp

def la_extreme_1_fixed_point(al, n, prec):
    al = ComplexField(prec)(al)
    params = myparams(al)
    return gmin(tht_fp_extreme(params, n))
```

The Code C.7 has the function "max_error_first_eigenvalue_gen_minus_fp" that tests Proposition 3.24, it prints on the command window the maximal error of the difference between the first eigenvalue of $L_{\alpha,n}$ computed by general algorithm and fixed iteration point.

Code C.7: Test of the extreme eigenvalue $\lambda_{\alpha,n,1}$ of $L_{\alpha,n}$ computed by fixed point iteration.

```
# File name: L_strong_test_extreme_eigenvalues_by_fixed_point.sage

load('L_eigenvalues_by_general_algorithm.sage')
```

```
load('L_strong_extreme_eigenvalues_by_fixed_point.sage')

def max_error_first_eigenvalue_gen_minus_fp(al, n, prec):
    eigvals_gen = eigenvalues_general_alg(al, n)
    extreme_1_eigbal_gen = eigvals_gen[0]
    extreme_1_eigval_fp = CDF(la_extreme_1_fixed_point(al, n, prec))
    eigvals_dif = extreme_1_eigval_gen-extreme_1_eigval_fp
    er = abs(eigvals_dif)
    str_er = '%.3e' % er
    print('test eigenvalues by fixed point iteration')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er + '\n')
```

The Code C.8 has the functions:

- "la_extreme_asymp" returns the first eigenvalue of $L_{\alpha,n}$ computed by asymptotic expansion (3.61).
- "extreme_eigenvalue_asymp" returns the firs eigenvalue of $L_{\alpha,n}$ computed by asymptotic expansion.

Code C.8: Extreme eigenvalue $\lambda_{\alpha,n,1}$ of $L_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: L_strong_extreme_eigenvalu
es_by_asymp.sage

load('L_strong_extreme_eigenvalues_by_fixed_point.sage')

def la_extreme_asymp(params, n):
    [al, al_real, coef_kappa] = params
    return params[0].parent()((log(abs(2*al_real-1))))

def extreme_eigenvalue_asymp(al, n, prec):
    al = ComplexField(prec)(al)
    params = myparams(al)
    return gmin(la_extreme_asymp(params, n))
```

The Code C.9 has the function "max_error_eigenvalues_gen_minus _fp" that tests Theorem 3.26, it prints on the command window the error and normalized error of the difference between, the first eigenvalue of $L_{\alpha,n}$ computed by asymptotic expansion and fixed point iteration.

Code C.9: Test of the extreme eigenvalue $\lambda_{\alpha,n,1}$ of $L_{\alpha,n}$ computed by asymptotic expansion.

```
# File name: L_strong_test_extreme_eigenvalues_by_asymp.sage

load('L_strong_extreme_eigenvalues_by_asymp.sage')

def max_error_eigenvalues_gen_minus_fp(al, n, prec):
    extreme_1_eigval_fp = la_extreme_1_fixed_point(al, n,prec)
    extreme_1_eigval_asymp = extreme_eigenvalue_asymp(al, n, prec)
    eigvals_dif = extreme_1_eigval_asymp-extreme_1_eigval_fp
    er = abs(eigvals_dif)
    er_norm = er*abs(2*al-1)**n
    str_er = '%.3e' % er
    str_norm_er = '%.3e' % er_norm
    print('test extreme eigenvalue by asymptotic expansion')
    print('n = ' + str(n))
    print('al = ' + str(al))
    print('maximal error = ' + str_er)
    print('normalized error = ' + str_norm_er + '\n')
```

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